Quantum Computation & Quantum Information

Book: Quantum Computation and Quantum Information. Nielsen, M. A., & Chuang, I. L.

Complex Numbers Short recap

Let $a, b \in \mathbb{R}$ and $z \in \mathbb{C}$ such that $z = a+bi$.

Real part	$\Re\mathfrak{e}(a+ib) = a$
Imaginary part	$\Im\mathfrak{m}(a+ib) = b$
Absolute Values	$\lvert z\rvert = \sqrt{zz^*}$ $= \lVert(a \;\; b)\rVert_2$
Complexe conjugate	$(a+ib)^* = a-ib$ $(a-ib)^* = a+ib$
Trig. Formulas	$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$
Trig. Formulas	$\cos z = \frac{e^{iz}+e^{-iz}}{2}$

Properties 1 (Complexe conjugate) • $(Z^*)^* = Z$ • $(Z+W)^* = Z^* + W^*$ • $(Z-W)^* = Z^* - W^*$ • $(ZW)^* = Z^*W^*$ • $Z^*Z = |Z|^2$ • $(Z^n)^* = (Z^*)^n$, *for* $n \in \mathbb{Z}$ • $ln(Z^*) = (ln(Z))^*$ *if Z is not 0 or a negative real number.*

Properties 2 (Absolute Values) • $|z_1 z_2| = |z_1|\,|z_2|$ • *The absolute value define the metric of the space* $\mathbb{C}$ *(*$\mathbb{C}$ *is complete).* • $|z_1+z_2|^2 = |z_1|^2+|z_2|^2+\Re\mathfrak{e}(z_1 z_2^*)$ • $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - \Re\mathfrak{e}(z_1 z_2^*)$ • $|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$

Linear Algebra

Short Definitions

Hermitian Operator	$A = A^\dagger$
Normal Operator	$AA^\dagger = A^\dagger A$
P Orthogonal Complement	$Q \equiv \mathbb{I} - P$

Linear Operator

A **linear operator** between the vector spaces V and W is define to be any function $\hat{A} : V \to W$ which satisfies:
$\hat{A}(\alpha\vec{v} + \beta\vec{w}) = \alpha\hat{A}\vec{v} + \beta\hat{A}\vec{w}$.

Properties 3 *Let* $\hat{A}$ *be a linear operator on* $V \to W$ *and* A *be the matrix representation of* $\hat{A}$. • $\hat{A}(\sum_i a_i \left|v_i\right\rangle) = \sum_i a_i \hat{A}\left|v_i\right\rangle$ • $\hat{A}\left|v_j\right\rangle = \sum_i A_{ij}\left|w_i\right\rangle$

Inner product

A Inner Product $\langle .,. \rangle$ is a function that output a complex number and satisfies the following conditions: Let $\vec{v} \in \mathbb{C}^n, \vec{w} \in \mathbb{C}^n$.

1. $\langle \vec{v}, \sum_i a_i \vec{w}_i \rangle = \sum_i a_i \langle \vec{v}, \vec{w}_i \rangle$
2. $\langle \vec{v}, \vec{w} \rangle = (\langle \vec{w}, \vec{v} \rangle)^*$
3. $\langle \vec{w}, \vec{w} \rangle > 0$ if and only if $w \neq 0$. Note $\forall \vec{w}(\langle \vec{w}, \vec{w} \rangle \geq 0)$.

In quantum mecanics the inner product is generaly noted $\langle .|. \rangle$.

Properties 4 • $\langle A, A \rangle = \lVert A \rVert^2$ • *if* $\langle A, B \rangle = 0$ *then* A *and* B *are orthogonal.*

Inner product Space

An inner product space is a vector space V equipped with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ (or $\mathbb{R}$).

An Inner product space with an orthonormal basis $|i\rangle$ such that $v = \sum_i v_i |i\rangle$ and $w = \sum_i w_i |i\rangle$, the inner product space of $\langle v, w \rangle$ is define by $(v^*)^T w$.

Hilbert Spaces

A Hilbert space is a vector space(generaly complex) equipped with an inner product, meaning every Cauchy sequence of vectors with respect to the induced norm converges to a vector within the space. **In finit dimentions hilbert spaces is exactly the same thing as Inner Product space.**

Dirac Notation (or Bra-Ket Notation)

Terminology: ket of A is $|A\rangle$ and bra of A is $\langle A|$. ket is a row vector and bra is a column vector.
Example, let $\Sigma = \{A, B, C\}$ then

$$|B\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \langle B| = (0, 1, 0)$$

Kronecker delta(δ_{ij})

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Properties 5 • $\langle i|j \rangle = \delta_{ij}$ • $\mathbb{I}_{ij} = \delta_{ij}$ • $\sum_i \delta_{ij} a_i = a_j$ • $\sum_k \delta_{ik}\delta_{kj} = \delta_{ij}$

Gram-Schmidt(in Dirac Notation)

$$|v_1\rangle = \frac{|w_1\rangle}{\lVert |w_1\rangle \rVert}$$

$$|v_{k+1}\rangle = \frac{|w_{k+1}\rangle - \sum_{i=1}^k \langle v_i|w_{k+1}\rangle |v_i\rangle}{\left\lVert |w_{k+1}\rangle - \sum_{i=1}^k \langle v_i|w_{k+1}\rangle |v_i\rangle \right\rVert}$$

Adjoin (Hermitian conjugate)

Let $\hat{A}$ be a linear operator on the hilbert space V. $\exists! \hat{A}^\dagger$ on V such that $\langle v, \hat{A}w \rangle = \langle \hat{A}^\dagger v, w \rangle$.

Properties 6 • $|v\rangle^\dagger = \langle v|$ • $(\hat{A}^\dagger)^\dagger = \hat{A}$ • $(\hat{A}\hat{B})^\dagger = \hat{A}^\dagger \hat{B}^\dagger$ • $(\sum_i a_i \hat{A}_i)^\dagger = \sum_i a_i^* \hat{A}_i^\dagger$

Outer Product

Let $|v\rangle \in V, |w\rangle \in W$ where V and W are **inner product spaces**. The **outer product** is define $|w\rangle \langle v|$ as to be a linear operator $V \Rightarrow W$ whose define by $(|w\rangle \langle v|)(|v'\rangle) \equiv |w\rangle \langle v|v'\rangle = \langle v|v'\rangle |w\rangle$

Properties 7 • $\sum_i |i\rangle \langle i| = \mathbb{I}$ *(completness relation)* • $|w\rangle \langle v| = \sum_i |w_i\rangle \langle v_i|$ • $|w\rangle \langle v|\, v'\rangle = \sum_i |w_i\rangle \langle v_i|\, v'\rangle$

The Cauchy-Schwarz Inequality

Let the vectors $|v\rangle, |w\rangle$ then $|\langle v|w\rangle|^2 \leq \langle v|v\rangle\langle w|w\rangle$

Tensor Product

Let V and W be vector spaces of dimension m and n respectively. Then $V \otimes W$ is an mn-dimension vector spaces. The elements of $V \otimes W$ are a linear combination of the tensor product $|v\rangle \otimes |w\rangle$.

Properties 8 *Let* z *be a scalar,* $|v\rangle \in V, |w\rangle \in W$, *where* V *and* W *are* ***Hilbert spaces***.
• $z(|v\rangle \otimes |w\rangle) = z|v\rangle \otimes |w\rangle = |v\rangle \otimes z|w\rangle$
• $|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$
• $(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$
• $(|w\rangle \otimes |v\rangle, |w'\rangle \otimes |v'\rangle) = \langle w|w'\rangle\langle v|v'\rangle$

Quantum mecanics

Pauli Matrices

• $\sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ • $\sigma_x = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ • $\sigma_y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ • $\sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Mathematical Concepts

Dirac Notation(or Bra-Ket Notation)

Terminology: ket of A is $|A\rangle$ and bra of A is $\langle A|$. Example, let $\Sigma = \{A, B, C\}$ then $|A\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $|B\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\langle A| = (1,0,0)$, $\langle B| = (0,1,0)$ Note that $|\Psi\rangle$ then $\langle\Psi| = |\Psi\rangle^T$.
bra-kets: we denote $\langle a|b\rangle$ the matrice product of

Cartesian product

Let $y = \{0, 1\}$ and $x = \{a, b\}$. Then,
$x \times y = \{(0, a), (0, b), (1, a), (1, b)\}$ and
$y \times x = \{(a, 0), (a, 1), (b, 0), (b, 1)\}$.

Tensor Product of vectors

$$\begin{bmatrix}a_0\\\vdots\\a_n\end{bmatrix} \otimes \begin{bmatrix}b_0\\\vdots\\b_n\end{bmatrix} = \begin{bmatrix}a_0b_0\\\vdots\\a_nb_n\end{bmatrix}$$

Tensor Product Properties

1. $(|\phi_1\rangle + |\phi_2\rangle) \otimes |\psi\rangle = |\phi_1\rangle \otimes |\psi\rangle + |\phi_2\rangle \otimes |\psi\rangle$
2. $(a\,|\phi\rangle) \otimes |\psi\rangle = a(|\phi\rangle \otimes |\psi\rangle)$
3. $|a\rangle \otimes |b\rangle = |b\rangle \otimes |a\rangle$

Quantum Information Systems

State Vector

Quantum State of a sytem is represented by a complex column vector. Let the quantum state vector v be equal to $\begin{bmatrix}a_0\\\vdots\\a_n\end{bmatrix}$, where $\sum_{i=0}^{n} |a_i|^2 = 1$ The **euclidean norm** of the $||v|| = \sqrt{\sum_{i=0}^{n} |a_i|^2}$

Common Quantum States

Plus State	$\|+\rangle = \frac{1}{\sqrt{2}}\|0\rangle + \frac{1}{\sqrt{2}}\|1\rangle$
Minus State	$\|-\rangle = \frac{1}{\sqrt{2}}\|0\rangle - \frac{1}{\sqrt{2}}\|1\rangle$
Other State	$\frac{1+2i}{3}\|0\rangle - \frac{2}{3}\|1\rangle$

Standart Basis Measurement

Let a quantum system be in the state $|\psi\rangle$, then the probability for the measure outcome to be a is $Pr(\text{outcome} = a) = |\langle a|\psi\rangle|^2$ If U is an unary matrice then the following Propertie hold, $||U\psi|| = ||\psi||$

Unary Operations

Unary Matrice

A squared matrix U having complex number entries is unitary if it satisfies the equations, $UU^\dagger = U^\dagger U = \mathbb{I}$ where $\mathbb{I}$ is the identity matrix.

Some unitary operations on qubits

Pauli operations:
$\mathbb{I} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ $\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ $\sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$
$\sigma_z = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$

Hadamard operation: $H = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0\\0 & \frac{1}{\sqrt{2}}\end{pmatrix}$

Phase operations: $P_\theta = \begin{pmatrix}1 & 0\\0 & e^{i\theta}\end{pmatrix}$

Quantum circtuit