1 Linear Systems and Matrix Equations

Linear Algebra from Elementary to Advanced

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Definition

Linear Functions	All terms are of degree 0 or 1. A solution of a system of linear equation is set of points that makes the equation system true.
Consistent	lin. systems is consistent if either 1 or ∞ solutions exist else inconsistent.
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Coefficient Matrix Example

$$\begin{cases} A_1x_1 + A_2x_2 + A_3x_3 = \alpha \\ B_1x_1 + B_2x_2 + B_3x_3 = \beta \end{cases} \Leftrightarrow \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix} \tag{1}$$

Augmented Matrix Example

$$\begin{cases} A_1x_1 + A_2x_2 + A_3x_3 = \alpha \\ B_1x_1 + B_2x_2 + B_3x_3 = \beta \end{cases} \Leftrightarrow \left[\begin{array}{ccc|c} A_1 & A_2 & A_3 & \alpha \\ B_1 & B_2 & B_3 & \beta \end{array}\right] \tag{2}$$

Row-Equivalence

Two matrce are row-equivalent if there is a sequence of **EROS** that transforms one into the other.

Elementary Row Operations (EROS)

1. [**Replacement**] Replace one row by sum of itself.
2. [**Interchange**] Swap position of 2 rows.
3. [**Scaling**] Multiply all entries in row by non-zero constant.

Echelon Form (ef)

1. All non-zero rows are above any rows of all-zero.
2. Each leading entry of a row is in a column to the right of the roe above it.
3. All entries in a column below a leading entry are 0.

Reduced Row Echelon Form (rref)

1. As to be in echelon form.
2. Leading entry in each row is 1.
3. Each leading 1 is the only non-zero entry in its column.

Theorems

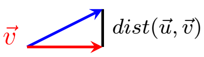
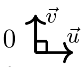
Theorem 1 *Every matrix is row equivalent to a unique row echelon form.*

Theorem 2 *Every matrix is row equivalent to a unique row echelon form.*

Matrix Algebra, Determinants, & Eigenvectors

Orthogonality and Diagonalization

Definition

Inner Product	$\vec{v} \cdot \vec{u} = \vec{v}^T\vec{u} = u_1v_1 + ... + u_nv_n$ (Also called dot product or scalar product)
Length of $\vec{x}$	$\|\|\vec{x}\|\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + ... + x_n^2}$ (Also called Norm or Magnitude)
Unit vectore	A vector with $\|\|\vec{x}\|\| = 1$
Normalization	The formula $\vec{u} = \frac{\vec{x}}{\|\|\vec{x}\|\|}$ creat a unit vector in the same direction as $\vec{x}$.
Distance	$dist(\vec{u}, \vec{v}) = \vec{u} - \vec{v}$ 
Orthogonality	$\vec{u} \cdot \vec{v} = 0$ 
Orthogonal Set	A set of vectors $\{\vec{u}_1, ..., \vec{u}_p\} \in \mathbb{R}^n$ such that each distinct vectors are orthogonal.
Orthogonal Basis	For a subspace W of R^n is a basis that is also an orthogonal set.
Orthonormal set	is a set of orthogonal unit vectors.
Orthogonal matrix	Is a squared matrix whose columns are orthonormal.
Normal equation	$A^TA\vec{x} = A^T\vec{b}$
Least-Squares Error	$\|\|\vec{b} - A\hat{x}\|\|$
Symetric	$A_{m\times n}$ is symetric if $A = A^T$
Spectrum	Set of eigenvalues of $A_{n\times n}\{\lambda_1, ..., \lambda_n\}$ is called the **spectrum** of A.
Quadratic Form	A **quadratic form** on $\mathbb{R}^n$ is a function Q from $\mathbb{R}^n \to \mathbb{R}$ of the form $Q(\vec{x}) = \vec{x}^TA\vec{x}$, where A is a symetric matrix.
Orthogonal Complements	Let W be a subspace of $\mathbb{R}^n$. The orthogonal complement of W is: $W^\perp = \{\vec{x} \in \mathbb{R}^n \| \vec{x} \cdot \vec{w} = \vec{0}, \forall \vec{w} \in W\}$
Orthogonal Projection	of $\vec{y} \in \mathbb{R}^n$ onto subspace W of $\mathbb{R}^n$ is the vector $\hat{y}$ in W such that $\vec{y} - \hat{y}$ is in $W^\perp$. (Noted $\text{proj}_W(\vec{y})$)
Least-Squares Problem	is to find $\vec{x}$ that makes $\|\|\vec{b} - A\vec{x}\|\| = \text{dist}(\vec{b}, A\vec{x})$ as small as possible.
Orthogonally Diagonalizable Matrix	is a square matrix for which there exist an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1} = PDP^T$.

Theorems

Properties 1 *Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$ then:*
• $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ • $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$ • $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = (c\vec{v}) \cdot \vec{u}$
• $\vec{u} \cdot \vec{u} = u_1^2 + ... + u_n^2 \geq 0$

Theorem 3 (Fundamental Subspaces Theorem) *Let $A_{M\times N}$ then:* • $(row(A))^\perp = nul(A)$ • $(COl(A))^\perp = nul(a^\perp)$

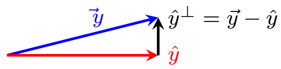
Theorem 4 *If $S = \{\vec{u}_1, ..., \vec{u}_p\}$ is an **orthogonal set** of non-zero vectors in $\mathbb{R}$ then S is a **linearly independant** set.*

Theorem 5 *let $\{\vec{u}_1, ..., \vec{u}_p\}$ be an **orthogonal basis** for $w \subset \mathbb{R}^n$. Let $y \in W$. Then $\vec{y} = C_1\vec{u}_1 + ... + C_p\vec{u}_p$, $C_j = \frac{\vec{y}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}$*

Theorem 6 *Let $u = (\vec{u}_1\vec{u}_2\vec{u}_3)$, where $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is orthonormal set. $u^T \cdot u = I$*

Theorem 7 *Let u be an **orthogonal matrix**, Let $\vec{x}, \vec{y} \in \mathbb{R}$: • $|u\vec{x}| = |\vec{x}|$ • $(u\vec{x}) \cdot (u\vec{y}) = \vec{x} \cdot \vec{y}$*

Theorem 8 (Orthogonal Decomposition Theorem) *Let W be a subspace with an orthogonal basis $\{\vec{u}_1, ..., \vec{u}_p\}$. let $\vec{y} \in \mathbb{R}^n$ then: $\hat{y} = \left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + ... + \left(\frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\right)\vec{u}_p$*



Theorem 9 (Best Approximation Theorem) *Let W be a subspace of $\mathbb{R}$. Let $\vec{y} \in \mathbb{R}^n$ and $\hat{y} = proj_W(\vec{y})$. Then $\hat{y}$ is closest point to $\vec{y}$ in W. That is $\forall \vec{v} \neq \vec{y}$ $||\vec{y} - \hat{y}|| < ||\vec{y} - \vec{v}||$*

Theorem 10 (Orth. Proj. with Orthonormal Bases) *Let W subspace in $\mathbb{R}^n$. Let $\{\vec{u}_1, ..., \vec{u}_p\}$ be an orthonormal basis of W. $proj_W(\vec{y}) = (\vec{y}\cdot\vec{u}_1)\vec{u}_1 + ... + (\vec{y}\cdot\vec{u}_p)\vec{u}_p$ Let $U = (\vec{u}_1\ \vec{u}_2\ ...\ \vec{u}_p)$. Then $proj_W(\vec{y}) = UU^T\vec{y}$*

Theorem 11 (Normal Equations and Least-Squares Sol.) *The set of least squares solutions of $A\vec{x} = \vec{b}$ coincide the non-empty solutions of the Normal equation.*

Theorem 12 (Criteria for a Unique Least-Squares Sol.) *Let $A_{M\times N}$ then the following are equivalent: • $A\vec{x} = \vec{b}$ has unique least-square solution $\forall\vec{b} \in \mathbb{R}^n$. • Columns of A are linearly independant. • The matrix A^TA is invertible. When the statements are true, the least squares solution is given by $\hat{x} = (A^TA)^{-1}A^T\vec{b}$*

Theorem 13 (Eigenvectors of a Symmetric Matrix) *Let A be a symmetric matrix. Then any two eigenvectors of A from different eigenspaces are orthogonal.*

Theorem 14 (Symmetric Matrices and Orth. Diag.) *A square matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.*

Theorem 15 (Spectral Theorem For Symetric Matrices) *Let $A_{n\times n}$ be symetric. Then the following properties hold. • A has n real eigenvalues (counting multiplicities). • Then dimension of the eigenspace for each eigenvalue λ is equal to the multiplicity of λ. • The eigenvectors of A corresponding to different eigenvalues are orthogonal. • A is orthogonally diagonalizable.*

Theorem 16 (Principal Axis Theorem) *Let $A_{N\times N}$ be symetric. Then there is an orthogonal change of variable $\vec{x} = P\vec{y}$ that transform the quadratic form $\vec{x}^TA\vec{x}$ to $\vec{y}^TA\vec{y}$ whith no cross terms(Ex. ofcross terms is x_1x_2). The columns of P are the eiganvectors of A.*

Theorem 17 (Eiganvalue of Definites Quadratic) *The quadratic form is: • Positive definite if and only if $\forall i \lambda_i > 0$. • Negative definite if and only if $\forall i \lambda_i < 0$. • Indefinite if and only if A as positive and negative eiganvalues.*

Fact about the Orthogonal Complements

• $\vec{0} \in W^\perp$ since $\vec{0} \cdot \vec{w} = 0$ • If $W \in W^\perp, c \in \mathbb{R}$ then $cW \in W^\perp$ • If $\vec{w}_1, \vec{w}_2 \in W^\perp$ so $(\vec{w}_1 + \vec{w}_2) \cdot \vec{x} = \vec{w}_1 \cdot \vec{x} + \vec{w}_2 \cdot \vec{x} \in W$

Gram-Schmit Process(G-S)

The Gram-Schmit Process is an algorithme producing an orthogonal basis.
Start whith $\{\vec{x}_1, ..., \vec{x}_n\}$ basis for noozero subspace w of $\mathbb{R}^n$ define:

$$\vec{v}_1 = \vec{x}_1$$
$$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{x}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1$$
$$\vdots$$
$$\vec{v}_n = \vec{x}_n - \left(\frac{\vec{x}_n\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 - ... - \left(\frac{\vec{x}_n\cdot\vec{v}_{n-1}}{\vec{v}_{n-1}\cdot\vec{v}_{n-1}}\right)\vec{v}_{n-1}$$

Finding the Matrix of a Quadratic Form

$$Q(\vec{x}) = Ax_1^2 + Bx_1x_2 + Cx_2^2 \Leftrightarrow \begin{bmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{bmatrix}$$

Quadratic Form Terminology

A quadratic for $Q(\vec{x})$ is: • Positive Definite If $Q(\vec{x}) > 0, \forall\vec{x} \neq \vec{0}$ • Positive Semidefinite If $Q(\vec{x}) \geq 0$ • Negative Definite If $Q(\vec{x}) < 0, \forall\vec{x} \neq \vec{0}$ • Negative Semidefinite If $Q(\vec{x}) \leq 0$