

Complex Analysis Short recap Quantum mechanics

Let $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $z \in \mathbb{C}$.

Real part $\Re(a + ib) = a$
 Real part $\Im(a + ib) = b$
 Absolute Values $|z| = \sqrt{zz^*} = \|(a \ b)\|_2$
 Complex conjugate $(a + ib)^* = a - ib$
 $(a - ib)^* = a + ib$

Trig. Formulas $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$
 Trig. Formulas $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

Pauli Matrices

$$\bullet \sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bullet \sigma_x = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\bullet \sigma_y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \bullet \sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Properties 1 (Complex conjugate)

- $(Z^*)^* = Z$ • $(Z + W)^* = Z^* + W^*$
- $(Z - W)^* = Z^* - W^*$ • $(ZW)^* = Z^*W^*$
- $Z^*Z = |Z|^2$ • $(Z^n)^* = (Z^*)^n$, for $n \in \mathbb{Z}$
- $\ln(Z^*) = (\ln(Z))^*$ if Z is not 0 or a negative real number.

Properties 2 (Complex Absolute Values)

- $|z_1 z_2| = |z_1| |z_2|$ • The absolute value define the metric of the space \mathbb{C} (\mathbb{C} is complete).
- $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\Re(z_1 z_2^*)$
- $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\Re(z_1 z_2^*)$
- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

Linear Algebra

Linear Operator

A **linear operator** between the vector spaces V and W is define to be any function $A : V \rightarrow W$ which is linear in its input.

Properties 3 Let \hat{A} be a linear operator on $V \rightarrow W$ and A be the matrix representation of \hat{A} .

- $\hat{A}(\sum_i a_i |v_i\rangle) = \sum_i a_i \hat{A}|v_i\rangle$
- $\hat{A}|v_j\rangle = \sum_i A_{ij} |w_i\rangle$

Inner product

A Inner Product $\langle \cdot, \cdot \rangle$ is a function that output a complex number and satisfies the following conditions: Let $\vec{v} \in \mathbb{C}^n, \vec{w} \in \mathbb{C}^n$.

1. $\langle \vec{v}, \sum_i a_i \vec{w}_i \rangle = \sum_i a_i \langle \vec{v}, \vec{w}_i \rangle$
2. $\langle \vec{v}, \vec{w} \rangle = (\langle \vec{w}, \vec{v} \rangle)^*$
3. if $x = 0$ and only if $\langle \vec{w}, \vec{w} \rangle \geq 0$

In quantum mechanics the inner product is generally noted $\langle \cdot | \cdot \rangle$.

Properties 4 • $\langle A, A \rangle = \|A\|^2$ • if $\langle A, B \rangle = 0$ then A and B are orthogonal.

Mathematical Concepts

Dirac Notation(or Bra-Ket Notation)

Terminology: ket of A is $|A\rangle$ and bra of A is $\langle A|$. Example, let $\Sigma = \{A, B, C\}$ then

$$|A\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, |B\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \langle A| = (1, 0, 0),$$

$\langle B| = (0, 1, 0)$ Note that $|\Psi\rangle$ then $\langle\Psi| = |\Psi\rangle^T$.

bra-kets: we denote $\langle a|b\rangle$ the matrix product of

Cartesian product

Let $y = \{0, 1\}$ and $x = \{a, b\}$. Then,
 $x \times y = \{(0, a), (0, b), (1, a), (1, b)\}$ and
 $y \times x = \{(a, 0), (a, 1), (b, 0), (b, 1)\}$.

Tensor Product of vectors

$$\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} \otimes \begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_0 b_0 \\ \vdots \\ a_n b_n \end{bmatrix}$$

Tensor Product Properties

1. $(|\phi_1\rangle + |\phi_2\rangle) \otimes |\psi\rangle = |\phi_1\rangle \otimes |\psi\rangle + |\phi_2\rangle \otimes |\psi\rangle$
2. $(a|\phi\rangle) \otimes |\psi\rangle = a(|\phi\rangle \otimes |\psi\rangle)$
3. $|a\rangle \otimes |b\rangle = |b\rangle \otimes |a\rangle$

Quantum Information Systems

State Vector

Quantum State of a system is represented by a complex column vector. Let the quantum

state vector v be equal to $\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$, where

$\sum_{i=0}^n |a_i|^2 = 1$ The **euclidean norm** of the $\|v\| = \sqrt{\sum_{i=0}^n |a_i|^2}$

Common Quantum States

Plus State $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
Minus State $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$
Other State $\frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle$

Standard Basis Measurement

Let a quantum system be in the state $|\psi\rangle$, then the probability for the measure

outcome to be a is $Pr(\text{outcome} = a) = |\langle a|\psi\rangle|^2$ If U is a unitary matrix then the following Properties hold, $\|U\psi\| = \|\psi\|$

Unary Operations

Unary Matrices

A square matrix U having complex number entries is unitary if it satisfies the equations, $UU^\dagger = U^\dagger U = \mathbb{I}$ where \mathbb{I} is the identity matrix.

Some unitary operations on qubits

Pauli operations:

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Hadamard operation: } H = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\text{Phase operations: } P_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

Quantum circuit