Linear Algebra from Elementary to Advanced

University: Johns Hopkins University

Linear Systems and Matrix Equations

Definition

Linear Functions	All terms are of degree 0 or 1.
Consistent	lin. systems is consistent if either 1 or ∞ solutions exist else inconsistent.
Row-Equivalence	Two matrce are row-equivalent if there is a sequence of **EROS** that transforms one into the other.
Pivot Position	is a position in a matrix that corresponds to a leading 1 in the matrix reduced echelon form.
Pivot Columns	is a culumn of A that contain a **pivot**.
Leading Entry	of a row in a matrix is the leftmost nonzero entry in the row.
Basic Variables	in a system of linear equations are the variables that correspond to pivot columns of the matrix for the system.
Free Variables	are the variables that do not correspond to pivot columns.
Column Vector	is a matrix with only one column.
Lin. Combination	$\begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix} = C_1 + ... + C_m$
Span	span($\vec{v}$) is the set of all scalar multiples of $\vec{v}$, where $\vec{c} \neq \vec{0}$.
Identity Matrix	$\begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 1 \end{bmatrix}_{n\times n}$
Homogeneous	A homogeneous system of linear equations can be written in the form $A\vec{x} = \vec{0}$.
Trivial Sol.	The trivial solution to a homogeneous system is $\vec{x} = \vec{0}$. Any other solutions are called nontrivial solutions.
Linearly Dependent	A set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} \in \mathbb{R}^n$ is **linearly dependent** if there exist weights such that $\vec{c}_1\vec{v}_1 + \vec{c}_2\vec{v}_2 + ... + \vec{c}_p\vec{v}_p = \vec{0}$.
Surjective(onto)	For a **linear transformation** T: for every vector $\vec{w} \in W$, there exist a vector $\vec{v} \in V$ such that $T(v) = W$.
Injective(one-to-one)	For a **linear transformation** T: with $V \to W$, if $T(\vec{v}_1) = T(\vec{v}_2)$, then $\vec{v}_1 = \vec{v}_2$ meaning that each vector in V maps to a unique vector in W.
Standard Vector	$\vec{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ The Standard vectors of $\mathbb{R}^n$ are $\{\vec{e}_1, ..., \vec{e}_n\}$
Linearly Independent	$\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ is **linearly independent** if the equation $x_1\vec{v}_1 + x_2\vec{v}_2 + ... + x_n\vec{v}_n$
Linear Transformation	is a transformation T that satisfies both the following conditions: • $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ • $T(c\vec{u}) = cT(\vec{u})$

Theorems

Properties 1 *Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$ then:* • $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ • $\vec{u}+\vec{0} = \vec{u}$ • $\vec{u}+(-\vec{u}) = \vec{0}$ • $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$ • $(c+d)\vec{w} = c\vec{w}+d\vec{w}$ • $c(d\vec{u}) = (cd)\vec{u}$

Theorem 1 (Uniqueness of Reduced Echelon Form) *Every matrix is row equivalent to a unique row echelon form.*

Theorem 2 (Existence of Solutions) *A system of linear equations is consistent if and only if the last (rightmost) column of its augmented matrix is not a pivot column.*

Theorem 3 (Matrix Eq., Vector Equ., and Lin. Systems) *If A is a $m \times n$ matrix with columns $\vec{a}_1, ..., \vec{a}_n$, $\vec{b} \in \mathbb{R}^n$ then the matrix equation $A\vec{x} = \vec{b}$ has same solution set as vector equation $x_1\vec{a}_1 + ... + x_n\vec{a}_n = \vec{b}$ wich is the same solution set to system of linear equation with augmented matrix $\left[A|\vec{b}\right] = \left[\vec{a}_1, ..., \vec{a}_n|\vec{b}\right]$*

Theorem 4 (Existence of Solutions) *Let $A_{n\times m}$ the following are equivalent:* • $\forall\vec{b} \in \mathbb{R}^n, A\vec{x} = \vec{b}$ *has a solution.* • *each $\vec{b}$ is the linear combination of the columns of A.* • *Columns of A span $\mathbb{R}^n$.* • *A has a pivot in every row.*

Theorem 5 *If $A_{m\times n}, \vec{u}, \vec{v} \in \mathbb{R}^n$ $c \in \mathbb{R}$, where $A\vec{u}$ is defined* • $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$ • $A(c\vec{u}) = c(A\vec{u})$

Theorem 6 (Nontrivial Sol. of Homogeneous Equ.) *The homogeneous equation $A\vec{x} = 0$ has a nontrivial solution if and only if the equation has at least one free variable.*

Theorem 7 (Solution Sets with Free Variables) *Suppose the equation $A\vec{x} = \vec{b}$ is consistent. Let p be a particular solution to the equation $A\vec{x} = \vec{b}$ Then the solution set of $A\vec{x} = \vec{b}$ is the set of all vectors of the form $\vec{w} = \vec{p} + \vec{v}_h$, where v_h is the solution of the homogeneous equation $A\vec{x} = \vec{0}$.*

Theorem 8 (Sets of Two or More Vectors) *The set of $s = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ of 2 or more vectors is linearly dependent if and only if at least 1 vector is a linear combination of the other.*

Theorem 9 (Sets with More Vectors than Entries) *If $p > n$, then the set $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} \in \mathbb{R}$ must be linearly dependent.*

Theorem 10 (Sets Containing the Zero Vector) *Any set of vectors that contains the zero vector is linearly dependent.*

Theorem 11 (The Standard Matrix of a Lin. Trans.) *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is injective if and only if $T(\vec{x}) = \vec{0}$ has only the trivial solution.*

Coefficient Matrix Example

$$\begin{cases} A_1x_1 + A_2x_2 + A_3x_3 = \alpha \\ B_1x_1 + B_2x_2 + B_3x_3 = \beta \end{cases} \Leftrightarrow \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix} \tag{1}$$

Augmented Matrix Example

$$\begin{cases} A_1x_1 + A_2x_2 + A_3x_3 = \alpha \\ B_1x_1 + B_2x_2 + B_3x_3 = \beta \end{cases} \Leftrightarrow \left[\begin{array}{ccc|c} A_1 & A_2 & A_3 & \alpha \\ B_1 & B_2 & B_3 & \beta \end{array}\right] \tag{2}$$

Elementary Row Operations (EROS)

1. **[Replacement]** Replace one row by sum of itself.
2. **[Interchange]** Swap position of 2 rows.
3. **[Scaling]** Multiply all entries in row by non-zero constant.

Echelon Form (ef)

1. All non-zero rows are above any rows of all-zero.
2. Each leading entry of a row is in a column to the right of the roe above it.
3. All entries in a column below a leading entry are 0.

Reduced Row Echelon Form (rref)

1. As to be in echelon form.
2. Leading entry in each row is 1.
3. Each leading 1 is the only non-zero entry in its column.

Vector Operations

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} A_1 + B_1 \\ A_2 + B_2 \end{pmatrix} \quad C\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} CA_1 \\ CA_2 \end{pmatrix}, C \in \mathbb{R}$$

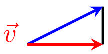
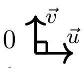
Matrix & Vector Multiplication

Let $A_{m\times n}, \vec{x} \in \mathbb{R}^n$. $A\vec{x} = [a_1, ..., a_n]\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = a_1x_1 + ... + a_nx_n$

Matrix Algebra, Determinants, & Eigenvectors

Orthogonality and Diagonalization

Definition

Inner Product	$\vec{v} \cdot \vec{u} = \vec{v}^T\vec{u} = u_1v_1 + ... + u_nv_n$ (Also called dot product or scalar product)
Length of $\vec{x}$	$\|\|\vec{x}\|\| = \sqrt{\vec{x}\cdot\vec{x}} = \sqrt{x_1^2 + ... + x_n^2}$ (Also called Norm or Magnitude)
Unit vectore	A vector with $\|\|\vec{x}\|\| = 1$
Normalization	The formula $\vec{u} = \frac{\vec{x}}{\|\|\vec{x}\|\|}$ creat a unit vector in the same direction as $\vec{x}$.
Distance	$dist(\vec{u}, \vec{v}) = \vec{u} - \vec{v}$ 
Orthogonality	$\vec{u}\cdot\vec{v} = 0$ 
Orthogonal Set	A set of vectors $\{\vec{u}_1, ..., \vec{u}_p\} \in \mathbb{R}^n$ such that each distinct vectors are orthogonal.
Orthogonal Basis	For a subspace W of R^n is a basis that is also an orthogonal set.
Orthonormal set	is a set of orthogonal unit vectors.
Orthogonal matrix	Is a squared matrix whose columns are orthonormal.
Normal equation	$A^TA\vec{x} = A^T\vec{b}$
Least-Squares Error	$\|\|\vec{b} - A\hat{x}\|\|$
Symetric	$A_{m\times n}$ is symetric if $A = A^T$
Spectrum	Set of eigenvalues of $A_{n\times n}\{\lambda_1, ..., \lambda_n\}$ is called the **spectrum** of A.
Quadratic Form	A **quadratic form** on $\mathbb{R}^n$ is a function Q from $\mathbb{R}^n \to \mathbb{R}$ of the form $Q(\vec{x}) = \vec{x}^TA\vec{x}$, where A is a symetric matrix.
Orthogonal Complements	Let W be a subspace of $\mathbb{R}^n$. The orthogonal complement of W is: $W^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x}\cdot\vec{w} = \vec{0}, \forall\vec{w} \in W\}$
Orthogonal Projection	of $\vec{y} \in \mathbb{R}^n$ onto subspace W of $\mathbb{R}^n$ is the vector $\hat{y}$ in W such that $\vec{y} - \hat{y}$ is in $W^\perp$. (Noted $\text{proj}_W(\vec{y})$)
Least-Squares Problem	is to find $\vec{x}$ that makes $\|\|\vec{b} - A\vec{x}\|\|$ $= \text{dist}(\vec{b}, A\vec{x})$ as small as possible.
Orthogonally Diagonalizable Matrix	is a square matrix for which there exist an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1} = PDP^T$.

Theorems

Properties 2 *Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$ then:*
• $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ • $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$ • $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v}) = (c\vec{v})\cdot\vec{u}$
• $\vec{u}\cdot\vec{u} = u_1^2 + ... + u_n^2 \geq 0$

Theorem 12 (Fundamental Subspaces Theorem) *Let $A_{M\times N}$ then:* • $(row(A))^\perp = nul(A)$ • $(COl(A))^\perp = nul(a^\perp)$

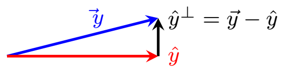
Theorem 13 *If $S = \{\vec{u}_1, ..., \vec{u}_p\}$ is an **orthogonal set** of non-zero vectors in $\mathbb{R}$ then S is a **linearly independant** set.*

Theorem 14 *let $\{\vec{u}_1, ..., \vec{u}_p\}$ be an **orthogonal basis** for $w \subset \mathbb{R}^n$. Let $y \in W$. Then $\vec{y} = C_1\vec{u}_1 + ... + C_p\vec{u}_p$, $C_j = \frac{\vec{y}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}$*

Theorem 15 *Let $u = (\vec{u}_1 \vec{u}_2 \vec{u}_3)$, where $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is orthonormal set. $u^T \cdot u = I$*

Theorem 16 *Let u be an **orthogonal matrix**, Let $\vec{x}, \vec{y} \in \mathbb{R}$:* • $|u\vec{x}| = |\vec{x}|$ • $(u\vec{x}) \cdot (u\vec{y}) = \vec{x} \cdot \vec{y}$

Theorem 17 (Orthogonal Decomposition Theorem) *Let W be a subspace with an orthogonal basis $\{\vec{u}_1, ..., \vec{u}_p\}$. let $\vec{y} \in \mathbb{R}^n$ then:* $\hat{y} = \left(\frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\right) \vec{u}_1 + ... + \left(\frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p}\right) \vec{u}_p$



Theorem 18 (Best Approximation Theorem) *Let W be a subspace of $\mathbb{R}$. Let $\vec{y} \in \mathbb{R}^n$ and $\hat{y} = proj_W(\vec{y})$. Then $\hat{y}$ is closest point to $\vec{y}$ in W. That is $\forall \vec{v} \neq \vec{y}$ $||\vec{y} - \hat{y}|| < ||\vec{y} - \vec{v}||$*

Theorem 19 (Orth. Proj. with Orthonormal Bases) *Let W subspace in $\mathbb{R}^n$. Let $\{\vec{u}_1, ..., \vec{u}_p\}$ be an orthonormal basis of W. $proj_W(\vec{y}) = (\vec{y} \cdot \vec{u}_1)\vec{u}_1 + ... + (\vec{y} \cdot \vec{u}_p)\vec{u}_p$ Let $U = (\vec{u}_1\ \vec{u}_2\ ...\ \vec{u}_p)$. Then $proj_W(\vec{y}) = UU^T\vec{y}$*

Theorem 20 (Normal Equations and Least-Squares Sol.) *The set of least squares solutions of $A\vec{x} = \vec{b}$ coincide the non-empty solutions of the Normal equation.*

Theorem 21 (Criteria for a Unique Least-Squares Sol.) *Let $A_{M \times N}$ then the following are equivalent:* • *$A\vec{x} = \vec{b}$ has unique least-square solution $\forall \vec{b} \in \mathbb{R}^n$.* • *Columns of A are linearly independant.* • *The matrix A^TA is invertible. When the statements are true, the least squares solution is given by $\hat{x} = (A^TA)^{-1}A^T\vec{b}$*

Theorem 22 (Eigenvectors of a Symmetric Matrix) *Let A be a symmetric matrix. Then any two eigenvectors of A from different eigenspaces are orthogonal.*

Theorem 23 (Symmetric Matrices and Orth. Diag.) *A square matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.*

Theorem 24 (Spectral Theorem For Symetric Matrices) *Let $A_{n \times n}$ be symetric. Then the following properties hold.* • *A has n real eigenvalues (counting multiplicities).* • *Then dimension of the eigenspace for each eigenvalue λ is equal to the multiplicity of λ.* • *The eigenvectors of A corresponding to different eigenvalues are orthogonal.* • *A is orthogonally diagonalizable.*

Theorem 25 (Principal Axis Theorem) *Let $A_{N \times N}$ be symetric. Then there is an orthogonal change of variable $\vec{x} = P\vec{y}$ that transform the quadratic form $\vec{x}^TA\vec{x}$ to $\vec{y}^TA\vec{y}$ whith no cross terms(Ex. ofcross terms is x_1x_2). The columns of P are the eiganvectors of A.*

Theorem 26 (Eiganvalue of Definites Quadratic) *The quadratic form is:* • *Positive definite if and only if $\forall i \lambda_i > 0$.* • *Negative definite if and only if $\forall i \lambda_i < 0$.* • *Indefinite if and only if A as positive and negative eiganvalues.*

Fact about the Orthogonal Complements

• $\vec{0} \in W^\perp$ since $\vec{0} \cdot \vec{w} = 0$ • If $W \in W^\perp, c \in \mathbb{R}$ then $cW \in W^\perp$ • If $\vec{w}_1, \vec{w}_2 \in W^\perp$ so $(\vec{w}_1 + \vec{w}_2) \cdot \vec{x} = \vec{w}_1 \cdot \vec{x} + \vec{w}_2 \cdot \vec{x} \in W$

Gram-Schmit Process(G-S)

The Gram-Schmit Process is an algorithme producing an orthogonal basis.
Start whith $\{\vec{x}_1, ..., \vec{x}_n\}$ basis for noozero subspace w of $\mathbb{R}^n$ define:

$\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right) \vec{v}_1$

$\vdots$

$\vec{v}_n = \vec{x}_n - \left(\frac{\vec{x}_n \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right) \vec{v}_1 - ... - \left(\frac{\vec{x}_n \cdot \vec{v}_{n-1}}{\vec{v}_{n-1} \cdot \vec{v}_{n-1}}\right) \vec{v}_{n-1}$

Finding the Matrix of a Quadratic Form

$$Q(\vec{x}) = Ax_1^2 + Bx_1x_2 + Cx_2^2 \Leftrightarrow \begin{bmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{bmatrix}$$

Quadratic Form Terminology

A quadratic for $Q(\vec{x})$ is: • Positive Definite If $Q(\vec{x}) > 0, \forall \vec{x} \neq \vec{0}$ • Positive Semidefinite If $Q(\vec{x}) \geq 0$ • Negative Definite If $Q(\vec{x}) < 0, \forall \vec{x} \neq \vec{0}$ • Negative Semidefinite If $Q(\vec{x}) \leq 0$