The Impact of Aggregate Shocks on Firms with Heterogeneous Productivities

Ignacio Rojas

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Abstract

This study develops a model of firms characterized by heterogeneous productivities, examining the effects of an aggregate productivity shock. The findings suggest that this model better aligns with aggregated investment and output data compared to alternative frameworks. Furthermore, it predicts that firms with lower productivity levels are more severely affected by aggregate shocks, both in terms of deviations from their steady state and in the persistence of these deviations.

1 Introduction

Microeconomic empirical evidence consistently provides evidence of persistent heterogeneity among households and firms. For households, heterogeneity in consumption, wealth, and income have been widely documented in works such as Achdou (2016), Auclert (2017), Buera (2015), Kaplan et al. (2018), and Moll (2014). Similarly, firms exhibit differences in productivity, investment levels, and other characteristics, as highlighted in studies by Bachmann et al. (2013), Khan and Thomas (2008, 2013), Terry (2017), and Winberry (2018). The common approach in economics to analyze these phenomena involves discrete-time modeling. While this method creates a familiar framework for economists, it is computationally expensive due to the need to have a state vector capturing the distribution of agents or firms.

This work develops a model for heterogeneous firms that exhibit idiosyncratic productivities dependent on their historical paths. The model simplifies firms' investment decisions by considering their capital and productivity states. First, the steady state of the economy is analyzed using comparative statics. Next, the study evaluates firms' responses to an aggregate productivity shock, employing the computationally efficient algorithm of Ahn et al. (2017). This approach significantly reduces computational costs.

Both the steady-state analysis and the shock simulations reveal various nonlinear and relevant effects across the continuum of firm productivities. These effects are not thoroughly explored in existing literature. This thesis uses an efficient method to analyze such heterogeneous effects along the entire distribution of productivities and capital. It also shows that lower-productivity firms are more exposed to aggregate shocks.

Earlier research on continuous-time heterogeneous-agent models builds on foundational works by Aiyagari (1994), Bewley (1986), and Huggett (1993). These models underpin the literature on incomplete markets, often incorporating a borrowing constraint that limits households' negative assets (debt). Achdou (2016), Buera (2015), Kaplan et al. (2018), and Moll (2014) extend these models to investigate questions about inequality, monetary policy, and misallocation. Recent work by Song et al. (2019) provides evidence that labor income inequality has increased, primarily due to wage disparities between firms, driven by rising segregation. This finding suggests that shocks affecting firms could have profound implications for inequality, opening a channel to study the effects of firms models to answer inequality questions.

Motivated by these insights, this thesis contributes to the literature by extending computational advancements to a model with heterogeneous firms. A simplified investment decision framework is used, featuring convex investment costs and a positive capital constraint in every period. Following the irrelevance result established by Khan and Thomas (2008), the aggregate investment can be represented within a model incorporating convex costs. This is an RBC model keeping nominal prices out of the scope as does Bachmann et al. (2013), Bloom (2007) and Kahn and Thomas (2008).

The model is written in continuous time, offering computational advantages over discrete-time alternatives. Using continuous-time formulations allows the adoption of Ahn et al.'s (2017) algorithm to efficiently solve for optimal policy functions and steady-state distributions. The time-continuous approach reduces the problem to two partial differential equations: the Hamilton-Jacobi-Bellman (HJB) equation for individual firm optimization and the Kolmogorov Forward Equation (KFE) for firm distribution dynamics.

Productivity levels in this framework follow a diffusion process, resulting in a continuum of production levels discretized into a sufficiently dense grid. While earlier models often rely on simplified agent types, such as the two-agent framework introduced by Campbell and Mankiw (1989), this thesis examines whether such simplifications adequately capture firm dynamics in both steady-state and aggregate shock scenarios. First I am going to consider firms with two discrete productivity levels, high and low, transitioning between states through a Poisson process. This can represent transitions between small-medium enterprises (SMEs) and large firms. The second approach incorporates a continuous distribution of productivity levels, allowing for greater granularity. While the Poisson process provides a reasonable approximation of the diffusion model in steady state, it exhibits limitations when incorporating aggregate shocks.

Parameterization closely follows Khan and Thomas (2008) and Bachmann et al. (2013), with adjustments to cost structure parameters. Convex investment costs are calibrated to achieve an average investment rate of approximately 20%, consistent with empirical data from King and Rebelo (1999). This calibration ensures that simulated aggregate variables align closely with observed data in terms of standard deviations and correlations.

Static analysis reveals nonlinear effects across the productivity spectrum. When aggregate productivity shocks are introduced, firms are grouped by productivity quintiles to observe disaggregated effects. These results highlight variations in magnitude and persistence of shock impacts across different firm types. This thesis quantifies the macroeconomic implications of incorporating multiple heterogeneous firms, acknowledging the empirical fact that firms exhibit varying productivity levels. Unlike ad hoc models that impose high and low productivity distinctions, this framework derives productivity differences endogenously through the accumulation of idiosyncratic shocks over time. Firms are identical ex ante but diverge ex post based on their unique productivity trajectories.

The work is structured as follows. Section 2 describes the firms' model with a diffusion process with an Ornstein-Uhlenbeck process as well as, Poisson option. Section 3 details the computation of these models and the application of the finite differences method. Section 4 compares a representative firm model with two cases of heterogeneous firm models. Section 5 presents the effects of parameter changes between steady states and their consequences for the distribution of firms. Section 6 discusses the computation of the algorithm for the method by Ahn et al. (2017), focusing on the inclusion of the aggregate shock, its calibration, and its effects. Finally, Section 7 concludes.

2 The Model of Firms with Different Productivity Levels in Steady State

2.1 Firms with a Diffusion Process

The model is formulated in continuous time, featuring a continuum of firms with varying productivity levels. These firms decide their current period investment levels while accounting for convex adjustment costs and the constraint of maintaining positive capital at all times. Firms also recognize that productivity follows a stochastic process.

The production function for a firm is defined as $F(k_t, z_t) = z_t k_t^{\alpha}$, where z_t represents the firm's idiosyncratic productivity in period t, k_t is the firm's capital, and α is the parameter representing the capital's marginal return. Firms are only required to have positive capital for production. Since the inclusion of a representative household would result in labor being supplied inelastically, it is omitted for simplicity.[1]

Formally, the problem faced by a firm i with productivity z, capital k, and a production function $F(k_t, z_t)$ exhibiting diminishing returns to scale can be expressed as:

[1] The positive capital constraint arises naturally from microeconomic foundations. Computationally, however, issues arise at zero capital, so the capital continuum is discretized with a minimum feasible capital level of 0.225 units. The choice of this minimum is further discussed in the text, as it depends on the discretization of both capital and productivity.

$$\max_{\{i\}} E_t \int_{t=0}^{\infty} e^{-\rho t}\Big[z_t k_t^{\alpha} - i_t - \frac{\chi}{2}\Big(\frac{i_t}{k_t}\Big)^2 k_t\Big]dt \tag{1}$$

$$s.a \quad \dot{k_t} = i_t - \delta k_t \tag{2}$$

$$dz_t = \mu(z_t)dt + \sigma(z_t)W_t \tag{3}$$

$$k_t > 0 \tag{4}$$

In the aforementioned problem, i_t represents the firm's chosen investment level in the current period, χ represents the convex cost of investment, and δ is the depreciation rate of the capital stock. The productivity process z_t follows a logarithmic Ornstein-Uhlenbeck process, expressed as:

$$dlog(z_t) = -\theta log(z_t)dt + \sigma^2 dW_t$$

where W_t represents a Wiener process. Using Ito's Lemma, the diffusion process for productivity in levels is derived as:

$$dz_t = \theta(-log(z_t) + \frac{\sigma^2}{2})z_t dt + \sigma^2 z_t^2 dW_t \tag{5}$$

this transformation implies that the productivity process converges to a lognormal distribution in levels. Lastly, the $k_t > 0$ constraint ensures the firm maintains positive capital to operate.

2.1.1 Hamilton-Jacobi-Bellman (HJB)

The Bellman equation for the problem defined in equations (1)-(3) is expressed as follows:

$$\rho V(k_t, z_t) = \max_{i_t} \Big\{ z_t k_t^{\alpha} - i_t - \frac{\chi}{2}\frac{i_t^2}{k_t} + \frac{E_t[dV(k_t, z_t)]}{dt} \Big\}$$

where $V(k_t, z_t)$ represents the value function of firm i, conditional on its capital and productivity states in the current period.

The full derivation of the HJB equation is provided in Appendix A.1. In summary, by applying Ito's Lemma to the value function in conjunction with the productivity diffusion process and taking the time interval to zero, the HJB equation becomes:

$$\rho V(k,z) \;=\; \max_{i}\left\{zk^{\alpha} - i - \frac{\chi}{2}\Big(\frac{i}{k}\Big)^{2}k + \frac{\partial V(k,z)}{\partial k}(i-\delta k) + \frac{\partial V(k,z)}{\partial z}\mu(z) + \frac{\partial^2 V(k,z)}{\partial z^2}\frac{\sigma^2(z)}{2}\right\} \tag{6}$$

The optimal investment decision in steady state for a generic firm is given by the following first-order condition:

$$\{i\} : i^{*} = \frac{(\partial_k V - 1)k}{\chi} \tag{7}$$

Substituting this optimal condition into the HJB equation simplifies it to:

$$\rho V(k,z) = \left\{zk^{\alpha} - \frac{(\partial_k V - 1)k}{\chi} - \frac{\chi}{2}\Big(\frac{(\partial_k V - 1)}{\chi}\Big)^{2}k + \partial_k V\big(\frac{(\partial_k V - 1)k}{\chi} - \delta k\big) + \partial_z V\mu(z) + \partial_{zz}V\frac{\sigma^2(z)}{2}\right\} \tag{8}$$

Equation (8) includes nonlinear terms, making the implicit or explicit finite difference methods unsuitable for solving it directly. Instead, a semi-implicit finite difference method is employed, which separates the nonlinear terms for simplification and solves the remaining terms implicitly.[2]

The algorithm employed to solve the HJB equation and its stability and convergence properties are discussed in Section 3.

2.1.2 Kolmogorov Forward Equation (KFE)

To characterize the general equilibrium of this model, it is essential to determine the density function of firms based on their state variables. This density function can be derived using one of two partial differential equations, commonly referred to in economics as the Kolmogorov Forward Equation (KFE) and the Kolmogorov Backward Equation. The KFE is particularly useful for describing the stationary distribution, as explained by Stokey (2009).

The diffusion process of productivity levels, as defined by equation (3), can be expressed generically as:

$$dz_t = \mu(z_t)dt + \sigma(z_t)dW$$

where $\mu(z_t) \equiv \theta(\log\bar{(}z_t) - \log(z) + \frac{\sigma^2}{2})z_t$ and $\sigma(z_t) \equiv \sigma^2 z_t^2$. The stationary distribution of firms, denoted by $g(z,t)$, can be found by solving the KFE:

[2] This method is similar to the Hopscotch algorithm proposed by Gourlay (1970), as adapted by Hartley (2010), who demonstrated its convergence. Achdou et al. (2020) applied a similar approach to heterogeneous-agent models.

$$\frac{\partial g(z,t)}{\partial t} = -\frac{\partial}{\partial z}(\mu(z)g(z,t)) + \frac{1}{2}\frac{\partial^2}{\partial z^2}(\sigma^2 g(z,t)) \tag{9}$$

If we are interested in the stationary distribution, we assume $g(z,t) = g(z)$ for $t > \tau$, which transforms the KFE into:

$$0 = -\frac{d}{dz}(\mu(z)g(z)) + \frac{1}{2}\frac{d}{dz^2}(\sigma^2 g(z)) \tag{10}$$

When the distribution function depends not only on z but also on the other state variable, k, the multivariate version of the KFE must be used. The joint density $g(k,z)$ of firms is then governed by the following partial differential equation:

$$0 = -\frac{\partial}{\partial k}((i^*(k,z) - \delta k)g(k,z)) - \frac{\partial}{\partial z}(\mu(z)g(k,z)) + \frac{1}{2}\frac{\partial^2}{\partial z^2}(\sigma^2 g(k,z)) \tag{11}$$

The solution to this equation provides the stationary density of firms over the joint distribution of capital and productivity. This distribution is essential for analyzing equilibrium outcomes and the impact of aggregate shocks.

2.2 Representative Household

The economy is populated by a representative household that receives income y_t, which can include both endogenous (profits from firms) and exogenous components. The household decides how much to consume and save, conditional on the assets it owns. The problem is formulated as:

$$\max_{c_t, s_t} E_t \int_{t=0}^{\infty} e^{-\rho t} u(c_t) dt \tag{12}$$

$$s.a.\ \dot{a}_t = r_t a_t + y_t - c_t \tag{13}$$

$$a_t \geq 0; c_t \geq 0 \tag{14}$$

An important assumption in this model is that the representative household is always in possession of positive assets. This aligns with equilibrium, where the representative household's assets equal the aggregate capital in the economy, which is strictly positive due to the structure of the firm sector. To obtain a closed-form solution for the household problem, we assume the utility function follows a

Constant Relative Risk Aversion (CRRA) form:

$$u(c_t) = \begin{cases} \frac{c_t^{1-\gamma}}{1-\gamma}\ , & \gamma \geq 0, \neq 1 \\ ln(c_t)\ , & \gamma = 1 \end{cases} \tag{15}$$

When $\gamma \neq 1$, the closed-form solution for consumption is given by:

$$c^* = (\frac{\gamma - 1}{\gamma} r + \frac{\rho}{\gamma})(a + \frac{y}{r}) \tag{16}$$

In steady-state equilibrium, the household's optimal savings level, s^*, is:

$$s^* = Y - c^* \tag{17}$$

2.3 Closing the Model

The steady-state equilibrium is defined by the equality of the aggregate investment market (firms' demand) and the savings market of the representative household (demand), thereby determining the equilibrium price corresponding to the interest rate. The definition of the model's equilibrium is given by the following equation:

$$0 = s - \int_0^{k_{max}} \int_{z_{min}}^{z_{max}} i^*_{i,j} g_{i,j}(k, z) dz dk \tag{18}$$

From this market-clearing equation for savings and investment, the equilibrium price corresponding to the economy's interest rate in steady state is determined.

To equalize the household's savings rate with aggregate investment, it is necessary to solve the system of equations for the two partial differential equations (PDEs) discussed in the previous sections, the HJB and KFE, as defined by equations (8) and (11). The method used to achieve this is discussed in Section 3.

From the previously determined equilibrium, where the density function of firms in terms of capital and productivity is obtained (along with the aggregates and price of the economy), the accumulated capital level of the economy can be derived (since $g(k, z)$ is known). The total assets held by the representative household in steady state must equal the total capital available to firms for production, ensuring that the capital market clears in equilibrium. This is formally defined by the following equation:

$$0 = a - \int_0^{k_{max}} \int_{z_{min}}^{z_{max}} K_{i,j} g_{i,j}(k,z) dz dk \tag{19}$$

The method for achieving this equilibrium is developed in the following section, which discusses the use of the semi-implicit finite differences method. Since the model includes nonlinear PDEs, this requires iterating over value functions, as proposed by Achdou et al. (2014) and Moll (2014) in their literature.

2.4 Two productivity level Firms

In the literature on heterogeneous agents, a common simplification is the assumption of two types of agents, as in Campbell and Mankiw (1989). For firms, this can also be a valid alternative. Thus, the model presented in the previous section is developed here for an environment with two types of firms.

The following outlines the system of equations for the firm sector when only two possible productivity levels exist. In this framework, there is a continuum of high- and low-productivity firms, enabling the steady-state equilibrium to be determined in the presence of a representative household. This type of model can be interpreted as the transition of a firm from small-medium size to large, and vice versa.

Let us consider the investment problem of a generic firm with productivity z_t, as defined in (1-2), without considering equation (3), which now becomes a Poisson process. As in the previous section, the problem is defined as:

$$\max_{\{i_t\}} E_t \int_{t=0}^{\infty} e^{-\rho t} \left[z_t k_t^{\alpha} - i_t - \frac{\chi}{2} \left(\frac{i_t}{k_t} \right)^2 k_t \right] dt$$

$$s.t \quad \dot{k}_t = i_t - \delta k_t$$

Here, the firm's productivity level z_t^j has only two possible values: high productivity (z_t^h) and low productivity (z_t^l). These productivity levels are random for the firm and follow a Poisson process with an intensity of change given by λ^j, depending on the current state.

To achieve the steady-state equilibrium, the Hamilton-Jacobi-Bellman (HJB) equation for firms' optimal decisions conditional on their states is required. This follows from the discrete-time Bellman equation by letting the time interval approach zero.

The discrete-time Bellman equation is defined as:

$$V(k_t, z_t) = \max_{i_t} \left\{ (z_t k_t^\alpha - i_t - \frac{\chi}{2} \frac{i_t^2}{k_t})dt + \beta(dt) E_t[V(k_{t+dt}, z_{t+dt})] \right\}$$

$$\text{s.a } k_{t+dt} = (i_t - \delta k_t) k_t dt$$

The full derivation of the HJB equation, representing the solution to the firm's problem in steady state, is provided in Appendix A.2. Since this is a steady-state equation, the time subscripts are omitted for notational simplicity. Additionally, the value function depends on the productivity level in each period due to the different probabilities of transitioning between productivity states. Therefore, there are two HJB equations, depending on the current state of the i-th firm:

$$\rho V^h(k, z^h) = \max_i \left\{ \pi(k, z^h) + \partial_k V^h(k)(i - \delta k)) + \lambda^h [V^l(k, z^l) - V^h(k, z^h)] \right\} \tag{20}$$

$$\rho V^l(k, z^l) = \max_i \left\{ \pi(k, z^l) \right\} + \partial_k V^l(k)(i - \delta k)) + \lambda^l [V^h(k, z^h) - V^l(k, z^l)] \Big\} \tag{21}$$

where $\pi(k, z^h) \equiv zk^\alpha - i - \frac{\chi}{2}\left(\frac{i^2}{k}\right)$.

The existence of two HJB equations arises from the asymmetric Poisson process governing firms' productivity levels. If a firm is in a high-productivity state, the value function includes the intensity parameter λ^h. Conversely, a low-productivity state requires an HJB equation with λ^l. Due to the asymmetry of the intensity parameter λ^p (with $p \in \{l, h\}$), two HJB equations are necessary.

As in the diffusion model, the density function for the continuum of firms at each productivity level must be determined to compute the aggregate levels of investment and capital for the firm sector, allowing the markets to clear with the representative household. To achieve this, the Kolmogorov Forward Equation (KFE) is used, as the focus is on the steady-state distribution of firms.

Formally, the density function for each productivity level is obtained from the following KFE for a Poisson process. The derivation of this equation is provided in Appendix A.3[3].

$$0 = -\partial_k (i^* - \delta k) g(k, z^p) - \lambda^p g(k, z^p) + \lambda^{-p} g(k, z^{-p}) \tag{22}$$

[3]This derivation may be more intuitive for readers.

With the PDE (22) and the HJB equations (20) and (21), the firm system is complete, allowing the aggregate investment and capital levels for the firm sector to be determined. To close the model, the other sector of the economy is required to clear the markets and determine the equilibrium price in steady state (the interest rate).

The market-clearing condition is defined as:

$$S_{household} = \int_0^\infty i^* dG^h(k) + \int_0^\infty i^* dG^l(k) = Y_{household} - C_{household} \tag{23}$$

Once the equilibrium price is determined, and thus the aggregate investment level in steady-state equilibrium, the firm's density function is obtained, and the aggregate capital in the economy is computed as:

$$K = \int_0^\infty k dG^h(k) + \int_0^\infty k dG^l(k) \tag{24}$$

In equilibrium, this capital level must equal the household's wealth, denoted by a, completing the steady-state equilibrium for the economy with two productivity levels.

3 Computation

The model is formulated in continuous time to to use its computational advantages. This approach allows for the implementation of an efficient algorithm for rapid numerical resolution, as demonstrated by Achdou et al. (2017), who employed the finite differences method with iterative steps to solve models with heterogeneous agents.

As mentioned in Section 2.1.1, substituting the optimal investment function (7) into the HJB equation (6) results in (8), which is a nonlinear partial differential equation. This equation cannot be solved in a single iteration, as is typically done with the finite differences method. Consequently, the algorithm developed by Achdou et al. (2017) is employed, with the steps outlined below[4]:

- Generate an initial guess for the value function sufficiently close to the steady state. This can be obtained by assuming $i - \delta k = 0$, corresponding to the point where V_k is not relevant, considering a price to allow the value function to converge.

- Utilize two finite difference discretizations to avoid the need for boundary conditions, applying the Upwind method.

[4]Discussion of the Upwind Method, sparse matrices and the firms' distributon are presented in Appendix B.

- Use the value function conditional on the given prices to determine the firm distribution using the KFE.
- Compute the aggregates and verify whether the markets in the economy clear. If they do not, adjust the price (interest rate) iteratively until the markets clear.

4 Parametrization

The choice of key parameters follows the values established in the literature on investment modeling with heterogeneous firms in discrete time, as in Khan and Thomas (2008). These parameters are also used in Bachmann et al. (2013), who adopt the same calibration, shown in the table below.

Table 1: Khan y Thomas (2008) parameters

α	δ	ρ
0.256	0.069	0.235

For the distribution parameters, productivity data for UK firms was obtained from the *Annual Business Survey* (ABS) of the UK Office for National Statistics (ONS). The dataset used is the *Firm-level labour productivity estimates from the Annual Business Survey*, which provides kernel density estimates for the mass of firms between 2006 and 2017. The kernel density function for firm-level idiosyncratic productivity was based on labor productivity per employee, weighted by firm size and expressed in constant 2016 prices. This density function is defined by the value added per worker in pounds. Since the points in the kernel density distribution are uniformly spaced, they were rescaled to match the probability functions of a lognormal distribution. As shown in Appendix C, these empirical distributions resemble a lognormal function.

Given that the lognormal function is defined over positive real numbers, the rescaled function was adjusted to remain within positive values, ensuring that the moments of the rescaled function match those of the distribution resulting from the productivity diffusion process in levels. The moments of the limiting function of the diffusion process and the estimated moments of the data from 2010–2017 are shown in the table below.

Table 2: Moments of Firm Distributions: Simulation vs. Empirical Data

	Mean	Variance	Skewness	Kurtosis
Simulation	1.1921	0.256	1.2186	1.7520
Data	[1.12, 1.16]	[0.254, 0.285]	[1.31, 1.37]	[1.53, 1.99]

This calibration assumes that the distributions of firms across economies are very similar. This assumption is supported by evidence from studies estimating firm productivity distributions in various economies, including Vanhala (2017), Di Mauro (2015), and Gouin-Bononfant (2018). These studies show that firm productivity distributions in Finland, Germany, Spain, Italy, and Canada follow a lognormal distribution, making this assumption reasonable to be used.

The diffusion process was designed to ensure a lognormal distribution. The diffusion process parameters were chosen to closely match the moments of the empirical distribution.[5] The following table shows these parameters, along with the quadratic cost of investment, which was calibrated to ensure the average investment in the simulation is approximately 20% of output.

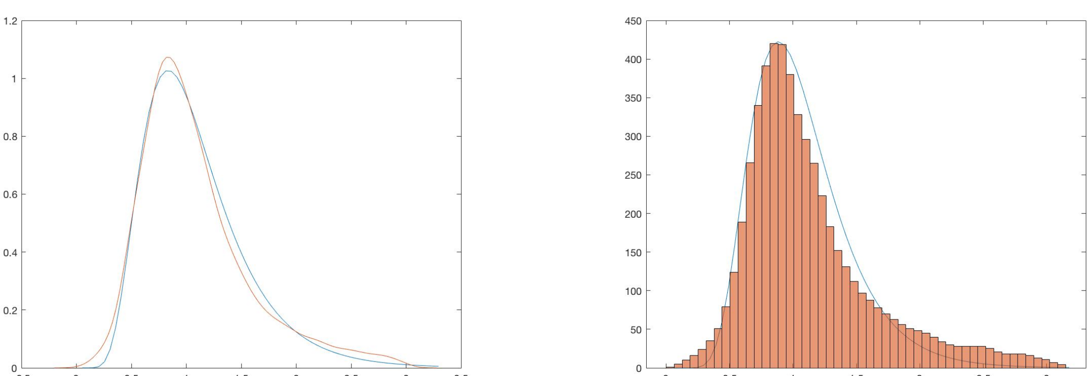
Table 3: Firms' Parameters

θ	σ^2	χ
0.1985	0.1395	0.265

Table 4: Household's Parameters

γ	ρ	Y
1.05	0.0235	0

Figure 1: Limiting distribution of diffusion process compare to data



((a)) Kernel

((b)) Histogram

Figure 3 provides a graphical representation of the limiting distribution of the diffusion process

[5]The main difference from the calibration proposed by Khan and Thomas (2008) is the parameter θ. In this study, θ is significantly smaller, which causes the productivity distribution to accumulate below 1. In contrast, a θ value closer to 1, as in Khan and Thomas (2008), results in a more centered distribution, resembling a normal distribution.

and the distribution derived from the data, shown as a kernel density (panel (a)) and as a histogram (panel (b)).

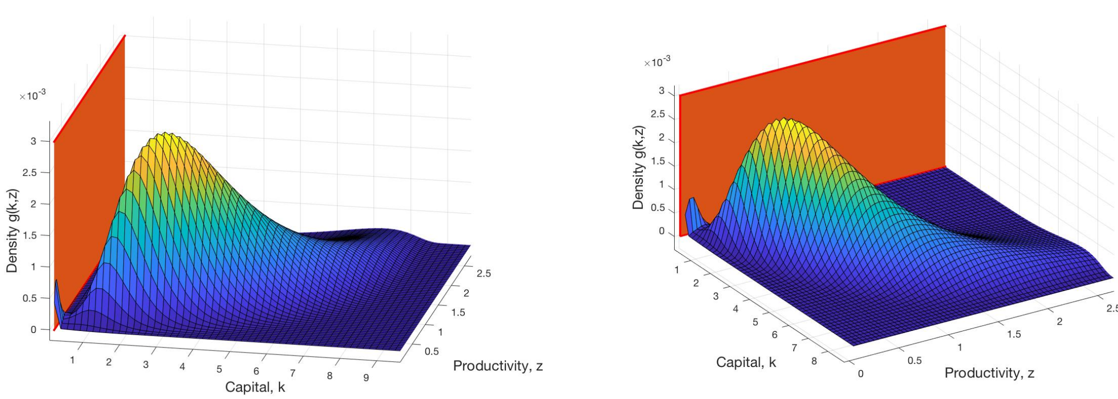
The representative household parameters are shown in Table 4. The discount rate is slightly higher than the value that would yield logarithmic utility for the household. The discount rate is the same as that used for firms, as households own the firms and receive all profits as income. Exogenous income is set to 0, making the household's income equal to the firms' profits.

Following the algorithm described in the previous section, which modifies the algorithm presented by Achdou et al. (2013), the firm distribution and steady-state moments can be computed. In the simulated economy, state variables—capital and productivity—take values $k \in [0.225; 35]$[6] and $z \in [0.053; 2.985]$.

4.1 Diffusion Process Model

The steady-state equilibrium for the model with a diffusion process is characterized in terms of its moments by an interest rate of $r = 3.47\%$, total investment of $I = 0.2937$, total capital of $K = 4.2702$, total profits distributed by firms to the representative household of $D = 1.2912$, and total consumption of $C = 0.9975$. The steady-state density distribution of firms across the continuum of capital and productivity is analyzed in Figure 2. This figure is presented in two panels to highlight different aspects of the distribution.

Figure 2: Ergodic distribution of firms, using a difussion process



((a)) ((b))

[6]As mentioned earlier, firms' capital is restricted to positive values. However, numerically starting the grid very close to 0 requires a very small Δk to capture capital levels near 0.2 units. This would require over 2000 grid points, significantly increasing computational costs in the aggregate shock algorithm. To balance this, the number of productivity points was reduced. This configuration aims to capture both aspects effectively.

In Figure 2, it can be observed that most of the firm mass is concentrated in the quadrant with low productivity and low capital. When fixing the productivity level, the steady-state distribution of capital follows a lognormal distribution, as previously presented. Additionally, in the low-capital sector near the minimum capital level (marked by the orange plane), firms with lower productivity exhibit a point of density accumulation. This arises due to the inability of these firms to hold capital below the minimum feasible level, as determined by the discretization of capital discussed in the previous subsection. This phenomenon occurs exclusively at the lower productivity levels. For higher productivity levels, the capital constraint is not active, and no firms endogenously hold such low levels of capital.

This model with a diffusion process can be simplified in two ways. The first simplification assumes that there are only two productivity levels, with a continuum of firms at each productivity level. In this case, changes in a firm's productivity level depend on a Poisson process. The second simplification assumes the existence of a representative firm in the economy. The following two subsections compare these two levels of simplification of the diffusion process model, discussing the benefits and challenges they entail.

4.2 Model with Two Types of Firms

As mentioned earlier, the first level of simplification of the model is to assume that firms can only have two productivity states: one high and one low. In each state, there is a continuum of firms, and changes in productivity levels are governed by an asymmetric Poisson jump process. Determining the rate of change between productivity levels is nontrivial.

When working with only two productivity levels, it is necessary to define high- and low-productivity firms. In this context, firms with productivity equal to or greater than 50% above the mean productivity are considered high-productivity firms. In the previous model, the mean productivity was set to 1.19, so high-productivity firms are defined as those with productivity levels ≥ 1.58. In the steady-state distribution, this group represents approximately 25% of all firms. Therefore, the Poisson process model is calibrated to ensure that low-productivity firms constitute 75% of the total mass of firms and high-productivity firms make up 25%.

To achieve this, the Poisson process jump rates λ^l and λ^h were selected such that λ^l is 1/3 of λ^h. Since λ in Poisson processes represents the probability of state change, the probability of transitioning from low to high productivity is one-third the probability of transitioning from high to low productivity. The selected rates are shown in the following table:

Table 5: Parameters

λ^l	λ^h	χ	α	δ	ρ	γ	Y
0.1	0.3	0.28	0.256	0.069	0.235	1.05	0

The remaining parameters are unchanged from the diffusion process model. Additionally, the required productivity levels to ensure the steady-state mean productivity matches the diffusion process model involve a continuum of options. A series of calibrations was conducted to determine the productivity levels. The table below summarizes the results for the primary variables, prices, and economic aggregates for different ratios of z^h/z^l.

Table 6: Prices and Aggregates, by different productivity ratios

z^h/z^l	≈4 (2.452/0.613)	≈3.5 (2.31/0.66)	≈3 (2.145/0.716)	≈2.5 (1.951/0.78)
r	3.44%	3.45%	3.47%	3.48%
Investment	0.2910	0.2913	0.2921	0.2922
Capital	4.2167	4.2214	4.2338	4.2353
Profits	1.3000	1.2915	1.2843	1.2751
Consumption	1.0090	1.0002	0.9921	0.9829

As shown in Table 6, the equilibrium price (interest rate) in the two-productivity-level model is a reasonable approximation of the diffusion process model. However, economic aggregates are smaller than in the continuous productivity-level model. Aggregate investment levels remain stable across all examined ratios, with values close to those in the diffusion process model. Similarly, total capital follows a stable trend but is approximately 2% lower in the two-productivity-level model. Profits and representative household consumption exhibit values close to those in the diffusion process model.

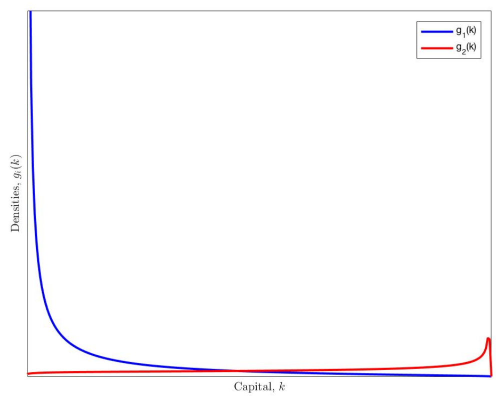
These results are quite similar to those of the diffusion process model, possibly because the Poisson process model is a good approximation of the diffusion process model or due to the calibration of the proportions of high- and low-productivity firms. To test this, a robustness analysis was conducted by examining whether alternative proportions of high- and low-productivity firms better approximate the economic aggregates in the diffusion process model[7].

As seen in the results across table 6 , the equilibrium interest rate remains stable across all

[7] The Robustness analysis for different Poisson processes are left in Appendix D.

parametrizations. However, aggregates such as household consumption are better approximated with a $z^h/z^l = 4$ ratio. For profits and household consumption, the $z^h/z^l = 3.5$ ratio yields results closest to the diffusion process model. Thus, the subsequent analysis uses this parametrization. Figure 3 shows the two distribution functions for the firm continuum (high and low productivity) in the steady-state Poisson model. For the low-productivity continuum ($g_1(k)$, blue function), there is an accumulation point. This accumulation point is endogenous to the low-productivity firms' productivity level and does not necessarily align with the minimum admissible capital level.

Figure 3: Ergodic distribution for how-high productivity firms



For the high-productivity continuum, a larger proportion of firms accumulates at higher capital levels. Additionally, positive density starts to appear from the accumulation point of low-productivity firms. While the Poisson model in steady-state equilibrium achieves moments similar to the diffusion process model, it fails to represent firms constrained by minimum capital. Under this parametrization, such firms do not have positive density.[8] Thus, while this simplification is effective for moment approximation, it cannot represent firms subject to additional constraints.

4.3 Model with a Representative Firm

Finally, to assess whether it is reasonable to apply a stronger simplification of the diffusion model, this section discusses a model with a representative firm and how it compares to the two models presented in the previous sections.

Consider an economy with a representative household and a representative firm, where the latter

[8] To represent this, excessively high productivity ratios near 70 are required, resulting in moments far from the diffusion model.

has a given and fixed productivity level. To compare these two models, it is necessary to equalize the total income received by the representative household in both models. This approach is analogous to comparing economies with the same level of wealth. Since exogenous income is fixed at 0, firm profits correspond to the household's total income. Setting household income levels effectively compares interest rates, investment levels, and accumulated capital in economies with similar wealth levels.

For a fair comparison, the parameters used in the diffusion model (with a continuum of productivity levels) and the Poisson process model (with two productivity levels) were fixed. Additionally, the productivity level (z) in the representative firm model is set to match the distribution obtained from the diffusion model data.

To summarize the three cases presented so far, the model with a continuum of productivity levels (and many firms within each level) can be considered the most comprehensive. This is followed by the model with two productivity levels (and many firms within each level). Lastly, the model with a single representative firm represents the simplest approach. Table 7 presents the moments, prices, and aggregates for these three models to facilitate comparison.

Table 7: Comparing aggregates between models

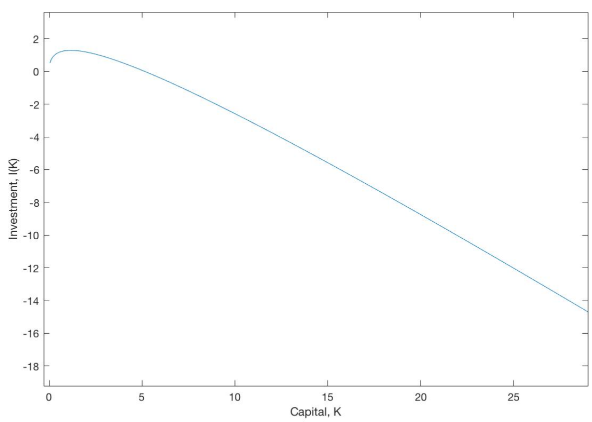
	Representative	Poisson	Difussion
r	3.15%	3.47%	3.47%
Investment	0.2230	0.2921	0.2937
Capital	3.9373	4.2338	4.2702
Profits	1.2912	1.2843	1.2912
Consumption	1.0839	0.9921	0.9975

From the comparison in Table 7, it can be observed that the interest rate is significantly lower in the model with a representative firm, being 32 basis points (or 9.25%) lower than in the diffusion model or the two-productivity-level model. The simplified heterogeneous firm model provides a good approximation of the diffusion model when comparing the representative firm model with the complete diffusion model.

To explain the differences between the representative firm model and models with varying firm productivity, it is necessary to examine the firms' policy functions. Figure 4 presents the policy function for the representative firm. Comparing Figure 4 to the policy functions from the diffusion model in Figure B.2, panel (a), it becomes evident that firms with high productivity levels exhibit greater curvature in their policy functions. This allows them to achieve higher levels of investment, which the representative firm simplification cannot capture.

This difference is reflected in the aggregate investment levels of each model. The investment level in the representative firm model is substantially lower than in both heterogeneous firm models. Consequently, the representative household's consumption is higher, as the required savings are lower than in the environments faced by households in models with varying productivity levels. This also implies that the equilibrium interest rate is lower, given the fixed firm profit levels.

Figure 4: Policy Function of the representative Firm ($z = 1.19$)



When comparing prices and aggregate steady-state levels, using the diffusion model as a benchmark, the two-productivity-level model provides an excellent approximation of the heterogeneous firm model. This holds true given the assumptions about the steady-state distribution of high- and low-productivity firms. However, the representative firm model does not perform as well. The equilibrium price (interest rate) is significantly lower due to underinvestment by the firm, leading to lower accumulated aggregate capital in the economy.

5 Comparative Statics

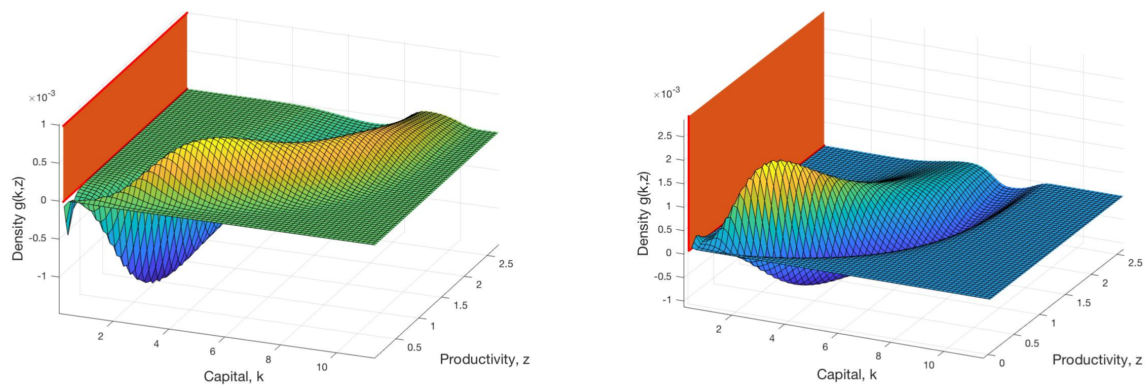
The following section present the differences in distributions between steady states when there is an increase or decrease in the curvature parameter of the production function (α) and the investment cost parameter (χ).

5.1 Change in α

This section discusses the differences in firms' policy functions and density distributions between different steady states, aiming to highlight non-linear aspects in a model with firms possessing various productivity levels. This is intended to underscore the advantages of using a heterogeneous firm model over one with a representative firm.

An interesting question to address is: what happens if the capital return parameter increases (or decreases) by 10% of the value used in previous sections? The parameter α was increased (or reduced) by the same proportion to evaluate potential non-linear effects of these changes, as well as to observe sufficiently significant impacts on firms' policy functions and distributions across the continuum of capital and idiosyncratic productivity.

Figure 5: Change in ergodic distribution, changing α



((a)) α increased by 10%

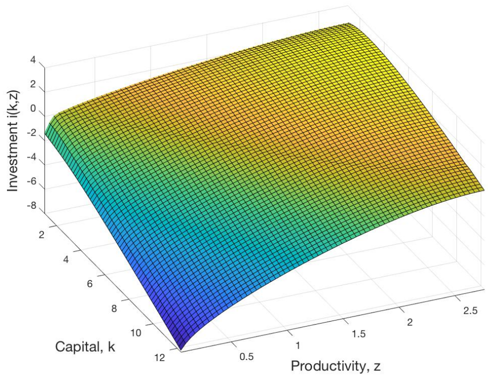
((b)) α reduced by 10%

Figure 5 shows the difference in firm distribution functions between the steady state with the original parameters presented in earlier sections and the firm distribution when the capital return parameter is 10% higher than in the original setting (panel (a)) and 10% lower in panel (b).

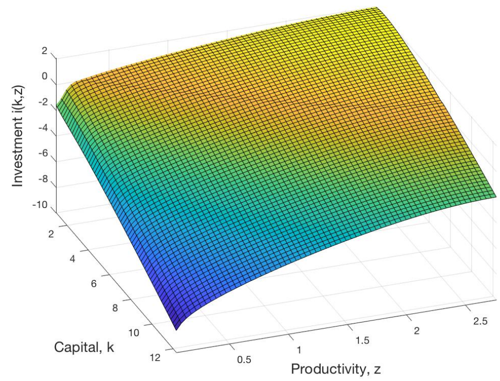
In panel (a), as expected, firms accumulate more capital at all productivity levels due to the increase in capital returns, resulting in a positive differential across higher capital levels. However, the decrease in density is concentrated in the low-productivity and low-capital region, while the increase is more evenly distributed across the range of productivity levels. Additionally, the mass accumulation at the minimum capital constraint significantly diminishes.

From panel (a) of Figure 7, it is evident that these aggregate changes are not uniform across

Figure 6: Policy functions of firms, different α values



((a)) α increase by 10%



((b)) α reduce by 10%

the productivity continuum. The reduction in firm density is concentrated in the quadrant of less productive, low-capital firms, while the increase in firm density is more uniformly distributed across the productivity continuum, resulting in a higher proportion of productive firms in the economy.

In panel (b) of Figure 7, there is a greater concentration of firms in the low-productivity and low-capital zone. Although the effects are essentially the inverse of those in panel (a), the magnitudes in panel (b) are greater. This means that the accumulation in the low-productivity, low-capital zone occurs more rapidly than the reduction in density caused by an increase in capital returns. The mass accumulation at the constrained point does not increase as sharply as the reduction in panel (a).

Regarding the policy functions in Figure 6 compared to the original policy function (Figure B.2, panel (a)), panel (a) shows fewer effects of the minimum capital constraint. Firms with lower productivity levels tend to maintain the shape of their policy functions. Conversely, in panel (b), the policy functions for less productive firms flatten out, converging to a specific investment level as capital approaches the constraint, preventing these firms from disinvesting as much as they might otherwise prefer.

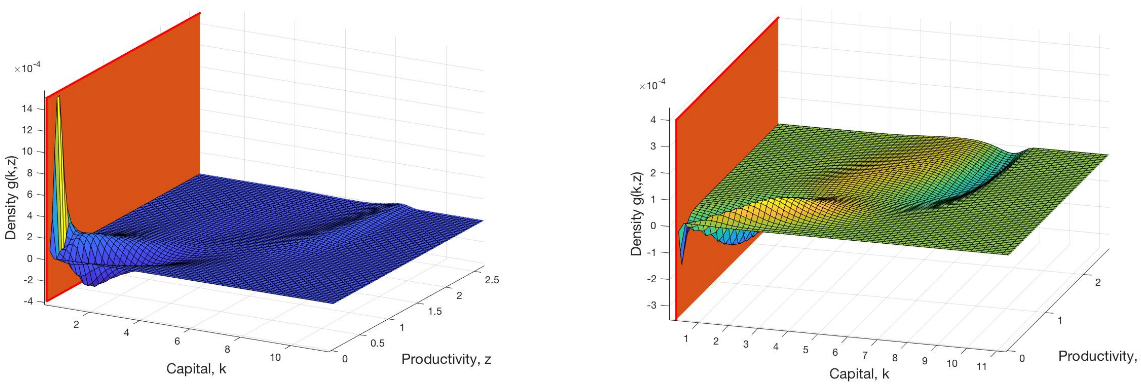
5.2 Change in χ

This section examines a 10% reduction (or increase) in the quadratic investment cost parameter (χ) in the heterogeneous firm diffusion model to illustrate the benefits of using a more complex and realistic firm model compared to a representative firm model.

Figure 7 shows the differences between steady-state distributions under altered χ parameterizations and the original model. Panel (a) shows the difference when the investment cost decreases by 10%, while panel (b) depicts the difference when χ increases by 10%.

One key observation from the changes in steady-state density functions with altered χ values is that the sign of the density change depends on the firm's productivity level. In panel (a), where the quadratic investment cost decreases, more productive firms accumulate greater capital for each productivity level. On the other hand, firms in the lower half of the productivity range reduce their capital accumulation in the new steady state. This occurs because the reduced friction allows investment (or disinvestment) to move closer to the firms' optimal productivity-state decisions without friction.

Figure 7: Change in ergodic distribution, changing χ



((a)) χ reduce by 10%

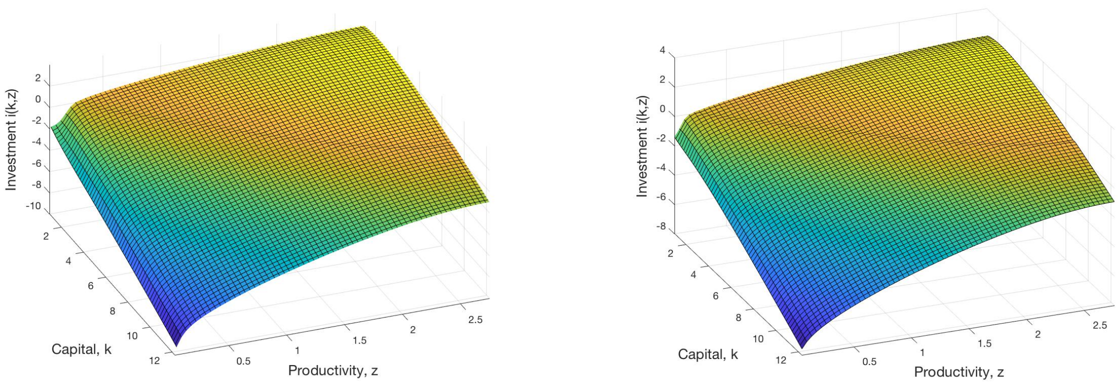
((b)) χ increase by 10%

In panel (b) of Figure 9, where the quadratic investment cost increases by 10%, the effects on the density function also depend on firm productivity levels. Higher-productivity firms accumulate less capital in the steady state because the increased disinvestment cost makes maintaining high capital levels less desirable, given the potential for lower productivity in the future. Conversely, lower-productivity firms accumulate more capital than in the original scenario.

Finally, changes in investment costs (χ) exhibit asymmetric scaling effects. A reduction in χ has a larger impact on firm distributions than an increase of the same magnitude.

Regarding firms' policy functions by productivity and capital level, the effects of a χ parameter change are shown in Figure 8. In panel (a), where the cost is lower, more firms exhibit flattened policy functions ($i(k|z)$), constrained by the minimum capital level. This reaffirms earlier explanations about how reduced friction in investment increases (or decreases) investment according to the productivity state. Conversely, when this cost increases, fewer constrained policy functions are observed.

Figure 8: Policy Functions, different (χ) values



((a)) ((b))

These last two subsections emphasize the analytical differences enabled by a heterogeneous firm model compared to a representative firm model, where it is not possible to derive a firm distribution function in the first place.

However, it remains an open question whether the Poisson process or diffusion models can replicate moments under an aggregate productivity shock in the modeled economy. This will be analyzed in Section 6, applying the algorithm developed by Ahn et al. (2017) and adapting it to this context, while comparing the results with data and models explored by King and Rebelo (1999), Khan and Thomas (2008), and Bachmann et al. (2013).

6 Aggregate Shock

As shown in the previous section, changes in global parameters such as χ, the quadratic cost of investment, and α, the return rate of capital, exhibit differentiated effects in magnitude depending on whether the change is positive or negative. Additionally, there are nonlinear changes throughout the spectrum of productivities. Considering these effects in the steady state, one can guess that differentiated effects exist along the continuum of productivities in response to aggregate shocks to firms' aggregate productivity in the economy. This is the focus of the present section.

When incorporating the aggregate shock into the production of firms, it can be expressed as follows:

$$Y_t = A_t z_t k_t^{\alpha} \tag{25}$$

where aggregate productivity, following prior literature, adheres to an AR(1) process in continuous time, specifically an Ornstein-Uhlenbeck process, as shown in Equation (26). The parameters used are consistent with those in Khan and Thomas (2008), Bachmann et al. (2006), and Winberry (2018): $\eta_A = 0.8254$ and $\sigma_A = 0.00953$.

$$dA_t = -\eta_A A_t dt + \sigma_A dW_t \tag{26}$$

Unlike the algorithm of Winberry (2018), where the steady state is perturbed, the equilibrium in this algorithm is linearized.

The algorithm presented by Ahn et al. (2017) consists of the following steps:

1. Solve for the steady state of the model without considering aggregate shocks.

2. Linearize the equilibrium conditions using a first-order Taylor expansion (around the steady state).

3. Reduce the model's dimensions for the value function and the distribution, if the model is sufficiently large.

4. Solve the linear system created.

5. Simulate the model and obtain impulse response functions and simulated moments.

For the first step, all procedures described in the previous sections are executed, considering the type of idiosyncratic uncertainty faced by firms, whether it involves only two levels of productivity or a continuum. As shown earlier, the steady state is characterized by a matrix system summarized as:

$$\rho V(k,z) = \pi(V(k,z)) + A(v(k,z)|p)V(k,z) \tag{27}$$

$$0 = A(v(k,z)|p)' g(k,z) \tag{28}$$

$$p = F(g(k,z)) \tag{29}$$

Here, the price in this reduced model is solely the interest rate, which, as shown in Equation (29), is a function of the distribution of firms, represented by $g(k,z)$. The above system contains $K \cdot N + 1$

equations, as the price is unique, while both the value function V and g are defined by N entries for the discretization of capital and K entries for the discretization of productivity levels.

For the steady state of these economies, the grid for the value function's states was discretized into $K = 65$ and $z = 70$ points for the model where idiosyncratic productivity follows a diffusion process. For the simplified model, where productivity follows a Poisson process, the capital space discretization is identical to the previous model, while the productivity space naturally consists of $z = 2$. As the second step stands, the steady state is linearized around its levels and price, adding the conditionality of the aggregate shock. Thus, the system of equations becomes dependent on time and an additional state variable, namely the level of aggregate productivity.

The most significant discussion of this methodology occurs in Step 3, as there is no optimal method for reducing the dimensions of the firms' value function and the distribution function. Given this, the authors developed code that verifies whether the simulated variables are internally consistent. A corresponding graph can be found in Appendix E.

In the model with a diffusion process, the space consisting of 5200 points (considering both states: capital and productivity) for the value function was reduced to $k = 960$ and $z = 254$. This ensures that the ex-post errors of internal consistency are sufficiently low, as shown in the figure in Appendix E. This reduction allows the algorithm to compute impulse response functions in approximately 16 seconds, achieving results comparable to the algorithm presented by Winberry (2018).

For the simplified model with two productivity levels, the algorithm runs faster due to fewer points in the distribution function, which reduces the necessity for dimensional reduction. In this case, the distribution function was reduced from 130 points to 58 points, and the value function was reduced to 24 points. The algorithm executes in just over 1 second.

6.1 Model's Moments

To compute the moments for these models, innovations are applied to aggregate productivity over a long horizon of 10,000 periods, with 4 sub-periods (dt) for each period, resulting in 40,000 simulations. These moments can then be compared to those obtained from data. King and Rebelo (1999) derived various moments for the US economy from 1947 to 1996 (on a quarterly basis), removing seasonality using the HP filter. These moments are shown in Table 8.

Table 8: US data Moments

	σ_j	σ_j (relative)	Corr	Persistence
Y	1.81	1	1	0.84
C	1.35	0.74	0.88	0.80
I	5.30	2.93	0.80	0.87
r	0.30	0.16	-0.35	0.60

Table 9: Difussion Process Model Moments

	σ_j	σ_j (relative)	Corr	Persistence
Y	2.11	1	1	0.90
C	1.64	0.77	0.89	0.98
I	6.27	2.97	0.94	0.93
r	0.35	0.17	-0.83	0.97

Table 9 presents the moments of the model with a diffusion process, including absolute standard deviations[9], in the first column, the relative standard deviations to the standard deviation of output in the second column, the correlation of σ_j with respect to σ_Y in the third column, and the first autocorrelations of each series in the fourth column.

Another algorithm for simplifying aggregate shocks is the one presented by Winberry (2018), which perturbs the steady state. This model is solved based on the work of Khan and Thomas (2008) with non-convex adjustment costs in investment and the parametrization used in this model, differing in the parameters for firms' idiosyncratic productivity and investment costs, as discussed earlier. Both King and Rebelo (1999) and Winberry (2018) provide moments derived from long-period simulations of their models. These moments are compared in the table below to better assess the results obtained using this linearization method.

Table 10: Moments comparison

	σ_j				relative σ_j (σ_j/σ_Y)			
	Y	C	I	r	Y	C	I	r
Data	1.81	1.35	5.30	0.30	1	0.74	2.93	0.16
King and Rebelo	1.39	0.61	4.09	0.05	1	0.44	2.95	0.04
Winberry	2.16	1.02	8.48	0.17	1	0.47	3.93	0.08
Poisson Model	1.85	1.95	2.76	0.63	1	1.05	1.49	0.34
Difussion Model	2.11	1.64	6.27	0.35	1	0.77	2.97	0.17

One key aspect to highlight about the moments of the diffusion model is the relative closeness of its relative standard deviations to those observed in the data. These moments improve upon those obtained by King and Rebelo (1999) in consumption and the interest rate and also upon those obtained by Winberry (2018) for the three aggregates analyzed. Regarding the relative standard deviation of over-investment to the standard deviation of output, achieving a σ_I/σ_Y ratio similar to the data supports the findings of Khan and Thomas (2008) and does not present an issue for this model.

[9] The value of the standard deviation of output can be adjusted by reducing the variance of the diffusion process of aggregate productivity. However, this is not presented to maintain comparability between models.

However, the Poisson process model does not achieve the same results as the diffusion model in terms of relative standard deviations, as consumption and the interest rate exhibit excessive variation compared to all other models, and the aggregate investment shows a very low variation.

Table 11: Comparison of Moments

	Persistence				Correlation			
	Y	C	I	r	Y	C	I	r
Data	0.84	0.8	0.87	0.6	1	0.88	0.80	-0.35
King and Rebelo	0.72	0.79	0.71	0.71	1	0.94	0.99	0.95
Winberry	-	-	-	-	1	0.90	0.97	0.79
Poisson Model	0.97	0.93	0.76	0.98	1	0.98	0.94	-0.99
Diffusion Model	0.90	0.98	0.93	0.97	1	0.89	0.94	-0.83

Table 11 presents the different persistence levels of each variable across models, given by the first autocorrelation of the standard deviations and the contemporaneous correlations between σ_j and σ_Y. Regarding persistence, models with convex costs exhibit higher persistence compared to the data; however, the persistence of investment is closer to that found in the data than in the model of King and Rebelo (1999). Focusing on the correlation between models, this model achieves a negative correlation between the interest rate and output, which distinguishes it from the other two compared models. Examining the rest of the correlations, the diffusion model provides a good approximation of the contemporaneous correlation in aggregate consumption and a closer match in investment correlation compared to the other models.

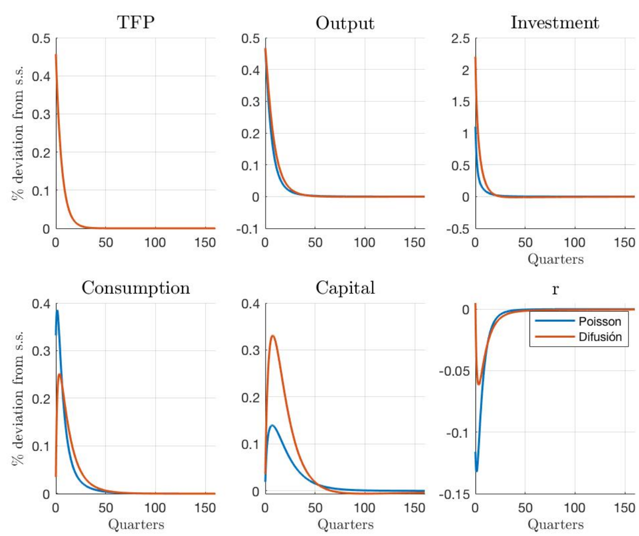
When comparing the models with two levels of productivity and many levels of productivity, the first aspect to analyze is the standard deviations obtained from the models. The two-productivity-level model produces moments that differ significantly from the data, with surprisingly low investment variation, both in absolute and relative terms. Additionally, the interest rate variation is twice as high as in the data and the diffusion model, while aggregate consumption follows the same pattern.

Second, in terms of persistence, both models exhibit higher persistence than reported by King and Rebelo (1999) for the data, except for the investment persistence in the Poisson model. Regarding the remaining moments, it is unclear which model consistently performs better across all metrics. Finally, comparing the correlations obtained from the diffusion model for interest rates and consumption, this model delivers correlations much closer to those observed in the data. In contrast, the investment correlation is indistinguishable between the models.

Considering the steady-state analysis where the Poisson process model closely approximates the model with a continuous range of productivity levels, the inclusion of an aggregate shock reveals significant differences in moments compared to the diffusion model. The latter successfully represents moments derived from the data, whereas the Poisson process model does not. Consequently, in the presence of an aggregate shock, the model with two productivity levels fails to approximate the model with many productivity levels effectively.

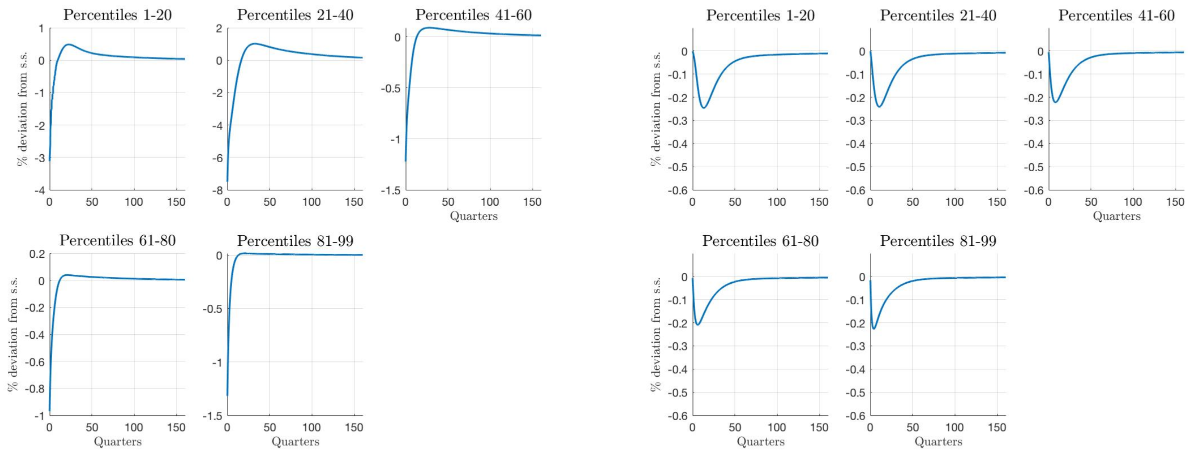
An alternative way to analyze the two models presented in this work is through impulse response functions (IRF) derived from a single shock of one standard deviation magnitude from the steady-state level of aggregate productivity, as defined previously. Figure 9 displays the IRFs for both models in response to the same aggregate shock, expressed as a percentage deviation from the steady state. As inferred from the moments, the IRFs naturally reflect that investment in the Poisson process model (blue lines) reacts significantly less than in the diffusion model. This also causes aggregate capital to react to a lesser extent.

Figure 9: IRF to a aggregate productiviy shock, 1 sd shock



In terms of persistence, the Poisson model's investment does not fall below the aggregate steady-state investment level, which contributes to the greater persistence of the aggregate shock's effect on the economy's aggregate capital. When addressing a question involving an aggregate shock, it can be concluded that the model with two productivity levels does not effectively approximate the model with many productivity levels. For such analyses, it is necessary to utilize the diffusion model to achieve

Figure 10: IRF to (a) Investment (b) Capital by quintile, 1 sd shock



(a) (b)

more accurate conclusions.

6.2 Disaggregation of the Aggregate Shock

An important and natural question that arises from an aggregate shock in an economy concerns the differing impacts this shock has on various types of firms and the potential implications of these effects. This subsection analyzes these effects using pseudo-IRFs derived from a single, negative aggregate shock on the variables and the simulated distribution function. These functions are not exact IRFs because, at any given time, different firms are at different productivity levels. Over the simulation periods, a specific firm may belong to different quintiles in each period.

First, it is important to note that this analysis is conducted within an environment with a minimum capital constraint, meaning firms with lower capital levels may face an active constraint. The firms in the continuum are divided into productivity quintiles, with the first quintile representing the sum of the investments of all firms that belong to the lowest quintile of the productivity distribution in a given period.

Figure 10 panel (a) shows the movement of investment for each quintile of firms, with each quintile representing the firms' productivity levels at a given period. The deviations are expressed as a percentage from the steady-state level.

One key observation is the substantial negative impact of the aggregate shock on firms in the two lowest quintiles. The effects in these quintiles range from 3–8% in the initial periods, whereas the largest deviation in the higher quintiles is slightly above 1% compared to the steady state. Regarding the duration of the aggregate shock's effects, it is evident that firms with lower productivity levels experience longer-lasting impacts, while the duration is shorter for firms in the highest productivity quintile. Thus, the less productive firms are affected more severely in terms of both intensity and duration.

Figure 10 panel (b) illustrates the movements of disaggregated capital for firms by productivity quintiles in each period following the aggregate shock. Similar to the investment analysis, the largest effects are observed for firms in the lowest productivity quintiles. However, for capital, the timing and peak of the deviation are more relevant than the magnitude of the initial effect.

Firms in the higher productivity quintiles experience a sharper capital depletion closer to the moment of the aggregate shock and recover more quickly to the steady-state level. In contrast, firms in the lower productivity quintiles take several periods to reach their maximum capital depletion before beginning to recover toward the steady-state value. Similar to investment, the aggregate shock has a significantly more prolonged effect on the lower quintiles.

7 Conclusions

Throughout this work, it has been demonstrated that employing a model in which firms exhibit heterogeneous idiosyncratic productivity levels allows for a broader analysis of firm behavior, such as nonlinear differences in the stationary distribution function when facing different global parameters. Additionally, this study illustrates the differences and limitations of a representative firm model in a steady-state analysis compared to models with heterogeneous firms. Even in steady-state analysis, the equilibrium price in the representative firm model is significantly lower than in heterogeneous models, due to the inability to capture varying investment levels across firm types, conditional on their states of k and z.

Building on the findings of Khan and Thomas (2008) regarding the variations of idiosyncratic and aggregate shocks, this study assumes a simplified structure of investment costs faced by firms in an economy with varying productivity levels. This simplification enables the use of the algorithm developed by Ahn et al. (2017) to analyze a continuous-time RBC model and efficiently study an aggregate productivity shock. This approach generates moments similar to those analyzed in King and Rebelo (1999).

This achievement relies on most parameters used in the discrete-time literature on heterogeneous firms, such as those in Bachmann et al. (2013), Khan and Thomas (2008), and Winberry (2018). This work modifies only the parameters related to quadratic investment costs due to the cost structure and employs a slightly different parameterization for idiosyncratic shocks to align the productivity distribution function more closely with empirical data.

Moreover, the study identifies scenarios where a model with heterogeneous firms possessing a large number of productivity levels can be simplified to a model with only two distinct idiosyncratic productivity levels. It concludes that this simplification is valid for questions related to the steady-state of the economy, where restrictions on certain state variables are not of interest. Subsequently, the nonlinear effects of the diffusion model were analyzed under scenarios in which firms face different global parameters in the steady state. The results revealed diverse effects depending on productivity, both in magnitude and direction.

To clarify the scenarios under which a Poisson process model is reasonable, an aggregate productivity shock was incorporated. For questions involving this type of shock, based on the moments and IRFs obtained, the simplification of productivity levels is no longer valid for the economy as a whole. The moments derived differ significantly from those of the data and the model with many productivity levels.

The central result emerges when disaggregating the effects of an aggregate shock of one standard deviation. Firms in the lowest quintiles are more severely affected in terms of investment and experience a more prolonged reaction. This outcome could have important implications for inequality within the economy, considering the findings of Song et al. (2019), which demonstrate a tendency for wage types to concentrate by firm type.

Finally, regarding potential extensions of this thesis, a theoretical direction would involve incorporating market imperfections, focusing on credit constraints. This could be developed through: (i) financing requirements as in the financial accelerator of Bernanke et al. (1999), (ii) an additional restriction on capital dynamics, or (iii) a non-negativity constraint on profits. On the empirical side, various studies have explored cross-sectional or short-panel firm data. In this case, the model allows for the joint identification of time series and panel data, which could be highly relevant for the design of macroprudential policies.

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A Appendix

A.1 HJB Diffusion Process

The firm's problem defined in (1-3) can be expressed in discrete time through the following Bellman equation:

$$v(k_t, z_t) = max\Big\{(z_t k_t^\alpha - i_t - \frac{\chi}{2}\frac{i_t^2}{k_t})dt + \beta(dt)E_t[v(k_{t+dt}, z_{t+dt})]\Big\}$$

$$k_{t+dt} = dt(i_t - \delta k_t)k_t$$

Given that the intertemporal discount rate can be expressed as $\beta(\Delta) = e^{-\rho\Delta}$, the second term can be represented by a first-order Taylor expansion, simplifying to $1 - \rho\Delta$. Thus:

$$v(k_t, z_t) = max\{z_t k_t^\alpha - i_t - \frac{\chi}{2}\frac{i_t^2}{k_t}\}dt + (1 - \rho dt)E_t[v(k_{t+dt}, z_{t+dt})]$$

By expanding the expectation term and rearranging, we get:

$$v(k_{t+dt}, z_{t+dt})\rho dt = max\{z_t k_t^\alpha - i_t - \frac{\chi}{2}\frac{i_t^2}{k_t}\}dt + E_t[v(k_{t+dt}, z_{t+dt}) - v(k_t, z_t)]$$

Dividing through by dt gives:

$$\rho v(k_{t+dt}, z_{t+dt}) = max\{z_t k_t^\alpha - i_t - \frac{\chi}{2}\frac{i_t^2}{k_t}\} + \frac{E_t[v(k_{t+dt}, z_{t+dt}) - v(k_t, z_t)]}{dt}$$

Taking the limit as $\Delta \to 0$, the resulting expression is:

$$\rho v(k_t, z_t) = \max_{i_t}\{z_t k_t^\alpha - i_t - \frac{\chi}{2}\frac{i_t^2}{k_t}\} + \frac{E_t[dv(k_t, z_t)]}{dt} \tag{30}$$

Under the diffusion process and applying the generalized Itô's Lemma:

$$dv(k_t, z_t) = [\partial_k v(k_t, z_t)(i_t - \delta k_t) + \partial_z v(k_t, z_t)\mu(z_t) + \frac{1}{2}\partial_{zz} v(k_t, z_t)\sigma^2(z_t)]dt + \partial_z v(k_t, z_t)\sigma(z_t)dW_t$$

Taking the expectation, where $E[dW_t] = 0$ as the expectation of a standard Brownian motion:

$$E[dv(k_t, z_t)] = [\partial_k v(k_t, z_t)(i_t - \delta k_t) + \partial_z v(k_t, z_t)\mu(z_t) + \frac{1}{2}\partial_{zz} v(k_t, z_t)\sigma^2(z_t)]dt$$

Substituting into (41):

$$\rho V(k,z) = \max_i \{zk^{\alpha} - i - \frac{\chi}{2}\left(\frac{i}{k}\right)^2 k\} + \partial_k V(k,z)(i-\delta k) + \partial_z V(k,z)\mu(z) + \partial_{zz}V(k,z)\frac{\sigma^2(z)}{2} \quad (31)$$

A.2 HJB Process with *Poisson*

Similar to the derivation presented in the previous appendix, we can obtain the HJB equation for the case where a firm's productivity can take two values, and the stochastic process follows a *Poisson* process.

Let us formulate the firm's problem for periods of length dt. Firms have an intertemporal discount rate equal to that of their owners, $\beta(dt) = e^{-\rho dt}$. Firms can have two levels of productivity, z^h or z^l. Firms retain their productivity level with probability $p^h(dt) = e^{-\lambda^h dt}$ ($p^l(dt) = e^{-\lambda^l dt}$) and transition to a different productivity level with probability $1 - p^h(dt)$ $(1 - p^l(dt))$.

$$v(k_t, z_t) = max\{z_t k_t^{\alpha} - i_t - \frac{\chi}{2}\frac{i_t^2}{k_t}\}dt + \beta(dt)E_t[v(k_{t+dt}, z_{t+dt})]$$

$$k_{t+dt} = (i_t - \delta k_t)k_t dt$$

The expectation term can be expressed in terms of probabilities since there are only two possible states for productivity:

$$v^h(k_t, z_t^h) = max\{z_t^h k_t^{\alpha} - i_t - \frac{\chi}{2}\frac{i_t^2}{k_t}\}dt + \beta(dt)[p^h(dt)v^h(k_{t+dt}, z_{t+dt}^h) + (1-p^h(dt))v^l(k_{t+dt}, z_{t+dt}^l)]$$

$$k_{t+dt} = (i_t - \delta k_t)k_t dt$$

Both the intertemporal discount rate and the probability of maintaining a given productivity level can be rewritten and approximated using a first-order Taylor expansion:

$$\beta(dt) = e^{-\rho dt} \approx 1 - \rho dt,\ p^h(dt) = e^{-\lambda^h dt} \approx 1 - \lambda^h dt$$

Substituting this into the Bellman equation:

$$v^h(k_t, z_t^h) = max\{z_t^h k_t^\alpha - i_t - \frac{\chi}{2}\frac{i_t^2}{k_t}\}dt + (1-\rho dt)[(1-\lambda^h dt)v^h(k_{t+dt}, z_{t+dt}^h) + \lambda^h dt v^l(k_{t+dt}, z_{t+dt}^l)]$$

Rearranging:

$$[(1-\lambda^h dt)v^h(k_{t+dt}, z_{t+dt}^h) + \lambda^h dt v^l(k_{t+dt}, z_{t+dt}^l)]\rho dt = max\{\pi(k_t, z_t^h)\}dt + [(1-\lambda^h dt)v^h(k_{t+dt}, z_{t+dt}^h) + \lambda^h dt v^l(k_{t+dt}, z_{t+dt}^l)] - v^h(k_t, z_t^h)$$

Dividing both sides by dt:

$$\rho v^h(k_{t+dt}, z_{t+dt}^h) = max\{\pi(k_t, z_t^h)\} + \frac{v^h(k_{t+dt}, z_{t+dt}^h) - v^h(k_t, z_t^h)}{dt} + \lambda^h[v^l(k_{t+dt}, z_{t+dt}^l) - v^h(k_{t+dt}, z_{t+dt}^h)]$$

Substituting the law of motion for k_{t+dt} into the firm's value function and taking $dt \to 0$:

$$\lim_{dt\to 0}\frac{v^h(k_{t+dt}) - v^h(k_t)}{dt} = \lim_{dt\to 0}\frac{v^h(dt(i_t - \delta k_t) + k_t) - v^h(k_t)}{dt} = \partial_k v^h(k_t)(i_t - \delta k_t))$$

Replacing this result into the Bellman equation and taking the limit as $dt \to 0$, the HJB becomes:

$$\rho v^h(k_t, z_t^h) = \max_{i_t}\{\pi(k_t, z_t^h)\} + \partial_k v^h(k_t)(i_t - \delta k_t)) + \lambda^h[v^l(k_t, z_t^l) - v^h(k_t, z_t^h)]$$

A.3 KFE Process with *Poisson*

This derivation is an adaptation of the Kolmogorov equation for a *Poisson* process as presented in Ahn (2017).

First, consider a continuum of firms, each with a different level of capital k and productivity z^p. The productivity level is a random variable that can take two values, z^h and z^l, and follows a *Poisson* process with intensities λ^h and λ^l.

The evolution of the capital for any given firm can be defined in discretized time as follows:

$$k_{t+dt} - k_t = (i_t - \delta k_t)dt \iff k_t = k_{t+dt} - (i_t - \delta k_t)dt$$

Following the investment decision and the resulting change in capital, the productivity level z_{t+dt}, which is a random variable, becomes known. The productivity level changes with a probability of $\lambda^p dt$.

The cumulative density function (CDF) related to capital accumulation is defined as:

$$G^p(k,t) = Pr(k_t \leq k, z_t = z^p)$$

This CDF represents the proportion of firms at time t with productivity level z^p and capital less than or equal to k. The CDF satisfies standard properties of cumulative distribution functions for a random variable, such as:

$$G^h(0,t) + G^l(0,t) = 0$$

$$G^h(\infty,t) + G^l(\infty,t) = 1$$

The probability density function is defined as the derivative of the CDF with respect to k: $g^p(k,t) \equiv \partial_k G^p(k,t)$.

To derive the law of motion for the CDF of accumulated capital, consider the capital motion equation and fix any point k. Then, determine the probability that k_{t+dt} is less than or equal to the given k. From the capital motion equation, this probability depends on the current level of capital k_t and whether net investment is positive or negative. There are three possible alternatives for the capital level at time $t + dt$ to fall below the fixed k, with only two cases occurring simultaneously:

$$Pr(k_{t+dt} \leq k | z^p_{t+dt} = z^p_t) = Pr(k_t \leq k) + Pr(k \leq k_t \leq k - (i_t - \delta k_t)dt)$$

(if $(i_t - \delta k_t)dt < 0$ and the productivity level does not change),

$$Pr(k_{t+dt} \leq k | z^p_{t+dt} = z^p_t) = Pr(k_t \leq k) + Pr(k - (i_t - \delta k_t)dt \leq k_t \leq k)$$

(if $(i_t - \delta k_t)dt > 0$ and the productivity level does not change).

The first probability corresponds to firms already below the threshold, while the second term represents firms transitioning into the range due to their investment policy. Investment is a smooth, continuous function of the current capital level, conditional on productivity, and thus capital moves in only one direction.

For simplicity, we focus on the case where net investment is negative. Using $Pr(k \leq k_t) = 1 - Pr(k_t \leq k)$, we obtain:

$$Pr(k_{t+dt} \leq k) = Pr(k_t \leq k - (i_t - \delta k_t)dt)$$

Considering that the next productivity level z_{t+dt} can be either z^p or z^{-p}, with probabilities $(1 - dt\lambda^p)$ and $dt\lambda^{-p}$ respectively, substituting the probability definitions gives:

$$G^p(k, t + dt) = (1 - dt\lambda^p)G^p(k - (i_t - \delta k_t)dt, t) + dt\lambda^{-p}(k - (i_t - \delta k_t)dt, t)$$

Subtracting $G^p(k, t)$ from both sides and dividing through by dt:

$$\frac{G^p(k, t + dt) - G^p(k, t)}{dt} = \frac{G^p(k - (i_t - \delta k_t)dt, t) - G^p(k, t)}{dt} - \lambda^p G^p(k - (i_t - \delta k_t)dt, t) \\ + \lambda^{-p}(k - (i_t - \delta k_t)dt, t)$$

Taking the limit as $dt \to 0$:

$$\partial_t G^p(k, t) = \partial_k G^p(k, t)(k - (i_t - \delta k_t)) - \lambda^p G^p(k, t) + \lambda^{-p} G^{-p}(k, t)$$

To obtain the Kolmogorov forward equation, differentiate $G^p(k, t)$ and $G^{-p}(k, t)$ with respect to k, focusing on the density $g^p(k, t)$:

$$\partial_t G^p(k, t) = \partial_k G^p(k, t)(k - (i_t - \delta k_t)) - \lambda^p \partial_k G^p(k, t) + \lambda^{-p} \partial_k G^{-p}(k, t)$$

In steady-state, where $\partial_t g^p(k, t) = 0$, the equation simplifies to:

$$0 = g^p(k,t)(k-(i_t-\delta k_t)) - \lambda^p g^p(k,t) + \lambda^{-p} g^{-p}(k,t)$$

B Upwind, sparse matrices and Firms' distribution

As shown in equation (8), the nonlinear PDE requires iterations of the value function to determine its limiting form. The first step involves generating an initial guess, enabling the computation of the optimal investment value V^n when $n = 0$. Using this initial investment level (i^0, a fixed value), the PDE transforms into a linear finite difference problem, which can be solved using the implicit finite differences method, as shown below:

$$\frac{V^{n+1}-V^n}{\Delta} + \rho V^{n+1} = zk^\alpha - i^n - \frac{\chi_1}{2}\left(\frac{i^n}{k}\right)^2 k + \partial_k V^{n+1}(i^n - \delta k) + \partial_z V^{n+1}\mu(z) + \partial_{zz} V^{n+1}\frac{\sigma^2(z)}{2} \tag{32}$$

In equation (25), components with superscripts n and $n+1$ represent the known value function from the current iteration (V^n) and the next iteration (V^{n+1}). The optimal investment level i^n is derived from the value function V^n, transforming equation (25) into a form suitable for the implicit finite differences method to compute V^{n+1} until convergence.[10] The remainder of this section elaborates on this process.

B.1 Upwind Method

To avoid the use of boundary conditions and achieve a more accurate solution, the Upwind method is employed. This method combines forward and backward finite difference discretizations of the value function.

The derivatives of the value function can be expressed in terms of a discretized grid as follows:

$$\text{Backward difference: } \partial_k V \approx \frac{V_{i,j} - V_{i-1,j}}{dk}$$
$$\text{Forward difference: } \partial_k V \approx \frac{V_{i+1,j} - V_{i,j}}{dk}$$

[10] As previously noted, V_k^{n+1}, V_z^{n+1}, and V_{zz}^{n+1} represent the first and second partial derivatives of the value function with respect to capital and productivity in iteration $n+1$.

Here, Δk represents the distance between discretized capital points. In $V_{i,j}$, the first subscript denotes the position on the capital grid, while the second subscript denotes the position on the productivity grid.

The Upwind method applies the forward difference when the capital differential is positive and the backward difference when the capital differential is negative. Thus, $V_k^{n+1}(i^n - \delta k)$ can be expressed as:

$$V_k^{n+1}(i^n - \delta k) = \frac{V_{i,j}^{n+1} - V_{i-1,j}^{n+1}}{dk}(i^n - \delta k)\mathbb{1}_{\{i^n - \delta k \leq 0\}} + \frac{V_{i+1,j}^{n+1} - V_{i,j}^{n+1}}{dk}(i^n - \delta k)\mathbb{1}_{\{i^n - \delta k \geq 0\}}$$

Once this process is complete, the implicit finite differences method is used, alongside bisection, to compute the next value function. This yields:

$$\begin{aligned}\frac{V^{n+1} - V^n}{\Delta} + \rho V^{n+1} = x^n + \frac{V_{i,j}^{n+1} - V_{i-1,j}^{n+1}}{dk}(i^n - \delta k)\mathbb{1}_{\{i^n - \delta k \leq 0\}} + \frac{V_{i+1,j}^{n+1} - V_{i,j}^{n+1}}{dk}(i^n - \delta k)\mathbb{1}_{\{i^n - \delta k \geq 0\}} + \\ \frac{V_{i,j+1}^{n+1} - V_{i,j}^{n+1}}{dz}(\mu(z))\mathbb{1}_{\{\mu(z_t) \geq 0\}} + \frac{V_{i,j}^{n+1} - V_{i,j-1}^{n+1}}{dz}(\mu(z_t))\mathbb{1}_{\{\mu(z_t) \leq 0\}} + \frac{V_{i,j+1}^{n+1} - 2V_{i,j}^{n+1} + V_{i,j-1}^{n+1}}{dz}\frac{\sigma^2(z)}{2}\end{aligned} \tag{33}$$

In this equation, $x^n \equiv zk^\alpha - i^n - \frac{\chi_1}{2}\left(\frac{i^n}{k}\right)^2 k$. The second derivative with respect to productivity uses central discretization.

Numerical experiments, such as those by Candler (1999), show that the backward difference converges to the exact solution for $i^n - \delta k < 0$, and the forward difference converges for $i^n - \delta k > 0$. Achdou et al. (2020) demonstrate that this algorithm is unconditionally stable, leveraging the implicit finite differences method's properties as established by Barles and Souganidis (1991).

B.2 Diagonal Matrices

The problem described in (26) can be simplified by defining a square matrix that transforms the value function into a diagonal square matrix of dimensions $I \times K$, where I represents the number of capital points in the grid, and K represents the number of possible productivity states used in the discretization.

It is important to note that the matrices K and Z, presented below, contain two boundary condition columns represented by $\vec{0}$, corresponding to $(x_{1,1}, 0, \ldots, 0)'$ and $(0, \ldots, 0, z_{I,K})'$. These columns are necessary to compute $V_{0,1}^{n+1}$ and $V_{I+1,K}^{n+1}$. However, since these boundary conditions are unknown and the Upwind method is used to avoid them, we can fix $x_{1,j} = 0$ and $z_{I,j} = 0$, with $j \in (1, \ldots, K)$. This ensures that these elements are not necessary for the existence of V^{n+1} and can therefore be set to 0.

The derivative with respect to capital can be represented in matrix form as follows:

$$
\mathrm{K} = \begin{bmatrix}
y_{1,1} & z_{1,1} & 0 & & & & & \cdots & & & & 0 \\
x_{2,1} & y_{2,1} & \ddots & 0 & & & \cdots & & & & \vdots & \\
0 & \ddots & \ddots & z_{I-2,1} & 0 & & & & & & & \\
\vdots & 0 & x_{I-1,1} & y_{I-1,1} & z_{I-1,1} & 0 & & & & & & \\
 & \vdots & 0 & x_{I,1} & y_{I,1} & 0 & 0 & & & & & \\
 & & \vdots & 0 & 0 & y_{1,2} & z_{1,2} & 0 & & & & \\
 & & & \vdots & 0 & x_{2,2} & y_{2,2} & z_{2,2} & 0 & & & \\
 & & & & \vdots & 0 & \ddots & \ddots & \ddots & 0 & & \\
 & & & & & \vdots & 0 & x_{I,2} & y_{I,2} & 0 & 0 & \\
 & & & & & & \vdots & 0 & \ddots & \ddots & 0 & 0 \\
0 & & & & & & & & \vdots & 0 & x_{I,K} & y_{I,K}
\end{bmatrix}
$$

where the elements of the matrix are defined as:

$$
x_{i,j} = -\frac{(i^n - \delta k)\mathbb{1}_{\{i^n - \delta k \leq 0\}}}{dk} \tag{34}
$$

$$
y_{i,j} = \frac{(i^n - \delta k)\mathbb{1}_{\{i^n - \delta k \leq 0\}}}{dk} - \frac{(i^n - \delta k)\mathbb{1}_{\{i^n - \delta k \geq 0\}}}{dk} \tag{35}
$$

$$
z_{i,j} = \frac{(i^n - \delta k)\mathbb{1}_{\{i^n - \delta k \geq 0\}}}{dk} \tag{36}
$$

Next, the matrix for transitions in the second dimension of the value function, corresponding to productivity, must be computed. The banded matrix for this dimension will have increments of I, as there are K points for each capital level. Once these K points are completed, the next productivity level begins.[11]

The Z matrix, representing productivity transitions, can be visualized as follows:

[11] In Matlab, an efficient way to construct these matrices is to use sparse matrices, which save memory by not storing the zeros in the diagonal bands.

$$
\mathrm{Z} = \begin{bmatrix}
\bullet & 0 & \cdot & \cdot & 0 & \diamond & 0 & \cdot & & & & & & & 0 \\
0 & \bullet & 0 & \cdot & \cdot & 0 & \diamond & 0 & \cdot & & & & & & \\
\cdot & 0 & \bullet & 0 & \cdot & \cdot & 0 & \diamond & 0 & \cdot & & & & & \\
\cdot & \cdot & 0 & \bullet & 0 & \cdot & \cdot & 0 & \diamond & 0 & \cdot & & & & \\
0 & \cdot & \cdot & 0 & \bullet & 0 & \cdot & \cdot & 0 & \diamond & 0 & \cdot & & & \\
\star & 0 & \cdot & \cdot & 0 & \bullet & 0 & \cdot & \cdot & 0 & \diamond & 0 & \cdot & & \\
0 & \star & 0 & \cdot & \cdot & 0 & \ddots & 0 & \cdot & \cdot & 0 & \diamond & 0 & \cdot & \\
\cdot & 0 & \star & 0 & \cdot & \cdot & 0 & \bullet & 0 & \cdot & \cdot & 0 & \diamond & 0 & \cdot \\
 & \cdot & 0 & \star & 0 & \cdot & \cdot & 0 & \bullet & 0 & \cdot & \cdot & 0 & \diamond & 0 \\
 & & \cdot & 0 & \star & 0 & \cdot & \cdot & 0 & \bullet & 0 & \cdot & \cdot & 0 & \diamond \\
 & & & \cdot & 0 & \star & 0 & \cdot & \cdot & 0 & \bullet & 0 & \cdot & \cdot & 0 \\
 & & & & \cdot & 0 & \star & 0 & \cdot & \cdot & 0 & \bullet & 0 & \cdot & \cdot \\
 & & & & & \cdot & 0 & \star & 0 & \cdot & \cdot & 0 & \bullet & 0 & \cdot \\
 & & & & & & \cdot & 0 & \star & 0 & \cdot & \cdot & 0 & \bullet & 0 \\
0 & & & & & & & \cdot & 0 & \star & 0 & \cdot & \cdot & 0 & \bullet
\end{bmatrix}
$$

where the elements are defined as:

$$\star = -\frac{(\mu(z_t))\mathbb{1}_{\{\mu(z_t)\leq 0\}}}{dz} + \frac{\sigma^2(z_t)}{2dz} \tag{37}$$

$$\bullet = \frac{\mu(z_t))\mathbb{1}_{\{\mu(z_t)\leq 0\}}}{dz} - \frac{\mu(z_t))\mathbb{1}_{\{\mu(z_t)\geq 0\}}}{dz} - \frac{\sigma^2(z_t)}{dz} \tag{38}$$

$$\diamond = \frac{\mu(z_t)\mathbb{1}_{\{\mu(z_t)\geq 0\}}}{dz} + \frac{\sigma^2(z_t)}{2dz} \tag{39}$$

Combining the K and Z matrices results in a linear system to be solved:

$$\frac{V^{n+1}}{\Delta} - \frac{V^n}{\Delta} + \rho V^{n+1} = x^n + (K^n + Z^n)V^{n+1}$$

Here, the sum of the matrices $K^n + Z^n$ results in a banded diagonal matrix. This matrix uses the main diagonal, two adjacent diagonals, and the diagonals I-steps away from the center, as illustrated in Figure 1.

Starting with an initial guess for V^n, the value function is updated to V^{n+1} using the implicit finite differences method. The equation above can then be rearranged to isolate the next value function:

$$V^{n+1} = \Big[\frac{\rho}{\Delta}I - (K^n + Z^n)\Big]^{-1}(x^n + \frac{V^n}{\Delta}) \tag{40}$$

After reorganizing the value function, the resulting matrix forms the actual value function for the given algorithm. Once convergence is achieved, the policy function for each firm can be obtained, as illustrated in Figure B.2, panel (a).

Figure B.1: Visualization matrix $K^n + Z^n$

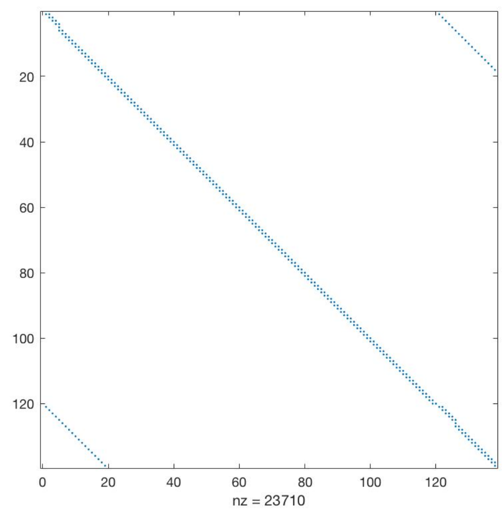
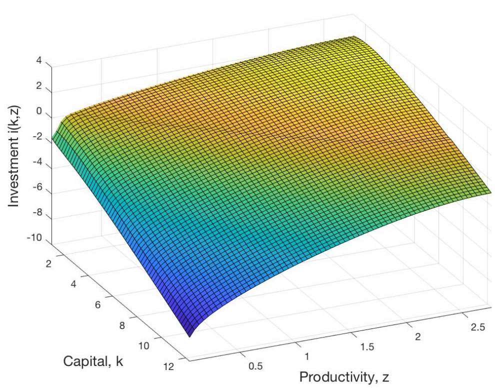
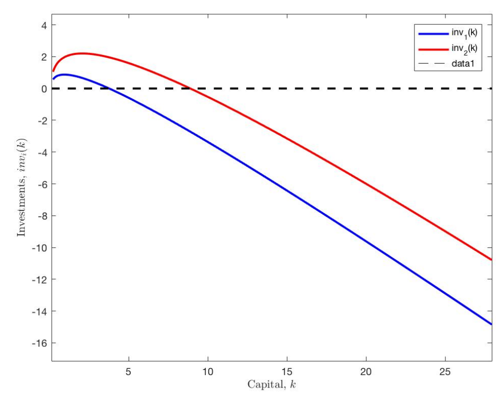


Figure B.2: Policy Function of different models



((a)) Difussion



((b)) *Poisson*

For the problem with two productivity types, the policy function follows the same algorithm. However, the matrix for productivity state changes is simpler, as there are only two possible states. Consequently, the system is represented by a single large matrix of size $2I \times 2I$, converging more quickly. Similarly to the previous model, the policy function is illustrated in Figure B.2, panel (b).

Due to the minimum capital constraint, the policy function $i(k|z)$ exhibits a more linear pattern for lower productivity levels, as firms cannot disinvest as much as they might prefer, given the requirement to maintain minimum capital levels.

B.3 Firm Distributions

To determine the steady-state distribution of firms, it is necessary to recall the KFE given by equation (11):

$$0 = -\frac{\partial}{\partial k}((i^*(k,z) - \delta k)g(k,z)) - \frac{\partial}{\partial z}(\mu(z)g(k,z)) + \frac{1}{2}\frac{\partial^2}{\partial z^2}(\sigma^2 g(k,z))$$

This equation can be written in terms of the matrices K^n and Z^n by discretizing the problem:

$$0 = (K^n + Z^n)^T g(k,z) \tag{41}$$

Here, $(K^n + Z^n)$[12] represent the terms described in equations (26–32).

For the numerical computation of this equation, $g(k,z)$ can be expressed as:

$$g(k,z) = (K^n + Z^n)^{T^{-1}}\vec{0}$$

The issue with expressing the distribution this way is that the matrix $(K^n + Z^n)^T$ is singular and therefore not invertible, which makes the above problem unsolvable. To address this, Achdou et al. (2017) propose a "dirty" replacement: substituting one component of $\vec{0}$ with an arbitrary value and then normalizing the resulting $\hat{g}$ to ensure the following integral equals 1, since g represents a cumulative probability function.[13]

$$1 = \int_{k_{min}}^{k_{max}} \int_{z_{min}}^{z_{max}} g(k,z)dzdk \tag{42}$$

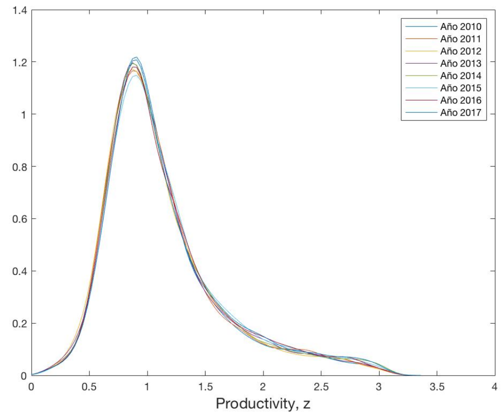
C Productivity Distributions

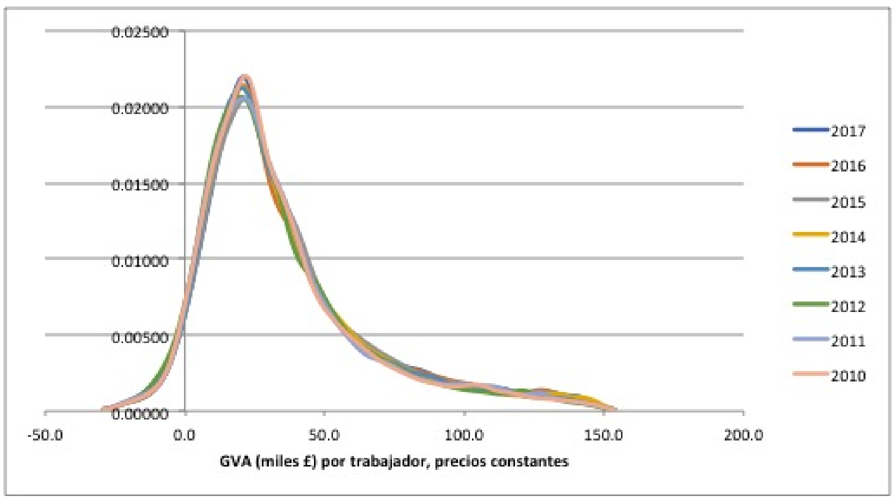
The figure below illustrates the *Kernel* distributions of rescaled firm data for the years 2010 to 2017.

It can be observed that the distribution of these firms remains consistent across the years and resembles a lognormal distribution. Additionally, the lower tails of the distributions show no significant changes, while the upper tails exhibit some volatility throughout the studied period.

[12] The transpose arises from the negative terms in the expression. Upon "unfolding" the matrix, it becomes evident that the terms involving the first partial derivative include indicator functions. These indicators switch due to the negative signs, making the matrix the transpose of the one presented in equation (33).

[13] It is crucial for the sum of the densities in $\hat{g}$ to be greater than 0. This ensures proper normalization of the true density and avoids negative values at any point in the density function.



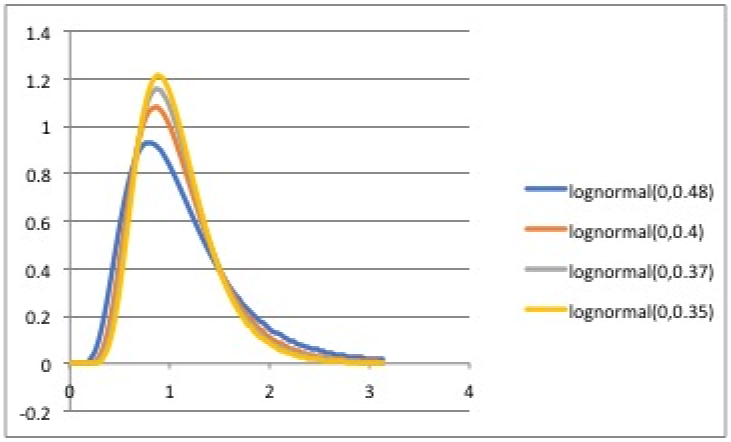


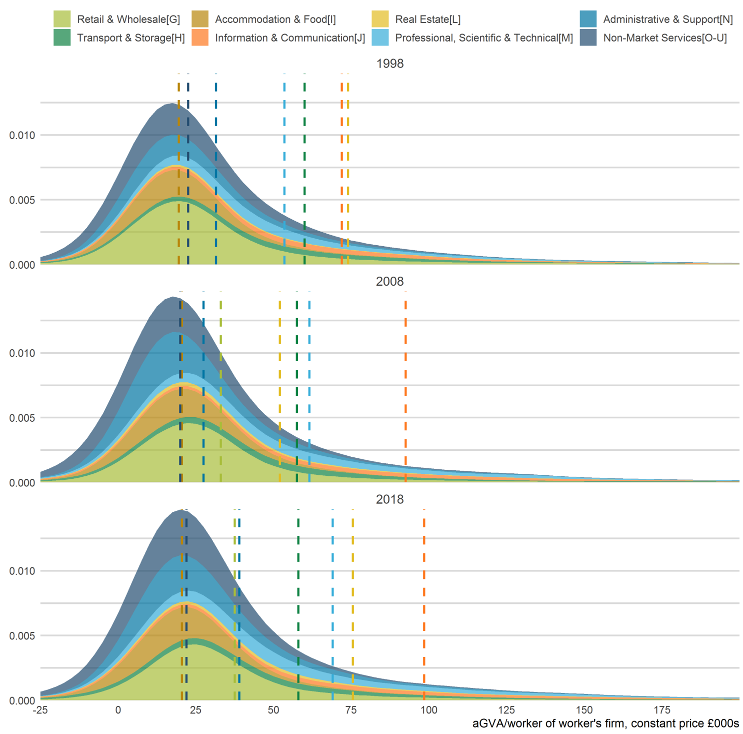
The above figure corresponds to the original $Kernel$ distribution, measured in thousands of £ per worker. Naturally, the shape mirrors the rescaled distribution. Since part of the original data is in the negatives, rescaling was necessary to define the distribution in the positive domain of a lognormal distribution.

The above figure shows the distribution of firms by type for three years from the database discussed in the main section of this study.

D Robustness analysis of Poisson Process Model

In Tables 12–15, the outputs of models with the same parameters as the previous cases are presented, but with different Poisson process intensities to evaluate the robustness of the two-productivity-level model. Table 12 shows the results for a fixed proportion of 90–10 (low-productivity to high-productivity





Fuente: Office for National Statistics - Annual Business Survey (ABS), Inter-Departmental Business Register (IDBR)

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firms) in steady state, with columns representing different ratios of z^h/z^l.

Table 12: Prices and Aggregates, by different productivity, ratio 90-10

z^h/z^l	$\approx$4 (3.33/0.825)	$\approx$3.5 (3/0.858)	$\approx$3 (2.685/0.895)	$\approx$2.5 (2.333/0.933)
r	3.46%	3.47%	3.48%	3.49%
Investment	0.292	0.2917	0.2933	0.2931
Capital	4.2322	4.2277	4.2505	4.2473
Profits	1.2915	1.2827	1.2782	1.2696
Consumption	0.9995	0.9910	0.9849	0.9766

Table 13: Prices and Aggregates, by different productivity, ratio 85-15

z^h/z^l	$\approx$4 (2.96/0.74)	$\approx$3.5 (2.744/0.784)	$\approx$3 (2.481/0.825)	$\approx$2.5 (2.189/0.876)
r	3.45%	3.46%	3.47%	3.48%
Investment	0.2916	0.2936	0.2928	0.2924
Capital	4.2265	4.2547	4.2435	4.2379
Profits	1.2981	1.2973	1.2816	1.2725
Consumption	1.0065	1.0038	0.9888	0.9800

Table 14: Prices and Aggregates, by different productivity, ratio 80-20

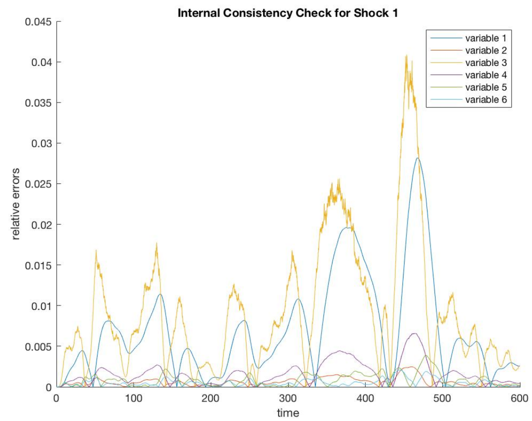
z^h/z^l	$\approx$4 (2.68/0.67)	$\approx$3.5 (2.503/0.715)	$\approx$3 (2.3/0.766)	$\approx$2.5 (2.063/0.825)
r	3.44%	3.45%	3.47%	3.48%
Investment	0.2909	0.2914	0.2920	0.2925
Capital	4.2155	4.2239	4.2321	4.2384
Profits	1.2987	1.2911	1.2827	1.2739
Consumption	1.0079	0.9996	0.9907	0.9815

Table 15: Prices and Aggregates, by different productivity, ratio 70-30

z^h/z^l	$\approx$4 (2.26/0.565)	$\approx$3.5 (2.146/0.613)	$\approx$3 (2.012/0.671)	$\approx$2.5 (1.85/0.74)
r	3.44%	3.45%	3.47%	3.48%
Investment	0.2914	0.2915	0.2919	0.2923
Capital	4.2227	4.2245	4.2310	4.2358
Profits	1.2999	1.2916	1.2843	1.2758
Consumption	1.0085	1.0001	0.9924	0.9835

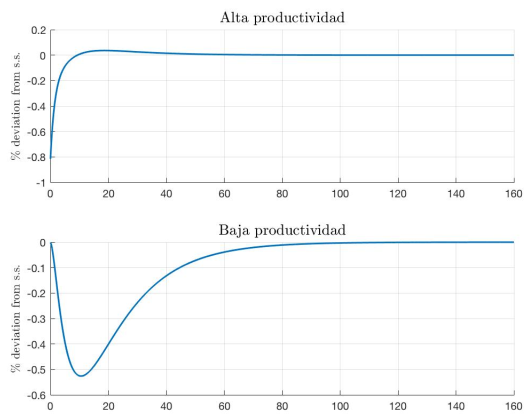
E Internal Consistency Graph of the Simulation

Below is the consistency check graph from the code by Ahn et al. (2017), which confirms that the errors in the linearized variables are minimal. The results are presented in the figure below.



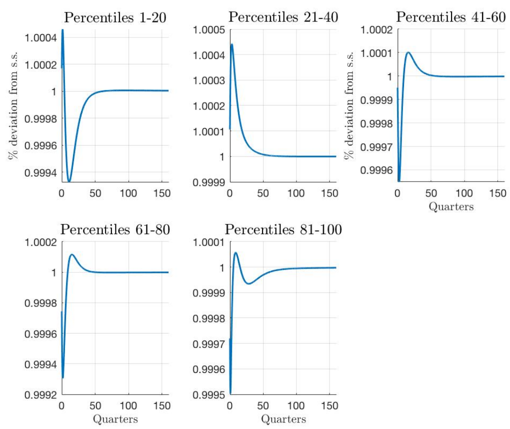
F Disaggregation of Investment in the *Poisson* Model

The following figure shows the disaggregated effects of a positive aggregate productivity shock, indicating changes in investment for firms that are classified as having high or low productivity in each period.



F.1 Disaggregation of Distribution Function Movement in the Diffusion Model

The following figure illustrates the variation in the distribution of different percentiles over time. It can be observed that changes in the distributions of firms are minimal in response to an aggregate shock.


Percentiles 1-20
Percentiles 21-40
Percentiles 41-60
Percentiles 61-80
Percentiles 81-100
% deviation from s.s.
Quarters