On the Anomalous Magnetic Moments of the Electron and Muon

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An Overview of the g-factor

- The g-factor is a proportionality constant that allows physicists to relate the different observed magnetic moments of a particle to their angular momentum quantum numbers
- Motivational example: a current-carrying loop
- Classical example: a charged particle in a circular orbit

$$\vec{\mu} = \frac{q}{2m}\vec{L}. \tag{1}$$

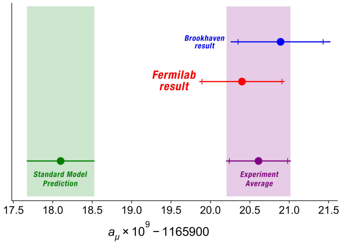
- Key insight: the magnetic moment is proportional to the angular momentum $\rightarrow$ this result is carried over in quantum mechanics as well
- Magnetic moment of the electron is proportional to its spin angular momentum

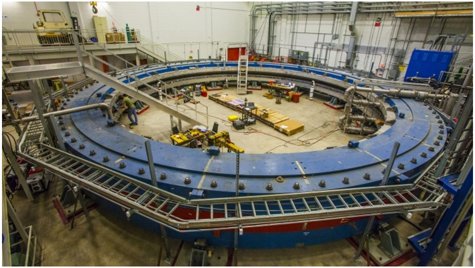
$$\vec{\mu_e} = g\frac{q}{2m}\vec{S}. \tag{2}$$

- Essential question: what is the value of g?

The Fermilab $g-2$ Experiment

- Designed to measure the $g-2$ factor of the muon to a very high accuracy
- The final measurement had a significant amount of disagreement with the theoretically calculated value using the Standard Model
- Postulated to point towards new physics beyond the Standard Model





[1]

Dirac Equation + Minimal Coupling

- Starting point: Dirac Equation

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0. \tag{3}$$

- Minimal coupling recipe

$$p^\mu \to p^\mu - qA^\mu \implies \partial_\mu \to \partial_\mu + iqA_\mu. \tag{4}$$

- Minimally coupled Dirac Equation

$$[\gamma^\mu(i\partial_\mu - qA_\mu) - m]\Psi = 0 \tag{5}$$

Dirac Equation + Minimal Coupling (Continued)

- Idea: take the non-relativistic limit of the minimally coupled Dirac Equation

$$\Psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix} e^{-iEt}. \tag{6}$$

- We now substitute this Ansatz into the minimally coupled Dirac Equation and obtain two coupled equations

$$(E - m - q\phi)\chi + \vec{\sigma} \cdot (i\vec{\nabla} + q\vec{A})\varphi = 0, \tag{7}$$

$$(E + m - q\phi)\varphi + \vec{\sigma} \cdot (i\vec{\nabla} + q\vec{A})\chi = 0. \tag{8}$$

- Idea: the bottom half of the Dirac Spinor is very suppressed in the non-relativistic limit $\rightarrow$ we only need to solve for the top spinor to gauge the behaviour of the electron

Dirac Equation + Minimal Coupling (Continued)

- Equation for the top half spinor

$$\left[\frac{1}{2m}\left[\vec{\sigma}\cdot(i\vec{\nabla}+q\vec{A})\right]^2+q\phi\right]\chi=(E-m)\chi. \tag{9}$$

- This equation can be re-written as

$$\left[\frac{1}{2m}\left(i\vec{\nabla}+q\vec{A}\right)^2+q\phi-\frac{q}{2m}\vec{\sigma}\cdot\vec{B}\right]\chi=(E-m)\chi. \tag{10}$$

- The above equation can be compared with the Schrodinger Equation derived by Pauli

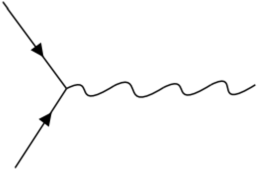
$$\left[\frac{1}{2m}(\vec{p}+q\vec{A})^2+q\phi-\vec{\mu}\cdot\vec{B}\right]\psi=E\psi \tag{11}$$

- Upon comparing we find that $g=2$.

Feynman Rules for QED

$$= -\frac{i\eta_{\mu\nu}}{p^2+i\epsilon}$$

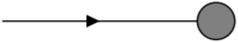
$$= \frac{i(\gamma^\mu p_\mu + m)}{p^2-m^2+i\epsilon}$$



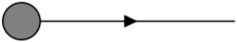
$$-ie\gamma^\mu$$

$$= \epsilon^\mu_{in}(p)$$

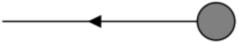
$$= \epsilon^\mu_{out}(p)$$



$$= u^s(p)$$



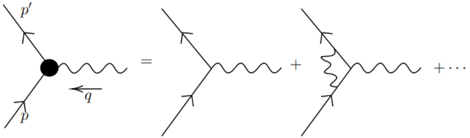
$$= \overline{u}^s(p)$$



$$= \overline{v}^s(p)$$

$$= v^s(p)$$

The relevant Feynman diagram is



We define the *electron vertex function* $\Gamma^\mu(p', p)$ as the sum of the vertex corrections such that $i\mathcal{M} = -ie\bar{u}(p')\Gamma^\mu(p', p)u(p)\tilde{A}^{cl}_\mu(q)$. Generally,

$$\Gamma^\mu = \gamma^\mu A(q^2) + (p'^\mu + p^\mu)B(q^2) + (p'^\mu - p^\mu)C(q^2). \tag{12}$$

Using the *Ward Identity* $q_\mu \mathcal{M}^\mu = 0$ and the *Gordon Identity*, we get

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2), \tag{13}$$

where F_1 and F_2 are *form factors*.

Meaning of the Form Factors

For electron scattering off a static magnetic field, $A_\mu^{cl}(x) = (0, -\vec{A}(x))$. The scattering amplitude becomes:

$$i\mathcal{M} = -ie\bar{u}(p')\left[\gamma^i F_1(q^2) + \frac{i\sigma^{i\nu}q_\nu}{2m}F_2(q^2)\right]u(p)\tilde{A}_i^{cl}(q). \tag{14}$$

Explicitly calculating the spinors and taking the non-relativistic limit $E \approx m$ and $q^0 \ll 1$,

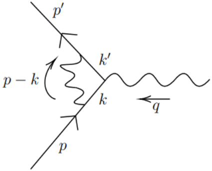
$$i\mathcal{M} = i\frac{e}{2m}2[F_1(0) + F_2(0)]\langle S_k\rangle\,\tilde{B}^k(q), \tag{15}$$

where $\langle S_k\rangle$ is the expectation value of the spin-operator in the k direction and $\tilde{B}^k(q)$ is the Fourier transform of the magnetic field. Interpreting $\mathcal{M}$ as the scattering potential, we get the *Landé g factor*:

$$g = 2[F_1(0) + F_2(0)]. \tag{16}$$

One-Loop Calculation

At one-loop level, the relevant Feynman diagram is



Using the Feynman rules for QED and denoting the one-loop correction to Γ^μ as $\delta\Gamma_1^\mu$,

$$\bar{u}(p')\delta\Gamma_1^\mu u(p) = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4}\bar{u}(p')ie\gamma^\nu \frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon}\gamma^\mu \times$$
$$\times \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\frac{-i\eta_{\nu\rho}}{(k-p)^2+i\epsilon}ie\gamma^\rho u(p)$$
$$= ie^2\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\frac{2\bar{u}(p')[\not{k}\gamma^\mu\not{k}'+m^2\gamma^\mu-2m(k^\mu+k'^\mu)]u(p)}{(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)[(k-p)^2+i\epsilon]}. \tag{17}$$

Feynman Parameter Trick

The relevant identity is

$$\frac{1}{A_1 A_2 A_3} = \int_0^1 dx_1 dx_2 dx_3 \, \frac{2\delta(x_1 + x_2 + x_3 - 1)}{[x_1 A_1 + x_2 A_2 + x_3 A_3]^3}. \tag{18}$$

Comparing with our integral, we get

$$\bar{u}(p')\delta\Gamma_1^{\mu} u(p) = ie^2 \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \int_0^1 dx_1 \, dx_2 \, dx_3 \, \frac{2\delta(x_1 + x_2 + x_3 - 1)N^{\mu}}{D^3}, \tag{19}$$

where N^{μ} is the numerator and
$D = x_1(k^2 - m^2) + x_2(k'^2 - m^2) + x_3(k-p)^2 + (x_1 + x_2 + x_3)i\epsilon$. To simplify the numerator, we use gamma matrix identities and the following integral identities:

$$\int \frac{d^4l}{(2\pi)^4} \frac{l^{\mu}}{D^3} = 0, \qquad \int \frac{d^4l}{(2\pi)^4} \frac{l^{\mu} l^{\nu}}{D^3} = \int \frac{d^4l}{(2\pi)^4} \frac{\frac{1}{4}\eta^{\mu\nu} l^2}{D^3}. \tag{20}$$

Feynman Parameter Trick (Continued)

We define $l = k + x_2 q - x_3 p$ and $\Delta = -x_1 x_2 q^2 + (1 - x_3)^2 m^2$. After simplification, the numerator and denominator become

$$N^\mu = 2\bar{u}(p')\Big[\gamma^\mu\left(-\frac{1}{2}l^2 + (1 - x_1)(1 - x_2)q^2 + (1 - 4x_3 + x_3^2)m^2\right) + \\ + i\sigma^{\mu\nu}q_\nu m x_3(x_3 - 1)\Big]u(p), \tag{21}$$

$$D = l^2 - \Delta + i\epsilon. \tag{22}$$

We now write out the full expression for the electron vertex function, correct to one-loop order, $\Gamma_1^\mu = \gamma^\mu + \delta\Gamma_1^\mu$ (see next slide).

Form Factor Integrals

$$
\begin{aligned}
\Gamma_1^\mu = \gamma^\mu \Bigg\{ 1 + 2ie^2 \int_0^1 dx_1\, dx_2\, dx_3 \int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4} \frac{2\delta(x_1 + x_2 + x_3 - 1)}{D^3} \times \\
\times \left(-\frac{1}{2} l^2 + (1 - x_1)(1 - x_2) q^2 + (1 - 4x_3 + x_3^2) m^2 \right) \Bigg\} + \\
+ \frac{i\sigma^{\mu\nu} q_\nu}{2m} \times 2ie^2 \int_0^1 dx_1\, dx_2\, dx_3 \int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4} \times \\
\times \frac{2\delta(x_1 + x_2 + x_3 - 1)}{D^3} 2m^2 x_3 (1 - x_3).
\end{aligned}
\tag{23}
$$

Comparing with the form factors expression, we get that the coefficient of γ^μ is $F_1(0)$ and the coefficient of $\frac{i\sigma^{\mu\nu} q_\nu}{2m}$ is $F_2(0)$, both correct to one-loop order. In particular, we want to solve:

$$
\int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - \Delta)^n}, \quad \int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4} \frac{l^2}{(l^2 - \Delta)^n}. \tag{24}
$$

Wick Rotation

We now perform what is known as a Wick rotation to solve our integrals.

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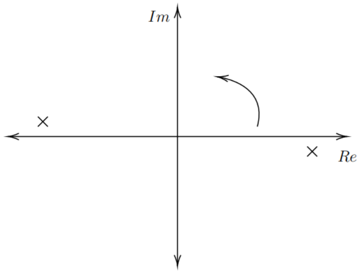
Wick Rotation (Continued)

The l^0 integral has poles at $l^0 = \pm\sqrt{|\vec{l}|^2 - \Delta} \mp i\epsilon$:



We rotate in the direction shown i.e. $l^0 \to il^0$. This simplifies the Minkowski space integrals to 4-D Euclidean ones:

$$\frac{i(-1)^n}{(2\pi)^4}\int_{-\infty}^{\infty} d^4 l_E \, \frac{1}{(l_E^2+\Delta)^n}, \quad \frac{i(-1)^{n-1}}{(2\pi)^4}\int_{-\infty}^{\infty} d^4 l_E \, \frac{l_E^2}{(l_E^2+\Delta)^n}. \tag{25}$$

Solution to the Integrals

Using 4-D polar coordinates and solving, we get

$$\int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - \Delta)^n} = \frac{i(-1)^n}{(4\pi)^2} \frac{1}{(n-1)(n-2)} \frac{1}{\Delta^{n-2}}. \quad (26)$$

$$\int_{-\infty}^{\infty} \frac{d^4 l}{(2\pi)^4} \frac{l^2}{(l^2 - \Delta)^n} = \frac{i(-1)^{n-1}}{(4\pi)^2} \frac{2}{(n-1)(n-2)(n-3)} \frac{1}{\Delta^{n-3}}. \quad (27)$$

The second integral diverges for $n = 3$. However, only $F_1(0)$ involves this divergent quantity and we know that $F_1(0) = 1$ to all orders in perturbation theory[3]. Putting everything together, we get, to one-loop order,

$$F_2(0) = \frac{\alpha}{2\pi} \quad (28)$$

Comparing with the Experimental Prediction

Comparing with the expression of the Landé g factor and using the experimental value of the fine structure constant[2],

$$a_e^{\text{theory}} = \frac{g-2}{2} = 0.00116140973. \tag{29}$$

The experimental data shows that

$$a_e^{\text{exp}} = 0.00115965218073(28) \tag{30}$$

The percentage difference between the two is approximately 0.15%, which suggests excellent agreement despite calculating only to one-loop order.

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