Example 33

6	7	8	10
15	80	78	15
15	5	5	

If we apply the minimum-cost method to this table, we set $x_{11} = 10$ and cross out row 1. This forces us to make x_{22} and x_{23} basic variables, thereby incurring their high shipping costs.

3. Vogel's Method

Step 1. Compute for each row (and column) a penalty equal to the difference between the two smallest costs in the row (column).

Step 2. Find the row or column with the largest penalty

Step 3. Choose the variable with the smallest shipping cost in this row or column

Example 34

6	7	8	10
15	80	78	15
15	5	5	

				Supply	Row Penalty
	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$	10	7 – 6 = 1
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	78 – 15 = 63
Demand	15	5	5		
Column Penalty	15 – 6 = 9	80 – 7 = 73	78 – 8 = 70		

				Supply	Row Penalty
	$\boxed{6}$	$\boxed{7}$ 5	$\boxed{8}$	5	8 – 6 = 2
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	78 – 15 = 63
Demand	15	×	5		
Column Penalty	9	—	70		

				Supply	Row Penalty
	$\boxed{6}$	$\boxed{7}$ 5	$\boxed{8}$ 5	0	—
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	—
Demand	15	×	×		
Column Penalty	9	—	—		

				Supply	Row Penalty
	$\boxed{6}$ 0	$\boxed{7}$ 5	$\boxed{8}$ 5	×	—
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	—
Demand	15	×	×		
Column Penalty	—	—	—		

0 $\boxed{6}$	5 $\boxed{7}$	5 $\boxed{8}$	10
15 $\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15
15	5	5	

<u>The Transportation Simplex Method</u>

Pivot Process

Step 1. Determine the variable that should enter the basis

Step 2. Find the loop (*it can be shown that there is only one loop*) involving this variable and some of the basic variables

Step 3. Count only cells in the loop and label those found in step 2 that are an even number of cells away from the entering variable as even cells. The others are odd cells.

Loop: It must

- (i) start and finish on the same nonbasic cell
- (ii) move alternatively horizontally and vertically on basic cells

Step 4. Find the odd cell whose variable assumes the smallest value. Call this value θ. The variable in this cell will leave the basis.

Step 5. Decrease the value of each odd cell by θ and increase the value of each even cell by θ

Summary of Transportation Simplex Method

Step 1. If the problem is unbalanced, balance it

Step 2. Find a bfs

Step 3. Use $u_1 = 0$ and $u_i + v_j = c_{ij}$ for all basic variables to find the u_i and v_j

Step 4. If $u_i + v_j - c_{ij} \le 0$ for all nonbasic variables, then the current bfs is optimal. Otherwise, the nonbasic variable with the most positive $u_i + v_j - c_{ij}$ will enter the basis. Using pivot process to find new bfs. Go to step 3.

Example 35

Shipping Costs, Supply, and Demand for Powerco

From	To				Supply (million kwh)
	City 1	City 2	City 3	City 4	
Plant 1	\$8	\$6	\$10	\$9	35
Plant 2	\$9	\$12	\$13	\$7	50
Plant 3	\$14	\$9	\$16	\$5	40
Demand (million kwh)	45	20	30	30	

Step 1. It is balanced.
Step 2. Use North-west corner method

8 35	6	10	9	35
9 10	12 20	13 20	7	50
14	9	16 10	5 30	40
45	20	30	30	

Step 3.

$$\left.\begin{aligned} &u_1 = 0 \\ &u_1 + v_1 = 8 \\ &u_2 + v_1 = 9 \\ &u_2 + v_2 = 12 \\ &u_2 + v_3 = 13 \\ &u_3 + v_3 = 16 \\ &u_3 + v_4 = 5 \end{aligned}\right\} u_1 = 0, v_1 = 8, u_2 = 1, v_2 = 11, v_3 = 12, u_3 = 4, v_4 = 1$$

Step 4

$$u_1 + v_2 - c_{12} = 0 + 11 - 6 = 5$$
$$u_1 + v_3 - c_{13} = 0 + 12 - 10 = 2$$
$$u_1 + v_4 - c_{14} = 0 + 1 - 9 = -8$$
$$u_2 + v_4 - c_{24} = 1 + 1 - 7 = -5$$
$$u_3 + v_1 - c_{31} = 4 + 8 - 14 = -2$$
$$u_3 + v_2 - c_{32} = 4 + 11 - 9 = 6$$

Thus, x_{32} should enter the basis

Step 5

8 35	6	10	9	35
9 10	12 20 odd	13 20 even	7	50
14	9 EBV even	16 10 odd	5 30	40
45	20	30	30	

$\theta = 10 < 20$. Thus, x_{33} leave the basis

[8] 35	[6]	[10]	[9]	35
[9] 10	[12] 10	[13] 30	[7]	50
[14]	[9] 10	[16]	[5] 30	40
45	20	30	30	

Step 3.

$$\left.\begin{array}{l} u_1 = 0 \\ u_1 + v_1 = 8 \\ u_2 + v_1 = 9 \\ u_2 + v_2 = 12 \\ u_2 + v_3 = 13 \\ u_3 + v_2 = 9 \\ u_3 + v_4 = 5 \end{array}\right\} u_1 = 0, v_1 = 8, u_2 = 1, v_2 = 11, v_3 = 12, u_3 = -2, v_4 = 7$$

Step 4.

$$u_1 + v_2 - c_{12} = 0 + 11 - 6 = 5$$
$$u_1 + v_3 - c_{13} = 0 + 12 - 10 = 2$$
$$u_1 + v_4 - c_{14} = 0 + 7 - 9 = -2$$
$$u_2 + v_4 - c_{24} = 1 + 7 - 7 = 1$$
$$u_3 + v_1 - c_{31} = -2 + 8 - 14 = -8$$
$$u_3 + v_3 - c_{33} = -2 + 12 - 16 = -6$$

We next enter x_{12} into the basis

	$v_j =$ 8	11	12	7	
$u_i = 0$	odd 35 [8]	even [6]	[10]	[9]	35
1	10 even [9]	10 odd [12]	30 [13]	[7]	50
−2	[14]	10 [9]	[16]	30 [5]	40
	45	20	30	30	

$\theta = 10 < 35$. Thus, x_{22} leave the basis

Step 5

25 [8]	10 [6]	[10]	[9]	35
20 [9]	[12]	30 [13]	[7]	50
[14]	10 [9]	[16]	30 [5]	40
45	20	30	30	

Step 3

	$v_j =$ 8	6	12	2	
$u_i = 0$	25 [8]	10 [6]	[10]	[9]	35
1	20 [9]	[12]	30 [13]	[7]	50
3	[14]	10 [9]	[16]	30 [5]	40
	45	20	30	30	

	$v_j =$ 8	6	12	2	
$u_i = 0$	8 25	6 10	10	9	35
1	9 20	12	13 30	7	50
3	14	9 10	16	5 30	40
	45	20	30	30	

Step 4.

$u_1 + v_3 - c_{13} = 0 + 12 - 10 = 2$

$u_1 + v_4 - c_{14} = 0 + 2 - 9 = -7$

$u_2 + v_2 - c_{22} = 1 + 6 - 12 = -5$

$u_2 + v_4 - c_{24} = 1 + 2 - 7 = -4$

$u_3 + v_1 - c_{31} = 3 + 8 - 14 = -3$

$u_3 + v_3 - c_{33} = 3 + 12 - 16 = -1$

We next enter x_{13} into the basis

	$v_j =$ 8	6	12	2	
$u_i = 0$	8 25 odd	6 10	10 even	9	35
1	9 20 even	12	13 30 odd	7	50
3	14	9 10	16	5 30	40
	45	20	30	30	

$\theta = 25 < 30$. Thus, x_{11} leave the basis

Step 5

$\boxed{8}$	10 $\boxed{6}$	25 $\boxed{10}$	$\boxed{9}$	35
45 $\boxed{9}$	$\boxed{12}$	5 $\boxed{13}$	$\boxed{7}$	50
$\boxed{14}$	10 $\boxed{9}$	$\boxed{16}$	30 $\boxed{5}$	40
45	20	30	30	

Step 3

	$v_j =$ 6	6	10	2	
$u_i = 0$	$\boxed{8}$	10 $\boxed{6}$	25 $\boxed{10}$	$\boxed{9}$	35
3	45 $\boxed{9}$	$\boxed{12}$	5 $\boxed{13}$	$\boxed{7}$	50
3	$\boxed{14}$	10 $\boxed{9}$	$\boxed{16}$	30 $\boxed{5}$	40
	45	20	30	30	

Step 4

$u_1 + v_1 - c_{11} = 0 + 6 - 8 = -2$

$u_1 + v_4 - c_{14} = 0 + 2 - 9 = -7$

$u_2 + v_2 - c_{22} = 3 + 6 - 12 = -3$

$u_2 + v_4 - c_{24} = 3 + 2 - 7 = -2$

$u_3 + v_1 - c_{31} = 3 + 6 - 14 = -5$

$u_3 + v_3 - c_{33} = 3 + 10 - 16 = -3$

Thus, it is optimal now

$x_{12}=10$, $x_{13}=25$, $x_{21}=45$, $x_{23}=5$, $x_{32}=10$, $x_{34}=30$, and

$z=6x_{12}+10x_{13}+9x_{21}+13x_{23}+9x_{32}+5x_{34}=1{,}020$

Remark:

For a maximization problem, step 4 is updated as follows:

Step 4: If $u_i + v_j - c_{ij} \geq 0$ for all nonbasic variables, then the current bfs is optimal. Otherwise, the nonbasic variable with the most negative $u_i + v_j - c_{ij}$ will enter the basis. Using pivot process to find new bfs. Go to step 3.

The Assignment Problems

LP $\supset$ Transportation $\supset$ Assignment

$$Min \quad z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \quad s.t.$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,\ldots,n \quad (s_i, \text{supply constraint})$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1,\ldots,n \quad (\text{d}_j, \text{demand constraint})$$

$$x_{ij} = 0 \text{ or } 1$$

where x_{ij} = 1 if person(machine) *i* does job *j*
=0 otherwise

The Hungarian Method

Step 1. Circle the smallest entry in each row and subtract it from all entries in the row

Step 2. Circle the smallest entry in each column and subtract it from all entries in the column

Step 3. Draw the minimum number of lines that are needed to cover all the zeros in the matrix. Stop when it takes n lines and use the position of zeros to make the assignment; otherwise, adjust the matrix as follows:

(i) find the smallest entry not crossed out

(ii) subtract if from all entries which not crossed out

(iii) add it to all entries crossed out twice

Return to Step 3.

Example 36

Setup Times for Machineco

	Time (Hours)			
Machine	Job 1	Job 2	Job 3	Job 4
1	14	5	8	7
2	2	12	6	5
3	7	8	3	9
4	2	4	6	10

				Row Minimum
14	5	8	7	5
2	12	6	5	2
7	8	3	9	3
2	4	6	10	2

	9	0	3	2
	0	10	4	3
	4	5	0	6
	0	2	4	8
Column Minimum	0	0	0	2

9	0	3	0
0	10	4	1
4	5	0	4
0	2	4	6

10	0	3	0
0	9	3	0
5	5	0	4
0	1	3	5

We have found the optimal assignment $x_{12} = 1$, $x_{24} = 1$, $x_{33} = 1$, and $x_{41} = 1$.

Why does this method work?
If a constant is added to each cost in a row (or column) of a balanced transportation problem, then the optimal solution to the problem is unchanged.

Remark: An intuitive explanation

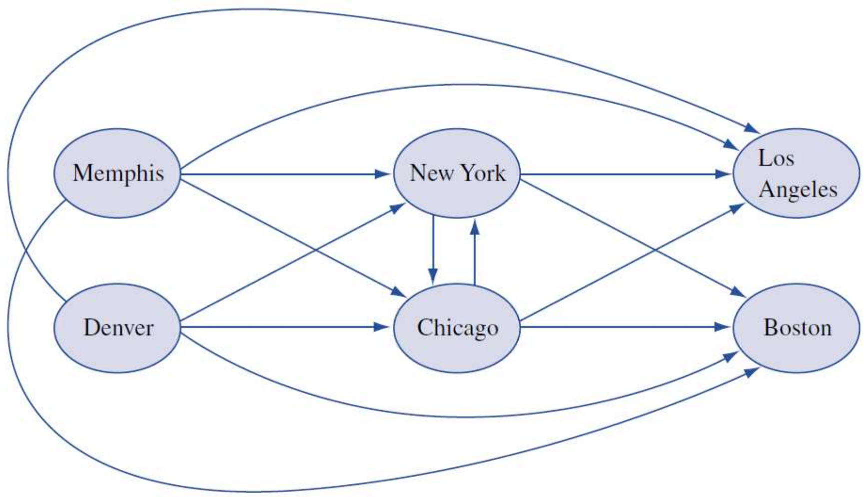
$$\text{New objective function} = \text{old objective function} + k(x_{11} + x_{12} + x_{13} + x_{14})$$

Because any feasible solution to the Machineco problem must have $x_{11} + x_{12} + x_{13} + x_{14} = 1$,

$$\text{New objective function} = \text{old objective function} + k$$

Transshipment Problem

In addition to supply and demand point, there are transshipment points that can both receive goods from other points and send goods to other points.



Shipping Costs for Transshipments

	To ($)					
From	**Memphis**	**Denver**	**N.Y.**	**Chicago**	**L.A.**	**Boston**
Memphis	0	—	8	13	25	28
Denver	—	0	15	12	26	25
N.Y.	—	—	0	6	16	17
Chicago	—	—	6	0	14	16
L.A.	—	—	—	—	0	—
Boston	—	—	—	—	—	0

The Memphis factory: 150 units
The Denver factory: 200 units
The customers in Los Angeles: 130 units
The customers in and Boston: 130 units

(1) Assume that total supply exceeds total demand
(2) Each transshipment point has zero supply and zero demand

Let S = total available capacity. Construct a transportation tableau as follows:

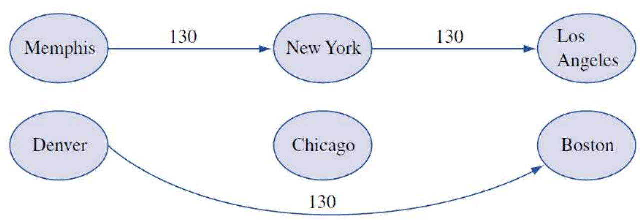
> A row is needed for each supply and transshipment point and a column is needed for each demand and transshipment point.
>
> Each supply (demand) point has a supply (demand) equal to its original supply (demand).
>
> Each transshipment point has a supply equal to S and a demand equal to S
>
> If necessary, add a dummy demand point. Shipments to the dummy and from a point to itself will have a zero shipping cost.

Although we don't know how much will be shipped through each transshipment point, we can be sure that the total amount will not exceed S.

Transshipment pint (N.Y., Chicago)

	N.Y.	Chicago	L.A.	Boston	Dummy	Supply
Memphis	8	13	25	28	0	150
Denver	15	12	26	25	0	200
N.Y.	0	6	16	17	0	350
Chicago	6	0	14	16	0	350
Demand	350	350	130	130	90	

	N.Y.	Chicago	L.A.	Boston	Dummy	Supply
Memphis	$\boxed{8}$ 130	$\boxed{13}$	$\boxed{25}$	$\boxed{28}$	$\boxed{0}$ 20	150
Denver	$\boxed{15}$	$\boxed{12}$	$\boxed{26}$	$\boxed{25}$ 130	$\boxed{0}$ 70	200
N.Y.	$\boxed{0}$ 220	$\boxed{6}$	$\boxed{16}$ 130	$\boxed{17}$	$\boxed{0}$	350
Chicago	$\boxed{6}$	$\boxed{0}$ 350	$\boxed{14}$	$\boxed{16}$	$\boxed{0}$	350
Demand	350	350	130	130	90	



Remark: We ignore the shipments to the dummy and from a point to itself.

Integer Programming

An integer program (IP) is an LP where some or all variables are required to be nonnegative integer.

(1) Pure IPs

(2) Mixed IPs

(3) Zero one IPs

Def. The LP obtained by omitting all integer or 0–1 constraints on variables is called the LP relaxation of the IP.

e.g.,

Pure IP

$Max \quad z = 21x_1 + 11x_2 \quad s.t.$

$7x_1 + 4x_2 \le 13$

$x_1 \ge 0, \quad x_2 \ge 0$

Both are integer

LP relaxation

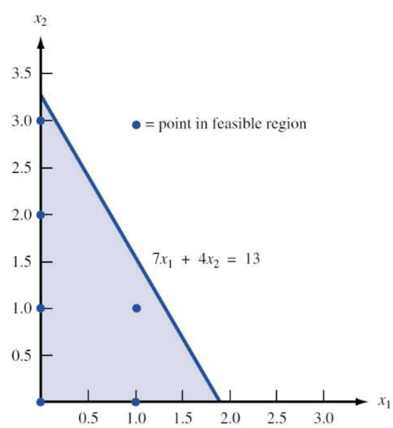
$Max \quad z = 21x_1 + 11x_2 \quad s.t.$

$7x_1 + 4x_2 \le 13$

$x_1 \ge 0, \quad x_2 \ge 0$

Remark:

1. Optimal z-value for LP relaxation ≧ optimal z-value for IP
2. The feasible region for an IP is a subset of the feasible region for the IP's LP relaxation. If LP relaxation is bounding, then the feasible solution of the IP is finite. As a result, we can enumerate that z-value for each point and determine the best one.



3. If the solution of the LP relaxation are all integers, then the optimal solution to the LP relaxation is also the optimal solution to the IP.

4. A native idea would be to round the LP solution to the nearest integers. But there are also problems: (1) non-feasible (2) suboptimal

Either–Or Constraints

$$f(x_1, x_2, \ldots, x_n) \leq 0$$
$$g(x_1, x_2, \ldots, x_n) \leq 0$$

$$f(x_1, x_2, \ldots, x_n) \leq My$$
$$g(x_1, x_2, \ldots, x_n) \leq M(1 - y)$$

y is a 0–1 variable, and M is a large number

If-Then Constraints

If a constraint $f > 0$ is satisfied, then the constraint $g \geqq 0$ must be satisfied

$$-g(x_1, x_2, \ldots, x_n) \leq My$$
$$f(x_1, x_2, \ldots, x_n) \leq M(1 - y)$$
$$y = 0 \text{ or } 1$$

y is a 0–1 variable, and M is a large number

Example 37

If customers in region 1 send their payments to city 1, then no other customers may send their payments to city 1.

Thus, we use binary variables.
If $x_{11} = 1$, then $x_{21} = x_{31} = x_{41} = 0$

If $x_{11} > 0$, then $x_{21} + x_{31} + x_{41} \leqq 0$, or $-(x_{21} + x_{31} + x_{41}) \geqq 0$

We can formulate as follows:

$$x_{21} + x_{31} + x_{41} \leq My$$
$$x_{11} \leq M(1 - y)$$
$$y = 0 \text{ or } 1$$