Discrete Mathematics

Class: MATH 2410 ©Maximilien Notz 2025

Mathematical Statement

Statement	is any declarative sentence which is either true or false.
Atomic	if it cannot be divided into smaller statements.
Molecular	if it can be divided into smaller statements.
conjunction	$p \wedge q$ equivalent to "p and q".
disjunction	$p \vee q$ equivalent to "p or q".
	where p is the hypothesis and q the conclusion.
Implication	$p \to q$ equivalent to "if p then q".
Biconditional	$p \leftrightarrow q$ equivalent to "if and only if p then q".
Negation	$\neg p$ equivalent to "not p".
Converse	
Contrapositive	
There is a x	$\exists x$
For all x	$\forall x$

Naive Set Theory

Set Notation

Universal set	$\mathbb{U}$
Empty set	$\emptyset = \{\}$, Remember: $\forall A(\emptyset \subset A)$
Power set	$\mathcal{P}(A)$ is the set of all the subsets of A.
Partition of A	A collection of nonempty, pairwise-disjoint subsets whose union is A.
Element of	$\in$. Example:$2 \in \{1,2,3\}$
Subset of	$\subseteq$. Example: $\{A,B,C\} \subseteq \{B,C,D\}$ $A \subseteq B \Leftrightarrow \forall x$
Proper subset of	$\subset$. Example: $\{A,B,C\} \subset \{A,B,C,D\}$
Intersection	$\bigcap_{i\in I} A_i = \{x \in \mathbb{U} \vert \forall i \in I, x \in A_i\}$ $A \cap B = \{x \in \mathbb{U} \vert x \in A \wedge x \in B\}$
Union	$\bigcup_{i\in I} A_i = \{x \in \mathbb{U} \vert \exists i \in I, x \in A_i\}$ $A \cup B = \{x \in \mathbb{U} \vert x \in A \vee x \in B\}$
Difference	$A \backslash B = \{x \in A \vert x \notin B\}$
Symmetric difference	$A \Delta B = (A\backslash B) \cup (B \backslash A)$
Cartesian Product	$A \times B = \{(x,y) \vert x \in A \wedge y \in B\}$
Complement of	$A^C = \bar{A} = \{x \in \mathbb{U} \vert x \notin A\}$

Cardinality

Cardinality($\vert A\vert$)	The number of elements in a set.
finite set	Let X be a finite set then $\vert X\vert \in \mathbb{N}$
countable set	A set S is countable if and only if that is finit or $\vert S\vert = \vert\mathbb{N}\vert$.
aleph null.	$\aleph_0 = \vert\mathbb{N}\vert$

Axiom 1 (Axiom of extensionality) *Two sets are equal if and only if they have the same elements.*

Theorem 1 *Let A and B be sets, then $|A| = |B|$ if and only if there is a one-to-one correspondence from A to B.*

Theorem 2 *If A and B are countable, then $A \cup B$ is countable.*

Theorem 3 (Cantor's Theorem) *For every set A, $|A| < |\mathcal{P}(A)|$.*

Theorem 4 (Schröder–Bernstein) *If there are injective function(one-to-one) functions $f : A \to B$ and $g : B \to A$, then there is a one-to-one correspondence between A and B. In other words If A and B are set with $|A| \neq |B|$ and $|B| \neq |A|$, then $|A| = |B|$.*

Properties 1 *Let S be the universal set.* • *if $A \subseteq B$ and $B \subseteq A$ then $A = b$.* • $\forall A, A \subseteq A$ • $|\mathcal{P}(A)| = 2^{|A|}$ • $A \cup A = A \cap A = A$ • $A \cup \emptyset = A$ • $A \cap \emptyset = \emptyset$ • $A \cup S = S$ • $A \cap S = A$ • $(A \cup B) \cup C = A \cup (B \cup C)$ • $(A \cap B) \cap C = A \cap (B \cap C)$ • $A \cup B = B \Leftrightarrow A \subseteq B$ • $A \cup B = A \Leftrightarrow A \subseteq B$ • $A \backslash B \neq B \backslash A$ • $A \backslash \emptyset = A$ • $A \backslash S = \emptyset$ • $A \backslash \emptyset = A \Leftrightarrow A \subseteq B$ • $A \backslash S = A^C$ • $A \times (B \cup C) = (A \times B) \cup (A \times C)$ • $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ • $(A \cup B)^C = A^C \cap B^C$ • $(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C) = A \cap (B \backslash C)$ • $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$

Functions

Functions	A rule that assigns each input exactly one output.
Domain	The set of all input of a function.(X in $f : X \to Y$)
Codomain	The set of all output a function.(Y in $f : X \to Y$)
Range	Is the subset of Y of elements that have an antecedent in X by f
$f : x \to y$	a function f with a domain x and a codomain y.
Recursive f.	
Injective	every element of the codomain is the image of $f(a) = f(b) \Rightarrow a = b$ **at most** one element from the domain.
Surjective	every element of the codomain is the image of **at least** one element from the domain.
Bijection	A function that is **Injective** and **Surjective**.
Image	$f(A) = \{f(a) \in Y : a \in A\}$, where $A \subset$ domain.
Inverse Image	$f^{-1}(B) = \{f(b) \in X : b \in B\}$, where $B \subset$ codomain.
Function Set	B^A contains all functions from A to B ($A \to B$).

Counting

n-bit string	
bit string weight	the number of **1** in a bit string.
$\mathbf{B^n_k}$	the set of all **n-bit strings** of weight k.
Factorial	$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$

Additive Principle

General Definition: if event A can occur in m ways, and even B can occur in n **disjoint** (A and B can't apen at the same time.) ways, then A and B can occur in $m + n$ ways.
Set Definition: Given 2 sets A and B, then $|A \cup B| = |A| + |B| - |A \cap B|$. Given 3 sets A, B and C, then $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |C \cap B| + |A \cap B \cap C|$.

Multiplicative Principle

General Definition: if event A can occur m ways, and each possibility for A allows for exactly n ways for event B, then the event "A and B" can occur $m \cdot n$ ways.
Set Definition: Given 2 sets A and B, we have $|A \times B| = |A| \cdot |B|$.

Binomial coefficient

Formula: n choose $k = \binom{n}{k} = C^n_k = \frac{n!}{(n-k)!k!}$

Theorem 5 (Binomial Theorem) $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Properties 2 • $\binom{n}{k}$ *is the number of subset of size n each of cardinality k.* • $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ • $\binom{n}{k} = |\mathbf{B_k^n}|$

Sequences

Symbolic Logic

Name	Symbol	Translate to
Disjonction	$A \wedge B$	A and B.
Conjunction	$A \vee B$	A or B.
Negation	$\neg A$	not A.
Condition/Implication	$A \Rightarrow B$	if A then B.
Bicondition	$A \Leftrightarrow B$	if and only if A then B.
Exclusive Disjunction	$A \oplus B$	Either A or B, but not both.
Universal	$\forall x$	For all x's.
Existential	$\exists x$	There is at least one x.
Unique Existential	$\exists! x$	There is exactly one x.
Equivalence	$A \equiv B$	A is identical to B.

Important Equivalences & Properties

• $\neg(\neg A) \equiv A$ • $p \wedge T \equiv p$ • $p \wedge \perp \equiv \perp$ • $p \vee T \equiv T$ • $p \vee \perp \equiv P$ • $A \oplus B \equiv (A \vee B) \wedge \neg(A \wedge B)$ • $p \Rightarrow q \equiv \neg p \Rightarrow \neg q$ • $p \Rightarrow q \equiv \neg p \vee q$ • $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$ • $\neg(p \Leftrightarrow q) \equiv \neg p \Leftrightarrow q \equiv p \Leftrightarrow \neg q$

Properties 3 • $A \vee B \equiv B \vee A$ • $A \vee (B \vee C) \equiv C \vee (A \vee B)$ • $A \wedge B \equiv B \wedge A$ • $A \wedge (B \wedge C) \equiv C \wedge (A \wedge B)$ • $A \oplus B \equiv B \oplus A$ • $A \oplus (B \oplus C) \equiv C \oplus (A \oplus B)$

deMorganLaws

• $\neg \forall x P(x) = \exists x P(\neg x)$ • $\neg \exists x P(x) = \forall x P(\neg x)$ • $\neg \exists x \exists y P(x, y) = \forall x \exists y P(\neg x, y)$ • $\neg \left(\bigwedge_{i=0}^{n} a_i\right) \equiv \bigvee_{i=0}^{n} \neg a_i$ • $\neg \left(\bigvee_{i=0}^{n} a_i\right) \equiv \bigwedge_{i=0}^{n} \neg a_i$

Proofs

Direct Proof

Goal: Prove $p \Rightarrow q$.
Idea: Assume p and use definitions/algebra to derive q.
Template: Assume p. [derive consequences] Therefore q.

Proof by Contrapositive

Goal: Prove $p \Rightarrow q$.
Idea: Instead of proving $p \Rightarrow q$, prove $\neg q \Rightarrow \neg p$.
Template: To prove $p \Rightarrow q$, assume $\neg q$ and derive $\neg p$; therefore $\neg q \Rightarrow \neg p$, so $p \Rightarrow q$.

Proof by Counter Example

Goal: Disprove $\forall x\, P(x)$ (show $\exists x\, \neg P(x)$).
Idea: Exhibit a specific counterexample x_0 with $\neg P(x_0)$.
Template: Identify the claim form (usually $\forall x\, P(x)$). Choose a concrete x_0 in the domain and verify $\neg P(x_0)$ holds by computation or definition checking.

Proof by Cases

Goal: Prove the claim.
Idea: Split into exhaustive, mutually exclusive cases and prove the claim in each case.
Template: Partition the domain into cases $C_1, \ldots, C_k$ that cover all possibilities. For each i, assume C_i and show the statement holds. Conclude it holds in all cases by exhaustion.

Proof by Contradiction

Goal: Prove a statement S.
Idea: Assume $\neg S$ and derive a contradiction; conclude S.
Template: Suppose $\neg S$. [Deduce an impossibility such as $P \wedge \neg P$ or a known falsehood.] Contradiction; therefore S.

Graph Theory