# Quantum Computation & Quantum Information

Book: Quantum Computation and Quantum Information. Nielsen, M. A., & Chuang, I. L.



## Complex Numbers Short recap

Let $a, b \in \mathbb{R}$ and $z \in \mathbb{C}$ such that $z = a+bi$.

| | |
|---|---|
| Real part | $\Re\mathfrak{e}(a+ib) = a$ |
| Imaginary part | $\Im\mathfrak{m}(a+ib) = b$ |
| Absolute Values | $\lvert z\rvert = \sqrt{zz^*}$ |
| | $= \lVert(a\ \ b)\rVert_2$ |
| Complexe conjugate | $(a+ib)^* = a - ib$ |
| | $(a-ib)^* = a + ib$ |
| Trig. Formulas | $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ |
| Trig. Formulas | $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ |

**Properties 1 (Complexe conjugate)** • $(Z^*)^* = Z$ • $(Z+W)^* = Z^* + W^*$ • $(Z-W)^* = Z^* - W^*$ • $(ZW)^* = Z^*W^*$ • $Z^*Z = \lvert Z\rvert^2$ • $(Z^n)^* = (Z^*)^n$, *for* $n \in \mathbb{Z}$ • $ln(Z^*) = (ln(Z))^*$ *if* $Z$ *is not 0 or a negative real number.*

**Properties 2 (Absolute Values)** • $\lvert z_1z_2\rvert = \lvert z_1\rvert\,\lvert z_2\rvert$ • *The absolute value define the metric of the space* $\mathbb{C}$ *(*$\mathbb{C}$ *is complete).* • $\lvert z_1+z_2\rvert^2 = \lvert z_1\rvert^2 + \lvert z_2\rvert^2 + \Re\mathfrak{e}(z_1z_2^*)$ • $\lvert z_1-z_2\rvert^2 = \lvert z_1\rvert^2 + \lvert z_2\rvert^2 - \Re\mathfrak{e}(z_1z_2^*)$ • $\lvert z_1+z_2\rvert^2 + \lvert z_1-z_2\rvert^2 = 2(\lvert z_1\rvert^2 + \lvert z_2\rvert^2)$

## Linear Algebra

### Short Definitions

| | |
|---|---|
| Hermitian Operator | $A = A^\dagger$ |
| Normal Operator | $AA^\dagger = A^\dagger A$ |
| Unitary Operator | $A^\dagger = A^{-1}$ |
| Identity Operator | $\mathbb{I}$ such that: |
| | $\forall \lvert v\rangle, \mathbb{I}\lvert v\rangle = \lvert v\rangle$ |
| P Orthogonal Complement | $Q \equiv \mathbb{I} - P$ |

### Linear Operator

A **linear operator** between the vector spaces $V$ and $W$ is define to be any function $\hat{A} : V \to W$ which satisfies:
$\hat{A}(\alpha\vec{v} + \beta\vec{w}) = \alpha\hat{A}\vec{v} + \beta\hat{A}\vec{w}$.

**Properties 3** *Let* $\hat{A}$ *be a linear operator on* $V \to W$ *and* $A$ *be the matrix representation of* $\hat{A}$. • $\hat{A}(\sum_i a_i \lvert v_i\rangle) = \sum_i a_i\hat{A}\lvert v_i\rangle$ • $\hat{A}\lvert v_j\rangle = \sum_i A_{ij}\lvert w_i\rangle$

### Inner product

A Inner Product $\langle .,. \rangle$ is a function that output a complex number and satisfies the following conditions: Let $\vec{v} \in \mathbb{C}^n, \vec{w} \in \mathbb{C}^n$.

1. $\langle \vec{v}, \sum_i a_i\vec{w}_i\rangle = \sum_i a_i\langle \vec{v}, \vec{w}_i\rangle$
2. $\langle \vec{v}, \vec{w}\rangle = (\langle \vec{w}, \vec{v}\rangle)^*$
3. $\langle \vec{w}, \vec{w}\rangle > 0$ if and only if $w \neq 0$. Note $\forall\vec{w}(\langle \vec{w}, \vec{w}\rangle \geq 0)$.

In quantum mecanics the inner product is generaly noted $\langle .|. \rangle$.

**Properties 4** • $\langle A, A\rangle = \lVert A\rVert^2$ • *if* $\langle A, B\rangle = 0$ *then* $A$ *and* $B$ *are orthogonal.*

### Inner product Space

An inner product space is a vector space $V$ equipped with an inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ (or $\mathbb{R}$).

An Inner product space with an orthonormal basis $\lvert i\rangle$ such that $v = \sum_i v_i\lvert i\rangle$ and $w = \sum_i w_i\lvert i\rangle$, the inner product space of $\langle v, w\rangle$ is define by $(v^*)^T w$.

### Hilbert Spaces

A Hilbert space is a vector space(generaly complex) equipped with an inner product, meaning every Cauchy sequence of vectors with respect to the induced norm converges to a vector within the space. **In finit dimentions hilbert spaces is exactly the same thing as Inner Product space.**

### Dirac Notation (or Bra-Ket Notation)

Terminology: ket of $A$ is $\lvert A\rangle$ and bra of $A$ is $\langle A\rvert$. ket is a row vector and bra is a column vector.
Example, let $\Sigma = \{A, B, C\}$ then

$$\lvert B\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix} \text{ and } \langle B\rvert = (0, 1, 0)$$

### Kronecker delta($\delta_{ij}$)

$$\delta_{ij} = \begin{cases}1 & \text{if } i = j,\\ 0 & \text{if } i \neq j.\end{cases}$$

**Properties 5** • $\langle i|j\rangle = \delta_{ij}$ • $\mathbb{I}_{ij} = \delta_{ij}$ • $\sum_i \delta_{ij}a_i = a_j$ • $\sum_k \delta_{ik}\delta_{kj} = \delta_{ij}$

### Gram-Schmidt(in Dirac Notation)

$$\lvert v_1\rangle = \frac{\lvert w_1\rangle}{\lVert\lvert w_1\rangle\rVert}$$

$$\lvert v_{k+1}\rangle = \frac{\lvert w_{k+1}\rangle - \sum_{i=1}^k \langle v_i|w_{k+1}\rangle\lvert v_i\rangle}{\left\lVert\lvert w_{k+1}\rangle - \sum_{i=1}^k \langle v_i|w_{k+1}\rangle\lvert v_i\rangle\right\rVert}$$

### Adjoin (Hermitian conjugate)

Let $\hat{A}$ be a linear operator on the hilbert space $V$. Then $\exists!\hat{A}^\dagger$ on $V$ such that $\langle v, \hat{A}w\rangle = \langle \hat{A}^\dagger v, w\rangle$.

**Properties 6** • $\lvert v\rangle^\dagger = \langle v\rvert$ • $(\hat{A}^\dagger)^\dagger = \hat{A}$ • $(\hat{A}\hat{B})^\dagger = \hat{A}^\dagger\hat{B}^\dagger$ • $(\sum_i a_i\hat{A}_i)^\dagger = \sum_i a_i^*\hat{A}_i^\dagger$

### Outer Product

Let $\lvert v\rangle \in V, \lvert w\rangle \in W$ where $V$ and $W$ are **inner product spaces**. The **outer product** is define $\lvert w\rangle\langle v\rvert$ as to be a linear operator $V \Rightarrow W$ whose define by $(\lvert w\rangle\langle v\rvert)(\lvert v'\rangle) \equiv \lvert w\rangle\langle v|v'\rangle = \langle v|v'\rangle\lvert w\rangle$

**Properties 7** • $\sum_i \lvert i\rangle\langle i\rvert = \mathbb{I}$ *(completness relation)* • $\lvert w\rangle\langle v\rvert = \sum_i \lvert w_i\rangle\langle v_i\rvert$ • $\lvert w\rangle\langle v|\,v'\rangle = \sum_i \lvert w_i\rangle\langle v_i|\,v'\rangle$

### The Cauchy-Schwarz Inequality

Let the vectors $\lvert v\rangle, \lvert w\rangle$ then $\lvert\langle v|w\rangle\rvert^2 \leq \langle v|v\rangle\langle w|w\rangle$

## Tensor Product

Let $V$ and $W$ be vector spaces of dimension $m$ and $n$ respectively. Then $V \otimes W$ is an $mn$-dimension vector spaces. The elements of $V \otimes W$ are a linear combination of the tensor product $\lvert v\rangle \otimes \lvert w\rangle$.

**Properties 8** *Let* $z$ *be a scalar,* $\lvert v\rangle \in V, \lvert w\rangle \in W$*, where* $V$ *and* $W$ *are* ***Hilbert spaces****, then the following must be satisfied.*
• $z(\lvert v\rangle \otimes \lvert w\rangle) = z\lvert v\rangle \otimes \lvert w\rangle = \lvert v\rangle \otimes z\lvert w\rangle$
• $\lvert v\rangle \otimes (\lvert w_1\rangle + \lvert w_2\rangle) = \lvert v\rangle \otimes \lvert w_1\rangle + \lvert v\rangle \otimes \lvert w_2\rangle$
• $(\lvert v_1\rangle + \lvert v_2\rangle) \otimes \lvert w\rangle = \lvert v_1\rangle \otimes \lvert w\rangle + \lvert v_2\rangle \otimes \lvert w\rangle$
• $(\lvert w\rangle \otimes \lvert v\rangle, \lvert w'\rangle \otimes \lvert v'\rangle) = \langle w|w'\rangle\langle v|v'\rangle$

### Kronecker Product

The **Kronecker product** is the matrix representation of the tensor product.
Let $A$ be a $m \times n$ matrix and $B$ be a $p \times q$ matrix. The **Kronecker product** $A \otimes B$ is a $mp \times nq$ matrix defined as:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

## Comutator and Anticomutator

| | |
|---|---|
| Comutator | $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ |
| Anticomutator | $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ |
| $\hat{A}$ comutes with $\hat{B}$ | $[\hat{A}, \hat{B}] = 0$, then $\hat{A}\hat{B} = \hat{B}\hat{A}$ |
| $\hat{A}$ anticomutes with $\hat{B}$ | $\{\hat{A}, \hat{B}\} = 0$, then |
| | $\hat{A}\hat{B} = -\hat{B}\hat{A}$ |

**Theorem 1 (Simultaneous Diagonalization)** *Two hermitian operators* $\hat{A}$ *and* $\hat{B}$ *can be diagonalized in the same basis if and only if they comute.*

# Quantum mechanics

## Short Definitions

| | |
|---|---|
| $h$ | Planck constant |
| $\hbar = \frac{h}{2\pi}$ | Reduced Planck constant |
| ground state | Lowest energy state of a quantum system. |
| eigenstate | State $\lvert E\rangle$ such that $\hat{A}\lvert E\rangle = E\lvert E\rangle$ where $E$ is the eigenvalue. |

## Long Definitions

**Hamiltonian operator** corresponding to the total energy of the system. $\boxed{\hat{H} = \hat{V} + \hat{K}}$, potential energy operator: $\hat{V}$, kinetic energy operator: $\hat{K}$.

**Hamiltonian Spectral Decomposition** $\hat{H} = \sum_i E_i \lvert E_i\rangle \langle E_i\rvert$, where $\lvert E_i\rangle$ are the eigenvectors (eigenstates) and $E_i$ are the eigenvalues.

**Postulates 1** *Every isolated physical system is associated with a Hilbert space, known as the **state space** of the system. The system is completely described by its state vector, which is a unit vector in the **state space**.*

**Postulates 2** *The evolution of a closed quantum system is described by a unitary transformation. Then $\lvert\psi_2\rangle = U\lvert\psi_1\rangle$ for **discrete time evolution** and $i\hbar\frac{d\lvert\psi\rangle}{dt} = \hat{H}\lvert\psi\rangle$ for **continuous time evolution**.*

## Pauli Matrices

- $\sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $\sigma_x = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $\sigma_y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
- $\sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

## Hadamard Transform

The Hadamard transform $H$ is a linear operator defined by its action on the computational basis states $\lvert 0\rangle$ and $\lvert 1\rangle$ as follows: