Differential Forms

Course Project for PHY442: General Relativity

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Abstract

The theory of differential m-forms in $\mathbb{R}^n$ is developed from the ground up. Concepts such as the tangent space and one-forms are first introduced alongside the wedge product operation. Then, the discussion moves onto differential m-forms and how they can be integrated. The exterior derivative and Hodge Star operator are then introduced, and their properties are discussed. Finally, Maxwell's Equations are rewritten using the language of differential forms.

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1 Introduction

The theory of differential forms is an immensely powerful tool in differential geometry. It was pioneered by Élie Joseph Cartan in the years around 1900, and provides a unified approach in defining integrands over curves, surfaces, solids, and higher-dimensional manifolds. Differential forms and the general Stokes Formula also generalise all of the classical formulas of Green, Gauss, Stokes that we are familiar with from vector calculus. The study of differential forms is also often termed the exterior differential calculus. A mathematically rigorous study of differential forms requires the machinery of multilinear algebra. However, it is entirely possible to acquire a solid working knowledge of differential forms without entering into this formalism. That is the objective of this paper. The development of this theory will involve a number of examples alongside analogies from multivariable and vector calculus in an attempt to familiarise the reader with the new theory as much as possible. At the end, as an application of the theory, we will see how Maxwell's Equations of Electromagnetism can be rewritten using the language of differential forms. Now, in the words of Khan from Star Trek into Darkness

2 Introducing Forms

2.1 Introducing the Tangent Space

Suppose we're in $\mathbb{R}^2$ and we have a curve C. We choose a point $p \in C$. We can then define the *tangent space* to C at p as the set of all vectors that are tangent to the curve at that point. The tangent space is denoted by T_pC [3]. A visual picture of what's going on is shown below.

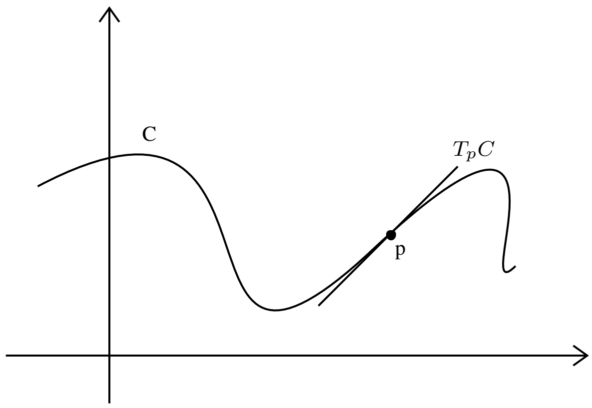

Figure 1: A curve C, a point p and the associated tangent space T_pC

In the above figure, it is clear that the tangent space is just a line, and by definition the line is the linear span of a vector on the line. What we can do now is ask ourselves if we can find such a vector, which would allow us to fully write the tangent space. This is a fairly simple problem. Since C is a curve in $\mathbb{R}^2$, it can be expressed in the form $y = f(x)$. Then, any point on C is of the form $(a, f(a))$ where $a \in \mathbb{R}$.

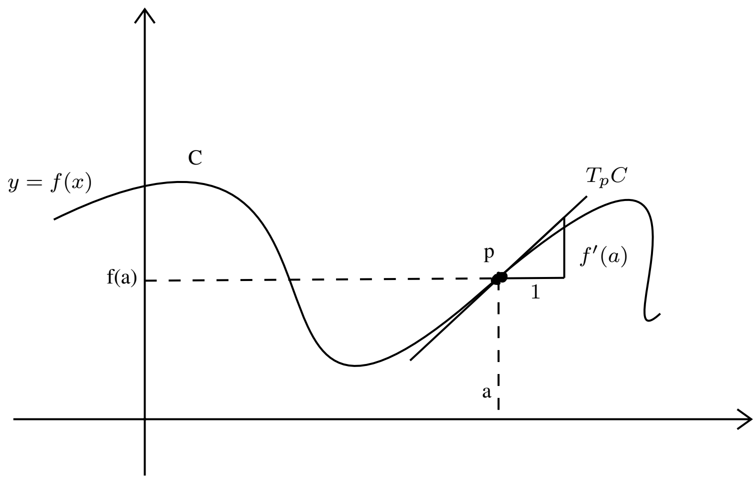

Figure 2: Evaluating for the vector on T_pC

Again, a picture of what's happening is shown in Figure 2. We have fixed a point p that has coordinates $(a, f(a))$ and are evaluating for the tangent space at that point. The idea is that if we walk 1 unit to the right of p then we know exactly what the vertical component of our vector is: it's just $f'(a)$! So now we have our vector that is parallel to the line that defines T_pC. That vector is $\vec{v} = \langle 1, f'(a) \rangle$. So the tangent space at p can be defined as

$$T_pC = \text{span}\{\langle 1, f'(a) \rangle\} \tag{2.1}$$

One important note is the way that I am using parentheses to denote points on C and angle brackets to denote elements of T_pC. However, this alludes to an important question that comes up once we have defined the tangent space: How we can go about distinguishing between points on the curve and vectors in the tangent space? Both objects live in the "ambient" space of $\mathbb{R}^2$ which is why it does become important to draw a distinction between them aside from using different notation for both. The answer to this question involves a more in-depth understanding about coordinates and coordinate systems [1]. A "coordinate" is a rule of assigning a number to a point in space. So we can think of a coordinate as a function which takes in as an input a point of a space and outputs a real number as an output. Consider $\mathbb{R}^2$ as an example. If we have a point p and we say that the "x-coordinate" of p is 5, what we are actually saying is that there's a function $x : \mathbb{R}^2 \to \mathbb{R}$ such that $x(p) = 5$. A "coordinate system" is then just a collection of such coordinate functions that allow us to map points of some space to the real numbers. An example would be when we say that the coordinates of a point $q \in \mathbb{R}^3$ are $(4, 8, 13)$. Here what we are saying is that there are in fact three coordinate functions: $x : \mathbb{R}^3 \to \mathbb{R}, y : \mathbb{R}^3 \to \mathbb{R}, z : \mathbb{R}^3 \to \mathbb{R}$ such that $x(q) = 4$, $y(q) = 8$ and $z(q) = 13$.

Equipped with this idea about coordinate functions, the way we distinguish between points in $\mathbb{R}^2$ that lie on a curve C and elements of T_pC is by defining different coordinate functions for them. For the curve C, the coordinate system is defined by the function $(x, y) : C \to \mathbb{R}^2$ such that $(x, y)(p) = (x(p), y(p))$ where $x(p), y(p) \in \mathbb{R}$. What coordinate system should we use for the tangent space T_pC? A good starting point would be to find a basis for T_pC. The basis is obvious provided that we recognise that

$$\begin{aligned} \frac{d}{dt}(x + t, y) &= \langle 1, 0 \rangle \\ \frac{d}{dt}(x, y + t) &= \langle 0, 1 \rangle \end{aligned} \tag{2.2}$$

The two vectors in Equation (2.2) can serve as a basis for T_pC which means that any vector $\vec{v} \in T_pC$ can be expressed in the form $\vec{v} = dx\langle 1, 0\rangle + dy\langle 0, 1\rangle$ where $dx, dy \in \mathbb{R}$. This seems to indicate that dx and dy are good candidates for the coordinate functions we can define on T_pC which is indeed the case. So the coordinate system for T_pC is defined by the function $\langle dx, dy\rangle : T_pC \to \mathbb{R}^2$ such that $\langle dx, dy\rangle(\vec{v}) = \langle dx(\vec{v}), dy(\vec{v})\rangle$ where $dx(\vec{v}), dy(\vec{v}) \in \mathbb{R}$. Saying that the coordinates of a vector $\vec{v}$ in T_pC are $\langle 2, 3\rangle$, for example, is the same thing as saying that $dx(\vec{v}) = 2$ and $dy(\vec{v}) = 3$. In general, we may refer to the coordinates of an arbitrary vector in T_pC as $\langle dx, dy\rangle$, just as we may refer to the coordinates of an arbitrary point in C as (x, y).

Another important distinction when talking about the tangent space is the point we are considering. In general, the tangent spaces at two different points p and q are fundamentally different, and thus so are the vectors. A common notation is to write $\vec{v}_p$ to mean a vector from T_pC and $\vec{v}_q$ to mean a vector from T_qC.

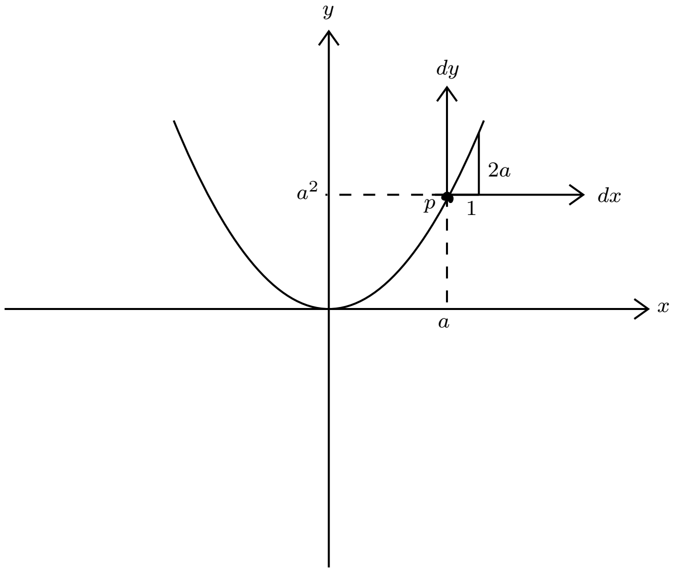

Figure 3: A curve with equation $y = x^2$ alongside the tangent space at a point p

The figure above shows a curve $C \subseteq \mathbb{R}^2$ with the equation $y = x^2$ as well as the tangent space T_pC drawn at an arbitrary point $p \in C$. Given that the equation of C is $y = x^2$, any point $p \in C$ is of the form (a, a^2) where $a \in \mathbb{R}$. Using the notation of our coordinate functions, this is the same as saying that

$$(x, y)(p) = (a, a^2) \tag{2.3}$$

Furthermore, for any $\vec{v} \in T_pC$ we have that

$$\langle dx, dy\rangle(\vec{v}) = \langle 1, 2a\rangle \tag{2.4}$$

Usually, in order to simplify things, we write the coordinates as

$$\begin{aligned} (x, y) &= (a, a^2) \in C \\ \langle dx, dy\rangle &= \langle 1, 2a\rangle \in T_pC \end{aligned} \tag{2.5}$$

Another important observation is that the "origin" of T_pC is not the same as the origin of the ambient space $\mathbb{R}^2$; rather the origin is wherever the point p is located.

2.2 $\mathbb{R}^n$ VS $T_p\mathbb{R}^n$

Up until now, we have talked about the tangent space to a curve $C \subseteq \mathbb{R}^2$ at a point $p \in C$. However, this idea can be generalised to talk about $T_p\mathbb{R}^n$, the tangent space of $\mathbb{R}^n$ at any arbitrary point p. This is defined to be the set of all vectors based at a point $p \in \mathbb{R}^n$. Although this is a fine definition, it makes imagining what this vector space looks like a little strange. The reason for this is because from a visual perspective, this definition seems to imply that there is a "copy" of $\mathbb{R}^n$ attached to every point on $\mathbb{R}^n$. However, it is here that we need to make a distinction between $\mathbb{R}^n$ and the *vector space* $T_p\mathbb{R}^n$ so that we don't end up confusing the two [2]. Elements of $\mathbb{R}^n$ are points in the regular n-dimensional Euclidean space. In contrast, elements of $T_p\mathbb{R}^n$ are n-dimensional vectors based on some point $p \in \mathbb{R}^n$. It is not immediate what the difference is between $\mathbb{R}^n$ and the vector space $T_p\mathbb{R}^n$, primarily because we are normally used to thinking of points and vectors in $\mathbb{R}^n$ as essentially being the same thing. In fact, the two spaces are isomorphic! However, from our previous discussion about coordinate systems, we do know in fact that these spaces are indeed different because the coordinate functions defined on them are different. For $\mathbb{R}^n$ the coordinate functions that are defined are

$$(x_1, x_2, \ldots, x_n)(p) = (x_1(p), x_2(p), \ldots x_n(p)) \tag{2.6}$$

where $x_1(p), x_2(p), \ldots, x_n(p) \in \mathbb{R}$. For $T_p\mathbb{R}^n$ the coordinate functions that are defined are

$$\langle dx_1, dx_2, \ldots, dx_n \rangle(\vec{v}) = \langle dx_1(\vec{v}), dx_2(\vec{v}), \ldots, dx_n(\vec{v}) \rangle \tag{2.7}$$

where $dx_1(\vec{v}), dx_2(\vec{v}), \ldots, dx_n(\vec{v}) \in \mathbb{R}$.
Visually, the way we can think of $\mathbb{R}^n$ and $T_p\mathbb{R}^n$ being different is by looking at Figure 4. There, we have two points defined on $\mathbb{R}^2$ called p_1 and p_2. Each point has its own tangent space denoted by $T_{p_1}\mathbb{R}^2$ and $T_{p_2}\mathbb{R}^2$ respectively. $\mathbb{R}^2$ in this case uses our traditional x-y coordinates to locate points, but each tangent space uses the dx-dy coordinates to locate vectors in each separate tangent space. So the points and the vectors essentially "live" in different spaces because we need different coordinates to find either one.

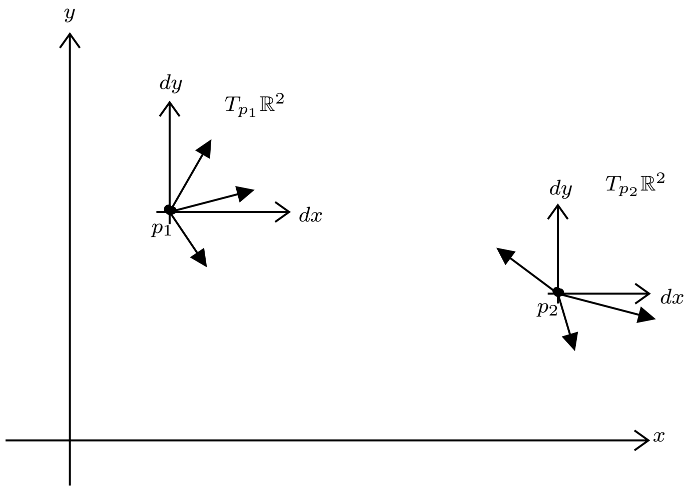

Figure 4: $\mathbb{R}^2$ alongside two tangent spaces defined on p_1 and p_2

2.3 Introducing One-Forms

Now that we have a good idea of what the tangent space is, it is fairly easy to define a one-form. This is simply a linear function that takes in a vector of the tangent space as an input, and gives a real number as an output [1, 3]. Mathematically, a one-form is a function $\omega : T_p\mathbb{R}^n \to \mathbb{R}$ that is linear. By "linear" we mean that

$$\omega(a\vec{v} + b\vec{w}) = a\omega(\vec{v}) + b\omega(\vec{w}) \tag{2.8}$$

where $a, b \in \mathbb{R}$.
Since a one-form acts on vectors to produce a number, it can also be thought of as an element of the dual space of the tangent space, which can be denoted by $(T_p\mathbb{R}^n)'$. However, for our purposes, the definition of a one-form as a linear function the acts on vectors of the tangent space to give numbers is more than sufficient.
The condition that ω must be linear imposes a huge constraint on the form that the function can take when acting on an arbitrary $\vec{v} \in T_p\mathbb{R}^n$. Any $\vec{v} \in T_p\mathbb{R}^n$ is of the form

$$\langle dx_1, dx_2, \ldots, dx_n \rangle = dx_1\langle 1, 0, 0, \ldots, 0 \rangle + dx_2\langle 0, 1, 0, \ldots, 0 \rangle + \ldots + dx_n\langle 0, 0, \ldots, 1 \rangle \tag{2.9}$$

This implies that

$$\begin{aligned}\omega(\vec{v}) &= dx_1\omega(\langle 1,0,0,\ldots,0\rangle) + dx_2\omega(\langle 0,1,0,\ldots,0\rangle) + \ldots dx_n\omega(\langle 0,0,\ldots,1\rangle) \\ &= a_1dx_1 + a_2dx_2 + \ldots + a_ndx_n\end{aligned} \tag{2.10}$$

This demonstrates a key idea: Evaluating a one-form on a vector is the same as projecting onto each coordinate axis, scaling each by some constant and adding the results.
Let's now look at an example to get a feel for how one-forms work. Let's say we have a one-form ω that acts in the following way

$$\omega(\langle dx, dy\rangle) = 3dx + 2dy \tag{2.11}$$

We can think of the action of ω as the following dot product

$$\omega(\langle dx, dy\rangle) = \langle 3,2\rangle \cdot \langle dx, dy\rangle \tag{2.12}$$

Now the dot product actually has a meaningful geometric interpretation. When we take the dot product of two vectors $\langle a,b\rangle$ and $\langle c,d\rangle$, we think of projecting $\langle c,d\rangle$ onto $\langle a,b\rangle$ and multiplying by $|\langle a,b\rangle|$. So we can think of evaluating a one-form on a vector as being the same as projecting the vector onto some line and then multiplying by some constant. In the context of the one-form defined in Equation (2.11), the line that ω projects vectors onto is $\text{span}\{\langle 3,2\rangle\}$.
So now we have two "pictures" of how a one-form acts on a vector $\vec{v} \in T_p\mathbb{R}^n$. One picture involves taking the scalar projection of a vector onto a line multiplying by a number. The other picture involves projecting the vector onto each of the coordinate axes, scaling each projection by some number and adding all of the results together. Although both pictures are equally good ways of looking at one-forms, the latter one is much more easily generalisable when we talk about multiplying one-forms together.
Another important thing when talking about one-forms are the so-called "elementary" one-forms. These essentially pick out the relevant coordinate component of the vector in the tangent space. The elementary one-forms are denoted by $\tilde{dx_i}$ where $i \in \{1,2,\ldots,n\}$. As an example, let's say we have a vector $\vec{v} = \langle dx_1, dx_2, \ldots, dx_n\rangle$. Then we have that $\tilde{dx_1}(\vec{v}) = dx_1$, $\tilde{dx_2}(\vec{v}) = dx_2$ and so so on.
It can be shown that the set of elementary one-forms $\{\tilde{dx_1}, \tilde{dx_2}, \ldots, \tilde{dx_n}\}$ actually form a basis for $(T_p\mathbb{R}^n)'$. In order to do this, we just need to show that the set of elementary one-forms is linearly independent. This is because $\{\tilde{dx_1}, \tilde{dx_2}, \ldots, \tilde{dx_n}\}$ is an n-dimensional set and $(T_p\mathbb{R}^n)'$ is also n-dimensional. In order to show that the set of elementary one-forms $\{\tilde{dx_1}, \tilde{dx_2}, \ldots, \tilde{dx_n}\}$ is linearly independent, we need to show that

$$a_1\tilde{dx_1} + a_2\tilde{dx_2} + \ldots + a_n\tilde{dx_n} = 0 \implies a_1 = a_2 = \ldots = a_n = 0 \tag{2.13}$$

Let $\vec{v} \in T_p\mathbb{R}^n$. We then have

$$\begin{aligned}(a_1\tilde{dx_1} + a_2\tilde{dx_2} + \ldots + a_n\tilde{dx_n})(\vec{v}) &= a_1\tilde{dx_1}(\vec{v}) + a_2\tilde{dx_2}(\vec{v}) + \ldots + a_n\tilde{dx_n}(\vec{v}) \\ &= a_1dx_1 + a_2dx_2 + \ldots + a_ndx_n\end{aligned} \tag{2.14}$$

Since $dx_1, dx_2, \ldots, dx_n$ are all generally assumed to be non-zero, Equation (2.14) implies that $a_1 = a_2 = \ldots = a_n = 0$. So we have proved that the elementary one-forms $\{\tilde{dx_1}, \tilde{dx_2}, \ldots, \tilde{dx_n}\}$ is linearly independent, which further implies that it forms a basis.

2.4 Introducing m-Forms

2.4.1 2-Forms on $T_p\mathbb{R}^n$

Let's take stock of what we have done so far. We have the idea of the tangent space $T_p\mathbb{R}^n$ and the idea of one-forms that act on vectors in $T_p\mathbb{R}^n$ to produce numbers. A question we can now ask is if there is a way we can "multiply" two one-forms together in a meaningful way. Our goal is essentially this: we have two one-forms α and β. We wish to define a product one form $\alpha \wedge \beta$ where the symbol "$\wedge$" denotes the multiplication (it is noramlly called the wedge product) such that $\alpha \wedge \beta$ is a linear function that takes in two vectors to produce a number (i.e. $\alpha \wedge \beta : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$) and this function has some meaningful geometric interpretation. As always, let's first look at a simple case where $n = 2$ [1, 3, 2]. In this scenario we have that $\alpha \wedge \beta : T_p\mathbb{R}^2 \times T_p\mathbb{R}^2 \to \mathbb{R}$ such that $\alpha \wedge \beta(\vec{v}, \vec{w}) =$ some number where $\vec{v}, \vec{w} \in T_p\mathbb{R}^2$. What we can notice is that the one-forms α and β can in a sense act on $\vec{v}$ individually to produce two different numbers $\alpha(\vec{v})$ and $\beta(\vec{v})$. Using these two numbers, we can form a vector $\langle \alpha(\vec{v}), \beta(\vec{v}) \rangle$ that lives in a plane spanned by α and β. All of this reasoning can be replicated for the vector $\vec{w}$ as well. So now we have two vectors $\langle \alpha(\vec{v}), \beta(\vec{v}) \rangle$ and $\langle \alpha(\vec{w}), \beta(\vec{w}) \rangle$ that live in the α-β plane instead of the usual dx-dy plane. The idea now is to associate a number that can be obtained from these two vectors, which we will define to be the action of $\alpha \wedge \beta$ on the vectors $\vec{v}$ and $\vec{w}$. Do we know of a way of obtaining a number from two vectors? Yes! We can take the signed area of the parallelogram that is spanned by the two vectors.

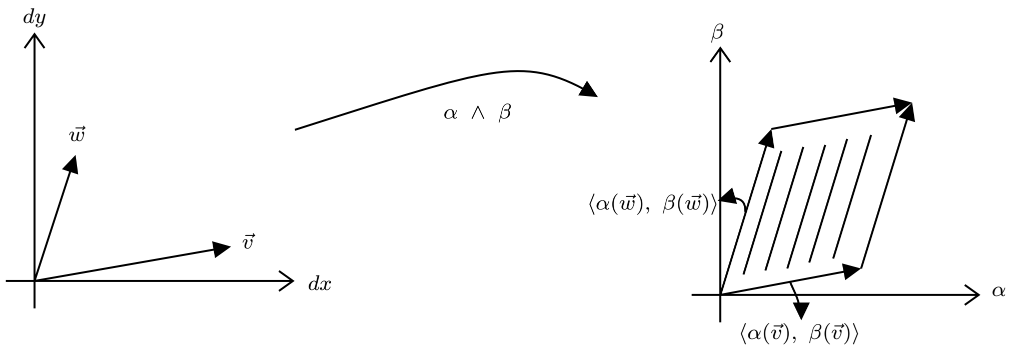

Figure 5: The action of $\alpha \wedge \beta$ on the vectors $\vec{v}$ and $\vec{w}$ is equal to the signed area of the parallelogram spanned by $\langle \alpha(\vec{v}), \beta(\vec{v}) \rangle$ and $\langle \alpha(\vec{w}), \beta(\vec{w}) \rangle$

Linear algebra tells us that the signed area of the parallelogram shown in Figure 5 is equal to

$$\begin{vmatrix} \alpha(\vec{v}) & \beta(\vec{v}) \\ \alpha(\vec{w}) & \beta(\vec{w}) \end{vmatrix} \tag{2.15}$$

where we are implicitly using the Right-Hand Rule for the cross product to construct the matrix whose determinant has to be evaluated.

So there we have it! We finally know how the product $\alpha \wedge \beta$, which we now define to be a two-form, acts on two vectors $\vec{v}, \vec{w} \in T_p\mathbb{R}^2$. Putting everything together we have

$$\alpha \wedge \beta(\vec{v}, \vec{w}) = \begin{vmatrix} \alpha(\vec{v}) & \beta(\vec{v}) \\ \alpha(\vec{w}) & \beta(\vec{w}) \end{vmatrix} \tag{2.16}$$

We can generalise Equation (2.16) fairly easily by simply letting $\vec{v}, \vec{w} \in T_p\mathbb{R}^n$.

Let's do a simple example to get a feel for how the wedge product of two one-forms acts on two vectors. Let's consider $T_p\mathbb{R}^3$. Take a vector $\vec{a} = \langle 1, 2, -5\rangle$ and another vector $\vec{b} = \langle 0, 3, -2\rangle$. Let $\omega_1 = 3\tilde{dx} - 2\tilde{dy} - \tilde{dz}$ and $\omega_2 = \tilde{dx} + 4\tilde{dy}$. Then

$$\begin{aligned}\omega_1 \wedge \omega_2(\vec{a}, \vec{b}) &= \begin{vmatrix} \omega_1(\vec{a}) & \omega_2(\vec{a}) \\ \omega_1(\vec{b}) & \omega_2(\vec{b}) \end{vmatrix} \\ &= \begin{vmatrix} 4 & 9 \\ -4 & 12 \end{vmatrix} \\ &= 84 \end{aligned} \tag{2.17}$$

2.4.2 Properties of the Wedge Product

Now that we have figured out how to combine two one-forms together to obtain a two-form, we can make some interesting observations about the wedge product operation that we have defined [1, 2, 3]. The first of these properties is called *skew symmetry* which basically means that if we interchange the two vectors that are being fed into the two form, we get an overall minus sign. Mathematically speaking all we are saying is

$$\alpha \wedge \beta(\vec{v}, \vec{w}) = -\alpha \wedge \beta(\vec{w}, \vec{v}) \tag{2.18}$$

Proving this property is very straightforward using the definition of the wedge product from Equation (2.16).

$$\alpha \wedge \beta(\vec{v}, \vec{w}) = \begin{vmatrix} \alpha(\vec{v}) & \beta(\vec{v}) \\ \alpha(\vec{w}) & \beta(\vec{w}) \end{vmatrix} = - \begin{vmatrix} \alpha(\vec{w}) & \beta(\vec{w}) \\ \alpha(\vec{v}) & \beta(\vec{v}) \end{vmatrix} = -\alpha \wedge \beta(\vec{w}, \vec{v}) \tag{2.19}$$

In the above manipulation, we have made use of the fact that when we interchange two rows of a matrix, the determinant changes sign. Another identity which is similar to the one in Equation (2.18) is the following

$$\alpha \wedge \beta(\vec{v}, \vec{w}) = -\beta \wedge \alpha(\vec{v}, \vec{w}) \tag{2.20}$$

This is again easy to prove using the definition of the wedge product

$$\alpha \wedge \beta(\vec{v}, \vec{w}) = \begin{vmatrix} \alpha(\vec{v}) & \beta(\vec{v}) \\ \alpha(\vec{w}) & \beta(\vec{w}) \end{vmatrix} = - \begin{vmatrix} \beta(\vec{v}) & \alpha(\vec{v}) \\ \beta(\vec{w}) & \alpha(\vec{w}) \end{vmatrix} = -\beta \wedge \alpha(\vec{v}, \vec{w}) \tag{2.21}$$

In the above manipulation, we have made use of the fact that when we interchange two columns of a matrix, the determinant changes sign. Equation (2.18) says that the two-form, $\alpha \wedge \beta$, is a skew-symmetric operator on pairs of vectors. Equation (2.20) says that $\wedge$ can be thought of as a skew-symmetric operator on one-forms. Two straightforward corollaries from Equations (2.18) and (2.20) are the following

$$\begin{aligned} \alpha \wedge \beta(\vec{v}, \vec{v}) &= 0 \\ \alpha \wedge \alpha(\vec{v}, \vec{w}) &= 0 \end{aligned} \tag{2.22}$$

Another property of the wedge product is that it is *bilinear*. This means that the following properties hold

$$\begin{aligned} \alpha \wedge \beta(\vec{v} + \vec{w}, \vec{a}) &= \alpha \wedge \beta(\vec{v}, \vec{a}) + \alpha \wedge \beta(\vec{w}, \vec{a}) \\ \alpha \wedge \beta(\vec{v}, \vec{w} + \vec{a}) &= \alpha \wedge \beta(\vec{v}, \vec{w}) + \alpha \wedge \beta(\vec{v}, \vec{a}) \\ \alpha \wedge \beta(c\vec{v}, \vec{w}) &= c\alpha \wedge \beta(\vec{v}, \vec{w}) \\ \alpha \wedge \beta(\vec{v}, c\vec{w}) &= c\alpha \wedge \beta(\vec{v}, \vec{w}) \end{aligned} \tag{2.23}$$

Proving these is straightforward from the definition of the wedge product in Equation (2.16). The proof for the first and third lines in Equation (3.23) will be shown here; the proofs for the other two will proceed very similarly.
Let's first show that $\alpha \wedge \beta(\vec{v} + \vec{w}, \vec{a}) = \alpha \wedge \beta(\vec{v}, \vec{a}) + \alpha \wedge \beta(\vec{w}, \vec{a})$.

$$\begin{aligned}
\alpha \wedge \beta(\vec{v} + \vec{w}, \vec{a}) &= \begin{vmatrix} \alpha(\vec{v} + \vec{w}) & \beta(\vec{v} + \vec{w}) \\ \alpha(\vec{a}) & \beta(\vec{a}) \end{vmatrix} \\
&= \begin{vmatrix} \alpha(\vec{v}) + \alpha(\vec{w}) & \beta(\vec{v}) + \beta(\vec{w}) \\ \alpha(\vec{a}) & \beta(\vec{a}) \end{vmatrix} \\
&= \begin{vmatrix} \alpha(\vec{v}) & \beta(\vec{v}) \\ \alpha(\vec{a}) & \beta(\vec{b}) \end{vmatrix} + \begin{vmatrix} \alpha(\vec{w}) & \beta(\vec{w}) \\ \alpha(\vec{a}) & \beta(\vec{b}) \end{vmatrix} \\
&= \alpha \wedge \beta(\vec{v}, \vec{a}) + \alpha \wedge \beta(\vec{w}, \vec{a})
\end{aligned} \tag{2.24}$$

In the above manipulation we have made use of the following fact for splitting up a determinant

$$\begin{vmatrix} a+b & c+d \\ e & f \end{vmatrix} = \begin{vmatrix} a & c \\ e & f \end{vmatrix} + \begin{vmatrix} b & d \\ e & f \end{vmatrix} \tag{2.25}$$

Let's now show that $\alpha \wedge \beta(c\vec{v}, \vec{w}) = c\alpha \wedge \beta(\vec{v}, \vec{w})$.

$$\begin{aligned}
\alpha \wedge \beta(c\vec{v}, \vec{w}) &= \begin{vmatrix} \alpha(c\vec{v}) & \beta(c\vec{v}) \\ \alpha(\vec{w}) & \beta(\vec{w}) \end{vmatrix} \\
&= \begin{vmatrix} c\alpha(\vec{v}) & c\beta(\vec{v}) \\ \alpha(\vec{w}) & \beta(\vec{w}) \end{vmatrix} \\
&= c \begin{vmatrix} \alpha(\vec{v}) & \beta(\vec{v}) \\ \alpha(\vec{w}) & \beta(\vec{w}) \end{vmatrix} \\
&= c\alpha \wedge \beta(\vec{v}, \vec{w})
\end{aligned} \tag{2.26}$$

In the above manipulation we have made use of the fact that if each row/column of a matrix is multiplied by the same constant k, then the determinant of the matrix is multiplied by k

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix} \tag{2.27}$$

One final property of the wedge product is that it is *distributive* over addition. What we mean is that

$$(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma \tag{2.28}$$

To prove this, let's first define $\delta \equiv \alpha + \beta$ and act the left-hand-side on two arbitrary vectors $\vec{v}$ and $\vec{w}$.

$$\begin{aligned}
\delta \wedge \gamma(\vec{v}, \vec{w}) &= \begin{vmatrix} \delta(\vec{v}) & \gamma(\vec{v}) \\ \delta(\vec{w}) & \gamma(\vec{w}) \end{vmatrix} \\
&= \begin{vmatrix} (\alpha + \beta)(\vec{v}) & \gamma(\vec{v}) \\ (\alpha + \beta)(\vec{w}) & \gamma(\vec{w}) \end{vmatrix} \\
&= \begin{vmatrix} \alpha(\vec{v}) + \beta(\vec{v}) & \gamma(\vec{v}) \\ \alpha(\vec{w}) + \beta(\vec{w}) & \gamma(\vec{w}) \end{vmatrix} \\
&= \begin{vmatrix} \alpha(\vec{v}) & \gamma(\vec{v}) \\ \alpha(\vec{w}) & \gamma(\vec{w}) \end{vmatrix} + \begin{vmatrix} \beta(\vec{v}) & \gamma(\vec{v}) \\ \beta(\vec{w}) & \gamma(\vec{w}) \end{vmatrix} \\
&= \alpha \wedge \gamma(\vec{v}, \vec{w}) + \beta \wedge \gamma(\vec{v}, \vec{w})
\end{aligned} \tag{2.29}$$

In the above manipulation, we have made use of the fact that since one-forms are linear functions then $(\alpha + \beta)(\vec{v}) = \alpha(\vec{v}) + \beta(\vec{v})$. Another fact that we have made use of is that

$$\begin{vmatrix} a+b & c \\ d+e & f \end{vmatrix} = \begin{vmatrix} a & c \\ d & f \end{vmatrix} + \begin{vmatrix} b & c \\ e & f \end{vmatrix} \tag{2.30}$$

The last line of Equation (2.29) implies that $(\alpha+\beta)\wedge\gamma(\vec{v},\vec{w}) = \alpha\wedge\gamma(\vec{v},\vec{w}) + \beta\wedge\gamma(\vec{v},\vec{w})$. Since $\vec{v}$ and $\vec{w}$ were arbitrary vectors, we have proven Equation (2.28).
The properties we have just talked about actually provide us with a way to "algebraically" work with one-forms that are being wedged together. Let's use our previous example in which we had a one-form $\omega_1 = 3\tilde{dx} - 2\tilde{dy} - \tilde{dz}$ and another one-form $\omega_2 = \tilde{dx} + 4\tilde{dy}$. We can now compute $\omega_1 \wedge \omega_2$ in the following way

$$\begin{aligned} \omega_1 \wedge \omega_2 &= (3\tilde{dx} - 2\tilde{dy} - \tilde{dz}) \wedge (\tilde{dx} + 4\tilde{dy}) \\ &= \cancel{3\tilde{dx} \wedge \tilde{dx}} + 12\tilde{dx}\wedge\tilde{dy} - 2\tilde{dy}\wedge\tilde{dx} - \cancel{8\tilde{dy}\wedge\tilde{dy}} - \tilde{dz}\wedge\tilde{dx} - 4\tilde{dz}\wedge\tilde{dy} \\ &= 14\tilde{dx}\wedge\tilde{dy} + \tilde{dx}\wedge\tilde{dz} + 4\tilde{dy}\wedge\tilde{dz} \end{aligned} \tag{2.31}$$

We can now ask the question about how the new expression for $\omega_1 \wedge \omega_2$ in the above equation will act on two vectors. Let's use the same two vectors as before: $\vec{a} = \langle 1, 2, -5\rangle$ and $\vec{b} = \langle 0, 3, -2\rangle$.

$$\begin{aligned} \omega_1 \wedge \omega_2(\vec{a},\vec{b}) &= (14\tilde{dx}\wedge\tilde{dy} + \tilde{dx}\wedge\tilde{dz} + 4\tilde{dy}\wedge\tilde{dz})(\vec{a},\vec{b}) \\ &= 14\tilde{dx}\wedge\tilde{dy}(\vec{a},\vec{b}) + \tilde{dx}\wedge\tilde{dz}(\vec{a},\vec{b}) + 4\tilde{dy}\wedge\tilde{dz}(\vec{a},\vec{b}) \\ &= 14\begin{vmatrix} \tilde{dx}(\vec{a}) & \tilde{dy}(\vec{a}) \\ \tilde{dx}(\vec{b}) & \tilde{dy}(\vec{b}) \end{vmatrix} + \begin{vmatrix} \tilde{dx}(\vec{a}) & \tilde{dz}(\vec{a}) \\ \tilde{dx}(\vec{b}) & \tilde{dz}(\vec{b}) \end{vmatrix} + 4\begin{vmatrix} \tilde{dy}(\vec{a}) & \tilde{dz}(\vec{a}) \\ \tilde{dy}(\vec{b}) & \tilde{dz}(\vec{b}) \end{vmatrix} \\ &= 14\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -5 \\ 0 & -2 \end{vmatrix} + 4\begin{vmatrix} 2 & -5 \\ 3 & -2 \end{vmatrix} \\ &= 14(3) - 2 + 4(11) = 84 \end{aligned} \tag{2.32}$$

So we've successfully replicated our previous result.

2.4.3 Generalising to m-forms on $T_p\mathbb{R}^n$

We now have all of the ingredients we know to talk about the most general types of forms: m-forms on $T_p\mathbb{R}^n$ [1, 3]. Building up to these is a very simple generalisation of the way we built two-forms from one-forms. Before we do that, let's try to get some hindsight into what properties we want our m-forms to have. Well, in a similar way as to how the two-form that was built from two one-forms was bilinear, we want our m-form to be what's called *multilinear*. What this means is

$$\omega(\vec{v_1}, \vec{v_2}, \ldots, a\vec{w} + b\vec{s}, \ldots, \vec{v_m}) = a\omega(\vec{v_1}, \vec{v_2}, \ldots, \vec{w}, \ldots, \vec{v_m}) + b\omega(\vec{v_1}, \vec{v_2}, \ldots, \vec{s}, \ldots, \vec{v_m}) \tag{2.33}$$

Another property we wish to retain is skew-symmetry, where if we perform an exchange between any two of the m vectors that are being fed into the m-form, we need to pick up an overall minus sign.

$$\omega(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_i}, \vec{v}_{i+1}, \ldots, \vec{v_m}) = -\omega(\vec{v_1}, \vec{v_2}, \ldots, \vec{v}_{i+1}, \vec{v_i}, \ldots, \vec{v_m}) \tag{2.34}$$

With these two properties in mind, as well as inspiration from how we built two-forms from one-forms, the way we can construct m-forms is simply by taking the wedge product of m

one-forms. So instead of a determinant of a two-dimensional matrix, we'll end up with the determinant of an m-dimensional matrix. Mathematically, what we are saying is

$$\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_m(\vec{v}_1, \ldots, \vec{v}_m) = \begin{vmatrix} \omega_1(\vec{v}_1) & \omega_2(\vec{v}_1) & \ldots & \omega_m(\vec{v}_1) \\ \omega_1(\vec{v}_2) & \omega_2(\vec{v}_2) & \ldots & \omega_m(\vec{v}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1(\vec{v}_m) & \omega_2(\vec{v}_m) & \ldots & \omega_m(\vec{v}_m) \end{vmatrix} \tag{2.35}$$

It's very easy to show that with the way an m-form is defined in Equation (2.35), the properties of multilinearity, skew-symmetry and even distributivity are retained. The proof is going to be very similar to the way it worked for the case in which we talked about two-forms in the previous section.

So there we have it! We finally have a full working definition for an m-form on $T_p\mathbb{R}^n$, which is a function $\omega : (T_p\mathbb{R}^n)^m \to \mathbb{R}$ that is multilinear and skew-symmetric and can be constructed by taking the wedge product of m one-forms that are functions of the type $\gamma : T_p\mathbb{R}^n \to \mathbb{R}$ using the prescription in Equation (2.35).

Now, just as we talked about how the set of elementary one-forms $\{\tilde{dx}_1, \tilde{dx}_2, \ldots, \tilde{dx}_n\}$ forms a basis for the space of one-forms on $T_p\mathbb{R}^n$, we can ask ourselves the question of what the basis would be for the space of m-forms on $T_p\mathbb{R}^n$. As always, let's first look at a simple case. Consider $T_p\mathbb{R}^4$ and see all of the different m-forms we can construct. The easiest are the zero-forms which are just the real numbers themselves. So there's no need to construct a basis for that. Now let's look at the one-forms. We already know what the basis is for this case: it's just the set $\{\tilde{dx}_1, \tilde{dx}_2, \tilde{dx}_3, \tilde{dx}_4\}$. Now let's think about the two-forms. Let intuition be your guide: If the two-forms are built from the wedge product of two one-forms, the basis two-forms would just be the wedge product of the elementary one-forms! However, there is a caveat here due to the fact that we have skew-symmetry. As a result of skew symmetry, we cannot have a basis element like $\tilde{dx}_1 \wedge \tilde{dx}_1$ because that's just zero. Furthermore, skew symmetry creates a bunch of redundant basis elements as well because $\tilde{dx}_1 \wedge \tilde{dx}_2 = -\tilde{dx}_2 \wedge \tilde{dx}_1$ (we only need one of these in our basis set since we can group up like terms). So instead of all possible wedge products, we actually end up with the following set as the basis for all two-forms on $T_p\mathbb{R}^4$: $\{\tilde{dx}_1 \wedge \tilde{dx}_2, \tilde{dx}_1 \wedge \tilde{dx}_3, \tilde{dx}_1 \wedge \tilde{dx}_4, \tilde{dx}_2 \wedge \tilde{dx}_3, \tilde{dx}_2 \wedge \tilde{dx}_4, \tilde{dx}_3 \wedge \tilde{dx}_4\}$. Now let's consider three-forms. Naturally, this will involve basis elements that are constructed by taking the wedge product of three basis one-forms. Keeping skew-symmetry in mind, we end up with the following set: $\{\tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \tilde{dx}_3, \tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \tilde{dx}_4, \tilde{dx}_1 \wedge \tilde{dx}_3 \wedge \tilde{dx}_4, \tilde{dx}_2 \wedge \tilde{dx}_3 \wedge \tilde{dx}_4\}$. Now we can look at the four-forms. However, since we are in $T_p\mathbb{R}^4$, there is only one wedge product we can construct keeping skew-symmetry in mind: $\tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \tilde{dx}_3 \wedge \tilde{dx}_4$. This is the only basis element for the four-forms in $T_p\mathbb{R}^4$. Now what about something like a five-form? Well, if we have only 4 basis one-forms to wedge together, then one of the 4 is going to be repeated by the Pigeonhole Principle and we'll end up with zero. In fact, this is why we can't really go any further than a four-form if we are considering $T_p\mathbb{R}^4$.

With this simple example, we have some idea about what the basis is for the space of m-forms on $T_p\mathbb{R}^n$. The idea is to look at the n basis one-forms on $T_p\mathbb{R}^n$, choose m of these and construct a basis m-form by performing the wedge product, all the while keeping skew-symmetry at the back of our minds. This leads us to the conclusion that the space of m-forms on $T_p\mathbb{R}^n$ has the following basis

$$\{\tilde{dx}_{i_1} \wedge \tilde{dx}_{i_2} \wedge \ldots \tilde{dx}_{i_m} | 1 \leq i_1 < i_2 < \ldots < i_m \leq n\} \tag{2.36}$$

A useful shorthand is to define what's called a *multi-index* of degree m denoted by $I = (i_1, i_2, \ldots, i_m)$ so that we can write $\tilde{dx}_{i_1} \wedge \tilde{dx}_{i_2} \wedge \ldots \tilde{dx}_{i_m} = \tilde{dx}_I$. Since this is a basis for the space of m-forms, we can write any arbitrary m-form in the following way

$$\alpha = \sum_I a_I \tilde{dx}_I \tag{2.37}$$

Let's just review everything we've done so far. We introduced one-forms, linear functions that act on vectors in the tangent space to produce numbers alongside the idea of elementary one-forms that are a basis for the space of all one-forms. Then, we introduced the wedge product $\wedge$ that allowed us to construct two-forms from one-forms, which we generalised to construct m-forms that act on m vectors. Two key properties that we have of our m-forms is multilinearity and skew-symmetry. Finally, we constructed a basis for the space of m-forms on $T_p\mathbb{R}^n$ by performing the wedge product on the basis one-forms of $T_p\mathbb{R}^n$. Ususally the space of m-forms on $T_p\mathbb{R}^n$ is denoted by $\bigwedge^m \mathbb{R}^n$.

One question we can now ask is what would the dimension of $\bigwedge^m \mathbb{R}^n$ be. This amounts to asking how many different basis elements of the form $\tilde{dx}_I$ we can construct from n elementary one-forms. However, this is a simple counting problem. We have n elementary one-forms, and we need to choose m one-forms to wedge. So we have n-choose-m basis elements. Hence the dimension of $\bigwedge^m \mathbb{R}^n$ is simply $\binom{n}{m}$.

Another question we can now try to answer is what happens if we wedge an m-form and a k-form. Let α be an m-form and β be a k-form where $m \neq k$. Expanding both in their respective bases, we have the following expressions

$$\alpha = \sum_I a_I \tilde{dx}_I \qquad \beta = \sum_J b_J \tilde{dx}_J \tag{2.38}$$

where $I = (i_1, i_2, \ldots i_m)$ and $J = (j_1, j_2, \ldots, j_k)$. Consider $\beta \wedge \alpha$

$$\beta \wedge \alpha = \sum_I \sum_J a_I b_J \tilde{dx}_J \wedge \tilde{dx}_I \tag{2.39}$$

In order to express $\beta \wedge \alpha$ in terms of $\alpha \wedge \beta$ we need to express $\tilde{dx}_J \wedge \tilde{dx}_I$ in terms of $\tilde{dx}_I \wedge \tilde{dx}_J$. In order to do this, it will be helpful to expand $\tilde{dx}_J \wedge \tilde{dx}_I$ fully

$$\tilde{dx}_J \wedge \tilde{dx}_I = \tilde{dx}_{j_1} \wedge \tilde{dx}_{j_2} \wedge \ldots \wedge \tilde{dx}_{j_k} \wedge \tilde{dx}_{i_1} \wedge \tilde{dx}_{i_2} \wedge \ldots \wedge \tilde{dx}_{i_m} \tag{2.40}$$

Now in order to express $\tilde{dx}_J \wedge \tilde{dx}_I$ in terms of $\tilde{dx}_I \wedge \tilde{dx}_J$, we need to shift the $\tilde{dx}_{i_1}$ to the leftmost side in the wedge product. This is easy to do using skew-symmetry. We just need to pick up a minus sign every time we shift it. Since there are k elementary forms behind $\tilde{dx}_{i_1}$ we pick up an overall factor of $(-1)^k$. All in all what we get is

$$\tilde{dx}_J \wedge \tilde{dx}_I = (-1)^k \tilde{dx}_{i_1} \wedge \tilde{dx}_{j_1} \wedge \tilde{dx}_{j_2} \wedge \ldots \wedge \tilde{dx}_{j_k} \wedge \tilde{dx}_{i_2} \wedge \tilde{dx}_{i_3} \wedge \ldots \wedge \tilde{dx}_{i_m} \tag{2.41}$$

We now need to shift the $\tilde{dx}_{i_2}$. This also needs to be shifted k times, so we'll pick up an additional $(-1)^k$ factor. This process will repeat itself for all of the m elementary one forms in the sequence $\tilde{dx}_I$. So we'll actually pick up a total factor of $(-1)^{km}$. In the end what we get is

$$\tilde{dx}_J \wedge \tilde{dx}_I = (-1)^{km} \tilde{dx}_{i_1} \wedge \tilde{dx}_{i_2} \wedge \ldots \wedge \tilde{dx}_{i_m} \wedge \tilde{dx}_{j_1} \wedge \tilde{dx}_{j_2} \wedge \ldots \wedge \tilde{dx}_{j_k} = (-1)^{km} \tilde{dx}_I \wedge \tilde{dx}_J \tag{2.42}$$

The above expression can now be substituted into Equation (2.39) to write

$$\beta \wedge \alpha = (-1)^{km} \sum_I \sum_J a_I b_J \tilde{dx}_I \wedge \tilde{dx}_I = (-1)^{km} \alpha \wedge \beta \tag{2.43}$$

One straightforward corollary from Equation (2.43) is that if α is a k-form where k is odd then $\alpha \wedge \alpha = 0$.

3 Differential Forms

3.1 Introducing Families of Forms

We've talked about m-forms on $T_p\mathbb{R}^n$. Any arbitrary m-form ω can be written in the following way

$$\omega = \sum_I a_I \tilde{dx}_I \tag{3.1}$$

where $\tilde{dx}_I = \tilde{dx}_{i_1} \wedge \tilde{dx}_{i_2} \wedge \ldots \tilde{dx}_{i_m}$ is the wedge product of m elementary one-forms and $I = (i_1, i_2, \ldots i_m)$ is a multi-index such that $1 \leq i_1 < i_2 < \ldots < i_m \leq n$. So far, our discussion has been restricted to talking about an m-form at a specific point $p \in \mathbb{R}^n$. However, we can generalise this idea fairly easily to consider families of m-forms $\omega_p : (T_p\mathbb{R}^n)^m \to \mathbb{R}$ where p ranges over all of $\mathbb{R}^n$ [3, 2]. For this generalisation to be useful, we can think of some nice properties that these families of forms should have which can be motivated from our usual understanding of calculus. For example, let's say we have two points on $\mathbb{R}^n$ (lets call them p and q) that are very close together. Then what we would like is that the ω_p and ω_q should be very similar. This is very similar to how we think about continuity. Another nice property we can think of our family of forms ω_p to have is the idea of differentiability. It will be useful to consider an example to get a feel for what differentiability means in this context. Let's consider two-forms on $T_p\mathbb{R}^3$. The elementary one-forms in this scenario is then just the set $\{\tilde{dx}, \tilde{dy}, \tilde{dz}\}$. If we fix a point $p \in \mathbb{R}^3$, then we can always write a two-form as $\omega_p = a_p\tilde{dx} \wedge \tilde{dy} + b_p\tilde{dy} \wedge \tilde{dz} + c_p\tilde{dx} \wedge \tilde{dz}$ where a_p, b_p and c_p are constants. Suppose we now let p vary over all of $\mathbb{R}^3$. Then, the constants will also vary. In other words, if we let p vary, then a_p, b_p and c_p will be functions of the form $f : \mathbb{R}^3 \to \mathbb{R}$. What we can now argue is that ω_p being differentiable is equivalent to saying that all three of our functions a_p, b_p and c_p are differentiable functions. Usually, instead of differentiable, we place the constraint that the functions are smooth i.e. they have an infinite number of continuous derivatives. We can then say that the family ω_p is a *differential two-form* on $\mathbb{R}^3$.

With this example, it is easy to generalise to talk about *differential m-forms* on $\mathbb{R}^n$ [1, 3]. They can be expressed in a similar way as in Equation (3.1), but the only extra condition we impose is that the coefficients a_I are actually smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$. One immediate observation that we can make from this generalisation is that if we have a differentiable m-form ω on $\mathbb{R}^n$, then if we actually evaluate the coefficients at a specific point $p \in \mathbb{R}^n$, we end up with an m-form on $T_p\mathbb{R}^n$. Furthermore, we define a differential zero-form to just be a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$, since such a function doesn't really need any vectors to give us a number.

Let's look at an example to get a better idea how differential forms work. Let's define $\alpha = x^2\tilde{dx} \wedge \tilde{dy} - x^3z\tilde{dy} \wedge \tilde{dz}$. Clearly, α is a differential two-form on $\mathbb{R}^3$. Let's choose a point $p = (2, 1, -1)$. We can now evaluate α at p to obtain a two-form that is defined on $T_p\mathbb{R}^3$. We denote this two-form by α_p which is given by $\alpha_p = 4\tilde{dx} \wedge \tilde{dy} + 8\tilde{dy} \wedge \tilde{dz}$. All we've done here is substitute the values of x, y and z from the given point p. Now as we've already mentioned, α_p is a regular two-form on $T_p\mathbb{R}^3$. So it's going to take in two vectors and give a number as an output. Lets choose $\vec{v}_1 = \langle 1, -2, 3\rangle$ and $\vec{v}_2 = \langle 2, 0, 1\rangle$. We can now evaluate $\alpha_p(\vec{v}_1, \vec{v}_2)$.

$$\begin{aligned}
\alpha_p(\vec{v_1}, \vec{v_2}) &= (4\tilde{dx} \wedge \tilde{dy} + 8\tilde{dy} \wedge \tilde{dz})(\vec{v}_1, \vec{v}_2) \\
&= 4\tilde{dx} \wedge \tilde{dy}(\vec{v}_1, \vec{v}_2) + 8\tilde{dy} \wedge \tilde{dz}(\vec{v}_1, \vec{v}_2) \\
&= 4\begin{vmatrix} \tilde{dx}(\vec{v}_1) & \tilde{dy}(\vec{v}_1) \\ \tilde{dx}(\vec{v}_2) & \tilde{dy}(\vec{v}_2) \end{vmatrix} + 8\begin{vmatrix} \tilde{dy}(\vec{v}_1) & \tilde{dz}(\vec{v}_1) \\ \tilde{dy}(\vec{v}_2) & \tilde{dz}(\vec{v}_2) \end{vmatrix} \\
&= 4\begin{vmatrix} 1 & -2 \\ 2 & 0 \end{vmatrix} + 8\begin{vmatrix} -2 & 3 \\ 0 & 1 \end{vmatrix} \\
&= 4(4) + 8(-2) = 0
\end{aligned} \tag{3.2}$$

From this example, what we can conclude is that for the full evaluation of a differential m-form on $\mathbb{R}^n$ we need two things: a "base point" p from $\mathbb{R}^n$, and m vectors from $T_p\mathbb{R}^n$. Now, what we can try to do is generalise these two ingredients. So instead of a specific base point, we can consider an arbitrary point in $\mathbb{R}^n$. Similarly, instead of a constant vector that lives in $T_p\mathbb{R}^n$, we can consider a vector field, which is simply a choice of vector in $T_p\mathbb{R}^n$ for each point $p \in \mathbb{R}^n$. Let's use an example. Define the differential two-form on $\mathbb{R}^3$ given by $\omega = x^2\tilde{dx}\wedge\tilde{dy} - x^3\tilde{dy}\wedge\tilde{dz}$. Now, our point $p \in \mathbb{R}^3$ is going to arbitrary, which means that it simply has the form (x, y, z) where $x, y, z \in \mathbb{R}$. Furthermore, instead of vectors, we're going to consider vector fields, which will take in a point p of the form (x, y, z) as an input to give a vector in $T_p\mathbb{R}^3$ as an output. Let $\vec{v}_1 = \langle x, 2yz, xy\rangle$ and $\vec{v}_2 = \langle y, xz, y^2\rangle$. Since our point p is general, we don't need to evaluate ω before acting it on two vectors. Instead, we can directly compute $\omega(\vec{v}_1, \vec{v}_2)$.

$$\begin{aligned}
\omega(\vec{v_1}, \vec{v_2}) &= (x^2\tilde{dx} \wedge \tilde{dy} - x^3\tilde{dy} \wedge \tilde{dz})(\vec{v}_1, \vec{v}_2) \\
&= x^2\tilde{dx} \wedge \tilde{dy}(\vec{v}_1, \vec{v}_2) - x^3\tilde{dy} \wedge \tilde{dz}(\vec{v}_1, \vec{v}_2) \\
&= x^2\begin{vmatrix} \tilde{dx}(\vec{v}_1) & \tilde{dy}(\vec{v}_1) \\ \tilde{dx}(\vec{v}_2) & \tilde{dy}(\vec{v}_2) \end{vmatrix} - x^3\begin{vmatrix} \tilde{dy}(\vec{v}_1) & \tilde{dz}(\vec{v}_1) \\ \tilde{dy}(\vec{v}_2) & \tilde{dz}(\vec{v}_2) \end{vmatrix} \\
&= x^2\begin{vmatrix} x & 2yz \\ y & xz \end{vmatrix} - x^3\begin{vmatrix} 2yz & xy \\ xz & y^2 \end{vmatrix} \\
&= x^2(x^2z - 2y^2z) - x^3(2y^3z - x^2yz)
\end{aligned} \tag{3.3}$$

So instead of a number $\omega(\vec{v}_1, \vec{v}_2)$ actually gave us a function from $\mathbb{R}^3$ to $\mathbb{R}$. We fed vector fields into our differential form, and got a function in the end. This is the "big picture" of what differential forms do [3]. More specifically, a differential m-form on $\mathbb{R}^n$ will take in m vector fields on $\mathbb{R}^n$ as an input and yield a function $f : \mathbb{R}^n \to \mathbb{R}$.

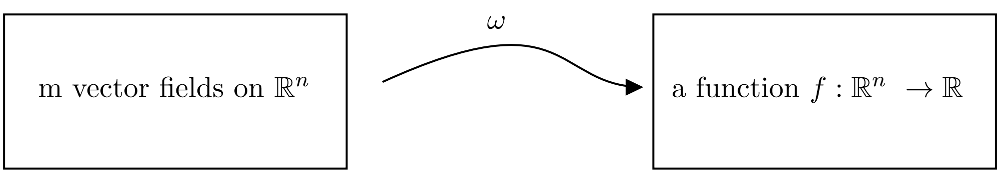

Figure 6: The "big picture" of how differential forms work

3.2 Integrating Differential Two-Forms

We now wish to look at how we can integrate differential forms [3, 1, 2]. This is where we see the real power of differential forms. We're going to see how integrating differential forms and functions are essentially the same thing, and this is what allows us to recast integration over

$\mathbb{R}^n$ in a much more natural way using differential forms.
We're not going to jump straight into looking at how to integrate differential m-forms on $\mathbb{R}^n$. It's going to be much more insightful to look at a simpler case from which we can try to draw some generalisations. The simple case we're going to consider integrating two-forms on $\mathbb{R}^3$. Let $D \subseteq \mathbb{R}^2$. Define a function $\phi : D \to \mathbb{R}^3$ such that $\phi(D) = S$ defines a two-dimensional surface in $\mathbb{R}^3$. Our goal is simple: define the integral of a differential two-form ω over S. In other words, we desire a meaningful expression of $\int_S \omega$.
The way we're going to proceed involves a lot of inspiration from how we define a double integral over a domain M in $\mathbb{R}^2$ when we studied multivariable calculus. So it's a good idea to review how that worked. The way we generally went about integrating a function $f : M \to \mathbb{R}$ was by partitioning the region M into a set of lattice points denoted by $\{(x_i, y_j)\}$ where i is an index that goes from 1 to n and j is an index that goes from 1 to m. What we can do now is define $\Delta x_i = x_{i+1} - x_i$ and $\Delta y_j = y_{j+1} - y_j$ and evaluate the quantity $f(x_i, y_j)\Delta x_i \Delta y_j$ for all values of i and j. A crude approximation of the integral of f over the region M is then given by

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\Delta x_i \Delta y_j \tag{3.4}$$

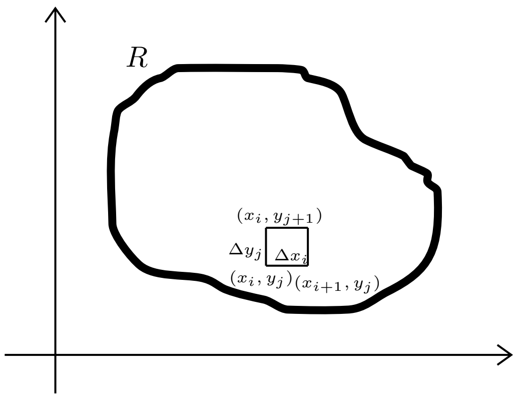

Figure 7: The integral of a function f over a region R can be approximated by evaluating f at each lattice point (x_i, y_i) multiplied by $\Delta x_i \Delta y_i$ and adding everything up

This approximation gets better and better if we increase the number of lattice points, which is the same as increasing the value of n and m. If we take the limit of Equation (3.4) as $n, m \to \infty$ which is the same as the limit of $\Delta x_i, \Delta y_j \to 0$ we obtain the value of the integral i.e.

$$\iint_M f(x, y)dA = \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_j \to 0}} \sum_i \sum_j f(x_i, y_j)\Delta x_i \Delta y_j \tag{3.5}$$

The way we are going to integrate our two-form is going to be very similar. The idea is to first define vectors tangent to our surface S at a lattice of points on S. We then evaluate the differential two-form at each point on the lattice to obtain a two-form; the two-form that will be defined at each lattice point can then act on the tangent vectors defined at each lattice point to give a number. The final result will involve adding up these results at each lattice point on our surface.

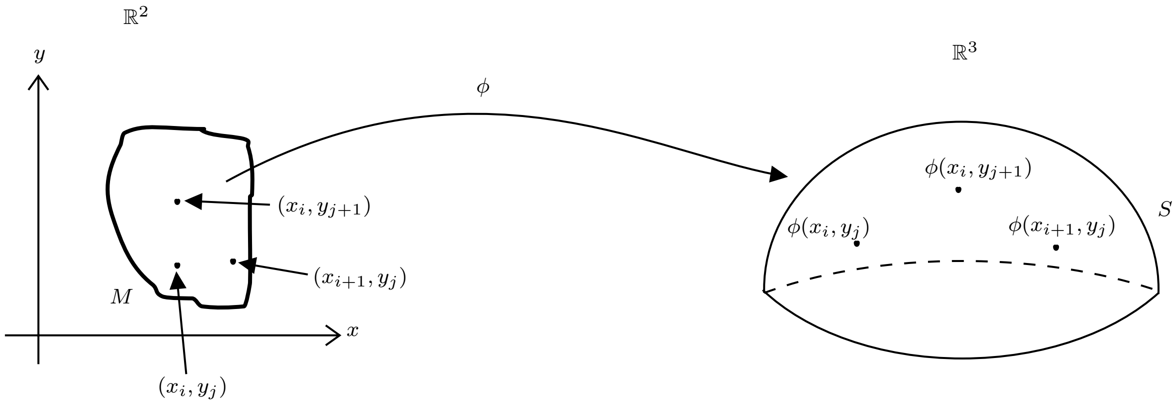

Figure 8: A picture of the domain M and the surface S

Figure 8 shows a picture of the domain M which is in $\mathbb{R}^2$ and the surface S which is in $\mathbb{R}^3$. On the domain M, we've defined a lattice of points, using which we can define a lattice of points on the surface S. Simply use the function ϕ to take each lattice point (x_i, y_j) on M to a lattice point $\phi(x_i, y_j)$ on S. What we can do now is define two vectors on $T_{(x_i,y_j)}\mathbb{R}^2$: $\vec{v}^1_{i,j} = (x_{i+1}, y_j) - (x_i, y_j)$ and $\vec{v}^2_{i,j} = (x_i, y_{j+1}) - (x_i, y_j)$. We can also similarly define two vectors on $T_{\phi(x_i,y_j)}\mathbb{R}^3$: $\vec{w}^1_{i,j} = \phi(x_{i+1}, y_j) - \phi(x_i, y_j)$ and $\vec{w}^2_{i,j} = \phi(x_i, y_{j+1}) - \phi(x_i, y_j)$.

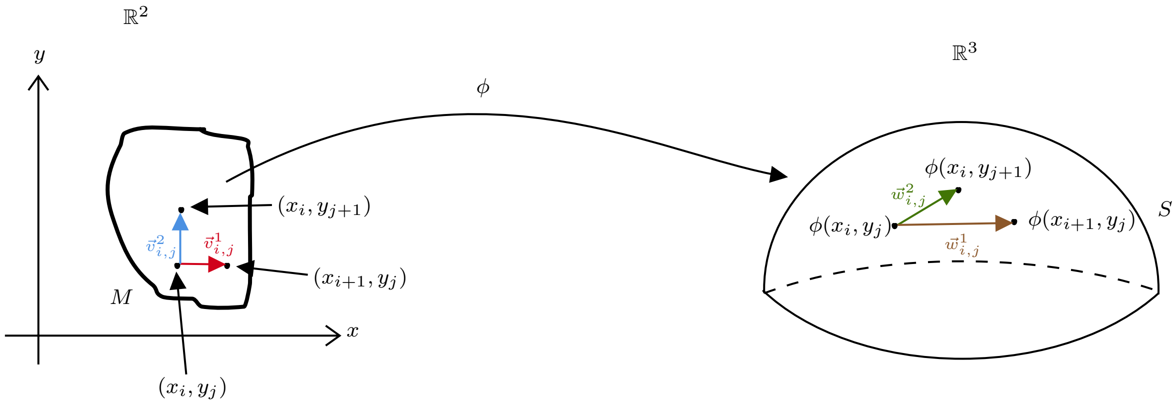

Figure 9: The vectors $\vec{v}^1_{i,j}$, $\vec{v}^2_{i,j}$, $\vec{w}^1_{i,j}$, and $\vec{w}^2_{i,j}$

We now have the tangent vectors we wanted to originally define on the surface S. What we can do now is evaluate our differential two-form ω at each lattice point to get an ordinary two-form $\omega_{\phi(x_i,y_j)}$. This two-form can then act on the two vectors defined on each lattice point on the surface which are $\vec{w}^1_{i,j}$ and $\vec{w}^2_{i,j}$. At the end, what we have is a number $\omega_{\phi(x_i,y_j)}(\vec{w}^1_{i,j}, \vec{w}^2_{i,j})$ for each lattice point on the surface. We can now think of integrating the differential two-form ω as summing all of these numbers up with all of the lattice points on our surface. So what we have is the following result

$$\int_S \omega \approx \sum_i \sum_j \omega_{\phi(x_i,y_j)}(\phi(x_{i+1}, y_j) - \phi(x_i, y_j), \phi(x_i, y_{j+1}) - \phi(x_i, y_j)) \tag{3.6}$$

where we have substituted the definitions of the vectors $\vec{w}^1_{i,j}$ and $\vec{w}^2_{i,j}$. The question we can ask now is how we can make the approximation in Equation (3.6) better. Well, we can just have the lattice points come closer together, very analogously to the way we took the limit of

$\Delta x_i, \Delta y_j \to 0$ in our original double integral problem. Here we define $\Delta x_i = |\vec{v}^1_{i,j}| = |x_{i+1} - x_i|$ and $\Delta y_j = |\vec{v}^2_{i,j}| = |y_{j+1} - y_j|$. With this we have the following result

$$\int_S \omega = \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_j \to 0}} \sum_i \sum_j \omega_{\phi(x_i,y_j)}(\phi(x_{i+1}, y_j) - \phi(x_i, y_j), \phi(x_i, y_{j+1}) - \phi(x_i, y_j)) \tag{3.7}$$

Can we make Equation (3.7) any neater? The answer is a resounding YES! The trick is to recognise the bilinearity of our two-form. So we can multiply and divide by the term $\Delta x_i \Delta y_j$ and obtain

$$\int_S \omega = \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_j \to 0}} \sum_i \sum_j \omega_{\phi(x_i,y_j)} \left(\frac{\phi(x_{i+1}, y_j) - \phi(x_i, y_j)}{\Delta x_i}, \frac{\phi(x_i, y_{j+1}) - \phi(x_i, y_j)}{\Delta y_j} \right) \Delta x_i \Delta y_j \tag{3.8}$$

We can now recognise that the two big terms in the parentheses in Equation (3.8) are simply partial derivatives once we take the limit of $\Delta x_i, \Delta y_j \to 0$. More specifically

$$\begin{aligned} \lim_{\Delta x_i \to 0} \frac{\phi(x_{i+1}, y_j) - \phi(x_i, y_j)}{\Delta x_i} &= \frac{\partial \phi}{\partial x}(x_i, y_j) \\ \lim_{\Delta y_j \to 0} \frac{\phi(x_i, y_{j+1}) - \phi(x_i, y_j)}{\Delta x_i} &= \frac{\partial \phi}{\partial y}(x_i, y_j) \end{aligned} \tag{3.9}$$

So what we end up with is the limit of the double sum of just some function that is evaluated at the lattice point (x_i, y_j) multiplied by $\Delta x_i \Delta y_j$, but this is just a double integral that we've seen before. All in all, the final result that we obtain is

$$\int_S \omega = \iint_M \omega_{\phi(x,y)} \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) dA \tag{3.10}$$

where $dA = dxdy$ [1]. Equation (3.10) is the key result we've been after. One important thing that should be noted here is that all of the theory that we've just developed assumed that we were able to define a function $\phi : D \to \mathbb{R}^3$ to parameterise our two-dimensional surface in $\mathbb{R}^3$. If we are not able to find such a parameterisation, we wouldn't have been able to do much.
A good idea now would be to look at an example to see how integrating two-forms actually works. Let's look at a cylinder that has a radius of 1 and a height of 2. Mathematically, this cylinder can be expressed as the set $S = \{(x, y, z)|x, y, z \in \mathbb{R}; x^2 + y^2 = 1; 0 \le z \le 2\}$. Consider the differential two-form $\omega = yz\tilde{dx} \wedge \tilde{dy} + \frac{z}{x}\tilde{dy} \wedge \tilde{dz}$. The first thing we need to do is parameterise S i.e. find some function $\phi : D \to \mathbb{R}^3$ where $D \subseteq \mathbb{R}^2$ such that $\phi(D) = S$. From our knowledge of cylindrical coordinates, finding such a parameterisation is not too difficult. Let $D = [0, 2\pi] \times [0, 2]$ and define $\phi : D \to \mathbb{R}^3$ such that $\phi(\theta, z) = (\cos\theta, \sin\theta, z)$. Then clearly $\phi(D) = S$. Now that we have a parameterisation, we can just simply use Equation (3.10) to integrate ω over the cylinder. We'll do this step-by-step. Let's first evaluate the vectors that are going to be fed into the two-form. For this, we simply need to take some partial derivatives.

$$\begin{aligned} \frac{\partial \phi}{\partial \theta} &= \langle -\sin\theta, \cos\theta, 0 \rangle \\ \frac{\partial \phi}{\partial z} &= \langle 0, 0, 1 \rangle \end{aligned} \tag{3.11}$$

Let's now evaluate $\omega_{\phi(\theta,z)}$.

$$\omega_{\phi(\theta,z)} = \omega_{(\cos\theta,\sin\theta,z)} = z\sin\theta\tilde{dx} \wedge \tilde{dy} + \frac{z}{\cos\theta}\tilde{dy} \wedge \tilde{dz} \tag{3.12}$$

We now need to evaluate $\omega_{\phi(\theta,z)}\left(\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right)$.

$$\begin{aligned}
\omega_{\phi(\theta,z)}\left(\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right) &= \left(z\sin\theta\tilde{dx}\wedge\tilde{dy} + \frac{z}{\cos\theta}\tilde{dy}\wedge\tilde{dz}\right)(\langle -\sin\theta, \cos\theta, 0\rangle, \langle 0,0,1\rangle) \\
&= z\sin\theta\tilde{dx}\wedge\tilde{dy}(\langle -\sin\theta, \cos\theta, 0\rangle, \langle 0,0,1\rangle) + \\
&\qquad \frac{z}{\cos\theta}\tilde{dy}\wedge\tilde{dz}(\langle -\sin\theta, \cos\theta, 0\rangle, \langle 0,0,1\rangle) \\
&= z\sin\theta\begin{vmatrix} -\sin\theta & \cos\theta \\ 0 & 0\end{vmatrix} + \frac{z}{\cos\theta}\begin{vmatrix}\cos\theta & 0 \\ 0 & 1\end{vmatrix} \\
&= z
\end{aligned} \tag{3.13}$$

So all in all what we get is the following expression

$$\int_S \omega = \iint_D z dA \tag{3.14}$$

Since $D = [0, 2\pi] \times [0, 2]$, we can write $dA = d\theta dz$ and write the integral as

$$\int_S \omega = \int_0^{2\pi}\int_0^2 z d\theta dz = 4\pi \tag{3.15}$$

From this example, what we can see is the integrating ω over S was basically the same thing as integration the function $f(z, \theta) = z$ over the domain D. So, in a way, integrating a differential forms is essentially the same as integrating functions.

3.3 Integrating Differential m-Forms on $\mathbb{R}^n$

We are now ready to generalise the recipe we developed in the previous section to talk about integrating a differential m-form on some m-dimensional surface in $\mathbb{R}^n$. In the previous section, we first defined a parameterisation i.e. a function $\phi : M \to \mathbb{R}^3$ where $M \subseteq \mathbb{R}^2$ such that $\phi(M) = S$ (here S is the two-dimensional surface in $\mathbb{R}^3$). In a similar fashion, to integrate a differential m-form on some m-dimensional surface in $\mathbb{R}^n$, we need a parameterisation $\phi : M \to \mathbb{R}^n$ such that $\phi(M) = S$; in this case however, $M \subseteq \mathbb{R}^m$ and the surface S is an m-dimensional surface in $\mathbb{R}^n$. Then the formula written in Equation (3.10) generalises to

$$\int_S \omega = \int\cdots\int_M \omega_{\phi(x_1,x_2,\ldots,x_m)}\left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \ldots, \frac{\partial\phi}{\partial x_m}\right) dV_m \tag{3.16}$$

where on the right-hand-side we have an m-integral and dV_m is the m-dimensional Euclidean volume element i.e. $dV_m = dx_1 dx_2 \ldots dx_m$ [1, 2].
Now that we have our general formula, a good idea will be to look at an example. Let's look at integrating a differential one-form on a one-dimensional surface in $\mathbb{R}^4$. The one-form we're going to consider is $\alpha = x_1\tilde{dx}_1 + (x_1^2 + x_2)\tilde{dx}_2 + x_3x_4\tilde{dx}_4$. A one-dimensional surface in $\mathbb{R}^4$ is essentially just a curve C. So what we are doing is integrating α on a curve C that's living in $\mathbb{R}^4$. Let's say that the curve is given by the parameterisation $f : [0, 3\pi] \to \mathbb{R}^4$ such that $f(t) = (\cos t, \sin t, t, -t)$. Using the general formula given by Equation (3.16), we obtain

$$\int_C \alpha = \int_0^{3\pi} \alpha_{f(t)}\left(\frac{\partial f}{\partial t}\right) dt \tag{3.17}$$

Evaluating $\alpha_{f(t)}$ we obtain

$$\alpha_{f(t)} = \alpha_{(\cos t, \sin t, t, -t)} = \cos t\tilde{dx}_1 + (\cos^2 t + \sin t)\tilde{dx}_2 - t^2\tilde{dx}_4 \tag{3.18}$$

Evaluating the partial derivative (technically in this scenario since our parameterisation is dependent on a single variable, the partial derivative is the same as the total derivative) we obtain

$$\frac{\partial f}{\partial t} = \langle -\sin t, \cos t, 1, -1\rangle \tag{3.19}$$

Evaluating the integrand beforehand we obtain

$$\begin{aligned}\alpha_{f(t)}\left(\frac{\partial f}{\partial t}\right) &= (\cos t\tilde{dx}_1 + (\cos^2 t + \sin t)\tilde{dx}_2 - t^2\tilde{dx}_4)\langle -\sin t, \cos t, 1, -1\rangle \\ &= \cos t\tilde{dx}_1\langle -\sin t, \cos t, 1, -1\rangle + (\cos^2 t + \sin t)\tilde{dx}_2\langle -\sin t, \cos t, 1, -1\rangle - \\ &\qquad t^2\tilde{dx}_4\langle -\sin t, \cos t, 1, -1\rangle \\ &= -\sin t\cos t + (\cos^2 t + \sin t)\cos t + t^2 \\ &= \cos^3 t + t^2\end{aligned} \tag{3.20}$$

Substituting the final expression from Equation (3.20) into Equation (3.17) we obtain

$$\int_C \alpha = \int_0^{3\pi} \cos^3 t + t^2 dt = 9\pi^3 \tag{3.21}$$

3.3.1 "Relearning" u-Substitution

One application of the generalised formula shown in Equation (3.16) is that we can re-cast the idea of performing a substitution of an integral using the language of differential forms [3]. The object we will need to consider here is a differential one-form in $\mathbb{R}$. The idea is that we can define a different parameterisation for the surface (in this context, since we are simply in $\mathbb{R}$ and are considering a differential one-form, the surface is simply going to be an interval on the real line) that we are integrating our differential one-form over, and the different parameterisation is essentially the same as performing a substitution of our integral. As an example, we can consider the following integral

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx \tag{3.22}$$

We can recast the integral above using the language of differential forms pretty easily. Define the one-form $\omega = \frac{1}{\sqrt{1-x^2}}\tilde{dx}$ and the surface S to be the closed interval $[0,1]$. One parameterisation we can define is $f : [0,1] \to \mathbb{R}$ such that $f(t) = t$. This is called the trivial parameterisation, because we're just mapping the interval $[0,1]$ back to itself. With this trivial parameterisation, the expression for the integral of ω over S just ends up being the same as the integral in Equation (3.22). Let's see how.

$$\begin{aligned}\int_S \omega &= \int_0^1 \omega_{f(t)}\left(\frac{\partial f}{\partial t}\right)dt \\ &= \int_0^1 \frac{1}{\sqrt{1-t^2}}\tilde{dx}(\langle 1\rangle)dt \\ &= \int_0^1 \frac{1}{\sqrt{1-t^2}}dt\end{aligned} \tag{3.23}$$

As promised we just get back Equation (3.22) using the trivial parameterisation. Now we can define a new paramterisation called $g : [0, \frac{\pi}{2}] \to \mathbb{R}$ such that $g(t) = \sin t$. This is clearly a parameterisation of our surface $[0,1]$. If we use this parameterisation we obtain

$$\begin{aligned}\int_S \omega &= \int_0^{\frac{\pi}{2}} \omega_{g(t)}\left(\frac{\partial g}{\partial t}\right) \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin^2 t}} \tilde{dx}(\langle \cos t\rangle)dt \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{1-\sin^2 t}} dt\end{aligned} \tag{3.24}$$

However, the final integral shown in Equation (3.24) is nothing more than just performing the substitution $x = \sin u$ on the original integral we had in Equation (3.22). So the final integral in Equation (3.24) is the same thing as the integral shown in Equation (3.22). Furthermore, the one in Equation (3.24) is simpler to calculate because

$$\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{1-\sin^2 t}} dt = \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{\cos^2 t}} dt = \int_0^{\frac{\pi}{2}} dt = \frac{\pi}{2} \tag{3.25}$$

So the conclusion is simple: when we look at integrating differential one-forms on $\mathbb{R}$, the process of substitution is essentially just choosing a different parameterisation for the surface we are integrating over. We could have used the trivial parameterisation or the fancier one, but the final result of the integral must be the same. So we can choose whatever parameterisation we want. This idea of the choice of parameterisation not changing the final answer we should get for our integral actually carries over to the general case as well where we have an m-form that needs to be integrated over an m-dimensional surface in $\mathbb{R}^n$.

3.3.2 The Change of Variables Formula in $\mathbb{R}^2$

Another application of the generalised formula shown in Equation (3.16) is that we can recast the change of variables formula we know from multivariable calculus in the language of differential forms as well [3]. The example we can consider is integrating the function $f(x,y) = x^2+y^2$ over the domain $D = \{(x,y)|x^2+y^2 \leq 4\}$. The expression for the integral in Cartesian coordinates is

$$\int_D (x^2+y^2)dxdy \tag{3.26}$$

We can rewrite this integral in terms of integrating a differential two-form on $\mathbb{R}^2$. Define the two-form $\omega = (x^2+y^2)\tilde{dx} \wedge \tilde{dy}$. If we choose the trivial parameterisation, where we're mapping the domain D to itself, we essentially get back Equation (3.26). However, we know that it's easier to solve an integral like this if we go to polar coordinates. The transformation from Cartesian to polar is going to be a reparameterisation of the domain D. More specifically, we can define the function $\phi : [0,2] \times [0,2\pi] \to \mathbb{R}$ such that $\phi(r,\theta) = (r\cos\theta, r\sin\theta)$. With this parameterisation we obtain

$$\int_D \omega = \int_0^2 \int_0^{2\pi} \omega_{\phi(r,\theta)}\left(\frac{\partial\phi}{\partial r}, \frac{\partial\phi}{\partial\theta}\right) d\theta dr \tag{3.27}$$

Evaluating $\omega_{\phi(r,\theta)}$ we obtain

$$\omega_{\phi(r,\theta)} = (r^2\cos^2\theta + r^2\sin^2\theta)\tilde{dx}\tilde{dy} = r^2\tilde{dx} \wedge \tilde{dy} \tag{3.28}$$

We now need to evaluate the partial derivatives to obtain the vectors we need to feed into the two-form

$$\begin{aligned} \frac{\partial \phi}{\partial r} &= \langle \cos\theta, \sin\theta \rangle \\ \frac{\partial \phi}{\partial \theta} &= \langle -r\sin\theta, r\cos\theta \rangle \end{aligned} \tag{3.29}$$

So the integral in Equation (3.27) transforms to

$$\int_D \omega = \int_0^2 \int_0^{2\pi} r^2 \tilde{dx} \wedge \tilde{dy}(\langle \cos\theta, \sin\theta \rangle, \langle -r\sin\theta, r\cos\theta \rangle) d\theta dr \tag{3.30}$$

Evaluating $\tilde{dx} \wedge \tilde{dy}(\langle \cos\theta, \sin\theta \rangle, \langle -r\sin\theta, r\cos\theta \rangle)$ we obtain

$$\begin{aligned} \tilde{dx} \wedge \tilde{dy}(\langle \cos\theta, \sin\theta \rangle, \langle -r\sin\theta, r\cos\theta \rangle) &= \begin{vmatrix} \tilde{dx}(\langle \cos\theta, \sin\theta \rangle) & \tilde{dy}(\langle \cos\theta, \sin\theta \rangle) \\ \tilde{dx}(\langle -r\sin\theta, r\cos\theta \rangle) & \tilde{dy}(\langle -r\sin\theta, r\cos\theta \rangle) \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta = r \end{aligned} \tag{3.31}$$

Notice that in Equation (3.31) the evaluation of the two-form on the two tangent vectors gives us the Jacobian of the transformation from Cartesian to polar coordinates. Substituting that result gives us the final integral we need to evaluate

$$\int_D \omega = \int_0^2 \int_0^{2\pi} r^3 d\theta dr = 8\pi \tag{3.32}$$

The upshot here is that if we're integrating two-forms on $\mathbb{R}^2$, then a reparameterisation of the domain we are integrating over is essentially the same as a coordinate transformation and transforming the original integral into a new one in terms of the new coordinates using the Jacobian.

3.3.3 A Small Note on Orientations

In the previous section, the parameterisation I defined was $\phi : [0,2] \times [0,2\pi] \to \mathbb{R}$ such that $\phi(r,\theta) = (r\cos\theta, r\sin\theta)$. However, I could have just as easily defined $\phi : [0,2\pi] \times [0,2] \to \mathbb{R}$ such that $\phi(\theta, r) = (r\cos\theta, r\sin\theta)$. However, if I chose the latter parameterisation, the final result I would've gotten was -8π. This seems to lead us towards a contradiction, because we've already talked about how the parameterisation we choose shouldn't really affect our final answer. So there's some missing piece of information that we need to remove this contradiction. This missing piece is the *orientation* of the surface we are integrating over [1]. Essentially the general formula that was written in Equation (3.16) should have been

$$\int_S \omega = \pm \int \cdots \int_M \omega_{\phi(x_1,x_2,\ldots,x_m)} \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \ldots, \frac{\partial \phi}{\partial x_m} \right) dV_m \tag{3.33}$$

The orientation on S is essentially the extra piece of information we need to choose between the signs from the above equation. In all of the integrals that I have computed so far, I've implicitly used the $+$ sign for whatever parameterisation I've defined. This is called the orientation on S that is *induced* by the parameterisation. It's basically the most "natural" orientation we can choose. However, the more detailed scheme of integrating differential forms involves determining the orientation of the surface, either algebraically or geometrically. A discussion on how to determine the orientation of a surface is present in Bachman.

4 The Exterior Derivative

4.1 Introducing the Exterior Derivative

Now that we have a good feel for how differential m-forms work, we can try to think of some operations that we can apply on them. With this idea in mind, we can ask ourselves the question of whether we can define an operation that allows us to obtain an $(m+1)$-form from an m-form. As always, let's first look at a simple case: obtaining a one-form from a zero-form. A differential zero-form on $\mathbb{R}^n$ is just a function $f(x_1, x_2, \ldots, x_n)$. We want to define an operator d on this zero-form such that df is a differential one-form. In order to figure out an explicit form of the d operator, it will be helpful to recall how we evaluated a differential one-form. We basically needed a point $p \in \mathbb{R}^n$ and a vector $\vec{v} \in T_p\mathbb{R}^n$. What we can do now is to think of a nice geometric operation we can do that involves these two ingredients, a function from $\mathbb{R}^n$ to $\mathbb{R}$ and it involves the derivative of the function. We already have an operation that involves all of these ingredients: the directional derivative! Recall from multivariable calculus that if f is a function from $\mathbb{R}^n$ to $\mathbb{R}$ then we can define the directional derivative of f at a point p in the direction of some vector $v = \langle v_1, v_2, \ldots, v_n \rangle$ in the following way

$$D_{\vec{u}}f|_p = \vec{\nabla} f|_p \cdot \vec{v} = \left.\frac{\partial f}{\partial x_1}\right|_p v_1 + \left.\frac{\partial f}{\partial x_2}\right|_p v_2 + \ldots + \left.\frac{\partial f}{\partial x_n}\right|_p v_n \tag{4.1}$$

However, we can think of the above equation as the action of a one-form on a vector. More specifically we can define a one-form df_p that acts on a vector v in the following way

$$(df_p)(\vec{v}) = \left(\left.\frac{\partial f}{\partial x_1}\right|_p \tilde{dx}_1 + \left.\frac{\partial f}{\partial x_2}\right|_p \tilde{dx}_2 + \ldots + \left.\frac{\partial f}{\partial x_n}\right|_p \tilde{dx}_n \right)(\vec{v}) \tag{4.2}$$

Then clearly Equations (4.1) and (4.2) are equivalent. More generally, we can then define the differential one-form df as

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \tilde{dx}_i \tag{4.3}$$

We can now ask ourselves the question of how to generalise to differential m-forms. We know that we can write a differential m-form as

$$\omega = \sum_I f_I \tilde{dx}_I \tag{4.4}$$

The coefficients f_I are all just differentiable functions, and we already know how the d operator acts on those. So we can naturally just define

$$d\omega = \sum_I df_I \tilde{dx}_I = \sum_I \sum_{j=1}^{n} \frac{\partial f_I}{\partial x_j} \tilde{dx}_j \wedge \tilde{dx}_I \tag{4.5}$$

Clearly then $d\omega$ is a differential $(m+1)$-form and we've achieved what we were after. The operation d as defined by Equation (4.5) is called *exterior differentiation* [4, 3].
Let's now look at an example. If $\alpha = f\tilde{dx} \wedge \tilde{dy} + g\tilde{dx} \wedge \tilde{dz} + h\tilde{dy} \wedge \tilde{dz}$ is a differential two-form

on $\mathbb{R}^3$ then

$$
\begin{aligned}
d\alpha &= \cancel{\frac{\partial f}{\partial x}\tilde{dx} \wedge \tilde{dx} \wedge \tilde{dy}} + \cancel{\frac{\partial f}{\partial y}\tilde{dy} \wedge \tilde{dx} \wedge \tilde{dy}} + \frac{\partial f}{\partial z}\tilde{dz} \wedge \tilde{dx} \wedge \tilde{dy} \\
&\quad + \cancel{\frac{\partial g}{\partial x}\tilde{dx} \wedge \tilde{dx} \wedge \tilde{dz}} + \frac{\partial g}{\partial y}\tilde{dy} \wedge \tilde{dx} \wedge \tilde{dz} + \cancel{\frac{\partial g}{\partial z}\tilde{dz} \wedge \tilde{dx} \wedge \tilde{dz}} \\
&\quad + \frac{\partial h}{\partial x}\tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} + \cancel{\frac{\partial h}{\partial y}\tilde{dy} \wedge \tilde{dy} \wedge \tilde{dz}} + \cancel{\frac{\partial h}{\partial z}\tilde{dz} \wedge \tilde{dy} \wedge \tilde{dz}} \\
&= \left(\frac{\partial f}{\partial z} - \frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\right)\tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz}
\end{aligned} \tag{4.6}
$$

Another example we can look at is a differential four-form on $\mathbb{R}^4$. However, the exterior derivative of such a differential form will just be zero. This is because if if $\omega = f\tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \tilde{dx}_3 \wedge \tilde{dx}_4$ is a four-form on $\mathbb{R}^4$ then

$$
\begin{aligned}
d\omega &= \cancel{\frac{\partial f}{\partial x_1}\tilde{dx}_1 \wedge \tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \tilde{dx}_3 \wedge \tilde{dx}_4} + \cancel{\frac{\partial f}{\partial x_2}\tilde{dx}_2 \wedge \tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \tilde{dx}_3 \wedge \tilde{dx}_4} \\
&\quad + \cancel{\frac{\partial f}{\partial x_3}\tilde{dx}_3 \wedge \tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \tilde{dx}_3 \wedge \tilde{dx}_4} + \cancel{\frac{\partial f}{\partial x_4}\tilde{dx}_4 \wedge \tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \tilde{dx}_3 \wedge \tilde{dx}_4} \\
&= 0
\end{aligned} \tag{4.7}
$$

4.2 Properties of the Exterior Derivative

Let's now look at some general properties of the exterior derivative operator [4]. One property is that it is linear i.e. $d(a\alpha + b\beta) = a\ d(\alpha) + b\ d(\beta)$ for all m-forms α and β and all scalars a and b. The proof for this is fairly straightforward. Let $\alpha = \sum_I f_I \tilde{dx}_I$ and $\beta = \sum_I g_I \tilde{dx}_I$. Then $a\alpha + b\beta = \sum_I (af_I + bg_J)\tilde{dx}_I$. Taking the exterior derivative

$$
\begin{aligned}
d(a\alpha + b\beta) &= \sum_I d(af_I + bg_I)\tilde{dx}_I \\
&= \sum_I \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(af_I + bg_I)\tilde{dx}_j \wedge \tilde{dx}_I \\
&= \sum_I \sum_{j=1}^{n} \left(a\frac{\partial f_I}{\partial x_j} + b\frac{\partial g_I}{\partial x_j}\right)\tilde{dx}_j \wedge \tilde{dx}_I \\
&= a\sum_I \sum_{j=1}^{n} \frac{\partial f_I}{\partial x_j}\tilde{dx}_j \wedge \tilde{dx}_I + b\sum_I \sum_{j=1}^{n} \frac{\partial g_I}{\partial x_j}\tilde{dx}_j \wedge \tilde{dx}_I \\
&= a\ d(\alpha) + b\ d(\beta)
\end{aligned} \tag{4.8}
$$

In the above manipulation, we made use of the fact that partial differentiation is linear. Another property is that if ω is a differential m-form and μ is a differential k-form then

$$d(\omega \wedge \mu) = (d\omega) \wedge \mu + (-1)^m \omega \wedge (d\mu) \tag{4.9}$$

The above formula is essentially the Leibniz Rule for differential forms. In order to prove it, we first write ω and μ in terms of the basis forms as

$$\omega = \sum_I f_I \tilde{dx}_I \qquad \mu = \sum_J g_J \tilde{dx}_J \tag{4.10}$$

where $I = (i_1, i_2, \ldots, i_m)$ and $J = (j_1, j_2, \ldots, j_m)$. Then we get

$$\omega \wedge \mu = \sum_{I,J} f_I g_J \tilde{dx}_I \wedge \tilde{dx}_J \tag{4.11}$$

Taking the exterior derivative of Equation (4.10) we get

$$\begin{aligned} d(\omega \wedge \mu) &= \sum_{I,J} d(f_I g_J)\tilde{dx}_I \wedge dx_J \\ &= \sum_{I,J}\sum_{r=1}^{n} \frac{\partial (f_I g_J)}{\partial x_r}\tilde{dx}_r \wedge \tilde{dx}_I \wedge \tilde{dx}_J \\ &= \sum_{I,J}\sum_{r=1}^{n} \left(\frac{\partial f_I}{\partial x_r} g_J + f_I \frac{\partial g_J}{\partial x_r}\right)\tilde{dx}_r \wedge \tilde{dx}_I \wedge \tilde{dx}_J \\ &= \sum_{I,J}\sum_{r=1}^{n} \frac{\partial f_I}{\partial x_r} g_J \tilde{dx}_r \wedge \tilde{dx}_I \wedge \tilde{dx}_J + \sum_{I,J}\sum_{r=1}^{n} f_I \frac{\partial g_J}{\partial x_r}\tilde{dx}_r \wedge \tilde{dx}_I \wedge \tilde{dx}_J \end{aligned} \tag{4.12}$$

Consider the first term in the last line of Equation (4.12)

$$\sum_{I,J}\sum_{r=1}^{n} \frac{\partial f_I}{\partial x_r} g_J \tilde{dx}_r \wedge \tilde{dx}_I \wedge \tilde{dx}_J = \sum_{I,J} df_I g_J \tilde{dx}_I \wedge \tilde{dx}_J = d\omega \wedge \mu \tag{4.13}$$

Now consider the second term

$$\begin{aligned} \sum_{I,J}\sum_{r=1}^{n} f_I \frac{\partial g_J}{\partial x_r}\tilde{dx}_r \wedge \tilde{dx}_I \wedge \tilde{dx}_J &= (-1)^m \sum_{I,J}\sum_{r=1}^{n} f_I \frac{\partial g_J}{\partial x_r}\tilde{dx}_I \wedge \tilde{dx}_r \wedge \tilde{dx}_J \\ &= (-1)^m \sum_{I,J} f_I dg_J \tilde{dx}_I \wedge \tilde{dx}_J \\ &= (-1)^m \omega \wedge (d\mu) \end{aligned} \tag{4.14}$$

Hence, we've proven Equation (4.9).
Another property of the exterior derivative is that if α is an n-form on $\mathbb{R}^n$ then $d\alpha = 0$. This makes sense because $d\alpha$ would be an $n+1$-form on $\mathbb{R}^n$ which would mean that $d\alpha$ is just zero. In order to prove this, we need to recognise that a differential n-form on $\mathbb{R}^n$ can be written as $\alpha = f\tilde{dx}_1\tilde{dx}_2 \wedge \ldots \wedge dx_n$ where f is some smooth function. If we take the exterior derivative, we obtain

$$d\alpha = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\tilde{dx}_i \wedge \tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \ldots \wedge \tilde{dx}_n = 0 \tag{4.15}$$

A third property is that for any differential form α

$$d^2\alpha = d(d\alpha) = 0 \tag{4.16}$$

In order to prove this, we first write an arbitrary differential m-form α as $\alpha = \sum_I f_I \tilde{dx}_I$. Upon applying the d operator twice, we end up with the expression

$$d^2\alpha = \sum_{I}\sum_{k=1}^{n}\sum_{j=1}^{n} \frac{\partial f_I}{\partial x_k \partial x_j}\tilde{dx}_k \wedge \tilde{dx}_j \wedge \tilde{dx}_I \tag{4.17}$$

One observation we can make is that in the above equation, we don't need to consider the case where $j = k$ because we would then have a $\tilde{dx}_j \wedge \tilde{dx}_j$ term which would give us zero. So we can rewrite Equation (4.17) as

$$d^2\alpha = \sum_I \sum_{k=1}^{n} \sum_{\substack{j=1 \\ j\neq k}}^{n} \frac{\partial f_I}{\partial x_k \partial x_j} \tilde{dx}_k \wedge \tilde{dx}_j \wedge \tilde{dx}_I \tag{4.18}$$

What we can do now is break up the summation from $j = 1$ to $j = k - 1$ and from $j = k + 1$ to $j = n$. With this we obtain

$$d^2\alpha = \sum_I \sum_{k=1}^{n} \sum_{j=1}^{k-1} \frac{\partial f_I}{\partial x_k \partial x_j} \tilde{dx}_k \wedge \tilde{dx}_j \wedge \tilde{dx}_I + \sum_I \sum_{k=1}^{n} \sum_{j=k+1}^{n} \frac{\partial f_I}{\partial x_k \partial x_j} \tilde{dx}_k \wedge \tilde{dx}_j \wedge \tilde{dx}_I \tag{4.19}$$

Now, what we can notice the are indices in the second summation of the above equation. The sum over k runs from 1 to n which means that $1 \leq k \leq n$. The sum over j runs from $k + 1$ to n which means that $k + 1 \leq j \leq n$. These two inequalities imply that $1 \leq k < j \leq n$. What we can do now is to change the order of summation. This results in the second summation in Equation (4.19) being equivalent to

$$\sum_I \sum_{j=1}^{n} \sum_{k=1}^{j-1} \frac{\partial f_I}{\partial x_k \partial x_j} \tilde{dx}_k \wedge \tilde{dx}_j \wedge \tilde{dx}_I \tag{4.20}$$

With this we can rewrite Equation (4.19) as

$$d^2\alpha = \sum_I \sum_{k=1}^{n} \sum_{j=1}^{k-1} \frac{\partial f_I}{\partial x_k \partial x_j} \tilde{dx}_k \wedge \tilde{dx}_j \wedge \tilde{dx}_I + \sum_I \sum_{j=1}^{n} \sum_{k=1}^{j-1} \frac{\partial f_I}{\partial x_k \partial x_j} \tilde{dx}_k \wedge \tilde{dx}_j \wedge \tilde{dx}_I \tag{4.21}$$

In order to proceed, we can notice that the two summations above are essentially the same, just with the indices switched. So we can switch the indices of the second summation to obtain

$$d^2\alpha = \sum_I \sum_{k=1}^{n} \sum_{j=1}^{k-1} \frac{\partial f_I}{\partial x_k \partial x_j} \tilde{dx}_k \wedge \tilde{dx}_j \wedge \tilde{dx}_I + \sum_I \sum_{k=1}^{n} \sum_{j=1}^{k-1} \frac{\partial f_I}{\partial x_j \partial x_k} \tilde{dx}_j \wedge \tilde{dx}_k \wedge \tilde{dx}_I \tag{4.22}$$

We can group the two summations together by interchanging the positions of $\tilde{dx}_j$ and $\tilde{dx}_k$ in the second summation. We just pick up a minus sign with this interchange. All in all we obtain

$$d^2\alpha = \sum_I \sum_{k=1}^{n} \sum_{j=1}^{k-1} \left(\frac{\partial f_I}{\partial x_k \partial x_j} - \frac{\partial f_I}{\partial x_j \partial x_k} \right) \tilde{dx}_k \wedge \tilde{dx}_j \wedge \tilde{dx}_I \tag{4.23}$$

Now, since the functions f_I are defined to be smooth, it follows from Clairaut's Theorem that the term in the parentheses is zero because mixed partial derivatives commute. This proves Equation (4.16).

4.3 Closed and Exact Forms

With the exterior derivative defined, we can make two more definitions [4]. A differential m-form α is defined to be *closed* if $d\alpha = 0$. A differential m-form α is defined to be *exact* if there exists a differential $m - 1$-form β such that $\alpha = d\beta$. As an example, consider the differential one-form $\alpha = y\tilde{dx} + x\tilde{dy}$. This form is exact. Consider the function $f(x, y) = xy$. Taking the exterior derivative of f yields

$$df = d(xy) = \frac{\partial f}{\partial x}\tilde{dx} + \frac{\partial f}{\partial y}\tilde{dy} = y\tilde{dx} + x\tilde{dy} \tag{4.24}$$

The form α is also closed because

$$\begin{aligned} d\alpha &= \frac{\partial y}{\partial x}\tilde{dx} \wedge \tilde{dx} + \frac{\partial y}{\partial y}\tilde{dy} \wedge \tilde{dx} + \frac{\partial x}{\partial x}\tilde{dx} \wedge \tilde{dy} + \frac{\partial x}{\partial y}\tilde{dy} \wedge \tilde{dy} \\ &= \tilde{dy} \wedge \tilde{dx} + \tilde{dx} \wedge \tilde{dy} = 0 \end{aligned} \tag{4.25}$$

A question we can ask ourselves is if there's a relationship between a closed and an exact form. There in fact is one: Every exact form is closed. This is very easy to prove. If α is exact then

$$\alpha = d\beta \implies d\alpha = d^2\beta = 0 \tag{4.26}$$

Is every closed form exact? The answer depends strongly on the topology (the qualitative "shape") of the space where the forms are defined. For $\mathbb{R}^n$, it turns out that if we measure distances between two points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ using the Euclidean distance function

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2} \tag{4.27}$$

then, $\mathbb{R}^n$ has a very "nice" topology that is induced onto it. With this induced topology, it can be shown that every closed form on $\mathbb{R}^n$ is exact. This is called the *Poincaré lemma* [4, 2].

5 The Hodge Star Operator

5.1 Introducing the Hodge Start Operator

We know that the dimension of $\bigwedge^m \mathbb{R}^n$, the space of m-forms on $\mathbb{R}^n$ is $\binom{n}{m}$. However, a well known result is that $\binom{n}{m} = \binom{n}{n-m}$. This implies that the dimension of $\bigwedge^m \mathbb{R}^n$ is equal to the dimension of $\bigwedge^{n-m} \mathbb{R}^n$. An obvious question we can ask ourselves is if there's a much deeper relationship between these two spaces, aside from their dimension being the same. It turns out that there is indeed a relationship, which is called *Hodge duality* [4, 3]. In order to see this, we need to define what's called the *Hodge star operator*. This is a function $* : \bigwedge^m \mathbb{R}^n \to \bigwedge^{n-m} \mathbb{R}^n$ that acts on the basis one-forms in the following way

$$*(\tilde{dx}_I) = \tilde{dx}_J \qquad \text{such that} \qquad \tilde{dx}_I \wedge \tilde{dx}_J = \tilde{dV}_n \tag{5.1}$$

where $\tilde{dx}_I$ is an m-form, $\tilde{dx}_J$ is an $n-m$-form and $\tilde{dV}_n = \tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \ldots \wedge \tilde{dx}_n$ is the volume form. We can think of $\tilde{dx}_J$ as a sort of "missing part" that is needed to be wedged to $\tilde{dx}_I$ in order to obtain $\tilde{dV}_n$.
Let's look at $\mathbb{R}^2$ to get an idea of how the Hodge star operator works. The volume form on $\mathbb{R}^2$ is simply $\tilde{dx} \wedge \tilde{dy}$. On $\mathbb{R}^2$ we can have zero-forms, one-forms and two-forms. From the definition of the Hodge star operator, we already know that it's going to map a zero-form to a two-form, a one-form to a one-form and a two-form to a zero form. Let's look at differential zero-forms first. These are just spanned by the number 1. So we have a very simple result

$$*1 = \tilde{dx} \wedge \tilde{dy} \tag{5.2}$$

Now let's look at one-forms. The basis for differential one-forms on $\mathbb{R}^2$ is just $\{\tilde{dx}, \tilde{dy}\}$. So we need to look at each basis element separately. If we consider $\tilde{dx}$ first, we have that

$$*(\tilde{dx}) = \tilde{dy} \tag{5.3}$$

because $\tilde{dx} \wedge \tilde{dy} =$ is the volume form. Now if we consider $\tilde{dy}$ we see that

$$*(\tilde{dy}) = -\tilde{dx} \tag{5.4}$$

because $\tilde{dy} \wedge (-\tilde{dx}) = \tilde{dx} \wedge \tilde{dy}$. Now if we consider differential two-forms on $\mathbb{R}^2$, they are spanned by the two-form $\tilde{dx} \wedge \tilde{dy}$. Since this is already the volume form on $\mathbb{R}^2$ we again have a very simple result

$$*(\tilde{dx} \wedge \tilde{dy}) = 1 \tag{5.5}$$

More generally, if we have a differential form $\alpha = \sum_I \tilde{dx}_I$ then

$$*\alpha = \sum_I f_I(*\tilde{dx}_I) \tag{5.6}$$

where $*\tilde{dx}_I$ is determined using Equation (5.1). When we looked at the example of $\mathbb{R}^2$ we saw that $*(\tilde{dx} \wedge \tilde{dy}) = 1$. This result is general in $\mathbb{R}^n$ i.e. $*d\tilde{V}_n = 1$. Similarly $*1 = d\tilde{V}_n$. The common nomenclature is to call $*\alpha$ the *Hodge dual* of α.

Let's look at another example. Let ω be a differential two-form on $\mathbb{R}^5$ given by $\omega = x_1^2\tilde{dx}_1 \wedge \tilde{dx}_2 + 2x_1x_3\tilde{dx}_3 \wedge \tilde{dx}_4 + 7x_4\tilde{dx}_1 \wedge \tilde{dx}_5$. Then we already know that $*\omega$ is going to be a differential three-form. Using Equation (5.6)

$$*\omega = x_1^2(*(\tilde{dx}_1 \wedge \tilde{dx}_2)) + 2x_1x_3(*(\tilde{dx}_3 \wedge \tilde{dx}_4)) + 7x_4(*(\tilde{dx}_1 \wedge \tilde{dx}_5)) \tag{5.7}$$

In order to evaluate, $*(\tilde{dx}_1 \wedge \tilde{dx}_2)$ we already know beforehand that the result will be of the form $\tilde{dx}_3 \wedge \tilde{dx}_4 \wedge \tilde{dx}_5$. The only thing we need to fix is the sign, which we can determine by looking at how many times we would need to move things around to obtain the volume form. In this case, it's pretty straightforward because wedging $\tilde{dx}_1 \wedge \tilde{dx}_2$ with $\tilde{dx}_3 \wedge \tilde{dx}_4 \wedge \tilde{dx}_5$ gives us the volume form with no shifting required. So we have that

$$*(\tilde{dx}_1 \wedge \tilde{dx}_2) = \tilde{dx}_3 \wedge \tilde{dx}_4 \wedge \tilde{dx}_5 \tag{5.8}$$

We evaluate $*(\tilde{dx}_3 \wedge \tilde{dx}_4)$ in a similar way. Already we know that the result is going to be of the form $\tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \tilde{dx}_5$. If we consider $\tilde{dx}_3 \wedge \tilde{dx}_4 \wedge \tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \tilde{dx}_5$, we can see that we need would 4 shifts in total (2 shifts to move the $\tilde{dx}_4$ and 2 more to shift the $\tilde{dx}_3$). So we obtain

$$*(\tilde{dx}_3 \wedge \tilde{dx}_4) = \tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \tilde{dx}_5 \tag{5.9}$$

Evaluating $*(\tilde{dx}_1 \wedge \tilde{dx}_5)$ similarly, we obtain

$$*(\tilde{dx}_1 \wedge \tilde{dx}_5) = -\tilde{dx}_2 \wedge \tilde{dx}_3 \wedge \tilde{dx}_4 \tag{5.10}$$

All in all, what we get is

$$*\omega = x_1^2\tilde{dx}_3 \wedge \tilde{dx}_4 \wedge \tilde{dx}_5 + 2x_1x_3\tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \tilde{dx}_5 - 7x_4\tilde{dx}_2 \wedge \tilde{dx}_3 \wedge \tilde{dx}_4 \tag{5.11}$$

5.2 Re-writing the Gradient, Divergence and Curl

With the exterior derivative and Hodge star operator now defined, we can now use them to rewrite the gradient, divergence and curl in terms of differential forms as well [3, 4]. For this, we will need to make the following identification (by identification, I essentially mean a mapping) between a vector field in $\mathbb{R}^3$ and a differential one-form on $\mathbb{R}^3$

$$\vec{F} = \langle P, Q, R\rangle \to \tilde{\omega} = P\tilde{dx} + Q\tilde{dy} + R\tilde{dz} \tag{5.12}$$

5.2.1 Rewriting the Gradient

Recall that the gradient of a function $f : \mathbb{R}^3 \to \mathbb{R}$ was defined in the following way

$$\text{grad}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \tag{5.13}$$

If we think of f as a zero-form, we can apply the exterior derivative operator on it to obtain

$$df = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \tilde{dx_i} = \frac{\partial f}{\partial x}\tilde{dx} + \frac{\partial f}{\partial y}\tilde{dy} + \frac{\partial f}{\partial z}\tilde{dz} \tag{5.14}$$

Now with the identification in Equation (5.12) it is easy to see that

$$\text{grad}(f) = df \tag{5.15}$$

5.2.2 Rewriting the Divergence

The divergence of a vector field $\vec{F} = \langle P, Q, R \rangle$ was defined in the following way

$$\text{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{5.16}$$

I now claim that with the identification defined in Equation (5.12)

$$\text{div}(\vec{F}) = *d*\omega \tag{5.17}$$

In order to prove this, we start by explicitly determining $*\omega$.

$$\begin{aligned} *\omega &= *(P\tilde{dx} + Q\tilde{dy} + R\tilde{dz}) \\ &= P(*\tilde{dx}) + Q(*\tilde{dy}) + R(*\tilde{dz}) \\ &= P\tilde{dy} \wedge \tilde{dz} - Q\tilde{dx} \wedge \tilde{dz} + R\tilde{dx} \wedge \tilde{dy} \end{aligned} \tag{5.18}$$

We now apply the exterior derivative onto the expression in the last line of Equation (5.18)

$$\begin{aligned} d(*\omega) &= d(P\tilde{dy} \wedge \tilde{dz} - Q\tilde{dx} \wedge \tilde{dz} + R\tilde{dx} \wedge \tilde{dy}) \\ &= \frac{\partial P}{\partial x}\tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} - \frac{\partial Q}{\partial y}\tilde{dy} \wedge \tilde{dx} \wedge \tilde{dz} + \frac{\partial R}{\partial z}\tilde{dz} \wedge \tilde{dx} \wedge \tilde{dy} \\ &= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} \end{aligned} \tag{5.19}$$

In the above manipulation, we have implicitly ignored terms that would have resulted in the wedge product of the same one-form twice or thrice, because those terms would just be zero. All that's left now is to apply the Hodge star operator on the last expression in Equation (5.19). This is going to be a trivial result however, because $d(*\omega)$ is a three form and we're in $\mathbb{R}^3$, which means

$$*(d*\omega) = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) * (\tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{5.20}$$

In the above manipulation we have used the fact that $*(\tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz}) = 1$ because $\tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz}$ is the volume form on $\mathbb{R}^3$. Hence, we have proven Equation (5.17).

5.2.3 Rewriting the Curl

The curl of a vector field $\vec{F} = \langle P, Q, R \rangle$ is defined as

$$\operatorname{curl}(\vec{F}) = \left\langle \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right), \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right), \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right\rangle \tag{5.21}$$

Using the identification in Equation (5.12), I claim that

$$\operatorname{curl}(\vec{F}) = *d\omega \tag{5.22}$$

In order to prove this, we first compute $d\omega$.

$$\begin{aligned} d\omega &= d(P\tilde{dx} + Q\tilde{dy} + R\tilde{dz}) \\ &= \frac{\partial P}{\partial y}\tilde{dy} \wedge \tilde{dx} + \frac{\partial P}{\partial z}\tilde{dz} \wedge \tilde{dx} + \frac{\partial Q}{\partial x}\tilde{dx} \wedge \tilde{dy} + \frac{\partial Q}{\partial z}\tilde{dz} \wedge \tilde{dy} \\ &\quad + \frac{\partial R}{\partial x}\tilde{dx} \wedge \tilde{dz} + \frac{\partial R}{\partial y}\tilde{dy} \wedge \tilde{dz} \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \tilde{dx} \wedge \tilde{dy} + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \tilde{dx} \wedge \tilde{dz} \\ &\quad + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \tilde{dy} \wedge \tilde{dz} \end{aligned} \tag{5.23}$$

In the above manipulation, we have again implicitly ignored terms that would have resulted in the wedge product of the same one-form twice. Applying the Hodge star operator on $d\omega$ we obtain

$$\begin{aligned} *d\omega &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) * (\tilde{dx} \wedge \tilde{dy}) + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) * (\tilde{dx} \wedge \tilde{dz}) \\ &\quad + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) * (\tilde{dy} \wedge \tilde{dz}) \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \tilde{dz} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \tilde{dy} + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \tilde{dx} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \tilde{dx} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \tilde{dy} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \tilde{dz} \end{aligned} \tag{5.24}$$

Upon comparing Equations (5.21) and the last term in (5.24) with the identification in Equation (5.12), we have proven Equation (5.22).

6 Rewriting Maxwell's Equations

We now proceed to work towards rewriting Maxwell's Equations in the language of differential forms [3, 2]. Unfortunately, we can't do that just yet and need to do some more work before getting there. The first thing we need to do is generalise the Hodge star operator in a way that's going to be helpful for us later on.

6.1 Introducing Bilinear Forms

The current definition for the Hodge star operator $* : \bigwedge^m \mathbb{R}^n \to \bigwedge^{n-m} \mathbb{R}^n$ is given by Equations (5.6) and (5.1). A better definition for the Hodge start operator, however, can be formulated by introducing the concept of a *bilinear form*. This is a function $\langle \cdot, \cdot \rangle : \bigwedge^m \mathbb{R}^n \times \bigwedge^m \mathbb{R}^n \to \mathbb{R}$

that takes in two m-forms and returns a number. In order to see how we can define a bi-linear form in an efficient way, we're going to consider a simple case first. Let's first consider $\bigwedge^1 \mathbb{R}^2$ i.e. one-forms on $\mathbb{R}^2$. The basis for this space is the set $\{\tilde{dx}, \tilde{dy}\}$. The most general bilinear form we can consider in this case is

$$\langle a\tilde{dx} + b\tilde{dy}, c\tilde{dx} + d\tilde{dy}\rangle \tag{6.1}$$

which we can break up using the fact that the bilinear form is, well, bilinear into

$$ac\langle \tilde{dx}, \tilde{dx}\rangle + ad\langle \tilde{dx}, \tilde{dy}\rangle + bc\langle \tilde{dy}, \tilde{dx}\rangle + bd\langle \tilde{dy}, \tilde{dy}\rangle \tag{6.2}$$

We can write Equation (6.2) in terms of matrix multiplication

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} \langle \tilde{dx}, \tilde{dx}\rangle & \langle \tilde{dx}, \tilde{dy}\rangle \\ \langle \tilde{dy}, \tilde{dx}\rangle & \langle \tilde{dy}, \tilde{dy}\rangle \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \tag{6.3}$$

The upshot here is that the bilinear form and the two-by-two matrix shown in Equation (6.3) are equivalent. The matrix encodes everything we need to know about the bilinear form. Furthermore, this idea can be easily generalised. If we consider $\bigwedge^1 \mathbb{R}^n$ i.e. one-forms on $\mathbb{R}^n$ then the bilinear form is given by the n-by$n$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \tag{6.4}$$

where $a_{ij} = \langle \tilde{dx}_i, \tilde{dx}_j\rangle$. What we can now ask is if there is a way to "lift" a bilinear form defined on one-forms into a bilinear form defined on more general m-forms. It turns out that there is a way to do this, although it can't really be motivated all that well. It also involves working with the symmetric group S_n. A discussion of this group is provided in Sjamaar. The way a bilinear form defined on one-forms is generalised is through the formula

$$\langle \tilde{dx}_I, \tilde{dx}_J\rangle = \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \langle \tilde{dx}_{i_1}, \tilde{dx}_{j_{\sigma(1)}}\rangle \langle \tilde{dx}_{i_2}, \tilde{dx}_{j_{\sigma(2)}}\rangle \dots \langle \tilde{dx}_{i_m}, \tilde{dx}_{j_{\sigma(m)}}\rangle \tag{6.5}$$

Equation (6.5) is a lot to take in. So it's going to be very fruitful to look at an example to see how the formula works.

Suppose we're working in $\mathbb{R}^4$. A bilinear form is defined by the following matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 0 & -2 \\ 3 & 0 & 1 & 2 \\ 4 & -2 & 2 & 3 \end{pmatrix} \tag{6.6}$$

We wish to calculate this bilinear form evaluated on the three-forms $\alpha = \tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \tilde{dx}_3$ and $\beta = \tilde{dx}_2 \wedge \tilde{dx}_3 \wedge \tilde{dx}_4$ i.e. we want to compute $\langle \tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \tilde{dx}_3, \tilde{dx}_2 \wedge \tilde{dx}_3 \wedge \tilde{dx}_4\rangle$. Since we're considering three-forms, the relevant group is S_3 which comprises of the elements $\{(1), (12), (13), (23), (123), (132)\}$. It's also going to be helpful to write the multi-indices of each three-form beforehand as well. For α the multi-index is $I = (1, 2, 3)$ and for β the multi-index is $J = (2, 3, 4)$. Since S_3 has six elements, we are going to have six terms to sum over in this scenario.

For this problem Equation (6.5) simplifies to

$$\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \langle \tilde{dx}_{i_1}, \tilde{dx}_{j_{\sigma(1)}}\rangle \langle \tilde{dx}_{i_2}, \tilde{dx}_{j_{\sigma(2)}}\rangle \langle \tilde{dx}_{i_3}, \tilde{dx}_{j_{\sigma(3)}}\rangle \tag{6.7}$$

We can now begin to sum over the elements of S_3. The first element is the permutation (1), under which $\sigma(1) = 1$, $\sigma(2) = 2$ and $\sigma(3) = 3$. Furthermore sgn(1) = 1. So the first term is just $\langle \tilde{dx}_{i_1}, \tilde{dx}_{j_1}\rangle\langle \tilde{dx}_{i_2}, \tilde{dx}_{j_2}\rangle\langle \tilde{dx}_{i_3}, \tilde{dx}_{j_3}\rangle = \langle \tilde{dx}_1, \tilde{dx}_2\rangle\langle \tilde{dx}_2, \tilde{dx}_3\rangle\langle \tilde{dx}_3, \tilde{dx}_4\rangle = 0$ where we just look at the relevant components of the matrix and multiply them together.
The second term in the summation involves the permutation (12), under which $\sigma(1) = 2$, $\sigma(2) = 1$ and $\sigma(3) = 3$. Furthermore, sgn(12) $= -1$. So the second term is the product $-\langle \tilde{dx}_{i_1}, \tilde{dx}_{j_2}\rangle\langle \tilde{dx}_{i_2}, \tilde{dx}_{j_1}\rangle\langle \tilde{dx}_{i_3}, \tilde{dx}_{j_3}\rangle = -\langle \tilde{dx}_1, \tilde{dx}_3\rangle\langle \tilde{dx}_2, \tilde{dx}_2\rangle\langle \tilde{dx}_3, \tilde{dx}_4\rangle = 6$.
The third term in the summation involves the permutation (13), under which $\sigma(1) = 3$, $\sigma(2) = 2$ and $\sigma(3) = 1$. Furthermore, sgn(13) $= -1$. So the third term is the product $-\langle \tilde{dx}_{i_1}, \tilde{dx}_{j_3}\rangle\langle \tilde{dx}_{i_2}, \tilde{dx}_{j_2}\rangle\langle \tilde{dx}_{i_3}, \tilde{dx}_{j_1}\rangle = -\langle \tilde{dx}_1, \tilde{dx}_4\rangle\langle \tilde{dx}_2, \tilde{dx}_3\rangle\langle \tilde{dx}_3, \tilde{dx}_2\rangle = 0$.
The fourth term in the summation involves the permutation (23), under which $\sigma(1) = 1$, $\sigma(2) = 3$ and $\sigma(3) = 2$. Furthermore, sgn(23) $= -1$. So the fourth term is the product $-\langle \tilde{dx}_{i_1}, \tilde{dx}_{j_1}\rangle\langle \tilde{dx}_{i_2}, \tilde{dx}_{j_3}\rangle\langle \tilde{dx}_{i_3}, \tilde{dx}_{j_2}\rangle = -\langle \tilde{dx}_1, \tilde{dx}_2\rangle\langle \tilde{dx}_2, \tilde{dx}_4\rangle\langle \tilde{dx}_3, \tilde{dx}_3\rangle = 4$.
The fifth term in the summation involves the permutation (123), under which $\sigma(1) = 2$, $\sigma(2) = 3$ and $\sigma(3) = 1$. Furthermore, sgn(123) = 1. So the fifth term is the product $\langle \tilde{dx}_{i_1}, \tilde{dx}_{j_2}\rangle\langle \tilde{dx}_{i_2}, \tilde{dx}_{j_3}\rangle\langle \tilde{dx}_{i_3}, \tilde{dx}_{j_1}\rangle = -\langle \tilde{dx}_1, \tilde{dx}_3\rangle\langle \tilde{dx}_2, \tilde{dx}_4\rangle\langle \tilde{dx}_3, \tilde{dx}_2\rangle = 0$.
The sixth term in the summation involves the permutation (132), under which $\sigma(1) = 3$, $\sigma(2) = 1$ and $\sigma(3) = 2$. Furthermore, sgn(132) = 1. So the sixth term is the product $\langle \tilde{dx}_{i_1}, \tilde{dx}_{j_3}\rangle\langle \tilde{dx}_{i_2}, \tilde{dx}_{j_1}\rangle\langle \tilde{dx}_{i_3}, \tilde{dx}_{j_2}\rangle = -\langle \tilde{dx}_1, \tilde{dx}_4\rangle\langle \tilde{dx}_2, \tilde{dx}_2\rangle\langle \tilde{dx}_3, \tilde{dx}_3\rangle = -4$.
Finally, we've evaluated all six terms in the summation. All that's left is adding them up which yields $\langle \alpha, \beta\rangle = 6$

6.2 Generalising the Hodge Star Operator

Hopefully, we now have a good idea about how bilinear forms work. It's now time to generalise the Hodge star operator. We define $*\alpha$ to be the unique $n-m$-form such that for all $\beta \in \bigwedge^m \mathbb{R}^n$

$$\beta \wedge (*\alpha) = \langle \beta, \alpha\rangle \tilde{dx}_1 \wedge \tilde{dx}_2 \wedge \ldots \wedge \tilde{dx}_n \tag{6.8}$$

where the bilinear form that is going to be utilised is given by

$$\langle \tilde{dx}_I, \tilde{dx}_J\rangle = \begin{cases} 1 & \text{if} \quad I = J \\ 0 & \text{if} \quad I \neq J \end{cases} \tag{6.9}$$

where $I = J$ means that $i_1 = j_1, i_2 = j_2$ and so on. Equations (6.8) and (6.9) essentially imply the definition of the Hodge star operator as highlighted in Equations (5.1) and (5.6). Let's look at a simple example to showcase this. Let's look at one-forms on $\mathbb{R}^3$. The basis for this space is $\{\tilde{dx}, \tilde{dy}, \tilde{dz}\}$. We're going to compute $*\tilde{dx}$ using the new definition and replicate the same result as we would have gotten if we used Equations (5.1) and (5.6). We know that $*\tilde{dx}$ is going to be a two-form. We can thus write it generally as

$$*\tilde{dx} = A\tilde{dx} \wedge \tilde{dy} + B\tilde{dx} \wedge \tilde{dz} + C\tilde{dy} \wedge \tilde{dz} \tag{6.10}$$

If we wedge both sides with $\tilde{dx}$ we get

$$\tilde{dx} \wedge *\tilde{dx} = C\tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} \tag{6.11}$$

However, if we perform the same wedge product using Equations (6.8) and (6.9) we get

$$\tilde{dx} \wedge *\tilde{dx} = \langle \tilde{dx}, \tilde{dx}\rangle \tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} = \tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} \tag{6.12}$$

Equations (6.12) and (6.11) imply that $C = 1$. We can similarly solve for A and B by wedging with $\tilde{dy}$ and $\tilde{dz}$. But these will simply show that $A = B = 0$ since $\langle \tilde{dy}, \tilde{dx}\rangle = \langle \tilde{dz}, \tilde{dx}\rangle = 0$

according to the bilinear form defined in Equation (6.9). So we obtain the result $*\tilde{dx} = \tilde{dx} \wedge \tilde{dy}$, the same result we would have gotten if we used Equations (5.1) and (5.6).
However, Equation (6.9) is not the only bilinear form we can define. Equations (6.4) and (6.5) allow us to define any arbitrary bilinear form on $\bigwedge^m \mathbb{R}^n$ and Equation (6.8) defines the Hodge star operator under that bilinear form. So we have indeed generalised the Hodge star operator.

6.3 Introducing the Minkowski Inner Product

In order to recast Maxwell's Equations using the language of differential forms, we need to define a particular bilinear form on the one-forms in $\mathbb{R}^4$. This bilinear form is called the *Minkowski Inner Product*. We first define $\tilde{dx_1} = \tilde{dt}$, $\tilde{dx_2} = \tilde{dx}$, $\tilde{dx_3} = \tilde{dy}$ and $\tilde{dx_4} = \tilde{dz}$. The Minkowski Inner Product is then defined by the following set of bilinear forms amongst the elementary one-forms

$$\begin{aligned} \langle \tilde{dt}, \tilde{dt} \rangle &= -1 \\ \langle \tilde{dx}, \tilde{dx} \rangle &= 1 \\ \langle \tilde{dy}, \tilde{dy} \rangle &= 1 \\ \langle \tilde{dz}, \tilde{dz} \rangle &= 1 \end{aligned} \tag{6.13}$$

where all of the other inner products between the elementary one-forms are just zero. The Minkowski Inner Product can be written as the following four-by-four matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.14}$$

In order to proceed further, a helpful thing which will come in handy later is to define a bilinear form on two-forms using the Minkowski Inner Product. This is going to involve the "lifting" scheme governed by Equation (6.5). Let's first identify the basis two-forms we're going to be working with. For this we just need to choose any two of the four one-forms from the set $\{\tilde{dt}, \tilde{dx}, \tilde{dy}, \tilde{dz}\}$. This yields the set $\{\tilde{dt} \wedge \tilde{dx}, \tilde{dt} \wedge \tilde{dy}, \tilde{dt} \wedge \tilde{dz}, \tilde{dx} \wedge \tilde{dy}, \tilde{dx} \wedge \tilde{dz}, \tilde{dy} \wedge \tilde{dz}\}$. Now, we need to take the inner product of all six of these two-forms with themselves and with each other i.e. we need to compute 36 inner products. If we apply Equation (6.5) for this scenario we obtain the following equation

$$\sum_{\sigma \in S_2} \operatorname{sgn}(\sigma) \langle \tilde{dx_{i_1}}, \tilde{dx_{j_{\sigma(1)}}} \rangle \langle \tilde{dx_{i_2}}, \tilde{dx_{j_{\sigma(2)}}} \rangle \tag{6.15}$$

where $S_2 = \{(1), (12)\}$. Furthermore, $\operatorname{sgn}(1) = 1$ and $\operatorname{sgn}(12) = -1$.
Let's start with computing $\langle \tilde{dt} \wedge \tilde{dx}, \tilde{dt} \wedge \tilde{dx} \rangle$.

$$\begin{aligned} \langle \tilde{dt} \wedge \tilde{dx}, \tilde{dt} \wedge \tilde{dx} \rangle &= \operatorname{sgn}(1) \langle \tilde{dx_{i_1}}, \tilde{dx_{j_1}} \rangle \langle \tilde{dx_{i_2}}, \tilde{dx_{j_2}} \rangle + \operatorname{sgn}(12) \langle \tilde{dx_{i_1}}, \tilde{dx_{j_2}} \rangle \langle \tilde{dx_{i_2}}, \tilde{dx_{j_1}} \rangle \\ &= \langle \tilde{dt}, \tilde{dt} \rangle \langle \tilde{dx}, \tilde{dx} \rangle - \langle \tilde{dt}, \tilde{dx} \rangle \langle \tilde{dx}, \tilde{dt} \rangle \\ &= -1 \end{aligned} \tag{6.16}$$

In a very similar way, we can show that $\langle \tilde{dt} \wedge \tilde{dy}, \tilde{dt} \wedge \tilde{dy} \rangle = \langle \tilde{dt} \wedge \tilde{dz}, \tilde{dt} \wedge \tilde{dz} \rangle = -1$.
Let's now consider $\langle \tilde{dt} \wedge \tilde{dx}, \tilde{dt} \wedge \tilde{dy} \rangle$.

$$\begin{aligned} \langle \tilde{dt} \wedge \tilde{dx}, \tilde{dt} \wedge \tilde{dy} \rangle &= \operatorname{sgn}(1) \langle \tilde{dx_{i_1}}, \tilde{dx_{j_1}} \rangle \langle \tilde{dx_{i_2}}, \tilde{dx_{j_2}} \rangle + \operatorname{sgn}(12) \langle \tilde{dx_{i_1}}, \tilde{dx_{j_2}} \rangle \langle \tilde{dx_{i_2}}, \tilde{dx_{j_1}} \rangle \\ &= \langle \tilde{dt}, \tilde{dt} \rangle \langle \tilde{dx}, \tilde{dy} \rangle - \langle \tilde{dt}, \tilde{dy} \rangle \langle \tilde{dx}, \tilde{dt} \rangle \\ &= 0 \end{aligned} \tag{6.17}$$

Similarly, we can show that $\langle \tilde{dt} \wedge \tilde{dx}, \tilde{dt} \wedge \tilde{dz} \rangle = 0$.
Let's now look at $\langle \tilde{dx} \wedge \tilde{dy}, \tilde{dx} \wedge \tilde{dy} \rangle$.

$$\begin{aligned} \langle \tilde{dx} \wedge \tilde{dy}, \tilde{dx} \wedge \tilde{dy} \rangle &= \operatorname{sgn}(1)\langle \tilde{dx_{i_1}}, \tilde{dx_{j_1}} \rangle \langle \tilde{dx_{i_2}}, \tilde{dx_{j_2}} \rangle + \operatorname{sgn}(12)\langle \tilde{dx_{i_1}}, \tilde{dx_{j_2}} \rangle \langle \tilde{dx_{i_2}}, \tilde{dx_{j_1}} \rangle \\ &= \langle \tilde{dx}, \tilde{dx} \rangle \langle \tilde{dy}, \tilde{dy} \rangle - \langle \tilde{dx}, \tilde{dy} \rangle \langle \tilde{dy}, \tilde{dx} \rangle \\ &= 1 \end{aligned} \tag{6.18}$$

In a similar fashion, we can show that $\langle \tilde{dx} \wedge \tilde{dz}, \tilde{dx} \wedge \tilde{dz} \rangle = \langle \tilde{dy} \wedge \tilde{dz}, \tilde{dy} \wedge \tilde{dz} \rangle = 1$.
Let's now look at $\langle \tilde{dx} \wedge \tilde{dy}, \tilde{dx} \wedge \tilde{dz} \rangle$.

$$\begin{aligned} \langle \tilde{dx} \wedge \tilde{dy}, \tilde{dx} \wedge \tilde{dz} \rangle = &= \operatorname{sgn}(1)\langle \tilde{dx_{i_1}}, \tilde{dx_{j_1}} \rangle \langle \tilde{dx_{i_2}}, \tilde{dx_{j_2}} \rangle + \operatorname{sgn}(12)\langle \tilde{dx_{i_1}}, \tilde{dx_{j_2}} \rangle \langle \tilde{dx_{i_2}}, \tilde{dx_{j_1}} \rangle \\ &= \langle \tilde{dx}, \tilde{dx} \rangle \langle \tilde{dy}, \tilde{dz} \rangle - \langle \tilde{dx}, \tilde{dz} \rangle \langle \tilde{dy}, \tilde{dx} \rangle \\ &= 0 \end{aligned} \tag{6.19}$$

From these computations, we can note some general patterns so that we don't need to compute all 36 inner products. The general picture is that if $\tilde{dt}$ is being wedged with either one of $\tilde{dx}, \tilde{dy}$ or $\tilde{dz}$ and we compute the bilinear form of the resultant two-form with itself, we get -1. Furthermore, if we wedge any two of $\tilde{dx}$, $\tilde{dy}$ or $\tilde{dz}$ with one another and compute the bilinear form of the resultant two-form with itself, we get 1. Furthermore, if $\tilde{dt}$ is wedged with two different forms from $\tilde{dx}, \tilde{dy}$ or, $\tilde{z}$ and the bilinear form between those two-forms is computed, we end up with zero. Similarly, if $\tilde{dx}, \tilde{dy}$ and $\tilde{dz}$ are mixed in the bilinear form, we get zero. All in all, we just end up with 6 inner products that are non-zero. These are

$$\begin{aligned} \langle \tilde{dt} \wedge \tilde{dx}, \tilde{dt} \wedge \tilde{dx} \rangle &= -1 \\ \langle \tilde{dt} \wedge \tilde{dy}, \tilde{dt} \wedge \tilde{dy} \rangle &= -1 \\ \langle \tilde{dt} \wedge \tilde{dz}, \tilde{dt} \wedge \tilde{dz} \rangle &= -1 \\ \langle \tilde{dx} \wedge \tilde{dy}, \tilde{dx} \wedge \tilde{dy} \rangle &= 1 \\ \langle \tilde{dx} \wedge \tilde{dz}, \tilde{dx} \wedge \tilde{dz} \rangle &= 1 \\ \langle \tilde{dy} \wedge \tilde{dz}, \tilde{dy} \wedge \tilde{dz} \rangle &= 1 \end{aligned} \tag{6.20}$$

We now move on towards defining a bilinear form on three-forms in $\mathbb{R}^4$ using the Minkowski Inner Product. The basis set for three-forms is $\{\tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy}, \tilde{dt} \wedge \tilde{dx} \wedge \tilde{dz}, \tilde{dt} \wedge \tilde{dy} \wedge \tilde{dz}, \tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz}\}$ so we need to compute 16 inner products. However, we are going to proceed similarly as before, where we compute a few and notice some patterns which we use to write all of the inner products. Applying Equation (6.5) for this case we have the following formula

$$\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \langle \tilde{dx_{i_1}}, \tilde{dx_{j_{\sigma(1)}}} \rangle \langle \tilde{dx_{i_2}}, \tilde{dx_{j_{\sigma(2)}}} \rangle \langle \tilde{dx_{i_3}}, \tilde{dx_{j_{\sigma(3)}}} \rangle \tag{6.21}$$

where $S_3 = \{(1), (12), (13), (23), (123), (132)\}$. Furthermore, $\operatorname{sgn}(1) = 1$, $\operatorname{sgn}(12) = -1$, $\operatorname{sgn}(13) = -1$, $\operatorname{sgn}(23) = -1$, $\operatorname{sgn}(123) = 1$ and $\operatorname{sgn}(132) = 1$.
Let's consider $\langle \tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz}, \tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} \rangle$.

$$\begin{aligned} \langle \tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz}, \tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} \rangle &= \langle \tilde{dx}, \tilde{dx} \rangle \langle \tilde{dy}, \tilde{dy} \rangle \langle \tilde{dz}, \tilde{dz} \rangle - \langle \tilde{dx}, \tilde{dy} \rangle \langle \tilde{dy}, \tilde{dx} \rangle \langle \tilde{dz}, \tilde{dz} \rangle \\ &\quad - \langle \tilde{dx}, \tilde{dz} \rangle \langle \tilde{dy}, \tilde{dy} \rangle \langle \tilde{dz}, \tilde{dx} \rangle - \langle \tilde{dx}, \tilde{dx} \rangle \langle \tilde{dy}, \tilde{dz} \rangle \langle \tilde{dz}, \tilde{dy} \rangle \\ &\quad + \langle \tilde{dx}, \tilde{dz} \rangle \langle \tilde{dy}, \tilde{dx} \rangle \langle \tilde{dz}, \tilde{dy} \rangle + \langle \tilde{dx}, \tilde{dy} \rangle \langle \tilde{dy}, \tilde{dz} \rangle \langle \tilde{dz}, \tilde{dx} \rangle \\ &= 1 \end{aligned} \tag{6.22}$$

From this one computation, the form of the Minkowski Inner Product, the previous results from determining the bilinear form on two-forms and some intuition, we can conclude that only 4 of the inner products we need to consider here are going to be non-zero, while the rest are going to be zero. The non-zero inner products in this case are

$$\begin{aligned}\langle \tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz}, \tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} \rangle &= 1 \\ \langle \tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy}, \tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy} \rangle &= -1 \\ \langle \tilde{dt} \wedge \tilde{dy} \wedge \tilde{dz}, \tilde{dt} \wedge \tilde{dy} \wedge \tilde{dz} \rangle &= -1 \\ \langle \tilde{dt} \wedge \tilde{dx} \wedge \tilde{dz}, \tilde{dt} \wedge \tilde{dx} \wedge \tilde{dz} \rangle &= -1\end{aligned} \tag{6.23}$$

Now that we have the bilinear form defined on two-forms and three-forms, our next goal is going to be to compute the Hodge star of the one-form, two-form basis elements and three-form basis elements. We've done most of the work already so this is going to be a straightforward computation.
Let's look at the Hodge duals of the one-form basis elements first. Let's compute $*\tilde{dt}$. This can be done by wedging it with $\tilde{dt}$. Technically, what we should do is expand $*\tilde{dt}$ in the basis of two-forms with some arbitrary constants, and wedge it with all four one-form basis elements to eliminate the constants. However, we have a huge simplification thanks to the Minkowski Inner Product, due to which we only need to wedge $*\tilde{dt}$ with $\tilde{dt}$ because all of the other constants will give us zero due to Equation (6.13).

$$\begin{aligned}\tilde{dt} \wedge *\tilde{dt} &= \langle \tilde{dt}, \tilde{dt} \rangle \tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} \\ &= -\tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} \\ &\implies *\tilde{dt} = -\tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz}\end{aligned} \tag{6.24}$$

In a very similar fashion, we can compute the Hodge duals of the other one-form basis elements as well. The final results we obtain are

$$\begin{aligned}*\tilde{dt} &= -\tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} \\ *\tilde{dx} &= -\tilde{dt} \wedge \tilde{dy} \wedge \tilde{dz} \\ *\tilde{dy} &= \tilde{dt} \wedge \tilde{dz} \wedge \tilde{dx} \\ *\tilde{dz} &= -\tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy}\end{aligned} \tag{6.25}$$

Now we have to deal with the two-form basis elements. However, we have a similar simplification in this case as well because of the general pattern that is demonstrated in Equation (6.20). As an example, let's consider finding $*(\tilde{dt} \wedge \tilde{dx})$. Its inner product with all other possible two-form basis elements is zero, and its inner product with itself is -1. So we simply need to wedge $*(\tilde{dt} \wedge \tilde{dx})$ with $\tilde{dt} \wedge \tilde{dx}$ to determine the Hodge dual. Computing this explicitly, we obtain

$$\begin{aligned}\tilde{dt} \wedge \tilde{dx} \wedge *(\tilde{dt} \wedge \tilde{dx}) &= \langle \tilde{dt} \wedge \tilde{dx}, \tilde{dt} \wedge \tilde{dx} \rangle \tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} \\ &= \tilde{dt} \wedge \tilde{dx} \wedge -(\tilde{dy} \wedge \tilde{dz}) \\ &\implies *(\tilde{dt} \wedge \tilde{dx}) = -\tilde{dy} \wedge \tilde{dz}\end{aligned} \tag{6.26}$$

In a very similar fashion, we can compute all 6 Hodge duals. The final results are shown below

$$\begin{aligned}
*(\tilde{dt} \wedge \tilde{dx}) &= -\tilde{dy} \wedge \tilde{dz} \\
*(\tilde{dt} \wedge \tilde{dy}) &= -\tilde{dz} \wedge \tilde{dx} \\
*(\tilde{dt} \wedge \tilde{dz}) &= -\tilde{dx} \wedge \tilde{dy} \\
*(\tilde{dx} \wedge \tilde{dy}) &= \tilde{dt} \wedge \tilde{dz} \\
*(\tilde{dy} \wedge \tilde{dz}) &= \tilde{dt} \wedge \tilde{dx} \\
*(\tilde{dz} \wedge \tilde{dx}) &= \tilde{dt} \wedge \tilde{dy}
\end{aligned} \tag{6.27}$$

Finally, we have the Hodge duals of the three-form basis elements. Here again, we have a simplification because the inner product of any three-form with any other three-form except itself is non-zero. As an example, let's consider finding $*(\tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy})$. Much like in the previous two computations, we simply need to wedge it with $\tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy}$ to obtain the Hodge dual. Doing this explicitly we obtain

$$\begin{aligned}
\tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy} \wedge *(\tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy}) &= \langle \tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy}, \tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy} \rangle \tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} \\
&= \tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy} \wedge (-\tilde{dz}) \\
&\implies *(\tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy}) = -\tilde{dz}
\end{aligned} \tag{6.28}$$

All 4 Hodge duals of the basis three-forms are given by

$$\begin{aligned}
*(\tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy}) &= -\tilde{dz} \\
*(\tilde{dt} \wedge \tilde{dz} \wedge \tilde{dx}) &= -\tilde{dy} \\
*(\tilde{dt} \wedge \tilde{dy} \wedge \tilde{dz}) &= -\tilde{dx} \\
*(\tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz}) &= -\tilde{dt}
\end{aligned} \tag{6.29}$$

6.4 The Grand Finale

In the words of Doctor Strange:

Let's first write down Maxwell's Equations. We'll be using units in which $\mu_o = \epsilon_o = 1$. In these units, Maxwell's Equations read

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho \qquad \text{Gauss's Law} \\ \vec{\nabla}\cdot\vec{B} &= 0 \qquad \text{Divergence of } \vec{B} \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \qquad \text{Faraday's Law} \\ \vec{\nabla}\times\vec{B} &= \vec{J} + \frac{\partial \vec{E}}{\partial t} \qquad \text{Ampere-Maxwell Law} \end{aligned} \tag{6.30}$$

where $\vec{E} = \langle E_1, E_2, E_3\rangle$ and $\vec{B} = \langle B_1, B_2, B_3\rangle$ are the electric and magnetic fields respectively, ρ is the charge density and $\vec{J} = \langle J_1, J_2, J_3\rangle$ is the current density vector. We first introduce the *Faraday two-form* $\tilde{F}$ given by

$$\tilde{F} = -E_1\tilde{dt}\wedge\tilde{dx} - E_2\tilde{dt}\wedge\tilde{dy} - E_3\tilde{dt}\wedge\tilde{dz} + B_1\tilde{dy}\wedge\tilde{dz} + B_2\tilde{dz}\wedge\tilde{dx} + B_3\tilde{dx}\wedge\tilde{dy} \tag{6.31}$$

We now compute the exterior derivative of $\tilde{F}$

$$\begin{aligned} d\tilde{F} = &-\left(\frac{\partial E_1}{\partial t}\tilde{dt} + \frac{\partial E_1}{\partial x}\tilde{dx} + \frac{\partial E_1}{\partial y}\tilde{dy} + \frac{\partial E_1}{\partial z}\tilde{dz}\right)\wedge\tilde{dt}\wedge\tilde{dx} \\ &-\left(\frac{\partial E_2}{\partial t}\tilde{dt} + \frac{\partial E_2}{\partial x}\tilde{dx} + \frac{\partial E_2}{\partial y}\tilde{dy} + \frac{\partial E_2}{\partial z}\tilde{dz}\right)\wedge\tilde{dt}\wedge\tilde{dy} \\ &-\left(\frac{\partial E_3}{\partial t}\tilde{dt} + \frac{\partial E_3}{\partial x}\tilde{dx} + \frac{\partial E_3}{\partial y}\tilde{dy} + \frac{\partial E_3}{\partial z}\tilde{dz}\right)\wedge\tilde{dt}\wedge\tilde{dz} \\ &+\left(\frac{\partial B_1}{\partial t}\tilde{dt} + \frac{\partial B_1}{\partial x}\tilde{dx} + \frac{\partial B_1}{\partial y}\tilde{dy} + \frac{\partial B_1}{\partial z}\tilde{dz}\right)\wedge\tilde{dy}\wedge\tilde{dz} \\ &+\left(\frac{\partial B_2}{\partial t}\tilde{dt} + \frac{\partial B_2}{\partial x}\tilde{dx} + \frac{\partial B_2}{\partial y}\tilde{dy} + \frac{\partial B_2}{\partial z}\tilde{dz}\right)\wedge\tilde{dz}\wedge\tilde{dx} \\ &+\left(\frac{\partial B_3}{\partial t}\tilde{dt} + \frac{\partial B_3}{\partial x}\tilde{dx} + \frac{\partial B_3}{\partial y}\tilde{dy} + \frac{\partial B_3}{\partial z}\tilde{dz}\right)\wedge\tilde{dx}\wedge\tilde{dy} \\ = &\left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} + \frac{\partial B_3}{\partial t}\right)\tilde{dt}\wedge\tilde{dx}\wedge\tilde{dy} \\ &+\left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z} + \frac{\partial B_1}{\partial t}\right)\tilde{dt}\wedge\tilde{dy}\wedge\tilde{dz} \\ &+\left(\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x} + \frac{\partial B_2}{\partial t}\right)\tilde{dt}\wedge\tilde{dz}\wedge\tilde{dx} \\ &+\left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)\tilde{dx}\wedge\tilde{dy}\wedge\tilde{dz} \end{aligned} \tag{6.32}$$

Upon setting $d\tilde{F} = 0$ we obtain

$$\begin{aligned} \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} + \frac{\partial B_3}{\partial t} &= 0 \\ \frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z} + \frac{\partial B_1}{\partial t} &= 0 \\ \frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x} + \frac{\partial B_2}{\partial t} &= 0 \\ \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} &= 0 \end{aligned} \tag{6.33}$$

Voila! The first three equations shown in Equation (6.33) are nothing more than just an explicit form Faraday's Law. The fourth equation is nothing more than the Divergence of $\vec{B}$ equation. So we have that

$$d\tilde{F} = 0 \iff \vec{\nabla} \cdot \vec{B} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{6.34}$$

We've recovered two Maxwell Equations. In order to recover the other two, we need to introduce the *current one-form* $\tilde{J} = -\rho\tilde{dt} + J_1\tilde{dx} + J_2\tilde{dy} + J_3\tilde{dz}$. I now claim that the other two Maxwell's Equations that are left are equivalent to

$$*d*\tilde{F} = \tilde{J} \tag{6.35}$$

Computing $*\tilde{F}$ we obtain

$$\begin{aligned} *\tilde{F} &= -E_1 * (\tilde{dt} \wedge \tilde{dx}) - E_2 * (\tilde{dt} \wedge \tilde{dy}) - E_3 * (\tilde{dt} \wedge \tilde{dz}) \\ &\quad + B_1 * (\tilde{dy} \wedge \tilde{dz}) + B_2 * (\tilde{dz} \wedge \tilde{dx}) + B_3 * (\tilde{dx} \wedge \tilde{dy}) \\ &= E_1\tilde{dy} \wedge \tilde{dz} + E_2\tilde{dz} \wedge \tilde{dx} + E_3\tilde{dx} \wedge \tilde{dy} \\ &\quad + B_1\tilde{dt} \wedge \tilde{dx} + B_2\tilde{dt} \wedge \tilde{dy} + B_3\tilde{dt} \wedge \tilde{dz} \end{aligned} \tag{6.36}$$

Taking the exterior derivative of $*\tilde{F}$ we obtain

$$\begin{aligned} d*\tilde{F} = &\left(\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}\right)\tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} \\ &+ \left(\frac{\partial E_3}{\partial t} + \frac{\partial B_1}{\partial y} - \frac{\partial B_2}{\partial x}\right)\tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy} \\ &+ \left(\frac{\partial E_1}{\partial t} + \frac{\partial B_2}{\partial z} - \frac{\partial B_3}{\partial y}\right)\tilde{dt} \wedge \tilde{dy} \wedge \tilde{dz} \\ &+ \left(\frac{\partial E_2}{\partial t} + \frac{\partial B_3}{\partial x} - \frac{\partial B_1}{\partial z}\right)\tilde{dt} \wedge \tilde{dz} \wedge \tilde{dx} \end{aligned} \tag{6.37}$$

We now take the Hodge dual of $d*\tilde{F}$ using the formulas from Equation (6.29). This yields

$$\begin{aligned} *d*F = &-\left(\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}\right)\tilde{dt} \\ &-\left(\frac{\partial E_1}{\partial t} + \frac{\partial B_2}{\partial z} - \frac{\partial B_3}{\partial y}\right)\tilde{dx} \\ &-\left(\frac{\partial E_2}{\partial t} + \frac{\partial B_3}{\partial x} - \frac{\partial B_1}{\partial z}\right)\tilde{dy} \\ &-\left(\frac{\partial E_3}{\partial t} + \frac{\partial B_1}{\partial y} - \frac{\partial B_2}{\partial x}\right)\tilde{dz} \end{aligned} \tag{6.38}$$

If we let $*d*\tilde{F} = \tilde{J}$ then we simply obtain the following equations

$$\begin{aligned} \frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z} &= \rho \\ -\frac{\partial E_1}{\partial t} - \frac{\partial B_2}{\partial z} + \frac{\partial B_3}{\partial y} &= J_1 \\ -\frac{\partial E_2}{\partial t} - \frac{\partial B_3}{\partial x} + \frac{\partial B_1}{\partial z} &= J_2 \\ -\frac{\partial E_3}{\partial t} - \frac{\partial B_1}{\partial y} + \frac{\partial B_2}{\partial x} &= J_3 \end{aligned} \tag{6.39}$$

Voila again! The first equation shown in Equation (6.39) is nothing more than Gauss's Law. The other three equations are just an explicit form of the Ampere-Maxwell Law. So we have that

$$*d*\tilde{F} = \tilde{J} \iff \vec{\nabla}\cdot\vec{E} = \rho \quad \text{and} \quad \vec{\nabla}\times\vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t} \tag{6.40}$$

In conclusion, Maxwell's Equations in terms of differential forms are given by

$$d\tilde{F} = 0 \quad \text{and} \quad *d*\tilde{F} = \tilde{J} \tag{6.41}$$

7 Conclusion

The study of differential forms can be thought of as a stepping stone for one to enter into the realm of differential geometry. Although the mathematics that was developed in this paper is just a tiny fraction of the actual depth in which this subject has been developed, the number of examples and applications mentioned throughout should only convince oneself of the sheer power of the subject.

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