Quantum Computation & Quantum Information

Book: Quantum Computation and Quantum Information. Nielsen, M. A., & Chuang, I. L.

Complex Numbers Short recap

Let $a, b \in \mathbb{R}$ and $z \in \mathbb{C}$ such that $z = a+bi$.

Real part	$\Re\mathfrak{e}(a+ib) = a$
Imaginary part	$\Im\mathfrak{m}(a+ib) = b$
Absolute Values	$\lvert z\rvert = \sqrt{zz^*}$
	$= \lVert(a\ \ b)\rVert_2$
Complexe conjugate	$(a+ib)^* = a - ib$
	$(a-ib)^* = a + ib$
Trig. Formulas	$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$
Trig. Formulas	$\cos z = \frac{e^{iz}+e^{-iz}}{2}$

Properties 1 (Complexe conjugate)
• $(Z^*)^* = Z$ • $(Z+W)^* = Z^* + W^*$ • $(Z-W)^* = Z^* - W^*$ • $(ZW)^* = Z^*W^*$ • $Z^*Z = |Z|^2$ • $(Z^n)^* = (Z^*)^n$*, for* $n \in \mathbb{Z}$ • $ln(Z^*) = (ln(Z))^*$ *if Z is not 0 or a negative real number.*

Properties 2 (Absolute Values)
• $|z_1z_2| = |z_1|\,|z_2|$ • *The absolute value define the metric of the space $\mathbb{C}$ ($\mathbb{C}$ is complete).* • $|z_1+z_2|^2 = |z_1|^2+|z_2|^2+\Re\mathfrak{e}(z_1z_2^*)$ • $|z_1-z_2|^2 = |z_1|^2+|z_2|^2-\Re\mathfrak{e}(z_1z_2^*)$ • $|z_1+z_2|^2+|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$

Linear Algebra

Short Definitions

Hermitian Operator	$A = A^\dagger$
Normal Operator	$AA^\dagger = A^\dagger A$
P Orthogonal Complement	$Q \equiv \mathbb{I} - P$

Linear Operator

A **linear operator** between the vector spaces V and W is define to be any function $\hat{A} : V \to W$ which satisfies:
$\hat{A}(\alpha\vec{v} + \beta\vec{w}) = \alpha\hat{A}\vec{v} + \beta\hat{A}\vec{w}$.

Properties 3 *Let $\hat{A}$ be a linear operator on $V \to W$ and A be the matrix representation of $\hat{A}$.* • $\hat{A}(\sum_i a_i |v_i\rangle) = \sum_i a_i\hat{A}|v_i\rangle$ • $\hat{A}|v_j\rangle = \sum_i A_{ij}|w_i\rangle$

Inner product

A Inner Product $\langle .,. \rangle$ is a function that output a complex number and satisfies the following conditions: Let $\vec{v} \in \mathbb{C}^n, \vec{w} \in \mathbb{C}^n$.

1. $\langle\vec{v}, \sum_i a_i\vec{w_i}\rangle = \sum_i a_i\langle\vec{v}, \vec{w_i}\rangle$
2. $\langle\vec{v}, \vec{w}\rangle = (\langle\vec{w}, \vec{v}\rangle)^*$
3. $\langle\vec{w}, \vec{w}\rangle > 0$ if and only if $w \neq 0$. Note $\forall\vec{w}(\langle\vec{w}, \vec{w}\rangle \geq 0)$.

In quantum mecanics the inner product is generaly noted $\langle .|. \rangle$.

Properties 4 • $\langle A, A\rangle = \lVert A\rVert^2$ • *if $\langle A, B\rangle = 0$ then A and B are orthogonal.*

Inner product Space

An inner product space is a vector space V equipped with an inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ (or $\mathbb{R}$).

An Inner product space with an orthonormal basis $|i\rangle$ such that $v = \sum_i v_i|i\rangle$ and $w = \sum_i w_i|i\rangle$, the inner product space of $\langle v, w\rangle$ is define by $(v^*)^T w$.

Hilbert Spaces

A Hilbert space is a vector space(generaly complex) equipped with an inner product, meaning every Cauchy sequence of vectors with respect to the induced norm converges to a vector within the space. **In finit dimentions hilbert spaces is exactly the same thing as Inner Product space.**

Dirac Notation (or Bra-Ket Notation)

Terminology: ket of A is $|A\rangle$ and bra of A is $\langle A|$. ket is a row vector and bra is a column vector.
Example, let $\Sigma = \{A, B, C\}$ then

$|B\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\langle B| = (0, 1, 0)$

Kronecker delta(δ_{ij})

$$\delta_{ij} = \begin{cases}1 & \text{if } i = j,\\ 0 & \text{if } i \neq j.\end{cases}$$

Properties 5 • $\langle i|j\rangle = \delta_{ij}$ • $\mathbb{I}_{ij} = \delta_{ij}$ • $\sum_i \delta_{ij}a_i = a_j$ • $\sum_k \delta_{ik}\delta_{kj} = \delta_{ij}$

Gram-Schmidt(in Dirac Notation)

$$|v_1\rangle = \frac{|w_1\rangle}{\lVert|w_1\rangle\rVert}$$

$$|v_{k+1}\rangle = \frac{|w_{k+1}\rangle - \sum_{i=1}^k \langle v_i|w_{k+1}\rangle\,|v_i\rangle}{\left\lVert|w_{k+1}\rangle - \sum_{i=1}^k \langle v_i|w_{k+1}\rangle\,|v_i\rangle\right\rVert}$$

Adjoin (Hermitian conjugate)

Let $\hat{A}$ be a linear operator on the hilbert space V. $\exists!\hat{A}^\dagger$ on V such that $\langle v, \hat{A}w\rangle = \langle\hat{A}^\dagger v, w\rangle$.

Properties 6 • $|v\rangle^\dagger = \langle v|$ • $(\hat{A}^\dagger)^\dagger = \hat{A}$ • $(\hat{A}\hat{B})^\dagger = \hat{A}^\dagger\hat{B}^\dagger$ • $(\sum_i a_i\hat{A}_i)^\dagger = \sum_i a_i^*\hat{A}_i^\dagger$

Outer Product

Let $|v\rangle \in V, |w\rangle \in W$ where V and W are **inner product spaces**. The **outer product** is define $|w\rangle\langle v|$ as to be a linear operator $V \Rightarrow W$ whose define by $(|w\rangle\langle v|)(|v'\rangle) \equiv |w\rangle\langle v|v'\rangle = \langle v|v'\rangle\,|w\rangle$

Properties 7 • $\sum_i |i\rangle\langle i| = \mathbb{I}$ *(completness relation)* • $|w\rangle\langle v| = \sum_i |w_i\rangle\langle v_i|$ • $|w\rangle\langle v|\,v'\rangle = \sum_i |w_i\rangle\langle v_i|\,v'\rangle$

The Cauchy-Schwarz Inequality

Let the vectors $|v\rangle, |w\rangle$ then $|\langle v|w\rangle|^2 \leq \langle v|v\rangle\langle w|w\rangle$

Tensor Product

Let V and W be vector spaces of dimension m and n respectively. Then $V \otimes W$ is an mn-dimension vector spaces. The elements of $V \otimes W$ are a linear combination of the tensor product $|v\rangle \otimes |w\rangle$.

Properties 8 *Let z be a scalar, $|v\rangle \in V, |w\rangle \in W$, where V and W are* ***Hilbert spaces****.*
• $z(|v\rangle \otimes |w\rangle) = z|v\rangle \otimes |w\rangle = |v\rangle \otimes z|w\rangle$
• $|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$
• $(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$
• $(|w\rangle \otimes |v\rangle, |w'\rangle \otimes |v'\rangle) = \langle w|w'\rangle\langle v|v'\rangle$

Quantum mecanics

Pauli Matrices

• $\sigma_0 = I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ • $\sigma_x = X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
• $\sigma_y = Y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$ • $\sigma_z = Z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$