Mathematical Modeling: Irrigation Problem

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Ms. Belledin

Academic Honesty Agreement:
We have neither given nor received unauthorized assistance on this assignment.

Abstract

In this paper, we present an irrigation system that optimizes the uniformity of water distribution through the field, and then calculated the amount of water each part of the field gets per hour. To design the system, we used a Python script that would compute the standard deviation of water distribution while iterating through the uniform distance between each nozzle. Our simulation concluded that 18 nozzles, with a nozzle distance of 22.74 ft, was most optimal for distributing water across the field most evenly. We also concluded that the irrigation system should move at a speed of 50 ft/hr to ensure that each part of the field gets an appropriate amount of water given the bounds in the problem.

Contents

1 Introduction

1.1 Problem Statement

The problem we were designated involves designing a moving irrigation system for a stretch of farmland. The irrigation system runs down a trough of water that supplies multiple nozzles running down the "arms" of the system. The two main points we approached were 1. finding the ideal placement of nozzles in order to uniformly water the field and 2. determining the speed of the system in order to ensure no piece of farmland receives more than 0.5 inches per hour of water, and that each piece of farmland receives at least 1 inch of water every 4 days (≈ 0.0104 in/hr).

1.2 Assumptions

1. Nozzles are equally spaced across the field. Since our goal with this problem is to distribute the water evenly across the field, it would make the most sense to space the nozzles evenly as well. Having certain areas where nozzles are closer or farther apart would most likely result in an uneven water distribution.

2. The irrigation system will run down the 1000 foot length of the field along a trough that is in the middle of the field, with 187.5 feet of farmland on both sides. We made this assumption as we were only given 400 feet of pipe, and therefore couldn't run the irrigation system along the width of the field. The irrigation system moves down the middle of the field since this optimizes the uniformity and efficiency of water distribution across the pipe-system; with the system in the middle of the field instead of on an edge, the farthest the water has to travel is reduced by one-half.

3. The irrigation system starts from before the field starts, and extends beyond the width of the field. We made this assumption because we are only accounting for the uniform distribution of water across the field, and we do not account for water waste that may appear on the edges of the field. Starting the irrigation system before the beginning of the field and extending the system beyond the edges ensures that we have optimized the uniformity of the distribution of water.

1.3 Preliminary Analysis

Consider our rectangular field of dimensions L by W, where $L = 1000$ft and $W = 375$ft. Define a **coordinate system** (x ft, y ft) for labeling points on the field as in the image below, such that the field is given by Field $= \{(x, y) \mid -\frac{L}{2} \leq x \leq \frac{L}{2}, -\frac{W}{2} \leq y \leq \frac{W}{2}\}$.

A **nozzle placement** consists of placing N nozzles of radius $R = 25$ft, where $1 \leq N \leq 20$ at distinct positions $(n_i)_{i=1}^{i=N}$ along the y-axis such that $n_1 \leq n_2 \leq \cdots \leq n_N$ and $n_N - n_1 < 400$ft, the length of the pipe. Although this allows nozzles to be placed arbitrarily close to each other, and at positions more than R away from either end of the field, these configurations are not optimal and are not considered in our analysis.

We focus on a single **traversal** of the pipe across the field, considering only the constant

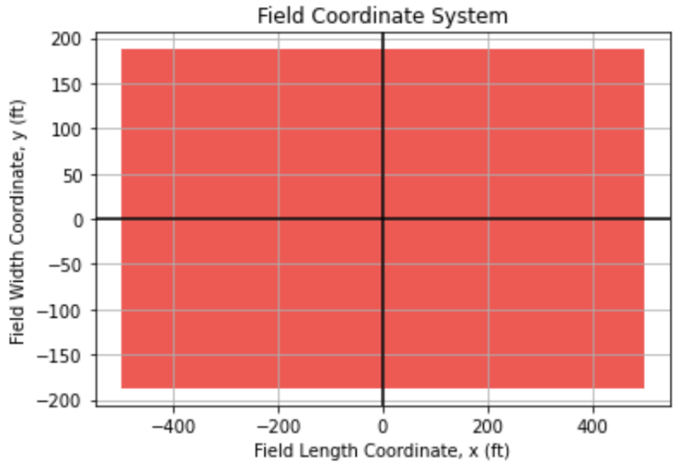

Figure 1: Coordinate system used throughout this paper.

velocity motion of the pipe, oriented parallel to the y-axis from $x = 0$ft to $x = L$. Thus, letting v denote the speed of the pipe in ft/hr, the positions of each nozzle t hours into a given traversal (from left to right, although the reverse direction is analogous) are given by $\big(x_i(t), y_i(t)\big)_{i=1}^{i=N} = (vt, n_i)$.

2 Methods

2.1 Definition of Chord Length

For this paper, we define chord length as the sum of all chord lengths at a certain x-coordinate.

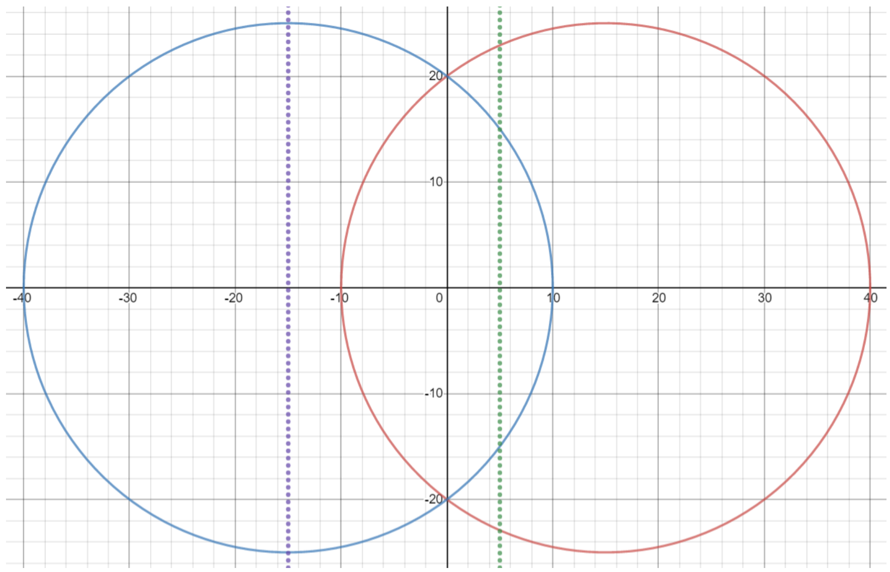

Figure 2: Chord measurement

In Figure 2, the purple dotted line only intersects one circle, so the chord length would be the distance between the points of intersection. The green dotted line intersects both circles, so chord length would be the length of the intersection with the red circle plus that of the blue circle.

2.2 Chord Graph

To determine the water fall on each vertical line in the field, we looked to the chord length at that line of the circle. Chord length and waterfall are directly proportional since when the irrigation system moves across, the distance divided by the speed gives time, or how long that point is under the nozzle spray.

We can create a graph of what water fall would look like by adding the functions of chord length. We will call this function $w(x)$ for future use. This helps with visualization.

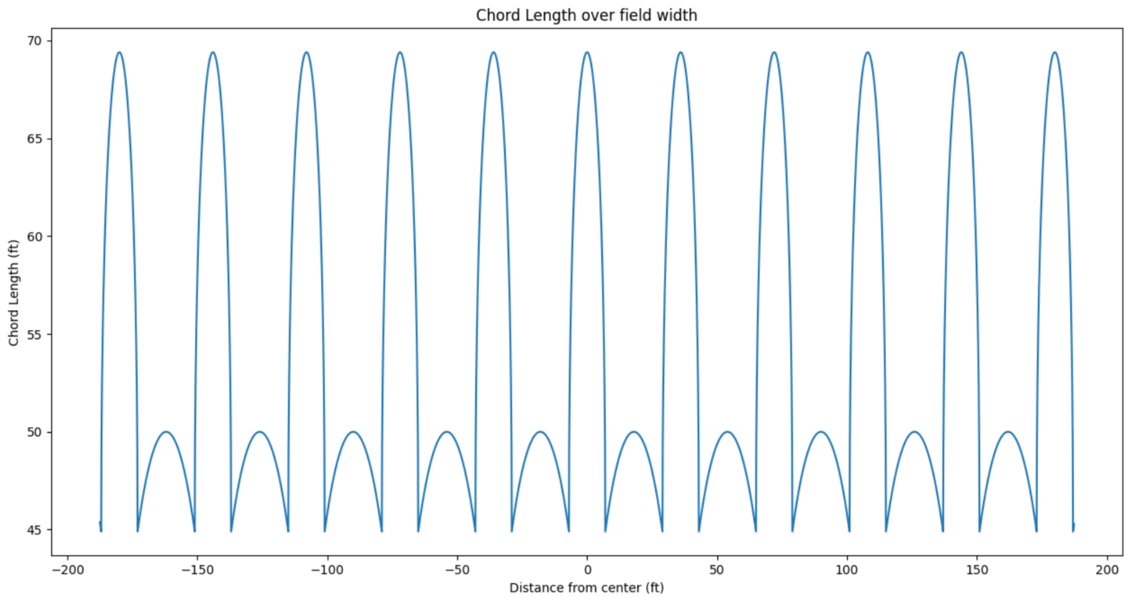

Figure 3: 12 nozzles with distance between equal to 36 ft

2.3 Uniformity

We defined uniformity as having the smallest standard deviation in summed chord length. In order to calculate this for a continuous distribution, we took multiple steps:

- We calculated the integral of $w(x)$ over the field and divided by 375 ft, the length of the field, to find the average value.

$$\mu = \frac{\int_{-187.5}^{187.5} w(x)dx}{375}$$

- We then calculated the integral of $(w(x) - \mu)^2$ and divided by 375 ft to find variance. Since we are minimizing standard deviation, and variance is the square of standard deviation, minimizing variance leads to the same result.

$$\sigma^2 = \frac{\int_{-187.5}^{187.5} (w(x) - \mu)^2 dx}{375}$$

2.4 Calculation of Optimal Spacing

Using our calculation of uniformity, we used a python script to find the best distance between nozzles. We had separate cases for even and odd amounts of nozzles as in the odd cases, there is a nozzle in a center, and in the even there is not. We iterated from 15 ft to 50 ft between nozzles with a step size of 0.01 ft and found where variance was the smallest, and recorded which nozzle placements led to the smallest variance.

2.5 Calculation of Speed

To calculate speed, we must first relate summed chord lengths to water fall. If the flow through the pipe is 40 gallons/min, the flow through a single nozzle is $40/n$ gallons/min, where n is the number of nozzles.

The depth of water that falls in a minute can be found by dividing the flow through the nozzle by the area of the circle. Since 1 gallon = 231 in^3 and the area of the circle is $\pi(25 * 12 \text{ in})^2 = 90000\pi$, the depth of water that falls in a minute is equal to:

$$\frac{\frac{40}{n} * 231 \text{ in}^3/\text{min}}{90000\pi \text{ in}^2} = \frac{.03268}{n} \text{ in/min}$$

Converting to inches per hour:

$$\frac{.03268}{n} \text{ in/min} * 60 = \frac{1.9608}{n} \text{ in/hour}$$

If we divide our summed chord length by speed, we get the number of hours a nozzle is over that portion of the field. Then, we can multiply this time by depth of water per hour to get the value for how much water falls on that portion of the field. To calculate the range of speeds for our irrigation system, we only need to look at the maximum, d_{max} and minimum, d_{min} of our graph of summed chord lengths, as they are the extrema:

- Maximum: the maximum amount of water that may fall in one spot during an hour is .5 in. If the number of nozzles is greater than 4, then $\frac{1.9608}{n}$ in/hour is greater than .5, so we do not need to worry about this constraint.
- Minimum: in order to make scheduling easier, we want the irrigation system to go across the field once a day. It has to traverse 1000ft + 25ft + 25ft, the length of the field and then starting and ending 25ft away from the edge to ensure the edges receive the same amount of water. The speed, v, must be greater than 1050ft/24hr = 43.75 ft/hour.

  Another condition we get is that each part of the field must receive at least $\frac{1}{7}$ of an inch since it needs 1 inch in a week.

  $$\frac{d_{min}}{v} * \frac{1.9608}{n} \text{ in/hour} > \frac{1}{7} \text{ in}$$

  or

  $$d_{min} * \frac{13.7256}{n} \text{ hour}^{-1} > v$$

  and we get v satisfies:

  $$43.75 \text{ ft/hour} < v < d_{min} * \frac{13.7256}{n} \text{ hour}^{-1} \tag{1}$$

3 Results

3.1 Optimal Spacing

Using our python script we determined that 18 nozzles with a spacing of around 22.74 ft was the most optimal placement, with $\sigma^2 = 10.640$. The maximum chord length is 91.549 ft and the minimum is 78.471 ft.

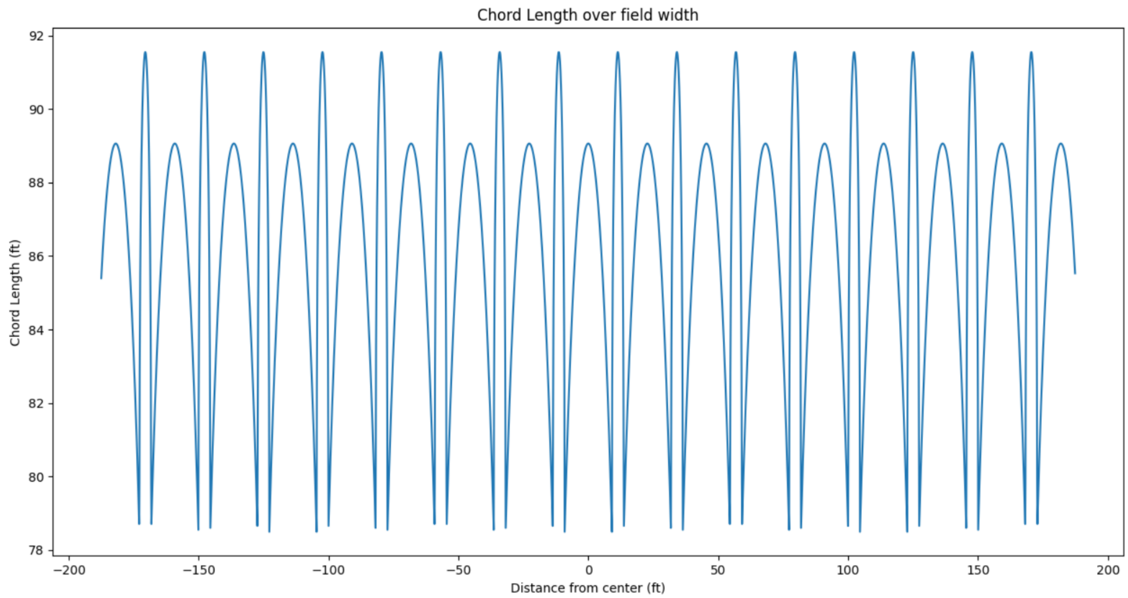

Figure 4: 18 nozzles with distance between equal to 22.74 ft

3.2 Speed

Using our max and min values and the number of nozzles, we can complete Equation 1.

$$43.75 \text{ ft/hour} < v < d_{min} * \frac{13.7256}{n} \text{ hour}^{-1}$$
$$43.75 \text{ ft/hour} < v < 78.471 \text{ ft} * \frac{13.7256}{18} \text{ hour}^{-1}$$
$$43.75 \text{ ft/hour} < v < 59.837 \text{ ft/hour}$$

Any speed in this range would work, but we chose **50 ft/hour** as our final solution due to its ability to divide into 1050 ft and its simplicity.

3.3 Waterfall Graph

Using our speed on nozzle count, we can multiply $w(x)$, our chord length function, by the constant $\frac{1.9608}{18*v}$, where $\frac{1.9608}{18}$ was the depth of water per hour and v is speed. Now, we have a graph of the amount of water a column receives per week:

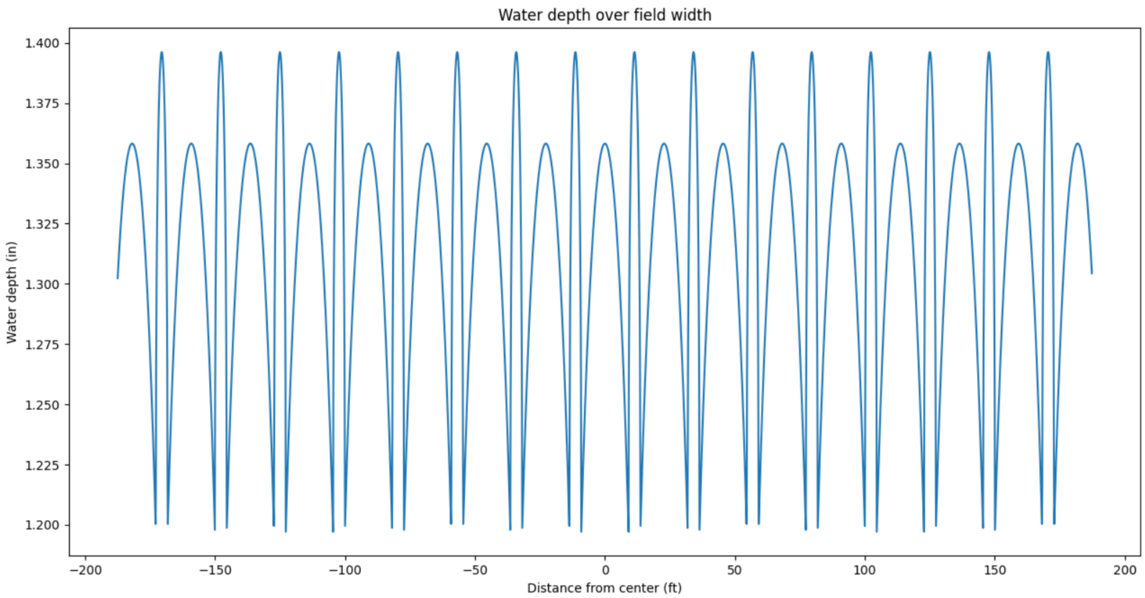

Figure 5: Precipitation of 18 nozzles with distance between equal to 22.74 ft over a week

4 Conclusion

We start by deriving expressions for the water received by each horizontal strip of the field for each nozzle placement. We then assume that evenly spaced nozzles produce more uniform watering distributions. Using our expressions and this assumption, we find the arrangement of nozzles that minimizes the variance in the amount of water each square foot of the field receives. We then found bounds for the speed of the irrigation system, where it had to complete a cycle in a day, and had to travel slow enough to provide ample water.

Our optimal nozzle placement consists of 18 nozzles, each 22.74 ft apart from the next. Adding too many nozzles produces too many overlapping regions between nozzles, making the overall distribution of water less uniform. Adding too few nozzles leaves areas of the field under-watered, as they are covered by an insufficient total chord length. Our solution finds a happy medium between these two extremes.

Minimizing the variance in the watering distribution proved advantageous. Alternative metrics of non-uniformity such as the range between the most and least watered parts of the field are only sensitive to specific, extreme patches of the field. The variance is affected by the water distribution over all parts of the field. Additionally, by squaring deviations from the mean, variance punishes large deviations extremely heavily, while minimizing smaller deviations. Lastly, the variance is a "smooth" (differentiable) function, enabling reliable optimization.

The speed we chose to run the irrigation system at is 50 ft/hour. This is within the bound of 43.75 ft/hr, completing a cycle in one day, and the upper bound of 59.837 ft/hr, which was the fastest speed that allowed the minimum chord length to receive just enough water.

4.1 Strengths and Weaknesses

Strengths - The main strength of our model is the robustness and precision of the values we were able to obtain. Our model utilized Python code to simulate the distribution of water as the modular of nozzles used and spacing between the nozzles changed. By using a robust simulation, we were able to obtain values with precision to the hundredths place for the distance between the nozzles, and simultaneously were able to figure out the optimal number of nozzles used with an even number of nozzles and an odd number of nozzles. By simulating virtually all possibilities, our model provides very accurate results that do not overlook potential edge cases.

Weaknesses - There are some visible weaknesses with our model, however. One of our main assumptions is that the nozzles are evenly spaced across the field. As our aim for this problem is to uniformly distribute water across the field, it is logical for the nozzles to be evenly spaced. Still, there is the possibility that one could achieve an even more optimal solution by simulating nozzle placement with uneven nozzle spacing; perhaps one possible way to utilize such a placement would be to place nozzles near the edges more unevenly. Furthermore, our model only considered how to distribute water most evenly across the field, and as such, it does not consider any wasted water on the edges. Further research should take into consideration the wasted water and devise a metric for taking the waste into account, and should utilize advanced integration techniques, and perhaps gradient descent and Monte Carlo simulations, as a method of optimizing nozzle placement even if the nozzles are not spaced evenly.

5 References

No external resources consulted— just the four of us and Ms. Belledin!

6 Member Contributions

Stokes contributed to the assumptions, problem statement, and editing.
Sreyas worked on the mathematics of variance and integrated total chord lengths. Also attempted gradient descent/Monte Carlo Markov Chain optimization and contributed to math-heavy sections of paper.
Jacob wrote the code to identify the arrangement of nozzles and found the way to find the standard deviation across a continuous distribution. Also, calculated the bounds on the speed of the irrigation system
Evan wrote the summary sheet, the assumptions, and the conclusion. Also contributed to the math and theory behind the optimal arrangement of nozzles.

A Appendix

Link to code used to evaluate standard deviation of a sprinkler arrangement:
`https://repl.it/@JacobGrinberg/Irrigation`.