Linear Algebra from Elementary to Advanced

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Definition

Linear Functions	All terms are of degree 0 or 1. A solution of a system of linear equation is set of points that makes the equation system true.
Consistent	lin. systems is consistent if either 1 or ∞ solutions exist else inconsistent.
Conist	

Coefficient Matrix

$$\begin{cases} A_1x_1 + A_2x_2 + A_3x_3 = \alpha \\ B_1x_1 + B_2x_2 + B_3x_3 = \beta \\ C_1x_1 + C_2x_2 + C_3x_3 = \gamma \end{cases} \Leftrightarrow \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \tag{1}$$

Augmented Matrix

$$\begin{cases} A_1x_1 + A_2x_2 + A_3x_3 = \alpha \\ B_1x_1 + B_2x_2 + B_3x_3 = \beta \\ C_1x_1 + C_2x_2 + C_3x_3 = \gamma \end{cases} \Leftrightarrow \left[\begin{array}{ccc|c} A_1 & A_2 & A_3 & \alpha \\ B_1 & B_2 & B_3 & \beta \\ C_1 & C_2 & C_3 & \gamma \end{array}\right] \tag{2}$$

Row-Equivalence

Two matrce are row-equivalent if there is a sequence of **EROS** that transforms one into the other.

Elementary Row Operations (EROS)

1. **[Replacement]** Replace one row by sum of itself.
2. **[Interchange]** Swap position of 2 rows.
3. **[Scaling]** Multiply all entries in row by non-zero constant.

Echelon Form (ef)

1. All non-zero rows are above any rows of all-zero.
2. Each leading entry of a row is in a column to the right of the roe above it.
3. All entries in a column below a leading entry are 0.

Reduced Row Echelon Form (rref)

1. As to be in echelon form.
2. Leading entry in each row is 1.
3. Each leading 1 is the only non-zero entry in its column.

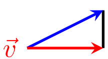
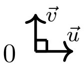
Theorems

Theorem 1 *Every matrix is row equivalent to a unique row echelon form.*

Theorem 2 *Every matrix is row equivalent to a unique row echelon form.*

Orthogonality and Diagonalization

Definition

Inner Product	$\vec{v} \cdot \vec{u} = \vec{v}^T \vec{u} = u_1v_1 + ... + u_nv_n$ (Also called dot product or scalar product)
Length of $\vec{x}$	$\|\|\vec{x}\|\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + ... + x_n^2}$ (Also called Norm or Magnitude)
Unit vectore	A vector with $\|\|\vec{x}\|\| = 1$
Normalization	The formula $\vec{u} = \frac{\vec{x}}{\|\|\vec{x}\|\|}$ creat a unit vector in th same direction as $\vec{x}$.
Distance	$dist(\vec{u}, \vec{v}) = \vec{u} - \vec{v}$ 
Orthogonality	$\vec{u} \cdot \vec{v} = 0$ 
Orthogonal Set	A set of vectors $\{\vec{u}_1, ..., \vec{u}_p\} \in \mathbb{R}^n$ such that each distinct vectors are orthogonal.
Orthogonal Basis	For a subspace W of R^n is a basis that is also an orthogonal set.
Orthonormal set	is a set of orthogonal unit vectors.
Orthogonal matrix	Is a squared matrix whose columns are orthon
Orthogonal Complements	Let W be a subspace of $\mathbb{R}^n$. The orthogonal complement of W is: $W^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{w} = \vec{0}, \forall \vec{w} \in W\}$

Fact about the Orthogonal Complements

• $\vec{0} \in W^\perp$ since $\vec{0} \cdot \vec{w} = 0$ • If $W \in W^\perp, c \in \mathbb{R}$ then $cW \in W^\perp$
• If $\vec{w}_1, \vec{w}_2 \in W^\perp$ so $(\vec{w}_1 + \vec{w}_2) \cdot \vec{x} = \vec{w}_1 \cdot \vec{x} + \vec{w}_2 \cdot \vec{x} \in W$

Theorems

Properties 1 *Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$ then:*
• $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ • $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$ • $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = (c\vec{v}) \cdot \vec{u}$
• $\vec{u} \cdot \vec{u} = u_1^2 + ... + u_n^2 \geq 0$

Theorem 3 (Fundamental Subspaces Theorem) *Let $A_{M \times N}$ then:* • $(row(A))^\perp = nul(A)$ • $(COl(A))^\perp = nul(a^\perp)$

Theorem 4 *If $S = \{\vec{u}_1, ..., \vec{u}_p\}$ is an **orthogonal set** of non-zero vectors in $\mathbb{R}$ then S is a **linearly independant** set.*

Theorem 5 *let $\{\vec{u}_1, ..., \vec{u}_p\}$ be an **orthogonal basis** for $w \subset \mathbb{R}^n$. Let $y \in W$. Then $\vec{y} = C_1\vec{u}_1 + ... + C_p\vec{u}_p$, $C_j = \frac{\vec{y} \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i}$*

Theorem 6 *Let $u = (\vec{u}_1\vec{u}_2\vec{u}_3)$, where $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is orthonormal set. $u^T \cdot u = I$*

Theorem 7 *Let u be an **orthogonal matrix**, Let $\vec{x}, \vec{y} \in \mathbb{R}$:*
• $|u\vec{x}| = |\vec{x}|$ • $(u\vec{x}) \cdot (u\vec{y}) = \vec{x} \cdot \vec{y}$