Dynamics of a Perturbed Circular Membrane

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1 Introduction

The sweet-sounding pitch of a plucked guitar string is not merely a subjective distinction, but rather founded on the physical fact that the standing waves, or modes, that form along the string possess frequencies that are an integral multiple of the lowest, fundamental frequency. Then, why is it that the natural two-dimensional extension of this system - a circular drum membrane - produces a cacophonous clash sound instead?

Clearly, the dynamics of a string cannot be willfully projected to that of the 2D case. In order to faithfully model the motion of a drum surface under any arbtirary pertubration, we must trace back to the governing principle of all wave systems: the wave equation.

2 Method

The 2D wave equation for a wave with propagation speed c is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

where u is the perpendicular displacement of the wave on the plane.

The symmetry of the circular membrane case justifies the expression of (1) in cylindrical form, as parametrised by polar coordinates (r, θ):

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) \tag{2}$$

Under the appropriate boundary conditions[1], the solutions to the wave equation above for a drum of radius a are modes given in terms of Bessel Functions of the first kind J_m and parametrised by integers (m, n) such that:

$$u_{mn}(r, \theta, t) = J_m(\lambda_{mn} r)(a_{mn} cos(m\theta) + b_{mn} sin(m\theta))(cos(c\lambda_{mn} t) + sin(c\lambda_{mn} t)) \tag{3}$$

Where $\lambda_{mn} = \frac{\alpha_{mn}}{a}$ and α_{mn} is the n^{th} zero of the m^{th} order Bessel Function J_m, so that $(m, n) \in \mathbb{Z} \times \mathbb{Z}^+$.

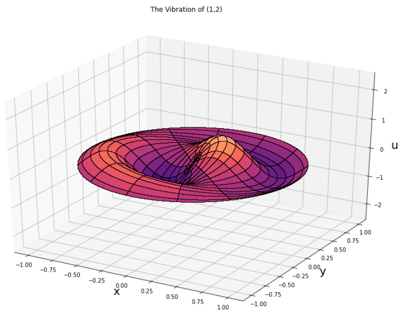
Provided the initial shape of the drum $u(r, \theta, 0) = f(r, \theta)$ and zero initial velocity $\dot{u}_{mn}(r, \theta, 0) = 0$, the general solution involves a superposition of all the normal modes of the drum [2] , and simplifies to:

$$u(r, \theta, t) = \sum_{m=0}^{\infty} a_{0n} J_0(\lambda_{0n} r) cos(c\lambda_{0n} t) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)(a_{mn} cos(m\theta) + b_{mn} sin(m\theta))(cos(c\lambda_{mn} t) \tag{4}$$

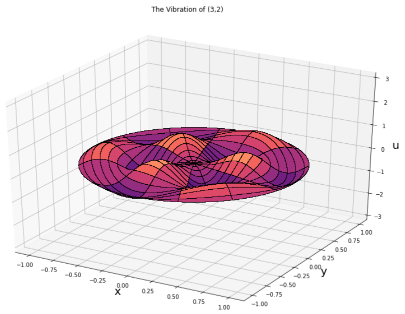
$$\begin{cases} a_{0n} = \frac{1}{\pi a^2 J_1^2(\alpha_{0n})} \int_0^{2\pi} \int_0^a f(r, \theta) J_0(\lambda_{0n} r) r \, \mathrm{d}r\mathrm{d}\theta \\ a_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a f(r, \theta) J_m(\lambda_{mn} r) cos(m\theta) r \, \mathrm{d}r\mathrm{d}\theta \\ b_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a f(r, \theta) J_m(\lambda_{mn} r) sin(m\theta) r \, \mathrm{d}r\mathrm{d}\theta \end{cases}$$

While it is compututionally impossible to encapsulate the infinite multiplicity of modes contributing to the general solution, an approximation may be obtained by summation of the first $(m_{\text{max}}, n_{\text{max}})$ terms. Additionally, for an axisymmetric disturbance $f(r, \theta) = f(r)$, the second term in (4) dissolves as the integrals defining the amplitude coefficients simplify to zero.

[1] Please refer to theory pdfs in the zip file for derivation details or alternatively this link
[2] General solution derivation



(a) $(m, n) = (1, 2)$



(b) $(m, n) = (3, 2)$

Figure 1: Two possible combinations of n circular and m diametral nodes that characterise a mode shape

3 Notable Results

3.1 Mode Parametrisation

We observe in Figure 1, that the parameters m and n greatly influence the structure of a vibrational mode. Specifically, m and n are shown to denote the number of diametral and circular nodes present respectively[3]. As such, 1a expectedly displays two concentric circular nodes (one at the rim; another around midway from the centre) being bisected by a single diametral node. The corresponding case also follows for 1b.

3.2 Inharmonicity

The natural frequencies on a guitar string are related to each other by integral multiples. In the case of the circular drum , however, we see in Figure 2 that the ratio of the first few frequencies to the fundamental $(1, 0)$ are not integers. This indicates why a drum does not have a distinct pitch: the frequency of its modes do not belong to the harmonic series and are instead, irregular and non-integer multiples of the fundamental. It is this inhamonicity of the natural frequencies that results in the characterstic discordant thump sound generated by a drum.

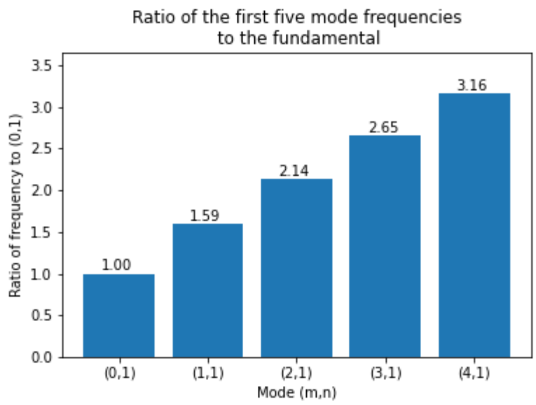


Figure 2

3.3 General Solution: Modes from a Pertubration

Recall from (4) that the general solution is comprised of the superposition of all modes. However, the amplitude or strength of a mode's contribution to that superposition is variable based on the initial condition. To discover the underlying relationship between the two, an initial condition was defined to simulate the striking of a drum at a localised point [2] located at radial position r from the centre and (without loss of generality) $\theta = 0$

The blue curve in Figure 3 shows the effect of changing this radial position on the strengths of four modes in contributing to the superposition of (4). For example, in the $(0, 2)$ case, we observe that the mode's intensity disappears when the drum is struck at approximately radial position $r = 0.43$m. The reason for this can be understood by comparing with the singular mode's natural oscillation at the point (orange), which in this case reveals a stationary node.

[3]please refer to the included Jupyter Notebook for detailed animations and a more thorough investigation

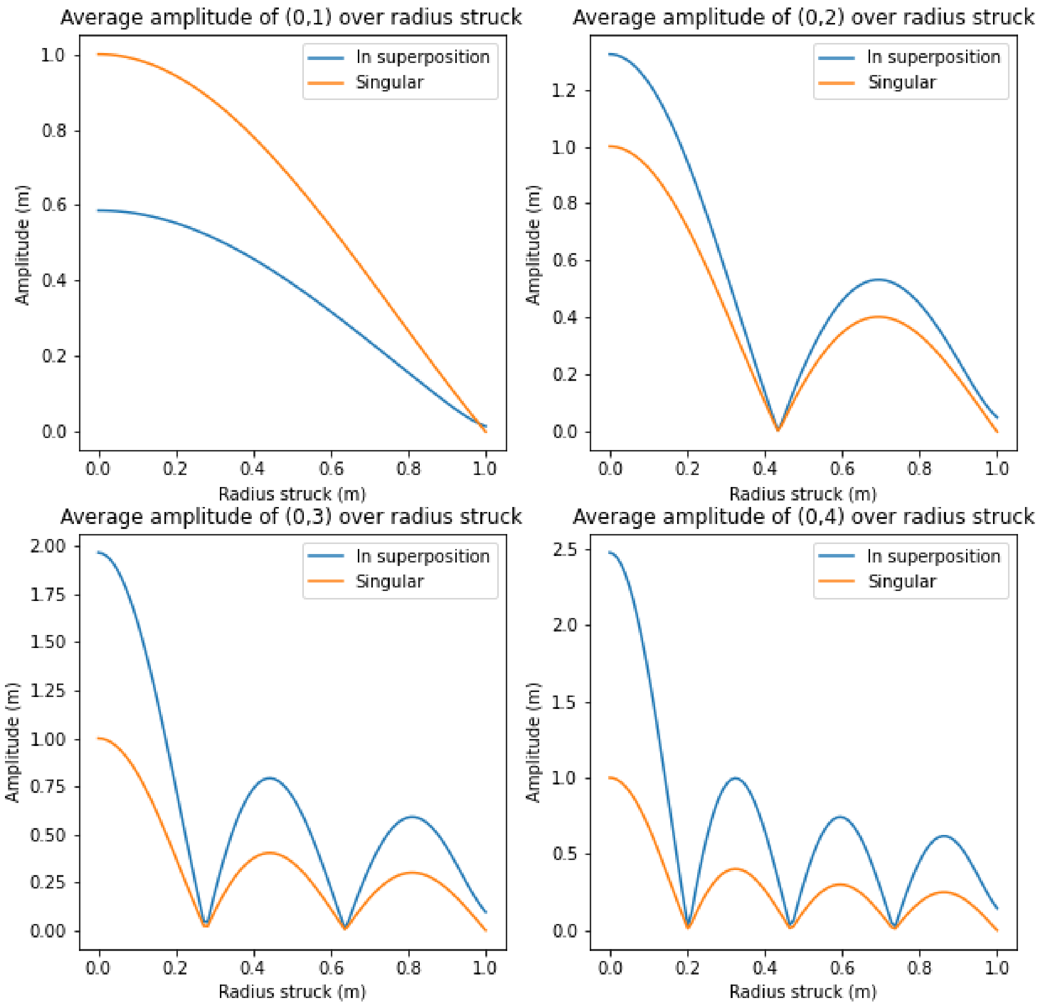


Figure 3: Amplitudes of several modes over a range of radial strike positions (orange) compared to the absolute value of the mode's inherent displacement over the radius of the drum. Drum radius = 1m

In fact, the equivalent shape of the two curves (up to only a dilation factor) reveals that the degree to which a mode is excited upon the drum being struck at a point depends on how intensely the mode naturally oscillates at that point. Moreover, this suggests that axisymmetric petrubrations excite only modes of the form $(0, n)$ as such pertubrations (by definition) cannot involve a stationary diameter, which is a condition required for any general (m, n) to be excited.

4 Conclusion

Simulation of a perturbed circular membrane reveals the presence of discrete modes that are characterised by the presence of circular and diametral nodes, and parameterised by terms n and m respectively. While analogous to standing waves on a plucked guitar string, these modes depart from the 1D case in that they are inharmonic - non-integer multiples of the fundamental freqeuncy - which explains the indistinct thump of a drum. Finally, while the general solution comprises a superposition of all modes, the extent to which any particular mode is activated during an arbitrary pertubration depends on how intensely that mode characteristically oscillates at the position of the strike.