Intersection of 2 lines in a circle

Paul Maynard

October 21, 2018

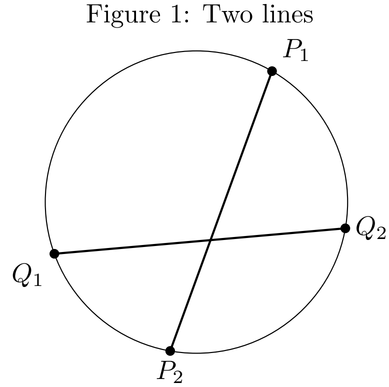
Abstract

This document outlines a method for determining probabilities how many slices two random lines will divide a circle into.

1 Setup

Consider two lines each defined by two points on a circle, P_1 and P_2, Q_1 and Q_2, as in Figure 1.

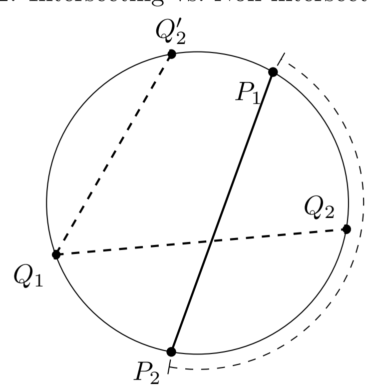
Figure 1: Two lines



These lines can obviously intersect or not. If they intersect, then the circle is cut into four regions. If they don't then it is cut into three. This intersection can be checked without reference to the lines, but by simply looking at the angles of the points.

If P_1 and P_2 are "next to" each other, that is, there is no point between them, then they do not intersect any point. On the other hand, if there is a point between them, but not that points counterpart, then that point's line will intersect it.

Figure 2: Intersecting vs. Non-intersecting lines



We can see in Figure 2 that the line $\overline{Q_1Q_2}$ intersects $\overline{P_1P_2}$, while $\overline{Q_1Q_2'}$ does not. In fact, any Q_2 within the dashed interval will result in an intersection, while any outside will not. Thus, in order to check fot intersection, it is sufficient to check the angles.

2 Probability

To calculate the probability of different amounts of spaces, consider the possible variables. Since the configuration is preserved under rotation, it is sufficient to consider the angles of the points from P_1. In Figure 3, the angle between P_1 and P_2 is α, and β_1 and β_2 are the angles of Q_1 and Q_2 from P_1. There are two possible configurations of these lines:

1. $\beta_1, \beta_2 \leq \alpha$ or $\beta_1, \beta_2 \geq \alpha$. In this case, there will be no intersection, so they divide the circle into

three regions

2. $\beta_1 < \alpha < \beta_2$ or $\beta_1 > \alpha > \beta_2$. In this case, the lines intersect, so there are four regions.

Figure 3: Angle Configuration

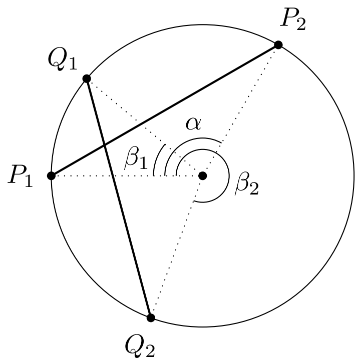
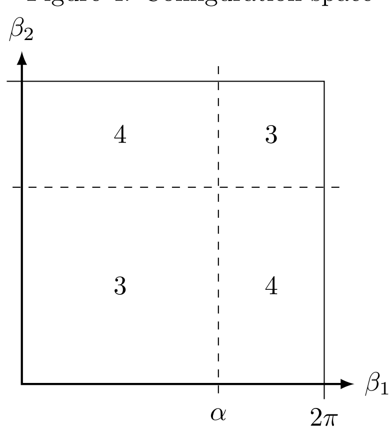


Figure 4: Configuration space



From Figure 4, we can see that, for any given α, the probabilities of a given number of regions for a randomly selected β_1 and β_2 are:

$$P_\alpha(3 \text{ regions}) = \frac{\alpha^2 + (2\pi - \alpha)^2}{4\pi^2} \tag{1}$$

$$P_\alpha(4 \text{ regions}) = \frac{2\alpha(2\pi - \alpha)}{4\pi^2} \tag{2}$$

We can then calculate the total probability of each by integrating over the entire posible range of α

$$P(3 \text{ regions}) = \int_0^{2\pi} \frac{\alpha^2 + (2\pi - \alpha)^2}{8\pi^3} \, d\alpha = \frac{2}{3} \tag{3}$$

$$P(4 \text{ regions}) = \int_0^{2\pi} \frac{2\alpha(2\pi - \alpha)}{8\pi^3} \, d\alpha = \frac{1}{3} \tag{4}$$

3 Conclusion

From these calculations, we can see that the odds are $\frac{2}{3}$ for 3 regions, and $\frac{1}{3}$ for 2 regions. This method can possibly be extended to n regions, with some more work.