Formalisation of Ground Resolution and CDCL in Isabelle/HOL

Mathias Fleury and Jasmin Blanchette

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theory *Prop-Abstract-Transformation*
imports *Main Prop-Logic Wellfounded-More*

begin

This file is devoted to abstract properties of the transformations, like consistency preservation and lifting from terms to proposition.

0.1 Rewrite systems and properties

0.1.1 Lifting of rewrite rules

We can lift a rewrite relation r over a full1 formula: the relation r works on terms, while *propo-rew-step* works on formulas.

inductive *propo-rew-step* :: $('v\ propo \Rightarrow 'v\ propo \Rightarrow bool) \Rightarrow 'v\ propo \Rightarrow 'v\ propo \Rightarrow bool$

for r :: $'v$ *propo* $\Rightarrow$ $'v$ *propo* $\Rightarrow$ *bool* **where**
global-rel: $r\ \varphi\ \psi \Longrightarrow$ *propo-rew-step* $r\ \varphi\ \psi$ |
propo-rew-one-step-lift: *propo-rew-step* $r\ \varphi\ \varphi' \Longrightarrow$ *wf-conn* c (ψs @ φ # $\psi s'$)
$\Longrightarrow$ *propo-rew-step* r (*conn* c (ψs @ φ # $\psi s'$)) (*conn* c (ψs @ φ'# $\psi s'$))

Here is a more precise link between the lifting and the subformulas: if a rewriting takes place between φ and φ', then there are two subformulas ψ in φ and ψ' in φ', ψ' is the result of the rewriting of r on ψ.

This lemma is only a health condition:

lemma *propo-rew-step-subformula-imp*:
shows *propo-rew-step* $r\ \varphi\ \varphi' \Longrightarrow \exists\ \psi\ \psi'.\ \psi \preceq \varphi \wedge \psi' \preceq \varphi' \wedge r\ \psi\ \psi'$
apply (*induct rule*: *propo-rew-step.induct*)
using *subformula.simps subformula-into-subformula* **apply** *blast*
using *wf-conn-no-arity-change subformula-into-subformula wf-conn-no-arity-change-helper*
in-set-conv-decomp **by** *metis*

The converse is moreover true: if there is a ψ and ψ', then every formula φ containing ψ, can be rewritten into a formula φ', such that it contains φ'.

lemma *propo-rew-step-subformula-rec*:
fixes $\psi\ \psi'\ \varphi$ :: $'v$ *propo*
shows $\psi \preceq \varphi \Longrightarrow r\ \psi\ \psi' \Longrightarrow (\exists \varphi'.\ \psi' \preceq \varphi' \wedge$ *propo-rew-step* $r\ \varphi\ \varphi')$
proof (*induct* φ *rule*: *subformula.induct*)
case *subformula-refl*
then have *propo-rew-step* $r\ \psi\ \psi'$ **using** *propo-rew-step.intros* **by** *auto*
moreover have $\psi' \preceq \psi'$ **using** *Prop-Logic.subformula-refl* **by** *auto*
ultimately show $\exists \varphi'.\ \psi' \preceq \varphi' \wedge$ *propo-rew-step* $r\ \psi\ \varphi'$ **by** *fastforce*
next
case (*subformula-into-subformula* $\psi''\ l\ c$)
note *IH* = *this*(*4*) **and** r = *this*(*5*) **and** ψ'' = *this*(*1*) **and** *wf* = *this*(*2*) **and** *incl* = *this*(*3*)
then obtain φ' **where** $*$: $\psi' \preceq \varphi' \wedge$ *propo-rew-step* $r\ \psi''\ \varphi'$ **by** *metis*
moreover obtain $\xi\ \xi'$ :: $'v$ *propo list* **where**
l: $l = \xi$ @ ψ'' # ξ' **using** *List.split-list* ψ'' **by** *metis*
ultimately have *propo-rew-step* r (*conn* $c\ l$) (*conn* c (ξ @ φ' # ξ'))
using *propo-rew-step.intros*(*2*) *wf* **by** *metis*
moreover have $\psi' \preceq$ *conn* c (ξ @ φ' # ξ')
using *wf* $*$ *wf-conn-no-arity-change Prop-Logic.subformula-into-subformula*
by (*metis* (*no-types*) *in-set-conv-decomp l wf-conn-no-arity-change-helper*)
ultimately show $\exists \varphi'.\ \psi' \preceq \varphi' \wedge$ *propo-rew-step* r (*conn* $c\ l$) φ' **by** *metis*
qed

lemma *propo-rew-step-subformula*:
$(\exists \psi\ \psi'.\ \psi \preceq \varphi \wedge r\ \psi\ \psi') \longleftrightarrow (\exists \varphi'.$ *propo-rew-step* $r\ \varphi\ \varphi')$
using *propo-rew-step-subformula-imp propo-rew-step-subformula-rec* **by** *metis*+

lemma *consistency-decompose-into-list*:
assumes *wf*: *wf-conn* $c\ l$ **and** *wf'*: *wf-conn* $c\ l'$
and *same*: $\forall n.\ A \models l\ !\ n \longleftrightarrow (A \models l'\ !\ n)$
shows $A \models$ *conn* $c\ l \longleftrightarrow A \models$ *conn* $c\ l'$
proof (*cases* c *rule*: *connective-cases-arity-2*)
case *nullary*
then show ($A \models$ *conn* $c\ l$) $\longleftrightarrow$ ($A \models$ *conn* $c\ l'$) **using** *wf wf'* **by** *auto*
next
case *unary* **note** c = *this*
then obtain a **where** l: $l = [a]$ **using** *wf-conn-Not-decomp wf* **by** *metis*

```
  obtain a' where l': l' = [a'] using wf-conn-Not-decomp wf' c by metis
  have A ⊨ a ⟷ A ⊨ a' using l l' by (metis nth-Cons-0 same)
  then show A ⊨ conn c l ⟷ A ⊨ conn c l' using  l l' c by auto
next
  case binary note c = this
  then obtain a b where l: l = [a, b]
    using wf-conn-bin-list-length list-length2-decomp wf by metis
  obtain a' b' where l': l' = [a', b']
    using wf-conn-bin-list-length list-length2-decomp wf' c by metis

  have p: A ⊨ a ⟷ A ⊨ a' A ⊨ b ⟷ A ⊨ b'
    using l l' same by (metis diff-Suc-1 nth-Cons' nat.distinct(2))+
  show A ⊨ conn c l ⟷ A ⊨ conn c l'
    using wf c p unfolding binary-connectives-def l l' by auto
qed
```

Relation between *propo-rew-step* and the rewriting we have seen before: *propo-rew-step* $r\ \varphi\ \varphi'$ means that we rewrite ψ inside φ (ie at a path p) into ψ'.

```
lemma propo-rew-step-rewrite:
  fixes φ φ' :: 'v propo and r :: 'v propo ⇒ 'v propo ⇒ bool
  assumes propo-rew-step r φ φ'
  shows ∃ψ ψ' p. r ψ ψ' ∧ path-to p φ ψ ∧ replace-at p φ ψ' = φ'
  using assms
proof (induct rule: propo-rew-step.induct)
  case(global-rel φ ψ)
  moreover have path-to [] φ φ by auto
  moreover have replace-at [] φ ψ = ψ by auto
  ultimately show ?case by metis
next
  case (propo-rew-one-step-lift φ φ' c ξ ξ') note rel = this(1) and IH0 = this(2) and corr = this(3)
  obtain ψ ψ' p where IH: r ψ ψ' ∧ path-to p φ ψ ∧ replace-at p φ ψ' = φ' using IH0 by metis

  {
    fix x :: 'v
    assume c = CT ∨ c = CF ∨ c = CVar x
    then have False using corr by auto
    then have ∃ψ ψ' p. r ψ ψ' ∧ path-to p (conn c (ξ@ (φ # ξ'))) ψ
                    ∧ replace-at p (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
      by fast
  }
  moreover {
    assume c: c = CNot
    then have empty: ξ =[] ξ'=[] using corr by auto
    have path-to (L#p) (conn c (ξ@ (φ # ξ'))) ψ
      using c empty IH wf-conn-unary path-to-l by fastforce
    moreover have replace-at (L#p) (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
      using c empty IH by auto
    ultimately have ∃ψ ψ' p. r ψ ψ' ∧ path-to p (conn c (ξ@ (φ # ξ'))) ψ
                          ∧ replace-at p (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
    using IH by metis
  }
  moreover {
    assume c: c ∈ binary-connectives
    have length (ξ@ φ # ξ') = 2 using wf-conn-bin-list-length corr c by metis
    then have length ξ + length ξ' = 1 by auto
    then have ld: (length ξ = 1 ∧ length ξ' = 0) ∨ (length ξ = 0 ∧ length ξ' = 1)  by arith
```

```
    obtain a b where ab: (ξ=[] ∧ ξ'=[b]) ∨ (ξ=[a] ∧ ξ'=[])
      using ld by (case-tac ξ, case-tac ξ', auto)
    {
      assume φ: ξ=[] ∧ ξ'=[b]
      have path-to (L#p) (conn c (ξ@ (φ # ξ'))) ψ
        using φ c IH ab  corr  by (simp add: path-to-l)
      moreover have replace-at (L#p) (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
        using c IH ab φ unfolding binary-connectives-def by auto
      ultimately have ∃ψ ψ' p. r ψ ψ' ∧ path-to p (conn c (ξ@ (φ # ξ'))) ψ
        ∧ replace-at p (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
        using IH by metis
    }
    moreover {
      assume φ: ξ=[a]  ξ'=[]
      then have path-to (R#p) (conn c (ξ@ (φ # ξ'))) ψ
        using c IH corr path-to-r corr φ  by (simp add: path-to-r)
      moreover have replace-at (R#p) (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
        using c IH ab φ unfolding binary-connectives-def by auto
      ultimately have ?case using IH by metis
    }
    ultimately have ?case using ab by blast
  }
  ultimately show ?case using connective-cases-arity by blast
qed
```

0.1.2 Consistency preservation

We define *preserves-un-sat*: it means that a relation preserves consistency.

```
definition preserves-un-sat where
preserves-un-sat r ⟷ (∀φ ψ. r φ ψ ⟶ (∀A. A ⊨ φ ⟷ A ⊨ ψ))


lemma propo-rew-step-preservers-val-explicit:
propo-rew-step r φ ψ ⟹ preserves-un-sat r ⟹ propo-rew-step r φ ψ ⟹ (∀A. A ⊨φ ⟷ A⊨ψ)
  unfolding preserves-un-sat-def
proof (induction rule: propo-rew-step.induct)
  case global-rel
  then show ?case by simp
next
  case (propo-rew-one-step-lift φ φ' c ξ ξ') note rel = this(1) and wf = this(2)
    and IH = this(3)[OF this(4) this(1)]  and consistent = this(4)
  {
    fix A
    from IH have ∀n. (A ⊨ (ξ @ φ # ξ') ! n) = (A ⊨ (ξ @ φ' # ξ') ! n)
      by (metis (mono-tags, hide-lams) list-update-length nth-Cons-0 nth-append-length-plus
        nth-list-update-neq)
    then have  (A ⊨ conn c (ξ @ φ # ξ')) = (A ⊨ conn c (ξ @ φ' # ξ'))
      by (meson consistency-decompose-into-list wf wf-conn-no-arity-change-helper
        wf-conn-no-arity-change)
  }
  then show ∀A. A ⊨ conn c (ξ @ φ # ξ') ⟷ A ⊨ conn c (ξ @ φ' # ξ') by auto
qed


lemma propo-rew-step-preservers-val':
```

assumes *preserves-un-sat r*
shows *preserves-un-sat* (*propo-rew-step r*)
using *assms* **by** (*simp add*: *preserves-un-sat-def propo-rew-step-preservers-val-explicit*)

lemma *preserves-un-sat-OO*[*intro*]:
preserves-un-sat f $\Longrightarrow$ *preserves-un-sat g* $\Longrightarrow$ *preserves-un-sat* (*f OO g*)
unfolding *preserves-un-sat-def* **by** *auto*

lemma *star-consistency-preservation-explicit*:
assumes (*propo-rew-step r*)$\hat{}$** φ ψ **and** *preserves-un-sat r*
shows $\forall A.\ A \models \varphi \longleftrightarrow A \models \psi$
using *assms* **by** (*induct rule*: *rtranclp-induct*)
(*auto simp add*: *propo-rew-step-preservers-val-explicit*)

lemma *star-consistency-preservation*:
preserves-un-sat r $\Longrightarrow$ *preserves-un-sat* (*propo-rew-step r*)$\hat{}$**
by (*simp add*: *star-consistency-preservation-explicit preserves-un-sat-def*)

0.1.3 Full Lifting

In the previous a relation was lifted to a formula, now we define the relation such it is applied as long as possible. The definition is thus simply: it can be derived and nothing more can be derived.

lemma *full-ropo-rew-step-preservers-val*[*simp*]:
preserves-un-sat r $\Longrightarrow$ *preserves-un-sat* (*full* (*propo-rew-step r*))
by (*metis full-def preserves-un-sat-def star-consistency-preservation*)

lemma *full-propo-rew-step-subformula*:
full (*propo-rew-step r*) φ' φ $\Longrightarrow$ $\neg(\exists\ \psi\ \psi'.\ \psi \preceq \varphi \wedge r\ \psi\ \psi')$
unfolding *full-def* **using** *propo-rew-step-subformula-rec* **by** *metis*

0.2 Transformation testing

0.2.1 Definition and first properties

To prove correctness of our transformation, we create a *all-subformula-st* predicate. It tests recursively all subformulas. At each step, the actual formula is tested. The aim of this *test-symb* function is to test locally some properties of the formulas (i.e. at the level of the connective or at first level). This allows a clause description between the rewrite relation and the *test-symb*

definition *all-subformula-st* :: ($'a$ *propo* $\Rightarrow$ *bool*) $\Rightarrow$ $'a$ *propo* $\Rightarrow$ *bool* **where**
all-subformula-st test-symb $\varphi \equiv \forall \psi.\ \psi \preceq \varphi \longrightarrow$ *test-symb* ψ

lemma *test-symb-imp-all-subformula-st*[*simp*]:
test-symb FT $\Longrightarrow$ *all-subformula-st test-symb FT*
test-symb FF $\Longrightarrow$ *all-subformula-st test-symb FF*
test-symb (*FVar x*) $\Longrightarrow$ *all-subformula-st test-symb* (*FVar x*)
unfolding *all-subformula-st-def* **using** *subformula-leaf* **by** *metis+*

lemma *all-subformula-st-test-symb-true-phi*:

all-subformula-st test-symb $\varphi \Longrightarrow$ *test-symb* φ
unfolding *all-subformula-st-def* **by** *auto*

lemma *all-subformula-st-decomp-imp*:
wf-conn c l $\Longrightarrow$ (*test-symb* (*conn c l*) $\wedge$ ($\forall \varphi \in$ *set l. all-subformula-st test-symb* φ))
$\Longrightarrow$ *all-subformula-st test-symb* (*conn c l*)
unfolding *all-subformula-st-def* **by** *auto*

To ease the finding of proofs, we give some explicit theorem about the decomposition.

lemma *all-subformula-st-decomp-rec*:
all-subformula-st test-symb (*conn c l*) $\Longrightarrow$ *wf-conn c l*
$\Longrightarrow$ (*test-symb* (*conn c l*) $\wedge$ ($\forall \varphi \in$ *set l. all-subformula-st test-symb* φ))
unfolding *all-subformula-st-def* **by** *auto*

lemma *all-subformula-st-decomp*:
fixes *c* :: *'v connective* **and** *l* :: *'v propo list*
assumes *wf-conn c l*
shows *all-subformula-st test-symb* (*conn c l*)
$\longleftrightarrow$ (*test-symb* (*conn c l*) $\wedge$ ($\forall \varphi \in$ *set l. all-subformula-st test-symb* φ))
using *assms all-subformula-st-decomp-rec all-subformula-st-decomp-imp* **by** *metis*

lemma *helper-fact*: $c \in$ *binary-connectives* $\longleftrightarrow$ (c = *COr* $\vee$ c = *CAnd* $\vee$ c = *CEq* $\vee$ c = *CImp*)
unfolding *binary-connectives-def* **by** *auto*
lemma *all-subformula-st-decomp-explicit*[*simp*]:
fixes φ ψ :: *'v propo*
shows *all-subformula-st test-symb* (*FAnd* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FAnd* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
and *all-subformula-st test-symb* (*FOr* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FOr* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
and *all-subformula-st test-symb* (*FNot* φ)
$\longleftrightarrow$ (*test-symb* (*FNot* φ) $\wedge$ *all-subformula-st test-symb* φ)
and *all-subformula-st test-symb* (*FEq* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FEq* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
and *all-subformula-st test-symb* (*FImp* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FImp* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
proof −
have *all-subformula-st test-symb* (*FAnd* φ ψ) $\longleftrightarrow$ *all-subformula-st test-symb* (*conn CAnd* [φ, ψ])
by *auto*
moreover have ... $\longleftrightarrow$*test-symb* (*conn CAnd* [φ, ψ])$\wedge$($\forall \xi \in$ *set* [φ, ψ]. *all-subformula-st test-symb* ξ)
using *all-subformula-st-decomp wf-conn-helper-facts*(*5*) **by** *metis*
finally show *all-subformula-st test-symb* (*FAnd* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FAnd* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
by *simp*

have *all-subformula-st test-symb* (*FOr* φ ψ) $\longleftrightarrow$ *all-subformula-st test-symb* (*conn COr* [φ, ψ])
by *auto*
moreover have ... $\longleftrightarrow$
(*test-symb* (*conn COr* [φ, ψ]) $\wedge$ ($\forall \xi \in$ *set* [φ, ψ]. *all-subformula-st test-symb* ξ))
using *all-subformula-st-decomp wf-conn-helper-facts*(*6*) **by** *metis*
finally show *all-subformula-st test-symb* (*FOr* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FOr* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
by *simp*

have *all-subformula-st test-symb* (*FEq* φ ψ) $\longleftrightarrow$ *all-subformula-st test-symb* (*conn CEq* [φ, ψ])
by *auto*

moreover have ...
$\longleftrightarrow$ (*test-symb* (*conn CEq* [φ, ψ]) $\wedge$ ($\forall \xi \in$ *set* [φ, ψ]. *all-subformula-st test-symb* ξ))
using *all-subformula-st-decomp wf-conn-helper-facts*(*8*) **by** *metis*
finally show *all-subformula-st test-symb* (*FEq* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FEq* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
by *simp*

have *all-subformula-st test-symb* (*FImp* φ ψ) $\longleftrightarrow$ *all-subformula-st test-symb* (*conn CImp* [φ, ψ])
by *auto*
moreover have ...
$\longleftrightarrow$(*test-symb* (*conn CImp* [φ, ψ]) $\wedge$ ($\forall \xi \in$ *set* [φ, ψ]. *all-subformula-st test-symb* ξ))
using *all-subformula-st-decomp wf-conn-helper-facts*(*7*) **by** *metis*
finally show *all-subformula-st test-symb* (*FImp* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FImp* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
by *simp*

have *all-subformula-st test-symb* (*FNot* φ) $\longleftrightarrow$ *all-subformula-st test-symb* (*conn CNot* [φ])
by *auto*
moreover have ... = (*test-symb* (*conn CNot* [φ]) $\wedge$ ($\forall \xi \in$ *set* [φ]. *all-subformula-st test-symb* ξ))
using *all-subformula-st-decomp wf-conn-helper-facts*(*1*) **by** *metis*
finally show *all-subformula-st test-symb* (*FNot* φ)
$\longleftrightarrow$ (*test-symb* (*FNot* φ) $\wedge$ *all-subformula-st test-symb* φ) **by** *simp*
qed

As *all-subformula-st* tests recursively, the function is true on every subformula.

lemma *subformula-all-subformula-st*:
$\psi \preceq \varphi \Longrightarrow$ *all-subformula-st test-symb* $\varphi \Longrightarrow$ *all-subformula-st test-symb* ψ
by (*induct rule*: *subformula.induct*, *auto simp add*: *all-subformula-st-decomp*)

The following theorem *no-test-symb-step-exists* shows the link between the *test-symb* function and the corresponding rewrite relation r: if we assume that if every time *test-symb* is true, then a r can be applied, finally as long as $\neg$ *all-subformula-st test-symb* φ, then something can be rewritten in φ.

lemma *no-test-symb-step-exists*:
fixes r:: *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **and** *test-symb*:: *'v propo* $\Rightarrow$ *bool* **and** x :: *'v*
and φ :: *'v propo*
assumes
test-symb-false-nullary: $\forall x$. *test-symb FF* $\wedge$ *test-symb FT* $\wedge$ *test-symb* (*FVar x*) **and**
$\forall \varphi'$. $\varphi' \preceq \varphi \longrightarrow$ ($\neg$*test-symb* φ') $\longrightarrow$ ($\exists$ ψ. r φ' ψ) **and**
$\neg$ *all-subformula-st test-symb* φ
shows $\exists \psi\ \psi'$. $\psi \preceq \varphi \wedge r\ \psi\ \psi'$
using *assms*
proof (*induct* φ *rule*: *propo-induct-arity*)
case (*nullary* φ x)
then show $\exists \psi\ \psi'$. $\psi \preceq \varphi \wedge r\ \psi\ \psi'$
using *wf-conn-nullary test-symb-false-nullary* **by** *fastforce*
next
case (*unary* φ) **note** *IH* = *this*(*1*)[*OF this*(*2*)] **and** r = *this*(*2*) **and** *nst* = *this*(*3*) **and** *subf* = *this*(*4*)
from *r IH nst* **have** *H*: $\neg$ *all-subformula-st test-symb* $\varphi \Longrightarrow \exists \psi$. $\psi \preceq \varphi \wedge (\exists \psi'$. $r\ \psi\ \psi')$
by (*metis subformula-in-subformula-not subformula-refl subformula-trans*)
{
assume *n*: $\neg$*test-symb* (*FNot* φ)
obtain ψ **where** r (*FNot* φ) ψ **using** *subformula-refl r n nst* **by** *blast*
moreover have *FNot* $\varphi \preceq$ *FNot* φ **using** *subformula-refl* **by** *auto*

ultimately have $\exists \psi\ \psi'.\ \psi \preceq FNot\ \varphi \wedge r\ \psi\ \psi'$ **by** *metis*
}
moreover {
assume *n*: *test-symb* (*FNot* φ)
then have $\neg$ *all-subformula-st test-symb* φ
using *all-subformula-st-decomp-explicit*(*3*) *nst subf* **by** *blast*
then have $\exists \psi\ \psi'.\ \psi \preceq FNot\ \varphi \wedge r\ \psi\ \psi'$
using *H subformula-in-subformula-not subformula-refl subformula-trans* **by** *blast*
}
ultimately show $\exists \psi\ \psi'.\ \psi \preceq FNot\ \varphi \wedge r\ \psi\ \psi'$ **by** *blast*
next
case (*binary* φ $\varphi 1$ $\varphi 2$)
note $IH\varphi 1\text{-}0 = this(1)[OF\ this(4)]$ **and** $IH\varphi 2\text{-}0 = this(2)[OF\ this(4)]$ **and** $r = this(4)$
and $\varphi = this(3)$ **and** $le = this(5)$ **and** $nst = this(6)$

obtain $c :: 'v\ connective$ **where**
c: $(c = CAnd \vee c = COr \vee c = CImp \vee c = CEq) \wedge conn\ c\ [\varphi 1, \varphi 2] = \varphi$
using φ **by** *fastforce*

then have *corr*: *wf-conn c* $[\varphi 1, \varphi 2]$ **using** *wf-conn.simps* **unfolding** *binary-connectives-def* **by** *auto*
have *inc*: $\varphi 1 \preceq \varphi\ \varphi 2 \preceq \varphi$ **using** *binary-connectives-def c subformula-in-binary-conn* **by** *blast+*
from *r* $IH\varphi 1\text{-}0$ **have** $IH\varphi 1$: $\neg$ *all-subformula-st test-symb* $\varphi 1 \Longrightarrow \exists \psi\ \psi'.\ \psi \preceq \varphi 1 \wedge r\ \psi\ \psi'$
using *inc*(*1*) *subformula-trans le* **by** *blast*
from *r* $IH\varphi 2\text{-}0$ **have** $IH\varphi 2$: $\neg$ *all-subformula-st test-symb* $\varphi 2 \Longrightarrow \exists \psi.\ \psi \preceq \varphi 2 \wedge (\exists \psi'.\ r\ \psi\ \psi')$
using *inc*(*2*) *subformula-trans le* **by** *blast*
have *cases*: $\neg$*test-symb* $\varphi \vee \neg$*all-subformula-st test-symb* $\varphi 1 \vee \neg$*all-subformula-st test-symb* $\varphi 2$
using *c nst* **by** *auto*
show $\exists \psi\ \psi'.\ \psi \preceq \varphi \wedge r\ \psi\ \psi'$
using $IH\varphi 1$ $IH\varphi 2$ *subformula-trans inc subformula-refl cases le* **by** *blast*
qed

0.2.2 Invariant conservation

If two rewrite relation are independant (or at least independant enough), then the property characterizing the first relation *all-subformula-st test-symb* remains true. The next show the same property, with changes in the assumptions.

The assumption $\forall \varphi'\ \psi.\ \varphi' \preceq \Phi \longrightarrow r\ \varphi'\ \psi \longrightarrow$ *all-subformula-st test-symb* $\varphi' \longrightarrow$ *all-subformula-st test-symb* ψ means that rewriting with r does not mess up the property we want to preserve locally.

The previous assumption is not enough to go from r to *propo-rew-step* r: we have to add the assumption that rewriting inside does not mess up the term: $\forall c\ \xi\ \varphi\ \xi'\ \varphi'.\ \varphi \preceq \Phi \longrightarrow$ *propo-rew-step* $r\ \varphi\ \varphi' \longrightarrow$ *wf-conn* $c\ (\xi\ @\ \varphi\ \#\ \xi') \longrightarrow$ *test-symb* (*conn* $c\ (\xi\ @\ \varphi\ \#\ \xi')) \longrightarrow$ *test-symb* $\varphi' \longrightarrow$ *test-symb* (*conn* $c\ (\xi\ @\ \varphi'\ \#\ \xi')$)

Invariant while lifting of the rewriting relation

The condition $\varphi \preceq \Phi$ (that will by used with $\Phi = \varphi$ most of the time) is here to ensure that the recursive conditions on Φ will moreover hold for the subterm we are rewriting. For example if there is no equivalence symbol in Φ, we do not have to care about equivalence symbols in the two previous assumptions.

lemma *propo-rew-step-inv-stay'*:
fixes $r :: 'v\ propo \Rightarrow 'v\ propo \Rightarrow bool$ **and** *test-symb*$:: 'v\ propo \Rightarrow bool$ **and** $x :: 'v$

and φ ψ Φ:: *'v propo*
assumes H: $\forall \varphi' \psi.\ \varphi' \preceq \Phi \longrightarrow r\ \varphi'\ \psi \longrightarrow$ *all-subformula-st test-symb* φ'
$\longrightarrow$ *all-subformula-st test-symb* ψ
and H': $\forall$ (c:: *'v connective*) $\xi\ \varphi\ \xi'\ \varphi'.\ \varphi \preceq \Phi \longrightarrow$ *propo-rew-step* $r\ \varphi\ \varphi'$
$\longrightarrow$ *wf-conn* c (ξ @ φ # ξ') $\longrightarrow$ *test-symb* (*conn* c (ξ @ φ # ξ')) $\longrightarrow$ *test-symb* φ'
$\longrightarrow$ *test-symb* (*conn* c (ξ @ φ' # ξ')) **and**
propo-rew-step $r\ \varphi\ \psi$ **and**
$\varphi \preceq \Phi$ **and**
all-subformula-st test-symb φ
shows *all-subformula-st test-symb* ψ
using *assms*(*3*−*5*)
proof (*induct rule*: *propo-rew-step.induct*)
case *global-rel*
then show *?case* **using** H **by** *simp*
next
case (*propo-rew-one-step-lift* $\varphi\ \varphi'\ c\ \xi\ \xi'$)
note *rel* = *this*(*1*) **and** φ = *this*(*2*) **and** *corr* = *this*(*3*) **and** Φ = *this*(*4*) **and** *nst* = *this*(*5*)
have *sq*: $\varphi \preceq \Phi$
using Φ *corr subformula-into-subformula subformula-refl subformula-trans*
by (*metis in-set-conv-decomp*)
from *corr* **have** $\forall\ \psi.\ \psi \in$ *set* (ξ @ φ # ξ') $\longrightarrow$ *all-subformula-st test-symb* ψ
using *all-subformula-st-decomp nst* **by** *blast*
then have $*$: $\forall \psi.\ \psi \in$ *set* (ξ @ φ' # ξ') $\longrightarrow$ *all-subformula-st test-symb* ψ **using** φ *sq* **by** *fastforce*
then have *test-symb* φ' **using** *all-subformula-st-test-symb-true-phi* **by** *auto*
moreover from *corr nst* **have** *test-symb* (*conn* c (ξ @ φ # ξ'))
using *all-subformula-st-decomp* **by** *blast*
ultimately have *test-symb*: *test-symb* (*conn* c (ξ @ φ' # ξ')) **using** H' *sq corr rel* **by** *blast*

have *wf-conn* c (ξ @ φ' # ξ')
by (*metis wf-conn-no-arity-change-helper corr wf-conn-no-arity-change*)
then show *all-subformula-st test-symb* (*conn* c (ξ @ φ' # ξ'))
using $*$ *test-symb* **by** (*metis all-subformula-st-decomp*)
qed

The need for $\varphi \preceq \Phi$ is not always necessary, hence we moreover have a version without inclusion.

lemma *propo-rew-step-inv-stay*:
fixes r:: *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **and** *test-symb*:: *'v propo* $\Rightarrow$ *bool* **and** x :: *'v*
and φ ψ :: *'v propo*
assumes
H: $\forall \varphi' \psi.\ r\ \varphi'\ \psi \longrightarrow$ *all-subformula-st test-symb* $\varphi' \longrightarrow$ *all-subformula-st test-symb* ψ **and**
H': $\forall$ (c:: *'v connective*) $\xi\ \varphi\ \xi'\ \varphi'.$ *wf-conn* c (ξ @ φ # ξ') $\longrightarrow$ *test-symb* (*conn* c (ξ @ φ # ξ'))
$\longrightarrow$ *test-symb* $\varphi' \longrightarrow$ *test-symb* (*conn* c (ξ @ φ' # ξ')) **and**
propo-rew-step $r\ \varphi\ \psi$ **and**
all-subformula-st test-symb φ
shows *all-subformula-st test-symb* ψ
using *propo-rew-step-inv-stay'*[*of* φ r *test-symb* φ ψ] *assms subformula-refl* **by** *metis*

The lemmas can be lifted to *propo-rew-step* $r^{\downarrow}$ instead of *propo-rew-step*

Invariant after all rewriting

lemma *full-propo-rew-step-inv-stay-with-inc*:
fixes r:: *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **and** *test-symb*:: *'v propo* $\Rightarrow$ *bool* **and** x :: *'v*
and φ ψ :: *'v propo*
assumes
H: $\forall\ \varphi\ \psi.$ *propo-rew-step* $r\ \varphi\ \psi \longrightarrow$ *all-subformula-st test-symb* φ

$\longrightarrow$ *all-subformula-st test-symb* ψ **and**
H': $\forall$ (*c*:: *'v connective*) ξ φ ξ' φ'. $\varphi \preceq \Phi \longrightarrow$ *propo-rew-step r* φ φ'
$\longrightarrow$ *wf-conn c* (ξ @ φ # ξ') $\longrightarrow$ *test-symb* (*conn c* (ξ @ φ # ξ')) $\longrightarrow$ *test-symb* φ'
$\longrightarrow$ *test-symb* (*conn c* (ξ @ φ' # ξ')) **and**
$\varphi \preceq \Phi$ **and**
full: *full* (*propo-rew-step r*) φ ψ **and**
init: *all-subformula-st test-symb* φ
shows *all-subformula-st test-symb* ψ
using *assms* **unfolding** *full-def*
proof –
have *rel*: (*propo-rew-step r*)** φ ψ
using *full* **unfolding** *full-def* **by** *auto*
then show *all-subformula-st test-symb* ψ
using *init*
proof (*induct rule*: *rtranclp-induct*)
case *base*
then show *all-subformula-st test-symb* φ **by** *blast*
next
case (*step b c*) **note** *star* = *this*(*1*) **and** *IH* = *this*(*3*) **and** *one* = *this*(*2*) **and** *all* = *this*(*4*)
then have *all-subformula-st test-symb b* **by** *metis*
then show *all-subformula-st test-symb c* **using** *propo-rew-step-inv-stay' H H' rel one* **by** *auto*
qed
qed

lemma *full-propo-rew-step-inv-stay'*:
fixes *r*:: *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **and** *test-symb*:: *'v propo* $\Rightarrow$ *bool* **and** *x* :: *'v*
and φ ψ :: *'v propo*
assumes
H: $\forall$ φ ψ. *propo-rew-step r* φ ψ $\longrightarrow$ *all-subformula-st test-symb* φ
$\longrightarrow$ *all-subformula-st test-symb* ψ **and**
H': $\forall$ (*c*:: *'v connective*) ξ φ ξ' φ'. *propo-rew-step r* φ φ' $\longrightarrow$ *wf-conn c* (ξ @ φ # ξ')
$\longrightarrow$ *test-symb* (*conn c* (ξ @ φ # ξ')) $\longrightarrow$ *test-symb* φ' $\longrightarrow$ *test-symb* (*conn c* (ξ @ φ' # ξ')) **and**
full: *full* (*propo-rew-step r*) φ ψ **and**
init: *all-subformula-st test-symb* φ
shows *all-subformula-st test-symb* ψ
using *full-propo-rew-step-inv-stay-with-inc*[*of r test-symb* φ] *assms subformula-refl* **by** *metis*

lemma *full-propo-rew-step-inv-stay*:
fixes *r*:: *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **and** *test-symb*:: *'v propo* $\Rightarrow$ *bool* **and** *x* :: *'v*
and φ ψ :: *'v propo*
assumes
H: $\forall \varphi$ ψ. *r* φ ψ $\longrightarrow$ *all-subformula-st test-symb* φ $\longrightarrow$ *all-subformula-st test-symb* ψ **and**
H': $\forall$ (*c*:: *'v connective*) ξ φ ξ' φ'. *wf-conn c* (ξ @ φ # ξ') $\longrightarrow$ *test-symb* (*conn c* (ξ @ φ # ξ'))
$\longrightarrow$ *test-symb* φ' $\longrightarrow$ *test-symb* (*conn c* (ξ @ φ' # ξ')) **and**
full: *full* (*propo-rew-step r*) φ ψ **and**
init: *all-subformula-st test-symb* φ
shows *all-subformula-st test-symb* ψ
unfolding *full-def*
proof –
have *rel*: (*propo-rew-step r*)^** φ ψ
using *full* **unfolding** *full-def* **by** *auto*
then show *all-subformula-st test-symb* ψ
using *init*
proof (*induct rule*: *rtranclp-induct*)
case *base*
then show *all-subformula-st test-symb* φ **by** *blast*

```
  next
    case (step b c)
    note star = this(1) and IH = this(3) and one = this(2) and all = this(4)
    then have all-subformula-st test-symb b by metis
    then show all-subformula-st test-symb c
      using propo-rew-step-inv-stay subformula-refl H H' rel one by auto
  qed
qed


lemma full-propo-rew-step-inv-stay-conn:
  fixes r:: 'v propo ⇒ 'v propo ⇒ bool and test-symb:: 'v propo ⇒ bool and x :: 'v
  and φ ψ :: 'v propo
  assumes
    H: ∀ φ ψ. r φ ψ ⟶ all-subformula-st test-symb φ ⟶ all-subformula-st test-symb ψ and
    H': ∀ (c:: 'v connective) l l'. wf-conn c l ⟶ wf-conn c l'
      ⟶ (test-symb (conn c l) ⟷ test-symb (conn c l')) and
    full: full (propo-rew-step r) φ ψ and
    init: all-subformula-st test-symb φ
  shows all-subformula-st test-symb ψ
proof −
  have ⋀(c:: 'v connective) ξ φ ξ' φ'. wf-conn c (ξ @ φ # ξ')
    ⟹ test-symb (conn c (ξ @ φ # ξ')) ⟹ test-symb φ' ⟹ test-symb (conn c (ξ @ φ' # ξ'))
    using H' by (metis wf-conn-no-arity-change-helper wf-conn-no-arity-change)
  then show all-subformula-st test-symb ψ
    using H full init full-propo-rew-step-inv-stay by blast
qed

end
theory Prop-Normalisation
imports Main Prop-Logic Prop-Abstract-Transformation ../lib/Multiset-More
begin
```

Given the previous definition about abstract rewriting and theorem about them, we now have the detailed rule making the transformation into CNF/DNF.

0.3 Rewrite Rules

The idea of Christoph Weidenbach's book is to remove gradually the operators: first equivalencies, then implication, after that the unused true/false and finally the reorganizing the or/and. We will prove each transformation seperately.

0.3.1 Elimination of the equivalences

The first transformation consists in removing every equivalence symbol.

```
inductive elim-equiv :: 'v propo ⇒ 'v propo ⇒ bool where
elim-equiv[simp]: elim-equiv (FEq φ ψ) (FAnd (FImp φ ψ)  (FImp ψ φ))

lemma elim-equiv-transformation-consistent:
A ⊨ FEq φ ψ ⟷ A ⊨ FAnd (FImp φ ψ) (FImp ψ φ)
  by auto

lemma elim-equiv-explicit: elim-equiv φ ψ ⟹ ∀ A. A ⊨ φ ⟷ A ⊨ ψ
```

by (*induct rule*: *elim-equiv.induct*, *auto*)

lemma *elim-equiv-consistent*: *preserves-un-sat elim-equiv*
unfolding *preserves-un-sat-def* **by** (*simp add*: *elim-equiv-explicit*)

lemma *elimEquv-lifted-consistant*:
preserves-un-sat (*full* (*propo-rew-step elim-equiv*))
by (*simp add*: *elim-equiv-consistent*)

This function ensures that there is no equivalencies left in the formula tested by *no-equiv-symb*.

fun *no-equiv-symb* :: *'v propo* ⇒ *bool* **where**
no-equiv-symb (*FEq* - -) = *False* |
no-equiv-symb - = *True*

Given the definition of *no-equiv-symb*, it does not depend on the formula, but only on the connective used.

lemma *no-equiv-symb-conn-characterization*[*simp*]:
fixes *c* :: *'v connective* **and** *l* :: *'v propo list*
assumes *wf*: *wf-conn c l*
shows *no-equiv-symb* (*conn c l*) ⟷ *c* ≠ *CEq*
by (*metis connective.distinct*(*13*,*25*,*35*,*43*) *wf no-equiv-symb.elims*(*3*) *no-equiv-symb.simps*(*1*)
wf-conn.cases wf-conn-list(*6*))

definition *no-equiv* **where** *no-equiv* = *all-subformula-st no-equiv-symb*

lemma *no-equiv-eq*[*simp*]:
fixes *φ ψ* :: *'v propo*
shows
¬*no-equiv* (*FEq φ ψ*)
no-equiv FT
no-equiv FF
using *no-equiv-symb.simps*(*1*) *all-subformula-st-test-symb-true-phi* **unfolding** *no-equiv-def* **by** *auto*

The following lemma helps to reconstruct *no-equiv* expressions: this representation is easier to use than the set definition.

lemma *all-subformula-st-decomp-explicit-no-equiv*[*iff*]:
fixes *φ ψ* :: *'v propo*
shows
no-equiv (*FNot φ*) ⟷ *no-equiv φ*
no-equiv (*FAnd φ ψ*) ⟷ (*no-equiv φ* ∧ *no-equiv ψ*)
no-equiv (*FOr φ ψ*) ⟷ (*no-equiv φ* ∧ *no-equiv ψ*)
no-equiv (*FImp φ ψ*) ⟷ (*no-equiv φ* ∧ *no-equiv ψ*)
by (*auto simp*: *no-equiv-def*)

A theorem to show the link between the rewrite relation *elim-equiv* and the function *no-equiv-symb*. This theorem is one of the assumption we need to characterize the transformation.

lemma *no-equiv-elim-equiv-step*:
fixes *φ* :: *'v propo*
assumes *no-equiv*: ¬ *no-equiv φ*
shows ∃*ψ ψ'*. *ψ* ≼ *φ* ∧ *elim-equiv ψ ψ'*
proof −
have *test-symb-false-nullary*:
∀*x*::*'v*. *no-equiv-symb FF* ∧ *no-equiv-symb FT* ∧ *no-equiv-symb* (*FVar x*)
unfolding *no-equiv-def* **by** *auto*

moreover {
fix *c*:: *'v connective* **and** *l* :: *'v propo list* **and** ψ :: *'v propo*
assume *a1*: *elim-equiv* (*conn c l*) ψ
have $\bigwedge p\ pa.\ \neg$ *elim-equiv* (*p*::*'v propo*) *pa* $\vee \neg$ *no-equiv-symb p*
using *elim-equiv.cases no-equiv-symb.simps*(*1*) **by** *blast*
then have *elim-equiv* (*conn c l*) $\psi \Longrightarrow \neg$*no-equiv-symb* (*conn c l*) **using** *a1* **by** *metis*
}
moreover have *H'*: $\forall\psi.\ \neg$*elim-equiv FT* ψ $\forall\psi.\ \neg$*elim-equiv FF* ψ $\forall\psi\ x.\ \neg$*elim-equiv* (*FVar x*) ψ
using *elim-equiv.cases* **by** *auto*
moreover have $\bigwedge\varphi.\ \neg$ *no-equiv-symb* $\varphi \Longrightarrow \exists\psi.$ *elim-equiv* φ ψ
by (*case-tac* φ, *auto simp*: *elim-equiv.simps*)
then have $\bigwedge\varphi'.\ \varphi' \preceq \varphi \Longrightarrow \neg$*no-equiv-symb* $\varphi' \Longrightarrow \exists\psi.$ *elim-equiv* φ' ψ **by** *force*
ultimately show *?thesis*
using *no-test-symb-step-exists no-equiv test-symb-false-nullary* **unfolding** *no-equiv-def* **by** *blast*
qed

Given all the previous theorem and the characterization, once we have rewritten everything, there is no equivalence symbol any more.

lemma *no-equiv-full-propo-rew-step-elim-equiv*:
full (*propo-rew-step elim-equiv*) φ $\psi \Longrightarrow$ *no-equiv* ψ
using *full-propo-rew-step-subformula no-equiv-elim-equiv-step* **by** *blast*

0.3.2 Eliminate Implication

After that, we can eliminate the implication symbols.

inductive *elim-imp* :: *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **where**
[*simp*]: *elim-imp* (*FImp* φ ψ) (*FOr* (*FNot* φ) ψ)

lemma *elim-imp-transformation-consistent*:
$A \models$ *FImp* φ $\psi \longleftrightarrow A \models$ *FOr* (*FNot* φ) ψ
by *auto*

lemma *elim-imp-explicit*: *elim-imp* φ $\psi \Longrightarrow \forall A.\ A \models \varphi \longleftrightarrow A \models \psi$
by (*induct* φ ψ *rule*: *elim-imp.induct*, *auto*)

lemma *elim-imp-consistent*: *preserves-un-sat elim-imp*
unfolding *preserves-un-sat-def* **by** (*simp add*: *elim-imp-explicit*)

lemma *elim-imp-lifted-consistant*:
preserves-un-sat (*full* (*propo-rew-step elim-imp*))
by (*simp add*: *elim-imp-consistent*)

fun *no-imp-symb* **where**
no-imp-symb (*FImp - -*) = *False* |
no-imp-symb - = True

lemma *no-imp-symb-conn-characterization*:
wf-conn c l $\Longrightarrow$ *no-imp-symb* (*conn c l*) $\longleftrightarrow c \neq$ *CImp*
by (*induction rule*: *wf-conn-induct*) *auto*

definition *no-imp* **where** *no-imp* $\equiv$ *all-subformula-st no-imp-symb*
declare *no-imp-def*[*simp*]

lemma *no-imp-Imp*[*simp*]:
$\neg$*no-imp* (*FImp* φ ψ)

no-imp FT
no-imp FF
unfolding *no-imp-def* **by** *auto*

lemma *all-subformula-st-decomp-explicit-imp*[*simp*]:
fixes φ ψ :: *'v propo*
shows
no-imp (*FNot* φ) $\longleftrightarrow$ *no-imp* φ
no-imp (*FAnd* φ ψ) $\longleftrightarrow$ (*no-imp* φ $\wedge$ *no-imp* ψ)
no-imp (*FOr* φ ψ) $\longleftrightarrow$ (*no-imp* φ $\wedge$ *no-imp* ψ)
by *auto*

Invariant of the *elim-imp* transformation

lemma *elim-imp-no-equiv*:
elim-imp φ ψ $\Longrightarrow$ *no-equiv* φ $\Longrightarrow$ *no-equiv* ψ
by (*induct* φ ψ *rule*: *elim-imp.induct*, *auto*)

lemma *elim-imp-inv*:
fixes φ ψ :: *'v propo*
assumes *full* (*propo-rew-step elim-imp*) φ ψ **and** *no-equiv* φ
shows *no-equiv* ψ
using *full-propo-rew-step-inv-stay-conn*[*of elim-imp no-equiv-symb* φ ψ] *assms elim-imp-no-equiv*
no-equiv-symb-conn-characterization **unfolding** *no-equiv-def* **by** *metis*

lemma *no-no-imp-elim-imp-step-exists*:
fixes φ :: *'v propo*
assumes *no-equiv*: $\neg$ *no-imp* φ
shows $\exists \psi\ \psi'.\ \psi \preceq \varphi \wedge$ *elim-imp* ψ ψ'
proof −
have *test-symb-false-nullary*: $\forall x.$ *no-imp-symb FF* $\wedge$ *no-imp-symb FT* $\wedge$ *no-imp-symb* (*FVar* (x:: *'v*))
by *auto*
moreover {
fix c:: *'v connective* **and** l :: *'v propo list* **and** ψ :: *'v propo*
have H: *elim-imp* (*conn c l*) ψ $\Longrightarrow$ $\neg$*no-imp-symb* (*conn c l*)
by (*auto elim*: *elim-imp.cases*)
}
moreover
have H': $\forall \psi.$ $\neg$*elim-imp FT* ψ $\forall \psi.$ $\neg$*elim-imp FF* ψ $\forall \psi$ $x.$ $\neg$*elim-imp* (*FVar x*) ψ
by (*auto elim*: *elim-imp.cases*)+
moreover
have $\bigwedge \varphi.$ $\neg$ *no-imp-symb* φ $\Longrightarrow$ $\exists \psi.$ *elim-imp* φ ψ
by (*case-tac* φ) (*force simp*: *elim-imp.simps*)+
then have $\bigwedge \varphi'.$ $\varphi' \preceq \varphi$ $\Longrightarrow$ $\neg$*no-imp-symb* φ' $\Longrightarrow$ $\exists$ $\psi.$ *elim-imp* φ' ψ **by** *force*
ultimately show *?thesis*
using *no-test-symb-step-exists no-equiv test-symb-false-nullary* **unfolding** *no-imp-def* **by** *blast*
qed

lemma *no-imp-full-propo-rew-step-elim-imp*: *full* (*propo-rew-step elim-imp*) φ ψ $\Longrightarrow$ *no-imp* ψ
using *full-propo-rew-step-subformula no-no-imp-elim-imp-step-exists* **by** *blast*

0.3.3 Eliminate all the True and False in the formula

Contrary to the book, we have to give the transformation and the "commutative" transformation. The latter is implicit in the book.

inductive *elimTB* **where**

ElimTB1: *elimTB* (*FAnd* φ *FT*) φ |
ElimTB1': *elimTB* (*FAnd FT* φ) φ |

ElimTB2: *elimTB* (*FAnd* φ *FF*) *FF* |
ElimTB2': *elimTB* (*FAnd FF* φ) *FF* |

ElimTB3: *elimTB* (*FOr* φ *FT*) *FT* |
ElimTB3': *elimTB* (*FOr FT* φ) *FT* |

ElimTB4: *elimTB* (*FOr* φ *FF*) φ |
ElimTB4': *elimTB* (*FOr FF* φ) φ |

ElimTB5: *elimTB* (*FNot FT*) *FF* |
ElimTB6: *elimTB* (*FNot FF*) *FT*

lemma *elimTB-consistent*: *preserves-un-sat elimTB*
proof −
{
fix φ ψ:: *'b propo*
have *elimTB* φ ψ $\Longrightarrow \forall A.\ A \models \varphi \longleftrightarrow A \models \psi$ **by** (*induction rule*: *elimTB.inducts*) *auto*
}
then show *?thesis* **using** *preserves-un-sat-def* **by** *auto*
qed

inductive *no-T-F-symb* :: *'v propo* ⇒ *bool* **where**
no-T-F-symb-comp: $c \neq CF \Longrightarrow c \neq CT \Longrightarrow$ *wf-conn c l* $\Longrightarrow (\forall \varphi \in$ *set l*. $\varphi \neq FT \wedge \varphi \neq FF)$
$\Longrightarrow$ *no-T-F-symb* (*conn c l*)

lemma *wf-conn-no-T-F-symb-iff*[*simp*]:
wf-conn c $\psi s \Longrightarrow$
no-T-F-symb (*conn c* ψs) $\longleftrightarrow (c \neq CF \wedge c \neq CT \wedge (\forall \psi \in$ *set* ψs. $\psi \neq FF \wedge \psi \neq FT))$
unfolding *no-T-F-symb.simps* **apply** (*cases c*)
using *wf-conn-list*(*1*) **apply** *fastforce*
using *wf-conn-list*(*2*) **apply** *fastforce*
using *wf-conn-list*(*3*) **apply** *fastforce*
apply (*metis* (*no-types*, *hide-lams*) *conn-inj connective.distinct*(*5*,*17*))
using *conn-inj* **apply** *blast+*
done

lemma *wf-conn-no-T-F-symb-iff-explicit*[*simp*]:
no-T-F-symb (*FAnd* φ ψ) $\longleftrightarrow (\forall \chi \in$ *set* $[\varphi, \psi]$. $\chi \neq FF \wedge \chi \neq FT)$
no-T-F-symb (*FOr* φ ψ) $\longleftrightarrow (\forall \chi \in$ *set* $[\varphi, \psi]$. $\chi \neq FF \wedge \chi \neq FT)$
no-T-F-symb (*FEq* φ ψ) $\longleftrightarrow (\forall \chi \in$ *set* $[\varphi, \psi]$. $\chi \neq FF \wedge \chi \neq FT)$
no-T-F-symb (*FImp* φ ψ) $\longleftrightarrow (\forall \chi \in$ *set* $[\varphi, \psi]$. $\chi \neq FF \wedge \chi \neq FT)$
apply (*metis conn.simps*(*36*) *conn.simps*(*37*) *conn.simps*(*5*) *propo.distinct*(*19*)
wf-conn-helper-facts(*5*) *wf-conn-no-T-F-symb-iff*)
apply (*metis conn.simps*(*36*) *conn.simps*(*37*) *conn.simps*(*6*) *propo.distinct*(*22*)
wf-conn-helper-facts(*6*) *wf-conn-no-T-F-symb-iff*)
using *wf-conn-no-T-F-symb-iff* **apply** *fastforce*
by (*metis conn.simps*(*36*) *conn.simps*(*37*) *conn.simps*(*7*) *propo.distinct*(*23*) *wf-conn-helper-facts*(*7*)
wf-conn-no-T-F-symb-iff)

lemma *no-T-F-symb-false*[*simp*]:

fixes c :: $'v$ *connective*
shows
$\neg$*no-T-F-symb* (*FT* :: $'v$ *propo*)
$\neg$*no-T-F-symb* (*FF* :: $'v$ *propo*)
by (*metis* (*no-types*) *conn.simps*(*1*,*2*) *wf-conn-no-T-F-symb-iff wf-conn-nullary*)+

lemma *no-T-F-symb-bool*[*simp*]:
fixes x :: $'v$
shows *no-T-F-symb* (*FVar* x)
using *no-T-F-symb-comp wf-conn-nullary* **by** (*metis connective.distinct*(*3*, *15*) *conn.simps*(*3*) *empty-iff list.set*(*1*))

lemma *no-T-F-symb-fnot-imp*:
$\neg$*no-T-F-symb* (*FNot* φ) $\Longrightarrow$ φ = *FT* $\vee$ φ = *FF*
proof (*rule ccontr*)
assume *n*: $\neg$ *no-T-F-symb* (*FNot* φ)
assume $\neg$ (φ = *FT* $\vee$ φ = *FF*)
then have $\forall \varphi' \in$ *set* [φ]. $\varphi' \neq$*FT* $\wedge$ $\varphi' \neq$*FF* **by** *auto*
moreover have *wf-conn CNot* [φ] **by** *simp*
ultimately have *no-T-F-symb* (*FNot* φ)
using *no-T-F-symb.intros* **by** (*metis conn.simps*(*4*) *connective.distinct*(*5*,*17*))
then show *False* **using** *n* **by** *blast*
qed

lemma *no-T-F-symb-fnot*[*simp*]:
no-T-F-symb (*FNot* φ) $\longleftrightarrow$ $\neg$(φ = *FT* $\vee$ φ = *FF*)
using *no-T-F-symb.simps no-T-F-symb-fnot-imp* **by** (*metis conn-inj-not*(*2*) *list.set-intros*(*1*))

Actually it is not possible to remover every *FT* and *FF*: if the formula is equal to true or false, we can not remove it.

inductive *no-T-F-symb-except-toplevel* **where**
no-T-F-symb-except-toplevel-true[*simp*]: *no-T-F-symb-except-toplevel FT* |
no-T-F-symb-except-toplevel-false[*simp*]: *no-T-F-symb-except-toplevel FF* |
noTrue-no-T-F-symb-except-toplevel[*simp*]: *no-T-F-symb* φ $\Longrightarrow$ *no-T-F-symb-except-toplevel* φ

lemma *no-T-F-symb-except-toplevel-bool*:
fixes x :: $'v$
shows *no-T-F-symb-except-toplevel* (*FVar* x)
by *simp*

lemma *no-T-F-symb-except-toplevel-not-decom*:
$\varphi \neq$ *FT* $\Longrightarrow$ $\varphi \neq$ *FF* $\Longrightarrow$ *no-T-F-symb-except-toplevel* (*FNot* φ)
by *simp*

lemma *no-T-F-symb-except-toplevel-bin-decom*:
fixes φ ψ :: $'v$ *propo*
assumes $\varphi \neq$ *FT* **and** $\varphi \neq$ *FF* **and** $\psi \neq$ *FT* **and** $\psi \neq$ *FF*
and *c*: *c*$\in$ *binary-connectives*
shows *no-T-F-symb-except-toplevel* (*conn c* [φ, ψ])
by (*metis* (*no-types*, *lifting*) *assms c conn.simps*(*4*) *list.discI noTrue-no-T-F-symb-except-toplevel wf-conn-no-T-F-symb-iff no-T-F-symb-fnot set-ConsD wf-conn-binary wf-conn-helper-facts*(*1*) *wf-conn-list-decomp*(*1*,*2*))

lemma *no-T-F-symb-except-toplevel-if-is-a-true-false*:
fixes l :: $'v$ *propo list* **and** c :: $'v$ *connective*

assumes *corr*: *wf-conn c l*
and $FT \in$ *set l* $\vee$ $FF \in$ *set l*
shows $\neg$*no-T-F-symb-except-toplevel* (*conn c l*)
by (*metis assms empty-iff no-T-F-symb-except-toplevel.simps wf-conn-no-T-F-symb-iff set-empty wf-conn-list*(*1*,*2*))

lemma *no-T-F-symb-except-top-level-false-example*[*simp*]:
fixes φ ψ :: *'v propo*
assumes $\varphi = FT \vee \psi = FT \vee \varphi = FF \vee \psi = FF$
shows
$\neg$ *no-T-F-symb-except-toplevel* (*FAnd* φ ψ)
$\neg$ *no-T-F-symb-except-toplevel* (*FOr* φ ψ)
$\neg$ *no-T-F-symb-except-toplevel* (*FImp* φ ψ)
$\neg$ *no-T-F-symb-except-toplevel* (*FEq* φ ψ)
using *assms no-T-F-symb-except-toplevel-if-is-a-true-false* **unfolding** *binary-connectives-def*
by (*metis* (*no-types*) *conn.simps*(*5*−*8*) *insert-iff list.simps*(*14*−*15*) *wf-conn-helper-facts*(*5*−*8*))+

lemma *no-T-F-symb-except-top-level-false-not*[*simp*]:
fixes φ ψ :: *'v propo*
assumes $\varphi = FT \vee \varphi = FF$
shows
$\neg$ *no-T-F-symb-except-toplevel* (*FNot* φ)
by (*simp add*: *assms no-T-F-symb-except-toplevel.simps*)

This is the local extension of *no-T-F-symb-except-toplevel*.

definition *no-T-F-except-top-level* **where**
no-T-F-except-top-level $\equiv$ *all-subformula-st no-T-F-symb-except-toplevel*

This is another property we will use. While this version might seem to be the one we want to prove, it is not since FT can not be reduced.

definition *no-T-F* **where**
no-T-F $\equiv$ *all-subformula-st no-T-F-symb*

lemma *no-T-F-except-top-level-false*:
fixes *l* :: *'v propo list* **and** *c* :: *'v connective*
assumes *wf-conn c l*
and $FT \in$ *set l* $\vee$ $FF \in$ *set l*
shows $\neg$*no-T-F-except-top-level* (*conn c l*)
by (*simp add*: *all-subformula-st-decomp assms no-T-F-except-top-level-def no-T-F-symb-except-toplevel-if-is-a-true-false*)

lemma *no-T-F-except-top-level-false-example*[*simp*]:
fixes φ ψ :: *'v propo*
assumes $\varphi = FT \vee \psi = FT \vee \varphi = FF \vee \psi = FF$
shows
$\neg$*no-T-F-except-top-level* (*FAnd* φ ψ)
$\neg$*no-T-F-except-top-level* (*FOr* φ ψ)
$\neg$*no-T-F-except-top-level* (*FEq* φ ψ)
$\neg$*no-T-F-except-top-level* (*FImp* φ ψ)
by (*metis all-subformula-st-test-symb-true-phi assms no-T-F-except-top-level-def no-T-F-symb-except-top-level-false-example*)+

lemma *no-T-F-symb-except-toplevel-no-T-F-symb*:

no-T-F-symb-except-toplevel $\varphi \Longrightarrow \varphi \neq FF \Longrightarrow \varphi \neq FT \Longrightarrow$ *no-T-F-symb* φ
by (*induct rule*: *no-T-F-symb-except-toplevel.induct*, *auto*)

The two following lemmas give the precise link between the two definitions.

lemma *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb*:
no-T-F-except-top-level $\varphi \Longrightarrow \varphi \neq FF \Longrightarrow \varphi \neq FT \Longrightarrow$ *no-T-F* φ
unfolding *no-T-F-except-top-level-def no-T-F-def* **apply** (*induct* φ)
using *no-T-F-symb-fnot* **by** *fastforce+*

lemma *no-T-F-no-T-F-except-top-level*:
no-T-F $\varphi \Longrightarrow$ *no-T-F-except-top-level* φ
unfolding *no-T-F-except-top-level-def no-T-F-def*
unfolding *all-subformula-st-def* **by** *auto*

lemma *no-T-F-except-top-level-simp*[*simp*]: *no-T-F-except-top-level FF no-T-F-except-top-level FT*
unfolding *no-T-F-except-top-level-def* **by** *auto*

lemma *no-T-F-no-T-F-except-top-level'*[*simp*]:
no-T-F-except-top-level $\varphi \longleftrightarrow (\varphi = FF \lor \varphi = FT \lor$ *no-T-F* $\varphi)$
using *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb no-T-F-no-T-F-except-top-level*
by *auto*

lemma *no-T-F-bin-decomp*[*simp*]:
assumes *c*: $c \in$ *binary-connectives*
shows *no-T-F* (*conn c* [φ, ψ]) $\longleftrightarrow$ (*no-T-F* $\varphi \land$ *no-T-F* ψ)
proof −
have *wf*: *wf-conn c* [φ, ψ] **using** *c* **by** *auto*
then have *no-T-F* (*conn c* [φ, ψ]) $\longleftrightarrow$ (*no-T-F-symb* (*conn c* [φ, ψ]) $\land$ *no-T-F* $\varphi \land$ *no-T-F* ψ)
by (*simp add*: *all-subformula-st-decomp no-T-F-def*)
then show *no-T-F* (*conn c* [φ, ψ]) $\longleftrightarrow$ (*no-T-F* $\varphi \land$ *no-T-F* ψ)
using *c wf all-subformula-st-decomp list.discI no-T-F-def no-T-F-symb-except-toplevel-bin-decom no-T-F-symb-except-toplevel-no-T-F-symb no-T-F-symb-false*(*1*,*2*) *wf-conn-helper-facts*(*2*,*3*) *wf-conn-list*(*1*,*2*) **by** *metis*
qed

lemma *no-T-F-bin-decomp-expanded*[*simp*]:
assumes *c*: $c = CAnd \lor c = COr \lor c = CEq \lor c = CImp$
shows *no-T-F* (*conn c* [φ, ψ]) $\longleftrightarrow$ (*no-T-F* $\varphi \land$ *no-T-F* ψ)
using *no-T-F-bin-decomp assms* **unfolding** *binary-connectives-def* **by** *blast*

lemma *no-T-F-comp-expanded-explicit*[*simp*]:
fixes φ ψ :: *'v propo*
shows
no-T-F (*FAnd* φ ψ) $\longleftrightarrow$ (*no-T-F* $\varphi \land$ *no-T-F* ψ)
no-T-F (*FOr* φ ψ) $\longleftrightarrow$ (*no-T-F* $\varphi \land$ *no-T-F* ψ)
no-T-F (*FEq* φ ψ) $\longleftrightarrow$ (*no-T-F* $\varphi \land$ *no-T-F* ψ)
no-T-F (*FImp* φ ψ) $\longleftrightarrow$ (*no-T-F* $\varphi \land$ *no-T-F* ψ)
using *conn.simps*(*5*−*8*) *no-T-F-bin-decomp-expanded* **by** (*metis* (*no-types*))+

lemma *no-T-F-comp-not*[*simp*]:
fixes φ ψ :: *'v propo*
shows *no-T-F* (*FNot* φ) $\longleftrightarrow$ *no-T-F* φ
by (*metis all-subformula-st-decomp-explicit*(*3*) *all-subformula-st-test-symb-true-phi no-T-F-def no-T-F-symb-false*(*1*,*2*) *no-T-F-symb-fnot-imp*)

lemma *no-T-F-decomp*:

fixes φ ψ :: *'v propo*
assumes φ: *no-T-F* (*FAnd* φ ψ) $\vee$ *no-T-F* (*FOr* φ ψ) $\vee$ *no-T-F* (*FEq* φ ψ) $\vee$ *no-T-F* (*FImp* φ ψ)
shows *no-T-F* ψ **and** *no-T-F* φ
using *assms* **by** *auto*

lemma *no-T-F-decomp-not*:
fixes φ :: *'v propo*
assumes φ: *no-T-F* (*FNot* φ)
shows *no-T-F* φ
using *assms* **by** *auto*

lemma *no-T-F-symb-except-toplevel-step-exists*:
fixes φ ψ :: *'v propo*
assumes *no-equiv* φ **and** *no-imp* φ
shows $\psi \preceq \varphi \Longrightarrow \neg$ *no-T-F-symb-except-toplevel* $\psi \Longrightarrow \exists \psi'$. *elimTB* ψ ψ'
proof (*induct* ψ *rule*: *propo-induct-arity*)
case (*nullary* φ' *x*)
then have *False* **using** *no-T-F-symb-except-toplevel-true no-T-F-symb-except-toplevel-false* **by** *auto*
then show *?case* **by** *blast*
next
case (*unary* ψ)
then have $\psi = FF \vee \psi = FT$ **using** *no-T-F-symb-except-toplevel-not-decom* **by** *blast*
then show *?case* **using** *ElimTB5 ElimTB6* **by** *blast*
next
case (*binary* φ' $\psi 1$ $\psi 2$)
note *IH1* = *this*(*1*) **and** *IH2* = *this*(*2*) **and** φ' = *this*(*3*) **and** $F\varphi$ = *this*(*4*) **and** *n* = *this*(*5*)
{
assume φ' = *FImp* $\psi 1$ $\psi 2$ $\vee$ φ' = *FEq* $\psi 1$ $\psi 2$
then have *False* **using** *n* $F\varphi$ *subformula-all-subformula-st assms*
by (*metis* (*no-types*) *no-equiv-eq*(*1*) *no-equiv-def no-imp-Imp*(*1*) *no-imp-def*)
then have *?case* **by** *blast*
}
moreover {
assume φ': φ' = *FAnd* $\psi 1$ $\psi 2$ $\vee$ φ' = *FOr* $\psi 1$ $\psi 2$
then have $\psi 1 = FT \vee \psi 2 = FT \vee \psi 1 = FF \vee \psi 2 = FF$
using *no-T-F-symb-except-toplevel-bin-decom conn.simps*(*5*,*6*) *n* **unfolding** *binary-connectives-def*
by *fastforce+*
then have *?case* **using** *elimTB.intros* φ' **by** *blast*
}
ultimately show *?case* **using** φ' **by** *blast*
qed

lemma *no-T-F-except-top-level-rew*:
fixes φ :: *'v propo*
assumes *noTB*: $\neg$ *no-T-F-except-top-level* φ **and** *no-equiv*: *no-equiv* φ **and** *no-imp*: *no-imp* φ
shows $\exists \psi$ ψ'. $\psi \preceq \varphi \wedge$ *elimTB* ψ ψ'
proof −
have *test-symb-false-nullary*: $\forall x$. *no-T-F-symb-except-toplevel* (*FF*:: *'v propo*)
$\wedge$ *no-T-F-symb-except-toplevel FT* $\wedge$ *no-T-F-symb-except-toplevel* (*FVar* (*x*:: *'v*)) **by** *auto*
moreover {
fix *c*:: *'v connective* **and** *l* :: *'v propo list* **and** ψ :: *'v propo*
have *H*: *elimTB* (*conn c l*) $\psi \Longrightarrow \neg$*no-T-F-symb-except-toplevel* (*conn c l*)
by (*cases conn c l rule*: *elimTB.cases*, *auto*)
}
moreover {
fix *x* :: *'v*

have H': *no-T-F-except-top-level FT* *no-T-F-except-top-level FF*
no-T-F-except-top-level (*FVar x*)
by (*auto simp*: *no-T-F-except-top-level-def test-symb-false-nullary*)
}
moreover {
fix ψ
have $\psi \preceq \varphi \Longrightarrow \neg$ *no-T-F-symb-except-toplevel* $\psi \Longrightarrow \exists \psi'$. *elimTB* ψ ψ'
using *no-T-F-symb-except-toplevel-step-exists no-equiv no-imp* **by** *auto*
}
ultimately show *?thesis*
using *no-test-symb-step-exists noTB* **unfolding** *no-T-F-except-top-level-def* **by** *blast*
qed

lemma *elimTB-inv*:
fixes φ ψ :: *'v propo*
assumes *full* (*propo-rew-step elimTB*) φ ψ
and *no-equiv* φ **and** *no-imp* φ
shows *no-equiv* ψ **and** *no-imp* ψ
proof −
{
fix φ ψ :: *'v propo*
have H: *elimTB* φ $\psi \Longrightarrow$ *no-equiv* $\varphi \Longrightarrow$ *no-equiv* ψ
by (*induct* φ ψ *rule*: *elimTB.induct*, *auto*)
}
then show *no-equiv* ψ
using *full-propo-rew-step-inv-stay-conn*[*of elimTB no-equiv-symb* φ ψ]
no-equiv-symb-conn-characterization assms **unfolding** *no-equiv-def* **by** *metis*
next
{
fix φ ψ :: *'v propo*
have H: *elimTB* φ $\psi \Longrightarrow$ *no-imp* $\varphi \Longrightarrow$ *no-imp* ψ
by (*induct* φ ψ *rule*: *elimTB.induct*, *auto*)
}
then show *no-imp* ψ
using *full-propo-rew-step-inv-stay-conn*[*of elimTB no-imp-symb* φ ψ] *assms*
no-imp-symb-conn-characterization **unfolding** *no-imp-def* **by** *metis*
qed

lemma *elimTB-full-propo-rew-step*:
fixes φ ψ :: *'v propo*
assumes *no-equiv* φ **and** *no-imp* φ **and** *full* (*propo-rew-step elimTB*) φ ψ
shows *no-T-F-except-top-level* ψ
using *full-propo-rew-step-subformula no-T-F-except-top-level-rew assms elimTB-inv* **by** *fastforce*

0.3.4 PushNeg

Push the negation inside the formula, until the litteral.

inductive *pushNeg* **where**
PushNeg1[*simp*]: *pushNeg* (*FNot* (*FAnd* φ ψ)) (*FOr* (*FNot* φ) (*FNot* ψ)) |
PushNeg2[*simp*]: *pushNeg* (*FNot* (*FOr* φ ψ)) (*FAnd* (*FNot* φ) (*FNot* ψ)) |
PushNeg3[*simp*]: *pushNeg* (*FNot* (*FNot* φ)) φ

lemma *pushNeg-transformation-consistent*:
$A \models$ *FNot* (*FAnd* φ ψ) $\longleftrightarrow A \models$ (*FOr* (*FNot* φ) (*FNot* ψ))

$A \models FNot\ (FOr\ \varphi\ \psi) \longleftrightarrow A \models (FAnd\ (FNot\ \varphi)\ (FNot\ \psi))$
$A \models FNot\ (FNot\ \varphi) \longleftrightarrow A \models \varphi$
by *auto*

lemma *pushNeg-explicit*: $pushNeg\ \varphi\ \psi \Longrightarrow \forall A.\ A \models \varphi \longleftrightarrow A \models \psi$
by (*induct* $\varphi\ \psi$ *rule*: *pushNeg.induct*, *auto*)

lemma *pushNeg-consistent*: *preserves-un-sat pushNeg*
unfolding *preserves-un-sat-def* **by** (*simp add*: *pushNeg-explicit*)

lemma *pushNeg-lifted-consistant*:
preserves-un-sat (*full* (*propo-rew-step pushNeg*))
by (*simp add*: *pushNeg-consistent*)

fun *simple* **where**
simple FT = *True* |
simple FF = *True* |
simple (*FVar* -) = *True* |
simple - = *False*

lemma *simple-decomp*:
$simple\ \varphi \longleftrightarrow (\varphi = FT \lor \varphi = FF \lor (\exists x.\ \varphi = FVar\ x))$
by (*cases* φ) *auto*

lemma *subformula-conn-decomp-simple*:
fixes $\varphi\ \psi$ :: *'v propo*
assumes *s*: *simple* ψ
shows $\varphi \preceq FNot\ \psi \longleftrightarrow (\varphi = FNot\ \psi \lor \varphi = \psi)$
proof −
have $\varphi \preceq conn\ CNot\ [\psi] \longleftrightarrow (\varphi = conn\ CNot\ [\psi] \lor (\exists\ \psi \in set\ [\psi].\ \varphi \preceq \psi))$
using *subformula-conn-decomp wf-conn-helper-facts*(*1*) **by** *metis*
then show $\varphi \preceq FNot\ \psi \longleftrightarrow (\varphi = FNot\ \psi \lor \varphi = \psi)$ **using** *s* **by** (*auto simp*: *simple-decomp*)
qed

lemma *subformula-conn-decomp-explicit*[*simp*]:
fixes φ :: *'v propo* **and** *x* :: *'v*
shows
$\varphi \preceq FNot\ FT \longleftrightarrow (\varphi = FNot\ FT \lor \varphi = FT)$
$\varphi \preceq FNot\ FF \longleftrightarrow (\varphi = FNot\ FF \lor \varphi = FF)$
$\varphi \preceq FNot\ (FVar\ x) \longleftrightarrow (\varphi = FNot\ (FVar\ x) \lor \varphi = FVar\ x)$
by (*auto simp*: *subformula-conn-decomp-simple*)

fun *simple-not-symb* **where**
simple-not-symb (*FNot* φ) = (*simple* φ) |
simple-not-symb - = *True*

definition *simple-not* **where**
simple-not = *all-subformula-st simple-not-symb*
declare *simple-not-def*[*simp*]

lemma *simple-not-Not*[*simp*]:
$\neg$ *simple-not* (*FNot* (*FAnd* $\varphi\ \psi$))
$\neg$ *simple-not* (*FNot* (*FOr* $\varphi\ \psi$))

by *auto*

lemma *simple-not-step-exists*:
fixes φ ψ :: *'v propo*
assumes *no-equiv* φ **and** *no-imp* φ
shows $\psi \preceq \varphi \Longrightarrow \neg$ *simple-not-symb* $\psi \Longrightarrow \exists \psi'$. *pushNeg* ψ ψ'
apply (*induct* ψ, *auto*)
apply (*rename-tac* ψ, *case-tac* ψ, *auto intro*: *pushNeg.intros*)
by (*metis assms*(*1*,*2*) *no-imp-Imp*(*1*) *no-equiv-eq*(*1*) *no-imp-def no-equiv-def*
subformula-in-subformula-not subformula-all-subformula-st)+

lemma *simple-not-rew*:
fixes φ :: *'v propo*
assumes *noTB*: $\neg$ *simple-not* φ **and** *no-equiv*: *no-equiv* φ **and** *no-imp*: *no-imp* φ
shows $\exists \psi$ ψ'. $\psi \preceq \varphi \wedge$ *pushNeg* ψ ψ'
proof −
have $\forall x$. *simple-not-symb* (*FF*:: *'v propo*) $\wedge$ *simple-not-symb FT* $\wedge$ *simple-not-symb* (*FVar* (*x*:: *'v*))
by *auto*
moreover {
fix *c*:: *'v connective* **and** *l* :: *'v propo list* **and** ψ :: *'v propo*
have *H*: *pushNeg* (*conn c l*) $\psi \Longrightarrow \neg$*simple-not-symb* (*conn c l*)
by (*cases conn c l rule*: *pushNeg.cases*) *auto*
}
moreover {
fix *x* :: *'v*
have *H'*: *simple-not FT simple-not FF simple-not* (*FVar x*)
by *simp-all*
}
moreover {
fix ψ :: *'v propo*
have $\psi \preceq \varphi \Longrightarrow \neg$ *simple-not-symb* $\psi \Longrightarrow \exists \psi'$. *pushNeg* ψ ψ'
using *simple-not-step-exists no-equiv no-imp* **by** *blast*
}
ultimately show *?thesis* **using** *no-test-symb-step-exists noTB* **unfolding** *simple-not-def* **by** *blast*
qed

lemma *no-T-F-except-top-level-pushNeg1*:
no-T-F-except-top-level (*FNot* (*FAnd* φ ψ)) $\Longrightarrow$ *no-T-F-except-top-level* (*FOr* (*FNot* φ) (*FNot* ψ))
using *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb no-T-F-comp-not no-T-F-decomp*(*1*)
no-T-F-decomp(*2*) *no-T-F-no-T-F-except-top-level* **by** (*metis no-T-F-comp-expanded-explicit*(*2*)
propo.distinct(*5*,*17*))

lemma *no-T-F-except-top-level-pushNeg2*:
no-T-F-except-top-level (*FNot* (*FOr* φ ψ)) $\Longrightarrow$ *no-T-F-except-top-level* (*FAnd* (*FNot* φ) (*FNot* ψ))
by *auto*

lemma *no-T-F-symb-pushNeg*:
no-T-F-symb (*FOr* (*FNot* φ') (*FNot* ψ'))
no-T-F-symb (*FAnd* (*FNot* φ') (*FNot* ψ'))
no-T-F-symb (*FNot* (*FNot* φ'))
by *auto*

lemma *propo-rew-step-pushNeg-no-T-F-symb*:
propo-rew-step pushNeg φ $\psi \Longrightarrow$ *no-T-F-except-top-level* $\varphi \Longrightarrow$ *no-T-F-symb* $\varphi \Longrightarrow$ *no-T-F-symb* ψ
apply (*induct rule*: *propo-rew-step.induct*)
apply (*cases rule*: *pushNeg.cases*)

```
  apply simp-all
  apply (metis no-T-F-symb-pushNeg(1))
  apply (metis no-T-F-symb-pushNeg(2))
  apply (simp, metis all-subformula-st-test-symb-true-phi no-T-F-def)
proof −
  fix φ φ'::'a propo and c::'a connective and ξ ξ'::'a propo list
  assume rel: propo-rew-step pushNeg φ φ'
  and IH: no-T-F φ ⟹ no-T-F-symb φ ⟹ no-T-F-symb φ'
  and wf: wf-conn c (ξ @ φ # ξ')
  and n: conn c (ξ @ φ # ξ') = FF ∨ conn c (ξ @ φ # ξ') = FT ∨ no-T-F (conn c (ξ @ φ # ξ'))
  and x: c ≠ CF ∧ c ≠ CT ∧ φ ≠ FF ∧ φ ≠ FT ∧ (∀ψ ∈ set ξ ∪ set ξ'. ψ ≠ FF ∧ ψ ≠ FT)
  then have c ≠ CF ∧ c ≠ CF ∧ wf-conn c (ξ @ φ' # ξ')
    using wf-conn-no-arity-change-helper  wf-conn-no-arity-change by metis
  moreover have n': no-T-F (conn c (ξ @ φ # ξ')) using n by (simp add: wf wf-conn-list(1,2))
  moreover
  {
    have no-T-F φ
      by (metis Un-iff all-subformula-st-decomp list.set-intros(1)  n' wf no-T-F-def set-append)
    moreover then have no-T-F-symb φ
      by (simp add: all-subformula-st-test-symb-true-phi no-T-F-def)
    ultimately have φ' ≠ FF ∧ φ' ≠ FT
      using IH no-T-F-symb-false(1) no-T-F-symb-false(2) by blast
    then have ∀ψ∈ set (ξ @ φ' # ξ'). ψ ≠ FF ∧ ψ ≠ FT using x by auto
  }
  ultimately show no-T-F-symb (conn c (ξ @ φ' # ξ')) by (simp add: x)
qed

lemma propo-rew-step-pushNeg-no-T-F:
  propo-rew-step pushNeg φ ψ ⟹ no-T-F φ ⟹ no-T-F ψ
proof (induct rule: propo-rew-step.induct)
  case global-rel
  then show ?case
    by (metis (no-types, lifting) no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb
      no-T-F-def no-T-F-except-top-level-pushNeg1 no-T-F-except-top-level-pushNeg2
      no-T-F-no-T-F-except-top-level all-subformula-st-decomp-explicit(3) pushNeg.simps
      simple.simps(1,2,5,6))
next
  case (propo-rew-one-step-lift φ φ' c ξ ξ')
  note rel = this(1) and IH = this(2) and wf = this(3) and no-T-F = this(4)
  moreover have wf': wf-conn c (ξ @ φ' # ξ')
    using wf-conn-no-arity-change wf-conn-no-arity-change-helper wf by metis
  ultimately show no-T-F (conn c (ξ @ φ' # ξ'))
    using all-subformula-st-test-symb-true-phi
    by (fastforce simp: no-T-F-def all-subformula-st-decomp wf wf')
qed


lemma pushNeg-inv:
  fixes φ ψ :: 'v propo
  assumes full (propo-rew-step pushNeg) φ ψ
  and no-equiv φ and no-imp φ and no-T-F-except-top-level φ
  shows no-equiv ψ and no-imp ψ and no-T-F-except-top-level ψ
proof −
  {
    fix φ ψ :: 'v propo
    assume rel: propo-rew-step pushNeg φ ψ
```

```
    and  no: no-T-F-except-top-level φ
    then have  no-T-F-except-top-level ψ
     proof −
       {
         assume φ = FT ∨ φ = FF
         from rel this have False
           apply (induct rule: propo-rew-step.induct)
             using pushNeg.cases apply blast
           using wf-conn-list(1) wf-conn-list(2) by auto
         then have no-T-F-except-top-level ψ by blast
       }
       moreover {
         assume φ ≠ FT ∧ φ ≠ FF
         then have no-T-F φ
           by (metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
         then have no-T-F ψ
           using propo-rew-step-pushNeg-no-T-F rel by auto
         then have no-T-F-except-top-level ψ by (simp add: no-T-F-no-T-F-except-top-level)
       }
       ultimately show no-T-F-except-top-level ψ by metis
     qed
  }
  moreover {
    fix c :: 'v connective and ξ ξ' :: 'v propo list and ζ ζ' :: 'v propo
    assume rel: propo-rew-step pushNeg ζ ζ'
    and incl: ζ ≼ φ
    and corr: wf-conn c (ξ @ ζ # ξ')
    and no-T-F: no-T-F-symb-except-toplevel (conn c (ξ @ ζ # ξ'))
    and n: no-T-F-symb-except-toplevel ζ'
    have no-T-F-symb-except-toplevel (conn c (ξ @ ζ' # ξ'))
    proof
      have p: no-T-F-symb (conn c (ξ @ ζ # ξ'))
        using corr wf-conn-list(1) wf-conn-list(2) no-T-F-symb-except-toplevel-no-T-F-symb no-T-F
        by blast
      have l: ∀φ∈set (ξ @ ζ # ξ'). φ ≠ FT ∧ φ ≠ FF
        using corr wf-conn-no-T-F-symb-iff p by blast
      from rel incl have ζ'≠FT ∧ζ'≠FF
        apply (induction ζ ζ' rule: propo-rew-step.induct)
        apply (cases rule: pushNeg.cases, auto)
        by (metis assms(4) no-T-F-symb-except-top-level-false-not no-T-F-except-top-level-def
          all-subformula-st-test-symb-true-phi subformula-in-subformula-not
          subformula-all-subformula-st append-is-Nil-conv list.distinct(1)
          wf-conn-no-arity-change-helper wf-conn-list(1,2) wf-conn-no-arity-change)+
      then have ∀φ ∈ set (ξ @ ζ' # ξ'). φ ≠ FT ∧ φ ≠ FF using l by auto
      moreover have c ≠ CT ∧ c ≠ CF using corr by auto
      ultimately show no-T-F-symb (conn c (ξ @ ζ' # ξ'))
        by (metis corr no-T-F-symb-comp wf-conn-no-arity-change wf-conn-no-arity-change-helper)
    qed
  }
  ultimately show no-T-F-except-top-level ψ
    using full-propo-rew-step-inv-stay-with-inc[of pushNeg no-T-F-symb-except-toplevel φ] assms
      subformula-refl unfolding no-T-F-except-top-level-def full-unfold by metis
next
  {
    fix φ ψ :: 'v propo
    have H: pushNeg φ ψ ⟹ no-equiv φ ⟹ no-equiv ψ
```

```
    by (induct φ ψ rule: pushNeg.induct, auto)
  }
  then show no-equiv ψ
    using full-propo-rew-step-inv-stay-conn[of pushNeg no-equiv-symb φ ψ]
    no-equiv-symb-conn-characterization assms unfolding no-equiv-def full-unfold by metis
next
  {
    fix φ ψ :: 'v propo
    have H: pushNeg φ ψ ⟹ no-imp φ ⟹  no-imp ψ
      by (induct φ ψ rule: pushNeg.induct, auto)
  }
  then show no-imp ψ
    using full-propo-rew-step-inv-stay-conn[of pushNeg no-imp-symb φ ψ] assms
      no-imp-symb-conn-characterization unfolding no-imp-def full-unfold by metis
qed


lemma pushNeg-full-propo-rew-step:
  fixes φ ψ :: 'v propo
  assumes
    no-equiv φ and
    no-imp φ and
    full (propo-rew-step pushNeg) φ ψ and
    no-T-F-except-top-level φ
  shows simple-not ψ
  using assms full-propo-rew-step-subformula pushNeg-inv(1,2) simple-not-rew by blast
```

0.3.5 Push inside

```
inductive push-conn-inside :: 'v connective ⇒ 'v connective ⇒ 'v propo ⇒ 'v propo ⇒ bool
  for c c':: 'v connective where
push-conn-inside-l[simp]: c = CAnd ∨ c = COr ⟹ c' = CAnd ∨ c' = COr
  ⟹ push-conn-inside c c' (conn c [conn c' [φ1, φ2], ψ])
       (conn c' [conn c [φ1, ψ], conn c [φ2, ψ]]) |
push-conn-inside-r[simp]: c = CAnd ∨ c = COr ⟹ c' = CAnd ∨ c' = COr
  ⟹ push-conn-inside c c' (conn c [ψ, conn c' [φ1, φ2]])
    (conn c' [conn c [ψ, φ1], conn c [ψ, φ2]])


lemma push-conn-inside-explicit: push-conn-inside c c' φ ψ ⟹ ∀ A. A⊨φ ⟷ A⊨ψ
  by (induct φ ψ rule: push-conn-inside.induct, auto)

lemma push-conn-inside-consistent: preserves-un-sat (push-conn-inside c c')
  unfolding preserves-un-sat-def by (simp add: push-conn-inside-explicit)

lemma propo-rew-step-push-conn-inside[simp]:
 ¬propo-rew-step (push-conn-inside c c') FT ψ ¬propo-rew-step (push-conn-inside c c') FF ψ
 proof −
  {
    {
      fix φ ψ
      have push-conn-inside c c' φ ψ ⟹ φ = FT ∨ φ = FF ⟹ False
        by (induct rule: push-conn-inside.induct, auto)
    } note H = this
    fix φ
    have propo-rew-step (push-conn-inside c c') φ ψ ⟹ φ = FT ∨ φ = FF ⟹ False
```

apply (*induct rule*: *propo-rew-step.induct*, *auto simp*: *wf-conn-list*(*1*) *wf-conn-list*(*2*))
using *H* **by** *blast+*
}
then show
¬*propo-rew-step* (*push-conn-inside c c′*) *FT ψ*
¬*propo-rew-step* (*push-conn-inside c c′*) *FF ψ* **by** *blast+*
qed

inductive *not-c-in-c′-symb*:: *′v connective* ⇒ *′v connective* ⇒ *′v propo* ⇒ *bool* **for** *c c′* **where**
not-c-in-c′-symb-l[*simp*]: *wf-conn c* [*conn c′* [*φ*, *φ′*], *ψ*] ⟹ *wf-conn c′* [*φ*, *φ′*]
⟹ *not-c-in-c′-symb c c′* (*conn c* [*conn c′* [*φ*, *φ′*], *ψ*]) |
not-c-in-c′-symb-r[*simp*]: *wf-conn c* [*ψ*, *conn c′* [*φ*, *φ′*]] ⟹ *wf-conn c′* [*φ*, *φ′*]
⟹ *not-c-in-c′-symb c c′* (*conn c* [*ψ*, *conn c′* [*φ*, *φ′*]])

abbreviation *c-in-c′-symb c c′ φ* ≡ ¬*not-c-in-c′-symb c c′ φ*

lemma *c-in-c′-symb-simp*:
not-c-in-c′-symb c c′ ξ ⟹ *ξ* = *FF* ∨ *ξ* = *FT* ∨ *ξ* = *FVar x* ∨ *ξ* = *FNot FF* ∨ *ξ* = *FNot FT*
∨ *ξ* = *FNot* (*FVar x*)⟹ *False*
apply (*induct rule*: *not-c-in-c′-symb.induct*, *auto simp*: *wf-conn.simps wf-conn-list*(*1*−*3*))
using *conn-inj-not*(*2*) *wf-conn-binary* **unfolding** *binary-connectives-def* **by** *fastforce+*

lemma *c-in-c′-symb-simp′*[*simp*]:
¬*not-c-in-c′-symb c c′ FF*
¬*not-c-in-c′-symb c c′ FT*
¬*not-c-in-c′-symb c c′* (*FVar x*)
¬*not-c-in-c′-symb c c′* (*FNot FF*)
¬*not-c-in-c′-symb c c′* (*FNot FT*)
¬*not-c-in-c′-symb c c′* (*FNot* (*FVar x*))
using *c-in-c′-symb-simp* **by** *metis+*

definition *c-in-c′-only* **where**
c-in-c′-only c c′ ≡ *all-subformula-st* (*c-in-c′-symb c c′*)

lemma *c-in-c′-only-simp*[*simp*]:
c-in-c′-only c c′ FF
c-in-c′-only c c′ FT
c-in-c′-only c c′ (*FVar x*)
c-in-c′-only c c′ (*FNot FF*)
c-in-c′-only c c′ (*FNot FT*)
c-in-c′-only c c′ (*FNot* (*FVar x*))
unfolding *c-in-c′-only-def* **by** *auto*

lemma *not-c-in-c′-symb-commute*:
not-c-in-c′-symb c c′ ξ ⟹ *wf-conn c* [*φ*, *ψ*] ⟹ *ξ* = *conn c* [*φ*, *ψ*]
⟹ *not-c-in-c′-symb c c′* (*conn c* [*ψ*, *φ*])
proof (*induct rule*: *not-c-in-c′-symb.induct*)
case (*not-c-in-c′-symb-r φ′ φ″ ψ′*) **note** *H* = *this*
then have *ψ*: *ψ* = *conn c′* [*φ″*, *ψ′*] **using** *conn-inj* **by** *auto*
have *wf-conn c* [*conn c′* [*φ″*, *ψ′*], *φ*]
using *H*(*1*) *wf-conn-no-arity-change length-Cons* **by** *metis*
then show *not-c-in-c′-symb c c′* (*conn c* [*ψ*, *φ*])
unfolding *ψ* **using** *not-c-in-c′-symb.intros*(*1*) *H* **by** *auto*

```
next
  case (not-c-in-c′-symb-l φ′ φ′′ ψ′) note H = this
  then have φ = conn c′ [φ′, φ′′] using conn-inj by auto
  moreover have wf-conn c [ψ′, conn c′ [φ′, φ′′]]
    using H(1) wf-conn-no-arity-change length-Cons by metis
  ultimately show  not-c-in-c′-symb c c′ (conn c [ψ, φ])
    using not-c-in-c′-symb.intros(2) conn-inj not-c-in-c′-symb-l.hyps
      not-c-in-c′-symb-l.prems(1,2) by blast
qed

lemma not-c-in-c′-symb-commute′:
  wf-conn c [φ, ψ] ⟹ c-in-c′-symb c c′ (conn c [φ, ψ]) ⟷ c-in-c′-symb c c′ (conn c [ψ, φ])
  using not-c-in-c′-symb-commute wf-conn-no-arity-change by (metis length-Cons)

lemma not-c-in-c′-comm:
  assumes wf: wf-conn c [φ, ψ]
  shows c-in-c′-only c c′ (conn c [φ, ψ]) ⟷ c-in-c′-only c c′ (conn c [ψ, φ]) (is ?A ⟷ ?B)
proof −
  have ?A ⟷ (c-in-c′-symb c c′ (conn c [φ, ψ])
              ∧ (∀ξ ∈ set [φ, ψ]. all-subformula-st (c-in-c′-symb c c′) ξ))
    using all-subformula-st-decomp wf unfolding c-in-c′-only-def by fastforce
  also have ... ⟷ (c-in-c′-symb c c′ (conn c [ψ, φ])
                  ∧ (∀ξ ∈ set [ψ, φ]. all-subformula-st (c-in-c′-symb c c′) ξ))
    using not-c-in-c′-symb-commute′ wf by auto
  also
    have wf-conn c [ψ, φ] using wf-conn-no-arity-change wf by (metis length-Cons)
    then have (c-in-c′-symb c c′ (conn c [ψ, φ])
            ∧ (∀ξ ∈ set [ψ, φ]. all-subformula-st (c-in-c′-symb c c′) ξ))
          ⟷ ?B
      using all-subformula-st-decomp  unfolding c-in-c′-only-def by fastforce
  finally show ?thesis .
qed

lemma not-c-in-c′-simp[simp]:
  fixes φ1 φ2 ψ :: ′v propo and x :: ′v
  shows
  c-in-c′-symb c c′ FT
  c-in-c′-symb c c′ FF
  c-in-c′-symb c c′ (FVar x)
  wf-conn c [conn c′ [φ1, φ2], ψ] ⟹ wf-conn c′ [φ1, φ2]
    ⟹ ¬ c-in-c′-only c c′ (conn c [conn c′ [φ1, φ2], ψ])
  apply (simp-all add: c-in-c′-only-def)
  using all-subformula-st-test-symb-true-phi not-c-in-c′-symb-l by blast

lemma c-in-c′-symb-not[simp]:
  fixes c c′ :: ′v connective and ψ :: ′v propo
  shows c-in-c′-symb c c′ (FNot ψ)
proof −
  {
    fix ξ :: ′v propo
    have not-c-in-c′-symb c c′ (FNot ψ) ⟹ False
      apply (induct FNot ψ rule: not-c-in-c′-symb.induct)
      using conn-inj-not(2) by blast+
  }
 then show ?thesis by auto
qed
```

```
lemma c-in-c′-symb-step-exists:
  fixes φ :: ′v propo
  assumes c: c = CAnd ∨ c = COr and c′: c′ = CAnd ∨ c′ = COr
  shows ψ ≼ φ ⟹ ¬ c-in-c′-symb c c′ ψ ⟹ ∃ψ′. push-conn-inside c c′ ψ ψ′
  apply (induct ψ rule: propo-induct-arity)
  apply auto[2]
proof −
  fix ψ1 ψ2 φ′:: ′v propo
  assume IHψ1: ψ1 ≼ φ ⟹ ¬ c-in-c′-symb c c′ ψ1 ⟹ Ex (push-conn-inside c c′ ψ1)
  and IHψ2: ψ1 ≼ φ ⟹ ¬ c-in-c′-symb c c′ ψ1 ⟹ Ex (push-conn-inside c c′ ψ1)
  and φ′: φ′ = FAnd ψ1 ψ2 ∨ φ′ = FOr ψ1 ψ2 ∨ φ′ = FImp ψ1 ψ2 ∨ φ′ = FEq ψ1 ψ2
  and inφ: φ′ ≼ φ and  n0: ¬c-in-c′-symb c c′ φ′
  then have n: not-c-in-c′-symb c c′ φ′ by auto
  {
    assume φ′: φ′ = conn c [ψ1, ψ2]
    obtain a b where ψ1 = conn c′ [a, b] ∨ ψ2 = conn c′ [a, b]
      using n φ′ apply (induct rule: not-c-in-c′-symb.induct)
      using c by force+
    then have Ex (push-conn-inside c c′ φ′)
      unfolding φ′ apply auto
      using push-conn-inside.intros(1) c c′ apply blast
      using push-conn-inside.intros(2) c c′ by blast
  }
  moreover {
     assume φ′: φ′ ≠ conn c [ψ1, ψ2]
     have ∀φ c ca. ∃φ1 ψ1 ψ2 ψ1′ ψ2′ φ2′. conn (c::′v connective) [φ1, conn ca [ψ1, ψ2]] = φ
           ∨ conn c [conn ca [ψ1′, ψ2′], φ2′] = φ ∨ c-in-c′-symb c ca φ
       by (metis not-c-in-c′-symb.cases)
     then have Ex (push-conn-inside c c′ φ′)
       by (metis (no-types) c c′ n push-conn-inside-l push-conn-inside-r)
  }
  ultimately show Ex (push-conn-inside c c′ φ′) by blast
qed


lemma c-in-c′-symb-rew:
  fixes φ :: ′v propo
  assumes noTB: ¬c-in-c′-only c c′ φ
  and c: c = CAnd ∨ c = COr and c′: c′ = CAnd ∨ c′ = COr
  shows ∃ψ ψ′. ψ ≼ φ ∧ push-conn-inside c c′ ψ ψ′
proof −
  have test-symb-false-nullary:
    ∀x. c-in-c′-symb c c′ (FF:: ′v propo) ∧ c-in-c′-symb c c′ FT
      ∧ c-in-c′-symb c c′ (FVar (x:: ′v))
    by auto
  moreover {
    fix x :: ′v
    have  H′: c-in-c′-symb c c′ FT c-in-c′-symb c c′ FF c-in-c′-symb c c′ (FVar x)
      by simp+
  }
  moreover {
    fix ψ :: ′v propo
    have ψ ≼ φ ⟹ ¬ c-in-c′-symb c c′ ψ ⟹ ∃ψ′. push-conn-inside c c′ ψ ψ′
      by (auto simp: assms(2) c′ c-in-c′-symb-step-exists)
  }
```

```
  ultimately show ?thesis using noTB no-test-symb-step-exists[of c-in-c'-symb c c']
    unfolding c-in-c'-only-def by metis
qed

lemma push-conn-insidec-in-c'-symb-no-T-F:
  fixes φ ψ :: 'v propo
  shows propo-rew-step (push-conn-inside c c') φ ψ ⟹ no-T-F φ ⟹ no-T-F ψ
proof (induct rule: propo-rew-step.induct)
  case (global-rel φ ψ)
  then show no-T-F ψ
    by (cases rule: push-conn-inside.cases, auto)
next
  case (propo-rew-one-step-lift φ φ' c ξ ξ')
  note rel = this(1) and IH = this(2) and wf = this(3) and no-T-F = this(4)
  have no-T-F φ
    using wf no-T-F  no-T-F-def subformula-into-subformula subformula-all-subformula-st
    subformula-refl by (metis (no-types) in-set-conv-decomp)
  then have φ': no-T-F φ' using IH by blast

  have ∀ζ ∈ set (ξ @ φ # ξ'). no-T-F ζ by (metis wf no-T-F no-T-F-def all-subformula-st-decomp)
  then have n: ∀ζ ∈ set (ξ @ φ' # ξ'). no-T-F ζ using φ' by auto
  then have n': ∀ζ ∈ set (ξ @ φ' # ξ'). ζ ≠  FF ∧ ζ ≠ FT
    using φ' by (metis no-T-F-symb-false(1) no-T-F-symb-false(2) no-T-F-def
      all-subformula-st-test-symb-true-phi)

  have wf': wf-conn c (ξ @ φ' # ξ')
    using wf wf-conn-no-arity-change by (metis wf-conn-no-arity-change-helper)
  {
    fix x :: 'v
    assume c = CT ∨ c = CF ∨ c = CVar x
    then have False using wf by auto
    then have no-T-F (conn c (ξ @ φ' # ξ')) by blast
  }
  moreover {
    assume c: c = CNot
    then have ξ = [] ξ' = [] using wf by auto
    then have no-T-F (conn c (ξ @ φ' # ξ'))
      using c by (metis φ' conn.simps(4) no-T-F-symb-false(1,2) no-T-F-symb-fnot no-T-F-def
        all-subformula-st-decomp-explicit(3) all-subformula-st-test-symb-true-phi self-append-conv2)
  }
  moreover {
    assume c: c ∈ binary-connectives
    then have no-T-F-symb (conn c (ξ @ φ' # ξ')) using wf' n' no-T-F-symb.simps by fastforce
    then have no-T-F (conn c (ξ @ φ' # ξ'))
      by (metis all-subformula-st-decomp-imp wf' n no-T-F-def)
  }
  ultimately show no-T-F (conn c (ξ @ φ' # ξ')) using connective-cases-arity by auto
qed


lemma simple-propo-rew-step-push-conn-inside-inv:
propo-rew-step (push-conn-inside c c') φ ψ ⟹ simple φ ⟹ simple ψ
  apply (induct rule: propo-rew-step.induct)
  apply (rename-tac φ, case-tac φ, auto simp: push-conn-inside.simps)[]
  by (metis append-is-Nil-conv list.distinct(1) simple.elims(2) wf-conn-list(1−3))
```

```
lemma simple-propo-rew-step-inv-push-conn-inside-simple-not:
  fixes c c' :: 'v connective and φ ψ :: 'v propo
  shows propo-rew-step (push-conn-inside c c') φ ψ ⟹ simple-not φ ⟹ simple-not ψ
proof (induct rule: propo-rew-step.induct)
  case (global-rel φ ψ)
  then show ?case by (cases φ, auto simp: push-conn-inside.simps)
next
  case (propo-rew-one-step-lift φ φ' ca ξ ξ') note rew = this(1) and IH = this(2) and wf = this(3)
   and simple = this(4)
  show ?case
    proof (cases ca rule: connective-cases-arity)
      case nullary
      then show ?thesis using propo-rew-one-step-lift by auto
    next
      case binary note ca = this
      obtain a b where ab: ξ @ φ' # ξ' = [a, b]
        using wf ca list-length2-decomp wf-conn-bin-list-length
        by (metis (no-types) wf-conn-no-arity-change-helper)
      have ∀ζ ∈ set (ξ @ φ # ξ'). simple-not ζ
        by (metis wf all-subformula-st-decomp simple simple-not-def)
      then have ∀ζ ∈ set (ξ @ φ' # ξ'). simple-not ζ using IH by simp
      moreover have simple-not-symb (conn ca (ξ @ φ' # ξ')) using ca
      by (metis ab conn.simps(5−8) helper-fact simple-not-symb.simps(5) simple-not-symb.simps(6)
          simple-not-symb.simps(7) simple-not-symb.simps(8))
      ultimately show ?thesis
        by (simp add: ab all-subformula-st-decomp ca)
    next
      case unary
      then show ?thesis
         using rew simple-propo-rew-step-push-conn-inside-inv[OF rew] IH local.wf simple by auto
    qed
qed

lemma propo-rew-step-push-conn-inside-simple-not:
  fixes φ φ' :: 'v propo and ξ ξ' :: 'v propo list and c :: 'v connective
  assumes
    propo-rew-step (push-conn-inside c c') φ φ' and
    wf-conn c (ξ @ φ # ξ') and
    simple-not-symb (conn c (ξ @ φ # ξ')) and
    simple-not-symb φ'
  shows simple-not-symb (conn c (ξ @ φ' # ξ'))
  using assms
proof (induction rule: propo-rew-step.induct)
print-cases
  case (global-rel)
  then show ?case
    by (metis conn.simps(12,17) list.discI push-conn-inside.cases simple-not-symb.elims(3)
      wf-conn-helper-facts(5) wf-conn-list(2) wf-conn-list(8) wf-conn-no-arity-change
      wf-conn-no-arity-change-helper)
next
  case (propo-rew-one-step-lift φ φ' c' χs χs') note tel = this(1) and wf = this(2) and
    IH = this(3) and wf' = this(4) and simple' = this(5) and simple = this(6)
  then show ?case
    proof (cases c' rule: connective-cases-arity)
      case nullary
```

```
      then show ?thesis using wf simple simple' by auto
    next
      case binary note c = this(1)
      have corr': wf-conn c (ξ @ conn c' (χs @ φ' # χs') # ξ')
        using wf wf-conn-no-arity-change
        by (metis wf' wf-conn-no-arity-change-helper)
      then show ?thesis
        using c propo-rew-one-step-lift wf
        by (metis conn.simps(17) connective.distinct(37) propo-rew-step-subformula-imp
          push-conn-inside.cases simple-not-symb.elims(3) wf-conn.simps wf-conn-list(2,8))
    next
      case unary
      then have empty: χs = [] χs' = [] using wf by auto
      then show ?thesis using simple unary simple' wf wf'
        by (metis connective.distinct(37) connective.distinct(39) propo-rew-step-subformula-imp
          push-conn-inside.cases simple-not-symb.elims(3) tel wf-conn-list(8)
          wf-conn-no-arity-change wf-conn-no-arity-change-helper)
    qed
qed

lemma push-conn-inside-not-true-false:
  push-conn-inside c c' φ ψ ⟹ ψ ≠ FT ∧ ψ ≠ FF
  by (induct rule: push-conn-inside.induct, auto)

lemma push-conn-inside-inv:
  fixes φ ψ :: 'v propo
  assumes full (propo-rew-step (push-conn-inside c c')) φ ψ
  and no-equiv φ and no-imp φ and no-T-F-except-top-level φ and simple-not φ
  shows no-equiv ψ and no-imp ψ and no-T-F-except-top-level ψ and simple-not ψ
proof −
  {
    {
      fix φ ψ :: 'v propo
      have H: push-conn-inside c c' φ ψ ⟹ all-subformula-st simple-not-symb φ
        ⟹ all-subformula-st simple-not-symb ψ
        by (induct φ ψ rule: push-conn-inside.induct, auto)
    } note H = this

    fix φ ψ :: 'v propo
    have H: propo-rew-step (push-conn-inside c c') φ ψ ⟹ all-subformula-st simple-not-symb φ
      ⟹ all-subformula-st simple-not-symb ψ
      apply (induct φ ψ rule: propo-rew-step.induct)
      using H apply simp
      proof (rename-tac φ φ' ca ψs ψs', case-tac ca rule: connective-cases-arity)
        fix φ φ' :: 'v propo and c:: 'v connective and ξ ξ':: 'v propo list
        and x:: 'v
        assume wf-conn c (ξ @ φ # ξ')
        and  c = CT ∨ c = CF ∨ c = CVar x
        then have ξ @ φ # ξ' = [] by auto
        then have False by auto
        then show all-subformula-st simple-not-symb (conn c (ξ @ φ' # ξ')) by blast
      next
        fix φ φ' :: 'v propo and ca:: 'v connective and ξ ξ':: 'v propo list
        and x :: 'v
        assume rel: propo-rew-step (push-conn-inside c c') φ φ'
        and φ-φ': all-subformula-st simple-not-symb φ ⟹ all-subformula-st simple-not-symb φ'
```

```
    and corr: wf-conn ca (ξ @ φ # ξ')
    and n: all-subformula-st simple-not-symb (conn ca (ξ @ φ # ξ'))
    and c: ca = CNot

    have empty: ξ = [] ξ' = [] using c corr by auto
    then have simple-not:all-subformula-st simple-not-symb (FNot φ) using corr c n by auto
    then have simple φ
      using all-subformula-st-test-symb-true-phi simple-not-symb.simps(1) by blast
    then have simple φ'
      using rel simple-propo-rew-step-push-conn-inside-inv by blast
    then show all-subformula-st simple-not-symb (conn ca (ξ @ φ' # ξ')) using c empty
      by (metis simple-not φ-φ' append-Nil conn.simps(4) all-subformula-st-decomp-explicit(3)
        simple-not-symb.simps(1))
  next
    fix φ φ' :: 'v propo and ca :: 'v connective and ξ ξ' :: 'v propo list
    and x :: 'v
    assume rel: propo-rew-step (push-conn-inside c c') φ φ'
    and nφ: all-subformula-st simple-not-symb φ ⟹ all-subformula-st simple-not-symb φ'
    and corr: wf-conn ca (ξ @ φ # ξ')
    and n: all-subformula-st simple-not-symb (conn ca (ξ @ φ # ξ'))
    and c: ca ∈ binary-connectives

    have all-subformula-st simple-not-symb φ
      using n c corr all-subformula-st-decomp by fastforce
    then have φ': all-subformula-st simple-not-symb φ' using nφ by blast
    obtain a b where ab: [a, b] = (ξ @ φ # ξ')
      using corr c list-length2-decomp wf-conn-bin-list-length by metis
    then have ξ @ φ' # ξ' = [a, φ'] ∨ (ξ @ φ' # ξ') = [φ', b]
      using ab by (metis (no-types, hide-lams) append-Cons append-Nil append-Nil2
        append-is-Nil-conv butlast.simps(2) butlast-append list.sel(3) tl-append2)
    moreover
    {
      fix χ :: 'v propo
      have wf': wf-conn ca [a, b]
        using ab corr by presburger
      have all-subformula-st simple-not-symb (conn ca [a, b])
        using ab n by presburger
      then have all-subformula-st simple-not-symb χ ∨ χ ∉ set (ξ @ φ' # ξ')
        using wf' by (metis (no-types) φ' all-subformula-st-decomp calculation insert-iff
          list.set(2))
    }
    then have ∀φ. φ ∈ set (ξ @ φ' # ξ') ⟶ all-subformula-st simple-not-symb φ
        by (metis (no-types))

    moreover have simple-not-symb (conn ca (ξ @ φ' # ξ'))
      using ab conn-inj-not(1) corr wf-conn-list-decomp(4) wf-conn-no-arity-change
        not-Cons-self2 self-append-conv2 simple-not-symb.elims(3) by (metis (no-types) c
        calculation(1) wf-conn-binary)
    moreover have wf-conn ca (ξ @ φ' # ξ') using c calculation(1) by auto
    ultimately show all-subformula-st simple-not-symb (conn ca (ξ @ φ' # ξ'))
      by (metis all-subformula-st-decomp-imp)
  qed
}
moreover {
  fix ca :: 'v connective and ξ ξ' :: 'v propo list and φ φ' :: 'v propo
  have propo-rew-step (push-conn-inside c c') φ φ' ⟹ wf-conn ca (ξ @ φ # ξ')
```

```
    ⟹ simple-not-symb (conn ca (ξ @ φ # ξ')) ⟹ simple-not-symb φ'
    ⟹ simple-not-symb (conn ca (ξ @ φ' # ξ'))
    by (metis append-self-conv2 conn.simps(4) conn-inj-not(1) simple-not-symb.elims(3)
      simple-not-symb.simps(1) simple-propo-rew-step-push-conn-inside-inv
      wf-conn-no-arity-change-helper wf-conn-list-decomp(4) wf-conn-no-arity-change)
  }
  ultimately show simple-not ψ
    using full-propo-rew-step-inv-stay'[of push-conn-inside c c' simple-not-symb] assms
    unfolding no-T-F-except-top-level-def simple-not-def full-unfold by metis
next
  {
    fix φ ψ :: 'v propo
    have H: propo-rew-step (push-conn-inside c c') φ ψ ⟹ no-T-F-except-top-level φ
      ⟹ no-T-F-except-top-level ψ
      proof −
        assume rel: propo-rew-step (push-conn-inside c c') φ ψ
        and no-T-F-except-top-level φ
        then have no-T-F φ ∨ φ = FF ∨ φ = FT
          by (metis no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
        moreover {
          assume φ = FF ∨ φ = FT
          then have False using rel propo-rew-step-push-conn-inside by blast
          then have no-T-F-except-top-level ψ by blast
        }
        moreover {
          assume no-T-F φ ∧ φ ≠ FF ∧ φ ≠ FT
          then have no-T-F ψ using rel push-conn-insidec-in-c'-symb-no-T-F by blast
          then have no-T-F-except-top-level ψ using no-T-F-no-T-F-except-top-level by blast
        }
        ultimately show no-T-F-except-top-level ψ by blast
      qed
  }
  moreover {
    fix ca :: 'v connective and ξ ξ' :: 'v propo list and φ φ' :: 'v propo
    assume rel: propo-rew-step (push-conn-inside c c') φ φ'
    assume corr: wf-conn ca (ξ @ φ # ξ')
    then have c: ca ≠ CT ∧ ca ≠ CF by auto
    assume no-T-F: no-T-F-symb-except-toplevel (conn ca (ξ @ φ # ξ'))
    have no-T-F-symb-except-toplevel (conn ca (ξ @ φ' # ξ'))
    proof
      have c: ca ≠ CT ∧ ca ≠ CF using corr by auto
      have ζ: ∀ζ∈ set (ξ @ φ # ξ'). ζ≠FT ∧ ζ ≠ FF
        using corr no-T-F no-T-F-symb-except-toplevel-if-is-a-true-false by blast
      then have φ ≠ FT ∧ φ ≠ FF by auto
      from rel this have  φ' ≠ FT ∧ φ' ≠ FF
        apply (induct rule: propo-rew-step.induct)
        by (metis append-is-Nil-conv conn.simps(2) conn-inj list.distinct(1)
          wf-conn-helper-facts(3) wf-conn-list(1) wf-conn-no-arity-change
          wf-conn-no-arity-change-helper push-conn-inside-not-true-false)+
      then have ∀ζ ∈ set (ξ @ φ' # ξ'). ζ ≠ FT ∧ ζ ≠ FF using ζ by auto
      moreover have wf-conn ca (ξ @ φ' # ξ')
        using corr wf-conn-no-arity-change by (metis wf-conn-no-arity-change-helper)
      ultimately show no-T-F-symb (conn ca (ξ @ φ' # ξ')) using no-T-F-symb.intros c by metis
    qed
  }
  ultimately show no-T-F-except-top-level ψ
```

using *full-propo-rew-step-inv-stay'*[*of push-conn-inside c c' no-T-F-symb-except-toplevel*]
assms **unfolding** *no-T-F-except-top-level-def full-unfold* **by** *metis*

next
{
fix φ ψ :: *'v propo*
have *H*: *push-conn-inside c c'* φ ψ $\Longrightarrow$ *no-equiv* φ $\Longrightarrow$ *no-equiv* ψ
by (*induct* φ ψ *rule*: *push-conn-inside.induct*, *auto*)
}
then show *no-equiv* ψ
using *full-propo-rew-step-inv-stay-conn*[*of push-conn-inside c c' no-equiv-symb*] *assms*
no-equiv-symb-conn-characterization **unfolding** *no-equiv-def* **by** *metis*

next
{
fix φ ψ :: *'v propo*
have *H*: *push-conn-inside c c'* φ ψ $\Longrightarrow$ *no-imp* φ $\Longrightarrow$ *no-imp* ψ
by (*induct* φ ψ *rule*: *push-conn-inside.induct*, *auto*)
}
then show *no-imp* ψ
using *full-propo-rew-step-inv-stay-conn*[*of push-conn-inside c c' no-imp-symb*] *assms*
no-imp-symb-conn-characterization **unfolding** *no-imp-def* **by** *metis*
qed

lemma *push-conn-inside-full-propo-rew-step*:
fixes φ ψ :: *'v propo*
assumes
no-equiv φ **and**
no-imp φ **and**
full (*propo-rew-step* (*push-conn-inside c c'*)) φ ψ **and**
no-T-F-except-top-level φ **and**
simple-not φ **and**
$c = CAnd \lor c = COr$ **and**
$c' = CAnd \lor c' = COr$
shows *c-in-c'-only c c'* ψ
using *c-in-c'-symb-rew assms full-propo-rew-step-subformula* **by** *blast*

Only one type of connective in the formula (+ not)

inductive *only-c-inside-symb* :: *'v connective* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **for** *c* :: *'v connective* **where**
simple-only-c-inside[*simp*]: *simple* φ $\Longrightarrow$ *only-c-inside-symb c* φ |
simple-cnot-only-c-inside[*simp*]: *simple* φ $\Longrightarrow$ *only-c-inside-symb c* (*FNot* φ) |
only-c-inside-into-only-c-inside: *wf-conn c l* $\Longrightarrow$ *only-c-inside-symb c* (*conn c l*)

lemma *only-c-inside-symb-simp*[*simp*]:
only-c-inside-symb c FF only-c-inside-symb c FT only-c-inside-symb c (*FVar x*) **by** *auto*

definition *only-c-inside* **where** *only-c-inside c* = *all-subformula-st* (*only-c-inside-symb c*)

lemma *only-c-inside-symb-decomp*:
only-c-inside-symb c ψ $\longleftrightarrow$ (*simple* ψ
$\lor$ ($\exists$ φ'. ψ = *FNot* φ' $\land$ *simple* φ')
$\lor$ ($\exists$ *l*. ψ = *conn c l* $\land$ *wf-conn c l*))

by (*auto simp*: *only-c-inside-symb.intros*(*3*)) (*induct rule*: *only-c-inside-symb.induct*, *auto*)

lemma *only-c-inside-symb-decomp-not*[*simp*]:
fixes c :: $'v$ *connective*
assumes c: $c \neq$ *CNot*
shows *only-c-inside-symb* c (*FNot* ψ) $\longleftrightarrow$ *simple* ψ
apply (*auto simp*: *only-c-inside-symb.intros*(*3*))
by (*induct FNot* ψ *rule*: *only-c-inside-symb.induct*, *auto simp*: *wf-conn-list*(*8*) *c*)

lemma *only-c-inside-decomp-not*[*simp*]:
assumes c: $c \neq$ *CNot*
shows *only-c-inside* c (*FNot* ψ) $\longleftrightarrow$ *simple* ψ
by (*metis* (*no-types*, *hide-lams*) *all-subformula-st-def all-subformula-st-test-symb-true-phi c*
only-c-inside-def only-c-inside-symb-decomp-not simple-only-c-inside
subformula-conn-decomp-simple)

lemma *only-c-inside-decomp*:
only-c-inside c φ $\longleftrightarrow$
$(\forall \psi.\ \psi \preceq \varphi \longrightarrow$ (*simple* ψ $\vee$ ($\exists$ φ'. ψ = *FNot* φ' $\wedge$ *simple* φ')
$\vee$ ($\exists$ l. ψ = *conn* c l $\wedge$ *wf-conn* c l)))
unfolding *only-c-inside-def* **by** (*auto simp*: *all-subformula-st-def only-c-inside-symb-decomp*)

lemma *only-c-inside-c-c′-false*:
fixes c c' :: $'v$ *connective* **and** l :: $'v$ *propo list* **and** φ :: $'v$ *propo*
assumes cc': $c \neq c'$ **and** c: c = *CAnd* $\vee$ c = *COr* **and** c': c' = *CAnd* $\vee$ c' = *COr*
and *only*: *only-c-inside* c φ **and** *incl*: *conn* c' l $\preceq$ φ **and** *wf*: *wf-conn* c' l
shows *False*
proof −
let $?\psi$ = *conn* c' l
have *simple* $?\psi$ $\vee$ ($\exists$ φ'. $?\psi$ = *FNot* φ' $\wedge$ *simple* φ') $\vee$ ($\exists$ l. $?\psi$ = *conn* c l $\wedge$ *wf-conn* c l)
using *only-c-inside-decomp only incl* **by** *blast*
moreover have $\neg$ *simple* $?\psi$
using *wf simple-decomp* **by** (*metis* c' *connective.distinct*(*19*) *connective.distinct*(*7*,*9*,*21*,*29*,*31*)
wf-conn-list(*1*−*3*))
moreover
{
fix φ'
have $?\psi \neq$ *FNot* φ' **using** c' *conn-inj-not*(*1*) *wf* **by** *blast*
}
ultimately obtain l :: $'v$ *propo list* **where** $?\psi$ = *conn* c l $\wedge$ *wf-conn* c l **by** *metis*
then have $c = c'$ **using** *conn-inj wf* **by** *metis*
then show *False* **using** cc' **by** *auto*
qed

lemma *only-c-inside-implies-c-in-c′-symb*:
assumes δ: $c \neq c'$ **and** c: c = *CAnd* $\vee$ c = *COr* **and** c': c' = *CAnd* $\vee$ c' = *COr*
shows *only-c-inside* c φ $\Longrightarrow$ *c-in-c′-symb* c c' φ
apply (*rule ccontr*)
apply (*cases rule*: *not-c-in-c′-symb.cases*, *auto*)
by (*metis* δ c c' *connective.distinct*(*37*,*39*) *list.distinct*(*1*) *only-c-inside-c-c′-false*
subformula-in-binary-conn(*1*,*2*) *wf-conn.simps*)+

lemma *c-in-c′-symb-decomp-level1*:
fixes l :: $'v$ *propo list* **and** c c' ca :: $'v$ *connective*
shows *wf-conn* ca l $\Longrightarrow$ $ca \neq c$ $\Longrightarrow$ *c-in-c′-symb* c c' (*conn* ca l)

proof −
  have *not-c-in-c′-symb c c′ (conn ca l)* $\Longrightarrow$ *wf-conn ca l* $\Longrightarrow$ *ca = c*
    by (*induct conn ca l rule*: *not-c-in-c′-symb.induct*, *auto simp*: *conn-inj*)
  then show *wf-conn ca l* $\Longrightarrow$ *ca* $\neq$ *c* $\Longrightarrow$ *c-in-c′-symb c c′ (conn ca l)* **by** *blast*
qed

lemma *only-c-inside-implies-c-in-c′-only*:
  assumes δ: *c* $\neq$ *c′* **and** *c*: *c = CAnd* $\vee$ *c = COr* **and** *c′*: *c′ = CAnd* $\vee$ *c′ = COr*
  shows *only-c-inside c* φ $\Longrightarrow$ *c-in-c′-only c c′* φ
  unfolding *c-in-c′-only-def all-subformula-st-def*
  using *only-c-inside-implies-c-in-c′-symb*
    by (*metis all-subformula-st-def assms*(*1*) *c c′ only-c-inside-def subformula-trans*)

lemma *c-in-c′-symb-c-implies-only-c-inside*:
  assumes δ: *c = CAnd* $\vee$ *c = COr c′ = CAnd* $\vee$ *c′ = COr c* $\neq$ *c′* **and** *wf*: *wf-conn c* [φ, ψ]
  and *inv*: *no-equiv* (*conn c l*) *no-imp* (*conn c l*) *simple-not* (*conn c l*)
  shows *wf-conn c l* $\Longrightarrow$ *c-in-c′-only c c′ (conn c l)* $\Longrightarrow$ ($\forall$ $\psi\in$ *set l. only-c-inside c* ψ)
using *inv*
proof (*induct conn c l arbitrary*: *l rule*: *propo-induct-arity*)
  case (*nullary x*)
  then show *?case* **by** (*auto simp*: *wf-conn-list assms*)
next
  case (*unary* φ *la*)
  then have *c = CNot* $\wedge$ *la* = [φ] **by** (*metis* (*no-types*) *wf-conn-list*(*8*))
  then show *?case* **using** *assms*(*2*) *assms*(*1*) **by** *blast*
next
  case (*binary* φ*1* φ*2*)
  note *IH*φ*1 = this*(*1*) **and** *IH*φ*2 = this*(*2*) **and** φ *= this*(*3*) **and** *only = this*(*5*) **and** *wf = this*(*4*)
    and *no-equiv = this*(*6*) **and** *no-imp = this*(*7*) **and** *simple-not = this*(*8*)
  then have *l*: *l* = [φ*1*, φ*2*] **by** (*meson wf-conn-list*(*4*−*7*))
  let *?*φ *= conn c l*

  obtain *c1 l1 c2 l2* **where** φ*1*: φ*1 = conn c1 l1* **and** *wf*φ*1*: *wf-conn c1 l1*
    and φ*2*: φ*2 = conn c2 l2* **and** *wf*φ*2*: *wf-conn c2 l2* **using** *exists-c-conn* **by** *metis*
  then have *c-in-only*φ*1*: *c-in-c′-only c c′ (conn c1 l1)* **and** *c-in-c′-only c c′ (conn c2 l2)*
    using *only l* **unfolding** *c-in-c′-only-def* **using** *assms*(*1*) **by** *auto*
  have *inc*φ*1*: φ*1* $\preceq$ *?*φ **and** *inc*φ*2*: φ*2* $\preceq$ *?*φ
    using φ*1* φ*2* φ *local.wf* **by** (*metis conn.simps*(*5*−*8*) *helper-fact subformula-in-binary-conn*(*1*,*2*))+

  have *c1-eq*: *c1* $\neq$ *CEq* **and** *c2-eq*: *c2* $\neq$ *CEq*
    unfolding *no-equiv-def* **using** *inc*φ*1 inc*φ*2* **by** (*metis* φ*1* φ*2 wf*φ*1 wf*φ*2 assms*(*1*) *no-equiv*
      no-equiv-eq(*1*) *no-equiv-symb.elims*(*3*) *no-equiv-symb-conn-characterization wf-conn-list*(*4*,*5*)
      no-equiv-def subformula-all-subformula-st)+
  have *c1-imp*: *c1* $\neq$ *CImp* **and** *c2-imp*: *c2* $\neq$ *CImp*
    using *no-imp* **by** (*metis* φ*1* φ*2 all-subformula-st-decomp-explicit-imp*(*2*,*3*) *assms*(*1*)
      conn.simps(*5*,*6*) *l no-imp-Imp*(*1*) *no-imp-symb.elims*(*3*) *no-imp-symb-conn-characterization*
      wfφ*1 wf*φ*2 all-subformula-st-decomp no-imp-symb-conn-characterization*)+
  have *c1c*: *c1* $\neq$ *c′*
    proof
      assume *c1c*: *c1 = c′*
      then obtain ξ*1* ξ*2* **where** *l1*: *l1* = [ξ*1*, ξ*2*]
        by (*metis assms*(*2*) *connective.distinct*(*37*,*39*) *helper-fact wf*φ*1 wf-conn.simps*
          wf-conn-list-decomp(*1*−*3*))
      have *c-in-c′-only c c′ (conn c* [*conn c′ l1*, φ*2*]) **using** *c1c l only* φ*1* **by** *auto*

```
    moreover have not-c-in-c′-symb c c′ (conn c [conn c′ l1, φ2])
      using l1 φ1 c1c l local.wf not-c-in-c′-symb-l wfφ1 by blast
    ultimately show False using φ1 c1c l l1 local.wf not-c-in-c′-simp(4) wfφ1 by blast
  qed
then have (φ1 = conn c l1 ∧ wf-conn c l1) ∨ (∃ψ1. φ1 = FNot ψ1) ∨ simple φ1
  by (metis φ1 assms(1−3) c1-eq c1-imp simple.elims(3) wfφ1 wf-conn-list(4) wf-conn-list(5−7))
moreover {
  assume φ1 = conn c l1 ∧ wf-conn c l1
  then have only-c-inside c φ1
    by (metis IHφ1 φ1 all-subformula-st-decomp-imp incφ1 no-equiv no-equiv-def no-imp no-imp-def
      c-in-onlyφ1 only-c-inside-def only-c-inside-into-only-c-inside simple-not simple-not-def
      subformula-all-subformula-st)
}
moreover {
  assume ∃ψ1. φ1 = FNot ψ1
  then obtain ψ1 where φ1 = FNot ψ1 by metis
  then have only-c-inside c φ1
    by (metis all-subformula-st-def assms(1) connective.distinct(37,39) incφ1
      only-c-inside-decomp-not simple-not simple-not-def simple-not-symb.simps(1))
}
moreover {
  assume simple φ1
  then have only-c-inside c φ1
    by (metis all-subformula-st-decomp-explicit(3) assms(1) connective.distinct(37,39)
      only-c-inside-decomp-not only-c-inside-def)
}
ultimately have only-c-insideφ1: only-c-inside c φ1 by metis

have c-in-onlyφ2: c-in-c′-only c c′ (conn c2 l2)
  using only l φ2 wfφ2 assms unfolding c-in-c′-only-def by auto
have c2c: c2 ≠ c′
  proof
    assume c2c: c2 = c′
    then obtain ξ1 ξ2 where l2: l2 = [ξ1, ξ2]
     by (metis assms(2) wfφ2 wf-conn.simps connective.distinct(7,9,19,21,29,31,37,39))
    then have c-in-c′-symb c c′ (conn c [φ1, conn c′ l2])
      using c2c l only φ2 all-subformula-st-test-symb-true-phi unfolding c-in-c′-only-def by auto
    moreover have not-c-in-c′-symb c c′ (conn c [φ1, conn c′ l2])
      using assms(1) c2c l2 not-c-in-c′-symb-r wfφ2 wf-conn-helper-facts(5,6) by metis
    ultimately show False by auto
  qed
then have (φ2 = conn c l2 ∧ wf-conn c l2) ∨ (∃ψ2. φ2 = FNot ψ2) ∨ simple φ2
  using c2-eq by (metis φ2 assms(1−3) c2-eq c2-imp simple.elims(3) wfφ2 wf-conn-list(4−7))
moreover {
  assume φ2 = conn c l2 ∧ wf-conn c l2
  then have only-c-inside c φ2
    by (metis IHφ2 φ2 all-subformula-st-decomp incφ2 no-equiv no-equiv-def no-imp no-imp-def
      c-in-onlyφ2 only-c-inside-def only-c-inside-into-only-c-inside simple-not simple-not-def
      subformula-all-subformula-st)
}
moreover {
  assume ∃ψ2. φ2 = FNot ψ2
  then obtain ψ2 where φ2 = FNot ψ2 by metis
  then have only-c-inside c φ2
    by (metis all-subformula-st-def assms(1−3) connective.distinct(38,40) incφ2
      only-c-inside-decomp-not simple-not simple-not-def simple-not-symb.simps(1))
```

```
  }
  moreover {
    assume simple φ2
    then have only-c-inside c φ2
      by (metis all-subformula-st-decomp-explicit(3) assms(1) connective.distinct(37,39)
        only-c-inside-decomp-not only-c-inside-def)
  }
  ultimately have only-c-insideφ2: only-c-inside c φ2 by metis
  show ?case using l only-c-insideφ1 only-c-insideφ2 by auto
qed
```

Push Conjunction

```
definition pushConj where pushConj = push-conn-inside CAnd COr

lemma pushConj-consistent: preserves-un-sat pushConj
  unfolding pushConj-def by (simp add: push-conn-inside-consistent)

definition and-in-or-symb where and-in-or-symb = c-in-c′-symb CAnd COr

definition and-in-or-only where
and-in-or-only = all-subformula-st (c-in-c′-symb CAnd COr)

lemma pushConj-inv:
  fixes φ ψ :: ′v propo
  assumes full (propo-rew-step pushConj) φ ψ
  and   no-equiv φ and no-imp φ and no-T-F-except-top-level φ and simple-not φ
  shows no-equiv ψ and no-imp ψ and no-T-F-except-top-level ψ and simple-not ψ
  using push-conn-inside-inv assms unfolding pushConj-def by metis+


lemma pushConj-full-propo-rew-step:
  fixes φ ψ :: ′v propo
  assumes
    no-equiv φ and
    no-imp φ and
    full (propo-rew-step pushConj) φ ψ and
    no-T-F-except-top-level φ and
    simple-not φ
  shows and-in-or-only ψ
  using assms push-conn-inside-full-propo-rew-step
  unfolding pushConj-def and-in-or-only-def c-in-c′-only-def by (metis (no-types))
```

Push Disjunction

```
definition pushDisj where pushDisj = push-conn-inside COr CAnd

lemma pushDisj-consistent: preserves-un-sat pushDisj
  unfolding pushDisj-def by (simp add: push-conn-inside-consistent)

definition or-in-and-symb where or-in-and-symb = c-in-c′-symb COr CAnd

definition or-in-and-only where
or-in-and-only = all-subformula-st (c-in-c′-symb COr CAnd)
```

lemma *not-or-in-and-only-or-and*[*simp*]:
  ~*or-in-and-only* (*FOr* (*FAnd* $\psi 1$ $\psi 2$) φ')
  unfolding *or-in-and-only-def*
  by (*metis all-subformula-st-test-symb-true-phi conn.simps*(5−6) *not-c-in-c′-symb-l*
    wf-conn-helper-facts(5) *wf-conn-helper-facts*(6))

lemma *pushDisj-inv*:
  fixes φ ψ :: *′v propo*
  assumes *full* (*propo-rew-step pushDisj*) φ ψ
  and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ **and** *simple-not* φ
  shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ **and** *simple-not* ψ
  using *push-conn-inside-inv assms* **unfolding** *pushDisj-def* **by** *metis*+

lemma *pushDisj-full-propo-rew-step*:
  fixes φ ψ :: *′v propo*
  assumes
    no-equiv φ **and**
    no-imp φ **and**
    full (*propo-rew-step pushDisj*) φ ψ **and**
    no-T-F-except-top-level φ **and**
    simple-not φ
  shows *or-in-and-only* ψ
  using *assms push-conn-inside-full-propo-rew-step*
  unfolding *pushDisj-def or-in-and-only-def c-in-c′-only-def* **by** (*metis* (*no-types*))

0.4 The full transformations

0.4.1 Abstract Property characterizing that only some connective are inside the others

Definition

The normal is a super group of groups

inductive *grouped-by* :: *′a connective* ⇒ *′a propo* ⇒ *bool* **for** *c* **where**
simple-is-grouped[*simp*]: *simple* φ ⟹ *grouped-by c* φ |
simple-not-is-grouped[*simp*]: *simple* φ ⟹ *grouped-by c* (*FNot* φ) |
connected-is-group[*simp*]: *grouped-by c* φ ⟹ *grouped-by c* ψ ⟹ *wf-conn c* [φ, ψ]
  ⟹ *grouped-by c* (*conn c* [φ, ψ])

lemma *simple-clause*[*simp*]:
  grouped-by c FT
  grouped-by c FF
  grouped-by c (*FVar x*)
  grouped-by c (*FNot FT*)
  grouped-by c (*FNot FF*)
  grouped-by c (*FNot* (*FVar x*))
  by *simp*+

lemma *only-c-inside-symb-c-eq-c′*:
  only-c-inside-symb c (*conn c′* [$\varphi 1$, $\varphi 2$]) ⟹ *c′* = *CAnd* ∨ *c′* = *COr* ⟹ *wf-conn c′* [$\varphi 1$, $\varphi 2$]
    ⟹ *c′* = *c*
  by (*induct conn c′* [$\varphi 1$, $\varphi 2$] *rule*: *only-c-inside-symb.induct*, *auto simp*: *conn-inj*)

lemma *only-c-inside-c-eq-c′*:

only-c-inside c (*conn c′* [*φ1*, *φ2*]) ⟹ *c′* = *CAnd* ∨ *c′* = *COr* ⟹ *wf-conn c′* [*φ1*, *φ2*] ⟹ *c* = *c′*
unfolding *only-c-inside-def all-subformula-st-def* **using** *only-c-inside-symb-c-eq-c′ subformula-refl*
by *blast*

lemma *only-c-inside-imp-grouped-by*:
assumes *c*: *c* ≠ *CNot* **and** *c′*: *c′* = *CAnd* ∨ *c′* = *COr*
shows *only-c-inside c φ* ⟹ *grouped-by c φ* (**is** *?O φ* ⟹ *?G φ*)
proof (*induct φ rule*: *propo-induct-arity*)
case (*nullary φ x*)
then show *?G φ* **by** *auto*
next
case (*unary ψ*)
then show *?G* (*FNot ψ*) **by** (*auto simp*: *c*)
next
case (*binary φ φ1 φ2*)
note *IHφ1* = *this*(*1*) **and** *IHφ2* = *this*(*2*) **and** *φ* = *this*(*3*) **and** *only* = *this*(*4*)
have *φ-conn*: *φ* = *conn c* [*φ1*, *φ2*] **and** *wf*: *wf-conn c* [*φ1*, *φ2*]
proof −
obtain *c″ l″* **where** *φ-c″*: *φ* = *conn c″ l″* **and** *wf*: *wf-conn c″ l″*
using *exists-c-conn* **by** *metis*
then have *l″*: *l″* = [*φ1*, *φ2*] **using** *φ* **by** (*metis wf-conn-list*(*4*−*7*))
have *only-c-inside-symb c* (*conn c″* [*φ1*, *φ2*])
using *only all-subformula-st-test-symb-true-phi*
unfolding *only-c-inside-def φ-c″ l″* **by** *metis*
then have *c* = *c″*
by (*metis φ φ-c″ conn-inj conn-inj-not*(*2*) *l″ list.distinct*(*1*) *list.inject wf*
only-c-inside-symb.cases simple.simps(*5*−*8*))
then show *φ* = *conn c* [*φ1*, *φ2*] **and** *wf-conn c* [*φ1*, *φ2*] **using** *φ-c″ wf l″* **by** *auto*
qed
have *grouped-by c φ1* **using** *wf IHφ1 IHφ2 φ-conn only φ* **unfolding** *only-c-inside-def* **by** *auto*
moreover have *grouped-by c φ2*
using *wf φ IHφ1 IHφ2 φ-conn only* **unfolding** *only-c-inside-def* **by** *auto*
ultimately show *?G φ* **using** *φ-conn connected-is-group local.wf* **by** *blast*
qed

lemma *grouped-by-false*:
grouped-by c (*conn c′* [*φ*, *ψ*]) ⟹ *c* ≠ *c′* ⟹ *wf-conn c′* [*φ*, *ψ*] ⟹ *False*
apply (*induct conn c′* [*φ*, *ψ*] *rule*: *grouped-by.induct*)
apply (*auto simp*: *simple-decomp wf-conn-list*, *auto simp*: *conn-inj*)
by (*metis list.distinct*(*1*) *list.sel*(*3*) *wf-conn-list*(*8*))+

Then the CNF form is a conjunction of clauses: every clause is in CNF form and two formulas in CNF form can be related by an and.

inductive *super-grouped-by*:: *′a connective* ⇒ *′a connective* ⇒ *′a propo* ⇒ *bool* **for** *c c′* **where**
grouped-is-super-grouped[*simp*]: *grouped-by c φ* ⟹ *super-grouped-by c c′ φ* |
connected-is-super-group: *super-grouped-by c c′ φ* ⟹ *super-grouped-by c c′ ψ* ⟹ *wf-conn c* [*φ*, *ψ*]
⟹ *super-grouped-by c c′* (*conn c′* [*φ*, *ψ*])

lemma *simple-cnf*[*simp*]:
super-grouped-by c c′ FT
super-grouped-by c c′ FF
super-grouped-by c c′ (*FVar x*)
super-grouped-by c c′ (*FNot FT*)
super-grouped-by c c′ (*FNot FF*)
super-grouped-by c c′ (*FNot* (*FVar x*))

by *auto*

lemma *c-in-c′-only-super-grouped-by*:
assumes *c*: $c = CAnd \lor c = COr$ **and** *c′*: $c' = CAnd \lor c' = COr$ **and** *cc′*: $c \neq c'$
shows *no-equiv* $\varphi \Longrightarrow$ *no-imp* $\varphi \Longrightarrow$ *simple-not* $\varphi \Longrightarrow$ *c-in-c′-only* $c\ c'\ \varphi$
$\Longrightarrow$ *super-grouped-by* $c\ c'\ \varphi$
(**is** *?NE* $\varphi \Longrightarrow$ *?NI* $\varphi \Longrightarrow$ *?SN* $\varphi \Longrightarrow$ *?C* $\varphi \Longrightarrow$ *?S* φ)
proof (*induct* φ *rule*: *propo-induct-arity*)
case (*nullary* φ *x*)
then show *?S* φ **by** *auto*
next
case (*unary* φ)
then have *simple-not-symb* (*FNot* φ)
using *all-subformula-st-test-symb-true-phi* **unfolding** *simple-not-def* **by** *blast*
then have $\varphi = FT \lor \varphi = FF \lor (\exists\ x.\ \varphi = FVar\ x)$ **by** (*cases* φ, *auto*)
then show *?S* (*FNot* φ) **by** *auto*
next
case (*binary* φ $\varphi 1$ $\varphi 2$)
note $IH\varphi 1 = this(1)$ **and** $IH\varphi 2 = this(2)$ **and** *no-equiv* = *this*(*4*) **and** *no-imp* = *this*(*5*)
and *simpleN* = *this*(*6*) **and** *c-in-c′-only* = *this*(*7*) **and** $\varphi' = this(3)$
{
assume $\varphi = FImp\ \varphi 1\ \varphi 2 \lor \varphi = FEq\ \varphi 1\ \varphi 2$
then have *False* **using** *no-equiv no-imp* **by** *auto*
then have *?S* φ **by** *auto*
}
moreover {
assume φ: $\varphi = conn\ c'\ [\varphi 1, \varphi 2] \land wf\text{-}conn\ c'\ [\varphi 1, \varphi 2]$
have *c-in-c′-only*: *c-in-c′-only* $c\ c'\ \varphi 1 \land$ *c-in-c′-only* $c\ c'\ \varphi 2 \land$ *c-in-c′-symb* $c\ c'\ \varphi$
using *c-in-c′-only* φ' **unfolding** *c-in-c′-only-def* **by** *auto*
have *super-grouped-by* $c\ c'\ \varphi 1$ **using** φ c' *no-equiv no-imp simpleN* $IH\varphi 1$ *c-in-c′-only* **by** *auto*
moreover have *super-grouped-by* $c\ c'\ \varphi 2$
using φ c' *no-equiv no-imp simpleN* $IH\varphi 2$ *c-in-c′-only* **by** *auto*
ultimately have *?S* φ
using *super-grouped-by.intros*(*2*) φ **by** (*metis c wf-conn-helper-facts*(*5*,*6*))
}
moreover {
assume φ: $\varphi = conn\ c\ [\varphi 1, \varphi 2] \land wf\text{-}conn\ c\ [\varphi 1, \varphi 2]$
then have *only-c-inside* $c\ \varphi 1 \land$ *only-c-inside* $c\ \varphi 2$
using *c-in-c′-symb-c-implies-only-c-inside* $c\ c'$ *c-in-c′-only* *list.set-intros*(*1*)
wf-conn-helper-facts(*5*,*6*) *no-equiv no-imp simpleN last-ConsL last-ConsR last-in-set*
list.distinct(*1*) **by** (*metis* (*no-types*, *hide-lams*) *cc′*)
then have *only-c-inside* c ($conn\ c\ [\varphi 1, \varphi 2]$)
unfolding *only-c-inside-def* **using** φ
by (*simp add*: *only-c-inside-into-only-c-inside all-subformula-st-decomp*)
then have *grouped-by* $c\ \varphi$ **using** φ *only-c-inside-imp-grouped-by* c **by** *blast*
then have *?S* φ **using** *super-grouped-by.intros*(*1*) **by** *metis*
}
ultimately show *?S* φ **by** (*metis* φ' c c' *cc′ conn.simps*(*5*,*6*) *wf-conn-helper-facts*(*5*,*6*))
qed

0.4.2 Conjunctive Normal Form

definition *is-conj-with-TF* **where** *is-conj-with-TF* == *super-grouped-by COr CAnd*

lemma *or-in-and-only-conjunction-in-disj*:
shows *no-equiv* $\varphi \Longrightarrow$ *no-imp* $\varphi \Longrightarrow$ *simple-not* $\varphi \Longrightarrow$ *or-in-and-only* $\varphi \Longrightarrow$ *is-conj-with-TF* φ

using *c-in-c′-only-super-grouped-by*
unfolding *is-conj-with-TF-def or-in-and-only-def c-in-c′-only-def*
by (*simp add*: *c-in-c′-only-def c-in-c′-only-super-grouped-by*)

definition *is-cnf* **where**
is-cnf $\varphi \equiv$ *is-conj-with-TF* $\varphi \wedge$ *no-T-F-except-top-level* φ

Full CNF transformation

The full1 CNF transformation consists simply in chaining all the transformation defined before.

definition *cnf-rew* **where** *cnf-rew* =
(*full* (*propo-rew-step elim-equiv*)) *OO*
(*full* (*propo-rew-step elim-imp*)) *OO*
(*full* (*propo-rew-step elimTB*)) *OO*
(*full* (*propo-rew-step pushNeg*)) *OO*
(*full* (*propo-rew-step pushDisj*))

lemma *cnf-rew-consistent*: *preserves-un-sat cnf-rew*
by (*simp add*: *cnf-rew-def elimEquv-lifted-consistant elim-imp-lifted-consistant elimTB-consistent preserves-un-sat-OO pushDisj-consistent pushNeg-lifted-consistant*)

lemma *cnf-rew-is-cnf*: *cnf-rew* φ φ' $\Longrightarrow$ *is-cnf* φ'
apply (*unfold cnf-rew-def OO-def*)
apply *auto*
proof −
fix φ φEq φImp φTB φNeg $\varphi Disj$ :: *′v propo*
assume *Eq*: *full* (*propo-rew-step elim-equiv*) φ φEq
then have *no-equiv*: *no-equiv* φEq **using** *no-equiv-full-propo-rew-step-elim-equiv* **by** *blast*

assume *Imp*: *full* (*propo-rew-step elim-imp*) φEq φImp
then have *no-imp*: *no-imp* φImp **using** *no-imp-full-propo-rew-step-elim-imp* **by** *blast*
have *no-imp-inv*: *no-equiv* φImp **using** *no-equiv Imp elim-imp-inv* **by** *blast*

assume *TB*: *full* (*propo-rew-step elimTB*) φImp φTB
then have *noTB*: *no-T-F-except-top-level* φTB
using *no-imp-inv no-imp elimTB-full-propo-rew-step* **by** *blast*
have *noTB-inv*: *no-equiv* φTB *no-imp* φTB **using** *elimTB-inv TB no-imp no-imp-inv* **by** *blast+*

assume *Neg*: *full* (*propo-rew-step pushNeg*) φTB φNeg
then have *noNeg*: *simple-not* φNeg
using *noTB-inv noTB pushNeg-full-propo-rew-step* **by** *blast*
have *noNeg-inv*: *no-equiv* φNeg *no-imp* φNeg *no-T-F-except-top-level* φNeg
using *pushNeg-inv Neg noTB noTB-inv* **by** *blast+*

assume *Disj*: *full* (*propo-rew-step pushDisj*) φNeg $\varphi Disj$
then have *no-Disj*: *or-in-and-only* $\varphi Disj$
using *noNeg-inv noNeg pushDisj-full-propo-rew-step* **by** *blast*
have *noDisj-inv*: *no-equiv* $\varphi Disj$ *no-imp* $\varphi Disj$ *no-T-F-except-top-level* $\varphi Disj$
simple-not $\varphi Disj$
using *pushDisj-inv Disj noNeg noNeg-inv* **by** *blast+*

moreover have *is-conj-with-TF* $\varphi Disj$
using *or-in-and-only-conjunction-in-disj noDisj-inv no-Disj* **by** *blast*
ultimately show *is-cnf* $\varphi Disj$ **unfolding** *is-cnf-def* **by** *blast*

qed

0.4.3 Disjunctive Normal Form

definition *is-disj-with-TF* **where** *is-disj-with-TF* $\equiv$ *super-grouped-by CAnd COr*

lemma *and-in-or-only-conjunction-in-disj*:
shows *no-equiv* $\varphi \Longrightarrow$ *no-imp* $\varphi \Longrightarrow$ *simple-not* $\varphi \Longrightarrow$ *and-in-or-only* $\varphi \Longrightarrow$ *is-disj-with-TF* φ
using *c-in-c′-only-super-grouped-by*
unfolding *is-disj-with-TF-def and-in-or-only-def c-in-c′-only-def*
by (*simp add*: *c-in-c′-only-def c-in-c′-only-super-grouped-by*)

definition *is-dnf* :: *′a propo* $\Rightarrow$ *bool* **where**
is-dnf $\varphi \longleftrightarrow$ *is-disj-with-TF* $\varphi \wedge$ *no-T-F-except-top-level* φ

Full DNF transform

The full1 DNF transformation consists simply in chaining all the transformation defined before.

definition *dnf-rew* **where** *dnf-rew* $\equiv$
(*full* (*propo-rew-step elim-equiv*)) *OO*
(*full* (*propo-rew-step elim-imp*)) *OO*
(*full* (*propo-rew-step elimTB*)) *OO*
(*full* (*propo-rew-step pushNeg*)) *OO*
(*full* (*propo-rew-step pushConj*))

lemma *dnf-rew-consistent*: *preserves-un-sat dnf-rew*
by (*simp add*: *dnf-rew-def elimEquv-lifted-consistant elim-imp-lifted-consistant elimTB-consistent preserves-un-sat-OO pushConj-consistent pushNeg-lifted-consistant*)

theorem *dnf-transformation-correction*:
dnf-rew φ φ' $\Longrightarrow$ *is-dnf* φ'
apply (*unfold dnf-rew-def OO-def*)
by (*meson and-in-or-only-conjunction-in-disj elimTB-full-propo-rew-step elimTB-inv*(*1*,*2*) *elim-imp-inv is-dnf-def no-equiv-full-propo-rew-step-elim-equiv no-imp-full-propo-rew-step-elim-imp pushConj-full-propo-rew-step pushConj-inv*(*1*−*4*) *pushNeg-full-propo-rew-step pushNeg-inv*(*1*−*3*))

0.5 More aggressive simplifications: Removing true and false at the beginning

0.5.1 Transformation

We should remove *FT* and *FF* at the beginning and not in the middle of the algorithm. To do this, we have to use more rules (one for each connective):

inductive *elimTBFull* **where**
ElimTBFull1[*simp*]: *elimTBFull* (*FAnd* φ *FT*) φ |
ElimTBFull1′[*simp*]: *elimTBFull* (*FAnd FT* φ) φ |

ElimTBFull2[*simp*]: *elimTBFull* (*FAnd* φ *FF*) *FF* |
ElimTBFull2′[*simp*]: *elimTBFull* (*FAnd FF* φ) *FF* |

ElimTBFull3[*simp*]: *elimTBFull* (*FOr* φ *FT*) *FT* |
ElimTBFull3′[*simp*]: *elimTBFull* (*FOr FT* φ) *FT* |

ElimTBFull4[*simp*]: *elimTBFull* (*FOr* φ *FF*) φ |
ElimTBFull4'[*simp*]: *elimTBFull* (*FOr FF* φ) φ |

ElimTBFull5[*simp*]: *elimTBFull* (*FNot FT*) *FF* |
ElimTBFull5'[*simp*]: *elimTBFull* (*FNot FF*) *FT* |

ElimTBFull6-l[*simp*]: *elimTBFull* (*FImp FT* φ) φ |
ElimTBFull6-l'[*simp*]: *elimTBFull* (*FImp FF* φ) *FT* |
ElimTBFull6-r[*simp*]: *elimTBFull* (*FImp* φ *FT*) *FT* |
ElimTBFull6-r'[*simp*]: *elimTBFull* (*FImp* φ *FF*) (*FNot* φ) |

ElimTBFull7-l[*simp*]: *elimTBFull* (*FEq FT* φ) φ |
ElimTBFull7-l'[*simp*]: *elimTBFull* (*FEq FF* φ) (*FNot* φ) |
ElimTBFull7-r[*simp*]: *elimTBFull* (*FEq* φ *FT*) φ |
ElimTBFull7-r'[*simp*]: *elimTBFull* (*FEq* φ *FF*) (*FNot* φ)

The transformation is still consistent.

lemma *elimTBFull-consistent*: *preserves-un-sat elimTBFull*
proof –
  {
    fix φ ψ:: *'b propo*
    have *elimTBFull* φ ψ $\Longrightarrow \forall A.\ A \models \varphi \longleftrightarrow A \models \psi$
      by (*induct-tac rule*: *elimTBFull.inducts*, *auto*)
  }
  then show *?thesis* **using** *preserves-un-sat-def* **by** *auto*
qed

Contrary to the theorem *no-T-F-symb-except-toplevel-step-exists*, we do not need the assumption *no-equiv* φ and *no-imp* φ, since our transformation is more general.

lemma *no-T-F-symb-except-toplevel-step-exists'*:
  fixes φ :: *'v propo*
  shows $\psi \preceq \varphi \Longrightarrow \neg$ *no-T-F-symb-except-toplevel* $\psi \Longrightarrow \exists \psi'.$ *elimTBFull* ψ ψ'
proof (*induct* ψ *rule*: *propo-induct-arity*)
  case (*nullary* φ')
  then have *False* **using** *no-T-F-symb-except-toplevel-true no-T-F-symb-except-toplevel-false* **by** *auto*
  then show *Ex* (*elimTBFull* φ') **by** *blast*
next
  case (*unary* ψ)
  then have $\psi = FF \vee \psi = FT$ **using** *no-T-F-symb-except-toplevel-not-decom* **by** *blast*
  then show *Ex* (*elimTBFull* (*FNot* ψ)) **using** *ElimTBFull5 ElimTBFull5'* **by** *blast*
next
  case (*binary* φ' $\psi 1$ $\psi 2$)
  then have $\psi 1 = FT \vee \psi 2 = FT \vee \psi 1 = FF \vee \psi 2 = FF$
    by (*metis binary-connectives-def conn.simps*(*5*−*8*) *insertI1 insert-commute*
      no-T-F-symb-except-toplevel-bin-decom binary.hyps(*3*))
  then show *Ex* (*elimTBFull* φ') **using** *elimTBFull.intros binary.hyps*(*3*) **by** *blast*
qed

The same applies here. We do not need the assumption, but the deep link between $\neg$ *no-T-F-except-top-level* φ and the existence of a rewriting step, still exists.

lemma *no-T-F-except-top-level-rew'*:
  fixes φ :: *'v propo*
  assumes *noTB*: $\neg$ *no-T-F-except-top-level* φ
  shows $\exists \psi\ \psi'.\ \psi \preceq \varphi \wedge$ *elimTBFull* ψ ψ'
proof –

have *test-symb-false-nullary*:
$\forall x.$ *no-T-F-symb-except-toplevel* (*FF*:: *'v propo*) $\wedge$ *no-T-F-symb-except-toplevel FT*
$\wedge$ *no-T-F-symb-except-toplevel* (*FVar* (*x*:: *'v*))
by *auto*
moreover {
fix *c*:: *'v connective* **and** *l* :: *'v propo list* **and** ψ :: *'v propo*
have *H*: *elimTBFull* (*conn c l*) $\psi \Longrightarrow \neg$*no-T-F-symb-except-toplevel* (*conn c l*)
by (*cases conn c l rule*: *elimTBFull.cases*) *auto*
}
ultimately show *?thesis*
using *no-test-symb-step-exists*[*of no-T-F-symb-except-toplevel* φ *elimTBFull*] *noTB*
no-T-F-symb-except-toplevel-step-exists' **unfolding** *no-T-F-except-top-level-def* **by** *metis*
qed

lemma *elimTBFull-full-propo-rew-step*:
fixes φ ψ :: *'v propo*
assumes *full* (*propo-rew-step elimTBFull*) φ ψ
shows *no-T-F-except-top-level* ψ
using *full-propo-rew-step-subformula no-T-F-except-top-level-rew' assms* **by** *fastforce*

0.5.2 More invariants

As the aim is to use the transformation as the first transformation, we have to show some more invariants for *elim-equiv* and *elim-imp*. For the other transformation, we have already proven it.

lemma *propo-rew-step-ElimEquiv-no-T-F*: *propo-rew-step elim-equiv* φ $\psi \Longrightarrow$ *no-T-F* $\varphi \Longrightarrow$ *no-T-F* ψ
proof (*induct rule*: *propo-rew-step.induct*)
fix φ' :: *'v propo* **and** ψ' :: *'v propo*
assume *a1*: *no-T-F* φ'
assume *a2*: *elim-equiv* φ' ψ'
have $\forall$ *x0 x1*. ($\neg$ *elim-equiv* (*x1* :: *'v propo*) *x0* $\vee$ ($\exists$ *v2 v3 v4 v5 v6 v7*. *x1* = *FEq v2 v3*
$\wedge$ *x0* = *FAnd* (*FImp v4 v5*) (*FImp v6 v7*) $\wedge$ *v2* = *v4* $\wedge$ *v4* = *v7* $\wedge$ *v3* = *v5* $\wedge$ *v3* = *v6*))
= ($\neg$ *elim-equiv x1 x0* $\vee$ ($\exists$ *v2 v3 v4 v5 v6 v7*. *x1* = *FEq v2 v3*
$\wedge$ *x0* = *FAnd* (*FImp v4 v5*) (*FImp v6 v7*) $\wedge$ *v2* = *v4* $\wedge$ *v4* = *v7* $\wedge$ *v3* = *v5* $\wedge$ *v3* = *v6*))
by *meson*
then have $\forall$ *p pa*. $\neg$ *elim-equiv* (*p* :: *'v propo*) *pa* $\vee$ ($\exists$ *pb pc pd pe pf pg*. *p* = *FEq pb pc*
$\wedge$ *pa* = *FAnd* (*FImp pd pe*) (*FImp pf pg*) $\wedge$ *pb* = *pd* $\wedge$ *pd* = *pg* $\wedge$ *pc* = *pe* $\wedge$ *pc* = *pf*)
using *elim-equiv.cases* **by** *force*
then show *no-T-F* ψ' **using** *a1 a2* **by** *fastforce*
next
fix φ φ' :: *'v propo* **and** ξ ξ' :: *'v propo list* **and** *c* :: *'v connective*
assume *rel*: *propo-rew-step elim-equiv* φ φ'
and *IH*: *no-T-F* $\varphi \Longrightarrow$ *no-T-F* φ'
and *corr*: *wf-conn c* (ξ @ φ # ξ')
and *no-T-F*: *no-T-F* (*conn c* (ξ @ φ # ξ'))
{
assume *c*: *c* = *CNot*
then have *empty*: ξ = [] ξ' = [] **using** *corr* **by** *auto*
then have *no-T-F* φ **using** *no-T-F c no-T-F-decomp-not* **by** *auto*
then have *no-T-F* (*conn c* (ξ @ φ' # ξ')) **using** *c empty no-T-F-comp-not IH* **by** *auto*
}
moreover {
assume *c*: *c* $\in$ *binary-connectives*

```
    obtain a b where ab: ξ @ φ # ξ' = [a, b]
      using corr c list-length2-decomp wf-conn-bin-list-length by metis
    then have φ: φ = a ∨ φ = b
      by (metis append.simps(1) append-is-Nil-conv list.distinct(1) list.sel(3) nth-Cons-0
        tl-append2)
    have ζ: ∀ζ∈ set (ξ @ φ # ξ'). no-T-F ζ
      using no-T-F unfolding no-T-F-def using corr all-subformula-st-decomp by blast

    then have φ': no-T-F φ' using ab IH φ by auto
    have l': ξ @ φ' # ξ' = [φ', b] ∨ ξ @ φ' # ξ' = [a, φ']
      by (metis (no-types, hide-lams) ab append-Cons append-Nil append-Nil2 butlast.simps(2)
        butlast-append list.distinct(1) list.sel(3))
    then have ∀ζ ∈ set (ξ @ φ' # ξ'). no-T-F ζ using ζ φ' ab by fastforce
    moreover
      have ∀ζ ∈ set (ξ @ φ # ξ'). ζ ≠ FT ∧ ζ ≠ FF
        using ζ corr no-T-F no-T-F-except-top-level-false no-T-F-no-T-F-except-top-level by blast
      then have no-T-F-symb (conn c (ξ @ φ' # ξ'))
        by (metis φ' l' ab all-subformula-st-test-symb-true-phi c list.distinct(1)
          list.set-intros(1,2) no-T-F-symb-except-toplevel-bin-decom
          no-T-F-symb-except-toplevel-no-T-F-symb no-T-F-symb-false(1,2) no-T-F-def wf-conn-binary
          wf-conn-list(1,2))
    ultimately have no-T-F (conn c (ξ @ φ' # ξ'))
      by (metis l' all-subformula-st-decomp-imp c no-T-F-def wf-conn-binary)
  }
  moreover {
    fix x
    assume c = CVar x ∨ c = CF ∨ c = CT
    then have False using corr by auto
    then have  no-T-F (conn c (ξ @ φ' # ξ')) by auto
  }
  ultimately show no-T-F (conn c (ξ @ φ' # ξ')) using corr wf-conn.cases by metis
qed

lemma elim-equiv-inv':
  fixes φ ψ :: 'v propo
  assumes full (propo-rew-step elim-equiv) φ ψ and no-T-F-except-top-level φ
  shows no-T-F-except-top-level ψ
proof −
  {
    fix φ ψ :: 'v propo
    have propo-rew-step elim-equiv φ ψ ⟹ no-T-F-except-top-level φ
      ⟹ no-T-F-except-top-level ψ
      proof −
        assume rel: propo-rew-step elim-equiv φ ψ
        and no: no-T-F-except-top-level φ
        {
          assume φ = FT ∨ φ = FF
          from rel this have False
            apply (induct rule: propo-rew-step.induct, auto simp: wf-conn-list(1,2))
            using elim-equiv.simps by blast+
          then have no-T-F-except-top-level ψ by blast
        }
        moreover {
          assume φ ≠ FT ∧ φ ≠ FF
          then have no-T-F φ
            by (metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
```

```
        then have no-T-F ψ using propo-rew-step-ElimEquiv-no-T-F rel by blast
        then have no-T-F-except-top-level ψ by (simp add: no-T-F-no-T-F-except-top-level)
      }
      ultimately show no-T-F-except-top-level ψ by metis
    qed
  }
  moreover {
    fix c :: 'v connective and ξ ξ' :: 'v propo list and ζ ζ' :: 'v propo
    assume rel: propo-rew-step elim-equiv ζ ζ'
    and incl: ζ ≼ φ
    and corr: wf-conn c (ξ @ ζ # ξ')
    and no-T-F: no-T-F-symb-except-toplevel (conn c (ξ @ ζ # ξ'))
    and n: no-T-F-symb-except-toplevel ζ'
    have no-T-F-symb-except-toplevel (conn c (ξ @ ζ' # ξ'))
    proof
      have p: no-T-F-symb (conn c (ξ @ ζ # ξ'))
        using corr wf-conn-list(1) wf-conn-list(2) no-T-F-symb-except-toplevel-no-T-F-symb no-T-F
        by blast
      have l: ∀φ∈set (ξ @ ζ # ξ'). φ ≠ FT ∧ φ ≠ FF
        using corr wf-conn-no-T-F-symb-iff p by blast
      from rel incl have ζ'≠FT ∧ζ'≠FF
        apply (induction ζ ζ' rule: propo-rew-step.induct)
        apply (cases rule: elim-equiv.cases, auto simp: elim-equiv.simps)
        by (metis append-is-Nil-conv list.distinct wf-conn-list(1,2) wf-conn-no-arity-change
          wf-conn-no-arity-change-helper)+
      then have ∀φ ∈ set (ξ @ ζ' # ξ'). φ ≠ FT ∧ φ ≠ FF using l by auto
      moreover have c ≠ CT ∧ c ≠ CF using corr by auto
      ultimately show no-T-F-symb (conn c (ξ @ ζ' # ξ'))
        by (metis corr wf-conn-no-arity-change wf-conn-no-arity-change-helper no-T-F-symb-comp)
    qed
  }
  ultimately show no-T-F-except-top-level ψ
    using full-propo-rew-step-inv-stay-with-inc[of elim-equiv no-T-F-symb-except-toplevel φ]
      assms subformula-refl unfolding no-T-F-except-top-level-def by metis
qed


lemma propo-rew-step-ElimImp-no-T-F: propo-rew-step elim-imp φ ψ ⟹ no-T-F φ ⟹  no-T-F ψ
proof (induct rule: propo-rew-step.induct)
  case (global-rel φ' ψ')
  then show no-T-F ψ'
    using elim-imp.cases no-T-F-comp-not no-T-F-decomp(1,2)
    by (metis no-T-F-comp-expanded-explicit(2))
next
  case (propo-rew-one-step-lift φ φ' c ξ ξ')
  note rel = this(1) and IH = this(2) and corr = this(3) and no-T-F = this(4)
  {
    assume c: c = CNot
    then have empty: ξ = [] ξ' = [] using corr by auto
    then have no-T-F φ using no-T-F c no-T-F-decomp-not by auto
    then have no-T-F (conn c (ξ @ φ' # ξ')) using c empty no-T-F-comp-not IH by auto
  }
  moreover {
    assume c: c ∈ binary-connectives
    then obtain a b where ab: ξ @ φ # ξ' = [a, b]
      using corr list-length2-decomp wf-conn-bin-list-length by metis
```

```
    then have φ: φ = a ∨ φ = b
      by (metis append-self-conv2 wf-conn-list-decomp(4) wf-conn-unary list.discI list.sel(3)
        nth-Cons-0 tl-append2)
    have ζ: ∀ζ ∈ set (ξ @ φ # ξ'). no-T-F ζ using ab c propo-rew-one-step-lift.prems by auto

    then have φ': no-T-F φ'
      using ab IH φ corr no-T-F no-T-F-def all-subformula-st-decomp-explicit by auto
    have χ: ξ @ φ' # ξ' = [φ', b] ∨ ξ @ φ' # ξ' = [a, φ']
      by (metis (no-types, hide-lams) ab append-Cons append-Nil append-Nil2 butlast.simps(2)
        butlast-append list.distinct(1) list.sel(3))
    then have ∀ζ∈ set (ξ @ φ' # ξ'). no-T-F ζ using ζ φ' ab by fastforce
    moreover
      have no-T-F (last (ξ @ φ' # ξ'))  by (simp add: calculation)
      then have no-T-F-symb (conn c (ξ @ φ' # ξ'))
        by (metis χ φ' ζ ab all-subformula-st-test-symb-true-phi c last.simps list.distinct(1)
          list.set-intros(1) no-T-F-bin-decomp no-T-F-def)
    ultimately have no-T-F (conn c (ξ @ φ' # ξ')) using c χ by fastforce
  }
  moreover {
    fix x
    assume c = CVar x ∨ c = CF ∨ c = CT
    then have False using corr by auto
    then have no-T-F (conn c (ξ @ φ' # ξ')) by auto
  }
  ultimately show no-T-F (conn c (ξ @ φ' # ξ')) using corr wf-conn.cases by blast
qed


lemma elim-imp-inv':
  fixes φ ψ :: 'v propo
  assumes full (propo-rew-step elim-imp) φ ψ and no-T-F-except-top-level φ
  showsno-T-F-except-top-level ψ
proof −
  {
    {
      fix φ ψ :: 'v propo
      have H: elim-imp φ ψ ⟹ no-T-F-except-top-level φ ⟹  no-T-F-except-top-level ψ
        by (induct φ ψ rule: elim-imp.induct, auto)
    } note H = this
    fix φ ψ :: 'v propo
    have propo-rew-step elim-imp φ ψ ⟹ no-T-F-except-top-level φ ⟹  no-T-F-except-top-level ψ
      proof −
        assume rel: propo-rew-step elim-imp φ ψ
        and no: no-T-F-except-top-level φ
        {
          assume φ = FT ∨ φ = FF
          from rel this have False
            apply (induct rule: propo-rew-step.induct)
            by (cases rule: elim-imp.cases, auto simp: wf-conn-list(1,2))
          then have no-T-F-except-top-level ψ by blast
        }
        moreover {
          assume φ ≠ FT ∧ φ ≠ FF
          then have no-T-F φ
            by (metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
          then have no-T-F ψ
```

```
          using rel propo-rew-step-ElimImp-no-T-F by blast
        then have no-T-F-except-top-level ψ by (simp add: no-T-F-no-T-F-except-top-level)
      }
      ultimately show no-T-F-except-top-level ψ by metis
    qed
  }
  moreover {
    fix c :: 'v connective and ξ ξ' :: 'v propo list and ζ ζ' :: 'v propo
    assume rel: propo-rew-step elim-imp ζ ζ'
    and incl: ζ ≼ φ
    and corr: wf-conn c (ξ @ ζ # ξ')
    and no-T-F: no-T-F-symb-except-toplevel (conn c (ξ @ ζ # ξ'))
    and n: no-T-F-symb-except-toplevel ζ'
    have no-T-F-symb-except-toplevel (conn c (ξ @ ζ' # ξ'))
    proof
      have p: no-T-F-symb (conn c (ξ @ ζ # ξ'))
        by (simp add: corr no-T-F no-T-F-symb-except-toplevel-no-T-F-symb wf-conn-list(1,2))

      have l: ∀φ∈set (ξ @ ζ # ξ'). φ ≠ FT ∧ φ ≠ FF
        using corr wf-conn-no-T-F-symb-iff p by blast
      from rel incl have ζ'≠FT ∧ζ'≠FF
        apply (induction ζ ζ' rule: propo-rew-step.induct)
        apply (cases rule: elim-imp.cases, auto)
        using wf-conn-list(1,2) wf-conn-no-arity-change wf-conn-no-arity-change-helper
        by (metis append-is-Nil-conv list.distinct(1))+
      then have ∀φ∈set (ξ @ ζ' # ξ'). φ ≠ FT ∧ φ ≠ FF using l by auto
      moreover have c ≠ CT ∧ c ≠ CF using corr by auto
      ultimately show no-T-F-symb (conn c (ξ @ ζ' # ξ'))
        using corr wf-conn-no-arity-change no-T-F-symb-comp
        by (metis wf-conn-no-arity-change-helper)
    qed
  }
  ultimately show no-T-F-except-top-level ψ
    using full-propo-rew-step-inv-stay-with-inc[of elim-imp no-T-F-symb-except-toplevel φ]
    assms subformula-refl unfolding no-T-F-except-top-level-def by metis
qed
```

0.5.3 The new CNF and DNF transformation

The transformation is the same as before, but the order is not the same.

```
definition dnf-rew' :: 'a propo ⇒ 'a propo ⇒ bool where
dnf-rew' =
  (full (propo-rew-step elimTBFull)) OO
  (full (propo-rew-step elim-equiv)) OO
  (full (propo-rew-step elim-imp)) OO
  (full (propo-rew-step pushNeg)) OO
  (full (propo-rew-step pushConj))

lemma dnf-rew'-consistent: preserves-un-sat dnf-rew'
  by (simp add: dnf-rew'-def elimEquv-lifted-consistant elim-imp-lifted-consistant
    elimTBFull-consistent preserves-un-sat-OO pushConj-consistent pushNeg-lifted-consistant)

theorem cnf-transformation-correction:
    dnf-rew' φ φ' ⟹ is-dnf φ'
  unfolding dnf-rew'-def OO-def
```

by (*meson and-in-or-only-conjunction-in-disj elimTBFull-full-propo-rew-step elim-equiv-inv′ elim-imp-inv elim-imp-inv′ is-dnf-def no-equiv-full-propo-rew-step-elim-equiv no-imp-full-propo-rew-step-elim-imp pushConj-full-propo-rew-step pushConj-inv*(*1*−*4*) *pushNeg-full-propo-rew-step pushNeg-inv*(*1*−*3*))

Given all the lemmas before the CNF transformation is easy to prove:

definition *cnf-rew′* :: *′a propo* ⇒ *′a propo* ⇒ *bool* **where**
cnf-rew′ =
(*full* (*propo-rew-step elimTBFull*)) *OO*
(*full* (*propo-rew-step elim-equiv*)) *OO*
(*full* (*propo-rew-step elim-imp*)) *OO*
(*full* (*propo-rew-step pushNeg*)) *OO*
(*full* (*propo-rew-step pushDisj*))

lemma *cnf-rew′-consistent*: *preserves-un-sat cnf-rew′*
by (*simp add*: *cnf-rew′-def elimEquv-lifted-consistant elim-imp-lifted-consistant elimTBFull-consistent preserves-un-sat-OO pushDisj-consistent pushNeg-lifted-consistant*)

theorem *cnf′-transformation-correction*:
cnf-rew′ φ φ′ ⟹ *is-cnf φ′*
unfolding *cnf-rew′-def OO-def*
by (*meson elimTBFull-full-propo-rew-step elim-equiv-inv′ elim-imp-inv elim-imp-inv′ is-cnf-def no-equiv-full-propo-rew-step-elim-equiv no-imp-full-propo-rew-step-elim-imp or-in-and-only-conjunction-in-disj pushDisj-full-propo-rew-step pushDisj-inv*(*1*−*4*) *pushNeg-full-propo-rew-step pushNeg-inv*(*1*) *pushNeg-inv*(*2*) *pushNeg-inv*(*3*))

end
theory *Prop-Logic-Multiset*
imports *../lib/Multiset-More Prop-Normalisation Partial-Clausal-Logic*
begin

0.6 Link with Multiset Version

0.6.1 Transformation to Multiset

fun *mset-of-conj* :: *′a propo* ⇒ *′a literal multiset* **where**
mset-of-conj (*FOr φ ψ*) = *mset-of-conj φ* + *mset-of-conj ψ* |
mset-of-conj (*FVar v*) = {# *Pos v* #} |
mset-of-conj (*FNot* (*FVar v*)) = {# *Neg v* #} |
mset-of-conj FF = {#}

fun *mset-of-formula* :: *′a propo* ⇒ *′a literal multiset set* **where**
mset-of-formula (*FAnd φ ψ*) = *mset-of-formula φ* ∪ *mset-of-formula ψ* |
mset-of-formula (*FOr φ ψ*) = {*mset-of-conj* (*FOr φ ψ*)} |
mset-of-formula (*FVar ψ*) = {*mset-of-conj* (*FVar ψ*)} |
mset-of-formula (*FNot ψ*) = {*mset-of-conj* (*FNot ψ*)} |
mset-of-formula FF = {{#}} |
mset-of-formula FT = {}

0.6.2 Equisatisfiability of the two Version

lemma *is-conj-with-TF-FNot*:
is-conj-with-TF (*FNot φ*) ⟷ (∃ *v*. *φ* = *FVar v* ∨ *φ* = *FF* ∨ *φ* = *FT*)
unfolding *is-conj-with-TF-def* **apply** (*rule iffI*)
apply (*induction FNot φ rule*: *super-grouped-by.induct*)

apply (*induction FNot* φ *rule*: *grouped-by.induct*)
apply *simp*
apply (*cases* φ; *simp*)
apply *auto*
done

lemma *grouped-by-COr-FNot*:
grouped-by COr (*FNot* φ) $\longleftrightarrow$ ($\exists$ *v*. φ = *FVar v* $\vee$ φ = *FF* $\vee$ φ = *FT*)
unfolding *is-conj-with-TF-def* **apply** (*rule iffI*)
apply (*induction FNot* φ *rule*: *grouped-by.induct*)
apply *simp*
apply (*cases* φ; *simp*)
apply *auto*
done

lemma
shows *no-T-F-FF*[*simp*]: $\neg$*no-T-F FF* **and**
no-T-F-FT[*simp*]: $\neg$*no-T-F FT*
unfolding *no-T-F-def all-subformula-st-def* **by** *auto*

lemma *grouped-by-CAnd-FAnd*:
grouped-by CAnd (*FAnd* φ*1* φ*2*) $\longleftrightarrow$ *grouped-by CAnd* φ*1* $\wedge$ *grouped-by CAnd* φ*2*
apply (*rule iffI*)
apply (*induction FAnd* φ*1* φ*2 rule*: *grouped-by.induct*)
using *connected-is-group*[*of CAnd* φ*1* φ*2*] **by** *auto*

lemma *grouped-by-COr-FOr*:
grouped-by COr (*FOr* φ*1* φ*2*) $\longleftrightarrow$ *grouped-by COr* φ*1* $\wedge$ *grouped-by COr* φ*2*
apply (*rule iffI*)
apply (*induction FOr* φ*1* φ*2 rule*: *grouped-by.induct*)
using *connected-is-group*[*of COr* φ*1* φ*2*] **by** *auto*

lemma *grouped-by-COr-FAnd*[*simp*]: $\neg$ *grouped-by COr* (*FAnd* φ*1* φ*2*)
apply *clarify*
apply (*induction FAnd* φ*1* φ*2 rule*: *grouped-by.induct*)
apply *auto*
done

lemma *grouped-by-COr-FEq*[*simp*]: $\neg$ *grouped-by COr* (*FEq* φ*1* φ*2*)
apply *clarify*
apply (*induction FEq* φ*1* φ*2 rule*: *grouped-by.induct*)
apply *auto*
done

lemma [*simp*]: $\neg$*grouped-by COr* (*FImp* φ ψ)
apply *clarify*
by (*induction FImp* φ ψ *rule*: *grouped-by.induct*) *simp-all*

lemma [*simp*]: $\neg$ *is-conj-with-TF* (*FImp* φ ψ)
unfolding *is-conj-with-TF-def* **apply** *clarify*
by (*induction FImp* φ ψ *rule*: *super-grouped-by.induct*) *simp-all*

lemma [*simp*]: $\neg$*grouped-by COr* (*FEq* φ ψ)
apply *clarify*
by (*induction FEq* φ ψ *rule*: *grouped-by.induct*) *simp-all*

lemma [*simp*]: $\neg$ *is-conj-with-TF* (*FEq* φ ψ)
  unfolding *is-conj-with-TF-def* **apply** *clarify*
  by (*induction FEq* φ ψ *rule*: *super-grouped-by.induct*) *simp-all*

lemma *is-conj-with-TF-Fand*:
  is-conj-with-TF (*FAnd* $\varphi 1$ $\varphi 2$) $\Longrightarrow$ *is-conj-with-TF* $\varphi 1$ $\wedge$ *is-conj-with-TF* $\varphi 2$
  unfolding *is-conj-with-TF-def*
  apply (*induction FAnd* $\varphi 1$ $\varphi 2$ *rule*: *super-grouped-by.induct*)
   apply (*auto simp*: *grouped-by-CAnd-FAnd intro*: *grouped-is-super-grouped*)[]
  apply *auto*[]
  done

lemma *is-conj-with-TF-FOr*:
  is-conj-with-TF (*FOr* $\varphi 1$ $\varphi 2$) $\Longrightarrow$ *grouped-by COr* $\varphi 1$ $\wedge$ *grouped-by COr* $\varphi 2$
  unfolding *is-conj-with-TF-def*
  apply (*induction FOr* $\varphi 1$ $\varphi 2$ *rule*: *super-grouped-by.induct*)
   apply (*auto simp*: *grouped-by-COr-FOr*)[]
  apply *auto*[]
  done

lemma *grouped-by-COr-mset-of-formula*:
  grouped-by COr φ $\Longrightarrow$ *mset-of-formula* φ = (*if* φ = *FT then* {} *else* {*mset-of-conj* φ})
  by (*induction* φ) (*auto simp add*: *grouped-by-COr-FNot*)

When a formula is in CNF form, then there is equisatisfiability between the multiset version and the CNF form. Remark that the definition for the entailment are slightly different: *op* $\models$ uses a function assigning *True* or *False*, while *op* $\models s$ uses a set where being in the list means entailment of a literal.

theorem
  fixes φ :: *'v propo*
  assumes *is-cnf* φ
  shows *eval A* φ $\longleftrightarrow$ *Partial-Clausal-Logic.true-clss* ({*Pos v*|*v*. *A v*} $\cup$ {*Neg v*|*v*. $\neg$*A v*})
    (*mset-of-formula* φ)
  using *assms*
proof (*induction* φ)
  case *FF*
  then show *?case* **by** *auto*
next
  case *FT*
  then show *?case* **by** *auto*
next
  case (*FVar v*)
  then show *?case* **by** *auto*
next
  case (*FAnd* φ ψ)
  then show *?case*
  unfolding *is-cnf-def* **by** (*auto simp*: *is-conj-with-TF-FNot dest*: *is-conj-with-TF-Fand*
    dest!:*is-conj-with-TF-FOr*)
next
  case (*FOr* φ ψ)
  then have [*simp*]: *mset-of-formula* φ = {*mset-of-conj* φ} *mset-of-formula* ψ = {*mset-of-conj* ψ}
    unfolding *is-cnf-def* **by** (*auto dest*!:*is-conj-with-TF-FOr simp*: *grouped-by-COr-mset-of-formula*
      split: *if-splits*)
  have *is-conj-with-TF* φ *is-conj-with-TF* ψ

```
    using FOr(3) unfolding is-cnf-def no-T-F-def
    by (metis grouped-is-super-grouped is-conj-with-TF-FOr is-conj-with-TF-def)+
  then show ?case using FOr
    unfolding is-cnf-def by simp
next
  case (FImp φ ψ)
  then show ?case
    unfolding is-cnf-def by auto
next
  case (FEq φ ψ)
  then show ?case
    unfolding is-cnf-def by auto
next
  case (FNot φ)
  then show ?case
    unfolding is-cnf-def by (auto simp: is-conj-with-TF-FNot)
qed

end
```