# Linear Algebra from Elementary to Advanced

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## Linear Systems and Matrix Equations

### Definition

| | |
|---|---|
| Linear Functions | All terms are of degree 0 or 1. |
| Consistent | lin. systems is consistent if either 1 or $\infty$ solutions exist else inconsistent. |
| Row-Equivalence | Two matrce are row-equivalent if there is a sequence of **EROS** that transforms one into the other. |
| Pivot Position | is a position in a matrix that corresponds to a leading 1 in the matrix reduced echelon form. |
| Pivot Columns | is a culumn of $A$ that contain a **pivot**. |
| Leading Entry | of a row in a matrix is the leftmost nonzero entry in the row. |
| Basic Variables | in a system of linear equations are the variables that correspond to pivot columns of the matrix for the system. |
| Free Variables | are the variables that do not correspond to pivot columns. |
| Column Vector | is a matrix with only one column. |
| Lin. Combination | $\begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix} = C_1 + ... + C_m$ |
| Span | span$(\vec{v})$ is the set of all scalar multiples of $\vec{v}$, where $\vec{c} \neq \vec{0}$. |
| Identity Matrix | $I_n = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 1 \end{bmatrix}_{n \times n}$ |
| Homogeneous | A homogeneous system of linear equations can be written in the form $A\vec{x} = \vec{0}$. |
| Trivial Sol. | The trivial solution to a homogeneous system is $\vec{x} = \vec{0}$. Any other solutions are called nontrivial solutions. |
| Linearly Dependent | A set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} \in \mathbb{R}^n$ is **linearly dependent** if there exist weights such that $\vec{c}_1\vec{v}_1 + \vec{c}_2\vec{v}_2 + ... + \vec{c}_p\vec{v}_p = \vec{0}$. |
| Surjective(onto) | For a **linear transformation** $T$: for every vector $\vec{w} \in W$, there exist a vector $\vec{v} \in V$ such that $T(v) = W$. |
| Injective(one-to-one) | For a **linear transformation** $T$: with $V \to W$, if $T(\vec{v}_1) = T(\vec{v}_2)$, then $\vec{v}_1 = \vec{v}_2$ meaning that each vector in $V$ maps to a unique vector in $W$. |
| Standard Vector | $\vec{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ |
| Linearly Independent | $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ is **linearly independent** if the equation $x_1\vec{v}_1 + x_2\vec{v}_2 + ... + x_n\vec{v}_n$ |
| Linear Transformation | is a transformation $T$ that satisfies both the following conditions: • $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ • $T(c\vec{u}) = cT(\vec{u})$ |

## Theorems

**Properties 1** *Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$ then:* • $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ • $\vec{u}+\vec{0} = \vec{u}$ • $\vec{u}+(-\vec{u}) = \vec{0}$ • $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$ • $(c+d)\vec{w} = c\vec{w}+d\vec{w}$ • $c(d\vec{u}) = (cd)\vec{u}$

**Theorem 1 (Uniqueness of Reduced Echelon Form)** *Every matrix is row equivalent to a unique row echelon form.*

**Theorem 2 (Existence of Solutions)** *A system of linear equations is consistent if and only if the last (rightmost) column of its augmented matrix is not a pivot column.*

**Theorem 3 (Matrix Eq., Vector Equ., and Lin. Systems)** *If $A$ is a $m \times n$ matrix with columns $\vec{a}_1, ..., \vec{a}_n$, $\vec{b} \in \mathbb{R}^n$ then the matrix equation $A\vec{x} = \vec{b}$ has same solution set as vector equation $x_1\vec{a}_1 + ... + x_n\vec{a}_n = \vec{b}$ wich is the same solution set to system of linear equation with augmented matrix $\left[A|\vec{b}\right] = \left[\vec{a}_1, ..., \vec{a}_n|\vec{b}\right]$*

**Theorem 4 (Existence of Solutions)** *Let $A_{n\times m}$ the following are equivalent:* • *$\forall \vec{b} \in \mathbb{R}^n, A\vec{x} = \vec{b}$ has a solution.* • *each $\vec{b}$ is the linear combination of the columns of $A$.* • *Columns of $A$ span $\mathbb{R}^n$.* • *$A$ has a pivot in every row.*

**Theorem 5** *If $A_{m\times n}, \vec{u}, \vec{v} \in \mathbb{R}^n$ $c \in \mathbb{R}$, where $A\vec{u}$ is defined* • $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$ • $A(c\vec{u}) = c(A\vec{u})$

**Theorem 6 (Nontrivial Sol. of Homogeneous Equ.)** *The homogeneous equation $A\vec{x} = 0$ has a nontrivial solution if and only if the equation has at least one free variable.*

**Theorem 7 (Solution Sets with Free Variables)** *Suppose the equation $A\vec{x} = \vec{b}$ is consistent. Let $p$ be a particular solution to the equation $A\vec{x} = \vec{b}$ Then the solution set of $A\vec{x} = \vec{b}$ is the set of all vectors of the form $\vec{w} = \vec{p} + \vec{v}_h$, where $v_h$ is the solution of the homogeneous equation $A\vec{x} = \vec{0}$.*

**Theorem 8 (Sets of Two or More Vectors)** *The set of $s = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ of 2 or more vectors is linearly dependent if and only if at least 1 vector is a linear combination of the other.*

**Theorem 9 (Sets with More Vectors than Entries)** *If $p > n$, then the set $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} \in \mathbb{R}$ must be linearly dependent.*

**Theorem 10 (Sets Containing the Zero Vector)** *Any set of vectors that contains the zero vector is linearly dependent.*

**Theorem 11 (One-to-One Transformations)** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T$ is injective if and only if $T(\vec{x}) = \vec{0}$ has only the trivial solution.*

**Theorem 12 (The Standard Matrix of a Lin. Trans.)** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is unique matrix $A$ such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. $A$ is the $m \times n$ matrix whose $j$th column is the vector $T(\vec{e}_j)$: $A = [T(\vec{e}_1), T(\vec{e}_2), ..., T(\vec{e}_n)]$.*

**Theorem 13 (One-to-One and Onto Tran. and Matrices)** *Let $A$ be the standard matrix for the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. Then:* • *$T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $A$ span $\mathbb{R}^m$* • *$T$ is* ***Injective*** *if and only if the columns of $A$ are linearly independent.*

## Coefficient Matrix Example

$$\begin{cases} A_1x_1 + A_2x_2 + A_3x_3 = \alpha \\ B_1x_1 + B_2x_2 + B_3x_3 = \beta \end{cases} \Leftrightarrow \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix} \tag{1}$$

## Augmented Matrix Example

$$\begin{cases} A_1x_1 + A_2x_2 + A_3x_3 = \alpha \\ B_1x_1 + B_2x_2 + B_3x_3 = \beta \end{cases} \Leftrightarrow \left[\begin{array}{ccc|c} A_1 & A_2 & A_3 & \alpha \\ B_1 & B_2 & B_3 & \beta \end{array}\right] \tag{2}$$

## Elementary Row Operations (EROS)

1. **[Replacement]** Replace one row by sum of itself.
2. **[Interchange]** Swap position of 2 rows.
3. **[Scaling]** Multiply all entries in row by non-zero constant.

## Echelon Form (ef)

1. All non-zero rows are above any rows of all-zero.
2. Each leading entry of a row is in a column to the right of the roe above it.
3. All entries in a column below a leading entry are 0.

## Reduced Row Echelon Form (rref)

1. As to be in echelon form.
2. Leading entry in each row is 1.
3. Each leading 1 is the only non-zero entry in its column.

## Vector Operations

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} A_1 + B_1 \\ A_2 + B_2 \end{pmatrix} \qquad C\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} CA_1 \\ CA_2 \end{pmatrix}, C \in \mathbb{R}$$

## Matrix & Vector Multiplication

Let $A_{m\times n}, \vec{x} \in \mathbb{R}^n$. $A\vec{x} = [a_1, ..., a_n]\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = a_1x_1 + ... + a_nx_n$

# Matrix Algebra, Determinants, & Eigenvectors

## Definition

| | |
|---|---|
| Transposes | Let $A$ be an $m \times n$ matrix. The **transpose** of $A$, denoted $A^T$, is the $n \times m$ matrix whose columns are formed from the corresponding rows of $A$. |
| Diagonal Matrix | A **diagonal matrix** is a matrix whose non-diagonal entries are all 0. |

## Theorems & Properties

**Properties 2 (Addition and Scalar Multiplication)** *Let $A_{n\times m}, B_{n\times m}, C_{n\times m}$ and $r, s \in \mathbb{R}$ then:* • $A + B = B + A$ • $A + (B + C) = (A + B) + C$ • $A + 0 = A$ • $r(A + B) = rA + rB$ • $(r + s)A = rA + sA$ • $r(sA) = (rs)A$

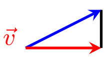
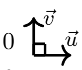
**Properties 3 (Properties of the Transpose)** *Let $A$ and $B$ be two matrices and let $r$ be a scalar. Then the following properties hold (as long as the sums and products are defined.)* • $(A^T)^T = A$ • $(A + B)^T = A^T + B^T$ • $(rA)^T = rA^T$ • $(AB)^T = A^T + B^T$

## matrix operations

$A = B$ if and only if their coresponding entries are equal.
Let $A_{n\times m}, \ , B_{n\times m}$ then $A+B$ is the $m\times n$ matrix obtained by adding the corresponding entries.
The scalar multiplication $cA$ is the $m \times n$ matrix obtained by multiplying each entry of $A$ by $c$.
Let $A_{n\times p}, B_{p\times m}$ such that $B = (\vec{b}_1, \vec{b}_2, \dots, \vec{b}_p)$ then $AB = (A\vec{b}_1, A\vec{b}_2, \dots, A\vec{b}_p)$

# Orthogonality and Diagonalization

## Definition

| | |
|---|---|
| Inner Product | $\vec{v} \cdot \vec{u} = \vec{v}^T\vec{u} = u_1v_1 + ... + u_nv_n$ (Also called dot product or scalar product) |
| Length of $\vec{x}$ | $\|\|\vec{x}\|\| = \sqrt{\vec{x}\cdot\vec{x}} = \sqrt{x_1^2 + ... + x_n^2}$ (Also called Norm or Magnitude) |
| Unit vectore | A vector with $\|\|\vec{x}\|\| = 1$ |
| Normalization | The formula $\vec{u} = \frac{\vec{x}}{\|\|\vec{x}\|\|}$ creat a unit vector in the same direction as $\vec{x}$. |
| Distance | $dist(\vec{u}, \vec{v}) = \vec{u} - \vec{v}$  $dist(\vec{u}, \vec{v})$ |
| Orthogonality | $\vec{u} \cdot \vec{v} = 0$  |
| Orthogonal Set | A set of vectors $\{\vec{u}_1, ..., \vec{u}_p\} \in \mathbb{R}^n$ such that each distinct vectors are orthogonal. |
| Orthogonal Basis | For a subspace $W$ of $R^n$ is a basis that is also an orthogonal set. |
| Orthonormal set | is a set of orthogonal unit vectors. |
| Orthogonal matrix | Is a squared matrix whose columns are orthonormal. |
| Normal equation | $A^TA\vec{x} = A^T\vec{b}$ |
| Least-Squares Error | $\|\|\vec{b} - A\hat{x}\|\|$ |
| Symetric | $A_{m\times n}$ is symetric if $A = A^T$ |
| Spectrum | Set of eigenvalues of $A_{n\times n}\{\lambda_1, ..., \lambda_n\}$ is called the **spectrum** of $A$. |
| Quadratic Form | A **quadratic form** on $\mathbb{R}^n$ is a function $Q$ from $\mathbb{R}^n \to \mathbb{R}$ of the form $Q(\vec{x}) = \vec{x}^TA\vec{x}$, where $A$ is a symetric matrix. |
| Orthogonal Complements | Let $W$ be a subspace of $\mathbb{R}^n$. The orthogonal complement of $W$ is: $W^\perp = \{\vec{x} \in \mathbb{R}^n \| \vec{x}\cdot\vec{w} = \vec{0}, \forall \vec{w} \in W\}$ |
| Orthogonal Projection | of $\vec{y} \in \mathbb{R}^n$ onto subspace $W$ of $\mathbb{R}^n$ is the vector $\hat{y}$ in $W$ such that $\vec{y} - \hat{y}$ is in $W^\perp$. (Noted $\text{proj}_W(\vec{y})$) |
| Least-Squares Problem | is to find $\vec{x}$ that makes $\|\|\vec{b} - A\vec{x}\|\|$ $= \text{dist}(\vec{b}, A\vec{x})$ as small as possible. |
| Orthogonally Diagonalizable Matrix | is a square matrix for which there exist an orthogonal matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1} = PDP^T$. |

## Theorems

**Properties 4** *Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$ then:* • $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ • $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$ • $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v}) = (c\vec{v})\cdot\vec{u}$ • $\vec{u}\cdot\vec{u} = u_1^2 + ... + u_n^2 \geq 0$

**Theorem 14 (Fundamental Subspaces Theorem)** *Let $A_{M\times N}$ then:* • $(row(A))^\perp = nul(A)$ • $(COl(A))^\perp = nul(a^\perp)$

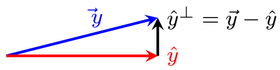
**Theorem 15** *If $S = \{\vec{u}_1, ..., \vec{u}_p\}$ is an **orthogonal set** of non-zero vectors in $\mathbb{R}$ then $S$ is a **linearly independant** set.*

**Theorem 16** *let $\{\vec{u}_1, ..., \vec{u}_p\}$ be an **orthogonal basis** for $w \subset \mathbb{R}^n$. Let $y \in W$. Then $\vec{y} = C_1\vec{u}_1 + ... + C_p\vec{u}_p$, $C_j = \frac{\vec{y}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}$*

**Theorem 17** *Let $u = (\vec{u}_1\vec{u}_2\vec{u}_3)$, where $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is orthonormal set.* $u^T \cdot u = I$

**Theorem 18** *Let $u$ be an* ***orthogonal matrix****, Let $\vec{x}, \vec{y} \in \mathbb{R}$:* • $|u\vec{x}| = |\vec{x}|$ • $(u\vec{x}) \cdot (u\vec{y}) = \vec{x} \cdot \vec{y}$

**Theorem 19 (Orthogonal Decomposition Theorem)** *Let $W$ be a subspace with an orthogonal basis $\{\vec{u}_1, ..., \vec{u}_p\}$. let $\vec{y} \in \mathbb{R}^n$ then:*
$\hat{y} = \left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + ... + \left(\frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\right)\vec{u}_p$




**Theorem 20 (Best Approximation Theorem)** *Let $W$ be a subspace of $\mathbb{R}$. Let $\vec{y} \in \mathbb{R}^n$ and $\hat{y} = proj_W(\vec{y})$. Then $\hat{y}$ is closest point to $\vec{y}$ in $W$. That is $\forall \vec{v} \neq \vec{y}$ $||\vec{y} - \hat{y}|| < ||\vec{y} - \vec{v}||$*

**Theorem 21 (Orth. Proj. with Orthonormal Bases)** *Let $W$ subspace in $\mathbb{R}^n$. Let $\{\vec{u}_1, ..., \vec{u}_p\}$ be an orthonormal basis of $W$. $proj_W(\vec{y}) = (\vec{y}\cdot\vec{u}_1)\vec{u}_1 + ... + (\vec{y}\cdot\vec{u}_p)\vec{u}_p$ Let $U = (\vec{u}_1\ \ \vec{u}_2\ \ ...\ \ \vec{u}_p)$. Then $proj_W(\vec{y}) = UU^T\vec{y}$*

**Theorem 22 (Normal Equations and Least-Squares Sol.)**
*The set of least squares solutions of $A\vec{x} = \vec{b}$ coincide the non-empty solutions of the Normal equation.*

**Theorem 23 (Criteria for a Unique Least-Squares Sol.)** *Let $A_{M\times N}$ then the following are equivalent:* • *$A\vec{x} = \vec{b}$ has unique least--square solution $\forall \vec{b} \in \mathbb{R}^n$.* • *Columns of $A$ are linearly independant.* • *The matrix $A^TA$ is invertible. When the statements are true, the least squares solution is given by $\hat{x} = (A^TA)^{-1}A^T\vec{b}$*

**Theorem 24 (Eigenvectors of a Symmetric Matrix)** *Let $A$ be a symmetric matrix. Then any two eigenvectors of $A$ from different eigenspaces are orthogonal.*

**Theorem 25 (Symmetric Matrices and Orth. Diag.)** *A square matrix $A$ is orthogonally diagonalizable if and only if $A$ is a symmetric matrix.*

**Theorem 26 (Spectral Theorem For Symetric Matrices)** *Let $A_{n\times n}$ be symetric. Then the following properties hold.* • *$A$ has $n$ real eigenvalues (counting multiplicities).* • *Then dimension of the eigenspace for each eigenvalue $\lambda$ is equal to the multiplicity of $\lambda$.* • *The eigenvectors of $A$ corresponding to different eigenvalues are orthogonal.* • *$A$ is orthogonally diagonalizable.*

**Theorem 27 (Principal Axis Theorem)** *Let $A_{N\times N}$ be symetric. Then there is an orthogonal change of variable $\vec{x} = P\vec{y}$ that transform the quadratic form $\vec{x}^TA\vec{x}$ to $\vec{y}^TA\vec{y}$ whith no cross terms(Ex. ofcross terms is $x_1x_2$). The columns of $P$ are the eiganvectors of $A$.*

**Theorem 28 (Eiganvalue of Definites Quadratic)** *The quadratic form is:* • *Positive definite if and only if $\forall i \lambda_i > 0$.* • *Negative definite if and only if $\forall i \lambda_i < 0$.* • *Indefinite if and only if $A$ as positive and negative eiganvalues.*

## Fact about the Orthogonal Complements

• $\vec{0} \in W^\perp$ since $\vec{0} \cdot \vec{w} = 0$ • If $W \in W^\perp, c \in \mathbb{R}$ then $cW \in W^\perp$ • If $\vec{w}_1, \vec{w}_2 \in W^\perp$ so $(\vec{w}_1 + \vec{w}_2) \cdot \vec{x} = \vec{w}_1 \cdot \vec{x} + \vec{w}_2 \cdot \vec{x} \in W$

## Gram-Schmit Process(G-S)

The Gram-Schmit Process is an algorithme producing an orthogonal basis.
Start whith $\{\vec{x}_1, ..., \vec{x}_n\}$ basis for noozero subspace w of $\mathbb{R}^n$ define:
$\vec{v}_1 = \vec{x}_1$
$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{x}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1$
$\vdots$
$\vec{v}_n = \vec{x}_n - \left(\frac{\vec{x}_n\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 - ... - \left(\frac{\vec{x}_n\cdot\vec{v}_{n-1}}{\vec{v}_{n-1}\cdot\vec{v}_{n-1}}\right)\vec{v}_{n-1}$

## Finding the Matrix of a Quadratic Form

$Q(\vec{x}) = Ax_1^2 + Bx_1x_2 + Cx_2^2 \Leftrightarrow \begin{bmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{bmatrix}$

## Quadratic Form Terminology

A quadratic for $Q(\vec{x})$ is: • Positive Definite If $Q(\vec{x}) > 0, \forall \vec{x} \neq \vec{0}$ • Positive Semidefinite If $Q(\vec{x}) \geq 0$ • Negative Definite If $Q(\vec{x}) < 0, \forall \vec{x} \neq \vec{0}$ • Negative Semidefinite If $Q(\vec{x}) \leq 0$