Report on Quantum Random Walks

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"A drunk man will find his way home, but a drunk bird may get lost forever."
– *Shizuo Kakutani*

1 Introduction to Random Walks

A random walk is a stochastic mathematical model which takes fixed, discrete steps in some space with random probability, for instance in 2 dimensions having equal probabilities of moving up, down, left, or right at any given step. One important feature of the random walk is that its trajectory of exploration is ungoverned by memory; the past actions of the walker will not influence the probability of the next step in any way.

This is particularly useful in scenarios where a graph search is required, but memory constraints prevent having complete knowledge of the graph or visited nodes as might be seen by DFS or BFS, such as Google's PageRank algorithm for ordering search results, which attempts to approximately model the probability distribution of pages using random walks, since keeping track of every possible page is computationally difficult. In fact, in order to attain even a 50% chance of hitting any particular vertex on the undirected graph starting from any other vertex, requires a random walk with at least V^3 steps, where V is the number of nodes in the graph (Chung). Thus, in quantifying our random walks, we prioritize certain metrics such as mixing time (the time it takes for random walks to reach a stationary probability distribution), sampling time (how quickly probability distribution can be sampled), and filling time (time it takes to reach different areas of the graph) (Qiang).

Another interesting problem I aim to explore within this paper is the viability of recurrence (returning to the point of origin), for both classical and quantum random walks, encapsulated by the quote of the drunk man (signifying a walk in 2d space) and the drunk bird (signifying a walk in 3d space). Moving into the quantum realm, we find that, with some mathematical analysis, quantum random walks show improvements across many of the stated metrics, while also producing interesting phenomena for the notion of recurrence.

2 The Quantum Random Walk

Our quantum random walk replaces the traditional paradigm of the coin flip in deciding the direction to walk with a repeatedly applied Hadamard gate, which induces a superposition upon the initial state. However, instead of having the Hadamard gate directly determine the direction of movement, we instead relegate this propensity to move in an additional degree of freedom known as chirality, represented by basis states (each one encoding a direction) (Nayak). By having the Hadamard apply to this internal state, which is then used to calculate the probability of moving in a direction at any given point, we are able to see the full effects of quantum entanglement in transforming the overall probability distributions, whereas acting directly on the direction with the Hadamard gate would produce symmetric probability every time.

3 Analysis of Quantum Random Walks

Consider a quantum walk along a single dimension. Applying our Hadamard gate to our chirality for each step will really distribute the components of the Hadamard to both the plus and minus basis states for our chirality, as follows, where L is left and R is right (Nayak).

$$|L\rangle \to \frac{1}{\sqrt{2}}\left(|L\rangle + |R\rangle\right)$$

$$|R\rangle \to \frac{1}{\sqrt{2}}\left(|L\rangle - |R\rangle\right)$$

Thus, the wave function of our particle at location n and time t is just a vector for the amplitudes attached to going left or right, as follows.

$$\Psi(n,t) = \begin{pmatrix} \psi_L(n,t) \\ \psi_R(n,t) \end{pmatrix}$$

For the duration of our quantum random walk, a Hadamard gate is applied at each step, taking the place of the coin flip, creating a quantum superposition each time. Thus, each subsequent state at t+1 arrives at that point by recursively applying the Hadamard gate to it's prior left and right states, as follows.

$$\Psi(n,t+1) = \begin{bmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \Psi(n-1,t) + \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix} \Psi(n+1,t)$$

Because our quantum walks have translational invariant, that their rules for motion are the same regardless of position, we can then perform a Fourier transformation, with spatial frequency k and e^{ikn} as our phase factor.

$$\widetilde{\Psi}(k,t) = \sum_n \Psi(n,t) e^{ikn}$$

$$\widetilde{\Psi}(k,t+1) = M_k \widetilde{\Psi}(k,t)$$

$$M_k = e^{ik} M_+ + e^{-ik} M_- = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-ik} & e^{-ik} \\ e^{ik} & -e^{ik} \end{bmatrix}$$

Calculating our eigenvalues and eigenvectors for M, we are able to derive our time evolution matrix.

$$M_k^t = (\lambda_k^1)^t \, |\Phi_k^1\rangle\langle\Phi_k^1| \; + \; (\lambda_k^2)^t \, |\Phi_k^2\rangle\langle\Phi_k^2|$$

$$\widetilde{\psi}_L(k,t) = \frac{1}{2}\left(1 + \frac{\cos k}{\sqrt{1+\cos^2 k}}\right) e^{-i\omega_k t} \; + \; \frac{(-1)^t}{2}\left(1 - \frac{\cos k}{\sqrt{1+\cos^2 k}}\right) e^{i\omega_k t}$$

$$\widetilde{\psi}_R(k,t) = \frac{i e^{ik}}{2\sqrt{1+\cos^2 k}}\left(e^{-i\omega_k t} - (-1)^t e^{i\omega_k t}\right)$$

And finally, we can inverse Fourier transform to see our wave function in real space again, in order to get the amplitudes of our right and left chirality.

$$\psi_L(n,t) = \frac{1+(-1)^{n+t}}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left(1 + \frac{\cos k}{\sqrt{1+\cos^2 k}}\right) e^{-i(\omega_k t + kn)}$$

$$\psi_R(n,t) = \frac{1+(-1)^{n+t}}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{e^{ik}}{\sqrt{1+\cos^2 k}} e^{-i(\omega_k t + kn)}$$

Then, it is just a matter of squaring each of the amplitudes and summing them up, which should net the probability of any particular position α at any time t.

$$P(\alpha,t) = |\psi_L(\alpha t,t)|^2 + |\psi_R(\alpha t,t)|^2$$

$$\approx \frac{1+(-1)^{(\alpha+1)t}}{\pi t \, |\omega'_{k_\alpha}|} \left[(1-\alpha)^2 \cos^2\!\big(\phi(\alpha)t + \pi/4\big) \; + \; (1-\alpha^2)\cos^2\!\big(\phi(\alpha)t + k_\alpha + \pi/4\big)\right] .$$

This final probability distribution, when plotted, appears as follows:

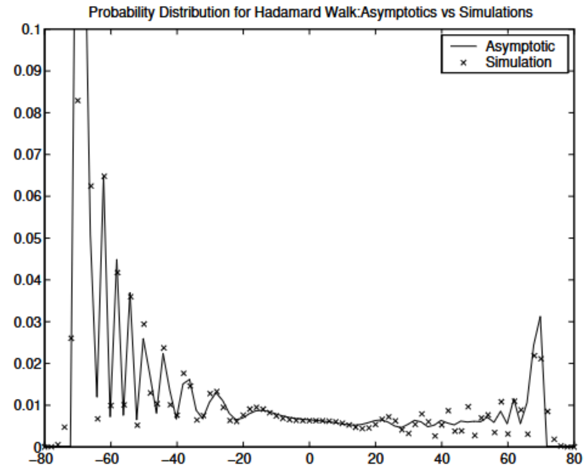

Figure 1: Quantum distribution at time t for position α (Nayak).

A number of interesting phenomenon can explain this probability distribution, so different from the normal distribution to be expected of a classical random walk. In our Hadamard walk, every step created a coherent superposition of the walker going left or right, based on our notion of chirality; after many steps, the distribution is formed from a superposition of many different paths, each with varying complex amplitudes. Near the origin at 0, there are many paths that might lead back to the starting point; many of these may destructively interfere with one another, meaning that when you sum the amplitudes, they cancel each other out, causing the plateau effect seen in the middle of the graph.

The opposite effect is true at the edges, particularly around $\pm t/\sqrt{2}$. Each application of the Hadamard gate creates an outward bias of $1/\sqrt{2}$, which becomes $\pm t/\sqrt{2}$ over t steps. As the walk evolves, the outward moving amplitudes composing the superposition undergoes constructive interference at $\pm t/\sqrt{2}$, creating the observed peaks, where many amplitudes converge as a result of this outward tendency. The fact that the left side is visibly higher is a result of the experiment itself, where the left state was arbitrarily chosen to be the initial chirality; the same would be true of the right if chosen, meaning that the overall graph remains symmetrical.

4 Advantages of Quantum Random Walks

We see that the edges of the distribution occur approximately at $\pm t/\sqrt{2}$, and that the probability mass in between these intervals is roughly uniform. What that tells us is that the standard deviation of our quantum distribution grows linearly with t, the number of steps, a quadratic speed up of the standard deviation for classical walks, $\sqrt{t}$. This entails a quadratic speed up in mixing time, sampling time, and filling time as well. Let us consider the problem discussed earlier, of attaining a 50% chance of hitting any particular node in a random walk, regardless of the starting position. To obtain the standard deviation where this was viable before required us to run V^3 steps, where V is the number of nodes. With our quadratic speed up, we now only require $V^{\frac{3}{2}}$ steps. Such a speed up can drastically improve runtime efficiency in algorithms that utilize these random walks to ascertain stationary probabilities, such as PageRank, or allow for a more thorough search with the saved time (Qiang).

5 Implementation

To implement our quantum random walk, some literature suggests using an ion trap, particularly on a beryllium ion, to confine the charged particle in a small, localized region of space, to observe its quantum state (Travaglione).

One can assemble a new equation,

$$\hat{U} = e^{i\hat{p}\hat{\sigma}_z}\hat{H}$$

a unitary operator assembled from applying the Hadamard gate to put the qubit into quantum superposition, along with a conditional displacement operator $e^{i\hat{p}\hat{\sigma}_z}$, where $\hat{p}$ is the momentum and $\hat{\sigma_z}$ is the Pauli operator on the qubit, determining whether the particle steps $+\hat{p}$ or -$\hat{p}$. This operation can be performed with the following Raman beams:

1. $\pi/2$-Pulse (Hadamard Gate)
$$|\downarrow\rangle \rightarrow \frac{1}{\sqrt{2}}\big(|\downarrow\rangle + |\uparrow\rangle\big).$$
2. First Displacement Beam
$$\frac{1}{\sqrt{2}}\big(|0\rangle|\downarrow\rangle + |\alpha\rangle|\uparrow\rangle\big),$$
where $|\alpha\rangle$ is the displaced coherent state.
3. π-Pulse (Internal State Swap):
$$|\downarrow\rangle \leftrightarrow |\uparrow\rangle.$$
4. Second Displacement Beam
$$\frac{1}{\sqrt{2}}\big(|\alpha\rangle|\downarrow\rangle + |-\alpha\rangle|\uparrow\rangle\big).$$

At each step, we first (1) apply the Hadamard gate to induce the quantum superposition, (2) shoot a displacement beam, which will move the ion so long as the qubit is in the $|\uparrow\rangle$ state, (3) swap the chirality of the particle, and (4) shoot a displacement beam in the opposite direction, which will move the ion so long as the qubit is in the $|\downarrow\rangle$ state. Running a walk with n steps will just be a matter of running this process n times (Travaglione).

In order to measure the results of our walk, we decouple the internal state from the motional state by measuring whether the ion is in the state $|\uparrow\rangle$ or $|\downarrow\rangle$. We apply the operator:

$$\hat{M}^{\pm} = e^{\pm i\hat{p}\hat{\sigma}_y}.$$

We apply the positive Hamiltonian if we get $|\uparrow\rangle$, and the negative otherwise; we measure the internal state of the ion again, and expect to measure the down state $|\downarrow\rangle$ with the probabilities shown in figure 1, dependent on its current position, or with probability 1/2 if decoherence has occurred.

The main setback to an implementation of this variety is the rapidity of decoherence after just a couple steps, such that the quantum random walk very quickly returns to just a standard random walk, with probability 1/2 of moving in either direction.

6 Drunk Birds and Quantum Computers

Finally, I wish to uncover if our drunk bird, the 3d random walk, can be recurrent using a quantum random walk, and thus can find her way home.

Firstly, let us prove that the drunk man doing a random walk in 2d won't get lost, but the drunk bird might. We might reduce the problem of the 2d random walk into a single dimension, and build it back from there. In a single dimension, after 2n steps, your probability of being back at your starting point is

$$\rho_1(2n) = \frac{\binom{2n}{n}}{2^{2n}}.$$

$\binom{2n}{n}$ means out of every step, choose exactly half to be all right or all left, such that the left and the right motions cancel each other out, and our walker returns to the origin; 2^{2n} is the number of all possible assortment of paths of size 2n.

We can use Stirling's rule to approximate our factorials:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Using this, we can approximate the binomial coefficient $\binom{2n}{n}$:

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \sim \frac{\sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n}}{\left[\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right]^2} = \frac{4^n}{\sqrt{\pi n}}.$$

Substituting that back into our probability estimation, we are left with

$$\rho_1(2n) = \frac{\binom{2n}{n}}{2^{2n}} \sim \frac{4^n}{\sqrt{\pi n}\cdot 4^n} = \frac{1}{\sqrt{\pi n}}.$$

This is for one particular value of n. As n approaches infinity, we sum each individual probability of being at the origin after 2n steps (removing the constant, as it is irrelevant as n approaches infinity).

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

This equation is a form of the p-series, with p = 1/2. A p-series is known to diverge if p ¿ 1, and converge otherwise. Since we know this sum diverges, it effectively tells us that even though we are walking for infinite steps, we also return to the origin an infinite number of times. Thus, no matter how long we have been walking, we will always return back to our starting location, at least in 1 dimension!

The same can be said for 2 dimensions; since our random walks are independent and identically distributed, extrapolating a 1d walk returning to the origin into a 2d walk returning to the origin is just a matter of squaring the probability, which as we know, still diverges!

$$\sum_{n=1}^{\infty}(\frac{1}{\sqrt{n}})^2 = \sum_{n=1}^{\infty}(\frac{1}{n})$$

The same is no longer true for 3 dimensions, which converges, meaning that for an infinite number of steps, there are only a finite number of times we return to the origin, meaning that it is not certain that a drunk bird will return to its nest successfully.

$$\sum_{n=1}^{\infty}(\frac{1}{\sqrt{n}})^3 = \sum_{n=1}^{\infty}(\frac{1}{n})^{\frac{3}{2}}$$

For our quantum random walks in 1 dimension our standard deviation grows quadratically faster, but this also means that we will escape from the origin quadratically faster as well. Since the bounds of our distribution grow around $\pm t/\sqrt{2}$ where t is the number of steps (Štefaňák), and distribution is roughly uniform outside the edges, we know that the probability of being at the origin decreases linearly with the number of steps, and thus follows

$$\sum_{n=1}^{\infty}(\frac{1}{n})$$

If one extrapolates this to 2 and 3 dimensions, we see that even the drunk man is not certain to find his way home, much less the drunk bird. As it turns out, access to a quantum computer was not adequate to helping the bird find her way home.

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