Example

Consider the function $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3$ defined on the set of all triples of numbers. Its first partials are

$$f'_1(x_1, x_2, x_3) = 2x_1 + 2x_2 + 2x_3$$
$$f'_2(x_1, x_2, x_3) = 4x_2 + 2x_1$$
$$f'_3(x_1, x_2, x_3) = 6x_3 + 2x_1.$$

So its Hessian is

$$\begin{pmatrix} f''_{11} & f''_{12} & f''_{13} \\ f''_{21} & f''_{22} & f''_{23} \\ f''_{31} & f''_{32} & f''_{33} \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 6 \end{pmatrix}.$$

The leading principal minors of the Hessian are $2 > 0$, $4 > 0$, and $8 > 0$. So the Hessian is positive definite, and f is strictly convex.

Homework:

negative semidefinite

positive semidefinite

Unconstrained Maximization and Minimization with Several Variables

one variable:

If $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is a local maximum. If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a local minimum.

multiple variables:

$$\begin{array}{ll} \max \text{ (or min) } f(x_1, x_2, \ldots, x_n) & \\ \text{s.t.} \quad (x_1, x_2, \ldots, x_n) \in R^n & \end{array}$$

A point $\bar{x}$ having $\frac{\partial f(\bar{x})}{\partial x_i} = 0$ for $i = 1, 2, \ldots, n$ is called a **stationary point** of f.

(1) $\bar{x}$ is a **local** maximum if f is strictly concave (i.e., H($\bar{x}$) is negative definite)

(2) $\bar{x}$ is a **local** minimum if f is strictly convex (i.e., H($\bar{x}$) is positive definite).

(3) $\bar{x}$ is a saddle point if H($\bar{x}$) is indefinite (it is not negative definite, negative semidefinite, positive definite, and positive semidefinite)

(4) otherwise, it is inconclusive.