Quantum Information

Class: IBM Quantum Learning

Complex Analysis Short recap

Let $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $z \in \mathbb{C}$.

Real part	$\Re\mathfrak{e}(a+ib) = a$
Real part	$\Im\mathfrak{m}(a+ib) = b$
Absolute Values	$\lvert z\rvert = \sqrt{zz^*} = \lVert(a\ \ b)\rVert_2$
Complexe conjugate	$(a+ib)^* = a - ib$
	$(a-ib)^* = a + ib$

Properties 1 (Complexe conjugate)
• $(Z^*)^* = Z$ • $(Z+W)^* = Z^* + W^*$
• $(Z-W)^* = Z^* - W^*$ • $(ZW)^* = Z^*W^*$
• $Z^*Z = \lvert Z\rvert^2$ • $(Z^n)^* = (Z^*)^n$, *for* $n \in \mathbb{Z}$
• $ln(Z^*) = (ln(Z))^*$ *if Z is not 0 or a negative real number.*

Properties 2 (Complexe Absolute Values)
• $\lvert z_1 z_2\rvert = \lvert z_1\rvert \lvert z_2\rvert$ • *The absolute value define the metric of the space $\mathbb{C}$ ($\mathbb{C}$ is complete).* • $\lvert z_1+z_2\rvert^2 = \lvert z_1\rvert^2 + \lvert z_2\rvert^2 + \Re\mathfrak{e}(z_1 z_2^*)$
• $\lvert z_1 - z_2\rvert^2 = \lvert z_1\rvert^2 + \lvert z_2\rvert^2 - \Re\mathfrak{e}(z_1 z_2^*)$
• $\lvert z_1 + z_2\rvert^2 + \lvert z_1 - z_2\rvert^2 = 2(\lvert z_1\rvert^2 + \lvert z_2\rvert^2)$

Linear Algebra

Linear Operator

A **linear operator** between the vector spaces V and W is define to be any function $A : V \to W$ which is linear in its input.

Properties 3 *Let $\hat{A}$ be a linear operator on $V \to W$ and A be the matrix representation of $\hat{A}$.* • $\hat{A}(\sum_i a_i \lvert v_i\rangle) = \sum_i a_i \hat{A} \lvert v_i\rangle$
• $\hat{A}\lvert v_j\rangle = \sum_i A_{ij} \lvert w_i\rangle$

Inner product

A Inner Product $\langle .,. \rangle$ is a function that output a complex number and satisfies the following conditions: Let $\vec{v} \in \mathbb{C}^n, \vec{w} \in \mathbb{C}^n$.

1. $\langle \vec{v}, \sum_i a_i \vec{w}_i \rangle = \sum_i a_i \langle \vec{v}, \vec{w}_i \rangle$
2. $\langle \vec{v}, \vec{w} \rangle = (\langle \vec{w}, \vec{v} \rangle)^*$
3. if $x = 0$ and only if $\langle \vec{w}, \vec{w} \rangle \geq 0$

In quantum mecanics the inner product is generaly noted $\langle .|. \rangle$.

Properties 4 $\langle A, A \rangle = \lVert A \rVert^2$

Quantum mecanics

Pauli Matrices

• $\sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ • $\sigma_x = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
• $\sigma_y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ • $\sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Mathematical Concepts

Dirac Notation(or Bra-Ket Notation)

Terminology: ket of A is $|A\rangle$ and bra of A is $\langle A|$. Example, let $\Sigma = \{A, B, C\}$ then $|A\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $|B\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\langle A| = (1, 0, 0)$, $\langle B| = (0, 1, 0)$ Note that $|\Psi\rangle$ then $\langle\Psi| = |\Psi\rangle^T$.
bra-kets: we denote $\langle a|b\rangle$ the matrice product of

Cartesian product

Let $y = \{0, 1\}$ and $x = \{a, b\}$. Then,
$x \times y = \{(0, a), (0, b), (1, a), (1, b)\}$ and
$y \times x = \{(a, 0), (a, 1), (b, 0), (b, 1)\}$.

Tensor Product of vectors

$$\begin{bmatrix}a_0\\\vdots\\a_n\end{bmatrix} \otimes \begin{bmatrix}b_0\\\vdots\\b_n\end{bmatrix} = \begin{bmatrix}a_0b_0\\\vdots\\a_nb_n\end{bmatrix}$$

Tensor Product Properties

1. $(|\phi_1\rangle + |\phi_2\rangle) \otimes |\psi\rangle = |\phi_1\rangle \otimes |\psi\rangle + |\phi_2\rangle \otimes |\psi\rangle$
2. $(a\,|\phi\rangle) \otimes |\psi\rangle = a(|\phi\rangle \otimes |\psi\rangle)$
3. $|a\rangle \otimes |b\rangle = |b\rangle \otimes |a\rangle$

Quantum Information Systems

State Vector

Quantum State of a sytem is represented by a complex column vector. Let the quantum state vector v be equal to $\begin{bmatrix}a_0\\\vdots\\a_n\end{bmatrix}$, where $\sum_{i=0}^{n} |a_i|^2 = 1$ The **euclidean norm** of the $||v|| = \sqrt{\sum_{i=0}^{n} |a_i|^2}$

Common Quantum States

Plus State	$\|+\rangle = \frac{1}{\sqrt{2}}\|0\rangle + \frac{1}{\sqrt{2}}\|1\rangle$
Minus State	$\|-\rangle = \frac{1}{\sqrt{2}}\|0\rangle - \frac{1}{\sqrt{2}}\|1\rangle$
Other State	$\frac{1+2i}{3}\|0\rangle - \frac{2}{3}\|1\rangle$

Standart Basis Measurement

Let a quantum system be in the state $|\psi\rangle$, then the probability for the measure outcome to be a is $Pr(\text{outcome} = a) = |\langle a|\psi\rangle|^2$ If U is an unary matrice then the following Propertie hold, $||U\psi|| = ||\psi||$

Unary Operations

Unary Matrice

A squared matrix U having complex number entries is unitary if it satisfies the equations, $UU^\dagger = U^\dagger U = \mathbb{I}$ where $\mathbb{I}$ is the identity matrix.

Some unitary operations on qubits

Pauli operations:
$\mathbb{I} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ $\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ $\sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$ $\sigma_z = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$

Hadamard operation: $H = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0\\0 & \frac{1}{\sqrt{2}}\end{pmatrix}$

Phase operations: $P_\theta = \begin{pmatrix}1 & 0\\0 & e^{i\theta}\end{pmatrix}$

Quantum circtuit