

Formalisation of Ground Resolution and CDCL in Isabelle/HOL

Mathias Fleury and Jasmin Blanchette

March 23, 2016

Contents

1	Transitions	5
1.1	More theorems about Closures	5
1.2	Full Transitions	7
1.3	Well-Foundedness and Full Transitions	8
1.4	More Well-Foundedness	9
2	Various Lemmas	11
3	More List	12
3.1	<i>upt</i>	12
3.2	Lexicographic Ordering	15
3.3	Remove	15
3.3.1	More lemmas about remove	15
3.3.2	Remove under condition	16
4	Logics	17
4.1	Definition and abstraction	17
4.2	properties of the abstraction	18
4.3	Subformulas and properties	21
4.4	Positions	24
5	Semantics over the syntax	27
6	Rewrite systems and properties	28
6.1	Lifting of rewrite rules	28
6.2	Consistency preservation	30
6.3	Full Lifting	31
7	Transformation testing	32
7.1	Definition and first properties	32
7.2	Invariant conservation	35
7.2.1	Invariant while lifting of the rewriting relation	35
7.2.2	Invariant after all rewriting	36

8	Rewrite Rules	38
8.1	Elimination of the equivalences	38
8.2	Eliminate Implication	39
8.3	Eliminate all the True and False in the formula	41
8.4	PushNeg	47
8.5	Push inside	52
8.5.1	Only one type of connective in the formula (+ not)	61
8.5.2	Push Conjunction	65
8.5.3	Push Disjunction	65
9	The full transformations	66
9.1	Abstract Property characterizing that only some connective are inside the others	66
9.1.1	Definition	66
9.2	Conjunctive Normal Form	68
9.2.1	Full CNF transformation	69
9.3	Disjunctive Normal Form	69
9.3.1	Full DNF transform	70
10	More aggressive simplifications: Removing true and false at the beginning	70
10.1	Transformation	70
10.2	More invariants	72
10.3	The new CNF and DNF transformation	76
11	Partial Clausal Logic	77
11.1	Clauses	77
11.2	Partial Interpretations	77
11.2.1	Consistency	77
11.2.2	Atoms	78
11.2.3	Totality	80
11.2.4	Interpretations	82
11.2.5	Satisfiability	84
11.2.6	Entailment for Multisets of Clauses	85
11.2.7	Tautologies	87
11.2.8	Entailment for clauses and propositions	88
11.3	Subsumptions	93
11.4	Removing Duplicates	94
11.5	Set of all Simple Clauses	94
11.6	Experiment: Expressing the Entailments as Locales	97
11.7	Entailment to be extended	98
12	Link with Multiset Version	99
12.1	Transformation to Multiset	99
12.2	Equisatisfiability of the two Version	100
13	Resolution	102
13.1	Simplification Rules	102
13.2	Unconstrained Resolution	104
13.2.1	Subsumption	104
13.3	Inference Rule	105
13.4	Lemma about the simplified state	120

13.5 Resolution and Invariants	123
13.5.1 Invariants	123
13.5.2 well-foundness if the relation	129
14 Partial Clausal Logic	143
14.1 Marked Literals	144
14.1.1 Definition	144
14.1.2 Entailment	145
14.1.3 Defined and undefined literals	147
14.2 Backtracking	148
14.3 Decomposition with respect to the marked literals	148
14.4 Negation of Clauses	155
14.5 Other	159
14.6 Abstract Clause Representation	161
15 Measure	163
16 NOT's CDCL	166
16.1 Auxiliary Lemmas and Measure	166
16.2 Initial definitions	167
16.2.1 The state	167
16.2.2 Definition of the operation	170
16.3 DPLL with backjumping	172
16.3.1 Definition	173
16.3.2 Basic properties	173
16.3.3 Termination	176
16.3.4 Normal Forms	181
16.4 CDCL	188
16.4.1 Learn and Forget	188
16.4.2 Definition of CDCL	190
16.5 CDCL with invariant	193
16.6 Termination	199
16.6.1 Restricting learn and forget	199
16.7 CDCL with restarts	211
16.7.1 Definition	211
16.7.2 Increasing restarts	212
16.8 Merging backjump and learning	219
16.8.1 Instantiations	232
17 DPLL as an instance of NOT	247
17.1 DPLL with simple backtrack	247
17.2 Adding restarts	252
18 DPLL	253
18.1 Rules	253
18.2 Invariants	253
18.3 Termination	262
18.4 Final States	264
18.5 Link with NOT's DPLL	265
18.5.1 Level of literals and clauses	267

18.5.2	Properties about the levels	271
19	Weidenbach's CDCL	273
19.1	The State	273
19.2	CDCL Rules	283
19.3	Invariants	290
19.3.1	Properties of the trail	290
19.3.2	Better-Suited Induction Principle	294
19.3.3	Compatibility with $op \sim$	298
19.3.4	Conservation of some Properties	303
19.3.5	Learned Clause	304
19.3.6	No alien atom in the state	305
19.3.7	No duplicates all around	308
19.3.8	Conflicts and co	309
19.3.9	Putting all the invariants together	318
19.3.10	No tautology is learned	321
19.4	CDCL Strong Completeness	322
19.5	Higher level strategy	323
19.5.1	Definition	323
19.5.2	Invariants	326
19.5.3	Literal of highest level in conflicting clauses	331
19.5.4	Literal of highest level in marked literals	335
19.5.5	Strong completeness	345
19.5.6	No conflict with only variables of level less than backtrack level	352
19.5.7	Final States are Conclusive	362
19.6	Termination	369
19.7	No Relearning of a clause	370
19.8	Decrease of a measure	386
20	Simple Implementation of the DPLL and CDCL	392
20.1	Common Rules	392
20.1.1	Propagation	392
20.1.2	Unit propagation for all clauses	393
20.1.3	Decide	394
20.2	Simple Implementation of DPLL	395
20.2.1	Combining the propagate and decide: a DPLL step	395
20.2.2	Adding invariants	398
20.2.3	Code export	404
20.3	CDCL Implementation	408
20.3.1	Definition of the rules	408
20.3.2	The Transitions	410
20.3.3	Code generation	422
21	Link between Weidenbach's and NOT's CDCL	437
21.1	Inclusion of the states	437
21.2	More lemmas conflict-propagate and backjumping	438
21.2.1	Termination	438
21.2.2	More backjumping	439
21.3	CDCL FW	452
21.4	FW with strategy	457

21.4.1 The intermediate step	457
21.5 Adding Restarts	492
22 Link between Weidenbach’s and NOT’s CDCL	503
22.1 Inclusion of the states	503
22.2 Additional Lemmas between NOT and W states	507
22.3 More lemmas conflict-propagate and backjumping	508
22.4 CDCL FW	508
23 Incremental SAT solving	516
24 2-Watched-Literal	526
24.1 Datastructure and Access Functions	526
24.2 Invariants	529
24.3 Abstract 2-WL	537
24.4 Instanciation of the previous locale	538
25 Invariants for 2 Watched-Literals	547
25.1 Interpretation for <i>conflict-driven-clause-learning_{W.cdcl_W}</i>	547
25.1.1 Direct Interpretation	547
25.1.2 Opaque Type with Invariant	548
26 Implementation for 2 Watched-Literals	555
27 Superposition	559
27.1 We can now define the rules of the calculus	565
theory <i>Wellfounded-More</i>	
imports <i>Main</i>	

begin

1 Transitions

This theory contains more facts about closure, the definition of full transformations, and well-foundedness.

1.1 More theorems about Closures

This is the equivalent of $?r \leq ?s \implies ?r^{**} \leq ?s^{**}$ for *tranclp*

lemma *tranclp-mono-explicit*:

$r^{++} \ a \ b \implies r \leq s \implies s^{++} \ a \ b$

using *rtranclp-mono* **by** (*auto dest!*: *tranclpD intro: rtranclp-into-tranclp2*)

lemma *tranclp-mono*:

assumes *mono*: $r \leq s$

shows $r^{++} \leq s^{++}$

using *rtranclp-mono[OF mono]* *mono* **by** (*auto dest!*: *tranclpD intro: rtranclp-into-tranclp2*)

lemma *tranclp-idemp-rel*:

$R^{++++} \ a \ b \longleftrightarrow R^{++} \ a \ b$

apply (*rule iffI*)

prefer 2 apply blast
by (*induction rule: tranclp-induct*) *auto*

Equivalent of $?r^{****} = ?r^{**}$

lemma *trancl-idemp*: $(r^+)^+ = r^+$
by *simp*

lemmas *tranclp-idemp[*simp*] = trancl-idemp[*to-pred*]*

This theorem already exists as $?r^{**} ?a ?b \equiv ?a = ?b \vee ?r^{++} ?a ?b$ (and sledgehammer uses it), but it makes sense to duplicate it, because it is unclear how stable the lemmas in Nitpick are.

lemma *rtranclp-unfold*: $rtranclp\ r\ a\ b \longleftrightarrow (a = b \vee tranclp\ r\ a\ b)$
by (*meson rtranclp.simps rtranclpD tranclp-into-rtranclp*)

lemma *tranclp-unfold-end*: $tranclp\ r\ a\ b \longleftrightarrow (\exists a'. rtranclp\ r\ a\ a' \wedge r\ a'\ b)$
by (*metis rtranclp.rtrancl-refl rtranclp-into-tranclp1 tranclp.cases tranclp-into-rtranclp*)

lemma *tranclp-unfold-begin*: $tranclp\ r\ a\ b \longleftrightarrow (\exists a'. r\ a\ a' \wedge rtranclp\ r\ a'\ b)$
by (*meson rtranclp-into-tranclp2 tranclpD*)

lemma *trancl-set-tranclp*: $(a, b) \in \{(b, a). P\ a\ b\}^+ \longleftrightarrow P^{++}\ b\ a$
apply (*rule iffI*)
apply (*induction rule: trancl-induct; simp*)
apply (*induction rule: tranclp-induct; auto simp: trancl-into-trancl2*)
done

lemma *tranclp-rtranclp-rtranclp-rel*: $R^{+++}\ a\ b \longleftrightarrow R^{**}\ a\ b$
by (*simp add: rtranclp-unfold*)

lemma *tranclp-rtranclp-rtranclp[*simp*]*: $R^{+++} = R^{**}$
by (*fastforce simp: rtranclp-unfold*)

lemma *rtranclp-exists-last-with-prop*:
assumes $R\ x\ z$
and $R^{**}\ z\ z'$ **and** $P\ x\ z$
shows $\exists y\ y'. R^{**}\ x\ y \wedge R\ y\ y' \wedge P\ y\ y' \wedge (\lambda a\ b. R\ a\ b \wedge \neg P\ a\ b)^{**}\ y'\ z'$
using *assms(2,1,3)*
proof (*induction arbitrary:*)
case *base*
then show *?case* **by** *auto*
next
case (*step* $z'\ z''$) **note** $z = \text{this}(2)$ **and** $IH = \text{this}(3)[OF\ \text{this}(4-5)]$
show *?case*
apply (*cases* $P\ z'\ z''$)
apply (*rule exI[of - z'], rule exI[of - z'']*)
using $z\ \text{assms}(1)\ \text{step.hyps}(1)\ \text{step.prem}(2)$ **apply** *auto[1]*
using $IH\ z\ rtranclp.rtrancl-into-rtrancl$ **by** *fastforce*
qed

lemma *rtranclp-and-rtranclp-left*: $(\lambda a\ b. P\ a\ b \wedge Q\ a\ b)^{**}\ S\ T \implies P^{**}\ S\ T$
by (*induction rule: rtranclp-induct*) *auto*

1.2 Full Transitions

We define here properties to define properties after all possible transitions.

abbreviation *no-step* *step* $S \equiv (\forall S'. \neg \text{step } S S')$

definition *full1* :: $(a \Rightarrow a \Rightarrow \text{bool}) \Rightarrow a \Rightarrow a \Rightarrow \text{bool}$ **where**
full1 transf = $(\lambda S S'. \text{trancpl transf } S S' \wedge (\forall S''. \neg \text{transf } S' S''))$

definition *full*:: $(a \Rightarrow a \Rightarrow \text{bool}) \Rightarrow a \Rightarrow a \Rightarrow \text{bool}$ **where**
full transf = $(\lambda S S'. \text{rtrancpl transf } S S' \wedge (\forall S''. \neg \text{transf } S' S''))$

lemma *rtrancpl-full1I*:

$R^{**} a b \implies \text{full1 } R b c \implies \text{full1 } R a c$
unfolding *full1-def* **by** *auto*

lemma *trancpl-full1I*:

$R^{++} a b \implies \text{full1 } R b c \implies \text{full1 } R a c$
unfolding *full1-def* **by** *auto*

lemma *rtrancpl-fullI*:

$R^{**} a b \implies \text{full } R b c \implies \text{full } R a c$
unfolding *full-def* **by** *auto*

lemma *trancpl-full-full1I*:

$R^{++} a b \implies \text{full } R b c \implies \text{full1 } R a c$
unfolding *full-def full1-def* **by** *auto*

lemma *full-fullI*:

$R a b \implies \text{full } R b c \implies \text{full1 } R a c$
unfolding *full-def full1-def* **by** *auto*

lemma *full-unfold*:

$\text{full } r S S' \iff ((S = S' \wedge \text{no-step } r S') \vee \text{full1 } r S S')$
unfolding *full-def full1-def* **by** (*auto simp add: rtrancpl-unfold*)

lemma *full1-is-full[intro]*: $\text{full1 } R S T \implies \text{full } R S T$

by (*simp add: full-unfold*)

lemma *not-full1-rtrancpl-relation*: $\neg \text{full1 } R^{**} a b$

by (*meson full1-def rtrancpl.rtrancpl-refl*)

lemma *not-full-rtrancpl-relation*: $\neg \text{full } R^{**} a b$

by (*meson full-fullI not-full1-rtrancpl-relation rtrancpl.rtrancpl-refl*)

lemma *full1-trancpl-relation-full*:

$\text{full1 } R^{++} a b \iff \text{full1 } R a b$
by (*metis converse-trancplE full1-def reflclp-trancpl rtrancplD rtrancpl-idemp rtrancpl-reflclp*
trancpl.r-into-trancpl trancpl-into-rtrancpl)

lemma *full-trancpl-relation-full*:

$\text{full } R^{++} a b \iff \text{full } R a b$
by (*metis full-unfold full1-trancpl-relation-full trancpl.r-into-trancpl trancplD*)

lemma *rtrancpl-full1-eq-or-full1*:

$(\text{full1 } R)^{**} a b \iff (a = b \vee \text{full1 } R a b)$

```

proof –
  have  $\forall p\ a\ aa.\ \neg\ p^{**}\ (a::'a)\ aa \vee a = aa \vee (\exists ab.\ p^{**}\ a\ ab \wedge p\ ab\ aa)$ 
    by (metis rtranclp.cases)
  then obtain  $aa :: ('a \Rightarrow 'a \Rightarrow \text{bool}) \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a$  where
     $f1: \forall p\ a\ ab.\ \neg\ p^{**}\ a\ ab \vee a = ab \vee p^{**}\ a\ (aa\ p\ a\ ab) \wedge p\ (aa\ p\ a\ ab)\ ab$ 
    by moura
  { assume  $a \neq b$ 
    { assume  $\neg\ full1\ R\ a\ b \wedge a \neq b$ 
      then have  $a \neq b \wedge a \neq b \wedge \neg\ full1\ R\ (aa\ (full1\ R)\ a\ b)\ b \vee \neg\ (full1\ R)^{**}\ a\ b \wedge a \neq b$ 
        using  $f1$  by (metis (no-types) full1-def full1-tranclp-relation-full)
      then have ?thesis
        using  $f1$  by blast }
    then have ?thesis
      by auto }
  then show ?thesis
    by fastforce
qed

```

```

lemma tranclp-full1-full1:
   $(full1\ R)^{++}\ a\ b \longleftrightarrow full1\ R\ a\ b$ 
  by (metis full1-def rtranclp-full1-eq-or-full1 tranclp-unfold-begin)

```

1.3 Well-Foundedness and Full Transitions

```

lemma wf-exists-normal-form:
  assumes  $wf:wf\ \{(x, y).\ R\ y\ x\}$ 
  shows  $\exists b.\ R^{**}\ a\ b \wedge no\text{-}step\ R\ b$ 
proof (rule ccontr)
  assume  $\neg\ ?thesis$ 
  then have  $H: \bigwedge b.\ \neg\ R^{**}\ a\ b \vee \neg\ no\text{-}step\ R\ b$ 
    by blast
  def  $F \equiv rec\text{-}nat\ a\ (\lambda i\ b.\ SOME\ c.\ R\ b\ c)$ 
  have [simp]:  $F\ 0 = a$ 
    unfolding  $F\text{-}def$  by auto
  have [simp]:  $\bigwedge i.\ F\ (Suc\ i) = (SOME\ b.\ R\ (F\ i)\ b)$ 
    using  $F\text{-}def$  by simp
  { fix  $i$ 
    have  $\forall j < i.\ R\ (F\ j)\ (F\ (Suc\ j))$ 
      proof (induction i)
        case  $0$ 
        then show ?case by auto
      next
        case  $(Suc\ i)$ 
        then have  $R^{**}\ a\ (F\ i)$ 
          by (induction i) auto
        then have  $R\ (F\ i)\ (SOME\ b.\ R\ (F\ i)\ b)$ 
          using  $H$  by (simp add: someI-ex)
        then have  $\forall j < Suc\ i.\ R\ (F\ j)\ (F\ (Suc\ j))$ 
          using  $H\ Suc$  by (simp add: less-Suc-eq)
        then show ?case by fast
      qed
    }
  then have  $\forall j.\ R\ (F\ j)\ (F\ (Suc\ j))$  by blast
  then show False
    using  $wf$  unfolding  $wfP\text{-}def\ wf\text{-}iff\text{-}no\text{-}infinite\text{-}down\text{-}chain$  by blast
qed

```


lemma *wf-exists-normal-form-full*:
assumes $wf:wf \ \{(x, y). \ R \ y \ x\}$
shows $\exists b. \ full \ R \ a \ b$
using *wf-exists-normal-form*[*OF assms*] **unfolding** *full-def* **by** *blast*

1.4 More Well-Foundedness

A little list of theorems that could be useful, but are hidden:

- link between *wf* and infinite chains: $wf \ ?r = (\nexists f. \ \forall i. \ (f \ (Suc \ i), f \ i) \in \ ?r), \llbracket wf \ ?r; \ \bigwedge k. \ (\ ?f \ (Suc \ k), \ ?f \ k) \notin \ ?r \implies \ ?thesis \rrbracket \implies \ ?thesis$

lemma *wf-if-measure-in-wf*:
 $wf \ R \implies (\bigwedge a \ b. \ (a, b) \in S \implies (\nu \ a, \nu \ b) \in R) \implies wf \ S$
by (*metis in-inv-image wfE-min wfI-min wf-inv-image*)

lemma *wfP-if-measure*: **fixes** $f :: 'a \Rightarrow nat$
shows $(\bigwedge x \ y. \ P \ x \implies g \ x \ y \implies f \ y < f \ x) \implies wf \ \{(y, x). \ P \ x \wedge g \ x \ y\}$
apply(*insert wf-measure[of f]*)
apply(*simp only: measure-def inv-image-def less-than-def less-eq*)
apply(*erule wf-subset*)
apply *auto*
done

lemma *wf-if-measure-f*:
assumes $wf \ r$
shows $wf \ \{(b, a). \ (f \ b, f \ a) \in r\}$
using *assms* **by** (*metis inv-image-def wf-inv-image*)

lemma *wf-wf-if-measure'*:
assumes $wf \ r$ **and** $H: (\bigwedge x \ y. \ P \ x \implies g \ x \ y \implies (f \ y, f \ x) \in r)$
shows $wf \ \{(y, x). \ P \ x \wedge g \ x \ y\}$
proof –
have $wf \ \{(b, a). \ (f \ b, f \ a) \in r\}$ **using** *assms(1) wf-if-measure-f* **by** *auto*
then have $wf \ \{(b, a). \ P \ a \wedge g \ a \ b \wedge (f \ b, f \ a) \in r\}$
using *wf-subset[of - \{(b, a). \ P \ a \wedge g \ a \ b \wedge (f \ b, f \ a) \in r\}]* **by** *auto*
moreover have $\{(b, a). \ P \ a \wedge g \ a \ b \wedge (f \ b, f \ a) \in r\} \subseteq \{(b, a). \ (f \ b, f \ a) \in r\}$ **by** *auto*
moreover have $\{(b, a). \ P \ a \wedge g \ a \ b \wedge (f \ b, f \ a) \in r\} = \{(b, a). \ P \ a \wedge g \ a \ b\}$ **using** *H* **by** *auto*
ultimately show *?thesis* **using** *wf-subset* **by** *simp*
qed

lemma *wf-lex-less*: $wf \ (\lex \ \{(a, b). \ (a::nat) < b\})$
proof –
have $m: \{(a, b). \ a < b\} = \text{measure } id$ **by** *auto*
show *?thesis* **apply** (*rule wf-lex*) **unfolding** *m* **by** *auto*
qed

lemma *wfP-if-measure2*: **fixes** $f :: 'a \Rightarrow nat$
shows $(\bigwedge x \ y. \ P \ x \ y \implies g \ x \ y \implies f \ x < f \ y) \implies wf \ \{(x, y). \ P \ x \ y \wedge g \ x \ y\}$
apply(*insert wf-measure[of f]*)
apply(*simp only: measure-def inv-image-def less-than-def less-eq*)
apply(*erule wf-subset*)
apply *auto*
done

lemma *lexord-on-finite-set-is-wf*:

assumes

P-finite: $\bigwedge U. P\ U \longrightarrow U \in A$ **and**

finite: *finite* *A* **and**

wf: *wf* *R* **and**

trans: *trans* *R*

shows *wf* $\{(T, S). (P\ S \wedge P\ T) \wedge (T, S) \in \text{lexord}\ R\}$

proof (*rule wfP-if-measure2*)

fix *T S*

assume *P*: $P\ S \wedge P\ T$ **and**

s-le-t: $(T, S) \in \text{lexord}\ R$

let *?f* = $\lambda S. \{U. (U, S) \in \text{lexord}\ R \wedge P\ U \wedge P\ S\}$

have *?f T* \subseteq *?f S*

using *s-le-t P lexord-trans trans* **by** *auto*

moreover **have** *T* \in *?f S*

using *s-le-t P* **by** *auto*

moreover **have** *T* \notin *?f T*

using *s-le-t* **by** (*auto simp add: lexord-irreflexive local.wf*)

ultimately **have** $\{U. (U, T) \in \text{lexord}\ R \wedge P\ U \wedge P\ T\} \subset \{U. (U, S) \in \text{lexord}\ R \wedge P\ U \wedge P\ S\}$
by *auto*

moreover **have** *finite* $\{U. (U, S) \in \text{lexord}\ R \wedge P\ U \wedge P\ S\}$

using *finite* **by** (*metis (no-types, lifting) P-finite finite-subset mem-Collect-eq subsetI*)

ultimately **show** $\text{card}\ (?\text{f}\ T) < \text{card}\ (?\text{f}\ S)$ **by** (*simp add: psubset-card-mono*)

qed

lemma *wf-fst-wf-pair*:

assumes *wf* $\{(M', M). R\ M'\ M\}$

shows *wf* $\{((M', N'), (M, N)). R\ M'\ M\}$

proof –

have *wf* $\{(M', M). R\ M'\ M\} <*\text{lex}*> \{\}$

using *assms* **by** *auto*

then **show** *?thesis*

by (*rule wf-subset*) *auto*

qed

lemma *wf-snd-wf-pair*:

assumes *wf* $\{(M', M). R\ M'\ M\}$

shows *wf* $\{((M', N'), (M, N)). R\ N'\ N\}$

proof –

have *wf*: *wf* $\{((M', N'), (M, N)). R\ M'\ M\}$

using *assms wf-fst-wf-pair* **by** *auto*

then **have** *wf*: $\bigwedge P. (\forall x. (\forall y. (y, x) \in \{((M', N'), M, N). R\ M'\ M\} \longrightarrow P\ y) \longrightarrow P\ x) \Longrightarrow \text{All}\ P$

unfolding *wf-def* **by** *auto*

show *?thesis*

unfolding *wf-def*

proof (*intro allI impI*)

fix *P* :: '*c* \times '*a* \Rightarrow bool **and** *x* :: '*c* \times '*a*

assume *H*: $\forall x. (\forall y. (y, x) \in \{((M', N'), M, y). R\ N'\ y\} \longrightarrow P\ y) \longrightarrow P\ x$

obtain *a b* **where** *x*: *x* = (*a*, *b*) **by** (*cases x*)

have *P*: $P\ x = (P\ o\ (\lambda(a, b). (b, a)))\ (b, a)$

unfolding *x* **by** *auto*

show *P x*

using *wf[of P o (λ(a, b). (b, a))]* **apply** *rule*

```

      using H apply simp
    unfolding P by blast
  qed
qed

```

```

lemma wf-if-measure-f-notation2:
  assumes wf r
  shows wf {(b, h a) | b a. (f b, f (h a)) ∈ r}
  apply (rule wf-subset)
  using wf-if-measure-f[OF assms, of f] by auto

```

```

lemma wf-wf-if-measure'-notation2:
  assumes wf r and H: (⋀ x y. P x ⟹ g x y ⟹ (f y, f (h x)) ∈ r)
  shows wf {(y, h x) | y x. P x ∧ g x y}
proof -
  have wf {(b, h a) | b a. (f b, f (h a)) ∈ r} using assms(1) wf-if-measure-f-notation2 by auto
  then have wf {(b, h a) | b a. P a ∧ g a b ∧ (f b, f (h a)) ∈ r}
    using wf-subset[of - {(b, h a) | b a. P a ∧ g a b ∧ (f b, f (h a)) ∈ r}] by auto
  moreover have {(b, h a) | b a. P a ∧ g a b ∧ (f b, f (h a)) ∈ r}
    ⊆ {(b, h a) | b a. (f b, f (h a)) ∈ r} by auto
  moreover have {(b, h a) | b a. P a ∧ g a b ∧ (f b, f (h a)) ∈ r} = {(b, h a) | b a. P a ∧ g a b}
    using H by auto
  ultimately show ?thesis using wf-subset by simp
qed

```

```

end
theory List-More
imports Main ../lib/Multiset-More
begin

```

```

Sledgehammer parameters
sledgehammer-params[debug]

```

2 Various Lemmas

Close to $(\bigwedge n. \forall m < n. ?P\ m \implies ?P\ n) \implies ?P\ ?n$, but with a separation between zero and non-zero, and case names.

```

thm nat-less-induct
lemma nat-less-induct-case[case-names 0 Suc]:
  assumes
    P 0 and
    ⋀ n. (∀ m < Suc n. P m) ⟹ P (Suc n)
  shows P n
  apply (induction rule: nat-less-induct)
  by (rename-tac n, case-tac n) (auto intro: assms)

```

This is only proved in simple cases by auto. In assumptions, nothing happens, and $?P$ (if $?Q$ then $?x$ else $?y$) = $(\neg (?Q \wedge \neg ?P\ ?x \vee \neg ?Q \wedge \neg ?P\ ?y))$ can blow up goals (because of other if expression).

```

lemma if-0-1-ge-0[simp]:
  0 < (if P then a else (0::nat)) ⟷ P ∧ 0 < a
  by auto

```

Bounded function have not been defined in Isabelle.

definition *bounded* **where**
 $\text{bounded } f \longleftrightarrow (\exists b. \forall n. f\ n \leq b)$

abbreviation *unbounded* :: ('a \Rightarrow 'b::ord) \Rightarrow bool **where**
 $\text{unbounded } f \equiv \neg \text{bounded } f$

lemma *not-bounded-nat-exists-larger*:

fixes $f :: \text{nat} \Rightarrow \text{nat}$
assumes *unbound*: *unbounded* f
shows $\exists n. f\ n > m \wedge n > n_0$

proof (*rule ccontr*)

assume $H: \neg ?thesis$

have *finite* $\{f\ n \mid n. n \leq n_0\}$

by *auto*

have $\bigwedge n. f\ n \leq \text{Max } (\{f\ n \mid n. n \leq n_0\} \cup \{m\})$

apply (*case-tac* $n \leq n_0$)

apply (*metis* (*mono-tags*, *lifting*) *Max-ge Un-insert-right* $\langle \text{finite } \{f\ n \mid n. n \leq n_0\} \rangle$
finite-insert insertCI mem-Collect-eq sup-bot.right-neutral)

by (*metis* (*no-types*, *lifting*) H *Max-less-iff Un-insert-right* $\langle \text{finite } \{f\ n \mid n. n \leq n_0\} \rangle$
finite-insert insertI1 insert-not-empty leI sup-bot.right-neutral)

then show *False*

using *unbound* **unfolding** *bounded-def* **by** *auto*

qed

lemma *bounded-const-product*:

fixes $k :: \text{nat}$ **and** $f :: \text{nat} \Rightarrow \text{nat}$

assumes $k > 0$

shows $\text{bounded } f \longleftrightarrow \text{bounded } (\lambda i. k * f\ i)$

unfolding *bounded-def* **apply** (*rule iffI*)

using *mult-le-mono2* **apply** *blast*

by (*meson* *assms* *le-less-trans* *less-or-eq-imp-le* *nat-mult-less-cancel-disj* *split-div-lemma*)

This lemma is not used, but here to show that a property that can be expected from *bounded* holds.

lemma *bounded-finite-linorder*:

fixes $f :: 'a \Rightarrow 'a :: \{\text{finite}, \text{linorder}\}$

shows *bounded* f

proof –

have $\bigwedge x. f\ x \leq \text{Max } \{f\ x \mid x. \text{True}\}$

by (*metis* (*mono-tags*) *Max-ge finite mem-Collect-eq*)

then show *?thesis*

unfolding *bounded-def* **by** *blast*

qed

3 More List

3.1 *upt*

The simplification rules are not very handy, because $[?i..<\text{Suc } ?j] = (\text{if } ?i \leq ?j \text{ then } [?i..<?j] @ [?j] \text{ else } [])$ leads to a case distinction, that we do not want if the condition is not in the context.

lemma *upt-Suc-le-append*: $\neg i \leq j \Longrightarrow [i..<\text{Suc } j] = []$

by *auto*

lemmas *upt-simps*[*simp*] = *upt-Suc-append* *upt-Suc-le-append*

declare *upt.simps*(2)[*simp del*]

lemma

assumes $i \leq n - m$

shows $\text{take } i \ [m..<n] = [m..<m+i]$

by (*metis* *Nat.le-diff-conv2* *add.commute* *assms* *diff-is-0-eq'* *linear* *take-upt* *upt-conv-Nil*)

The counterpart for this lemma when $n - m < i$ is $\text{length } ?xs \leq ?n \implies \text{take } ?n \ ?xs = ?xs$. It is close to $?i + ?m \leq ?n \implies \text{take } ?m \ [?i..<?n] = [?i..<?i + ?m]$, but seems more general.

lemma *take-upt-bound-minus*[*simp*]:

assumes $i \leq n - m$

shows $\text{take } i \ [m..<n] = [m..<m+i]$

using *assms* **by** (*induction* *i*) *auto*

lemma *append-cons-eq-upt*:

assumes $A @ B = [m..<n]$

shows $A = [m..<m+\text{length } A]$ **and** $B = [m + \text{length } A..<n]$

proof –

have $\text{take } (\text{length } A) \ (A @ B) = A$ **by** *auto*

moreover

have $\text{length } A \leq n - m$ **using** *assms* *linear* *calculation* **by** *fastforce*

then have $\text{take } (\text{length } A) \ [m..<n] = [m..<m+\text{length } A]$ **by** *auto*

ultimately show $A = [m..<m+\text{length } A]$ **using** *assms* **by** *auto*

show $B = [m + \text{length } A..<n]$ **using** *assms* **by** (*metis* *append-eq-conv-conj* *drop-upt*)

qed

lemma *length-list-Suc-0*:

$\text{length } W = \text{Suc } 0 \longleftrightarrow (\exists L. W = [L])$

apply (*cases* *W*)

apply *simp*

apply (*rename-tac* *a* *W'*, *case-tac* *W'*)

apply *auto*

done

lemma *length-list-2*: $\text{length } S = 2 \longleftrightarrow (\exists a \ b. S = [a, b])$

apply (*cases* *S*)

apply *simp*

apply (*rename-tac* *a* *S'*)

apply (*case-tac* *S'*)

by *simp-all*

The converse of $?A @ ?B = [?m..<?n] \implies ?A = [?m..<?m + \text{length } ?A]$

$?A @ ?B = [?m..<?n] \implies ?B = [?m + \text{length } ?A..<?n]$ does not hold, for example if *B* is empty and *A* is $[0::'a]$:

lemma $A @ B = [m..<n] \longleftrightarrow A = [m..<m+\text{length } A] \wedge B = [m + \text{length } A..<n]$

oops

A more restrictive version holds:

lemma $B \neq [] \implies A @ B = [m..<n] \longleftrightarrow A = [m..<m+\text{length } A] \wedge B = [m + \text{length } A..<n]$

(**is** $?P \implies ?A = ?B$)

```

proof
  assume ?A then show ?B by (auto simp add: append-cons-eq-upt)
next
  assume ?P and ?B
  then show ?A using append-eq-conv-conj by fastforce
qed

lemma append-cons-eq-upt-length-i:
  assumes  $A @ i \# B = [m..<n]$ 
  shows  $A = [m ..<i]$ 
proof -
  have  $A = [m ..< m + \text{length } A]$  using assms append-cons-eq-upt by auto
  have  $(A @ i \# B) ! (\text{length } A) = i$  by auto
  moreover have  $n - m = \text{length } (A @ i \# B)$ 
    using assms length-upt by presburger
  then have  $[m..<n] ! (\text{length } A) = m + \text{length } A$  by simp
  ultimately have  $i = m + \text{length } A$  using assms by auto
  then show ?thesis using  $\langle A = [m ..< m + \text{length } A] \rangle$  by auto
qed

lemma append-cons-eq-upt-length:
  assumes  $A @ i \# B = [m..<n]$ 
  shows  $\text{length } A = i - m$ 
  using assms
proof (induction A arbitrary: m)
  case Nil
  then show ?case by (metis append-Nil diff-is-0-eq list.size(3) order-refl upt-eq-Cons-conv)
next
  case (Cons a A)
  then have  $A: A @ i \# B = [m + 1..<n]$  by (metis append-Cons upt-eq-Cons-conv)
  then have  $m < i$  by (metis Cons.premis append-cons-eq-upt-length-i upt-eq-Cons-conv)
  with Cons.IH[OF A] show ?case by auto
qed

lemma append-cons-eq-upt-length-i-end:
  assumes  $A @ i \# B = [m..<n]$ 
  shows  $B = [\text{Suc } i ..<n]$ 
proof -
  have  $B = [\text{Suc } m + \text{length } A..<n]$  using assms append-cons-eq-upt[of  $A @ [i] B m n$ ] by auto
  have  $(A @ i \# B) ! (\text{length } A) = i$  by auto
  moreover have  $n - m = \text{length } (A @ i \# B)$ 
    using assms length-upt by auto
  then have  $[m..<n] ! (\text{length } A) = m + \text{length } A$  by simp
  ultimately have  $i = m + \text{length } A$  using assms by auto
  then show ?thesis using  $\langle B = [\text{Suc } m + \text{length } A..<n] \rangle$  by auto
qed

lemma Max-n-upt:  $\text{Max } (\text{insert } 0 \{ \text{Suc } 0..<n \}) = n - \text{Suc } 0$ 
proof (induct n)
  case 0
  then show ?case by simp
next
  case (Suc n) note IH = this
  have  $i: \text{insert } 0 \{ \text{Suc } 0..<\text{Suc } n \} = \text{insert } 0 \{ \text{Suc } 0..<n \} \cup \{n\}$  by auto
  show ?case using IH unfolding i by auto

```

qed

lemma *upt-decomp-lt*:

assumes H : $xs @ i \# ys @ j \# zs = [m ..< n]$

shows $i < j$

proof –

have xs : $xs = [m ..< i]$ **and** ys : $ys = [Suc\ i ..< j]$ **and** zs : $zs = [Suc\ j ..< n]$

using H **by** (*auto dest: append-cons-eq-upt-length-i append-cons-eq-upt-length-i-end*)

show *?thesis*

by (*metis append-cons-eq-upt-length-i-end assms lessI less-trans self-append-conv2
upt-eq-Cons-conv upt-rec ys*)

qed

3.2 Lexicographic Ordering

lemma *lexn-Suc*:

$(x \# xs, y \# ys) \in \text{lexn } r \ (Suc\ n) \longleftrightarrow$

$(\text{length } xs = n \wedge \text{length } ys = n) \wedge ((x, y) \in r \vee (x = y \wedge (xs, ys) \in \text{lexn } r\ n))$

by (*auto simp: map-prod-def image-iff lex-prod-def*)

lemma *lexn-n*:

$n > 0 \implies (x \# xs, y \# ys) \in \text{lexn } r\ n \longleftrightarrow$

$(\text{length } xs = n-1 \wedge \text{length } ys = n-1) \wedge ((x, y) \in r \vee (x = y \wedge (xs, ys) \in \text{lexn } r\ (n-1)))$

apply (*cases n*)

apply *simp*

by (*auto simp: map-prod-def image-iff lex-prod-def*)

There is some subtle point in the proof here. 1 is converted to $Suc\ 0$, but 2 is not: meaning that 1 is automatically simplified by default using the default simplification rule $\text{lexn } ?r\ 0 = \{\}$

$\text{lexn } ?r\ (Suc\ ?n) = \text{map-prod } (\lambda(x, xs). x \# xs) (\lambda(x, xs). x \# xs) \text{ ' } (?r < * \text{lex} > \text{lexn } ?r\ ?n) \cap \{(xs, ys). \text{length } xs = Suc\ ?n \wedge \text{length } ys = Suc\ ?n\}$. However, the latter needs additional simplification rule.

lemma *lexn2-conv*:

$([a, b], [c, d]) \in \text{lexn } r\ 2 \longleftrightarrow (a, c) \in r \vee (a = c \wedge (b, d) \in r)$

by (*auto simp: lexn-n simp del: lexn.simps(2)*)

lemma *lexn3-conv*:

$([a, b, c], [a', b', c']) \in \text{lexn } r\ 3 \longleftrightarrow$

$(a, a') \in r \vee (a = a' \wedge (b, b') \in r) \vee (a = a' \wedge b = b' \wedge (c, c') \in r)$

by (*auto simp: lexn-n simp del: lexn.simps(2)*)

3.3 Remove

3.3.1 More lemmas about remove

lemma *remove1-nil*:

remove1 $(- L)$ $W = [] \longleftrightarrow (W = [] \vee W = [-L])$

by (*cases W*) *auto*

lemma *remove1-mset-single-add*:

$a \neq b \implies \text{remove1-mset } a\ (\{\#b\# \} + C) = \{\#b\# \} + \text{remove1-mset } a\ C$

$\text{remove1-mset } a\ (\{\#a\# \} + C) = C$

by (*auto simp: multiset-eq-iff*)

3.3.2 Remove under condition

This function removes the first element when the condition f holds. It generalises *remove1*.

```
fun remove1-cond where
  remove1-cond  $f$  [] = [] |
  remove1-cond  $f$  ( $C' \# L$ ) = (if  $f$   $C'$  then  $L$  else  $C' \#$  remove1-cond  $f$   $L$ )
```

```
lemma remove1  $x$   $xs$  = remove1-cond (( $op =$ )  $x$ )  $xs$ 
by (induction  $xs$ ) auto
```

```
lemma mset-map-mset-remove1-cond:
  mset (map mset (remove1-cond ( $\lambda L$ . mset  $L =$  mset  $a$ )  $C$ )) =
    remove1-mset (mset  $a$ ) (mset (map mset  $C$ ))
by (induction  $C$ ) (auto simp: ac-simps remove1-mset-single-add)
```

We can also generalise *removeAll*, which is close to *filter*:

```
fun removeAll-cond where
  removeAll-cond  $f$  [] = [] |
  removeAll-cond  $f$  ( $C' \# L$ ) =
    (if  $f$   $C'$  then removeAll-cond  $f$   $L$  else  $C' \#$  removeAll-cond  $f$   $L$ )
```

```
lemma removeAll  $x$   $xs$  = removeAll-cond (( $op =$ )  $x$ )  $xs$ 
by (induction  $xs$ ) auto
```

```
lemma removeAll-cond  $P$   $xs$  = filter ( $\lambda x$ .  $\neg P$   $x$ )  $xs$ 
by (induction  $xs$ ) auto
```

```
lemma mset-map-mset-removeAll-cond:
  mset (map mset (removeAll-cond ( $\lambda b$ . mset  $b =$  mset  $a$ )  $C$ ))
    = removeAll-mset (mset  $a$ ) (mset (map mset  $C$ ))
by (induction  $C$ ) (auto simp: ac-simps mset-less-eqI multiset-diff-union-assoc)
```

Take from `../lib/Multiset_More.thy`, but named:

```
abbreviation union-mset-list where
  union-mset-list  $xs$   $ys$   $\equiv$  case-prod append (fold ( $\lambda x$  ( $ys$ ,  $zs$ ). (remove1  $x$   $ys$ ,  $x \#$   $zs$ ))  $xs$  ( $ys$ , []))
```

```
lemma union-mset-list:
  mset  $xs \# \cup$  mset  $ys$  = mset (union-mset-list  $xs$   $ys$ )
proof –
  have  $\bigwedge zs$ . mset (case-prod append (fold ( $\lambda x$  ( $ys$ ,  $zs$ ). (remove1  $x$   $ys$ ,  $x \#$   $zs$ ))  $xs$  ( $ys$ ,  $zs$ ))) =
    (mset  $xs \# \cup$  mset  $ys$ ) + mset  $zs$ 
    by (induct  $xs$  arbitrary:  $ys$ ) (simp-all add: multiset-eq-iff)
  then show ?thesis by simp
qed
```

```
end
theory Prop-Logic
```

```
imports Main
```

```
begin
```


4 Logics

In this section we define the syntax of the formula and an abstraction over it to have simpler proofs. After that we define some properties like subformula and rewriting.

4.1 Definition and abstraction

The propositional logic is defined inductively. The type parameter is the type of the variables.

datatype *'v propo* =
 $FT \mid FF \mid FVar \ 'v \mid FNot \ 'v \ propo \mid FAnd \ 'v \ propo \ 'v \ propo \mid FOr \ 'v \ propo \ 'v \ propo$
 $\mid FImp \ 'v \ propo \ 'v \ propo \mid FEq \ 'v \ propo \ 'v \ propo$

We do not define any notation for the formula, to distinguish properly between the formulas and Isabelle's logic.

To ease the proofs, we will write the the formula on a homogeneous manner, namely a connecting argument and a list of arguments.

datatype *'v connective* = $CT \mid CF \mid CVar \ 'v \mid CNot \mid CAnd \mid COr \mid CImp \mid CEq$

abbreviation *nullary-connective* $\equiv \{CF\} \cup \{CT\} \cup \{CVar \ x \mid x. \ True\}$

definition *binary-connectives* $\equiv \{CAnd, COr, CImp, CEq\}$

We define our own induction principal: instead of distinguishing every constructor, we group them by arity.

lemma *propo-induct-arity*[*case-names nullary unary binary*]:

fixes $\varphi \ \psi :: 'v \ propo$
assumes *nullary*: $(\bigwedge \varphi \ x. \ \varphi = FF \vee \varphi = FT \vee \varphi = FVar \ x \implies P \ \varphi)$
and *unary*: $(\bigwedge \psi. \ P \ \psi \implies P \ (FNot \ \psi))$
and *binary*: $(\bigwedge \varphi \ \psi1 \ \psi2. \ P \ \psi1 \implies P \ \psi2 \implies \varphi = FAnd \ \psi1 \ \psi2 \vee \varphi = FOr \ \psi1 \ \psi2 \vee \varphi = FImp \ \psi1 \ \psi2$
 $\vee \varphi = FEq \ \psi1 \ \psi2 \implies P \ \varphi)$
shows $P \ \psi$
apply (*induct rule: propo.induct*)
using *assms* **by** *metis+*

The function *conn* is the interpretation of our representation (connective and list of arguments). We define any thing that has no sense to be false

fun *conn* :: *'v connective* \Rightarrow *'v propo list* \Rightarrow *'v propo* **where**

conn $CT \ [] = FT \mid$
conn $CF \ [] = FF \mid$
conn $(CVar \ v) \ [] = FVar \ v \mid$
conn $CNot \ [\varphi] = FNot \ \varphi \mid$
conn $CAnd \ (\varphi \# [\psi]) = FAnd \ \varphi \ \psi \mid$
conn $COr \ (\varphi \# [\psi]) = FOr \ \varphi \ \psi \mid$
conn $CImp \ (\varphi \# [\psi]) = FImp \ \varphi \ \psi \mid$
conn $CEq \ (\varphi \# [\psi]) = FEq \ \varphi \ \psi \mid$
conn $- = FF$

We will often use case distinction, based on the arity of the *'v connective*, thus we define our own splitting principle.

lemma *connective-cases-arity*[*case-names nullary binary unary*]:

assumes *nullary*: $\bigwedge x. \ c = CT \vee c = CF \vee c = CVar \ x \implies P$
and *binary*: $c \in \text{binary-connectives} \implies P$

and unary: $c = CNot \implies P$
shows P
using *assms* **by** (*cases* c) (*auto simp: binary-connectives-def*)

lemma *connective-cases-arity-2*[*case-names nullary unary binary*]:
assumes *nullary*: $c \in \text{nullary-connective} \implies P$
and *unary*: $c = CNot \implies P$
and *binary*: $c \in \text{binary-connectives} \implies P$
shows P
using *assms* **by** (*cases* c , *auto simp add: binary-connectives-def*)

Our previous definition is not necessary correct (connective and list of arguments) , so we define an inductive predicate.

inductive *wf-conn* :: '*v* connective \Rightarrow '*v* propo list \Rightarrow bool **for** $c ::$ '*v* connective **where**
wf-conn-nullary[*simp*]: $(c = CT \vee c = CF \vee c = CVar\ v) \implies \text{wf-conn } c\ [] \mid$
wf-conn-unary[*simp*]: $c = CNot \implies \text{wf-conn } c\ [\psi] \mid$
wf-conn-binary[*simp*]: $c \in \text{binary-connectives} \implies \text{wf-conn } c\ (\psi \# \psi' \# [])$
thm *wf-conn.induct*

lemma *wf-conn-induct*[*consumes 1, case-names CT CF CVar CNot COr CAnd CImp CEq*]:
assumes *wf-conn* $c\ x$ **and**
 $(\bigwedge v. c = CT \implies P\ [])$ **and**
 $(\bigwedge v. c = CF \implies P\ [])$ **and**
 $(\bigwedge v. c = CVar\ v \implies P\ [])$ **and**
 $(\bigwedge \psi. c = CNot \implies P\ [\psi])$ **and**
 $(\bigwedge \psi\ \psi'. c = COr \implies P\ [\psi, \psi'])$ **and**
 $(\bigwedge \psi\ \psi'. c = CAnd \implies P\ [\psi, \psi'])$ **and**
 $(\bigwedge \psi\ \psi'. c = CImp \implies P\ [\psi, \psi'])$ **and**
 $(\bigwedge \psi\ \psi'. c = CEq \implies P\ [\psi, \psi'])$
shows $P\ x$
using *assms* **by** *induction* (*auto simp: binary-connectives-def*)

4.2 properties of the abstraction

First we can define simplification rules.

lemma *wf-conn-conn*[*simp*]:
 $\text{wf-conn } CT\ l \implies \text{conn } CT\ l = FT$
 $\text{wf-conn } CF\ l \implies \text{conn } CF\ l = FF$
 $\text{wf-conn } (CVar\ x)\ l \implies \text{conn } (CVar\ x)\ l = FVar\ x$
apply (*simp-all add: wf-conn.simps*)
unfolding *binary-connectives-def* **by** *simp-all*

lemma *wf-conn-list-decomp*[*simp*]:
 $\text{wf-conn } CT\ l \longleftrightarrow l = []$
 $\text{wf-conn } CF\ l \longleftrightarrow l = []$
 $\text{wf-conn } (CVar\ x)\ l \longleftrightarrow l = []$
 $\text{wf-conn } CNot\ (\xi @ \varphi \# \xi') \longleftrightarrow \xi = [] \wedge \xi' = []$
apply (*simp-all add: wf-conn.simps*)
unfolding *binary-connectives-def* **apply** *simp-all*
by (*metis append-Nil append-is-Nil-conv list.distinct(1) list.sel(3) tl-append2*)

lemma *wf-conn-list*:
 $\text{wf-conn } c\ l \implies \text{conn } c\ l = FT \longleftrightarrow (c = CT \wedge l = [])$

```

wf-conn c l  $\implies$  conn c l = FF  $\longleftrightarrow$  (c = CF  $\wedge$  l = [])
wf-conn c l  $\implies$  conn c l = FVar x  $\longleftrightarrow$  (c = CVar x  $\wedge$  l = [])
wf-conn c l  $\implies$  conn c l = FAnd a b  $\longleftrightarrow$  (c = CAnd  $\wedge$  l = a # b # [])
wf-conn c l  $\implies$  conn c l = FOr a b  $\longleftrightarrow$  (c = COr  $\wedge$  l = a # b # [])
wf-conn c l  $\implies$  conn c l = FEq a b  $\longleftrightarrow$  (c = CEq  $\wedge$  l = a # b # [])
wf-conn c l  $\implies$  conn c l = FImp a b  $\longleftrightarrow$  (c = CImp  $\wedge$  l = a # b # [])
wf-conn c l  $\implies$  conn c l = FNot a  $\longleftrightarrow$  (c = CNot  $\wedge$  l = a # [])
apply (induct l rule: wf-conn.induct)
unfolding binary-connectives-def by auto

```

In the binary connective cases, we will often decompose the list of arguments (of length 2) into two elements.

```

lemma list-length2-decomp: length l = 2  $\implies$  ( $\exists$  a b. l = a # b # [])
apply (induct l, auto)
by (rename-tac l, case-tac l, auto)

```

wf-conn for binary operators means that there are two arguments.

```

lemma wf-conn-bin-list-length:
  fixes l :: 'v propo list
  assumes conn: c  $\in$  binary-connectives
  shows length l = 2  $\longleftrightarrow$  wf-conn c l
proof
  assume length l = 2
  then show wf-conn c l using wf-conn-binary list-length2-decomp using conn by metis
next
  assume wf-conn c l
  then show length l = 2 (is ?P l)
  proof (cases rule: wf-conn.induct)
    case wf-conn-nullary
    then show ?P [] using conn binary-connectives-def
    using connective.distinct(11) connective.distinct(13) connective.distinct(9) by blast
  next
    fix  $\psi$  :: 'v propo
    case wf-conn-unary
    then show ?P [ $\psi$ ] using conn binary-connectives-def
    using connective.distinct by blast
  next
    fix  $\psi$   $\psi'$  :: 'v propo
    show ?P [ $\psi$ ,  $\psi'$ ] by auto
  qed
qed

```

```

lemma wf-conn-not-list-length[iff]:
  fixes l :: 'v propo list
  shows wf-conn CNot l  $\longleftrightarrow$  length l = 1
  apply auto
  apply (metis append-Nil connective.distinct(5,17,27) length-Cons list.size(3) wf-conn.simps
    wf-conn-list-decomp(4))
  by (simp add: length-Suc-conv wf-conn.simps)

```

Decomposing the Not into an element is moreover very useful.

```

lemma wf-conn-Not-decomp:
  fixes l :: 'v propo list and a :: 'v
  assumes corr: wf-conn CNot l
  shows  $\exists$  a. l = [a]

```

by (*metis* (*no-types*, *lifting*) *One-nat-def Suc-length-conv corr length-0-conv*
wf-conn-not-list-length)

The *wf-conn* remains correct if the length of list does not change. This lemma is very useful when we do one rewriting step

lemma *wf-conn-no-arity-change*:

$\text{length } l = \text{length } l' \implies \text{wf-conn } c \ l \longleftrightarrow \text{wf-conn } c \ l'$

proof –

```
{
  fix l l'
  have length l = length l'  $\implies$  wf-conn c l  $\implies$  wf-conn c l'
    apply (cases c l rule: wf-conn.induct, auto)
    by (metis wf-conn-bin-list-length)
}
```

then show $\text{length } l = \text{length } l' \implies \text{wf-conn } c \ l = \text{wf-conn } c \ l'$ **by** *metis*

qed

lemma *wf-conn-no-arity-change-helper*:

$\text{length } (\xi @ \varphi \# \xi') = \text{length } (\xi @ \varphi' \# \xi')$

by *auto*

The injectivity of *conn* is useful to prove equality of the connectives and the lists.

lemma *conn-inj-not*:

```
assumes correct: wf-conn c l
and conn: conn c l = FNot  $\psi$ 
shows c = CNot and l = [ $\psi$ ]
apply (cases c l rule: wf-conn.cases)
using correct conn unfolding binary-connectives-def apply auto
apply (cases c l rule: wf-conn.cases)
using correct conn unfolding binary-connectives-def by auto
```

lemma *conn-inj*:

fixes *c ca* :: '*v* connective **and** *l ψ s* :: '*v* propo list

assumes *corr*: *wf-conn ca l*

and *corr'*: *wf-conn c ψ s*

and *eq*: *conn ca l = conn c ψ s*

shows *ca = c $\wedge \psi$ s = l*

using *corr*

proof (*cases ca l rule: wf-conn.cases*)

case (*wf-conn-nullary v*)

then show *ca = c $\wedge \psi$ s = l* **using** *assms*

by (*metis conn.simps(1) conn.simps(2) conn.simps(3) wf-conn-list(1-3)*)

next

case (*wf-conn-unary ψ '*)

then have *: *FNot ψ' = conn c ψ s* **using** *conn-inj-not eq assms* **by** *auto*

then have *c = ca* **by** (*metis conn-inj-not(1) corr' wf-conn-unary(2)*)

moreover have *ψ s = l* **using** * *conn-inj-not(2) corr' wf-conn-unary(1)* **by** *force*

ultimately show *ca = c $\wedge \psi$ s = l* **by** *auto*

next

case (*wf-conn-binary $\psi' \psi''$*)

then show *ca = c $\wedge \psi$ s = l*

using *eq corr' unfolding binary-connectives-def apply (cases ca, auto simp add: wf-conn-list)*

using *wf-conn-list(4-7) corr' by metis+*

qed

4.3 Subformulas and properties

A characterization using sub-formulas is interesting for rewriting: we will define our relation on the sub-term level, and then lift the rewriting on the term-level. So the rewriting takes place on a subformula.

inductive *subformula* :: 'v propo \Rightarrow 'v propo \Rightarrow bool (infix \preceq 45) for φ **where**
subformula-refl[simp]: $\varphi \preceq \varphi$ |
subformula-into-subformula: $\psi \in \text{set } l \Rightarrow \text{wf-conn } c \ l \Rightarrow \varphi \preceq \psi \Rightarrow \varphi \preceq \text{conn } c \ l$

On the *subformula-into-subformula*, we can see why we use our *conn* representation: one case is enough to express the subformulas property instead of listing all the cases.

This is an example of a property related to subformulas.

lemma *subformula-in-subformula-not*:
shows $b: F\text{Not } \varphi \preceq \psi \Rightarrow \varphi \preceq \psi$
apply (induct rule: *subformula.induct*)
using *subformula-into-subformula wf-conn-unary subformula-refl list.set-intros(1) subformula-refl*
by (fastforce intro: *subformula-into-subformula*)+

lemma *subformula-in-binary-conn*:
assumes $\text{conn}: c \in \text{binary-connectives}$
shows $f \preceq \text{conn } c \ [f, g]$
and $g \preceq \text{conn } c \ [f, g]$
proof –
have $a: \text{wf-conn } c \ (f \# [g])$ **using** *conn wf-conn-binary binary-connectives-def* **by** auto
moreover **have** $b: f \preceq f$ **using** *subformula-refl* **by** auto
ultimately show $f \preceq \text{conn } c \ [f, g]$
by (*metis append-Nil in-set-conv-decomp subformula-into-subformula*)
next
have $a: \text{wf-conn } c \ ([f] @ [g])$ **using** *conn wf-conn-binary binary-connectives-def* **by** auto
moreover **have** $b: g \preceq g$ **using** *subformula-refl* **by** auto
ultimately show $g \preceq \text{conn } c \ [f, g]$ **using** *subformula-into-subformula* **by** force
qed

lemma *subformula-trans*:
 $\psi \preceq \psi' \Rightarrow \varphi \preceq \psi \Rightarrow \varphi \preceq \psi'$
apply (induct ψ' rule: *subformula.inducts*)
by (*auto simp: subformula-into-subformula*)

lemma *subformula-leaf*:
fixes $\varphi \ \psi :: 'v \text{ propo}$
assumes *incl*: $\varphi \preceq \psi$
and *simple*: $\psi = FT \vee \psi = FF \vee \psi = FVar \ x$
shows $\varphi = \psi$
using *incl simple*
by (induct rule: *subformula.induct*, *auto simp: wf-conn-list*)

lemma *subformula-not-incl-eq*:
assumes $\varphi \preceq \text{conn } c \ l$
and *wf-conn* $c \ l$
and $\forall \psi. \psi \in \text{set } l \longrightarrow \neg \varphi \preceq \psi$
shows $\varphi = \text{conn } c \ l$
using *assms* **apply** (induction *conn c l* rule: *subformula.induct*, *auto*)
using *conn-inj* **by** blast

lemma *wf-subformula-conn-cases*:

wf-conn $c\ l \implies \varphi \preceq \text{conn } c\ l \longleftrightarrow (\varphi = \text{conn } c\ l \vee (\exists \psi. \psi \in \text{set } l \wedge \varphi \preceq \psi))$
apply *standard*
using *subformula-not-incl-eq* **apply** *metis*
by (*auto simp add: subformula-into-subformula*)

lemma *subformula-decomp-explicit[simp]*:

$\varphi \preceq \text{FAnd } \psi\ \psi' \longleftrightarrow (\varphi = \text{FAnd } \psi\ \psi' \vee \varphi \preceq \psi \vee \varphi \preceq \psi') \text{ (is ?P FAnd)}$
 $\varphi \preceq \text{FOr } \psi\ \psi' \longleftrightarrow (\varphi = \text{FOr } \psi\ \psi' \vee \varphi \preceq \psi \vee \varphi \preceq \psi')$
 $\varphi \preceq \text{FEq } \psi\ \psi' \longleftrightarrow (\varphi = \text{FEq } \psi\ \psi' \vee \varphi \preceq \psi \vee \varphi \preceq \psi')$
 $\varphi \preceq \text{FImp } \psi\ \psi' \longleftrightarrow (\varphi = \text{FImp } \psi\ \psi' \vee \varphi \preceq \psi \vee \varphi \preceq \psi')$

proof –

have *wf-conn CAnd* $[\psi, \psi']$ **by** (*simp add: binary-connectives-def*)
then have $\varphi \preceq \text{conn CAnd } [\psi, \psi'] \longleftrightarrow$
 $(\varphi = \text{conn CAnd } [\psi, \psi'] \vee (\exists \psi''. \psi'' \in \text{set } [\psi, \psi'] \wedge \varphi \preceq \psi''))$
using *wf-subformula-conn-cases* **by** *metis*
then show *?P FAnd* **by** *auto*

next

have *wf-conn COr* $[\psi, \psi']$ **by** (*simp add: binary-connectives-def*)
then have $\varphi \preceq \text{conn COr } [\psi, \psi'] \longleftrightarrow$
 $(\varphi = \text{conn COr } [\psi, \psi'] \vee (\exists \psi''. \psi'' \in \text{set } [\psi, \psi'] \wedge \varphi \preceq \psi''))$
using *wf-subformula-conn-cases* **by** *metis*
then show *?P FOr* **by** *auto*

next

have *wf-conn CEq* $[\psi, \psi']$ **by** (*simp add: binary-connectives-def*)
then have $\varphi \preceq \text{conn CEq } [\psi, \psi'] \longleftrightarrow$
 $(\varphi = \text{conn CEq } [\psi, \psi'] \vee (\exists \psi''. \psi'' \in \text{set } [\psi, \psi'] \wedge \varphi \preceq \psi''))$
using *wf-subformula-conn-cases* **by** *metis*
then show *?P FEq* **by** *auto*

next

have *wf-conn CImp* $[\psi, \psi']$ **by** (*simp add: binary-connectives-def*)
then have $\varphi \preceq \text{conn CImp } [\psi, \psi'] \longleftrightarrow$
 $(\varphi = \text{conn CImp } [\psi, \psi'] \vee (\exists \psi''. \psi'' \in \text{set } [\psi, \psi'] \wedge \varphi \preceq \psi''))$
using *wf-subformula-conn-cases* **by** *metis*
then show *?P FImp* **by** *auto*

qed

lemma *wf-conn-helper-facts[iff]*:

wf-conn CNot $[\varphi]$
wf-conn CT $[]$
wf-conn CF $[]$
wf-conn (CVar x) $[]$
wf-conn CAnd $[\varphi, \psi]$
wf-conn COr $[\varphi, \psi]$
wf-conn CImp $[\varphi, \psi]$
wf-conn CEq $[\varphi, \psi]$
using *wf-conn.intros* **unfolding** *binary-connectives-def* **by** *fastforce+*

lemma *exists-c-conn*: $\exists\ c\ l. \varphi = \text{conn } c\ l \wedge \text{wf-conn } c\ l$

by (*cases* φ) *force+*

lemma *subformula-conn-decomp[simp]*:

assumes *wf*: *wf-conn* $c\ l$
shows $\varphi \preceq \text{conn } c\ l \longleftrightarrow (\varphi = \text{conn } c\ l \vee (\exists \psi \in \text{set } l. \varphi \preceq \psi)) \text{ (is ?A } \longleftrightarrow \text{ ?B)}$

proof (*rule iffI*)

```

{
  fix  $\xi$ 
  have  $\varphi \preceq \xi \implies \xi = \text{conn } c \ l \implies \text{wf-conn } c \ l \implies \forall x::'a \ \text{propo} \in \text{set } l. \neg \varphi \preceq x \implies \varphi = \text{conn } c \ l$ 
    apply (induct rule: subformula.induct)
    apply simp
    using conn-inj by blast
}
moreover assume ?A
ultimately show ?B using wf by metis
next
assume ?B
then show  $\varphi \preceq \text{conn } c \ l$  using wf wf-subformula-conn-cases by blast
qed

```

lemma *subformula-leaf-explicit*[simp]:

```

 $\varphi \preceq FT \longleftrightarrow \varphi = FT$ 
 $\varphi \preceq FF \longleftrightarrow \varphi = FF$ 
 $\varphi \preceq FVar \ x \longleftrightarrow \varphi = FVar \ x$ 
apply auto
using subformula-leaf by metis +

```

The variables inside the formula gives precisely the variables that are needed for the formula.

primrec *vars-of-prop*:: $'v \ \text{propo} \Rightarrow 'v \ \text{set}$ **where**

```

vars-of-prop FT = {} |
vars-of-prop FF = {} |
vars-of-prop (FVar x) = {x} |
vars-of-prop (FNot  $\varphi$ ) = vars-of-prop  $\varphi$  |
vars-of-prop (FAnd  $\varphi \ \psi$ ) = vars-of-prop  $\varphi \cup \text{vars-of-prop } \psi$  |
vars-of-prop (FOr  $\varphi \ \psi$ ) = vars-of-prop  $\varphi \cup \text{vars-of-prop } \psi$  |
vars-of-prop (FImp  $\varphi \ \psi$ ) = vars-of-prop  $\varphi \cup \text{vars-of-prop } \psi$  |
vars-of-prop (FEq  $\varphi \ \psi$ ) = vars-of-prop  $\varphi \cup \text{vars-of-prop } \psi$ 

```

lemma *vars-of-prop-incl-conn*:

```

fixes  $\xi \ \xi' :: 'v \ \text{propo list}$  and  $\psi :: 'v \ \text{propo}$  and  $c :: 'v \ \text{connective}$ 
assumes corr: wf-conn c l and incl:  $\psi \in \text{set } l$ 
shows vars-of-prop  $\psi \subseteq \text{vars-of-prop } (\text{conn } c \ l)$ 

```

proof (cases c rule: connective-cases-arity-2)

```

case nullary
then have False using corr incl by auto
then show vars-of-prop  $\psi \subseteq \text{vars-of-prop } (\text{conn } c \ l)$  by blast

```

next

```

case binary note c = this
then obtain a b where ab: l = [a, b]
  using wf-conn-bin-list-length list-length2-decomp corr by metis
then have  $\psi = a \vee \psi = b$  using incl by auto
then show vars-of-prop  $\psi \subseteq \text{vars-of-prop } (\text{conn } c \ l)$ 
  using ab c unfolding binary-connectives-def by auto

```

next

```

case unary note c = this
fix  $\varphi :: 'v \ \text{propo}$ 
have l = [ $\psi$ ] using corr c incl split-list by force
then show vars-of-prop  $\psi \subseteq \text{vars-of-prop } (\text{conn } c \ l)$  using c by auto

```

qed

The set of variables is compatible with the subformula order.

lemma *subformula-vars-of-prop*:
 $\varphi \preceq \psi \implies \text{vars-of-prop } \varphi \subseteq \text{vars-of-prop } \psi$
apply (*induct rule: subformula.induct*)
apply *simp*
using *vars-of-prop-incl-conn* **by** *blast*

4.4 Positions

Instead of 1 or 2 we use L or R

datatype *sign* = $L \mid R$

We use *nil* instead of ε .

fun *pos* :: '*v* *propo* \Rightarrow *sign* *list* *set* **where**
pos *FF* = $\{\square\}$ |
pos *FT* = $\{\square\}$ |
pos (*FVar* *x*) = $\{\square\}$ |
pos (*FAnd* $\varphi \psi$) = $\{\square\} \cup \{L \# p \mid p. p \in \text{pos } \varphi\} \cup \{R \# p \mid p. p \in \text{pos } \psi\}$ |
pos (*FOr* $\varphi \psi$) = $\{\square\} \cup \{L \# p \mid p. p \in \text{pos } \varphi\} \cup \{R \# p \mid p. p \in \text{pos } \psi\}$ |
pos (*FEq* $\varphi \psi$) = $\{\square\} \cup \{L \# p \mid p. p \in \text{pos } \varphi\} \cup \{R \# p \mid p. p \in \text{pos } \psi\}$ |
pos (*FImp* $\varphi \psi$) = $\{\square\} \cup \{L \# p \mid p. p \in \text{pos } \varphi\} \cup \{R \# p \mid p. p \in \text{pos } \psi\}$ |
pos (*FNot* φ) = $\{\square\} \cup \{L \# p \mid p. p \in \text{pos } \varphi\}$

lemma *finite-pos*: *finite* (*pos* φ)
by (*induct* φ , *auto*)

lemma *finite-inj-comp-set*:
fixes *s* :: '*v* *set*
assumes *finite*: *finite* *s*
and *inj*: *inj* *f*
shows *card* ($\{f p \mid p. p \in s\}$) = *card* *s*
using *finite*
proof (*induct s rule: finite-induct*)
show *card* $\{f p \mid p. p \in \{\}\} = \text{card } \{\}$ **by** *auto*
next
fix *x* :: '*v* **and** *s*:: '*v* *set*
assume *f*: *finite* *s* **and** *notin*: $x \notin s$
and *IH*: *card* $\{f p \mid p. p \in s\} = \text{card } s$
have *f'*: *finite* $\{f p \mid p. p \in \text{insert } x s\}$ **using** *f* **by** *auto*
have *notin'*: $f x \notin \{f p \mid p. p \in s\}$ **using** *notin* *inj* *injD* **by** *fastforce*
have $\{f p \mid p. p \in \text{insert } x s\} = \text{insert } (f x) \{f p \mid p. p \in s\}$ **by** *auto*
then have *card* $\{f p \mid p. p \in \text{insert } x s\} = 1 + \text{card } \{f p \mid p. p \in s\}$
using *finite* *card-insert-disjoint* *f'* *notin'* **by** *auto*
moreover have $\dots = \text{card } (\text{insert } x s)$ **using** *notin* *f* *IH* **by** *auto*
finally show *card* $\{f p \mid p. p \in \text{insert } x s\} = \text{card } (\text{insert } x s)$.
qed

lemma *cons-inject*:
inj (*op* $\#$ *s*)
by (*meson* *injI* *list.inject*)

lemma *finite-insert-nil-cons*:
finite *s* $\implies \text{card } (\text{insert } \square \{L \# p \mid p. p \in s\}) = 1 + \text{card } \{L \# p \mid p. p \in s\}$
using *card-insert-disjoint* **by** *auto*

lemma *cord-not[simp]*:

$\text{card } (\text{pos } (\text{FNot } \varphi)) = 1 + \text{card } (\text{pos } \varphi)$

by (*simp add: cons-inject finite-inj-comp-set finite-pos*)

lemma *card-seperate*:

assumes *finite s1 and finite s2*

shows $\text{card } (\{L \# p \mid p. p \in s1\} \cup \{R \# p \mid p. p \in s2\}) = \text{card } (\{L \# p \mid p. p \in s1\})$
 $+ \text{card } (\{R \# p \mid p. p \in s2\})$ (**is** $\text{card } (?L \cup ?R) = \text{card } ?L + \text{card } ?R$)

proof –

have *finite ?L using assms by auto*

moreover have *finite ?R using assms by auto*

moreover have $?L \cap ?R = \{\}$ **by** *blast*

ultimately show *?thesis using assms card-Un-disjoint by blast*

qed

definition *prop-size* **where** $\text{prop-size } \varphi = \text{card } (\text{pos } \varphi)$

lemma *prop-size-vars-of-prop*:

fixes $\varphi :: 'v \text{ propo}$

shows $\text{card } (\text{vars-of-prop } \varphi) \leq \text{prop-size } \varphi$

unfolding *prop-size-def* **apply** (*induct* φ , *auto simp add: cons-inject finite-inj-comp-set finite-pos*)

proof –

fix $\varphi1 \varphi2 :: 'v \text{ propo}$

assume *IH1: card (vars-of-prop $\varphi1$) \leq card (pos $\varphi1$)*

and *IH2: card (vars-of-prop $\varphi2$) \leq card (pos $\varphi2$)*

let $?L = \{L \# p \mid p. p \in \text{pos } \varphi1\}$

let $?R = \{R \# p \mid p. p \in \text{pos } \varphi2\}$

have $\text{card } (?L \cup ?R) = \text{card } ?L + \text{card } ?R$

using *card-seperate finite-pos by blast*

moreover have $\dots = \text{card } (\text{pos } \varphi1) + \text{card } (\text{pos } \varphi2)$

by (*simp add: cons-inject finite-inj-comp-set finite-pos*)

moreover have $\dots \geq \text{card } (\text{vars-of-prop } \varphi1) + \text{card } (\text{vars-of-prop } \varphi2)$ **using** *IH1 IH2 by arith*

then have $\dots \geq \text{card } (\text{vars-of-prop } \varphi1 \cup \text{vars-of-prop } \varphi2)$ **using** *card-Un-le le-trans by blast*

ultimately

show $\text{card } (\text{vars-of-prop } \varphi1 \cup \text{vars-of-prop } \varphi2) \leq \text{Suc } (\text{card } (?L \cup ?R))$

$\text{card } (\text{vars-of-prop } \varphi1 \cup \text{vars-of-prop } \varphi2) \leq \text{Suc } (\text{card } (?L \cup ?R))$

$\text{card } (\text{vars-of-prop } \varphi1 \cup \text{vars-of-prop } \varphi2) \leq \text{Suc } (\text{card } (?L \cup ?R))$

$\text{card } (\text{vars-of-prop } \varphi1 \cup \text{vars-of-prop } \varphi2) \leq \text{Suc } (\text{card } (?L \cup ?R))$

by *auto*

qed

value *pos* (*FImp* (*FAnd* (*FVar* *P*) (*FVar* *Q*)) (*FOr* (*FVar* *P*) (*FVar* *Q*)))

inductive *path-to* $:: \text{sign list} \Rightarrow 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ **where**

path-to-refl[intro]: path-to $[\] \varphi \varphi \mid$

path-to-l: c $\in \text{binary-connectives} \vee c = \text{CNot} \implies \text{wf-conn } c (\varphi \# l) \implies \text{path-to } p \varphi \varphi' \implies$

path-to (*L* $\# p$) (*conn* *c* ($\varphi \# l$)) $\varphi' \mid$

path-to-r: c $\in \text{binary-connectives} \implies \text{wf-conn } c (\psi \# \varphi \# [\]) \implies \text{path-to } p \varphi \varphi' \implies$

path-to (*R* $\# p$) (*conn* *c* ($\psi \# \varphi \# [\]$)) φ'

There is a deep link between subformulas and pathes: a (correct) path leads to a subformula and a subformula is associated to a given path.

lemma *path-to-subformula*:

path-to $p \varphi \varphi' \implies \varphi' \preceq \varphi$

```

apply (induct rule: path-to.induct)
  apply simp
  apply (metis list.set-intros(1) subformula-into-subformula)
  using subformula-trans subformula-in-binary-conn(2) by metis

lemma subformula-path-exists:
  fixes  $\varphi \varphi' :: 'v \text{ propo}$ 
  shows  $\varphi' \preceq \varphi \implies \exists p. \text{path-to } p \varphi \varphi'$ 
proof (induct rule: subformula.induct)
  case subformula-refl
  have path-to []  $\varphi' \varphi'$  by auto
  then show  $\exists p. \text{path-to } p \varphi' \varphi'$  by metis
next
  case (subformula-into-subformula  $\psi \ l \ c$ )
  note wf = this(2) and IH = this(4) and  $\psi = \text{this}(1)$ 
  then obtain  $p$  where  $p: \text{path-to } p \psi \varphi'$  by metis
  {
    fix  $x :: 'v$ 
    assume  $c = CT \vee c = CF \vee c = CVar \ x$ 
    then have False using subformula-into-subformula by auto
    then have  $\exists p. \text{path-to } p (\text{conn } c \ l) \varphi'$  by blast
  }
  moreover {
    assume  $c: c = CNot$ 
    then have  $l = [\psi]$  using wf  $\psi$  wf-conn-Not-decomp by fastforce
    then have path-to  $(L \# p) (\text{conn } c \ l) \varphi'$  by (metis c wf-conn-unary p path-to-l)
    then have  $\exists p. \text{path-to } p (\text{conn } c \ l) \varphi'$  by blast
  }
  moreover {
    assume  $c: c \in \text{binary-connectives}$ 
    obtain  $a \ b$  where  $ab: [a, b] = l$  using subformula-into-subformula  $c$  wf-conn-bin-list-length
      list-length2-decomp by metis
    then have  $a = \psi \vee b = \psi$  using  $\psi$  by auto
    then have path-to  $(L \# p) (\text{conn } c \ l) \varphi' \vee \text{path-to } (R \# p) (\text{conn } c \ l) \varphi'$  using  $c$  path-to-l
      path-to-r  $p \ ab$  by (metis wf-conn-binary)
    then have  $\exists p. \text{path-to } p (\text{conn } c \ l) \varphi'$  by blast
  }
  ultimately show  $\exists p. \text{path-to } p (\text{conn } c \ l) \varphi'$  using connective-cases-arity by metis
qed

```

```

fun replace-at ::  $\text{sign list} \Rightarrow 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow 'v \text{ propo}$  where
  replace-at []  $\psi = \psi$  |
  replace-at  $(L \# l) (FAnd \ \varphi \ \varphi') \psi = FAnd (\text{replace-at } l \ \varphi \ \psi) \ \varphi'$  |
  replace-at  $(R \# l) (FAnd \ \varphi \ \varphi') \psi = FAnd \ \varphi (\text{replace-at } l \ \varphi' \ \psi)$  |
  replace-at  $(L \# l) (FOr \ \varphi \ \varphi') \psi = FOr (\text{replace-at } l \ \varphi \ \psi) \ \varphi'$  |
  replace-at  $(R \# l) (FOr \ \varphi \ \varphi') \psi = FOr \ \varphi (\text{replace-at } l \ \varphi' \ \psi)$  |
  replace-at  $(L \# l) (FEq \ \varphi \ \varphi') \psi = FEq (\text{replace-at } l \ \varphi \ \psi) \ \varphi'$  |
  replace-at  $(R \# l) (FEq \ \varphi \ \varphi') \psi = FEq \ \varphi (\text{replace-at } l \ \varphi' \ \psi)$  |
  replace-at  $(L \# l) (FImp \ \varphi \ \varphi') \psi = FImp (\text{replace-at } l \ \varphi \ \psi) \ \varphi'$  |
  replace-at  $(R \# l) (FImp \ \varphi \ \varphi') \psi = FImp \ \varphi (\text{replace-at } l \ \varphi' \ \psi)$  |
  replace-at  $(L \# l) (FNot \ \varphi) \psi = FNot (\text{replace-at } l \ \varphi \ \psi)$ 

```

5 Semantics over the syntax

Given the syntax defined above, we define a semantics, by defining an evaluation function *eval*. This function is the bridge between the logic as we define it here and the built-in logic of Isabelle.

```
fun eval :: ('v  $\Rightarrow$  bool)  $\Rightarrow$  'v propo  $\Rightarrow$  bool (infix  $\models$  50) where
 $\mathcal{A} \models FT = True$  |
 $\mathcal{A} \models FF = False$  |
 $\mathcal{A} \models FVar\ v = (\mathcal{A}\ v)$  |
 $\mathcal{A} \models FNot\ \varphi = (\neg(\mathcal{A} \models \varphi))$  |
 $\mathcal{A} \models FAnd\ \varphi_1\ \varphi_2 = (\mathcal{A} \models \varphi_1 \wedge \mathcal{A} \models \varphi_2)$  |
 $\mathcal{A} \models FOr\ \varphi_1\ \varphi_2 = (\mathcal{A} \models \varphi_1 \vee \mathcal{A} \models \varphi_2)$  |
 $\mathcal{A} \models FImp\ \varphi_1\ \varphi_2 = (\mathcal{A} \models \varphi_1 \longrightarrow \mathcal{A} \models \varphi_2)$  |
 $\mathcal{A} \models FEq\ \varphi_1\ \varphi_2 = (\mathcal{A} \models \varphi_1 \longleftrightarrow \mathcal{A} \models \varphi_2)$ 
```

```
definition evalf (infix  $\models_f$  50) where
evalf  $\varphi\ \psi = (\forall A. A \models \varphi \longrightarrow A \models \psi)$ 
```

The deduction rule is in the book. And the proof looks like to the one of the book.

theorem *deduction-theorem*:

$(\varphi \models_f \psi) \longleftrightarrow (\forall A. (A \models FImp\ \varphi\ \psi))$

proof

```
assume H:  $\varphi \models_f \psi$ 
{
  fix A
  have  $A \models FImp\ \varphi\ \psi$ 
  proof (cases  $A \models \varphi$ )
    case True
      then have  $A \models \psi$  using H unfolding evalf-def by metis
      then show  $A \models FImp\ \varphi\ \psi$  by auto
    next
      case False
      then show  $A \models FImp\ \varphi\ \psi$  by auto
  qed
}
then show  $\forall A. A \models FImp\ \varphi\ \psi$  by blast
next
assume A:  $\forall A. A \models FImp\ \varphi\ \psi$ 
show  $\varphi \models_f \psi$ 
proof (rule ccontr)
  assume  $\neg \varphi \models_f \psi$ 
  then obtain A where  $A \models \varphi$  and  $\neg A \models \psi$  using evalf-def by metis
  then have  $\neg A \models FImp\ \varphi\ \psi$  by auto
  then show False using A by blast
qed
qed
```

A shorter proof:

```
lemma  $\varphi \models_f \psi \longleftrightarrow (\forall A. A \models FImp\ \varphi\ \psi)$ 
by (simp add: evalf-def)
```

```
definition same-over-set:: ('v  $\Rightarrow$  bool)  $\Rightarrow$  ('v  $\Rightarrow$  bool)  $\Rightarrow$  'v set  $\Rightarrow$  bool where
same-over-set A B S =  $(\forall c \in S. A\ c = B\ c)$ 
```

If two mapping *A* and *B* have the same value over the variables, then the same formula are satisfiable.

```

lemma same-over-set-eval:
  assumes same-over-set  $A\ B$  (vars-of-prop  $\varphi$ )
  shows  $A \models \varphi \longleftrightarrow B \models \varphi$ 
  using assms unfolding same-over-set-def by (induct  $\varphi$ , auto)

```

```

end
theory Prop-Abstract-Transformation
imports Main Prop-Logic Wellfounded-More

```

```

begin

```

This file is devoted to abstract properties of the transformations, like consistency preservation and lifting from terms to proposition.

6 Rewrite systems and properties

6.1 Lifting of rewrite rules

We can lift a rewrite relation r over a full formula: the relation r works on terms, while *propo-rew-step* works on formulas.

```

inductive propo-rew-step :: ('v propo  $\Rightarrow$  'v propo  $\Rightarrow$  bool)  $\Rightarrow$  'v propo  $\Rightarrow$  'v propo  $\Rightarrow$  bool
  for  $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$  where
    global-rel:  $r\ \varphi\ \psi \Longrightarrow \text{propo-rew-step}\ r\ \varphi\ \psi$  |
    propo-rew-one-step-lift:  $\text{propo-rew-step}\ r\ \varphi\ \varphi' \Longrightarrow \text{wf-conn}\ c\ (\psi s\ @\ \varphi\ \# \psi s') \Longrightarrow \text{propo-rew-step}\ r\ (\text{conn}\ c\ (\psi s\ @\ \varphi\ \# \psi s'))\ (\text{conn}\ c\ (\psi s\ @\ \varphi' \# \psi s'))$ 

```

Here is a more precise link between the lifting and the subformulas: if a rewriting takes place between φ and φ' , then there are two subformulas ψ in φ and ψ' in φ' , ψ' is the result of the rewriting of r on ψ .

This lemma is only a health condition:

```

lemma propo-rew-step-subformula-imp:
shows  $\text{propo-rew-step}\ r\ \varphi\ \varphi' \Longrightarrow \exists\ \psi\ \psi'.\ \psi \preceq \varphi \wedge \psi' \preceq \varphi' \wedge r\ \psi\ \psi'$ 
  apply (induct rule: propo-rew-step.induct)
  using subformula.simps subformula-into-subformula apply blast
  using wf-conn-no-arity-change subformula-into-subformula wf-conn-no-arity-change-helper in-set-conv-decomp by metis

```

The converse is moreover true: if there is a ψ and ψ' , then every formula φ containing ψ , can be rewritten into a formula φ' , such that it contains φ' .

```

lemma propo-rew-step-subformula-rec:
  fixes  $\psi\ \psi'\ \varphi :: 'v \text{ propo}$ 
  shows  $\psi \preceq \varphi \Longrightarrow r\ \psi\ \psi' \Longrightarrow (\exists\ \varphi'.\ \psi' \preceq \varphi' \wedge \text{propo-rew-step}\ r\ \varphi\ \varphi')$ 
proof (induct  $\varphi$  rule: subformula.induct)
  case subformula-refl
  hence  $\text{propo-rew-step}\ r\ \psi\ \psi'$  using propo-rew-step.intros by auto
  moreover have  $\psi' \preceq \psi'$  using Prop-Logic.subformula-refl by auto
  ultimately show  $\exists\ \varphi'.\ \psi' \preceq \varphi' \wedge \text{propo-rew-step}\ r\ \psi\ \varphi'$  by fastforce
next
  case (subformula-into-subformula  $\psi''\ l\ c$ )
  note  $IH = \text{this}(4)$  and  $r = \text{this}(5)$  and  $\psi'' = \text{this}(1)$  and  $\text{wf} = \text{this}(2)$  and  $\text{incl} = \text{this}(3)$ 
  then obtain  $\varphi'$  where  $\psi' \preceq \varphi' \wedge \text{propo-rew-step}\ r\ \psi''\ \varphi'$  by metis
  moreover obtain  $\xi\ \xi' :: 'v \text{ propo list}$  where

```

$l: l = \xi @ \psi'' \# \xi' \text{ using } \text{List.split-list } \psi'' \text{ by } \text{metis}$
ultimately have $\text{propo-rew-step } r \text{ (conn } c \text{ l) (conn } c \text{ (}\xi @ \varphi' \# \xi'))$
using $\text{propo-rew-step.intros}(2) \text{ wf by metis}$
moreover have $\psi' \preceq \text{conn } c \text{ (}\xi @ \varphi' \# \xi')$
using $\text{wf} * \text{wf-conn-no-arity-change Prop-Logic.subformula-into-subformula}$
by $(\text{metis (no-types) in-set-conv-decomp l wf-conn-no-arity-change-helper})$
ultimately show $\exists \varphi'. \psi' \preceq \varphi' \wedge \text{propo-rew-step } r \text{ (conn } c \text{ l) } \varphi' \text{ by metis}$
qed

lemma *propo-rew-step-subformula*:
 $(\exists \psi \psi'. \psi \preceq \varphi \wedge r \psi \psi') \longleftrightarrow (\exists \varphi'. \text{propo-rew-step } r \varphi \varphi')$
using *propo-rew-step-subformula-imp propo-rew-step-subformula-rec* **by metis+**

lemma *consistency-decompose-into-list*:
assumes $\text{wf}: \text{wf-conn } c \text{ l}$ **and** $\text{wf}': \text{wf-conn } c \text{ l'}$
and $\text{same}: \forall n. (A \models l ! n \longleftrightarrow (A \models l' ! n))$
shows $(A \models \text{conn } c \text{ l}) = (A \models \text{conn } c \text{ l'})$
proof (*cases c rule: connective-cases-arity-2*)
case *nullary*
thus $(A \models \text{conn } c \text{ l}) \longleftrightarrow (A \models \text{conn } c \text{ l'})$ **using** wf wf' by auto
next
case *unary note c = this*
then obtain a **where** $l: l = [a]$ **using** $\text{wf-conn-Not-decomp wf by metis}$
obtain a' **where** $l': l' = [a']$ **using** $\text{wf-conn-Not-decomp wf' c by metis}$
have $A \models a \longleftrightarrow A \models a'$ **using** $l \text{ l' by (metis nth-Cons-0 same)}$
thus $A \models \text{conn } c \text{ l} \longleftrightarrow A \models \text{conn } c \text{ l'}$ **using** $l \text{ l' c by auto}$
next
case *binary note c = this*
then obtain $a \text{ b}$ **where** $l: l = [a, b]$
using $\text{wf-conn-bin-list-length list-length2-decomp wf by metis}$
obtain $a' \text{ b'}$ **where** $l': l' = [a', b']$
using $\text{wf-conn-bin-list-length list-length2-decomp wf' c by metis}$

have $p: A \models a \longleftrightarrow A \models a' \wedge A \models b \longleftrightarrow A \models b'$
using $l \text{ l' same by (metis diff-Suc-1 nth-Cons' nat.distinct(2))}$
show $A \models \text{conn } c \text{ l} \longleftrightarrow A \models \text{conn } c \text{ l'}$
using $\text{wf c p unfolding binary-connectives-def l l' by auto}$
qed

Relation between *propo-rew-step* and the rewriting we have seen before: *propo-rew-step* $r \varphi \varphi'$ means that we rewrite ψ inside φ (ie at a path p) into ψ' .

lemma *propo-rew-step-rewrite*:
fixes $\varphi \varphi' :: 'v \text{ propo}$ **and** $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$
assumes *propo-rew-step* $r \varphi \varphi'$
shows $\exists \psi \psi' p. r \psi \psi' \wedge \text{path-to } p \varphi \psi \wedge \text{replace-at } p \varphi \psi' = \varphi'$
using *assms*
proof (*induct rule: propo-rew-step.induct*)
case (*global-rel* $\varphi \psi$)
moreover have $\text{path-to } [] \varphi \varphi$ **by auto**
moreover have $\text{replace-at } [] \varphi \psi = \psi$ **by auto**
ultimately show *?case* **by metis**
next
case (*propo-rew-one-step-lift* $\varphi \varphi' c \xi \xi'$) **note** $\text{rel} = \text{this}(1)$ **and** $\text{IH0} = \text{this}(2)$ **and** $\text{corr} = \text{this}(3)$
obtain $\psi \psi' p$ **where** $\text{IH}: r \psi \psi' \wedge \text{path-to } p \varphi \psi \wedge \text{replace-at } p \varphi \psi' = \varphi' \text{ using IH0 by metis}$

```

{
  fix x :: 'v
  assume c = CT ∨ c = CF ∨ c = CVar x
  hence False using corr by auto
  hence  $\exists \psi \psi' p. r \psi \psi' \wedge \text{path-to } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi$ 
     $\wedge \text{replace-at } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi' = \text{conn } c (\xi @ (\varphi' \# \xi'))$ 
    by fast
}
moreover {
  assume c: c = CNot
  hence empty:  $\xi = [] \ \xi' = []$  using corr by auto
  have path-to (L#p) (conn c ( $\xi @ (\varphi \# \xi')$ ))  $\psi$ 
    using c empty IH wf-conn-unary path-to-l by fastforce
  moreover have replace-at (L#p) (conn c ( $\xi @ (\varphi \# \xi')$ ))  $\psi' = \text{conn } c (\xi @ (\varphi' \# \xi'))$ 
    using c empty IH by auto
  ultimately have  $\exists \psi \psi' p. r \psi \psi' \wedge \text{path-to } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi$ 
     $\wedge \text{replace-at } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi' = \text{conn } c (\xi @ (\varphi' \# \xi'))$ 
    using IH by metis
}
moreover {
  assume c: c ∈ binary-connectives
  have length ( $\xi @ \varphi \# \xi'$ ) = 2 using wf-conn-bin-list-length corr c by metis
  hence length  $\xi$  + length  $\xi' = 1$  by auto
  hence ld: (length  $\xi = 1 \wedge \text{length } \xi' = 0$ ) ∨ (length  $\xi = 0 \wedge \text{length } \xi' = 1$ ) by arith
  obtain a b where ab: ( $\xi = [] \wedge \xi' = [b]$ ) ∨ ( $\xi = [a] \wedge \xi' = []$ )
    using ld by (case-tac  $\xi$ , case-tac  $\xi'$ , auto)
  {
    assume  $\varphi: \xi = [] \wedge \xi' = [b]$ 
    have path-to (L#p) (conn c ( $\xi @ (\varphi \# \xi')$ ))  $\psi$ 
      using  $\varphi$  c IH ab corr by (simp add: path-to-l)
    moreover have replace-at (L#p) (conn c ( $\xi @ (\varphi \# \xi')$ ))  $\psi' = \text{conn } c (\xi @ (\varphi' \# \xi'))$ 
      using c IH ab  $\varphi$  unfolding binary-connectives-def by auto
    ultimately have  $\exists \psi \psi' p. r \psi \psi' \wedge \text{path-to } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi$ 
       $\wedge \text{replace-at } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi' = \text{conn } c (\xi @ (\varphi' \# \xi'))$ 
      using IH by metis
  }
  moreover {
    assume  $\varphi: \xi = [a] \ \xi' = []$ 
    hence path-to (R#p) (conn c ( $\xi @ (\varphi \# \xi')$ ))  $\psi$ 
      using c IH corr path-to-r corr  $\varphi$  by (simp add: path-to-r)
    moreover have replace-at (R#p) (conn c ( $\xi @ (\varphi \# \xi')$ ))  $\psi' = \text{conn } c (\xi @ (\varphi' \# \xi'))$ 
      using c IH ab  $\varphi$  unfolding binary-connectives-def by auto
    ultimately have ?case using IH by metis
  }
  ultimately have ?case using ab by blast
}
ultimately show ?case using connective-cases-arity by blast
qed

```

6.2 Consistency preservation

We define *preserves-un-sat*: it means that a relation preserves consistency.

definition *preserves-un-sat* **where**

preserves-un-sat $r \longleftrightarrow (\forall \varphi \psi. r \varphi \psi \longrightarrow (\forall A. A \models \varphi \longleftrightarrow A \models \psi))$

lemma *propo-rew-step-preservers-val-explicit*:
 $\text{propo-rew-step } r \ \varphi \ \psi \implies \text{preserves-un-sat } r \implies \text{propo-rew-step } r \ \varphi \ \psi \implies (\forall A. A \models \varphi \longleftrightarrow A \models \psi)$
unfolding *preserves-un-sat-def*
proof (*induction rule: propo-rew-step.induct*)
case *global-rel*
thus *?case* **by** *simp*
next
case (*propo-rew-one-step-lift* $\varphi \ \varphi' \ c \ \xi \ \xi'$) **note** $\text{rel} = \text{this}(1)$ **and** $\text{wf} = \text{this}(2)$
and $\text{IH} = \text{this}(3)[\text{OF } \text{this}(4) \ \text{this}(1)]$ **and** $\text{consistent} = \text{this}(4)$
{
fix A
from IH **have** $\forall n. (A \models (\xi @ \varphi \# \xi') ! n) = (A \models (\xi @ \varphi' \# \xi') ! n)$
by (*metis* (*mono-tags*, *hide-lams*) *list-update-length* *nth-Cons-0* *nth-append-length-plus* *nth-list-update-neq*)
hence $(A \models \text{conn } c \ (\xi @ \varphi \# \xi')) = (A \models \text{conn } c \ (\xi @ \varphi' \# \xi'))$
by (*meson* *consistency-decompose-into-list* *wf* *wf-conn-no-arity-change-helper* *wf-conn-no-arity-change*)
}
thus $\forall A. A \models \text{conn } c \ (\xi @ \varphi \# \xi') \longleftrightarrow A \models \text{conn } c \ (\xi @ \varphi' \# \xi')$ **by** *auto*
qed

lemma *propo-rew-step-preservers-val'*:
assumes *preserves-un-sat* r
shows *preserves-un-sat* (*propo-rew-step* r)
using *assms* **by** (*simp* *add: preserves-un-sat-def propo-rew-step-preservers-val-explicit*)

lemma *preserves-un-sat-OO[intro]*:
 $\text{preserves-un-sat } f \implies \text{preserves-un-sat } g \implies \text{preserves-un-sat } (f \text{ OO } g)$
unfolding *preserves-un-sat-def* **by** *auto*

lemma *star-consistency-preservation-explicit*:
assumes $(\text{propo-rew-step } r)^{**} \ \varphi \ \psi$ **and** *preserves-un-sat* r
shows $\forall A. A \models \varphi \longleftrightarrow A \models \psi$
using *assms* **by** (*induct rule: rtranclp-induct*)
(auto simp add: propo-rew-step-preservers-val-explicit)

lemma *star-consistency-preservation*:
 $\text{preserves-un-sat } r \implies \text{preserves-un-sat } (\text{propo-rew-step } r)^{**}$
by (*simp* *add: star-consistency-preservation-explicit preserves-un-sat-def*)

6.3 Full Lifting

In the previous a relation was lifted to a formula, now we define the relation such it is applied as long as possible. The definition is thus simply: it can be derived and nothing more can be derived.

lemma *full-ropo-rew-step-preservers-val[simp]*:
 $\text{preserves-un-sat } r \implies \text{preserves-un-sat } (\text{full } (\text{propo-rew-step } r))$
by (*metis* *full-def preserves-un-sat-def star-consistency-preservation*)

lemma *full-propo-rew-step-subformula*:
 $\text{full } (\text{propo-rew-step } r) \ \varphi' \ \varphi \implies \neg(\exists \psi \ \psi'. \psi \preceq \varphi \wedge r \ \psi \ \psi')$

unfolding full-def using propo-rew-step-subformula-rec by metis

7 Transformation testing

7.1 Definition and first properties

To prove correctness of our transformation, we create a *all-subformula-st* predicate. It tests recursively all subformulas. At each step, the actual formula is tested. The aim of this *test-symb* function is to test locally some properties of the formulas (i.e. at the level of the connective or at first level). This allows a clause description between the rewrite relation and the *test-symb*

definition *all-subformula-st* :: ('a propo \Rightarrow bool) \Rightarrow 'a propo \Rightarrow bool **where**
all-subformula-st test-symb $\varphi \equiv \forall \psi. \psi \preceq \varphi \longrightarrow \text{test-symb } \psi$

lemma *test-symb-imp-all-subformula-st[simp]*:
test-symb FT \Longrightarrow *all-subformula-st test-symb FT*
test-symb FF \Longrightarrow *all-subformula-st test-symb FF*
test-symb (FVar x) \Longrightarrow *all-subformula-st test-symb (FVar x)*
unfolding *all-subformula-st-def* **using** *subformula-leaf* **by** *metis+*

lemma *all-subformula-st-test-symb-true-phi*:
all-subformula-st test-symb $\varphi \Longrightarrow \text{test-symb } \varphi$
unfolding *all-subformula-st-def* **by** *auto*

lemma *all-subformula-st-decomp-imp*:
wf-conn c l \Longrightarrow (*test-symb (conn c l)* \wedge ($\forall \varphi \in \text{set } l. \text{all-subformula-st test-symb } \varphi$))
 \Longrightarrow *all-subformula-st test-symb (conn c l)*
unfolding *all-subformula-st-def* **by** *auto*

To ease the finding of proofs, we give some explicit theorem about the decomposition.

lemma *all-subformula-st-decomp-rec*:
all-subformula-st test-symb (conn c l) \Longrightarrow *wf-conn c l*
 \Longrightarrow (*test-symb (conn c l)* \wedge ($\forall \varphi \in \text{set } l. \text{all-subformula-st test-symb } \varphi$))
unfolding *all-subformula-st-def* **by** *auto*

lemma *all-subformula-st-decomp*:
fixes *c* :: 'v connective **and** *l* :: 'v propo list
assumes *wf-conn c l*
shows *all-subformula-st test-symb (conn c l)*
 \longleftrightarrow (*test-symb (conn c l)* \wedge ($\forall \varphi \in \text{set } l. \text{all-subformula-st test-symb } \varphi$))
using *assms all-subformula-st-decomp-rec all-subformula-st-decomp-imp* **by** *metis*

lemma *helper-fact*: $c \in \text{binary-connectives} \longleftrightarrow (c = COr \vee c = CAnd \vee c = CEq \vee c = CImp)$
unfolding *binary-connectives-def* **by** *auto*

lemma *all-subformula-st-decomp-explicit[simp]*:
fixes $\varphi \psi$:: 'v propo
shows *all-subformula-st test-symb (FAnd $\varphi \psi$)*
 \longleftrightarrow (*test-symb (FAnd $\varphi \psi$)* \wedge *all-subformula-st test-symb φ* \wedge *all-subformula-st test-symb ψ*)
and *all-subformula-st test-symb (FOr $\varphi \psi$)*
 \longleftrightarrow (*test-symb (FOr $\varphi \psi$)* \wedge *all-subformula-st test-symb φ* \wedge *all-subformula-st test-symb ψ*)
and *all-subformula-st test-symb (FNot φ)*
 \longleftrightarrow (*test-symb (FNot φ)* \wedge *all-subformula-st test-symb φ*)
and *all-subformula-st test-symb (FEq $\varphi \psi$)*

$\longleftrightarrow (test-symb (FEq \varphi \psi) \wedge all-subformula-st test-symb \varphi \wedge all-subformula-st test-symb \psi)$
and $all-subformula-st test-symb (FImp \varphi \psi)$
 $\longleftrightarrow (test-symb (FImp \varphi \psi) \wedge all-subformula-st test-symb \varphi \wedge all-subformula-st test-symb \psi)$
proof –
have $all-subformula-st test-symb (FAnd \varphi \psi) \longleftrightarrow all-subformula-st test-symb (conn CAnd [\varphi, \psi])$
by *auto*
moreover have $\dots \longleftrightarrow test-symb (conn CAnd [\varphi, \psi]) \wedge (\forall \xi \in set [\varphi, \psi]. all-subformula-st test-symb \xi)$
using $all-subformula-st-decomp wf-conn-helper-facts(5)$ **by** *metis*
finally show $all-subformula-st test-symb (FAnd \varphi \psi)$
 $\longleftrightarrow (test-symb (FAnd \varphi \psi) \wedge all-subformula-st test-symb \varphi \wedge all-subformula-st test-symb \psi)$
by *simp*

have $all-subformula-st test-symb (FOr \varphi \psi) \longleftrightarrow all-subformula-st test-symb (conn COr [\varphi, \psi])$
by *auto*
moreover have $\dots \longleftrightarrow (test-symb (conn COr [\varphi, \psi]) \wedge (\forall \xi \in set [\varphi, \psi]. all-subformula-st test-symb \xi))$
using $all-subformula-st-decomp wf-conn-helper-facts(6)$ **by** *metis*
finally show $all-subformula-st test-symb (FOr \varphi \psi)$
 $\longleftrightarrow (test-symb (FOr \varphi \psi) \wedge all-subformula-st test-symb \varphi \wedge all-subformula-st test-symb \psi)$
by *simp*

have $all-subformula-st test-symb (FEq \varphi \psi) \longleftrightarrow all-subformula-st test-symb (conn CEq [\varphi, \psi])$
by *auto*
moreover have $\dots \longleftrightarrow (test-symb (conn CEq [\varphi, \psi]) \wedge (\forall \xi \in set [\varphi, \psi]. all-subformula-st test-symb \xi))$
using $all-subformula-st-decomp wf-conn-helper-facts(8)$ **by** *metis*
finally show $all-subformula-st test-symb (FEq \varphi \psi)$
 $\longleftrightarrow (test-symb (FEq \varphi \psi) \wedge all-subformula-st test-symb \varphi \wedge all-subformula-st test-symb \psi)$
by *simp*

have $all-subformula-st test-symb (FImp \varphi \psi) \longleftrightarrow all-subformula-st test-symb (conn CImp [\varphi, \psi])$
by *auto*
moreover have $\dots \longleftrightarrow (test-symb (conn CImp [\varphi, \psi]) \wedge (\forall \xi \in set [\varphi, \psi]. all-subformula-st test-symb \xi))$
using $all-subformula-st-decomp wf-conn-helper-facts(7)$ **by** *metis*
finally show $all-subformula-st test-symb (FImp \varphi \psi)$
 $\longleftrightarrow (test-symb (FImp \varphi \psi) \wedge all-subformula-st test-symb \varphi \wedge all-subformula-st test-symb \psi)$
by *simp*

have $all-subformula-st test-symb (FNot \varphi) \longleftrightarrow all-subformula-st test-symb (conn CNot [\varphi])$
by *auto*
moreover have $\dots = (test-symb (conn CNot [\varphi]) \wedge (\forall \xi \in set [\varphi]. all-subformula-st test-symb \xi))$
using $all-subformula-st-decomp wf-conn-helper-facts(1)$ **by** *metis*
finally show $all-subformula-st test-symb (FNot \varphi)$
 $\longleftrightarrow (test-symb (FNot \varphi) \wedge all-subformula-st test-symb \varphi)$ **by** *simp*
qed

As *all-subformula-st* tests recursively, the function is true on every subformula.

lemma *subformula-all-subformula-st*:

$\psi \preceq \varphi \implies all-subformula-st test-symb \varphi \implies all-subformula-st test-symb \psi$
by (*induct rule: subformula.induct*, *auto simp add: all-subformula-st-decomp*)

The following theorem *no-test-symb-step-exists* shows the link between the *test-symb* function and the corresponding rewrite relation *r*: if we assume that if every time *test-symb* is true, then a *r* can be applied, finally as long as $\neg all-subformula-st test-symb \varphi$, then something can be

rewritten in φ .

lemma *no-test-symb-step-exists*:

fixes $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ **and** $\text{test-symb} :: 'v \text{ propo} \Rightarrow \text{bool}$ **and** $x :: 'v$
and $\varphi :: 'v \text{ propo}$

assumes *test-symb-false-nullary*: $\forall x. \text{test-symb } FF \wedge \text{test-symb } FT \wedge \text{test-symb } (FVar\ x)$

and $\forall \varphi'. \varphi' \preceq \varphi \longrightarrow (\neg \text{test-symb } \varphi') \longrightarrow (\exists \psi. r\ \varphi'\ \psi)$ **and**

$\neg \text{all-subformula-st test-symb } \varphi$

shows $(\exists \psi\ \psi'. \psi \preceq \varphi \wedge r\ \psi\ \psi')$

using *assms*

proof (*induct* φ *rule*: *propo-induct-arity*)

case (*nullary* $\varphi\ x$)

thus $\exists \psi\ \psi'. \psi \preceq \varphi \wedge r\ \psi\ \psi'$

using *wf-conn-nullary test-symb-false-nullary* **by** *fastforce*

next

case (*unary* φ) **note** $IH = \text{this}(1)[OF\ \text{this}(2)]$ **and** $r = \text{this}(2)$ **and** $nst = \text{this}(3)$ **and** $\text{subf} = \text{this}(4)$

from $r\ IH\ nst$ **have** $H: \neg \text{all-subformula-st test-symb } \varphi \Longrightarrow \exists \psi. \psi \preceq \varphi \wedge (\exists \psi'. r\ \psi\ \psi')$

by (*metis subformula-in-subformula-not subformula-refl subformula-trans*)

{

assume $n: \neg \text{test-symb } (FNot\ \varphi)$

obtain ψ **where** $r\ (FNot\ \varphi)\ \psi$ **using** *subformula-refl* $r\ n\ nst$ **by** *blast*

moreover **have** $FNot\ \varphi \preceq FNot\ \varphi$ **using** *subformula-refl* **by** *auto*

ultimately **have** $\exists \psi\ \psi'. \psi \preceq FNot\ \varphi \wedge r\ \psi\ \psi'$ **by** *metis*

}

moreover {

assume $n: \text{test-symb } (FNot\ \varphi)$

hence $\neg \text{all-subformula-st test-symb } \varphi$

using *all-subformula-st-decomp-explicit*(3) $nst\ \text{subf}$ **by** *blast*

hence $\exists \psi\ \psi'. \psi \preceq FNot\ \varphi \wedge r\ \psi\ \psi'$

using $H\ \text{subformula-in-subformula-not subformula-refl subformula-trans}$ **by** *blast*

}

ultimately **show** $\exists \psi\ \psi'. \psi \preceq FNot\ \varphi \wedge r\ \psi\ \psi'$ **by** *blast*

next

case (*binary* $\varphi\ \varphi1\ \varphi2$)

note $IH\varphi1-0 = \text{this}(1)[OF\ \text{this}(4)]$ **and** $IH\varphi2-0 = \text{this}(2)[OF\ \text{this}(4)]$ **and** $r = \text{this}(4)$

and $\varphi = \text{this}(3)$ **and** $le = \text{this}(5)$ **and** $nst = \text{this}(6)$

obtain $c :: 'v \text{ connective}$ **where**

$c: (c = CAnd \vee c = COr \vee c = CImp \vee c = CEq) \wedge \text{conn } c\ [\varphi1, \varphi2] = \varphi$

using φ **by** *fastforce*

hence *corr*: $\text{wf-conn } c\ [\varphi1, \varphi2]$ **using** *wf-conn.simps* **unfolding** *binary-connectives-def* **by** *auto*

have *inc*: $\varphi1 \preceq \varphi\ \varphi2 \preceq \varphi$ **using** *binary-connectives-def* $c\ \text{subformula-in-binary-conn}$ **by** *blast*+

from $r\ IH\varphi1-0$ **have** $IH\varphi1: \neg \text{all-subformula-st test-symb } \varphi1 \Longrightarrow \exists \psi\ \psi'. \psi \preceq \varphi1 \wedge r\ \psi\ \psi'$

using *inc*(1) *subformula-trans* le **by** *blast*

from $r\ IH\varphi2-0$ **have** $IH\varphi2: \neg \text{all-subformula-st test-symb } \varphi2 \Longrightarrow \exists \psi. \psi \preceq \varphi2 \wedge (\exists \psi'. r\ \psi\ \psi')$

using *inc*(2) *subformula-trans* le **by** *blast*

have *cases*: $\neg \text{test-symb } \varphi \vee \neg \text{all-subformula-st test-symb } \varphi1 \vee \neg \text{all-subformula-st test-symb } \varphi2$

using $c\ nst$ **by** *auto*

show $\exists \psi\ \psi'. \psi \preceq \varphi \wedge r\ \psi\ \psi'$

using $IH\varphi1\ IH\varphi2\ \text{subformula-trans } inc\ \text{subformula-refl } cases\ le$ **by** *blast*

qed

7.2 Invariant conservation

If two rewrite relation are independant (or at least independant enough), then the property characterizing the first relation *all-subformula-st test-symb* remains true. The next show the same property, with changes in the assumptions.

The assumption $\forall \varphi' \psi. \varphi' \preceq \Phi \longrightarrow r \varphi' \psi \longrightarrow \text{all-subformula-st test-symb } \varphi' \longrightarrow \text{all-subformula-st test-symb } \psi$ means that rewriting with r does not mess up the property we want to preserve locally.

The previous assumption is not enough to go from r to *propo-rew-step* r : we have to add the assumption that rewriting inside does not mess up the term: $\forall c \xi \varphi \xi' \varphi'. \varphi \preceq \Phi \longrightarrow \text{propo-rew-step } r \varphi \varphi' \longrightarrow \text{wf-conn } c (\xi @ \varphi \# \xi') \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi')) \longrightarrow \text{test-symb } \varphi' \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$

7.2.1 Invariant while lifting of the rewriting relation

The condition $\varphi \preceq \Phi$ (that will be used with $\Phi = \varphi$ most of the time) is here to ensure that the recursive conditions on Φ will moreover hold for the subterm we are rewriting. For example if there is no equivalence symbol in Φ , we do not have to care about equivalence symbols in the two previous assumptions.

lemma *propo-rew-step-inv-stay*:

```

fixes  $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$  and  $\text{test-symb} :: 'v \text{ propo} \Rightarrow \text{bool}$  and  $x :: 'v$ 
and  $\varphi \psi \Phi :: 'v \text{ propo}$ 
assumes  $H: \forall \varphi' \psi. \varphi' \preceq \Phi \longrightarrow r \varphi' \psi \longrightarrow \text{all-subformula-st test-symb } \varphi'$ 
   $\longrightarrow \text{all-subformula-st test-symb } \psi$ 
and  $H': \forall (c :: 'v \text{ connective}) \xi \varphi \xi' \varphi'. \varphi \preceq \Phi \longrightarrow \text{propo-rew-step } r \varphi \varphi'$ 
   $\longrightarrow \text{wf-conn } c (\xi @ \varphi \# \xi') \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi')) \longrightarrow \text{test-symb } \varphi'$ 
   $\longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$  and
   $\text{propo-rew-step } r \varphi \psi$  and
   $\varphi \preceq \Phi$  and
   $\text{all-subformula-st test-symb } \varphi$ 
shows  $\text{all-subformula-st test-symb } \psi$ 
using assms(3-5)
proof (induct rule: propo-rew-step.induct)
  case global-rel
  thus ?case using  $H$  by simp
next
  case (propo-rew-one-step-lift  $\varphi \varphi' c \xi \xi'$ )
  note  $\text{rel} = \text{this}(1)$  and  $\varphi = \text{this}(2)$  and  $\text{corr} = \text{this}(3)$  and  $\Phi = \text{this}(4)$  and  $\text{nst} = \text{this}(5)$ 
  have  $\text{sq}: \varphi \preceq \Phi$ 
    using  $\Phi \text{ corr subformula-into-subformula subformula-refl subformula-trans}$ 
    by (metis in-set-conv-decomp)
  from  $\text{corr}$  have  $\forall \psi. \psi \in \text{set } (\xi @ \varphi \# \xi') \longrightarrow \text{all-subformula-st test-symb } \psi$ 
    using  $\text{all-subformula-st-decomp nst}$  by blast
  hence *:  $\forall \psi. \psi \in \text{set } (\xi @ \varphi' \# \xi') \longrightarrow \text{all-subformula-st test-symb } \psi$  using  $\varphi \text{ sq}$  by fastforce
  hence  $\text{test-symb } \varphi'$  using  $\text{all-subformula-st-test-symb-true-phi}$  by auto
  moreover from  $\text{corr nst}$  have  $\text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi'))$ 
    using  $\text{all-subformula-st-decomp}$  by blast
  ultimately have  $\text{test-symb: test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$  using  $H' \text{ sq corr rel}$  by blast

  have  $\text{wf-conn } c (\xi @ \varphi' \# \xi')$ 
    by (metis wf-conn-no-arity-change-helper corr wf-conn-no-arity-change)
  thus  $\text{all-subformula-st test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$ 

```

using * test-symb by (metis all-subformula-st-decomp)
qed

The need for $\varphi \preceq \Phi$ is not always necessary, hence we moreover have a version without inclusion.

lemma propo-rew-step-inv-stay:

fixes $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ and test-symb:: $'v \text{ propo} \Rightarrow \text{bool}$ and $x :: 'v$
and $\varphi \psi :: 'v \text{ propo}$
assumes
 $H: \forall \varphi' \psi. r \varphi' \psi \longrightarrow \text{all-subformula-st test-symb } \varphi' \longrightarrow \text{all-subformula-st test-symb } \psi$ and
 $H': \forall (c :: 'v \text{ connective}) \xi \varphi \xi' \varphi'. \text{wf-conn } c (\xi @ \varphi \# \xi') \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi'))$
 $\longrightarrow \text{test-symb } \varphi' \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$ and
 propo-rew-step $r \varphi \psi$ and
 all-subformula-st test-symb φ
shows all-subformula-st test-symb ψ
using propo-rew-step-inv-stay'[of $\varphi r \text{ test-symb } \varphi \psi$] assms subformula-refl by metis

The lemmas can be lifted to full (propo-rew-step r) instead of propo-rew-step

7.2.2 Invariant after all rewriting

lemma full-propo-rew-step-inv-stay-with-inc:

fixes $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ and test-symb:: $'v \text{ propo} \Rightarrow \text{bool}$ and $x :: 'v$
and $\varphi \psi :: 'v \text{ propo}$
assumes
 $H: \forall \varphi \psi. \text{propo-rew-step } r \varphi \psi \longrightarrow \text{all-subformula-st test-symb } \varphi$
 $\longrightarrow \text{all-subformula-st test-symb } \psi$ and
 $H': \forall (c :: 'v \text{ connective}) \xi \varphi \xi' \varphi'. \varphi \preceq \Phi \longrightarrow \text{propo-rew-step } r \varphi \varphi'$
 $\longrightarrow \text{wf-conn } c (\xi @ \varphi \# \xi') \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi')) \longrightarrow \text{test-symb } \varphi'$
 $\longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$ and
 $\varphi \preceq \Phi$ and
 full: full (propo-rew-step r) $\varphi \psi$ and
 init: all-subformula-st test-symb φ
shows all-subformula-st test-symb ψ
using assms unfolding full-def

proof –

have rel: (propo-rew-step r)** $\varphi \psi$
 using full unfolding full-def by auto
thus all-subformula-st test-symb ψ
 using init
 proof (induct rule: rtrancpl-induct)
 case base
 then show all-subformula-st test-symb φ **by** blast
 next
 case (step $b \ c$) **note** star = this(1) and IH = this(3) and one = this(2) and all = this(4)
 then have all-subformula-st test-symb b **by** metis
 then show all-subformula-st test-symb c **using** propo-rew-step-inv-stay' $H \ H'$ rel one **by** auto
 qed

qed

lemma full-propo-rew-step-inv-stay':

fixes $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ and test-symb:: $'v \text{ propo} \Rightarrow \text{bool}$ and $x :: 'v$
and $\varphi \psi :: 'v \text{ propo}$
assumes
 $H: \forall \varphi \psi. \text{propo-rew-step } r \varphi \psi \longrightarrow \text{all-subformula-st test-symb } \varphi$
 $\longrightarrow \text{all-subformula-st test-symb } \psi$ and

$H': \forall (c:: 'v \text{ connective}) \xi \varphi \xi' \varphi'. \text{propo-rew-step } r \varphi \varphi' \longrightarrow \text{wf-conn } c (\xi @ \varphi \# \xi')$
 $\longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi')) \longrightarrow \text{test-symb } \varphi' \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$ **and**
full: *full* (propo-rew-step r) $\varphi \psi$ **and**
init: *all-subformula-st test-symb* φ
shows *all-subformula-st test-symb* ψ
using *full-propo-rew-step-inv-stay-with-inc*[of r *test-symb* φ] *assms subformula-refl* **by** *metis*

lemma *full-propo-rew-step-inv-stay*:

fixes $r:: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ **and** *test-symb*:: $'v \text{ propo} \Rightarrow \text{bool}$ **and** $x:: 'v$
and $\varphi \psi:: 'v \text{ propo}$
assumes
 $H: \forall \varphi \psi. r \varphi \psi \longrightarrow \text{all-subformula-st test-symb } \varphi \longrightarrow \text{all-subformula-st test-symb } \psi$ **and**
 $H': \forall (c:: 'v \text{ connective}) \xi \varphi \xi' \varphi'. \text{wf-conn } c (\xi @ \varphi \# \xi') \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi'))$
 $\longrightarrow \text{test-symb } \varphi' \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$ **and**
full: *full* (propo-rew-step r) $\varphi \psi$ **and**
init: *all-subformula-st test-symb* φ
shows *all-subformula-st test-symb* ψ
unfolding *full-def*

proof –

have *rel*: $(\text{propo-rew-step } r)^{**} \varphi \psi$
using *full unfolding full-def* **by** *auto*
thus *all-subformula-st test-symb* ψ
using *init*
proof (*induct rule: rtrancpl-induct*)
case *base*
thus *all-subformula-st test-symb* φ **by** *blast*
next
case (*step b c*)
note *star* = *this*(1) **and** *IH* = *this*(3) **and** *one* = *this*(2) **and** *all* = *this*(4)
hence *all-subformula-st test-symb b* **by** *metis*
thus *all-subformula-st test-symb c*
using *propo-rew-step-inv-stay subformula-refl H H' rel one* **by** *auto*
qed
qed

lemma *full-propo-rew-step-inv-stay-conn*:

fixes $r:: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ **and** *test-symb*:: $'v \text{ propo} \Rightarrow \text{bool}$ **and** $x:: 'v$
and $\varphi \psi:: 'v \text{ propo}$
assumes
 $H: \forall \varphi \psi. r \varphi \psi \longrightarrow \text{all-subformula-st test-symb } \varphi \longrightarrow \text{all-subformula-st test-symb } \psi$ **and**
 $H': \forall (c:: 'v \text{ connective}) l l'. \text{wf-conn } c l \longrightarrow \text{wf-conn } c l'$
 $\longrightarrow (\text{test-symb } (\text{conn } c l) \longleftrightarrow \text{test-symb } (\text{conn } c l'))$ **and**
full: *full* (propo-rew-step r) $\varphi \psi$ **and**
init: *all-subformula-st test-symb* φ
shows *all-subformula-st test-symb* ψ

proof –

have $\bigwedge (c:: 'v \text{ connective}) \xi \varphi \xi' \varphi'. \text{wf-conn } c (\xi @ \varphi \# \xi')$
 $\implies \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi')) \implies \text{test-symb } \varphi' \implies \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$
using H' **by** (*metis wf-conn-no-arity-change-helper wf-conn-no-arity-change*)
thus *all-subformula-st test-symb* ψ
using H *full init full-propo-rew-step-inv-stay* **by** *blast*
qed

end

```

theory Prop-Normalisation
imports Main Prop-Logic Prop-Abstract-Transformation ../lib/Multiset-More
begin

```

Given the previous definition about abstract rewriting and theorem about them, we now have the detailed rule making the transformation into CNF/DNF.

8 Rewrite Rules

The idea of Christoph Weidenbach's book is to remove gradually the operators: first equivalencies, then implication, after that the unused true/false and finally the reorganizing the or/and. We will prove each transformation separately.

8.1 Elimination of the equivalences

The first transformation consists in removing every equivalence symbol.

```

inductive elim-equiv :: 'v propo  $\Rightarrow$  'v propo  $\Rightarrow$  bool where
elim-equiv[simp]: elim-equiv (FEq  $\varphi$   $\psi$ ) (FAnd (FImp  $\varphi$   $\psi$ ) (FImp  $\psi$   $\varphi$ ))

```

```

lemma elim-equiv-transformation-consistent:
A  $\models$  FEq  $\varphi$   $\psi \iff A \models$  FAnd (FImp  $\varphi$   $\psi$ ) (FImp  $\psi$   $\varphi$ )
by auto

```

```

lemma elim-equiv-explicit: elim-equiv  $\varphi$   $\psi \implies \forall A. A \models \varphi \iff A \models \psi$ 
by (induct rule: elim-equiv.induct, auto)

```

```

lemma elim-equiv-consistent: preserves-un-sat elim-equiv
unfolding preserves-un-sat-def by (simp add: elim-equiv-explicit)

```

```

lemma elimEquiv-lifted-consistent:
preserves-un-sat (full (propo-rew-step elim-equiv))
by (simp add: elim-equiv-consistent)

```

This function ensures that there is no equivalencies left in the formula tested by *no-equiv-symb*.

```

fun no-equiv-symb :: 'v propo  $\Rightarrow$  bool where
no-equiv-symb (FEq -) = False |
no-equiv-symb - = True

```

Given the definition of *no-equiv-symb*, it does not depend on the formula, but only on the connective used.

```

lemma no-equiv-symb-conn-characterization[simp]:
fixes c :: 'v connective and l :: 'v propo list
assumes wf: wf-conn c l
shows no-equiv-symb (conn c l)  $\iff c \neq$  CEq
by (metis connective.distinct(13,25,35,43) wf no-equiv-symb.elims(3) no-equiv-symb.simps(1)
wf-conn.cases wf-conn-list(6))

```

```

definition no-equiv where no-equiv = all-subformula-st no-equiv-symb

```

```

lemma no-equiv-eq[simp]:
fixes  $\varphi$   $\psi$  :: 'v propo
shows

```

```

    ¬no-equiv (FEq  $\varphi$   $\psi$ )
    no-equiv FT
    no-equiv FF
using no-equiv-symb.simps(1) all-subformula-st-test-symb-true-phi unfolding no-equiv-def by auto

```

The following lemma helps to reconstruct *no-equiv* expressions: this representation is easier to use than the set definition.

lemma *all-subformula-st-decomp-explicit-no-equiv[iff]*:

fixes $\varphi \psi :: 'v$ propo

shows

```

    no-equiv (FNot  $\varphi$ )  $\longleftrightarrow$  no-equiv  $\varphi$ 
    no-equiv (FAnd  $\varphi \psi$ )  $\longleftrightarrow$  (no-equiv  $\varphi \wedge$  no-equiv  $\psi$ )
    no-equiv (FOr  $\varphi \psi$ )  $\longleftrightarrow$  (no-equiv  $\varphi \wedge$  no-equiv  $\psi$ )
    no-equiv (FImp  $\varphi \psi$ )  $\longleftrightarrow$  (no-equiv  $\varphi \wedge$  no-equiv  $\psi$ )
by (auto simp: no-equiv-def)

```

A theorem to show the link between the rewrite relation *elim-equiv* and the function *no-equiv-symb*. This theorem is one of the assumption we need to characterize the transformation.

lemma *no-equiv-elim-equiv-step*:

fixes $\varphi :: 'v$ propo

assumes no-equiv: \neg no-equiv φ

shows $\exists \psi \psi'. \psi \preceq \varphi \wedge$ elim-equiv $\psi \psi'$

proof –

have test-symb-false-nullary:

$\forall x::'v. \text{no-equiv-symb } FF \wedge \text{no-equiv-symb } FT \wedge \text{no-equiv-symb } (FVar\ x)$

unfolding no-equiv-def **by** auto

moreover {

fix $c::'v$ connective **and** $l::'v$ propo list **and** $\psi::'v$ propo

assume a1: elim-equiv (conn c l) ψ

have $\bigwedge p\ pa. \neg$ elim-equiv ($p::'v$ propo) $pa \vee \neg$ no-equiv-symb p

using elim-equiv.cases no-equiv-symb.simps(1) **by** blast

then have elim-equiv (conn c l) $\psi \implies \neg$ no-equiv-symb (conn c l) **using** a1 **by** metis

}

moreover have $H': \forall \psi. \neg$ elim-equiv FT $\psi \forall \psi. \neg$ elim-equiv FF $\psi \forall \psi\ x. \neg$ elim-equiv (FVar x) ψ

using elim-equiv.cases **by** auto

moreover have $\bigwedge \varphi. \neg$ no-equiv-symb $\varphi \implies \exists \psi. \text{elim-equiv } \varphi \psi$

by (case-tac φ , auto simp: elim-equiv.simps)

then have $\bigwedge \varphi'. \varphi' \preceq \varphi \implies \neg$ no-equiv-symb $\varphi' \implies \exists \psi. \text{elim-equiv } \varphi' \psi$ **by** force

ultimately show ?thesis

using no-test-symb-step-exists no-equiv test-symb-false-nullary **unfolding** no-equiv-def **by** blast

qed

Given all the previous theorem and the characterization, once we have rewritten everything, there is no equivalence symbol any more.

lemma *no-equiv-full-propo-rew-step-elim-equiv*:

full (propo-rew-step elim-equiv) $\varphi \psi \implies$ no-equiv ψ

using full-propo-rew-step-subformula no-equiv-elim-equiv-step **by** blast

8.2 Eliminate Implication

After that, we can eliminate the implication symbols.

inductive *elim-imp* $:: 'v$ propo $\Rightarrow 'v$ propo \Rightarrow bool **where**

[simp]: *elim-imp* (FImp $\varphi \psi$) (FOr (FNot φ) ψ)

lemma *elim-imp-transformation-consistent*:
 $A \models FImp \varphi \psi \longleftrightarrow A \models FOr (FNot \varphi) \psi$
by *auto*

lemma *elim-imp-explicit*: $elim-imp \varphi \psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$
by (*induct* $\varphi \psi$ *rule*: *elim-imp.induct*, *auto*)

lemma *elim-imp-consistent*: *preserves-un-sat elim-imp*
unfolding *preserves-un-sat-def* **by** (*simp* *add*: *elim-imp-explicit*)

lemma *elim-imp-lifted-consistent*:
preserves-un-sat (full (propo-rew-step elim-imp))
by (*simp* *add*: *elim-imp-consistent*)

fun *no-imp-symb* **where**
no-imp-symb (*FImp* -) = *False* |
no-imp-symb - = *True*

lemma *no-imp-symb-conn-characterization*:
 $wf-conn \ c \ l \implies no-imp-symb \ (conn \ c \ l) \longleftrightarrow c \neq CImp$
by (*induction* *rule*: *wf-conn-induct*) *auto*

definition *no-imp* **where** *no-imp* $\equiv all-subformula-st \ no-imp-symb$
declare *no-imp-def* [*simp*]

lemma *no-imp-Imp* [*simp*]:
 $\neg no-imp \ (FImp \ \varphi \ \psi)$
no-imp *FT*
no-imp *FF*
unfolding *no-imp-def* **by** *auto*

lemma *all-subformula-st-decomp-explicit-imp* [*simp*]:
fixes $\varphi \ \psi :: 'v \ propo$
shows
 $no-imp \ (FNot \ \varphi) \longleftrightarrow no-imp \ \varphi$
 $no-imp \ (FAnd \ \varphi \ \psi) \longleftrightarrow (no-imp \ \varphi \wedge no-imp \ \psi)$
 $no-imp \ (FOr \ \varphi \ \psi) \longleftrightarrow (no-imp \ \varphi \wedge no-imp \ \psi)$
by *auto*

Invariant of the *elim-imp* transformation

lemma *elim-imp-no-equiv*:
 $elim-imp \ \varphi \ \psi \implies no-equiv \ \varphi \implies no-equiv \ \psi$
by (*induct* $\varphi \ \psi$ *rule*: *elim-imp.induct*, *auto*)

lemma *elim-imp-inv*:
fixes $\varphi \ \psi :: 'v \ propo$
assumes *full (propo-rew-step elim-imp) $\varphi \ \psi$* **and** *no-equiv φ*
shows *no-equiv ψ*
using *full-propo-rew-step-inv-stay-conn* [*of elim-imp no-equiv-symb $\varphi \ \psi$*] *assms elim-imp-no-equiv*
no-equiv-symb-conn-characterization **unfolding** *no-equiv-def* **by** *metis*

lemma *no-no-imp-elim-imp-step-exists*:
fixes $\varphi :: 'v \ propo$
assumes *no-equiv*: $\neg no-imp \ \varphi$
shows $\exists \psi \ \psi'. \ \psi \preceq \varphi \wedge elim-imp \ \psi \ \psi'$

proof –
have *test-symb-false-nullary*: $\forall x. \text{no-imp-symb } FF \wedge \text{no-imp-symb } FT \wedge \text{no-imp-symb } (FVar (x:: 'v))$
by *auto*
moreover {
fix *c*:: 'v *connective* **and** *l*:: 'v *propo list* **and** *ψ* :: 'v *propo*
have *H*: $\text{elim-imp } (\text{conn } c \ l) \ \psi \implies \neg \text{no-imp-symb } (\text{conn } c \ l)$
by (*auto elim: elim-imp.cases*)
}
moreover
have *H'*: $\forall \psi. \neg \text{elim-imp } FT \ \psi \ \forall \psi. \neg \text{elim-imp } FF \ \psi \ \forall \psi \ x. \neg \text{elim-imp } (FVar \ x) \ \psi$
by (*auto elim: elim-imp.cases*) +
moreover
have $\bigwedge \varphi. \neg \text{no-imp-symb } \varphi \implies \exists \psi. \text{elim-imp } \varphi \ \psi$
by (*case-tac φ*) (*force simp: elim-imp.simps*) +
then have $(\bigwedge \varphi'. \varphi' \preceq \varphi \implies \neg \text{no-imp-symb } \varphi' \implies \exists \psi. \text{elim-imp } \varphi' \ \psi)$ **by** *force*
ultimately show *?thesis*
using *no-test-symb-step-exists no-equiv test-symb-false-nullary unfolding no-imp-def* **by** *blast*
qed

lemma *no-imp-full-propo-rew-step-elim-imp*: $\text{full } (\text{propo-rew-step elim-imp}) \ \varphi \ \psi \implies \text{no-imp } \psi$
using *full-propo-rew-step-subformula no-no-imp-elim-imp-step-exists* **by** *blast*

8.3 Eliminate all the True and False in the formula

Contrary to the book, we have to give the transformation and the “commutative” transformation. The latter is implicit in the book.

inductive *elimTB* **where**

ElimTB1: $\text{elimTB } (FAnd \ \varphi \ FT) \ \varphi \mid$
ElimTB1': $\text{elimTB } (FAnd \ FT \ \varphi) \ \varphi \mid$

ElimTB2: $\text{elimTB } (FAnd \ \varphi \ FF) \ FF \mid$
ElimTB2': $\text{elimTB } (FAnd \ FF \ \varphi) \ FF \mid$

ElimTB3: $\text{elimTB } (FOr \ \varphi \ FT) \ FT \mid$
ElimTB3': $\text{elimTB } (FOr \ FT \ \varphi) \ FT \mid$

ElimTB4: $\text{elimTB } (FOr \ \varphi \ FF) \ \varphi \mid$
ElimTB4': $\text{elimTB } (FOr \ FF \ \varphi) \ \varphi \mid$

ElimTB5: $\text{elimTB } (FNot \ FT) \ FF \mid$
ElimTB6: $\text{elimTB } (FNot \ FF) \ FT$

lemma *elimTB-consistent*: *preserves-un-sat elimTB*

proof –
{
fix *$\varphi \ \psi$* :: 'b *propo*
have $\text{elimTB } \varphi \ \psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$ **by** (*induction rule: elimTB.inducts*) *auto*
}
then show *?thesis* **using** *preserves-un-sat-def* **by** *auto*
qed

inductive *no-T-F-symb* :: 'v *propo* \Rightarrow *bool* **where**

no-T-F-symb-comp: $c \neq CF \implies c \neq CT \implies \text{wf-conn } c \ l \implies (\forall \varphi \in \text{set } l. \varphi \neq FT \wedge \varphi \neq FF)$
 $\implies \text{no-T-F-symb } (\text{conn } c \ l)$

lemma *wf-conn-no-T-F-symb-iff*[simp]:

wf-conn *c* ψ *s* \implies
 $\text{no-T-F-symb } (\text{conn } c \ \psi s) \longleftrightarrow (c \neq CF \wedge c \neq CT \wedge (\forall \psi \in \text{set } \psi s. \psi \neq FF \wedge \psi \neq FT))$
unfolding *no-T-F-symb.simps* **apply** (*cases* *c*)
using *wf-conn-list*(1) **apply** *fastforce*
using *wf-conn-list*(2) **apply** *fastforce*
using *wf-conn-list*(3) **apply** *fastforce*
apply (*metis* (*no-types*, *hide-lams*) *conn-inj connective.distinct*(5,17))
using *conn-inj* **apply** *blast* +
done

lemma *wf-conn-no-T-F-symb-iff-explicit*[simp]:

$\text{no-T-F-symb } (FAnd \ \varphi \ \psi) \longleftrightarrow (\forall \chi \in \text{set } [\varphi, \psi]. \chi \neq FF \wedge \chi \neq FT)$
 $\text{no-T-F-symb } (FOr \ \varphi \ \psi) \longleftrightarrow (\forall \chi \in \text{set } [\varphi, \psi]. \chi \neq FF \wedge \chi \neq FT)$
 $\text{no-T-F-symb } (FEq \ \varphi \ \psi) \longleftrightarrow (\forall \chi \in \text{set } [\varphi, \psi]. \chi \neq FF \wedge \chi \neq FT)$
 $\text{no-T-F-symb } (FImp \ \varphi \ \psi) \longleftrightarrow (\forall \chi \in \text{set } [\varphi, \psi]. \chi \neq FF \wedge \chi \neq FT)$
apply (*metis* *conn.simps*(36) *conn.simps*(37) *conn.simps*(5) *propo.distinct*(19)
wf-conn-helper-facts(5) *wf-conn-no-T-F-symb-iff*)
apply (*metis* *conn.simps*(36) *conn.simps*(37) *conn.simps*(6) *propo.distinct*(22)
wf-conn-helper-facts(6) *wf-conn-no-T-F-symb-iff*)
using *wf-conn-no-T-F-symb-iff* **apply** *fastforce*
by (*metis* *conn.simps*(36) *conn.simps*(37) *conn.simps*(7) *propo.distinct*(23) *wf-conn-helper-facts*(7)
wf-conn-no-T-F-symb-iff)

lemma *no-T-F-symb-false*[simp]:

fixes *c* :: '*v* *connective*
shows
 $\neg \text{no-T-F-symb } (FT :: 'v \text{ propo})$
 $\neg \text{no-T-F-symb } (FF :: 'v \text{ propo})$
by (*metis* (*no-types*) *conn.simps*(1,2) *wf-conn-no-T-F-symb-iff* *wf-conn-nullary*) +

lemma *no-T-F-symb-bool*[simp]:

fixes *x* :: '*v*
shows *no-T-F-symb* (*FVar* *x*)
using *no-T-F-symb-comp* *wf-conn-nullary* **by** (*metis* *connective.distinct*(3, 15) *conn.simps*(3)
empty-iff *list.set*(1))

lemma *no-T-F-symb-fnot-imp*:

$\neg \text{no-T-F-symb } (FNot \ \varphi) \implies \varphi = FT \vee \varphi = FF$

proof (*rule* *ccontr*)

assume *n*: $\neg \text{no-T-F-symb } (FNot \ \varphi)$

assume $\neg (\varphi = FT \vee \varphi = FF)$

then have $\forall \varphi' \in \text{set } [\varphi]. \varphi' \neq FT \wedge \varphi' \neq FF$ **by** *auto*

moreover have *wf-conn* *CNot* $[\varphi]$ **by** *simp*

ultimately have *no-T-F-symb* (*FNot* φ)

using *no-T-F-symb.intros* **by** (*metis* *conn.simps*(4) *connective.distinct*(5,17))

then show *False* **using** *n* **by** *blast*

qed

lemma *no-T-F-symb-fnot*[simp]:

$\text{no-T-F-symb } (FNot \ \varphi) \longleftrightarrow \neg (\varphi = FT \vee \varphi = FF)$

using *no-T-F-symb.simps no-T-F-symb-fnot-imp* **by** (*metis conn-inj-not(2) list.set-intros(1)*)

Actually it is not possible to remove every *FT* and *FF*: if the formula is equal to true or false, we can not remove it.

inductive *no-T-F-symb-except-toplevel* **where**

no-T-F-symb-except-toplevel-true[*simp*]: *no-T-F-symb-except-toplevel FT* |

no-T-F-symb-except-toplevel-false[*simp*]: *no-T-F-symb-except-toplevel FF* |

noTrue-no-T-F-symb-except-toplevel[*simp*]: *no-T-F-symb $\varphi \implies$ no-T-F-symb-except-toplevel φ*

lemma *no-T-F-symb-except-toplevel-bool*:

fixes *x* :: 'v

shows *no-T-F-symb-except-toplevel (FVar x)*

by *simp*

lemma *no-T-F-symb-except-toplevel-not-decom*:

$\varphi \neq FT \implies \varphi \neq FF \implies$ no-T-F-symb-except-toplevel (FNot φ)

by *simp*

lemma *no-T-F-symb-except-toplevel-bin-decom*:

fixes *$\varphi \psi$* :: 'v *propo*

assumes *$\varphi \neq FT$ and $\varphi \neq FF$ and $\psi \neq FT$ and $\psi \neq FF$*

and *c*: *c* ∈ *binary-connectives*

shows *no-T-F-symb-except-toplevel (conn c [φ , ψ])*

by (*metis (no-types, lifting) assms c conn.simps(4) list.discI noTrue-no-T-F-symb-except-toplevel*

wf-conn-no-T-F-symb-iff no-T-F-symb-fnot set.ConsD wf-conn-binary wf-conn-helper-facts(1)

wf-conn-list-decomp(1,2))

lemma *no-T-F-symb-except-toplevel-if-is-a-true-false*:

fixes *l* :: 'v *propo list* **and** *c* :: 'v *connective*

assumes *corr*: *wf-conn c l*

and *FT* ∈ *set l* ∨ *FF* ∈ *set l*

shows *\neg no-T-F-symb-except-toplevel (conn c l)*

by (*metis assms empty-iff no-T-F-symb-except-toplevel.simps wf-conn-no-T-F-symb-iff set-empty*

wf-conn-list(1,2))

lemma *no-T-F-symb-except-top-level-false-example*[*simp*]:

fixes *$\varphi \psi$* :: 'v *propo*

assumes *$\varphi = FT \vee \psi = FT \vee \varphi = FF \vee \psi = FF$*

shows

\neg no-T-F-symb-except-toplevel (FAnd $\varphi \psi$)

\neg no-T-F-symb-except-toplevel (FOr $\varphi \psi$)

\neg no-T-F-symb-except-toplevel (FImp $\varphi \psi$)

\neg no-T-F-symb-except-toplevel (FEq $\varphi \psi$)

using *assms no-T-F-symb-except-toplevel-if-is-a-true-false unfolding binary-connectives-def*

by (*metis (no-types) conn.simps(5-8) insert-iff list.simps(14-15) wf-conn-helper-facts(5-8)*) +

lemma *no-T-F-symb-except-top-level-false-not*[*simp*]:

fixes *$\varphi \psi$* :: 'v *propo*

assumes *$\varphi = FT \vee \varphi = FF$*

shows

\neg no-T-F-symb-except-toplevel (FNot φ)

by (*simp add: assms no-T-F-symb-except-toplevel.simps*)

This is the local extension of *no-T-F-symb-except-toplevel*.

definition *no-T-F-except-top-level* **where**

no-T-F-except-top-level \equiv *all-subformula-st no-T-F-symb-except-toplevel*

This is another property we will use. While this version might seem to be the one we want to prove, it is not since *FT* can not be reduced.

definition *no-T-F* **where**

no-T-F \equiv *all-subformula-st no-T-F-symb*

lemma *no-T-F-except-top-level-false*:

fixes *l* :: 'v propo list **and** *c* :: 'v connective

assumes *wf-conn c l*

and *FT* \in *set l* \vee *FF* \in *set l*

shows \neg *no-T-F-except-top-level* (*conn c l*)

by (*simp add: all-subformula-st-decomp assms no-T-F-except-top-level-def no-T-F-symb-except-toplevel-if-is-a-true-false*)

lemma *no-T-F-except-top-level-false-example*[*simp*]:

fixes $\varphi \psi$:: 'v propo

assumes $\varphi = FT \vee \psi = FT \vee \varphi = FF \vee \psi = FF$

shows

\neg *no-T-F-except-top-level* (*FAnd* $\varphi \psi$)

\neg *no-T-F-except-top-level* (*FOr* $\varphi \psi$)

\neg *no-T-F-except-top-level* (*FEq* $\varphi \psi$)

\neg *no-T-F-except-top-level* (*FImp* $\varphi \psi$)

by (*metis all-subformula-st-test-symb-true-phi assms no-T-F-except-top-level-def no-T-F-symb-except-top-level-false-example*)+

lemma *no-T-F-symb-except-toplevel-no-T-F-symb*:

no-T-F-symb-except-toplevel $\varphi \implies \varphi \neq FF \implies \varphi \neq FT \implies$ *no-T-F-symb* φ

by (*induct rule: no-T-F-symb-except-toplevel.induct, auto*)

The two following lemmas give the precise link between the two definitions.

lemma *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb*:

no-T-F-except-top-level $\varphi \implies \varphi \neq FF \implies \varphi \neq FT \implies$ *no-T-F* φ

unfolding *no-T-F-except-top-level-def no-T-F-def* **apply** (*induct* φ)

using *no-T-F-symb-fnot* **by** *fastforce*+

lemma *no-T-F-no-T-F-except-top-level*:

no-T-F $\varphi \implies$ *no-T-F-except-top-level* φ

unfolding *no-T-F-except-top-level-def no-T-F-def*

unfolding *all-subformula-st-def* **by** *auto*

lemma *no-T-F-except-top-level-simp*[*simp*]: *no-T-F-except-top-level* *FF* *no-T-F-except-top-level* *FT*

unfolding *no-T-F-except-top-level-def* **by** *auto*

lemma *no-T-F-no-T-F-except-top-level'*[*simp*]:

no-T-F-except-top-level $\varphi \longleftrightarrow (\varphi = FF \vee \varphi = FT \vee$ *no-T-F* $\varphi)$

using *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb no-T-F-no-T-F-except-top-level* **by** *auto*

lemma *no-T-F-bin-decomp*[*simp*]:

assumes *c*: *c* \in *binary-connectives*

shows *no-T-F* (*conn c* [φ , ψ]) \longleftrightarrow (*no-T-F* $\varphi \wedge$ *no-T-F* ψ)

proof –

```

have wf: wf-conn c [ $\varphi$ ,  $\psi$ ] using c by auto
then have no-T-F (conn c [ $\varphi$ ,  $\psi$ ])  $\longleftrightarrow$  (no-T-F-symb (conn c [ $\varphi$ ,  $\psi$ ])  $\wedge$  no-T-F  $\varphi \wedge$  no-T-F  $\psi$ )
  by (simp add: all-subformula-st-decomp no-T-F-def)
then show no-T-F (conn c [ $\varphi$ ,  $\psi$ ])  $\longleftrightarrow$  (no-T-F  $\varphi \wedge$  no-T-F  $\psi$ )
  using c wf all-subformula-st-decomp list.discI no-T-F-def no-T-F-symb-except-toplevel-bin-decom
    no-T-F-symb-except-toplevel-no-T-F-symb no-T-F-symb-false(1,2) wf-conn-helper-facts(2,3)
    wf-conn-list(1,2) by metis
qed

```

```

lemma no-T-F-bin-decomp-expanded[simp]:
  assumes c: c = CAnd  $\vee$  c = COr  $\vee$  c = CEq  $\vee$  c = CImp
  shows no-T-F (conn c [ $\varphi$ ,  $\psi$ ])  $\longleftrightarrow$  (no-T-F  $\varphi \wedge$  no-T-F  $\psi$ )
  using no-T-F-bin-decomp assms unfolding binary-connectives-def by blast

```

```

lemma no-T-F-comp-expanded-explicit[simp]:
  fixes  $\varphi \psi :: 'v$  propo
  shows
    no-T-F (FAnd  $\varphi \psi$ )  $\longleftrightarrow$  (no-T-F  $\varphi \wedge$  no-T-F  $\psi$ )
    no-T-F (FOr  $\varphi \psi$ )  $\longleftrightarrow$  (no-T-F  $\varphi \wedge$  no-T-F  $\psi$ )
    no-T-F (FEq  $\varphi \psi$ )  $\longleftrightarrow$  (no-T-F  $\varphi \wedge$  no-T-F  $\psi$ )
    no-T-F (FImp  $\varphi \psi$ )  $\longleftrightarrow$  (no-T-F  $\varphi \wedge$  no-T-F  $\psi$ )
  using assms conn.simps(5-8) no-T-F-bin-decomp-expanded by (metis (no-types))+

```

```

lemma no-T-F-comp-not[simp]:
  fixes  $\varphi \psi :: 'v$  propo
  shows no-T-F (FNot  $\varphi$ )  $\longleftrightarrow$  no-T-F  $\varphi$ 
  by (metis all-subformula-st-decomp-explicit(3) all-subformula-st-test-symb-true-phi no-T-F-def
    no-T-F-symb-false(1,2) no-T-F-symb-fnot-imp)

```

```

lemma no-T-F-decomp:
  fixes  $\varphi \psi :: 'v$  propo
  assumes  $\varphi$ : no-T-F (FAnd  $\varphi \psi$ )  $\vee$  no-T-F (FOr  $\varphi \psi$ )  $\vee$  no-T-F (FEq  $\varphi \psi$ )  $\vee$  no-T-F (FImp  $\varphi \psi$ )
  shows no-T-F  $\psi$  and no-T-F  $\varphi$ 
  using assms by auto

```

```

lemma no-T-F-decomp-not:
  fixes  $\varphi :: 'v$  propo
  assumes  $\varphi$ : no-T-F (FNot  $\varphi$ )
  shows no-T-F  $\varphi$ 
  using assms by auto

```

```

lemma no-T-F-symb-except-toplevel-step-exists:
  fixes  $\varphi \psi :: 'v$  propo
  assumes no-equiv  $\varphi$  and no-imp  $\varphi$ 
  shows  $\psi \preceq \varphi \implies \neg$  no-T-F-symb-except-toplevel  $\psi \implies \exists \psi'. \text{elimTB } \psi \psi'$ 
proof (induct  $\psi$  rule: propo-induct-arity)
  case (nullary  $\varphi' x$ )
  then have False using no-T-F-symb-except-toplevel-true no-T-F-symb-except-toplevel-false by auto
  then show ?case by blast
next
  case (unary  $\psi$ )
  then have  $\psi = FF \vee \psi = FT$  using no-T-F-symb-except-toplevel-not-decom by blast
  then show ?case using ElimTB5 ElimTB6 by blast
next
  case (binary  $\varphi' \psi1 \psi2$ )

```

```

note IH1 = this(1) and IH2 = this(2) and  $\varphi' = \text{this}(3)$  and  $F\varphi = \text{this}(4)$  and  $n = \text{this}(5)$ 
{
  assume  $\varphi' = FImp\ \psi1\ \psi2 \vee \varphi' = FEq\ \psi1\ \psi2$ 
  then have False using n F $\varphi$  subformula-all-subformula-st assms
    by (metis (no-types) no-equiv-eq(1) no-equiv-def no-imp-Imp(1) no-imp-def)
  then have ?case by blast
}
moreover {
  assume  $\varphi'$ :  $\varphi' = FAnd\ \psi1\ \psi2 \vee \varphi' = FOr\ \psi1\ \psi2$ 
  then have  $\psi1 = FT \vee \psi2 = FT \vee \psi1 = FF \vee \psi2 = FF$ 
    using no-T-F-symb-except-toplevel-bin-decom conn.simps(5,6) n unfolding binary-connectives-def
    by fastforce+
  then have ?case using elimTB.intros  $\varphi'$  by blast
}
ultimately show ?case using  $\varphi'$  by blast
qed

```

lemma no-T-F-except-top-level-rew:

```

fixes  $\varphi :: 'v\ propo$ 
assumes noTB:  $\neg \text{no-T-F-except-top-level}\ \varphi$  and no-equiv: no-equiv  $\varphi$  and no-imp: no-imp  $\varphi$ 
shows  $\exists \psi\ \psi'. \psi \preceq \varphi \wedge \text{elimTB}\ \psi\ \psi'$ 
proof -
  have test-symb-false-nullary:  $\forall x. \text{no-T-F-symb-except-toplevel}\ (FF:: 'v\ propo)$ 
     $\wedge \text{no-T-F-symb-except-toplevel}\ FT \wedge \text{no-T-F-symb-except-toplevel}\ (FVar\ (x:: 'v))$  by auto
  moreover {
    fix c:: 'v connective and l:: 'v propo list and  $\psi :: 'v\ propo$ 
    have H:  $\text{elimTB}\ (\text{conn}\ c\ l)\ \psi \implies \neg \text{no-T-F-symb-except-toplevel}\ (\text{conn}\ c\ l)$ 
      by (cases (conn c l) rule: elimTB.cases, auto)
  }
  moreover {
    fix x:: 'v
    have H':  $\text{no-T-F-except-top-level}\ FT\ \text{no-T-F-except-top-level}\ FF$ 
       $\text{no-T-F-except-top-level}\ (FVar\ x)$ 
      by (auto simp: no-T-F-except-top-level-def test-symb-false-nullary)
  }
  moreover {
    fix  $\psi$ 
    have  $\psi \preceq \varphi \implies \neg \text{no-T-F-symb-except-toplevel}\ \psi \implies \exists \psi'. \text{elimTB}\ \psi\ \psi'$ 
      using no-T-F-symb-except-toplevel-step-exists no-equiv no-imp by auto
  }
  ultimately show ?thesis
    using no-test-symb-step-exists noTB unfolding no-T-F-except-top-level-def by blast
qed

```

lemma elimTB-inv:

```

fixes  $\varphi\ \psi :: 'v\ propo$ 
assumes full (propo-rew-step elimTB)  $\varphi\ \psi$ 
and no-equiv  $\varphi$  and no-imp  $\varphi$ 
shows no-equiv  $\psi$  and no-imp  $\psi$ 
proof -
  {
    fix  $\varphi\ \psi :: 'v\ propo$ 
    have H:  $\text{elimTB}\ \varphi\ \psi \implies \text{no-equiv}\ \varphi \implies \text{no-equiv}\ \psi$ 
      by (induct  $\varphi\ \psi$  rule: elimTB.induct, auto)
  }

```

```

then show no-equiv  $\psi$ 
  using full-propo-rew-step-inv-stay-conn[of elimTB no-equiv-symb  $\varphi \psi$ ]
    no-equiv-symb-conn-characterization assms unfolding no-equiv-def by metis
next
{
  fix  $\varphi \psi :: 'v \text{ propo}$ 
  have  $H: \text{elimTB } \varphi \psi \implies \text{no-imp } \varphi \implies \text{no-imp } \psi$ 
    by (induct  $\varphi \psi$  rule: elimTB.induct, auto)
}
then show no-imp  $\psi$ 
  using full-propo-rew-step-inv-stay-conn[of elimTB no-imp-symb  $\varphi \psi$ ] assms
    no-imp-symb-conn-characterization unfolding no-imp-def by metis
qed

```

```

lemma elimTB-full-propo-rew-step:
  fixes  $\varphi \psi :: 'v \text{ propo}$ 
  assumes no-equiv  $\varphi$  and no-imp  $\varphi$  and full (propo-rew-step elimTB)  $\varphi \psi$ 
  shows no-T-F-except-top-level  $\psi$ 
  using full-propo-rew-step-subformula no-T-F-except-top-level-rew assms elimTB-inv by fastforce

```

8.4 PushNeg

Push the negation inside the formula, until the litteral.

```

inductive pushNeg where
  PushNeg1[simp]: pushNeg (FNot (FAnd  $\varphi \psi$ )) (FOr (FNot  $\varphi$ ) (FNot  $\psi$ )) |
  PushNeg2[simp]: pushNeg (FNot (FOr  $\varphi \psi$ )) (FAnd (FNot  $\varphi$ ) (FNot  $\psi$ )) |
  PushNeg3[simp]: pushNeg (FNot (FNot  $\varphi$ ))  $\varphi$ 

```

```

lemma pushNeg-transformation-consistent:
   $A \models \text{FNot } (\text{FAnd } \varphi \psi) \longleftrightarrow A \models (\text{FOr } (\text{FNot } \varphi) (\text{FNot } \psi))$ 
   $A \models \text{FNot } (\text{FOr } \varphi \psi) \longleftrightarrow A \models (\text{FAnd } (\text{FNot } \varphi) (\text{FNot } \psi))$ 
   $A \models \text{FNot } (\text{FNot } \varphi) \longleftrightarrow A \models \varphi$ 
  by auto

```

```

lemma pushNeg-explicit: pushNeg  $\varphi \psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$ 
  by (induct  $\varphi \psi$  rule: pushNeg.induct, auto)

```

```

lemma pushNeg-consistent: preserves-un-sat pushNeg
  unfolding preserves-un-sat-def by (simp add: pushNeg-explicit)

```

```

lemma pushNeg-lifted-consistant:
  preserves-un-sat (full (propo-rew-step pushNeg))
  by (simp add: pushNeg-consistent)

```

```

fun simple where
  simple FT = True |
  simple FF = True |
  simple (FVar -) = True |
  simple - = False

```

```

lemma simple-decomp:
   $\text{simple } \varphi \longleftrightarrow (\varphi = \text{FT} \vee \varphi = \text{FF} \vee (\exists x. \varphi = \text{FVar } x))$ 

```

```

by (cases  $\varphi$ ) auto

lemma subformula-conn-decomp-simple:
  fixes  $\varphi \ \psi :: 'v \text{ propo}$ 
  assumes  $s: \text{simple } \psi$ 
  shows  $\varphi \preceq \text{FNot } \psi \longleftrightarrow (\varphi = \text{FNot } \psi \vee \varphi = \psi)$ 
proof -
  have  $\varphi \preceq \text{conn CNot } [\psi] \longleftrightarrow (\varphi = \text{conn CNot } [\psi] \vee (\exists \psi \in \text{set } [\psi]. \varphi \preceq \psi))$ 
    using subformula-conn-decomp wf-conn-helper-facts(1) by metis
  then show  $\varphi \preceq \text{FNot } \psi \longleftrightarrow (\varphi = \text{FNot } \psi \vee \varphi = \psi)$  using  $s$  by (auto simp: simple-decomp)
qed

lemma subformula-conn-decomp-explicit[simp]:
  fixes  $\varphi :: 'v \text{ propo}$  and  $x :: 'v$ 
  shows
     $\varphi \preceq \text{FNot } \text{FT} \longleftrightarrow (\varphi = \text{FNot } \text{FT} \vee \varphi = \text{FT})$ 
     $\varphi \preceq \text{FNot } \text{FF} \longleftrightarrow (\varphi = \text{FNot } \text{FF} \vee \varphi = \text{FF})$ 
     $\varphi \preceq \text{FNot } (\text{FVar } x) \longleftrightarrow (\varphi = \text{FNot } (\text{FVar } x) \vee \varphi = \text{FVar } x)$ 
  by (auto simp: subformula-conn-decomp-simple)

fun simple-not-symb where
  simple-not-symb (FNot  $\varphi$ ) = (simple  $\varphi$ ) |
  simple-not-symb - = True

definition simple-not where
  simple-not = all-subformula-st simple-not-symb
declare simple-not-def[simp]

lemma simple-not-Not[simp]:
   $\neg \text{simple-not } (\text{FNot } (\text{FAnd } \varphi \ \psi))$ 
   $\neg \text{simple-not } (\text{FNot } (\text{FOr } \varphi \ \psi))$ 
  by auto

lemma simple-not-step-exists:
  fixes  $\varphi \ \psi :: 'v \text{ propo}$ 
  assumes  $\text{no-equiv } \varphi$  and  $\text{no-imp } \varphi$ 
  shows  $\psi \preceq \varphi \implies \neg \text{simple-not-symb } \psi \implies \exists \psi'. \text{pushNeg } \psi \ \psi'$ 
  apply (induct  $\psi$ , auto)
  apply (rename-tac  $\psi$ , case-tac  $\psi$ , auto intro: pushNeg.intros)
  by (metis assms(1,2) no-imp-Imp(1) no-equiv-eq(1) no-imp-def no-equiv-def
      subformula-in-subformula-not subformula-all-subformula-st)+

lemma simple-not-rew:
  fixes  $\varphi :: 'v \text{ propo}$ 
  assumes  $\text{noTB}: \neg \text{simple-not } \varphi$  and  $\text{no-equiv}: \text{no-equiv } \varphi$  and  $\text{no-imp}: \text{no-imp } \varphi$ 
  shows  $\exists \psi \ \psi'. \psi \preceq \varphi \wedge \text{pushNeg } \psi \ \psi'$ 
proof -
  have  $\forall x. \text{simple-not-symb } (\text{FF}:: 'v \text{ propo}) \wedge \text{simple-not-symb } \text{FT} \wedge \text{simple-not-symb } (\text{FVar } (x:: 'v))$ 
    by auto
  moreover {
    fix  $c:: 'v \text{ connective}$  and  $l :: 'v \text{ propo list}$  and  $\psi :: 'v \text{ propo}$ 
    have  $H: \text{pushNeg } (\text{conn } c \ l) \ \psi \implies \neg \text{simple-not-symb } (\text{conn } c \ l)$ 
      by (cases (conn  $c \ l$ ) rule: pushNeg.cases) auto
  }

```



```

moreover {
  fix  $x :: 'v$ 
  have  $H': \text{simple-not } FT \text{ simple-not } FF \text{ simple-not } (FVar\ x)$ 
  by simp-all
}
moreover {
  fix  $\psi :: 'v \text{ propo}$ 
  have  $\psi \preceq \varphi \implies \neg \text{simple-not-symb } \psi \implies \exists \psi'. \text{pushNeg } \psi \psi'$ 
  using simple-not-step-exists no-equiv no-imp by blast
}
ultimately show ?thesis using no-test-symb-step-exists noTB unfolding simple-not-def by blast
qed

```

lemma *no-T-F-except-top-level-pushNeg1*:

```

 $\text{no-T-F-except-top-level } (FNot\ (FAnd\ \varphi\ \psi)) \implies \text{no-T-F-except-top-level } (FOr\ (FNot\ \varphi)\ (FNot\ \psi))$ 
using no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb no-T-F-comp-not no-T-F-decomp(1)
 $\text{no-T-F-decomp(2) no-T-F-no-T-F-except-top-level}$  by (metis no-T-F-comp-expanded-explicit(2)
 $\text{propo.distinct(5,17)}$ )

```

lemma *no-T-F-except-top-level-pushNeg2*:

```

 $\text{no-T-F-except-top-level } (FNot\ (FOr\ \varphi\ \psi)) \implies \text{no-T-F-except-top-level } (FAnd\ (FNot\ \varphi)\ (FNot\ \psi))$ 
by auto

```

lemma *no-T-F-symb-pushNeg*:

```

 $\text{no-T-F-symb } (FOr\ (FNot\ \varphi')\ (FNot\ \psi'))$ 
 $\text{no-T-F-symb } (FAnd\ (FNot\ \varphi')\ (FNot\ \psi'))$ 
 $\text{no-T-F-symb } (FNot\ (FNot\ \varphi'))$ 
by auto

```

lemma *propo-rew-step-pushNeg-no-T-F-symb*:

```

 $\text{propo-rew-step pushNeg } \varphi\ \psi \implies \text{no-T-F-except-top-level } \varphi \implies \text{no-T-F-symb } \varphi \implies \text{no-T-F-symb } \psi$ 
apply (induct rule: propo-rew-step.induct)
apply (cases rule: pushNeg.cases)
apply simp-all
apply (metis no-T-F-symb-pushNeg(1))
apply (metis no-T-F-symb-pushNeg(2))
apply (simp, metis all-subformula-st-test-symb-true-phi no-T-F-def)

```

proof –

```

fix  $\varphi\ \varphi':: 'a \text{ propo}$  and  $c:: 'a \text{ connective}$  and  $\xi\ \xi':: 'a \text{ propo list}$ 
assume rel: propo-rew-step pushNeg  $\varphi\ \varphi'$ 
and IH:  $\text{no-T-F } \varphi \implies \text{no-T-F-symb } \varphi \implies \text{no-T-F-symb } \varphi'$ 
and wf:  $\text{wf-conn } c\ (\xi @ \varphi \# \xi')$ 
and  $n: \text{conn } c\ (\xi @ \varphi \# \xi') = FF \vee \text{conn } c\ (\xi @ \varphi \# \xi') = FT \vee \text{no-T-F } (\text{conn } c\ (\xi @ \varphi \# \xi'))$ 
and  $x: c \neq CF \wedge c \neq CT \wedge \varphi \neq FF \wedge \varphi \neq FT \wedge (\forall \psi \in \text{set } \xi \cup \text{set } \xi'. \psi \neq FF \wedge \psi \neq FT)$ 
then have  $c \neq CF \wedge c \neq CT \wedge \text{wf-conn } c\ (\xi @ \varphi' \# \xi')$ 
  using wf-conn-no-arity-change-helper wf-conn-no-arity-change by metis
moreover have  $n': \text{no-T-F } (\text{conn } c\ (\xi @ \varphi' \# \xi'))$  using  $n$  by (simp add: wf wf-conn-list(1,2))
moreover
{
  have  $\text{no-T-F } \varphi$ 
  by (metis Un-iff all-subformula-st-decomp list.set-intros(1) n' wf no-T-F-def set-append)
  moreover then have  $\text{no-T-F-symb } \varphi$ 
  by (simp add: all-subformula-st-test-symb-true-phi no-T-F-def)
  ultimately have  $\varphi' \neq FF \wedge \varphi' \neq FT$ 
  using IH no-T-F-symb-false(1) no-T-F-symb-false(2) by blast
}

```

then have $\forall \psi \in \text{set } (\xi @ \varphi' \# \xi'). \psi \neq FF \wedge \psi \neq FT$ using x by *auto*
 }
 ultimately show *no-T-F-symb* (*conn* c ($\xi @ \varphi' \# \xi'$)) by (*simp add: x*)
 qed

lemma *propo-rew-step-pushNeg-no-T-F*:

propo-rew-step pushNeg $\varphi \psi \implies \text{no-T-F } \varphi \implies \text{no-T-F } \psi$

proof (*induct rule: propo-rew-step.induct*)

case *global-rel*

then show *?case*

by (*metis* (*no-types, lifting*) *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb*
no-T-F-def no-T-F-except-top-level-pushNeg1 no-T-F-except-top-level-pushNeg2
no-T-F-no-T-F-except-top-level all-subformula-st-decomp-explicit(3) pushNeg.simps
simple.simps(1,2,5,6))

next

case (*propo-rew-one-step-lift* $\varphi \varphi' c \xi \xi'$)

note *rel = this(1)* and *IH = this(2)* and *wf = this(3)* and *no-T-F = this(4)*

moreover have *wf'*: *wf-conn* c ($\xi @ \varphi' \# \xi'$)

using *wf-conn-no-arity-change wf-conn-no-arity-change-helper wf* by *metis*

ultimately show *no-T-F* (*conn* c ($\xi @ \varphi' \# \xi'$))

using *all-subformula-st-test-symb-true-phi*

by (*fastforce simp: no-T-F-def all-subformula-st-decomp wf wf'*)

qed

lemma *pushNeg-inv*:

fixes $\varphi \psi :: 'v \text{ propo}$

assumes *full* (*propo-rew-step pushNeg*) $\varphi \psi$

and *no-equiv* φ and *no-imp* φ and *no-T-F-except-top-level* φ

shows *no-equiv* ψ and *no-imp* ψ and *no-T-F-except-top-level* ψ

proof –

{

fix $\varphi \psi :: 'v \text{ propo}$

assume *rel*: *propo-rew-step pushNeg* $\varphi \psi$

and *no*: *no-T-F-except-top-level* φ

then have *no-T-F-except-top-level* ψ

proof –

{

assume $\varphi = FT \vee \varphi = FF$

from *rel* **this** **have** *False*

apply (*induct rule: propo-rew-step.induct*)

using *pushNeg.cases* **apply** *blast*

using *wf-conn-list(1) wf-conn-list(2)* **by** *auto*

then have *no-T-F-except-top-level* ψ **by** *blast*

}

moreover {

assume $\varphi \neq FT \wedge \varphi \neq FF$

then have *no-T-F* φ

by (*metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb*)

then have *no-T-F* ψ

using *propo-rew-step-pushNeg-no-T-F rel* **by** *auto*

then have *no-T-F-except-top-level* ψ **by** (*simp add: no-T-F-no-T-F-except-top-level*)

}

ultimately show *no-T-F-except-top-level* ψ **by** *metis*

qed

```

}
moreover {
  fix c :: 'v connective and  $\xi \xi' :: 'v \text{ propo list}$  and  $\zeta \zeta' :: 'v \text{ propo}$ 
  assume rel: propo-rew-step pushNeg  $\zeta \zeta'$ 
  and incl:  $\zeta \preceq \varphi$ 
  and corr: wf-conn c ( $\xi @ \zeta \# \xi'$ )
  and no-T-F: no-T-F-symb-except-toplevel (conn c ( $\xi @ \zeta \# \xi'$ ))
  and n: no-T-F-symb-except-toplevel  $\zeta'$ 
  have no-T-F-symb-except-toplevel (conn c ( $\xi @ \zeta' \# \xi'$ ))
  proof
    have p: no-T-F-symb (conn c ( $\xi @ \zeta \# \xi'$ ))
      using corr wf-conn-list(1) wf-conn-list(2) no-T-F-symb-except-toplevel-no-T-F-symb no-T-F
      by blast
    have l:  $\forall \varphi \in \text{set } (\xi @ \zeta \# \xi'). \varphi \neq FT \wedge \varphi \neq FF$ 
      using corr wf-conn-no-T-F-symb-iff p by blast
    from rel incl have  $\zeta' \neq FT \wedge \zeta' \neq FF$ 
    apply (induction  $\zeta \zeta'$  rule: propo-rew-step.induct)
    apply (cases rule: pushNeg.cases, auto)
    by (metis assms(4) no-T-F-symb-except-top-level-false-not no-T-F-except-top-level-def
      all-subformula-st-test-symb-true-phi subformula-in-subformula-not
      subformula-all-subformula-st append-is-Nil-conv list.distinct(1)
      wf-conn-no-arity-change-helper wf-conn-list(1,2) wf-conn-no-arity-change) +
    then have  $\forall \varphi \in \text{set } (\xi @ \zeta' \# \xi'). \varphi \neq FT \wedge \varphi \neq FF$  using l by auto
    moreover have  $c \neq CT \wedge c \neq CF$  using corr by auto
    ultimately show no-T-F-symb (conn c ( $\xi @ \zeta' \# \xi'$ ))
      by (metis corr no-T-F-symb-comp wf-conn-no-arity-change wf-conn-no-arity-change-helper)
  qed
}
ultimately show no-T-F-except-top-level  $\psi$ 
  using full-propo-rew-step-inv-stay-with-inc[of pushNeg no-T-F-symb-except-toplevel  $\varphi$ ] assms
  subformula-refl unfolding no-T-F-except-top-level-def full-unfold by metis
next
{
  fix  $\varphi \psi :: 'v \text{ propo}$ 
  have H: pushNeg  $\varphi \psi \implies \text{no-equiv } \varphi \implies \text{no-equiv } \psi$ 
    by (induct  $\varphi \psi$  rule: pushNeg.induct, auto)
}
then show no-equiv  $\psi$ 
  using full-propo-rew-step-inv-stay-conn[of pushNeg no-equiv-symb  $\varphi \psi$ ]
  no-equiv-symb-conn-characterization assms unfolding no-equiv-def full-unfold by metis
next
{
  fix  $\varphi \psi :: 'v \text{ propo}$ 
  have H: pushNeg  $\varphi \psi \implies \text{no-imp } \varphi \implies \text{no-imp } \psi$ 
    by (induct  $\varphi \psi$  rule: pushNeg.induct, auto)
}
then show no-imp  $\psi$ 
  using full-propo-rew-step-inv-stay-conn[of pushNeg no-imp-symb  $\varphi \psi$ ]
  no-imp-symb-conn-characterization unfolding no-imp-def full-unfold by metis
qed

```

lemma *pushNeg-full-propo-rew-step*:
 fixes $\varphi \psi :: 'v \text{ propo}$
 assumes

$no-equiv \varphi$ **and**
 $no-imp \varphi$ **and**
 $full (propo-rew-step pushNeg) \varphi \psi$ **and**
 $no-T-F-except-top-level \varphi$
shows $simple-not \psi$
using $assms full-propo-rew-step-subformula pushNeg-inv(1,2) simple-not-rew$ **by** $blast$

8.5 Push inside

inductive $push-conn-inside :: 'v connective \Rightarrow 'v connective \Rightarrow 'v propo \Rightarrow 'v propo \Rightarrow bool$
for $c c' :: 'v connective$ **where**
 $push-conn-inside-l[simp]: c = CAnd \vee c = COr \Longrightarrow c' = CAnd \vee c' = COr$
 $\Longrightarrow push-conn-inside c c' (conn c [conn c' [\varphi 1, \varphi 2], \psi])$
 $(conn c' [conn c [\varphi 1, \psi], conn c [\varphi 2, \psi]]) \mid$
 $push-conn-inside-r[simp]: c = CAnd \vee c = COr \Longrightarrow c' = CAnd \vee c' = COr$
 $\Longrightarrow push-conn-inside c c' (conn c [\psi, conn c' [\varphi 1, \varphi 2]])$
 $(conn c' [conn c [\psi, \varphi 1], conn c [\psi, \varphi 2]])$

lemma $push-conn-inside-explicit: push-conn-inside c c' \varphi \psi \Longrightarrow \forall A. A \models \varphi \longleftrightarrow A \models \psi$
by ($induct \varphi \psi$ rule: $push-conn-inside.induct$, $auto$)

lemma $push-conn-inside-consistent: preserves-un-sat (push-conn-inside c c')$
unfolding $preserves-un-sat-def$ **by** ($simp$ add: $push-conn-inside-explicit$)

lemma $propo-rew-step-push-conn-inside[simp]:$
 $\neg propo-rew-step (push-conn-inside c c') FT \psi \neg propo-rew-step (push-conn-inside c c') FF \psi$
proof –
 $\{$
 $\{$
 $\text{fix } \varphi \psi$
 $\text{have } push-conn-inside c c' \varphi \psi \Longrightarrow \varphi = FT \vee \varphi = FF \Longrightarrow False$
 $\text{by } (induct \text{ rule: } push-conn-inside.induct, auto)$
 $\}$ **note** $H = this$
 $\text{fix } \varphi$
 $\text{have } propo-rew-step (push-conn-inside c c') \varphi \psi \Longrightarrow \varphi = FT \vee \varphi = FF \Longrightarrow False$
 $\text{apply } (induct \text{ rule: } propo-rew-step.induct, auto simp: wf-conn-list(1) wf-conn-list(2))$
 $\text{using } H \text{ by } blast+$
 $\}$
then show
 $\neg propo-rew-step (push-conn-inside c c') FT \psi$
 $\neg propo-rew-step (push-conn-inside c c') FF \psi$ **by** $blast+$
qed

inductive $not-c-in-c'-symb :: 'v connective \Rightarrow 'v connective \Rightarrow 'v propo \Rightarrow bool$ **for** $c c'$ **where**
 $not-c-in-c'-symb-l[simp]: wf-conn c [conn c' [\varphi, \varphi'], \psi] \Longrightarrow wf-conn c' [\varphi, \varphi']$
 $\Longrightarrow not-c-in-c'-symb c c' (conn c [conn c' [\varphi, \varphi'], \psi]) \mid$
 $not-c-in-c'-symb-r[simp]: wf-conn c [\psi, conn c' [\varphi, \varphi']] \Longrightarrow wf-conn c' [\varphi, \varphi']$
 $\Longrightarrow not-c-in-c'-symb c c' (conn c [\psi, conn c' [\varphi, \varphi']])$

abbreviation $c-in-c'-symb c c' \varphi \equiv \neg not-c-in-c'-symb c c' \varphi$

lemma $c-in-c'-symb-simp:$
 $not-c-in-c'-symb c c' \xi \Longrightarrow \xi = FF \vee \xi = FT \vee \xi = FVar x \vee \xi = FNot FF \vee \xi = FNot FT$

$\vee \xi = FNot (FVar x) \implies False$
apply (induct rule: *not-c-in-c'-symb.induct*, auto simp: *wf-conn.simps wf-conn-list(1-3)*)
using *conn-inj-not(2) wf-conn-binary unfolding binary-connectives-def by fastforce+*

lemma *c-in-c'-symb-simp'[simp]:*
 $\neg not-c-in-c'-symb\ c\ c'\ FF$
 $\neg not-c-in-c'-symb\ c\ c'\ FT$
 $\neg not-c-in-c'-symb\ c\ c'\ (FVar\ x)$
 $\neg not-c-in-c'-symb\ c\ c'\ (FNot\ FF)$
 $\neg not-c-in-c'-symb\ c\ c'\ (FNot\ FT)$
 $\neg not-c-in-c'-symb\ c\ c'\ (FNot\ (FVar\ x))$
using *c-in-c'-symb-simp by metis+*

definition *c-in-c'-only where*
c-in-c'-only $c\ c' \equiv all-subformula-st\ (c-in-c'-symb\ c\ c')$

lemma *c-in-c'-only-simp[simp]:*
 $c-in-c'-only\ c\ c'\ FF$
 $c-in-c'-only\ c\ c'\ FT$
 $c-in-c'-only\ c\ c'\ (FVar\ x)$
 $c-in-c'-only\ c\ c'\ (FNot\ FF)$
 $c-in-c'-only\ c\ c'\ (FNot\ FT)$
 $c-in-c'-only\ c\ c'\ (FNot\ (FVar\ x))$
unfolding *c-in-c'-only-def by auto*

lemma *not-c-in-c'-symb-commute:*
 $not-c-in-c'-symb\ c\ c'\ \xi \implies wf-conn\ c\ [\varphi,\ \psi] \implies \xi = conn\ c\ [\varphi,\ \psi]$
 $\implies not-c-in-c'-symb\ c\ c'\ (conn\ c\ [\psi,\ \varphi])$
proof (induct rule: *not-c-in-c'-symb.induct*)
case (*not-c-in-c'-symb-r* $\varphi'\ \varphi''\ \psi'$) **note** $H = this$
then have $\psi = conn\ c'\ [\varphi'',\ \psi']$ **using** *conn-inj by auto*
have $wf-conn\ c\ [conn\ c'\ [\varphi'',\ \psi'],\ \varphi]$
using $H(1)$ *wf-conn-no-arity-change length-Cons by metis*
then show $not-c-in-c'-symb\ c\ c'\ (conn\ c\ [\psi,\ \varphi])$
unfolding ψ **using** *not-c-in-c'-symb.intros(1) H by auto*
next
case (*not-c-in-c'-symb-l* $\varphi'\ \varphi''\ \psi'$) **note** $H = this$
then have $\varphi = conn\ c'\ [\varphi',\ \varphi'']$ **using** *conn-inj by auto*
moreover have $wf-conn\ c\ [\psi',\ conn\ c'\ [\varphi',\ \varphi'']]$
using $H(1)$ *wf-conn-no-arity-change length-Cons by metis*
ultimately show $not-c-in-c'-symb\ c\ c'\ (conn\ c\ [\psi,\ \varphi])$
using *not-c-in-c'-symb.intros(2) conn-inj not-c-in-c'-symb-l.hyps*
 $not-c-in-c'-symb-l.prem(1,2)$ **by blast**
qed

lemma *not-c-in-c'-symb-commute':*
 $wf-conn\ c\ [\varphi,\ \psi] \implies c-in-c'-symb\ c\ c'\ (conn\ c\ [\varphi,\ \psi]) \longleftrightarrow c-in-c'-symb\ c\ c'\ (conn\ c\ [\psi,\ \varphi])$
using *not-c-in-c'-symb-commute wf-conn-no-arity-change by (metis length-Cons)*

lemma *not-c-in-c'-comm:*
assumes $wf: wf-conn\ c\ [\varphi,\ \psi]$
shows $c-in-c'-only\ c\ c'\ (conn\ c\ [\varphi,\ \psi]) \longleftrightarrow c-in-c'-only\ c\ c'\ (conn\ c\ [\psi,\ \varphi])$ (**is** $?A \longleftrightarrow ?B$)
proof –
have $?A \longleftrightarrow (c-in-c'-symb\ c\ c'\ (conn\ c\ [\varphi,\ \psi]))$

```

       $\wedge (\forall \xi \in \text{set } [\varphi, \psi]. \text{all-subformula-st } (c\text{-in-}c'\text{-symb } c \ c') \ \xi))$ 
    using all-subformula-st-decomp wf unfolding c-in-c'-only-def by fastforce
  also have ...  $\longleftrightarrow (c\text{-in-}c'\text{-symb } c \ c' (\text{conn } c \ [\psi, \varphi])$ 
       $\wedge (\forall \xi \in \text{set } [\psi, \varphi]. \text{all-subformula-st } (c\text{-in-}c'\text{-symb } c \ c') \ \xi))$ 
    using not-c-in-c'-symb-commute' wf by auto
  also
  have wf-conn c  $[\psi, \varphi]$  using wf-conn-no-arity-change wf by (metis length-Cons)
  then have (c-in-c'-symb c c' (conn c  $[\psi, \varphi]$ )
       $\wedge (\forall \xi \in \text{set } [\psi, \varphi]. \text{all-subformula-st } (c\text{-in-}c'\text{-symb } c \ c') \ \xi))$ 
     $\longleftrightarrow ?B$ 
    using all-subformula-st-decomp unfolding c-in-c'-only-def by fastforce
  finally show ?thesis .
qed

```

```

lemma not-c-in-c'-simp[simp]:
  fixes  $\varphi 1 \ \varphi 2 \ \psi :: 'v \text{ propo}$  and  $x :: 'v$ 
  shows
    c-in-c'-symb c c' FT
    c-in-c'-symb c c' FF
    c-in-c'-symb c c' (FVar x)
    wf-conn c [conn c'  $[\varphi 1, \varphi 2], \psi] \implies \text{wf-conn } c' [\varphi 1, \varphi 2]$ 
       $\implies \neg c\text{-in-}c'\text{-only } c \ c' (\text{conn } c \ [\text{conn } c' [\varphi 1, \varphi 2], \psi])$ 
  apply (simp-all add: c-in-c'-only-def)
  using all-subformula-st-test-symb-true-phi not-c-in-c'-symb-l by blast

```

```

lemma c-in-c'-symb-not[simp]:
  fixes c c' :: 'v connective and  $\psi :: 'v \text{ propo}$ 
  shows c-in-c'-symb c c' (FNot  $\psi$ )
proof -
  {
    fix  $\xi :: 'v \text{ propo}$ 
    have not-c-in-c'-symb c c' (FNot  $\psi$ )  $\implies \text{False}$ 
      apply (induct FNot  $\psi$  rule: not-c-in-c'-symb.induct)
      using conn-inj-not(2) by blast+
  }
  then show ?thesis by auto
qed

```

```

lemma c-in-c'-symb-step-exists:
  fixes  $\varphi :: 'v \text{ propo}$ 
  assumes c:  $c = C\text{And } \vee \ c = C\text{Or}$  and c':  $c' = C\text{And } \vee \ c' = C\text{Or}$ 
  shows  $\psi \preceq \varphi \implies \neg c\text{-in-}c'\text{-symb } c \ c' \ \psi \implies \exists \psi'. \text{push-conn-inside } c \ c' \ \psi \ \psi'$ 
  apply (induct  $\psi$  rule: propo-induct-arity)
  apply auto[2]
proof -
  fix  $\psi 1 \ \psi 2 \ \varphi' :: 'v \text{ propo}$ 
  assume IH $\psi 1$ :  $\psi 1 \preceq \varphi \implies \neg c\text{-in-}c'\text{-symb } c \ c' \ \psi 1 \implies \text{Ex } (\text{push-conn-inside } c \ c' \ \psi 1)$ 
  and IH $\psi 2$ :  $\psi 2 \preceq \varphi \implies \neg c\text{-in-}c'\text{-symb } c \ c' \ \psi 2 \implies \text{Ex } (\text{push-conn-inside } c \ c' \ \psi 2)$ 
  and  $\varphi'$ :  $\varphi' = F\text{And } \psi 1 \ \psi 2 \vee \varphi' = F\text{Or } \psi 1 \ \psi 2 \vee \varphi' = F\text{Imp } \psi 1 \ \psi 2 \vee \varphi' = F\text{Eq } \psi 1 \ \psi 2$ 
  and in $\varphi$ :  $\varphi' \preceq \varphi$  and n0:  $\neg c\text{-in-}c'\text{-symb } c \ c' \ \varphi'$ 
  then have n: not-c-in-c'-symb c c'  $\varphi'$  by auto
  {
    assume  $\varphi'$ :  $\varphi' = \text{conn } c \ [\psi 1, \psi 2]$ 
    obtain a b where  $\psi 1 = \text{conn } c' \ [a, b] \vee \psi 2 = \text{conn } c' \ [a, b]$ 
      using n  $\varphi'$  apply (induct rule: not-c-in-c'-symb.induct)

```

```

    using c by force+
  then have Ex (push-conn-inside c c'  $\varphi'$ )
    unfolding  $\varphi'$  apply auto
    using push-conn-inside.intros(1) c c' apply blast
    using push-conn-inside.intros(2) c c' by blast
}
moreover {
  assume  $\varphi'$ :  $\varphi' \neq \text{conn } c [\psi 1, \psi 2]$ 
  have  $\forall \varphi \ c \ ca. \exists \varphi 1 \ \psi 1 \ \psi 2 \ \psi 1' \ \psi 2' \ \varphi 2'. \text{conn } (c::'v \text{ connective}) [\varphi 1, \text{conn } ca [\psi 1, \psi 2]] = \varphi$ 
     $\vee \text{conn } c [\text{conn } ca [\psi 1', \psi 2'], \varphi 2'] = \varphi \vee c\text{-in-}c'\text{-symb } c \ ca \ \varphi$ 
    by (metis not-c-in-c'-symb.cases)
  then have Ex (push-conn-inside c c'  $\varphi'$ )
    by (metis (no-types) c c' n push-conn-inside-l push-conn-inside-r)
}
ultimately show Ex (push-conn-inside c c'  $\varphi'$ ) by blast
qed

```

lemma *c-in-c'-symb-rew*:

```

  fixes  $\varphi :: 'v \text{ propo}$ 
  assumes noTB:  $\neg c\text{-in-}c'\text{-only } c \ c' \ \varphi$ 
  and c:  $c = CAnd \vee c = COr$  and c':  $c' = CAnd \vee c' = COr$ 
  shows  $\exists \psi \ \psi'. \psi \preceq \varphi \wedge \text{push-conn-inside } c \ c' \ \psi \ \psi'$ 
proof -
  have test-symb-false-nullary:
     $\forall x. c\text{-in-}c'\text{-symb } c \ c' \ (FF:: 'v \text{ propo}) \wedge c\text{-in-}c'\text{-symb } c \ c' \ FT$ 
     $\wedge c\text{-in-}c'\text{-symb } c \ c' \ (FVar \ (x:: 'v))$ 
    by auto
  moreover {
    fix x :: 'v
    have H':  $c\text{-in-}c'\text{-symb } c \ c' \ FT \ c\text{-in-}c'\text{-symb } c \ c' \ FF \ c\text{-in-}c'\text{-symb } c \ c' \ (FVar \ x)$ 
      by simp+
  }
  moreover {
    fix  $\psi :: 'v \text{ propo}$ 
    have  $\psi \preceq \varphi \implies \neg c\text{-in-}c'\text{-symb } c \ c' \ \psi \implies \exists \psi'. \text{push-conn-inside } c \ c' \ \psi \ \psi'$ 
      by (auto simp: asms(2) c' c-in-c'-symb-step-exists)
  }
  ultimately show ?thesis using noTB no-test-symb-step-exists[of c-in-c'-symb c c']
    unfolding c-in-c'-only-def by metis
qed

```

lemma *push-conn-insidec-in-c'-symb-no-T-F*:

```

  fixes  $\varphi \ \psi :: 'v \text{ propo}$ 
  shows propo-rew-step (push-conn-inside c c')  $\varphi \ \psi \implies \text{no-T-F } \varphi \implies \text{no-T-F } \psi$ 
proof (induct rule: propo-rew-step.induct)
  case (global-rel  $\varphi \ \psi$ )
  then show no-T-F  $\psi$ 
    by (cases rule: push-conn-inside.cases, auto)
next
  case (propo-rew-one-step-lift  $\varphi \ \varphi' \ c \ \xi \ \xi'$ )
  note rel = this(1) and IH = this(2) and wf = this(3) and no-T-F = this(4)
  have no-T-F  $\varphi$ 
    using wf no-T-F no-T-F-def subformula-into-subformula subformula-all-subformula-st
    subformula-refl by (metis (no-types) in-set-conv-decomp)

```

```

then have  $\varphi'$ : no-T-F  $\varphi'$  using IH by blast

have  $\forall \zeta \in \text{set } (\xi @ \varphi \# \xi')$ . no-T-F  $\zeta$  by (metis wf no-T-F no-T-F-def all-subformula-st-decomp)
then have  $n$ :  $\forall \zeta \in \text{set } (\xi @ \varphi' \# \xi')$ . no-T-F  $\zeta$  using  $\varphi'$  by auto
then have  $n'$ :  $\forall \zeta \in \text{set } (\xi @ \varphi' \# \xi')$ .  $\zeta \neq FF \wedge \zeta \neq FT$ 
  using  $\varphi'$  by (metis no-T-F-symb-false(1) no-T-F-symb-false(2) no-T-F-def
    all-subformula-st-test-symb-true-phi)

have  $wf'$ : wf-conn  $c$  ( $\xi @ \varphi' \# \xi'$ )
  using wf wf-conn-no-arity-change by (metis wf-conn-no-arity-change-helper)
{
  fix  $x :: 'v$ 
  assume  $c = CT \vee c = CF \vee c = CVar\ x$ 
  then have False using wf by auto
  then have no-T-F (conn  $c$  ( $\xi @ \varphi' \# \xi'$ )) by blast
}
moreover {
  assume  $c: c = CNot$ 
  then have  $\xi = [] \ \xi' = []$  using wf by auto
  then have no-T-F (conn  $c$  ( $\xi @ \varphi' \# \xi'$ ))
    using  $c$  by (metis  $\varphi'$  conn.simps(4) no-T-F-symb-false(1,2) no-T-F-symb-fnot no-T-F-def
      all-subformula-st-decomp-explicit(3) all-subformula-st-test-symb-true-phi self-append-conv2)
}
moreover {
  assume  $c: c \in \text{binary-connectives}$ 
  then have no-T-F-symb (conn  $c$  ( $\xi @ \varphi' \# \xi'$ )) using  $wf' \ n' \ \text{no-T-F-symb.simps}$  by fastforce
  then have no-T-F (conn  $c$  ( $\xi @ \varphi' \# \xi'$ ))
    by (metis all-subformula-st-decomp-imp wf' n no-T-F-def)
}
ultimately show no-T-F (conn  $c$  ( $\xi @ \varphi' \# \xi'$ )) using connective-cases-arity by auto
qed

```

```

lemma simple-propo-rew-step-push-conn-inside-inv:
  propo-rew-step (push-conn-inside  $c \ c'$ )  $\varphi \ \psi \implies \text{simple } \varphi \implies \text{simple } \psi$ 
  apply (induct rule: propo-rew-step.induct)
  apply (rename-tac  $\varphi$ , case-tac  $\varphi$ , auto simp: push-conn-inside.simps)[]
  by (metis append-is-Nil-conv list.distinct(1) simple.elims(2) wf-conn-list(1-3))

```

```

lemma simple-propo-rew-step-inv-push-conn-inside-simple-not:
  fixes  $c \ c' :: 'v$  connective and  $\varphi \ \psi :: 'v$  propo
  shows propo-rew-step (push-conn-inside  $c \ c'$ )  $\varphi \ \psi \implies \text{simple-not } \varphi \implies \text{simple-not } \psi$ 
proof (induct rule: propo-rew-step.induct)
  case (global-rel  $\varphi \ \psi$ )
  then show ?case by (cases  $\varphi$ , auto simp: push-conn-inside.simps)
next
  case (propo-rew-one-step-lift  $\varphi \ \varphi' \ ca \ \xi \ \xi'$ ) note rew = this(1) and IH = this(2) and wf = this(3)
  and simple = this(4)
  show ?case
  proof (cases  $ca$  rule: connective-cases-arity)
  case nullary
  then show ?thesis using propo-rew-one-step-lift by auto
  next
  case binary note  $ca = \text{this}$ 

```



```

obtain  $a\ b$  where  $ab: \xi @ \varphi' \# \xi' = [a, b]$ 
  using  $wf\ ca\ list\ length2\ decomp\ wf\ conn\ bin\ list\ length$ 
  by ( $metis\ (no\ types)\ wf\ conn\ no\ arity\ change\ helper$ )
have  $\forall \zeta \in set\ (\xi @ \varphi \# \xi').\ simple\ not\ \zeta$ 
  by ( $metis\ wf\ all\ subformula\ st\ decomp\ simple\ simple\ not\ def$ )
then have  $\forall \zeta \in set\ (\xi @ \varphi' \# \xi').\ simple\ not\ \zeta$  using  $IH$  by  $simp$ 
moreover have  $simple\ not\ symb\ (conn\ ca\ (\xi @ \varphi' \# \xi'))$  using  $ca$ 
by ( $metis\ ab\ conn.simps(5-8)\ helper\ fact\ simple\ not\ symb.simps(5)\ simple\ not\ symb.simps(6)$ 
   $simple\ not\ symb.simps(7)\ simple\ not\ symb.simps(8)$ )
ultimately show  $?thesis$ 
  by ( $simp\ add: ab\ all\ subformula\ st\ decomp\ ca$ )
next
case  $unary$ 
then show  $?thesis$ 
  using  $rew\ simple\ propo\ rew\ step\ push\ conn\ inside\ inv[OF\ rew]\ IH\ local.wf\ simple$  by  $auto$ 
qed
qed

```

lemma $propo\ rew\ step\ push\ conn\ inside\ simple\ not:$

```

fixes  $\varphi\ \varphi' :: 'v\ propo$  and  $\xi\ \xi' :: 'v\ propo\ list$  and  $c :: 'v\ connective$ 
assumes
   $propo\ rew\ step\ (push\ conn\ inside\ c\ c')\ \varphi\ \varphi'$  and
   $wf\ conn\ c\ (\xi @ \varphi \# \xi')$  and
   $simple\ not\ symb\ (conn\ c\ (\xi @ \varphi \# \xi'))$  and
   $simple\ not\ symb\ \varphi'$ 
shows  $simple\ not\ symb\ (conn\ c\ (\xi @ \varphi' \# \xi'))$ 
using  $assms$ 
proof ( $induction\ rule: propo\ rew\ step.induct$ )
print-cases
case ( $global\ rel$ )
then show  $?case$ 
  by ( $metis\ conn.simps(12,17)\ list.discI\ push\ conn\ inside.cases\ simple\ not\ symb.elims(3)$ 
   $wf\ conn\ helper\ facts(5)\ wf\ conn\ list(2)\ wf\ conn\ list(8)\ wf\ conn\ no\ arity\ change$ 
   $wf\ conn\ no\ arity\ change\ helper$ )
next
case ( $propo\ rew\ one\ step\ lift\ \varphi\ \varphi'\ c'\ \chi s\ \chi s'$ ) note  $tel = this(1)$  and  $wf = this(2)$  and
   $IH = this(3)$  and  $wf' = this(4)$  and  $simple' = this(5)$  and  $simple = this(6)$ 
then show  $?case$ 
proof ( $cases\ c'\ rule: connective-cases-arity$ )
  case  $nullary$ 
  then show  $?thesis$  using  $wf\ simple\ simple'$  by  $auto$ 
next
  case  $binary$  note  $c = this(1)$ 
  have  $corr': wf\ conn\ c\ (\xi @ conn\ c'\ (\chi s @ \varphi' \# \chi s')) \# \xi'$ 
    using  $wf\ wf\ conn\ no\ arity\ change$ 
    by ( $metis\ wf'\ wf\ conn\ no\ arity\ change\ helper$ )
  then show  $?thesis$ 
    using  $c\ propo\ rew\ one\ step\ lift\ wf$ 
    by ( $metis\ conn.simps(17)\ connective.distinct(37)\ propo\ rew\ step\ subformula\ imp$ 
     $push\ conn\ inside.cases\ simple\ not\ symb.elims(3)\ wf\ conn.simps\ wf\ conn\ list(2,8)$ )
next
case  $unary$ 
then have  $empty: \chi s = []\ \chi s' = []$  using  $wf$  by  $auto$ 
then show  $?thesis$  using  $simple\ unary\ simple'\ wf\ wf'$ 
  by ( $metis\ connective.distinct(37)\ connective.distinct(39)\ propo\ rew\ step\ subformula\ imp$ 

```

```

push-conn-inside.cases simple-not-symb.elims(3) tel wf-conn-list(8)
wf-conn-no-arity-change wf-conn-no-arity-change-helper)
qed
qed

lemma push-conn-inside-not-true-false:
  push-conn-inside c c'  $\varphi \psi \implies \psi \neq FT \wedge \psi \neq FF$ 
  by (induct rule: push-conn-inside.induct, auto)

lemma push-conn-inside-inv:
  fixes  $\varphi \psi :: 'v \text{ propo}$ 
  assumes full (propo-rew-step (push-conn-inside c c'))  $\varphi \psi$ 
  and no-equiv  $\varphi$  and no-imp  $\varphi$  and no-T-F-except-top-level  $\varphi$  and simple-not  $\varphi$ 
  shows no-equiv  $\psi$  and no-imp  $\psi$  and no-T-F-except-top-level  $\psi$  and simple-not  $\psi$ 
proof -
  {
    {
      fix  $\varphi \psi :: 'v \text{ propo}$ 
      have H: push-conn-inside c c'  $\varphi \psi \implies \text{all-subformula-st simple-not-symb } \varphi$ 
         $\implies \text{all-subformula-st simple-not-symb } \psi$ 
        by (induct  $\varphi \psi$  rule: push-conn-inside.induct, auto)
    } note H = this

    fix  $\varphi \psi :: 'v \text{ propo}$ 
    have H: propo-rew-step (push-conn-inside c c')  $\varphi \psi \implies \text{all-subformula-st simple-not-symb } \varphi$ 
       $\implies \text{all-subformula-st simple-not-symb } \psi$ 
    apply (induct  $\varphi \psi$  rule: propo-rew-step.induct)
    using H apply simp
    proof (rename-tac  $\varphi \varphi'$  ca  $\psi s \psi s'$ , case-tac ca rule: connective-cases-arity)
      fix  $\varphi \varphi' :: 'v \text{ propo}$  and c:: ' $v \text{ connective}$  and  $\xi \xi' :: 'v \text{ propo list}$ 
      and x:: ' $v$ 
      assume wf-conn c ( $\xi @ \varphi \# \xi'$ )
      and c = CT  $\vee$  c = CF  $\vee$  c = CVar x
      then have  $\xi @ \varphi \# \xi' = []$  by auto
      then have False by auto
      then show all-subformula-st simple-not-symb (conn c ( $\xi @ \varphi' \# \xi'$ )) by blast
    next
      fix  $\varphi \varphi' :: 'v \text{ propo}$  and ca:: ' $v \text{ connective}$  and  $\xi \xi' :: 'v \text{ propo list}$ 
      and x:: ' $v$ 
      assume rel: propo-rew-step (push-conn-inside c c')  $\varphi \varphi'$ 
      and  $\varphi\text{-}\varphi'$ : all-subformula-st simple-not-symb  $\varphi \implies \text{all-subformula-st simple-not-symb } \varphi'$ 
      and corr: wf-conn ca ( $\xi @ \varphi \# \xi'$ )
      and n: all-subformula-st simple-not-symb (conn ca ( $\xi @ \varphi \# \xi'$ ))
      and c: ca = CNot

      have empty:  $\xi = [] \wedge \xi' = []$  using c corr by auto
      then have simple-not:all-subformula-st simple-not-symb (FNot  $\varphi$ ) using corr c n by auto
      then have simple  $\varphi$ 
        using all-subformula-st-test-symb-true-phi simple-not-symb.simps(1) by blast
      then have simple  $\varphi'$ 
        using rel simple-propo-rew-step-push-conn-inside-inv by blast
      then show all-subformula-st simple-not-symb (conn ca ( $\xi @ \varphi' \# \xi'$ )) using c empty
        by (metis simple-not  $\varphi\text{-}\varphi'$  append-Nil conn.simps(4) all-subformula-st-decomp-explicit(3)
          simple-not-symb.simps(1))
    next

```

```

fix  $\varphi \varphi' :: 'v \text{ propo}$  and  $ca :: 'v \text{ connective}$  and  $\xi \xi' :: 'v \text{ propo list}$ 
and  $x :: 'v$ 
assume  $rel: \text{propo-rew-step } (\text{push-conn-inside } c \ c') \ \varphi \ \varphi'$ 
and  $n\varphi: \text{all-subformula-st simple-not-symb } \varphi \implies \text{all-subformula-st simple-not-symb } \varphi'$ 
and  $corr: \text{wf-conn } ca \ (\xi @ \varphi \# \xi')$ 
and  $n: \text{all-subformula-st simple-not-symb } (\text{conn } ca \ (\xi @ \varphi \# \xi'))$ 
and  $c: ca \in \text{binary-connectives}$ 

have  $\text{all-subformula-st simple-not-symb } \varphi$ 
  using  $n \ c \ corr \ \text{all-subformula-st-decomp}$  by fastforce
then have  $\varphi': \text{all-subformula-st simple-not-symb } \varphi'$  using  $n\varphi$  by blast
obtain  $a \ b$  where  $ab: [a, b] = (\xi @ \varphi \# \xi')$ 
  using  $corr \ c \ \text{list-length2-decomp wf-conn-bin-list-length}$  by metis
then have  $\xi @ \varphi' \# \xi' = [a, \varphi'] \vee (\xi @ \varphi' \# \xi') = [\varphi', b]$ 
  using  $ab$  by (metis (no-types, hide-lams) append-Cons append-Nil append-Nil2
    append-is-Nil-conv butlast.simps(2) butlast-append list.sel(3) tl-append2)
moreover
{
  fix  $\chi :: 'v \text{ propo}$ 
  have  $wf': \text{wf-conn } ca \ [a, b]$ 
    using  $ab \ corr$  by presburger
  have  $\text{all-subformula-st simple-not-symb } (\text{conn } ca \ [a, b])$ 
    using  $ab \ n$  by presburger
  then have  $\text{all-subformula-st simple-not-symb } \chi \vee \chi \notin \text{set } (\xi @ \varphi' \# \xi')$ 
    using  $wf'$  by (metis (no-types)  $\varphi'$   $\text{all-subformula-st-decomp}$  calculation insert-iff
      list.set(2))
}
then have  $\forall \varphi. \varphi \in \text{set } (\xi @ \varphi' \# \xi') \longrightarrow \text{all-subformula-st simple-not-symb } \varphi$ 
  by (metis (no-types))

moreover have  $\text{simple-not-symb } (\text{conn } ca \ (\xi @ \varphi' \# \xi'))$ 
  using  $ab \ \text{conn-inj-not}(1) \ corr \ \text{wf-conn-list-decomp}(4) \ \text{wf-conn-no-arity-change}$ 
   $\text{not-Cons-self2 self-append-conv2 simple-not-symb.elims}(3)$  by (metis (no-types)  $c$ 
  calculation(1)  $\text{wf-conn-binary}$ )
moreover have  $\text{wf-conn } ca \ (\xi @ \varphi' \# \xi')$  using  $c \ \text{calculation}(1)$  by auto
ultimately show  $\text{all-subformula-st simple-not-symb } (\text{conn } ca \ (\xi @ \varphi' \# \xi'))$ 
  by (metis  $\text{all-subformula-st-decomp-imp}$ )
qed
}
moreover {
  fix  $ca :: 'v \text{ connective}$  and  $\xi \xi' :: 'v \text{ propo list}$  and  $\varphi \varphi' :: 'v \text{ propo}$ 
  have  $\text{propo-rew-step } (\text{push-conn-inside } c \ c') \ \varphi \ \varphi' \implies \text{wf-conn } ca \ (\xi @ \varphi \# \xi')$ 
     $\implies \text{simple-not-symb } (\text{conn } ca \ (\xi @ \varphi \# \xi')) \implies \text{simple-not-symb } \varphi'$ 
     $\implies \text{simple-not-symb } (\text{conn } ca \ (\xi @ \varphi' \# \xi'))$ 
  by (metis  $\text{append-self-conv2 conn.simps}(4) \ \text{conn-inj-not}(1) \ \text{simple-not-symb.elims}(3)$ 
     $\text{simple-not-symb.simps}(1) \ \text{simple-propo-rew-step-push-conn-inside-inv}$ 
     $\text{wf-conn-no-arity-change-helper wf-conn-list-decomp}(4) \ \text{wf-conn-no-arity-change}$ )
}
ultimately show  $\text{simple-not } \psi$ 
  using  $\text{full-propo-rew-step-inv-stay'}$  [of  $\text{push-conn-inside } c \ c' \ \text{simple-not-symb}$ ]  $\text{assms}$ 
  unfolding  $\text{no-T-F-except-top-level-def simple-not-def full-unfold}$  by metis
next
{
  fix  $\varphi \psi :: 'v \text{ propo}$ 
  have  $H: \text{propo-rew-step } (\text{push-conn-inside } c \ c') \ \varphi \ \psi \implies \text{no-T-F-except-top-level } \varphi$ 

```

```

⇒ no-T-F-except-top-level ψ
proof -
  assume rel: propo-rew-step (push-conn-inside c c') φ ψ
  and no-T-F-except-top-level φ
  then have no-T-F φ ∨ φ = FF ∨ φ = FT
    by (metis no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
  moreover {
    assume φ = FF ∨ φ = FT
    then have False using rel propo-rew-step-push-conn-inside by blast
    then have no-T-F-except-top-level ψ by blast
  }
  moreover {
    assume no-T-F φ ∧ φ ≠ FF ∧ φ ≠ FT
    then have no-T-F ψ using rel push-conn-insidec-in-c'-symb-no-T-F by blast
    then have no-T-F-except-top-level ψ using no-T-F-no-T-F-except-top-level by blast
  }
  ultimately show no-T-F-except-top-level ψ by blast
qed
}
moreover {
  fix ca :: 'v connective and ξ ξ' :: 'v propo list and φ φ' :: 'v propo
  assume rel: propo-rew-step (push-conn-inside c c') φ φ'
  assume corr: wf-conn ca (ξ @ φ # ξ')
  then have c: ca ≠ CT ∧ ca ≠ CF by auto
  assume no-T-F: no-T-F-symb-except-toplevel (conn ca (ξ @ φ # ξ'))
  have no-T-F-symb-except-toplevel (conn ca (ξ @ φ' # ξ'))
  proof
    have c: ca ≠ CT ∧ ca ≠ CF using corr by auto
    have ζ: ∀ ζ ∈ set (ξ @ φ # ξ'). ζ ≠ FT ∧ ζ ≠ FF
      using corr no-T-F no-T-F-symb-except-toplevel-if-is-a-true-false by blast
    then have φ ≠ FT ∧ φ ≠ FF by auto
    from rel this have φ' ≠ FT ∧ φ' ≠ FF
    apply (induct rule: propo-rew-step.induct)
    by (metis append-is-Nil-conv conn.simps(2) conn-inj list.distinct(1)
        wf-conn-helper-facts(3) wf-conn-list(1) wf-conn-no-arity-change
        wf-conn-no-arity-change-helper push-conn-inside-not-true-false)+
    then have ∀ ζ ∈ set (ξ @ φ' # ξ'). ζ ≠ FT ∧ ζ ≠ FF using ζ by auto
    moreover have wf-conn ca (ξ @ φ' # ξ')
      using corr wf-conn-no-arity-change by (metis wf-conn-no-arity-change-helper)
    ultimately show no-T-F-symb (conn ca (ξ @ φ' # ξ')) using no-T-F-symb.intros c by metis
  qed
}
ultimately show no-T-F-except-top-level ψ
  using full-propo-rew-step-inv-stay'[of push-conn-inside c c' no-T-F-symb-except-toplevel]
  assms unfolding no-T-F-except-top-level-def full-unfold by metis

next
{
  fix φ ψ :: 'v propo
  have H: push-conn-inside c c' φ ψ ⇒ no-equiv φ ⇒ no-equiv ψ
    by (induct φ ψ rule: push-conn-inside.induct, auto)
}
then show no-equiv ψ
  using full-propo-rew-step-inv-stay-conn[of push-conn-inside c c' no-equiv-symb] assms
  no-equiv-symb-conn-characterization unfolding no-equiv-def by metis

```

```

next
{
  fix  $\varphi \psi :: 'v \text{ propo}$ 
  have  $H: \text{push-conn-inside } c \ c' \ \varphi \ \psi \implies \text{no-imp } \varphi \implies \text{no-imp } \psi$ 
    by (induct  $\varphi \ \psi$  rule: push-conn-inside.induct, auto)
}
then show no-imp  $\psi$ 
  using full-propo-rew-step-inv-stay-conn[of push-conn-inside  $c \ c'$  no-imp-symb] assms
  no-imp-symb-conn-characterization unfolding no-imp-def by metis
qed

```

lemma *push-conn-inside-full-propo-rew-step*:

```

fixes  $\varphi \psi :: 'v \text{ propo}$ 
assumes
  no-equiv  $\varphi$  and
  no-imp  $\varphi$  and
  full (propo-rew-step (push-conn-inside  $c \ c'$ ))  $\varphi \ \psi$  and
  no-T-F-except-top-level  $\varphi$  and
  simple-not  $\varphi$  and
   $c = CAnd \vee c = COr$  and
   $c' = CAnd \vee c' = COr$ 
shows c-in-c'-only  $c \ c' \ \psi$ 
using c-in-c'-symb-rew assms full-propo-rew-step-subformula by blast

```

8.5.1 Only one type of connective in the formula (+ not)

inductive *only-c-inside-symb* :: $'v \text{ connective} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ **for** $c :: 'v \text{ connective}$ **where**
simple-only-c-inside[*simp*]: *simple* $\varphi \implies \text{only-c-inside-symb } c \ \varphi$ |
simple-cnot-only-c-inside[*simp*]: *simple* $\varphi \implies \text{only-c-inside-symb } c \ (FNot \ \varphi)$ |
only-c-inside-into-only-c-inside: *wf-conn* $c \ l \implies \text{only-c-inside-symb } c \ (\text{conn } c \ l)$

lemma *only-c-inside-symb-simp*[*simp*]:
only-c-inside-symb $c \ FF$ *only-c-inside-symb* $c \ FT$ *only-c-inside-symb* $c \ (FVar \ x)$ **by** *auto*

definition *only-c-inside* **where** *only-c-inside* $c = \text{all-subformula-st } (\text{only-c-inside-symb } c)$

lemma *only-c-inside-symb-decomp*:
only-c-inside-symb $c \ \psi \longleftrightarrow (\text{simple } \psi$
 $\vee (\exists \ \varphi'. \ \psi = FNot \ \varphi' \wedge \text{simple } \varphi')$
 $\vee (\exists \ l. \ \psi = \text{conn } c \ l \wedge \text{wf-conn } c \ l))$
by (*auto simp: only-c-inside-symb.intros(3)*) (*induct rule: only-c-inside-symb.induct, auto*)

lemma *only-c-inside-symb-decomp-not*[*simp*]:
fixes $c :: 'v \text{ connective}$
assumes $c: c \neq CNot$
shows *only-c-inside-symb* $c \ (FNot \ \psi) \longleftrightarrow \text{simple } \psi$
apply (*auto simp: only-c-inside-symb.intros(3)*)
by (*induct FNot* ψ *rule: only-c-inside-symb.induct, auto simp: wf-conn-list(8) c*)

lemma *only-c-inside-decomp-not*[*simp*]:
assumes $c: c \neq CNot$
shows *only-c-inside* $c \ (FNot \ \psi) \longleftrightarrow \text{simple } \psi$

by (metis (no-types, hide-lams) all-subformula-st-def all-subformula-st-test-symb-true-phi c
only-c-inside-def only-c-inside-symb-decomp-not simple-only-c-inside
subformula-conn-decomp-simple)

lemma only-c-inside-decomp:

only-c-inside c $\varphi \longleftrightarrow$

($\forall \psi. \psi \preceq \varphi \longrightarrow$ (simple $\psi \vee (\exists \varphi'. \psi = FNot \varphi' \wedge$ simple $\varphi')$
 $\vee (\exists l. \psi = conn\ c\ l \wedge$ wf-conn $c\ l)))$)

unfolding only-c-inside-def **by** (auto simp: all-subformula-st-def only-c-inside-symb-decomp)

lemma only-c-inside-c-c'-false:

fixes c c' :: 'v connective **and** l :: 'v propo list **and** $\varphi ::$ 'v propo

assumes cc': $c \neq c'$ **and** c: $c = CAnd \vee c = COr$ **and** c': $c' = CAnd \vee c' = COr$

and only: only-c-inside c φ **and** incl: $conn\ c'\ l \preceq \varphi$ **and** wf: wf-conn $c'\ l$

shows False

proof –

let $? \psi = conn\ c'\ l$

have simple $? \psi \vee (\exists \varphi'. ? \psi = FNot \varphi' \wedge$ simple $\varphi') \vee (\exists l. ? \psi = conn\ c\ l \wedge$ wf-conn $c\ l)$

using only-c-inside-decomp only incl **by** blast

moreover **have** \neg simple $? \psi$

using wf simple-decomp **by** (metis c' connective.distinct(19) connective.distinct(7,9,21,29,31)
wf-conn-list(1-3))

moreover

{

fix φ'

have $? \psi \neq FNot \varphi'$ **using** c' conn-inj-not(1) wf **by** blast

}

ultimately obtain l :: 'v propo list **where** $? \psi = conn\ c\ l \wedge$ wf-conn $c\ l$ **by** metis

then **have** $c = c'$ **using** conn-inj wf **by** metis

then **show** False **using** cc' **by** auto

qed

lemma only-c-inside-implies-c-in-c'-symb:

assumes $\delta: c \neq c'$ **and** c: $c = CAnd \vee c = COr$ **and** c': $c' = CAnd \vee c' = COr$

shows only-c-inside c $\varphi \implies$ c-in-c'-symb c c' φ

apply (rule ccontr)

apply (cases rule: not-c-in-c'-symb.cases, auto)

by (metis $\delta\ c\ c'$ connective.distinct(37,39) list.distinct(1) only-c-inside-c-c'-false

subformula-in-binary-conn(1,2) wf-conn.simps)+

lemma c-in-c'-symb-decomp-level1:

fixes l :: 'v propo list **and** c c' ca :: 'v connective

shows wf-conn ca l \implies $ca \neq c \implies$ c-in-c'-symb c c' (conn ca l)

proof –

have not-c-in-c'-symb c c' (conn ca l) \implies wf-conn ca l \implies $ca = c$

by (induct conn ca l rule: not-c-in-c'-symb.induct, auto simp: conn-inj)

then **show** wf-conn ca l \implies $ca \neq c \implies$ c-in-c'-symb c c' (conn ca l) **by** blast

qed

lemma only-c-inside-implies-c-in-c'-only:

assumes $\delta: c \neq c'$ **and** c: $c = CAnd \vee c = COr$ **and** c': $c' = CAnd \vee c' = COr$

shows only-c-inside c $\varphi \implies$ c-in-c'-only c c' φ

unfolding c-in-c'-only-def all-subformula-st-def

```

using only-c-inside-implies-c-in-c'-symb
  by (metis all-subformula-st-def assms(1) c c' only-c-inside-def subformula-trans)

lemma c-in-c'-symb-c-implies-only-c-inside:
  assumes  $\delta$ :  $c = CAnd \vee c = COr$   $c' = CAnd \vee c' = COr$   $c \neq c'$  and wf: wf-conn  $c$   $[\varphi, \psi]$ 
  and inv: no-equiv (conn  $c$   $l$ ) no-imp (conn  $c$   $l$ ) simple-not (conn  $c$   $l$ )
  shows wf-conn  $c$   $l \implies c\text{-in-}c'\text{-only } c$   $c' \text{ (conn } c$   $l) \implies (\forall \psi \in \text{set } l. \text{ only-c-inside } c$   $\psi)$ 
using inv
proof (induct conn c l arbitrary: l rule: propo-induct-arity)
  case (nullary x)
  then show ?case by (auto simp: wf-conn-list assms)
next
  case (unary  $\varphi$  la)
  then have  $c = CNot \wedge la = [\varphi]$  by (metis (no-types) wf-conn-list(8))
  then show ?case using assms(2) assms(1) by blast
next
  case (binary  $\varphi1$   $\varphi2$ )
  note  $IH\varphi1 = \text{this}(1)$  and  $IH\varphi2 = \text{this}(2)$  and  $\varphi = \text{this}(3)$  and  $\text{only} = \text{this}(5)$  and  $\text{wf} = \text{this}(4)$ 
  and  $\text{no-equiv} = \text{this}(6)$  and  $\text{no-imp} = \text{this}(7)$  and  $\text{simple-not} = \text{this}(8)$ 
  then have  $l: l = [\varphi1, \varphi2]$  by (meson wf-conn-list(4-7))
  let ? $\varphi = \text{conn } c$   $l$ 

  obtain  $c1$   $l1$   $c2$   $l2$  where  $\varphi1: \varphi1 = \text{conn } c1$   $l1$  and  $\text{wf}\varphi1: \text{wf-conn } c1$   $l1$ 
  and  $\varphi2: \varphi2 = \text{conn } c2$   $l2$  and  $\text{wf}\varphi2: \text{wf-conn } c2$   $l2$  using exists-c-conn by metis
  then have  $c\text{-in-}\text{only}\varphi1: c\text{-in-}c'\text{-only } c$   $c' \text{ (conn } c1$   $l1)$  and  $c\text{-in-}c'\text{-only } c$   $c' \text{ (conn } c2$   $l2)$ 
  using only l unfolding c-in-c'-only-def using assms(1) by auto
  have  $\text{inc}\varphi1: \varphi1 \preceq ?\varphi$  and  $\text{inc}\varphi2: \varphi2 \preceq ?\varphi$ 
  using  $\varphi1$   $\varphi2$   $\varphi$  local.wf by (metis conn.simps(5-8) helper-fact subformula-in-binary-conn(1,2))+

  have  $c1\text{-eq}: c1 \neq CEq$  and  $c2\text{-eq}: c2 \neq CEq$ 
  unfolding no-equiv-def using  $\text{inc}\varphi1$   $\text{inc}\varphi2$  by (metis  $\varphi1$   $\varphi2$   $\text{wf}\varphi1$   $\text{wf}\varphi2$  assms(1) no-equiv
no-equiv-eq(1) no-equiv-symb.elims(3) no-equiv-symb-conn-characterization wf-conn-list(4,5)
no-equiv-def subformula-all-subformula-st)+
  have  $c1\text{-imp}: c1 \neq CImp$  and  $c2\text{-imp}: c2 \neq CImp$ 
  using no-imp by (metis  $\varphi1$   $\varphi2$  all-subformula-st-decomp-explicit-imp(2,3) assms(1)
conn.simps(5,6) l no-imp-Imp(1) no-imp-symb.elims(3) no-imp-symb-conn-characterization
wf $\varphi1$  wf $\varphi2$  all-subformula-st-decomp no-imp-symb-conn-characterization)+
  have  $c1c: c1 \neq c'$ 
  proof
    assume  $c1c: c1 = c'$ 
    then obtain  $\xi1$   $\xi2$  where  $l1: l1 = [\xi1, \xi2]$ 
    by (metis assms(2) connective.distinct(37,39) helper-fact wf $\varphi1$  wf-conn.simps
wf-conn-list-decomp(1-3))
    have  $c\text{-in-}c'\text{-only } c$   $c' \text{ (conn } c$   $[\text{conn } c' l1, \varphi2])$  using  $c1c$   $l$  only  $\varphi1$  by auto
    moreover have  $\text{not-}c\text{-in-}c'\text{-symb } c$   $c' \text{ (conn } c$   $[\text{conn } c' l1, \varphi2])$ 
    using  $l1$   $\varphi1$   $c1c$   $l$  local.wf not-c-in-c'-symb-l wf $\varphi1$  by blast
    ultimately show False using  $\varphi1$   $c1c$   $l$   $l1$  local.wf not-c-in-c'-simp(4) wf $\varphi1$  by blast
  qed
  then have  $(\varphi1 = \text{conn } c$   $l1 \wedge \text{wf-conn } c$   $l1) \vee (\exists \psi1. \varphi1 = FNot \psi1) \vee \text{simple } \varphi1$ 
  by (metis  $\varphi1$  assms(1-3) c1-eq c1-imp simple.elims(3) wf $\varphi1$  wf-conn-list(4) wf-conn-list(5-7))
  moreover {
    assume  $\varphi1 = \text{conn } c$   $l1 \wedge \text{wf-conn } c$   $l1$ 
    then have only-c-inside  $c$   $\varphi1$ 
    by (metis IH $\varphi1$   $\varphi1$  all-subformula-st-decomp-imp inc $\varphi1$  no-equiv no-equiv-def no-imp no-imp-def

```

```

    c-in-only  $\varphi 1$  only-c-inside-def only-c-inside-into-only-c-inside simple-not simple-not-def
    subformula-all-subformula-st)
  }
  moreover {
    assume  $\exists \psi 1. \varphi 1 = FNot \psi 1$ 
    then obtain  $\psi 1$  where  $\varphi 1 = FNot \psi 1$  by metis
    then have only-c-inside  $c \varphi 1$ 
      by (metis all-subformula-st-def assms(1) connective.distinct(37,39) inc  $\varphi 1$ 
        only-c-inside-decomp-not simple-not simple-not-def simple-not-symb.simps(1))
    }
  moreover {
    assume simple  $\varphi 1$ 
    then have only-c-inside  $c \varphi 1$ 
      by (metis all-subformula-st-decomp-explicit(3) assms(1) connective.distinct(37,39)
        only-c-inside-decomp-not only-c-inside-def)
    }
  }
  ultimately have only-c-inside  $\varphi 1$ : only-c-inside  $c \varphi 1$  by metis

  have c-in-only  $\varphi 2$ : c-in-c'-only  $c \ c'$  (conn  $c2 \ l2$ )
    using only  $l \ \varphi 2 \ wf \ \varphi 2 \ assms$  unfolding c-in-c'-only-def by auto
  have c2c:  $c2 \neq c'$ 
  proof
    assume c2c:  $c2 = c'$ 
    then obtain  $\xi 1 \ \xi 2$  where  $l2: l2 = [\xi 1, \xi 2]$ 
      by (metis assms(2) wf  $\varphi 2 \ wf\text{-}conn.simps connective.distinct(7,9,19,21,29,31,37,39))
    then have c-in-c'-symb  $c \ c'$  (conn  $c \ [\varphi 1, \text{conn } c' \ l2]$ )
      using c2c  $l \ \text{only } \varphi 2 \ \text{all-subformula-st-test-symb-true-phi} unfolding c-in-c'-only-def by auto
    moreover have not-c-in-c'-symb  $c \ c'$  (conn  $c \ [\varphi 1, \text{conn } c' \ l2]$ )
      using assms(1) c2c  $l2 \ \text{not-c-in-c'-symb-r}$  wf  $\varphi 2 \ wf\text{-}conn\text{-helper-facts}(5,6) by metis
    ultimately show False by auto
  qed
  then have  $(\varphi 2 = \text{conn } c \ l2 \wedge wf\text{-}conn \ c \ l2) \vee (\exists \psi 2. \varphi 2 = FNot \psi 2) \vee \text{simple } \varphi 2$ 
    using c2-eq by (metis  $\varphi 2 \ assms(1-3) \ c2\text{-eq} \ c2\text{-imp} \ \text{simple.elims}$ (3) wf  $\varphi 2 \ wf\text{-}conn\text{-list}$ (4-7))
  moreover {
    assume  $\varphi 2 = \text{conn } c \ l2 \wedge wf\text{-}conn \ c \ l2$ 
    then have only-c-inside  $c \varphi 2$ 
      by (metis IH  $\varphi 2 \ \varphi 2 \ \text{all-subformula-st-decomp} \ inc \ \varphi 2 \ \text{no-equiv} \ \text{no-equiv-def} \ \text{no-imp} \ \text{no-imp-def}
        c-in-only  $\varphi 2 \ \text{only-c-inside-def} \ \text{only-c-inside-into-only-c-inside} \ \text{simple-not} \ \text{simple-not-def}
        subformula-all-subformula-st)
    }
  }
  moreover {
    assume  $\exists \psi 2. \varphi 2 = FNot \psi 2$ 
    then obtain  $\psi 2$  where  $\varphi 2 = FNot \psi 2$  by metis
    then have only-c-inside  $c \varphi 2$ 
      by (metis all-subformula-st-def assms(1-3) connective.distinct(38,40) inc  $\varphi 2$ 
        only-c-inside-decomp-not simple-not simple-not-def simple-not-symb.simps(1))
    }
  }
  moreover {
    assume simple  $\varphi 2$ 
    then have only-c-inside  $c \varphi 2$ 
      by (metis all-subformula-st-decomp-explicit(3) assms(1) connective.distinct(37,39)
        only-c-inside-decomp-not only-c-inside-def)
    }
  }
  ultimately have only-c-inside  $\varphi 2$ : only-c-inside  $c \varphi 2$  by metis
  show ?case using  $l \ \text{only-c-inside} \ \varphi 1 \ \text{only-c-inside} \ \varphi 2$  by auto$$$$$ 
```


qed

8.5.2 Push Conjunction

definition *pushConj* **where** *pushConj* = *push-conn-inside* CAnd COr

lemma *pushConj-consistent: preserves-un-sat pushConj*
unfolding *pushConj-def* **by** (*simp* add: *push-conn-inside-consistent*)

definition *and-in-or-symb* **where** *and-in-or-symb* = *c-in-c'-symb* CAnd COr

definition *and-in-or-only* **where**
and-in-or-only = *all-subformula-st* (*c-in-c'-symb* CAnd COr)

lemma *pushConj-inv*:
fixes $\varphi \psi :: 'v \text{ propo}$
assumes *full* (*propo-rew-step* *pushConj*) $\varphi \psi$
and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ **and** *simple-not* φ
shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ **and** *simple-not* ψ
using *push-conn-inside-inv* *assms* **unfolding** *pushConj-def* **by** *metis+*

lemma *pushConj-full-propo-rew-step*:
fixes $\varphi \psi :: 'v \text{ propo}$
assumes
 no-equiv φ **and**
 no-imp φ **and**
 full (*propo-rew-step* *pushConj*) $\varphi \psi$ **and**
 no-T-F-except-top-level φ **and**
 simple-not φ
shows *and-in-or-only* ψ
using *assms* *push-conn-inside-full-propo-rew-step*
unfolding *pushConj-def* *and-in-or-only-def* *c-in-c'-only-def* **by** (*metis* (*no-types*))

8.5.3 Push Disjunction

definition *pushDisj* **where** *pushDisj* = *push-conn-inside* COr CAnd

lemma *pushDisj-consistent: preserves-un-sat pushDisj*
unfolding *pushDisj-def* **by** (*simp* add: *push-conn-inside-consistent*)

definition *or-in-and-symb* **where** *or-in-and-symb* = *c-in-c'-symb* COr CAnd

definition *or-in-and-only* **where**
or-in-and-only = *all-subformula-st* (*c-in-c'-symb* COr CAnd)

lemma *not-or-in-and-only-or-and[simp]*:
 $\sim \text{or-in-and-only } (FOr (FAnd \psi1 \psi2) \varphi')$
unfolding *or-in-and-only-def*
by (*metis* *all-subformula-st-test-symb-true-phi* *conn.simps(5-6)* *not-c-in-c'-symb-l* *wf-conn-helper-facts(5)* *wf-conn-helper-facts(6)*)

lemma *pushDisj-inv*:
fixes $\varphi \psi :: 'v \text{ propo}$
assumes *full* (*propo-rew-step* *pushDisj*) $\varphi \psi$

and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ **and** *simple-not* φ
shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ **and** *simple-not* ψ
using *push-conn-inside-inv* *assms* **unfolding** *pushDisj-def* **by** *metis+*

lemma *pushDisj-full-propo-rew-step*:

fixes $\varphi \psi :: 'v \text{ propo}$

assumes

no-equiv φ **and**

no-imp φ **and**

full (*propo-rew-step* *pushDisj*) $\varphi \psi$ **and**

no-T-F-except-top-level φ **and**

simple-not φ

shows *or-in-and-only* ψ

using *assms* *push-conn-inside-full-propo-rew-step*

unfolding *pushDisj-def* *or-in-and-only-def* *c-in-c'-only-def* **by** (*metis* (*no-types*))

9 The full transformations

9.1 Abstract Property characterizing that only some connective are inside the others

9.1.1 Definition

The normal is a super group of groups

inductive *grouped-by* :: *'a* *connective* \Rightarrow *'a* *propo* \Rightarrow *bool* **for** *c* **where**

simple-is-grouped[*simp*]: *simple* $\varphi \Longrightarrow$ *grouped-by* *c* φ |

simple-not-is-grouped[*simp*]: *simple* $\varphi \Longrightarrow$ *grouped-by* *c* (*FNot* φ) |

connected-is-group[*simp*]: *grouped-by* *c* $\varphi \Longrightarrow$ *grouped-by* *c* $\psi \Longrightarrow$ *wf-conn* *c* [φ , ψ]
 \Longrightarrow *grouped-by* *c* (*conn* *c* [φ , ψ])

lemma *simple-clause*[*simp*]:

grouped-by *c* *FT*

grouped-by *c* *FF*

grouped-by *c* (*FVar* *x*)

grouped-by *c* (*FNot* *FT*)

grouped-by *c* (*FNot* *FF*)

grouped-by *c* (*FNot* (*FVar* *x*))

by *simp+*

lemma *only-c-inside-symb-c-eq-c'*:

only-c-inside-symb *c* (*conn* *c'* [$\varphi 1$, $\varphi 2$]) \Longrightarrow $c' = CAnd \vee c' = COr \Longrightarrow$ *wf-conn* *c'* [$\varphi 1$, $\varphi 2$]
 \Longrightarrow $c' = c$

by (*induct* *conn* *c'* [$\varphi 1$, $\varphi 2$] *rule*: *only-c-inside-symb.induct*, *auto* *simp*: *conn-inj*)

lemma *only-c-inside-c-eq-c'*:

only-c-inside *c* (*conn* *c'* [$\varphi 1$, $\varphi 2$]) \Longrightarrow $c' = CAnd \vee c' = COr \Longrightarrow$ *wf-conn* *c'* [$\varphi 1$, $\varphi 2$] \Longrightarrow $c = c'$

unfolding *only-c-inside-def* *all-subformula-st-def* **using** *only-c-inside-symb-c-eq-c'* *subformula-refl*

by *blast*

lemma *only-c-inside-imp-grouped-by*:

assumes *c*: $c \neq CNot$ **and** *c'*: $c' = CAnd \vee c' = COr$

shows *only-c-inside* *c* $\varphi \Longrightarrow$ *grouped-by* *c* φ (**is** *?O* $\varphi \Longrightarrow$ *?G* φ)

proof (*induct* φ *rule*: *propo-induct-arity*)

case (*nullary* φ *x*)

```

then show ?G  $\varphi$  by auto
next
case (unary  $\psi$ )
then show ?G (FNot  $\psi$ ) by (auto simp: c)
next
case (binary  $\varphi$   $\varphi 1$   $\varphi 2$ )
note IH $\varphi 1 = \text{this}(1)$  and IH $\varphi 2 = \text{this}(2)$  and  $\varphi = \text{this}(3)$  and only =  $\text{this}(4)$ 
have  $\varphi\text{-conn}$ :  $\varphi = \text{conn } c [\varphi 1, \varphi 2]$  and wf: wf-conn c  $[\varphi 1, \varphi 2]$ 
proof -
  obtain  $c'' l''$  where  $\varphi\text{-c''}$ :  $\varphi = \text{conn } c'' l''$  and wf: wf-conn  $c'' l''$ 
  using exists-c-conn by metis
  then have  $l''$ :  $l'' = [\varphi 1, \varphi 2]$  using  $\varphi$  by (metis wf-conn-list(4-7))
  have only-c-inside-symb c (conn  $c'' [\varphi 1, \varphi 2]$ )
  using only all-subformula-st-test-symb-true-phi
  unfolding only-c-inside-def  $\varphi\text{-c'' } l''$  by metis
  then have  $c = c''$ 
  by (metis  $\varphi$   $\varphi\text{-c''}$  conn-inj conn-inj-not(2)  $l''$  list.distinct(1) list.inject wf
    only-c-inside-symb.cases simple.simps(5-8))
  then show  $\varphi = \text{conn } c [\varphi 1, \varphi 2]$  and wf-conn c  $[\varphi 1, \varphi 2]$  using  $\varphi\text{-c''}$  wf  $l''$  by auto
qed
have grouped-by c  $\varphi 1$  using wf IH $\varphi 1$  IH $\varphi 2$   $\varphi\text{-conn}$  only  $\varphi$  unfolding only-c-inside-def by auto
moreover have grouped-by c  $\varphi 2$ 
using wf  $\varphi$  IH $\varphi 1$  IH $\varphi 2$   $\varphi\text{-conn}$  only unfolding only-c-inside-def by auto
ultimately show ?G  $\varphi$  using  $\varphi\text{-conn}$  connected-is-group local.wf by blast
qed

```

lemma grouped-by-false:

```

grouped-by c (conn  $c' [\varphi, \psi]$ )  $\implies c \neq c' \implies \text{wf-conn } c' [\varphi, \psi] \implies \text{False}$ 
apply (induct conn  $c' [\varphi, \psi]$  rule: grouped-by.induct)
apply (auto simp: simple-decomp wf-conn-list, auto simp: conn-inj)
by (metis list.distinct(1) list.sel(3) wf-conn-list(8))+

```

Then the CNF form is a conjunction of clauses: every clause is in CNF form and two formulas in CNF form can be related by an and.

inductive super-grouped-by:: 'a connective \Rightarrow 'a connective \Rightarrow 'a propo \Rightarrow bool **for** c c' **where**
 grouped-is-super-grouped[simp]: grouped-by c $\varphi \implies \text{super-grouped-by } c c' \varphi$ |
 connected-is-super-group: super-grouped-by c c' $\varphi \implies \text{super-grouped-by } c c' \psi \implies \text{wf-conn } c [\varphi, \psi]$
 $\implies \text{super-grouped-by } c c' (\text{conn } c' [\varphi, \psi])$

lemma simple-cnf[simp]:

```

super-grouped-by c c' FT
super-grouped-by c c' FF
super-grouped-by c c' (FVar x)
super-grouped-by c c' (FNot FT)
super-grouped-by c c' (FNot FF)
super-grouped-by c c' (FNot (FVar x))
by auto

```

lemma c-in-c'-only-super-grouped-by:

```

assumes c: c = CAnd  $\vee$  c = COr and c': c' = CAnd  $\vee$  c' = COr and cc': c  $\neq$  c'
shows no-equiv  $\varphi \implies \text{no-imp } \varphi \implies \text{simple-not } \varphi \implies \text{c-in-c'-only } c c' \varphi$ 
 $\implies \text{super-grouped-by } c c' \varphi$ 
(is ?NE  $\varphi \implies ?NI \varphi \implies ?SN \varphi \implies ?C \varphi \implies ?S \varphi$ )

```

proof (induct φ rule: propo-induct-arity)

```

case (nullary  $\varphi$   $x$ )
then show ?S  $\varphi$  by auto
next
case (unary  $\varphi$ )
then have simple-not-symb (FNot  $\varphi$ )
  using all-subformula-st-test-symb-true-phi unfolding simple-not-def by blast
then have  $\varphi = FT \vee \varphi = FF \vee (\exists x. \varphi = FVar x)$  by (cases  $\varphi$ , auto)
then show ?S (FNot  $\varphi$ ) by auto
next
case (binary  $\varphi$   $\varphi1$   $\varphi2$ )
note IH $\varphi1 = this(1)$  and IH $\varphi2 = this(2)$  and no-equiv = this(4) and no-imp = this(5)
  and simpleN = this(6) and c-in-c'-only = this(7) and  $\varphi' = this(3)$ 
{
  assume  $\varphi = FImp \varphi1 \varphi2 \vee \varphi = FEq \varphi1 \varphi2$ 
  then have False using no-equiv no-imp by auto
  then have ?S  $\varphi$  by auto
}
moreover {
  assume  $\varphi: \varphi = conn\ c' [\varphi1, \varphi2] \wedge wf\text{-}conn\ c' [\varphi1, \varphi2]$ 
  have c-in-c'-only: c-in-c'-only  $c\ c' \varphi1 \wedge c\text{-in-}c'\text{-only } c\ c' \varphi2 \wedge c\text{-in-}c'\text{-symb } c\ c' \varphi$ 
    using c-in-c'-only  $\varphi'$  unfolding c-in-c'-only-def by auto
  have super-grouped-by  $c\ c' \varphi1$  using  $\varphi\ c'$  no-equiv no-imp simpleN IH $\varphi1$  c-in-c'-only by auto
  moreover have super-grouped-by  $c\ c' \varphi2$ 
    using  $\varphi\ c'$  no-equiv no-imp simpleN IH $\varphi2$  c-in-c'-only by auto
  ultimately have ?S  $\varphi$ 
    using super-grouped-by.intros(2)  $\varphi$  by (metis c wf-conn-helper-facts(5,6))
}
moreover {
  assume  $\varphi: \varphi = conn\ c [\varphi1, \varphi2] \wedge wf\text{-}conn\ c [\varphi1, \varphi2]$ 
  then have only-c-inside  $c\ \varphi1 \wedge only\text{-}c\text{-inside } c\ \varphi2$ 
    using c-in-c'-symb-c-implies-only-c-inside  $c\ c' c\text{-in-}c'\text{-only list.set-intros(1)$ 
      wf-conn-helper-facts(5,6) no-equiv no-imp simpleN last-ConsL last-ConsR last-in-set
      list.distinct(1) by (metis (no-types, hide-lams) cc')
  then have only-c-inside  $c\ (conn\ c [\varphi1, \varphi2])$ 
    unfolding only-c-inside-def using  $\varphi$ 
    by (simp add: only-c-inside-into-only-c-inside all-subformula-st-decomp)
  then have grouped-by  $c\ \varphi$  using  $\varphi$  only-c-inside-imp-grouped-by  $c$  by blast
  then have ?S  $\varphi$  using super-grouped-by.intros(1) by metis
}
ultimately show ?S  $\varphi$  by (metis  $\varphi' c\ c' cc'$  conn.simps(5,6) wf-conn-helper-facts(5,6))
qed

```

9.2 Conjunctive Normal Form

definition *is-conj-with-TF* where *is-conj-with-TF* == *super-grouped-by COr CAnd*

lemma *or-in-and-only-conjunction-in-disj*:

shows *no-equiv* $\varphi \implies no\text{-}imp\ \varphi \implies simple\text{-}not\ \varphi \implies or\text{-}in\text{-}and\text{-}only\ \varphi \implies is\text{-}conj\text{-}with\text{-}TF\ \varphi$
using *c-in-c'-only-super-grouped-by*
unfolding *is-conj-with-TF-def or-in-and-only-def c-in-c'-only-def*
by (*simp add: c-in-c'-only-def c-in-c'-only-super-grouped-by*)

definition *is-cnf* where

is-cnf $\varphi \equiv is\text{-}conj\text{-}with\text{-}TF\ \varphi \wedge no\text{-}T\text{-}F\text{-}except\text{-}top\text{-}level\ \varphi$

9.2.1 Full CNF transformation

The full CNF transformation consists simply in chaining all the transformation defined before.

definition *cnf-rew* **where** *cnf-rew* =
 (full (propo-rew-step elim-equiv)) OO
 (full (propo-rew-step elim-imp)) OO
 (full (propo-rew-step elimTB)) OO
 (full (propo-rew-step pushNeg)) OO
 (full (propo-rew-step pushDisj))

lemma *cnf-rew-consistent: preserves-un-sat cnf-rew*
by (simp add: cnf-rew-def elimEquiv-lifted-consistant elim-imp-lifted-consistant elimTB-consistent
 preserves-un-sat-OO pushDisj-consistent pushNeg-lifted-consistant)

lemma *cnf-rew-is-cnf: cnf-rew φ φ' \implies is-cnf φ'*

apply (unfold cnf-rew-def OO-def)

apply auto

proof –

fix φ φEq φImp φTB φNeg \varphiDisj :: 'v propo

assume *Eq*: full (propo-rew-step elim-equiv) φ φEq

then have *no-equiv*: no-equiv φEq **using** no-equiv-full-propo-rew-step-elim-equiv **by** blast

assume *Imp*: full (propo-rew-step elim-imp) φEq φImp

then have *no-imp*: no-imp φImp **using** no-imp-full-propo-rew-step-elim-imp **by** blast

have *no-imp-inv*: no-equiv φImp **using** no-equiv Imp elim-imp-inv **by** blast

assume *TB*: full (propo-rew-step elimTB) φImp φTB

then have *noTB*: no-T-F-except-top-level φTB

using no-imp-inv no-imp elimTB-full-propo-rew-step **by** blast

have *noTB-inv*: no-equiv φTB no-imp φTB **using** elimTB-inv TB no-imp no-imp-inv **by** blast+

assume *Neg*: full (propo-rew-step pushNeg) φTB φNeg

then have *noNeg*: simple-not φNeg

using noTB-inv noTB pushNeg-full-propo-rew-step **by** blast

have *noNeg-inv*: no-equiv φNeg no-imp φNeg no-T-F-except-top-level φNeg

using pushNeg-inv Neg noTB noTB-inv **by** blast+

assume *Disj*: full (propo-rew-step pushDisj) φNeg \varphiDisj

then have *no-Disj*: or-in-and-only \varphiDisj

using noNeg-inv noNeg pushDisj-full-propo-rew-step **by** blast

have *noDisj-inv*: no-equiv \varphiDisj no-imp \varphiDisj no-T-F-except-top-level \varphiDisj

simple-not \varphiDisj

using pushDisj-inv Disj noNeg noNeg-inv **by** blast+

moreover have *is-conj-with-TF* \varphiDisj

using or-in-and-only-conjunction-in-disj noDisj-inv no-Disj **by** blast

ultimately show *is-cnf* \varphiDisj **unfolding** *is-cnf-def* **by** blast

qed

9.3 Disjunctive Normal Form

definition *is-disj-with-TF* **where** *is-disj-with-TF* \equiv super-grouped-by CAnd COr

lemma *and-in-or-only-conjunction-in-disj*:

shows $\text{no-equiv } \varphi \implies \text{no-imp } \varphi \implies \text{simple-not } \varphi \implies \text{and-in-or-only } \varphi \implies \text{is-disj-with-TF } \varphi$
using $c\text{-in-}c'\text{-only-super-grouped-by}$
unfolding $\text{is-disj-with-TF-def}$ $\text{and-in-or-only-def}$ $c\text{-in-}c'\text{-only-def}$
by ($\text{simp add: } c\text{-in-}c'\text{-only-def } c\text{-in-}c'\text{-only-super-grouped-by}$)

definition $\text{is-dnf} :: 'a \text{ propo} \Rightarrow \text{bool}$ **where**
 $\text{is-dnf } \varphi \longleftrightarrow \text{is-disj-with-TF } \varphi \wedge \text{no-T-F-except-top-level } \varphi$

9.3.1 Full DNF transform

The full DNF transformation consists simply in chaining all the transformation defined before.

definition dnf-rew **where** $\text{dnf-rew} \equiv$
 $(\text{full } (\text{propo-rew-step elim-equiv})) \text{ OO}$
 $(\text{full } (\text{propo-rew-step elim-imp})) \text{ OO}$
 $(\text{full } (\text{propo-rew-step elimTB})) \text{ OO}$
 $(\text{full } (\text{propo-rew-step pushNeg})) \text{ OO}$
 $(\text{full } (\text{propo-rew-step pushConj}))$

lemma $\text{dnf-rew-consistent: preserves-un-sat dnf-rew}$
by ($\text{simp add: dnf-rew-def elimEquiv-lifted-consistant elim-imp-lifted-consistant elimTB-consistent}$
 $\text{preserves-un-sat-OO pushConj-consistent pushNeg-lifted-consistant}$)

theorem $\text{dnf-transformation-correction:}$
 $\text{dnf-rew } \varphi \varphi' \implies \text{is-dnf } \varphi'$
apply ($\text{unfold dnf-rew-def OO-def}$)
by ($\text{meson and-in-or-only-conjunction-in-disj elimTB-full-propo-rew-step elimTB-inv}(1,2)$
 $\text{elim-imp-inv is-dnf-def no-equiv-full-propo-rew-step-elim-equiv}$
 $\text{no-imp-full-propo-rew-step-elim-imp pushConj-full-propo-rew-step pushConj-inv}(1-4)$
 $\text{pushNeg-full-propo-rew-step pushNeg-inv}(1-3)$)

10 More aggressive simplifications: Removing true and false at the beginning

10.1 Transformation

We should remove FT and FF at the beginning and not in the middle of the algorithm. To do this, we have to use more rules (one for each connective):

inductive elimTBFull **where**
 $\text{ElimTBFull1}[\text{simp}]: \text{elimTBFull } (F\text{And } \varphi \text{ FT}) \varphi \mid$
 $\text{ElimTBFull1}'[\text{simp}]: \text{elimTBFull } (F\text{And } \text{FT } \varphi) \varphi \mid$
 $\text{ElimTBFull2}[\text{simp}]: \text{elimTBFull } (F\text{And } \varphi \text{ FF}) \text{ FF} \mid$
 $\text{ElimTBFull2}'[\text{simp}]: \text{elimTBFull } (F\text{And } \text{FF } \varphi) \text{ FF} \mid$
 $\text{ElimTBFull3}[\text{simp}]: \text{elimTBFull } (F\text{Or } \varphi \text{ FT}) \text{ FT} \mid$
 $\text{ElimTBFull3}'[\text{simp}]: \text{elimTBFull } (F\text{Or } \text{FT } \varphi) \text{ FT} \mid$
 $\text{ElimTBFull4}[\text{simp}]: \text{elimTBFull } (F\text{Or } \varphi \text{ FF}) \varphi \mid$
 $\text{ElimTBFull4}'[\text{simp}]: \text{elimTBFull } (F\text{Or } \text{FF } \varphi) \varphi \mid$
 $\text{ElimTBFull5}[\text{simp}]: \text{elimTBFull } (F\text{Not } \text{FT}) \text{ FF} \mid$
 $\text{ElimTBFull5}'[\text{simp}]: \text{elimTBFull } (F\text{Not } \text{FF}) \text{ FT} \mid$

$\text{ElimTBFull6-l[simp]}: \text{elimTBFull } (F\text{Imp } FT \ \varphi) \ \varphi \mid$
 $\text{ElimTBFull6-l'[simp]}: \text{elimTBFull } (F\text{Imp } FF \ \varphi) \ FT \mid$
 $\text{ElimTBFull6-r[simp]}: \text{elimTBFull } (F\text{Imp } \varphi \ FT) \ FT \mid$
 $\text{ElimTBFull6-r'[simp]}: \text{elimTBFull } (F\text{Imp } \varphi \ FF) \ (F\text{Not } \varphi) \mid$

$\text{ElimTBFull7-l[simp]}: \text{elimTBFull } (F\text{Eq } FT \ \varphi) \ \varphi \mid$
 $\text{ElimTBFull7-l'[simp]}: \text{elimTBFull } (F\text{Eq } FF \ \varphi) \ (F\text{Not } \varphi) \mid$
 $\text{ElimTBFull7-r[simp]}: \text{elimTBFull } (F\text{Eq } \varphi \ FT) \ \varphi \mid$
 $\text{ElimTBFull7-r'[simp]}: \text{elimTBFull } (F\text{Eq } \varphi \ FF) \ (F\text{Not } \varphi) \mid$

The transformation is still consistent.

lemma *elimTBFull-consistent: preserves-un-sat elimTBFull*

proof –

{
 fix $\varphi \ \psi:: 'b \text{ propo}$
 have $\text{elimTBFull } \varphi \ \psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$
 by (*induct-tac rule: elimTBFull.inducts, auto*)
 }
then show *?thesis* **using** *preserves-un-sat-def* **by** *auto*
qed

Contrary to the theorem $\llbracket \text{no-equiv } ?\varphi; \text{no-imp } ?\varphi; ?\psi \preceq ?\varphi; \neg \text{no-T-F-symb-except-toplevel } ?\psi \rrbracket \implies \exists \psi'. \text{elimTB } ?\psi \ \psi'$, we do not need the assumption *no-equiv* φ and *no-imp* φ , since our transformation is more general.

lemma *no-T-F-symb-except-toplevel-step-exists'*:

fixes $\varphi:: 'v \text{ propo}$
shows $\psi \preceq \varphi \implies \neg \text{no-T-F-symb-except-toplevel } \psi \implies \exists \psi'. \text{elimTBFull } \psi \ \psi'$
proof (*induct* ψ *rule: propo-induct-arity*)
 case (*nullary* φ')
 then have *False* **using** *no-T-F-symb-except-toplevel-true no-T-F-symb-except-toplevel-false* **by** *auto*
 then show $\text{Ex } (\text{elimTBFull } \varphi')$ **by** *blast*
next
 case (*unary* ψ)
 then have $\psi = FF \vee \psi = FT$ **using** *no-T-F-symb-except-toplevel-not-decom* **by** *blast*
 then show $\text{Ex } (\text{elimTBFull } (F\text{Not } \psi))$ **using** *ElimTBFull5 ElimTBFull5'* **by** *blast*
next
 case (*binary* $\varphi' \ \psi1 \ \psi2$)
 then have $\psi1 = FT \vee \psi2 = FT \vee \psi1 = FF \vee \psi2 = FF$
 by (*metis binary-connectives-def conn.simps(5-8) insertI1 insert-commute*
 no-T-F-symb-except-toplevel-bin-decom binary.hyps(3))
 then show $\text{Ex } (\text{elimTBFull } \varphi')$ **using** *elimTBFull.intros binary.hyps(3)* **by** *blast*
qed

The same applies here. We do not need the assumption, but the deep link between $\neg \text{no-T-F-except-top-level}$ φ and the existence of a rewriting step, still exists.

lemma *no-T-F-except-top-level-rew'*:

fixes $\varphi:: 'v \text{ propo}$
assumes *noTB: $\neg \text{no-T-F-except-top-level } \varphi$*
shows $\exists \psi \ \psi'. \psi \preceq \varphi \wedge \text{elimTBFull } \psi \ \psi'$
proof –
 have *test-symb-false-nullary*:
 $\forall x. \text{no-T-F-symb-except-toplevel } (FF:: 'v \text{ propo}) \wedge \text{no-T-F-symb-except-toplevel } FT$
 $\wedge \text{no-T-F-symb-except-toplevel } (F\text{Var } (x:: 'v))$
 by *auto*
 moreover {

```

fix c :: 'v connective and l :: 'v propo list and ψ :: 'v propo
have H: elimTBFull (conn c l) ψ ⇒ ¬no-T-F-symb-except-toplevel (conn c l)
  by (cases (conn c l) rule: elimTBFull.cases) auto
}
ultimately show ?thesis
  using no-test-symb-step-exists[of no-T-F-symb-except-toplevel φ elimTBFull] noTB
  no-T-F-symb-except-toplevel-step-exists' unfolding no-T-F-except-top-level-def by metis
qed

```

```

lemma elimTBFull-full-propo-rew-step:
  fixes φ ψ :: 'v propo
  assumes full (propo-rew-step elimTBFull) φ ψ
  shows no-T-F-except-top-level ψ
  using full-propo-rew-step-subformula no-T-F-except-top-level-rew' assms by fastforce

```

10.2 More invariants

As the aim is to use the transformation as the first transformation, we have to show some more invariants for *elim-equiv* and *elim-imp*. For the other transformation, we have already proven it.

lemma *propo-rew-step-ElimEquiv-no-T-F*: *propo-rew-step elim-equiv φ ψ ⇒ no-T-F φ ⇒ no-T-F ψ*
proof (*induct rule: propo-rew-step.induct*)

```

fix φ' :: 'v propo and ψ' :: 'v propo
assume a1: no-T-F φ'
assume a2: elim-equiv φ' ψ'
have ∀ x0 x1. (¬ elim-equiv (x1 :: 'v propo) x0 ∨ (∃ v2 v3 v4 v5 v6 v7. x1 = FEq v2 v3
  ∧ x0 = FAnd (FImp v4 v5) (FImp v6 v7) ∧ v2 = v4 ∧ v4 = v7 ∧ v3 = v5 ∧ v3 = v6))
  = (¬ elim-equiv x1 x0 ∨ (∃ v2 v3 v4 v5 v6 v7. x1 = FEq v2 v3
  ∧ x0 = FAnd (FImp v4 v5) (FImp v6 v7) ∧ v2 = v4 ∧ v4 = v7 ∧ v3 = v5 ∧ v3 = v6))
  by meson
then have ∀ p pa. ¬ elim-equiv (p :: 'v propo) pa ∨ (∃ pb pc pd pe pf pg. p = FEq pb pc
  ∧ pa = FAnd (FImp pd pe) (FImp pf pg) ∧ pb = pd ∧ pd = pg ∧ pc = pe ∧ pc = pf)
  using elim-equiv.cases by force
then show no-T-F ψ' using a1 a2 by fastforce

```

next

```

fix φ φ' :: 'v propo and ξ ξ' :: 'v propo list and c :: 'v connective
assume rel: propo-rew-step elim-equiv φ φ'
and IH: no-T-F φ ⇒ no-T-F φ'
and corr: wf-conn c (ξ @ φ # ξ')
and no-T-F: no-T-F (conn c (ξ @ φ # ξ'))
{
  assume c: c = CNot
  then have empty: ξ = [] ξ' = [] using corr by auto
  then have no-T-F φ using no-T-F c no-T-F-decomp-not by auto
  then have no-T-F (conn c (ξ @ φ' # ξ')) using c empty no-T-F-comp-not IH by auto
}
moreover {
  assume c: c ∈ binary-connectives
  obtain a b where ab: ξ @ φ # ξ' = [a, b]
  using corr c list-length2-decomp wf-conn-bin-list-length by metis
  then have φ: φ = a ∨ φ = b
  by (metis append.simps(1) append-is-Nil-conv list.distinct(1) list.sel(3) nth-Cons-0
    tl-append2)

```



```

have ζ: ∀ ζ ∈ set (ξ @ φ # ξ'). no-T-F ζ
  using no-T-F unfolding no-T-F-def using corr all-subformula-st-decomp by blast

then have φ': no-T-F φ' using ab IH φ by auto
have l': ξ @ φ' # ξ' = [φ', b] ∨ ξ @ φ' # ξ' = [a, φ']
  by (metis (no-types, hide-lams) ab append-Cons append-Nil append-Nil2 butlast.simps(2)
      butlast-append list.distinct(1) list.sel(3))
then have ∀ ζ ∈ set (ξ @ φ' # ξ'). no-T-F ζ using ζ φ' ab by fastforce
moreover
  have ∀ ζ ∈ set (ξ @ φ # ξ'). ζ ≠ FT ∧ ζ ≠ FF
    using ζ corr no-T-F no-T-F-except-top-level-false no-T-F-no-T-F-except-top-level by blast
  then have no-T-F-symb (conn c (ξ @ φ' # ξ'))
    by (metis φ' l' ab all-subformula-st-test-symb-true-phi c list.distinct(1)
        list.set-intros(1,2) no-T-F-symb-except-toplevel-bin-decom
        no-T-F-symb-except-toplevel-no-T-F-symb no-T-F-symb-false(1,2) no-T-F-def wf-conn-binary
        wf-conn-list(1,2))
  ultimately have no-T-F (conn c (ξ @ φ' # ξ'))
    by (metis l' all-subformula-st-decomp-imp c no-T-F-def wf-conn-binary)
}
moreover {
  fix x
  assume c = CVar x ∨ c = CF ∨ c = CT
  then have False using corr by auto
  then have no-T-F (conn c (ξ @ φ' # ξ')) by auto
}
ultimately show no-T-F (conn c (ξ @ φ' # ξ')) using corr wf-conn.cases by metis
qed

```

lemma *elim-equiv-inv'*:

```

fixes φ ψ :: 'v propo
assumes full (propo-rew-step elim-equiv) φ ψ and no-T-F-except-top-level φ
shows no-T-F-except-top-level ψ
proof -
{
  fix φ ψ :: 'v propo
  have propo-rew-step elim-equiv φ ψ ⇒ no-T-F-except-top-level φ
    ⇒ no-T-F-except-top-level ψ
  proof -
    assume rel: propo-rew-step elim-equiv φ ψ
    and no: no-T-F-except-top-level φ
    {
      assume φ = FT ∨ φ = FF
      from rel this have False
      apply (induct rule: propo-rew-step.induct, auto simp: wf-conn-list(1,2))
      using elim-equiv.simps by blast+
      then have no-T-F-except-top-level ψ by blast
    }
  moreover {
    assume φ ≠ FT ∧ φ ≠ FF
    then have no-T-F φ
      by (metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
    then have no-T-F ψ using propo-rew-step-ElimEquiv-no-T-F rel by blast
    then have no-T-F-except-top-level ψ by (simp add: no-T-F-no-T-F-except-top-level)
  }
  ultimately show no-T-F-except-top-level ψ by metis
}

```

```

    qed
  }
  moreover {
    fix c :: 'v connective and  $\xi \xi' :: 'v \text{ propo list}$  and  $\zeta \zeta' :: 'v \text{ propo}$ 
    assume rel: propo-rew-step elim-equiv  $\zeta \zeta'$ 
    and incl:  $\zeta \preceq \varphi$ 
    and corr: wf-conn c ( $\xi @ \zeta \# \xi'$ )
    and no-T-F: no-T-F-symb-except-toplevel (conn c ( $\xi @ \zeta \# \xi'$ ))
    and n: no-T-F-symb-except-toplevel  $\zeta'$ 
    have no-T-F-symb-except-toplevel (conn c ( $\xi @ \zeta' \# \xi'$ ))
    proof
      have p: no-T-F-symb (conn c ( $\xi @ \zeta \# \xi'$ ))
        using corr wf-conn-list(1) wf-conn-list(2) no-T-F-symb-except-toplevel-no-T-F-symb no-T-F
        by blast
      have l:  $\forall \varphi \in \text{set } (\xi @ \zeta \# \xi'). \varphi \neq FT \wedge \varphi \neq FF$ 
        using corr wf-conn-no-T-F-symb-iff p by blast
      from rel incl have  $\zeta' \neq FT \wedge \zeta' \neq FF$ 
        apply (induction  $\zeta \zeta'$  rule: propo-rew-step.induct)
        apply (cases rule: elim-equiv.cases, auto simp: elim-equiv.simps)
        by (metis append-is-Nil-conv list.distinct wf-conn-list(1,2) wf-conn-no-arity-change
            wf-conn-no-arity-change-helper) +
      then have  $\forall \varphi \in \text{set } (\xi @ \zeta' \# \xi'). \varphi \neq FT \wedge \varphi \neq FF$  using l by auto
      moreover have  $c \neq CT \wedge c \neq CF$  using corr by auto
      ultimately show no-T-F-symb (conn c ( $\xi @ \zeta' \# \xi'$ ))
        by (metis corr wf-conn-no-arity-change wf-conn-no-arity-change-helper no-T-F-symb-comp)
    qed
  }
  ultimately show no-T-F-except-top-level  $\psi$ 
    using full-propo-rew-step-inv-stay-with-inc[of elim-equiv no-T-F-symb-except-toplevel  $\varphi$ ]
    assms subformula-refl unfolding no-T-F-except-top-level-def by metis
  qed

```

lemma *propo-rew-step-ElimImp-no-T-F*: *propo-rew-step elim-imp* $\varphi \psi \implies \text{no-T-F } \varphi \implies \text{no-T-F } \psi$

proof (*induct* rule: *propo-rew-step.induct*)

case (*global-rel* $\varphi' \psi'$)

then show *no-T-F* ψ'

using *elim-imp.cases* *no-T-F-comp-not* *no-T-F-decomp*(1,2)

by (*metis* *no-T-F-comp-expanded-explicit*(2))

next

case (*propo-rew-one-step-lift* $\varphi \varphi' c \xi \xi'$)

note rel = *this*(1) and IH = *this*(2) and corr = *this*(3) and *no-T-F* = *this*(4)

{

assume c: $c = CNot$

then have *empty*: $\xi = [] \wedge \xi' = []$ using corr by auto

then have *no-T-F* φ using *no-T-F* c *no-T-F-decomp-not* by auto

then have *no-T-F* (*conn* c ($\xi @ \varphi' \# \xi'$)) using c *empty* *no-T-F-comp-not* IH by auto

}

moreover {

assume c: $c \in \text{binary-connectives}$

then obtain a b where $ab: \xi @ \varphi \# \xi' = [a, b]$

using corr *list-length2-decomp* *wf-conn-bin-list-length* by *metis*

then have $\varphi: \varphi = a \vee \varphi = b$

by (*metis* *append-self-conv2* *wf-conn-list-decomp*(4) *wf-conn-unary* *list.discI* *list.sel*(3)
 nth-Cons-0 *tl-append2*)

```

have ζ: ∀ ζ ∈ set (ξ @ φ # ξ'). no-T-F ζ using ab c propo-rew-one-step-lift.prem by auto

then have φ': no-T-F φ'
  using ab IH φ corr no-T-F no-T-F-def all-subformula-st-decomp-explicit by auto
have χ: ξ @ φ' # ξ' = [φ', b] ∨ ξ @ φ' # ξ' = [a, φ']
  by (metis (no-types, hide-lams) ab append-Cons append-Nil append-Nil2 butlast.simps(2)
      butlast-append list.distinct(1) list.sel(3))
then have ∀ ζ ∈ set (ξ @ φ' # ξ'). no-T-F ζ using ζ φ' ab by fastforce
moreover
  have no-T-F (last (ξ @ φ' # ξ')) by (simp add: calculation)
  then have no-T-F-symb (conn c (ξ @ φ' # ξ'))
    by (metis χ φ' ζ ab all-subformula-st-test-symb-true-phi c last.simps list.distinct(1)
        list.set-intros(1) no-T-F-bin-decomp no-T-F-def)
  ultimately have no-T-F (conn c (ξ @ φ' # ξ')) using c χ by fastforce
}
moreover {
  fix x
  assume c = CVar x ∨ c = CF ∨ c = CT
  then have False using corr by auto
  then have no-T-F (conn c (ξ @ φ' # ξ')) by auto
}
ultimately show no-T-F (conn c (ξ @ φ' # ξ')) using corr wf-conn.cases by blast
qed

```

```

lemma elim-imp-inv':
  fixes φ ψ :: 'v propo
  assumes full (propo-rew-step elim-imp) φ ψ and no-T-F-except-top-level φ
  shows no-T-F-except-top-level ψ
proof -
  {
    {
      fix φ ψ :: 'v propo
      have H: elim-imp φ ψ ⟹ no-T-F-except-top-level φ ⟹ no-T-F-except-top-level ψ
        by (induct φ ψ rule: elim-imp.induct, auto)
    } note H = this
    fix φ ψ :: 'v propo
    have propo-rew-step elim-imp φ ψ ⟹ no-T-F-except-top-level φ ⟹ no-T-F-except-top-level ψ
    proof -
      assume rel: propo-rew-step elim-imp φ ψ
      and no: no-T-F-except-top-level φ
      {
        assume φ = FT ∨ φ = FF
        from rel this have False
        apply (induct rule: propo-rew-step.induct)
        by (cases rule: elim-imp.cases, auto simp: wf-conn-list(1,2))
        then have no-T-F-except-top-level ψ by blast
      }
    moreover {
      assume φ ≠ FT ∧ φ ≠ FF
      then have no-T-F φ
        by (metis no no-T-F-symb-except-top-level-all-subformula-st-no-T-F-symb)
      then have no-T-F ψ
        using rel propo-rew-step-ElimImp-no-T-F by blast
      then have no-T-F-except-top-level ψ by (simp add: no-T-F-no-T-F-except-top-level)
    }
  }

```

```

    }
    ultimately show no-T-F-except-top-level  $\psi$  by metis
  qed
}
moreover {
  fix  $c :: 'v$  connective and  $\xi \xi' :: 'v$  propo list and  $\zeta \zeta' :: 'v$  propo
  assume rel: propo-rew-step elim-imp  $\zeta \zeta'$ 
  and incl:  $\zeta \preceq \varphi$ 
  and corr: wf-conn  $c (\xi @ \zeta \# \xi')$ 
  and no-T-F: no-T-F-symb-except-toplevel (conn  $c (\xi @ \zeta \# \xi')$ )
  and n: no-T-F-symb-except-toplevel  $\zeta'$ 
  have no-T-F-symb-except-toplevel (conn  $c (\xi @ \zeta' \# \xi')$ )
  proof
    have  $p$ : no-T-F-symb (conn  $c (\xi @ \zeta \# \xi')$ )
    by (simp add: corr no-T-F no-T-F-symb-except-toplevel-no-T-F-symb wf-conn-list(1,2))

    have  $l$ :  $\forall \varphi \in \text{set } (\xi @ \zeta \# \xi'). \varphi \neq FT \wedge \varphi \neq FF$ 
    using corr wf-conn-no-T-F-symb-iff  $p$  by blast
    from rel incl have  $\zeta' \neq FT \wedge \zeta' \neq FF$ 
    apply (induction  $\zeta \zeta'$  rule: propo-rew-step.induct)
    apply (cases rule: elim-imp.cases, auto)
    using wf-conn-list(1,2) wf-conn-no-arity-change wf-conn-no-arity-change-helper
    by (metis append-is-Nil-conv list.distinct(1))+
    then have  $\forall \varphi \in \text{set } (\xi @ \zeta' \# \xi'). \varphi \neq FT \wedge \varphi \neq FF$  using  $l$  by auto
    moreover have  $c \neq CT \wedge c \neq CF$  using corr by auto
    ultimately show no-T-F-symb (conn  $c (\xi @ \zeta' \# \xi')$ )
    using corr wf-conn-no-arity-change no-T-F-symb-comp
    by (metis wf-conn-no-arity-change-helper)
  qed
}
ultimately show no-T-F-except-top-level  $\psi$ 
using full-propo-rew-step-inv-stay-with-inc[of elim-imp no-T-F-symb-except-toplevel  $\varphi$ ]
assms subformula-refl unfolding no-T-F-except-top-level-def by metis
qed

```

10.3 The new CNF and DNF transformation

The transformation is the same as before, but the order is not the same.

definition $\text{dnf-rew}' :: 'a \text{ propo} \Rightarrow 'a \text{ propo} \Rightarrow \text{bool}$ where

```

dnf-rew' =
  (full (propo-rew-step elimTBFULL)) OO
  (full (propo-rew-step elim-equiv)) OO
  (full (propo-rew-step elim-imp)) OO
  (full (propo-rew-step pushNeg)) OO
  (full (propo-rew-step pushConj))

```

lemma $\text{dnf-rew}'$ -consistent: preserves-un-sat $\text{dnf-rew}'$

```

by (simp add: dnf-rew'-def elimEquiv-lifted-consistant elim-imp-lifted-consistant
  elimTBFULL-consistent preserves-un-sat-OO pushConj-consistent pushNeg-lifted-consistant)

```

theorem cnf-transformation-correction:

```

  dnf-rew'  $\varphi \varphi' \implies \text{is-dnf } \varphi'$ 
  unfolding dnf-rew'-def OO-def
  by (meson and-in-or-only-conjunction-in-disj elimTBFULL-full-propo-rew-step elim-equiv-inv'
    elim-imp-inv elim-imp-inv' is-dnf-def no-equiv-full-propo-rew-step-elim-equiv)

```

no-imp-full-propo-rew-step-elim-imp pushConj-full-propo-rew-step pushConj-inv(1-4)
pushNeg-full-propo-rew-step pushNeg-inv(1-3))

Given all the lemmas before the CNF transformation is easy to prove:

definition *cnf-rew'* :: 'a propo \Rightarrow 'a propo \Rightarrow bool **where**

cnf-rew' =
(full (propo-rew-step elimTBFull)) OO
(full (propo-rew-step elim-equiv)) OO
(full (propo-rew-step elim-imp)) OO
(full (propo-rew-step pushNeg)) OO
(full (propo-rew-step pushDisj))

lemma *cnf-rew'-consistent: preserves-un-sat cnf-rew'*

by (*simp add: cnf-rew'-def elimEquiv-lifted-consistant elim-imp-lifted-consistant*
elimTBFull-consistent preserves-un-sat-OO pushDisj-consistent pushNeg-lifted-consistant)

theorem *cnf'-transformation-correction:*

cnf-rew' φ φ' \Longrightarrow is-cnf φ'

unfolding *cnf-rew'-def OO-def*

by (*meson elimTBFull-full-propo-rew-step elim-equiv-inv' elim-imp-inv elim-imp-inv' is-cnf-def*
no-equiv-full-propo-rew-step-elim-equiv no-imp-full-propo-rew-step-elim-imp
or-in-and-only-conjunction-in-disj pushDisj-full-propo-rew-step pushDisj-inv(1-4)
pushNeg-full-propo-rew-step pushNeg-inv(1) pushNeg-inv(2) pushNeg-inv(3))

end

11 Partial Clausal Logic

theory *Partial-Clausal-Logic*

imports *../lib/Clausal-Logic List-More*

begin

11.1 Clauses

Clauses are (finite) multisets of literals.

type-synonym 'a clause = 'a literal multiset

type-synonym 'v clauses = 'v clause set

11.2 Partial Interpretations

type-synonym 'a interp = 'a literal set

definition *true-lit* :: 'a interp \Rightarrow 'a literal \Rightarrow bool (**infix** \models_l 50) **where**

$I \models_l L \longleftrightarrow L \in I$

declare *true-lit-def[simp]*

11.2.1 Consistency

definition *consistent-interp* :: 'a literal set \Rightarrow bool **where**

consistent-interp I = ($\forall L. \neg(L \in I \wedge \neg L \in I)$)

lemma *consistent-interp-empty[simp]:*

consistent-interp {} unfolding consistent-interp-def by auto

lemma *consistent-interp-single[simp]*:
consistent-interp $\{L\}$ **unfolding** *consistent-interp-def* **by** *auto*

lemma *consistent-interp-subset*:
assumes
 $A \subseteq B$ **and**
consistent-interp B
shows *consistent-interp* A
using *assms* **unfolding** *consistent-interp-def* **by** *auto*

lemma *consistent-interp-change-insert*:
 $a \notin A \implies -a \notin A \implies \text{consistent-interp } (\text{insert } (-a) A) \longleftrightarrow \text{consistent-interp } (\text{insert } a A)$
unfolding *consistent-interp-def* **by** *fastforce*

lemma *consistent-interp-insert-pos[simp]*:
 $a \notin A \implies \text{consistent-interp } (\text{insert } a A) \longleftrightarrow \text{consistent-interp } A \wedge -a \notin A$
unfolding *consistent-interp-def* **by** *auto*

lemma *consistent-interp-insert-not-in*:
consistent-interp $A \implies a \notin A \implies -a \notin A \implies \text{consistent-interp } (\text{insert } a A)$
unfolding *consistent-interp-def* **by** *auto*

11.2.2 Atoms

definition *atms-of-ms* :: '*a* literal multiset set \Rightarrow '*a* set **where**
atms-of-ms $\psi s = \bigcup (\text{atms-of } ' \psi s)$

lemma *atms-of-mmltiset[simp]*:
atms-of (*mset* a) = *atm-of* '*set* a
by (*induct* a) *auto*

lemma *atms-of-ms-mset-unfold*:
atms-of-ms (*mset* ' b) = $(\bigcup x \in b. \text{atm-of } ' \text{set } x)$
unfolding *atms-of-ms-def* **by** *simp*

definition *atms-of-s* :: '*a* literal set \Rightarrow '*a* set **where**
atms-of-s $C = \text{atm-of } ' C$

lemma *atms-of-ms-empty-set[simp]*:
atms-of-ms $\{\}$ = $\{\}$
unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-mempty[simp]*:
atms-of-ms $\{\{\#\}\}$ = $\{\}$
unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-mono*:
 $A \subseteq B \implies \text{atms-of-ms } A \subseteq \text{atms-of-ms } B$
unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-finite[simp]*:
finite $\psi s \implies \text{finite } (\text{atms-of-ms } \psi s)$
unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-union[simp]*:

$atms\text{-}of\text{-}ms (\psi s \cup \chi s) = atms\text{-}of\text{-}ms \psi s \cup atms\text{-}of\text{-}ms \chi s$
unfolding $atms\text{-}of\text{-}ms\text{-}def$ **by** *auto*

lemma $atms\text{-}of\text{-}ms\text{-}insert[simp]$:
 $atms\text{-}of\text{-}ms (insert \psi s \chi s) = atms\text{-}of \psi s \cup atms\text{-}of\text{-}ms \chi s$
unfolding $atms\text{-}of\text{-}ms\text{-}def$ **by** *auto*

lemma $atms\text{-}of\text{-}ms\text{-}singleton[simp]$: $atms\text{-}of\text{-}ms \{L\} = atms\text{-}of L$
unfolding $atms\text{-}of\text{-}ms\text{-}def$ **by** *auto*

lemma $atms\text{-}of\text{-}atms\text{-}of\text{-}ms\text{-}mono[simp]$:
 $A \in \psi \implies atms\text{-}of A \subseteq atms\text{-}of\text{-}ms \psi$
unfolding $atms\text{-}of\text{-}ms\text{-}def$ **by** *fastforce*

lemma $atms\text{-}of\text{-}ms\text{-}single\text{-}set\text{-}mset\text{-}atms\text{-}of[simp]$:
 $atms\text{-}of\text{-}ms (single \text{ ' } set\text{-}mset B) = atms\text{-}of B$
unfolding $atms\text{-}of\text{-}ms\text{-}def$ $atms\text{-}of\text{-}def$ **by** *auto*

lemma $atms\text{-}of\text{-}ms\text{-}remove\text{-}incl$:
shows $atms\text{-}of\text{-}ms (Set.remove a \psi) \subseteq atms\text{-}of\text{-}ms \psi$
unfolding $atms\text{-}of\text{-}ms\text{-}def$ **by** *auto*

lemma $atms\text{-}of\text{-}ms\text{-}remove\text{-}subset$:
 $atms\text{-}of\text{-}ms (\varphi - \psi) \subseteq atms\text{-}of\text{-}ms \varphi$
unfolding $atms\text{-}of\text{-}ms\text{-}def$ **by** *auto*

lemma $finite\text{-}atms\text{-}of\text{-}ms\text{-}remove\text{-}subset[simp]$:
 $finite (atms\text{-}of\text{-}ms A) \implies finite (atms\text{-}of\text{-}ms (A - C))$
using $atms\text{-}of\text{-}ms\text{-}remove\text{-}subset[of A C]$ $finite\text{-}subset$ **by** *blast*

lemma $atms\text{-}of\text{-}ms\text{-}empty\text{-}iff$:
 $atms\text{-}of\text{-}ms A = \{\} \longleftrightarrow A = \{\{\#\}\} \vee A = \{\}$
apply (*rule iffI*)
apply (*metis (no-types, lifting) atms-empty-iff-empty atms-of-atms-of-ms-mono insert-absorb singleton-iff singleton-insert-inj-eq' subsetI subset-empty*)
apply *auto*[]
done

lemma $in\text{-}implies\text{-}atm\text{-}of\text{-}on\text{-}atms\text{-}of\text{-}ms$:
assumes $L \in \# C$ **and** $C \in N$
shows $atm\text{-}of L \in atms\text{-}of\text{-}ms N$
using $atms\text{-}of\text{-}atms\text{-}of\text{-}ms\text{-}mono[of C N]$ *assms* **by** (*simp add: atm-of-lit-in-atms-of subset-iff*)

lemma $in\text{-}plus\text{-}implies\text{-}atm\text{-}of\text{-}on\text{-}atms\text{-}of\text{-}ms$:
assumes $C + \{\#L\# \} \in N$
shows $atm\text{-}of L \in atms\text{-}of\text{-}ms N$
using $in\text{-}implies\text{-}atm\text{-}of\text{-}on\text{-}atms\text{-}of\text{-}ms[of - C + \{\#L\# \}]$ *assms* **by** *auto*

lemma $in\text{-}m\text{-}in\text{-}literals$:
assumes $\{\#A\# \} + D \in \psi s$
shows $atm\text{-}of A \in atms\text{-}of\text{-}ms \psi s$
using *assms* **by** (*auto dest: atms-of-atms-of-ms-mono*)

lemma $atms\text{-}of\text{-}s\text{-}union[simp]$:
 $atms\text{-}of\text{-}s (Ia \cup Ib) = atms\text{-}of\text{-}s Ia \cup atms\text{-}of\text{-}s Ib$

unfolding *atms-of-s-def* **by** *auto*

lemma *atms-of-s-single[simp]*:

atms-of-s $\{L\} = \{atm-of\ L\}$

unfolding *atms-of-s-def* **by** *auto*

lemma *atms-of-s-insert[simp]*:

atms-of-s (*insert* *L Ib*) = $\{atm-of\ L\} \cup atms-of-s\ Ib$

unfolding *atms-of-s-def* **by** *auto*

lemma *in-atms-of-s-decomp[iff]*:

$P \in atms-of-s\ I \longleftrightarrow (Pos\ P \in I \vee Neg\ P \in I)$ (**is** $?P \longleftrightarrow ?Q$)

proof

assume $?P$

then show $?Q$ **unfolding** *atms-of-s-def* **by** (*metis image-iff literal.exhaust-sel*)

next

assume $?Q$

then show $?P$ **unfolding** *atms-of-s-def* **by** *force*

qed

lemma *atm-of-in-atm-of-set-in-uminus*:

atm-of $L' \in atm-of\ 'B \implies L' \in B \vee -\ L' \in B$

using *atms-of-s-def* **by** (*cases* L') *fastforce+*

11.2.3 Totality

definition *total-over-set* :: $'a\ interp \Rightarrow 'a\ set \Rightarrow bool$ **where**

total-over-set $I\ S = (\forall l \in S. Pos\ l \in I \vee Neg\ l \in I)$

definition *total-over-m* :: $'a\ literal\ set \Rightarrow 'a\ clause\ set \Rightarrow bool$ **where**

total-over-m $I\ \psi s = total-over-set\ I\ (atms-of-ms\ \psi s)$

lemma *total-over-set-empty[simp]*:

total-over-set $I\ \{\}$

unfolding *total-over-set-def* **by** *auto*

lemma *total-over-m-empty[simp]*:

total-over-m $I\ \{\}$

unfolding *total-over-m-def* **by** *auto*

lemma *total-over-set-single[iff]*:

total-over-set $I\ \{L\} \longleftrightarrow (Pos\ L \in I \vee Neg\ L \in I)$

unfolding *total-over-set-def* **by** *auto*

lemma *total-over-set-insert[iff]*:

total-over-set $I\ (insert\ L\ Ls) \longleftrightarrow ((Pos\ L \in I \vee Neg\ L \in I) \wedge total-over-set\ I\ Ls)$

unfolding *total-over-set-def* **by** *auto*

lemma *total-over-set-union[iff]*:

total-over-set $I\ (Ls \cup Ls') \longleftrightarrow (total-over-set\ I\ Ls \wedge total-over-set\ I\ Ls')$

unfolding *total-over-set-def* **by** *auto*

lemma *total-over-m-subset*:

$A \subseteq B \implies total-over-m\ I\ B \implies total-over-m\ I\ A$

using *atms-of-ms-mono[of A]* **unfolding** *total-over-m-def* *total-over-set-def* **by** *auto*

lemma *total-over-m-sum*[iff]:
shows $\text{total-over-m } I \{C + D\} \longleftrightarrow (\text{total-over-m } I \{C\} \wedge \text{total-over-m } I \{D\})$
using *assms unfolding total-over-m-def total-over-set-def* **by** *auto*

lemma *total-over-m-union*[iff]:
 $\text{total-over-m } I (A \cup B) \longleftrightarrow (\text{total-over-m } I A \wedge \text{total-over-m } I B)$
unfolding *total-over-m-def total-over-set-def* **by** *auto*

lemma *total-over-m-insert*[iff]:
 $\text{total-over-m } I (\text{insert } a A) \longleftrightarrow (\text{total-over-set } I (\text{atms-of } a) \wedge \text{total-over-m } I A)$
unfolding *total-over-m-def total-over-set-def* **by** *fastforce*

lemma *total-over-m-extension*:
fixes $I :: 'v \text{ literal set}$ **and** $A :: 'v \text{ clauses}$
assumes *total: total-over-m I A*
shows $\exists I'. \text{total-over-m } (I \cup I') (A \cup B)$
 $\wedge (\forall x \in I'. \text{atm-of } x \in \text{atms-of-ms } B \wedge \text{atm-of } x \notin \text{atms-of-ms } A)$

proof –
let $?I' = \{\text{Pos } v \mid v. v \in \text{atms-of-ms } B \wedge v \notin \text{atms-of-ms } A\}$
have $(\forall x \in ?I'. \text{atm-of } x \in \text{atms-of-ms } B \wedge \text{atm-of } x \notin \text{atms-of-ms } A)$ **by** *auto*
moreover have $\text{total-over-m } (I \cup ?I') (A \cup B)$
using *total unfolding total-over-m-def total-over-set-def* **by** *auto*
ultimately show *?thesis* **by** *blast*

qed

lemma *total-over-m-consistent-extension*:
fixes $I :: 'v \text{ literal set}$ **and** $A :: 'v \text{ clauses}$
assumes *total: total-over-m I A*
and *cons: consistent-interp I*
shows $\exists I'. \text{total-over-m } (I \cup I') (A \cup B)$
 $\wedge (\forall x \in I'. \text{atm-of } x \in \text{atms-of-ms } B \wedge \text{atm-of } x \notin \text{atms-of-ms } A) \wedge \text{consistent-interp } (I \cup I')$

proof –
let $?I' = \{\text{Pos } v \mid v. v \in \text{atms-of-ms } B \wedge v \notin \text{atms-of-ms } A \wedge \text{Pos } v \notin I \wedge \text{Neg } v \notin I\}$
have $(\forall x \in ?I'. \text{atm-of } x \in \text{atms-of-ms } B \wedge \text{atm-of } x \notin \text{atms-of-ms } A)$ **by** *auto*
moreover have $\text{total-over-m } (I \cup ?I') (A \cup B)$
using *total unfolding total-over-m-def total-over-set-def* **by** *auto*
moreover have $\text{consistent-interp } (I \cup ?I')$
using *cons unfolding consistent-interp-def* **by** $(\text{intro allI}) (\text{rename-tac } L, \text{case-tac } L, \text{auto})$
ultimately show *?thesis* **by** *blast*

qed

lemma *total-over-set-atms-of-m*[simp]:
 $\text{total-over-set } Ia (\text{atms-of-s } Ia)$
unfolding *total-over-set-def atms-of-s-def* **by** $(\text{metis image-iff literal.exhaust-sel})$

lemma *total-over-set-literal-defined*:
assumes $\{\#A\# \} + D \in \psi s$
and $\text{total-over-set } I (\text{atms-of-ms } \psi s)$
shows $A \in I \vee -A \in I$
using *assms unfolding total-over-set-def* **by** $(\text{metis (no-types) Neg-atm-of-iff in-m-in-literals literal.collapse(1) uminus-Neg uminus-Pos})$

lemma *tot-over-m-remove*:
assumes $\text{total-over-m } (I \cup \{L\}) \{\psi\}$
and $L: \neg L \in \# \psi - L \notin \# \psi$

```

shows total-over-m I {ψ}
unfolding total-over-m-def total-over-set-def
proof
  fix l
  assume l: l ∈ atms-of-ms {ψ}
  then have Pos l ∈ I ∨ Neg l ∈ I ∨ l = atm-of L
    using assms unfolding total-over-m-def total-over-set-def by auto
  moreover have atm-of L ∉ atms-of-ms {ψ}
  proof (rule ccontr)
    assume ¬ ?thesis
    then have atm-of L ∈ atms-of ψ by auto
    then have Pos (atm-of L) ∈# ψ ∨ Neg (atm-of L) ∈# ψ
      using atm-imp-pos-or-neg-lit by metis
    then have L ∈# ψ ∨ ¬ L ∈# ψ by (cases L) auto
    then show False using L by auto
  qed
  ultimately show Pos l ∈ I ∨ Neg l ∈ I using l by metis
qed

```

```

lemma total-union:
  assumes total-over-m I ψ
  shows total-over-m (I ∪ I') ψ
  using assms unfolding total-over-m-def total-over-set-def by auto

```

```

lemma total-union-2:
  assumes total-over-m I ψ
  and total-over-m I' ψ'
  shows total-over-m (I ∪ I') (ψ ∪ ψ')
  using assms unfolding total-over-m-def total-over-set-def by auto

```

11.2.4 Interpretations

definition *true-cls* :: 'a interp \Rightarrow 'a clause \Rightarrow bool (infix \models 50) where
 $I \models C \longleftrightarrow (\exists L \in \# C. I \models L)$

```

lemma true-cls-empty[iff]: ¬ I ⊨ {#}
  unfolding true-cls-def by auto

```

```

lemma true-cls-singleton[iff]: I ⊨ {#L#} ⟷ I ⊨ L
  unfolding true-cls-def by (auto split:if-split-asm)

```

```

lemma true-cls-union[iff]: I ⊨ C + D ⟷ I ⊨ C ∨ I ⊨ D
  unfolding true-cls-def by auto

```

```

lemma true-cls-mono-set-mset: set-mset C ⊆ set-mset D ⟹ I ⊨ C ⟹ I ⊨ D
  unfolding true-cls-def subset-eq Bex-def by metis

```

```

lemma true-cls-mono-leD[dest]: A ⊆# B ⟹ I ⊨ A ⟹ I ⊨ B
  unfolding true-cls-def by auto

```

```

lemma
  assumes I ⊨ ψ
  shows true-cls-union-increase[simp]: I ∪ I' ⊨ ψ
  and true-cls-union-increase'[simp]: I' ∪ I ⊨ ψ
  using assms unfolding true-cls-def by auto

```

lemma *true-cls-mono-set-mset-l*:
 assumes $A \models \psi$
 and $A \subseteq B$
 shows $B \models \psi$
 using *assms unfolding true-cls-def* by *auto*

lemma *true-cls-replicate-mset[iff]*: $I \models \text{replicate-mset } n \ L \longleftrightarrow n \neq 0 \wedge I \models_l L$
 by (*induct n*) *auto*

lemma *true-cls-empty-entails[iff]*: $\neg \{\} \models N$
 by (*auto simp add: true-cls-def*)

lemma *true-cls-not-in-remove*:
 assumes $L \notin \# \chi$
 and $I \cup \{L\} \models \chi$
 shows $I \models \chi$
 using *assms unfolding true-cls-def* by *auto*

definition *true-clss* :: '*a interp* \Rightarrow '*a clauses* \Rightarrow *bool* (*infix* \models_s 50) **where**
 $I \models_s CC \longleftrightarrow (\forall C \in CC. I \models C)$

lemma *true-clss-empty[simp]*: $I \models_s \{\}$
 unfolding *true-clss-def* by *blast*

lemma *true-clss-singleton[iff]*: $I \models_s \{C\} \longleftrightarrow I \models C$
 unfolding *true-clss-def* by *blast*

lemma *true-clss-empty-entails-empty[iff]*: $\{\} \models_s N \longleftrightarrow N = \{\}$
 unfolding *true-clss-def* by (*auto simp add: true-cls-def*)

lemma *true-cls-insert-l [simp]*:
 $M \models A \implies \text{insert } L \ M \models A$
 unfolding *true-cls-def* by *auto*

lemma *true-clss-union[iff]*: $I \models_s CC \cup DD \longleftrightarrow I \models_s CC \wedge I \models_s DD$
 unfolding *true-clss-def* by *blast*

lemma *true-clss-insert[iff]*: $I \models_s \text{insert } C \ DD \longleftrightarrow I \models C \wedge I \models_s DD$
 unfolding *true-clss-def* by *blast*

lemma *true-clss-mono*: $DD \subseteq CC \implies I \models_s CC \implies I \models_s DD$
 unfolding *true-clss-def* by *blast*

lemma *true-clss-union-increase[simp]*:
 assumes $I \models_s \psi$
 shows $I \cup I' \models_s \psi$
 using *assms unfolding true-clss-def* by *auto*

lemma *true-clss-union-increase'[simp]*:
 assumes $I' \models_s \psi$
 shows $I \cup I' \models_s \psi$
 using *assms* by (*auto simp add: true-clss-def*)

lemma *true-clss-commute-l*:
 $(I \cup I' \models_s \psi) \longleftrightarrow (I' \cup I \models_s \psi)$

by (simp add: Un-commute)

lemma *model-remove[simp]*: $I \models_s N \implies I \models_s \text{Set.remove } a \ N$
 by (simp add: true-clss-def)

lemma *model-remove-minus[simp]*: $I \models_s N \implies I \models_s N - A$
 by (simp add: true-clss-def)

lemma *notin-vars-union-true-cls-true-cls*:
 assumes $\forall x \in I'. \text{atm-of } x \notin \text{atms-of-ms } A$
 and $\text{atms-of } L \subseteq \text{atms-of-ms } A$
 and $I \cup I' \models L$
 shows $I \models L$
 using assms **unfolding** true-cls-def true-lit-def Bex-def
 by (metis Un-iff atm-of-lit-in-atms-of contra-subsetD)

lemma *notin-vars-union-true-clss-true-clss*:
 assumes $\forall x \in I'. \text{atm-of } x \notin \text{atms-of-ms } A$
 and $\text{atms-of-ms } L \subseteq \text{atms-of-ms } A$
 and $I \cup I' \models_s L$
 shows $I \models_s L$
 using assms **unfolding** true-clss-def true-lit-def Ball-def
 by (meson atms-of-atms-of-ms-mono notin-vars-union-true-cls-true-cls subset-trans)

11.2.5 Satisfiability

definition *satisfiable* :: 'a clause set \Rightarrow bool **where**
satisfiable $CC \equiv \exists I. (I \models_s CC \wedge \text{consistent-interp } I \wedge \text{total-over-m } I \ CC)$

lemma *satisfiable-single[simp]*:
satisfiable $\{\{\#L\#\}\}$
unfolding *satisfiable-def* **by** fastforce

abbreviation *unsatisfiable* :: 'a clause set \Rightarrow bool **where**
unsatisfiable $CC \equiv \neg \text{satisfiable } CC$

lemma *satisfiable-decreasing*:
 assumes *satisfiable* $(\psi \cup \psi')$
 shows *satisfiable* ψ
 using assms *total-over-m-union* **unfolding** *satisfiable-def* **by** blast

lemma *satisfiable-def-min*:
satisfiable CC
 $\iff (\exists I. I \models_s CC \wedge \text{consistent-interp } I \wedge \text{total-over-m } I \ CC \wedge \text{atm-of } I = \text{atms-of-ms } CC)$
 (is ?sat \iff ?B)

proof

assume ?B **then show** ?sat **by** (auto simp add: *satisfiable-def*)

next

assume ?sat

then obtain I **where**

$I \models_s CC$ **and**

cons: *consistent-interp* I **and**

tot: *total-over-m* $I \ CC$

unfolding *satisfiable-def* **by** auto

let $?I = \{P. P \in I \wedge \text{atm-of } P \in \text{atms-of-ms } CC\}$

```

have I-CC: ?I  $\models_s$  CC
  using I-CC in-implies-atm-of-on-atms-of-ms unfolding true-clss-def Ball-def true-cls-def
  Bex-def true-lit-def
  by blast

moreover have cons: consistent-interp ?I
  using cons unfolding consistent-interp-def by auto
moreover have total-over-m ?I CC
  using tot unfolding total-over-m-def total-over-set-def by auto
moreover
  have atms-CC-incl: atms-of-ms CC  $\subseteq$  atm-of I
    using tot unfolding total-over-m-def total-over-set-def atms-of-ms-def
    by (auto simp add: atms-of-def atms-of-s-def[symmetric])
  have atm-of ' ?I = atms-of-ms CC
    using atms-CC-incl unfolding atms-of-ms-def by force
  ultimately show ?B by auto
qed

```

11.2.6 Entailment for Multisets of Clauses

definition *true-cls-mset* :: 'a interp \Rightarrow 'a clause multiset \Rightarrow bool (*infix* \models_m 50) **where**
 $I \models_m CC \longleftrightarrow (\forall C \in \# CC. I \models C)$

lemma *true-cls-mset-empty[simp]*: $I \models_m \{\#\}$
 unfolding *true-cls-mset-def* by auto

lemma *true-cls-mset-singleton[iff]*: $I \models_m \{\#C\# \} \longleftrightarrow I \models C$
 unfolding *true-cls-mset-def* by (auto split: if-split-asm)

lemma *true-cls-mset-union[iff]*: $I \models_m CC + DD \longleftrightarrow I \models_m CC \wedge I \models_m DD$
 unfolding *true-cls-mset-def* by fastforce

lemma *true-cls-mset-image-mset[iff]*: $I \models_m \text{image-mset } f A \longleftrightarrow (\forall x \in \# A. I \models f x)$
 unfolding *true-cls-mset-def* by fastforce

lemma *true-cls-mset-mono*: $\text{set-mset } DD \subseteq \text{set-mset } CC \Longrightarrow I \models_m CC \Longrightarrow I \models_m DD$
 unfolding *true-cls-mset-def* subset-iff by auto

lemma *true-clss-set-mset[iff]*: $I \models_s \text{set-mset } CC \longleftrightarrow I \models_m CC$
 unfolding *true-clss-def* *true-cls-mset-def* by auto

lemma *true-cls-mset-increasing-r[simp]*:
 $I \models_m CC \Longrightarrow I \cup J \models_m CC$
 unfolding *true-cls-mset-def* by auto

theorem *true-cls-remove-unused*:
 assumes $I \models \psi$
 shows $\{v \in I. \text{atm-of } v \in \text{atms-of } \psi\} \models \psi$
 using *assms* unfolding *true-cls-def* *atms-of-def* by auto

theorem *true-clss-remove-unused*:
 assumes $I \models_s \psi$
 shows $\{v \in I. \text{atm-of } v \in \text{atms-of-ms } \psi\} \models_s \psi$
 unfolding *true-clss-def* *atms-of-def* *Ball-def*
proof (*intro allI impI*)
 fix x

assume $x \in \psi$
then have $I \models x$
using *assms unfolding true-clss-def atms-of-def Ball-def* **by** *auto*

then have $\{v \in I. \text{atm-of } v \in \text{atms-of } x\} \models x$
by (*simp only: true-cls-remove-unused[of I]*)
moreover have $\{v \in I. \text{atm-of } v \in \text{atms-of } x\} \subseteq \{v \in I. \text{atm-of } v \in \text{atms-of-ms } \psi\}$
using $\langle x \in \psi \rangle$ **by** (*auto simp add: atms-of-ms-def*)
ultimately show $\{v \in I. \text{atm-of } v \in \text{atms-of-ms } \psi\} \models x$
using *true-cls-mono-set-mset-l* **by** *blast*
qed

A simple application of the previous theorem:

lemma *true-clss-union-decrease*:
assumes $II': I \cup I' \models \psi$
and $H: \forall v \in I'. \text{atm-of } v \notin \text{atms-of } \psi$
shows $I \models \psi$
proof –
let $?I = \{v \in I \cup I'. \text{atm-of } v \in \text{atms-of } \psi\}$
have $?I \models \psi$ **using** *true-cls-remove-unused II'* **by** *blast*
moreover have $?I \subseteq I$ **using** H **by** *auto*
ultimately show *?thesis* **using** *true-cls-mono-set-mset-l* **by** *blast*
qed

lemma *multiset-not-empty*:
assumes $M \neq \{\#\}$
and $x \in \# M$
shows $\exists A. x = \text{Pos } A \vee x = \text{Neg } A$
using *assms literal.exhaust-sel* **by** *blast*

lemma *atms-of-ms-empty*:
fixes $\psi :: 'v \text{ clauses}$
assumes $\text{atms-of-ms } \psi = \{\}$
shows $\psi = \{\} \vee \psi = \{\{\#\}\}$
using *assms* **by** (*auto simp add: atms-of-ms-def*)

lemma *consistent-interp-disjoint*:
assumes *consI: consistent-interp I*
and *disj: atms-of-s A \cap atms-of-s I = $\{\}$*
and *consA: consistent-interp A*
shows *consistent-interp (A \cup I)*
proof (*rule ccontr*)
assume $\neg ?thesis$
moreover have $\bigwedge L. \neg (L \in A \wedge \neg L \in I)$
using *disj unfolding atms-of-s-def* **by** (*auto simp add: rev-image-eqI*)
ultimately show *False*
using *consA consI unfolding consistent-interp-def* **by** (*metis (full-types) Un-iff literal.exhaust-sel uminus-Neg uminus-Pos*)
qed

lemma *total-remove-unused*:
assumes *total-over-m I ψ*
shows *total-over-m $\{v \in I. \text{atm-of } v \in \text{atms-of-ms } \psi\} \psi$*
using *assms unfolding total-over-m-def total-over-set-def*
by (*metis (lifting) literal.sel(1,2) mem-Collect-eq*)

lemma *true-cls-remove-hd-if-notin-vars*:
assumes *insert a M' ⊨ D*
and *atm-of a ∉ atms-of D*
shows *M' ⊨ D*
using *assms* **by** (*auto simp add: atm-of-lit-in-atms-of true-cls-def*)

lemma *total-over-set-atm-of*:
fixes *I :: 'v interp* **and** *K :: 'v set*
shows *total-over-set I K ⟷ (∀ l ∈ K. l ∈ (atm-of ' I))*
unfolding *total-over-set-def* **by** (*metis atms-of-s-def in-atms-of-s-decomp*)

11.2.7 Tautologies

definition *tautology (ψ:: 'v clause) ≡ ∀ I. total-over-set I (atms-of ψ) ⟶ I ⊨ ψ*

lemma *tautology-Pos-Neg[intro]*:
assumes *Pos p ∈# A* **and** *Neg p ∈# A*
shows *tautology A*
using *assms* **unfolding** *tautology-def total-over-set-def true-cls-def Bex-def*
by (*meson atm-iff-pos-or-neg-lit true-lit-def*)

lemma *tautology-minus[simp]*:
assumes *L ∈# A* **and** *¬L ∈# A*
shows *tautology A*
by (*metis assms literal.exhaust tautology-Pos-Neg uminus-Neg uminus-Pos*)

lemma *tautology-exists-Pos-Neg*:
assumes *tautology ψ*
shows *∃ p. Pos p ∈# ψ ∧ Neg p ∈# ψ*
proof (*rule ccontr*)
assume *p: ¬ (∃ p. Pos p ∈# ψ ∧ Neg p ∈# ψ)*
let *?I = {¬L | L. L ∈# ψ}*
have *total-over-set ?I (atms-of ψ)*
unfolding *total-over-set-def* **using** *atm-imp-pos-or-neg-lit* **by** *force*
moreover **have** *¬ ?I ⊨ ψ*
unfolding *true-cls-def true-lit-def Bex-def* **apply** *clarify*
using *p* **by** (*rename-tac x L, case-tac L*) *fastforce+*
ultimately show *False* **using** *assms* **unfolding** *tautology-def* **by** *auto*
qed

lemma *tautology-decomp*:
tautology ψ ⟷ (∃ p. Pos p ∈# ψ ∧ Neg p ∈# ψ)
using *tautology-exists-Pos-Neg* **by** *auto*

lemma *tautology-false[simp]*: *¬tautology {#}*
unfolding *tautology-def* **by** *auto*

lemma *tautology-add-single*:
tautology ({#a#} + L) ⟷ tautology L ∨ ¬a ∈# L
unfolding *tautology-decomp* **by** (*cases a*) *auto*

lemma *minus-interp-tautology*:
assumes *{¬L | L. L ∈# χ} ⊨ χ*
shows *tautology χ*
proof *—*

obtain L where $L \in \# \chi \wedge -L \in \# \chi$
 using *assms unfolding true-cls-def by auto*
 then show *?thesis using tautology-decomp literal.exhaust uminus-Neg uminus-Pos by metis*
 qed

lemma *remove-literal-in-model-tautology*:
 assumes $I \cup \{Pos\ P\} \models \varphi$
 and $I \cup \{Neg\ P\} \models \varphi$
 shows $I \models \varphi \vee \text{tautology } \varphi$
 using *assms unfolding true-cls-def by auto*

lemma *tautology-imp-tautology*:
 fixes $\chi \ \chi' :: 'v \text{ clause}$
 assumes $\forall I. \text{total-over-m } I \ \{\chi\} \longrightarrow I \models \chi \longrightarrow I \models \chi'$ and *tautology* χ
 shows *tautology* χ' **unfolding** *tautology-def*

proof (*intro allI HOL.impI*)
 fix $I :: 'v \text{ literal set}$
 assume *totI*: *total-over-set* I (*atms-of* χ')
 let $?I' = \{Pos\ v \mid v. v \in \text{atms-of } \chi \wedge v \notin \text{atms-of-s } I\}$
 have *totI'*: *total-over-m* $(I \cup ?I')$ $\{\chi\}$ **unfolding** *total-over-m-def total-over-set-def by auto*
 then have $\chi: I \cup ?I' \models \chi$ **using** *assms(2) unfolding total-over-m-def tautology-def by simp*
 then have $I \cup (?I' - I) \models \chi'$ **using** *assms(1) totI' by auto*
 moreover have $\bigwedge L. L \in \# \chi' \implies L \notin ?I'$
 using *totI unfolding total-over-set-def by (auto dest: pos-lit-in-atms-of)*
 ultimately show $I \models \chi'$ **unfolding** *true-cls-def by auto*
 qed

11.2.8 Entailment for clauses and propositions

definition *true-cls-cls* :: $'a \text{ clause} \Rightarrow 'a \text{ clause} \Rightarrow \text{bool}$ (**infix** \models_f 49) **where**
 $\psi \models_f \chi \iff (\forall I. \text{total-over-m } I \ (\{\psi\} \cup \{\chi\}) \longrightarrow \text{consistent-interp } I \longrightarrow I \models \psi \longrightarrow I \models \chi)$

definition *true-cls-clss* :: $'a \text{ clause} \Rightarrow 'a \text{ clauses} \Rightarrow \text{bool}$ (**infix** \models_{fs} 49) **where**
 $\psi \models_{fs} \chi \iff (\forall I. \text{total-over-m } I \ (\{\psi\} \cup \chi) \longrightarrow \text{consistent-interp } I \longrightarrow I \models \psi \longrightarrow I \models_s \chi)$

definition *true-clss-cls* :: $'a \text{ clauses} \Rightarrow 'a \text{ clause} \Rightarrow \text{bool}$ (**infix** \models_p 49) **where**
 $N \models_p \chi \iff (\forall I. \text{total-over-m } I \ (N \cup \{\chi\}) \longrightarrow \text{consistent-interp } I \longrightarrow I \models_s N \longrightarrow I \models \chi)$

definition *true-clss-clss* :: $'a \text{ clauses} \Rightarrow 'a \text{ clauses} \Rightarrow \text{bool}$ (**infix** \models_{ps} 49) **where**
 $N \models_{ps} N' \iff (\forall I. \text{total-over-m } I \ (N \cup N') \longrightarrow \text{consistent-interp } I \longrightarrow I \models_s N \longrightarrow I \models_s N')$

lemma *true-cls-cls-refl[simp]*:
 $A \models_f A$
unfolding *true-cls-cls-def by auto*

lemma *true-cls-cls-insert-l[simp]*:
 $a \models_f C \implies \text{insert } a\ A \models_p C$
unfolding *true-cls-cls-def true-clss-cls-def true-clss-def by fastforce*

lemma *true-cls-clss-empty[iff]*:
 $N \models_{fs} \{\}$
unfolding *true-cls-clss-def by auto*

lemma *true-prop-true-clause[iff]*:
 $\{\varphi\} \models_p \psi \iff \varphi \models_f \psi$
unfolding *true-cls-cls-def true-clss-cls-def by auto*

lemma *true-clss-clss-true-clss-cls*[iff]:
 $N \models_{ps} \{\psi\} \longleftrightarrow N \models_p \psi$
unfolding *true-clss-clss-def true-clss-cls-def* **by** *auto*

lemma *true-clss-clss-true-clss-clss*[iff]:
 $\{\chi\} \models_{ps} \psi \longleftrightarrow \chi \models_{fs} \psi$
unfolding *true-clss-clss-def true-clss-clss-def* **by** *auto*

lemma *true-clss-clss-empty*[simp]:
 $N \models_{ps} \{\}$
unfolding *true-clss-clss-def* **by** *auto*

lemma *true-clss-cls-subset*:
 $A \subseteq B \implies A \models_p CC \implies B \models_p CC$
unfolding *true-clss-cls-def total-over-m-union* **by** (*simp add: total-over-m-subset true-clss-mono*)

lemma *true-clss-clss-mono-l*[simp]:
 $A \models_p CC \implies A \cup B \models_p CC$
by (*auto intro: true-clss-cls-subset*)

lemma *true-clss-clss-mono-l2*[simp]:
 $B \models_p CC \implies A \cup B \models_p CC$
by (*auto intro: true-clss-cls-subset*)

lemma *true-clss-cls-mono-r*[simp]:
 $A \models_p CC \implies A \models_p CC + CC'$
unfolding *true-clss-cls-def total-over-m-union total-over-m-sum* **by** *blast*

lemma *true-clss-cls-mono-r'*[simp]:
 $A \models_p CC' \implies A \models_p CC + CC'$
unfolding *true-clss-cls-def total-over-m-union total-over-m-sum* **by** *blast*

lemma *true-clss-clss-union-l*[simp]:
 $A \models_{ps} CC \implies A \cup B \models_{ps} CC$
unfolding *true-clss-clss-def total-over-m-union* **by** *fastforce*

lemma *true-clss-clss-union-l-r*[simp]:
 $B \models_{ps} CC \implies A \cup B \models_{ps} CC$
unfolding *true-clss-clss-def total-over-m-union* **by** *fastforce*

lemma *true-clss-cls-in*[simp]:
 $CC \in A \implies A \models_p CC$
unfolding *true-clss-cls-def true-clss-def total-over-m-union* **by** *fastforce*

lemma *true-clss-cls-insert-l*[simp]:
 $A \models_p C \implies \text{insert } a \ A \models_p C$
unfolding *true-clss-cls-def true-clss-def* **using** *total-over-m-union*
by (*metis Un-iff insert-is-Un sup commute*)

lemma *true-clss-clss-insert-l*[simp]:
 $A \models_{ps} C \implies \text{insert } a \ A \models_{ps} C$
unfolding *true-clss-cls-def true-clss-clss-def true-clss-def* **by** *blast*

lemma *true-clss-clss-union-and*[iff]:

$A \models_{ps} C \cup D \longleftrightarrow (A \models_{ps} C \wedge A \models_{ps} D)$
proof
{
 fix $A \ C \ D :: 'a \ clauses$
 assume $A: A \models_{ps} C \cup D$
 have $A \models_{ps} C$
 unfolding *true-clss-clss-def true-clss-clss-def insert-def total-over-m-insert*
 proof (*intro allI impI*)
 fix I
 assume
 totAC: total-over-m I (A \cup C) and
 cons: consistent-interp I and
 I: I \models_s A
 then have *tot: total-over-m I A and tot': total-over-m I C* **by** *auto*
 obtain I' **where**
 tot': total-over-m (I \cup I') (A \cup C \cup D) and
 cons': consistent-interp (I \cup I') and
 H: $\forall x \in I'. atm\text{-of } x \in atm\text{-of-}ms \ D \wedge atm\text{-of } x \notin atm\text{-of-}ms \ (A \cup C)$
 using *total-over-m-consistent-extension[OF - cons, of A \cup C] tot tot'* **by** *blast*
 moreover have $I \cup I' \models_s A$ **using** I **by** *simp*
 ultimately have $I \cup I' \models_s C \cup D$ **using** A **unfolding** *true-clss-clss-def* **by** *auto*
 then have $I \cup I' \models_s C \cup D$ **by** *auto*
 then show $I \models_s C$ **using** *notin-vars-union-true-clss-true-clss[of I'] H* **by** *auto*
 qed
} **note** $H = this$
assume $A \models_{ps} C \cup D$
then show $A \models_{ps} C \wedge A \models_{ps} D$ **using** $H[of \ A]$ *Un-commute[of C D]* **by** *metis*
next
 assume $A \models_{ps} C \wedge A \models_{ps} D$
 then show $A \models_{ps} C \cup D$
 unfolding *true-clss-clss-def* **by** *auto*
qed

lemma *true-clss-clss-insert[iff]:*
 $A \models_{ps} insert \ L \ Ls \longleftrightarrow (A \models_p L \wedge A \models_{ps} Ls)$
using *true-clss-clss-union-and[of A {L} Ls]* **by** *auto*

lemma *true-clss-clss-subset:*
 $A \subseteq B \implies A \models_{ps} CC \implies B \models_{ps} CC$
by (*metis subset-Un-eq true-clss-clss-union-l*)

lemma *union-trus-clss-clss[simp]:* $A \cup B \models_{ps} B$
unfolding *true-clss-clss-def* **by** *auto*

lemma *true-clss-clss-remove[simp]:*
 $A \models_{ps} B \implies A \models_{ps} B - C$
by (*metis Un-Diff-Int true-clss-clss-union-and*)

lemma *true-clss-clss-subsetE:*
 $N \models_{ps} B \implies A \subseteq B \implies N \models_{ps} A$
by (*metis sup.orderE true-clss-clss-union-and*)

lemma *true-clss-clss-in-imp-true-clss-clss:*
assumes $N \models_{ps} U$
and $A \in U$

```

shows  $N \models_p A$ 
using assms mk-disjoint-insert by fastforce

lemma all-in-true-clss-clss:  $\forall x \in B. x \in A \implies A \models_{ps} B$ 
  unfolding true-clss-clss-def true-clss-def by auto

lemma true-clss-clss-left-right:
  assumes  $A \models_{ps} B$ 
  and  $A \cup B \models_{ps} M$ 
  shows  $A \models_{ps} M \cup B$ 
  using assms unfolding true-clss-clss-def by auto

lemma true-clss-clss-generalise-true-clss-clss:
   $A \cup C \models_{ps} D \implies B \models_{ps} C \implies A \cup B \models_{ps} D$ 
proof -
  assume a1:  $A \cup C \models_{ps} D$ 
  assume  $B \models_{ps} C$ 
  then have f2:  $\bigwedge M. M \cup B \models_{ps} C$ 
    by (meson true-clss-clss-union-l-r)
  have  $\bigwedge M. C \cup (M \cup A) \models_{ps} D$ 
    using a1 by (simp add: Un-commute sup-left-commute)
  then show ?thesis
    using f2 by (metis (no-types) Un-commute true-clss-clss-left-right true-clss-clss-union-and)
qed

lemma true-clss-clss-or-true-clss-clss-or-not-true-clss-clss-or:
  assumes  $D: N \models_p D + \{\#- L\# \}$ 
  and  $C: N \models_p C + \{\#L\# \}$ 
  shows  $N \models_p D + C$ 
  unfolding true-clss-clss-def
proof (intro allI impI)
  fix I
  assume
    tot: total-over-m I ( $N \cup \{D + C\}$ ) and
    consistent-interp I and
     $I \models_s N$ 
  {
    assume L:  $L \in I \vee -L \in I$ 
    then have total-over-m I  $\{D + \{\#- L\# \}\}$ 
      using tot by (cases L) auto
    then have  $I \models D + \{\#- L\# \}$  using D  $\langle I \models_s N \rangle$  tot  $\langle$ consistent-interp I $\rangle$ 
      unfolding true-clss-clss-def by auto
    moreover
      have total-over-m I  $\{C + \{\#L\# \}\}$ 
        using L tot by (cases L) auto
      then have  $I \models C + \{\#L\# \}$ 
        using C  $\langle I \models_s N \rangle$  tot  $\langle$ consistent-interp I $\rangle$  unfolding true-clss-clss-def by auto
      ultimately have  $I \models D + C$  using  $\langle$ consistent-interp I $\rangle$  consistent-interp-def by fastforce
    }
  moreover {
    assume L:  $L \notin I \wedge -L \notin I$ 
    let ?I' =  $I \cup \{L\}$ 
    have consistent-interp ?I' using L  $\langle$ consistent-interp I $\rangle$  by auto
    moreover have total-over-m ?I'  $\{D + \{\#- L\# \}\}$ 
      using tot unfolding total-over-m-def total-over-set-def by (auto simp add: atms-of-def)
  }

```

```

moreover have total-over-m  $?I' \models N$  using tot using total-union by blast
moreover have  $?I' \models_s N$  using  $\langle I \models_s N \rangle$  using true-clss-union-increase by blast
ultimately have  $?I' \models D + \{\#- L\# \}$ 
  using D unfolding true-clss-clb-def by blast
then have  $?I' \models D$  using L by auto
moreover
  have total-over-set I (atms-of ( $D + C$ )) using tot by auto
  then have  $L \notin \# D \wedge -L \notin \# D$ 
    using L unfolding total-over-set-def atms-of-def by (cases L) force+
  ultimately have  $I \models D + C$  unfolding true-clb-def by auto
}
ultimately show  $I \models D + C$  by blast
qed

lemma true-clb-union-mset[iff]:  $I \models C \# \cup D \longleftrightarrow I \models C \vee I \models D$ 
  unfolding true-clb-def by force

lemma true-clss-clb-union-mset-true-clss-clb-or-not-true-clss-clb-or:
  assumes
     $D: N \models_p D + \{\#- L\# \}$  and
     $C: N \models_p C + \{\#L\# \}$ 
  shows  $N \models_p D \# \cup C$ 
  unfolding true-clss-clb-def
proof (intro allI impI)
  fix I
  assume
    tot: total-over-m I ( $N \cup \{D \# \cup C\}$ ) and
    consistent-interp I and
     $I \models_s N$ 
  {
    assume L:  $L \in I \vee -L \in I$ 
    then have total-over-m I  $\{D + \{\#- L\# \}\}$ 
      using tot by (cases L) auto
    then have  $I \models D + \{\#- L\# \}$ 
      using D  $\langle I \models_s N \rangle$  tot  $\langle$ consistent-interp I $\rangle$  unfolding true-clss-clb-def by auto
    moreover
      have total-over-m I  $\{C + \{\#L\# \}\}$ 
        using L tot by (cases L) auto
      then have  $I \models C + \{\#L\# \}$ 
        using C  $\langle I \models_s N \rangle$  tot  $\langle$ consistent-interp I $\rangle$  unfolding true-clss-clb-def by auto
      ultimately have  $I \models D \# \cup C$  using  $\langle$ consistent-interp I $\rangle$  unfolding consistent-interp-def
        by auto
    }
  moreover {
    assume L:  $L \notin I \wedge -L \notin I$ 
    let  $?I' = I \cup \{L\}$ 
    have consistent-interp  $?I'$  using L  $\langle$ consistent-interp I $\rangle$  by auto
    moreover have total-over-m  $?I' \{D + \{\#- L\# \}\}$ 
      using tot unfolding total-over-m-def total-over-set-def by (auto simp add: atms-of-def)
    moreover have total-over-m  $?I' N$  using tot using total-union by blast
    moreover have  $?I' \models_s N$  using  $\langle I \models_s N \rangle$  using true-clss-union-increase by blast
    ultimately have  $?I' \models D + \{\#- L\# \}$ 
      using D unfolding true-clss-clb-def by blast
    then have  $?I' \models D$  using L by auto
    moreover

```

```

    have total-over-set  $I$  (atms-of ( $D + C$ )) using tot by auto
    then have  $L \notin \# D \wedge \neg L \notin \# D$ 
      using  $L$  unfolding total-over-set-def atms-of-def by (cases  $L$ ) force+
    ultimately have  $I \models D \# \cup C$  unfolding true-cls-def by auto
  }
  ultimately show  $I \models D \# \cup C$  by blast
qed

```

lemma *satisfiable-carac*[iff]:

$(\exists I. \text{consistent-interp } I \wedge I \models_s \varphi) \longleftrightarrow \text{satisfiable } \varphi$ (is $(\exists I. ?Q I) \longleftrightarrow ?S$)

proof

assume $?S$

then show $\exists I. ?Q I$ unfolding satisfiable-def by auto

next

assume $\exists I. ?Q I$

then obtain I where cons: consistent-interp I and $I: I \models_s \varphi$ by metis

let $?I' = \{Pos\ v \mid v. v \notin \text{atms-of-s } I \wedge v \in \text{atms-of-ms } \varphi\}$

have consistent-interp $(I \cup ?I')$

using cons unfolding consistent-interp-def by (intro allI) (rename-tac L , case-tac L , auto)

moreover have total-over-m $(I \cup ?I') \varphi$

unfolding total-over-m-def total-over-set-def by auto

moreover have $I \cup ?I' \models_s \varphi$

using I unfolding Ball-def true-clss-def true-cls-def by auto

ultimately show $?S$ unfolding satisfiable-def by blast

qed

lemma *satisfiable-carac*'[simp]: consistent-interp $I \implies I \models_s \varphi \implies \text{satisfiable } \varphi$

using satisfiable-carac by metis

11.3 Subsumptions

lemma *subsumption-total-over-m*:

assumes $A \subseteq \# B$

shows total-over-m $I \{B\} \implies \text{total-over-m } I \{A\}$

using assms unfolding subset-mset-def total-over-m-def total-over-set-def

by (auto simp add: mset-le-exists-conv)

lemma *atms-of-replicate-mset-replicate-mset-uminus*[simp]:

atms-of $(D - \text{replicate-mset } (\text{count } D\ L)\ L - \text{replicate-mset } (\text{count } D\ (-L))\ (-L))$

= atms-of $D - \{atm\text{-of } L\}$

by (fastforce simp: atm-of-eq-atm-of atms-of-def)

lemma *subsumption-chained*:

assumes

$\forall I. \text{total-over-m } I \{D\} \longrightarrow I \models D \longrightarrow I \models \varphi$ and

$C \subseteq \# D$

shows $(\forall I. \text{total-over-m } I \{C\} \longrightarrow I \models C \longrightarrow I \models \varphi) \vee \text{tautology } \varphi$

using assms

proof (induct card $\{Pos\ v \mid v. v \in \text{atms-of } D \wedge v \notin \text{atms-of } C\}$ arbitrary: D

rule: nat-less-induct-case)

case 0 note $n = \text{this}(1)$ and $H = \text{this}(2)$ and $\text{incl} = \text{this}(3)$

then have atms-of $D \subseteq \text{atms-of } C$ by auto

then have $\forall I. \text{total-over-m } I \{C\} \longrightarrow \text{total-over-m } I \{D\}$

unfolding total-over-m-def total-over-set-def by auto

moreover have $\forall I. I \models C \longrightarrow I \models D$ using incl true-cls-mono-leD by blast

ultimately show $?case$ using H by auto

```

next
case (Suc n D) note IH = this(1) and card = this(2) and H = this(3) and incl = this(4)
let ?atms = {Pos v | v. v ∈ atms-of D ∧ v ∉ atms-of C}
have finite ?atms by auto
then obtain L where L: L ∈ ?atms
  using card by (metis (no-types, lifting) Collect-empty-eq card-0-eq mem-Collect-eq
    nat.simps(3))
let ?D' = D - replicate-mset (count D L) L - replicate-mset (count D (-L)) (-L)
have atms-of-D: atms-of-ms {D} ⊆ atms-of-ms {?D'} ∪ {atm-of L} by auto

{
  fix I
  assume total-over-m I {?D'}
  then have tot: total-over-m (I ∪ {L}) {D}
    unfolding total-over-m-def total-over-set-def using atms-of-D by auto

  assume IDL: I ⊨ ?D'
  then have I ∪ {L} ⊨ D unfolding true-cls-def by force
  then have I ∪ {L} ⊨ φ using H tot by auto

  moreover
  have tot': total-over-m (I ∪ {-L}) {D}
    using tot unfolding total-over-m-def total-over-set-def by auto
  have I ∪ {-L} ⊨ D using IDL unfolding true-cls-def by force
  then have I ∪ {-L} ⊨ φ using H tot' by auto
  ultimately have I ⊨ φ ∨ tautology φ
    using L remove-literal-in-model-tautology by force
} note H' = this

have L ∉# C and -L ∉# C using L atm-iff-pos-or-neg-lit by force+
then have C-in-D': C ⊆# ?D' using ⟨C ⊆# D⟩ by (auto simp: subseteq-mset-def not-in-iff)
have card {Pos v | v. v ∈ atms-of ?D' ∧ v ∉ atms-of C} <
  card {Pos v | v. v ∈ atms-of D ∧ v ∉ atms-of C}
  using L by (auto intro!: psubset-card-mono)
then show ?case
  using IH C-in-D' H' unfolding card[symmetric] by blast
qed

```

11.4 Removing Duplicates

```

lemma tautology-remdups-mset[iff]:
  tautology (remdups-mset C) ⟷ tautology C
  unfolding tautology-decomp by auto

lemma atms-of-remdups-mset[simp]: atms-of (remdups-mset C) = atms-of C
  unfolding atms-of-def by auto

lemma true-cls-remdups-mset[iff]: I ⊨ remdups-mset C ⟷ I ⊨ C
  unfolding true-cls-def by auto

lemma true-clss-cls-remdups-mset[iff]: A ⊨p remdups-mset C ⟷ A ⊨p C
  unfolding true-clss-cls-def total-over-m-def by auto

```

11.5 Set of all Simple Clauses

definition *simple-clss* :: 'v set ⇒ 'v clause set **where**

$simple-clss\ atoms = \{C. \text{atms-of } C \subseteq \text{atms} \wedge \neg \text{tautology } C \wedge \text{distinct-mset } C\}$

lemma *simple-clss-empty[simp]*:

simple-clss $\{\}$ = $\{\{\#\}\}$

unfolding *simple-clss-def* **by** *auto*

lemma *simple-clss-insert*:

assumes $l \notin \text{atms}$

shows *simple-clss* (*insert* l *atms*) =

$(\text{op} + \{\#Pos\ l\#\}) \text{ ' } (simple-clss\ \text{atms})$

$\cup (\text{op} + \{\#Neg\ l\#\}) \text{ ' } (simple-clss\ \text{atms})$

$\cup simple-clss\ \text{atms}(\text{is } ?I = ?U)$

proof (*standard*; *standard*)

fix C

assume $C \in ?I$

then have

atms: *atms-of* $C \subseteq \text{insert } l\ \text{atms}$ **and**

taut: $\neg \text{tautology } C$ **and**

dist: *distinct-mset* C

unfolding *simple-clss-def* **by** *auto*

have $H: \bigwedge x. x \in \# C \implies \text{atm-of } x \in \text{insert } l\ \text{atms}$

using *atm-of-lit-in-atms-of atms* **by** *blast*

consider

(*Add*) $L \text{ where } L \in \# C \text{ and } L = Neg\ l \vee L = Pos\ l$

| (*No*) $Pos\ l \notin \# C \ Neg\ l \notin \# C$

by *auto*

then show $C \in ?U$

proof *cases*

case *Add*

then have $L \notin \# C - \{\#L\#\}$

using *dist* **unfolding** *distinct-mset-def* **by** (*auto simp: not-in-iff*)

moreover have $-L \notin \# C$

using *taut Add* **by** *auto*

ultimately have *atms-of* $(C - \{\#L\#\}) \subseteq \text{atms}$

using *atms Add* **by** (*smt H atms-of-def imageE in-diffD insertE literal.exhaust-sel subset-iff uminus-Neg uminus-Pos*)

moreover have $\neg \text{tautology } (C - \{\#L\#\})$

using *taut* **by** (*metis Add(1) insert-DiffM tautology-add-single*)

moreover have *distinct-mset* $(C - \{\#L\#\})$

using *dist* **by** *auto*

ultimately have $(C - \{\#L\#\}) \in simple-clss\ \text{atms}$

using *Add* **unfolding** *simple-clss-def* **by** *auto*

moreover have $C = \{\#L\#\} + (C - \{\#L\#\})$

using *Add* **by** (*auto simp: multiset-eq-iff*)

ultimately show *?thesis* **using** *Add* **by** *auto*

next

case *No*

then have $C \in simple-clss\ \text{atms}$

using *taut atms dist* **unfolding** *simple-clss-def*

by (*auto simp: atm-iff-pos-or-neg-lit split: if-split-asm dest!: H*)

then show *?thesis* **by** *blast*

qed

next

fix C

```

assume  $C \in ?U$ 
then consider
  (Add)  $L \ C'$  where  $C = \{\#L\# \} + C'$  and  $C' \in \text{simple-clss } \text{atms}$  and
     $L = \text{Pos } l \vee L = \text{Neg } l$ 
  | (No)  $C \in \text{simple-clss } \text{atms}$ 
by auto
then show  $C \in ?I$ 
proof cases
  case No
    then show  $?thesis$  unfolding simple-clss-def by auto
  next
    case (Add  $L \ C'$ ) note  $C' = \text{this}(1)$  and  $C = \text{this}(2)$  and  $L = \text{this}(3)$ 
    then have
       $\text{atms: atms-of } C' \subseteq \text{atms}$  and
       $\text{taut: } \neg \text{tautology } C'$  and
       $\text{dist: distinct-mset } C'$ 
      unfolding simple-clss-def by auto
    have  $\text{atms-of } C \subseteq \text{insert } l \ \text{atms}$ 
      using  $\text{atms } C' \ L$  by auto
    moreover have  $\neg \text{tautology } C$ 
      using  $\text{taut } C' \ L$  by ( $\text{metis assms atm-of-lit-in-atms-of atms literal.sel}(1,2)$   $\text{subset-eq}$ 
         $\text{tautology-add-single uminus-Neg uminus-Pos}$ )
    moreover have  $\text{distinct-mset } C$ 
      using  $\text{dist } C' \ L$ 
      by ( $\text{metis assms atm-of-lit-in-atms-of atms contra-subsetD distinct-mset-add-single}$ 
         $\text{literal.sel}(1,2)$ )
    ultimately show  $?thesis$  unfolding simple-clss-def by blast
qed
qed

lemma simple-clss-finite:
  fixes  $\text{atms} :: 'v \ \text{set}$ 
  assumes  $\text{finite } \text{atms}$ 
  shows  $\text{finite } (\text{simple-clss } \text{atms})$ 
  using  $\text{assms}$  by ( $\text{induction rule: finite-induct}$ ) ( $\text{auto simp: simple-clss-insert}$ )

lemma simple-clssE:
  assumes
     $x \in \text{simple-clss } \text{atms}$ 
  shows  $\text{atms-of } x \subseteq \text{atms} \wedge \neg \text{tautology } x \wedge \text{distinct-mset } x$ 
  using  $\text{assms}$  unfolding simple-clss-def by auto

lemma cls-in-simple-clss:
  shows  $\{\#\} \in \text{simple-clss } s$ 
  unfolding simple-clss-def by auto

lemma simple-clss-card:
  fixes  $\text{atms} :: 'v \ \text{set}$ 
  assumes  $\text{finite } \text{atms}$ 
  shows  $\text{card } (\text{simple-clss } \text{atms}) \leq (3::\text{nat}) \wedge (\text{card } \text{atms})$ 
  using  $\text{assms}$ 
proof ( $\text{induct atms rule: finite-induct}$ )
  case empty
    then show  $?case$  by auto
next

```



```

case (insert l C) note fin = this(1) and l = this(2) and IH = this(3)
have notin:
   $\wedge C'. \{\#Pos\ l\# \} + C' \notin \text{simple-clss } C$ 
   $\wedge C'. \{\#Neg\ l\# \} + C' \notin \text{simple-clss } C$ 
  using l unfolding simple-clss-def by auto
have H:  $\wedge C' D. \{\#Pos\ l\# \} + C' = \{\#Neg\ l\# \} + D \implies D \in \text{simple-clss } C \implies \text{False}$ 
proof -
  fix C' D
  assume C'D:  $\{\#Pos\ l\# \} + C' = \{\#Neg\ l\# \} + D$  and D:  $D \in \text{simple-clss } C$ 
  then have Pos l  $\in \#$  D by (metis insert-noteq-member literal.distinct(1) union-commute)
  then have l  $\in$  atms-of D
    by (simp add: atm-iff-pos-or-neg-lit)
  then show False using D l unfolding simple-clss-def by auto
qed
let ?P = (op +  $\{\#Pos\ l\# \}$ ) ' (simple-clss C)
let ?N = (op +  $\{\#Neg\ l\# \}$ ) ' (simple-clss C)
let ?O = simple-clss C
have card (?P  $\cup$  ?N  $\cup$  ?O) = card (?P  $\cup$  ?N) + card ?O
  apply (subst card-Un-disjoint)
  using l fin by (auto simp: simple-clss-finite notin)
moreover have card (?P  $\cup$  ?N) = card ?P + card ?N
  apply (subst card-Un-disjoint)
  using l fin H by (auto simp: simple-clss-finite notin)
moreover
  have card ?P = card ?O
    using inj-on-iff-eq-card[of ?O op +  $\{\#Pos\ l\# \}$ ]
    by (auto simp: fin simple-clss-finite inj-on-def)
  moreover have card ?N = card ?O
    using inj-on-iff-eq-card[of ?O op +  $\{\#Neg\ l\# \}$ ]
    by (auto simp: fin simple-clss-finite inj-on-def)
  moreover have  $(3::nat) \wedge \text{card (insert l C)} = 3 \wedge (\text{card } C) + 3 \wedge (\text{card } C) + 3 \wedge (\text{card } C)$ 
    using l by (simp add: fin mult-2-right numeral-3-eq-3)
  ultimately show ?case using IH l by (auto simp: simple-clss-insert)
qed

```

```

lemma simple-clss-mono:
  assumes incl:  $\text{atms} \subseteq \text{atms}'$ 
  shows simple-clss  $\text{atms} \subseteq \text{simple-clss } \text{atms}'$ 
  using assms unfolding simple-clss-def by auto

```

```

lemma distinct-mset-not-tautology-implies-in-simple-clss:
  assumes distinct-mset  $\chi$  and  $\neg \text{tautology } \chi$ 
  shows  $\chi \in \text{simple-clss (atms-of } \chi)$ 
  using assms unfolding simple-clss-def by auto

```

```

lemma simplified-in-simple-clss:
  assumes distinct-mset-set  $\psi$  and  $\forall \chi \in \psi. \neg \text{tautology } \chi$ 
  shows  $\psi \subseteq \text{simple-clss (atms-of-ms } \psi)$ 
  using assms unfolding simple-clss-def
  by (auto simp: distinct-mset-set-def atms-of-ms-def)

```

11.6 Experiment: Expressing the Entailments as Locales

```

locale entail =
  fixes entail :: 'a set  $\Rightarrow$  'b  $\Rightarrow$  bool (infix  $\models_e$  50)
  assumes entail-insert[simp]:  $I \neq \{\} \implies \text{insert } L\ I \models_e x \longleftrightarrow \{L\} \models_e x \vee I \models_e x$ 

```

assumes *entail-union*[*simp*]: $I \models_e A \implies I \cup I' \models_e A$
begin
definition *entails* :: 'a set \Rightarrow 'b set \Rightarrow bool (**infix** \models_{es} 50) **where**
 $I \models_{es} A \longleftrightarrow (\forall a \in A. I \models_e a)$
lemma *entails-empty*[*simp*]:
 $I \models_{es} \{\}$
unfolding *entails-def* **by** *auto*
lemma *entails-single*[*iff*]:
 $I \models_{es} \{a\} \longleftrightarrow I \models_e a$
unfolding *entails-def* **by** *auto*
lemma *entails-insert-l*[*simp*]:
 $M \models_{es} A \implies \text{insert } L \ M \models_{es} A$
unfolding *entails-def* **by** (*metis Un-commute entail-union insert-is-Un*)
lemma *entails-union*[*iff*]: $I \models_{es} CC \cup DD \longleftrightarrow I \models_{es} CC \wedge I \models_{es} DD$
unfolding *entails-def* **by** *blast*
lemma *entails-insert*[*iff*]: $I \models_{es} \text{insert } C \ DD \longleftrightarrow I \models_e C \wedge I \models_{es} DD$
unfolding *entails-def* **by** *blast*
lemma *entails-insert-mono*: $DD \subseteq CC \implies I \models_{es} CC \implies I \models_{es} DD$
unfolding *entails-def* **by** *blast*
lemma *entails-union-increase*[*simp*]:
assumes $I \models_{es} \psi$
shows $I \cup I' \models_{es} \psi$
using *assms* **unfolding** *entails-def* **by** *auto*
lemma *true-clss-commute-l*:
 $(I \cup I' \models_{es} \psi) \longleftrightarrow (I' \cup I \models_{es} \psi)$
by (*simp add: Un-commute*)
lemma *entails-remove*[*simp*]: $I \models_{es} N \implies I \models_{es} \text{Set.remove } a \ N$
by (*simp add: entails-def*)
lemma *entails-remove-minus*[*simp*]: $I \models_{es} N \implies I \models_{es} N - A$
by (*simp add: entails-def*)
end
interpretation *true-cls*: *entail true-cls*
by *standard* (*auto simp add: true-cls-def*)

11.7 Entailment to be extended

definition *true-clss-ext* :: 'a literal set \Rightarrow 'a literal multiset set \Rightarrow bool (**infix** \models_{sext} 49)
where
 $I \models_{sext} N \longleftrightarrow (\forall J. I \subseteq J \longrightarrow \text{consistent-interp } J \longrightarrow \text{total-over-m } J \ N \longrightarrow J \models_s N)$
lemma *true-clss-imp-true-cls-ext*:
 $I \models_s N \implies I \models_{sext} N$
unfolding *true-clss-ext-def* **by** (*metis sup.orderE true-clss-union-increase'*)

```

lemma true-clss-ext-decrease-right-remove-r:
  assumes  $I \models_{\text{sext}} N$ 
  shows  $I \models_{\text{sext}} N - \{C\}$ 
  unfolding true-clss-ext-def
proof (intro allI impI)
  fix  $J$ 
  assume
     $I \subseteq J$  and
    cons: consistent-interp J and
    tot: total-over-m J (N - {C})
  let  $?J = J \cup \{Pos \text{ (atm-of } P) | P. P \in \# C \wedge \text{atm-of } P \notin \text{atm-of ' } J\}$ 
  have  $I \subseteq ?J$  using  $\langle I \subseteq J \rangle$  by auto
  moreover have consistent-interp ?J
    using cons unfolding consistent-interp-def apply (intro allI)
    by (rename-tac L, case-tac L) (fastforce simp add: image-iff)+
  moreover have total-over-m ?J N
    using tot unfolding total-over-m-def total-over-set-def atms-of-ms-def
    apply clarify
    apply (rename-tac l a, case-tac a \in N - {C})
    apply auto
    using atms-of-s-def atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
    by (fastforce simp: atms-of-def)
  ultimately have  $?J \models_s N$ 
    using assms unfolding true-clss-ext-def by blast
  then have  $?J \models_s N - \{C\}$  by auto
  have  $\{v \in ?J. \text{atm-of } v \in \text{atms-of-ms } (N - \{C\})\} \subseteq J$ 
    using tot unfolding total-over-m-def total-over-set-def
    by (auto intro!: rev-image-eqI)
  then show  $J \models_s N - \{C\}$ 
    using true-clss-remove-unused[OF  $\langle ?J \models_s N - \{C\} \rangle$ ] unfolding true-clss-def
    by (meson true-clss-mono-set-mset-l)
qed

```

```

lemma consistent-true-clss-ext-satisfiable:
  assumes consistent-interp I and  $I \models_{\text{sext}} A$ 
  shows satisfiable A
  by (metis Un-empty-left assms satisfiable-carac subset-Un-eq sup.left-idem
    total-over-m-consistent-extension total-over-m-empty true-clss-ext-def)

```

```

lemma not-consistent-true-clss-ext:
  assumes  $\neg \text{consistent-interp } I$ 
  shows  $I \models_{\text{sext}} A$ 
  by (meson assms consistent-interp-subset true-clss-ext-def)
end
theory Prop-Logic-Multiset
imports ../lib/Multiset-More Prop-Normalisation Partial-Clausal-Logic
begin

```

12 Link with Multiset Version

12.1 Transformation to Multiset

```

fun mset-of-conj :: 'a propo  $\Rightarrow$  'a literal multiset where
mset-of-conj (FOr  $\varphi \psi$ ) = mset-of-conj  $\varphi$  + mset-of-conj  $\psi$  |

```

$mset-of-conj (FVar v) = \{\# \text{ Pos } v \# \} \mid$
 $mset-of-conj (FNot (FVar v)) = \{\# \text{ Neg } v \# \} \mid$
 $mset-of-conj FF = \{\#\}$

fun *mset-of-formula* :: 'a *propo* \Rightarrow 'a *literal multiset set* **where**
 $mset-of-formula (FAnd \varphi \psi) = mset-of-formula \varphi \cup mset-of-formula \psi \mid$
 $mset-of-formula (FOr \varphi \psi) = \{mset-of-conj (FOr \varphi \psi)\} \mid$
 $mset-of-formula (FVar \psi) = \{mset-of-conj (FVar \psi)\} \mid$
 $mset-of-formula (FNot \psi) = \{mset-of-conj (FNot \psi)\} \mid$
 $mset-of-formula FF = \{\{\#\}\} \mid$
 $mset-of-formula FT = \{\}$

12.2 Equisatisfiability of the two Version

lemma *is-conj-with-TF-FNot*:

$is-conj-with-TF (FNot \varphi) \longleftrightarrow (\exists v. \varphi = FVar v \vee \varphi = FF \vee \varphi = FT)$
unfolding *is-conj-with-TF-def* **apply** (rule *iffI*)
apply (induction *FNot* φ rule: *super-grouped-by.induct*)
apply (induction *FNot* φ rule: *grouped-by.induct*)
apply *simp*
apply (cases φ ; *simp*)
apply *auto*
done

lemma *grouped-by-COr-FNot*:

$grouped-by COr (FNot \varphi) \longleftrightarrow (\exists v. \varphi = FVar v \vee \varphi = FF \vee \varphi = FT)$
unfolding *is-conj-with-TF-def* **apply** (rule *iffI*)
apply (induction *FNot* φ rule: *grouped-by.induct*)
apply *simp*
apply (cases φ ; *simp*)
apply *auto*
done

lemma

shows *no-T-F-FF*[*simp*]: $\neg no-T-F FF$ **and**
no-T-F-FT[*simp*]: $\neg no-T-F FT$
unfolding *no-T-F-def all-subformula-st-def* **by** *auto*

lemma *grouped-by-CAnd-FAnd*:

$grouped-by CAnd (FAnd \varphi1 \varphi2) \longleftrightarrow grouped-by CAnd \varphi1 \wedge grouped-by CAnd \varphi2$
apply (rule *iffI*)
apply (induction *FAnd* $\varphi1 \varphi2$ rule: *grouped-by.induct*)
using *connected-is-group*[of *CAnd* $\varphi1 \varphi2$] **by** *auto*

lemma *grouped-by-COr-FOr*:

$grouped-by COr (FOr \varphi1 \varphi2) \longleftrightarrow grouped-by COr \varphi1 \wedge grouped-by COr \varphi2$
apply (rule *iffI*)
apply (induction *FOr* $\varphi1 \varphi2$ rule: *grouped-by.induct*)
using *connected-is-group*[of *COr* $\varphi1 \varphi2$] **by** *auto*

lemma *grouped-by-COr-FAnd*[*simp*]: $\neg grouped-by COr (FAnd \varphi1 \varphi2)$

apply *clarify*
apply (induction *FAnd* $\varphi1 \varphi2$ rule: *grouped-by.induct*)
apply *auto*
done

```

lemma grouped-by-COr-FEq[simp]:  $\neg$  grouped-by COr (FEq  $\varphi 1$   $\varphi 2$ )
  apply clarify
  apply (induction FEq  $\varphi 1$   $\varphi 2$  rule: grouped-by.induct)
  apply auto
  done

lemma [simp]:  $\neg$ grouped-by COr (FImp  $\varphi$   $\psi$ )
  apply clarify
  by (induction FImp  $\varphi$   $\psi$  rule: grouped-by.induct) simp-all

lemma [simp]:  $\neg$  is-conj-with-TF (FImp  $\varphi$   $\psi$ )
  unfolding is-conj-with-TF-def apply clarify
  by (induction FImp  $\varphi$   $\psi$  rule: super-grouped-by.induct) simp-all

lemma [simp]:  $\neg$ grouped-by COr (FEq  $\varphi$   $\psi$ )
  apply clarify
  by (induction FEq  $\varphi$   $\psi$  rule: grouped-by.induct) simp-all

lemma [simp]:  $\neg$  is-conj-with-TF (FEq  $\varphi$   $\psi$ )
  unfolding is-conj-with-TF-def apply clarify
  by (induction FEq  $\varphi$   $\psi$  rule: super-grouped-by.induct) simp-all

lemma is-conj-with-TF-Fand:
  is-conj-with-TF (FAnd  $\varphi 1$   $\varphi 2$ )  $\implies$  is-conj-with-TF  $\varphi 1 \wedge$  is-conj-with-TF  $\varphi 2$ 
  unfolding is-conj-with-TF-def
  apply (induction FAnd  $\varphi 1$   $\varphi 2$  rule: super-grouped-by.induct)
  apply (auto simp: grouped-by-CAnd-FAnd intro: grouped-is-super-grouped)[]
  apply auto[]
  done

lemma is-conj-with-TF-FOr:
  is-conj-with-TF (FOr  $\varphi 1$   $\varphi 2$ )  $\implies$  grouped-by COr  $\varphi 1 \wedge$  grouped-by COr  $\varphi 2$ 
  unfolding is-conj-with-TF-def
  apply (induction FOr  $\varphi 1$   $\varphi 2$  rule: super-grouped-by.induct)
  apply (auto simp: grouped-by-COr-FOr)[]
  apply auto[]
  done

lemma grouped-by-COr-mset-of-formula:
  grouped-by COr  $\varphi \implies$  mset-of-formula  $\varphi =$  (if  $\varphi = FT$  then  $\{\}$  else  $\{mset-of-conj \varphi\}$ )
  by (induction  $\varphi$ ) (auto simp add: grouped-by-COr-FNot)

```

When a formula is in CNF form, then there is equisatisfiability between the multiset version and the CNF form. Remark that the definition for the entailment are slightly different: $op \models$ uses a function assigning *True* or *False*, while $op \models s$ uses a set where being in the list means entailment of a literal.

```

theorem
  fixes  $\varphi :: 'v$  propo
  assumes is-cnf  $\varphi$ 
  shows eval  $A \varphi \longleftrightarrow$  Partial-Clausal-Logic.true-clss ( $\{Pos\ v|v. A\ v\} \cup \{Neg\ v|v. \neg A\ v\}$ )
    (mset-of-formula  $\varphi$ )
  using assms
proof (induction  $\varphi$ )
  case FF

```

```

    then show ?case by auto
next
  case FT
  then show ?case by auto
next
  case (FVar v)
  then show ?case by auto
next
  case (FAnd  $\varphi$   $\psi$ )
  then show ?case
  unfolding is-cnf-def by (auto simp: is-conj-with-TF-FNot dest: is-conj-with-TF-Fand
    dest!:is-conj-with-TF-FOr)
next
  case (FOr  $\varphi$   $\psi$ )
  then have [simp]: mset-of-formula  $\varphi = \{mset-of-conj \varphi\}$  mset-of-formula  $\psi = \{mset-of-conj \psi\}$ 
    unfolding is-cnf-def by (auto dest!:is-conj-with-TF-FOr simp: grouped-by-COr-mset-of-formula
      split: if-splits)
  have is-conj-with-TF  $\varphi$  is-conj-with-TF  $\psi$ 
    using FOr(3) unfolding is-cnf-def no-T-F-def
    by (metis grouped-is-super-grouped is-conj-with-TF-FOr is-conj-with-TF-def)+
  then show ?case using FOr
    unfolding is-cnf-def by simp
next
  case (FImp  $\varphi$   $\psi$ )
  then show ?case
    unfolding is-cnf-def by auto
next
  case (FEq  $\varphi$   $\psi$ )
  then show ?case
    unfolding is-cnf-def by auto
next
  case (FNot  $\varphi$ )
  then show ?case
    unfolding is-cnf-def by (auto simp: is-conj-with-TF-FNot)
qed

end
theory Prop-Resolution
imports Partial-Clausal-Logic List-More Wellfounded-More

begin

```

13 Resolution

13.1 Simplification Rules

inductive *simplify* :: '*v* clauses \Rightarrow '*v* clauses \Rightarrow bool **for** *N* :: '*v* clause set **where**

tautology-deletion:

$$(A + \{\#Pos P\# \} + \{\#Neg P\# \}) \in N \Longrightarrow simplify \ N \ (N - \{A + \{\#Pos P\# \} + \{\#Neg P\# \} \})$$

condensation:

$$(A + \{\#L\# \} + \{\#L\# \}) \in N \Longrightarrow simplify \ N \ (N - \{A + \{\#L\# \} + \{\#L\# \} \} \cup \{A + \{\#L\# \} \})$$

subsumption:

$$A \in N \Longrightarrow A \subset\# B \Longrightarrow B \in N \Longrightarrow simplify \ N \ (N - \{B\})$$

lemma *simplify-preserves-un-sat'*:

```

fixes  $N\ N' :: 'v\ clauses$ 
assumes simplify  $N\ N'$ 
and total-over-m  $I\ N$ 
shows  $I \models_s N' \longrightarrow I \models_s N$ 
using assms
proof (induct rule: simplify.induct)
  case (tautology-deletion  $A\ P$ )
  then have  $I \models A + \{\#Pos\ P\} + \{\#Neg\ P\}$ 
    by (metis total-over-m-def total-over-set-literal-defined true-clss-singleton true-clss-union
      true-lit-def uminus-Neg union-commute)
  then show ?case by (metis Un-Diff-cancel2 true-clss-singleton true-clss-union)
next
  case (condensation  $A\ P$ )
  then show ?case by (metis Diff-insert-absorb Set.set-insert insertE true-clss-union true-clss-def
    true-clss-singleton true-clss-union)
next
  case (subsumption  $A\ B$ )
  have  $A \neq B$  using subsumption.hyps(2) by auto
  then have  $I \models_s N - \{B\} \implies I \models A$  using  $\langle A \in N \rangle$  by (simp add: true-clss-def)
  moreover have  $I \models A \implies I \models B$  using  $\langle A < \# B \rangle$  by auto
  ultimately show ?case by (metis insert-Diff-single true-clss-insert)
qed

lemma simplify-preserves-un-sat:
  fixes  $N\ N' :: 'v\ clauses$ 
  assumes simplify  $N\ N'$ 
  and total-over-m  $I\ N$ 
  shows  $I \models_s N \longrightarrow I \models_s N'$ 
  using assms apply (induct rule: simplify.induct)
  using true-clss-def by fastforce+
```

lemma *simplify-preserves-un-sat''*:

```

  fixes  $N\ N' :: 'v\ clauses$ 
  assumes simplify  $N\ N'$ 
  and total-over-m  $I\ N'$ 
  shows  $I \models_s N \longrightarrow I \models_s N'$ 
  using assms apply (induct rule: simplify.induct)
  using true-clss-def by fastforce+
```

lemma *simplify-preserves-un-sat-eq*:

```

  fixes  $N\ N' :: 'v\ clauses$ 
  assumes simplify  $N\ N'$ 
  and total-over-m  $I\ N$ 
  shows  $I \models_s N \longleftrightarrow I \models_s N'$ 
  using simplify-preserves-un-sat simplify-preserves-un-sat' assms by blast
```

lemma *simplify-preserves-finite*:

```

assumes simplify  $\psi\ \psi'$ 
shows finite  $\psi \longleftrightarrow \text{finite } \psi'$ 
using assms by (induct rule: simplify.induct, auto simp add: remove-def)
```

lemma *rtranclp-simplify-preserves-finite*:

```

assumes rtranclp simplify  $\psi\ \psi'$ 
shows finite  $\psi \longleftrightarrow \text{finite } \psi'$ 
using assms by (induct rule: rtranclp-induct) (auto simp add: simplify-preserves-finite)
```

lemma *simplify-atms-of-ms*:
assumes *simplify* ψ ψ'
shows *atms-of-ms* $\psi' \subseteq \text{atms-of-ms } \psi$
using *assms* **unfolding** *atms-of-ms-def*
proof (*induct* rule: *simplify.induct*)
case (*tautology-deletion* A P)
then show ?*case* **by** *auto*
next
case (*condensation* A P)
moreover have $A + \{\#P\# \} + \{\#P\# \} \in \psi \implies \exists x \in \psi. \text{atm-of } P \in \text{atm-of } ' \text{set-mset } x$
by (*metis* *Un-iff* *atms-of-def* *atms-of-plus* *atms-of-singleton* *insert-iff*)
ultimately show ?*case* **by** (*auto* *simp* *add*: *atms-of-def*)
next
case (*subsumption* A P)
then show ?*case* **by** *auto*
qed

lemma *rtranclp-simplify-atms-of-ms*:
assumes *rtranclp* *simplify* ψ ψ'
shows *atms-of-ms* $\psi' \subseteq \text{atms-of-ms } \psi$
using *assms* **apply** (*induct* rule: *rtranclp-induct*)
apply (*fastforce* *intro*: *simplify-atms-of-ms*)
using *simplify-atms-of-ms* **by** *blast*

lemma *factoring-imp-simplify*:
assumes $\{\#L\# \} + \{\#L\# \} + C \in N$
shows $\exists N'. \text{simplify } N N'$
proof –
have $C + \{\#L\# \} + \{\#L\# \} \in N$ **using** *assms* **by** (*simp* *add*: *add.commute* *union-lcomm*)
from *condensation*[*OF* *this*] **show** ?*thesis* **by** *blast*
qed

13.2 Unconstrained Resolution

type-synonym *'v uncon-state* = *'v clauses*
inductive *uncon-res* :: *'v uncon-state* \Rightarrow *'v uncon-state* \Rightarrow *bool* **where**
resolution:
 $\{\#Pos\ p\# \} + C \in N \implies \{\#Neg\ p\# \} + D \in N \implies (\{\#Pos\ p\# \} + C, \{\#Neg\ p\# \} + D) \notin \text{already-used}$
 $\implies \text{uncon-res } (N) (N \cup \{C + D\}) \mid$
factoring: $\{\#L\# \} + \{\#L\# \} + C \in N \implies \text{uncon-res } N (N \cup \{C + \{\#L\# \}\})$

lemma *uncon-res-increasing*:
assumes *uncon-res* $S S'$ **and** $\psi \in S$
shows $\psi \in S'$
using *assms* **by** (*induct* rule: *uncon-res.induct*) *auto*

lemma *rtranclp-uncon-inference-increasing*:
assumes *rtranclp* *uncon-res* $S S'$ **and** $\psi \in S$
shows $\psi \in S'$
using *assms* **by** (*induct* rule: *rtranclp-induct*) (*auto* *simp* *add*: *uncon-res-increasing*)

13.2.1 Subsumption

definition *subsumes* :: *'a literal multiset* \Rightarrow *'a literal multiset* \Rightarrow *bool* **where**

$subsumes \chi \chi' \longleftrightarrow$
 $(\forall I. total-over-m I \{\chi'\} \longrightarrow total-over-m I \{\chi\})$
 $\wedge (\forall I. total-over-m I \{\chi\} \longrightarrow I \models \chi \longrightarrow I \models \chi')$

lemma *subsumes-refl[simp]*:
subsumes $\chi \chi$
unfolding *subsumes-def* **by** *auto*

lemma *subsumes-subsumption*:
assumes *subsumes* $D \chi$
and $C \subset\# D$ **and** $\neg tautology \chi$
shows *subsumes* $C \chi$ **unfolding** *subsumes-def*
using *assms* *subsumption-total-over-m* *subsumption-chained* **unfolding** *subsumes-def*
by (*blast intro!*: *subset-mset.less-imp-le*)

lemma *subsumes-tautology*:
assumes *subsumes* $(C + \{\#Pos P\# \} + \{\#Neg P\# \}) \chi$
shows *tautology* χ
using *assms* **unfolding** *subsumes-def* **by** (*simp add: tautology-def*)

13.3 Inference Rule

type-synonym *'v state* = *'v clauses* \times (*'v clause* \times *'v clause*) *set*

inductive *inference-clause* :: *'v state* \Rightarrow *'v clause* \times (*'v clause* \times *'v clause*) *set* \Rightarrow *bool*

(**infix** \Rightarrow_{Res} 100) **where**

resolution:

$\{\#Pos p\# \} + C \in N \Longrightarrow \{\#Neg p\# \} + D \in N \Longrightarrow (\{\#Pos p\# \} + C, \{\#Neg p\# \} + D) \notin$
already-used
 $\Longrightarrow inference-clause (N, already-used) (C + D, already-used \cup \{(\{\#Pos p\# \} + C, \{\#Neg p\# \} + D)\}) \mid$

factoring: $\{\#L\# \} + \{\#L\# \} + C \in N \Longrightarrow inference-clause (N, already-used) (C + \{\#L\# \}, already-used)$

inductive *inference* :: *'v state* \Rightarrow *'v state* \Rightarrow *bool* **where**

inference-step: *inference-clause* S (*clause*, *already-used*)

$\Longrightarrow inference S (fst S \cup \{clause\}, already-used)$

abbreviation *already-used-inv*

:: *'a literal multiset set* \times (*'a literal multiset* \times *'a literal multiset*) *set* \Rightarrow *bool* **where**

already-used-inv state \equiv

$(\forall (A, B) \in snd\ state. \exists p. Pos\ p \in\# A \wedge Neg\ p \in\# B \wedge$
 $((\exists \chi \in fst\ state. subsumes\ \chi ((A - \{\#Pos\ p\# \}) + (B - \{\#Neg\ p\# \})))$
 $\vee tautology ((A - \{\#Pos\ p\# \}) + (B - \{\#Neg\ p\# \}))))$

lemma *inference-clause-preserves-already-used-inv*:

assumes *inference-clause* $S S'$

and *already-used-inv* S

shows *already-used-inv* ($fst\ S \cup \{fst\ S'\}$, *snd* S')

using *assms* **apply** (*induct rule: inference-clause.induct*)

by *fastforce*+

lemma *inference-preserves-already-used-inv*:

assumes *inference* $S S'$

and *already-used-inv* S

shows *already-used-inv* S'

```

using assms
proof (induct rule: inference.induct)
  case (inference-step S clause already-used)
  then show ?case
    using inference-clause-preserves-already-used-inv[of S (clause, already-used)] by simp
qed

lemma rtrancplp-inference-preserves-already-used-inv:
  assumes rtrancplp inference S S'
  and already-used-inv S
  shows already-used-inv S'
  using assms apply (induct rule: rtrancplp-induct, simp)
  using inference-preserves-already-used-inv unfolding tautology-def by fast

lemma subsumes-condensation:
  assumes subsumes (C + {#L#} + {#L#}) D
  shows subsumes (C + {#L#}) D
  using assms unfolding subsumes-def by simp

lemma simplify-preserves-already-used-inv:
  assumes simplify N N'
  and already-used-inv (N, already-used)
  shows already-used-inv (N', already-used)
  using assms
proof (induct rule: simplify.induct)
  case (condensation C L)
  then show ?case
    using subsumes-condensation by simp fast
next
  {
    fix a:: 'a and A :: 'a set and P
    have  $(\exists x \in \text{Set.remove } a \ A. P \ x) \longleftrightarrow (\exists x \in A. x \neq a \wedge P \ x)$  by auto
  } note ex-member-remove = this
  {
    fix a a0 :: 'v clause and A :: 'v clauses and y
    assume  $a \in A$  and  $a0 \subset\# a$ 
    then have  $(\exists x \in A. \text{subsumes } x \ y) \longleftrightarrow (\text{subsumes } a \ y \ \vee (\exists x \in A. x \neq a \wedge \text{subsumes } x \ y))$ 
    by auto
  } note tt2 = this
case (subsumption A B) note  $A = \text{this}(1)$  and  $AB = \text{this}(2)$  and  $B = \text{this}(3)$  and  $\text{inv} = \text{this}(4)$ 
show ?case
  proof (standard, standard)
    fix x a b
    assume  $x: x \in \text{snd } (N - \{B\}, \text{already-used})$  and [simp]:  $x = (a, b)$ 
    obtain p where  $p: \text{Pos } p \in\# a \wedge \text{Neg } p \in\# b$  and
       $q: (\exists \chi \in N. \text{subsumes } \chi (a - \{\#\text{Pos } p\# \} + (b - \{\#\text{Neg } p\# \})))$ 
       $\vee \text{tautology } (a - \{\#\text{Pos } p\# \} + (b - \{\#\text{Neg } p\# \}))$ 
    using inv x by fastforce
    consider (taut)  $\text{tautology } (a - \{\#\text{Pos } p\# \} + (b - \{\#\text{Neg } p\# \})) \mid$ 
       $(\chi) \chi$  where  $\chi \in N$   $\text{subsumes } \chi (a - \{\#\text{Pos } p\# \} + (b - \{\#\text{Neg } p\# \}))$ 
       $\neg \text{tautology } (a - \{\#\text{Pos } p\# \} + (b - \{\#\text{Neg } p\# \}))$ 
    using q by auto
  then show
     $\exists p. \text{Pos } p \in\# a \wedge \text{Neg } p \in\# b$ 
     $\wedge ((\exists \chi \in \text{fst } (N - \{B\}, \text{already-used}). \text{subsumes } \chi (a - \{\#\text{Pos } p\# \} + (b - \{\#\text{Neg } p\# \})))$ 

```

```

       $\vee \text{tautology } (a - \{\#Pos\ p\# \} + (b - \{\#Neg\ p\# \}))$ 
proof cases
  case taut
    then show ?thesis using p by auto
  next
    case  $\chi$  note  $H = \text{this}$ 
    show ?thesis using  $p\ A\ AB\ B\ \text{subsumes-subsumption}[OF - AB\ H(3)]\ H(1,2)$  by auto
  qed
qed
next
case (tautology-deletion  $C\ P$ )
then show ?case apply clarify
proof  $-$ 
  fix  $a\ b$ 
  assume  $C + \{\#Pos\ P\# \} + \{\#Neg\ P\# \} \in N$ 
  assume already-used-inv ( $N$ , already-used)
  and  $(a, b) \in \text{snd } (N - \{C + \{\#Pos\ P\# \} + \{\#Neg\ P\# \}\}, \text{already-used})$ 
  then obtain  $p$  where
     $Pos\ p \in \# a \wedge Neg\ p \in \# b \wedge$ 
     $((\exists \chi \in \text{fst } (N \cup \{C + \{\#Pos\ P\# \} + \{\#Neg\ P\# \}\}, \text{already-used}).$ 
       $\text{subsumes } \chi\ (a - \{\#Pos\ p\# \} + (b - \{\#Neg\ p\# \})))$ 
     $\vee \text{tautology } (a - \{\#Pos\ p\# \} + (b - \{\#Neg\ p\# \})))$ 
  by fastforce
moreover have tautology  $(C + \{\#Pos\ P\# \} + \{\#Neg\ P\# \})$  by auto
ultimately show
   $\exists p. Pos\ p \in \# a \wedge Neg\ p \in \# b$ 
   $\wedge ((\exists \chi \in \text{fst } (N - \{C + \{\#Pos\ P\# \} + \{\#Neg\ P\# \}\}, \text{already-used}).$ 
     $\text{subsumes } \chi\ (a - \{\#Pos\ p\# \} + (b - \{\#Neg\ p\# \})))$ 
     $\vee \text{tautology } (a - \{\#Pos\ p\# \} + (b - \{\#Neg\ p\# \})))$ 
  by (metis (no-types) Diff-iff Un-insert-right empty-iff fst-conv insertE subsumes-tautology
    sup-bot.right-neutral)
qed
qed

```

lemma

factoring-satisfiable: $I \models \{\#L\# \} + \{\#L\# \} + C \longleftrightarrow I \models \{\#L\# \} + C$ **and**

resolution-satisfiable:

consistent-interp $I \implies I \models \{\#Pos\ p\# \} + C \implies I \models \{\#Neg\ p\# \} + D \implies I \models C + D$ **and**

factoring-same-vars: $\text{atms-of } (\{\#L\# \} + \{\#L\# \} + C) = \text{atms-of } (\{\#L\# \} + C)$

unfolding *true-cls-def consistent-interp-def* **by** (*fastforce split: if-split-asm*) $+$

lemma *inference-increasing*:

assumes *inference* $S\ S'$ **and** $\psi \in \text{fst } S$

shows $\psi \in \text{fst } S'$

using *assms* **by** (*induct rule: inference.induct, auto*)

lemma *rtranclp-inference-increasing*:

assumes *rtranclp inference* $S\ S'$ **and** $\psi \in \text{fst } S$

shows $\psi \in \text{fst } S'$

using *assms* **by** (*induct rule: rtranclp-induct, auto simp add: inference-increasing*)

lemma *inference-clause-already-used-increasing*:

assumes *inference-clause* $S\ S'$

shows $\text{snd } S \subseteq \text{snd } S'$

using *assms* **by** (*induct rule:inference-clause.induct*, *auto*)

lemma *inference-already-used-increasing:*

assumes *inference S S'*
shows *snd S* \subseteq *snd S'*
using *assms* **apply** (*induct rule:inference.induct*)
using *inference-clause-already-used-increasing* **by** *fastforce*

lemma *inference-clause-preserves-un-sat:*

fixes *N N' :: 'v clauses*
assumes *inference-clause T T'*
and *total-over-m I (fst T)*
and *consistent: consistent-interp I*
shows $I \models_s \text{fst } T \longleftrightarrow I \models_s \text{fst } T \cup \{\text{fst } T'\}$
using *assms* **apply** (*induct rule: inference-clause.induct*)
unfolding *consistent-interp-def true-clss-def* **by** *auto force+*

lemma *inference-preserves-un-sat:*

fixes *N N' :: 'v clauses*
assumes *inference T T'*
and *total-over-m I (fst T)*
and *consistent: consistent-interp I*
shows $I \models_s \text{fst } T \longleftrightarrow I \models_s \text{fst } T'$
using *assms* **apply** (*induct rule: inference.induct*)
using *inference-clause-preserves-un-sat* **by** *fastforce*

lemma *inference-clause-preserves-atms-of-ms:*

assumes *inference-clause S S'*
shows *atms-of-ms (fst (fst S \cup {fst S'}), snd S')* \subseteq *atms-of-ms (fst S)*
using *assms* **apply** (*induct rule: inference-clause.induct*)
apply *auto*
apply (*metis Set.set-insert UnCI atms-of-ms-insert atms-of-plus*)
apply (*metis Set.set-insert UnCI atms-of-ms-insert atms-of-plus*)
apply (*simp add: in-m-in-literals union-assoc*)
unfolding *atms-of-ms-def* **using** *assms* **by** *fastforce*

lemma *inference-preserves-atms-of-ms:*

fixes *N N' :: 'v clauses*
assumes *inference T T'*
shows *atms-of-ms (fst T')* \subseteq *atms-of-ms (fst T)*
using *assms* **apply** (*induct rule: inference.induct*)
using *inference-clause-preserves-atms-of-ms* **by** *fastforce*

lemma *inference-preserves-total:*

fixes *N N' :: 'v clauses*
assumes *inference (N, already-used) (N', already-used')*
shows *total-over-m I N* \implies *total-over-m I N'*
using *assms* *inference-preserves-atms-of-ms* **unfolding** *total-over-m-def total-over-set-def*
by *fastforce*

lemma *rtrancplp-inference-preserves-total:*

assumes *rtrancplp inference T T'*

shows *total-over-m I (fst T) \implies total-over-m I (fst T')*
using *assms* **by** (*induct rule: rtranclp-induct, auto simp add: inference-preserves-total*)

lemma *rtranclp-inference-preserves-un-sat*:
assumes *rtranclp inference N N'*
and *total-over-m I (fst N)*
and *consistent: consistent-interp I*
shows *I \models_s fst N \longleftrightarrow I \models_s fst N'*
using *assms* **apply** (*induct rule: rtranclp-induct*)
apply (*simp add: inference-preserves-un-sat*)
using *inference-preserves-un-sat rtranclp-inference-preserves-total* **by** *blast*

lemma *inference-preserves-finite*:
assumes *inference ψ ψ' and finite (fst ψ)*
shows *finite (fst ψ')*
using *assms* **by** (*induct rule: inference.induct, auto simp add: simplify-preserves-finite*)

lemma *inference-clause-preserves-finite-snd*:
assumes *inference-clause ψ ψ' and finite (snd ψ)*
shows *finite (snd ψ')*
using *assms* **by** (*induct rule: inference-clause.induct, auto*)

lemma *inference-preserves-finite-snd*:
assumes *inference ψ ψ' and finite (snd ψ)*
shows *finite (snd ψ')*
using *assms inference-clause-preserves-finite-snd* **by** (*induct rule: inference.induct, fastforce*)

lemma *rtranclp-inference-preserves-finite*:
assumes *rtranclp inference ψ ψ' and finite (fst ψ)*
shows *finite (fst ψ')*
using *assms* **by** (*induct rule: rtranclp-induct*)
(auto simp add: simplify-preserves-finite inference-preserves-finite)

lemma *consistent-interp-insert*:
assumes *consistent-interp I*
and *atm-of P \notin atm-of 'I*
shows *consistent-interp (insert P I)*
proof –
have *P: insert P I = I \cup {P}* **by** *auto*
show *?thesis* **unfolding** *P*
apply (*rule consistent-interp-disjoint*)
using *assms* **by** (*auto simp: image-iff*)
qed

lemma *simplify-clause-preserves-sat*:
assumes *simp: simplify ψ ψ'*
and *satisfiable ψ'*
shows *satisfiable ψ*
using *assms*
proof *induction*
case (*tautology-deletion A P*) **note** *AP = this(1) and sat = this(2)*
let *?A' = A + {#Pos P#} + {#Neg P#}*

```

let ? $\psi'$  =  $\psi - \{?A'\}$ 
obtain  $I$  where
   $I: I \models_s ?\psi'$  and
  cons: consistent-interp  $I$  and
  tot: total-over-m  $I ?\psi'$ 
  using sat unfolding satisfiable-def by auto
{ assume Pos  $P \in I \vee$  Neg  $P \in I$ 
  then have  $I \models ?A'$  by auto
  then have  $I \models_s \psi$  using  $I$  by (metis insert-Diff tautology-deletion.hyps true-clss-insert)
  then have ?case using cons tot by auto
}
moreover {
  assume Pos: Pos  $P \notin I$  and Neg: Neg  $P \notin I$ 
  then have consistent-interp ( $I \cup \{Pos P\}$ ) using cons by simp
  moreover have  $I'A: I \cup \{Pos P\} \models ?A'$  by auto
  have  $\{Pos P\} \cup I \models_s \psi - \{A + \{\#Pos P\}\} + \{\#Neg P\}\}$ 
    using  $\langle I \models_s \psi - \{A + \{\#Pos P\}\} + \{\#Neg P\}\rangle$  true-clss-union-increase' by blast
  then have  $I \cup \{Pos P\} \models_s \psi$ 
    by (metis (no-types) Un-empty-right Un-insert-left Un-insert-right  $I'A$  insert-Diff
      sup-bot.left-neutral tautology-deletion.hyps true-clss-insert)
  ultimately have ?case using satisfiable-carac' by blast
}
ultimately show ?case by blast
next
case (condensation  $A L$ ) note  $AL = this(1)$  and sat = this(2)
have f3: simplify  $\psi (\psi - \{A + \{\#L\#\} + \{\#L\#\}\} \cup \{A + \{\#L\#\}\})$ 
  using  $AL$  simplify.condensation by blast
obtain  $LL :: 'a$  literal multiset set  $\Rightarrow 'a$  literal set where
  f4:  $LL (\psi - \{A + \{\#L\#\} + \{\#L\#\}\} \cup \{A + \{\#L\#\}\}) \models_s \psi - \{A + \{\#L\#\} + \{\#L\#\}\} \cup \{A + \{\#L\#\}\}$ 
     $\wedge$  consistent-interp ( $LL (\psi - \{A + \{\#L\#\} + \{\#L\#\}\} \cup \{A + \{\#L\#\}\})$ )
     $\wedge$  total-over-m ( $LL (\psi - \{A + \{\#L\#\} + \{\#L\#\}\} \cup \{A + \{\#L\#\}\}) (\psi - \{A + \{\#L\#\} + \{\#L\#\}\} \cup \{A + \{\#L\#\}\})$ )
  using sat by (meson satisfiable-def)
have f5: insert ( $A + \{\#L\#\} + \{\#L\#\}$ ) ( $\psi - \{A + \{\#L\#\} + \{\#L\#\}\}) = \psi$ 
  using  $AL$  by fastforce
have atms-of ( $A + \{\#L\#\} + \{\#L\#\}$ ) = atms-of ( $\{\#L\#\} + A$ )
  by simp
then show ?case
  using f5 f4 f3 by (metis (no-types) add commute satisfiable-def simplify-preserves-un-sat'
    total-over-m-insert total-over-m-union)
next
case (subsumption  $A B$ ) note  $A = this(1)$  and  $AB = this(2)$  and  $B = this(3)$  and sat = this(4)
let ? $\psi'$  =  $\psi - \{B\}$ 
obtain  $I$  where  $I: I \models_s ?\psi'$  and cons: consistent-interp  $I$  and tot: total-over-m  $I ?\psi'$ 
  using sat unfolding satisfiable-def by auto
have  $I \models A$  using  $A I$  by (metis  $AB$  Diff-iff subset-mset.less-irrefl singletonD true-clss-def)
then have  $I \models B$  using  $AB$  subset-mset.less-imp-le true-clss-mono-leD by blast
then have  $I \models_s \psi$  using  $I$  by (metis insert-Diff-single true-clss-insert)
then show ?case using cons satisfiable-carac' by blast
qed

lemma simplify-preserves-unsat:
  assumes inference  $\psi \psi'$ 
  shows satisfiable (fst  $\psi'$ )  $\longrightarrow$  satisfiable (fst  $\psi$ )

```

using *assms* **apply** (*induct rule: inference.induct*)
using *satisfiable-decreasing* **by** (*metis fst-conv*)**+**

lemma *inference-preserves-unsat*:
assumes *inference** S S'*
shows *satisfiable (fst S') \longrightarrow satisfiable (fst S)*
using *assms* **apply** (*induct rule: rtrancplp-induct*)
apply *simp-all*
using *simplify-preserves-unsat* **by** *blast*

datatype *'v sem-tree* = *Node 'v 'v sem-tree 'v sem-tree | Leaf*

fun *sem-tree-size* :: *'v sem-tree \Rightarrow nat* **where**
sem-tree-size Leaf = 0 |
sem-tree-size (Node - ag ad) = 1 + sem-tree-size ag + sem-tree-size ad

lemma *sem-tree-size[case-names bigger]*:
 $(\bigwedge xs:: 'v \text{ sem-tree. } (\bigwedge ys:: 'v \text{ sem-tree. } \text{sem-tree-size } ys < \text{sem-tree-size } xs \implies P \text{ } ys) \implies P \text{ } xs)$
 $\implies P \text{ } xs$
by (*fact Nat.measure-induct-rule*)

fun *partial-interps* :: *'v sem-tree \Rightarrow 'v interp \Rightarrow 'v clauses \Rightarrow bool* **where**
partial-interps Leaf I ψ = $(\exists \chi. \neg I \models \chi \wedge \chi \in \psi \wedge \text{total-over-m } I \{ \chi \})$ |
partial-interps (Node v ag ad) I ψ \longleftrightarrow
 $(\text{partial-interps } ag \text{ } (I \cup \{ \text{Pos } v \}) \text{ } \psi \wedge \text{partial-interps } ad \text{ } (I \cup \{ \text{Neg } v \}) \text{ } \psi)$

lemma *simplify-preserve-partial-leaf*:
simplify N N' \implies partial-interps Leaf I N \implies partial-interps Leaf I N'
apply (*induct rule: simplify.induct*)
using *union-lcomm* **apply** *auto[1]*
apply (*simp, metis atms-of-plus total-over-set-union true-cls-union*)
apply *simp*
by (*metis atms-of-ms-singleton mset-le-exists-conv subset-mset-def true-cls-mono-leD total-over-m-def total-over-m-sum*)

lemma *simplify-preserve-partial-tree*:
assumes *simplify N N'*
and *partial-interps t I N*
shows *partial-interps t I N'*
using *assms* **apply** (*induct t arbitrary: I, simp*)
using *simplify-preserve-partial-leaf* **by** *metis*

lemma *inference-preserve-partial-tree*:
assumes *inference S S'*
and *partial-interps t I (fst S)*
shows *partial-interps t I (fst S')*
using *assms* **apply** (*induct t arbitrary: I, simp-all*)
by (*meson inference-increasing*)

lemma *rtrancplp-inference-preserve-partial-tree*:

assumes *rtranclp inference* $N\ N'$
and *partial-interps* $t\ I\ (\text{fst } N)$
shows *partial-interps* $t\ I\ (\text{fst } N')$
using *assms* **apply** (*induct rule: rtranclp-induct, auto*)
using *inference-preserve-partial-tree* **by** *force*

function *build-sem-tree* :: $'v :: \text{linorder set} \Rightarrow 'v \text{ clauses} \Rightarrow 'v \text{ sem-tree}$ **where**
build-sem-tree $\text{atms } \psi =$
 (*if* $\text{atms} = \{\}$ $\vee \neg \text{finite atms}$
 then *Leaf*
 else *Node* (*Min* atms) (*build-sem-tree* (*Set.remove* (*Min* atms) atms) ψ)
 (*build-sem-tree* (*Set.remove* (*Min* atms) atms) ψ))
by *auto*
termination
 apply (*relation measure* $(\lambda(A, -). \text{card } A), \text{simp-all}$)
 apply (*metis Min-in card-Diff1-less remove-def*) +
done
declare *build-sem-tree.induct*[*case-names tree*]

lemma *unsatisfiable-empty[simp]*:
 $\neg \text{unsatisfiable } \{\}$
unfolding *satisfiable-def* **apply** *auto*
using *consistent-interp-def* **unfolding** *total-over-m-def total-over-set-def atms-of-ms-def* **by** *blast*

lemma *partial-interps-build-sem-tree-atms-general*:
fixes $\psi :: 'v :: \text{linorder clauses}$ **and** $p :: 'v \text{ literal list}$
assumes *unsat: unsatisfiable* ψ **and** *finite* ψ **and** *consistent-interp* I
and *finite* atms
and $\text{atms-of-ms } \psi = \text{atms} \cup \text{atms-of-s } I$ **and** $\text{atms} \cap \text{atms-of-s } I = \{\}$
shows *partial-interps* (*build-sem-tree* $\text{atms } \psi$) $I\ \psi$
using *assms*
proof (*induct arbitrary: I rule: build-sem-tree.induct*)
case ($1\ \text{atms } \psi\ Ia$) **note** $IH1 = \text{this}(1)$ **and** $IH2 = \text{this}(2)$ **and** $\text{unsat} = \text{this}(3)$ **and** $\text{finite} = \text{this}(4)$
and $\text{cons} = \text{this}(5)$ **and** $f = \text{this}(6)$ **and** $\text{un} = \text{this}(7)$ **and** $\text{disj} = \text{this}(8)$
{
 assume $\text{atms}: \text{atms} = \{\}$
 then have $\text{atmsIa}: \text{atms-of-ms } \psi = \text{atms-of-s } Ia$ **using** un **by** *auto*
 then have *total-over-m* $Ia\ \psi$ **unfolding** *total-over-m-def* atmsIa **by** *auto*
 then have $\chi: \exists \chi \in \psi. \neg Ia \models \chi$
 using *unsat cons* **unfolding** *true-clss-def satisfiable-def* **by** *auto*
 then have *build-sem-tree* $\text{atms } \psi = \text{Leaf}$ **using** atms **by** *auto*
 moreover
 have $\text{tot}: \bigwedge \chi. \chi \in \psi \implies \text{total-over-m } Ia\ \{\chi\}$
 unfolding *total-over-m-def total-over-set-def atms-of-ms-def atms-of-s-def*
 using atmsIa *atms-of-ms-def* **by** *fastforce*
 have *partial-interps* *Leaf* $Ia\ \psi$
 using $\chi\ \text{tot}$ **by** (*auto simp add: total-over-m-def total-over-set-def atms-of-ms-def*)

 ultimately have *?case* **by** *metis*
}
moreover **{**
 assume $\text{atms}: \text{atms} \neq \{\}$
 have *build-sem-tree* $\text{atms } \psi = \text{Node}$ (*Min* atms) (*build-sem-tree* (*Set.remove* (*Min* atms) atms) ψ)
 (*build-sem-tree* (*Set.remove* (*Min* atms) atms) ψ)
}


```

using build-sem-tree.simps[of atms  $\psi$ ] f atms by metis

have consistent-interp ( $Ia \cup \{Pos (Min\ atms)\}$ ) unfolding consistent-interp-def
  by (metis Int-iff Min-in Un-iff atm-of-uminus atms cons consistent-interp-def disj empty-iff
    f in-atms-of-s-decomp insert-iff literal.distinct(1) literal.exhaust-sel literal.sel(2)
    uminus-Neg uminus-Pos)
moreover have atms-of-ms  $\psi = Set.remove (Min\ atms)\ atms \cup atms-of-s (Ia \cup \{Pos (Min\ atms)\})$ 
  using Min-in atms f un by fastforce
moreover have disj':  $Set.remove (Min\ atms)\ atms \cap atms-of-s (Ia \cup \{Pos (Min\ atms)\}) = \{\}$ 
  by simp (metis disj disjoint-iff-not-equal member-remove)
moreover have finite ( $Set.remove (Min\ atms)\ atms$ ) using f by (simp add: remove-def)
ultimately have subtree1: partial-interps (build-sem-tree ( $Set.remove (Min\ atms)\ atms$ )  $\psi$ )
  ( $Ia \cup \{Pos (Min\ atms)\}$ )  $\psi$ 
  using IH1[of  $Ia \cup \{Pos (Min\ (atms))\}$ ] atms f unsat finite by metis

have consistent-interp ( $Ia \cup \{Neg (Min\ atms)\}$ ) unfolding consistent-interp-def
  by (metis Int-iff Min-in Un-iff atm-of-uminus atms cons consistent-interp-def disj empty-iff
    f in-atms-of-s-decomp insert-iff literal.distinct(1) literal.exhaust-sel literal.sel(2)
    uminus-Neg)
moreover have atms-of-ms  $\psi = Set.remove (Min\ atms)\ atms \cup atms-of-s (Ia \cup \{Neg (Min\ atms)\})$ 
  using  $\langle atms-of-ms\ \psi = Set.remove (Min\ atms)\ atms \cup atms-of-s (Ia \cup \{Pos (Min\ atms)\}) \rangle$  by
blast

moreover have disj':  $Set.remove (Min\ atms)\ atms \cap atms-of-s (Ia \cup \{Neg (Min\ atms)\}) = \{\}$ 
  using disj by auto
moreover have finite ( $Set.remove (Min\ atms)\ atms$ ) using f by (simp add: remove-def)
ultimately have subtree2: partial-interps (build-sem-tree ( $Set.remove (Min\ atms)\ atms$ )  $\psi$ )
  ( $Ia \cup \{Neg (Min\ atms)\}$ )  $\psi$ 
  using IH2[of  $Ia \cup \{Neg (Min\ (atms))\}$ ] atms f unsat finite by metis

then have ?case
  using IH1 subtree1 subtree2 f local.finite unsat atms by simp
}
ultimately show ?case by metis
qed

```

```

lemma partial-interps-build-sem-tree-atms:
  fixes  $\psi :: 'v :: linorder\ clauses$  and  $p :: 'v\ literal\ list$ 
  assumes unsat: unsatisfiable  $\psi$  and finite: finite  $\psi$ 
  shows partial-interps (build-sem-tree (atms-of-ms  $\psi$ )  $\psi$ )  $\{\}$   $\psi$ 
proof –
  have consistent-interp  $\{\}$  unfolding consistent-interp-def by auto
  moreover have atms-of-ms  $\psi = atms-of-ms\ \psi \cup atms-of-s\ \{\}$  unfolding atms-of-s-def by auto
  moreover have atms-of-ms  $\psi \cap atms-of-s\ \{\} = \{\}$  unfolding atms-of-s-def by auto
  moreover have finite (atms-of-ms  $\psi$ ) unfolding atms-of-ms-def using finite by simp
  ultimately show partial-interps (build-sem-tree (atms-of-ms  $\psi$ )  $\psi$ )  $\{\}$   $\psi$ 
    using partial-interps-build-sem-tree-atms-general[of  $\psi\ \{\}$  atms-of-ms  $\psi$ ] assms by metis
qed

```

```

lemma can-decrease-count:
  fixes  $\psi'' :: 'v\ clauses \times ('v\ clause \times 'v\ clause \times 'v)\ set$ 
  assumes count  $\chi\ L = n$ 
  and  $L \in \# \chi$  and  $\chi \in fst\ \psi$ 
  shows  $\exists \psi'\ \chi'. inference^{**}\ \psi\ \psi' \wedge \chi' \in fst\ \psi' \wedge (\forall L. L \in \# \chi \longleftrightarrow L \in \# \chi')$ 

```

```

     $\wedge \text{count } \chi' L = 1$ 
     $\wedge (\forall \varphi. \varphi \in \text{fst } \psi \longrightarrow \varphi \in \text{fst } \psi')$ 
     $\wedge (I \models \chi \longleftrightarrow I \models \chi')$ 
     $\wedge (\forall I'. \text{total-over-}m \ I' \{\chi\} \longrightarrow \text{total-over-}m \ I' \{\chi'\})$ 
  using assms
proof (induct n arbitrary:  $\chi \ \psi$ )
  case 0
  then show ?case by (simp add: not-in-iff[symmetric])
next
case (Suc n  $\chi$ )
note IH = this(1) and count = this(2) and L = this(3) and  $\chi = \text{this}(4)$ 
{
  assume  $n = 0$ 
  then have inference**  $\psi \ \psi$ 
  and  $\chi \in \text{fst } \psi$ 
  and  $\forall L. (L \in \# \ \chi) \longleftrightarrow (L \in \# \ \chi)$ 
  and  $\text{count } \chi \ L = (1::\text{nat})$ 
  and  $\forall \varphi. \varphi \in \text{fst } \psi \longrightarrow \varphi \in \text{fst } \psi$ 
  by (auto simp add: count L  $\chi$ )
  then have ?case by metis
}
moreover {
  assume  $n > 0$ 
  then have  $\exists C. \chi = C + \{\#L, L\# \}$ 
  by (smt L Suc-eq-plus1-left add.left-commute add-diff-cancel-left' add-diff-cancel-right'
    count-greater-zero-iff count-single local.count multi-member-split plus-multiset.rep-eq)
  then obtain C where  $C: \chi = C + \{\#L, L\# \}$  by metis
  let  $? \chi' = C + \{\#L\# \}$ 
  let  $? \psi' = (\text{fst } \psi \cup \{? \chi'\}, \text{snd } \psi)$ 
  have  $\varphi: \forall \varphi \in \text{fst } \psi. (\varphi \in \text{fst } \psi \vee \varphi \neq ? \chi') \longleftrightarrow \varphi \in \text{fst } ? \psi'$  unfolding C by auto
  have inf: inference  $\psi \ ? \psi'$ 
  using C factoring  $\chi$  prod.collapse union-commute inference-step by metis
  moreover have count':  $\text{count } ? \chi' L = n$  using C count by auto
  moreover have  $L \chi': L \in \# \ ? \chi'$  by auto
  moreover have  $\chi' \psi': ? \chi' \in \text{fst } ? \psi'$  by auto
  ultimately obtain  $\psi''$  and  $\chi''$ 
  where
    inference**  $? \psi' \ \psi''$  and
     $\alpha: \chi'' \in \text{fst } \psi''$  and
     $\forall La. (La \in \# \ ? \chi') \longleftrightarrow (La \in \# \ \chi'')$  and
     $\beta: \text{count } \chi'' L = (1::\text{nat})$  and
     $\varphi': \forall \varphi. \varphi \in \text{fst } ? \psi' \longrightarrow \varphi \in \text{fst } \psi''$  and
     $I \chi: I \models ? \chi' \longleftrightarrow I \models \chi''$  and
    tot:  $\forall I'. \text{total-over-}m \ I' \{? \chi'\} \longrightarrow \text{total-over-}m \ I' \{\chi''\}$ 
    using IH[of  $? \chi' \ ? \psi'$ ] count'  $L \chi' \ \chi' \psi'$  by blast

  then have inference**  $\psi \ \psi''$ 
  and  $\forall La. (La \in \# \ \chi) \longleftrightarrow (La \in \# \ \chi'')$ 
  using inf unfolding C by auto
  moreover have  $\forall \varphi. \varphi \in \text{fst } \psi \longrightarrow \varphi \in \text{fst } \psi''$  using  $\varphi \ \varphi'$  by metis
  moreover have  $I \models \chi \longleftrightarrow I \models \chi''$  using  $I \chi$  unfolding true-cls-def C by auto
  moreover have  $\forall I'. \text{total-over-}m \ I' \{\chi\} \longrightarrow \text{total-over-}m \ I' \{\chi''\}$ 
  using tot unfolding C total-over-m-def by auto
  ultimately have ?case using  $\varphi \ \varphi' \ \alpha \ \beta$  by metis
}

```

ultimately show ?case by auto
qed

lemma can-decrease-tree-size:

fixes $\psi :: 'v$ state and $tree :: 'v$ sem-tree
assumes finite (fst ψ) and already-used-inv ψ
and partial-interps tree I (fst ψ)
shows $\exists (tree' :: 'v$ sem-tree) ψ' . inference** $\psi \psi' \wedge$ partial-interps tree' I (fst ψ')
 \wedge (sem-tree-size tree' < sem-tree-size tree \vee sem-tree-size tree = 0)
using assms

proof (induct arbitrary: I rule: sem-tree-size)

case (bigger xs I) note $IH = \text{this}(1)$ and finite = this(2) and a-u-i = this(3) and part = this(4)

{
 assume sem-tree-size xs = 0
 then have ?case using part by blast
}

moreover {

 assume sn0: sem-tree-size xs > 0
 obtain ag ad v where xs: xs = Node v ag ad using sn0 by (cases xs, auto)
 {
 assume sem-tree-size ag = 0 and sem-tree-size ad = 0
 then have ag: ag = Leaf and ad: ad = Leaf by (cases ag, auto) (cases ad, auto)

 then obtain $\chi \chi'$ where

$\chi: \neg I \cup \{Pos\ v\} \models \chi$ and
 tot χ : total-over-m ($I \cup \{Pos\ v\}$) { χ } and
 $\chi\psi$: $\chi \in \text{fst } \psi$ and
 $\chi': \neg I \cup \{Neg\ v\} \models \chi'$ and
 tot χ' : total-over-m ($I \cup \{Neg\ v\}$) { χ' } and
 $\chi'\psi$: $\chi' \in \text{fst } \psi$

 using part unfolding xs by auto

 have Posv: $\neg Pos\ v \in \# \chi$ using χ unfolding true-cls-def true-lit-def by auto

 have Negv: $\neg Neg\ v \in \# \chi'$ using χ' unfolding true-cls-def true-lit-def by auto

{
 assume Neg χ : $\neg Neg\ v \in \# \chi$
 have $\neg I \models \chi$ using χ Posv unfolding true-cls-def true-lit-def by auto
 moreover have total-over-m I { χ }
 using Posv Neg χ atm-imp-pos-or-neg-lit tot χ unfolding total-over-m-def total-over-set-def
 by fastforce
 ultimately have partial-interps Leaf I (fst ψ)
 and sem-tree-size Leaf < sem-tree-size xs
 and inference** $\psi \psi$
 unfolding xs by (auto simp add: $\chi\psi$)
 }

moreover {

 assume Pos χ : $\neg Pos\ v \in \# \chi'$
 then have $I\chi$: $\neg I \models \chi'$ using χ' Posv unfolding true-cls-def true-lit-def by auto
 moreover have total-over-m I { χ' }
 using Negv Pos χ atm-imp-pos-or-neg-lit tot χ'
 unfolding total-over-m-def total-over-set-def by fastforce
 ultimately have partial-interps Leaf I (fst ψ) and
 sem-tree-size Leaf < sem-tree-size xs and
 inference** $\psi \psi$

```

    using  $\chi' \psi$   $I_\chi$  unfolding  $xs$  by auto
  }
  moreover {
    assume neg:  $Neg\ v \in \# \chi$  and pos:  $Pos\ v \in \# \chi'$ 
    then obtain  $\psi' \chi^2$  where inf:  $rtranclp$  inference  $\psi\ \psi'$  and  $\chi^2 incl$ :  $\chi^2 \in fst\ \psi'$ 
      and  $\chi \chi^2 incl$ :  $\forall L. L \in \# \chi \longleftrightarrow L \in \# \chi^2$ 
      and  $count \chi^2$ :  $count\ \chi^2\ (Neg\ v) = 1$ 
      and  $\varphi$ :  $\forall \varphi::'v$  literal multiset.  $\varphi \in fst\ \psi \longrightarrow \varphi \in fst\ \psi'$ 
      and  $I_\chi$ :  $I \models \chi \longleftrightarrow I \models \chi^2$ 
      and  $tot\_imp_\chi$ :  $\forall I'. total\_over\_m\ I'\ \{\chi\} \longrightarrow total\_over\_m\ I'\ \{\chi^2\}$ 
      using can-decrease-count[of  $\chi\ Neg\ v\ count\ \chi\ (Neg\ v)\ \psi\ I$ ]  $\chi \psi\ \chi' \psi$  by auto

    have  $\chi' \in fst\ \psi'$  by (simp add:  $\chi' \psi\ \varphi$ )
    with pos
    obtain  $\psi'' \chi^{2'}$  where
      inf': inference**  $\psi'\ \psi''$ 
      and  $\chi^{2'} incl$ :  $\chi^{2'} \in fst\ \psi''$ 
      and  $\chi' \chi^{2'} incl$ :  $\forall L::'v$  literal.  $(L \in \# \chi') = (L \in \# \chi^{2'})$ 
      and  $count \chi^{2'}$ :  $count\ \chi^{2'}\ (Pos\ v) = (1::nat)$ 
      and  $\varphi'$ :  $\forall \varphi::'v$  literal multiset.  $\varphi \in fst\ \psi' \longrightarrow \varphi \in fst\ \psi''$ 
      and  $I_{\chi'}$ :  $I \models \chi' \longleftrightarrow I \models \chi^{2'}$ 
      and  $tot\_imp_{\chi'}$ :  $\forall I'. total\_over\_m\ I'\ \{\chi'\} \longrightarrow total\_over\_m\ I'\ \{\chi^{2'}\}$ 
      using can-decrease-count[of  $\chi'\ Pos\ v\ count\ \chi'\ (Pos\ v)\ \psi'\ I$ ] by auto

    obtain  $C$  where  $\chi^2$ :  $\chi^2 = C + \{\#Neg\ v\# \}$  and  $negC$ :  $Neg\ v \notin \# C$  and  $posC$ :  $Pos\ v \notin \# C$ 
    proof -
      have  $\bigwedge m. Suc\ 0 - count\ m\ (Neg\ v) = count\ (\chi^2 - m)\ (Neg\ v)$ 
        by (simp add:  $count \chi^2$ )
      then show ?thesis
        using that by (metis (no-types) One-nat-def Posv Suc-inject Suc-pred  $\chi \chi^2 incl$ 
          count-diff count-single insert-DiffM2 mem-Collect-eq multi-member-skip neg
          not-gr0 set-mset-def union-commute)
    qed

    obtain  $C'$  where
       $\chi^{2'}$ :  $\chi^{2'} = C' + \{\#Pos\ v\# \}$  and
       $posC'$ :  $Pos\ v \notin \# C'$  and
       $negC'$ :  $Neg\ v \notin \# C'$ 
    proof -
      assume a1:  $\bigwedge C'. [\chi^{2'} = C' + \{\#Pos\ v\# \}; Pos\ v \notin \# C'; Neg\ v \notin \# C'] \implies thesis$ 
      have f2:  $\bigwedge n. (n::nat) - n = 0$ 
        by simp
      have  $Neg\ v \notin \# \chi^{2'} - \{\#Pos\ v\# \}$ 
        using Negv  $\chi' \chi^{2'} incl$  by (auto simp: not-in-iff)
      have  $count\ \{\#Pos\ v\# \}\ (Pos\ v) = 1$ 
        by simp
      then show ?thesis
        by (metis  $\chi' \chi^{2'} incl$   $(Neg\ v \notin \# \chi^{2'} - \{\#Pos\ v\# \})$  a1  $count \chi^{2'}$  count-diff f2
          insert-DiffM2 less-numeral-extra(3) mem-Collect-eq pos set-mset-def)
    qed

    have already-used-inv  $\psi'$ 
      using  $rtranclp$ -inference-preserves-already-used-inv[of  $\psi\ \psi'$ ] a-u-i inf by blast
    then have a-u-i- $\psi''$ : already-used-inv  $\psi''$ 
      using  $rtranclp$ -inference-preserves-already-used-inv a-u-i inf' unfolding tautology-def

```

```

by simp

have totC: total-over-m I {C}
  using tot-imp $\chi$  tot $\chi$  tot-over-m-remove[of I Pos v C] negC posC unfolding  $\chi^2$ 
  by (metis total-over-m-sum uminus-Neg uminus-of-uminus-id)
have totC': total-over-m I {C'}
  using tot-imp $\chi'$  tot $\chi'$  total-over-m-sum tot-over-m-remove[of I Neg v C'] negC' posC'
  unfolding  $\chi^{2'}$  by (metis total-over-m-sum uminus-Neg)
have  $\neg I \models C + C'$ 
  using  $\chi$  I $\chi$   $\chi'$  I $\chi'$  unfolding  $\chi^2$   $\chi^{2'}$  true-cls-def by auto
then have part-I- $\psi'''$ : partial-interps Leaf I (fst  $\psi'' \cup \{C + C'\}$ )
  using totC totC' by simp
  (metis  $\neg I \models C + C'$  atms-of-ms-singleton total-over-m-def total-over-m-sum)
{
  assume ({#Pos v#} + C', {#Neg v#} + C)  $\notin$  snd  $\psi''$ 
  then have inf'': inference  $\psi''$  (fst  $\psi'' \cup \{C + C'\}$ , snd  $\psi'' \cup \{(\chi^{2'}, \chi^2)\}$ )
    using add.commute  $\varphi'$   $\chi^{2\text{incl}}$  ( $\chi^{2'} \in \text{fst } \psi''$ ) unfolding  $\chi^2$   $\chi^{2'}$ 
    by (metis prod.collapse inference-step resolution)
  have inference**  $\psi$  (fst  $\psi'' \cup \{C + C'\}$ , snd  $\psi'' \cup \{(\chi^{2'}, \chi^2)\}$ )
    using inf inf' inf'' rtranclp-trans by auto
  moreover have sem-tree-size Leaf < sem-tree-size xs unfolding xs by auto
  ultimately have ?case using part-I- $\psi'''$  by (metis fst-conv)
}
moreover {
  assume a: ({#Pos v#} + C', {#Neg v#} + C)  $\in$  snd  $\psi''$ 
  then have ( $\exists \chi \in \text{fst } \psi''$ . ( $\forall I$ . total-over-m I {C+C'}  $\longrightarrow$  total-over-m I { $\chi$ }
     $\wedge$  ( $\forall I$ . total-over-m I { $\chi$ }  $\longrightarrow$  I  $\models \chi \longrightarrow$  I  $\models C' + C$ ))
     $\vee$  tautology (C' + C))
  proof -
    obtain p where p: Pos p  $\in$  # ({#Pos v#} + C') and
      n: Neg p  $\in$  # ({#Neg v#} + C) and
      decomp: (( $\exists \chi \in \text{fst } \psi''$ .
        ( $\forall I$ . total-over-m I ({#Pos v#} + C') - {#Pos p#}
          + (({#Neg v#} + C) - {#Neg p#}))
           $\longrightarrow$  total-over-m I { $\chi$ }
           $\wedge$  ( $\forall I$ . total-over-m I { $\chi$ }  $\longrightarrow$  I  $\models \chi$ 
             $\longrightarrow$  I  $\models$  ({#Pos v#} + C') - {#Pos p#} + (({#Neg v#} + C) - {#Neg p#})))
           $\vee$  tautology (({#Pos v#} + C') - {#Pos p#} + (({#Neg v#} + C) - {#Neg p#})))
        using a by (blast intro: allE[OF a-u-i- $\psi''$ [unfolded subsumes-def Ball-def],
          of ({#Pos v#} + C', {#Neg v#} + C)])
    { assume p  $\neq$  v
      then have Pos p  $\in$  # C'  $\wedge$  Neg p  $\in$  # C using p n by force
      then have ?thesis unfolding Bex-def by auto
    }
  }
  moreover {
    assume p = v
    then have ?thesis using decomp by (metis add.commute add-diff-cancel-left)
  }
  ultimately show ?thesis by auto
qed
moreover {
  assume  $\exists \chi \in \text{fst } \psi''$ . ( $\forall I$ . total-over-m I {C+C'}  $\longrightarrow$  total-over-m I { $\chi$ }
     $\wedge$  ( $\forall I$ . total-over-m I { $\chi$ }  $\longrightarrow$  I  $\models \chi \longrightarrow$  I  $\models C' + C$ )
  then obtain  $\vartheta$  where  $\vartheta$ :  $\vartheta \in \text{fst } \psi''$  and

```

```

    tot- $\vartheta$ -CC':  $\forall I. \text{total-over-}m \ I \ \{C+C'\} \longrightarrow \text{total-over-}m \ I \ \{\vartheta\}$  and
     $\vartheta$ -inv:  $\forall I. \text{total-over-}m \ I \ \{\vartheta\} \longrightarrow I \models \vartheta \longrightarrow I \models C' + C$  by blast
  have partial-interps Leaf I (fst  $\psi''$ )
    using tot- $\vartheta$ -CC'  $\vartheta$ -inv totC totC'  $\langle \neg I \models C + C' \rangle$  total-over-m-sum by fastforce
  moreover have sem-tree-size Leaf < sem-tree-size xs unfolding xs by auto
  ultimately have ?case by (metis inf inf' rtranclp-trans)
}
moreover {
  assume tautCC': tautology (C' + C)
  have total-over-m I {C'+C} using totC totC' total-over-m-sum by auto
  then have  $\neg$ tautology (C' + C)
    using  $\langle \neg I \models C + C' \rangle$  unfolding add.commute[of C C'] total-over-m-def
    unfolding tautology-def by auto
  then have False using tautCC' unfolding tautology-def by auto
}
ultimately have ?case by auto
}
ultimately have ?case by auto
}
ultimately have ?case using part by (metis (no-types) sem-tree-size.simps(1))
}
moreover {
  assume size-ag: sem-tree-size ag > 0
  have sem-tree-size ag < sem-tree-size xs unfolding xs by auto
  moreover have partial-interps ag (I  $\cup$  {Pos v}) (fst  $\psi$ )
    and partad: partial-interps ad (I  $\cup$  {Neg v}) (fst  $\psi$ )
    using part partial-interps.simps(2) unfolding xs by metis+
  moreover have sem-tree-size ag < sem-tree-size xs  $\longrightarrow$  finite (fst  $\psi$ )  $\longrightarrow$  already-used-inv  $\psi$ 
 $\longrightarrow$  ( partial-interps ag (I  $\cup$  {Pos v}) (fst  $\psi$ )  $\longrightarrow$ 
    ( $\exists$  tree'  $\psi'. \text{inference}^{**} \ \psi \ \psi' \wedge \text{partial-interps tree}' (I \cup \{\text{Pos } v\})$  (fst  $\psi'$ )
       $\wedge$  (sem-tree-size tree' < sem-tree-size ag  $\vee$  sem-tree-size ag = 0)))
    using IH by auto
  ultimately obtain  $\psi' :: 'v$  state and tree' :: 'v sem-tree where
    inf: inference**  $\psi \ \psi'$ 
    and part: partial-interps tree' (I  $\cup$  {Pos v}) (fst  $\psi'$ )
    and size: sem-tree-size tree' < sem-tree-size ag  $\vee$  sem-tree-size ag = 0
    using finite part rtranclp.rtrancl-refl a-u-i by blast

  have partial-interps ad (I  $\cup$  {Neg v}) (fst  $\psi'$ )
    using rtranclp-inference-preserve-partial-tree inf partad by metis
  then have partial-interps (Node v tree' ad) I (fst  $\psi'$ ) using part by auto
  then have ?case using inf size size-ag part unfolding xs by fastforce
}
moreover {
  assume size-ad: sem-tree-size ad > 0
  have sem-tree-size ad < sem-tree-size xs unfolding xs by auto
  moreover have partag: partial-interps ag (I  $\cup$  {Pos v}) (fst  $\psi$ ) and
    partial-interps ad (I  $\cup$  {Neg v}) (fst  $\psi$ )
    using part partial-interps.simps(2) unfolding xs by metis+
  moreover have sem-tree-size ad < sem-tree-size xs  $\longrightarrow$  finite (fst  $\psi$ )  $\longrightarrow$  already-used-inv  $\psi$ 
 $\longrightarrow$  ( partial-interps ad (I  $\cup$  {Neg v}) (fst  $\psi$ )
 $\longrightarrow$  ( $\exists$  tree'  $\psi'. \text{inference}^{**} \ \psi \ \psi' \wedge \text{partial-interps tree}' (I \cup \{\text{Neg } v\})$  (fst  $\psi'$ )
       $\wedge$  (sem-tree-size tree' < sem-tree-size ad  $\vee$  sem-tree-size ad = 0)))
    using IH by auto
  ultimately obtain  $\psi' :: 'v$  state and tree' :: 'v sem-tree where

```

```

  inf: inference**  $\psi \psi'$ 
  and part: partial-interps tree' ( $I \cup \{Neg\ v\}$ ) (fst  $\psi'$ )
  and size: sem-tree-size tree' < sem-tree-size ad  $\vee$  sem-tree-size ad = 0
  using finite part rtranclp.rtrancl-refl a-u-i by blast

  have partial-interps ag ( $I \cup \{Pos\ v\}$ ) (fst  $\psi'$ )
    using rtranclp-inference-preserve-partial-tree inf partag by metis
  then have partial-interps (Node v ag tree') I (fst  $\psi'$ ) using part by auto
  then have ?case using inf size size-ad unfolding xs by fastforce
}
ultimately have ?case by auto
}
ultimately show ?case by auto
qed

lemma inference-completeness-inv:
  fixes  $\psi :: 'v :: linorder\ state$ 
  assumes
    unsat:  $\neg$ satisfiable (fst  $\psi$ ) and
    finite: finite (fst  $\psi$ ) and
    a-u-v: already-used-inv  $\psi$ 
  shows  $\exists \psi'. (inference^{**} \psi \psi' \wedge \{\#\} \in fst \psi')$ 
proof -
  obtain tree where partial-interps tree {} (fst  $\psi$ )
    using partial-interps-build-sem-tree-atms assms by metis
  then show ?thesis
    using unsat finite a-u-v
  proof (induct tree arbitrary:  $\psi$  rule: sem-tree-size)
    case (bigger tree  $\psi$ ) note H = this
    {
      fix  $\chi$ 
      assume tree: tree = Leaf
      obtain  $\chi$  where  $\chi: \neg \{\} \models \chi$  and tot $\chi$ : total-over-m {} { $\chi$ } and  $\chi\psi$ :  $\chi \in fst \psi$ 
        using H unfolding tree by auto
      moreover have  $\{\#\} = \chi$ 
        using tot $\chi$  unfolding total-over-m-def total-over-set-def by fastforce
      moreover have inference**  $\psi \psi$  by auto
      ultimately have ?case by metis
    }
  moreover {
    fix v tree1 tree2
    assume tree: tree = Node v tree1 tree2
    obtain
      tree'  $\psi'$  where inf: inference**  $\psi \psi'$  and
      part': partial-interps tree' {} (fst  $\psi'$ ) and
      decrease: sem-tree-size tree' < sem-tree-size tree  $\vee$  sem-tree-size tree = 0
      using can-decrease-tree-size[of  $\psi$ ] H(2,4,5) unfolding tautology-def by meson
    have sem-tree-size tree' < sem-tree-size tree using decrease unfolding tree by auto
    moreover have finite (fst  $\psi'$ ) using rtranclp-inference-preserves-finite inf H(4) by metis
    moreover have unsatisfiable (fst  $\psi'$ )
      using inference-preserves-unsat inf bigger.prem(2) by blast
    moreover have already-used-inv  $\psi'$ 
      using H(5) inf rtranclp-inference-preserves-already-used-inv[of  $\psi \psi'$ ] by auto
    ultimately have ?case using inf rtranclp-trans part' H(1) by fastforce
  }
}

```

```

      ultimately show ?case by (cases tree, auto)
    qed
  qed

lemma inference-completeness:
  fixes  $\psi :: 'v :: \text{linorder}$  state
  assumes unsat:  $\neg \text{satisfiable } (\text{fst } \psi)$ 
  and finite: finite (fst  $\psi$ )
  and snd  $\psi = \{\}$ 
  shows  $\exists \psi'. (\text{rtranclp inference } \psi \ \psi' \wedge \{\#\} \in \text{fst } \psi')$ 
proof -
  have already-used-inv  $\psi$  unfolding assms by auto
  then show ?thesis using assms inference-completeness-inv by blast
qed

```

```

lemma inference-soundness:
  fixes  $\psi :: 'v :: \text{linorder}$  state
  assumes rtranclp inference  $\psi \ \psi'$  and  $\{\#\} \in \text{fst } \psi'$ 
  shows unsatisfiable (fst  $\psi$ )
using assms by (meson rtranclp-inference-preserves-un-sat satisfiable-def true-cls-empty
  true-clss-def)

```

```

lemma inference-soundness-and-completeness:
  fixes  $\psi :: 'v :: \text{linorder}$  state
  assumes finite: finite (fst  $\psi$ )
  and snd  $\psi = \{\}$ 
  shows  $(\exists \psi'. (\text{inference}^{**} \ \psi \ \psi' \wedge \{\#\} \in \text{fst } \psi')) \longleftrightarrow \text{unsatisfiable } (\text{fst } \psi)$ 
using assms inference-completeness inference-soundness by metis

```

13.4 Lemma about the simplified state

abbreviation *simplified* $\psi \equiv (\text{no-step simplify } \psi)$

```

lemma simplified-count:
  assumes simp: simplified  $\psi$  and  $\chi: \chi \in \psi$ 
  shows count  $\chi \ L \leq 1$ 
proof -
  {
    let ? $\chi' = \chi - \{\#L, L\# \}$ 
    assume count  $\chi \ L \geq 2$ 
    then have f1: count  $(\chi - \{\#L, L\# \} + \{\#L, L\# \}) \ L = \text{count } \chi \ L$ 
      by simp
    then have  $L \in \# \ \chi - \{\#L\# \}$ 
      by (metis (no-types) add.left-neutral add-diff-cancel-left' count-union diff-diff-add
        diff-single-trivial insert-DiffM mem-Collect-eq multi-member-this not-gr0 set-mset-def)
    then have  $\chi': ?\chi' + \{\#L\# \} + \{\#L\# \} = \chi$ 
      using f1 by (metis diff-diff-add diff-single-eq-union in-diffD)

    have  $\exists \psi'. \text{simplify } \psi \ \psi'$ 
      by (metis (no-types, hide-lams)  $\chi \ \chi'$  add commute factoring-imp-simplify union-assoc)
    then have False using simp by auto
  }
  then show ?thesis by arith
qed

```

```

lemma simplified-no-both:

```



```

assumes simp: simplified  $\psi$  and  $\chi$ :  $\chi \in \psi$ 
shows  $\neg (L \in \# \chi \wedge \neg L \in \# \chi)$ 
proof (rule ccontr)
  assume  $\neg \neg (L \in \# \chi \wedge \neg L \in \# \chi)$ 
  then have  $L \in \# \chi \wedge \neg L \in \# \chi$  by metis
  then obtain  $\chi'$  where  $\chi = \chi' + \{\#Pos\ (atm-of\ L)\#\} + \{\#Neg\ (atm-of\ L)\#\}$ 
    by (metis Neg-atm-of-iff Pos-atm-of-iff diff-union-swap insert-DiffM2 uminus-Neg uminus-Pos)
  then show False using  $\chi$  simp tautology-deletion by fastforce
qed

```

```

lemma simplified-not-tautology:
  assumes simplified  $\{\psi\}$ 
  shows  $\sim$  tautology  $\psi$ 
proof (rule ccontr)
  assume  $\sim$  ?thesis
  then obtain  $p$  where  $Pos\ p \in \# \psi \wedge Neg\ p \in \# \psi$  using tautology-decomp by metis
  then obtain  $\chi$  where  $\psi = \chi + \{\#Pos\ p\#\} + \{\#Neg\ p\#\}$ 
    by (metis insert-noteq-member literal.distinct(1) multi-member-split)
  then have  $\sim$  simplified  $\{\psi\}$  by (auto intro: tautology-deletion)
  then show False using assms by auto
qed

```

```

lemma simplified-remove:
  assumes simplified  $\{\psi\}$ 
  shows simplified  $\{\psi - \{\#l\#\}\}$ 
proof (rule ccontr)
  assume  $ns$ :  $\neg$  simplified  $\{\psi - \{\#l\#\}\}$ 
  {
    assume  $\neg l \in \# \psi$ 
    then have  $\psi - \{\#l\#\} = \psi$  by simp
    then have False using  $ns$  assms by auto
  }
  moreover {
    assume  $l\psi$ :  $l \in \# \psi$ 
    have  $A$ :  $\bigwedge A. A \in \{\psi - \{\#l\#\}\} \longleftrightarrow A + \{\#l\#\} \in \{\psi\}$  by (auto simp add: lψ)
    obtain  $l'$  where  $l'$ : simplify  $\{\psi - \{\#l\#\}\}$   $l'$  using  $ns$  by metis
    then have  $\exists l'. \textit{simplify}\ \{\psi\}\ l'$ 
    proof (induction rule: simplify.induct)
      case (tautology-deletion  $A\ P$ )
      have  $\{\#Neg\ P\#\} + (\{\#Pos\ P\#\} + (A + \{\#l\#\})) \in \{\psi\}$ 
        by (metis (no-types) A add.commute tautology-deletion.hyps union-lcomm)
      then show ?thesis
        by (metis simplify.tautology-deletion[of A + \{\#l\#\} P \{\psi\}] add.commute)
    next
      case (condensation  $A\ L$ )
      have  $A + \{\#L\#\} + \{\#L\#\} + \{\#l\#\} \in \{\psi\}$ 
        using  $A$  condensation.hyps by blast
      then have  $\{\#L, L\#\} + (A + \{\#l\#\}) \in \{\psi\}$ 
        by (metis (no-types) union-assoc union-commute)
      then show ?case
        using factoring-imp-simplify by blast
    next
      case (subsumption  $A\ B$ )
      then show ?case by blast
  }
qed

```

```

    then have False using assms(1) by blast
  }
  ultimately show False by auto
qed

```

lemma *in-simplified-simplified*:

```

  assumes simp: simplified  $\psi$  and incl:  $\psi' \subseteq \psi$ 
  shows simplified  $\psi'$ 
proof (rule ccontr)
  assume  $\neg$  ?thesis
  then obtain  $\psi''$  where simplify  $\psi' \psi''$  by metis
  then have  $\exists l'. \textit{simplify } \psi \ l'$ 
  proof (induction rule: simplify.induct)
    case (tautology-deletion A P)
    then show ?thesis using simplify.tautology-deletion[of A P  $\psi$ ] incl by blast
  next
    case (condensation A L)
    then show ?case using simplify.condensation[of A L  $\psi$ ] incl by blast
  next
    case (subsumption A B)
    then show ?case using simplify.subsumption[of A  $\psi$  B] incl by auto
  qed
  then show False using assms(1) by blast
qed

```

lemma *simplified-in*:

```

  assumes simplified  $\psi$ 
  and  $N \in \psi$ 
  shows simplified  $\{N\}$ 
  using assms by (metis Set.set-insert empty-subsetI in-simplified-simplified insert-mono)

```

lemma *subsumes-imp-formula*:

```

  assumes  $\psi \leq \# \varphi$ 
  shows  $\{\psi\} \models_p \varphi$ 
  unfolding true-clss-clss-def apply auto
  using assms true-clss-mono-leD by blast

```

lemma *simplified-imp-distinct-mset-tauto*:

```

  assumes simp: simplified  $\psi'$ 
  shows distinct-mset-set  $\psi'$  and  $\forall \chi \in \psi'. \neg \textit{tautology } \chi$ 
proof -
  show  $\forall \chi \in \psi'. \neg \textit{tautology } \chi$ 
    using simp by (auto simp add: simplified-in simplified-not-tautology)

```

```

  show distinct-mset-set  $\psi'$ 

```

```

  proof (rule ccontr)
    assume  $\neg$  ?thesis
    then obtain  $\chi$  where  $\chi \in \psi'$  and  $\neg \textit{distinct-mset } \chi$  unfolding distinct-mset-set-def by auto
    then obtain L where count  $\chi \ L \geq 2$ 
      unfolding distinct-mset-def
      by (meson count-greater-eq-one-iff le-antisym simp simplified-count)
    then show False by (metis Suc-1  $\langle \chi \in \psi' \rangle$  not-less-eq-eq simp simplified-count)
  qed
qed

```

lemma *simplified-no-more-full1-simplified*:
assumes *simplified* ψ
shows $\neg \text{full1 simplify } \psi \ \psi'$
using *assms* **unfolding** *full1-def* **by** (*meson* *trancpD*)

13.5 Resolution and Invariants

inductive *resolution* :: '*v* state \Rightarrow '*v* state \Rightarrow bool **where**
full1-simp: *full1 simplify* $N \ N' \Longrightarrow \text{resolution } (N, \text{already-used}) \ (N', \text{already-used}) \mid$
inferring: *inference* $(N, \text{already-used}) \ (N', \text{already-used}') \Longrightarrow \text{simplified } N$
 $\Longrightarrow \text{full simplify } N' \ N'' \Longrightarrow \text{resolution } (N, \text{already-used}) \ (N'', \text{already-used}')$

13.5.1 Invariants

lemma *resolution-finite*:
assumes *resolution* $\psi \ \psi'$ **and** *finite* (*fst* ψ)
shows *finite* (*fst* ψ')
using *assms* **by** (*induct* *rule*: *resolution.induct*)
(*auto simp add*: *full1-def* *full-def* *rtrancp-simplify-preserved-finite*
dest: *trancp-into-rtrancp* *inference-preserved-finite*)

lemma *rtrancp-resolution-finite*:
assumes *resolution*** $\psi \ \psi'$ **and** *finite* (*fst* ψ)
shows *finite* (*fst* ψ')
using *assms* **by** (*induct* *rule*: *rtrancp-induct*, *auto simp add*: *resolution-finite*)

lemma *resolution-finite-snd*:
assumes *resolution* $\psi \ \psi'$ **and** *finite* (*snd* ψ)
shows *finite* (*snd* ψ')
using *assms* **apply** (*induct* *rule*: *resolution.induct*, *auto simp add*: *inference-preserved-finite-snd*)
using *inference-preserved-finite-snd* *snd-conv* **by** *metis*

lemma *rtrancp-resolution-finite-snd*:
assumes *resolution*** $\psi \ \psi'$ **and** *finite* (*snd* ψ)
shows *finite* (*snd* ψ')
using *assms* **by** (*induct* *rule*: *rtrancp-induct*, *auto simp add*: *resolution-finite-snd*)

lemma *resolution-always-simplified*:
assumes *resolution* $\psi \ \psi'$
shows *simplified* (*fst* ψ')
using *assms* **by** (*induct* *rule*: *resolution.induct*)
(*auto simp add*: *full1-def* *full-def*)

lemma *trancp-resolution-always-simplified*:
assumes *trancp resolution* $\psi \ \psi'$
shows *simplified* (*fst* ψ')
using *assms* **by** (*induct* *rule*: *trancp.induct*, *auto simp add*: *resolution-always-simplified*)

lemma *resolution-atms-of*:
assumes *resolution* $\psi \ \psi'$ **and** *finite* (*fst* ψ)
shows *atms-of-ms* (*fst* ψ') \subseteq *atms-of-ms* (*fst* ψ)
using *assms* **apply** (*induct* *rule*: *resolution.induct*)
apply(*simp add*: *rtrancp-simplify-atms-of-ms* *trancp-into-rtrancp* *full1-def*)
by (*metis* (*no-types*, *lifting*) *contra-subsetD* *fst-conv* *full-def*
inference-preserved-atms-of-ms *rtrancp-simplify-atms-of-ms* *subsetI*)

lemma *rtranclp-resolution-atms-of*:
assumes *resolution*** $\psi \ \psi'$ **and** *finite* (*fst* ψ)
shows *atms-of-ms* (*fst* ψ') \subseteq *atms-of-ms* (*fst* ψ)
using *assms* **apply** (*induct* rule: *rtranclp-induct*)
using *resolution-atms-of* *rtranclp-resolution-finite* **by** *blast*+

lemma *resolution-include*:
assumes *res*: *resolution* $\psi \ \psi'$ **and** *finite*: *finite* (*fst* ψ)
shows *fst* $\psi' \subseteq$ *simple-clss* (*atms-of-ms* (*fst* ψ))

proof –
have *finite'*: *finite* (*fst* ψ') **using** *local.finite* *res* *resolution-finite* **by** *blast*
have *simplified* (*fst* ψ') **using** *res* *finite'* *resolution-always-simplified* **by** *blast*
then have *fst* $\psi' \subseteq$ *simple-clss* (*atms-of-ms* (*fst* ψ'))
using *simplified-in-simple-clss* *finite'* *simplified-imp-distinct-mset-tauto*[*of* *fst* ψ'] **by** *auto*
moreover have *atms-of-ms* (*fst* ψ') \subseteq *atms-of-ms* (*fst* ψ)
using *res* *finite* *resolution-atms-of*[*of* $\psi \ \psi'$] **by** *auto*
ultimately show *?thesis* **by** (*meson* *atms-of-ms-finite* *local.finite* *order.trans* *rev-finite-subset* *simple-clss-mono*)

qed

lemma *rtranclp-resolution-include*:
assumes *res*: *trancpl* *resolution* $\psi \ \psi'$ **and** *finite*: *finite* (*fst* ψ)
shows *fst* $\psi' \subseteq$ *simple-clss* (*atms-of-ms* (*fst* ψ))
using *assms* **apply** (*induct* rule: *trancpl.induct*)
apply (*simp* *add*: *resolution-include*)
by (*meson* *simple-clss-mono* *order-class.le-trans* *resolution-include* *rtranclp-resolution-atms-of* *rtranclp-resolution-finite* *trancpl-into-rtranclp*)

abbreviation *already-used-all-simple*
 $:: ('a \text{ literal multiset} \times 'a \text{ literal multiset}) \text{ set} \Rightarrow 'a \text{ set} \Rightarrow \text{bool}$ **where**
already-used-all-simple *already-used* *vars* \equiv
 $(\forall (A, B) \in \text{already-used}. \text{simplified } \{A\} \wedge \text{simplified } \{B\} \wedge \text{atms-of } A \subseteq \text{vars} \wedge \text{atms-of } B \subseteq \text{vars})$

lemma *already-used-all-simple-vars-incl*:
assumes *vars* \subseteq *vars'*
shows *already-used-all-simple* *a* *vars* \implies *already-used-all-simple* *a* *vars'*
using *assms* **by** *fast*

lemma *inference-clause-preserves-already-used-all-simple*:
assumes *inference-clause* *S* *S'*
and *already-used-all-simple* (*snd* *S*) *vars*
and *simplified* (*fst* *S*)
and *atms-of-ms* (*fst* *S*) \subseteq *vars*
shows *already-used-all-simple* (*snd* (*fst* $S \cup \{\text{fst } S'\}$, *snd* *S'*)) *vars*
using *assms*

proof (*induct* rule: *inference-clause.induct*)
case (*factoring* *L* *C* *N* *already-used*)
then show *?case* **by** (*simp* *add*: *simplified-in* *factoring-imp-simplify*)

next
case (*resolution* *P* *C* *N* *D* *already-used*) **note** *H* = *this*
show *?case* **apply** *clarify*
proof –
fix *A* *B* *v*
assume (*A*, *B*) \in *snd* (*fst* (*N*, *already-used*))

```

     $\cup \{fst (C + D, already-used \cup \{(\{ \#Pos P \# \} + C, \{ \#Neg P \# \} + D)\}),$ 
       $snd (C + D, already-used \cup \{(\{ \#Pos P \# \} + C, \{ \#Neg P \# \} + D)\})\}$ 
  then have  $(A, B) \in already-used \vee (A, B) = (\{ \#Pos P \# \} + C, \{ \#Neg P \# \} + D)$  by auto
  moreover {
    assume  $(A, B) \in already-used$ 
    then have  $simplified \{A\} \wedge simplified \{B\} \wedge atms-of A \subseteq vars \wedge atms-of B \subseteq vars$ 
      using  $H(4)$  by auto
  }
  moreover {
    assume eq:  $(A, B) = (\{ \#Pos P \# \} + C, \{ \#Neg P \# \} + D)$ 
    then have  $simplified \{A\}$  using  $simplified-in H(1,5)$  by auto
    moreover have  $simplified \{B\}$  using eq  $simplified-in H(2,5)$  by auto
    moreover have  $atms-of A \subseteq atms-of-ms N$ 
      using eq  $H(1)$ 
      using  $atms-of-atms-of-ms-mono[of A N]$  by auto
    moreover have  $atms-of B \subseteq atms-of-ms N$ 
      using eq  $H(2)$   $atms-of-atms-of-ms-mono[of B N]$  by auto
    ultimately have  $simplified \{A\} \wedge simplified \{B\} \wedge atms-of A \subseteq vars \wedge atms-of B \subseteq vars$ 
      using  $H(6)$  by auto
  }
  ultimately show  $simplified \{A\} \wedge simplified \{B\} \wedge atms-of A \subseteq vars \wedge atms-of B \subseteq vars$ 
    by fast
qed
qed

```

lemma *inference-preserves-already-used-all-simple:*

```

  assumes inference  $S S'$ 
  and already-used-all-simple  $(snd S) vars$ 
  and simplified  $(fst S)$ 
  and  $atms-of-ms (fst S) \subseteq vars$ 
  shows already-used-all-simple  $(snd S') vars$ 
  using assms
proof (induct rule: inference.induct)
  case (inference-step  $S clause already-used$ )
  then show ?case
    using inference-clause-preserves-already-used-all-simple $[of S (clause, already-used) vars]$ 
    by auto
qed

```

lemma *already-used-all-simple-inv:*

```

  assumes resolution  $S S'$ 
  and already-used-all-simple  $(snd S) vars$ 
  and  $atms-of-ms (fst S) \subseteq vars$ 
  shows already-used-all-simple  $(snd S') vars$ 
  using assms
proof (induct rule: resolution.induct)
  case (full1-simp  $N N'$ )
  then show ?case by simp
next
  case (inferring  $N already-used N' already-used' N''$ )
  then show already-used-all-simple  $(snd (N'', already-used')) vars$ 
    using inference-preserves-already-used-all-simple $[of (N, already-used)]$  by simp
qed

```

lemma *rtrancplp-already-used-all-simple-inv:*

assumes *resolution*** $S\ S'$
and *already-used-all-simple* ($\text{snd } S$) *vars*
and *atms-of-ms* ($\text{fst } S$) $\subseteq \text{vars}$
and *finite* ($\text{fst } S$)
shows *already-used-all-simple* ($\text{snd } S'$) *vars*
using *assms*
proof (*induct rule: rtranclp-induct*)
case *base*
then show ?*case* **by** *simp*
next
case (*step* $S' S''$) **note** $\text{infstar} = \text{this}(1)$ **and** $\text{IH} = \text{this}(3)$ **and** $\text{res} = \text{this}(2)$ **and**
 $\text{already} = \text{this}(4)$ **and** $\text{atms} = \text{this}(5)$ **and** $\text{finite} = \text{this}(6)$
have *already-used-all-simple* ($\text{snd } S'$) *vars* **using** IH *already* *atms* *finite* **by** *simp*
moreover **have** *atms-of-ms* ($\text{fst } S'$) $\subseteq \text{atms-of-ms } (\text{fst } S)$
by (*simp add: infstar local.finite rtranclp-resolution-atms-of*)
then have *atms-of-ms* ($\text{fst } S'$) $\subseteq \text{vars}$ **using** *atms* **by** *auto*
ultimately show ?*case*
using *already-used-all-simple-inv*[OF res] **by** *simp*
qed

lemma *inference-clause-simplified-already-used-subset*:
assumes *inference-clause* $S\ S'$
and *simplified* ($\text{fst } S$)
shows $\text{snd } S \subset \text{snd } S'$
using *assms* **apply** (*induct rule: inference-clause.induct, auto*)
using *factoring-imp-simplify* **by** *blast*

lemma *inference-simplified-already-used-subset*:
assumes *inference* $S\ S'$
and *simplified* ($\text{fst } S$)
shows $\text{snd } S \subset \text{snd } S'$
using *assms* **apply** (*induct rule: inference.induct*)
by (*metis inference-clause-simplified-already-used-subset snd-conv*)

lemma *resolution-simplified-already-used-subset*:
assumes *resolution* $S\ S'$
and *simplified* ($\text{fst } S$)
shows $\text{snd } S \subset \text{snd } S'$
using *assms* **apply** (*induct rule: resolution.induct, simp-all add: full1-def*)
apply (*meson tranclpD*)
by (*metis inference-simplified-already-used-subset fst-conv snd-conv*)

lemma *tranclp-resolution-simplified-already-used-subset*:
assumes *tranclp resolution* $S\ S'$
and *simplified* ($\text{fst } S$)
shows $\text{snd } S \subset \text{snd } S'$
using *assms* **apply** (*induct rule: tranclp.induct*)
using *resolution-simplified-already-used-subset* **apply** *metis*
by (*meson tranclp-resolution-always-simplified resolution-simplified-already-used-subset less-trans*)

abbreviation *already-used-top vars* $\equiv \text{simple-clss vars} \times \text{simple-clss vars}$

lemma *already-used-all-simple-in-already-used-top*:
assumes *already-used-all-simple* $s\ \text{vars}$ **and** *finite* *vars*

shows $s \subseteq \text{already-used-top vars}$
proof
 fix x
 assume $x-s: x \in s$
 obtain $A B$ where $x: x = (A, B)$ by (cases x , auto)
 then have *simplified* $\{A\}$ and $\text{atms-of } A \subseteq \text{vars}$ using $\text{assms}(1)$ $x-s$ by *fastforce+*
 then have $A: A \in \text{simple-clss vars}$
 using *simple-clss-mono*[of $\text{atms-of } A \text{ vars}$] x $\text{assms}(2)$
 simplified-imp-distinct-mset-tauto[of $\{A\}$]
 distinct-mset-not-tautology-implies-in-simple-clss by *fast*
 moreover have *simplified* $\{B\}$ and $\text{atms-of } B \subseteq \text{vars}$ using $\text{assms}(1)$ $x-s$ x by *fast+*
 then have $B: B \in \text{simple-clss vars}$
 using *simplified-imp-distinct-mset-tauto*[of $\{B\}$]
 distinct-mset-not-tautology-implies-in-simple-clss
 simple-clss-mono[of $\text{atms-of } B \text{ vars}$] x $\text{assms}(2)$ by *fast*
 ultimately show $x \in \text{simple-clss vars} \times \text{simple-clss vars}$
 unfolding x by *auto*
qed

lemma *already-used-top-finite*:
 assumes *finite vars*
 shows *finite (already-used-top vars)*
 using *simple-clss-finite assms* by *auto*

lemma *already-used-top-increasing*:
 assumes $\text{var} \subseteq \text{var}'$ and *finite var'*
 shows *already-used-top var* \subseteq *already-used-top var'*
 using *assms simple-clss-mono* by *auto*

lemma *already-used-all-simple-finite*:
 fixes $s :: ('a \text{ literal multiset} \times 'a \text{ literal multiset}) \text{ set}$ and $\text{vars} :: 'a \text{ set}$
 assumes *already-used-all-simple s vars* and *finite vars*
 shows *finite s*
 using *assms already-used-all-simple-in-already-used-top*[OF $\text{assms}(1)$]
rev-finite-subset[OF *already-used-top-finite*[of vars]] by *auto*

abbreviation *card-simple vars* $\psi \equiv \text{card (already-used-top vars} - \psi)$

lemma *resolution-card-simple-decreasing*:
 assumes *res: resolution $\psi \psi'$*
 and *a-u-s: already-used-all-simple (snd ψ) vars*
 and *finite-v: finite vars*
 and *finite-fst: finite (fst ψ)*
 and *finite-snd: finite (snd ψ)*
 and *simp: simplified (fst ψ)*
 and $\text{atms-of-ms (fst } \psi) \subseteq \text{vars}$
 shows *card-simple vars (snd ψ')* $<$ *card-simple vars (snd ψ)*

proof –
 let $?vars = \text{vars}$
 let $?top = \text{simple-clss } ?vars \times \text{simple-clss } ?vars$
 have 1: *card-simple vars (snd ψ)* $= \text{card } ?top - \text{card (snd } \psi)$
 using *card-Diff-subset finite-snd already-used-all-simple-in-already-used-top*[OF *a-u-s*]
 finite-v by *metis*
 have *a-u-s': already-used-all-simple (snd ψ') vars*
 using *already-used-all-simple-inv res a-u-s assms(7)* by *blast*

have f : *finite* ($\text{snd } \psi'$) **using** *already-used-all-simple-finite* $a-u-s'$ *finite-v* **by** *auto*
have 2 : *card-simple vars* ($\text{snd } \psi'$) = *card* ?*top* – *card* ($\text{snd } \psi'$)
using *card-Diff-subset*[*OF* f] *already-used-all-simple-in-already-used-top*[*OF* $a-u-s'$ *finite-v*]
by *auto*
have *card* (*already-used-top vars*) \geq *card* ($\text{snd } \psi'$)
using *already-used-all-simple-in-already-used-top*[*OF* $a-u-s'$ *finite-v*]
card-mono[*of* *already-used-top vars* $\text{snd } \psi'$] *already-used-top-finite*[*OF* *finite-v*] **by** *metis*
then show ?*thesis*
using *psubset-card-mono*[*OF* f *resolution-simplified-already-used-subset*[*OF* *res simp*]]
unfolding $1\ 2$ **by** *linarith*
qed

lemma *trancpl-resolution-card-simple-decreasing*:

assumes *trancpl resolution* $\psi\ \psi'$ **and** *finite-fst*: *finite* ($\text{fst } \psi$)
and *already-used-all-simple* ($\text{snd } \psi$) *vars*
and *atms-of-ms* ($\text{fst } \psi$) \subseteq *vars*
and *finite-v*: *finite vars*
and *finite-snd*: *finite* ($\text{snd } \psi$)
and *simplified* ($\text{fst } \psi$)
shows *card-simple vars* ($\text{snd } \psi'$) < *card-simple vars* ($\text{snd } \psi$)
using *assms*
proof (*induct rule*: *trancpl-induct*)
case (*base* ψ')
then show ?*case* **by** (*simp add*: *resolution-card-simple-decreasing*)
next
case (*step* $\psi'\ \psi''$) **note** $\text{res} = \text{this}(1)$ **and** $\text{res}' = \text{this}(2)$ **and** $a-u-s = \text{this}(5)$ **and**
atms = *this*(6) **and** $f-v = \text{this}(7)$ **and** $f\text{-fst} = \text{this}(4)$ **and** $H = \text{this}$
then have *card-simple vars* ($\text{snd } \psi'$) < *card-simple vars* ($\text{snd } \psi$) **by** *auto*
moreover have $a-u-s'$: *already-used-all-simple* ($\text{snd } \psi'$) *vars*
using *rtrancpl-already-used-all-simple-inv*[*OF* *trancpl-into-rtrancpl*[*OF* res] $a-u-s$ *atms* $f\text{-fst}$] .
have *finite* ($\text{fst } \psi'$)
by (*meson* *finite-fst* res *rtrancpl-resolution-finite* *trancpl-into-rtrancpl*)
moreover have *finite* ($\text{snd } \psi'$) **using** *already-used-all-simple-finite*[*OF* $a-u-s'$ $f-v$] .
moreover have *simplified* ($\text{fst } \psi'$) **using** res *trancpl-resolution-always-simplified* **by** *blast*
moreover have *atms-of-ms* ($\text{fst } \psi'$) \subseteq *vars*
by (*meson* *atms* $f\text{-fst}$ *order.trans* res *rtrancpl-resolution-atms-of* *trancpl-into-rtrancpl*)
ultimately show ?*case*
using *resolution-card-simple-decreasing*[*OF* res' $a-u-s'$ $f-v$] $f-v$
less-trans[*of* *card-simple vars* ($\text{snd } \psi''$) *card-simple vars* ($\text{snd } \psi'$)
card-simple vars ($\text{snd } \psi$)]
by *blast*
qed

lemma *trancpl-resolution-card-simple-decreasing-2*:

assumes *trancpl resolution* $\psi\ \psi'$
and *finite-fst*: *finite* ($\text{fst } \psi$)
and *empty-snd*: $\text{snd } \psi = \{\}$
and *simplified* ($\text{fst } \psi$)
shows *card-simple* (*atms-of-ms* ($\text{fst } \psi$)) ($\text{snd } \psi'$) < *card-simple* (*atms-of-ms* ($\text{fst } \psi$)) ($\text{snd } \psi$)
proof –
let ?*vars* = (*atms-of-ms* ($\text{fst } \psi$))
have *already-used-all-simple* ($\text{snd } \psi$) ?*vars* **unfolding** *empty-snd* **by** *auto*
moreover have *atms-of-ms* ($\text{fst } \psi$) \subseteq ?*vars* **by** *auto*

moreover have *finite-v: finite ?vars using finite-fst by auto*
moreover have *finite-snd: finite (snd ψ) unfolding empty-snd by auto*
ultimately show *?thesis*
using assms(1,2,4) tranclp-resolution-card-simple-decreasing[of ψ ψ] by presburger
qed

13.5.2 well-foundness if the relation

lemma *wf-simplified-resolution:*

assumes *f-vars: finite vars*

shows *wf $\{(y:: 'v:: \text{linorder state}, x). (\text{atms-of-ms } (fst\ x) \subseteq \text{vars} \wedge \text{simplified } (fst\ x) \wedge \text{finite } (snd\ x) \wedge \text{finite } (fst\ x) \wedge \text{already-used-all-simple } (snd\ x)\ \text{vars}) \wedge \text{resolution } x\ y\}$*

proof –

{
fix *a b :: 'v::linorder state*
assume *(b, a) $\in \{(y, x). (\text{atms-of-ms } (fst\ x) \subseteq \text{vars} \wedge \text{simplified } (fst\ x) \wedge \text{finite } (snd\ x) \wedge \text{finite } (fst\ x) \wedge \text{already-used-all-simple } (snd\ x)\ \text{vars}) \wedge \text{resolution } x\ y\}$*
then have
atms-of-ms (fst a) \subseteq vars and
simp: simplified (fst a) and
finite (snd a) and
finite (fst a) and
a-u-v: already-used-all-simple (snd a) vars and
res: resolution a b by auto
have *finite (already-used-top vars) using f-vars already-used-top-finite by blast*
moreover have *already-used-top vars \subseteq already-used-top vars by auto*
moreover have *snd b \subseteq already-used-top vars*
using already-used-all-simple-in-already-used-top[of snd b vars]
a-u-v already-used-all-simple-inv[OF res] (finite (fst a)) (atms-of-ms (fst a) \subseteq vars) f-vars
by presburger
moreover have *snd a \subset snd b using resolution-simplified-already-used-subset[OF res simp] .*
ultimately have *finite (already-used-top vars) \wedge already-used-top vars \subseteq already-used-top vars*
 \wedge snd b \subseteq already-used-top vars \wedge snd a \subset snd b by metis
}
then show *?thesis using wf-bounded-set[of $\{(y:: 'v:: \text{linorder state}, x). (\text{atms-of-ms } (fst\ x) \subseteq \text{vars} \wedge \text{simplified } (fst\ x) \wedge \text{finite } (snd\ x) \wedge \text{finite } (fst\ x) \wedge \text{already-used-all-simple } (snd\ x)\ \text{vars}) \wedge \text{resolution } x\ y\}$ $\lambda\cdot$. already-used-top vars snd] by auto*

qed

lemma *wf-simplified-resolution':*

assumes *f-vars: finite vars*

shows *wf $\{(y:: 'v:: \text{linorder state}, x). (\text{atms-of-ms } (fst\ x) \subseteq \text{vars} \wedge \neg \text{simplified } (fst\ x) \wedge \text{finite } (snd\ x) \wedge \text{finite } (fst\ x) \wedge \text{already-used-all-simple } (snd\ x)\ \text{vars}) \wedge \text{resolution } x\ y\}$*

unfolding *wf-def*

apply *(simp add: resolution-always-simplified)*

by *(metis (mono-tags, hide-lams) fst-conv resolution-always-simplified)*

lemma *wf-resolution:*

assumes *f-vars: finite vars*

shows *wf $\{(y:: 'v:: \text{linorder state}, x). (\text{atms-of-ms } (fst\ x) \subseteq \text{vars} \wedge \text{simplified } (fst\ x) \wedge \text{finite } (snd\ x) \wedge \text{finite } (fst\ x) \wedge \text{already-used-all-simple } (snd\ x)\ \text{vars}) \wedge \text{resolution } x\ y\}$*
 $\cup \{(y, x). (\text{atms-of-ms } (fst\ x) \subseteq \text{vars} \wedge \neg \text{simplified } (fst\ x) \wedge \text{finite } (snd\ x) \wedge \text{finite } (fst\ x) \wedge \text{already-used-all-simple } (snd\ x)\ \text{vars}) \wedge \text{resolution } x\ y\}$ **(is wf ($?R \cup ?S$))**

proof –

have *Domain ?R Int Range ?S = {} using resolution-always-simplified by auto blast*

then show $wf\ (\ ?R \cup \ ?S)$
using $wf\text{-simplified-resolution}[OF\ f\text{-vars}]\ wf\text{-simplified-resolution}'[OF\ f\text{-vars}]\ wf\text{-Un}[of\ \ ?R\ \ ?S]$
by *fast*
qed

lemma *rtrancp-simplify-already-used-inv*:
assumes $simplify^{**}\ S\ S'$
and $already\text{-used}\text{-inv}\ (S, N)$
shows $already\text{-used}\text{-inv}\ (S', N)$
using *assms apply induction*
using *simplify-preserves-already-used-inv by fast+*

lemma *full1-simplify-already-used-inv*:
assumes $full1\ simplify\ S\ S'$
and $already\text{-used}\text{-inv}\ (S, N)$
shows $already\text{-used}\text{-inv}\ (S', N)$
using *assms trancp-into-rtrancp[of simplify S S'] rtrancp-simplify-already-used-inv*
unfolding *full1-def by fast*

lemma *full-simplify-already-used-inv*:
assumes $full\ simplify\ S\ S'$
and $already\text{-used}\text{-inv}\ (S, N)$
shows $already\text{-used}\text{-inv}\ (S', N)$
using *assms rtrancp-simplify-already-used-inv unfolding full-def by fast*

lemma *resolution-already-used-inv*:

assumes $resolution\ S\ S'$
and $already\text{-used}\text{-inv}\ S$
shows $already\text{-used}\text{-inv}\ S'$
using *assms*

proof *induction*

case (*full1-simp N N' already-used*)
then show $\ ?case$ **using** *full1-simplify-already-used-inv by fast*

next

case (*inferring N already-used N' already-used' N''*) **note** $inf = this(1)$ **and** $full = this(3)$ **and**
 $a\text{-}u\text{-}v = this(4)$

then show $\ ?case$

using *inference-preserves-already-used-inv[OF inf a-u-v] full-simplify-already-used-inv full*
by *fast*

qed

lemma *rtrancp-resolution-already-used-inv*:
assumes $resolution^{**}\ S\ S'$
and $already\text{-used}\text{-inv}\ S$
shows $already\text{-used}\text{-inv}\ S'$
using *assms apply induction*
using *resolution-already-used-inv by fast+*

lemma *rtanclp-simplify-preserves-unsat*:
assumes $simplify^{**}\ \psi\ \psi'$
shows $satisfiable\ \psi' \longrightarrow satisfiable\ \psi$
using *assms apply induction*
using *simplify-clause-preserves-sat by blast+*

lemma *full1-simplify-preserves-unsat*:
assumes $full1\ simplify\ \psi\ \psi'$

shows *satisfiable* $\psi' \longrightarrow \text{satisfiable } \psi$
using *assms* *rtranclp-simplify-preserves-unsat*[*of* $\psi \ \psi'$] *tranclp-into-rtranclp*
unfolding *full1-def* **by** *metis*

lemma *full-simplify-preserves-unsat*:
assumes *full simplify* $\psi \ \psi'$
shows *satisfiable* $\psi' \longrightarrow \text{satisfiable } \psi$
using *assms* *rtranclp-simplify-preserves-unsat*[*of* $\psi \ \psi'$] **unfolding** *full-def* **by** *metis*

lemma *resolution-preserves-unsat*:
assumes *resolution* $\psi \ \psi'$
shows *satisfiable* $(fst \ \psi') \longrightarrow \text{satisfiable } (fst \ \psi)$
using *assms* **apply** (*induct* *rule*: *resolution.induct*)
using *full1-simplify-preserves-unsat* **apply** (*metis* *fst-conv*)
using *full-simplify-preserves-unsat* *simplify-preserves-unsat* **by** *fastforce*

lemma *rtranclp-resolution-preserves-unsat*:
assumes *resolution*** $\psi \ \psi'$
shows *satisfiable* $(fst \ \psi') \longrightarrow \text{satisfiable } (fst \ \psi)$
using *assms* **apply** *induction*
using *resolution-preserves-unsat* **by** *fast+*

lemma *rtranclp-simplify-preserve-partial-tree*:
assumes *simplify*** $N \ N'$
and *partial-interps* $t \ I \ N$
shows *partial-interps* $t \ I \ N'$
using *assms* **apply** (*induction*, *simp*)
using *simplify-preserve-partial-tree* **by** *metis*

lemma *full1-simplify-preserve-partial-tree*:
assumes *full1 simplify* $N \ N'$
and *partial-interps* $t \ I \ N$
shows *partial-interps* $t \ I \ N'$
using *assms* *rtranclp-simplify-preserve-partial-tree*[*of* $N \ N' \ t \ I$] *tranclp-into-rtranclp*
unfolding *full1-def* **by** *fast*

lemma *full-simplify-preserve-partial-tree*:
assumes *full simplify* $N \ N'$
and *partial-interps* $t \ I \ N$
shows *partial-interps* $t \ I \ N'$
using *assms* *rtranclp-simplify-preserve-partial-tree*[*of* $N \ N' \ t \ I$] *tranclp-into-rtranclp*
unfolding *full-def* **by** *fast*

lemma *resolution-preserve-partial-tree*:
assumes *resolution* $S \ S'$
and *partial-interps* $t \ I \ (fst \ S)$
shows *partial-interps* $t \ I \ (fst \ S')$
using *assms* **apply** *induction*
using *full1-simplify-preserve-partial-tree* *fst-conv* **apply** *metis*
using *full-simplify-preserve-partial-tree* *inference-preserve-partial-tree* **by** *fastforce*

lemma *rtranclp-resolution-preserve-partial-tree*:
assumes *resolution*** $S \ S'$
and *partial-interps* $t \ I \ (fst \ S)$
shows *partial-interps* $t \ I \ (fst \ S')$

```

using assms apply induction
using resolution-preserve-partial-tree by fast+
thm nat-less-induct nat.induct

lemma nat-ge-induct[case-names 0 Suc]:
  assumes P 0
  and  $(\bigwedge n. (\bigwedge m. m < \text{Suc } n \implies P\ m) \implies P\ (\text{Suc } n))$ 
  shows P n
  using assms apply (induct rule: nat-less-induct)
  by (rename-tac n, case-tac n) auto

lemma wf-always-more-step-False:
  assumes wf R
  shows  $(\forall x. \exists z. (z, x) \in R) \implies \text{False}$ 
  using assms unfolding wf-def by (meson Domain.DomainI assms wfE-min)

lemma finite-finite-mset-element-of-mset[simp]:
  assumes finite N
  shows finite  $\{f\ \varphi\ L\ |\ \varphi\ L. \varphi \in N \wedge L \in \# \varphi \wedge P\ \varphi\ L\}$ 
  using assms
proof (induction N rule: finite-induct)
  case empty
  show ?case by auto
next
  case (insert x N) note finite = this(1) and IH = this(3)
  have  $\{f\ \varphi\ L\ |\ \varphi\ L. (\varphi = x \vee \varphi \in N) \wedge L \in \# \varphi \wedge P\ \varphi\ L\} \subseteq \{f\ x\ L\ |\ L. L \in \# x \wedge P\ x\ L\}$ 
   $\cup \{f\ \varphi\ L\ |\ \varphi\ L. \varphi \in N \wedge L \in \# \varphi \wedge P\ \varphi\ L\}$  by auto
  moreover have finite  $\{f\ x\ L\ |\ L. L \in \# x\}$  by auto
  ultimately show ?case using IH finite-subset by fastforce
qed

value card
value filter-mset
value  $\{\# \text{count } \varphi\ L\ |\ L \in \# \varphi. 2 \leq \text{count } \varphi\ L\ \#\}$ 
value  $(\lambda \varphi. \text{msetsum } \{\# \text{count } \varphi\ L\ |\ L \in \# \varphi. 2 \leq \text{count } \varphi\ L\ \#\})$ 

syntax
  -comprehension1'-mset  $:: 'a \Rightarrow 'b \Rightarrow 'b\ \text{multiset} \Rightarrow 'a\ \text{multiset}$ 
   $((\{\# \cdot / \cdot. - : \text{setof } \cdot \#\}))$ 
translations
   $\{\# e. x : \text{setof } M\ \#\} == \text{CONST set-mset } (\text{CONST image-mset } (\%x. e)\ M)$ 
value  $\{\# a. a : \text{setof } \{\# 1, 1, 2 :: \text{int}\} \#\} = \{1, 2\}$ 

definition sum-count-ge-2  $:: 'a\ \text{multiset set} \Rightarrow \text{nat } (\Xi)$  where
sum-count-ge-2  $\equiv \text{folding.F } (\lambda \varphi. \text{op } + (\text{msetsum } \{\# \text{count } \varphi\ L\ |\ L \in \# \varphi. 2 \leq \text{count } \varphi\ L\ \#\}))\ 0$ 

interpretation sum-count-ge-2:
  folding  $(\lambda \varphi. \text{op } + (\text{msetsum } \{\# \text{count } \varphi\ L\ |\ L \in \# \varphi. 2 \leq \text{count } \varphi\ L\ \#\}))\ 0$ 
rewrites
  folding.F  $(\lambda \varphi. \text{op } + (\text{msetsum } \{\# \text{count } \varphi\ L\ |\ L \in \# \varphi. 2 \leq \text{count } \varphi\ L\ \#\}))\ 0 = \text{sum-count-ge-2}$ 
proof –
  show folding  $(\lambda \varphi. \text{op } + (\text{msetsum } (\text{image-mset } (\text{count } \varphi)\ \{\# L \in \# \varphi. 2 \leq \text{count } \varphi\ L\ \#\}))\ 0$ 
  by standard auto

```

then interpret *sum-count-ge-2*:
folding ($\lambda\varphi. op + (msetsum \{ \#count \varphi L \mid L \in \# \varphi. 2 \leq count \varphi L \# \}) 0$).
show *folding.F* ($\lambda\varphi. op + (msetsum (image-mset (count \varphi) \{ \# L \in \# \varphi. 2 \leq count \varphi L \# \})) 0$
 $= sum-count-ge-2$ **by** (*auto simp add: sum-count-ge-2-def*)
qed

lemma *finite-incl-le-setsum*:

finite ($B :: 'a$ multiset set) $\implies A \subseteq B \implies \Xi A \leq \Xi B$

proof (*induction arbitrary:A rule: finite-induct*)

case *empty*

then show *?case* **by** *simp*

next

case (*insert a F*) **note** *finite = this(1)* **and** *aF = this(2)* **and** *IH = this(3)* **and** *AF = this(4)*

show *?case*

proof (*cases a ∈ A*)

assume $a \notin A$

then have $A \subseteq F$ **using** *AF* **by** *auto*

then show *?case* **using** *IH[of A]* **by** (*simp add: aF local.finite*)

next

assume $aA: a \in A$

then have $A - \{a\} \subseteq F$ **using** *AF* **by** *auto*

then have $\Xi (A - \{a\}) \leq \Xi F$ **using** *IH* **by** *blast*

then show *?case*

proof –

obtain $nn :: nat \Rightarrow nat \Rightarrow nat$ **where**

$\forall x0\ x1. (\exists v2. x0 = x1 + v2) = (x0 = x1 + nn\ x0\ x1)$

by *moura*

then have $\Xi F = \Xi (A - \{a\}) + nn\ (\Xi F)\ (\Xi (A - \{a\}))$

by (*meson* $\langle \Xi (A - \{a\}) \leq \Xi F \rangle$ *le-iff-add*)

then show *?thesis*

by (*metis* (*no-types*) *le-iff-add aA aF add.assoc finite.insertI finite-subset insert.premis local.finite sum-count-ge-2.insert sum-count-ge-2.remove*)

qed

qed

qed

lemma *simplify-finite-measure-decrease*:

simplify $N\ N' \implies finite\ N \implies card\ N' + \Xi N' < card\ N + \Xi N$

proof (*induction rule: simplify.induct*)

case (*tautology-deletion A P*) **note** *an = this(1)* **and** *fin = this(2)*

let $?N' = N - \{A + \{ \#Pos\ P \# \} + \{ \#Neg\ P \# \} \}$

have $card\ ?N' < card\ N$

by (*meson card-Diff1-less tautology-deletion.hyps tautology-deletion.premis*)

moreover have $?N' \subseteq N$ **by** *auto*

then have $sum-count-ge-2\ ?N' \leq sum-count-ge-2\ N$ **using** *finite-incl-le-setsum[OF fin]* **by** *blast*

ultimately show *?case* **by** *linarith*

next

case (*condensation A L*) **note** *AN = this(1)* **and** *fin = this(2)*

let $?C' = A + \{ \#L \# \}$

let $?C = A + \{ \#L \# \} + \{ \#L \# \}$

let $?N' = N - \{ ?C \} \cup \{ ?C' \}$

have $card\ ?N' \leq card\ N$

using *AN* **by** (*metis* (*no-types, lifting*) *Diff-subset Un-empty-right Un-insert-right card.remove card-insert-if card-mono fin finite-Diff order-refl*)

moreover have $\Xi \{ ?C' \} < \Xi \{ ?C \}$

```

proof –
  have mset-decomp:
     $\{\# \text{ La} \in \# A. (L = \text{La} \longrightarrow \text{La} \in \# A) \wedge (L \neq \text{La} \longrightarrow 2 \leq \text{count } A \text{ La})\# \}$ 
    =  $\{\# \text{ La} \in \# A. L \neq \text{La} \wedge 2 \leq \text{count } A \text{ La}\# \} +$ 
     $\{\# \text{ La} \in \# A. L = \text{La} \wedge \text{Suc } 0 \leq \text{count } A \text{ L}\# \}$ 
    by (auto simp: multiset-eq-iff ac-simps)
  have mset-decomp2:  $\{\# \text{ La} \in \# A. L \neq \text{La} \longrightarrow 2 \leq \text{count } A \text{ La}\# \} =$ 
     $\{\# \text{ La} \in \# A. L \neq \text{La} \wedge 2 \leq \text{count } A \text{ La}\# \} + \text{replicate-mset } (\text{count } A \text{ L}) \text{ L}$ 
    by (auto simp: multiset-eq-iff)
  show ?thesis
    by (auto simp: mset-decomp mset-decomp2 filter-mset-eq ac-simps)
qed
have  $\exists ?N' < \exists N$ 
proof cases
  assume a1:  $?C' \in N$ 
  then show ?thesis
    proof –
      have f2:  $\bigwedge m M. \text{insert } (m::'a \text{ literal multiset}) (M - \{m\}) = M \cup \{ \} \vee m \notin M$ 
        using Un-empty-right insert-Diff by blast
      have f3:  $\bigwedge m M Ma. \text{insert } (m::'a \text{ literal multiset}) M - \text{insert } m Ma = M - \text{insert } m Ma$ 
        by simp
      then have f4:  $\bigwedge m m. M - \{m::'a \text{ literal multiset}\} = M \cup \{ \} \vee m \in M$ 
        using Diff-insert-absorb Un-empty-right by fastforce
      have f5:  $\text{insert } (A + \{\#L\# \} + \{\#L\# \}) N = N$ 
        using f3 f2 Un-empty-right condensation.hyps insert-iff by fastforce
      have  $\bigwedge m M. \text{insert } (m::'a \text{ literal multiset}) M = M \cup \{ \} \vee m \notin M$ 
        using f3 f2 Un-empty-right add.right-neutral insert-iff by fastforce
      then have  $\exists (N - \{A + \{\#L\# \} + \{\#L\# \}) < \exists N$ 
        using f5 f4 by (metis Un-empty-right  $\langle \exists \{A + \{\#L\# \} \} < \exists \{A + \{\#L\# \} + \{\#L\# \} \}$ 
        add.right-neutral add-diff-cancel-left' add-gr-0 diff-less fin finite.emptyI not-le
        sum-count-ge-2.empty sum-count-ge-2.insert-remove trans-le-add2)
      then show ?thesis
        using f3 f2 a1 by (metis (no-types) Un-empty-right Un-insert-right condensation.hyps
        insert-iff multi-self-add-other-not-self)
    qed
  next
    assume  $?C' \notin N$ 
    have mset-decomp:
       $\{\# \text{ La} \in \# A. (L = \text{La} \longrightarrow \text{Suc } 0 \leq \text{count } A \text{ La}) \wedge (L \neq \text{La} \longrightarrow 2 \leq \text{count } A \text{ La})\# \}$ 
      =  $\{\# \text{ La} \in \# A. L \neq \text{La} \wedge 2 \leq \text{count } A \text{ La}\# \} +$ 
       $\{\# \text{ La} \in \# A. L = \text{La} \wedge \text{Suc } 0 \leq \text{count } A \text{ L}\# \}$ 
      by (auto simp: multiset-eq-iff ac-simps)
    have mset-decomp2:  $\{\# \text{ La} \in \# A. L \neq \text{La} \longrightarrow 2 \leq \text{count } A \text{ La}\# \} =$ 
       $\{\# \text{ La} \in \# A. L \neq \text{La} \wedge 2 \leq \text{count } A \text{ La}\# \} + \text{replicate-mset } (\text{count } A \text{ L}) \text{ L}$ 
      by (auto simp: multiset-eq-iff)

    show ?thesis
      using  $\langle \exists \{A + \{\#L\# \} \} < \exists \{A + \{\#L\# \} + \{\#L\# \} \}$  condensation.hyps fin
      sum-count-ge-2.remove[of - A + \{\#L\# \} + \{\#L\# \}]  $\langle ?C' \notin N \rangle$ 
      by (auto simp: mset-decomp mset-decomp2 filter-mset-eq)
    qed
  ultimately show ?case by linarith
next
case (subsumption A B) note AN = this(1) and AB = this(2) and BN = this(3) and fin = this(4)
have  $\text{card } (N - \{B\}) < \text{card } N$  using BN by (meson card-Diff1-less subsumption.premis)

```

moreover have $\Xi (N - \{B\}) \leq \Xi N$
by (*simp add: Diff-subset finite-incl-le-setsum subsumption.premis*)
ultimately show ?case **by** *linarith*
qed

lemma *simplify-terminates*:

wf $\{(N', N). \text{finite } N \wedge \text{simplify } N N'\}$
using *assms* **apply** (*rule wfP-if-measure[of finite simplify $\lambda N. \text{card } N + \Xi N$]*)
using *simplify-finite-measure-decrease* **by** *blast*

lemma *wf-terminates*:

assumes *wf r*
shows $\exists N'. (N', N) \in r^* \wedge (\forall N''. (N'', N') \notin r)$

proof –

let $?P = \lambda N. (\exists N'. (N', N) \in r^* \wedge (\forall N''. (N'', N') \notin r))$

have $(\forall x. (\forall y. (y, x) \in r \longrightarrow ?P y) \longrightarrow ?P x)$

proof *clarify*

fix *x*

assume $H: \forall y. (y, x) \in r \longrightarrow ?P y$

{ assume $\exists y. (y, x) \in r$

then obtain *y* **where** $y: (y, x) \in r$ **by** *blast*

then have $?P y$ **using** *H* **by** *blast*

then have $?P x$ **using** *y* **by** (*meson rtrancl.rtrancl-into-rtrancl*)

}

moreover {

assume $\neg(\exists y. (y, x) \in r)$

then have $?P x$ **by** *auto*

}

ultimately show $?P x$ **by** *blast*

qed

moreover have $(\forall x. (\forall y. (y, x) \in r \longrightarrow ?P y) \longrightarrow ?P x) \longrightarrow \text{All } ?P$

using *assms* **unfolding** *wf-def* **by** (*rule allE*)

ultimately have $\text{All } ?P$ **by** *blast*

then show $?P N$ **by** *blast*

qed

lemma *rtrancl-simplify-terminates*:

assumes *fin: finite N*

shows $\exists N'. \text{simplify}^{**} N N' \wedge \text{simplified } N'$

proof –

have $H: \{(N', N). \text{finite } N \wedge \text{simplify } N N'\} = \{(N', N). \text{simplify } N N' \wedge \text{finite } N\}$ **by** *auto*

then have *wf*: *wf* $\{(N', N). \text{simplify } N N' \wedge \text{finite } N\}$

using *simplify-terminates* **by** (*simp add: H*)

obtain N' **where** $N': (N', N) \in \{(b, a). \text{simplify } a b \wedge \text{finite } a\}^*$ **and**

more: $(\forall N''. (N'', N') \notin \{(b, a). \text{simplify } a b \wedge \text{finite } a\})$

using *Prop-Resolution.wf-terminates[OF wf, of N]* **by** *blast*

have $1: \text{simplify}^{**} N N'$

using N' **by** (*induction rule: rtrancl.induct*) *auto*

then have *finite N'* **using** *fin rtrancl-simplify-preserves-finite* **by** *blast*

then have $2: \forall N''. \neg \text{simplify } N' N''$ **using** *more* **by** *auto*

show ?thesis **using** $1\ 2$ **by** *blast*

qed

```

lemma finite-simplified-full1-simp:
  assumes finite N
  shows simplified N  $\vee$  ( $\exists N'. \text{full1 simplify } N N'$ )
  using rtrancp-simplify-terminates[OF assms] unfolding full1-def
  by (metis Nitpick.rtrancp-unfold)

lemma finite-simplified-full-simp:
  assumes finite N
  shows  $\exists N'. \text{full simplify } N N'$ 
  using rtrancp-simplify-terminates[OF assms] unfolding full-def by metis

lemma can-decrease-tree-size-resolution:
  fixes  $\psi :: 'v \text{ state}$  and  $\text{tree} :: 'v \text{ sem-tree}$ 
  assumes finite (fst  $\psi$ ) and already-used-inv  $\psi$ 
  and partial-interps tree I (fst  $\psi$ )
  and simplified (fst  $\psi$ )
  shows  $\exists (\text{tree}' :: 'v \text{ sem-tree}) \psi'. \text{resolution}^{**} \psi \psi' \wedge \text{partial-interps tree}' I (\text{fst } \psi')$ 
     $\wedge (\text{sem-tree-size tree}' < \text{sem-tree-size tree} \vee \text{sem-tree-size tree} = 0)$ 
  using assms
proof (induct arbitrary: I rule: sem-tree-size)
  case (bigger xs I) note IH = this(1) and finite = this(2) and a-u-i = this(3) and part = this(4)
  and simp = this(5)

  { assume sem-tree-size xs = 0
    then have ?case using part by blast
  }

  moreover {
    assume sn0: sem-tree-size xs > 0
    obtain ag ad v where xs: xs = Node v ag ad using sn0 by (cases xs, auto)
    {
      assume sem-tree-size ag = 0  $\wedge$  sem-tree-size ad = 0
      then have ag: ag = Leaf and ad: ad = Leaf by (cases ag, auto, cases ad, auto)

      then obtain  $\chi \chi'$  where
         $\chi: \neg I \cup \{\text{Pos } v\} \models \chi$  and
        tot $\chi$ : total-over-m ( $I \cup \{\text{Pos } v\}$ )  $\{\chi\}$  and
         $\chi\psi$ :  $\chi \in \text{fst } \psi$  and
         $\chi': \neg I \cup \{\text{Neg } v\} \models \chi'$  and
        tot $\chi'$ : total-over-m ( $I \cup \{\text{Neg } v\}$ )  $\{\chi'\}$  and  $\chi'\psi$ :  $\chi' \in \text{fst } \psi$ 
        using part unfolding xs by auto
      have Posv: Pos v  $\notin \# \chi$  using  $\chi$  unfolding true-cls-def true-lit-def by auto
      have Negv: Neg v  $\notin \# \chi'$  using  $\chi'$  unfolding true-cls-def true-lit-def by auto
      {
        assume Neg $\chi$ :  $\neg \text{Neg } v \in \# \chi$ 
        then have  $\neg I \models \chi$  using  $\chi$  Posv unfolding true-cls-def true-lit-def by auto
        moreover have total-over-m I  $\{\chi\}$ 
          using Posv Neg $\chi$  atm-imp-pos-or-neg-lit tot $\chi$  unfolding total-over-m-def total-over-set-def
          by fastforce
        ultimately have partial-interps Leaf I (fst  $\psi$ )
          and sem-tree-size Leaf < sem-tree-size xs
          and resolution $^{**} \psi \psi$ 
          unfolding xs by (auto simp add:  $\chi\psi$ )
      }
    }
  }

```



```

moreover {
  assume  $Pos\chi: \neg Pos\ v \in \# \chi'$ 
  then have  $I\chi: \neg I \models \chi'$  using  $\chi' Posv$  unfolding true-cls-def true-lit-def by auto
  moreover have total-over-m  $I \{\chi'\}$ 
    using  $Negv\ Pos\chi\ atm\ imp\ pos\ or\ neg\ lit\ tot\chi'$ 
    unfolding total-over-m-def total-over-set-def by fastforce
  ultimately have partial-interps  $Leaf\ I\ (fst\ \psi)$ 
  and sem-tree-size  $Leaf < sem-tree-size\ xs$ 
  and resolution**  $\psi\ \psi$  using  $\chi'\psi\ I\chi$  unfolding xs by auto
}
moreover {
  assume  $neg: Neg\ v \in \# \chi$  and  $pos: Pos\ v \in \# \chi'$ 
  have count  $\chi\ (Neg\ v) = 1$ 
    using simplified-count[OF simp  $\chi\psi$ ] neg
    by (simp add: dual-order.antisym)
  have count  $\chi'\ (Pos\ v) = 1$ 
    using simplified-count[OF simp  $\chi'\psi$ ] pos
    by (simp add: dual-order.antisym)

  obtain  $C$  where  $\chi C: \chi = C + \{\#Neg\ v\#\}$  and  $negC: Neg\ v \notin \# C$  and  $posC: Pos\ v \notin \# C$ 
    by (metis (no-types, lifting) One-nat-def Posv Suc-eq-plus1-left  $\langle count\ \chi\ (Neg\ v) = 1 \rangle$ 
      add-diff-cancel-left' count-diff count-greater-eq-one-iff count-single insert-DiffM
      insert-DiffM2 less-numeral-extra(3) multi-member-skip not-le not-less-eq-eq)

  obtain  $C'$  where
     $\chi C': \chi' = C' + \{\#Pos\ v\#\}$  and
     $posC': Pos\ v \notin \# C'$  and
     $negC': Neg\ v \notin \# C'$ 
    by (metis (no-types, lifting) One-nat-def Negv Suc-eq-plus1-left  $\langle count\ \chi'\ (Pos\ v) = 1 \rangle$ 
      add-diff-cancel-left' count-diff count-greater-eq-one-iff count-single insert-DiffM
      insert-DiffM2 less-numeral-extra(3) multi-member-skip not-le not-less-eq-eq)

  have totC: total-over-m  $I \{C\}$ 
    using tot $\chi\ tot-over-m-remove$ [of  $I\ Pos\ v\ C$ ] negC posC unfolding  $\chi C$ 
    by (metis total-over-m-sum uminus-Neg uminus-of-uminus-id)
  have totC': total-over-m  $I \{C'\}$ 
    using tot $\chi'\ tot-over-m-sum tot-over-m-remove$ [of  $I\ Neg\ v\ C'$ ] negC' posC'
    unfolding  $\chi C'$  by (metis total-over-m-sum uminus-Neg)
  have  $\neg I \models C + C'$ 
    using  $\chi\ \chi'\ \chi C\ \chi C'$  by auto
  then have part-I- $\psi'''$ : partial-interps  $Leaf\ I\ (fst\ \psi \cup \{C + C'\})$ 
    using totC totC'  $\neg I \models C + C'$  by (metis Un-insert-right insertI1
      partial-interps.simps(1) total-over-m-sum)
  {
    assume  $(\{\#Pos\ v\#\} + C', \{\#Neg\ v\#\} + C) \notin snd\ \psi$ 
    then have inf'': inference  $\psi\ (fst\ \psi \cup \{C + C'\}, snd\ \psi \cup \{(\chi', \chi)\})$ 
      by (metis  $\chi'\psi\ \chi C\ \chi C'\ \chi\psi\ add.commute\ inference-step\ prod.collapse\ resolution$ )
    obtain  $N'$  where full: full simplify  $(fst\ \psi \cup \{C + C'\})\ N'$ 
      by (metis finite-simplified-full-simp fst-conv inf'' inference-preserves-finite
        local.finite)
    have resolution  $\psi\ (N', snd\ \psi \cup \{(\chi', \chi)\})$ 
      using resolution.intros(2)[OF - simp full, of  $snd\ \psi\ snd\ \psi \cup \{(\chi', \chi)\}$ ] inf''
      by (metis surjective-pairing)
    moreover have partial-interps  $Leaf\ I\ N'$ 
      using full-simplify-preserve-partial-tree[OF full part-I- $\psi'''$ ] .
  }
}

```

```

moreover have sem-tree-size Leaf < sem-tree-size xs unfolding xs by auto
ultimately have ?case
  by (metis (no-types) prod.sel(1) rtranclp.rtrancl-into-rtrancl rtranclp.rtrancl-refl)
}
moreover {
  assume a: ( $\{\#Pos\ v\# \} + C', \{\#Neg\ v\# \} + C$ )  $\in$  snd  $\psi$ 
  then have ( $\exists \chi \in fst\ \psi. (\forall I. total-over-m\ I\ \{C+C'\} \longrightarrow total-over-m\ I\ \{\chi\})$ 
     $\wedge (\forall I. total-over-m\ I\ \{\chi\} \longrightarrow I \models \chi \longrightarrow I \models C' + C)) \vee tautology\ (C' + C)$ 
  proof -
    obtain p where p:  $Pos\ p \in \# (\{\#Pos\ v\# \} + C') \wedge Neg\ p \in \# (\{\#Neg\ v\# \} + C)$ 
       $\wedge ((\exists \chi \in fst\ \psi. (\forall I. total-over-m\ I\ (\{\#Pos\ v\# \} + C' - \{\#Pos\ p\# \} + ((\{\#Neg\ v\# \} + C) - \{\#Neg\ p\# \})))$ 
 $\longrightarrow total-over-m\ I\ \{\chi\}) \wedge (\forall I. total-over-m\ I\ \{\chi\} \longrightarrow I \models \chi \longrightarrow I \models (\{\#Pos\ v\# \} + C' - \{\#Pos\ p\# \} + ((\{\#Neg\ v\# \} + C) - \{\#Neg\ p\# \}))) \vee tautology\ ((\{\#Pos\ v\# \} + C' - \{\#Pos\ p\# \} + ((\{\#Neg\ v\# \} + C) - \{\#Neg\ p\# \})))$ 
    using a by (blast intro: allE[OF a-u-i[unfolded subsumes-def Ball-def],
      of ( $\{\#Pos\ v\# \} + C', \{\#Neg\ v\# \} + C$ )]])
    { assume  $p \neq v$ 
      then have  $Pos\ p \in \# C' \wedge Neg\ p \in \# C$  using p by force
      then have ?thesis by auto
    }
  moreover {
    assume  $p = v$ 
    then have ?thesis using p by (metis add.commute add-diff-cancel-left)
  }
  ultimately show ?thesis by auto
qed
moreover {
  assume  $\exists \chi \in fst\ \psi. (\forall I. total-over-m\ I\ \{C+C'\} \longrightarrow total-over-m\ I\ \{\chi\})$ 
 $\wedge (\forall I. total-over-m\ I\ \{\chi\} \longrightarrow I \models \chi \longrightarrow I \models C' + C)$ 
  then obtain  $\vartheta$  where
     $\vartheta$ :  $\vartheta \in fst\ \psi$  and
     $tot\ \vartheta\ CC'$ :  $\forall I. total-over-m\ I\ \{C+C'\} \longrightarrow total-over-m\ I\ \{\vartheta\}$  and
     $\vartheta$ -inv:  $\forall I. total-over-m\ I\ \{\vartheta\} \longrightarrow I \models \vartheta \longrightarrow I \models C' + C$  by blast
  have partial-interps Leaf I (fst  $\psi$ )
    using  $tot\ \vartheta\ CC'\ \vartheta\ \vartheta$ -inv  $totC\ totC'$   $\hookrightarrow I \models C + C'$  total-over-m-sum by fastforce
  moreover have sem-tree-size Leaf < sem-tree-size xs unfolding xs by auto
  ultimately have ?case by blast
}
moreover {
  assume tautCC': tautology ( $C' + C$ )
  have total-over-m I  $\{C'+C\}$  using  $totC\ totC'$  total-over-m-sum by auto
  then have  $\neg tautology\ (C' + C)$ 
    using  $\hookrightarrow I \models C + C'$  unfolding add.commute[of  $C\ C'$ ] total-over-m-def
    unfolding tautology-def by auto
  then have False using tautCC' unfolding tautology-def by auto
}
ultimately have ?case by auto
}
ultimately have ?case by auto
}
ultimately have ?case using part by (metis (no-types) sem-tree-size.simps(1))
}
moreover {
  assume size-ag: sem-tree-size ag > 0
  have sem-tree-size ag < sem-tree-size xs unfolding xs by auto

```

```

moreover have partial-interps ag ( $I \cup \{Pos\ v\}$ ) (fst  $\psi$ )
and partad: partial-interps ad ( $I \cup \{Neg\ v\}$ ) (fst  $\psi$ )
  using part partial-interps.simps(2) unfolding xs by metis+
moreover
  have sem-tree-size ag < sem-tree-size xs  $\implies$  finite (fst  $\psi$ )  $\implies$  already-used-inv  $\psi$ 
     $\implies$  partial-interps ag ( $I \cup \{Pos\ v\}$ ) (fst  $\psi$ )  $\implies$  simplified (fst  $\psi$ )
     $\implies \exists tree' \psi'. resolution^{**} \psi \psi' \wedge partial-interps\ tree' (I \cup \{Pos\ v\}) (fst\ \psi')$ 
       $\wedge (sem-tree-size\ tree' < sem-tree-size\ ag \vee sem-tree-size\ ag = 0)$ 
    using IH[of ag I  $\cup$  {Pos v}] by auto
ultimately obtain  $\psi' :: 'v\ state$  and  $tree' :: 'v\ sem-tree$  where
  inf: resolution^{**}  $\psi \psi'$ 
  and part: partial-interps tree' ( $I \cup \{Pos\ v\}$ ) (fst  $\psi'$ )
  and size: sem-tree-size tree' < sem-tree-size ag  $\vee sem-tree-size\ ag = 0$ 
  using finite part rtranclp.rtrancl_refl a-u-i simp by blast

have partial-interps ad ( $I \cup \{Neg\ v\}$ ) (fst  $\psi'$ )
  using rtranclp-resolution-preserve-partial-tree inf partad by fast
then have partial-interps (Node v tree' ad) I (fst  $\psi'$ ) using part by auto
then have ?case using inf size size-ag part unfolding xs by fastforce
}
moreover {
  assume size-ad: sem-tree-size ad > 0
  have sem-tree-size ad < sem-tree-size xs unfolding xs by auto
moreover
  have
    partag: partial-interps ag ( $I \cup \{Pos\ v\}$ ) (fst  $\psi$ ) and
    partial-interps ad ( $I \cup \{Neg\ v\}$ ) (fst  $\psi$ )
    using part partial-interps.simps(2) unfolding xs by metis+
  moreover have sem-tree-size ad < sem-tree-size xs  $\longrightarrow$  finite (fst  $\psi$ )  $\longrightarrow$  already-used-inv  $\psi$ 
     $\longrightarrow$  (partial-interps ad ( $I \cup \{Neg\ v\}$ ) (fst  $\psi$ )  $\longrightarrow$  simplified (fst  $\psi$ )
     $\longrightarrow (\exists tree' \psi'. resolution^{**} \psi \psi' \wedge partial-interps\ tree' (I \cup \{Neg\ v\}) (fst\ \psi')$ 
       $\wedge (sem-tree-size\ tree' < sem-tree-size\ ad \vee sem-tree-size\ ad = 0)))$ 
    using IH by blast
  ultimately obtain  $\psi' :: 'v\ state$  and  $tree' :: 'v\ sem-tree$  where
    inf: resolution^{**}  $\psi \psi'$ 
    and part: partial-interps tree' ( $I \cup \{Neg\ v\}$ ) (fst  $\psi'$ )
    and size: sem-tree-size tree' < sem-tree-size ad  $\vee sem-tree-size\ ad = 0$ 
    using finite part rtranclp.rtrancl_refl a-u-i simp by blast

  have partial-interps ag ( $I \cup \{Pos\ v\}$ ) (fst  $\psi'$ )
    using rtranclp-resolution-preserve-partial-tree inf partag by fast
  then have partial-interps (Node v ag tree') I (fst  $\psi'$ ) using part by auto
  then have ?case using inf size size-ad unfolding xs by fastforce
}
ultimately have ?case by auto
}
ultimately show ?case by auto
qed

```

```

lemma resolution-completeness-inv:
fixes  $\psi :: 'v :: linorder\ state$ 
assumes
  unsat:  $\neg satisfiable (fst\ \psi)$  and
  finite: finite (fst  $\psi$ ) and
  a-u-v: already-used-inv  $\psi$ 

```

```

shows  $\exists \psi'. (\text{resolution}^{**} \psi \psi' \wedge \{\#\} \in \text{fst } \psi')$ 
proof -
  obtain tree where partial-interps tree {} (fst  $\psi$ )
  using partial-interps-build-sem-tree-atms assms by metis
then show ?thesis
  using unsat finite a-u-v
proof (induct tree arbitrary:  $\psi$  rule: sem-tree-size)
  case (bigger tree  $\psi$ ) note  $H = \text{this}$ 
  {
    fix  $\chi$ 
    assume tree: tree = Leaf
    obtain  $\chi$  where  $\chi: \neg \{\} \models \chi$  and tot $\chi$ : total-over-m {} { $\chi$ } and  $\chi\psi: \chi \in \text{fst } \psi$ 
    using  $H$  unfolding tree by auto
    moreover have  $\{\#\} = \chi$ 
    using  $H$  atms-empty-iff-empty tot $\chi$ 
    unfolding true-cls-def total-over-m-def total-over-set-def by fastforce
    moreover have resolution $^{**} \psi \psi$  by auto
    ultimately have ?case by metis
  }
  moreover {
    fix v tree1 tree2
    assume tree: tree = Node v tree1 tree2
    obtain  $\psi_0$  where  $\psi_0$ : resolution $^{**} \psi \psi_0$  and simp: simplified (fst  $\psi_0$ )
    proof -
      { assume simplified (fst  $\psi$ )
        moreover have resolution $^{**} \psi \psi$  by auto
        ultimately have thesis using that by blast
      }
      moreover {
        assume  $\neg \text{simplified (fst } \psi)$ 
        then have  $\exists \psi'. \text{full1 simplify (fst } \psi) \psi'$ 
          by (metis Nitpick.rtranclp-unfold bigger.prem(3) full1-def
            rtranclp-simplify-terminates)
        then obtain  $N$  where full1 simplify (fst  $\psi$ )  $N$  by metis
        then have resolution  $\psi (N, \text{snd } \psi)$ 
          using resolution.intros(1)[of fst  $\psi$   $N$  snd  $\psi$ ] by auto
        moreover have simplified  $N$ 
          using  $\langle \text{full1 simplify (fst } \psi) N \rangle$  unfolding full1-def by blast
        ultimately have ?thesis using that by force
      }
      ultimately show ?thesis by auto
    qed
  }

```

```

have p: partial-interps tree {} (fst  $\psi_0$ )
and uns: unsatisfiable (fst  $\psi_0$ )
and f: finite (fst  $\psi_0$ )
and a-u-v: already-used-inv  $\psi_0$ 
  using  $\psi_0$  bigger.prem(1) rtranclp-resolution-preserve-partial-tree apply blast
  using  $\psi_0$  bigger.prem(2) rtranclp-resolution-preserves-unsat apply blast
  using  $\psi_0$  bigger.prem(3) rtranclp-resolution-finite apply blast
  using rtranclp-resolution-already-used-inv[OF  $\psi_0$  bigger.prem(4)] by blast
obtain tree'  $\psi'$  where
  inf: resolution $^{**} \psi_0 \psi'$  and
  part': partial-interps tree' {} (fst  $\psi'$ ) and

```

```

    decrease: sem-tree-size tree' < sem-tree-size tree  $\vee$  sem-tree-size tree = 0
    using can-decrease-tree-size-resolution[OF f a-u-v p simp] unfolding tautology-def
    by meson
  have s: sem-tree-size tree' < sem-tree-size tree using decrease unfolding tree by auto
  have fin: finite (fst  $\psi'$ )
    using f inf rtrancpl-resolution-finite by blast
  have unsat: unsatisfiable (fst  $\psi'$ )
    using rtrancpl-resolution-preserves-unsat inf uns by metis
  have a-u-i': already-used-inv  $\psi'$ 
    using a-u-v inf rtrancpl-resolution-already-used-inv[of  $\psi_0 \psi'$ ] by auto
  have ?case
    using inf rtrancpl-trans[of resolution] H(1)[OF s part' unsat fin a-u-i']  $\psi_0$  by blast
}
ultimately show ?case by (cases tree, auto)
qed
qed

```

```

lemma resolution-preserves-already-used-inv:
  assumes resolution S S'
  and already-used-inv S
  shows already-used-inv S'
  using assms
  apply (induct rule: resolution.induct)
  apply (rule full1-simplify-already-used-inv; simp)
  apply (rule full-simplify-already-used-inv, simp)
  apply (rule inference-preserves-already-used-inv, simp)
  apply blast
done

```

```

lemma rtrancpl-resolution-preserves-already-used-inv:
  assumes resolution** S S'
  and already-used-inv S
  shows already-used-inv S'
  using assms
  apply (induct rule: rtrancpl-induct)
  apply simp
  using resolution-preserves-already-used-inv by fast

```

```

lemma resolution-completeness:
  fixes  $\psi :: 'v :: \text{linorder}$  state
  assumes unsat:  $\neg$ satisfiable (fst  $\psi$ )
  and finite: finite (fst  $\psi$ )
  and snd  $\psi$  = {}
  shows  $\exists \psi'. (\text{resolution}^{**} \psi \psi' \wedge \{\#\} \in \text{fst } \psi')$ 
proof -
  have already-used-inv  $\psi$  unfolding assms by auto
  then show ?thesis using assms resolution-completeness-inv by blast
qed

```

```

lemma rtrancpl-preserves-sat:
  assumes simplify** S S'
  and satisfiable S
  shows satisfiable S'
  using assms apply induction
  apply simp

```

by (meson satisfiable-carac satisfiable-def simplify-preserves-un-sat-eq)

lemma *resolution-preserves-sat*:

assumes *resolution* $S\ S'$
and *satisfiable* (fst S)
shows *satisfiable* (fst S')
using *assms* **apply** (induction rule: *resolution.induct*)
using *rtrancplp-preserves-sat* *trancplp-into-rtrancplp* **unfolding** *full1-def* **apply** *fastforce*
by (metis *fst-conv* *full-def* *inference-preserves-un-sat* *rtrancplp-preserves-sat*
satisfiable-carac' *satisfiable-def*)

lemma *rtrancplp-resolution-preserves-sat*:

assumes *resolution*** $S\ S'$
and *satisfiable* (fst S)
shows *satisfiable* (fst S')
using *assms* **apply** (induction rule: *rtrancplp-induct*)
apply *simp*
using *resolution-preserves-sat* **by** *blast*

lemma *resolution-soundness*:

fixes $\psi :: 'v :: \text{linorder state}$
assumes *resolution*** $\psi\ \psi'$ **and** $\{\#\} \in \text{fst } \psi'$
shows *unsatisfiable* (fst ψ)
using *assms* **by** (meson *rtrancplp-resolution-preserves-sat* *satisfiable-def* *true-cls-empty*
true-cls-def)

lemma *resolution-soundness-and-completeness*:

fixes $\psi :: 'v :: \text{linorder state}$
assumes *finite*: *finite* (fst ψ)
and *snd*: *snd* $\psi = \{\}$
shows $(\exists \psi'. (\text{resolution** } \psi\ \psi' \wedge \{\#\} \in \text{fst } \psi')) \longleftrightarrow \text{unsatisfiable (fst } \psi)$
using *assms* *resolution-completeness* *resolution-soundness* **by** *metis*

lemma *simplified-falsity*:

assumes *simp*: *simplified* ψ
and $\{\#\} \in \psi$
shows $\psi = \{\{\#\}\}$
proof (rule *ccontr*)
assume $H: \neg ?thesis$
then obtain χ **where** $\chi \in \psi$ **and** $\chi \neq \{\#\}$ **using** *assms*(2) **by** *blast*
then have $\{\#\} \subsetneq \chi$ **by** (*simp* *add*: *mset-less-empty-nonempty*)
then have *simplify* ψ ($\psi - \{\chi\}$)
using *simplify.subsumption*[*OF* *assms*(2) $\langle \{\#\} \subsetneq \chi \rangle \langle \chi \in \psi \rangle$] **by** *blast*
then show *False* **using** *simp* **by** *blast*
qed

lemma *simplify-falsity-in-preserved*:

assumes *simplify* $\chi s\ \chi s'$
and $\{\#\} \in \chi s$
shows $\{\#\} \in \chi s'$
using *assms*
by *induction auto*

lemma *rtrancplp-simplify-falsity-in-preserved*:

```

assumes simplify**  $\chi s \chi s'$ 
and  $\{\#\} \in \chi s$ 
shows  $\{\#\} \in \chi s'$ 
using assms
by induction (auto intro: simplify-falsity-in-preserved)

lemma resolution-falsity-get-falsity-alone:
  assumes finite (fst  $\psi$ )
  shows  $(\exists \psi'. (\text{resolution}^{**} \psi \psi' \wedge \{\#\} \in \text{fst } \psi')) \longleftrightarrow (\exists a-u-v. \text{resolution}^{**} \psi (\{\{\#\}\}, a-u-v))$ 
  (is  $?A \longleftrightarrow ?B$ )
proof
  assume  $?B$ 
  then show  $?A$  by auto
next
  assume  $?A$ 
  then obtain  $\chi s$  a-u-v where  $\chi s: \text{resolution}^{**} \psi (\chi s, a-u-v)$  and  $F: \{\#\} \in \chi s$  by auto
  { assume simplified  $\chi s$ 
    then have  $?B$  using simplified-falsity[OF - F]  $\chi s$  by blast
  }
  moreover {
    assume  $\neg \text{simplified } \chi s$ 
    then obtain  $\chi s'$  where full1 simplify  $\chi s \chi s'$ 
    by (metis  $\chi s$  assms finite-simplified-full1-simp fst-conv rtranclp-resolution-finite)
    then have  $\{\#\} \in \chi s'$ 
    unfolding full1-def by (meson  $F$  rtranclp-simplify-falsity-in-preserved
      trancplp-into-rtranclp)
    then have  $?B$ 
    by (metis  $\chi s$  (full1 simplify  $\chi s \chi s'$ ) fst-conv full1-simp resolution-always-simplified
      rtranclp.rtrancl-into-rtrancl simplified-falsity)
  }
  ultimately show  $?B$  by blast
qed

lemma resolution-soundness-and-completeness':
  fixes  $\psi :: 'v :: \text{linorder state}$ 
  assumes
    finite: finite (fst  $\psi$ ) and
    snd: snd  $\psi = \{\}$ 
  shows  $(\exists a-u-v. (\text{resolution}^{**} \psi (\{\{\#\}\}, a-u-v))) \longleftrightarrow \text{unsatisfiable } (\text{fst } \psi)$ 
  using assms resolution-completeness resolution-soundness resolution-falsity-get-falsity-alone
  by metis

end

theory Partial-Annotated-Clausal-Logic
imports Partial-Clausal-Logic

begin

```

14 Partial Clausal Logic

We here define marked literals (that will be used in both DPLL and CDCL) and the entailment corresponding to it.

14.1 Marked Literals

14.1.1 Definition

datatype ('v, 'lvl, 'mark) marked-lit =
is-marked: Marked (lit-of: 'v literal) (level-of: 'lvl) |
is-proped: Propagated (lit-of: 'v literal) (mark-of: 'mark)

lemma marked-lit-list-induct[case-names nil marked proped]:
assumes $P \ []$ **and**
 $\bigwedge L \ l \ xs. P \ xs \implies P \ (\text{Marked } L \ l \ \# \ xs)$ **and**
 $\bigwedge L \ m \ xs. P \ xs \implies P \ (\text{Propagated } L \ m \ \# \ xs)$
shows $P \ xs$
using *assms* **apply** (induction *xs*, *simp*)
by (rename-tac *a xs*, case-tac *a*) *auto*

lemma is-marked-ex-Marked:
 $\text{is-marked } L \implies \exists K \ lvl. L = \text{Marked } K \ lvl$
by (cases *L*) *auto*

type-synonym ('v, 'l, 'm) marked-lits = ('v, 'l, 'm) marked-lit list

definition lits-of :: ('a, 'b, 'c) marked-lit set \Rightarrow 'a literal set **where**
lits-of *Ls* = *lit-of* ' *Ls*

abbreviation lits-of-l :: ('a, 'b, 'c) marked-lit list \Rightarrow 'a literal set **where**
lits-of-l *Ls* \equiv *lits-of* (set *Ls*)

lemma lits-of-l-empty[*simp*]:
lits-of {} = {}
unfolding *lits-of-def* **by** *auto*

lemma lits-of-insert[*simp*]:
lits-of (insert *L Ls*) = insert (lit-of *L*) (*lits-of* *Ls*)
unfolding *lits-of-def* **by** *auto*

lemma lits-of-l-Un[*simp*]:
lits-of ($l \cup l'$) = *lits-of* *l* \cup *lits-of* *l'*
unfolding *lits-of-def* **by** *auto*

lemma finite-lits-of-def[*simp*]:
finite (*lits-of-l* *L*)
unfolding *lits-of-def* **by** *auto*

abbreviation unmark **where**
unmark $\equiv (\lambda a. \{\# \text{lit-of } a \# \})$

abbreviation unmark-s **where**
unmark-s *M* \equiv unmark ' *M*

abbreviation unmark-l **where**
unmark-l *M* \equiv unmark-s (set *M*)

lemma atms-of-ms-lambda-lit-of-is-atm-of-lit-of[*simp*]:
atms-of-ms (unmark-l *M'*) = atm-of ' *lits-of-l* *M'*
unfolding *atms-of-ms-def* *lits-of-def* **by** *auto*

lemma *lits-of-l-empty-is-empty*[*iff*]:
 $\text{lits-of-l } M = \{\} \longleftrightarrow M = []$
by (*induct* M) (*auto simp: lits-of-def*)

14.1.2 Entailment

definition *true-annot* :: ('a, 'l, 'm) *marked-lits* \Rightarrow 'a *clause* \Rightarrow bool (**infix** \models_a 49) **where**
 $I \models_a C \longleftrightarrow (\text{lits-of-l } I) \models C$

definition *true-annots* :: ('a, 'l, 'm) *marked-lits* \Rightarrow 'a *clauses* \Rightarrow bool (**infix** \models_{as} 49) **where**
 $I \models_{as} CC \longleftrightarrow (\forall C \in CC. I \models_a C)$

lemma *true-annot-empty-model*[*simp*]:
 $\neg [] \models_a \psi$
unfolding *true-annot-def* *true-cls-def* **by** *simp*

lemma *true-annot-empty*[*simp*]:
 $\neg I \models_a \{\#\}$
unfolding *true-annot-def* *true-cls-def* **by** *simp*

lemma *empty-true-annots-def*[*iff*]:
 $[] \models_{as} \psi \longleftrightarrow \psi = \{\}$
unfolding *true-annots-def* **by** *auto*

lemma *true-annots-empty*[*simp*]:
 $I \models_{as} \{\}$
unfolding *true-annots-def* **by** *auto*

lemma *true-annots-single-true-annot*[*iff*]:
 $I \models_{as} \{C\} \longleftrightarrow I \models_a C$
unfolding *true-annots-def* **by** *auto*

lemma *true-annot-insert-l*[*simp*]:
 $M \models_a A \Longrightarrow L \# M \models_a A$
unfolding *true-annot-def* **by** *auto*

lemma *true-annots-insert-l* [*simp*]:
 $M \models_{as} A \Longrightarrow L \# M \models_{as} A$
unfolding *true-annots-def* **by** *auto*

lemma *true-annots-union*[*iff*]:
 $M \models_{as} A \cup B \longleftrightarrow (M \models_{as} A \wedge M \models_{as} B)$
unfolding *true-annots-def* **by** *auto*

lemma *true-annots-insert*[*iff*]:
 $M \models_{as} \text{insert } a \ A \longleftrightarrow (M \models_a a \wedge M \models_{as} A)$
unfolding *true-annots-def* **by** *auto*

Link between \models_{as} and \models_s :

lemma *true-annots-true-cls*:
 $I \models_{as} CC \longleftrightarrow \text{lits-of-l } I \models_s CC$
unfolding *true-annots-def* *Ball-def* *true-annot-def* *true-clss-def* **by** *auto*

lemma *in-lit-of-true-annot*:

$a \in \text{lits-of-l } M \longleftrightarrow M \models_a \{\#a\# \}$
unfolding *true-annot-def lits-of-def* **by** *auto*

lemma *true-annot-lit-of-notin-skip*:
 $L \# M \models_a A \implies \text{lit-of } L \not\subseteq \# A \implies M \models_a A$
unfolding *true-annot-def true-clss-def* **by** *auto*

lemma *true-clss-singleton-lit-of-implies-incl*:
 $I \models_s \text{unmark-l MLs} \implies \text{lits-of-l MLs} \subseteq I$
unfolding *true-clss-def lits-of-def* **by** *auto*

lemma *true-annot-true-clss-clss*:
 $\text{MLs} \models_a \psi \implies \text{set } (\text{map unmark MLs}) \models_p \psi$
unfolding *true-annot-def true-clss-clss-def true-clss-def*
by (*auto dest: true-clss-singleton-lit-of-implies-incl*)

lemma *true-annots-true-clss-clss*:
 $\text{MLs} \models_{as} \psi \implies \text{set } (\text{map unmark MLs}) \models_{ps} \psi$
by (*auto*
dest: true-clss-singleton-lit-of-implies-incl
simp add: true-clss-def true-annots-def true-annot-def lits-of-def true-clss-def
true-clss-clss-def)

lemma *true-annots-marked-true-clss[iff]*:
 $\text{map } (\lambda M. \text{Marked } M \ a) \ M \models_{as} N \longleftrightarrow \text{set } M \models_s N$

proof –

have *: *lit-of* ‘ $(\lambda M. \text{Marked } M \ a)$ ’ *set* $M = \text{set } M$ **unfolding** *lits-of-def* **by** *force*
show ?thesis **by** (*simp add: true-annots-true-clss * lits-of-def*)

qed

lemma *true-annot-singleton[iff]*: $M \models_a \{\#L\# \} \longleftrightarrow L \in \text{lits-of-l } M$
unfolding *true-annot-def lits-of-def* **by** *auto*

lemma *true-annots-true-clss-clss*:
 $A \models_{as} \Psi \implies \text{unmark-l } A \models_{ps} \Psi$
unfolding *true-clss-clss-def true-annots-def true-clss-def*
by (*auto*
dest!: true-clss-singleton-lit-of-implies-incl
simp add: lits-of-def true-annot-def true-clss-def)

lemma *true-annot-commute*:
 $M @ M' \models_a D \longleftrightarrow M' @ M \models_a D$
unfolding *true-annot-def* **by** (*simp add: Un-commute*)

lemma *true-annots-commute*:
 $M @ M' \models_{as} D \longleftrightarrow M' @ M \models_{as} D$
unfolding *true-annots-def* **by** (*auto simp add: true-annot-commute*)

lemma *true-annot-mono[dest]*:
 $\text{set } I \subseteq \text{set } I' \implies I \models_a N \implies I' \models_a N$
using *true-clss-mono-set-mset-l* **unfolding** *true-annot-def lits-of-def*
by (*metis (no-types) Un-commute Un-upper1 image-Un sup.orderE*)

lemma *true-annots-mono*:
 $\text{set } I \subseteq \text{set } I' \implies I \models_{as} N \implies I' \models_{as} N$

unfolding *true-annots-def* **by** *auto*

14.1.3 Defined and undefined literals

definition *defined-lit* :: ('a, 'l, 'm) *marked-lit list* \Rightarrow 'a *literal* \Rightarrow bool
where

defined-lit *I L* \longleftrightarrow ($\exists l. \text{Marked } L \ l \in \text{set } I$) \vee ($\exists P. \text{Propagated } L \ P \in \text{set } I$)
 \vee ($\exists l. \text{Marked } (-L) \ l \in \text{set } I$) \vee ($\exists P. \text{Propagated } (-L) \ P \in \text{set } I$)

abbreviation *undefined-lit* :: ('a, 'l, 'm) *marked-lit list* \Rightarrow 'a *literal* \Rightarrow bool
where *undefined-lit* *I L* $\equiv \neg \text{defined-lit } I \ L$

lemma *defined-lit-rev[simp]*:
defined-lit (rev *M*) *L* \longleftrightarrow *defined-lit* *M L*
unfolding *defined-lit-def* **by** *auto*

lemma *atm-imp-marked-or-proped*:
assumes $x \in \text{set } I$
shows
 $(\exists l. \text{Marked } (- \text{lit-of } x) \ l \in \text{set } I)$
 $\vee (\exists l. \text{Marked } (\text{lit-of } x) \ l \in \text{set } I)$
 $\vee (\exists l. \text{Propagated } (- \text{lit-of } x) \ l \in \text{set } I)$
 $\vee (\exists l. \text{Propagated } (\text{lit-of } x) \ l \in \text{set } I)$
using *assms marked-lit.exhaust-sel* **by** *metis*

lemma *literal-is-lit-of-marked*:
assumes $L = \text{lit-of } x$
shows $(\exists l. x = \text{Marked } L \ l) \vee (\exists l'. x = \text{Propagated } L \ l')$
using *assms* **by** (*cases x*) *auto*

lemma *true-annot-iff-marked-or-true-lit*:
defined-lit *I L* \longleftrightarrow ($(\text{lits-of-l } I) \models L \vee (\text{lits-of-l } I) \models -L$)
unfolding *defined-lit-def* **by** (*auto simp add: lits-of-def rev-image-eqI dest!: literal-is-lit-of-marked*)

lemma *consistent-interp (lits-of-l I) \Longrightarrow I \models_{as} N \Longrightarrow satisfiable N*
by (*simp add: true-annots-true-cls*)

lemma *defined-lit-map*:
defined-lit *Ls L* \longleftrightarrow *atm-of* *L* $\in (\lambda l. \text{atm-of } (\text{lit-of } l)) \text{ ' set } Ls$
unfolding *defined-lit-def* **apply** (*rule iffI*)
using *image-iff* **apply** *fastforce*
by (*fastforce simp add: atm-of-eq-atm-of dest: atm-imp-marked-or-proped*)

lemma *defined-lit-uminus[iff]*:
defined-lit *I* $(-L)$ \longleftrightarrow *defined-lit* *I L*
unfolding *defined-lit-def* **by** *auto*

lemma *Marked-Propagated-in-iff-in-lits-of-l*:
defined-lit *I L* \longleftrightarrow ($L \in \text{lits-of-l } I \vee -L \in \text{lits-of-l } I$)
unfolding *lits-of-def* *defined-lit-def*
by (*auto simp: rev-image-eqI (rename-tac x, case-tac x, auto)+*)

lemma *consistent-add-undefined-lit-consistent[simp]*:
assumes
consistent-interp (lits-of-l Ls) **and**

$\text{undefined-lit } Ls \ L$
shows *consistent-interp* (*insert* L (*lits-of-l* Ls))
using *assms* **unfolding** *consistent-interp-def* **by** (*auto simp: Marked-Propagated-in-iff-in-lits-of-l*)

lemma *decided-empty[simp]*:
 $\neg \text{defined-lit } [] \ L$
unfolding *defined-lit-def* **by** *simp*

14.2 Backtracking

fun *backtrack-split* :: ($'v, 'l, 'm$) *marked-lits*
 $\Rightarrow ('v, 'l, 'm) \text{ marked-lits} \times ('v, 'l, 'm) \text{ marked-lits}$ **where**
backtrack-split $[] = ([], [])$ |
backtrack-split (*Propagated* $L \ P \ \# \ mlits$) = *apfst* (*(op #)* (*Propagated* $L \ P$)) (*backtrack-split* $mlits$) |
backtrack-split (*Marked* $L \ l \ \# \ mlits$) = ($[], \text{Marked } L \ l \ \# \ mlits$)

lemma *backtrack-split-fst-not-marked*: $a \in \text{set } (\text{fst } (\text{backtrack-split } l)) \implies \neg \text{is-marked } a$
by (*induct l rule: marked-lit-list-induct*) *auto*

lemma *backtrack-split-snd-hd-marked*:
 $\text{snd } (\text{backtrack-split } l) \neq [] \implies \text{is-marked } (\text{hd } (\text{snd } (\text{backtrack-split } l)))$
by (*induct l rule: marked-lit-list-induct*) *auto*

lemma *backtrack-split-list-eq[simp]*:
 $\text{fst } (\text{backtrack-split } l) @ (\text{snd } (\text{backtrack-split } l)) = l$
by (*induct l rule: marked-lit-list-induct*) *auto*

lemma *backtrack-snd-empty-not-marked*:
 $\text{backtrack-split } M = (M'', []) \implies \forall l \in \text{set } M. \neg \text{is-marked } l$
by (*metis append-Nil2 backtrack-split-fst-not-marked backtrack-split-list-eq snd-conv*)

lemma *backtrack-split-some-is-marked-then-snd-has-hd*:
 $\exists l \in \text{set } M. \text{is-marked } l \implies \exists M'' \ L' \ M''. \text{backtrack-split } M = (M'', L' \ \# \ M')$
by (*metis backtrack-snd-empty-not-marked list.exhaust prod.collapse*)

Another characterisation of the result of *backtrack-split*. This view allows some simpler proofs, since *takeWhile* and *dropWhile* are highly automated:

lemma *backtrack-split-takeWhile-dropWhile*:
 $\text{backtrack-split } M = (\text{takeWhile } (\text{Not } o \text{ is-marked}) \ M, \text{dropWhile } (\text{Not } o \text{ is-marked}) \ M)$

proof (*induct M*)
case *Nil* **show** *?case* **by** *simp*
next
case (*Cons* $L \ M$) **then show** *?case* **by** (*cases L*) *auto*
qed

14.3 Decomposition with respect to the marked literals

The pattern *get-all-marked-decomposition* $[] = [([], [])]$ is necessary otherwise, we can call the *hd* function in the other pattern.

fun *get-all-marked-decomposition* :: ($'a, 'l, 'm$) *marked-lits*
 $\Rightarrow (('a, 'l, 'm) \text{ marked-lits} \times ('a, 'l, 'm) \text{ marked-lits}) \text{ list}$ **where**
get-all-marked-decomposition (*Marked* $L \ l \ \# \ Ls$) =
 $(\text{Marked } L \ l \ \# \ Ls, []) \ \# \ \text{get-all-marked-decomposition } Ls$ |
get-all-marked-decomposition (*Propagated* $L \ P \ \# \ Ls$) =
 $(\text{apsnd } ((\text{op } \#) (\text{Propagated } L \ P)) (\text{hd } (\text{get-all-marked-decomposition } Ls)))$

tl (get-all-marked-decomposition Ls) |
 get-all-marked-decomposition [] = [([], [])]

value get-all-marked-decomposition [Propagated A5 B5, Marked C4 D4, Propagated A3 B3,
 Propagated A2 B2, Marked C1 D1, Propagated A0 B0]

lemma get-all-marked-decomposition-never-empty[iff]:
 get-all-marked-decomposition M = [] \longleftrightarrow False
 by (induct M, simp) (rename-tac a xs, case-tac a, auto)

lemma get-all-marked-decomposition-never-empty-sym[iff]:
 [] = get-all-marked-decomposition M \longleftrightarrow False
 using get-all-marked-decomposition-never-empty[of M] by presburger

lemma get-all-marked-decomposition-decomp:
 hd (get-all-marked-decomposition S) = (a, c) \implies S = c @ a

proof (induct S arbitrary: a c)

case Nil

then show ?case by simp

next

case (Cons x A)

then show ?case by (cases x; cases hd (get-all-marked-decomposition A)) auto

qed

lemma get-all-marked-decomposition-backtrack-split:

backtrack-split S = (M, M') \longleftrightarrow hd (get-all-marked-decomposition S) = (M', M)

proof (induction S arbitrary: M M')

case Nil

then show ?case by auto

next

case (Cons a S)

then show ?case using backtrack-split-takeWhile-dropWhile by (cases a) force+

qed

lemma get-all-marked-decomposition-nil-backtrack-split-snd-nil:

get-all-marked-decomposition S = [([], A)] \implies snd (backtrack-split S) = []

by (simp add: get-all-marked-decomposition-backtrack-split sndI)

lemma get-all-marked-decomposition-length-1-fst-empty-or-length-1:

assumes get-all-marked-decomposition M = (a, b) # []

shows a = [] \vee (length a = 1 \wedge is-marked (hd a) \wedge hd a \in set M)

using assms

proof (induct M arbitrary: a b rule: marked-lit-list-induct)

case nil then show ?case by simp

next

case (marked L mark M)

then show ?case by simp

next

case (proped L mark M)

then show ?case by (cases get-all-marked-decomposition M) force+

qed

lemma get-all-marked-decomposition-fst-empty-or-hd-in-M:

assumes get-all-marked-decomposition M = (a, b) # l

```

shows  $a = [] \vee (\text{is-marked } (hd\ a) \wedge hd\ a \in \text{set } M)$ 
using assms apply (induct M arbitrary: a b rule: marked-lit-list-induct)
apply auto[2]
by (metis UnCI backtrack-split-snd-hd-marked get-all-marked-decomposition-backtrack-split
get-all-marked-decomposition-decomp hd-in-set list.sel(1) set-append snd-conv)

lemma get-all-marked-decomposition-snd-not-marked:
assumes  $(a, b) \in \text{set } (\text{get-all-marked-decomposition } M)$ 
and  $L \in \text{set } b$ 
shows  $\neg \text{is-marked } L$ 
using assms apply (induct M arbitrary: a b rule: marked-lit-list-induct, simp)
by (rename-tac L' l xs a b, case-tac get-all-marked-decomposition xs; fastforce)+

lemma tl-get-all-marked-decomposition-skip-some:
assumes  $x \in \text{set } (tl\ (\text{get-all-marked-decomposition } M1))$ 
shows  $x \in \text{set } (tl\ (\text{get-all-marked-decomposition } (M0\ @\ M1)))$ 
using assms
by (induct M0 rule: marked-lit-list-induct)
(auto simp add: list.set-sel(2))

lemma hd-get-all-marked-decomposition-skip-some:
assumes  $(x, y) = hd\ (\text{get-all-marked-decomposition } M1)$ 
shows  $(x, y) \in \text{set } (\text{get-all-marked-decomposition } (M0\ @\ \text{Marked } K\ i\ \# M1))$ 
using assms
proof (induct M0)
case Nil
then show ?case by auto
next
case (Cons L M0)
then have xy:  $(x, y) \in \text{set } (\text{get-all-marked-decomposition } (M0\ @\ \text{Marked } K\ i\ \# M1))$  by blast
show ?case
proof (cases L)
case (Marked l m)
then show ?thesis using xy by auto
next
case (Propagated l m)
then show ?thesis
using xy Cons.prems
by (cases get-all-marked-decomposition (M0 @ Marked K i # M1))
(auto dest!: get-all-marked-decomposition-decomp
arg-cong[of get-all-marked-decomposition - - hd])
qed
qed

lemma get-all-marked-decomposition-snd-union:
 $\text{set } M = \bigcup (\text{set } 'snd\ 'set\ (\text{get-all-marked-decomposition } M)) \cup \{L \mid L.\ \text{is-marked } L \wedge L \in \text{set } M\}$ 
(is ?M M = ?U M  $\cup$  ?Ls M)
proof (induct M arbitrary:)
case Nil
then show ?case by simp
next
case (Cons L M)
show ?case
proof (cases L)
case (Marked a l) note  $L = \text{this}$ 

```

```

    then have  $L \in ?Ls (L \# M)$  by auto
    moreover have  $?U (L \# M) = ?U M$  unfolding  $L$  by auto
    moreover have  $?M M = ?U M \cup ?Ls M$  using  $Cons.hyps$  by auto
    ultimately show  $?thesis$  by auto
  next
    case  $(Propagated\ a\ P)$ 
    then show  $?thesis$  using  $Cons.hyps$  by (cases (get-all-marked-decomposition  $M$ )) auto
  qed
qed

lemma in-get-all-marked-decomposition-in-get-all-marked-decomposition-prepend:
   $(a, b) \in set (get-all-marked-decomposition\ M') \implies$ 
   $\exists b'. (a, b' @ b) \in set (get-all-marked-decomposition\ (M @ M'))$ 
  apply (induction  $M$  rule: marked-lit-list-induct)
  apply (metis append-Nil)
  apply auto[]
  by (rename-tac  $L' m xs$ , case-tac get-all-marked-decomposition  $(xs @ M')$ ) auto

lemma get-all-marked-decomposition-remove-unmark-ssed-length:
  assumes  $\forall l \in set\ M'. \neg is-marked\ l$ 
  shows  $length (get-all-marked-decomposition\ (M' @ M''))$ 
     $= length (get-all-marked-decomposition\ M'')$ 
  using assms by (induct  $M'$  arbitrary:  $M''$  rule: marked-lit-list-induct) auto

lemma get-all-marked-decomposition-not-is-marked-length:
  assumes  $\forall l \in set\ M'. \neg is-marked\ l$ 
  shows  $1 + length (get-all-marked-decomposition\ (Propagated\ (-L)\ P \# M))$ 
     $= length (get-all-marked-decomposition\ (M' @ Marked\ L\ l \# M))$ 
  using assms get-all-marked-decomposition-remove-unmark-ssed-length by fastforce

lemma get-all-marked-decomposition-last-choice:
  assumes  $tl (get-all-marked-decomposition\ (M' @ Marked\ L\ l \# M)) \neq []$ 
  and  $\forall l \in set\ M'. \neg is-marked\ l$ 
  and  $hd (tl (get-all-marked-decomposition\ (M' @ Marked\ L\ l \# M))) = (M0', M0)$ 
  shows  $hd (get-all-marked-decomposition\ (Propagated\ (-L)\ P \# M)) = (M0', Propagated\ (-L)\ P \# M0)$ 
  using assms by (induct  $M'$  rule: marked-lit-list-induct) auto

lemma get-all-marked-decomposition-except-last-choice-equal:
  assumes  $\forall l \in set\ M'. \neg is-marked\ l$ 
  shows  $tl (get-all-marked-decomposition\ (Propagated\ (-L)\ P \# M))$ 
     $= tl (tl (get-all-marked-decomposition\ (M' @ Marked\ L\ l \# M)))$ 
  using assms by (induct  $M'$  rule: marked-lit-list-induct) auto

lemma get-all-marked-decomposition-hd-hd:
  assumes  $get-all-marked-decomposition\ Ls = (M, C) \# (M0, M0') \# l$ 
  shows  $tl\ M = M0' @ M0 \wedge is-marked\ (hd\ M)$ 
  using assms
proof (induct  $Ls$  arbitrary:  $M\ C\ M0\ M0'\ l$ )
  case Nil
  then show ?case by simp
next
  case  $(Cons\ a\ Ls\ M\ C\ M0\ M0'\ l)$  note  $IH = this(1)$  and  $g = this(2)$ 
  { fix  $L\ level$ 
    assume  $a: a = Marked\ L\ level$ 

```

```

have Ls = M0' @ M0
  using g a by (force intro: get-all-marked-decomposition-decomp)
then have tl M = M0' @ M0 ∧ is-marked (hd M) using g a by auto
}
moreover {
  fix L P
  assume a: a = Propagated L P
  have tl M = M0' @ M0 ∧ is-marked (hd M)
    using IH Cons.premis unfolding a by (cases get-all-marked-decomposition Ls) auto
}
ultimately show ?case by (cases a) auto
qed

```

```

lemma get-all-marked-decomposition-exists-prepend[dest]:
  assumes (a, b) ∈ set (get-all-marked-decomposition M)
  shows ∃ c. M = c @ b @ a
  using assms apply (induct M rule: marked-lit-list-induct)
  apply simp
  by (rename-tac L' m xs, case-tac get-all-marked-decomposition xs;
    auto dest!: arg-cong[of get-all-marked-decomposition - - hd]
    get-all-marked-decomposition-decomp)+

```

```

lemma get-all-marked-decomposition-incl:
  assumes (a, b) ∈ set (get-all-marked-decomposition M)
  shows set b ⊆ set M and set a ⊆ set M
  using assms get-all-marked-decomposition-exists-prepend by fastforce+

```

```

lemma get-all-marked-decomposition-exists-prepend':
  assumes (a, b) ∈ set (get-all-marked-decomposition M)
  obtains c where M = c @ b @ a
  using assms apply (induct M rule: marked-lit-list-induct)
  apply auto[1]
  by (rename-tac L' m xs, case-tac hd (get-all-marked-decomposition xs),
    auto dest!: get-all-marked-decomposition-decomp simp add: list.set-sel(2))+

```

```

lemma union-in-get-all-marked-decomposition-is-subset:
  assumes (a, b) ∈ set (get-all-marked-decomposition M)
  shows set a ∪ set b ⊆ set M
  using assms by force

```

```

lemma Marked-cons-in-get-all-marked-decomposition-append-Marked-cons:
  ∃ M1 M2. (Marked K i # M1, M2) ∈ set (get-all-marked-decomposition (c @ Marked K i # c'))
  apply (induction c rule: marked-lit-list-induct)
  apply auto[2]
  apply (rename-tac L m xs,
    case-tac hd (get-all-marked-decomposition (xs @ Marked K i # c')))
  apply (case-tac get-all-marked-decomposition (xs @ Marked K i # c'))
  by auto

```

```

definition all-decomposition-implies :: 'a literal multiset set
  ⇒ (('a, 'l, 'm) marked-lit list × ('a, 'l, 'm) marked-lit list) list ⇒ bool where
  all-decomposition-implies N S
    ⇔ (∀ (Ls, seen) ∈ set S. unmark-l Ls ∪ N ⊨ps unmark-l seen)

```

```

lemma all-decomposition-implies-empty[iff]:

```


all-decomposition-implies N \square **unfolding** *all-decomposition-implies-def* **by** *auto*

lemma *all-decomposition-implies-single*[*iff*]:
all-decomposition-implies N $[(Ls, \text{seen})]$
 $\longleftrightarrow \text{unmark-}l \ Ls \cup N \models_{ps} \text{unmark-}l \ \text{seen}$
unfolding *all-decomposition-implies-def* **by** *auto*

lemma *all-decomposition-implies-append*[*iff*]:
all-decomposition-implies N $(S @ S')$
 $\longleftrightarrow (\text{all-decomposition-implies } N \ S \wedge \text{all-decomposition-implies } N \ S')$
unfolding *all-decomposition-implies-def* **by** *auto*

lemma *all-decomposition-implies-cons-pair*[*iff*]:
all-decomposition-implies N $((Ls, \text{seen}) \# S')$
 $\longleftrightarrow (\text{all-decomposition-implies } N \ [(Ls, \text{seen})] \wedge \text{all-decomposition-implies } N \ S')$
unfolding *all-decomposition-implies-def* **by** *auto*

lemma *all-decomposition-implies-cons-single*[*iff*]:
all-decomposition-implies N $(l \# S') \longleftrightarrow$
 $(\text{unmark-}l \ (\text{fst } l) \cup N \models_{ps} \text{unmark-}l \ (\text{snd } l) \wedge$
 $\text{all-decomposition-implies } N \ S')$
unfolding *all-decomposition-implies-def* **by** *auto*

lemma *all-decomposition-implies-trail-is-implied*:
assumes *all-decomposition-implies* N $(\text{get-all-marked-decomposition } M)$
shows $N \cup \{\text{unmark } L \mid L. \text{is-marked } L \wedge L \in \text{set } M\}$
 $\models_{ps} \text{unmark } ' \bigcup (\text{set } ' \text{snd } ' \text{set } (\text{get-all-marked-decomposition } M))$
using *assms*
proof (*induct length (get-all-marked-decomposition M) arbitrary: M*)
case 0
then show ?*case* **by** *auto*
next
case (*Suc n*) **note** $IH = \text{this}(1)$ **and** $\text{length} = \text{this}(2)$ **and** $\text{decomp} = \text{this}(3)$
consider
 $(le1) \text{length } (\text{get-all-marked-decomposition } M) \leq 1$
 $\mid (gt1) \text{length } (\text{get-all-marked-decomposition } M) > 1$
by *arith*
then show ?*case*
proof *cases*
case *le1*
then obtain $a \ b$ **where** $g: \text{get-all-marked-decomposition } M = (a, b) \# \square$
by (*cases get-all-marked-decomposition M*) *auto*
moreover {
assume $a = \square$
then have ?*thesis* **using** *Suc.prem*s g **by** *auto*
}
moreover {
assume $l: \text{length } a = 1$ **and** $m: \text{is-marked } (\text{hd } a)$ **and** $hd: \text{hd } a \in \text{set } M$
then have $\text{unmark } (\text{hd } a) \in \{\text{unmark } L \mid L. \text{is-marked } L \wedge L \in \text{set } M\}$ **by** *auto*
then have $H: \text{unmark-}l \ a \cup N \subseteq N \cup \{\text{unmark } L \mid L. \text{is-marked } L \wedge L \in \text{set } M\}$
using l **by** (*cases a*) *auto*
have $f1: \text{unmark-}l \ a \cup N \models_{ps} \text{unmark-}l \ b$
using *decomp* **unfolding** *all-decomposition-implies-def* g **by** *simp*
have ?*thesis*
apply (*rule true-cls-cls-subset*) **using** $f1 \ H \ g$ **by** *auto*

```

}
ultimately show ?thesis
  using get-all-marked-decomposition-length-1-fst-empty-or-length-1 by blast
next
case gt1
then obtain Ls0 seen0 M' where
  Ls0: get-all-marked-decomposition M = (Ls0, seen0) # get-all-marked-decomposition M' and
  length': length (get-all-marked-decomposition M') = n and
  M'-in-M: set M' ⊆ set M
  using length by (induct M rule: marked-lit-list-induct) (auto simp: subset-insertI2)
let ?d = ⋃ (set 'snd ' set (get-all-marked-decomposition M'))
let ?unM = {unmark L | L. is-marked L ∧ L ∈ set M}
let ?unM' = {unmark L | L. is-marked L ∧ L ∈ set M'}
{
  assume n = 0
  then have get-all-marked-decomposition M' = [] using length' by auto
  then have ?thesis using Suc.premis unfolding all-decomposition-implies-def Ls0 by auto
}
moreover {
  assume n: n > 0
  then obtain Ls1 seen1 l where
    Ls1: get-all-marked-decomposition M' = (Ls1, seen1) # l
    using length' by (induct M' rule: marked-lit-list-induct) auto

  have all-decomposition-implies N (get-all-marked-decomposition M')
    using decomp unfolding Ls0 by auto
  then have N: N ∪ ?unM' ⊨ps unmark-s ?d
    using IH length' by auto
  have l: N ∪ ?unM' ⊆ N ∪ ?unM
    using M'-in-M by auto
  from true-clss-clss-subset[OF this N]
  have ΨN: N ∪ ?unM ⊨ps unmark-s ?d by auto
  have is-marked (hd Ls0) and LS: tl Ls0 = seen1 @ Ls1
    using get-all-marked-decomposition-hd-hd[of M] unfolding Ls0 Ls1 by auto

  have LSM: seen1 @ Ls1 = M' using get-all-marked-decomposition-decomp[of M'] Ls1 by auto
  have M': set M' = ?d ∪ {L | L. is-marked L ∧ L ∈ set M'}
    using get-all-marked-decomposition-snd-union by auto

  {
    assume Ls0 ≠ []
    then have hd Ls0 ∈ set M
      using get-all-marked-decomposition-fst-empty-or-hd-in-M Ls0 by blast
    then have N ∪ ?unM ⊨p unmark (hd Ls0)
      using ⟨is-marked (hd Ls0)⟩ by (metis (mono-tags, lifting) UnCI mem-Collect-eq
        true-clss-clss-in)
  } note hd-Ls0 = this

  have l: unmark ' (?d ∪ {L | L. is-marked L ∧ L ∈ set M'}) = unmark-s ?d ∪ ?unM'
    by auto
  have N ∪ ?unM' ⊨ps unmark ' (?d ∪ {L | L. is-marked L ∧ L ∈ set M'})
    unfolding l using N by (auto simp: all-in-true-clss-clss)
  then have t: N ∪ ?unM' ⊨ps unmark-l (tl Ls0)
    using M' unfolding LS LSM by auto
  then have N ∪ ?unM ⊨ps unmark-l (tl Ls0)

```

```

    using  $M'$ -in- $M$  true-clss-clss-subset[ $OF - t$ , of  $N \cup ?unM$ ] by auto
  then have  $N \cup ?unM \models_{ps} unmark-l Ls0$ 
    using  $hd-Ls0$  by (cases  $Ls0$ ) auto

  moreover have  $unmark-l Ls0 \cup N \models_{ps} unmark-l seen0$ 
    using decomp unfolding  $Ls0$  by simp
  moreover have  $\bigwedge M Ma. (M::'a \text{ literal multiset set}) \cup Ma \models_{ps} M$ 
    by (simp add: all-in-true-clss-clss)
  ultimately have  $\Psi: N \cup ?unM \models_{ps} unmark-l seen0$ 
    by (meson true-clss-clss-left-right true-clss-clss-union-and true-clss-clss-union-l-r)

  moreover have  $unmark \text{ ' (set seen0 } \cup ?d) = unmark-l seen0 \cup unmark-s ?d$ 
    by auto
  ultimately have  $?thesis$  using  $\Psi N$  unfolding  $Ls0$  by simp
}
ultimately show  $?thesis$  by auto
qed
qed

lemma all-decomposition-implies-propagated-lits-are-implied:
  assumes all-decomposition-implies  $N$  (get-all-marked-decomposition  $M$ )
  shows  $N \cup \{unmark L \mid L. \text{ is-marked } L \wedge L \in \text{set } M\} \models_{ps} unmark-l M$ 
    (is  $?I \models_{ps} ?A$ )
proof -
  have  $?I \models_{ps} unmark-s \{L \mid L. \text{ is-marked } L \wedge L \in \text{set } M\}$ 
    by (auto intro: all-in-true-clss-clss)
  moreover have  $?I \models_{ps} unmark \text{ ' } \bigcup (\text{set ' snd ' set (get-all-marked-decomposition } M))$ 
    using all-decomposition-implies-trail-is-implied assms by blast
  ultimately have  $N \cup \{unmark m \mid m. \text{ is-marked } m \wedge m \in \text{set } M\}$ 
     $\models_{ps} unmark \text{ ' } \bigcup (\text{set ' snd ' set (get-all-marked-decomposition } M))$ 
     $\cup unmark \text{ ' } \{m \mid m. \text{ is-marked } m \wedge m \in \text{set } M\}$ 
    by blast
  then show  $?thesis$ 
    by (metis (no-types) get-all-marked-decomposition-snd-union[of M] image-Un)
qed

lemma all-decomposition-implies-insert-single:
  all-decomposition-implies  $N M \implies \text{all-decomposition-implies (insert } C N) M$ 
  unfolding all-decomposition-implies-def by auto

```

14.4 Negation of Clauses

definition $CNot :: 'v \text{ clause} \Rightarrow 'v \text{ clauses}$ **where**
 $CNot \psi = \{ \{ \# - L \# \} \mid L. L \in \# \psi \}$

lemma *in-CNot-uminus*[*iff*]:
 shows $\{ \# L \# \} \in CNot \psi \longleftrightarrow -L \in \# \psi$
unfolding $CNot-def$ by *force*

lemma
 shows
CNot-singleton[*simp*]: $CNot \{ \# L \# \} = \{ \{ \# - L \# \} \}$ **and**
CNot-empty[*simp*]: $CNot \{ \# \} = \{ \}$ **and**
CNot-plus[*simp*]: $CNot (A + B) = CNot A \cup CNot B$
unfolding $CNot-def$ by *auto*

lemma *CNot-eq-empty[iff]*:
 $CNot\ D = \{\} \longleftrightarrow D = \{\#\}$
unfolding *CNot-def* **by** (*auto simp add: multiset-eqI*)

lemma *in-CNot-implies-uminus*:
assumes $L \in\# D$ **and** $M \models_{as} CNot\ D$
shows $M \models_a \{\#-L\# \}$ **and** $-L \in lits-of-l\ M$
using *assms* **by** (*auto simp: true-annots-def true-annot-def CNot-def*)

lemma *CNot-remdups-mset[simp]*:
 $CNot\ (remdups-mset\ A) = CNot\ A$
unfolding *CNot-def* **by** *auto*

lemma *Ball-CNot-Ball-mset[simp]* :
 $(\forall x \in CNot\ D. P\ x) \longleftrightarrow (\forall L \in\# D. P\ \{\#-L\# \})$
unfolding *CNot-def* **by** *auto*

lemma *consistent-CNot-not*:
assumes *consistent-interp I*
shows $I \models_s CNot\ \varphi \implies \neg I \models \varphi$
using *assms* **unfolding** *consistent-interp-def true-clss-def true-clss-def* **by** *auto*

lemma *total-not-true-clss-true-clss-CNot*:
assumes *total-over-m I {φ}* **and** $\neg I \models \varphi$
shows $I \models_s CNot\ \varphi$
using *assms* **unfolding** *total-over-m-def total-over-set-def true-clss-def true-clss-def CNot-def*
apply *clarify*
by (*rename-tac x L, case-tac L*) (*force intro: pos-lit-in-atms-of neg-lit-in-atms-of*)**+**

lemma *total-not-CNot*:
assumes *total-over-m I {φ}* **and** $\neg I \models_s CNot\ \varphi$
shows $I \models \varphi$
using *assms* *total-not-true-clss-true-clss-CNot* **by** *auto*

lemma *atms-of-ms-CNot-atms-of[simp]*:
 $atms-of-ms\ (CNot\ C) = atms-of\ C$
unfolding *atms-of-ms-def atms-of-def CNot-def* **by** *fastforce*

lemma *true-clss-clss-contradiction-true-clss-clss-false*:
 $C \in D \implies D \models_{ps} CNot\ C \implies D \models_p \{\#\}$
unfolding *true-clss-clss-def true-clss-clss-def total-over-m-def*
by (*metis Un-commute atms-of-empty atms-of-ms-CNot-atms-of atms-of-ms-insert atms-of-ms-union*
consistent-CNot-not insert-absorb sup-bot.left-neutral true-clss-def)

lemma *true-annots-CNot-all-atms-defined*:
assumes $M \models_{as} CNot\ T$ **and** $a1: L \in\# T$
shows $atm-of\ L \in atm-of\ 'lits-of-l\ M$
by (*metis assms atm-of-uminus image-eqI in-CNot-implies-uminus(1) true-annot-singleton*)

lemma *true-annots-CNot-all-uminus-atms-defined*:
assumes $M \models_{as} CNot\ T$ **and** $a1: -L \in\# T$
shows $atm-of\ L \in atm-of\ 'lits-of-l\ M$
by (*metis assms atm-of-uminus image-eqI in-CNot-implies-uminus(1) true-annot-singleton*)

lemma *true-clss-clss-false-left-right*:

```

assumes  $\{\{ \#L\#\} \cup B \models_p \{\#\}$ 
shows  $B \models_{ps} CNot \{\#L\#\}$ 
unfolding true-clss-clss-def true-clss-clss-def
proof (intro allI impI)
  fix  $I$ 
  assume
     $tot: total-over-m\ I\ (B \cup CNot \{\#L\#\})$  and
     $cons: consistent-interp\ I$  and
     $I: I \models_s B$ 
  have  $total-over-m\ I\ (\{\{ \#L\#\} \cup B)$  using  $tot$  by auto
  then have  $\neg I \models_s insert\ \{\#L\#\}\ B$ 
    using  $assms\ cons$  unfolding true-clss-clss-def by simp
  then show  $I \models_s CNot \{\#L\#\}$ 
    using  $tot\ I$  by (cases L) auto
qed

```

lemma *true-annots-true-clss-def-iff-negation-in-model*:
 $M \models_{as} CNot\ C \longleftrightarrow (\forall L \in \# C. \neg L \in lits-of-l\ M)$
unfolding *CNot-def true-annots-true-clss true-clss-def* **by** *auto*

lemma *true-annot-CNot-diff*:
 $I \models_{as} CNot\ C \implies I \models_{as} CNot\ (C - C')$
by (*auto simp: true-annots-true-clss-def-iff-negation-in-model dest: in-diffD*)

lemma *consistent-CNot-not-tautology*:
 $consistent-interp\ M \implies M \models_s CNot\ D \implies \neg tautology\ D$
by (*metis atms-of-ms-CNot-atms-of consistent-CNot-not satisfiable-carac' satisfiable-def tautology-def total-over-m-def*)

lemma *atms-of-ms-CNot-atms-of-ms*: $atms-of-ms\ (CNot\ CC) = atms-of-ms\ \{CC\}$
by *simp*

lemma *total-over-m-CNot-toal-over-m[simp]*:
 $total-over-m\ I\ (CNot\ C) = total-over-set\ I\ (atms-of\ C)$
unfolding *total-over-m-def total-over-set-def* **by** *auto*

The following lemma is very useful when in the goal appears an axioms like $\neg L = K$: this lemma allows the simplifier to rewrite L.

lemma *uminus-lit-swap*: $\neg(a::'a\ literal) = i \longleftrightarrow a = \neg i$
by *auto*

lemma *true-clss-clss-plus-CNot*:
assumes $CC-L: A \models_p CC + \{\#L\#\}$
and $CNot-CC: A \models_{ps} CNot\ CC$
shows $A \models_p \{\#L\#\}$
unfolding *true-clss-clss-def true-clss-clss-def CNot-def total-over-m-def*
proof (*intro allI impI*)
fix I
assume
 $tot: total-over-set\ I\ (atms-of-ms\ (A \cup \{\{ \#L\#\}\}))$ **and**
 $cons: consistent-interp\ I$
 $I: I \models_s A$
let $?I = I \cup \{Pos\ P \mid P. P \in atms-of\ CC \wedge P \notin atm-of\ 'I\}$
have $cons': consistent-interp\ ?I$

using *cons unfolding consistent-interp-def*
 by (*auto simp: uminus-lit-swap atms-of-def rev-image-eqI*)
 have $I': ?I \models_s A$
 using *I true-clss-union-increase by blast*
 have *tot-CNot: total-over-m ?I (A \cup CNot CC)*
 using *tot atms-of-s-def by (fastforce simp: total-over-m-def total-over-set-def)*

 then have *tot-I-A-CC-L: total-over-m ?I (A \cup {CC + {#L#}})*
 using *tot unfolding total-over-m-def total-over-set-atm-of by auto*
 then have $?I \models CC + \{\#L\# \}$ using *CC-L cons' I' unfolding true-clss-clss-def by blast*
 moreover
 have $?I \models_s CNot CC$ using *CNot-CC cons' I' tot-CNot unfolding true-clss-clss-def by auto*
 then have $\neg A \models_p CC$
 by (*metis (no-types, lifting) I' atms-of-ms-CNot-atms-of-ms atms-of-ms-union cons'*
consistent-CNot-not tot-CNot total-over-m-def true-clss-clss-def)
 then have $\neg ?I \models CC$ using $\langle ?I \models_s CNot CC \rangle cons'$ *consistent-CNot-not by blast*
 ultimately have $?I \models \{\#L\# \}$ by *blast*
 then show $I \models \{\#L\# \}$
 by (*metis (no-types, lifting) atms-of-ms-union cons' consistent-CNot-not tot total-not-CNot*
total-over-m-def total-over-set-union true-clss-union-increase)
 qed

lemma *true-annots-CNot-lit-of-notin-skip:*
 assumes *LM: L # M \models_{as} CNot A* and *LA: lit-of L $\notin \# A$ \neg lit-of L $\notin \# A$*
 shows $M \models_{as} CNot A$
 using *LM unfolding true-annots-def Ball-def*
proof (*intro allI impI*)
 fix l
 assume $H: \forall x. x \in CNot A \longrightarrow L \# M \models_a x$ and $l: l \in CNot A$
 then have $L \# M \models_a l$ by *auto*
 then show $M \models_a l$ using *LA l by (cases L) (auto simp: CNot-def)*
 qed

lemma *true-clss-clss-union-false-true-clss-clss-cnot:*
 $A \cup \{B\} \models_{ps} \{\{\# \}\} \longleftrightarrow A \models_{ps} CNot B$
 using *total-not-CNot consistent-CNot-not unfolding total-over-m-def true-clss-clss-def*
 by *fastforce*

lemma *true-annot-remove-hd-if-notin-vars:*
 assumes $a \# M' \models_a D$ and *atm-of (lit-of a) \notin atms-of D*
 shows $M' \models_a D$
 using *assms true-clss-remove-hd-if-notin-vars unfolding true-annot-def by auto*

lemma *true-annot-remove-if-notin-vars:*
 assumes $M @ M' \models_a D$ and $\forall x \in \text{atms-of } D. x \notin \text{atm-of ' lits-of-l } M$
 shows $M' \models_a D$
 using *assms apply (induct M, simp)*
 using *true-annot-remove-hd-if-notin-vars by force+*

lemma *true-annots-remove-if-notin-vars:*
 assumes $M @ M' \models_{as} D$ and $\forall x \in \text{atms-of-ms } D. x \notin \text{atm-of ' lits-of-l } M$
 shows $M' \models_{as} D$ unfolding *true-annots-def*
 using *assms true-annot-remove-if-notin-vars[of M M']*
 unfolding *true-annots-def atms-of-ms-def by force*

lemma *all-variables-defined-not-imply-cnot*:
assumes
 $\forall s \in \text{atms-of-ms } \{B\}. s \in \text{atm-of } \text{' lits-of-l } A$ **and**
 $\neg A \models_a B$
shows $A \models_{as} CNot\ B$
unfolding *true-annot-def true-annots-def Ball-def CNot-def true-lit-def*
proof (*clarify, rule ccontr*)
fix L
assume $LB: L \in \# B$ **and** $\neg \text{ lits-of-l } A \models_l - L$
then have $\text{atm-of } L \in \text{atm-of } \text{' lits-of-l } A$
using *assms(1)* **by** (*simp add: atm-of-lit-in-atms-of lits-of-def*)
then have $L \in \text{ lits-of-l } A \vee -L \in \text{ lits-of-l } A$
using *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set* **by** *metis*
then have $L \in \text{ lits-of-l } A$ **using** $\langle \neg \text{ lits-of-l } A \models_l - L \rangle$ **by** *auto*
then show *False*
using LB *assms(2)* **unfolding** *true-annot-def true-lit-def true-cls-def Bex-def*
by *blast*
qed

lemma *CNot-union-mset[simp]*:
 $CNot\ (A \# \cup B) = CNot\ A \cup CNot\ B$
unfolding *CNot-def* **by** *auto*

14.5 Other

abbreviation *no-dup* $L \equiv \text{distinct } (\text{map } (\lambda l. \text{atm-of } (\text{lit-of } l))\ L)$

lemma *no-dup-rev[simp]*:
 $\text{no-dup } (\text{rev } M) \longleftrightarrow \text{no-dup } M$
by (*auto simp: rev-map[symmetric]*)

lemma *no-dup-length-eq-card-atm-of-lits-of-l*:
assumes *no-dup* M
shows $\text{length } M = \text{card } (\text{atm-of } \text{' lits-of-l } M)$
using *assms* **unfolding** *lits-of-def* **by** (*induct M*) (*auto simp add: image-image*)

lemma *distinct-consistent-interp*:
 $\text{no-dup } M \implies \text{consistent-interp } (\text{ lits-of-l } M)$
proof (*induct M*)
case *Nil*
show *?case* **by** *auto*
next
case (*Cons L M*)
then have $a1: \text{consistent-interp } (\text{ lits-of-l } M)$ **by** *auto*
have $a2: \text{atm-of } (\text{lit-of } L) \notin (\lambda l. \text{atm-of } (\text{lit-of } l))\ \text{' set } M$ **using** *Cons.prem*s **by** *auto*
have *undefined-lit* M (*lit-of L*)
using $a2$ *image-iff* **unfolding** *defined-lit-def* **by** *fastforce*
then show *?case*
using $a1$ **by** *simp*
qed

lemma *distinct-get-all-marked-decomposition-no-dup*:
assumes $(a, b) \in \text{set } (\text{get-all-marked-decomposition } M)$
and *no-dup* M
shows *no-dup* $(a @ b)$
using *assms* **by** *force*

lemma *true-annots-lit-of-notin-skip*:

assumes $L \# M \models_{as} CNot\ A$

and $\neg lit\text{-}of\ L \notin \# A$

and *no-dup* $(L \# M)$

shows $M \models_{as} CNot\ A$

proof –

have $\forall l \in \# A. \neg l \in lits\text{-}of\text{-}l\ (L \# M)$

using *assms*(1) *in-CNot-implies-uminus*(2) **by** *blast*

moreover

have $atm\text{-}of\ (lit\text{-}of\ L) \notin atm\text{-}of\ 'lits\text{-}of\text{-}l\ M$

using *assms*(3) **unfolding** *lits-of-def* **by** *force*

then have $\neg lit\text{-}of\ L \notin lits\text{-}of\text{-}l\ M$ **unfolding** *lits-of-def*

by (*metis* (*no-types*) *atm-of-uminus imageI*)

ultimately have $\forall l \in \# A. \neg l \in lits\text{-}of\text{-}l\ M$

using *assms*(2) **by** (*metis insert-iff list.simps*(15) *lits-of-insert uminus-of-uminus-id*)

then show *?thesis* **by** (*auto simp add: true-annots-def*)

qed

abbreviation *true-annots-mset* (**infix** \models_{asm} 50) **where**

$I \models_{asm} C \equiv I \models_{as} (set\text{-}mset\ C)$

abbreviation *true-clss-clss-m:: 'v clause multiset \Rightarrow 'v clause multiset \Rightarrow bool* (**infix** \models_{psm} 50)

where

$I \models_{psm} C \equiv set\text{-}mset\ I \models_{ps} (set\text{-}mset\ C)$

Analog of $\llbracket ?N \models_{ps} ?B; ?A \subseteq ?B \rrbracket \Longrightarrow ?N \models_{ps} ?A$

lemma *true-clss-clssm-subsetE*: $N \models_{psm} B \Longrightarrow A \subseteq \# B \Longrightarrow N \models_{psm} A$

using *set-mset-mono true-clss-clss-subsetE* **by** *blast*

abbreviation *true-clss-clss-m:: 'a clause multiset \Rightarrow 'a clause \Rightarrow bool* (**infix** \models_{pm} 50) **where**

$I \models_{pm} C \equiv set\text{-}mset\ I \models_p C$

abbreviation *distinct-mset-mset :: 'a multiset multiset \Rightarrow bool* **where**

distinct-mset-mset $\Sigma \equiv distinct\text{-}mset\text{-}set\ (set\text{-}mset\ \Sigma)$

abbreviation *all-decomposition-implies-m* **where**

all-decomposition-implies-m $A\ B \equiv all\text{-}decomposition\text{-}implies\ (set\text{-}mset\ A)\ B$

abbreviation *atms-of-mm :: 'a literal multiset multiset \Rightarrow 'a set* **where**

atms-of-mm $U \equiv atms\text{-}of\text{-}ms\ (set\text{-}mset\ U)$

Other definition using *Union-mset*

lemma *atms-of-mm* $U \equiv set\text{-}mset\ (\bigcup \# image\text{-}mset\ (image\text{-}mset\ atm\text{-}of)\ U)$

unfolding *atms-of-ms-def* **by** (*auto simp: atms-of-def*)

abbreviation *true-clss-m:: 'a interp \Rightarrow 'a clause multiset \Rightarrow bool* (**infix** \models_{sm} 50) **where**

$I \models_{sm} C \equiv I \models_s set\text{-}mset\ C$

abbreviation *true-clss-ext-m* (**infix** \models_{sextm} 49) **where**

$I \models_{sextm} C \equiv I \models_{sext}\ set\text{-}mset\ C$

end

theory *CDCL-Abstract-Clause-Representation*

imports *Main Partial-Clausal-Logic*

begin

type-synonym *'v clause* = *'v literal multiset*

type-synonym *'v clauses* = *'v clause multiset*

14.6 Abstract Clause Representation

We will abstract the representation of clause and clauses via two locales. We expect our representation to behave like multiset, but the internal representation can be done using list or whatever other representation.

We assume the following:

- there is an equivalent to adding and removing a literal and to taking the union of clauses.

locale *raw-cls* =

fixes

mset-cls:: *'cls* \Rightarrow *'v clause* **and**

insert-cls :: *'v literal* \Rightarrow *'cls* \Rightarrow *'cls* **and**

remove-lit :: *'v literal* \Rightarrow *'cls* \Rightarrow *'cls*

assumes

insert-cls[*simp*]: *mset-cls* (*insert-cls* *L C*) = *mset-cls* *C* + {*#L#*} **and**

remove-lit[*simp*]: *mset-cls* (*remove-lit* *L C*) = *remove1-mset* *L* (*mset-cls* *C*)

begin

end

locale *raw-ccls-union* =

fixes

mset-cls:: *'cls* \Rightarrow *'v clause* **and**

union-cls :: *'cls* \Rightarrow *'cls* \Rightarrow *'cls* **and**

insert-cls :: *'v literal* \Rightarrow *'cls* \Rightarrow *'cls* **and**

remove-lit :: *'v literal* \Rightarrow *'cls* \Rightarrow *'cls*

assumes

insert-ccls[*simp*]: *mset-cls* (*insert-cls* *L C*) = *mset-cls* *C* + {*#L#*} **and**

mset-ccls-union-cls[*simp*]: *mset-cls* (*union-cls* *C D*) = *mset-cls* *C* $\# \cup$ *mset-cls* *D* **and**

remove-clit[*simp*]: *mset-cls* (*remove-lit* *L C*) = *remove1-mset* *L* (*mset-cls* *C*)

begin

end

Instantiation of the previous locale, in an unnamed context to avoid polluting with simp rules

context

begin

interpretation *list-cls*: *raw-cls mset*

op $\#$ *remove1*

by *unfold-locales* (*auto simp: union-mset-list ex-mset*)

interpretation *cls-cls*: *raw-cls id*

$\lambda L C. C + \{ \#L\# \}$ *remove1-mset*

by *unfold-locales* (*auto simp: union-mset-list*)

interpretation *list-cls*: *raw-ccls-union mset*

union-mset-list

op $\#$ *remove1*

by *unfold-locales* (*auto simp: union-mset-list ex-mset*)

```

interpretation cls-cls: raw-ccls-union id
  op  $\# \cup \lambda L \ C. \ C + \{\#L\# \}$  remove1-mset
  by unfold-locales (auto simp: union-mset-list)
end

```

Over the abstract clauses, we have the following properties:

- We can insert a clause
- We can take the union (used only in proofs for the definition of *clauses*)
- there is an operator indicating whether the abstract clause is contained or not
- if a concrete clause is contained the abstract clauses, then there is an abstract clause

```

locale raw-clss =
  raw-cls mset-cls insert-cls remove-lit
for
  mset-cls:: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls +
fixes
  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss
assumes
  insert-clss[simp]: mset-clss (insert-clss L C) = mset-clss C +  $\{\#mset-cls \ L\#\}$  and
  union-clss[simp]: mset-clss (union-clss C D) = mset-clss C + mset-clss D and
  mset-clss-union-clss[simp]: mset-clss (insert-clss C' D) =  $\{\#mset-cls \ C'\#\} + mset-clss \ D$  and
  in-clss-mset-clss[dest]: in-clss a C  $\Longrightarrow$  mset-cls a  $\in \#$  mset-clss C and
  in-mset-clss-exists-preimage: b  $\in \#$  mset-clss C  $\Longrightarrow$   $\exists b'. \text{in-clss } b' \ C \wedge mset-cls \ b' = b$  and
  remove-from-clss-mset-clss[simp]:
    mset-clss (remove-from-clss a C) = mset-clss C -  $\{\#mset-cls \ a\#\}$  and
  in-clss-union-clss[simp]:
    in-clss a (union-clss C D)  $\longleftrightarrow$  in-clss a C  $\vee$  in-clss a D
begin

end

```

```

experiment
begin
  fun remove-first where
    remove-first - [] = [] |
    remove-first C (C' # L) = (if mset C = mset C' then L else C' # remove-first C L)

```

```

lemma mset-map-mset-remove-first:
  mset (map mset (remove-first a C)) = remove1-mset (mset a) (mset (map mset C))
by (induction C) (auto simp: ac-simps remove1-mset-single-add)

```

```

interpretation clss-clss: raw-clss id  $\lambda L \ C. \ C + \{\#L\# \}$  remove1-mset
  id op + op  $\in \# \lambda L \ C. \ C + \{\#L\# \}$  remove1-mset
by unfold-locales (auto simp: ac-simps)

```

```

interpretation list-clss: raw-clss mset

```

```

    op # remove1 λL. mset (map mset L) op @ λL C. L ∈ set C op #
    remove-first
  by unfold-locales (auto simp: ac-simps union-mset-list mset-map-mset-remove-first ex-mset)
end

```

```

end
theory CDCL-WNOT-Measure
imports Main
begin

```

15 Measure

This measure show the termination of the core of CDCL: each step improves the number of literals we know for sure.

This measure can also be seen as the increasing lexicographic order: it is an order on bounded sequences, when each element is bounded. The proof involves a measure like the one defined here (the same?).

definition $\mu_C :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat list} \Rightarrow \text{nat}$ **where**
 $\mu_C s b M \equiv (\sum i=0..<\text{length } M. M!i * b^\wedge (s + i - \text{length } M))$

lemma $\mu_C\text{-nil}[simp]$:
 $\mu_C s b [] = 0$
unfolding $\mu_C\text{-def}$ **by** *auto*

lemma $\mu_C\text{-single}[simp]$:
 $\mu_C s b [L] = L * b^\wedge (s - \text{Suc } 0)$
unfolding $\mu_C\text{-def}$ **by** *auto*

lemma *set-sum-atLeastLessThan-add*:
 $(\sum i=k..<k+(b::\text{nat}). f i) = (\sum i=0..<b. f (k + i))$
by (*induction b*) *auto*

lemma *set-sum-atLeastLessThan-Suc*:
 $(\sum i=1..<\text{Suc } j. f i) = (\sum i=0..<j. f (\text{Suc } i))$
using *set-sum-atLeastLessThan-add*[*of - 1 j*] **by** *force*

lemma $\mu_C\text{-cons}$:
 $\mu_C s b (L \# M) = L * b^\wedge (s - 1 - \text{length } M) + \mu_C s b M$

proof –

have $\mu_C s b (L \# M) = (\sum i=0..<\text{length } (L\#M). (L\#M)!i * b^\wedge (s + i - \text{length } (L\#M)))$
unfolding $\mu_C\text{-def}$ **by** *blast*

also have $\dots = (\sum i=0..<1. (L\#M)!i * b^\wedge (s + i - \text{length } (L\#M)))$
 $+ (\sum i=1..<\text{length } (L\#M). (L\#M)!i * b^\wedge (s + i - \text{length } (L\#M)))$

by (*rule setsum-add-nat-ivl[symmetric]*) *simp-all*

finally have $\mu_C s b (L \# M) = L * b^\wedge (s - 1 - \text{length } M)$
 $+ (\sum i=1..<\text{length } (L\#M). (L\#M)!i * b^\wedge (s + i - \text{length } (L\#M)))$

by *auto*

moreover {

have $(\sum i=1..<\text{length } (L\#M). (L\#M)!i * b^\wedge (s + i - \text{length } (L\#M))) =$
 $(\sum i=0..<\text{length } (M). (L\#M)!(\text{Suc } i) * b^\wedge (s + (\text{Suc } i) - \text{length } (L\#M)))$

unfolding *length-Cons set-sum-atLeastLessThan-Suc* **by** *blast*

also have $\dots = (\sum i=0..<\text{length } (M). M!i * b^\wedge (s + i - \text{length } M))$

by *auto*

finally have $(\sum i=1..<\text{length } (L\#M). (L\#M)!i * b^\wedge (s + i - \text{length } (L\#M))) = \mu_C s b M$

unfolding $\mu_C\text{-def}$.
 }
 ultimately show ?thesis by presburger
 qed

lemma $\mu_C\text{-append}$:

assumes $s \geq \text{length } (M @ M')$
 shows $\mu_C s b (M @ M') = \mu_C (s - \text{length } M') b M + \mu_C s b M'$
 proof -
 have $\mu_C s b (M @ M') = (\sum i=0..<\text{length } (M @ M'). (M @ M')!i * b^\wedge (s + i - \text{length } (M @ M')))$
 unfolding $\mu_C\text{-def}$ by blast
 moreover then have ... = $(\sum i=0..<\text{length } M. (M @ M')!i * b^\wedge (s + i - \text{length } (M @ M')))$
 + $(\sum i=\text{length } M..<\text{length } (M @ M'). (M @ M')!i * b^\wedge (s + i - \text{length } (M @ M')))$
 by (auto intro!: setsum-add-nat-ivl[symmetric])
 moreover
 have $\forall i \in \{0..<\text{length } M\}. (M @ M')!i * b^\wedge (s + i - \text{length } (M @ M')) = M ! i * b^\wedge (s - \text{length } M' + i - \text{length } M)$
 using $\langle s \geq \text{length } (M @ M') \rangle$ by (auto simp add: nth-append ac-simps)
 then have $\mu_C (s - \text{length } M') b M = (\sum i=0..<\text{length } M. (M @ M')!i * b^\wedge (s + i - \text{length } (M @ M')))$
 (M @ M'))
 unfolding $\mu_C\text{-def}$ by auto
 ultimately have $\mu_C s b (M @ M') = \mu_C (s - \text{length } M') b M$
 + $(\sum i=\text{length } M..<\text{length } (M @ M'). (M @ M')!i * b^\wedge (s + i - \text{length } (M @ M')))$
 by auto
 moreover {
 have $(\sum i=\text{length } M..<\text{length } (M @ M'). (M @ M')!i * b^\wedge (s + i - \text{length } (M @ M')) =$
 $(\sum i=0..<\text{length } M'. M'!i * b^\wedge (s + i - \text{length } M'))$
 unfolding length-append set-sum-atLeastLessThan-add by auto
 then have $(\sum i=\text{length } M..<\text{length } (M @ M'). (M @ M')!i * b^\wedge (s + i - \text{length } (M @ M')) = \mu_C s b$
 M'
 unfolding $\mu_C\text{-def}$.
 }
 ultimately show ?thesis by presburger
 qed

lemma $\mu_C\text{-cons-non-empty-inf}$:

assumes $M\text{-ge-1}: \forall i \in \text{set } M. i \geq 1$ and $M: M \neq []$
 shows $\mu_C s b M \geq b^\wedge (s - \text{length } M)$
 using assms by (cases M) (auto simp: mult-eq-if $\mu_C\text{-cons}$)

Copy of `~~/src/HOL/ex/NatSum.thy` (but generalized to $0 \leq k$)

lemma sum-of-powers: $0 \leq k \implies (k - 1) * (\sum i=0..<n. k^\wedge i) = k^\wedge n - (1::nat)$
 apply (cases $k = 0$)
 apply (cases n ; simp)
 by (induct n) (auto simp: Nat.nat-distrib)

In the degenerated cases, we only have the large inequality holds. In the other cases, the following strict inequality holds:

lemma $\mu_C\text{-bounded-non-degenerated}$:

fixes $b :: nat$
 assumes
 $b > 0$ and
 $M \neq []$ and
 $M\text{-le}: \forall i < \text{length } M. M!i < b$ and
 $s \geq \text{length } M$

```

shows  $\mu_C \ s \ b \ M < b^\wedge s$ 
proof -
  consider (b1)  $b = 1 \mid (b) \ b > 1$  using  $\langle b > 0 \rangle$  by (cases b) auto
  then show ?thesis
  proof cases
    case b1
    then have  $\forall i < \text{length } M. M!i = 0$  using M-le by auto
    then have  $\mu_C \ s \ b \ M = 0$  unfolding  $\mu_C$ -def by auto
    then show ?thesis using  $\langle b > 0 \rangle$  by auto
  next
  case b
  have  $\forall i \in \{0..<\text{length } M\}. M!i * b^\wedge (s+i - \text{length } M) \leq (b-1) * b^\wedge (s+i - \text{length } M)$ 
    using M-le  $\langle b > 1 \rangle$  by auto
  then have  $\mu_C \ s \ b \ M \leq (\sum i=0..<\text{length } M. (b-1) * b^\wedge (s+i - \text{length } M))$ 
    using  $\langle M \neq [] \rangle \langle b > 0 \rangle$  unfolding  $\mu_C$ -def by (auto intro: setsum-mono)
  also
  have  $\forall i \in \{0..<\text{length } M\}. (b-1) * b^\wedge (s+i - \text{length } M) = (b-1) * b^\wedge i * b^\wedge (s - \text{length } M)$ 
    by (metis Nat.add-diff-assoc2 add.commute assms(4) mult.assoc power-add)
  then have  $(\sum i=0..<\text{length } M. (b-1) * b^\wedge (s+i - \text{length } M))$ 
    =  $(\sum i=0..<\text{length } M. (b-1) * b^\wedge i * b^\wedge (s - \text{length } M))$ 
    by (auto simp add: ac-simps)
  also have  $\dots = (\sum i=0..<\text{length } M. b^\wedge i) * b^\wedge (s - \text{length } M) * (b-1)$ 
    by (simp add: setsum-left-distrib setsum-right-distrib ac-simps)
  finally have  $\mu_C \ s \ b \ M \leq (\sum i=0..<\text{length } M. b^\wedge i) * (b-1) * b^\wedge (s - \text{length } M)$ 
    by (simp add: ac-simps)

  also
  have  $(\sum i=0..<\text{length } M. b^\wedge i) * (b-1) = b^\wedge (\text{length } M) - 1$ 
    using sum-of-powers[of b length M]  $\langle b > 1 \rangle$ 
    by (auto simp add: ac-simps)
  finally have  $\mu_C \ s \ b \ M \leq (b^\wedge (\text{length } M) - 1) * b^\wedge (s - \text{length } M)$ 
    by auto
  also have  $\dots < b^\wedge (\text{length } M) * b^\wedge (s - \text{length } M)$ 
    using  $\langle b > 1 \rangle$  by auto
  also have  $\dots = b^\wedge s$ 
    by (metis assms(4) le-add-diff-inverse power-add)
  finally show ?thesis unfolding  $\mu_C$ -def by (auto simp add: ac-simps)
qed
qed

```

In the degenerate case $b = (0::'a)$, the list M is empty (since the list cannot contain any element).

lemma μ_C -bounded:

fixes $b :: \text{nat}$

assumes

$M\text{-le}: \forall i < \text{length } M. M!i < b$ and

$s \geq \text{length } M$

$b > 0$

shows $\mu_C \ s \ b \ M < b^\wedge s$

proof -

consider (M0) $M = [] \mid (M) \ b > 0$ and $M \neq []$

using M-le by (cases b, cases M) auto

then show ?thesis

proof cases

case M0

```

    then show ?thesis using M-le <b > 0> by auto
  next
    case M
    show ?thesis using  $\mu_C$ -bounded-non-degenerated[OF M assms(1,2)] by arith
  qed
qed

```

When $b = 0$, we cannot show that the measure is empty, since $0^0 = 1$.

```

lemma  $\mu_C$ -base-0:
  assumes length M  $\leq$  s
  shows  $\mu_C$  s 0 M  $\leq$  M!0
proof -
  {
    assume s = length M
    moreover {
      fix n
      have  $(\sum_{i=0..<n.} M ! i * (0::nat) ^ i) \leq M ! 0$ 
      apply (induction n rule: nat-induct)
      by simp (rename-tac n, case-tac n, auto)
    }
    ultimately have ?thesis unfolding  $\mu_C$ -def by auto
  }
  moreover
  {
    assume length M < s
    then have  $\mu_C$  s 0 M = 0 unfolding  $\mu_C$ -def by auto
    ultimately show ?thesis using assms unfolding  $\mu_C$ -def by linarith
  }
qed

```

```

end
theory CDCL-NOT
imports CDCL-Abstract-Clause-Representation List-More Wellfounded-More CDCL-WNOT-Measure
Partial-Annotated-Clausal-Logic
begin

```

16 NOT's CDCL

16.1 Auxiliary Lemmas and Measure

```

lemma no-dup-cannot-not-lit-and-uminus:
  no-dup M  $\implies$   $\neg$  lit-of xa = lit-of x  $\implies$   $x \in \text{set } M \implies xa \notin \text{set } M$ 
  by (metis atm-of-uminus distinct-map inj-on-eq-iff uminus-not-id')

```

```

lemma atms-of-ms-single-atm-of[simp]:
  atms-of-ms {unmark L | L. P L} = atm-of ' {lit-of L | L. P L}
  unfolding atms-of-ms-def by force

```

```

lemma atms-of-uminus-lit-atm-of-lit-of:
  atms-of {#  $\neg$ lit-of x. x  $\in$  # A#} = atm-of ' (lit-of ' (set-mset A))
  unfolding atms-of-def by (auto simp add: Fun.image-comp)

```

```

lemma atms-of-ms-single-image-atm-of-lit-of:
  atms-of-ms (unmark-s A) = atm-of ' (lit-of ' A)
  unfolding atms-of-ms-def by auto

```

16.2 Initial definitions

16.2.1 The state

We define here an abstraction over operation on the state we are manipulating.

```

locale dpll-state-ops =
  raw-clss mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
for
  mset-cls:: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss +
fixes
  trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
  raw-clauses :: 'st  $\Rightarrow$  'clss and
  prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
  tl-trail :: 'st  $\Rightarrow$  'st and
  add-clNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  remove-clNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st
begin

notation insert-cls (infix !++ 50)

notation in-clss (infix ! $\in$ ! 50)
notation union-clss (infix  $\oplus$  50)
notation insert-clss (infix !++! 50)

abbreviation clausesNOT where
clausesNOT S  $\equiv$  mset-clss (raw-clauses S)

end

locale dpll-state =
  dpll-state-ops mset-cls insert-cls remove-lit — related to each clause
  mset-clss union-clss in-clss insert-clss remove-from-clss — related to the clauses

  trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT — related to the state
for
  mset-cls:: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
  raw-clauses :: 'st  $\Rightarrow$  'clss and
  prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
  tl-trail :: 'st  $\Rightarrow$  'st and
  add-clNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and

```

$remove_cls_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st +$
assumes
 $trail_prepend_trail[simp]:$
 $\bigwedge st L. \text{undefined-lit } (trail\ st) \ (lit\text{-of } L) \implies trail\ (prepend_trail\ L\ st) = L \# trail\ st$
and
 $tl_trail[simp]: trail\ (tl_trail\ S) = tl\ (trail\ S)$ **and**
 $trail_add_cls_{NOT}[simp]: \bigwedge st C. no_dup\ (trail\ st) \implies trail\ (add_cls_{NOT}\ C\ st) = trail\ st$ **and**
 $trail_remove_cls_{NOT}[simp]: \bigwedge st C. trail\ (remove_cls_{NOT}\ C\ st) = trail\ st$ **and**

 $clauses_prepend_trail[simp]:$
 $\bigwedge st L. \text{undefined-lit } (trail\ st) \ (lit\text{-of } L) \implies$
 $clauses_{NOT}\ (prepend_trail\ L\ st) = clauses_{NOT}\ st$
and
 $clauses_tl_trail[simp]: \bigwedge st. clauses_{NOT}\ (tl_trail\ st) = clauses_{NOT}\ st$ **and**
 $clauses_add_cls_{NOT}[simp]:$
 $\bigwedge st C. no_dup\ (trail\ st) \implies clauses_{NOT}\ (add_cls_{NOT}\ C\ st) = \{\#mset_cls\ C\} + clauses_{NOT}\ st$
and
 $clauses_remove_cls_{NOT}[simp]:$
 $\bigwedge st C. clauses_{NOT}\ (remove_cls_{NOT}\ C\ st) = removeAll_mset\ (mset_cls\ C)\ (clauses_{NOT}\ st)$
begin

function $reduce_trail_to_{NOT} :: 'a\ list \Rightarrow 'st \Rightarrow 'st$ **where**
 $reduce_trail_to_{NOT}\ F\ S =$
 $(if\ length\ (trail\ S) = length\ F \vee trail\ S = []\ then\ S\ else\ reduce_trail_to_{NOT}\ F\ (tl_trail\ S))$
by $fast+$
termination by $(relation\ measure\ (\lambda(-, S). length\ (trail\ S)))\ auto$
declare $reduce_trail_to_{NOT}.simps[simp\ del]$

lemma
shows
 $reduce_trail_to_{NOT}\ nil[simp]: trail\ S = [] \implies reduce_trail_to_{NOT}\ F\ S = S$ **and**
 $reduce_trail_to_{NOT}\ eq_length[simp]: length\ (trail\ S) = length\ F \implies reduce_trail_to_{NOT}\ F\ S = S$
by $(auto\ simp: reduce_trail_to_{NOT}.simps)$

lemma $reduce_trail_to_{NOT}\ length_ne[simp]:$
 $length\ (trail\ S) \neq length\ F \implies trail\ S \neq [] \implies$
 $reduce_trail_to_{NOT}\ F\ S = reduce_trail_to_{NOT}\ F\ (tl_trail\ S)$
by $(auto\ simp: reduce_trail_to_{NOT}.simps)$

lemma $trail_reduce_trail_to_{NOT}\ length_le:$
assumes $length\ F > length\ (trail\ S)$
shows $trail\ (reduce_trail_to_{NOT}\ F\ S) = []$
using $assms$ **by** $(induction\ F\ S\ rule: reduce_trail_to_{NOT}.induct)$
 $(simp\ add: less_imp_diff_less\ reduce_trail_to_{NOT}.simps)$

lemma $trail_reduce_trail_to_{NOT}\ nil[simp]:$
 $trail\ (reduce_trail_to_{NOT}\ []\ S) = []$
by $(induction\ []\ S\ rule: reduce_trail_to_{NOT}.induct)$
 $(simp\ add: less_imp_diff_less\ reduce_trail_to_{NOT}.simps)$

lemma $clauses_reduce_trail_to_{NOT}\ nil:$
 $clauses_{NOT}\ (reduce_trail_to_{NOT}\ []\ S) = clauses_{NOT}\ S$
by $(induction\ []\ S\ rule: reduce_trail_to_{NOT}.induct)$
 $(simp\ add: less_imp_diff_less\ reduce_trail_to_{NOT}.simps)$

lemma *trail-reduce-trail-to_{NOT}-drop*:
trail (*reduce-trail-to_{NOT}* *F S*) =
 (if *length* (*trail S*) ≥ *length F*
 then *drop* (*length* (*trail S*) − *length F*) (*trail S*)
 else [])
apply (*induction F S rule: reduce-trail-to_{NOT}.induct*)
apply (*rename-tac F S, case-tac trail S*)
apply *auto*[]
apply (*rename-tac list, case-tac Suc* (*length list*) > *length F*)
prefer 2 **apply** *simp*
apply (*subgoal-tac Suc* (*length list*) − *length F* = *Suc* (*length list* − *length F*))
apply *simp*
apply *simp*
done

lemma *reduce-trail-to_{NOT}-skip-beginning*:
assumes *trail S = F' @ F*
shows *trail* (*reduce-trail-to_{NOT}* *F S*) = *F*
using *assms* **by** (*auto simp: trail-reduce-trail-to_{NOT}-drop*)

lemma *reduce-trail-to_{NOT}-clauses[*simp*]*:
clauses_{NOT} (*reduce-trail-to_{NOT}* *F S*) = *clauses_{NOT}* *S*
by (*induction F S rule: reduce-trail-to_{NOT}.induct*)
(*simp add: less-imp-diff-less reduce-trail-to_{NOT}.simps*)

abbreviation *trail-weight* **where**
trail-weight S ≡ *map* ((λ*l*. 1 + *length l*) o *snd*) (*get-all-marked-decomposition* (*trail S*))

definition *state-eq_{NOT}* :: '*st* ⇒ '*st* ⇒ *bool* (**infix** ~ 50) **where**
S ~ *T* ⟷ *trail S* = *trail T* ∧ *clauses_{NOT}* *S* = *clauses_{NOT}* *T*

lemma *state-eq_{NOT}-ref[*simp*]*:
S ~ *S*
unfolding *state-eq_{NOT}-def* **by** *auto*

lemma *state-eq_{NOT}-sym*:
S ~ *T* ⟷ *T* ~ *S*
unfolding *state-eq_{NOT}-def* **by** *auto*

lemma *state-eq_{NOT}-trans*:
S ~ *T* ⟹ *T* ~ *U* ⟹ *S* ~ *U*
unfolding *state-eq_{NOT}-def* **by** *auto*

lemma
shows
state-eq_{NOT}-trail: *S* ~ *T* ⟹ *trail S* = *trail T* **and**
state-eq_{NOT}-clauses: *S* ~ *T* ⟹ *clauses_{NOT}* *S* = *clauses_{NOT}* *T*
unfolding *state-eq_{NOT}-def* **by** *auto*

lemmas *state-simp_{NOT}[*simp*]* = *state-eq_{NOT}-trail state-eq_{NOT}-clauses*

lemma *trail-eq-reduce-trail-to_{NOT}-eq*:
trail S = *trail T* ⟹ *trail* (*reduce-trail-to_{NOT}* *F S*) = *trail* (*reduce-trail-to_{NOT}* *F T*)
apply (*induction F S arbitrary: T rule: reduce-trail-to_{NOT}.induct*)
by (*metis tl-trail reduce-trail-to_{NOT}-eq-length reduce-trail-to_{NOT}-length-ne reduce-trail-to_{NOT}-nil*)

lemma *reduce-trail-to_{NOT}-state-eq_{NOT}-compatible*:
assumes *ST*: $S \sim T$
shows *reduce-trail-to_{NOT}* $F S \sim \text{reduce-trail-to}_{NOT} F T$
proof –
have *clauses_{NOT}* (*reduce-trail-to_{NOT}* $F S$) = *clauses_{NOT}* (*reduce-trail-to_{NOT}* $F T$)
using *ST* **by** *auto*
moreover have *trail* (*reduce-trail-to_{NOT}* $F S$) = *trail* (*reduce-trail-to_{NOT}* $F T$)
using *trail-eq-reduce-trail-to_{NOT}-eq*[*of S T F*] *ST* **by** *auto*
ultimately show *?thesis* **by** (*auto simp del: state-simp_{NOT} simp: state-eq_{NOT}-def*)
qed

lemma *trail-reduce-trail-to_{NOT}-add-cl_{NOT}*[*simp*]:
no-dup (*trail* S) \implies
trail (*reduce-trail-to_{NOT}* F (*add-cl_{NOT}* $C S$)) = *trail* (*reduce-trail-to_{NOT}* $F S$)
by (*rule trail-eq-reduce-trail-to_{NOT}-eq*) *simp*

lemma *reduce-trail-to_{NOT}-trail-tl-trail-decomp*[*simp*]:
trail $S = F' @ \text{Marked } K () \# F \implies$
trail (*reduce-trail-to_{NOT}* F (*tl-trail* S)) = F
apply (*rule reduce-trail-to_{NOT}-skip-beginning*[*of - tl (F' @ Marked K () # [])*])
by (*cases F'*) (*auto simp add:tl-append reduce-trail-to_{NOT}-skip-beginning*)

lemma *reduce-trail-to_{NOT}-length*:
length $M = \text{length } M' \implies \text{reduce-trail-to}_{NOT} M S = \text{reduce-trail-to}_{NOT} M' S$
apply (*induction M S arbitrary: rule: reduce-trail-to_{NOT}.induct*)
by (*simp add: reduce-trail-to_{NOT}.sims*)

end

16.2.2 Definition of the operation

locale *propagate-ops* =
dpll-state mset-cl_s insert-cl_s remove-lit
mset-cl_{ss} union-cl_{ss} in-cl_{ss} insert-cl_{ss} remove-from-cl_{ss}
trail raw-clauses prepend-trail tl-trail add-cl_{NOT} remove-cl_{NOT}
for
mset-cl_s:: '*cl_s* \Rightarrow '*v* clause **and**
insert-cl_s :: '*v* literal \Rightarrow '*cl_s* \Rightarrow '*cl_s* **and**
remove-lit :: '*v* literal \Rightarrow '*cl_s* \Rightarrow '*cl_s* **and**
mset-cl_{ss}:: '*cl_{ss}* \Rightarrow '*v* clauses **and**
union-cl_{ss} :: '*cl_{ss}* \Rightarrow '*cl_{ss}* \Rightarrow '*cl_{ss}* **and**
in-cl_{ss} :: '*cl_s* \Rightarrow '*cl_{ss}* \Rightarrow bool **and**
insert-cl_{ss} :: '*cl_s* \Rightarrow '*cl_{ss}* \Rightarrow '*cl_{ss}* **and**
remove-from-cl_{ss} :: '*cl_s* \Rightarrow '*cl_{ss}* \Rightarrow '*cl_{ss}* **and**
trail :: '*st* \Rightarrow ('*v*, unit, unit) marked-lits **and**
raw-clauses :: '*st* \Rightarrow '*cl_{ss}* **and**
prepend-trail :: ('*v*, unit, unit) marked-lit \Rightarrow '*st* \Rightarrow '*st* **and**
tl-trail :: '*st* \Rightarrow '*st* **and**
add-cl_{NOT} :: '*cl_s* \Rightarrow '*st* \Rightarrow '*st* **and**
remove-cl_{NOT} :: '*cl_s* \Rightarrow '*st* \Rightarrow '*st* +
fixes
propagate-cond :: ('*v*, unit, unit) marked-lit \Rightarrow '*st* \Rightarrow bool
begin
inductive *propagate_{NOT}* :: '*st* \Rightarrow '*st* \Rightarrow bool **where**
propagate_{NOT}[*intro*]: $C + \{\#L\} \in \# \text{clauses}_{NOT} S \implies \text{trail } S \models_{as} C \text{Not } C$

```

     $\Rightarrow$  undefined-lit (trail S) L
     $\Rightarrow$  propagate-cond (Propagated L ()) S
     $\Rightarrow$  T  $\sim$  prepend-trail (Propagated L ()) S
     $\Rightarrow$  propagateNOT S T
inductive-cases propagateNOTE[elim]: propagateNOT S T

end

locale decide-ops =
  dpll-state mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clssNOT remove-clssNOT
for
  mset-clss:: 'cls  $\Rightarrow$  'v clause and
  insert-clss :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
  raw-clauses :: 'st  $\Rightarrow$  'clss and
  prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
  tl-trail :: 'st  $\Rightarrow$  'st and
  add-clssNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  remove-clssNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st
begin
inductive decideNOT :: 'st  $\Rightarrow$  'st  $\Rightarrow$  bool where
decideNOT[intro]: undefined-lit (trail S) L  $\Rightarrow$  atm-of L  $\in$  atms-of-mm (clausesNOT S)
     $\Rightarrow$  T  $\sim$  prepend-trail (Marked L ()) S
     $\Rightarrow$  decideNOT S T

inductive-cases decideNOTE[elim]: decideNOT S S'
end

locale backjumping-ops =
  dpll-state mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clssNOT remove-clssNOT
for
  mset-clss:: 'cls  $\Rightarrow$  'v clause and
  insert-clss :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
  raw-clauses :: 'st  $\Rightarrow$  'clss and
  prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
  tl-trail :: 'st  $\Rightarrow$  'st and
  add-clssNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  remove-clssNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st +

```

fixes
 $backjump\text{-}conds :: 'v\ clause \Rightarrow 'v\ clause \Rightarrow 'v\ literal \Rightarrow 'st \Rightarrow 'st \Rightarrow bool$
begin

inductive *backjump* **where**
 $trail\ S = F' @\ Marked\ K\ ()\# F$
 $\Rightarrow T \sim prepend\text{-}trail\ (Propagated\ L\ ())\ (reduce\text{-}trail\text{-}to_{NOT}\ F\ S)$
 $\Rightarrow C \in\# clauses_{NOT}\ S$
 $\Rightarrow trail\ S \models_{as}\ CNot\ C$
 $\Rightarrow undefined\text{-}lit\ F\ L$
 $\Rightarrow atm\text{-}of\ L \in\ atm\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \cup atm\text{-}of\ ' (lits\text{-}of\text{-}l\ (trail\ S))$
 $\Rightarrow clauses_{NOT}\ S \models_{pm}\ C' + \{\#L\#\}$
 $\Rightarrow F \models_{as}\ CNot\ C'$
 $\Rightarrow backjump\text{-}conds\ C\ C'\ L\ S\ T$
 $\Rightarrow backjump\ S\ T$

inductive-cases *backjumpE*: *backjump* *S* *T*

The condition $atm\text{-}of\ L \in\ atm\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \cup atm\text{-}of\ ' lits\text{-}of\text{-}l\ (trail\ S)$ is not implied by the condition $clauses_{NOT}\ S \models_{pm}\ C' + \{\#L\#\}$ (no negation).

end

16.3 DPLL with backjumping

locale *dp11-with-backjumping-ops* =
 $propagate\text{-}ops\ mset\text{-}cls\ insert\text{-}cls\ remove\text{-}lit$
 $mset\text{-}clss\ union\text{-}clss\ in\text{-}clss\ insert\text{-}clss\ remove\text{-}from\text{-}clss$
 $trail\ raw\text{-}clauses\ prepend\text{-}trail\ tl\text{-}trail\ add\text{-}cls_{NOT}\ remove\text{-}cls_{NOT}\ propagate\text{-}conds +$
 $decide\text{-}ops\ mset\text{-}cls\ insert\text{-}cls\ remove\text{-}lit$
 $mset\text{-}clss\ union\text{-}clss\ in\text{-}clss\ insert\text{-}clss\ remove\text{-}from\text{-}clss$
 $trail\ raw\text{-}clauses\ prepend\text{-}trail\ tl\text{-}trail\ add\text{-}cls_{NOT}\ remove\text{-}cls_{NOT} +$
 $backjumping\text{-}ops\ mset\text{-}cls\ insert\text{-}cls\ remove\text{-}lit$
 $mset\text{-}clss\ union\text{-}clss\ in\text{-}clss\ insert\text{-}clss\ remove\text{-}from\text{-}clss$
 $trail\ raw\text{-}clauses\ prepend\text{-}trail\ tl\text{-}trail\ add\text{-}cls_{NOT}\ remove\text{-}cls_{NOT}\ backjump\text{-}conds$
for
 $mset\text{-}cls :: 'cls \Rightarrow 'v\ clause\ \mathbf{and}$
 $insert\text{-}cls :: 'v\ literal \Rightarrow 'cls \Rightarrow 'cls\ \mathbf{and}$
 $remove\text{-}lit :: 'v\ literal \Rightarrow 'cls \Rightarrow 'cls\ \mathbf{and}$
 $mset\text{-}clss :: 'clss \Rightarrow 'v\ clauses\ \mathbf{and}$
 $union\text{-}clss :: 'clss \Rightarrow 'clss \Rightarrow 'clss\ \mathbf{and}$
 $in\text{-}clss :: 'cls \Rightarrow 'clss \Rightarrow bool\ \mathbf{and}$
 $insert\text{-}clss :: 'cls \Rightarrow 'clss \Rightarrow 'clss\ \mathbf{and}$
 $remove\text{-}from\text{-}clss :: 'cls \Rightarrow 'clss \Rightarrow 'clss\ \mathbf{and}$
 $trail :: 'st \Rightarrow ('v, unit, unit)\ marked\text{-}lits\ \mathbf{and}$
 $raw\text{-}clauses :: 'st \Rightarrow 'clss\ \mathbf{and}$
 $prepend\text{-}trail :: ('v, unit, unit)\ marked\text{-}lit \Rightarrow 'st \Rightarrow 'st\ \mathbf{and}$
 $tl\text{-}trail :: 'st \Rightarrow 'st\ \mathbf{and}$
 $add\text{-}cls_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st\ \mathbf{and}$
 $remove\text{-}cls_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st\ \mathbf{and}$
 $inv :: 'st \Rightarrow bool\ \mathbf{and}$
 $backjump\text{-}conds :: 'v\ clause \Rightarrow 'v\ clause \Rightarrow 'v\ literal \Rightarrow 'st \Rightarrow 'st \Rightarrow bool\ \mathbf{and}$
 $propagate\text{-}conds :: ('v, unit, unit)\ marked\text{-}lit \Rightarrow 'st \Rightarrow bool +$
assumes
 $bj\text{-}can\text{-}jump:$
 $\bigwedge S\ C\ F'\ K\ F\ L.$
 $inv\ S \Rightarrow$

$no_dup (trail S) \implies$
 $trail S = F' @ Marked K () \# F \implies$
 $C \in \# clauses_{NOT} S \implies$
 $trail S \models_{as} CNot C \implies$
 $undefined_lit F L \implies$
 $atm_of L \in atms_of_mm (clauses_{NOT} S) \cup atm_of ' (lits_of_l (F' @ Marked K () \# F)) \implies$
 $clauses_{NOT} S \models_{pm} C' + \{\#L\# \} \implies$
 $F \models_{as} CNot C' \implies$
 $\neg no_step_backjump S$

begin

We cannot add a like condition $atms_of C' \subseteq atms_of_ms N$ because to ensure that we can backjump even if the last decision variable has disappeared.

The part of the condition $atm_of L \in atm_of ' lits_of_l (F' @ Marked K () \# F)$ is important, otherwise you are not sure that you can backtrack.

16.3.1 Definition

We define dpll with backjumping:

inductive $dpll_bj :: 'st \Rightarrow 'st \Rightarrow bool$ **for** $S :: 'st$ **where**
 $bj_decide_{NOT}: decide_{NOT} S S' \implies dpll_bj S S' \mid$
 $bj_propagate_{NOT}: propagate_{NOT} S S' \implies dpll_bj S S' \mid$
 $bj_backjump: backjump S S' \implies dpll_bj S S'$

lemmas $dpll_bj_induct = dpll_bj.induct[split_format(complete)]$

thm $dpll_bj_induct[OF dpll_with_backjumping_ops_axioms]$

lemma $dpll_bj_all_induct[consumes 2, case_names decide_{NOT} propagate_{NOT} backjump]:$

fixes $S T :: 'st$

assumes

$dpll_bj S T$ **and**

$inv S$

$\bigwedge L T. undefined_lit (trail S) L \implies atm_of L \in atms_of_mm (clauses_{NOT} S)$

$\implies T \sim prepend_trail (Marked L ()) S$

$\implies P S T$ **and**

$\bigwedge C L T. C + \{\#L\# \} \in \# clauses_{NOT} S \implies trail S \models_{as} CNot C \implies undefined_lit (trail S) L$

$\implies T \sim prepend_trail (Propagated L ()) S$

$\implies P S T$ **and**

$\bigwedge C F' K F L C' T. C \in \# clauses_{NOT} S \implies F' @ Marked K () \# F \models_{as} CNot C$

$\implies trail S = F' @ Marked K () \# F$

$\implies undefined_lit F L$

$\implies atm_of L \in atms_of_mm (clauses_{NOT} S) \cup atm_of ' (lits_of_l (F' @ Marked K () \# F))$

$\implies clauses_{NOT} S \models_{pm} C' + \{\#L\# \}$

$\implies F \models_{as} CNot C'$

$\implies T \sim prepend_trail (Propagated L ()) (reduce_trail_to_{NOT} F S)$

$\implies P S T$

shows $P S T$

apply ($induct T$ rule: $dpll_bj_induct[OF local.dpll_with_backjumping_ops_axioms]$)

apply ($rule assms(1)$)

using $assms(3)$ **apply** $blast$

apply ($elim propagate_{NOT} E$) **using** $assms(4)$ **apply** $blast$

apply ($elim backjump E$) **using** $assms(5)$ $\langle inv S \rangle$ **by** $simp$

16.3.2 Basic properties

First, some better suited induction principle lemma $dpll_bj_clauses$:

assumes *dpll-bj S T and inv S*
shows $\text{clauses}_{\text{NOT}} S = \text{clauses}_{\text{NOT}} T$
using *assms by (induction rule: dpll-bj-all-induct) auto*

No duplicates in the trail lemma *dpll-bj-no-dup:*

assumes *dpll-bj S T and inv S*
and *no-dup (trail S)*
shows *no-dup (trail T)*
using *assms by (induction rule: dpll-bj-all-induct)*
(auto simp add: defined-lit-map reduce-trail-to_{NOT}-skip-beginning)

Valuations lemma *dpll-bj-sat-iff:*

assumes *dpll-bj S T and inv S*
shows $I \models_{\text{sm}} \text{clauses}_{\text{NOT}} S \longleftrightarrow I \models_{\text{sm}} \text{clauses}_{\text{NOT}} T$
using *assms by (induction rule: dpll-bj-all-induct) auto*

Clauses lemma *dpll-bj-atms-of-ms-clauses-inv:*

assumes
dpll-bj S T and
inv S
shows $\text{atms-of-mm} (\text{clauses}_{\text{NOT}} S) = \text{atms-of-mm} (\text{clauses}_{\text{NOT}} T)$
using *assms by (induction rule: dpll-bj-all-induct) auto*

lemma *dpll-bj-atms-in-trail:*

assumes
dpll-bj S T and
inv S and
atm-of ' (lits-of-l (trail S)) \subseteq atms-of-mm (clauses_{NOT} S)
shows *atm-of ' (lits-of-l (trail T)) \subseteq atms-of-mm (clauses_{NOT} S)*
using *assms by (induction rule: dpll-bj-all-induct)*
(auto simp: in-plus-implies-atm-of-on-atms-of-ms reduce-trail-to_{NOT}-skip-beginning)

lemma *dpll-bj-atms-in-trail-in-set:*

assumes *dpll-bj S T and*
inv S and
atms-of-mm (clauses_{NOT} S) \subseteq A and
atm-of ' (lits-of-l (trail S)) \subseteq A
shows *atm-of ' (lits-of-l (trail T)) \subseteq A*
using *assms by (induction rule: dpll-bj-all-induct)*
(auto simp: in-plus-implies-atm-of-on-atms-of-ms)

lemma *dpll-bj-all-decomposition-implies-inv:*

assumes
dpll-bj S T and
inv: inv S and
decomp: all-decomposition-implies-m (clauses_{NOT} S) (get-all-marked-decomposition (trail S))
shows *all-decomposition-implies-m (clauses_{NOT} T) (get-all-marked-decomposition (trail T))*
using *assms(1,2)*

proof *(induction rule: dpll-bj-all-induct)*

case *decide_{NOT}*

then show *?case using decomp by auto*

next

case *(propagate_{NOT} C L T) note propa = this(1) and undef = this(3) and T = this(4)*
let *?M' = trail (prepend-trail (Propagated L ())) S*
let *?N = clauses_{NOT} S*

```

obtain  $a\ y\ l$  where  $ay$ : get-all-marked-decomposition  $?M' = (a, y) \# l$ 
  by (cases get-all-marked-decomposition  $?M'$ ) fastforce+
then have  $M'$ :  $?M' = y @ a$  using get-all-marked-decomposition-decomp[of  $?M'$ ] by auto
have  $M$ : get-all-marked-decomposition (trail  $S$ ) =  $(a, tl\ y) \# l$ 
  using  $ay$  undef by (cases get-all-marked-decomposition (trail  $S$ )) auto
have  $y_0$ :  $y = (Propagated\ L\ ()) \# (tl\ y)$ 
  using  $ay$  undef by (auto simp add:  $M$ )
from arg-cong[OF this, of set] have  $y[simp]$ :  $set\ y = insert\ (Propagated\ L\ ())\ (set\ (tl\ y))$ 
  by simp
have  $tr-S$ : trail  $S = tl\ y @ a$ 
  using arg-cong[OF  $M'$ , of tl]  $y_0\ M$  get-all-marked-decomposition-decomp by force
have  $a-Un-N-M$ : unmark-l  $a \cup set-mset\ ?N \models_{ps} unmark-l\ (tl\ y)$ 
  using decomp  $ay$  unfolding all-decomposition-implies-def by (simp add:  $M$ )+

moreover have unmark-l  $a \cup set-mset\ ?N \models_p \{\#L\#\}$  (is  $?I \models_p -$ )
proof (rule true-clss-clss-plus-CNot)
  show  $?I \models_p C + \{\#L\#\}$ 
  using propa propagateNOT.prems by (auto dest!: true-clss-clss-in-imp-true-clss-clss)
next
  have unmark-l  $?M' \models_{ps} CNot\ C$ 
  using (trail  $S \models_{as} CNot\ C$ ) undef by (auto simp add: true-annots-true-clss-clss)
  have  $a1$ : unmark-l  $a \cup unmark-l\ (tl\ y) \models_{ps} CNot\ C$ 
  using propagateNOT.hyps(2)  $tr-S$  true-annots-true-clss-clss
  by (force simp add: image-Un sup-commute)
  then have unmark-l  $a \cup set-mset\ (clauses_{NOT}\ S) \models_{ps} unmark-l\ a \cup unmark-l\ (tl\ y)$ 
  using  $a-Un-N-M$  true-clss-clss-def by blast
  then show unmark-l  $a \cup set-mset\ (clauses_{NOT}\ S) \models_{ps} CNot\ C$ 
  using  $a1$  by (meson true-clss-clss-left-right true-clss-clss-union-and
    true-clss-clss-union-l-r)
  qed
ultimately have unmark-l  $a \cup set-mset\ ?N \models_{ps} unmark-l\ ?M'$ 
  unfolding  $M'$  by (auto simp add: all-in-true-clss-clss image-Un)
then show  $?case$ 
  using decomp  $T\ M$  undef unfolding  $ay$  all-decomposition-implies-def by (auto simp add:  $ay$ )
next
case (backjump  $C\ F'\ K\ F\ L\ D\ T$ ) note  $confl = this(2)$  and  $tr = this(3)$  and  $undef = this(4)$ 
and  $L = this(5)$  and  $N-C = this(6)$  and  $vars-D = this(5)$  and  $T = this(8)$ 
have decomp: all-decomposition-implies-m (clausesNOT  $S$ ) (get-all-marked-decomposition  $F$ )
  using decomp unfolding  $tr$  all-decomposition-implies-def
  by (metis (no-types, lifting) get-all-marked-decomposition.simps(1)
    get-all-marked-decomposition-never-empty hd-Cons-tl insert-iff list.sel(3) list.set(2)
    tl-get-all-marked-decomposition-skip-some)

obtain  $a\ b\ li$  where  $F$ : get-all-marked-decomposition  $F = (a, b) \# li$ 
  by (cases get-all-marked-decomposition  $F$ ) auto
have  $F = b @ a$ 
  using get-all-marked-decomposition-decomp[of  $F\ a\ b$ ]  $F$  by auto
have  $a-N-b$ : unmark-l  $a \cup set-mset\ (clauses_{NOT}\ S) \models_{ps} unmark-l\ b$ 
  using decomp unfolding all-decomposition-implies-def by (auto simp add:  $F$ )

have  $F-D$ : unmark-l  $F \models_{ps} CNot\ D$ 
  using ( $F \models_{as} CNot\ D$ ) by (simp add: true-annots-true-clss-clss)
then have unmark-l  $a \cup unmark-l\ b \models_{ps} CNot\ D$ 
  unfolding ( $F = b @ a$ ) by (simp add: image-Un sup-commute)
have  $a-N-CNot-D$ : unmark-l  $a \cup set-mset\ (clauses_{NOT}\ S) \models_{ps} CNot\ D \cup unmark-l\ b$ 

```

```

apply (rule true-clss-clss-left-right)
using a-N-b F-D unfolding (F = b @ a) by (auto simp add: image-Un ac-simps)

have a-N-D-L: unmark-l a  $\cup$  set-mset (clausesNOT S)  $\models_p$  D+{#L#}
  by (simp add: N-C)
have unmark-l a  $\cup$  set-mset (clausesNOT S)  $\models_p$  {#L#}
  using a-N-D-L a-N-CNot-D by (blast intro: true-clss-clss-plus-CNot)
then show ?case
  using decomp T tr undef unfolding all-decomposition-implies-def by (auto simp add: F)
qed

```

16.3.3 Termination

Using a proper measure lemma *length-get-all-marked-decomposition-append-Marked*:

```

length (get-all-marked-decomposition (F' @ Marked K () # F)) =
  length (get-all-marked-decomposition F')
+ length (get-all-marked-decomposition (Marked K () # F))
- 1
by (induction F' rule: marked-lit-list-induct) auto

```

lemma *take-length-get-all-marked-decomposition-marked-sandwich*:

```

take (length (get-all-marked-decomposition F))
  (map (f o snd) (rev (get-all-marked-decomposition (F' @ Marked K () # F))))
=
  map (f o snd) (rev (get-all-marked-decomposition F))

```

proof (induction F' rule: marked-lit-list-induct)

```

case nil
then show ?case by auto
next
case (marked K)
then show ?case by (simp add: length-get-all-marked-decomposition-append-Marked)
next
case (proped L m F') note IH = this(1)
obtain a b l where F': get-all-marked-decomposition (F' @ Marked K () # F) = (a, b) # l
  by (cases get-all-marked-decomposition (F' @ Marked K () # F)) auto
have length (get-all-marked-decomposition F) - length l = 0
  using length-get-all-marked-decomposition-append-Marked[of F' K F]
  unfolding F' by (cases get-all-marked-decomposition F') auto
then show ?case
  using IH by (simp add: F')
qed

```

lemma *length-get-all-marked-decomposition-length*:

```

length (get-all-marked-decomposition M)  $\leq$  1 + length M
by (induction M rule: marked-lit-list-induct) auto

```

lemma *length-in-get-all-marked-decomposition-bounded*:

```

assumes i:i  $\in$  set (trail-weight S)
shows i  $\leq$  Suc (length (trail S))
proof -
obtain a b where
  (a, b)  $\in$  set (get-all-marked-decomposition (trail S)) and
  ib: i = Suc (length b)
  using i by auto
then obtain c where trail S = c @ b @ a

```


using *get-all-marked-decomposition-exists-prepend'* **by** *metis*
from *arg-cong[OF this, of length]* **show** *?thesis* **using** *i ib* **by** *auto*
qed

Well-foundedness The bounds are the following:

- $1 + \text{card}(\text{atms-of-ms } A)$: $\text{card}(\text{atms-of-ms } A)$ is an upper bound on the length of the list. As *get-all-marked-decomposition* appends an possibly empty couple at the end, adding one is needed.
- $2 + \text{card}(\text{atms-of-ms } A)$: $\text{card}(\text{atms-of-ms } A)$ is an upper bound on the number of elements, where adding one is necessary for the same reason as for the bound on the list, and one is needed to have a strict bound.

abbreviation *unassigned-lit* :: '*b* literal multiset set \Rightarrow '*a* list \Rightarrow nat **where**
unassigned-lit *N M* $\equiv \text{card}(\text{atms-of-ms } N) - \text{length } M$

lemma *dpll-bj-trail-mes-increasing-prop*:

fixes *M* :: ('*v*, unit, unit) marked-lits **and** *N* :: '*v* clauses

assumes

dpll-bj *S T* **and**

inv *S* **and**

NA: $\text{atms-of-mm}(\text{clauses}_{\text{NOT}} S) \subseteq \text{atms-of-ms } A$ **and**

MA: $\text{atm-of } \text{'lits-of-l}(\text{trail } S) \subseteq \text{atms-of-ms } A$ **and**

n-d: $\text{no-dup}(\text{trail } S)$ **and**

finite: $\text{finite } A$

shows $\mu_C(1 + \text{card}(\text{atms-of-ms } A))(2 + \text{card}(\text{atms-of-ms } A))(\text{trail-weight } T)$

$> \mu_C(1 + \text{card}(\text{atms-of-ms } A))(2 + \text{card}(\text{atms-of-ms } A))(\text{trail-weight } S)$

using *assms(1,2)*

proof (*induction rule: dpll-bj-all-induct*)

case (*propagate*_{NOT} *C L*) **note** *CLN* = *this(1)* **and** *MC* = *this(2)* **and** *undef-L* = *this(3)* **and** *T* = *this(4)*

have *incl*: $\text{atm-of } \text{'lits-of-l}(\text{Propagated } L ()) \# \text{trail } S \subseteq \text{atms-of-ms } A$

using *propagate*_{NOT} *dpll-bj-atms-in-trail-in-set* *bj-propagate*_{NOT} *NA MA CLN*

by (*auto simp: in-plus-implies-atm-of-on-atms-of-ms*)

have *no-dup*: $\text{no-dup}(\text{Propagated } L ()) \# \text{trail } S$

using *defined-lit-map* *n-d* *undef-L* **by** *auto*

obtain *a b l* **where** *M*: $\text{get-all-marked-decomposition}(\text{trail } S) = (a, b) \# l$

by (*cases* *get-all-marked-decomposition* (*trail S*)) *auto*

have *b-le-M*: $\text{length } b \leq \text{length}(\text{trail } S)$

using *get-all-marked-decomposition-decomp*[*of trail S*] **by** (*simp add: M*)

have *finite* (*atms-of-ms A*) **using** *finite* **by** *simp*

then have $\text{length}(\text{Propagated } L ()) \# \text{trail } S \leq \text{card}(\text{atms-of-ms } A)$

using *incl* *finite* **unfolding** *no-dup-length-eq-card-atm-of-lits-of-l*[*OF no-dup*]

by (*simp add: card-mono*)

then have *latm*: $\text{unassigned-lit } A b = \text{Suc}(\text{unassigned-lit } A(\text{Propagated } L d \# b))$

using *b-le-M* **by** *auto*

then show *?case* **using** *T* *undef-L* **by** (*auto simp: latm M* $\mu_C\text{-cons}$)

next

case (*decide*_{NOT} *L*) **note** *undef-L* = *this(1)* **and** *MC* = *this(2)* **and** *T* = *this(3)*

have *incl*: $\text{atm-of } \text{'lits-of-l}(\text{Marked } L ()) \# (\text{trail } S) \subseteq \text{atms-of-ms } A$

using *dpll-bj-atms-in-trail-in-set* *bj-decide*_{NOT} *decide*_{NOT}.*decide*_{NOT}[*OF decide*_{NOT}.*hyps*] *NA MA*

MC

```

by auto

have no-dup: no-dup (Marked L () # (trail S))
  using defined-lit-map n-d undef-L by auto
obtain a b l where M: get-all-marked-decomposition (trail S) = (a, b) # l
  by (cases get-all-marked-decomposition (trail S)) auto

then have length (Marked L () # (trail S)) ≤ card (atms-of-ms A)
  using incl finite unfolding no-dup-length-eq-card-atm-of-lits-of-l[OF no-dup]
  by (simp add: card-mono)
show ?case using T undef-L by (simp add: μC-cons)
next
case (backjump C F' K F L C' T) note undef-L = this(4) and MC = this(1) and tr-S = this(3)
and
  L = this(5) and T = this(8)
have incl: atm-of ' lits-of-l (Propagated L () # F) ⊆ atms-of-ms A
  using dpll-bj-atms-in-trail-in-set NA MA L by (auto simp: tr-S)

have no-dup: no-dup (Propagated L () # F)
  using defined-lit-map n-d undef-L tr-S by auto
obtain a b l where M: get-all-marked-decomposition (trail S) = (a, b) # l
  by (cases get-all-marked-decomposition (trail S)) auto
have b-le-M: length b ≤ length (trail S)
  using get-all-marked-decomposition-decomp[of trail S] by (simp add: M)
have fin-atms-A: finite (atms-of-ms A) using finite by simp

then have F-le-A: length (Propagated L () # F) ≤ card (atms-of-ms A)
  using incl finite unfolding no-dup-length-eq-card-atm-of-lits-of-l[OF no-dup]
  by (simp add: card-mono)
have tr-S-le-A: length (trail S) ≤ (card (atms-of-ms A))
  using n-d MA by (metis fin-atms-A card-mono no-dup-length-eq-card-atm-of-lits-of-l)
obtain a b l where F: get-all-marked-decomposition F = (a, b) # l
  by (cases get-all-marked-decomposition F) auto
then have F = b @ a
  using get-all-marked-decomposition-decomp[of Propagated L () # F a
    Propagated L () # b] by simp
then have latm: unassigned-lit A b = Suc (unassigned-lit A (Propagated L () # b))
  using F-le-A by simp
obtain rem where
  rem: map (λa. Suc (length (snd a))) (rev (get-all-marked-decomposition (F' @ Marked K () # F)))
  = map (λa. Suc (length (snd a))) (rev (get-all-marked-decomposition F)) @ rem
  using take-length-get-all-marked-decomposition-marked-sandwich[of F λa. Suc (length a) F' K]
  unfolding o-def by (metis append-take-drop-id)
then have rem: map (λa. Suc (length (snd a)))
  (get-all-marked-decomposition (F' @ Marked K () # F))
  = rev rem @ map (λa. Suc (length (snd a))) ((get-all-marked-decomposition F))
  by (simp add: rev-map[symmetric] rev-swap)
have length (rev rem @ map (λa. Suc (length (snd a))) (get-all-marked-decomposition F))
  ≤ Suc (card (atms-of-ms A))
  using arg-cong[OF rem, of length] tr-S-le-A
  length-get-all-marked-decomposition-length[of F' @ Marked K () # F] tr-S by auto
moreover
{ fix i :: nat and xs :: 'a list
  have i < length xs ⇒ length xs - Suc i < length xs
    by auto

```

then have $H: i < \text{length } xs \implies \text{rev } xs ! i \in \text{set } xs$
 using $\text{rev-nth}[of\ i\ xs]$ unfolding in-set-conv-nth by (force simp add: in-set-conv-nth)
 } note $H = \text{this}$
 have $\forall i < \text{length } \text{rem}. \text{rev } \text{rem} ! i < \text{card } (\text{atms-of-ms } A) + 2$
 using tr-S-le-A length-in-get-all-marked-decomposition-bounded[$of\ -\ S$] unfolding tr-S
 by (force simp add: $o\text{-def } \text{rem } \text{dest}!$; H intro: length-get-all-marked-decomposition-length)
 ultimately show ?case
 using $\mu_C\text{-bounded}[of\ \text{rev } \text{rem } \text{card } (\text{atms-of-ms } A) + 2\ \text{unassigned-lit } A\ l]\ T\ \text{undef-L}$
 by (simp add: $\text{rem } \mu_C\text{-append } \mu_C\text{-cons } F\ \text{tr-S}$)
 qed

lemma $\text{dpll-bj-trail-mes-decreasing-prop}$:

assumes dpll : $\text{dpll-bj } S\ T$ and inv : $\text{inv } S$ and
 $N\text{-A}$: $\text{atms-of-mm } (\text{clauses}_{NOT}\ S) \subseteq \text{atms-of-ms } A$ and
 $M\text{-A}$: $\text{atm-of } ' \text{lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A$ and
 nd : $\text{no-dup } (\text{trail } S)$ and
 fin-A : $\text{finite } A$

shows $(2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $\quad - \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } T)$
 $\quad < (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $\quad - \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } S)$

proof –

let $?b = 2 + \text{card } (\text{atms-of-ms } A)$
 let $?s = 1 + \text{card } (\text{atms-of-ms } A)$
 let $? \mu = \mu_C\ ?s\ ?b$
 have $M'\text{-A}$: $\text{atm-of } ' \text{lits-of-l } (\text{trail } T) \subseteq \text{atms-of-ms } A$
 by (meson $M\text{-A } N\text{-A } \text{dpll } \text{dpll-bj-atms-in-trail-in-set } \text{inv}$)
 have nd' : $\text{no-dup } (\text{trail } T)$
 using $\langle \text{dpll-bj } S\ T \rangle$ $\text{dpll-bj-no-dup } nd\ \text{inv}$ by blast
 { fix $i :: \text{nat}$ and $xs :: 'a\ \text{list}$
 have $i < \text{length } xs \implies \text{length } xs - \text{Suc } i < \text{length } xs$
 by auto
 then have $H: i < \text{length } xs \implies xs ! i \in \text{set } xs$
 using $\text{rev-nth}[of\ i\ xs]$ unfolding in-set-conv-nth by (force simp add: in-set-conv-nth)
 } note $H = \text{this}$

have $l\text{-M-A}$: $\text{length } (\text{trail } S) \leq \text{card } (\text{atms-of-ms } A)$
 by (simp add: $\text{fin-A } M\text{-A } \text{card-mono } \text{no-dup-length-eq-card-atm-of-lits-of-l } nd$)
 have $l\text{-M}'\text{-A}$: $\text{length } (\text{trail } T) \leq \text{card } (\text{atms-of-ms } A)$
 by (simp add: $\text{fin-A } M'\text{-A } \text{card-mono } \text{no-dup-length-eq-card-atm-of-lits-of-l } nd'$)
 have $l\text{-trail-weight-M}$: $\text{length } (\text{trail-weight } T) \leq 1 + \text{card } (\text{atms-of-ms } A)$
 using $l\text{-M}'\text{-A}$ length-get-all-marked-decomposition-length[$of\ \text{trail } T$] by auto
 have bounded-M : $\forall i < \text{length } (\text{trail-weight } T). (\text{trail-weight } T) ! i < \text{card } (\text{atms-of-ms } A) + 2$
 using length-in-get-all-marked-decomposition-bounded[$of\ -\ T$] $l\text{-M}'\text{-A}$
 by (metis (no-types, lifting) H $\text{Nat.le-trans } \text{add-2-eq-Suc}'\ \text{not-le not-less-eq-eq}$)

from $\text{dpll-bj-trail-mes-increasing-prop}[OF\ \text{dpll } \text{inv } N\text{-A } M\text{-A } nd\ \text{fin-A}]$

have $\mu_C\ ?s\ ?b (\text{trail-weight } S) < \mu_C\ ?s\ ?b (\text{trail-weight } T)$ by simp

moreover from $\mu_C\text{-bounded}[OF\ \text{bounded-M } l\text{-trail-weight-M}]$

have $\mu_C\ ?s\ ?b (\text{trail-weight } T) \leq ?b \wedge ?s$ by auto

ultimately show ?thesis by linarith

qed

lemma wf-dpll-bj :

assumes fin : $\text{finite } A$

```

shows wf {(T, S). dpll-bj S T
  ∧ atms-of-mm (clausesNOT S) ⊆ atms-of-ms A ∧ atm-of ' lits-of-l (trail S) ⊆ atms-of-ms A
  ∧ no-dup (trail S) ∧ inv S}
(is wf ?A)
proof (rule wf-bounded-measure[of -
  λ-. (2 + card (atms-of-ms A))^(1 + card (atms-of-ms A))
  λS. μC (1+card (atms-of-ms A)) (2+card (atms-of-ms A)) (trail-weight S)])
fix a b :: 'st
let ?b = 2+card (atms-of-ms A)
let ?s = 1+card (atms-of-ms A)
let ?μ = μC ?s ?b
assume ab: (b, a) ∈ ?A

have fin-A: finite (atms-of-ms A)
  using fin by auto
have
  dpll-bj: dpll-bj a b and
  N-A: atms-of-mm (clausesNOT a) ⊆ atms-of-ms A and
  M-A: atm-of ' lits-of-l (trail a) ⊆ atms-of-ms A and
  nd: no-dup (trail a) and
  inv: inv a
  using ab by auto

have M'-A: atm-of ' lits-of-l (trail b) ⊆ atms-of-ms A
  by (meson M-A N-A ⟨dpll-bj a b⟩ dpll-bj-atms-in-trail-in-set inv)
have nd': no-dup (trail b)
  using ⟨dpll-bj a b⟩ dpll-bj-no-dup nd inv by blast
{ fix i :: nat and xs :: 'a list
  have i < length xs ⟹ length xs - Suc i < length xs
  by auto
  then have H: i < length xs ⟹ xs ! i ∈ set xs
  using rev-nth[of i xs] unfolding in-set-conv-nth by (force simp add: in-set-conv-nth)
} note H = this

have l-M-A: length (trail a) ≤ card (atms-of-ms A)
  by (simp add: fin-A M-A card-mono no-dup-length-eq-card-atm-of-lits-of-l nd)
have l-M'-A: length (trail b) ≤ card (atms-of-ms A)
  by (simp add: fin-A M'-A card-mono no-dup-length-eq-card-atm-of-lits-of-l nd')
have l-trail-weight-M: length (trail-weight b) ≤ 1+card (atms-of-ms A)
  using l-M'-A length-get-all-marked-decomposition-length[of trail b] by auto
have bounded-M: ∀ i < length (trail-weight b). (trail-weight b)! i < card (atms-of-ms A) + 2
  using length-in-get-all-marked-decomposition-bounded[of - b] l-M'-A
  by (metis (no-types, lifting) Nat.le-trans One-nat-def Suc-1 add.right-neutral add-Suc-right
    le-imp-less-Suc less-eq-Suc-le nth-mem)

from dpll-bj-trail-mes-increasing-prop[OF dpll-bj inv N-A M-A nd fin]
have μC ?s ?b (trail-weight a) < μC ?s ?b (trail-weight b) by simp
moreover from μC-bounded[OF bounded-M l-trail-weight-M]
  have μC ?s ?b (trail-weight b) ≤ ?b ^ ?s by auto
ultimately show ?b ^ ?s ≤ ?b ^ ?s ∧
  μC ?s ?b (trail-weight b) ≤ ?b ^ ?s ∧
  μC ?s ?b (trail-weight a) < μC ?s ?b (trail-weight b)
  by blast
qed

```

16.3.4 Normal Forms

We prove that given a normal form of DPLL, with some invariants, the either N is satisfiable and the built valuation M is a model; or N is unsatisfiable.

Idea of the proof: We have to prove that *satisfiable* N , $\neg M \models_{as} N$ and there is no remaining step is incompatible.

1. The *decide* rules tells us that every variable in N has a value.
2. $\neg M \models_{as} N$ tells us that there is conflict.
3. There is at least one decision in the trail (otherwise, M is a model of N).
4. Now if we build the clause with all the decision literals of the trail, we can apply the *backjump* rule.

The assumption are saying that we have a finite upper bound A for the literals, that we cannot do any step *no-step dpll-bj* S

theorem *dpll-backjump-final-state*:

fixes $A :: 'v \text{ literal multiset set}$ **and** $S \ T :: 'st$

assumes

atms-of-mm (*clauses*_{NOT} S) \subseteq *atms-of-ms* A **and**

atm-of ' *lits-of-l* (*trail* S) \subseteq *atms-of-ms* A **and**

no-dup (*trail* S) **and**

finite A **and**

inv: *inv* S **and**

n-s: *no-step dpll-bj* S **and**

decomp: *all-decomposition-implies-m* (*clauses*_{NOT} S) (*get-all-marked-decomposition* (*trail* S))

shows *unsatisfiable* (*set-mset* (*clauses*_{NOT} S))

\vee (*trail* $S \models_{asm}$ *clauses*_{NOT} $S \wedge$ *satisfiable* (*set-mset* (*clauses*_{NOT} S)))

proof –

let $?N = \text{set-mset} (\text{clauses}_{NOT} S)$

let $?M = \text{trail } S$

consider

(*sat*) *satisfiable* $?N$ **and** $?M \models_{as} ?N$

| (*sat'*) *satisfiable* $?N$ **and** $\neg ?M \models_{as} ?N$

| (*unsat*) *unsatisfiable* $?N$

by *auto*

then show *?thesis*

proof *cases*

case *sat'* **note** $\text{sat} = \text{this}(1)$ **and** $M = \text{this}(2)$

obtain C **where** $C \in ?N$ **and** $\neg ?M \models_a C$ **using** M **unfolding** *true-annots-def* **by** *auto*

obtain $I :: 'v \text{ literal set}$ **where**

$I \models_s ?N$ **and**

cons: *consistent-interp* I **and**

tot: *total-over-m* I $?N$ **and**

atm-I-N: *atm-of* ' $I \subseteq$ *atms-of-ms* $?N$

using *sat* **unfolding** *satisfiable-def-min* **by** *auto*

let $?I = I \cup \{P \mid P. P \in \text{lits-of-l } ?M \wedge \text{atm-of } P \notin \text{atm-of ' } I\}$

let $?O = \{\text{unmark } L \mid L. \text{is-marked } L \wedge L \in \text{set } ?M \wedge \text{atm-of } (\text{lit-of } L) \notin \text{atms-of-ms } ?N\}$

have *cons-I'*: *consistent-interp* $?I$

using *cons* **using** (*no-dup* $?M$) **unfolding** *consistent-interp-def*

by (*auto simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set lits-of-def*

dest!: *no-dup-cannot-not-lit-and-uminus*)

```

have tot-I': total-over-m ?I (?N ∪ unmark-l ?M)
  using tot atm-I-N unfolding total-over-m-def total-over-set-def
  by (fastforce simp: image-iff lits-of-def)
have {P | P. P ∈ lits-of-l ?M ∧ atm-of P ∉ atm-of 'I} ⊨s ?O
  using ⟨I ⊨s ?N⟩ atm-I-N by (auto simp add: atm-of-eq-atm-of true-clss-def lits-of-def)
then have I'-N: ?I ⊨s ?N ∪ ?O
  using ⟨I ⊨s ?N⟩ true-clss-union-increase by force
have tot': total-over-m ?I (?N ∪ ?O)
  using atm-I-N tot unfolding total-over-m-def total-over-set-def
  by (force simp: lits-of-def dest!: is-marked-ex-Marked)

have atms-N-M: atms-of-ms ?N ⊆ atm-of 'lits-of-l ?M
  proof (rule ccontr)
    assume ¬ ?thesis
    then obtain l :: 'v where
      l-N: l ∈ atms-of-ms ?N and
      l-M: l ∉ atm-of 'lits-of-l ?M
    by auto
    have undefined-lit ?M (Pos l)
      using l-M by (metis Marked-Propagated-in-iff-in-lits-of-l
        atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set literal.sel(1))
    from bj-decideNOT[OF decideNOT[OF this]] show False
      using l-N n-s by (metis literal.sel(1) state-eqNOT-ref)
  qed
have ?M ⊨as CNot C
  apply (rule all-variables-defined-not-imply-cnot)
  using ⟨C ∈ set-mset (clausesNOT S)⟩ ⟨¬ trail S ⊨a C⟩
    atms-N-M by (auto dest: atms-of-atms-of-ms-mono)
have ∃ l ∈ set ?M. is-marked l
  proof (rule ccontr)
    let ?O = {unmark L | L. is-marked L ∧ L ∈ set ?M ∧ atm-of (lit-of L) ∉ atms-of-ms ?N}
    have ∅[iff]: ⋀ I. total-over-m I (?N ∪ ?O ∪ unmark-l ?M)
      ⟷ total-over-m I (?N ∪ unmark-l ?M)
      unfolding total-over-set-def total-over-m-def atms-of-ms-def by blast
    assume ¬ ?thesis
    then have [simp]: {unmark L | L. is-marked L ∧ L ∈ set ?M}
      = {unmark L | L. is-marked L ∧ L ∈ set ?M ∧ atm-of (lit-of L) ∉ atms-of-ms ?N}
    by auto
    then have ?N ∪ ?O ⊨ps unmark-l ?M
      using all-decomposition-implies-propagated-lits-are-implied[OF decomp] by auto

    then have ?I ⊨s unmark-l ?M
      using cons-I' I'-N tot-I' ⟨?I ⊨s ?N ∪ ?O⟩ unfolding ∅ true-clss-clss-def by blast
    then have lits-of-l ?M ⊆ ?I
      unfolding true-clss-def lits-of-def by auto
    then have ?M ⊨as ?N
      using I'-N ⟨C ∈ ?N⟩ ⟨¬ ?M ⊨a C⟩ cons-I' atms-N-M
      by (meson (trail S ⊨as CNot C) consistent-CNot-not rev-subsetD sup-ge1 true-annot-def
        true-annots-def true-clss-mono-set-mset-l true-clss-def)
    then show False using M by fast
  qed
from List.split-list-first-propE[OF this] obtain K :: 'v literal and
  F F' :: ('v, unit, unit) marked-lit list where
  M-K: ?M = F' @ Marked K () # F and
  nm: ∀ f ∈ set F'. ¬ is-marked f

```

```

  unfolding is-marked-def by (metis (full-types) old.unit.exhaust)
let ?K = Marked K () :: ('v, unit, unit) marked-lit
have ?K ∈ set ?M
  unfolding M-K by auto
let ?C = image-mset lit-of {#L∈#mset ?M. is-marked L ∧ L≠?K#} :: 'v literal multiset
let ?C' = set-mset (image-mset (λL::'v literal. {#L#}) (?C + unmark ?K))
have ?N ∪ {unmark L | L. is-marked L ∧ L ∈ set ?M} ⊨ps unmark-l ?M
  using all-decomposition-implies-propagated-lits-are-implied[OF decomp] .
moreover have C': ?C' = {unmark L | L. is-marked L ∧ L ∈ set ?M}
  unfolding M-K by standard force+
ultimately have N-C-M: ?N ∪ ?C' ⊨ps unmark-l ?M
  by auto
have N-M-False: ?N ∪ (λL. unmark L) ' (set ?M) ⊨ps {{#}}
  using M ⟨?M ⊨as CNot C⟩ ⟨C∈?N⟩ unfolding true-clss-clss-def true-annot-def Ball-def
  true-annot-def by (metis consistent-CNot-not sup.orderE sup-commute true-clss-def
    true-clss-singleton-lit-of-implies-incl true-clss-union true-clss-union-increase)

have undefined-lit F K using ⟨no-dup ?M⟩ unfolding M-K by (simp add: defined-lit-map)
moreover
  have ?N ∪ ?C' ⊨ps {{#}}
  proof -
    have A: ?N ∪ ?C' ∪ unmark-l ?M = ?N ∪ unmark-l ?M
      unfolding M-K by auto
    show ?thesis
      using true-clss-clss-left-right[OF N-C-M, of {{#}}] N-M-False unfolding A by auto
  qed
have ?N ⊨p image-mset uminus ?C + {#-K#}
  unfolding true-clss-clss-def true-clss-clss-def total-over-m-def
  proof (intro allI impI)
    fix I
    assume
      tot: total-over-set I (atms-of-ms (?N ∪ {image-mset uminus ?C + {#-K#}})) and
      cons: consistent-interp I and
      I ⊨s ?N
    have (K ∈ I ∧ -K ∉ I) ∨ (-K ∈ I ∧ K ∉ I)
      using cons tot unfolding consistent-interp-def by (cases K) auto
    have {a ∈ set (trail S). is-marked a ∧ a ≠ Marked K ()} =
      set (trail S) ∩ {L. is-marked L ∧ L ≠ Marked K ()}
    by auto
    then have tot': total-over-set I
      (atm-of ' lit-of ' (set ?M ∩ {L. is-marked L ∧ L ≠ Marked K ()}))
      using tot by (auto simp add: atms-of-uminus-lit-atm-of-lit-of)
    { fix x :: ('v, unit, unit) marked-lit
      assume
        a3: lit-of x ∉ I and
        a1: x ∈ set ?M and
        a4: is-marked x and
        a5: x ≠ Marked K ()
      then have Pos (atm-of (lit-of x)) ∈ I ∨ Neg (atm-of (lit-of x)) ∈ I
        using a5 a4 tot' a1 unfolding total-over-set-def atms-of-s-def by blast
      moreover have f6: Neg (atm-of (lit-of x)) = - Pos (atm-of (lit-of x))
        by simp
      ultimately have - lit-of x ∈ I
        using f6 a3 by (metis (no-types) atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
          literal.sel(1))
    }
  }

```

```

} note  $H = this$ 

have  $\neg I \models_s ?C'$ 
  using  $\langle ?N \cup ?C' \models_{ps} \{\{\#\}\} \rangle \text{ tot cons } \langle I \models_s ?N \rangle$ 
  unfolding true-clss-clss-def total-over-m-def
  by (simp add: atms-of-uminus-lit-atm-of-lit-of atms-of-ms-single-image-atm-of-lit-of)
then show  $I \models \text{image-mset } \text{uminus } ?C + \{\#\} - K\#$ 
  unfolding true-clss-def true-clss-def using  $\langle (K \in I \wedge -K \notin I) \vee (-K \in I \wedge K \notin I) \rangle$ 
  by (auto dest!: H)
qed
moreover have  $F \models_{as} CNot (\text{image-mset } \text{uminus } ?C)$ 
  using nm unfolding true-annots-def CNot-def M-K by (auto simp add: lits-of-def)
ultimately have False
  using bj-can-jump[of  $S F' K F C -K$ 
     $\text{image-mset } \text{uminus } (\text{image-mset lit-of } \{\#\} L : \# \text{ mset } ?M. \text{is-marked } L \wedge L \neq \text{Marked } K ()\#)$ ]
     $\langle C \in ?N \rangle n-s \langle ?M \models_{as} CNot C \rangle \text{bj-backjump inv } \langle \text{no-dup } (\text{trail } S) \rangle$ 
    unfolding M-K by auto
  then show ?thesis by fast
qed auto
qed

end

locale dpll-with-backjumping =
  dpll-with-backjumping-ops mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clss_{NOT} remove-clss_{NOT} inv backjump-conds
  propagate-conds
for
  mset-clss ::  $'cls \Rightarrow 'v \text{ clause}$  and
  insert-clss ::  $'v \text{ literal} \Rightarrow 'cls \Rightarrow 'cls$  and
  remove-lit ::  $'v \text{ literal} \Rightarrow 'cls \Rightarrow 'cls$  and
  mset-clss ::  $'clss \Rightarrow 'v \text{ clauses}$  and
  union-clss ::  $'clss \Rightarrow 'clss \Rightarrow 'clss$  and
  in-clss ::  $'cls \Rightarrow 'clss \Rightarrow \text{bool}$  and
  insert-clss ::  $'cls \Rightarrow 'clss \Rightarrow 'clss$  and
  remove-from-clss ::  $'cls \Rightarrow 'clss \Rightarrow 'clss$  and
  trail ::  $'st \Rightarrow ('v, \text{unit}, \text{unit}) \text{ marked-lits}$  and
  raw-clauses ::  $'st \Rightarrow 'clss$  and
  prepend-trail ::  $('v, \text{unit}, \text{unit}) \text{ marked-lit} \Rightarrow 'st \Rightarrow 'st$  and
  tl-trail ::  $'st \Rightarrow 'st$  and
  add-clss_{NOT} ::  $'cls \Rightarrow 'st \Rightarrow 'st$  and
  remove-clss_{NOT} ::  $'cls \Rightarrow 'st \Rightarrow 'st$  and
  inv ::  $'st \Rightarrow \text{bool}$  and
  backjump-conds ::  $'v \text{ clause} \Rightarrow 'v \text{ clause} \Rightarrow 'v \text{ literal} \Rightarrow 'st \Rightarrow 'st \Rightarrow \text{bool}$  and
  propagate-conds ::  $('v, \text{unit}, \text{unit}) \text{ marked-lit} \Rightarrow 'st \Rightarrow \text{bool}$ 
+
assumes dpll-bj-inv:  $\bigwedge S T. \text{dpll-bj } S T \implies \text{inv } S \implies \text{inv } T$ 
begin

lemma rtranclp-dpll-bj-inv:
  assumes dpll-bj*  $S T$  and inv S
  shows inv T
  using assms by (induction rule: rtranclp-induct)
  (auto simp add: dpll-bj-no-dup intro: dpll-bj-inv)

```


lemma *rtranclp-dpll-bj-no-dup*:

assumes *dpll-bj** S T and inv S*

and *no-dup (trail S)*

shows *no-dup (trail T)*

using *assms by (induction rule: rtranclp-induct)*

(auto simp add: dpll-bj-no-dup dest: rtranclp-dpll-bj-inv dpll-bj-inv)

lemma *rtranclp-dpll-bj-atms-of-ms-clauses-inv*:

assumes

*dpll-bj** S T and inv S*

shows *atms-of-mm (clauses_{NOT} S) = atms-of-mm (clauses_{NOT} T)*

using *assms by (induction rule: rtranclp-induct)*

(auto dest: rtranclp-dpll-bj-inv dpll-bj-atms-of-ms-clauses-inv)

lemma *rtranclp-dpll-bj-atms-in-trail*:

assumes

*dpll-bj** S T and*

inv S and

atm-of ' (lits-of-l (trail S)) \subseteq atms-of-mm (clauses_{NOT} S)

shows *atm-of ' (lits-of-l (trail T)) \subseteq atms-of-mm (clauses_{NOT} T)*

using *assms apply (induction rule: rtranclp-induct)*

using *dpll-bj-atms-in-trail dpll-bj-atms-of-ms-clauses-inv rtranclp-dpll-bj-inv by auto*

lemma *rtranclp-dpll-bj-sat-iff*:

assumes *dpll-bj** S T and inv S*

shows *$I \models_{sm} \text{clauses}_{NOT} S \longleftrightarrow I \models_{sm} \text{clauses}_{NOT} T$*

using *assms by (induction rule: rtranclp-induct)*

(auto dest!: dpll-bj-sat-iff simp: rtranclp-dpll-bj-inv)

lemma *rtranclp-dpll-bj-atms-in-trail-in-set*:

assumes

*dpll-bj** S T and*

inv S

atms-of-mm (clauses_{NOT} S) \subseteq A and

atm-of ' (lits-of-l (trail S)) \subseteq A

shows *atm-of ' (lits-of-l (trail T)) \subseteq A*

using *assms by (induction rule: rtranclp-induct)*

(auto dest: rtranclp-dpll-bj-inv

simp: dpll-bj-atms-in-trail-in-set rtranclp-dpll-bj-atms-of-ms-clauses-inv rtranclp-dpll-bj-inv)

lemma *rtranclp-dpll-bj-all-decomposition-implies-inv*:

assumes

*dpll-bj** S T and*

inv S

all-decomposition-implies-m (clauses_{NOT} S) (get-all-marked-decomposition (trail S))

shows *all-decomposition-implies-m (clauses_{NOT} T) (get-all-marked-decomposition (trail T))*

using *assms by (induction rule: rtranclp-induct)*

(auto intro: dpll-bj-all-decomposition-implies-inv simp: rtranclp-dpll-bj-inv)

lemma *rtranclp-dpll-bj-inv-incl-dpll-bj-inv-trancl*:

{(T, S). dpll-bj⁺⁺ S T

\wedge atms-of-mm (clauses_{NOT} S) \subseteq atms-of-ms A \wedge atm-of ' lits-of-l (trail S) \subseteq atms-of-ms A

\wedge no-dup (trail S) \wedge inv S}

\subseteq {(T, S). dpll-bj S T \wedge atms-of-mm (clauses_{NOT} S) \subseteq atms-of-ms A

\wedge atm-of ' lits-of-l (trail S) \subseteq atms-of-ms A \wedge no-dup (trail S) \wedge inv S}⁺

```

  (is ?A  $\subseteq$  ?B+)
proof standard
  fix x
  assume x-A: x  $\in$  ?A
  obtain S T :: 'st where
    x[simp]: x = (T, S) by (cases x) auto
  have
    dpll-bj++ S T and
    atms-of-mm (clausesNOT S)  $\subseteq$  atms-of-ms A and
    atm-of ' lits-of-l (trail S)  $\subseteq$  atms-of-ms A and
    no-dup (trail S) and
    inv S
  using x-A by auto
  then show x  $\in$  ?B+ unfolding x
  proof (induction rule: tranclp-induct)
    case base
    then show ?case by auto
  next
  case (step T U) note step = this(1) and ST = this(2) and IH = this(3)[OF this(4-7)]
    and N-A = this(4) and M-A = this(5) and nd = this(6) and inv = this(7)

    have [simp]: atms-of-mm (clausesNOT S) = atms-of-mm (clausesNOT T)
    using step rtranclp-dpll-bj-atms-of-ms-clauses-inv tranclp-into-rtranclp inv by fastforce
    have no-dup (trail T)
    using local.step nd rtranclp-dpll-bj-no-dup tranclp-into-rtranclp inv by fastforce
    moreover have atm-of ' (lits-of-l (trail T))  $\subseteq$  atms-of-ms A
    by (metis inv M-A N-A local.step rtranclp-dpll-bj-atms-in-trail-in-set
      tranclp-into-rtranclp)
    moreover have inv T
    using inv local.step rtranclp-dpll-bj-inv tranclp-into-rtranclp by fastforce
    ultimately have (U, T)  $\in$  ?B using ST N-A M-A inv by auto
    then show ?case using IH by (rule trancl-into-trancl2)
  qed
qed

```

```

lemma wf-tranclp-dpll-bj:
  assumes fin: finite A
  shows wf {(T, S). dpll-bj++ S T
     $\wedge$  atms-of-mm (clausesNOT S)  $\subseteq$  atms-of-ms A  $\wedge$  atm-of ' lits-of-l (trail S)  $\subseteq$  atms-of-ms A
     $\wedge$  no-dup (trail S)  $\wedge$  inv S}
  using wf-trancl[OF wf-dpll-bj[OF fin]] rtranclp-dpll-bj-inv-incl-dpll-bj-inv-trancl
  by (rule wf-subset)

```

```

lemma dpll-bj-sat-ext-iff:
  dpll-bj S T  $\implies$  inv S  $\implies$  Isextm clausesNOT S  $\longleftrightarrow$  Isextm clausesNOT T
  by (simp add: dpll-bj-clauses)

```

```

lemma rtranclp-dpll-bj-sat-ext-iff:
  dpll-bj** S T  $\implies$  inv S  $\implies$  Isextm clausesNOT S  $\longleftrightarrow$  Isextm clausesNOT T
  by (induction rule: rtranclp-induct) (simp-all add: rtranclp-dpll-bj-inv dpll-bj-sat-ext-iff)

```

```

theorem full-dpll-backjump-final-state:
  fixes A :: 'v literal multiset set and S T :: 'st
  assumes
    full: full dpll-bj S T and

```

atms-S: *atms-of-mm* (*clauses*_{NOT} *S*) \subseteq *atms-of-ms* *A* **and**
atms-trail: *atm-of* ‘*lits-of-l* (*trail S*) \subseteq *atms-of-ms* *A* **and**
n-d: *no-dup* (*trail S*) **and**
finite A **and**
inv: *inv S* **and**
decomp: *all-decomposition-implies-m* (*clauses*_{NOT} *S*) (*get-all-marked-decomposition* (*trail S*))
shows *unsatisfiable* (*set-mset* (*clauses*_{NOT} *S*))
 \vee (*trail T* \models_{asm} *clauses*_{NOT} *S* \wedge *satisfiable* (*set-mset* (*clauses*_{NOT} *S*)))
proof –
have *st*: *dpll-bj*** *S T* **and** *no-step dpll-bj T*
using *full unfolding full-def* **by** *fast+*
moreover have *atms-of-mm* (*clauses*_{NOT} *T*) \subseteq *atms-of-ms* *A*
using *atms-S inv rtranclp-dpll-bj-atms-of-ms-clauses-inv st* **by** *blast*
moreover have *atm-of* ‘*lits-of-l* (*trail T*) \subseteq *atms-of-ms* *A*
using *atms-S atms-trail inv rtranclp-dpll-bj-atms-in-trail-in-set st* **by** *auto*
moreover have *no-dup* (*trail T*)
using *n-d inv rtranclp-dpll-bj-no-dup st* **by** *blast*
moreover have *inv*: *inv T*
using *inv rtranclp-dpll-bj-inv st* **by** *blast*
moreover
have *decomp*: *all-decomposition-implies-m* (*clauses*_{NOT} *T*) (*get-all-marked-decomposition* (*trail T*))
using $\langle inv S \rangle$ *decomp rtranclp-dpll-bj-all-decomposition-implies-inv st* **by** *blast*
ultimately have *unsatisfiable* (*set-mset* (*clauses*_{NOT} *T*))
 \vee (*trail T* \models_{asm} *clauses*_{NOT} *T* \wedge *satisfiable* (*set-mset* (*clauses*_{NOT} *T*)))
using $\langle finite A \rangle$ *dpll-backjump-final-state* **by** *force*
then show *?thesis*
by (*meson* $\langle inv S \rangle$ *rtranclp-dpll-bj-sat-iff satisfiable-carac st true-annots-true-cls*)
qed

corollary *full-dpll-backjump-final-state-from-init-state*:

fixes *A* :: ‘*v* literal multiset set **and** *S T* :: ‘*st*
assumes
full: *full dpll-bj S T* **and**
trail S = [] **and**
*clauses*_{NOT} *S* = *N* **and**
inv S
shows *unsatisfiable* (*set-mset N*) \vee (*trail T* \models_{asm} *N* \wedge *satisfiable* (*set-mset N*))
using *assms full-dpll-backjump-final-state[of S T set-mset N]* **by** *auto*

lemma *tranclp-dpll-bj-trail-mes-decreasing-prop*:

assumes *dpll*: *dpll-bj⁺⁺ S T* **and** *inv*: *inv S* **and**
N-A: *atms-of-mm* (*clauses*_{NOT} *S*) \subseteq *atms-of-ms* *A* **and**
M-A: *atm-of* ‘*lits-of-l* (*trail S*) \subseteq *atms-of-ms* *A* **and**
n-d: *no-dup* (*trail S*) **and**
fin-A: *finite A*
shows $(2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $\quad - \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } T)$
 $\quad < (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $\quad - \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } S)$

using *dpll*

proof (*induction*)

case *base*

then show *?case*

using *N-A M-A n-d dpll-bj-trail-mes-decreasing-prop fin-A inv* **by** *blast*

next

case (*step* T U) **note** $st = \text{this}(1)$ **and** $dpll = \text{this}(2)$ **and** $IH = \text{this}(3)$
have $\text{atms-of-mm} (\text{clauses}_{NOT} S) = \text{atms-of-mm} (\text{clauses}_{NOT} T)$
using $\text{rtrancpl-dpll-bj-atms-of-ms-clauses-inv}$ **by** ($\text{metis dpll-bj-clauses dpll-bj-inv inv st trancplD}$)
then have $N\text{-}A'$: $\text{atms-of-mm} (\text{clauses}_{NOT} T) \subseteq \text{atms-of-ms } A$
using $N\text{-}A$ **by** *auto*
moreover have $M\text{-}A'$: $\text{atm-of } ' \text{ lits-of-l } (\text{trail } T) \subseteq \text{atms-of-ms } A$
by ($\text{meson } M\text{-}A \text{ } N\text{-}A \text{ inv rtrancpl-dpll-bj-atms-in-trail-in-set st dpll trancpl.r-into-trancpl trancpl-into-rtrancpl trancpl-trans}$)
moreover have nd : $\text{no-dup} (\text{trail } T)$
by ($\text{metis inv n-d rtrancpl-dpll-bj-no-dup st trancpl-into-rtrancpl}$)
moreover have $\text{inv } T$
by ($\text{meson dpll dpll-bj-inv inv rtrancpl-dpll-bj-inv st trancpl-into-rtrancpl}$)
ultimately show $?case$
using $IH \text{ dpll-bj-trail-mes-decreasing-prop[of } T \text{ } U \text{ } A] \text{ dpll fin-}A$ **by** *linarith*
qed
end

16.4 CDCL

16.4.1 Learn and Forget

locale *learn-ops* =
 $\text{dpll-state mset-cls insert-cls remove-lit}$
 $\text{mset-clss union-clss in-clss insert-clss remove-from-clss}$
 $\text{trail raw-clauses prepend-trail tl-trail add-clss}_{NOT} \text{ remove-clss}_{NOT}$
for
 $\text{mset-clss} :: 'cls \Rightarrow 'v \text{ clause and}$
 $\text{insert-clss} :: 'v \text{ literal} \Rightarrow 'cls \Rightarrow 'cls \text{ and}$
 $\text{remove-lit} :: 'v \text{ literal} \Rightarrow 'cls \Rightarrow 'cls \text{ and}$
 $\text{mset-clss} :: 'clss \Rightarrow 'v \text{ clauses and}$
 $\text{union-clss} :: 'clss \Rightarrow 'clss \Rightarrow 'clss \text{ and}$
 $\text{in-clss} :: 'cls \Rightarrow 'clss \Rightarrow \text{bool and}$
 $\text{insert-clss} :: 'cls \Rightarrow 'clss \Rightarrow 'clss \text{ and}$
 $\text{remove-from-clss} :: 'cls \Rightarrow 'clss \Rightarrow 'clss \text{ and}$
 $\text{trail} :: 'st \Rightarrow ('v, \text{unit}, \text{unit}) \text{ marked-lits and}$
 $\text{raw-clauses} :: 'st \Rightarrow 'clss \text{ and}$
 $\text{prepend-trail} :: ('v, \text{unit}, \text{unit}) \text{ marked-lit} \Rightarrow 'st \Rightarrow 'st \text{ and}$
 $\text{tl-trail} :: 'st \Rightarrow 'st \text{ and}$
 $\text{add-clss}_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st \text{ and}$
 $\text{remove-clss}_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st +$
fixes
 $\text{learn-cond} :: 'cls \Rightarrow 'st \Rightarrow \text{bool}$
begin
inductive $\text{learn} :: 'st \Rightarrow 'st \Rightarrow \text{bool where}$
 $\text{learn}_{NOT}\text{-rule: clauses}_{NOT} S \models_{pm} \text{mset-clss } C \implies$
 $\text{atms-of } (\text{mset-clss } C) \subseteq \text{atms-of-mm} (\text{clauses}_{NOT} S) \cup \text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \implies$
 $\text{learn-cond } C S \implies$
 $T \sim \text{add-clss}_{NOT} C S \implies$
 $\text{learn } S T$
inductive-cases $\text{learn}_{NOT}E$: $\text{learn } S T$

lemma $\text{learn-}\mu_C\text{-stable}$:
assumes $\text{learn } S T$ **and** $\text{no-dup} (\text{trail } S)$
shows $\mu_C A B (\text{trail-weight } S) = \mu_C A B (\text{trail-weight } T)$

```

using assms by (auto elim: learnNOTE)
end

locale forget-ops =
  dpll-state mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clssNOT remove-clssNOT
for
  mset-clss:: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
  raw-clauses :: 'st  $\Rightarrow$  'clss and
  prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
  tl-trail :: 'st  $\Rightarrow$  'st and
  add-clssNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  remove-clssNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st +
fixes
  forget-cond :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  bool
begin
inductive forgetNOT :: 'st  $\Rightarrow$  'st  $\Rightarrow$  bool where
forgetNOT:
  removeAll-mset (mset-cls C)(clausesNOT S)  $\models_{pm}$  mset-cls C  $\Rightarrow$ 
  forget-cond C S  $\Rightarrow$ 
  C ! $\in$ ! raw-clauses S  $\Rightarrow$ 
  T  $\sim$  remove-clssNOT C S  $\Rightarrow$ 
  forgetNOT S T
inductive-cases forgetNOTE: forgetNOT S T

lemma forget- $\mu_C$ -stable:
  assumes forgetNOT S T
  shows  $\mu_C A B$  (trail-weight S) =  $\mu_C A B$  (trail-weight T)
  using assms by (auto elim!: forgetNOTE)
end

locale learn-and-forgetNOT =
  learn-ops mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clssNOT remove-clssNOT learn-cond +
  forget-ops mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clssNOT remove-clssNOT forget-cond
for
  mset-clss:: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and

```

```

remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
trail :: 'st ⇒ ('v, unit, unit) marked-lits and
raw-clauses :: 'st ⇒ 'clss and
prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
tl-trail :: 'st ⇒ 'st and
add-clssNOT :: 'cls ⇒ 'st ⇒ 'st and
remove-clssNOT :: 'cls ⇒ 'st ⇒ 'st and
learn-cond forget-cond :: 'cls ⇒ 'st ⇒ bool
begin
inductive learn-and-forgetNOT :: 'st ⇒ 'st ⇒ bool
where
lf-learn: learn S T ⇒ learn-and-forgetNOT S T |
lf-forget: forgetNOT S T ⇒ learn-and-forgetNOT S T
end

```

16.4.2 Definition of CDCL

```

locale conflict-driven-clause-learning-ops =
  dpll-with-backjumping-ops mset-clss insert-clss remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clssNOT remove-clssNOT
  inv backjump-conds propagate-conds +
  learn-and-forgetNOT mset-clss insert-clss remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clssNOT remove-clssNOT learn-cond
  forget-cond
for
  mset-clss:: 'cls ⇒ 'v clause and
  insert-clss :: 'v literal ⇒ 'cls ⇒ 'cls and
  remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
  mset-clss:: 'clss ⇒ 'v clauses and
  union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
  in-clss :: 'cls ⇒ 'clss ⇒ bool and
  insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
  remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
  trail :: 'st ⇒ ('v, unit, unit) marked-lits and
  raw-clauses :: 'st ⇒ 'clss and
  prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
  tl-trail :: 'st ⇒ 'st and
  add-clssNOT :: 'cls ⇒ 'st ⇒ 'st and
  remove-clssNOT :: 'cls ⇒ 'st ⇒ 'st and
  inv :: 'st ⇒ bool and
  backjump-conds :: 'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool and
  propagate-conds :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ bool and
  learn-cond forget-cond :: 'cls ⇒ 'st ⇒ bool
begin

inductive cdclNOT :: 'st ⇒ 'st ⇒ bool for S :: 'st where
c-dpll-bj: dpll-bj S S' ⇒ cdclNOT S S' |
c-learn: learn S S' ⇒ cdclNOT S S' |
c-forgetNOT: forgetNOT S S' ⇒ cdclNOT S S'

lemma cdclNOT-all-induct[consumes 1, case-names dpll-bj learn forgetNOT]:
  fixes S T :: 'st
  assumes cdclNOT S T and
  dpll: ∧T. dpll-bj S T ⇒ P S T and

```

learning:

$\bigwedge C T. \text{ clauses}_{NOT} S \models_{pm} \text{ mset-cls } C \implies$
 $\text{atms-of } (\text{mset-cls } C) \subseteq \text{atms-of-mm } (\text{clauses}_{NOT} S) \cup \text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \implies$
 $T \sim \text{add-cl}_{NOT} C S \implies$

P S T and

forgetting: $\bigwedge C T. \text{ removeAll-mset } (\text{mset-cls } C) (\text{clauses}_{NOT} S) \models_{pm} \text{ mset-cls } C \implies$

$C !\in! \text{ raw-clauses } S \implies$

$T \sim \text{remove-cl}_{NOT} C S \implies$

P S T

shows *P S T*

using *assms(1)* **by** (*induction rule: cdcl_{NOT}.induct*)

(*auto intro: assms(2, 3, 4) elim!: learn_{NOT}E forget_{NOT}E*)+

lemma *cdcl_{NOT}-no-dup:*

assumes

cdcl_{NOT} S T and

inv S and

no-dup (trail S)

shows *no-dup (trail T)*

using *assms* **by** (*induction rule: cdcl_{NOT}-all-induct*) (*auto intro: dpll-bj-no-dup*)

Consistency of the trail lemma *cdcl_{NOT}-consistent:*

assumes

cdcl_{NOT} S T and

inv S and

no-dup (trail S)

shows *consistent-interp (lits-of-l (trail T))*

using *cdcl_{NOT}-no-dup[OF assms] distinct-consistent-interp* **by** *fast*

The subtle problem here is that tautologies can be removed, meaning that some variable can disappear of the problem. It is also possible that some variable of the trail are not in the clauses anymore.

lemma *cdcl_{NOT}-atms-of-ms-clauses-decreasing:*

assumes *cdcl_{NOT} S T and inv S and no-dup (trail S)*

shows $\text{atms-of-mm } (\text{clauses}_{NOT} T) \subseteq \text{atms-of-mm } (\text{clauses}_{NOT} S) \cup \text{atm-of } ' (\text{lits-of-l } (\text{trail } S))$

using *assms* **by** (*induction rule: cdcl_{NOT}-all-induct*)

(*auto dest!: dpll-bj-atms-of-ms-clauses-inv set-mp simp add: atms-of-ms-def Union-eq*)

lemma *cdcl_{NOT}-atms-in-trail:*

assumes *cdcl_{NOT} S T and inv S and no-dup (trail S)*

and $\text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-mm } (\text{clauses}_{NOT} S)$

shows $\text{atm-of } ' (\text{lits-of-l } (\text{trail } T)) \subseteq \text{atms-of-mm } (\text{clauses}_{NOT} S)$

using *assms* **by** (*induction rule: cdcl_{NOT}-all-induct*) (*auto simp add: dpll-bj-atms-in-trail*)

lemma *cdcl_{NOT}-atms-in-trail-in-set:*

assumes

cdcl_{NOT} S T and inv S and no-dup (trail S) and

atms-of-mm (clauses_{NOT} S) \subseteq A and

atm-of ' (lits-of-l (trail S)) \subseteq A

shows $\text{atm-of } ' (\text{lits-of-l } (\text{trail } T)) \subseteq A$

using *assms*

by (*induction rule: cdcl_{NOT}-all-induct*)

(*simp-all add: dpll-bj-atms-in-trail-in-set dpll-bj-atms-of-ms-clauses-inv*)

lemma *cdcl_{NOT}-all-decomposition-implies:*

```

assumes  $cdcl_{NOT} S T$  and  $inv S$  and  $n-d[simp]: no-dup (trail S)$  and
   $all-decomposition-implies-m (clauses_{NOT} S) (get-all-marked-decomposition (trail S))$ 
shows
   $all-decomposition-implies-m (clauses_{NOT} T) (get-all-marked-decomposition (trail T))$ 
using  $assms(1,2,4)$ 
proof (induction rule:  $cdcl_{NOT}$ -all-induct)
  case  $dpll-bj$ 
  then show  $?case$ 
    using  $dpll-bj-all-decomposition-implies-inv n-d$  by  $blast$ 
next
  case  $learn$ 
  then show  $?case$  by ( $auto simp add: all-decomposition-implies-def$ )
next
  case ( $forget_{NOT} C T$ ) note  $cls-C = this(1)$  and  $C = this(2)$  and  $T = this(3)$  and  $iniv = this(4)$ 
and
   $decomp = this(5)$ 
show  $?case$ 
  unfolding  $all-decomposition-implies-def Ball-def$ 
proof (intro allI, clarify)
  fix  $a b$ 
  assume  $(a, b) \in set (get-all-marked-decomposition (trail T))$ 
  then have  $unmark-l a \cup set-mset (clauses_{NOT} S) \models_{ps} unmark-l b$ 
    using  $decomp T$  by ( $auto simp add: all-decomposition-implies-def$ )
  moreover
    have  $a1:mset-cls C \in set-mset (clauses_{NOT} S)$ 
      using  $C$  by  $blast$ 
    have  $clauses_{NOT} T = clauses_{NOT} (remove-cls_{NOT} C S)$ 
      using  $T state-eq_{NOT}-clauses$  by  $blast$ 
    then have  $set-mset (clauses_{NOT} T) \models_{ps} set-mset (clauses_{NOT} S)$ 
      using  $a1$  by ( $metis (no-types) clauses-remove-cls_{NOT} cls-C insert-Diff order-refl$ 
         $set-mset-minus-replicate-mset(1) true-clss-clss-def true-clss-clss-insert$ )
    ultimately show  $unmark-l a \cup set-mset (clauses_{NOT} T) \models_{ps} unmark-l b$ 
      using  $true-clss-clss-generalise-true-clss-clss$  by  $blast$ 
  qed
qed

```

Extension of models lemma $cdcl_{NOT}-bj-sat-ext-iff$:

```

assumes  $cdcl_{NOT} S T$  and  $inv S$  and  $n-d: no-dup (trail S)$ 
shows  $I \models_{sextm} clauses_{NOT} S \longleftrightarrow I \models_{sextm} clauses_{NOT} T$ 
using  $assms$ 
proof (induction rule:  $cdcl_{NOT}$ -all-induct)
  case  $dpll-bj$ 
  then show  $?case$  by ( $simp add: dpll-bj-clauses$ )
next
  case ( $learn C T$ ) note  $T = this(3)$ 
  { fix  $J$ 
    assume
       $I \models_{sextm} clauses_{NOT} S$  and
       $I \subseteq J$  and
       $tot: total-over-m J (set-mset (\{ \#mset-cls C \# \} + clauses_{NOT} S))$  and
       $cons: consistent-interp J$ 
    then have  $J \models_{sm} clauses_{NOT} S$  unfolding  $true-clss-ext-def$  by  $auto$ 

    moreover

```



```

    with  $\langle \text{clauses}_{NOT} S \models_{pm} \text{mset-cl} C \rangle$  have  $J \models \text{mset-cl} C$ 
    using tot cons unfolding true-clss-cl-def by auto
    ultimately have  $J \models_{sm} \{\# \text{mset-cl} C\} + \text{clauses}_{NOT} S$  by auto
  }
then have  $H: I \models_{sextm} (\text{clauses}_{NOT} S) \implies I \models_{sext} \text{insert} (\text{mset-cl} C) (\text{set-mset} (\text{clauses}_{NOT} S))$ 
  unfolding true-clss-ext-def by auto
show ?case
  apply standard
  using  $T$  n-d apply (auto simp add: H)[]
  using  $T$  n-d apply simp
  by (metis Diff-insert-absorb insert-subset subsetI subset-antisym
    true-clss-ext-decrease-right-remove-r)
next
case (forgetNOT  $C$   $T$ ) note  $\text{cls-}C = \text{this}(1)$  and  $T = \text{this}(3)$ 
{ fix  $J$ 
  assume
     $I \models_{sext} \text{set-mset} (\text{clauses}_{NOT} S) - \{\text{mset-cl} C\}$  and
     $I \subseteq J$  and
    tot: total-over-m  $J$  (set-mset ( $\text{clauses}_{NOT} S$ )) and
    cons: consistent-interp  $J$ 
  then have  $J \models_s \text{set-mset} (\text{clauses}_{NOT} S) - \{\text{mset-cl} C\}$ 
    unfolding true-clss-ext-def by (meson Diff-subset total-over-m-subset)

  moreover
    with  $\text{cls-}C$  have  $J \models \text{mset-cl} C$ 
    using tot cons unfolding true-clss-cl-def
    by (metis Un-commute forgetNOT.hypos(2) in-clss-mset-clss insert-Diff insert-is-Un order-refl
      set-mset-minus-replicate-mset(1))
    ultimately have  $J \models_{sm} (\text{clauses}_{NOT} S)$  by (metis insert-Diff-single true-clss-insert)
  }
  then have  $H: I \models_{sext} \text{set-mset} (\text{clauses}_{NOT} S) - \{\text{mset-cl} C\} \implies I \models_{sextm} (\text{clauses}_{NOT} S)$ 
    unfolding true-clss-ext-def by blast
  show ?case using  $T$  by (auto simp: true-clss-ext-decrease-right-remove-r H)
qed

end — end of conflict-driven-clause-learning-ops

```

16.5 CDCL with invariant

```

locale conflict-driven-clause-learning =
  conflict-driven-clause-learning-ops +
  assumes cdclNOT-inv:  $\bigwedge S T. \text{cdcl}_{NOT} S T \implies \text{inv} S \implies \text{inv} T$ 
begin
sublocale dpll-with-backjumping
  apply unfold-locales
  using cdclNOT.simps cdclNOT-inv by auto

lemma rtranclp-cdclNOT-inv:
  cdclNOT**  $S T \implies \text{inv} S \implies \text{inv} T$ 
  by (induction rule: rtranclp-induct) (auto simp add: cdclNOT-inv)

lemma rtranclp-cdclNOT-no-dup:
  assumes cdclNOT**  $S T$  and inv  $S$ 
  and no-dup (trail  $S$ )
  shows no-dup (trail  $T$ )
  using assms by (induction rule: rtranclp-induct) (auto intro: cdclNOT-no-dup rtranclp-cdclNOT-inv)

```

lemma *rtrancpl-cdcl_{NOT}-trail-clauses-bound*:

assumes

cdcl: *cdcl_{NOT}** S T* **and**

inv: *inv S* **and**

n-d: *no-dup (trail S)* **and**

atms-clauses-S: *atms-of-mm (clauses_{NOT} S) ⊆ A* **and**

atms-trail-S: *atm-of ‘(lits-of-l (trail S)) ⊆ A*

shows *atm-of ‘(lits-of-l (trail T)) ⊆ A ∧ atms-of-mm (clauses_{NOT} T) ⊆ A*

using *cdcl*

proof (*induction rule: rtrancpl-induct*)

case *base*

then show *?case* **using** *atms-clauses-S atms-trail-S* **by** *simp*

next

case (*step T U*) **note** *st = this(1)* **and** *cdcl_{NOT} = this(2)* **and** *IH = this(3)*

have *inv T* **using** *inv st rtrancpl-cdcl_{NOT}-inv* **by** *blast*

have *no-dup (trail T)*

using *rtrancpl-cdcl_{NOT}-no-dup[of S T]* *st cdcl_{NOT} inv n-d* **by** *blast*

then have *atms-of-mm (clauses_{NOT} U) ⊆ A*

using *cdcl_{NOT}-atms-of-ms-clauses-decreasing[OF cdcl_{NOT}] IH n-d ⟨inv T⟩* **by** *fast*

moreover

have *atm-of ‘(lits-of-l (trail U)) ⊆ A*

using *cdcl_{NOT}-atms-in-trail-in-set[OF cdcl_{NOT}, of A] ⟨no-dup (trail T)⟩*

by (*meson atms-trail-S atms-clauses-S IH ⟨inv T⟩ cdcl_{NOT}*)

ultimately show *?case* **by** *fast*

qed

lemma *rtrancpl-cdcl_{NOT}-all-decomposition-implies*:

assumes *cdcl_{NOT}** S T* **and** *inv S* **and** *no-dup (trail S)* **and**

all-decomposition-implies-m (clauses_{NOT} S) (get-all-marked-decomposition (trail S))

shows

all-decomposition-implies-m (clauses_{NOT} T) (get-all-marked-decomposition (trail T))

using *assms* **by** (*induction*)

(*auto intro: rtrancpl-cdcl_{NOT}-inv cdcl_{NOT}-all-decomposition-implies rtrancpl-cdcl_{NOT}-no-dup*)

lemma *rtrancpl-cdcl_{NOT}-bj-sat-ext-iff*:

assumes *cdcl_{NOT}** S T* **and** *inv S* **and** *no-dup (trail S)*

shows $I \models_{\text{sextm}} \text{clauses}_{\text{NOT}} S \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{\text{NOT}} T$

using *assms* **apply** (*induction rule: rtrancpl-induct*)

using *cdcl_{NOT}-bj-sat-ext-iff* **by** (*auto intro: rtrancpl-cdcl_{NOT}-inv rtrancpl-cdcl_{NOT}-no-dup*)

definition *cdcl_{NOT}-NOT-all-inv* **where**

$\text{cdcl}_{\text{NOT}}\text{-NOT-all-inv } A \ S \longleftrightarrow (\text{finite } A \wedge \text{inv } S \wedge \text{atms-of-mm } (\text{clauses}_{\text{NOT}} S) \subseteq \text{atms-of-ms } A$
 $\wedge \text{atm-of ‘lits-of-l (trail S) } \subseteq \text{atms-of-ms } A \wedge \text{no-dup (trail S)})$

lemma *cdcl_{NOT}-NOT-all-inv*:

assumes *cdcl_{NOT}** S T* **and** *cdcl_{NOT}-NOT-all-inv A S*

shows *cdcl_{NOT}-NOT-all-inv A T*

using *assms* **unfolding** *cdcl_{NOT}-NOT-all-inv-def*

by (*simp add: rtrancpl-cdcl_{NOT}-inv rtrancpl-cdcl_{NOT}-no-dup rtrancpl-cdcl_{NOT}-trail-clauses-bound*)

abbreviation *learn-or-forget* **where**

$\text{learn-or-forget } S \ T \equiv \text{learn } S \ T \vee \text{forget}_{\text{NOT}} S \ T$

lemma *rtrancpl-learn-or-forget-cdcl_{NOT}*:

*learn-or-forget*** $S \ T \implies \text{cdcl}_{NOT}$ ** $S \ T$
using *rtrancpl-mono*[*of learn-or-forget cdcl_{NOT}*] **by** (*blast intro: cdcl_{NOT}.c-learn cdcl_{NOT}.c-forget_{NOT}*)

lemma *learn-or-forget-dpll- μ_C* :

assumes

*l-f: learn-or-forget*** $S \ T$ **and**

dpll: dpll-bj $T \ U$ **and**

inv: cdcl_{NOT}-NOT-all-inv $A \ S$

shows $(2 + \text{card}(\text{atms-of-ms } A)) \wedge (1 + \text{card}(\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card}(\text{atms-of-ms } A)) (2 + \text{card}(\text{atms-of-ms } A)) (\text{trail-weight } U)$
 $< (2 + \text{card}(\text{atms-of-ms } A)) \wedge (1 + \text{card}(\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card}(\text{atms-of-ms } A)) (2 + \text{card}(\text{atms-of-ms } A)) (\text{trail-weight } S)$
(is $?_{\mu} U < ?_{\mu} S$ **)**

proof –

have $?_{\mu} S = ?_{\mu} T$

using *l-f*

proof (*induction*)

case *base*

then show *?case* **by** *simp*

next

case (*step* $T \ U$)

moreover then have *no-dup* (*trail* T)

using *rtrancpl-cdcl_{NOT}-no-dup*[*of S T*] *cdcl_{NOT}-NOT-all-inv-def inv*

rtrancpl-learn-or-forget-cdcl_{NOT} **by** *auto*

ultimately show *?case*

using *forget- μ_C -stable learn- μ_C -stable inv* **unfolding** *cdcl_{NOT}-NOT-all-inv-def* **by** *presburger*

qed

moreover have *cdcl_{NOT}-NOT-all-inv* $A \ T$

using *rtrancpl-learn-or-forget-cdcl_{NOT}* *cdcl_{NOT}-NOT-all-inv l-f inv* **by** *blast*

ultimately show *?thesis*

using *dpll-bj-trail-mes-decreasing-prop*[*of T U A, OF dpll*] *finite*

unfolding *cdcl_{NOT}-NOT-all-inv-def* **by** *presburger*

qed

lemma *infinite-cdcl_{NOT}-exists-learn-and-forget-infinite-chain*:

assumes

$\bigwedge i. \text{cdcl}_{NOT} (f \ i) (f (Suc \ i))$ **and**

inv: cdcl_{NOT}-NOT-all-inv $A \ (f \ 0)$

shows $\exists j. \forall i \geq j. \text{learn-or-forget} (f \ i) (f (Suc \ i))$

using *assms*

proof (*induction* $(2 + \text{card}(\text{atms-of-ms } A)) \wedge (1 + \text{card}(\text{atms-of-ms } A))$)

$- \mu_C (1 + \text{card}(\text{atms-of-ms } A)) (2 + \text{card}(\text{atms-of-ms } A)) (\text{trail-weight } (f \ 0))$

arbitrary: f

rule: nat-less-induct-case)

case (*Suc* n) **note** $IH = \text{this}(1)$ **and** $\mu = \text{this}(2)$ **and** $\text{cdcl}_{NOT} = \text{this}(3)$ **and** $\text{inv} = \text{this}(4)$

consider

(*dpll-end*) $\exists j. \forall i \geq j. \text{learn-or-forget} (f \ i) (f (Suc \ i))$

| (*dpll-more*) $\neg(\exists j. \forall i \geq j. \text{learn-or-forget} (f \ i) (f (Suc \ i)))$

by *blast*

then show *?case*

proof *cases*

case *dpll-end*

then show *?thesis* **by** *auto*

next

case *dpll-more*

then have $j: \exists i. \neg \text{learn } (f \ i) \ (f \ (\text{Suc } i)) \wedge \neg \text{forget}_{NOT} \ (f \ i) \ (f \ (\text{Suc } i))$
by *blast*
obtain i **where**
 $\neg \text{learn } (f \ i) \ (f \ (\text{Suc } i)) \wedge \neg \text{forget}_{NOT} \ (f \ i) \ (f \ (\text{Suc } i))$ **and**
 $\forall k < i. \text{learn-or-forget} \ (f \ k) \ (f \ (\text{Suc } k))$
proof –
obtain i_0 **where** $\neg \text{learn } (f \ i_0) \ (f \ (\text{Suc } i_0)) \wedge \neg \text{forget}_{NOT} \ (f \ i_0) \ (f \ (\text{Suc } i_0))$
using j **by** *auto*
then have $\{i. i \leq i_0 \wedge \neg \text{learn } (f \ i) \ (f \ (\text{Suc } i)) \wedge \neg \text{forget}_{NOT} \ (f \ i) \ (f \ (\text{Suc } i))\} \neq \{\}$
by *auto*
let $?I = \{i. i \leq i_0 \wedge \neg \text{learn } (f \ i) \ (f \ (\text{Suc } i)) \wedge \neg \text{forget}_{NOT} \ (f \ i) \ (f \ (\text{Suc } i))\}$
let $?i = \text{Min } ?I$
have *finite* $?I$
by *auto*
have $\neg \text{learn } (f \ ?i) \ (f \ (\text{Suc } ?i)) \wedge \neg \text{forget}_{NOT} \ (f \ ?i) \ (f \ (\text{Suc } ?i))$
using *Min-in[OF <finite ?I> <?I ≠ {}>]* **by** *auto*
moreover have $\forall k < ?i. \text{learn-or-forget} \ (f \ k) \ (f \ (\text{Suc } k))$
using *Min.coboundedI[of {i. i ≤ i₀ ∧ ¬ learn (f i) (f (Suc i)) ∧ ¬ forget_{NOT} (f i) (f (Suc i))}, simplified]*
by (*meson* $\neg \text{learn } (f \ i_0) \ (f \ (\text{Suc } i_0)) \wedge \neg \text{forget}_{NOT} \ (f \ i_0) \ (f \ (\text{Suc } i_0))$) *less-imp-le dual-order.trans not-le*
ultimately show *?thesis* **using** *that* **by** *blast*
qed
def $g \equiv \lambda n. f \ (n + \text{Suc } i)$
have *dpll-bj* $(f \ i) \ (g \ 0)$
using $\neg \text{learn } (f \ i) \ (f \ (\text{Suc } i)) \wedge \neg \text{forget}_{NOT} \ (f \ i) \ (f \ (\text{Suc } i))$ *cdcl_{NOT} cdcl_{NOT}.cases*
g-def **by** *auto*
{
fix j
assume $j \leq i$
then have *learn-or-forget*** $(f \ 0) \ (f \ j)$
apply (*induction* j)
apply *simp*
by (*metis* (*no-types*, *lifting*) *Suc-leD Suc-le-lessD rtranclp.simps*
 $\langle \forall k < i. \text{learn } (f \ k) \ (f \ (\text{Suc } k)) \vee \text{forget}_{NOT} \ (f \ k) \ (f \ (\text{Suc } k)) \rangle$)
}
then have *learn-or-forget*** $(f \ 0) \ (f \ i)$ **by** *blast*
then have $(2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $- \mu_C \ (1 + \text{card } (\text{atms-of-ms } A)) \ (2 + \text{card } (\text{atms-of-ms } A)) \ (\text{trail-weight } (g \ 0))$
 $< (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $- \mu_C \ (1 + \text{card } (\text{atms-of-ms } A)) \ (2 + \text{card } (\text{atms-of-ms } A)) \ (\text{trail-weight } (f \ 0))$
using *learn-or-forget-dpll- μ_C [of f 0 f i g 0 A]* *inv* $\langle \text{dpll-bj } (f \ i) \ (g \ 0) \rangle$
unfolding *cdcl_{NOT}-NOT-all-inv-def* **by** *linarith*
moreover have *cdcl_{NOT}-i*: *cdcl_{NOT}*** $(f \ 0) \ (g \ 0)$
using *rtranclp-learn-or-forget-cdcl_{NOT}[of f 0 f i]* $\langle \text{learn-or-forget** } (f \ 0) \ (f \ i) \rangle$
cdcl_{NOT}[of i] **unfolding** *g-def* **by** *auto*
moreover have $\bigwedge i. \text{cdcl}_{NOT} \ (g \ i) \ (g \ (\text{Suc } i))$
using *cdcl_{NOT} g-def* **by** *auto*
moreover have *cdcl_{NOT}-NOT-all-inv* $A \ (g \ 0)$
using *inv cdcl_{NOT}-i rtranclp-cdcl_{NOT}-trail-clauses-bound g-def cdcl_{NOT}-NOT-all-inv* **by** *auto*
ultimately obtain j **where** $j: \bigwedge i. i \geq j \implies \text{learn-or-forget} \ (g \ i) \ (g \ (\text{Suc } i))$
using *IH* **unfolding** $\mu[\text{symmetric}]$ **by** *presburger*
show *?thesis*
proof

```

    {
      fix k
      assume  $k \geq j + \text{Suc } i$ 
      then have learn-or-forget (f k) (f (Suc k))
        using j[of k-Suc i] unfolding g-def by auto
    }
    then show  $\forall k \geq j + \text{Suc } i. \text{learn-or-forget } (f k) (f (Suc k))$ 
      by auto
  qed
next
case 0 note H = this(1) and cdclNOT = this(2) and inv = this(3)
show ?case
proof (rule ccontr)
  assume  $\neg ?case$ 
  then have j:  $\exists i. \neg \text{learn } (f i) (f (Suc i)) \wedge \neg \text{forget}_{NOT} (f i) (f (Suc i))$ 
    by blast
  obtain i where
     $\neg \text{learn } (f i) (f (Suc i)) \wedge \neg \text{forget}_{NOT} (f i) (f (Suc i))$  and
     $\forall k < i. \text{learn-or-forget } (f k) (f (Suc k))$ 
  proof -
    obtain i0 where  $\neg \text{learn } (f i_0) (f (Suc i_0)) \wedge \neg \text{forget}_{NOT} (f i_0) (f (Suc i_0))$ 
      using j by auto
    then have {i.  $i \leq i_0 \wedge \neg \text{learn } (f i) (f (Suc i)) \wedge \neg \text{forget}_{NOT} (f i) (f (Suc i))$ }  $\neq \{\}$ 
      by auto
    let ?I = {i.  $i \leq i_0 \wedge \neg \text{learn } (f i) (f (Suc i)) \wedge \neg \text{forget}_{NOT} (f i) (f (Suc i))$ }
    let ?i = Min ?I
    have finite ?I
      by auto
    have  $\neg \text{learn } (f ?i) (f (Suc ?i)) \wedge \neg \text{forget}_{NOT} (f ?i) (f (Suc ?i))$ 
      using Min-in[OF (finite ?I) (?I  $\neq \{\}$ )] by auto
    moreover have  $\forall k < ?i. \text{learn-or-forget } (f k) (f (Suc k))$ 
      using Min.coboundedI[of {i.  $i \leq i_0 \wedge \neg \text{learn } (f i) (f (Suc i)) \wedge \neg \text{forget}_{NOT} (f i) (f (Suc i))$ }, simplified]
      by (meson ( $\neg \text{learn } (f i_0) (f (Suc i_0)) \wedge \neg \text{forget}_{NOT} (f i_0) (f (Suc i_0))$ )> less-imp-le
        dual-order.trans not-le)
    ultimately show ?thesis using that by blast
  qed
  have dpll-bj (f i) (f (Suc i))
    using ( $\neg \text{learn } (f i) (f (Suc i)) \wedge \neg \text{forget}_{NOT} (f i) (f (Suc i))$ )> cdclNOT cdclNOT.cases
    by blast
  {
    fix j
    assume  $j \leq i$ 
    then have learn-or-forget** (f 0) (f j)
      apply (induction j)
      apply simp
      by (metis (no-types, lifting) Suc-leD Suc-le-lessD rtranclp.simps
        ( $\forall k < i. \text{learn } (f k) (f (Suc k)) \vee \text{forget}_{NOT} (f k) (f (Suc k))$ ))
  }
  then have learn-or-forget** (f 0) (f i) by blast

  then show False
    using learn-or-forget-dpll- $\mu_C$ [of f 0 f i f (Suc i) A] inv 0
    ( $\text{dpll-bj } (f i) (f (Suc i))$ )> unfolding cdclNOT-NOT-all-inv-def by linarith

```

qed
qed

lemma *wf-cdcl_{NOT}-no-learn-and-forget-infinite-chain*:

assumes

no-infinite-lf: $\bigwedge f j. \neg (\forall i \geq j. \text{learn-or-forget } (f i) (f (Suc i)))$

shows $wf \{(T, S). cdcl_{NOT} S T \wedge cdcl_{NOT-}NOT\text{-all-inv } A S\}$ **(is** $wf \{(T, S). cdcl_{NOT} S T$
 $\wedge ?inv S\}$ **)**

unfolding *wf-iff-no-infinite-down-chain*

proof (*rule ccontr*)

assume $\neg \neg (\exists f. \forall i. (f (Suc i), f i) \in \{(T, S). cdcl_{NOT} S T \wedge ?inv S\})$

then obtain *f* **where**

$\forall i. cdcl_{NOT} (f i) (f (Suc i)) \wedge ?inv (f i)$

by *fast*

then have $\exists j. \forall i \geq j. \text{learn-or-forget } (f i) (f (Suc i))$

using *infinite-cdcl_{NOT}-exists-learn-and-forget-infinite-chain*[*of f*] **by** *meson*

then show *False* **using** *no-infinite-lf* **by** *blast*

qed

lemma *inv-and-tranclp-cdcl_{NOT}-tranclp-cdcl_{NOT}-and-inv*:

$cdcl_{NOT}^{++} S T \wedge cdcl_{NOT-}NOT\text{-all-inv } A S \longleftrightarrow (\lambda S T. cdcl_{NOT} S T \wedge cdcl_{NOT-}NOT\text{-all-inv } A S)^{++} S T$

(is $?A \wedge ?I \longleftrightarrow ?B$ **)**

proof

assume $?A \wedge ?I$

then have $?A$ **and** $?I$ **by** *blast+*

then show $?B$

apply *induction*

apply (*simp add: tranclp.r-into-trancl*)

by (*subst tranclp.simps*) (*auto intro: cdcl_{NOT}-NOT-all-inv tranclp-into-rtranclp*)

next

assume $?B$

then have $?A$ **by** *induction auto*

moreover have $?I$ **using** $\langle ?B \rangle$ *tranclpD* **by** *fastforce*

ultimately show $?A \wedge ?I$ **by** *blast*

qed

lemma *wf-tranclp-cdcl_{NOT}-no-learn-and-forget-infinite-chain*:

assumes

no-infinite-lf: $\bigwedge f j. \neg (\forall i \geq j. \text{learn-or-forget } (f i) (f (Suc i)))$

shows $wf \{(T, S). cdcl_{NOT}^{++} S T \wedge cdcl_{NOT-}NOT\text{-all-inv } A S\}$

using *wf-tranclp*[*OF wf-cdcl_{NOT}-no-learn-and-forget-infinite-chain*[*OF no-infinite-lf*]]

apply (*rule wf-subset*)

by (*auto simp: trancl-set-tranclp inv-and-tranclp-cdcl_{NOT}-tranclp-cdcl_{NOT}-and-inv*)

lemma *cdcl_{NOT}-final-state*:

assumes

n-s: *no-step* $cdcl_{NOT} S$ **and**

inv: $cdcl_{NOT-}NOT\text{-all-inv } A S$ **and**

decomp: *all-decomposition-implies-m* ($clauses_{NOT} S$) (*get-all-marked-decomposition* ($trail S$))

shows *unsatisfiable* (*set-mset* ($clauses_{NOT} S$))

$\vee (trail S \models_{asm} clauses_{NOT} S \wedge \text{satisfiable } (set\text{-mset } (clauses_{NOT} S)))$

proof –

have *n-s'*: *no-step* *dpll-bj* S

using *n-s* **by** (*auto simp: cdcl_{NOT}.simps*)

show *?thesis*
apply (rule *dpll-backjump-final-state*[of *S A*])
using *inv decomp n-s'* **unfolding** *cdcl_{NOT}-NOT-all-inv-def* **by** *auto*
qed

lemma *full-cdcl_{NOT}-final-state*:

assumes
full: *full cdcl_{NOT} S T and*
inv: *cdcl_{NOT}-NOT-all-inv A S and*
n-d: *no-dup (trail S) and*
decomp: *all-decomposition-implies-m (clauses_{NOT} S) (get-all-marked-decomposition (trail S))*
shows *unsatisfiable (set-mset (clauses_{NOT} T))*
 \vee (*trail T* \models_{asm} *clauses_{NOT} T* \wedge *satisfiable (set-mset (clauses_{NOT} T))*)
proof –
have *st*: *cdcl_{NOT}** S T and n-s*: *no-step cdcl_{NOT} T*
using *full unfolding full-def* **by** *blast+*
have *n-s'*: *cdcl_{NOT}-NOT-all-inv A T*
using *cdcl_{NOT}-NOT-all-inv inv st* **by** *blast*
moreover have *all-decomposition-implies-m (clauses_{NOT} T) (get-all-marked-decomposition (trail T))*
using *cdcl_{NOT}-NOT-all-inv-def decomp inv rtranclp-cdcl_{NOT}-all-decomposition-implies st* **by** *auto*
ultimately show *?thesis*
using *cdcl_{NOT}-final-state n-s* **by** *blast*
qed

end — end of *conflict-driven-clause-learning*

16.6 Termination

16.6.1 Restricting learn and forget

locale *conflict-driven-clause-learning-learning-before-backjump-only-distinct-learn* =
dpll-state mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-cls_{NOT} remove-cls_{NOT} +
conflict-driven-clause-learning mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-cls_{NOT} remove-cls_{NOT}
inv backjump-conds propagate-conds
 $\lambda C S. \text{distinct-mset } (mset\text{-cls } C) \wedge \neg \text{tautology } (mset\text{-cls } C) \wedge \text{learn-restrictions } C S \wedge$
 $(\exists F K d F' C' L. \text{trail } S = F' @ \text{Marked } K () \# F \wedge mset\text{-cls } C = C' + \{\#L\# \} \wedge F \models_{as} CNot$
 C'
 $\wedge C' + \{\#L\# \} \notin \# \text{clauses}_{NOT} S)$
 $\lambda C S. \neg (\exists F' F K d L. \text{trail } S = F' @ \text{Marked } K () \# F \wedge F \models_{as} CNot (\text{remove1-mset } L (mset\text{-cls}$
 $C)))$
 $\wedge \text{forget-restrictions } C S$
for
mset-cls:: *'cls* \Rightarrow *'v clause and*
insert-cls:: *'v literal* \Rightarrow *'cls* \Rightarrow *'cls and*
remove-lit:: *'v literal* \Rightarrow *'cls* \Rightarrow *'cls and*
mset-clss:: *'clss* \Rightarrow *'v clauses and*
union-clss:: *'clss* \Rightarrow *'clss* \Rightarrow *'clss and*
in-clss:: *'cls* \Rightarrow *'clss* \Rightarrow *bool and*
insert-clss:: *'cls* \Rightarrow *'clss* \Rightarrow *'clss and*
remove-from-clss:: *'cls* \Rightarrow *'clss* \Rightarrow *'clss and*
trail:: *'st* \Rightarrow (*'v, unit, unit*) *marked-lits and*
raw-clauses:: *'st* \Rightarrow *'clss and*

```

prepend-trail :: ('v, unit, unit) marked-lit => 'st => 'st and
tl-trail :: 'st => 'st and
add-clsNOT :: 'cls => 'st => 'st and
remove-clsNOT :: 'cls => 'st => 'st and
inv :: 'st => bool and
backjump-conds :: 'v clause => 'v clause => 'v literal => 'st => 'st => bool and
propagate-conds :: ('v, unit, unit) marked-lit => 'st => bool and
learn-restrictions forget-restrictions :: 'cls => 'st => bool
begin

lemma cdclNOT-learn-all-induct[consumes 1, case-names dpll-bj learn forgetNOT]:
fixes S T :: 'st
assumes cdclNOT S T and
dpll:  $\bigwedge T. \text{dpll-bj } S \ T \implies P \ S \ T$  and
learning:
 $\bigwedge C \ F \ K \ F' \ C' \ L \ T. \text{clauses}_{NOT} \ S \models_{pm} \text{mset-cls } C \implies$ 
 $\text{atms-of } (\text{mset-cls } C) \subseteq \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \cup \text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \implies$ 
 $\text{distinct-mset } (\text{mset-cls } C) \implies$ 
 $\neg \text{tautology } (\text{mset-cls } C) \implies$ 
 $\text{learn-restrictions } C \ S \implies$ 
 $\text{trail } S = F' @ \text{Marked } K \ () \ \# \ F \implies$ 
 $\text{mset-cls } C = C' + \{\#L\# \} \implies$ 
 $F \models_{as} CNot \ C' \implies$ 
 $C' + \{\#L\# \} \not\models_{\#} \text{clauses}_{NOT} \ S \implies$ 
 $T \sim \text{add-cls}_{NOT} \ C \ S \implies$ 
 $P \ S \ T$  and
forgetting:  $\bigwedge C \ T. \text{removeAll-mset } (\text{mset-cls } C) \ (\text{clauses}_{NOT} \ S) \models_{pm} \text{mset-cls } C \implies$ 
 $C \ !\in \text{raw-clauses } S \implies$ 
 $\neg(\exists F' \ F \ K \ L. \text{trail } S = F' @ \text{Marked } K \ () \ \# \ F \wedge F \models_{as} CNot \ (\text{mset-cls } C - \{\#L\# \})) \implies$ 
 $T \sim \text{remove-cls}_{NOT} \ C \ S \implies$ 
 $\text{forget-restrictions } C \ S \implies$ 
 $P \ S \ T$ 
shows P S T
using assms(1)
apply (induction rule: cdclNOT.induct)
  apply (auto dest: assms(2) simp add: learn-ops-axioms)[]
  apply (auto elim!: learn-ops.learn.cases[OF learn-ops-axioms] dest: assms(3))[]
  apply (auto elim!: forget-ops.forgetNOT.cases[OF forget-ops-axioms] dest!: assms(4))
done

lemma rtranclp-cdclNOT-inv:
cdclNOT** S T  $\implies$  inv S  $\implies$  inv T
apply (induction rule: rtranclp-induct)
  apply simp
using cdclNOT-inv unfolding conflict-driven-clause-learning-def
conflict-driven-clause-learning-axioms-def by blast

lemma learn-always-simple-clauses:
assumes
learn: learn S T and
n-d: no-dup (trail S)
shows set-mset (clausesNOT T - clausesNOT S)
 $\subseteq$  simple-cls (atms-of-mm (clausesNOT S)  $\cup$  atm-of ' lits-of-l (trail S))
proof
fix C assume C: C  $\in$  set-mset (clausesNOT T - clausesNOT S)

```


have *distinct-mset* $C \neg \text{tautology } C$ **using** *learn* C *n-d* **by** (*elim learn*_{NOT} E ; *auto*) +
then have $C \in \text{simple-clss } (\text{atms-of } C)$
using *distinct-mset-not-tautology-implies-in-simple-clss* **by** *blast*
moreover have $\text{atms-of } C \subseteq \text{atms-of-mm } (\text{clauses}_{\text{NOT}} S) \cup \text{atm-of ' lits-of-l } (\text{trail } S)$
using *learn* C *n-d* **by** (*elim learn*_{NOT} E) (*auto simp: atms-of-ms-def atms-of-def image-Un true-annots-CNot-all-atms-defined*)
moreover have *finite* $(\text{atms-of-mm } (\text{clauses}_{\text{NOT}} S) \cup \text{atm-of ' lits-of-l } (\text{trail } S))$
by *auto*
ultimately show $C \in \text{simple-clss } (\text{atms-of-mm } (\text{clauses}_{\text{NOT}} S) \cup \text{atm-of ' lits-of-l } (\text{trail } S))$
using *simple-clss-mono* **by** (*metis (no-types) insert-subset mk-disjoint-insert*)
qed

definition *conflicting-bj-clss* $S \equiv$
 $\{C + \{\#L\# \} \mid C \text{ L. } C + \{\#L\# \} \in \# \text{ clauses}_{\text{NOT}} S \wedge \text{distinct-mset } (C + \{\#L\# \})$
 $\wedge \neg \text{tautology } (C + \{\#L\# \})$
 $\wedge (\exists F' K F. \text{trail } S = F' @ \text{Marked } K () \# F \wedge F \models_{\text{as}} C \text{Not } C)\}$

lemma *conflicting-bj-clss-remove-cl*_{NOT} $[simp]$:
 $\text{conflicting-bj-clss } (\text{remove-cl}_{\text{NOT}} C S) = \text{conflicting-bj-clss } S - \{\text{mset-cl } C\}$
unfolding *conflicting-bj-clss-def* **by** *fastforce*

lemma *conflicting-bj-clss-remove-cl*_{NOT} $'[simp]$:
 $T \sim \text{remove-cl}_{\text{NOT}} C S \implies \text{conflicting-bj-clss } T = \text{conflicting-bj-clss } S - \{\text{mset-cl } C\}$
unfolding *conflicting-bj-clss-def* **by** *fastforce*

lemma *conflicting-bj-clss-add-cl*_{NOT} -state-eq :

assumes

$T: T \sim \text{add-cl}_{\text{NOT}} C' S$ **and**

$n\text{-d: no-dup } (\text{trail } S)$

shows *conflicting-bj-clss* T

$= \text{conflicting-bj-clss } S$

$\cup (\text{if } \exists C \text{ L. } \text{mset-cl } C' = C + \{\#L\# \} \wedge \text{distinct-mset } (C + \{\#L\# \}) \wedge \neg \text{tautology } (C + \{\#L\# \})$
 $\wedge (\exists F' K d F. \text{trail } S = F' @ \text{Marked } K () \# F \wedge F \models_{\text{as}} C \text{Not } C)$
 $\text{then } \{\text{mset-cl } C'\} \text{ else } \{\})$

proof –

def $P \equiv \lambda C \text{ L } T. \text{distinct-mset } (C + \{\#L\# \}) \wedge \neg \text{tautology } (C + \{\#L\# \}) \wedge$
 $(\exists F' K F. \text{trail } T = F' @ \text{Marked } K () \# F \wedge F \models_{\text{as}} C \text{Not } C)$

have *conf*: $\bigwedge T. \text{conflicting-bj-clss } T = \{C + \{\#L\# \} \mid C \text{ L. } C + \{\#L\# \} \in \# \text{ clauses}_{\text{NOT}} T \wedge P C \text{ L } T\}$

unfolding *conflicting-bj-clss-def* $P\text{-def}$ **by** *auto*

have $P\text{-S-T}$: $\bigwedge C \text{ L. } P C \text{ L } T = P C \text{ L } S$

using T *n-d* **unfolding** $P\text{-def}$ **by** *auto*

have P : $\text{conflicting-bj-clss } T = \{C + \{\#L\# \} \mid C \text{ L. } C + \{\#L\# \} \in \# \text{ clauses}_{\text{NOT}} S \wedge P C \text{ L } T\} \cup$
 $\{C + \{\#L\# \} \mid C \text{ L. } C + \{\#L\# \} \in \# \{\# \text{mset-cl } C'\# \} \wedge P C \text{ L } T\}$

using T *n-d* **unfolding** *conf* **by** *auto*

moreover have $\{C + \{\#L\# \} \mid C \text{ L. } C + \{\#L\# \} \in \# \text{ clauses}_{\text{NOT}} S \wedge P C \text{ L } T\} = \text{conflicting-bj-clss } S$

using T *n-d* **unfolding** $P\text{-def}$ *conflicting-bj-clss-def* **by** *auto*

moreover have $\{C + \{\#L\# \} \mid C \text{ L. } C + \{\#L\# \} \in \# \{\# \text{mset-cl } C'\# \} \wedge P C \text{ L } T\} =$
 $(\text{if } \exists C \text{ L. } \text{mset-cl } C' = C + \{\#L\# \} \wedge P C \text{ L } S \text{ then } \{\text{mset-cl } C'\} \text{ else } \{\})$

using $n\text{-d } T$ **by** (*force simp: P-S-T*)

ultimately show *?thesis* **unfolding** $P\text{-def}$ **by** *presburger*

qed

lemma *conflicting-bj-clss-add-cl*_{NOT}:

$no_dup (trail\ S) \implies$
 $conflicting_bj_clss\ (add_cls_{NOT}\ C'\ S)$
 $= conflicting_bj_clss\ S$
 $\cup (if\ \exists\ C\ L.\ mset_cls\ C' = C + \{\#L\# \} \wedge distinct_mset\ (C + \{\#L\# \}) \wedge \neg tautology\ (C + \{\#L\# \})$
 $\wedge (\exists\ F'\ K\ d\ F.\ trail\ S = F' @\ Marked\ K\ ()\ \# F \wedge F \models_{as}\ CNot\ C)$
 $then\ \{mset_cls\ C'\}\ else\ \{\})$
using $conflicting_bj_clss_add_cls_{NOT}\ state_eq$ **by** $auto$

lemma $conflicting_bj_clss_incl_clauses$:
 $conflicting_bj_clss\ S \subseteq set_mset\ (clauses_{NOT}\ S)$
unfolding $conflicting_bj_clss_def$ **by** $auto$

lemma $finite_conflicting_bj_clss[simp]$:
 $finite\ (conflicting_bj_clss\ S)$
using $conflicting_bj_clss_incl_clauses[of\ S]$ rev_finite_subset **by** $blast$

lemma $learn_conflicting_increasing$:
 $no_dup (trail\ S) \implies learn\ S\ T \implies conflicting_bj_clss\ S \subseteq conflicting_bj_clss\ T$
apply $(elim\ learn_{NOT}E)$
by $(subst\ conflicting_bj_clss_add_cls_{NOT}\ state_eq[of\ T])\ auto$

abbreviation $conflicting_bj_clss_yet\ b\ S \equiv$
 $\exists\ ^\wedge b - card\ (conflicting_bj_clss\ S)$

abbreviation $\mu_L :: nat \Rightarrow 'st \Rightarrow nat \times nat$ **where**
 $\mu_L\ b\ S \equiv (conflicting_bj_clss_yet\ b\ S,\ card\ (set_mset\ (clauses_{NOT}\ S)))$

lemma $do_not_forget_before_backtrack_rule_clause_learned_clause_untouched$:
assumes $forget_{NOT}\ S\ T$
shows $conflicting_bj_clss\ S = conflicting_bj_clss\ T$
using $assms$ **apply** $(elim\ forget_{NOT}E)$
apply $auto$
unfolding $conflicting_bj_clss_def$
apply $clarify$
using $diff_union_cancelR$ **by** $(metis\ diff_union_cancelR)$

lemma $forget_mu_L_decrease$:
assumes $forget_{NOT}: forget_{NOT}\ S\ T$
shows $(\mu_L\ b\ T,\ \mu_L\ b\ S) \in less_than\ <*\textit{lex}*\>\ less_than$
proof –
have $card\ (set_mset\ (clauses_{NOT}\ S)) > 0$
using $forget_{NOT}$ **by** $(elim\ forget_{NOT}E)\ (auto\ simp:\ size_mset_removeAll_mset_le_iff\ card_gt_0_iff)$
then have $card\ (set_mset\ (clauses_{NOT}\ T)) < card\ (set_mset\ (clauses_{NOT}\ S))$
using $forget_{NOT}$ **by** $(elim\ forget_{NOT}E)\ (auto\ simp:\ size_mset_removeAll_mset_le_iff)$
then show $?thesis$
unfolding $do_not_forget_before_backtrack_rule_clause_learned_clause_untouched[OF\ forget_{NOT}]$
by $auto$
qed

lemma $set_condition_or_split$:
 $\{a.\ (a = b \vee Q\ a) \wedge S\ a\} = (if\ S\ b\ then\ \{b\}\ else\ \{\}) \cup \{a.\ Q\ a \wedge S\ a\}$
by $auto$

lemma set_insert_neq :
 $A \neq insert\ a\ A \longleftrightarrow a \notin A$

by auto

lemma *learn- μ_L -decrease*:

assumes *learnST*: *learn* *S* *T* **and** *n-d*: *no-dup* (*trail* *S*) **and**
A: *atms-of-mm* (*clauses*_{NOT} *S*) \cup *atm-of* ‘ *lits-of-l* (*trail* *S*) \subseteq *A* **and**
fin-A: *finite* *A*
shows (μ_L (*card* *A*) *T*, μ_L (*card* *A*) *S*) \in *less-than* $\langle *lex* \rangle$ *less-than*

proof –

have [*simp*]: (*atms-of-mm* (*clauses*_{NOT} *T*) \cup *atm-of* ‘ *lits-of-l* (*trail* *T*))
 $=$ (*atms-of-mm* (*clauses*_{NOT} *S*) \cup *atm-of* ‘ *lits-of-l* (*trail* *S*))
using *learnST* *n-d* **by** (*elim learn*_{NOT} *E*) *auto*

then have *card* (*atms-of-mm* (*clauses*_{NOT} *T*) \cup *atm-of* ‘ *lits-of-l* (*trail* *T*))
 $=$ *card* (*atms-of-mm* (*clauses*_{NOT} *S*) \cup *atm-of* ‘ *lits-of-l* (*trail* *S*))
by (*auto intro!*: *card-mono*)

then have \exists : ($\exists::nat$) \wedge *card* (*atms-of-mm* (*clauses*_{NOT} *T*) \cup *atm-of* ‘ *lits-of-l* (*trail* *T*))
 $=$ $\exists \wedge$ *card* (*atms-of-mm* (*clauses*_{NOT} *S*) \cup *atm-of* ‘ *lits-of-l* (*trail* *S*))
by (*auto intro*: *power-mono*)

moreover have *conflicting-bj-clss* *S* \subseteq *conflicting-bj-clss* *T*
using *learnST* *n-d* **by** (*simp add*: *learn-conflicting-increasing*)

moreover have *conflicting-bj-clss* *S* \neq *conflicting-bj-clss* *T*
using *learnST*

proof (*elim learn*_{NOT} *E*, *goal-cases*)

case (*1 C*) **note** *clss-S* = *this*(*1*) **and** *atms-C* = *this*(*2*) **and** *inv* = *this*(*3*) **and** *T* = *this*(*4*)

then obtain *F K F' C' L* **where**

tr-S: *trail* *S* = *F' @ Marked* *K* () # *F* **and**

C: *mset-cls* *C* = *C' + {#L#}* **and**

F: *F* \models_{as} *CNot* *C'* **and**

C-S: *C' + {#L#}* \notin *clauses*_{NOT} *S*

by *blast*

moreover have *distinct-mset* (*mset-cls* *C*) \neg *tautology* (*mset-cls* *C*) **using** *inv* **by** *blast+*

ultimately have *C' + {#L#}* \in *conflicting-bj-clss* *T*

using *T n-d unfolding* *conflicting-bj-clss-def* **by** *fastforce*

moreover have *C' + {#L#}* \notin *conflicting-bj-clss* *S*

using *C-S unfolding* *conflicting-bj-clss-def* **by** *auto*

ultimately show *?case* **by** *blast*

qed

moreover have *fin-T*: *finite* (*conflicting-bj-clss* *T*)

using *learnST* **by** *induction* (*auto simp add*: *conflicting-bj-clss-add-clss*_{NOT})

ultimately have *card* (*conflicting-bj-clss* *T*) \geq *card* (*conflicting-bj-clss* *S*)

using *card-mono* **by** *blast*

moreover

have *fin'*: *finite* (*atms-of-mm* (*clauses*_{NOT} *T*) \cup *atm-of* ‘ *lits-of-l* (*trail* *T*))
by *auto*

have *1*: *atms-of-ms* (*conflicting-bj-clss* *T*) \subseteq *atms-of-mm* (*clauses*_{NOT} *T*)
unfolding *conflicting-bj-clss-def* *atms-of-ms-def* **by** *auto*

have *2*: $\bigwedge x. x \in$ *conflicting-bj-clss* *T* $\implies \neg$ *tautology* *x* \wedge *distinct-mset* *x*
unfolding *conflicting-bj-clss-def* **by** *auto*

have *T*: *conflicting-bj-clss* *T*

\subseteq *simple-clss* (*atms-of-mm* (*clauses*_{NOT} *T*) \cup *atm-of* ‘ *lits-of-l* (*trail* *T*))

by *standard* (*meson* *1 2 fin'* \langle *finite* (*conflicting-bj-clss* *T*) \rangle *simple-clss-mono*
distinct-mset-set-def *simplified-in-simple-clss subsetCE* *sup.coboundedI1*)

moreover

then have $\#$: $\exists \wedge$ *card* (*atms-of-mm* (*clauses*_{NOT} *T*) \cup *atm-of* ‘ *lits-of-l* (*trail* *T*))

```

    ≥ card (conflicting-bj-clss T)
  by (meson Nat.le-trans simple-clss-card simple-clss-finite card-mono fin')
have atms-of-mm (clausesNOT T) ∪ atm-of ' lits-of-l (trail T) ⊆ A
  using learnNOTE[OF learnST] A by simp
then have 3 ^ (card A) ≥ card (conflicting-bj-clss T)
  using # fin-A by (meson simple-clss-card simple-clss-finite
    simple-clss-mono calculation(2) card-mono dual-order.trans)
ultimately show ?thesis
  using psubset-card-mono[OF fin-T ]
  unfolding less-than-iff lex-prod-def by clarify
  (meson ⟨conflicting-bj-clss S ≠ conflicting-bj-clss T⟩
    ⟨conflicting-bj-clss S ⊆ conflicting-bj-clss T⟩
    diff-less-mono2 le-less-trans not-le psubsetI)
qed

```

We have to assume the following:

- *inv S*: the invariant holds in the initial state.
- *A* is a (finite *finite A*) superset of the literals in the trail *atm-of ' lits-of-l (trail S) ⊆ atms-of-ms A* and in the clauses *atms-of-mm (clauses_{NOT} S) ⊆ atms-of-ms A*. This can be the set of all the literals in the starting set of clauses.
- *no-dup (trail S)*: no duplicate in the trail. This is invariant along the path.

definition μ_{CDCL} **where**

$$\mu_{CDCL} A T \equiv ((2 + \text{card}(\text{atms-of-ms } A)) \wedge (1 + \text{card}(\text{atms-of-ms } A)) \\ - \mu_C (1 + \text{card}(\text{atms-of-ms } A)) (2 + \text{card}(\text{atms-of-ms } A)) (\text{trail-weight } T), \\ \text{conflicting-bj-clss-yet}(\text{card}(\text{atms-of-ms } A)) T, \text{card}(\text{set-mset}(\text{clauses}_{NOT} T)))$$

lemma *cdcl_{NOT}-decreasing-measure*:

```

assumes
  cdclNOT S T and
  inv: inv S and
  atm-clss: atms-of-mm (clausesNOT S) ⊆ atms-of-ms A and
  atm-lits: atm-of ' lits-of-l (trail S) ⊆ atms-of-ms A and
  n-d: no-dup (trail S) and
  fin-A: finite A
shows (μCDCL A T, μCDCL A S)
  ∈ less-than <*lex*> (less-than <*lex*> less-than)
using assms(1)
proof induction
case (c-dpll-bj T)
from dpll-bj-trail-mes-decreasing-prop[OF this(1) inv atm-clss atm-lits n-d fin-A]
show ?case unfolding μCDCL-def
  by (meson in-lex-prod less-than-iff)
next
case (c-learn T) note learn = this(1)
then have S: trail S = trail T
  using inv atm-clss atm-lits n-d fin-A
  by (elim learnNOTE) auto
show ?case
  using learn-μL-decrease[OF learn n-d, of atms-of-ms A] atm-clss atm-lits fin-A n-d
  unfolding S μCDCL-def by auto
next
case (c-forgetNOT T) note forgetNOT = this(1)

```

have $\text{trail } S = \text{trail } T$ **using** forget_{NOT} **by** induction auto
then show $?case$
using $\text{forget-}\mu_L\text{-decrease}[OF \text{ forget}_{NOT}]$ **unfolding** $\mu_{CDCL}\text{-def}$ **by** $auto$
qed

lemma $\text{wf-cdcl}_{NOT}\text{-restricted-learning}$:

assumes $\text{finite } A$
shows $\text{wf } \{(T, S).$
 $(\text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A$
 $\wedge \text{no-dup } (\text{trail } S)$
 $\wedge \text{inv } S)$
 $\wedge \text{cdcl}_{NOT} S T \}$
by $(\text{rule wf-wf-if-measure}'[\text{of less-than } <*lex*> (\text{less-than } <*lex*> \text{less-than})])$
 $(\text{auto intro: cdcl}_{NOT}\text{-decreasing-measure}[OF \text{ - - - - assms}])$

definition $\mu_C' :: 'v \text{ literal multiset set} \Rightarrow 'st \Rightarrow \text{nat}$ **where**

$\mu_C' A T \equiv \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } T)$

definition $\mu_{CDCL}' :: 'v \text{ literal multiset set} \Rightarrow 'st \Rightarrow \text{nat}$ **where**

$\mu_{CDCL}' A T \equiv$
 $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A T) * (1 + 3^{\text{card } (\text{atms-of-ms } A)}) * 2$
 $+ \text{conflicting-bj-clss-yet } (\text{card } (\text{atms-of-ms } A)) T * 2$
 $+ \text{card } (\text{set-mset } (\text{clauses}_{NOT} T))$

lemma $\text{cdcl}_{NOT}\text{-decreasing-measure}'$:

assumes
 $\text{cdcl}_{NOT} S T$ **and**
 $\text{inv: inv } S$ **and**
 $\text{atms-clss: atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A$ **and**
 $\text{atms-trail: atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A$ **and**
 $\text{n-d: no-dup } (\text{trail } S)$ **and**
 $\text{fin-A: finite } A$

shows $\mu_{CDCL}' A T < \mu_{CDCL}' A S$

using $\text{assms}(1)$

proof $(\text{induction rule: cdcl}_{NOT}\text{-learn-all-induct})$

case $(\text{dpll-bj } T)$

then have $(2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A T$
 $< (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A S$

using $\text{dpll-bj-trail-mes-decreasing-prop fin-A inv n-d atms-clss atms-trail}$

unfolding $\mu_C'\text{-def}$ **by** blast

then have $XX: ((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A T) + 1$
 $\leq (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A S$

by $auto$

from $\text{mult-le-mono1}[OF \text{ this, of } (1 + 3^{\text{card } (\text{atms-of-ms } A)})]$

have $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A T) * (1 + 3^{\text{card } (\text{atms-of-ms } A)}) + (1 + 3^{\text{card } (\text{atms-of-ms } A)})$
 $\leq ((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A S)$
 $* (1 + 3^{\text{card } (\text{atms-of-ms } A)})$

unfolding $\text{Nat.add-mult-distrib}$

by presburger

moreover

have $\text{cl-T-S: clauses}_{NOT} T = \text{clauses}_{NOT} S$

using $\text{dpll-bj.hyps inv dpll-bj-clauses}$ **by** $auto$

have $\text{conflicting-bj-clss-yet } (\text{card } (\text{atms-of-ms } A)) S < 1 + 3^{\text{card } (\text{atms-of-ms } A)}$

```

    by simp
  ultimately have  $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A T)$ 
    *  $(1 + 3 \wedge \text{card } (\text{atms-of-ms } A)) + \text{conflicting-bj-clss-yet } (\text{card } (\text{atms-of-ms } A)) T$ 
    <  $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A S) * (1 + 3 \wedge \text{card } (\text{atms-of-ms } A))$ 
  A))
  by linarith
  then have  $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A T)$ 
    *  $(1 + 3 \wedge \text{card } (\text{atms-of-ms } A))$ 
    +  $\text{conflicting-bj-clss-yet } (\text{card } (\text{atms-of-ms } A)) T$ 
    <  $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A S)$ 
    *  $(1 + 3 \wedge \text{card } (\text{atms-of-ms } A))$ 
    +  $\text{conflicting-bj-clss-yet } (\text{card } (\text{atms-of-ms } A)) S$ 
  by linarith
  then have  $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A T)$ 
    *  $(1 + 3 \wedge \text{card } (\text{atms-of-ms } A)) * 2$ 
    +  $\text{conflicting-bj-clss-yet } (\text{card } (\text{atms-of-ms } A)) T * 2$ 
    <  $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A S)$ 
    *  $(1 + 3 \wedge \text{card } (\text{atms-of-ms } A)) * 2$ 
    +  $\text{conflicting-bj-clss-yet } (\text{card } (\text{atms-of-ms } A)) S * 2$ 
  by linarith
  then show ?case unfolding  $\mu_{CDCL}'\text{-def cl-T-S}$  by presburger
next
case (learn C F' K F C' L T) note  $\text{clss-S-C} = \text{this}(1)$  and  $\text{atms-C} = \text{this}(2)$  and  $\text{dist} = \text{this}(3)$ 
  and  $\text{tauto} = \text{this}(4)$  and  $\text{learn-restr} = \text{this}(5)$  and  $\text{tr-S} = \text{this}(6)$  and  $C' = \text{this}(7)$  and
   $F-C = \text{this}(8)$  and  $C\text{-new} = \text{this}(9)$  and  $T = \text{this}(10)$ 
have insert (mset-cls C) (conflicting-bj-clss S)  $\subseteq$  simple-clss (atms-of-ms A)
proof -
  have mset-cls C  $\in$  simple-clss (atms-of-ms A)
  using C'
  by (metis (no-types, hide-lams) Un-subset-iff simple-clss-mono
    contra-subsetD dist distinct-mset-not-tautology-implies-in-simple-clss
    dual-order.trans atms-C atms-clss atms-trail tauto)
  moreover have conflicting-bj-clss S  $\subseteq$  simple-clss (atms-of-ms A)
  proof
    fix x :: 'v literal multiset
    assume x  $\in$  conflicting-bj-clss S
    then have x  $\in \# \text{clauses}_{NOT} S \wedge \text{distinct-mset } x \wedge \neg \text{tautology } x$ 
    unfolding conflicting-bj-clss-def by blast
    then show x  $\in$  simple-clss (atms-of-ms A)
    by (meson atms-clss atms-of-atms-of-ms-mono atms-of-ms-finite simple-clss-mono
      distinct-mset-not-tautology-implies-in-simple-clss fin-A finite-subset
      set-rev-mp)
  qed
  ultimately show ?thesis
  by auto
qed
then have card (insert (mset-cls C) (conflicting-bj-clss S))  $\leq 3 \wedge (\text{card } (\text{atms-of-ms } A))$ 
  by (meson Nat.le-trans atms-of-ms-finite simple-clss-card simple-clss-finite
    card-mono fin-A)
moreover have [simp]: card (insert (mset-cls C) (conflicting-bj-clss S))
  = Suc (card ((conflicting-bj-clss S)))
  by (metis (no-types) C' C-new card-insert-if conflicting-bj-clss-incl-clauses contra-subsetD
    finite-conflicting-bj-clss)
moreover have [simp]: conflicting-bj-clss (add-clsNOT C S) = conflicting-bj-clss S  $\cup \{\text{mset-cls } C\}$ 
  using dist tauto F-C by (subst conflicting-bj-clss-add-clsNOT[OF n-d]) (force simp: C' tr-S n-d)

```

ultimately have [simp]: *conflicting-bj-clss-yet* (*card* (*atms-of-ms* *A*)) *S*
 = *Suc* (*conflicting-bj-clss-yet* (*card* (*atms-of-ms* *A*)) (*add-cls*_{NOT} *C S*))
by *simp*
have 1: *clauses*_{NOT} *T* = *clauses*_{NOT} (*add-cls*_{NOT} *C S*) **using** *T* **by** *auto*
have 2: *conflicting-bj-clss-yet* (*card* (*atms-of-ms* *A*)) *T*
 = *conflicting-bj-clss-yet* (*card* (*atms-of-ms* *A*)) (*add-cls*_{NOT} *C S*)
using *T* **unfolding** *conflicting-bj-clss-def* **by** *auto*
have 3: $\mu_C' A T = \mu_C' A (\text{add-cls}_{NOT} C S)$
using *T* **unfolding** μ_C' -def **by** *auto*
have $((2 + \text{card} (\text{atms-of-ms } A)) \wedge (1 + \text{card} (\text{atms-of-ms } A)) - \mu_C' A (\text{add-cls}_{NOT} C S))$
 $* (1 + 3 \wedge \text{card} (\text{atms-of-ms } A)) * 2$
 = $((2 + \text{card} (\text{atms-of-ms } A)) \wedge (1 + \text{card} (\text{atms-of-ms } A)) - \mu_C' A S)$
 $* (1 + 3 \wedge \text{card} (\text{atms-of-ms } A)) * 2$
using *n-d* **unfolding** μ_C' -def **by** *auto*
moreover
have *conflicting-bj-clss-yet* (*card* (*atms-of-ms* *A*)) (*add-cls*_{NOT} *C S*)
 * 2
 + *card* (*set-mset* (*clauses*_{NOT} (*add-cls*_{NOT} *C S*)))
 < *conflicting-bj-clss-yet* (*card* (*atms-of-ms* *A*)) *S* * 2
 + *card* (*set-mset* (*clauses*_{NOT} *S*))
by (*simp* *add*: *C' C-new n-d*)
ultimately show ?*case* **unfolding** μ_{CDCL}' -def 1 2 3 **by** *presburger*
next
case (*forget*_{NOT} *C T*) **note** *T* = *this*(4)
have [simp]: $\mu_C' A (\text{remove-cls}_{NOT} C S) = \mu_C' A S$
unfolding μ_C' -def **by** *auto*
have *forget*_{NOT} *S T*
apply (*rule* *forget*_{NOT}.*intros*) **using** *forget*_{NOT} **by** *auto*
then have *conflicting-bj-clss* *T* = *conflicting-bj-clss* *S*
using *do-not-forget-before-backtrack-rule-clause-learned-clause-untouched* **by** *blast*
moreover have *card* (*set-mset* (*clauses*_{NOT} *T*)) < *card* (*set-mset* (*clauses*_{NOT} *S*))
by (*metis* *T* *card-Diff1-less clauses-remove-cls*_{NOT} *finite-set-mset forget*_{NOT}.*hyps*(2)
in-clss-mset-clss order-refl set-mset-minus-replicate-mset(1) *state-eq*_{NOT}-*clauses*)
ultimately show ?*case* **unfolding** μ_{CDCL}' -def
using *T* $\mu_C' A (\text{remove-cls}_{NOT} C S) = \mu_C' A S$ **by** (*metis* (*no-types*) *add-le-cancel-left*
 μ_C' -def *not-le state-eq*_{NOT}-*trail*)
qed

lemma *cdcl*_{NOT}-*clauses-bound*:

assumes
*cdcl*_{NOT} *S T* **and**
inv *S* **and**
atms-of-mm (*clauses*_{NOT} *S*) $\subseteq A$ **and**
atm-of '(*lits-of-l* (*trail* *S*)) $\subseteq A$ **and**
n-d: *no-dup* (*trail* *S*) **and**
fin-A[*simp*]: *finite* *A*
shows *set-mset* (*clauses*_{NOT} *T*) $\subseteq \text{set-mset} (\text{clauses}_{NOT} S) \cup \text{simple-clss } A$
using *assms*
proof (*induction rule*: *cdcl*_{NOT}-*learn-all-induct*)
case *dpll-bj*
then show ?*case* **using** *dpll-bj-clauses* **by** *simp*
next
case *forget*_{NOT}
then show ?*case* **using** *clauses-remove-cls*_{NOT} **unfolding** *state-eq*_{NOT}-def **by** *auto*
next

case (*learn C F K d F' C' L*) **note** *atms-C = this(2)* **and** *dist = this(3)* **and** *tauto = this(4)* **and**
T = this(10) **and** *atms-clss-S = this(12)* **and** *atms-trail-S = this(13)*
have *atms-of (mset-cls C) ⊆ A*
using *atms-C atms-clss-S atms-trail-S* **by** *fast*
then have *simple-clss (atms-of (mset-cls C)) ⊆ simple-clss A*
by (*simp add: simple-clss-mono*)
then have *mset-cls C ∈ simple-clss A*
using *finite dist tauto* **by** (*auto dest: distinct-mset-not-tautology-implies-in-simple-clss*)
then show ?*case* **using** *T n-d* **by** *auto*
qed

lemma *rtrancpl-cdcl_{NOT}-clauses-bound:*

assumes
*cdcl_{NOT}** S T* **and**
inv S **and**
atms-of-mm (clauses_{NOT} S) ⊆ A **and**
atm-of '(lits-of-l (trail S)) ⊆ A **and**
n-d: no-dup (trail S) **and**
finite: finite A
shows *set-mset (clauses_{NOT} T) ⊆ set-mset (clauses_{NOT} S) ∪ simple-clss A*
using *assms(1-5)*
proof *induction*
case *base*
then show ?*case* **by** *simp*
next
case (*step T U*) **note** *st = this(1)* **and** *cdcl_{NOT} = this(2)* **and** *IH = this(3)[OF this(4-7)]* **and**
inv = this(4) **and** *atms-clss-S = this(5)* **and** *atms-trail-S = this(6)* **and** *finite-clss-S = this(7)*
have *inv T*
using *rtrancpl-cdcl_{NOT}-inv st inv* **by** *blast*
moreover have *atms-of-mm (clauses_{NOT} T) ⊆ A* **and** *atm-of '(lits-of-l (trail T)) ⊆ A*
using *rtrancpl-cdcl_{NOT}-trail-clauses-bound[OF st] inv atms-clss-S atms-trail-S n-d* **by** *auto*
moreover have *no-dup (trail T)*
using *rtrancpl-cdcl_{NOT}-no-dup[OF st (inv S) n-d]* **by** *simp*
ultimately have *set-mset (clauses_{NOT} U) ⊆ set-mset (clauses_{NOT} T) ∪ simple-clss A*
using *cdcl_{NOT} finite n-d* **by** (*auto simp: cdcl_{NOT}-clauses-bound*)
then show ?*case* **using** *IH* **by** *auto*
qed

lemma *rtrancpl-cdcl_{NOT}-card-clauses-bound:*

assumes
*cdcl_{NOT}** S T* **and**
inv S **and**
atms-of-mm (clauses_{NOT} S) ⊆ A **and**
atm-of '(lits-of-l (trail S)) ⊆ A **and**
n-d: no-dup (trail S) **and**
finite: finite A
shows *card (set-mset (clauses_{NOT} T)) ≤ card (set-mset (clauses_{NOT} S)) + 3 ^ (card A)*
using *rtrancpl-cdcl_{NOT}-clauses-bound[OF assms] finite* **by** (*meson Nat.le-trans*
simple-clss-card simple-clss-finite card-Un-le card-mono finite-UnI
finite-set-mset nat-add-left-cancel-le)

lemma *rtrancpl-cdcl_{NOT}-card-clauses-bound':*

assumes
*cdcl_{NOT}** S T* **and**
inv S **and**

$atms\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \subseteq A$ and
 $atm\text{-}of\ (lits\text{-}of\text{-}l\ (trail\ S)) \subseteq A$ and
 $n\text{-}d\text{-}no\text{-}dup\ (trail\ S)$ and
 $finite\text{-}finite\ A$
shows $card\ \{C \mid C.\ C \in \# clauses_{NOT}\ T \wedge (tautology\ C \vee \neg distinct\text{-}mset\ C)\}$
 $\leq card\ \{C \mid C.\ C \in \# clauses_{NOT}\ S \wedge (tautology\ C \vee \neg distinct\text{-}mset\ C)\} + 3 \wedge (card\ A)$
 $(is\ card\ ?T \leq card\ ?S + -)$
using $rtranclp\text{-}cdcl_{NOT}\text{-}clauses\text{-}bound[OF\ assms]\ finite$
proof –
have $?T \subseteq ?S \cup simple\text{-}clss\ A$
using $rtranclp\text{-}cdcl_{NOT}\text{-}clauses\text{-}bound[OF\ assms]$ **by** *force*
then have $card\ ?T \leq card\ (?S \cup simple\text{-}clss\ A)$
using $finite$ **by** $(simp\ add\text{-}assms(5)\ simple\text{-}clss\text{-}finite\ card\text{-}mono)$
then show $?thesis$
by $(meson\ le\text{-}trans\ simple\text{-}clss\text{-}card\ card\text{-}Un\text{-}le\ local.\ finite\ nat\text{-}add\text{-}left\text{-}cancel\text{-}le)$
qed

lemma $rtranclp\text{-}cdcl_{NOT}\text{-}card\text{-}simple\text{-}clauses\text{-}bound$:

assumes
 $cdcl_{NOT}^{**}\ S\ T$ and
 $inv\ S$ and
 $NA\text{-}atms\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \subseteq A$ and
 $MA\text{-}atm\text{-}of\ (lits\text{-}of\text{-}l\ (trail\ S)) \subseteq A$ and
 $n\text{-}d\text{-}no\text{-}dup\ (trail\ S)$ and
 $finite\text{-}finite\ A$
shows $card\ (set\text{-}mset\ (clauses_{NOT}\ T))$
 $\leq card\ \{C.\ C \in \# clauses_{NOT}\ S \wedge (tautology\ C \vee \neg distinct\text{-}mset\ C)\} + 3 \wedge (card\ A)$
 $(is\ card\ ?T \leq card\ ?S + -)$
using $rtranclp\text{-}cdcl_{NOT}\text{-}clauses\text{-}bound[OF\ assms]\ finite$
proof –
have $\bigwedge x.\ x \in \# clauses_{NOT}\ T \implies \neg tautology\ x \implies distinct\text{-}mset\ x \implies x \in simple\text{-}clss\ A$
using $rtranclp\text{-}cdcl_{NOT}\text{-}clauses\text{-}bound[OF\ assms]$ **by** $(metis\ (no\text{-}types,\ hide\text{-}lams)\ Un\text{-}iff\ NA\ atms\text{-}of\text{-}atms\text{-}of\text{-}ms\text{-}mono\ simple\text{-}clss\text{-}mono\ contra\text{-}subsetD\ subset\text{-}trans\ distinct\text{-}mset\text{-}not\text{-}tautology\text{-}implies\text{-}in\text{-}simple\text{-}clss)$
then have $set\text{-}mset\ (clauses_{NOT}\ T) \subseteq ?S \cup simple\text{-}clss\ A$
using $rtranclp\text{-}cdcl_{NOT}\text{-}clauses\text{-}bound[OF\ assms]$ **by** *auto*
then have $card\ (set\text{-}mset\ (clauses_{NOT}\ T)) \leq card\ (?S \cup simple\text{-}clss\ A)$
using $finite$ **by** $(simp\ add\text{-}assms(5)\ simple\text{-}clss\text{-}finite\ card\text{-}mono)$
then show $?thesis$
by $(meson\ le\text{-}trans\ simple\text{-}clss\text{-}card\ card\text{-}Un\text{-}le\ local.\ finite\ nat\text{-}add\text{-}left\text{-}cancel\text{-}le)$
qed

definition $\mu_{CDCL}'\text{-}bound :: 'v\ literal\ multiset\ set \Rightarrow 'st \Rightarrow nat$ **where**

$\mu_{CDCL}'\text{-}bound\ A\ S =$
 $((2 + card\ (atms\text{-}of\text{-}ms\ A)) \wedge (1 + card\ (atms\text{-}of\text{-}ms\ A))) * (1 + 3 \wedge card\ (atms\text{-}of\text{-}ms\ A)) * 2$
 $+ 2 * 3 \wedge (card\ (atms\text{-}of\text{-}ms\ A))$
 $+ card\ \{C.\ C \in \# clauses_{NOT}\ S \wedge (tautology\ C \vee \neg distinct\text{-}mset\ C)\} + 3 \wedge (card\ (atms\text{-}of\text{-}ms\ A))$

lemma $\mu_{CDCL}'\text{-}bound\text{-}reduce\text{-}trail\text{-}to_{NOT}[simp]$:

$\mu_{CDCL}'\text{-}bound\ A\ (reduce\text{-}trail\text{-}to_{NOT}\ M\ S) = \mu_{CDCL}'\text{-}bound\ A\ S$
unfolding $\mu_{CDCL}'\text{-}bound\text{-}def$ **by** *auto*

lemma $rtranclp\text{-}cdcl_{NOT}\text{-}\mu_{CDCL}'\text{-}bound\text{-}reduce\text{-}trail\text{-}to_{NOT}$:

assumes
 $cdcl_{NOT}^{**}\ S\ T$ and

inv S and
atms-of-mm (clauses_{NOT} S) \subseteq atms-of-ms A and
atm-of '(lits-of-l (trail S)) \subseteq atms-of-ms A and
n-d: no-dup (trail S) and
finite: finite (atms-of-ms A) and
U: $U \sim \text{reduce-trail-to}_{NOT} M T$
shows $\mu_{CDCL}' A U \leq \mu_{CDCL}'\text{-bound } A S$
proof –
have $((2 + \text{card (atms-of-ms A)}) \wedge (1 + \text{card (atms-of-ms A)}) - \mu_C' A U)$
 $\leq (2 + \text{card (atms-of-ms A)}) \wedge (1 + \text{card (atms-of-ms A)})$
by *auto*
then have $((2 + \text{card (atms-of-ms A)}) \wedge (1 + \text{card (atms-of-ms A)}) - \mu_C' A U)$
 $* (1 + 3 \wedge \text{card (atms-of-ms A)}) * 2$
 $\leq (2 + \text{card (atms-of-ms A)}) \wedge (1 + \text{card (atms-of-ms A)}) * (1 + 3 \wedge \text{card (atms-of-ms A)}) * 2$
using *mult-le-mono1* **by** *blast*
moreover
have *conflicting-bj-clss-yet (card (atms-of-ms A)) T * 2 \leq 2 * 3 \wedge card (atms-of-ms A)*
by *linarith*
moreover have *card (set-mset (clauses_{NOT} U))*
 $\leq \text{card } \{C. C \in \# \text{ clauses}_{NOT} S \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\} + 3 \wedge \text{card (atms-of-ms A)}$
using *rtranclp-cdcl_{NOT}-card-simple-clauses-bound[OF assms(1-6)] U* **by** *auto*
ultimately show *?thesis*
unfolding $\mu_{CDCL}'\text{-def}$ $\mu_{CDCL}'\text{-bound-def}$ **by** *linarith*
qed

lemma *rtranclp-cdcl_{NOT}- μ_{CDCL}' -bound:*

assumes
*cdcl_{NOT}** S T and*
inv S and
atms-of-mm (clauses_{NOT} S) \subseteq atms-of-ms A and
atm-of '(lits-of-l (trail S)) \subseteq atms-of-ms A and
n-d: no-dup (trail S) and
finite: finite (atms-of-ms A)
shows $\mu_{CDCL}' A T \leq \mu_{CDCL}'\text{-bound } A S$
proof –
have $\mu_{CDCL}' A (\text{reduce-trail-to}_{NOT} (\text{trail } T) T) = \mu_{CDCL}' A T$
unfolding $\mu_{CDCL}'\text{-def}$ $\mu_C'\text{-def}$ *conflicting-bj-clss-def* **by** *auto*
then show *?thesis using rtranclp-cdcl_{NOT}- μ_{CDCL}' -bound-reduce-trail-to_{NOT}[OF assms, of - trail T]*
state-eq_{NOT}-ref **by** *fastforce*
qed

lemma *rtranclp- μ_{CDCL}' -bound-decreasing:*

assumes
*cdcl_{NOT}** S T and*
inv S and
atms-of-mm (clauses_{NOT} S) \subseteq atms-of-ms A and
atm-of '(lits-of-l (trail S)) \subseteq atms-of-ms A and
n-d: no-dup (trail S) and
finite[simp]: finite (atms-of-ms A)
shows $\mu_{CDCL}'\text{-bound } A T \leq \mu_{CDCL}'\text{-bound } A S$
proof –
have $\{C. C \in \# \text{ clauses}_{NOT} T \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\}$
 $\subseteq \{C. C \in \# \text{ clauses}_{NOT} S \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\}$ **(is ?T \subseteq ?S)**
proof *(rule Set.subsetI)*
fix *C* **assume** *C \in ?T*

```

then have  $C \vdash T$ :  $C \in \# \text{ clauses}_{NOT} T$  and  $t \vdash$ : tautology  $C \vee \neg \text{distinct-mset } C$ 
  by auto
then have  $C \notin \text{simple-clss}$  (atms-of-ms  $A$ )
  by (auto dest: simple-clssE)
then show  $C \in ?S$ 
  using  $C \vdash T$  rtracp-cdclNOT-clauses-bound[OF assms]  $t \vdash$  by force
qed
then have  $\text{card } \{C. C \in \# \text{ clauses}_{NOT} T \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\} \leq$ 
 $\text{card } \{C. C \in \# \text{ clauses}_{NOT} S \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\}$ 
by (simp add: card-mono)
then show ?thesis
  unfolding  $\mu_{CDCL}$ '-bound-def by auto
qed

end — end of conflict-driven-clause-learning-learning-before-backjump-only-distinct-learn

```

16.7 CDCL with restarts

16.7.1 Definition

```

locale restart-ops =
  fixes
     $\text{cdcl}_{NOT} :: 'st \Rightarrow 'st \Rightarrow \text{bool}$  and
     $\text{restart} :: 'st \Rightarrow 'st \Rightarrow \text{bool}$ 
  begin
inductive  $\text{cdcl}_{NOT}\text{-raw-restart} :: 'st \Rightarrow 'st \Rightarrow \text{bool}$  where
 $\text{cdcl}_{NOT} S T \Longrightarrow \text{cdcl}_{NOT}\text{-raw-restart } S T \mid$ 
 $\text{restart } S T \Longrightarrow \text{cdcl}_{NOT}\text{-raw-restart } S T$ 
  end

locale conflict-driven-clause-learning-with-restarts =
  conflict-driven-clause-learning mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT
  inv backjump-conds propagate-conds learn-cond forget-cond
  for
     $\text{mset-clss} :: 'cls \Rightarrow 'v \text{ clause}$  and
     $\text{insert-clss} :: 'v \text{ literal} \Rightarrow 'cls \Rightarrow 'cls$  and
     $\text{remove-lit} :: 'v \text{ literal} \Rightarrow 'cls \Rightarrow 'cls$  and
     $\text{mset-clss} :: 'clss \Rightarrow 'v \text{ clauses}$  and
     $\text{union-clss} :: 'clss \Rightarrow 'clss \Rightarrow 'clss$  and
     $\text{in-clss} :: 'cls \Rightarrow 'clss \Rightarrow \text{bool}$  and
     $\text{insert-clss} :: 'cls \Rightarrow 'clss \Rightarrow 'clss$  and
     $\text{remove-from-clss} :: 'cls \Rightarrow 'clss \Rightarrow 'clss$  and
     $\text{trail} :: 'st \Rightarrow ('v, \text{unit}, \text{unit}) \text{ marked-lits}$  and
     $\text{raw-clauses} :: 'st \Rightarrow 'clss$  and
     $\text{prepend-trail} :: ('v, \text{unit}, \text{unit}) \text{ marked-lit} \Rightarrow 'st \Rightarrow 'st$  and
     $\text{tl-trail} :: 'st \Rightarrow 'st$  and
     $\text{add-cl}_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st$  and
     $\text{remove-cl}_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st$  and
     $\text{inv} :: 'st \Rightarrow \text{bool}$  and
     $\text{backjump-conds} :: 'v \text{ clause} \Rightarrow 'v \text{ clause} \Rightarrow 'v \text{ literal} \Rightarrow 'st \Rightarrow 'st \Rightarrow \text{bool}$  and
     $\text{propagate-conds} :: ('v, \text{unit}, \text{unit}) \text{ marked-lit} \Rightarrow 'st \Rightarrow \text{bool}$  and
     $\text{learn-cond forget-cond} :: 'cls \Rightarrow 'st \Rightarrow \text{bool}$ 
  begin

```

```

lemma cdclNOT-iff-cdclNOT-raw-restart-no-restarts:
  cdclNOT S T  $\longleftrightarrow$  restart-ops.cdclNOT-raw-restart cdclNOT ( $\lambda$ - . False) S T
  (is ?C S T  $\longleftrightarrow$  ?R S T)
proof
  fix S T
  assume ?C S T
  then show ?R S T by (simp add: restart-ops.cdclNOT-raw-restart.intros(1))
next
  fix S T
  assume ?R S T
  then show ?C S T
  apply (cases rule: restart-ops.cdclNOT-raw-restart.cases)
  using ⟨?R S T⟩ by fast+
qed

lemma cdclNOT-cdclNOT-raw-restart:
  cdclNOT S T  $\implies$  restart-ops.cdclNOT-raw-restart cdclNOT restart S T
  by (simp add: restart-ops.cdclNOT-raw-restart.intros(1))
end

```

16.7.2 Increasing restarts

To add restarts we need some assumptions on the predicate (called *cdcl_{NOT}* here):

- a function f that is strictly monotonic. The first step is actually only used as a restart to clean the state (e.g. to ensure that the trail is empty). Then we assume that $(1::'a) \leq f$ n for $(1::'a) \leq n$: it means that between two consecutive restarts, at least one step will be done. This is necessary to avoid sequence. like: full – restart – full – ...
- a measure μ : it should decrease under the assumptions *bound-inv*, whenever a *cdcl_{NOT}* or a *restart* is done. A parameter is given to μ : for conflict- driven clause learning, it is an upper-bound of the clauses. We are assuming that such a bound can be found after a restart whenever the invariant holds.
- we also assume that the measure decrease after any *cdcl_{NOT}* step.
- an invariant on the states *cdcl_{NOT}-inv* that also holds after restarts.
- it is *not required* that the measure decrease with respect to restarts, but the measure has to be bound by some function μ -*bound* taking the same parameter as μ and the initial state of the considered *cdcl_{NOT}* chain.

```

locale cdclNOT-increasing-restarts-ops =
  restart-ops cdclNOT restart for
    restart :: 'st  $\Rightarrow$  'st  $\Rightarrow$  bool and
    cdclNOT :: 'st  $\Rightarrow$  'st  $\Rightarrow$  bool +
fixes
  f :: nat  $\Rightarrow$  nat and
  bound-inv :: 'bound  $\Rightarrow$  'st  $\Rightarrow$  bool and
   $\mu$  :: 'bound  $\Rightarrow$  'st  $\Rightarrow$  nat and
  cdclNOT-inv :: 'st  $\Rightarrow$  bool and
   $\mu$ -bound :: 'bound  $\Rightarrow$  'st  $\Rightarrow$  nat
assumes

```

f : unbounded f **and**
 $f\text{-ge-1}$: $\bigwedge n. n \geq 1 \implies f\ n \neq 0$ **and**
 bound-inv : $\bigwedge A\ S\ T. \text{cdcl}_{NOT}\text{-inv}\ S \implies \text{bound-inv}\ A\ S \implies \text{cdcl}_{NOT}\ S\ T \implies \text{bound-inv}\ A\ T$ **and**
 $\text{cdcl}_{NOT}\text{-measure}$: $\bigwedge A\ S\ T. \text{cdcl}_{NOT}\text{-inv}\ S \implies \text{bound-inv}\ A\ S \implies \text{cdcl}_{NOT}\ S\ T \implies \mu\ A\ T < \mu$
 $A\ S$ **and**
 measure-bound2 : $\bigwedge A\ T\ U. \text{cdcl}_{NOT}\text{-inv}\ T \implies \text{bound-inv}\ A\ T \implies \text{cdcl}_{NOT}^{**}\ T\ U$
 $\implies \mu\ A\ U \leq \mu\text{-bound}\ A\ T$ **and**
 measure-bound4 : $\bigwedge A\ T\ U. \text{cdcl}_{NOT}\text{-inv}\ T \implies \text{bound-inv}\ A\ T \implies \text{cdcl}_{NOT}^{**}\ T\ U$
 $\implies \mu\text{-bound}\ A\ U \leq \mu\text{-bound}\ A\ T$ **and**
 $\text{cdcl}_{NOT}\text{-restart-inv}$: $\bigwedge A\ U\ V. \text{cdcl}_{NOT}\text{-inv}\ U \implies \text{restart}\ U\ V \implies \text{bound-inv}\ A\ U \implies \text{bound-inv}$
 $A\ V$
and
 exists-bound : $\bigwedge R\ S. \text{cdcl}_{NOT}\text{-inv}\ R \implies \text{restart}\ R\ S \implies \exists A. \text{bound-inv}\ A\ S$ **and**
 $\text{cdcl}_{NOT}\text{-inv}$: $\bigwedge S\ T. \text{cdcl}_{NOT}\text{-inv}\ S \implies \text{cdcl}_{NOT}\ S\ T \implies \text{cdcl}_{NOT}\text{-inv}\ T$ **and**
 $\text{cdcl}_{NOT}\text{-inv-restart}$: $\bigwedge S\ T. \text{cdcl}_{NOT}\text{-inv}\ S \implies \text{restart}\ S\ T \implies \text{cdcl}_{NOT}\text{-inv}\ T$
begin

lemma $\text{cdcl}_{NOT}\text{-cdcl}_{NOT}\text{-inv}$:

assumes

$(\text{cdcl}_{NOT} \rightsquigarrow n)\ S\ T$ **and**

$\text{cdcl}_{NOT}\text{-inv}\ S$

shows $\text{cdcl}_{NOT}\text{-inv}\ T$

using *assms* **by** (*induction* n *arbitrary*: T) (*auto intro*: $\text{bound-inv}\ \text{cdcl}_{NOT}\text{-inv}$)

lemma $\text{cdcl}_{NOT}\text{-bound-inv}$:

assumes

$(\text{cdcl}_{NOT} \rightsquigarrow n)\ S\ T$ **and**

$\text{cdcl}_{NOT}\text{-inv}\ S$

$\text{bound-inv}\ A\ S$

shows $\text{bound-inv}\ A\ T$

using *assms* **by** (*induction* n *arbitrary*: T) (*auto intro*: $\text{bound-inv}\ \text{cdcl}_{NOT}\text{-cdcl}_{NOT}\text{-inv}$)

lemma $\text{rtrancplp-cdcl}_{NOT}\text{-cdcl}_{NOT}\text{-inv}$:

assumes

$\text{cdcl}_{NOT}^{**}\ S\ T$ **and**

$\text{cdcl}_{NOT}\text{-inv}\ S$

shows $\text{cdcl}_{NOT}\text{-inv}\ T$

using *assms* **by** *induction* (*auto intro*: $\text{cdcl}_{NOT}\text{-inv}$)

lemma $\text{rtrancplp-cdcl}_{NOT}\text{-bound-inv}$:

assumes

$\text{cdcl}_{NOT}^{**}\ S\ T$ **and**

$\text{bound-inv}\ A\ S$ **and**

$\text{cdcl}_{NOT}\text{-inv}\ S$

shows $\text{bound-inv}\ A\ T$

using *assms* **by** *induction* (*auto intro*: $\text{bound-inv}\ \text{rtrancplp-cdcl}_{NOT}\text{-cdcl}_{NOT}\text{-inv}$)

lemma $\text{cdcl}_{NOT}\text{-comp-n-le}$:

assumes

$(\text{cdcl}_{NOT} \rightsquigarrow (\text{Suc}\ n))\ S\ T$ **and**

$\text{bound-inv}\ A\ S$

$\text{cdcl}_{NOT}\text{-inv}\ S$

shows $\mu\ A\ T < \mu\ A\ S - n$

using *assms*

proof (*induction* n *arbitrary*: T)

case 0
then show ?case **using** $cdcl_{NOT}$ -measure **by** auto
next
case (Suc n) **note** $IH = this(1)[OF - this(3) this(4)]$ **and** $S-T = this(2)$ **and** $b-inv = this(3)$ **and** $c-inv = this(4)$
obtain $U :: 'st$ **where** $S-U: (cdcl_{NOT} \rightsquigarrow (Suc\ n))\ S\ U$ **and** $U-T: cdcl_{NOT}\ U\ T$ **using** $S-T$ **by** auto
then have $\mu\ A\ U < \mu\ A\ S - n$ **using** $IH[of\ U]$ **by** simp
moreover
have bound-inv A U
using $S-U\ b-inv\ cdcl_{NOT}$ -bound-inv $c-inv$ **by** blast
then have $\mu\ A\ T < \mu\ A\ U$ **using** $cdcl_{NOT}$ -measure[OF - - U-T] $S-U\ c-inv\ cdcl_{NOT}$ - $cdcl_{NOT}$ -inv
by auto
ultimately show ?case **by** linarith
qed

lemma wf-cdcl_{NOT}:
 wf {(T, S). $cdcl_{NOT}\ S\ T \wedge cdcl_{NOT}$ -inv S \wedge bound-inv A S} (is wf ?A)
apply (rule wfP-if-measure2[of - - $\mu\ A$])
using $cdcl_{NOT}$ -comp-n-le[of 0 - - A] **by** auto

lemma rtranclp-cdcl_{NOT}-measure:
assumes
 $cdcl_{NOT}^{**}\ S\ T$ **and**
 bound-inv A S **and**
 $cdcl_{NOT}$ -inv S
shows $\mu\ A\ T \leq \mu\ A\ S$
using assms
proof (induction rule: rtranclp-induct)
case base
then show ?case **by** auto
next
case (step T U) **note** $IH = this(3)[OF\ this(4)\ this(5)]$ **and** $st = this(1)$ **and** $cdcl_{NOT} = this(2)$
and
 $b-inv = this(4)$ **and** $c-inv = this(5)$
have bound-inv A T
by (meson $cdcl_{NOT}$ -bound-inv rtranclp-imp-relpoup st step.prem)
moreover have $cdcl_{NOT}$ -inv T
using $c-inv\ rtranclp$ - $cdcl_{NOT}$ - $cdcl_{NOT}$ -inv st **by** blast
ultimately have $\mu\ A\ U < \mu\ A\ T$ **using** $cdcl_{NOT}$ -measure[OF - - $cdcl_{NOT}$] **by** auto
then show ?case **using** IH **by** linarith
qed

lemma $cdcl_{NOT}$ -comp-bounded:
assumes
 bound-inv A S **and** $cdcl_{NOT}$ -inv S **and** $m \geq 1 + \mu\ A\ S$
shows $\neg(cdcl_{NOT} \rightsquigarrow m)\ S\ T$
using assms $cdcl_{NOT}$ -comp-n-le[of m-1 S T A] **by** fastforce

- $f\ n < m$ ensures that at least one step has been done.

inductive $cdcl_{NOT}$ -restart **where**
 restart-step: $(cdcl_{NOT} \rightsquigarrow m)\ S\ T \implies m \geq f\ n \implies restart\ T\ U$
 $\implies cdcl_{NOT}$ -restart (S, n) (U, Suc n) |
 restart-full: full1 $cdcl_{NOT}\ S\ T \implies cdcl_{NOT}$ -restart (S, n) (T, Suc n)

lemmas $cdcl_{NOT}\text{-with-restart-induct} = cdcl_{NOT}\text{-restart.induct}[split\text{-format}(complete),$
 $OF\ cdcl_{NOT}\text{-increasing-restarts-ops-axioms}]$

lemma $cdcl_{NOT}\text{-restart-cdcl}_{NOT}\text{-raw-restart}:$

$cdcl_{NOT}\text{-restart } S\ T \implies cdcl_{NOT}\text{-raw-restart}^{**} (fst\ S)\ (fst\ T)$

proof (*induction rule*: $cdcl_{NOT}\text{-restart.induct}$)

case ($restart\text{-step } m\ S\ T\ n\ U$)

then have $cdcl_{NOT}^{**} S\ T$ **by** (*meson relpowp-imp-rtrancpl*)

then have $cdcl_{NOT}\text{-raw-restart}^{**} S\ T$ **using** $cdcl_{NOT}\text{-raw-restart.intros}(1)$

$rtrancpl\text{-mono}[of\ cdcl_{NOT}\ cdcl_{NOT}\text{-raw-restart}]$ **by blast**

moreover have $cdcl_{NOT}\text{-raw-restart } T\ U$

using ($restart\ T\ U$) $cdcl_{NOT}\text{-raw-restart.intros}(2)$ **by blast**

ultimately show $?case$ **by auto**

next

case ($restart\text{-full } S\ T$)

then have $cdcl_{NOT}^{**} S\ T$ **unfolding** $full1\text{-def}$ **by auto**

then show $?case$ **using** $cdcl_{NOT}\text{-raw-restart.intros}(1)$

$rtrancpl\text{-mono}[of\ cdcl_{NOT}\ cdcl_{NOT}\text{-raw-restart}]$ **by auto**

qed

lemma $cdcl_{NOT}\text{-with-restart-bound-inv}:$

assumes

$cdcl_{NOT}\text{-restart } S\ T$ **and**

$bound\text{-inv } A\ (fst\ S)$ **and**

$cdcl_{NOT}\text{-inv } (fst\ S)$

shows $bound\text{-inv } A\ (fst\ T)$

using *assms* **apply** (*induction rule*: $cdcl_{NOT}\text{-restart.induct}$)

prefer 2 **apply** (*metis* $rtrancpl\text{-unfold } fstI\ full1\text{-def } rtrancpl\text{-cdcl}_{NOT}\text{-bound-inv}$)

by (*metis* $cdcl_{NOT}\text{-bound-inv } cdcl_{NOT}\text{-cdcl}_{NOT}\text{-inv } cdcl_{NOT}\text{-restart-inv } fst\text{-conv}$)

lemma $cdcl_{NOT}\text{-with-restart-cdcl}_{NOT}\text{-inv}:$

assumes

$cdcl_{NOT}\text{-restart } S\ T$ **and**

$cdcl_{NOT}\text{-inv } (fst\ S)$

shows $cdcl_{NOT}\text{-inv } (fst\ T)$

using *assms* **apply** *induction*

apply (*metis* $cdcl_{NOT}\text{-cdcl}_{NOT}\text{-inv } cdcl_{NOT}\text{-inv-restart } fst\text{-conv}$)

apply (*metis* $fstI\ full\text{-def } full\text{-unfold } rtrancpl\text{-cdcl}_{NOT}\text{-cdcl}_{NOT}\text{-inv}$)

done

lemma $rtrancpl\text{-cdcl}_{NOT}\text{-with-restart-cdcl}_{NOT}\text{-inv}:$

assumes

$cdcl_{NOT}\text{-restart}^{**} S\ T$ **and**

$cdcl_{NOT}\text{-inv } (fst\ S)$

shows $cdcl_{NOT}\text{-inv } (fst\ T)$

using *assms* **by** *induction* (*auto intro*: $cdcl_{NOT}\text{-with-restart-cdcl}_{NOT}\text{-inv}$)

lemma $rtrancpl\text{-cdcl}_{NOT}\text{-with-restart-bound-inv}:$

assumes

$cdcl_{NOT}\text{-restart}^{**} S\ T$ **and**

$cdcl_{NOT}\text{-inv } (fst\ S)$ **and**

$bound\text{-inv } A\ (fst\ S)$

shows $bound\text{-inv } A\ (fst\ T)$

using *assms* **apply** *induction*

apply (*simp add*: $cdcl_{NOT}\text{-cdcl}_{NOT}\text{-inv } cdcl_{NOT}\text{-with-restart-bound-inv}$)

using *cdcl_{NOT}-with-restart-bound-inv rtrancpl-cdcl_{NOT}-with-restart-cdcl_{NOT}-inv* **by** *blast*

lemma *cdcl_{NOT}-with-restart-increasing-number:*

cdcl_{NOT}-restart S T \implies snd T = 1 + snd S

by (*induction rule: cdcl_{NOT}-restart.induct*) *auto*

end

locale *cdcl_{NOT}-increasing-restarts =*

cdcl_{NOT}-increasing-restarts-ops restart cdcl_{NOT} f bound-inv μ cdcl_{NOT}-inv μ -bound +

dpll-state mset-cls insert-cls remove-lit

mset-clss union-clss in-clss insert-clss remove-from-clss

trail raw-clauses prepend-trail tl-trail add-cls_{NOT} remove-cls_{NOT}

for

mset-cls:: 'cls \Rightarrow 'v clause and

insert-cls :: 'v literal \Rightarrow 'cls \Rightarrow 'cls and

remove-lit :: 'v literal \Rightarrow 'cls \Rightarrow 'cls and

mset-clss:: 'clss \Rightarrow 'v clauses and

union-clss :: 'clss \Rightarrow 'clss \Rightarrow 'clss and

in-clss :: 'cls \Rightarrow 'clss \Rightarrow bool and

insert-clss :: 'cls \Rightarrow 'clss \Rightarrow 'clss and

remove-from-clss :: 'cls \Rightarrow 'clss \Rightarrow 'clss and

trail :: 'st \Rightarrow ('v, unit, unit) marked-lits and

raw-clauses :: 'st \Rightarrow 'clss and

prepend-trail :: ('v, unit, unit) marked-lit \Rightarrow 'st \Rightarrow 'st and

tl-trail :: 'st \Rightarrow 'st and

add-cls_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st and

remove-cls_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st and

f :: nat \Rightarrow nat and

restart :: 'st \Rightarrow 'st \Rightarrow bool and

bound-inv :: 'bound \Rightarrow 'st \Rightarrow bool and

μ :: 'bound \Rightarrow 'st \Rightarrow nat and

cdcl_{NOT} :: 'st \Rightarrow 'st \Rightarrow bool and

cdcl_{NOT}-inv :: 'st \Rightarrow bool and

μ -bound :: 'bound \Rightarrow 'st \Rightarrow nat +

assumes

measure-bound: $\bigwedge A T V n. cdcl_{NOT}\text{-inv } T \implies bound\text{-inv } A T$

$\implies cdcl_{NOT}\text{-restart } (T, n) (V, Suc\ n) \implies \mu\ A\ V \leq \mu\text{-bound } A\ T$ and

cdcl_{NOT}-raw-restart- μ -bound:

$cdcl_{NOT}\text{-restart } (T, a) (V, b) \implies cdcl_{NOT}\text{-inv } T \implies bound\text{-inv } A\ T$

$\implies \mu\text{-bound } A\ V \leq \mu\text{-bound } A\ T$

begin

lemma *rtrancpl-cdcl_{NOT}-raw-restart- μ -bound:*

*cdcl_{NOT}-restart** (T, a) (V, b) $\implies cdcl_{NOT}\text{-inv } T \implies bound\text{-inv } A\ T$*

$\implies \mu\text{-bound } A\ V \leq \mu\text{-bound } A\ T$

apply (*induction rule: rtrancpl-induct2*)

apply *simp*

by (*metis cdcl_{NOT}-raw-restart- μ -bound dual-order.trans fst-conv*

rtrancpl-cdcl_{NOT}-with-restart-bound-inv rtrancpl-cdcl_{NOT}-with-restart-cdcl_{NOT}-inv)

lemma *cdcl_{NOT}-raw-restart-measure-bound:*

cdcl_{NOT}-restart (T, a) (V, b) $\implies cdcl_{NOT}\text{-inv } T \implies bound\text{-inv } A\ T$

$\implies \mu\ A\ V \leq \mu\text{-bound } A\ T$

apply (*cases rule: cdcl_{NOT}-restart.cases*)

apply *simp*

using *measure-bound relpoup-imp-rtrancpl* **apply** *fastforce*
by (*metis full-def full-unfold measure-bound2 prod.inject*)

lemma *rtrancpl-cdcl_{NOT}-raw-restart-measure-bound*:

*cdcl_{NOT}-restart*** (*T*, *a*) (*V*, *b*) \implies *cdcl_{NOT}-inv* *T* \implies *bound-inv* *A* *T*
 $\implies \mu$ *A* *V* $\leq \mu$ -*bound* *A* *T*

apply (*induction rule: rtrancpl-induct2*)

apply (*simp add: measure-bound2*)

by (*metis dual-order.trans fst-conv measure-bound2 r-into-rtrancpl rtrancpl.rtrancpl-refl*
rtrancpl-cdcl_{NOT}-with-restart-bound-inv rtrancpl-cdcl_{NOT}-with-restart-cdcl_{NOT}-inv
rtrancpl-cdcl_{NOT}-raw-restart-μ-bound)

lemma *wf-cdcl_{NOT}-restart*:

wf {(*T*, *S*). *cdcl_{NOT}-restart* *S* *T* \wedge *cdcl_{NOT}-inv* (*fst* *S*)} (**is** *wf* ?*A*)

proof (*rule ccontr*)

assume \neg ?*thesis*

then obtain *g* **where**

g: $\bigwedge i.$ *cdcl_{NOT}-restart* (*g* *i*) (*g* (*Suc* *i*)) **and**

cdcl_{NOT}-inv-g: $\bigwedge i.$ *cdcl_{NOT}-inv* (*fst* (*g* *i*))

unfolding *wf-iff-no-infinite-down-chain* **by** *fast*

have *snd-g*: $\bigwedge i.$ *snd* (*g* *i*) = *i* + *snd* (*g* 0)

apply (*induct-tac i*)

apply *simp*

by (*metis Suc-eq-plus1-left add.commute add.left-commute*
cdcl_{NOT}-with-restart-increasing-number g)

then have *snd-g-0*: $\bigwedge i.$ *i* > 0 \implies *snd* (*g* *i*) = *i* + *snd* (*g* 0)

by *blast*

have *unbounded-f-g*: *unbounded* ($\lambda i.$ *f* (*snd* (*g* *i*)))

using *f* **unfolding** *bounded-def* **by** (*metis add.commute f less-or-eq-imp-le snd-g*
not-bounded-nat-exists-larger not-le le-iff-add)

{ **fix** *i*

have *H*: $\bigwedge T$ *Ta* *m*. (*cdcl_{NOT}* \rightsquigarrow *m*) *T* *Ta* \implies *no-step cdcl_{NOT}* *T* \implies *m* = 0

apply (*case-tac m*) **by** *simp* (*meson relpoup-E2*)

have \exists *T* *m*. (*cdcl_{NOT}* \rightsquigarrow *m*) (*fst* (*g* *i*)) *T* \wedge *m* \geq *f* (*snd* (*g* *i*))

using *g*[*of i*] **apply** (*cases rule: cdcl_{NOT}-restart.cases*)

apply *auto*[]

using *g*[*of Suc i*] *f-ge-1* **apply** (*cases rule: cdcl_{NOT}-restart.cases*)

apply (*auto simp add: full1-def full-def dest: H dest: rtrancplD*)

using *H Suc-leI leD* **by** *blast*

} **note** *H* = *this*

obtain *A* **where** *bound-inv* *A* (*fst* (*g* 1))

using *g*[*of 0*] *cdcl_{NOT}-inv-g*[*of 0*] **apply** (*cases rule: cdcl_{NOT}-restart.cases*)

apply (*metis One-nat-def cdcl_{NOT}-inv exists-bound fst-conv relpoup-imp-rtrancpl*
rtrancpl-induct)

using *H*[*of 1*] **unfolding** *full1-def* **by** (*metis One-nat-def Suc-eq-plus1 diff-is-0-eq' diff-zero*
f-ge-1 fst-conv le-add2 relpoup-E2 snd-conv)

let ?*j* = μ -*bound* *A* (*fst* (*g* 1)) + 1

obtain *j* **where**

j: *f* (*snd* (*g* *j*)) > ?*j* **and** *j* > 1

using *unbounded-f-g not-bounded-nat-exists-larger* **by** *blast*

{

fix *i* *j*

have *cdcl_{NOT}-with-restart*: *j* \geq *i* \implies *cdcl_{NOT}-restart*** (*g* *i*) (*g* *j*)

```

    apply (induction j)
    apply simp
    by (metis g le-Suc-eq rtrancpl.rtrancle-into-rtrancle rtrancpl.rtrancle-refl)
  } note cdclNOT-restart = this
  have cdclNOT-inv (fst (g (Suc 0)))
    by (simp add: cdclNOT-inv-g)
  have cdclNOT-restart** (fst (g 1), snd (g 1)) (fst (g j), snd (g j))
    using ⟨j > 1⟩ by (simp add: cdclNOT-restart)
  have μ A (fst (g j)) ≤ μ-bound A (fst (g 1))
    apply (rule rtrancpl-cdclNOT-raw-restart-measure-bound)
    using ⟨cdclNOT-restart** (fst (g 1), snd (g 1)) (fst (g j), snd (g j))⟩ apply blast
    apply (simp add: cdclNOT-inv-g)
    using ⟨bound-inv A (fst (g 1))⟩ apply simp
  done
  then have μ A (fst (g j)) ≤ ?j
    by auto
  have inv: bound-inv A (fst (g j))
    using ⟨bound-inv A (fst (g 1))⟩ ⟨cdclNOT-inv (fst (g (Suc 0)))⟩
    ⟨cdclNOT-restart** (fst (g 1), snd (g 1)) (fst (g j), snd (g j))⟩
    rtrancpl-cdclNOT-with-restart-bound-inv by auto
  obtain T m where
    cdclNOT-m: (cdclNOT  $\rightsquigarrow$  m) (fst (g j)) T and
    f-m: f (snd (g j)) ≤ m
    using H[of j] by blast
  have ?j < m
    using f-m j Nat.le-trans by linarith

  then show False
    using ⟨μ A (fst (g j)) ≤ μ-bound A (fst (g 1))⟩
    cdclNOT-comp-bounded[OF inv cdclNOT-inv-g, of ] cdclNOT-inv-g cdclNOT-m
    ⟨?j < m⟩ by auto
qed

```

lemma *cdcl_{NOT}-restart-steps-bigger-than-bound:*

```

  assumes
    cdclNOT-restart S T and
    bound-inv A (fst S) and
    cdclNOT-inv (fst S) and
    f (snd S) > μ-bound A (fst S)
  shows full1 cdclNOT (fst S) (fst T)
  using assms
proof (induction rule: cdclNOT-restart.induct)
  case restart-full
  then show ?case by auto
next
  case (restart-step m S T n U) note st = this(1) and f = this(2) and bound-inv = this(4) and
    cdclNOT-inv = this(5) and μ = this(6)
  then obtain m' where m: m = Suc m' by (cases m) auto
  have μ A S - m' = 0
    using f bound-inv cdclNOT-inv μ m rtrancpl-cdclNOT-raw-restart-measure-bound by fastforce
  then have False using cdclNOT-comp-n-le[of m' S T A] restart-step unfolding m by simp
  then show ?case by fast
qed

```

lemma *rtrancpl-cdcl_{NOT}-with-inv-inv-rtrancpl-cdcl_{NOT}:*

```

assumes
  inv: cdclNOT-inv S and
  binv: bound-inv A S
shows ( $\lambda S T. \text{cdcl}_{NOT} S T \wedge \text{cdcl}_{NOT}\text{-inv } S \wedge \text{bound-inv } A S$ )** S T  $\longleftrightarrow \text{cdcl}_{NOT}$ ** S T
  (is ?A** S T  $\longleftrightarrow$  ?B** S T)
apply (rule iffI)
  using rtrancpl-mono[of ?A ?B] apply blast
apply (induction rule: rtrancpl-induct)
  using inv binv apply simp
by (metis (mono-tags, lifting) binv inv rtrancpl.simps rtrancpl-cdclNOT-bound-inv
  rtrancpl-cdclNOT-cdclNOT-inv)

lemma no-step-cdclNOT-restart-no-step-cdclNOT:
assumes
  n-s: no-step cdclNOT-restart S and
  inv: cdclNOT-inv (fst S) and
  binv: bound-inv A (fst S)
shows no-step cdclNOT (fst S)
proof (rule ccontr)
assume  $\neg$  ?thesis
then obtain T where T: cdclNOT (fst S) T
  by blast
then obtain U where U: full ( $\lambda S T. \text{cdcl}_{NOT} S T \wedge \text{cdcl}_{NOT}\text{-inv } S \wedge \text{bound-inv } A S$ ) T U
  using wf-exists-normal-form-full[OF wf-cdclNOT, of A T] by auto
moreover have inv-T: cdclNOT-inv T
  using  $\langle \text{cdcl}_{NOT} (\text{fst } S) T \rangle \text{cdcl}_{NOT}\text{-inv } inv$  by blast
moreover have b-inv-T: bound-inv A T
  using  $\langle \text{cdcl}_{NOT} (\text{fst } S) T \rangle binv \text{bound-inv } inv$  by blast
ultimately have full cdclNOT T U
  using rtrancpl-cdclNOT-with-inv-inv-rtrancpl-cdclNOT rtrancpl-cdclNOT-bound-inv
  rtrancpl-cdclNOT-cdclNOT-inv unfolding full-def by blast
then have full1 cdclNOT (fst S) U
  using T full-fullI by metis
then show False by (metis n-s prod.collapse restart-full)
qed

end

```

16.8 Merging backjump and learning

```

locale cdclNOT-merge-bj-learn-ops =
  decide-ops mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT +
  forget-ops mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT forget-cond +
  propagate-ops mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT propagate-conds
for
  mset-cls:: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and

```

```

in-clss :: 'cls ⇒ 'clss ⇒ bool and
insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
trail :: 'st ⇒ ('v, unit, unit) marked-lits and
raw-clauses :: 'st ⇒ 'clss and
prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
tl-trail :: 'st ⇒ 'st and
add-clNOT :: 'cls ⇒ 'st ⇒ 'st and
remove-clNOT :: 'cls ⇒ 'st ⇒ 'st and
propagate-conds :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ bool and
forget-cond :: 'cls ⇒ 'st ⇒ bool +
fixes backjump-l-cond :: 'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool
begin
inductive backjump-l where
backjump-l: trail S = F' @ Marked K () # F
  ⇒ no-dup (trail S)
  ⇒ T ~ prepend-trail (Propagated L ()) (reduce-trail-toNOT F (add-clNOT C'' S))
  ⇒ C ∈ # clausesNOT S
  ⇒ trail S ⊨as CNot C
  ⇒ undefined-lit F L
  ⇒ atm-of L ∈ atms-of-mm (clausesNOT S) ∪ atm-of ' (lits-of-l (trail S))
  ⇒ clausesNOT S ⊨pm C' + {#L#}
  ⇒ mset-cls C'' = C' + {#L#}
  ⇒ F ⊨as CNot C'
  ⇒ backjump-l-cond C C' L S T
  ⇒ backjump-l S T

```

Avoid (meaningless) simplification:

```

declare reduce-trail-toNOT-length-ne[simp del] Set.Un-iff[simp del] Set.insert-iff[simp del]
inductive-cases backjump-lE: backjump-l S T
thm backjump-lE
declare reduce-trail-toNOT-length-ne[simp] Set.Un-iff[simp] Set.insert-iff[simp]

```

```

inductive cdclNOT-merged-bj-learn :: 'st ⇒ 'st ⇒ bool for S :: 'st where
cdclNOT-merged-bj-learn-decideNOT: decideNOT S S' ⇒ cdclNOT-merged-bj-learn S S' |
cdclNOT-merged-bj-learn-propagateNOT: propagateNOT S S' ⇒ cdclNOT-merged-bj-learn S S' |
cdclNOT-merged-bj-learn-backjump-l: backjump-l S S' ⇒ cdclNOT-merged-bj-learn S S' |
cdclNOT-merged-bj-learn-forgetNOT: forgetNOT S S' ⇒ cdclNOT-merged-bj-learn S S'

```

```

lemma cdclNOT-merged-bj-learn-no-dup-inv:
  cdclNOT-merged-bj-learn S T ⇒ no-dup (trail S) ⇒ no-dup (trail T)
apply (induction rule: cdclNOT-merged-bj-learn.induct)
  using defined-lit-map apply fastforce
  using defined-lit-map apply fastforce
  apply (force simp: defined-lit-map elim!: backjump-lE)[]
  using forgetNOT.simps apply auto[1]
done
end

```

```

locale cdclNOT-merge-bj-learn-proxy =
  cdclNOT-merge-bj-learn-ops mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT propagate-conds
  forget-cond
  λC C' L' S T. backjump-l-cond C C' L' S T

```

$\wedge \text{distinct-mset } (C' + \{\#L'\#\}) \wedge \neg\text{tautology } (C' + \{\#L'\#\})$
for
mset-cl :: 'cls \Rightarrow 'v clause **and**
insert-cl :: 'v literal \Rightarrow 'cls \Rightarrow 'cls **and**
remove-lit :: 'v literal \Rightarrow 'cls \Rightarrow 'cls **and**
mset-clss :: 'clss \Rightarrow 'v clauses **and**
union-clss :: 'clss \Rightarrow 'clss \Rightarrow 'clss **and**
in-clss :: 'cls \Rightarrow 'clss \Rightarrow bool **and**
insert-clss :: 'cls \Rightarrow 'clss \Rightarrow 'clss **and**
remove-from-clss :: 'cls \Rightarrow 'clss \Rightarrow 'clss **and**
trail :: 'st \Rightarrow ('v, unit, unit) marked-lits **and**
raw-clauses :: 'st \Rightarrow 'clss **and**
prepend-trail :: ('v, unit, unit) marked-lit \Rightarrow 'st \Rightarrow 'st **and**
tl-trail :: 'st \Rightarrow 'st **and**
add-cl_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st **and**
remove-cl_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st **and**
propagate-conds :: ('v, unit, unit) marked-lit \Rightarrow 'st \Rightarrow bool **and**
forget-cond :: 'cls \Rightarrow 'st \Rightarrow bool **and**
backjump-l-cond :: 'v clause \Rightarrow 'v clause \Rightarrow 'v literal \Rightarrow 'st \Rightarrow 'st \Rightarrow bool +
fixes
inv :: 'st \Rightarrow bool
assumes
bj-merge-can-jump:
 $\bigwedge S \ C \ F' \ K \ F \ L.$
inv *S*
 $\Rightarrow \text{trail } S = F' @ \text{Marked } K \ () \# F$
 $\Rightarrow C \in \# \text{ clauses}_{\text{NOT}} S$
 $\Rightarrow \text{trail } S \models_{\text{as}} C \text{Not } C$
 $\Rightarrow \text{undefined-lit } F \ L$
 $\Rightarrow \text{atm-of } L \in \text{atms-of-mm } (\text{clauses}_{\text{NOT}} S) \cup \text{atm-of ' (lits-of-l (F' @ Marked } K \ () \# F))$
 $\Rightarrow \text{clauses}_{\text{NOT}} S \models_{\text{pm}} C' + \{\#L'\#\}$
 $\Rightarrow F \models_{\text{as}} C \text{Not } C'$
 $\Rightarrow \neg \text{no-step backjump-l } S$ **and**
cdcl-merged-inv: $\bigwedge S \ T. \text{cdcl}_{\text{NOT}}\text{-merged-bj-learn } S \ T \Rightarrow \text{inv } S \Rightarrow \text{inv } T$
begin

abbreviation *backjump-conds* :: 'v clause \Rightarrow 'v clause \Rightarrow 'v literal \Rightarrow 'st \Rightarrow 'st \Rightarrow bool
where
backjump-conds $\equiv \lambda C \ C' \ L' \ S \ T. \text{distinct-mset } (C' + \{\#L'\#\}) \wedge \neg\text{tautology } (C' + \{\#L'\#\})$

Without additional knowledge on *backjump-l-cond*, it is impossible to have the same invariant.

sublocale *dpll-with-backjumping-ops* *mset-cl* *insert-cl* *remove-lit*
mset-clss *union-clss* *in-clss* *insert-clss* *remove-from-clss*
trail *raw-clauses* *prepend-trail* *tl-trail* *add-cl_{NOT}* *remove-cl_{NOT}* *inv*
backjump-conds *propagate-conds*

proof (*unfold-locales*, *goal-cases*)

case 1

{ **fix** *S* *S'*

assume *bj*: *backjump-l* *S* *S'* **and** *no-dup* (*trail* *S*)

then obtain *F' K F L C' C D* **where**

S': *S' ~ prepend-trail (Propagated L ()) (reduce-trail-to_{NOT} F (add-cl_{NOT} D S))*

and

tr-S: *trail* *S* = *F' @ Marked K () # F* **and**

C: *C* $\in \# \text{ clauses}_{\text{NOT}} S$ **and**

tr-S-C: *trail* *S* $\models_{\text{as}} C \text{Not } C$ **and**

```

undef-L: undefined-lit F L and
atm-L:
  atm-of L ∈ insert (atm-of K) (atms-of-mm (clausesNOT S) ∪ atm-of ‘ (lits-of-l F' ∪ lits-of-l F))
  and
cls-S-C': clausesNOT S ⊨pm C' + {#L#} and
F-C': F ⊨as CNot C' and
dist: distinct-mset (C' + {#L#}) and
not-tauto: ¬ tautology (C' + {#L#}) and
cond: backjump-l-cond C C' L S S'
mset-cls D = C' + {#L#}
by (elim backjump-lE) metis
interpret backjumping-ops mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-clsNOT remove-clsNOT
backjump-conds
  by unfold-locales
have ∃ T. backjump S T
  apply rule
  apply (rule backjump.intros)
    using tr-S apply simp
    apply (rule state-eqNOT-ref)
    using C apply simp
    using tr-S-C apply simp
    using undef-L apply simp
    using atm-L tr-S apply simp
    using cls-S-C' apply simp
    using F-C' apply simp
    using dist not-tauto cond apply simp
  done
} note H = this(1)
then show ?case using 1 bj-merge-can-jump by meson
qed

end

locale cdclNOT-merge-bj-learn-proxy2 =
  cdclNOT-merge-bj-learn-proxy mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clsNOT remove-clsNOT
  propagate-conds forget-cond backjump-l-cond inv
for
  mset-cls:: 'cls ⇒ 'v clause and
  insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
  remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
  mset-clss:: 'clss ⇒ 'v clauses and
  union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
  in-clss :: 'cls ⇒ 'clss ⇒ bool and
  insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
  remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
  trail :: 'st ⇒ ('v, unit, unit) marked-lits and
  raw-clauses :: 'st ⇒ 'clss and
  prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
  tl-trail :: 'st ⇒ 'st and
  add-clsNOT :: 'cls ⇒ 'st ⇒ 'st and
  remove-clsNOT :: 'cls ⇒ 'st ⇒ 'st and

```

```

propagate-conds :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  bool and
forget-cond :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  bool and
backjump-l-cond :: 'v clause  $\Rightarrow$  'v clause  $\Rightarrow$  'v literal  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  bool and
inv :: 'st  $\Rightarrow$  bool
begin

sublocale conflict-driven-clause-learning-ops mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clsNOT remove-clsNOT
  inv backjump-conds propagate-conds
   $\lambda C \cdot$ . distinct-mset (mset-cls C)  $\wedge$   $\neg$ tautology (mset-cls C)
  forget-cond
by unfold-locales
end

locale cdclNOT-merge-bj-learn =
  cdclNOT-merge-bj-learn-proxy2 mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clsNOT remove-clsNOT
  propagate-conds forget-cond backjump-l-cond inv
for
  mset-cls:: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
  raw-clauses :: 'st  $\Rightarrow$  'clss and
  prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
  tl-trail :: 'st  $\Rightarrow$  'st and
  add-clsNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  remove-clsNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  backjump-l-cond :: 'v clause  $\Rightarrow$  'v clause  $\Rightarrow$  'v literal  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  bool and
  propagate-conds :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  bool and
  forget-cond :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  bool and
  inv :: 'st  $\Rightarrow$  bool +
assumes
  dpll-merge-bj-inv:  $\bigwedge S T. \text{dpll-bj } S T \Longrightarrow \text{inv } S \Longrightarrow \text{inv } T$  and
  learn-inv:  $\bigwedge S T. \text{learn } S T \Longrightarrow \text{inv } S \Longrightarrow \text{inv } T$ 
begin

sublocale
  conflict-driven-clause-learning mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clsNOT remove-clsNOT
  inv backjump-conds propagate-conds
   $\lambda C \cdot$ . distinct-mset (mset-cls C)  $\wedge$   $\neg$ tautology (mset-cls C)
  forget-cond
apply unfold-locales
using cdclNOT-merged-bj-learn-forgetNOT cdcl-merged-inv learn-inv
by (auto simp add: cdclNOT.simps dpll-merge-bj-inv)

```

lemma *backjump-l-learn-backjump*:

assumes *bt*: *backjump-l S T* **and** *inv*: *inv S* **and** *n-d*: *no-dup (trail S)*

shows $\exists C' L D. \text{learn } S \text{ (add-cl}_{\text{NOT}} D S)$

$\wedge \text{mset-cl}_{\text{S}} D = (C' + \{\#L\# \})$

$\wedge \text{backjump (add-cl}_{\text{NOT}} D S) T$

$\wedge \text{atms-of } (C' + \{\#L\# \}) \subseteq \text{atms-of-mm (clauses}_{\text{NOT}} S) \cup \text{atm-of ' (lits-of-l (trail S))}$

proof –

obtain *C F' K F L l C' D* **where**

tr-S: *trail S = F' @ Marked K () # F* **and**

T: *T ~ prepend-trail (Propagated L l) (reduce-trail-to_{NOT} F (add-cl_{NOT} D S))* **and**

C-cl_S: *C ∈ # clauses_{NOT} S* **and**

tr-S-CNot-C: *trail S ⊨_{as} CNot C* **and**

undef: *undefined-lit F L* **and**

atm-L: *atm-of L ∈ atms-of-mm (clauses_{NOT} S) ∪ atm-of ' (lits-of-l (trail S))* **and**

clss-C: *clauses_{NOT} S ⊨_{pm} mset-cl_S D* **and**

D: *mset-cl_S D = C' + {#L#}*

F ⊨_{as} CNot C' **and**

distinct: *distinct-mset (mset-cl_S D)* **and**

not-tauto: $\neg \text{tautology (mset-cl}_{\text{S}} D)$

using *bt inv* **by** (*elim backjump-lE*) *simp*

have *atms-C'*: *atms-of C' ⊆ atm-of ' (lits-of-l F)*

by (*metis D(2) atms-of-def image-subsetI true-annots-CNot-all-atms-defined*)

then have *atms-of (C' + {#L#}) ⊆ atms-of-mm (clauses_{NOT} S) ∪ atm-of ' (lits-of-l (trail S))*

using *atm-L tr-S* **by** *auto*

moreover have *learn*: *learn S (add-cl_{NOT} D S)*

apply (*rule learn.intros*)

apply (*rule clss-C*)

using *atms-C' atm-L D* **apply** (*fastforce simp add: tr-S in-plus-implies-atm-of-on-atms-of-ms*)

apply *standard*

apply (*rule distinct*)

apply (*rule not-tauto*)

apply *simp*

done

moreover have *bj*: *backjump (add-cl_{NOT} D S) T*

apply (*rule backjump.intros*)

using $\langle F \models_{\text{as}} \text{CNot } C' \rangle$ *C-cl_S tr-S-CNot-C undef T distinct not-tauto n-d D*

by (*auto simp: tr-S state-eq_{NOT}-def simp del: state-simp_{NOT}*)

ultimately show *?thesis* **using** *D* **by** *blast*

qed

lemma *cdcl_{NOT}-merged-bj-learn-is-tranclp-cdcl_{NOT}*:

cdcl_{NOT}-merged-bj-learn S T ⇒ inv S ⇒ no-dup (trail S) ⇒ cdcl_{NOT}⁺⁺ S T

proof (*induction rule: cdcl_{NOT}-merged-bj-learn.induct*)

case (*cdcl_{NOT}-merged-bj-learn-decide_{NOT} T*)

then have *cdcl_{NOT} S T*

using *bj-decide_{NOT} cdcl_{NOT}.sims* **by** *fastforce*

then show *?case* **by** *auto*

next

case (*cdcl_{NOT}-merged-bj-learn-propagate_{NOT} T*)

then have *cdcl_{NOT} S T*

using *bj-propagate_{NOT} cdcl_{NOT}.sims* **by** *fastforce*

then show *?case* **by** *auto*

next

case (*cdcl_{NOT}-merged-bj-learn-forget_{NOT} T*)

then have *cdcl_{NOT} S T*

using $c\text{-forget}_{NOT}$ **by** $blast$
then show $?case$ **by** $auto$
next
case $(cdcl_{NOT}\text{-merged-bj-learn-backjump-l } T)$ **note** $bt = this(1)$ **and** $inv = this(2)$ **and**
 $n\text{-d} = this(3)$
obtain $C' :: 'v \text{ literal multiset}$ **and** $L :: 'v \text{ literal}$ **and** $D :: 'cls$ **where**
 $f3: learn\ S\ (add\ cls_{NOT}\ D\ S) \wedge$
 $backjump\ (add\ cls_{NOT}\ D\ S)\ T \wedge$
 $atms\text{-of}\ (C' + \{\#L\}) \subseteq atms\text{-of}\text{-mm}\ (clauses_{NOT}\ S) \cup atm\text{-of}\ ' \text{ lits-of-l}\ (trail\ S)$ **and**
 $D: mset\text{-cls}\ D = C' + \{\#L\}$
using $n\text{-d}\ backjump\text{-l-learn-backjump}[OF\ bt\ inv]$ **by** $blast$
then have $f4: cdcl_{NOT}\ S\ (add\ cls_{NOT}\ D\ S)$
using $n\text{-d}\ c\text{-learn}$ **by** $blast$
have $cdcl_{NOT}\ (add\ cls_{NOT}\ D\ S)\ T$
using $f3\ n\text{-d}\ bj\text{-backjump}\ c\text{-dpll-bj}$ **by** $blast$
then show $?case$
using $f4$ **by** $(meson\ tranclp.r\text{-into-trancl}\ tranclp.trancl\text{-into-trancl})$
qed

lemma $rtranclp\text{-}cdcl_{NOT}\text{-merged-bj-learn-is-rtranclp-cdcl_{NOT}\text{-and-inv}$:
 $cdcl_{NOT}\text{-merged-bj-learn}^{**}\ S\ T \implies inv\ S \implies no\text{-dup}\ (trail\ S) \implies cdcl_{NOT}^{**}\ S\ T \wedge inv\ T$
proof (*induction rule: rtranclp-induct*)
case $base$
then show $?case$ **by** $auto$
next
case $(step\ T\ U)$ **note** $st = this(1)$ **and** $cdcl_{NOT} = this(2)$ **and** $IH = this(3)[OF\ this(4-)]$ **and**
 $inv = this(4)$ **and** $n\text{-d} = this(5)$
have $cdcl_{NOT}^{**}\ T\ U$
using $cdcl_{NOT}\text{-merged-bj-learn-is-tranclp-cdcl_{NOT}}[OF\ cdcl_{NOT}]\ IH$
 $rtranclp\text{-}cdcl_{NOT}\text{-no-dup}\ inv\ n\text{-d}$ **by** $auto$
then have $cdcl_{NOT}^{**}\ S\ U$ **using** IH **by** $fastforce$
moreover have $inv\ U$ **using** $n\text{-d}\ IH\ \langle cdcl_{NOT}^{**}\ T\ U \rangle\ rtranclp\text{-}cdcl_{NOT}\text{-inv}$ **by** $blast$
ultimately show $?case$ **using** st **by** $fast$
qed

lemma $rtranclp\text{-}cdcl_{NOT}\text{-merged-bj-learn-is-rtranclp-cdcl_{NOT}}$:
 $cdcl_{NOT}\text{-merged-bj-learn}^{**}\ S\ T \implies inv\ S \implies no\text{-dup}\ (trail\ S) \implies cdcl_{NOT}^{**}\ S\ T$
using $rtranclp\text{-}cdcl_{NOT}\text{-merged-bj-learn-is-rtranclp-cdcl_{NOT}\text{-and-inv}}$ **by** $blast$

lemma $rtranclp\text{-}cdcl_{NOT}\text{-merged-bj-learn-inv}$:
 $cdcl_{NOT}\text{-merged-bj-learn}^{**}\ S\ T \implies inv\ S \implies no\text{-dup}\ (trail\ S) \implies inv\ T$
using $rtranclp\text{-}cdcl_{NOT}\text{-merged-bj-learn-is-rtranclp-cdcl_{NOT}\text{-and-inv}}$ **by** $blast$

definition $\mu_C' :: 'v \text{ literal multiset set} \Rightarrow 'st \Rightarrow nat$ **where**
 $\mu_C' A\ T \equiv \mu_C\ (1 + card\ (atms\text{-of}\text{-ms}\ A))\ (2 + card\ (atms\text{-of}\text{-ms}\ A))\ (trail\text{-weight}\ T)$

definition $\mu_{CDCL}'\text{-merged} :: 'v \text{ literal multiset set} \Rightarrow 'st \Rightarrow nat$ **where**
 $\mu_{CDCL}'\text{-merged}\ A\ T \equiv$
 $((2 + card\ (atms\text{-of}\text{-ms}\ A)) \wedge (1 + card\ (atms\text{-of}\text{-ms}\ A)) - \mu_C' A\ T) * 2 + card\ (set\text{-mset}\ (clauses_{NOT}\ T))$

lemma $cdcl_{NOT}\text{-decreasing-measure}'$:
assumes
 $cdcl_{NOT}\text{-merged-bj-learn}\ S\ T$ **and**
 $inv: inv\ S$ **and**

$atm-clss$: $atms-of-mm \ (clauses_{NOT} \ S) \subseteq atms-of-ms \ A$ **and**
 $atm-trail$: $atm-of \ ' \ lits-of-l \ (trail \ S) \subseteq atms-of-ms \ A$ **and**
 $n-d$: $no-dup \ (trail \ S)$ **and**
 $fin-A$: $finite \ A$
shows $\mu_{CDCL}'\text{-merged} \ A \ T < \mu_{CDCL}'\text{-merged} \ A \ S$
using $assms(1)$
proof *induction*
case $(cdcl_{NOT}\text{-merged-bj-learn-decide}_{NOT} \ T)$
have $clauses_{NOT} \ S = clauses_{NOT} \ T$
using $cdcl_{NOT}\text{-merged-bj-learn-decide}_{NOT}.hyps$ **by** *auto*
moreover **have**
 $(2 + card \ (atms-of-ms \ A)) \wedge (1 + card \ (atms-of-ms \ A))$
 $- \mu_C \ (1 + card \ (atms-of-ms \ A)) \ (2 + card \ (atms-of-ms \ A)) \ (trail\text{-weight} \ T)$
 $< (2 + card \ (atms-of-ms \ A)) \wedge (1 + card \ (atms-of-ms \ A))$
 $- \mu_C \ (1 + card \ (atms-of-ms \ A)) \ (2 + card \ (atms-of-ms \ A)) \ (trail\text{-weight} \ S)$
apply $(rule \ dpll\text{-bj-trail-mes-decreasing-prop})$
using $cdcl_{NOT}\text{-merged-bj-learn-decide}_{NOT} \ fin-A \ atm-clss \ atm-trail \ n-d \ inv$
by $(simp\text{-all} \ add: \ bj\text{-decide}_{NOT} \ cdcl_{NOT}\text{-merged-bj-learn-decide}_{NOT}.hyps)$
ultimately show $?case$
unfolding $\mu_{CDCL}'\text{-merged-def} \ \mu_C'\text{-def}$ **by** *simp*
next
case $(cdcl_{NOT}\text{-merged-bj-learn-propagate}_{NOT} \ T)$
have $clauses_{NOT} \ S = clauses_{NOT} \ T$
using $cdcl_{NOT}\text{-merged-bj-learn-propagate}_{NOT}.hyps$
by $(simp \ add: \ bj\text{-propagate}_{NOT} \ inv \ dpll\text{-bj-clauses})$
moreover **have**
 $(2 + card \ (atms-of-ms \ A)) \wedge (1 + card \ (atms-of-ms \ A))$
 $- \mu_C \ (1 + card \ (atms-of-ms \ A)) \ (2 + card \ (atms-of-ms \ A)) \ (trail\text{-weight} \ T)$
 $< (2 + card \ (atms-of-ms \ A)) \wedge (1 + card \ (atms-of-ms \ A))$
 $- \mu_C \ (1 + card \ (atms-of-ms \ A)) \ (2 + card \ (atms-of-ms \ A)) \ (trail\text{-weight} \ S)$
apply $(rule \ dpll\text{-bj-trail-mes-decreasing-prop})$
using $inv \ n-d \ atm-clss \ atm-trail \ fin-A$ **by** $(simp\text{-all} \ add: \ bj\text{-propagate}_{NOT} \ cdcl_{NOT}\text{-merged-bj-learn-propagate}_{NOT}.hyps)$
ultimately show $?case$
unfolding $\mu_{CDCL}'\text{-merged-def} \ \mu_C'\text{-def}$ **by** *simp*
next
case $(cdcl_{NOT}\text{-merged-bj-learn-forget}_{NOT} \ T)$
have $card \ (set\text{-mset} \ (clauses_{NOT} \ T)) < card \ (set\text{-mset} \ (clauses_{NOT} \ S))$
using $\langle forget_{NOT} \ S \ T \rangle$ **by** $(metis \ card\text{-Diff1-less} \ clauses\text{-remove-cls}_{NOT} \ finite\text{-set-mset} \ forget_{NOT}.cases \ in\text{-clss-mset-clss} \ linear \ set\text{-mset-minus-replicate-mset}(1) \ state\text{-eq}_{NOT}\text{-def})$
moreover
have $trail \ S = trail \ T$
using $\langle forget_{NOT} \ S \ T \rangle$ **by** $(auto \ elim: \ forget_{NOT}E)$
then **have**
 $(2 + card \ (atms-of-ms \ A)) \wedge (1 + card \ (atms-of-ms \ A))$
 $- \mu_C \ (1 + card \ (atms-of-ms \ A)) \ (2 + card \ (atms-of-ms \ A)) \ (trail\text{-weight} \ T)$
 $= (2 + card \ (atms-of-ms \ A)) \wedge (1 + card \ (atms-of-ms \ A))$
 $- \mu_C \ (1 + card \ (atms-of-ms \ A)) \ (2 + card \ (atms-of-ms \ A)) \ (trail\text{-weight} \ S)$
by *auto*
ultimately show $?case$
unfolding $\mu_{CDCL}'\text{-merged-def} \ \mu_C'\text{-def}$ **by** *simp*
next
case $(cdcl_{NOT}\text{-merged-bj-learn-backjump-l} \ T)$ **note** $bj-l = this(1)$
obtain $C' \ L \ D$ **where**
 $learn: \ learn \ S \ (add\text{-cls}_{NOT} \ D \ S)$ **and**

bj: *backjump* (*add-cl*_{NOT} *D S*) *T* **and**
atms-C: *atms-of* (*C'* + {#*L*#}) \subseteq *atms-of-mm* (*clauses*_{NOT} *S*) \cup *atm-of* ‘ (*lits-of-l* (*trail S*)) **and**
D: *mset-cl* *D* = *C'* + {#*L*#}
using *bj-l inv backjump-l-learn-backjump n-d atm-clss atm-trail* **by** *meson*
have *card-T-S*: *card* (*set-mset* (*clauses*_{NOT} *T*)) \leq 1 + *card* (*set-mset* (*clauses*_{NOT} *S*))
using *bj-l inv* **by** (*force elim!*: *backjump-lE simp*: *card-insert-if*)
have
 $((2 + \text{card}(\text{atms-of-ms } A)) \wedge (1 + \text{card}(\text{atms-of-ms } A))$
 $\quad - \mu_C (1 + \text{card}(\text{atms-of-ms } A)) (2 + \text{card}(\text{atms-of-ms } A)) (\text{trail-weight } T))$
 $< ((2 + \text{card}(\text{atms-of-ms } A)) \wedge (1 + \text{card}(\text{atms-of-ms } A))$
 $\quad - \mu_C (1 + \text{card}(\text{atms-of-ms } A)) (2 + \text{card}(\text{atms-of-ms } A))$
 $\quad (\text{trail-weight}(\text{add-cl}_{\text{NOT}} D S)))$
apply (*rule dpll-bj-trail-mes-decreasing-prop*)
using *bj bj-backjump* **apply** *blast*
using *cdcl*_{NOT}.*c-learn cdcl*_{NOT}.*inv inv learn* **apply** *blast*
using *atms-C atm-clss atm-trail D* **apply** (*simp add: n-d*) **apply** *fast*
using *atm-trail n-d* **apply** *simp*
apply (*simp add: n-d*)
using *fin-A* **apply** *simp*
done
then have $((2 + \text{card}(\text{atms-of-ms } A)) \wedge (1 + \text{card}(\text{atms-of-ms } A))$
 $\quad - \mu_C (1 + \text{card}(\text{atms-of-ms } A)) (2 + \text{card}(\text{atms-of-ms } A)) (\text{trail-weight } T))$
 $< ((2 + \text{card}(\text{atms-of-ms } A)) \wedge (1 + \text{card}(\text{atms-of-ms } A))$
 $\quad - \mu_C (1 + \text{card}(\text{atms-of-ms } A)) (2 + \text{card}(\text{atms-of-ms } A)) (\text{trail-weight } S))$
using *n-d* **by** *auto*
then show *?case*
using *card-T-S unfolding* μ_{CDCL}' -*merged-def* μ_C' -*def* **by** *linarith*
qed

lemma *wf-cdcl*_{NOT}-*merged-bj-learn*:

assumes

fin-A: *finite A*

shows *wf* {(*T*, *S*).

(*inv S* \wedge *atms-of-mm* (*clauses*_{NOT} *S*) \subseteq *atms-of-ms A* \wedge *atm-of* ‘ *lits-of-l* (*trail S*) \subseteq *atms-of-ms A*
 \wedge *no-dup* (*trail S*))

\wedge *cdcl*_{NOT}-*merged-bj-learn S T*}

apply (*rule wfP-if-measure*[*of - -* μ_{CDCL}' -*merged A*])

using *cdcl*_{NOT}-*decreasing-measure'* *fin-A* **by** *simp*

lemma *tranclp-cdcl*_{NOT}-*cdcl*_{NOT}-*tranclp*:

assumes

*cdcl*_{NOT}-*merged-bj-learn*⁺⁺ *S T* **and**

inv: *inv S* **and**

atm-clss: *atms-of-mm* (*clauses*_{NOT} *S*) \subseteq *atms-of-ms A* **and**

atm-trail: *atm-of* ‘ *lits-of-l* (*trail S*) \subseteq *atms-of-ms A* **and**

n-d: *no-dup* (*trail S*) **and**

fin-A[*simp*]: *finite A*

shows (*T*, *S*) \in {(*T*, *S*).

(*inv S* \wedge *atms-of-mm* (*clauses*_{NOT} *S*) \subseteq *atms-of-ms A* \wedge *atm-of* ‘ *lits-of-l* (*trail S*) \subseteq *atms-of-ms A*
 \wedge *no-dup* (*trail S*))

\wedge *cdcl*_{NOT}-*merged-bj-learn S T*}⁺ (*is -* \in *?P*⁺)

using *assms*(1)

proof (*induction rule: tranclp-induct*)

case *base*

then show *?case* **using** *n-d atm-clss atm-trail inv* **by** *auto*

next

case (*step* $T\ U$) **note** $st = this(1)$ **and** $cdcl_{NOT} = this(2)$ **and** $IH = this(3)$
have $cdcl_{NOT}^{**} S\ T$
 apply (*rule* $rtranclp\text{-}cdcl_{NOT}\text{-merged}\text{-bj}\text{-learn}\text{-is}\text{-}rtranclp\text{-}cdcl_{NOT}$)
 using $st\ cdcl_{NOT}\ inv\ n\text{-}d\ atm\text{-}clss\ atm\text{-}trail\ inv$ **by** *auto*
have $inv\ T$
 apply (*rule* $rtranclp\text{-}cdcl_{NOT}\text{-merged}\text{-bj}\text{-learn}\text{-inv}$)
 using $inv\ st\ cdcl_{NOT}\ n\text{-}d\ atm\text{-}clss\ atm\text{-}trail\ inv$ **by** *auto*
moreover have $atms\text{-}of\text{-}mm\ (clauses_{NOT}\ T) \subseteq atms\text{-}of\text{-}ms\ A$
 using $rtranclp\text{-}cdcl_{NOT}\text{-trail}\text{-clauses}\text{-}bound[OF\ \langle cdcl_{NOT}^{**} S\ T \rangle\ inv\ n\text{-}d\ atm\text{-}clss\ atm\text{-}trail]$
 by *fast*
moreover have $atm\text{-}of\ ' (lits\text{-}of\text{-}l\ (trail\ T)) \subseteq atms\text{-}of\text{-}ms\ A$
 using $rtranclp\text{-}cdcl_{NOT}\text{-trail}\text{-clauses}\text{-}bound[OF\ \langle cdcl_{NOT}^{**} S\ T \rangle\ inv\ n\text{-}d\ atm\text{-}clss\ atm\text{-}trail]$
 by *fast*
moreover have $no\text{-}dup\ (trail\ T)$
 using $rtranclp\text{-}cdcl_{NOT}\text{-no}\text{-}dup[OF\ \langle cdcl_{NOT}^{**} S\ T \rangle\ inv\ n\text{-}d]$ **by** *fast*
ultimately have $(U, T) \in ?P$
 using $cdcl_{NOT}$ **by** *auto*
then show $?case$ **using** IH **by** (*simp add: trancl\text{-}into\text{-}trancl2*)
qed

lemma *wf\text{-}tranclp\text{-}cdcl_{NOT}\text{-merged}\text{-bj}\text{-learn*:

assumes *finite* A
shows $wf\ \{(T, S).$
 $(inv\ S \wedge atms\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \subseteq atms\text{-}of\text{-}ms\ A \wedge atm\text{-}of\ ' lits\text{-}of\text{-}l\ (trail\ S) \subseteq atms\text{-}of\text{-}ms\ A$
 $\wedge no\text{-}dup\ (trail\ S))$
 $\wedge cdcl_{NOT}\text{-merged}\text{-bj}\text{-learn}^{++} S\ T\}$
apply (*rule* *wf\text{-}subset*)
apply (*rule* $wf\text{-}trancl[OF\ wf\text{-}cdcl_{NOT}\text{-merged}\text{-bj}\text{-learn}]$)
using *assms* **apply** *simp*
using $tranclp\text{-}cdcl_{NOT}\text{-}cdcl_{NOT}\text{-}tranclp[OF\ \text{---}\ \langle finite\ A \rangle]$ **by** *auto*

lemma *backjump\text{-}no\text{-}step\text{-}backjump\text{-}l*:

$backjump\ S\ T \implies inv\ S \implies \neg no\text{-}step\ backjump\text{-}l\ S$
apply (*elim* *backjumpE*)
apply (*rule* *bj\text{-}merge\text{-}can\text{-}jump*)
apply *auto*[7]
by *blast*

lemma *cdcl_{NOT}\text{-merged}\text{-bj}\text{-learn}\text{-final}\text{-state*:

fixes $A :: 'v\ literal\ multiset\ set$ **and** $S\ T :: 'st$
assumes
 $n\text{-}s$: $no\text{-}step\ cdcl_{NOT}\text{-merged}\text{-bj}\text{-learn}\ S$ **and**
 $atms\text{-}S$: $atms\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \subseteq atms\text{-}of\text{-}ms\ A$ **and**
 $atms\text{-}trail$: $atm\text{-}of\ ' lits\text{-}of\text{-}l\ (trail\ S) \subseteq atms\text{-}of\text{-}ms\ A$ **and**
 $n\text{-}d$: $no\text{-}dup\ (trail\ S)$ **and**
 $finite\ A$ **and**
 inv : $inv\ S$ **and**
 $decomp$: $all\text{-}decomposition\text{-}implies\text{-}m\ (clauses_{NOT}\ S)\ (get\text{-}all\text{-}marked\text{-}decomposition\ (trail\ S))$
shows $unsatisfiable\ (set\text{-}mset\ (clauses_{NOT}\ S))$
 $\vee (trail\ S \models_{asm} clauses_{NOT}\ S \wedge satisfiable\ (set\text{-}mset\ (clauses_{NOT}\ S)))$

proof –

let $?N = set\text{-}mset\ (clauses_{NOT}\ S)$
let $?M = trail\ S$
consider

```

  (sat) satisfiable ?N and ?M  $\models_{as}$  ?N
| (sat') satisfiable ?N and  $\neg$  ?M  $\models_{as}$  ?N
| (unsat) unsatisfiable ?N
by auto
then show ?thesis
proof cases
  case sat' note sat = this(1) and M = this(2)
  obtain C where C  $\in$  ?N and  $\neg$  ?M  $\models_a$  C using M unfolding true-annots-def by auto
  obtain I :: 'v literal set where
    I  $\models_s$  ?N and
    cons: consistent-interp I and
    tot: total-over-m I ?N and
    atm-I-N: atm-of 'I  $\subseteq$  atms-of-ms ?N
  using sat unfolding satisfiable-def-min by auto
  let ?I = I  $\cup$  {P | P. P  $\in$  lits-of-l ?M  $\wedge$  atm-of P  $\notin$  atm-of 'I}
  let ?O = {unmark L | L. is-marked L  $\wedge$  L  $\in$  set ?M  $\wedge$  atm-of (lit-of L)  $\notin$  atms-of-ms ?N}
  have cons-I': consistent-interp ?I
    using cons using (no-dup ?M) unfolding consistent-interp-def
    by (auto simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set lits-of-def
      dest!: no-dup-cannot-not-lit-and-uminus)
  have tot-I': total-over-m ?I (?N  $\cup$  unmark-l ?M)
    using tot atms-of-s-def unfolding total-over-m-def total-over-set-def
    by (fastforce simp: image-iff)
  have {P | P. P  $\in$  lits-of-l ?M  $\wedge$  atm-of P  $\notin$  atm-of 'I}  $\models_s$  ?O
    using (I  $\models_s$  ?N) atm-I-N by (auto simp add: atm-of-eq-atm-of true-clss-def lits-of-def)
  then have I'-N: ?I  $\models_s$  ?N  $\cup$  ?O
    using (I  $\models_s$  ?N) true-clss-union-increase by force
  have tot': total-over-m ?I (?N  $\cup$  ?O)
    using atm-I-N tot unfolding total-over-m-def total-over-set-def
    by (force simp: image-iff lits-of-def dest!: is-marked-ex-Marked)

  have atms-N-M: atms-of-ms ?N  $\subseteq$  atm-of 'lits-of-l ?M
  proof (rule ccontr)
    assume  $\neg$  ?thesis
    then obtain l :: 'v where
      l-N: l  $\in$  atms-of-ms ?N and
      l-M: l  $\notin$  atm-of 'lits-of-l ?M
    by auto
    have undefined-lit ?M (Pos l)
      using l-M by (metis Marked-Propagated-in-iff-in-lits-of-l
        atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set literal.sel(1))
    have decideNOT S (prepend-trail (Marked (Pos l) ()) S)
      by (metis (undefined-lit ?M (Pos l) decideNOT.intros l-N literal.sel(1)
        state-eqNOT-ref))
    then show False
      using cdclNOT-merged-bj-learn-decideNOT n-s by blast
  qed

  have ?M  $\models_{as}$  CNot C
  apply (rule all-variables-defined-not-imply-cnot)
    using atms-N-M (C  $\in$  ?N) ( $\neg$  ?M  $\models_a$  C) atms-of-atms-of-ms-mono[OF (C  $\in$  ?N)]
    by (auto dest: atms-of-atms-of-ms-mono)
  have  $\exists l \in$  set ?M. is-marked l
  proof (rule ccontr)
    let ?O = {unmark L | L. is-marked L  $\wedge$  L  $\in$  set ?M  $\wedge$  atm-of (lit-of L)  $\notin$  atms-of-ms ?N}

```

```

have  $\vartheta[\text{iff}]$ :  $\bigwedge I. \text{total-over-m } I \ (\ ?N \cup \ ?O \cup \text{unmark-l } ?M )$ 
 $\longleftrightarrow \text{total-over-m } I \ (\ ?N \cup \text{unmark-l } ?M )$ 
unfolding total-over-set-def total-over-m-def atms-of-ms-def by blast
assume  $\neg \ ?thesis$ 
then have  $[simp]: \{ \text{unmark } L \mid L. \text{is-marked } L \wedge L \in \text{set } ?M \}$ 
 $= \{ \text{unmark } L \mid L. \text{is-marked } L \wedge L \in \text{set } ?M \wedge \text{atm-of } (\text{lit-of } L) \notin \text{atms-of-ms } ?N \}$ 
by auto
then have  $?N \cup \ ?O \models_{ps} \text{unmark-l } ?M$ 
using all-decomposition-implies-propagated-lits-are-implied[OF decomp] by auto

then have  $?I \models_s \text{unmark-l } ?M$ 
using cons-I' I'-N tot-I'  $\langle ?I \models_s ?N \cup \ ?O \rangle$  unfolding  $\vartheta$  true-clss-clss-def by blast
then have  $\text{lits-of-l } ?M \subseteq ?I$ 
unfolding true-clss-def lits-of-def by auto
then have  $?M \models_{as} ?N$ 
using  $I'-N \langle C \in ?N \rangle \langle \neg \ ?M \models_a C \rangle \text{cons-I' atms-N-M}$ 
by (meson  $\langle \text{trail } S \models_{as} CNot \ C \rangle$  consistent-CNot-not rev-subsetD sup-ge1 true-annot-def
true-annots-def true-clss-mono-set-mset-l true-clss-def)
then show False using M by fast
qed
from List.split-list-first-propE[OF this] obtain  $K :: 'v \text{ literal and } d :: \text{unit and}$ 
 $F \ F' :: ('v, \text{unit}, \text{unit}) \text{ marked-lit list where}$ 
 $M-K: ?M = F' @ \text{Marked } K \ () \ \# \ F \text{ and}$ 
 $nm: \forall f \in \text{set } F'. \neg \text{is-marked } f$ 
unfolding is-marked-def by (metis (full-types) old.unit.exhaust)
let  $?K = \text{Marked } K \ () :: ('v, \text{unit}, \text{unit}) \text{ marked-lit}$ 
have  $?K \in \text{set } ?M$ 
unfolding M-K by auto
let  $?C = \text{image-mset lit-of } \{ \#L \in \#mset \ ?M. \text{is-marked } L \wedge L \neq ?K \# \} :: 'v \text{ literal multiset}$ 
let  $?C' = \text{set-mset } (\text{image-mset } (\lambda L :: 'v \text{ literal. } \{ \#L \# \}) \ (\ ?C + \text{unmark } ?K ))$ 
have  $?N \cup \{ \text{unmark } L \mid L. \text{is-marked } L \wedge L \in \text{set } ?M \} \models_{ps} \text{unmark-l } ?M$ 
using all-decomposition-implies-propagated-lits-are-implied[OF decomp] .
moreover have  $C': ?C' = \{ \text{unmark } L \mid L. \text{is-marked } L \wedge L \in \text{set } ?M \}$ 
unfolding M-K apply standard
apply force
using IntI by auto
ultimately have  $N-C-M: ?N \cup \ ?C' \models_{ps} \text{unmark-l } ?M$ 
by auto
have  $N-M-False: ?N \cup (\lambda L. \text{unmark } L) \ (\ \text{set } ?M ) \models_{ps} \{ \{ \# \} \}$ 
using  $M \ \langle ?M \models_{as} CNot \ C \rangle \ \langle C \in ?N \rangle$  unfolding true-clss-clss-def true-annots-def Ball-def
true-annot-def by (metis consistent-CNot-not sup.orderE sup-commute true-clss-def
true-clss-singleton-lit-of-implies-incl true-clss-union true-clss-union-increase)

have undefined-lit F K using  $\langle \text{no-dup } ?M \rangle$  unfolding M-K by (simp add: defined-lit-map)
moreover
have  $?N \cup \ ?C' \models_{ps} \{ \{ \# \} \}$ 
proof –
have  $A: ?N \cup \ ?C' \cup \text{unmark-l } ?M = ?N \cup \text{unmark-l } ?M$ 
unfolding M-K by auto
show ?thesis
using true-clss-clss-left-right[OF N-C-M, of  $\{ \{ \# \} \}$ ] N-M-False unfolding A by auto
qed
have  $?N \models_p \text{image-mset uminus } ?C + \{ \# - K \# \}$ 
unfolding true-clss-clss-def true-clss-clss-def total-over-m-def
proof (intro allI impI)

```

```

fix I
assume
  tot: total-over-set I (atms-of-ms (?N ∪ {image-mset uminus ?C + {#- K#}})) and
  cons: consistent-interp I and
  I ⊨s ?N
have (K ∈ I ∧ -K ∉ I) ∨ (-K ∈ I ∧ K ∉ I)
  using cons tot unfolding consistent-interp-def by (cases K) auto
have {a ∈ set (trail S). is-marked a ∧ a ≠ Marked K ()} =
  set (trail S) ∩ {L. is-marked L ∧ L ≠ Marked K ()}
by auto
then have tot': total-over-set I
  (atm-of 'lit-of ' (set ?M ∩ {L. is-marked L ∧ L ≠ Marked K ()}))
using tot by (auto simp add: atms-of-uminus-lit-atm-of-lit-of)
{ fix x :: ('v, unit, unit) marked-lit
  assume
    a3: lit-of x ∉ I and
    a1: x ∈ set ?M and
    a4: is-marked x and
    a5: x ≠ Marked K ()
  then have Pos (atm-of (lit-of x)) ∈ I ∨ Neg (atm-of (lit-of x)) ∈ I
    using a5 a4 tot' a1 unfolding total-over-set-def atms-of-s-def by blast
  moreover have f6: Neg (atm-of (lit-of x)) = - Pos (atm-of (lit-of x))
    by simp
  ultimately have - lit-of x ∈ I
    using f6 a3 by (metis (no-types) atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
      literal.sel(1))
} note H = this

have ¬I ⊨s ?C'
  using ⟨?N ∪ ?C' ⊨ps {{#}}⟩ tot cons ⟨I ⊨s ?N⟩
  unfolding true-clss-clss-def total-over-m-def
  by (simp add: atms-of-uminus-lit-atm-of-lit-of atms-of-ms-single-image-atm-of-lit-of)
then show I ⊨ image-mset uminus ?C + {#- K#}
  unfolding true-clss-def true-clss-def Bex-def
  using ⟨(K ∈ I ∧ -K ∉ I) ∨ (-K ∈ I ∧ K ∉ I)⟩
  by (auto dest!: H)
qed
moreover have F ⊨as CNot (image-mset uminus ?C)
  using nm unfolding true-annots-def CNot-def M-K by (auto simp add: lits-of-def)
ultimately have False
  using bj-merge-can-jump[of S F' K F C -K
    image-mset uminus (image-mset lit-of {# L :# mset ?M. is-marked L ∧ L ≠ Marked K ()#})]
    ⟨C ∈ ?N⟩ n-s ⟨?M ⊨as CNot C⟩ bj-backjump inv unfolding M-K
  by (auto simp: cdclNOT-merged-bj-learn.simps)
then show ?thesis by fast
qed auto
qed

lemma full-cdclNOT-merged-bj-learn-final-state:
  fixes A :: 'v literal multiset set and S T :: 'st
assumes
  full: full cdclNOT-merged-bj-learn S T and
  atms-S: atms-of-mm (clausesNOT S) ⊆ atms-of-ms A and
  atms-trail: atm-of 'lits-of-l (trail S) ⊆ atms-of-ms A and
  n-d: no-dup (trail S) and

```

finite A **and**
inv: inv S **and**
decomp: all-decomposition-implies-m (clauses_{NOT} S) (get-all-marked-decomposition (trail S))
shows *unsatisfiable (set-mset (clauses_{NOT} T))*
 \vee (*trail T* \models_{asm} *clauses_{NOT} T* \wedge *satisfiable (set-mset (clauses_{NOT} T))*)
proof –
have *st: cdcl_{NOT}-merged-bj-learn** S T* **and** *n-s: no-step cdcl_{NOT}-merged-bj-learn T*
using *full unfolding full-def* **by** *blast+*
then have *st: cdcl_{NOT}** S T*
using *inv rtrancpl-cdcl_{NOT}-merged-bj-learn-is-rtrancpl-cdcl_{NOT}-and-inv n-d* **by** *auto*
have *atms-of-mm (clauses_{NOT} T) \subseteq atms-of-ms A* **and** *atm-of ' lits-of-l (trail T) \subseteq atms-of-ms A*
using *rtrancpl-cdcl_{NOT}-trail-clauses-bound[OF st inv n-d atms-S atms-trail]* **by** *blast+*
moreover have *no-dup (trail T)*
using *rtrancpl-cdcl_{NOT}-no-dup inv n-d st* **by** *blast*
moreover have *inv T*
using *rtrancpl-cdcl_{NOT}-inv inv st* **by** *blast*
moreover have *all-decomposition-implies-m (clauses_{NOT} T) (get-all-marked-decomposition (trail T))*
using *rtrancpl-cdcl_{NOT}-all-decomposition-implies inv st decomp n-d* **by** *blast*
ultimately show *?thesis*
using *cdcl_{NOT}-merged-bj-learn-final-state[of T A] (finite A) n-s* **by** *fast*
qed
end

16.8.1 Instantiations

locale *cdcl_{NOT}-with-backtrack-and-restarts =*
conflict-driven-clause-learning-learning-before-backjump-only-distinct-learnt
mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-cl_{NOT} remove-cl_{NOT}
inv backjump-conds propagate-conds learn-restrictions forget-restrictions
for
mset-cls:: 'cls \Rightarrow 'v clause **and**
insert-cls :: 'v literal \Rightarrow 'cls \Rightarrow 'cls **and**
remove-lit :: 'v literal \Rightarrow 'cls \Rightarrow 'cls **and**
mset-clss:: 'clss \Rightarrow 'v clauses **and**
union-clss :: 'clss \Rightarrow 'clss \Rightarrow 'clss **and**
in-clss :: 'cls \Rightarrow 'clss \Rightarrow bool **and**
insert-clss :: 'cls \Rightarrow 'clss \Rightarrow 'clss **and**
remove-from-clss :: 'cls \Rightarrow 'clss \Rightarrow 'clss **and**
trail :: 'st \Rightarrow ('v, unit, unit) marked-lits **and**
raw-clauses :: 'st \Rightarrow 'clss **and**
prepend-trail :: ('v, unit, unit) marked-lit \Rightarrow 'st \Rightarrow 'st **and**
tl-trail :: 'st \Rightarrow 'st **and**
add-cl_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st **and**
remove-cl_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st **and**
inv :: 'st \Rightarrow bool **and**
backjump-conds :: 'v clause \Rightarrow 'v clause \Rightarrow 'v literal \Rightarrow 'st \Rightarrow 'st \Rightarrow bool **and**
propagate-conds :: ('v, unit, unit) marked-lit \Rightarrow 'st \Rightarrow bool **and**
learn-restrictions forget-restrictions :: 'cls \Rightarrow 'st \Rightarrow bool
+
fixes *f :: nat \Rightarrow nat*
assumes
unbounded: unbounded f **and** *f-ge-1: $\bigwedge n. n \geq 1 \Rightarrow f n \geq 1$* **and**
inv-restart: $\bigwedge S T. inv S \Rightarrow T \sim \text{reduce-trail-to}_{NOT} ([:: 'a \text{ list}) S \Rightarrow inv T$

begin

lemma *bound-inv-inv*:

assumes

inv S **and**

n-d: no-dup (trail S) **and**

atms-clss-S-A: atms-of-mm (clauses_{NOT} S) ⊆ atms-of-ms A **and**

atms-trail-S-A: atm-of ' lits-of-l (trail S) ⊆ atms-of-ms A **and**

finite A **and**

cdcl_{NOT}: cdcl_{NOT} S T

shows

atms-of-mm (clauses_{NOT} T) ⊆ atms-of-ms A **and**

atm-of ' lits-of-l (trail T) ⊆ atms-of-ms A **and**

finite A

proof –

have *cdcl_{NOT} S T*

using *⟨inv S⟩ cdcl_{NOT}* **by** *linarith*

then have *atms-of-mm (clauses_{NOT} T) ⊆ atms-of-mm (clauses_{NOT} S) ∪ atm-of ' lits-of-l (trail S)*

using *⟨inv S⟩*

by (*meson conflict-driven-clause-learning-ops.cdcl_{NOT}-atms-of-ms-clauses-decreasing*
conflict-driven-clause-learning-ops-axioms n-d)

then show *atms-of-mm (clauses_{NOT} T) ⊆ atms-of-ms A*

using *atms-clss-S-A atms-trail-S-A* **by** *blast*

next

show *atm-of ' lits-of-l (trail T) ⊆ atms-of-ms A*

by (*meson ⟨inv S⟩ atms-clss-S-A atms-trail-S-A cdcl_{NOT} cdcl_{NOT}-atms-in-trail-in-set n-d*)

next

show *finite A*

using *⟨finite A⟩* **by** *simp*

qed

sublocale *cdcl_{NOT}-increasing-restarts-ops* $\lambda S T. T \sim \text{reduce-trail-to}_{NOT} ([::'a \text{ list}) S \text{ cdcl}_{NOT} f$
 $\lambda A S. \text{atms-of-mm (clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of ' lits-of-l (trail S) } \subseteq \text{atms-of-ms } A \wedge$
finite A

$\mu_{CDCL}' \lambda S. \text{inv } S \wedge \text{no-dup (trail S)}$

$\mu_{CDCL}'\text{-bound}$

apply *unfold-locales*

apply (*simp add: unbounded*)

using *f-ge-1* **apply** *force*

using *bound-inv-inv* **apply** *meson*

apply (*rule cdcl_{NOT}-decreasing-measure'; simp*)

apply (*rule rtrancpl-cdcl_{NOT}-μ_{CDCL}'-bound; simp*)

apply (*rule rtrancpl-μ_{CDCL}'-bound-decreasing; simp*)

apply *auto* []

apply *auto* []

using *cdcl_{NOT}-inv cdcl_{NOT}-no-dup* **apply** *blast*

using *inv-restart* **apply** *auto* []

done

lemma *cdcl_{NOT}-with-restart-μ_{CDCL}'-le-μ_{CDCL}'-bound*:

assumes

cdcl_{NOT}: cdcl_{NOT}-restart (T, a) (V, b) **and**

cdcl_{NOT}-inv:

inv T

no-dup (trail T) **and**

```

bound-inv:
  atms-of-mm (clausesNOT T) ⊆ atms-of-ms A
  atm-of ' lits-of-l (trail T) ⊆ atms-of-ms A
  finite A
shows μCDCL' A V ≤ μCDCL'-bound A T
using cdclNOT-inv bound-inv
proof (induction rule: cdclNOT-with-restart-induct[OF cdclNOT])
case (1 m S T n U) note U = this(3)
show ?case
apply (rule rtrancpl-cdclNOT-μCDCL'-bound-reduce-trail-toNOT[of S T])
using ⟨(cdclNOT ~ m) S T⟩ apply (fastforce dest!: relpowp-imp-rtrancpl)
using 1 by auto
next
case (2 S T n) note full = this(2)
show ?case
apply (rule rtrancpl-cdclNOT-μCDCL'-bound)
using full 2 unfolding full1-def by force+
qed

lemma cdclNOT-with-restart-μCDCL'-bound-le-μCDCL'-bound:
assumes
  cdclNOT: cdclNOT-restart (T, a) (V, b) and
  cdclNOT-inv:
    inv T
    no-dup (trail T) and
  bound-inv:
    atms-of-mm (clausesNOT T) ⊆ atms-of-ms A
    atm-of ' lits-of-l (trail T) ⊆ atms-of-ms A
    finite A
shows μCDCL'-bound A V ≤ μCDCL'-bound A T
using cdclNOT-inv bound-inv
proof (induction rule: cdclNOT-with-restart-induct[OF cdclNOT])
case (1 m S T n U) note U = this(3)
have μCDCL'-bound A T ≤ μCDCL'-bound A S
apply (rule rtrancpl-μCDCL'-bound-decreasing)
using ⟨(cdclNOT ~ m) S T⟩ apply (fastforce dest!: relpowp-imp-rtrancpl)
using 1 by auto
then show ?case using U unfolding μCDCL'-bound-def by auto
next
case (2 S T n) note full = this(2)
show ?case
apply (rule rtrancpl-μCDCL'-bound-decreasing)
using full 2 unfolding full1-def by force+
qed

sublocale cdclNOT-increasing-restarts - - - - -
  f
  λS T. T ~ reduce-trail-toNOT ([::'a list) S
  λA S. atms-of-mm (clausesNOT S) ⊆ atms-of-ms A
    ∧ atm-of ' lits-of-l (trail S) ⊆ atms-of-ms A ∧ finite A
  μCDCL' cdclNOT
  λS. inv S ∧ no-dup (trail S)
  μCDCL'-bound
apply unfold-locales
using cdclNOT-with-restart-μCDCL'-le-μCDCL'-bound apply simp

```

using *cdcl_{NOT}-with-restart- μ_{CDCL} '-bound-le- μ_{CDCL} '-bound* **apply** *simp*
done

lemma *cdcl_{NOT}-restart-all-decomposition-implies*:

assumes *cdcl_{NOT}-restart* *S T* **and**
inv (*fst S*) **and**
no-dup (*trail* (*fst S*))
all-decomposition-implies-m (*clauses_{NOT}* (*fst S*)) (*get-all-marked-decomposition* (*trail* (*fst S*)))
shows
all-decomposition-implies-m (*clauses_{NOT}* (*fst T*)) (*get-all-marked-decomposition* (*trail* (*fst T*)))
using *assms* **apply** (*induction*)
using *rtrancpl-cdcl_{NOT}-all-decomposition-implies* **by** (*auto dest!:* *trancpl-into-rtrancpl*
simp: full1-def)

lemma *rtrancpl-cdcl_{NOT}-restart-all-decomposition-implies*:

assumes *cdcl_{NOT}-restart*** *S T* **and**
inv: *inv* (*fst S*) **and**
n-d: *no-dup* (*trail* (*fst S*)) **and**
decomp:
all-decomposition-implies-m (*clauses_{NOT}* (*fst S*)) (*get-all-marked-decomposition* (*trail* (*fst S*)))
shows
all-decomposition-implies-m (*clauses_{NOT}* (*fst T*)) (*get-all-marked-decomposition* (*trail* (*fst T*)))
using *assms*(1)

proof (*induction rule: rtrancpl-induct*)

case *base*

then show *?case* **using** *decomp* **by** *simp*

next

case (*step T u*) **note** *st = this(1)* **and** *r = this(2)* **and** *IH = this(3)*

have *inv* (*fst T*)

using *rtrancpl-cdcl_{NOT}-with-restart-cdcl_{NOT}-inv[OF st]* *inv n-d* **by** *blast*

moreover have *no-dup* (*trail* (*fst T*))

using *rtrancpl-cdcl_{NOT}-with-restart-cdcl_{NOT}-inv[OF st]* *inv n-d* **by** *blast*

ultimately show *?case*

using *cdcl_{NOT}-restart-all-decomposition-implies* *r IH n-d* **by** *fast*

qed

lemma *cdcl_{NOT}-restart-sat-ext-iff*:

assumes

st: *cdcl_{NOT}-restart* *S T* **and**

n-d: *no-dup* (*trail* (*fst S*)) **and**

inv: *inv* (*fst S*)

shows $I \models_{\text{sextm}} \text{clauses}_{\text{NOT}}(\text{fst } S) \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{\text{NOT}}(\text{fst } T)$

using *assms*

proof (*induction*)

case (*restart-step m S T n U*)

then show *?case*

using *rtrancpl-cdcl_{NOT}-bj-sat-ext-iff n-d* **by** (*fastforce dest!:* *relpowp-imp-rtrancpl*)

next

case *restart-full*

then show *?case* **using** *rtrancpl-cdcl_{NOT}-bj-sat-ext-iff* **unfolding** *full1-def*

by (*fastforce dest!:* *trancpl-into-rtrancpl*)

qed

lemma *rtrancpl-cdcl_{NOT}-restart-sat-ext-iff*:

assumes

st: $cdcl_{NOT}\text{-restart}^{**} S T$ **and**
n-d: $no\text{-dup} (trail (fst S))$ **and**
inv: $inv (fst S)$
shows $I \models_{sextm} clauses_{NOT} (fst S) \longleftrightarrow I \models_{sextm} clauses_{NOT} (fst T)$
using *st*
proof (*induction*)
case *base*
then show ?*case* **by** *simp*
next
case (*step* $T U$) **note** $st = this(1)$ **and** $r = this(2)$ **and** $IH = this(3)$
have $inv (fst T)$
using $rtranclp\text{-}cdcl_{NOT}\text{-with-restart-cdcl}_{NOT}\text{-inv}[OF\ st]\ inv\ n\text{-d}$ **by** *blast*+
moreover have $no\text{-dup} (trail (fst T))$
using $rtranclp\text{-}cdcl_{NOT}\text{-with-restart-cdcl}_{NOT}\text{-inv}\ rtranclp\text{-}cdcl_{NOT}\text{-no-dup}\ st\ inv\ n\text{-d}$ **by** *blast*
ultimately show ?*case*
using $cdcl_{NOT}\text{-restart-sat-ext-iff}[OF\ r]\ IH$ **by** *blast*
qed

theorem *full-cdcl_{NOT}-restart-backjump-final-state*:

fixes $A :: 'v\ literal\ multiset\ set$ **and** $S\ T :: 'st$

assumes

full: $full\ cdcl_{NOT}\text{-restart} (S, n) (T, m)$ **and**

atms-S: $atms\text{-of-mm} (clauses_{NOT} S) \subseteq atms\text{-of-ms}\ A$ **and**

atms-trail: $atm\text{-of}\ 'lits\text{-of-l} (trail S) \subseteq atms\text{-of-ms}\ A$ **and**

n-d: $no\text{-dup} (trail S)$ **and**

*fin-A[*simp*]*: *finite* A **and**

inv: $inv\ S$ **and**

decomp: $all\text{-decomposition-implies-m} (clauses_{NOT} S) (get\text{-all-marked-decomposition} (trail S))$

shows $unsatisfiable (set\text{-mset} (clauses_{NOT} S))$

$\vee (lits\text{-of-l} (trail T) \models_{sextm} clauses_{NOT} S \wedge satisfiable (set\text{-mset} (clauses_{NOT} S)))$

proof –

have *st*: $cdcl_{NOT}\text{-restart}^{**} (S, n) (T, m)$ **and**

n-s: $no\text{-step}\ cdcl_{NOT}\text{-restart} (T, m)$

using *full unfolding full-def* **by** *fast+*

have *binv-T*: $atms\text{-of-mm} (clauses_{NOT} T) \subseteq atms\text{-of-ms}\ A$

atm-of ' *lits-of-l* (*trail* T) $\subseteq atms\text{-of-ms}\ A$

using $rtranclp\text{-}cdcl_{NOT}\text{-with-restart-bound-inv}[OF\ st, of\ A]\ inv\ n\text{-d}\ atms\text{-}S\ atms\text{-}trail$
by *auto*

moreover have *inv-T*: $no\text{-dup} (trail T)$ *inv* T

using $rtranclp\text{-}cdcl_{NOT}\text{-with-restart-cdcl}_{NOT}\text{-inv}[OF\ st]\ inv\ n\text{-d}$ **by** *auto*

moreover have $all\text{-decomposition-implies-m} (clauses_{NOT} T) (get\text{-all-marked-decomposition} (trail T))$

using $rtranclp\text{-}cdcl_{NOT}\text{-restart-all-decomposition-implies}[OF\ st]\ inv\ n\text{-d}$

decomp **by** *auto*

ultimately have T : $unsatisfiable (set\text{-mset} (clauses_{NOT} T))$

$\vee (trail\ T \models_{asm} clauses_{NOT} T \wedge satisfiable (set\text{-mset} (clauses_{NOT} T)))$

using $no\text{-step-cdcl}_{NOT}\text{-restart-no-step-cdcl}_{NOT}[of\ (T, m)\ A]\ n\text{-s}$

$cdcl_{NOT}\text{-final-state}[of\ T\ A]$ **unfolding** $cdcl_{NOT}\text{-NOT-all-inv-def}$ **by** *auto*

have $eq\text{-sat-S-T} : \bigwedge I. I \models_{sextm} clauses_{NOT} S \longleftrightarrow I \models_{sextm} clauses_{NOT} T$

using $rtranclp\text{-}cdcl_{NOT}\text{-restart-sat-ext-iff}[OF\ st]\ inv\ n\text{-d}\ atms\text{-}S$

atms-trail **by** *auto*

have *cons-T*: $consistent\text{-interp} (lits\text{-of-l} (trail T))$

using $inv\text{-}T(1)\ distinct\text{-consistent-interp}$ **by** *blast*

consider

(*unsat*) $unsatisfiable (set\text{-mset} (clauses_{NOT} T))$

| (*sat*) $trail\ T \models_{asm} clauses_{NOT} T$ **and** $satisfiable (set\text{-mset} (clauses_{NOT} T))$

```

    using  $T$  by blast
  then show ?thesis
  proof cases
    case unsat
    then have unsatisfiable (set-mset (clausesNOT  $S$ ))
      using eq-sat- $S$ - $T$  consistent-true-clss-ext-satisfiable true-clss-imp-true-clss-ext
      unfolding satisfiable-def by blast
    then show ?thesis by fast
  next
  case sat
  then have lits-of-l (trail  $T$ )  $\models_{\text{sextm}}$  clausesNOT  $S$ 
    using rtrancpl-cdclNOT-restart-sat-ext-iff[OF st] inv n-d atms- $S$ 
    atms-trail by (auto simp: true-clss-imp-true-clss-ext true-annots-true-clss)
  moreover then have satisfiable (set-mset (clausesNOT  $S$ ))
    using cons- $T$  consistent-true-clss-ext-satisfiable by blast
  ultimately show ?thesis by blast
qed
end — end of cdclNOT-with-backtrack-and-restarts locale

```

The restart does only reset the trail, contrary to Weidenbach's version. But there is a forget rule.

```

locale cdclNOT-merge-bj-learn-with-backtrack-restarts =
  cdclNOT-merge-bj-learn mset-clss insert-clss remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clssNOT remove-clssNOT
   $\lambda C C' L' S T.$  distinct-mset ( $C' + \{\#L'\# \} \wedge \text{backjump-l-cond } C C' L' S T$ 
  propagate-conds forget-conds inv
for
  mset-clss :: 'cls  $\Rightarrow$  'v clause and
  insert-clss :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss :: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
  raw-clauses :: 'st  $\Rightarrow$  'clss and
  prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
  tl-trail :: 'st  $\Rightarrow$  'st and
  add-clssNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  remove-clssNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  propagate-conds :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  bool and
  inv :: 'st  $\Rightarrow$  bool and
  forget-conds :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  bool and
  backjump-l-cond :: 'v clause  $\Rightarrow$  'v clause  $\Rightarrow$  'v literal  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  bool
  +
fixes f :: nat  $\Rightarrow$  nat
assumes
  unbounded: unbounded f and f-ge-1:  $\bigwedge n. n \geq 1 \implies f\ n \geq 1$  and
  inv-restart:  $\bigwedge S T. \text{inv } S \implies T \sim \text{reduce-trail-to}_{\text{NOT}} \ []\ S \implies \text{inv } T$ 
begin

```

definition not-simplified-clss $A = \{\#C \in \# A. \text{tautology } C \vee \neg \text{distinct-mset } C\# \}$

lemma *simple-clss-or-not-simplified-cls*:
assumes *atms-of-mm* (*clauses*_{NOT} *S*) \subseteq *atms-of-ms* *A* **and**
 $x \in \#$ *clauses*_{NOT} *S* **and** *finite* *A*
shows $x \in \text{simple-clss } (\text{atms-of-ms } A) \vee x \in \# \text{ not-simplified-cls } (\text{clauses}_{\text{NOT}} S)$
proof –
consider
 (*simpl*) \neg *tautology* *x* **and** *distinct-mset* *x*
 | (*n-simp*) *tautology* *x* \vee \neg *distinct-mset* *x*
by *auto*
then show ?thesis
proof *cases*
case *simpl*
then have $x \in \text{simple-clss } (\text{atms-of-ms } A)$
by (*meson* *assms* *atms-of-atms-of-ms-mono* *atms-of-ms-finite* *simple-clss-mono*
distinct-mset-not-tautology-implies-in-simple-clss *finite-subset*
subsetCE)
then show ?thesis **by** *blast*
next
case *n-simp*
then have $x \in \# \text{ not-simplified-cls } (\text{clauses}_{\text{NOT}} S)$
using $\langle x \in \# \text{ clauses}_{\text{NOT}} S \rangle$ **unfolding** *not-simplified-cls-def* **by** *auto*
then show ?thesis **by** *blast*
qed
qed

lemma *cdcl_{NOT}-merged-bj-learn-clauses-bound*:
assumes
cdcl_{NOT}-merged-bj-learn *S* *T* **and**
inv: *inv* *S* **and**
atms-clss: *atms-of-mm* (*clauses*_{NOT} *S*) \subseteq *atms-of-ms* *A* **and**
atms-trail: *atm-of* '(*lits-of-l* (*trail* *S*)) \subseteq *atms-of-ms* *A* **and**
n-d: *no-dup* (*trail* *S*) **and**
fin-A[*simp*]: *finite* *A*
shows *set-mset* (*clauses*_{NOT} *T*) \subseteq *set-mset* (*not-simplified-cls* (*clauses*_{NOT} *S*))
 \cup *simple-clss* (*atms-of-ms* *A*)
using *assms*
proof (*induction* *rule*: *cdcl_{NOT}-merged-bj-learn.induct*)
case *cdcl_{NOT}-merged-bj-learn-decide_{NOT}*
then show ?case **using** *dpll-bj-clauses* **by** (*force* *dest*!: *simple-clss-or-not-simplified-cls*)
next
case *cdcl_{NOT}-merged-bj-learn-propagate_{NOT}*
then show ?case **using** *dpll-bj-clauses* **by** (*force* *dest*!: *simple-clss-or-not-simplified-cls*)
next
case *cdcl_{NOT}-merged-bj-learn-forget_{NOT}*
then show ?case **using** *clauses-remove-cls_{NOT}* **unfolding** *state-eq_{NOT}-def*
by (*force* *elim*!: *forget_{NOT}E* *dest*: *simple-clss-or-not-simplified-cls*)
next
case (*cdcl_{NOT}-merged-bj-learn-backjump-l* *T*) **note** *bj* = *this*(1) **and** *inv* = *this*(2) **and**
atms-clss = *this*(3) **and** *atms-trail* = *this*(4) **and** *n-d* = *this*(5)

have *cdcl_{NOT}*** *S* *T*
apply (*rule* *rtranclp-cdcl_{NOT}-merged-bj-learn-is-rtranclp-cdcl_{NOT}*)
using $\langle \text{backjump-l } S \ T \rangle$ *inv* *cdcl_{NOT}-merged-bj-learn.simps* *n-d* **by** *blast* +
have *atm-of* '(*lits-of-l* (*trail* *T*)) \subseteq *atms-of-ms* *A*

```

using rtrancp-cdclNOT-trail-clauses-bound[OF  $\langle \text{cdcl}_{NOT}^{**} S T \rangle$ ] inv atms-trail atms-clss
n-d by auto
have atms-of-mm (clausesNOT T)  $\subseteq$  atms-of-ms A
using rtrancp-cdclNOT-trail-clauses-bound[OF  $\langle \text{cdcl}_{NOT}^{**} S T \rangle$ ] inv n-d atms-clss atms-trail
by fast
moreover have no-dup (trail T)
using rtrancp-cdclNOT-no-dup[OF  $\langle \text{cdcl}_{NOT}^{**} S T \rangle$ ] inv n-d by fast

obtain F' K F L l C' C D where
  tr-S: trail S = F' @ Marked K () # F and
  T: T  $\sim$  prepend-trail (Propagated L l) (reduce-trail-toNOT F (add-clNOT D S)) and
  C  $\in$  # clausesNOT S and
  trail S  $\models_{as}$  CNot C and
  undef: undefined-lit F L and
  clausesNOT S  $\models_{pm}$  C' + \{\#L\# and
  F  $\models_{as}$  CNot C' and
  D: mset-clS D = C' + \{\#L\# and
  dist: distinct-mset (C' + \{\#L\#) and
  tauto:  $\neg$  tautology (C' + \{\#L\#) and
  backjump-l-cond C C' L S T
using  $\langle \text{backjump-l } S T \rangle$  apply (elim backjump-lE) by auto

have atms-of C'  $\subseteq$  atm-of ' (lits-of-l F)
using  $\langle F \models_{as} CNot C' \rangle$  by (simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
atms-of-def image-subset-iff in-CNot-implies-uminus(2))
then have atms-of (C' + \{\#L\#)  $\subseteq$  atms-of-ms A
using T  $\langle \text{atm-of ' lits-of-l (trail } T) \subseteq \text{atms-of-ms } A \rangle$  tr-S undef n-d by auto
then have simple-clss (atms-of (C' + \{\#L\#)))  $\subseteq$  simple-clss (atms-of-ms A)
apply – by (rule simple-clss-mono) (simp-all)
then have C' + \{\#L\#  $\in$  simple-clss (atms-of-ms A)
using distinct-mset-not-tautology-implies-in-simple-clss[OF dist tauto]
by auto
then show ?case
using T inv atms-clss undef tr-S n-d D by (force dest!: simple-clss-or-not-simplified-clss)
qed

lemma cdclNOT-merged-bj-learn-not-simplified-decreasing:
assumes cdclNOT-merged-bj-learn S T
shows (not-simplified-clS (clausesNOT T))  $\subseteq$  # (not-simplified-clS (clausesNOT S))
using assms apply induction
prefer 4
unfolding not-simplified-clS-def apply (auto elim!: backjump-lE forgetNOTE)[3]
by (elim backjump-lE) auto

lemma rtrancp-cdclNOT-merged-bj-learn-not-simplified-decreasing:
assumes cdclNOT-merged-bj-learn** S T
shows (not-simplified-clS (clausesNOT T))  $\subseteq$  # (not-simplified-clS (clausesNOT S))
using assms apply induction
apply simp
by (drule cdclNOT-merged-bj-learn-not-simplified-decreasing) auto

lemma rtrancp-cdclNOT-merged-bj-learn-clauses-bound:
assumes
  cdclNOT-merged-bj-learn** S T and
  inv S and

```

$atms\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \subseteq atms\text{-}of\text{-}ms\ A$ **and**
 $atm\text{-}of\ (lits\text{-}of\text{-}l\ (trail\ S)) \subseteq atms\text{-}of\text{-}ms\ A$ **and**
 $n\text{-}d\text{-}no\text{-}dup\ (trail\ S)$ **and**
 $finite[simp]: finite\ A$
shows $set\text{-}mset\ (clauses_{NOT}\ T) \subseteq set\text{-}mset\ (not\text{-}simplified\text{-}cls\ (clauses_{NOT}\ S))$
 $\cup simple\text{-}clss\ (atms\text{-}of\text{-}ms\ A)$
using $assms(1-5)$
proof *induction*
case *base*
then show $?case$ **by** $(auto\ dest!:\ simple\text{-}clss\text{-}or\text{-}not\text{-}simplified\text{-}cls)$
next
case $(step\ T\ U)$ **note** $st = this(1)$ **and** $cdcl_{NOT} = this(2)$ **and** $IH = this(3)[OF\ this(4-7)]$ **and**
 $inv = this(4)$ **and** $atms\text{-}clss\text{-}S = this(5)$ **and** $atms\text{-}trail\text{-}S = this(6)$ **and** $finite\text{-}cls\text{-}S = this(7)$
have $st': cdcl_{NOT}^{**}\ S\ T$
using $inv\ rtranclp\text{-}cdcl_{NOT}\text{-}merged\text{-}bj\text{-}learn\text{-}is\text{-}rtranclp\text{-}cdcl_{NOT}\text{-}and\text{-}inv\ st\ n\text{-}d$ **by** *blast*
have $inv\ T$
using $inv\ rtranclp\text{-}cdcl_{NOT}\text{-}merged\text{-}bj\text{-}learn\text{-}inv\ st\ n\text{-}d$ **by** *blast*
moreover
have $atms\text{-}of\text{-}mm\ (clauses_{NOT}\ T) \subseteq atms\text{-}of\text{-}ms\ A$ **and**
 $atm\text{-}of\ (lits\text{-}of\text{-}l\ (trail\ T)) \subseteq atms\text{-}of\text{-}ms\ A$
using $rtranclp\text{-}cdcl_{NOT}\text{-}trail\text{-}clauses\text{-}bound[OF\ st']\ inv\ atms\text{-}clss\text{-}S\ atms\text{-}trail\text{-}S\ n\text{-}d$
by *blast+*
moreover moreover have $no\text{-}dup\ (trail\ T)$
using $rtranclp\text{-}cdcl_{NOT}\text{-}no\text{-}dup[OF\ (cdcl_{NOT}^{**}\ S\ T)\ inv\ n\text{-}d]$ **by** *fast*
ultimately have $set\text{-}mset\ (clauses_{NOT}\ U)$
 $\subseteq set\text{-}mset\ (not\text{-}simplified\text{-}cls\ (clauses_{NOT}\ T)) \cup simple\text{-}clss\ (atms\text{-}of\text{-}ms\ A)$
using $cdcl_{NOT}\ finite\ cdcl_{NOT}\text{-}merged\text{-}bj\text{-}learn\text{-}clauses\text{-}bound$
by $(auto\ intro!:\ cdcl_{NOT}\text{-}merged\text{-}bj\text{-}learn\text{-}clauses\text{-}bound)$
moreover have $set\text{-}mset\ (not\text{-}simplified\text{-}cls\ (clauses_{NOT}\ T))$
 $\subseteq set\text{-}mset\ (not\text{-}simplified\text{-}cls\ (clauses_{NOT}\ S))$
using $rtranclp\text{-}cdcl_{NOT}\text{-}merged\text{-}bj\text{-}learn\text{-}not\text{-}simplified\text{-}decreasing[OF\ st]$ **by** *auto*
ultimately show $?case$ **using** $IH\ inv\ atms\text{-}clss\text{-}S$
by $(auto\ dest!:\ simple\text{-}clss\text{-}or\text{-}not\text{-}simplified\text{-}cls)$
qed

abbreviation $\mu_{CDCL}'\text{-bound}$ **where**
 $\mu_{CDCL}'\text{-bound}\ A\ T \equiv ((2 + card\ (atms\text{-}of\text{-}ms\ A)) \wedge (1 + card\ (atms\text{-}of\text{-}ms\ A))) * 2$
 $+ card\ (set\text{-}mset\ (not\text{-}simplified\text{-}cls\ (clauses_{NOT}\ T)))$
 $+ 3 \wedge card\ (atms\text{-}of\text{-}ms\ A)$

lemma $rtranclp\text{-}cdcl_{NOT}\text{-}merged\text{-}bj\text{-}learn\text{-}clauses\text{-}bound\text{-}card$:

assumes

$cdcl_{NOT}\text{-}merged\text{-}bj\text{-}learn^{**}\ S\ T$ **and**
 $inv\ S$ **and**
 $atms\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \subseteq atms\text{-}of\text{-}ms\ A$ **and**
 $atm\text{-}of\ (lits\text{-}of\text{-}l\ (trail\ S)) \subseteq atms\text{-}of\text{-}ms\ A$ **and**
 $n\text{-}d\text{-}no\text{-}dup\ (trail\ S)$ **and**
 $finite: finite\ A$

shows $\mu_{CDCL}'\text{-merged}\ A\ T \leq \mu_{CDCL}'\text{-bound}\ A\ S$

proof –

have $set\text{-}mset\ (clauses_{NOT}\ T) \subseteq set\text{-}mset\ (not\text{-}simplified\text{-}cls\ (clauses_{NOT}\ S))$
 $\cup simple\text{-}clss\ (atms\text{-}of\text{-}ms\ A)$
using $rtranclp\text{-}cdcl_{NOT}\text{-}merged\text{-}bj\text{-}learn\text{-}clauses\text{-}bound[OF\ assms]$.
moreover have $card\ (set\text{-}mset\ (not\text{-}simplified\text{-}cls\ (clauses_{NOT}\ S))$
 $\cup simple\text{-}clss\ (atms\text{-}of\text{-}ms\ A))$

$\leq \text{card } (\text{set-mset } (\text{not-simplified-cls}(\text{clauses}_{NOT} S))) + 3 \wedge \text{card } (\text{atms-of-ms } A)$
by (*meson* *Nat.le-trans* *atms-of-ms-finite* *simple-clss-card* *card-Un-le* *finite* *nat-add-left-cancel-le*)
ultimately have $\text{card } (\text{set-mset } (\text{clauses}_{NOT} T))$
 $\leq \text{card } (\text{set-mset } (\text{not-simplified-cls}(\text{clauses}_{NOT} S))) + 3 \wedge \text{card } (\text{atms-of-ms } A)$
by (*meson* *Nat.le-trans* *atms-of-ms-finite* *simple-clss-finite* *card-mono* *finite-UnI* *finite-set-mset* *local.finite*)
moreover have $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A T) * 2$
 $\leq (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) * 2$
by *auto*
ultimately show *?thesis unfolding* $\mu_{CDCL}'\text{-merged-def}$ **by** *auto*
qed

sublocale *cdcl_{NOT}-increasing-restarts-ops* $\lambda S T. T \sim \text{reduce-trail-to}_{NOT} ([::'a \text{ list}) S$
cdcl_{NOT}-merged-bj-learn *f*
 $\lambda A S. \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A$
 $\wedge \text{atm-of ' lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \wedge \text{finite } A$
 $\mu_{CDCL}'\text{-merged}$
 $\lambda S. \text{inv } S \wedge \text{no-dup } (\text{trail } S)$
 $\mu_{CDCL}'\text{-bound}$
apply *unfold-locales*
using *unbounded apply simp*
using *f-ge-1 apply force*
apply (*blast* *dest!:* *cdcl_{NOT}-merged-bj-learn-is-tranclp-cdcl_{NOT}* *tranclp-into-rtranclp* *rtranclp-cdcl_{NOT}-trail-clauses-bound*)
apply (*simp* *add:* *cdcl_{NOT}-decreasing-measure'*)
using *rtranclp-cdcl_{NOT}-merged-bj-learn-clauses-bound-card* **apply** *blast*
apply (*drule* *rtranclp-cdcl_{NOT}-merged-bj-learn-not-simplified-decreasing*)
apply (*auto* *dest!:* *simp:* *card-mono* *set-mset-mono*)
apply *simp*
apply *auto*
using *cdcl_{NOT}-merged-bj-learn-no-dup-inv* *cdcl-merged-inv* **apply** *blast*
apply (*auto* *simp:* *inv-restart*)
done

lemma *cdcl_{NOT}-restart- μ_{CDCL}' -merged-le- μ_{CDCL}' -bound:*

assumes
cdcl_{NOT}-restart *T V*
inv (*fst* *T*) **and**
no-dup (*trail* (*fst* *T*)) **and**
 $\text{atms-of-mm } (\text{clauses}_{NOT} (\text{fst } T)) \subseteq \text{atms-of-ms } A$ **and**
 $\text{atm-of ' lits-of-l } (\text{trail } (\text{fst } T)) \subseteq \text{atms-of-ms } A$ **and**
finite *A*
shows $\mu_{CDCL}'\text{-merged } A (\text{fst } V) \leq \mu_{CDCL}'\text{-bound } A (\text{fst } T)$
using *assms*
proof *induction*
case (*restart-full* *S T n*)
show *?case*
unfolding *fst-conv*
apply (*rule* *rtranclp-cdcl_{NOT}-merged-bj-learn-clauses-bound-card*)
using *restart-full* **unfolding** *full1-def* **by** (*force* *dest!:* *tranclp-into-rtranclp*)
next
case (*restart-step* *m S T n U*) **note** *st = this(1)* **and** *U = this(3)* **and** *inv = this(4)* **and**
n-d = this(5) **and** *atms-clss = this(6)* **and** *atms-trail = this(7)* **and** *finite = this(8)*
then have *st': cdcl_{NOT}-merged-bj-learn** S T*

by (blast dest: relpoup-imp-rtrancpl)
 then have st'' : $cdcl_{NOT}^{**} S T$
 using inv n-d apply – by (rule rtrancpl-cdcl_{NOT}-merged-bj-learn-is-rtrancpl-cdcl_{NOT}) auto
 have inv T
 apply (rule rtrancpl-cdcl_{NOT}-merged-bj-learn-inv)
 using inv st' n-d by auto
 then have inv U
 using U by (auto simp: inv-restart)
 have atms-of-mm (clauses_{NOT} T) \subseteq atms-of-ms A
 using rtrancpl-cdcl_{NOT}-trail-clauses-bound[OF st''] inv atms-clss atms-trail n-d
 by simp
 then have atms-of-mm (clauses_{NOT} U) \subseteq atms-of-ms A
 using U by simp
 have not-simplified-cls (clauses_{NOT} U) \subseteq # not-simplified-cls (clauses_{NOT} T)
 using $\langle U \sim \text{reduce-trail-to}_{NOT} [] T \rangle$ by auto
 moreover have not-simplified-cls (clauses_{NOT} T) \subseteq # not-simplified-cls (clauses_{NOT} S)
 apply (rule rtrancpl-cdcl_{NOT}-merged-bj-learn-not-simplified-decreasing)
 using $\langle (cdcl_{NOT}\text{-merged-bj-learn } \widetilde{m}) S T \rangle$ by (auto dest!: relpoup-imp-rtrancpl)
 ultimately have U-S: not-simplified-cls (clauses_{NOT} U) \subseteq # not-simplified-cls (clauses_{NOT} S)
 by auto

 have (set-mset (clauses_{NOT} U))
 \subseteq set-mset (not-simplified-cls (clauses_{NOT} U)) \cup simple-clss (atms-of-ms A)
 apply (rule rtrancpl-cdcl_{NOT}-merged-bj-learn-clauses-bound)
 apply simp
 using $\langle inv U \rangle$ apply simp
 using $\langle \text{atms-of-mm (clauses}_{NOT} U) \subseteq \text{atms-of-ms A} \rangle$ apply simp
 using U apply simp
 using U apply simp
 using finite apply simp
 done
 then have f1: card (set-mset (clauses_{NOT} U)) \leq card (set-mset (not-simplified-cls (clauses_{NOT} U))
 \cup simple-clss (atms-of-ms A))
 by (simp add: simple-clss-finite card-mono local.finite)

 moreover have set-mset (not-simplified-cls (clauses_{NOT} U)) \cup simple-clss (atms-of-ms A)
 \subseteq set-mset (not-simplified-cls (clauses_{NOT} S)) \cup simple-clss (atms-of-ms A)
 using U-S by auto
 then have f2:
 card (set-mset (not-simplified-cls (clauses_{NOT} U)) \cup simple-clss (atms-of-ms A))
 \leq card (set-mset (not-simplified-cls (clauses_{NOT} S)) \cup simple-clss (atms-of-ms A))
 by (simp add: simple-clss-finite card-mono local.finite)

 moreover have card (set-mset (not-simplified-cls (clauses_{NOT} S))
 \cup simple-clss (atms-of-ms A))
 \leq card (set-mset (not-simplified-cls (clauses_{NOT} S))) + card (simple-clss (atms-of-ms A))
 using card-Un-le by blast
 moreover have card (simple-clss (atms-of-ms A)) \leq 3 \wedge card (atms-of-ms A)
 using atms-of-ms-finite simple-clss-card local.finite by blast
 ultimately have card (set-mset (clauses_{NOT} U))
 \leq card (set-mset (not-simplified-cls (clauses_{NOT} S))) + 3 \wedge card (atms-of-ms A)
 by linarith
 then show ?case unfolding μ_{CDCL}' -merged-def by auto
 qed

inv (fst S)
shows $I \models_{sextm} clauses_{NOT} (fst S) \longleftrightarrow I \models_{sextm} clauses_{NOT} (fst T)$
using $assms$
proof (*induction rule: cdcl_{NOT}-restart.induct*)
case (*restart-full* $S T n$)
then have $cdcl_{NOT}$ -merged-bj-learn** $S T$
by (*simp add: tranclp-into-rtranclp full1-def*)
then show ?*case*
using $rtranclp$ - $cdcl_{NOT}$ -bj-sat-ext-iff *restart-full.prem*s(1,2)
 $rtranclp$ - $cdcl_{NOT}$ -merged-bj-learn-is- $rtranclp$ - $cdcl_{NOT}$ **by** *auto*
next
case (*restart-step* $m S T n U$)
then have $cdcl_{NOT}$ -merged-bj-learn** $S T$
by (*auto simp: tranclp-into-rtranclp full1-def dest!: relpowp-imp-rtranclp*)
then have $I \models_{sextm} clauses_{NOT} S \longleftrightarrow I \models_{sextm} clauses_{NOT} T$
using $rtranclp$ - $cdcl_{NOT}$ -bj-sat-ext-iff *restart-step.prem*s(1,2)
 $rtranclp$ - $cdcl_{NOT}$ -merged-bj-learn-is- $rtranclp$ - $cdcl_{NOT}$ **by** *auto*
moreover have $I \models_{sextm} clauses_{NOT} T \longleftrightarrow I \models_{sextm} clauses_{NOT} U$
using *restart-step.hyps*(3) **by** *auto*
ultimately show ?*case* **by** *auto*
qed

lemma $rtranclp$ - $cdcl_{NOT}$ -restart-eq-sat-iff:
assumes
 $cdcl_{NOT}$ -restart** $S T$ **and**
 inv : inv ($fst S$) **and** n -d: $no_dup(trail (fst S))$
shows $I \models_{sextm} clauses_{NOT} (fst S) \longleftrightarrow I \models_{sextm} clauses_{NOT} (fst T)$
using $assms$ (1)
proof (*induction rule: rtranclp-induct*)
case *base*
then show ?*case* **by** *simp*
next
case (*step* $T U$) **note** $st = this(1)$ **and** $cdcl = this(2)$ **and** $IH = this(3)$
have inv ($fst T$) **and** $no_dup(trail (fst T))$
using $rtranclp$ - $cdcl_{NOT}$ -with-restart- $cdcl_{NOT}$ - inv **using** st inv n -d **by** *blast+*
then have $I \models_{sextm} clauses_{NOT} (fst T) \longleftrightarrow I \models_{sextm} clauses_{NOT} (fst U)$
using $cdcl_{NOT}$ -restart-eq-sat-iff $cdcl$ **by** *blast*
then show ?*case* **using** IH **by** *blast*
qed

lemma $cdcl_{NOT}$ -restart-all-decomposition-implies-m:
assumes
 $cdcl_{NOT}$ -restart $S T$ **and**
 inv : inv ($fst S$) **and** n -d: $no_dup(trail (fst S))$ **and**
all-decomposition-implies-m ($clauses_{NOT} (fst S)$)
(get -all-marked-decomposition ($trail (fst S)$))
shows all-decomposition-implies-m ($clauses_{NOT} (fst T)$)
(get -all-marked-decomposition ($trail (fst T)$))
using $assms$
proof (*induction*)
case (*restart-full* $S T n$) **note** $full = this(1)$ **and** $inv = this(2)$ **and** n -d = $this(3)$ **and**
 $decomp = this(4)$
have st : $cdcl_{NOT}$ -merged-bj-learn** $S T$ **and**
 n -s: no -step $cdcl_{NOT}$ -merged-bj-learn T
using $full$ **unfolding** *full1-def* **by** (*fast dest: tranclp-into-rtranclp*)+

have st' : $cdcl_{NOT}^{**} S T$
using inv $rtrancp-cdcl_{NOT}$ -merged-bj-learn-is-rtrancp-cdcl $_{NOT}$ -and- inv st $n-d$ **by** $auto$
have $inv T$
using $rtrancp-cdcl_{NOT}$ - $cdcl_{NOT}$ - inv [$OF st$] inv $n-d$ **by** $auto$
then show $?case$
using $rtrancp-cdcl_{NOT}$ -all-decomposition-implies[$OF - - n-d decomp$] $st' inv$ **by** $auto$
next
case ($restart-step m S T n U$) **note** $st = this(1)$ **and** $U = this(3)$ **and** $inv = this(4)$ **and**
 $n-d = this(5)$ **and** $decomp = this(6)$
show $?case$ **using** U **by** $auto$
qed

lemma $rtrancp-cdcl_{NOT}$ -restart-all-decomposition-implies- m :

assumes
 $cdcl_{NOT}$ -restart $^{**} S T$ **and**
 inv : $inv (fst S)$ **and** $n-d$: $no-dup(trail (fst S))$ **and**
 $decomp$: $all-decomposition-implies-m (clauses_{NOT} (fst S))$
 $(get-all-marked-decomposition (trail (fst S)))$
shows $all-decomposition-implies-m (clauses_{NOT} (fst T))$
 $(get-all-marked-decomposition (trail (fst T)))$
using $assms$
proof ($induction$)
case $base$
then show $?case$ **using** $decomp$ **by** $simp$
next
case ($step T U$) **note** $st = this(1)$ **and** $cdcl = this(2)$ **and** $IH = this(3)[OF this(4-)]$ **and**
 $inv = this(4)$ **and** $n-d = this(5)$ **and** $decomp = this(6)$
have $inv (fst T)$ **and** $no-dup (trail (fst T))$
using $rtrancp-cdcl_{NOT}$ -with-restart- $cdcl_{NOT}$ - inv **using** $st inv n-d$ **by** $blast+$
then show $?case$
using $cdcl_{NOT}$ -restart-all-decomposition-implies- m [$OF cdcl$] IH **by** $auto$
qed

lemma $full-cdcl_{NOT}$ -restart-normal-form:

assumes
 $full$: $full cdcl_{NOT}$ -restart $S T$ **and**
 inv : $inv (fst S)$ **and** $n-d$: $no-dup(trail (fst S))$ **and**
 $decomp$: $all-decomposition-implies-m (clauses_{NOT} (fst S))$
 $(get-all-marked-decomposition (trail (fst S)))$ **and**
 $atms-cls$: $atms-of-mm (clauses_{NOT} (fst S)) \subseteq atms-of-ms A$ **and**
 $atms-trail$: $atm-of ' lits-of-l (trail (fst S)) \subseteq atms-of-ms A$ **and**
 fin : $finite A$
shows $unsatisfiable (set-mset (clauses_{NOT} (fst S)))$
 $\vee lits-of-l (trail (fst T)) \models_{sextm} clauses_{NOT} (fst S) \wedge satisfiable (set-mset (clauses_{NOT} (fst S)))$
proof –
have $inv-T$: $inv (fst T)$ **and** $n-d-T$: $no-dup (trail (fst T))$
using $rtrancp-cdcl_{NOT}$ -with-restart- $cdcl_{NOT}$ - inv **using** $full inv n-d$ **unfolding** $full-def$ **by** $blast+$
moreover have
 $atms-cls-T$: $atms-of-mm (clauses_{NOT} (fst T)) \subseteq atms-of-ms A$ **and**
 $atms-trail-T$: $atm-of ' lits-of-l (trail (fst T)) \subseteq atms-of-ms A$
using $rtrancp-cdcl_{NOT}$ -with-restart-bound- inv [$of S T A$] $full atms-cls atms-trail fin inv n-d$
unfolding $full-def$ **by** $blast+$
ultimately have $no-step cdcl_{NOT}$ -merged-bj-learn ($fst T$)
apply –
apply ($rule no-step-cdcl_{NOT}$ -restart-no-step- $cdcl_{NOT}$ [$of - A$])

```

    using full unfolding full-def apply simp
    apply simp
    using fin apply simp
    done
  moreover have all-decomposition-implies-m (clausesNOT (fst T))
    (get-all-marked-decomposition (trail (fst T)))
    using rtrancpl-cdclNOT-restart-all-decomposition-implies-m[of S T] inv n-d decomp
    full unfolding full-def by auto
  ultimately have unsatisfiable (set-mset (clausesNOT (fst T)))
    ∨ trail (fst T) ⊨asm clausesNOT (fst T) ∧ satisfiable (set-mset (clausesNOT (fst T)))
    apply -
    apply (rule cdclNOT-merged-bj-learn-final-state)
    using atms-cls-T atms-trail-T fin n-d-T fin inv-T by blast+
  then consider
    (unsat) unsatisfiable (set-mset (clausesNOT (fst T)))
  | (sat) trail (fst T) ⊨asm clausesNOT (fst T) and satisfiable (set-mset (clausesNOT (fst T)))
  by auto
  then show unsatisfiable (set-mset (clausesNOT (fst S)))
    ∨ lits-of-l (trail (fst T)) ⊨sextm clausesNOT (fst S) ∧ satisfiable (set-mset (clausesNOT (fst S)))
  proof cases
    case unsat
    then have unsatisfiable (set-mset (clausesNOT (fst S)))
      unfolding satisfiable-def apply auto
      using rtrancpl-cdclNOT-restart-eq-sat-iff[of S T] full inv n-d
      consistent-true-clss-ext-satisfiable true-clss-imp-true-clss-ext
      unfolding satisfiable-def full-def by blast
    then show ?thesis by blast
  next
    case sat
    then have lits-of-l (trail (fst T)) ⊨sextm clausesNOT (fst T)
      using true-clss-imp-true-clss-ext by (auto simp: true-annots-true-clss)
    then have lits-of-l (trail (fst T)) ⊨sextm clausesNOT (fst S)
      using rtrancpl-cdclNOT-restart-eq-sat-iff[of S T] full inv n-d unfolding full-def by blast
    moreover then have satisfiable (set-mset (clausesNOT (fst S)))
      using consistent-true-clss-ext-satisfiable distinct-consistent-interp n-d-T by fast
    ultimately show ?thesis by fast
  qed
qed

corollary full-cdclNOT-restart-normal-form-init-state:
  assumes
    init-state: trail S = [] clausesNOT S = N and
    full: full cdclNOT-restart (S, 0) T and
    inv: inv S
  shows unsatisfiable (set-mset N)
    ∨ lits-of-l (trail (fst T)) ⊨sextm N ∧ satisfiable (set-mset N)
  using full-cdclNOT-restart-normal-form[of (S, 0) T] assms by auto

end

end
theory DPLL-NOT
imports CDCL-NOT
begin

```

17 DPLL as an instance of NOT

17.1 DPLL with simple backtrack

We are using a concrete couple instead of an abstract state.

locale *dpll-with-backtrack*

begin

inductive *backtrack* :: ('v, unit, unit) marked-lit list \times 'v clauses

\Rightarrow ('v, unit, unit) marked-lit list \times 'v clauses \Rightarrow bool **where**

backtrack-split (*fst* *S*) = (*M'*, *L* # *M*) \Longrightarrow *is-marked* *L* \Longrightarrow *D* \in # *snd* *S*

\Longrightarrow *fst* *S* \models_{as} *CNot* *D* \Longrightarrow *backtrack* *S* (*Propagated* (\neg (*lit-of* *L*)) () # *M*, *snd* *S*)

inductive-cases *backtrackE*[*elim*]: *backtrack* (*M*, *N*) (*M'*, *N'*)

lemma *backtrack-is-backjump*:

fixes *M* *M'* :: ('v, unit, unit) marked-lit list

assumes

backtrack: *backtrack* (*M*, *N*) (*M'*, *N'*) **and**

no-dup: (*no-dup* \circ *fst*) (*M*, *N*) **and**

decomp: *all-decomposition-implies-m* *N* (*get-all-marked-decomposition* *M*)

shows

$\exists C F' K F L l C'$.

$M = F' @ \text{Marked } K () \# F \wedge$

$M' = \text{Propagated } L l \# F \wedge N = N' \wedge C \in \# N \wedge F' @ \text{Marked } K d \# F \models_{as} \text{CNot } C \wedge$

undefined-lit *F* *L* \wedge *atm-of* *L* \in *atms-of-mm* *N* \cup *atm-of* ' *lits-of-l* (*F'* @ *Marked* *K* *d* # *F*) \wedge

$N \models_{pm} C' + \{\#L\} \wedge F \models_{as} \text{CNot } C'$

proof –

let ?*S* = (*M*, *N*)

let ?*T* = (*M'*, *N'*)

obtain *F* *F'* *P* *L* *D* **where**

b-sp: *backtrack-split* *M* = (*F'*, *L* # *F*) **and**

is-marked *L* **and**

D \in # *snd* ?*S* **and**

M \models_{as} *CNot* *D* **and**

bt: *backtrack* ?*S* (*Propagated* (\neg (*lit-of* *L*)) *P* # *F*, *N*) **and**

M': *M'* = *Propagated* (\neg (*lit-of* *L*)) *P* # *F* **and**

[*simp*]: *N'* = *N*

using *backtrackE*[*OF* *backtrack*] **by** (*metis* *backtrack* *fstI* *sndI*)

let ?*K* = *lit-of* *L*

let ?*C* = *image-mset* *lit-of* {#*K* \in #*mset* *M*. *is-marked* *K* \wedge *K* \neq *L* #} :: 'v literal multiset

let ?*C'* = *set-mset* (*image-mset* *single* (?*C* + {#?*K* #}))

obtain *K* **where** *L*: *L* = *Marked* *K* () **using** ' *is-marked* *L* ' **by** (*cases* *L*) *auto*

have *M*: *M* = *F'* @ *Marked* *K* () # *F*

using *b-sp* **by** (*metis* *L* *backtrack-split-list-eq* *fst-conv* *snd-conv*)

moreover **have** *F'* @ *Marked* *K* () # *F* \models_{as} *CNot* *D*

using ' *M* \models_{as} *CNot* *D* ' **unfolding** *M* .

moreover **have** *undefined-lit* *F* (\neg ?*K*)

using *no-dup* **unfolding** *M* *L* **by** (*simp* *add*: *defined-lit-map*)

moreover **have** *atm-of* (\neg *K*) \in *atms-of-mm* *N* \cup *atm-of* ' *lits-of-l* (*F'* @ *Marked* *K* *d* # *F*)

by *auto*

moreover

have *set-mset* *N* \cup ?*C'* \models_{ps} {# #}

proof –

have *A*: *set-mset* *N* \cup ?*C'* \cup *unmark-l* *M* =

set-mset *N* \cup *unmark-l* *M*

```

    unfolding M L by auto
  have set-mset N  $\cup$   $\{\{\#lit-of L\# \} \mid L. is-marked L \wedge L \in set M\}$ 
     $\models_{ps} unmark-l M$ 
    using all-decomposition-implies-propagated-lits-are-implied[OF decomp] .
  moreover have C':  $?C' = \{\{\#lit-of L\# \} \mid L. is-marked L \wedge L \in set M\}$ 
    unfolding M L apply standard
    apply force
    using IntI by auto
  ultimately have N-C-M: set-mset N  $\cup$   $?C' \models_{ps} unmark-l M$ 
    by auto
  have set-mset N  $\cup$   $(\lambda L. \{\#lit-of L\# \}) ' (set M) \models_{ps} \{\{\#\}\}$ 
    unfolding true-clss-clss-def
  proof (intro allI impI, goal-cases)
    case (1 I) note tot = this(1) and cons = this(2) and I-N-M = this(3)
    have I  $\models D$ 
      using I-N-M  $\langle D \in \# \text{ snd } ?S \rangle$  unfolding true-clss-def by auto
    moreover have I  $\models_s CNot D$ 
      using  $\langle M \models_{as} CNot D \rangle$  unfolding M by (metis 1(3)  $\langle M \models_{as} CNot D \rangle$ 
        true-annots-true-clss true-clss-mono-set-mset-l true-clss-def
        true-clss-singleton-lit-of-implies-incl true-clss-union)
    ultimately show ?case using cons consistent-CNot-not by blast
  qed
  then show ?thesis
    using true-clss-clss-left-right[OF N-C-M, of  $\{\{\#\}\}$ ] unfolding A by auto
  qed
  have N  $\models_{pm} image-mset uminus ?C + \{\#-?K\# \}$ 
    unfolding true-clss-clss-def true-clss-clss-def total-over-m-def
  proof (intro allI impI)
    fix I
    assume
      tot: total-over-set I (atms-of-ms (set-mset N  $\cup$   $\{image-mset uminus ?C + \{\#-?K\# \}\}$ )) and
      cons: consistent-interp I and
      I  $\models_{sm} N$ 
    have  $(K \in I \wedge -K \notin I) \vee (-K \in I \wedge K \notin I)$ 
      using cons tot unfolding consistent-interp-def L by (cases K) auto
    have  $\{a \in set M. is-marked a \wedge a \neq Marked K ()\} =$ 
       $set M \cap \{L. is-marked L \wedge L \neq Marked K ()\}$ 
      by auto
    then have
      tI: total-over-set I (atm-of 'lit-of ' (set M  $\cap$   $\{L. is-marked L \wedge L \neq Marked K d\}$ ))
      using tot by (auto simp add: L atms-of-uminus-lit-atm-of-lit-of)

  then have H:  $\bigwedge x.
    lit-of x \notin I \implies x \in set M \implies is-marked x
    \implies x \neq Marked K d \implies -lit-of x \in I$ 
  proof -
    fix x :: ('v, unit, unit) marked-lit
    assume a1:  $x \neq Marked K d$ 
    assume a2: is-marked x
    assume a3:  $x \in set M$ 
    assume a4:  $lit-of x \notin I$ 
    have atm-of (lit-of x)  $\in atm-of 'lit-of '
      (set M \cap \{m. is-marked m \wedge m \neq Marked K d\})$ 
      using a3 a2 a1 by blast
    then have Pos (atm-of (lit-of x))  $\in I \vee Neg (atm-of (lit-of x)) \in I$ 

```



```

    using tI unfolding total-over-set-def by blast
  then show  $\neg \text{lit-of } x \in I$ 
    using a4 by (metis (no-types) atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
      literal.sel(1,2))
  qed
  have  $\neg I \models_s ?C'$ 
    using  $\langle \text{set-mset } N \cup ?C' \models_{ps} \{\{\#\}\} \rangle \text{ tot cons } \langle I \models_{sm} N \rangle$ 
    unfolding true-clss-clss-def total-over-m-def
    by (simp add: atms-of-uminus-lit-atm-of-lit-of atms-of-ms-single-image-atm-of-lit-of)
  then show  $I \models \text{image-mset } \text{uminus } ?C + \{\#\text{-lit-of } L\#\}$ 
    unfolding true-clss-def true-clss-def
    using  $\langle (K \in I \wedge \neg K \notin I) \vee (\neg K \in I \wedge K \notin I) \rangle$ 
    unfolding L by (auto dest!: H)
  qed
  moreover
  have  $\text{set } F' \cap \{K. \text{is-marked } K \wedge K \neq L\} = \{\}$ 
    using backtrack-split-fst-not-marked[of - M] b-sp by auto
  then have  $F \models_{as} CNot (\text{image-mset } \text{uminus } ?C)$ 
    unfolding M CNot-def true-annots-def by (auto simp add: L lits-of-def)
  ultimately show ?thesis
    using  $M' \langle D \in \# \text{ snd } ?S \rangle L$  by force
  qed

```

lemma *backtrack-is-backjump'*:

fixes $M M' :: ('v, \text{unit}, \text{unit}) \text{ marked-lit list}$

assumes

backtrack: *backtrack S T* **and**

no-dup: $(\text{no-dup} \circ \text{fst}) S$ **and**

decomp: *all-decomposition-implies-m* (*snd S*) (*get-all-marked-decomposition* (*fst S*))

shows

$\exists C F' K F L l C'.$

$\text{fst } S = F' @ \text{Marked } K () \# F \wedge$

$T = (\text{Propagated } L l \# F, \text{snd } S) \wedge C \in \# \text{ snd } S \wedge \text{fst } S \models_{as} CNot C$

$\wedge \text{undefined-lit } F L \wedge \text{atm-of } L \in \text{atms-of-mm } (\text{snd } S) \cup \text{atm-of 'lits-of-l } (\text{fst } S) \wedge$

$\text{snd } S \models_{pm} C' + \{\#L\#\} \wedge F \models_{as} CNot C'$

apply (*cases S*, *cases T*)

using *backtrack-is-backjump*[*of fst S snd S fst T snd T*] *assms* **by** *fastforce*

sublocale *dpll-state*

id $\lambda L C. C + \{\#L\#\} \text{ remove1-mset}$

id $op + op \in \# \lambda L C. C + \{\#L\#\} \text{ remove1-mset}$

fst snd $\lambda L (M, N). (L \# M, N) \lambda (M, N). (tl M, N)$

$\lambda C (M, N). (M, \{\#C\#\} + N) \lambda C (M, N). (M, \text{removeAll-mset } C N)$

by *unfold-locales* (*auto simp: ac-simps*)

sublocale *backjumping-ops*

id $\lambda L C. C + \{\#L\#\} \text{ remove1-mset}$

id $op + op \in \# \lambda L C. C + \{\#L\#\} \text{ remove1-mset}$

fst snd $\lambda L (M, N). (L \# M, N) \lambda (M, N). (tl M, N)$

$\lambda C (M, N). (M, \{\#C\#\} + N) \lambda C (M, N). (M, \text{removeAll-mset } C N) \lambda - - S T. \text{backtrack } S T$

by *unfold-locales*

lemma *reduce-trail-to_{NOT}-snd*:

snd (*reduce-trail-to_{NOT}* *F S*) = *snd S*

apply (*induction F S rule: reduce-trail-to_{NOT}.induct*)

by (cases S , rename-tac F Sa , case-tac Sa)
 (simp add: less-imp-diff-less reduce-trail-to_{NOT}.simps)

lemma reduce-trail-to_{NOT}:
 reduce-trail-to_{NOT} F S =
 (if length (fst S) \geq length F
 then drop (length (fst S) - length F) (fst S)
 else [],
 snd S) (is ? R = ? C)

proof -
 have ? R = (fst ? R , snd ? R)
 by auto
 also have (fst ? R , snd ? R) = ? C
 by (auto simp: trail-reduce-trail-to_{NOT}-drop reduce-trail-to_{NOT}-snd)
 finally show ?thesis .
qed

lemma backtrack-is-backjump'':
 fixes M M' :: (' v , unit, unit) marked-lit list
 assumes
 backtrack: backtrack S T and
 no-dup: (no-dup \circ fst) S and
 decomp: all-decomposition-implies-m (snd S) (get-all-marked-decomposition (fst S))
 shows backjump S T

proof -
 obtain C F' K F L l C' where
 1: fst S = F' @ Marked K () # F and
 2: T = (Propagated L l # F , snd S) and
 3: $C \in \#$ snd S and
 4: fst S \models_{as} CNot C and
 5: undefined-lit F L and
 6: atm-of $L \in$ atms-of-mm (snd S) \cup atm-of ' l lits-of- l (fst S) and
 7: snd S \models_{pm} $C' + \{\#L\#$ and
 8: $F \models_{as}$ CNot C'
 using backtrack-is-backjump'[OF assms] by force
 show ?thesis
 apply (cases S)
 using backjump.intros[OF 1 - 4 5 - 8, of T] 2 backtrack 1 5 3 6 7
 by (auto simp: state-eq_{NOT}-def trail-reduce-trail-to_{NOT}-drop
 reduce-trail-to_{NOT} simp del: state-simp_{NOT})
qed

lemma can-do-bt-step:
 assumes
 M : fst S = F' @ Marked K d # F and
 $C \in \#$ snd S and
 C : fst S \models_{as} CNot C
 shows \neg no-step backtrack S

proof -
 obtain L G' G where
 backtrack-split (fst S) = (G' , L # G)
 unfolding M by (induction F' rule: marked-lit-list-induct) auto
 moreover then have is-marked L
 by (metis backtrack-split-snd-hd-marked list.distinct(1) list.sel(1) snd-conv)
 ultimately show ?thesis

```

    using backtrack.intros[of S G' L G C] (C ∈# snd S) C unfolding M by auto
qed

end

sublocale dpll-with-backtrack ⊆ dpll-with-backjumping-ops
  id λL C. C + {#L#} remove1-mset
  id op + op ∈# λL C. C + {#L#} remove1-mset
  fst snd λL (M, N). (L # M, N)
  λ(M, N). (tl M, N) λC (M, N). (M, {#C#} + N) λC (M, N). (M, removeAll-mset C N)
  λ(M, N). no-dup M ∧ all-decomposition-implies-m N (get-all-marked-decomposition M)
  λ- - S T. backtrack S T
  λ- -. True
  apply unfold-locales
  by (metis (mono-tags, lifting) case-prod-beta comp-def dpll-with-backtrack.backtrack-is-backjump''
      dpll-with-backtrack.can-do-bt-step id-apply)

sublocale dpll-with-backtrack ⊆ dpll-with-backjumping
  id λL C. C + {#L#} remove1-mset
  id op + op ∈# λL C. C + {#L#} remove1-mset
  fst snd λL (M, N). (L # M, N)
  λ(M, N). (tl M, N) λC (M, N). (M, {#C#} + N) λC (M, N). (M, removeAll-mset C N)
  λ(M, N). no-dup M ∧ all-decomposition-implies-m N (get-all-marked-decomposition M)
  λ- - S T. backtrack S T
  λ- -. True
  apply unfold-locales
  using dpll-bj-no-dup dpll-bj-all-decomposition-implies-inv apply fastforce
done

context dpll-with-backtrack
begin
term learn
end

context dpll-with-backtrack
begin
lemma wf-tranclp-dpll-inital-state:
  assumes fin: finite A
  shows wf {((M'::('v, unit, unit) marked-lits, N'::'v clauses), ([], N)) | M' N' N.
    dpll-bj++ ([], N) (M', N') ∧ atms-of-mm N ⊆ atms-of-ms A}
  using wf-tranclp-dpll-bj[OF assms(1)] by (rule wf-subset) auto

corollary full-dpll-final-state-conclusive:
  fixes M M' :: ('v, unit, unit) marked-lit list
  assumes
    full: full dpll-bj ([], N) (M', N')
  shows unsatisfiable (set-mset N) ∨ (M' ⊨asm N ∧ satisfiable (set-mset N))
  using assms full-dpll-backjump-final-state[of ([], N) (M', N') set-mset N] by auto

corollary full-dpll-normal-form-from-init-state:
  fixes M M' :: ('v, unit, unit) marked-lit list
  assumes
    full: full dpll-bj ([], N) (M', N')
  shows M' ⊨asm N ↔ satisfiable (set-mset N)

```

proof –

```

have no-dup M'
  using rtrancpl-dpll-bj-no-dup[of (], N) (M', N')]
  full unfolding full-def by auto
then have M'  $\models_{asm}$  N  $\implies$  satisfiable (set-mset N)
  using distinct-consistent-interp satisfiable-carac' true-annots-true-cls by blast
then show ?thesis
using full-dpll-final-state-conclusive[OF full] by auto
qed

```

interpretation *conflict-driven-clause-learning-ops*

```

  id  $\lambda L C. C + \{\#L\# \}$  remove1-mset
  id  $op + op \in \# \lambda L C. C + \{\#L\# \}$  remove1-mset
  fst snd  $\lambda L (M, N). (L \# M, N)$ 
 $\lambda (M, N). (tl M, N) \lambda C (M, N). (M, \{\#C\# \} + N) \lambda C (M, N). (M, removeAll-mset C N)$ 
 $\lambda (M, N). no-dup M \wedge all-decomposition-implies-m N (get-all-marked-decomposition M)$ 
 $\lambda - - S T. backtrack S T$ 
 $\lambda - -. True \lambda - -. False \lambda - -. False$ 
by unfold-locales

```

interpretation *conflict-driven-clause-learning*

```

  id  $\lambda L C. C + \{\#L\# \}$  remove1-mset
  id  $op + op \in \# \lambda L C. C + \{\#L\# \}$  remove1-mset
  fst snd  $\lambda L (M, N). (L \# M, N)$ 
 $\lambda (M, N). (tl M, N) \lambda C (M, N). (M, \{\#C\# \} + N) \lambda C (M, N). (M, removeAll-mset C N)$ 
 $\lambda (M, N). no-dup M \wedge all-decomposition-implies-m N (get-all-marked-decomposition M)$ 
 $\lambda - - S T. backtrack S T$ 
 $\lambda - -. True \lambda - -. False \lambda - -. False$ 
apply unfold-locales
using cdclNOT-all-decomposition-implies cdclNOT-no-dup by fastforce

```

lemma cdcl_{NOT}-is-dpll:

```

  cdclNOT S T  $\longleftrightarrow$  dpll-bj S T
by (auto simp: cdclNOT.simps learn.simps forgetNOT.simps)

```

Another proof of termination:

```

lemma wf {(T, S). dpll-bj S T  $\wedge$  cdclNOT-NOT-all-inv A S}
  unfolding cdclNOT-is-dpll[symmetric]
  by (rule wf-cdclNOT-no-learn-and-forget-infinite-chain)
  (auto simp: learn.simps forgetNOT.simps)
end

```

17.2 Adding restarts

This was mainly a test whether it was possible to instantiate the assumption of the locale.

```

locale dpll-withbacktrack-and-restarts =
  dpll-with-backtrack +
  fixes f :: nat  $\Rightarrow$  nat
  assumes unbounded: unbounded f and f-ge-1:  $\bigwedge n. n \geq 1 \implies f n \geq 1$ 
begin
  sublocale cdclNOT-increasing-restarts
    id  $\lambda L C. C + \{\#L\# \}$  remove1-mset
    id  $op + op \in \# \lambda L C. C + \{\#L\# \}$  remove1-mset
    fst snd  $\lambda L (M, N). (L \# M, N) \lambda (M, N). (tl M, N)$ 
     $\lambda C (M, N). (M, \{\#C\# \} + N) \lambda C (M, N). (M, removeAll-mset C N) f \lambda (-, N) S. S = ([], N)$ 

```

```

λA (M, N). atms-of-mm N ⊆ atms-of-ms A ∧ atm-of ' lits-of-l M ⊆ atms-of-ms A ∧ finite A
  ∧ all-decomposition-implies-m N (get-all-marked-decomposition M)
λA T. (2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A))
  - μC (1+card (atms-of-ms A)) (2+card (atms-of-ms A)) (trail-weight T) dpll-bj
λ(M, N). no-dup M ∧ all-decomposition-implies-m N (get-all-marked-decomposition M)
λA -. (2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A))
apply unfold-locales
  apply (rule unbounded)
  using f-ge-1 apply fastforce
  apply (smt dpll-bj-all-decomposition-implies-inv dpll-bj-atms-in-trail-in-set
    dpll-bj-clauses id-apply prod.case-eq-if)
  apply (rule dpll-bj-trail-mes-decreasing-prop; auto)
  apply (rename-tac A T U, case-tac T, simp)
  apply (rename-tac A T U, case-tac U, simp)
  using dpll-bj-clauses dpll-bj-all-decomposition-implies-inv dpll-bj-no-dup by fastforce+
end

end
theory DPLL-W
imports Main Partial-Clausal-Logic Partial-Annotated-Clausal-Logic List-More Wellfounded-More
  DPLL-NOT
begin

```

18 DPLL

18.1 Rules

```

type-synonym 'a dpllW-marked-lit = ('a, unit, unit) marked-lit
type-synonym 'a dpllW-marked-lits = ('a, unit, unit) marked-lits
type-synonym 'v dpllW-state = 'v dpllW-marked-lits × 'v clauses

```

```

abbreviation trail :: 'v dpllW-state ⇒ 'v dpllW-marked-lits where
  trail ≡ fst
abbreviation clauses :: 'v dpllW-state ⇒ 'v clauses where
  clauses ≡ snd

```

The definition of DPLL is given in figure 2.13 page 70 of CW.

```

inductive dpllW :: 'v dpllW-state ⇒ 'v dpllW-state ⇒ bool where
  propagate: C + {#L#} ∈ # clauses S ⇒ trail S ⊨as CNot C ⇒ undefined-lit (trail S) L
    ⇒ dpllW S (Propagated L () # trail S, clauses S) |
  decided: undefined-lit (trail S) L ⇒ atm-of L ∈ atms-of-mm (clauses S)
    ⇒ dpllW S (Marked L () # trail S, clauses S) |
  backtrack: backtrack-split (trail S) = (M', L # M) ⇒ is-marked L ⇒ D ∈ # clauses S
    ⇒ trail S ⊨as CNot D ⇒ dpllW S (Propagated (- (lit-of L)) () # M, clauses S)

```

18.2 Invariants

```

lemma dpllW-distinct-inv:
  assumes dpllW S S'
  and no-dup (trail S)
  shows no-dup (trail S')
  using assms
proof (induct rule: dpllW.induct)
case (decided L S)
  then show ?case using defined-lit-map by force

```

```

next
  case (propagate C L S)
  then show ?case using defined-lit-map by force
next
  case (backtrack S M' L M D) note extracted = this(1) and no-dup = this(5)
  show ?case
    using no-dup backtrack-split-list-eq[of trail S, symmetric] unfolding extracted by auto
qed

lemma dpllW-consistent-interp-inv:
  assumes dpllW S S'
  and consistent-interp (lits-of-l (trail S))
  and no-dup (trail S)
  shows consistent-interp (lits-of-l (trail S'))
  using assms
proof (induct rule: dpllW.induct)
  case (backtrack S M' L M D) note extracted = this(1) and marked = this(2) and D = this(4) and
    cons = this(5) and no-dup = this(6)
  have no-dup': no-dup M
    by (metis (no-types) backtrack-split-list-eq distinct.simps(2) distinct-append extracted
      list.simps(9) map-append no-dup snd-conv)
  then have insert (lit-of L) (lits-of-l M)  $\subseteq$  lits-of-l (trail S)
    using backtrack-split-list-eq[of trail S, symmetric] unfolding extracted by auto
  then have cons: consistent-interp (insert (lit-of L) (lits-of-l M))
    using consistent-interp-subset cons by blast
  moreover
    have lit-of L  $\notin$  lits-of-l M
      using no-dup backtrack-split-list-eq[of trail S, symmetric] extracted
      unfolding lits-of-def by force
  moreover
    have atm-of ( $\neg$ lit-of L)  $\notin$  ( $\lambda m.$  atm-of (lit-of m)) ' set M
      using no-dup backtrack-split-list-eq[of trail S, symmetric] unfolding extracted by force
    then have  $\neg$ lit-of L  $\notin$  lits-of-l M
      unfolding lits-of-def by force
  ultimately show ?case by simp
qed (auto intro: consistent-add-undefined-lit-consistent)

lemma dpllW-vars-in-snd-inv:
  assumes dpllW S S'
  and atm-of ' (lits-of-l (trail S))  $\subseteq$  atms-of-mm (clauses S)
  shows atm-of ' (lits-of-l (trail S'))  $\subseteq$  atms-of-mm (clauses S')
  using assms
proof (induct rule: dpllW.induct)
  case (backtrack S M' L M D)
  then have atm-of (lit-of L)  $\in$  atms-of-mm (clauses S)
    using backtrack-split-list-eq[of trail S, symmetric] by auto
  moreover
    have atm-of ' lits-of-l (trail S)  $\subseteq$  atms-of-mm (clauses S)
      using backtrack(5) by simp
    then have  $\bigwedge xb. xb \in \text{set } M \implies \text{atm-of (lit-of } xb) \in \text{atms-of-mm (clauses S)}$ 
      using backtrack-split-list-eq[symmetric, of trail S] backtrack.hyps(1)
      unfolding lits-of-def by auto
  ultimately show ?case by (auto simp : lits-of-def)
qed (auto simp: in-plus-implies-atm-of-on-atms-of-ms)

```

lemma *atms-of-ms-lit-of-atms-of*: *atms-of-ms* $((\lambda a. \{\#lit\text{-of } a\# \}) \text{ ' } c) = \textit{atm-of ' lit-of ' } c$
unfolding *atms-of-ms-def* **using** *image-iff* **by** *force*

Lemma theorem 2.8.2 page 71 of CW

lemma *dpll_W-propagate-is-conclusion*:

assumes *dpll_W* *S S'*
and *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
and *atm-of ' lits-of-l* (*trail S*) \subseteq *atms-of-mm* (*clauses S*)
shows *all-decomposition-implies-m* (*clauses S'*) (*get-all-marked-decomposition* (*trail S'*))
using *assms*
proof (*induct rule: dpll_W.induct*)
case (*decided L S*)
then show *?case* **unfolding** *all-decomposition-implies-def* **by** *simp*
next
case (*propagate C L S*) **note** *inS = this(1)* **and** *cnot = this(2)* **and** *IH = this(4)* **and** *undef = this(3)* **and** *atms-incl = this(5)*
let *?I = set (map ($\lambda a. \{\#lit\text{-of } a\# \})$ (*trail S*)) \cup *set-mset* (*clauses S*)
have *?I* $\models_p C + \{\#L\# \}$ **by** (*auto simp add: inS*)
moreover have *?I* $\models_{ps} CNot\ C$ **using** *true-annots-true-clss-cl* *cnot* **by** *fastforce*
ultimately have *?I* $\models_p \{\#L\# \}$ **using** *true-clss-cl* *plus-CNot* [*of ?I C L*] *inS* **by** *blast*
{
assume *get-all-marked-decomposition* (*trail S*) = []
then have *?case* **by** *blast*
}
moreover {
assume *n: get-all-marked-decomposition* (*trail S*) $\neq []$
have *1: $\bigwedge a\ b. (a, b) \in \text{set} (\text{tl} (\text{get-all-marked-decomposition} (\text{trail } S)))$*
\implies (*unmark-l a* \cup *set-mset* (*clauses S*)) \models_{ps} *unmark-l b*
****using** *IH* **unfolding** *all-decomposition-implies-def* **by** (*fastforce simp add: list.set-sel(2) n*)**
moreover have *2: $\bigwedge a\ c. \text{hd} (\text{get-all-marked-decomposition} (\text{trail } S)) = (a, c)$*
\implies (*unmark-l a* \cup *set-mset* (*clauses S*)) \models_{ps} (*unmark-l c*)
****by** (*metis IH all-decomposition-implies-cons-pair all-decomposition-implies-single list.collapse n*)**
moreover have *3: $\bigwedge a\ c. \text{hd} (\text{get-all-marked-decomposition} (\text{trail } S)) = (a, c)$*
\implies (*unmark-l a* \cup *set-mset* (*clauses S*)) $\models_p \{\#L\# \}$
proof –
fix *a c*
assume *h: hd* (*get-all-marked-decomposition* (*trail S*)) = (*a, c*)
have *h': trail S = c @ a* **using** *get-all-marked-decomposition-decomp* *h* **by** *blast*
have *I: set (map ($\lambda a. \{\#lit\text{-of } a\# \})$ *a*) \cup *set-mset* (*clauses S*)
\cup *unmark-l c* $\models_{ps} CNot\ C$
****using** $\langle ?I \models_{ps} CNot\ C \rangle$ **unfolding** *h'* **by** (*simp add: Un-commute Un-left-commute*)**
have
atms-of-ms* (*CNot C*) \subseteq *atms-of-ms* (*set (map ($\lambda a. \{\#lit\text{-of } a\# \})$ *a*) \cup *set-mset* (*clauses S*))
****and****
***atms-of-ms* (*unmark-l c*) \subseteq *atms-of-ms* (*set (map ($\lambda a. \{\#lit\text{-of } a\# \})$ *a*)*
\cup *set-mset* (*clauses S*))
****apply** (*metis CNot-plus Un-subset-iff atms-of-atms-of-ms-mono atms-of-ms-CNot-atms-of***
***atms-of-ms-union inS sup.coboundedI2*)**
****using** *inS atms-of-atms-of-ms-mono atms-incl* **by** (*fastforce simp: h'*)**
then have *unmark-l a* \cup *set-mset* (*clauses S*) $\models_{ps} CNot\ C$
****using** *true-clss-clss-left-right* [*OF - I*] *h 2* **by** *auto***
then show *unmark-l a* \cup *set-mset* (*clauses S*) $\models_p \{\#L\# \}$
****by** (*metis (no-types) Un-insert-right inS insertI1 mk-disjoint-insert inS*)******

```

      true-clss-cll-in true-clss-cll-plus-CNot)
    qed
  ultimately have ?case
    by (cases hd (get-all-marked-decomposition (trail S)))
      (auto simp: all-decomposition-implies-def)
  }
  ultimately show ?case by auto
next
case (backtrack S M' L M D) note extracted = this(1) and marked = this(2) and D = this(3) and
  cnot = this(4) and cons = this(4) and IH = this(5) and atms-incl = this(6)
have S: trail S = M' @ L # M
  using backtrack-split-list-eq[of trail S] unfolding extracted by auto
have M':  $\forall l \in \text{set } M'. \neg \text{is-marked } l$ 
  using extracted backtrack-split-fst-not-marked[of - trail S] by simp
have n: get-all-marked-decomposition (trail S)  $\neq []$  by auto
then have all-decomposition-implies-m (clauses S) ((L # M, M')
  # tl (get-all-marked-decomposition (trail S)))
  by (metis (no-types) IH extracted get-all-marked-decomposition-backtrack-split list.exhaust-sel)
then have 1: unmark-l (L # M)  $\cup$  set-mset (clauses S)  $\models_{ps} (\lambda a. \{\# \text{lit-of } a\# \})$  ' set M'
  by simp
moreover
have unmark-l (L # M)  $\cup$  unmark-l M'  $\models_{ps}$  CNot D
  by (metis (mono-tags, lifting) S Un-commute cons image-Un set-append
    true-annots-true-clss-clss)
then have 2: unmark-l (L # M)  $\cup$  set-mset (clauses S)  $\cup$  unmark-l M'
   $\models_{ps}$  CNot D
  by (metis (no-types, lifting) Un-assoc Un-left-commute true-clss-clss-union-l-r)
ultimately
have set (map ( $\lambda a. \{\# \text{lit-of } a\# \}$ ) (L # M))  $\cup$  set-mset (clauses S)  $\models_{ps}$  CNot D
  using true-clss-clss-left-right by fastforce
then have set (map ( $\lambda a. \{\# \text{lit-of } a\# \}$ ) (L # M))  $\cup$  set-mset (clauses S)  $\models_p \{\#\}$ 
  by (metis (mono-tags, lifting) D Un-def mem-Collect-eq
    true-clss-clss-contradiction-true-clss-cll-false)
then have IL: unmark-l M  $\cup$  set-mset (clauses S)  $\models_p \{\# - \text{lit-of } L\# \}$ 
  using true-clss-clss-false-left-right by auto
show ?case unfolding S all-decomposition-implies-def
proof
  fix x P level
  assume x:  $x \in \text{set } (get-all-marked-decomposition$ 
    (fst (Propagated ( $- \text{lit-of } L$ ) P # M, clauses S)))
  let ?M' = Propagated ( $- \text{lit-of } L$ ) P # M
  let ?hd = hd (get-all-marked-decomposition ?M')
  let ?tl = tl (get-all-marked-decomposition ?M')
  have x = ?hd  $\vee x \in \text{set } ?tl$ 
    using x
    by (cases get-all-marked-decomposition ?M')
      auto
  moreover {
    assume x':  $x \in \text{set } ?tl$ 
    have L': Marked (lit-of L) () = L using marked by (cases L, auto)
    have x  $\in \text{set } (get-all-marked-decomposition (M' @ L # M))$ 
      using x' get-all-marked-decomposition-except-last-choice-equal[of M' lit-of L P M]
      L' by (metis (no-types) M' list.set-sel(2) tl-Nil)
    then have case x of (Ls, seen)  $\Rightarrow$  unmark-l Ls  $\cup$  set-mset (clauses S)
       $\models_{ps}$  unmark-l seen
  }

```



```

    using marked IH by (cases L) (auto simp add: S all-decomposition-implies-def)
  }
  moreover {
    assume x': x = ?hd
    have tl: tl (get-all-marked-decomposition (M' @ L # M)) ≠ []
    proof -
      have f1:  $\bigwedge ms.$  length (get-all-marked-decomposition (M' @ ms))
        = length (get-all-marked-decomposition ms)
      by (simp add: M' get-all-marked-decomposition-remove-unmark-ssed-length)
      have Suc (length (get-all-marked-decomposition M)) ≠ Suc 0
      by blast
      then show ?thesis
      using f1 marked by (metis (no-types) get-all-marked-decomposition.simps(1) length-tl
        list.sel(3) list.size(3) marked-lit.collapse(1))
    qed
    obtain M0' M0 where
      L0: hd (tl (get-all-marked-decomposition (M' @ L # M))) = (M0, M0')
      by (cases hd (tl (get-all-marked-decomposition (M' @ L # M))))
    have x'': x = (M0, Propagated (−lit-of L) P # M0')
      unfolding x' using get-all-marked-decomposition-last-choice tl M' L0
      by (metis marked marked-lit.collapse(1))
    obtain l-get-all-marked-decomposition where
      get-all-marked-decomposition (trail S) = (L # M, M') # (M0, M0') #
      l-get-all-marked-decomposition
      using get-all-marked-decomposition-backtrack-split extracted by (metis (no-types) L0 S
        hd-Cons-tl n tl)
    then have M = M0' @ M0 using get-all-marked-decomposition-hd-hd by fastforce
    then have IL': unmark-l M0 ∪ set-mset (clauses S)
      ∪ unmark-l M0' ⊢ps { {# − lit-of L # } }
      using IL by (simp add: Un-commute Un-left-commute image-Un)
    moreover have H: unmark-l M0 ∪ set-mset (clauses S)
      ⊢ps unmark-l M0'
      using IH x'' unfolding all-decomposition-implies-def by (metis (no-types, lifting) L0 S
        list.set-sel(1) list.set-sel(2) old.prod.case tl tl-Nil)
    ultimately have case x of (Ls, seen) ⇒ unmark-l Ls ∪ set-mset (clauses S)
      ⊢ps unmark-l seen
      using true-clss-clss-left-right unfolding x'' by auto
  }
  ultimately show case x of (Ls, seen) ⇒
    unmark-l Ls ∪ set-mset (snd (?M', clauses S))
    ⊢ps unmark-l seen
    unfolding snd-conv by blast
  qed
qed

```

Lemma theorem 2.8.3 page 72 of CW

theorem *dpll_W-propagate-is-conclusion-of-decided:*

assumes *dpll_W S S'*
and *all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S))*
and *atm-of ' lits-of-l (trail S) ⊆ atms-of-mm (clauses S)*
shows *set-mset (clauses S') ∪ { {# lit-of L # } | L. is-marked L ∧ L ∈ set (trail S') }*
⊢_{ps} (λa. {# lit-of a # } ' ⋃ (set ' snd ' set (get-all-marked-decomposition (trail S'))))
using *all-decomposition-implies-trail-is-implied[OF dpll_W-propagate-is-conclusion[OF assms]]* .

Lemma theorem 2.8.4 page 72 of CW

lemma *only-propagated-vars-unsat*:
assumes *marked*: $\forall x \in \text{set } M. \neg \text{is-marked } x$
and *DN*: $D \in N$ **and** $D: M \models_{\text{as}} \text{CNot } D$
and *inv*: *all-decomposition-implies* N (*get-all-marked-decomposition* M)
and *atm-incl*: *atm-of* ‘*lits-of-l* $M \subseteq \text{atms-of-ms } N$
shows *unsatisfiable* N
proof (*rule ccontr*)
assume $\neg \text{unsatisfiable } N$
then obtain I **where**
 $I: I \models_s N$ **and**
cons: *consistent-interp* I **and**
tot: *total-over-m* $I N$
unfolding *satisfiable-def* **by** *auto*
then have $I-D: I \models D$
using *DN* **unfolding** *true-clss-def* **by** *auto*

have $l0: \{\{\# \text{lit-of } L\# \} \mid L. \text{is-marked } L \wedge L \in \text{set } M\} = \{\}$ **using** *marked* **by** *auto*
have *atms-of-ms* $(N \cup \text{unmark-l } M) = \text{atms-of-ms } N$
using *atm-incl* **unfolding** *atms-of-ms-def lits-of-def* **by** *auto*

then have *total-over-m* $I (N \cup (\lambda a. \{\# \text{lit-of } a\# \}) \text{ ‘ } (\text{set } M))$
using *tot* **unfolding** *total-over-m-def* **by** *auto*
then have $I \models_s (\lambda a. \{\# \text{lit-of } a\# \}) \text{ ‘ } (\text{set } M)$
using *all-decomposition-implies-propagated-lits-are-implied*[*OF inv*] *cons I*
unfolding *true-clss-clss-def l0* **by** *auto*
then have $IM: I \models_s \text{unmark-l } M$ **by** *auto*
{
fix K
assume $K \in \# D$
then have $-K \in \text{lits-of-l } M$
by (*auto split: if-split-asm*
intro: allE[*OF D*[*unfolded true-annots-def Ball-def*], *of* $\{\# -K\# \}$])
then have $-K \in I$ **using** IM *true-clss-singleton-lit-of-implies-incl* **by** *fastforce*
}
then have $\neg I \models D$ **using** *cons* **unfolding** *true-clss-def consistent-interp-def* **by** *auto*
then show *False* **using** $I-D$ **by** *blast*
qed

lemma *dpll_W-same-clauses*:
assumes *dpll_W* $S S'$
shows *clauses* $S = \text{clauses } S'$
using *assms* **by** (*induct rule: dpll_W.induct, auto*)

lemma *rtranclp-dpll_W-inv*:
assumes *rtranclp dpll_W* $S S'$
and *inv*: *all-decomposition-implies-m* (*clauses* S) (*get-all-marked-decomposition* (*trail* S))
and *atm-incl*: *atm-of* ‘*lits-of-l* (*trail* S) $\subseteq \text{atms-of-mm}$ (*clauses* S)
and *consistent-interp* (*lits-of-l* (*trail* S))
and *no-dup* (*trail* S)
shows *all-decomposition-implies-m* (*clauses* S') (*get-all-marked-decomposition* (*trail* S'))
and *atm-of* ‘*lits-of-l* (*trail* S') $\subseteq \text{atms-of-mm}$ (*clauses* S')
and *clauses* $S = \text{clauses } S'$
and *consistent-interp* (*lits-of-l* (*trail* S'))
and *no-dup* (*trail* S')
using *assms*

proof (*induct rule: rtrancpl-induct*)

case *base*

show

all-decomposition-implies-m (*clauses S*) (*get-all-marked-decomposition* (*trail S*)) **and**

atm-of ‘ *lits-of-l* (*trail S*) \subseteq *atms-of-mm* (*clauses S*) **and**

clauses S = *clauses S* **and**

consistent-interp (*lits-of-l* (*trail S*)) **and**

no-dup (*trail S*) **using** *assms* **by** *auto*

next

case (*step S' S''*) **note** *dpll_WStar* = *this*(1) **and** *IH* = *this*(3,4,5,6,7) **and**

dpll_W = *this*(2)

moreover

assume

inv: *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*)) **and**

atm-incl: *atm-of* ‘ *lits-of-l* (*trail S*) \subseteq *atms-of-mm* (*clauses S*) **and**

cons: *consistent-interp* (*lits-of-l* (*trail S*)) **and**

no-dup (*trail S*)

ultimately have *decomp*: *all-decomposition-implies-m* (*clauses S'*)

(*get-all-marked-decomposition* (*trail S'*)) **and**

atm-incl': *atm-of* ‘ *lits-of-l* (*trail S'*) \subseteq *atms-of-mm* (*clauses S'*) **and**

snd: *clauses S* = *clauses S'* **and**

cons': *consistent-interp* (*lits-of-l* (*trail S'*)) **and**

no-dup': *no-dup* (*trail S'*) **by** *blast+*

show *clauses S* = *clauses S''* **using** *dpll_W-same-clauses*[*OF dpll_W*] *snd* **by** *metis*

show *all-decomposition-implies-m* (*clauses S''*) (*get-all-marked-decomposition* (*trail S''*))

using *dpll_W-propagate-is-conclusion*[*OF dpll_W*] *decomp atm-incl'* **by** *auto*

show *atm-of* ‘ *lits-of-l* (*trail S''*) \subseteq *atms-of-mm* (*clauses S''*)

using *dpll_W-vars-in-snd-inv*[*OF dpll_W*] *atm-incl atm-incl'* **by** *auto*

show *no-dup* (*trail S''*) **using** *dpll_W-distinct-inv*[*OF dpll_W*] *no-dup' dpll_W* **by** *auto*

show *consistent-interp* (*lits-of-l* (*trail S''*))

using *cons' no-dup' dpll_W-consistent-interp-inv*[*OF dpll_W*] **by** *auto*

qed

definition *dpll_W-all-inv S* \equiv

(*all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*)))

\wedge *atm-of* ‘ *lits-of-l* (*trail S*) \subseteq *atms-of-mm* (*clauses S*)

\wedge *consistent-interp* (*lits-of-l* (*trail S*))

\wedge *no-dup* (*trail S*))

lemma *dpll_W-all-inv-dest*[*dest*]:

assumes *dpll_W-all-inv S*

shows *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*))

and *atm-of* ‘ *lits-of-l* (*trail S*) \subseteq *atms-of-mm* (*clauses S*)

and *consistent-interp* (*lits-of-l* (*trail S*)) \wedge *no-dup* (*trail S*)

using *assms* **unfolding** *dpll_W-all-inv-def lits-of-def* **by** *auto*

lemma *rtrancpl-dpll_W-all-inv*:

assumes *rtrancpl dpll_W S S'*

and *dpll_W-all-inv S*

shows *dpll_W-all-inv S'*

using *assms rtrancpl-dpll_W-inv*[*OF assms*(1)] **unfolding** *dpll_W-all-inv-def lits-of-def* **by** *blast*

lemma *dpll_W-all-inv*:

assumes *dpll_W S S'*

and $dpll_W\text{-all-inv } S$
 shows $dpll_W\text{-all-inv } S'$
 using *assms rtrancpl-dpll_W-all-inv* by *blast*

lemma *rtrancpl-dpll_W-inv-starting-from-0*:

assumes *rtrancpl dpll_W S S'*

and *inv: trail S = []*

shows $dpll_W\text{-all-inv } S'$

proof –

have $dpll_W\text{-all-inv } S$

using *assms unfolding all-decomposition-implies-def dpll_W-all-inv-def* by *auto*

then show *?thesis* using *rtrancpl-dpll_W-all-inv[OF assms(1)]* by *blast*

qed

lemma *dpll_W-can-do-step*:

assumes *consistent-interp (set M)*

and *distinct M*

and *atm-of ' (set M) ⊆ atms-of-mm N*

shows *rtrancpl dpll_W ([], N) (map (λM. Marked M ()) M, N)*

using *assms*

proof (*induct M*)

case *Nil*

then show *?case* by *auto*

next

case (*Cons L M*)

then have *undefined-lit (map (λM. Marked M ()) M) L*

unfolding *defined-lit-def consistent-interp-def* by *auto*

moreover have *atm-of L ∈ atms-of-mm N* using *Cons.prem(3)* by *auto*

ultimately have *dpll_W (map (λM. Marked M ()) M, N) (map (λM. Marked M ()) (L # M), N)*

using *dpll_W.decided* by *auto*

moreover have *consistent-interp (set M)* and *distinct M* and *atm-of ' set M ⊆ atms-of-mm N*

using *Cons.prem* *unfolding consistent-interp-def* by *auto*

ultimately show *?case* using *Cons.hyps* by *auto*

qed

definition *conclusive-dpll_W-state (S:: 'v dpll_W-state) ⟷*

(trail S ⊨_{asm} clauses S ∨ ((∀ L ∈ set (trail S). ¬is-marked L)

∧ (∃ C ∈# clauses S. trail S ⊨_{as} CNot C)))

lemma *dpll_W-strong-completeness*:

assumes *set M ⊨_{sm} N*

and *consistent-interp (set M)*

and *distinct M*

and *atm-of ' (set M) ⊆ atms-of-mm N*

shows *dpll_W** ([], N) (map (λM. Marked M ()) M, N)*

and *conclusive-dpll_W-state (map (λM. Marked M ()) M, N)*

proof –

show *rtrancpl dpll_W ([], N) (map (λM. Marked M ()) M, N)* using *dpll_W-can-do-step assms* by *auto*

have *map (λM. Marked M ()) M ⊨_{asm} N* using *assms(1) true-annots-marked-true-cl* by *auto*

then show *conclusive-dpll_W-state (map (λM. Marked M ()) M, N)*

unfolding *conclusive-dpll_W-state-def* by *auto*

qed

```

lemma dpllW-sound:
  assumes
    rtrancpl dpllW ([], N) (M, N) and
     $\forall S. \neg dpll_W (M, N) S$ 
  shows  $M \models_{asm} N \longleftrightarrow \text{satisfiable } (set\text{-mset } N) \text{ (is } ?A \longleftrightarrow ?B)$ 
proof
  let  $?M' = \text{lits-of-}l\ M$ 
  assume  $?A$ 
  then have  $?M' \models_{sm} N$  by (simp add: true-annots-true-cls)
  moreover have consistent-interp  $?M'$ 
    using rtrancpl-dpllW-inv-starting-from-0[OF assms(1)] by auto
  ultimately show  $?B$  by auto
next
  assume  $?B$ 
  show  $?A$ 
  proof (rule ccontr)
    assume  $n: \neg ?A$ 
    have  $(\exists L. \text{undefined-lit } M\ L \wedge \text{atm-of } L \in \text{atms-of-mm } N) \vee (\exists D \in \#N. M \models_{as} CNot\ D)$ 
    proof -
      obtain  $D :: 'a\ \text{clause}$  where  $D: D \in \# N$  and  $\neg M \models_a D$ 
      using  $n$  unfolding true-annots-def Ball-def by auto
      then have  $(\exists L. \text{undefined-lit } M\ L \wedge \text{atm-of } L \in \text{atms-of } D) \vee M \models_{as} CNot\ D$ 
      unfolding true-annots-def Ball-def CNot-def true-annot-def
      using atm-of-lit-in-atms-of true-annot-iff-marked-or-true-lit true-cls-def by blast
      then show ?thesis
        by (metis Bex-def D atms-of-atms-of-ms-mono rev-subsetD)
    qed
    moreover {
      assume  $\exists L. \text{undefined-lit } M\ L \wedge \text{atm-of } L \in \text{atms-of-mm } N$ 
      then have False using assms(2) decided by fastforce
    }
    moreover {
      assume  $\exists D \in \#N. M \models_{as} CNot\ D$ 
      then obtain  $D$  where  $DN: D \in \# N$  and  $MD: M \models_{as} CNot\ D$  by auto
      {
        assume  $\forall l \in set\ M. \neg \text{is-marked } l$ 
        moreover have dpllW-all-inv  $([], N)$ 
          using assms unfolding all-decomposition-implies-def dpllW-all-inv-def by auto
        ultimately have unsatisfiable  $(set\text{-mset } N)$ 
          using only-propagated-vars-unsat[of M D set-mset N]  $DN\ MD$ 
          rtrancpl-dpllW-all-inv[OF assms(1)] by force
        then have False using  $\langle ?B \rangle$  by blast
      }
    }
    moreover {
      assume  $l: \exists l \in set\ M. \text{is-marked } l$ 
      then have False
        using backtrack[of (M, N) - - - D]  $DN\ MD\ assms(2)$ 
        backtrack-split-some-is-marked-then-snd-has-hd[OF l]
        by (metis backtrack-split-snd-hd-marked fst-conv list.distinct(1) list.sel(1) snd-conv)
    }
    ultimately have False by blast
  }
  ultimately show False by blast
qed

```

18.3 Termination

definition $dpll_W\text{-mes } M n =$

$\text{map } (\lambda l. \text{ if is-marked } l \text{ then } 2 \text{ else } (1::\text{nat})) (\text{rev } M) @ \text{replicate } (n - \text{length } M) \ 3$

lemma $\text{length-dpll}_W\text{-mes}:$

assumes $\text{length } M \leq n$

shows $\text{length } (dpll_W\text{-mes } M n) = n$

using *assms unfolding dpll_W-mes-def by auto*

lemma $\text{distinctcard-atm-of-lit-of-eq-length}:$

assumes *no-dup S*

shows $\text{card } (\text{atm-of } \text{' lits-of-l } S) = \text{length } S$

using *assms by (induct S) (auto simp add: image-image lits-of-def)*

lemma $dpll_W\text{-card-decrease}:$

assumes *dpll: dpll_W S S' and length (trail S') ≤ card vars*

and $\text{length } (\text{trail } S) \leq \text{card vars}$

shows $(dpll_W\text{-mes } (\text{trail } S') (\text{card vars}), dpll_W\text{-mes } (\text{trail } S) (\text{card vars}))$

$\in \text{lexn } \{(a, b). a < b\} (\text{card vars})$

using *assms*

proof (*induct rule: dpll_W.induct*)

case (*propagate C L S*)

have *m*: $\text{map } (\lambda l. \text{ if is-marked } l \text{ then } 2 \text{ else } 1) (\text{rev } (\text{trail } S))$

$@ \text{replicate } (\text{card vars} - \text{length } (\text{trail } S)) \ 3$

$= \text{map } (\lambda l. \text{ if is-marked } l \text{ then } 2 \text{ else } 1) (\text{rev } (\text{trail } S)) @ 3$

$\# \text{replicate } (\text{card vars} - \text{Suc } (\text{length } (\text{trail } S))) \ 3$

using *propagate.prem[simplified] using Suc-diff-le by fastforce*

then show *?case*

using *propagate.prem[s(1) unfolding dpll_W-mes-def by (fastforce simp add: lexn-conv assms(2))*

next

case (*decided S L*)

have *m*: $\text{map } (\lambda l. \text{ if is-marked } l \text{ then } 2 \text{ else } 1) (\text{rev } (\text{trail } S))$

$@ \text{replicate } (\text{card vars} - \text{length } (\text{trail } S)) \ 3$

$= \text{map } (\lambda l. \text{ if is-marked } l \text{ then } 2 \text{ else } 1) (\text{rev } (\text{trail } S)) @ 3$

$\# \text{replicate } (\text{card vars} - \text{Suc } (\text{length } (\text{trail } S))) \ 3$

using *decided.prem[simplified] using Suc-diff-le by fastforce*

then show *?case*

using *decided.prem[s] unfolding dpll_W-mes-def by (force simp add: lexn-conv assms(2))*

next

case (*backtrack S M' L M D*)

have *L*: *is-marked L* **using** *backtrack.hyps(2) by auto*

have *S*: $\text{trail } S = M' @ L \# M$

using *backtrack.hyps(1) backtrack-split-list-eq[of trail S] by auto*

show *?case*

using *backtrack.prem[s L unfolding dpll_W-mes-def S by (fastforce simp add: lexn-conv assms(2))*

qed

Proposition theorem 2.8.7 page 73 of CW

lemma $dpll_W\text{-card-decrease}':$

assumes *dpll: dpll_W S S'*

and *atm-incl: atm-of ' lits-of-l (trail S) ⊆ atms-of-mm (clauses S)*

and *no-dup: no-dup (trail S)*

shows $(dpll_W\text{-mes } (\text{trail } S') (\text{card } (\text{atms-of-mm } (\text{clauses } S'))),$

$dpll_W\text{-mes } (\text{trail } S) (\text{card } (\text{atms-of-mm } (\text{clauses } S)))) \in \text{lex } \{(a, b). a < b\}$

proof —

```

have finite (atms-of-mm (clauses S)) unfolding atms-of-ms-def by auto
then have 1: length (trail S) ≤ card (atms-of-mm (clauses S))
  using distinctcard-atm-of-lit-of-eq-length[OF no-dup] atm-incl card-mono by metis

moreover
  have no-dup': no-dup (trail S') using dpll dpllW-distinct-inv no-dup by blast
  have SS': clauses S' = clauses S using dpll by (auto dest!: dpllW-same-clauses)
  have atm-incl': atm-of ' lits-of-l (trail S') ⊆ atms-of-mm (clauses S')
    using atm-incl dpll dpllW-vars-in-snd-inv[OF dpll] by force
  have finite (atms-of-mm (clauses S'))
    unfolding atms-of-ms-def by auto
  then have 2: length (trail S') ≤ card (atms-of-mm (clauses S'))
    using distinctcard-atm-of-lit-of-eq-length[OF no-dup'] atm-incl' card-mono SS' by metis

ultimately have (dpllW-mes (trail S') (card (atms-of-mm (clauses S'))),
  dpllW-mes (trail S) (card (atms-of-mm (clauses S'))))
  ∈ lexn {(a, b). a < b} (card (atms-of-mm (clauses S)))
  using dpllW-card-decrease[OF assms(1), of atms-of-mm (clauses S)] by blast
then have (dpllW-mes (trail S') (card (atms-of-mm (clauses S'))),
  dpllW-mes (trail S) (card (atms-of-mm (clauses S')))) ∈ lex {(a, b). a < b}
  unfolding lex-def by auto
then show (dpllW-mes (trail S') (card (atms-of-mm (clauses S'))),
  dpllW-mes (trail S) (card (atms-of-mm (clauses S')))) ∈ lex {(a, b). a < b}
  using dpllW-same-clauses[OF assms(1)] by auto
qed

lemma wf-lexn: wf (lexn {(a, b). (a::nat) < b} (card (atms-of-mm (clauses S))))
proof -
  have m: {(a, b). a < b} = measure id by auto
  show ?thesis apply (rule wf-lexn) unfolding m by auto
qed

lemma dpllW-wf:
wf {(S', S). dpllW-all-inv S ∧ dpllW S S'}
apply (rule wf-wf-if-measure'[OF wf-lex-less, of - -
  λS. dpllW-mes (trail S) (card (atms-of-mm (clauses S)))])
using dpllW-card-decrease' by fast

lemma dpllW-trancpl-star-commute:
{(S', S). dpllW-all-inv S ∧ dpllW S S'}+ = {(S', S). dpllW-all-inv S ∧ trancpl dpllW S S'}
(is ?A = ?B)
proof
  { fix S S'
    assume (S, S') ∈ ?A
    then have (S, S') ∈ ?B
      by (induct rule: trancpl.induct, auto)
  }
then show ?A ⊆ ?B by blast
  { fix S S'
    assume (S, S') ∈ ?B
    then have dpllW++ S' S and dpllW-all-inv S' by auto
    then have (S, S') ∈ ?A
      proof (induct rule: trancpl.induct)
        case r-into-trancpl

```

```

    then show ?case by (simp-all add: r-into-trancl')
  next
    case (trancl-into-trancl S S' S'')
    then have  $(S', S) \in \{a. \text{case } a \text{ of } (S', S) \Rightarrow \text{dpll}_W\text{-all-inv } S \wedge \text{dpll}_W S S'\}^+$  by blast
    moreover have  $\text{dpll}_W\text{-all-inv } S'$ 
      using  $\text{rtranclp-dpll}_W\text{-all-inv}[OF \text{tranclp-into-rtranclp}[OF \text{trancl-into-trancl.hyps}(1)]]$ 
       $\text{trancl-into-trancl.premis}$  by auto
    ultimately have  $(S'', S') \in \{(pa, p). \text{dpll}_W\text{-all-inv } p \wedge \text{dpll}_W p pa\}^+$ 
      using  $\langle \text{dpll}_W\text{-all-inv } S' \rangle \text{trancl-into-trancl.hyps}(3)$  by blast
    then show ?case
      using  $\langle (S', S) \in \{a. \text{case } a \text{ of } (S', S) \Rightarrow \text{dpll}_W\text{-all-inv } S \wedge \text{dpll}_W S S'\}^+ \rangle$  by auto
  qed
}
then show  $?B \subseteq ?A$  by blast
qed

```

lemma $\text{dpll}_W\text{-wf-tranclp}$: $\text{wf } \{(S', S). \text{dpll}_W\text{-all-inv } S \wedge \text{dpll}_W^{++} S S'\}$
unfolding $\text{dpll}_W\text{-tranclp-star-commute[symmetric]}$ **by** (simp add: $\text{dpll}_W\text{-wf wf-trancl}$)

lemma $\text{dpll}_W\text{-wf-plus}$:
shows $\text{wf } \{(S', ([], N)) \mid S'. \text{dpll}_W^{++} ([], N) S'\}$ (is $\text{wf } ?P$)
apply (rule $\text{wf-subset}[OF \text{dpll}_W\text{-wf-tranclp, of } ?P]$)
using *assms* **unfolding** $\text{dpll}_W\text{-all-inv-def}$ **by** auto

18.4 Final States

lemma $\text{dpll}_W\text{-no-more-step-is-a-conclusive-state}$:

assumes $\forall S'. \neg \text{dpll}_W S S'$

shows $\text{conclusive-dpll}_W\text{-state } S$

proof –

have *vars*: $\forall s \in \text{atms-of-mm (clauses } S). s \in \text{atm-of ' lits-of-l (trail } S)$

proof (rule *ccontr*)

assume $\neg (\forall s \in \text{atms-of-mm (clauses } S). s \in \text{atm-of ' lits-of-l (trail } S))$

then obtain L **where**

$L\text{-in-atms}$: $L \in \text{atms-of-mm (clauses } S)$ **and**

$L\text{-notin-trail}$: $L \notin \text{atm-of ' lits-of-l (trail } S)$ **by** *metis*

obtain L' **where** $L': \text{atm-of } L' = L$ **by** (*meson literal.sel(2)*)

then have $\text{undefined-lit (trail } S) L'$

unfolding *Marked-Propagated-in-iff-in-lits-of-l* **by** (*metis L-notin-trail atm-of-uminus imageI*)

then show *False* **using** $\text{dpll}_W.\text{decided assms}(1)$ $L\text{-in-atms } L'$ **by** *blast*

qed

show *?thesis*

proof (rule *ccontr*)

assume *not-final*: $\neg ?thesis$

then have

$\neg \text{trail } S \models_{\text{asm}} \text{clauses } S$ **and**

$(\exists L \in \text{set (trail } S). \text{is-marked } L) \vee (\forall C \in \# \text{clauses } S. \neg \text{trail } S \models_{\text{as}} C \text{Not } C)$

unfolding $\text{conclusive-dpll}_W\text{-state-def}$ **by** *auto*

moreover {

assume $\exists L \in \text{set (trail } S). \text{is-marked } L$

then obtain $L M' M$ **where** $L: \text{backtrack-split (trail } S) = (M', L \# M)$

using *backtrack-split-some-is-marked-then-snd-has-hd* **by** *blast*

obtain D **where** $D \in \# \text{clauses } S$ **and** $\neg \text{trail } S \models_a D$

using $\langle \neg \text{trail } S \models_{\text{asm}} \text{clauses } S \rangle$ **unfolding** *true-annots-def* **by** *auto*

then have $\forall s \in \text{atms-of-ms } \{D\}. s \in \text{atm-of ' lits-of-l (trail } S)$

using *vars* **unfolding** *atms-of-ms-def* **by** *auto*


```

    then have trail S  $\models_{as}$  CNot D
      using all-variables-defined-not-imply-cnot[of D]  $\langle \neg \text{trail } S \models_a D \rangle$  by auto
    moreover have is-marked L
      using L by (metis backtrack-split-snd-hd-marked list.distinct(1) list.sel(1) snd-conv)
    ultimately have False
      using assms(1) dpllW.backtrack L  $\langle D \in \# \text{ clauses } S \rangle \langle \text{trail } S \models_{as} \text{CNot } D \rangle$  by blast
  }
  moreover {
    assume tr:  $\forall C \in \# \text{ clauses } S. \neg \text{trail } S \models_{as} \text{CNot } C$ 
    obtain C where C-in-cl:  $C \in \# \text{ clauses } S$  and trC:  $\neg \text{trail } S \models_a C$ 
      using  $\langle \neg \text{trail } S \models_{asm} \text{ clauses } S \rangle$  unfolding true-annots-def by auto
    have  $\forall s \in \text{atms-of-ms } \{C\}. s \in \text{atm-of ' lits-of-l } (\text{trail } S)$ 
      using vars  $\langle C \in \# \text{ clauses } S \rangle$  unfolding atms-of-ms-def by auto
    then have trail S  $\models_{as}$  CNot C
      by (meson C-in-cl tr trC all-variables-defined-not-imply-cnot)
    then have False using tr C-in-cl by auto
  }
  ultimately show False by blast
qed
qed

lemma dpllW-conclusive-state-correct:
  assumes dpllW** ( $\square$ , N) (M, N) and conclusive-dpllW-state (M, N)
  shows  $M \models_{asm} N \longleftrightarrow \text{satisfiable } (\text{set-mset } N)$  (is ?A  $\longleftrightarrow$  ?B)
proof
  let ?M' = lits-of-l M
  assume ?A
  then have ?M'  $\models_{sm}$  N by (simp add: true-annots-true-cl)
  moreover have consistent-interp ?M'
    using rtrancp-dpllW-inv-starting-from-0[OF assms(1)] by auto
  ultimately show ?B by auto
next
  assume ?B
  show ?A
  proof (rule ccontr)
    assume n:  $\neg ?A$ 
    have no-mark:  $\forall L \in \text{set } M. \neg \text{is-marked } L \exists C \in \# N. M \models_{as} \text{CNot } C$ 
      using n assms(2) unfolding conclusive-dpllW-state-def by auto
    moreover obtain D where DN:  $D \in \# N$  and MD:  $M \models_{as} \text{CNot } D$  using no-mark by auto
    ultimately have unsatisfiable (set-mset N)
      using only-propagated-vars-unsat rtrancp-dpllW-all-inv[OF assms(1)]
      unfolding dpllW-all-inv-def by force
    then show False using  $\langle ?B \rangle$  by blast
  qed
qed
qed

```

18.5 Link with NOT's DPLL

interpretation dpll_W-NOT: dpll-with-backtrack .

```

declare dpllW-NOT.state-simpNOT[simp del]
lemma state-eqNOT-iff-eq[iff, simp]: dpllW-NOT.state-eqNOT S T  $\longleftrightarrow S = T$ 
  unfolding dpllW-NOT.state-eqNOT-def by (cases S, cases T) auto
lemma dpllW-dpllW-bj:
  assumes inv: dpllW-all-inv S and dpll: dpllW S T
  shows dpllW-NOT.dpll-bj S T

```

```

using dpll inv
apply (induction rule: dpllW.induct)
  apply (rule dpllW-NOT.bj-propagateNOT)
  apply (rule dpllW-NOT.propagateNOT.propagateNOT; simp?)
  apply fastforce
  apply (rule dpllW-NOT.bj-decideNOT)
  apply (rule dpllW-NOT.decideNOT.decideNOT; simp?)
  apply fastforce
  apply (frule dpllW-NOT.backtrack.intros[of - - - -], simp-all)
  apply (rule dpllW-NOT.dpll-bj.bj-backjump)
  apply (rule dpllW-NOT.backtrack-is-backjump'',
    simp-all add: dpllW-all-inv-def)
done

lemma dpllW-bj-dpll:
  assumes inv: dpllW-all-inv S and dpll: dpllW-NOT.dpll-bj S T
  shows dpllW S T
  using dpll
  apply (induction rule: dpllW-NOT.dpll-bj.induct)
    apply (elim dpllW-NOT.decideNOTE, cases S)
    apply (frule decided; simp)

    apply (elim dpllW-NOT.propagateNOTE, cases S)
    apply (auto intro!: propagate[of - - (-, -), simplified])[]
  apply (elim dpllW-NOT.backjumpWE, cases S)
  by (simp add: dpllW.simps dpll-with-backtrack.backtrack.simps)

lemma rtrancpl-dpllW-rtrancpl-dpllW-NOT:
  assumes dpllW** S T and dpllW-all-inv S
  shows dpllW-NOT.dpll-bj** S T
  using assms apply (induction)
  apply simp
  by (auto intro: rtrancpl-dpllW-all-inv dpllW-dpllW-bj rtrancpl.rtrancpl-into-rtrancpl)

lemma rtrancpl-dpll-rtrancpl-dpllW:
  assumes dpllW-NOT.dpll-bj** S T and dpllW-all-inv S
  shows dpllW** S T
  using assms apply (induction)
  apply simp
  by (auto intro: dpllW-bj-dpll rtrancpl.rtrancpl-into-rtrancpl rtrancpl-dpllW-all-inv)

lemma dpll-conclusive-state-correctness:
  assumes dpllW-NOT.dpll-bj** ([], N) (M, N) and conclusive-dpllW-state (M, N)
  shows M ⊨asm N ⟷ satisfiable (set-mset N)
proof -
  have dpllW-all-inv ([], N)
    unfolding dpllW-all-inv-def by auto
  show ?thesis
    apply (rule dpllW-conclusive-state-correct)
      apply (simp add: ⟨dpllW-all-inv ([], N)⟩ assms(1) rtrancpl-dpll-rtrancpl-dpllW)
      using assms(2) by simp
qed

end
theory CDCL-W-Level

```

```

imports Partial-Annotated-Clausal-Logic
begin

```

18.5.1 Level of literals and clauses

Getting the level of a variable, implies that the list has to be reversed. Here is the function after reversing.

```

fun get-rev-level :: ('v, nat, 'a) marked-lits  $\Rightarrow$  nat  $\Rightarrow$  'v literal  $\Rightarrow$  nat where
  get-rev-level [] - = 0 |
  get-rev-level (Marked l level # Ls) n L =
    (if atm-of l = atm-of L then level else get-rev-level Ls level L) |
  get-rev-level (Propagated l - # Ls) n L =
    (if atm-of l = atm-of L then n else get-rev-level Ls n L)

```

abbreviation get-level $M\ L \equiv$ get-rev-level (rev M) 0 L

lemma get-rev-level-uminus[simp]: get-rev-level $M\ n(-L) =$ get-rev-level $M\ n\ L$
by (induct arbitrary: n rule: get-rev-level.induct) auto

lemma atm-of-notin-get-rev-level-eq-0:
assumes atm-of $L \notin$ atm-of ' lits-of-l M
shows get-rev-level $M\ n\ L = 0$
using assms **by** (induct M arbitrary: n rule: marked-lit-list-induct) auto

lemma get-rev-level-ge-0-atm-of-in:
assumes get-rev-level $M\ n\ L > n$
shows atm-of $L \in$ atm-of ' lits-of-l M
using assms **by** (induct M arbitrary: n rule: marked-lit-list-induct)
 (fastforce simp: atm-of-notin-get-rev-level-eq-0)+

In *get-rev-level* (resp. *get-level*), the beginning (resp. the end) can be skipped if the literal is not in the beginning (resp. the end).

lemma get-rev-level-skip[simp]:
assumes atm-of $L \notin$ atm-of ' lits-of-l M
shows get-rev-level ($M @$ Marked $K\ i \# M'$) $n\ L =$ get-rev-level (Marked $K\ i \# M'$) $i\ L$
using assms **by** (induct M arbitrary: $n\ i$ rule: marked-lit-list-induct) auto

lemma get-rev-level-notin-end[simp]:
assumes atm-of $L \notin$ atm-of ' lits-of-l M'
shows get-rev-level ($M @ M'$) $n\ L =$ get-rev-level $M\ n\ L$
using assms **by** (induct M arbitrary: n rule: marked-lit-list-induct)
 (auto simp: atm-of-notin-get-rev-level-eq-0)

If the literal is at the beginning, then the end can be skipped

lemma get-rev-level-skip-end[simp]:
assumes atm-of $L \in$ atm-of ' lits-of-l M
shows get-rev-level ($M @ M'$) $n\ L =$ get-rev-level $M\ n\ L$
using assms **by** (induct arbitrary: n rule: marked-lit-list-induct) auto

lemma get-level-skip-beginning:
assumes atm-of $L' \neq$ atm-of (lit-of K)
shows get-level ($K \# M$) $L' =$ get-level $M\ L'$
using assms **by** auto

lemma *get-level-skip-beginning-not-marked-rev*:
assumes *atm-of* $L \notin \text{atm-of } \text{'lit-of '}(set\ S)$
and $\forall s \in set\ S. \neg is\text{-marked } s$
shows *get-level* $(M @ rev\ S)\ L = \text{get-level } M\ L$
using *assms* **by** (*induction* S *rule*: *marked-lit-list-induct*) *auto*

lemma *get-level-skip-beginning-not-marked[simp]*:
assumes *atm-of* $L \notin \text{atm-of } \text{'lit-of '}(set\ S)$
and $\forall s \in set\ S. \neg is\text{-marked } s$
shows *get-level* $(M @ S)\ L = \text{get-level } M\ L$
using *get-level-skip-beginning-not-marked-rev*[*of* $L\ rev\ S\ M$] *assms* **by** *auto*

lemma *get-rev-level-skip-beginning-not-marked[simp]*:
assumes *atm-of* $L \notin \text{atm-of } \text{'lit-of '}(set\ S)$
and $\forall s \in set\ S. \neg is\text{-marked } s$
shows *get-rev-level* $(rev\ S @ rev\ M)\ 0\ L = \text{get-level } M\ L$
using *get-level-skip-beginning-not-marked-rev*[*of* $L\ rev\ S\ M$] *assms* **by** *auto*

lemma *get-level-skip-in-all-not-marked*:
fixes $M :: ('a, nat, 'b)\ \text{marked-lit list}$ **and** $L :: 'a\ \text{literal}$
assumes $\forall m \in set\ M. \neg is\text{-marked } m$
and *atm-of* $L \in \text{atm-of } \text{'lit-of '}(set\ M)$
shows *get-rev-level* $M\ n\ L = n$
using *assms* **by** (*induction* M *rule*: *marked-lit-list-induct*) *auto*

lemma *get-level-skip-all-not-marked[simp]*:
fixes M
defines $M' \equiv rev\ M$
assumes $\forall m \in set\ M. \neg is\text{-marked } m$
shows *get-level* $M\ L = 0$
proof –
have $M: M = rev\ M'$
unfolding $M'\text{-def}$ **by** *auto*
show *?thesis*
using *assms* **unfolding** M **by** (*induction* M' *rule*: *marked-lit-list-induct*) *auto*
qed

abbreviation $MMax\ M \equiv Max\ (set\text{-mset } M)$

the $\{\#0::'a\#\}$ is there to ensures that the set is not empty.

definition *get-maximum-level* $:: ('a, nat, 'b)\ \text{marked-lit list} \Rightarrow 'a\ \text{literal multiset} \Rightarrow nat$
where
get-maximum-level $M\ D = MMax\ (\{\#0\#\} + image\text{-mset } (get\text{-level } M)\ D)$

lemma *get-maximum-level-ge-get-level*:
 $L \in \# D \implies \text{get-maximum-level } M\ D \geq \text{get-level } M\ L$
unfolding *get-maximum-level-def* **by** *auto*

lemma *get-maximum-level-empty[simp]*:
 $\text{get-maximum-level } M\ \{\#\} = 0$
unfolding *get-maximum-level-def* **by** *auto*

lemma *get-maximum-level-exists-lit-of-max-level*:
 $D \neq \{\#\} \implies \exists L \in \# D. \text{get-level } M\ L = \text{get-maximum-level } M\ D$
unfolding *get-maximum-level-def*

```

apply (induct D)
apply simp
by (rename-tac D x, case-tac D = {#}) (auto simp add: max-def)

lemma get-maximum-level-empty-list[simp]:
  get-maximum-level [] D = 0
unfolding get-maximum-level-def by (simp add: image-constant-conv)

lemma get-maximum-level-single[simp]:
  get-maximum-level M {#L#} = get-level M L
unfolding get-maximum-level-def by simp

lemma get-maximum-level-plus:
  get-maximum-level M (D + D') = max (get-maximum-level M D) (get-maximum-level M D')
by (induct D) (auto simp add: get-maximum-level-def)

lemma get-maximum-level-exists-lit:
  assumes n: n > 0
  and max: get-maximum-level M D = n
  shows  $\exists L \in \#D. \text{get-level } M L = n$ 
proof –
  have f: finite (insert 0 (( $\lambda L. \text{get-level } M L$ ) ‘ set-mset D)) by auto
  then have n  $\in$  (( $\lambda L. \text{get-level } M L$ ) ‘ set-mset D)
    using n max Max-in[OF f] unfolding get-maximum-level-def by simp
  then show  $\exists L \in \# D. \text{get-level } M L = n$  by auto
qed

lemma get-maximum-level-skip-first[simp]:
  assumes atm-of L  $\notin$  atms-of D
  shows get-maximum-level (Propagated L C # M) D = get-maximum-level M D
  using asms unfolding get-maximum-level-def atms-of-def
    atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
  by (smt atm-of-in-atm-of-set-in-uminus get-level-skip-beginning image-iff marked-lit.sel(2)
    multiset.map-cong0)

lemma get-maximum-level-skip-beginning:
  assumes DH: atms-of D  $\subseteq$  atm-of ‘lits-of-l H
  shows get-maximum-level (c @ Marked Kh i # H) D = get-maximum-level H D
proof –
  have (get-rev-level (rev H @ Marked Kh i # rev c) 0) ‘ set-mset D
    = (get-rev-level (rev H) 0) ‘ set-mset D
    using DH unfolding atms-of-def
    by (metis (no-types, lifting) get-rev-level-skip-end image-cong image-subset-iff set-rev)
  then show ?thesis using DH unfolding get-maximum-level-def by auto
qed

lemma get-maximum-level-D-single-propagated:
  get-maximum-level [Propagated x21 x22] D = 0
proof –
  have A: insert 0 (( $\lambda L. 0$ ) ‘ (set-mset D  $\cap$  {L. atm-of x21 = atm-of L}))
     $\cup$  (( $\lambda L. 0$ ) ‘ (set-mset D  $\cap$  {L. atm-of x21  $\neq$  atm-of L})) = {0}
    by auto
  show ?thesis unfolding get-maximum-level-def by (simp add: A)
qed

```

lemma *get-maximum-level-skip-notin*:
assumes $D: \forall L \in \#D. \text{atm-of } L \in \text{atm-of 'lits-of-l } M$
shows $\text{get-maximum-level } M \ D = \text{get-maximum-level } (\text{Propagated } x21 \ x22 \ \# \ M) \ D$
proof –
have $A: (\text{get-rev-level } (\text{rev } M \ @ \ [\text{Propagated } x21 \ x22]) \ 0) \ ' \ \text{set-mset } D$
 $= (\text{get-rev-level } (\text{rev } M) \ 0) \ ' \ \text{set-mset } D$
using D **by** (*auto intro!*: *image-cong simp add: lits-of-def*)
show ?thesis **unfolding** *get-maximum-level-def* **by** (*auto simp: A*)
qed

lemma *get-maximum-level-skip-un-marked-not-present*:
assumes $\forall L \in \#D. \text{atm-of } L \in \text{atm-of 'lits-of-l } aa$ **and**
 $\forall m \in \text{set } M. \neg \text{is-marked } m$
shows $\text{get-maximum-level } aa \ D = \text{get-maximum-level } (M \ @ \ aa) \ D$
using *assms* **by** (*induction M rule: marked-lit-list-induct*)
(*auto intro!*: *get-maximum-level-skip-notin[of D - @ aa] simp add: image-Un*)

lemma *get-maximum-level-union-mset*:
 $\text{get-maximum-level } M \ (A \ \#\cup \ B) = \text{get-maximum-level } M \ (A + B)$
unfolding *get-maximum-level-def* **by** (*auto simp: image-Un*)

fun *get-maximum-possible-level*:: (*'b*, *nat*, *'c*) *marked-lit list* \Rightarrow *nat* **where**
 $\text{get-maximum-possible-level } [] = 0 \mid$
 $\text{get-maximum-possible-level } (\text{Marked } K \ i \ \# \ l) = \max \ i \ (\text{get-maximum-possible-level } l) \mid$
 $\text{get-maximum-possible-level } (\text{Propagated } - \ - \ \# \ l) = \text{get-maximum-possible-level } l$

lemma *get-maximum-possible-level-append[simp]*:
 $\text{get-maximum-possible-level } (M \ @ \ M')$
 $= \max (\text{get-maximum-possible-level } M) (\text{get-maximum-possible-level } M')$
by (*induct M rule: marked-lit-list-induct*) *auto*

lemma *get-maximum-possible-level-rev[simp]*:
 $\text{get-maximum-possible-level } (\text{rev } M) = \text{get-maximum-possible-level } M$
by (*induct M rule: marked-lit-list-induct*) *auto*

lemma *get-maximum-possible-level-ge-get-rev-level*:
 $\max (\text{get-maximum-possible-level } M) \ i \geq \text{get-rev-level } M \ i \ L$
by (*induct M arbitrary: i rule: marked-lit-list-induct*) (*auto simp add: le-max-iff-disj*)

lemma *get-maximum-possible-level-ge-get-level[simp]*:
 $\text{get-maximum-possible-level } M \geq \text{get-level } M \ L$
using *get-maximum-possible-level-ge-get-rev-level* [*of rev - 0*] **by** *auto*

lemma *get-maximum-possible-level-ge-get-maximum-level[simp]*:
 $\text{get-maximum-possible-level } M \geq \text{get-maximum-level } M \ D$
using *get-maximum-level-exists-lit-of-max-level* **unfolding** *Bex-def*
by (*metis get-maximum-level-empty get-maximum-possible-level-ge-get-level le0*)

fun *get-all-mark-of-propagated* **where**
 $\text{get-all-mark-of-propagated } [] = [] \mid$
 $\text{get-all-mark-of-propagated } (\text{Marked } - \ - \ \# \ L) = \text{get-all-mark-of-propagated } L \mid$
 $\text{get-all-mark-of-propagated } (\text{Propagated } - \ \text{mark} \ \# \ L) = \text{mark} \ \# \ \text{get-all-mark-of-propagated } L$

lemma *get-all-mark-of-propagated-append[simp]*:

get-all-mark-of-propagated ($A @ B$) = *get-all-mark-of-propagated* $A @$ *get-all-mark-of-propagated* B
by (*induct* A *rule*: *marked-lit-list-induct*) *auto*

18.5.2 Properties about the levels

fun *get-all-levels-of-marked* :: ('b, 'a, 'c) *marked-lit list* \Rightarrow 'a *list* **where**
get-all-levels-of-marked [] = [] |
get-all-levels-of-marked (*Marked* l *level* # Ls) = *level* # *get-all-levels-of-marked* Ls |
get-all-levels-of-marked (*Propagated* - - # Ls) = *get-all-levels-of-marked* Ls

lemma *get-all-levels-of-marked-nil-iff-not-is-marked*:
get-all-levels-of-marked xs = [] $\longleftrightarrow (\forall x \in \text{set } xs. \neg \text{is-marked } x)$
using *assms* **by** (*induction* xs *rule*: *marked-lit-list-induct*) *auto*

lemma *get-all-levels-of-marked-cons*:
get-all-levels-of-marked ($a \# b$) =
 (if *is-marked* a then [*level-of* a] else []) @ *get-all-levels-of-marked* b
by (*cases* a) *simp-all*

lemma *get-all-levels-of-marked-append[simp]*:
get-all-levels-of-marked ($a @ b$) = *get-all-levels-of-marked* $a @$ *get-all-levels-of-marked* b
by (*induct* a) (*simp-all* *add*: *get-all-levels-of-marked-cons*)

lemma *in-get-all-levels-of-marked-iff-decomp*:
 $i \in \text{set } (\text{get-all-levels-of-marked } M) \longleftrightarrow (\exists c K c'. M = c @ \text{Marked } K i \# c') \text{ (is } ?A \longleftrightarrow ?B)$

proof

assume $?B$

then show $?A$ **by** *auto*

next

assume $?A$

then show $?B$

apply (*induction* M *rule*: *marked-lit-list-induct*)

apply *auto*[]

apply (*metis* *append-Cons* *append-Nil* *get-all-levels-of-marked.simps*(2) *set-ConsD*)

by (*metis* *append-Cons* *get-all-levels-of-marked.simps*(3))

qed

lemma *get-rev-level-less-max-get-all-levels-of-marked*:
get-rev-level M n $L \leq \text{Max } (\text{set } (n \# \text{get-all-levels-of-marked } M))$
by (*induct* M *arbitrary*: n *rule*: *get-all-levels-of-marked.induct*)
 (*simp-all* *add*: *max.coboundedI2*)

lemma *get-rev-level-ge-min-get-all-levels-of-marked*:
assumes *atm-of* $L \in \text{atm-of ' lits-of-l } M$
shows *get-rev-level* M n $L \geq \text{Min } (\text{set } (n \# \text{get-all-levels-of-marked } M))$
using *assms* **by** (*induct* M *arbitrary*: n *rule*: *get-all-levels-of-marked.induct*)
 (*auto* *simp* *add*: *min-le-iff-disj*)

lemma *get-all-levels-of-marked-rev-eq-rev-get-all-levels-of-marked[simp]*:
get-all-levels-of-marked (*rev* M) = *rev* (*get-all-levels-of-marked* M)
by (*induct* M *rule*: *get-all-levels-of-marked.induct*)
 (*simp-all* *add*: *max.coboundedI2*)

lemma *get-maximum-possible-level-max-get-all-levels-of-marked*:
get-maximum-possible-level $M = \text{Max } (\text{insert } 0 (\text{set } (\text{get-all-levels-of-marked } M)))$
by (*induct* M *rule*: *marked-lit-list-induct*) (*auto* *simp*: *insert-commute*)

lemma *get-rev-level-in-levels-of-marked:*

get-rev-level M n $L \in \{0, n\} \cup \text{set } (\text{get-all-levels-of-marked } M)$
by (*induction* M *arbitrary*: n *rule*: *marked-lit-list-induct*) (*force simp add*: *atm-of-eq-atm-of*) $+$

lemma *get-rev-level-in-atms-in-levels-of-marked:*

atm-of $L \in \text{atm-of } ' (\text{lits-of-l } M) \implies$
get-rev-level M n $L \in \{n\} \cup \text{set } (\text{get-all-levels-of-marked } M)$
by (*induction* M *arbitrary*: n *rule*: *marked-lit-list-induct*) (*auto simp add*: *atm-of-eq-atm-of*)

lemma *get-all-levels-of-marked-no-marked:*

$(\forall l \in \text{set } Ls. \neg \text{is-marked } l) \longleftrightarrow \text{get-all-levels-of-marked } Ls = []$
by (*induction* Ls) (*auto simp add*: *get-all-levels-of-marked-cons*)

lemma *get-level-in-levels-of-marked:*

get-level M $L \in \{0\} \cup \text{set } (\text{get-all-levels-of-marked } M)$
using *get-rev-level-in-levels-of-marked*[*of rev M 0 L*] **by** *auto*

The zero is here to avoid empty-list issues with *last*:

lemma *get-level-get-rev-level-get-all-levels-of-marked:*

assumes *atm-of* $L \notin \text{atm-of } ' (\text{lits-of-l } M)$

shows

get-level $(K @ M)$ $L = \text{get-rev-level } (\text{rev } K) (\text{last } (0 \# \text{get-all-levels-of-marked } (\text{rev } M)))$ L

using *assms*

proof (*induct* M *arbitrary*: K)

case *Nil*

then show *?case* **by** *auto*

next

case $(\text{Cons } a \ M)$

then have $H: \bigwedge K. \text{get-level } (K @ M) \ L$

$= \text{get-rev-level } (\text{rev } K) (\text{last } (0 \# \text{get-all-levels-of-marked } (\text{rev } M)))$ L

by *auto*

have *get-level* $((K @ [a]) @ M) \ L$

$= \text{get-rev-level } (a \# \text{rev } K) (\text{last } (0 \# \text{get-all-levels-of-marked } (\text{rev } M)))$ L

using $H[\text{of } K @ [a]]$ **by** *simp*

then show *?case* **using** *Cons(2)* **by** (*cases* a) *auto*

qed

lemma *get-rev-level-can-skip-correctly-ordered:*

assumes

no-dup M **and**

atm-of $L \notin \text{atm-of } ' (\text{lits-of-l } M)$ **and**

get-all-levels-of-marked $M = \text{rev } [\text{Suc } 0..<\text{Suc } (\text{length } (\text{get-all-levels-of-marked } M))]$

shows *get-rev-level* $(\text{rev } M @ K)$ $0 \ L = \text{get-rev-level } K (\text{length } (\text{get-all-levels-of-marked } M))$ L

using *assms*

proof (*induct* M *arbitrary*: K *rule*: *marked-lit-list-induct*)

case *nil*

then show *?case* **by** *simp*

next

case $(\text{marked } L' \ i \ M \ K)$

then have

i: $i = \text{Suc } (\text{length } (\text{get-all-levels-of-marked } M))$ **and**

get-all-levels-of-marked $M = \text{rev } [\text{Suc } 0..<\text{Suc } (\text{length } (\text{get-all-levels-of-marked } M))]$

by *auto*

then have *get-rev-level* $(\text{rev } M @ (\text{Marked } L' \ i \ \# \ K))$ $0 \ L$


```

    = get-rev-level (Marked L' i # K) (length (get-all-levels-of-marked M)) L
    using marked by auto
  then show ?case using marked unfolding i by auto
next
case (proped L' D M K)
then have get-all-levels-of-marked M = rev [Suc 0.. $\leq$  Suc (length (get-all-levels-of-marked M))]
  by auto
then have get-rev-level (rev M @ (Propagated L' D # K)) 0 L
  = get-rev-level (Propagated L' D # K) (length (get-all-levels-of-marked M)) L
  using proped by auto
then show ?case using proped by auto
qed

lemma get-level-skip-beginning-hd-get-all-levels-of-marked:
  assumes atm-of L  $\notin$  atm-of ' lits-of-l S and get-all-levels-of-marked S  $\neq$  []
  shows get-level (M@ S) L = get-rev-level (rev M) (hd (get-all-levels-of-marked S)) L
  using assms
proof (induction S arbitrary: M rule: marked-lit-list-induct)
  case nil
  then show ?case by (auto simp add: lits-of-def)
next
  case (marked K m) note notin = this(2)
  then show ?case by (auto simp add: lits-of-def)
next
  case (proped L l) note IH = this(1) and L = this(2) and neq = this(3)
  show ?case using IH[of M@[Propagated L l]] L neq by (auto simp add: atm-of-eq-atm-of)
qed

end
theory CDCL-W
imports CDCL-Abstract-Clause-Representation List-More CDCL-W-Level Wellfounded-More

begin

```

19 Weidenbach's CDCL

```

declare upt.simps(2)[simp del]

```

19.1 The State

We will abstract the representation of clause and clauses via two locales. We expect our representation to behave like multiset, but the internal representation can be done using list or whatever other representation.

```

locale stateW-ops =
  raw-clss mset-cl insert-cl remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  +
  raw-ccls-union mset-ccls union-ccls insert-ccls remove-clit
for
  — Clause
  mset-clss :: 'cls  $\Rightarrow$  'v clause and
  insert-clss :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and

```

— Multiset of Clauses

mset-clss :: 'clss \Rightarrow 'v clauses **and**
union-clss :: 'clss \Rightarrow 'clss \Rightarrow 'clss **and**
in-clss :: 'cls \Rightarrow 'clss \Rightarrow bool **and**
insert-clss :: 'cls \Rightarrow 'clss \Rightarrow 'clss **and**
remove-from-clss :: 'cls \Rightarrow 'clss \Rightarrow 'clss **and**

mset-ccls :: 'ccls \Rightarrow 'v clause **and**
union-ccls :: 'ccls \Rightarrow 'ccls \Rightarrow 'ccls **and**
insert-ccls :: 'v literal \Rightarrow 'ccls \Rightarrow 'ccls **and**
remove-clit :: 'v literal \Rightarrow 'ccls \Rightarrow 'ccls

+

fixes

ccls-of-cls :: 'cls \Rightarrow 'ccls **and**
cls-of-ccls :: 'ccls \Rightarrow 'cls **and**

trail :: 'st \Rightarrow ('v, nat, 'v clause) marked-lits **and**
hd-raw-trail :: 'st \Rightarrow ('v, nat, 'cls) marked-lit **and**
raw-init-clss :: 'st \Rightarrow 'clss **and**
raw-learned-clss :: 'st \Rightarrow 'clss **and**
backtrack-lvl :: 'st \Rightarrow nat **and**
raw-conflicting :: 'st \Rightarrow 'ccls option **and**

cons-trail :: ('v, nat, 'cls) marked-lit \Rightarrow 'st \Rightarrow 'st **and**
tl-trail :: 'st \Rightarrow 'st **and**
add-init-cls :: 'cls \Rightarrow 'st \Rightarrow 'st **and**
add-learned-cls :: 'cls \Rightarrow 'st \Rightarrow 'st **and**
remove-cls :: 'cls \Rightarrow 'st \Rightarrow 'st **and**
update-backtrack-lvl :: nat \Rightarrow 'st \Rightarrow 'st **and**
update-conflicting :: 'ccls option \Rightarrow 'st \Rightarrow 'st **and**

init-state :: 'clss \Rightarrow 'st **and**
restart-state :: 'st \Rightarrow 'st

assumes

mset-ccls-ccls-of-cls[simp]:
mset-ccls (ccls-of-cls C) = *mset-cls* C **and**
mset-cls-cls-of-ccls[simp]:
mset-cls (cls-of-ccls D) = *mset-ccls* D **and**
ex-mset-cls: $\exists a. \text{mset-cls } a = E$

begin

fun *mmset-of-mlit* :: ('a, 'b, 'cls) marked-lit \Rightarrow ('a, 'b, 'v clause) marked-lit
where

mmset-of-mlit (Propagated L C) = Propagated L (*mset-cls* C) |
mmset-of-mlit (Marked L i) = Marked L i

lemma *lit-of-mmset-of-mlit*[simp]:
lit-of (mmset-of-mlit a) = *lit-of* a
by (cases a) auto

lemma *lit-of-mmset-of-mlit-set-lit-of-l*[simp]:
lit-of ' mmset-of-mlit ' set M' = *lits-of-l* M'
by (induction M') auto

lemma *map-mmset-of-mlit-true-annot-true-cls*[simp]:
map mmset-of-mlit M' \models_{as} C \longleftrightarrow M' \models_{as} C

by (simp add: true-annots-true-cls lits-of-def)

abbreviation *init-clss* $\equiv \lambda S. \text{mset-clss } (\text{raw-init-clss } S)$

abbreviation *learned-clss* $\equiv \lambda S. \text{mset-clss } (\text{raw-learned-clss } S)$

abbreviation *conflicting* $\equiv \lambda S. \text{map-option mset-ccls } (\text{raw-conflicting } S)$

notation *insert-cls* (**infix** $!++$ 50)

notation *in-clss* (**infix** $!\in$ 50)

notation *union-clss* (**infix** \oplus 50)

notation *insert-clss* (**infix** $!++!$ 50)

notation *union-ccls* (**infix** $!\cup$ 50)

definition *raw-clauses* $:: 'st \Rightarrow 'clss$ **where**

raw-clauses $S = \text{union-clss } (\text{raw-init-clss } S) (\text{raw-learned-clss } S)$

abbreviation *clauses* $:: 'st \Rightarrow 'v \text{ clauses}$ **where**

clauses $S \equiv \text{mset-clss } (\text{raw-clauses } S)$

end

locale *state_W* =

state_W-ops

— functions for clauses:

mset-cls insert-cls remove-lit

mset-clss union-clss in-clss insert-clss remove-from-clss

— functions for the conflicting clause:

mset-ccls union-ccls insert-ccls remove-clit

— Conversion between conflicting and non-conflicting

ccls-of-cls cls-of-ccls

— functions for the state:

— access functions:

trail hd-raw-trail raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting

— changing state:

cons-trail tl-trail add-init-cls add-learned-cls remove-cls update-backtrack-lvl

update-conflicting

— get state:

init-state

restart-state

for

mset-cls $:: 'cls \Rightarrow 'v \text{ clause}$ **and**

insert-cls $:: 'v \text{ literal} \Rightarrow 'cls \Rightarrow 'cls$ **and**

remove-lit $:: 'v \text{ literal} \Rightarrow 'cls \Rightarrow 'cls$ **and**

mset-clss $:: 'clss \Rightarrow 'v \text{ clauses}$ **and**

union-clss $:: 'clss \Rightarrow 'clss \Rightarrow 'clss$ **and**

in-clss $:: 'cls \Rightarrow 'clss \Rightarrow \text{bool}$ **and**

insert-clss $:: 'cls \Rightarrow 'clss \Rightarrow 'clss$ **and**

remove-from-clss $:: 'cls \Rightarrow 'clss \Rightarrow 'clss$ **and**

$mset-ccls :: 'ccls \Rightarrow 'v \text{ clause and}$
 $union-ccls :: 'ccls \Rightarrow 'ccls \Rightarrow 'ccls \text{ and}$
 $insert-ccls :: 'v \text{ literal} \Rightarrow 'ccls \Rightarrow 'ccls \text{ and}$
 $remove-clit :: 'v \text{ literal} \Rightarrow 'ccls \Rightarrow 'ccls \text{ and}$

$ccls-of-cl :: 'cls \Rightarrow 'ccls \text{ and}$
 $cls-of-ccls :: 'ccls \Rightarrow 'cls \text{ and}$

$trail :: 'st \Rightarrow ('v, nat, 'v \text{ clause}) \text{ marked-lits and}$
 $hd-raw-trail :: 'st \Rightarrow ('v, nat, 'cls) \text{ marked-lit and}$
 $raw-init-clss :: 'st \Rightarrow 'clss \text{ and}$
 $raw-learned-clss :: 'st \Rightarrow 'clss \text{ and}$
 $backtrack-lvl :: 'st \Rightarrow nat \text{ and}$
 $raw-conflicting :: 'st \Rightarrow 'ccls \text{ option and}$

$cons-trail :: ('v, nat, 'cls) \text{ marked-lit} \Rightarrow 'st \Rightarrow 'st \text{ and}$
 $tl-trail :: 'st \Rightarrow 'st \text{ and}$
 $add-init-cls :: 'cls \Rightarrow 'st \Rightarrow 'st \text{ and}$
 $add-learned-cls :: 'cls \Rightarrow 'st \Rightarrow 'st \text{ and}$
 $remove-cls :: 'cls \Rightarrow 'st \Rightarrow 'st \text{ and}$
 $update-backtrack-lvl :: nat \Rightarrow 'st \Rightarrow 'st \text{ and}$
 $update-conflicting :: 'ccls \text{ option} \Rightarrow 'st \Rightarrow 'st \text{ and}$

$init-state :: 'clss \Rightarrow 'st \text{ and}$
 $restart-state :: 'st \Rightarrow 'st +$

assumes

$hd-raw-trail: trail\ S \neq [] \implies mset-of-mlit\ (hd-raw-trail\ S) = hd\ (trail\ S) \text{ and}$
 $trail-cons-trail[simp]:$

$\bigwedge L\ st. \text{ undefined-lit } (trail\ st)\ (lit-of\ L) \implies$
 $trail\ (cons-trail\ L\ st) = mset-of-mlit\ L\ \# \ trail\ st \text{ and}$

$trail-tl-trail[simp]: \bigwedge st. trail\ (tl-trail\ st) = tl\ (trail\ st) \text{ and}$

$trail-add-init-cls[simp]:$

$\bigwedge st\ C. no-dup\ (trail\ st) \implies trail\ (add-init-cls\ C\ st) = trail\ st \text{ and}$

$trail-add-learned-cls[simp]:$

$\bigwedge C\ st. no-dup\ (trail\ st) \implies trail\ (add-learned-cls\ C\ st) = trail\ st \text{ and}$

$trail-remove-cls[simp]:$

$\bigwedge C\ st. trail\ (remove-cls\ C\ st) = trail\ st \text{ and}$

$trail-update-backtrack-lvl[simp]: \bigwedge st\ C. trail\ (update-backtrack-lvl\ C\ st) = trail\ st \text{ and}$

$trail-update-conflicting[simp]: \bigwedge C\ st. trail\ (update-conflicting\ C\ st) = trail\ st \text{ and}$

$init-clss-cons-trail[simp]:$

$\bigwedge M\ st. \text{ undefined-lit } (trail\ st)\ (lit-of\ M) \implies$

$init-clss\ (cons-trail\ M\ st) = init-clss\ st$

and

$init-clss-tl-trail[simp]:$

$\bigwedge st. init-clss\ (tl-trail\ st) = init-clss\ st \text{ and}$

$init-clss-add-init-cls[simp]:$

$\bigwedge st\ C. no-dup\ (trail\ st) \implies init-clss\ (add-init-cls\ C\ st) = \{\#mset-cls\ C\} + init-clss\ st$

and

$init-clss-add-learned-cls[simp]:$

$\bigwedge C\ st. no-dup\ (trail\ st) \implies init-clss\ (add-learned-cls\ C\ st) = init-clss\ st \text{ and}$

$init-clss-remove-cls[simp]:$

$\bigwedge C\ st. init-clss\ (remove-cls\ C\ st) = removeAll-mset\ (mset-cls\ C)\ (init-clss\ st) \text{ and}$

$init-clss-update-backtrack-lvl[simp]:$

$\bigwedge st\ C. init-clss\ (update-backtrack-lvl\ C\ st) = init-clss\ st \text{ and}$

$\text{init-clss-update-conflicting[simp]}:$
 $\bigwedge C \text{ st. } \text{init-clss} (\text{update-conflicting } C \text{ st}) = \text{init-clss st and}$

$\text{learned-clss-cons-trail[simp]}:$
 $\bigwedge M \text{ st. } \text{undefined-lit} (\text{trail st}) (\text{lit-of } M) \implies$
 $\text{learned-clss} (\text{cons-trail } M \text{ st}) = \text{learned-clss st and}$

$\text{learned-clss-tl-trail[simp]}:$
 $\bigwedge \text{st. } \text{learned-clss} (\text{tl-trail st}) = \text{learned-clss st and}$

$\text{learned-clss-add-init-cl[simp]}:$
 $\bigwedge \text{st } C. \text{no-dup} (\text{trail st}) \implies \text{learned-clss} (\text{add-init-cl } C \text{ st}) = \text{learned-clss st and}$

$\text{learned-clss-add-learned-cl[simp]}:$
 $\bigwedge C \text{ st. no-dup} (\text{trail st}) \implies$
 $\text{learned-clss} (\text{add-learned-cl } C \text{ st}) = \{\#mset-cl } C \# \} + \text{learned-clss st and}$

$\text{learned-clss-remove-cl[simp]}:$
 $\bigwedge C \text{ st. } \text{learned-clss} (\text{remove-cl } C \text{ st}) = \text{removeAll-mset} (\text{mset-cl } C) (\text{learned-clss st}) \text{ and}$

$\text{learned-clss-update-backtrack-lvl[simp]}:$
 $\bigwedge \text{st } C. \text{learned-clss} (\text{update-backtrack-lvl } C \text{ st}) = \text{learned-clss st and}$

$\text{learned-clss-update-conflicting[simp]}:$
 $\bigwedge C \text{ st. } \text{learned-clss} (\text{update-conflicting } C \text{ st}) = \text{learned-clss st and}$

$\text{backtrack-lvl-cons-trail[simp]}:$
 $\bigwedge M \text{ st. } \text{undefined-lit} (\text{trail st}) (\text{lit-of } M) \implies$
 $\text{backtrack-lvl} (\text{cons-trail } M \text{ st}) = \text{backtrack-lvl st and}$

$\text{backtrack-lvl-tl-trail[simp]}:$
 $\bigwedge \text{st. } \text{backtrack-lvl} (\text{tl-trail st}) = \text{backtrack-lvl st and}$

$\text{backtrack-lvl-add-init-cl[simp]}:$
 $\bigwedge \text{st } C. \text{no-dup} (\text{trail st}) \implies \text{backtrack-lvl} (\text{add-init-cl } C \text{ st}) = \text{backtrack-lvl st and}$

$\text{backtrack-lvl-add-learned-cl[simp]}:$
 $\bigwedge C \text{ st. no-dup} (\text{trail st}) \implies \text{backtrack-lvl} (\text{add-learned-cl } C \text{ st}) = \text{backtrack-lvl st and}$

$\text{backtrack-lvl-remove-cl[simp]}:$
 $\bigwedge C \text{ st. } \text{backtrack-lvl} (\text{remove-cl } C \text{ st}) = \text{backtrack-lvl st and}$

$\text{backtrack-lvl-update-backtrack-lvl[simp]}:$
 $\bigwedge \text{st } k. \text{backtrack-lvl} (\text{update-backtrack-lvl } k \text{ st}) = k \text{ and}$

$\text{backtrack-lvl-update-conflicting[simp]}:$
 $\bigwedge C \text{ st. } \text{backtrack-lvl} (\text{update-conflicting } C \text{ st}) = \text{backtrack-lvl st and}$

$\text{conflicting-cons-trail[simp]}:$
 $\bigwedge M \text{ st. } \text{undefined-lit} (\text{trail st}) (\text{lit-of } M) \implies$
 $\text{conflicting} (\text{cons-trail } M \text{ st}) = \text{conflicting st and}$

$\text{conflicting-tl-trail[simp]}:$
 $\bigwedge \text{st. } \text{conflicting} (\text{tl-trail st}) = \text{conflicting st and}$

$\text{conflicting-add-init-cl[simp]}:$
 $\bigwedge \text{st } C. \text{no-dup} (\text{trail st}) \implies \text{conflicting} (\text{add-init-cl } C \text{ st}) = \text{conflicting st and}$

$\text{conflicting-add-learned-cl[simp]}:$
 $\bigwedge C \text{ st. no-dup} (\text{trail st}) \implies \text{conflicting} (\text{add-learned-cl } C \text{ st}) = \text{conflicting st}$
and

$\text{conflicting-remove-cl[simp]}:$
 $\bigwedge C \text{ st. } \text{conflicting} (\text{remove-cl } C \text{ st}) = \text{conflicting st and}$

$\text{conflicting-update-backtrack-lvl[simp]}:$
 $\bigwedge \text{st } C. \text{conflicting} (\text{update-backtrack-lvl } C \text{ st}) = \text{conflicting st and}$

$\text{conflicting-update-conflicting[simp]}:$
 $\bigwedge C \text{ st. } \text{raw-conflicting} (\text{update-conflicting } C \text{ st}) = C \text{ and}$

$\text{init-state-trail[simp]}: \bigwedge N. \text{trail} (\text{init-state } N) = [] \text{ and}$
 $\text{init-state-clss[simp]}: \bigwedge N. (\text{init-clss} (\text{init-state } N)) = \text{mset-clss } N \text{ and}$

init-state-learned-clss[simp]: $\bigwedge N. \text{learned-clss } (\text{init-state } N) = \{\#\}$ **and**
init-state-backtrack-lvl[simp]: $\bigwedge N. \text{backtrack-lvl } (\text{init-state } N) = 0$ **and**
init-state-conflicting[simp]: $\bigwedge N. \text{conflicting } (\text{init-state } N) = \text{None}$ **and**

trail-restart-state[simp]: $\text{trail } (\text{restart-state } S) = []$ **and**
init-clss-restart-state[simp]: $\text{init-clss } (\text{restart-state } S) = \text{init-clss } S$ **and**
learned-clss-restart-state[intro]:
 $\text{learned-clss } (\text{restart-state } S) \subseteq \# \text{learned-clss } S$ **and**
backtrack-lvl-restart-state[simp]: $\text{backtrack-lvl } (\text{restart-state } S) = 0$ **and**
conflicting-restart-state[simp]: $\text{conflicting } (\text{restart-state } S) = \text{None}$

begin

lemma

shows

clauses-cons-trail[simp]:
 $\text{undefined-lit } (\text{trail } S) (\text{lit-of } M) \implies \text{clauses } (\text{cons-trail } M S) = \text{clauses } S$ **and**

clss-tl-trail[simp]: $\text{clauses } (\text{tl-trail } S) = \text{clauses } S$ **and**

clauses-add-learned-clss-unfolded:

$\text{no-dup } (\text{trail } S) \implies \text{clauses } (\text{add-learned-clss } U S) =$
 $\{\# \text{mset-clss } U \# \} + \text{learned-clss } S + \text{init-clss } S$

and

clauses-add-init-clss[simp]:

$\text{no-dup } (\text{trail } S) \implies$

$\text{clauses } (\text{add-init-clss } N S) = \{\# \text{mset-clss } N \# \} + \text{init-clss } S + \text{learned-clss } S$ **and**

clauses-update-backtrack-lvl[simp]: $\text{clauses } (\text{update-backtrack-lvl } k S) = \text{clauses } S$ **and**

clauses-update-conflicting[simp]: $\text{clauses } (\text{update-conflicting } D S) = \text{clauses } S$ **and**

clauses-remove-clss[simp]:

$\text{clauses } (\text{remove-clss } C S) = \text{removeAll-mset } (\text{mset-clss } C) (\text{clauses } S)$ **and**

clauses-add-learned-clss[simp]:

$\text{no-dup } (\text{trail } S) \implies \text{clauses } (\text{add-learned-clss } C S) = \{\# \text{mset-clss } C \# \} + \text{clauses } S$ **and**

clauses-restart[simp]: $\text{clauses } (\text{restart-state } S) \subseteq \# \text{clauses } S$ **and**

clauses-init-state[simp]: $\bigwedge N. \text{clauses } (\text{init-state } N) = \text{mset-clss } N$

prefer 9 using raw-clauses-def learned-clss-restart-state apply fastforce

by (auto simp: ac-simps replicate-mset-plus raw-clauses-def intro: multiset-eqI)

abbreviation *state* :: $'st \Rightarrow ('v, \text{nat}, 'v \text{ clause}) \text{ marked-lit list} \times 'v \text{ clauses} \times 'v \text{ clauses}$
 $\times \text{nat} \times 'v \text{ clause option}$ **where**

state *S* $\equiv (\text{trail } S, \text{init-clss } S, \text{learned-clss } S, \text{backtrack-lvl } S, \text{conflicting } S)$

abbreviation *incr-lvl* :: $'st \Rightarrow 'st$ **where**

incr-lvl *S* $\equiv \text{update-backtrack-lvl } (\text{backtrack-lvl } S + 1) S$

definition *state-eq* :: $'st \Rightarrow 'st \Rightarrow \text{bool}$ (**infix** \sim 50) **where**

$S \sim T \iff \text{state } S = \text{state } T$

lemma *state-eq-ref*[simp, intro]:

$S \sim S$

unfolding *state-eq-def* **by** *auto*

lemma *state-eq-sym*:

$S \sim T \iff T \sim S$

unfolding *state-eq-def* **by** *auto*

lemma *state-eq-trans*:

$S \sim T \implies T \sim U \implies S \sim U$
unfolding *state-eq-def* **by** *auto*

lemma

shows

state-eq-trail: $S \sim T \implies \text{trail } S = \text{trail } T$ **and**
state-eq-init-clss: $S \sim T \implies \text{init-clss } S = \text{init-clss } T$ **and**
state-eq-learned-clss: $S \sim T \implies \text{learned-clss } S = \text{learned-clss } T$ **and**
state-eq-backtrack-lvl: $S \sim T \implies \text{backtrack-lvl } S = \text{backtrack-lvl } T$ **and**
state-eq-conflicting: $S \sim T \implies \text{conflicting } S = \text{conflicting } T$ **and**
state-eq-clauses: $S \sim T \implies \text{clauses } S = \text{clauses } T$ **and**
state-eq-undefined-lit: $S \sim T \implies \text{undefined-lit } (\text{trail } S) L = \text{undefined-lit } (\text{trail } T) L$

unfolding *state-eq-def* *raw-clauses-def* **by** *auto*

lemma *state-eq-raw-conflicting-None*:

$S \sim T \implies \text{conflicting } T = \text{None} \implies \text{raw-conflicting } S = \text{None}$

unfolding *state-eq-def* *raw-clauses-def* **by** *auto*

lemmas *state-simp*[*simp*] = *state-eq-trail* *state-eq-init-clss* *state-eq-learned-clss*

state-eq-backtrack-lvl *state-eq-conflicting* *state-eq-clauses* *state-eq-undefined-lit*
state-eq-raw-conflicting-None

lemma *atms-of-ms-learned-clss-restart-state-in-atms-of-ms-learned-clssI*[*intro*]:

$x \in \text{atms-of-mm } (\text{learned-clss } (\text{restart-state } S)) \implies x \in \text{atms-of-mm } (\text{learned-clss } S)$

by (*meson* *atms-of-ms-mono* *learned-clss-restart-state* *set-mset-mono* *subsetCE*)

function *reduce-trail-to* :: '*a list* \Rightarrow '*st* \Rightarrow '*st* **where**

reduce-trail-to *F S* =

(*if* *length* (*trail S*) = *length F* \vee *trail S* = [] *then S* *else* *reduce-trail-to F* (*tl-trail S*))

by *fast+*

termination

by (*relation measure* ($\lambda(-, S). \text{length } (\text{trail } S)$)) *simp-all*

declare *reduce-trail-to.simps*[*simp del*]

lemma

shows

reduce-trail-to-nil[*simp*]: *trail S* = [] \implies *reduce-trail-to F S* = *S* **and**

reduce-trail-to-eq-length[*simp*]: *length* (*trail S*) = *length F* \implies *reduce-trail-to F S* = *S*

by (*auto simp: reduce-trail-to.simps*)

lemma *reduce-trail-to-length-ne*:

length (*trail S*) \neq *length F* \implies *trail S* \neq [] \implies

reduce-trail-to F S = *reduce-trail-to F* (*tl-trail S*)

by (*auto simp: reduce-trail-to.simps*)

lemma *trail-reduce-trail-to-length-le*:

assumes *length F* > *length* (*trail S*)

shows *trail* (*reduce-trail-to F S*) = []

using *assms* **apply** (*induction F S* *rule: reduce-trail-to.induct*)

by (*metis* (*no-types*, *hide-lams*) *length-tl* *less-imp-diff-less* *less-irrefl* *trail-tl-trail*

reduce-trail-to.simps)

lemma *trail-reduce-trail-to-nil*[*simp*]:

trail (*reduce-trail-to* [] *S*) = []

apply (induction $\llbracket :: ('v, nat, 'v \text{ clause}) \text{ marked-lits } S \text{ rule: reduce-trail-to.induct} \rrbracket$)
by (metis length-0-conv reduce-trail-to-length-ne reduce-trail-to-nil)

lemma clauses-reduce-trail-to-nil:

clauses (reduce-trail-to $\llbracket S \rrbracket$) = clauses S

proof (induction $\llbracket S \rrbracket$ rule: reduce-trail-to.induct)

case (1 Sa)

then have clauses (reduce-trail-to ($\llbracket :: 'a \text{ list} \rrbracket$ (tl-trail Sa))) = clauses (tl-trail Sa)

\vee trail $Sa = \llbracket$

by fastforce

then show clauses (reduce-trail-to ($\llbracket :: 'a \text{ list} \rrbracket$ Sa)) = clauses Sa

by (metis (no-types) length-0-conv reduce-trail-to-eq-length clss-tl-trail
reduce-trail-to-length-ne)

qed

lemma reduce-trail-to-skip-beginning:

assumes trail $S = F' @ F$

shows trail (reduce-trail-to $F S$) = F

using assms **by** (induction F' arbitrary: S) (auto simp: reduce-trail-to-length-ne)

lemma clauses-reduce-trail-to[simp]:

clauses (reduce-trail-to $F S$) = clauses S

apply (induction $F S$ rule: reduce-trail-to.induct)

by (metis clss-tl-trail reduce-trail-to.simps)

lemma conflicting-update-trail[simp]:

conflicting (reduce-trail-to $F S$) = conflicting S

apply (induction $F S$ rule: reduce-trail-to.induct)

by (metis conflicting-tl-trail reduce-trail-to.simps)

lemma backtrack-lvl-update-trail[simp]:

backtrack-lvl (reduce-trail-to $F S$) = backtrack-lvl S

apply (induction $F S$ rule: reduce-trail-to.induct)

by (metis backtrack-lvl-tl-trail reduce-trail-to.simps)

lemma init-clss-update-trail[simp]:

init-clss (reduce-trail-to $F S$) = init-clss S

apply (induction $F S$ rule: reduce-trail-to.induct)

by (metis init-clss-tl-trail reduce-trail-to.simps)

lemma learned-clss-update-trail[simp]:

learned-clss (reduce-trail-to $F S$) = learned-clss S

apply (induction $F S$ rule: reduce-trail-to.induct)

by (metis learned-clss-tl-trail reduce-trail-to.simps)

lemma raw-conflicting-reduce-trail-to[simp]:

raw-conflicting (reduce-trail-to $F S$) = None \longleftrightarrow raw-conflicting S = None

apply (induction $F S$ rule: reduce-trail-to.induct)

by (metis conflicting-update-trail map-option-is-None)

lemma trail-eq-reduce-trail-to-eq:

trail $S = \text{trail } T \implies \text{trail (reduce-trail-to } F S) = \text{trail (reduce-trail-to } F T)$

apply (induction $F S$ arbitrary: T rule: reduce-trail-to.induct)

by (metis trail-tl-trail reduce-trail-to.simps)


```

lemma reduce-trail-to-state-eqNOT-compatible:
  assumes ST:  $S \sim T$ 
  shows reduce-trail-to  $F S \sim \text{reduce-trail-to } F T$ 
proof –
  have trail (reduce-trail-to  $F S$ ) = trail (reduce-trail-to  $F T$ )
    using trail-eq-reduce-trail-to-eq[of  $S T F$ ] ST by auto
  then show ?thesis using ST by (auto simp del: state-simp simp: state-eq-def)
qed

lemma reduce-trail-to-trail-tl-trail-decomp[simp]:
  trail S = F' @ Marked K d # F  $\implies$  (trail (reduce-trail-to  $F S$ )) =  $F$ 
  apply (rule reduce-trail-to-skip-beginning[of -  $F' @ \text{Marked } K d \# []$ ])
  by (cases  $F'$ ) (auto simp add:tl-append reduce-trail-to-skip-beginning)

lemma reduce-trail-to-add-learned-cls[simp]:
  no-dup (trail S)  $\implies$ 
    trail (reduce-trail-to  $F$  (add-learned-cls  $C S$ )) = trail (reduce-trail-to  $F S$ )
  by (rule trail-eq-reduce-trail-to-eq) auto

lemma reduce-trail-to-add-init-cls[simp]:
  no-dup (trail S)  $\implies$ 
    trail (reduce-trail-to  $F$  (add-init-cls  $C S$ )) = trail (reduce-trail-to  $F S$ )
  by (rule trail-eq-reduce-trail-to-eq) auto

lemma reduce-trail-to-remove-learned-cls[simp]:
  trail (reduce-trail-to  $F$  (remove-cls  $C S$ )) = trail (reduce-trail-to  $F S$ )
  by (rule trail-eq-reduce-trail-to-eq) auto

lemma reduce-trail-to-update-conflicting[simp]:
  trail (reduce-trail-to  $F$  (update-conflicting  $C S$ )) = trail (reduce-trail-to  $F S$ )
  by (rule trail-eq-reduce-trail-to-eq) auto

lemma reduce-trail-to-update-backtrack-lvl[simp]:
  trail (reduce-trail-to  $F$  (update-backtrack-lvl  $C S$ )) = trail (reduce-trail-to  $F S$ )
  by (rule trail-eq-reduce-trail-to-eq) auto

lemma in-get-all-marked-decomposition-marked-or-empty:
  assumes  $(a, b) \in \text{set } (\text{get-all-marked-decomposition } M)$ 
  shows  $a = [] \vee (\text{is-marked } (\text{hd } a))$ 
  using assms
proof (induct  $M$  arbitrary: a b)
  case Nil then show ?case by simp
next
  case (Cons  $m M$ )
  show ?case
  proof (cases  $m$ )
  case (Marked  $l \text{ mark}$ )
  then show ?thesis using Cons by auto
  next
  case (Propagated  $l \text{ mark}$ )
  then show ?thesis using Cons by (cases get-all-marked-decomposition  $M$ ) force+
qed
qed

```

lemma *reduce-trail-to-length*:
 $\text{length } M = \text{length } M' \implies \text{reduce-trail-to } M \ S = \text{reduce-trail-to } M' \ S$
apply (*induction* $M \ S$ *arbitrary*: *rule*: *reduce-trail-to.induct*)
by (*simp add*: *reduce-trail-to.simps*)

lemma *trail-reduce-trail-to-drop*:
 $\text{trail } (\text{reduce-trail-to } F \ S) =$
 (*if* $\text{length } (\text{trail } S) \geq \text{length } F$
then $\text{drop } (\text{length } (\text{trail } S) - \text{length } F) (\text{trail } S)$
else \square)
apply (*induction* $F \ S$ *rule*: *reduce-trail-to.induct*)
apply (*rename-tac* $F \ S$, *case-tac* *trail* S)
apply *auto*
apply (*rename-tac* *list*, *case-tac* *Suc* ($\text{length } \text{list}$) $> \text{length } F$)
prefer 2 **apply** (*metis* *diff-is-0-eq* *drop-Cons'* *length-Cons* *nat-le-linear* *nat-less-le*
reduce-trail-to-eq-length *trail-reduce-trail-to-length-le*)
apply (*subgoal-tac* *Suc* ($\text{length } \text{list}$) $- \text{length } F = \text{Suc } (\text{length } \text{list} - \text{length } F)$)
by (*auto simp add*: *reduce-trail-to-length-ne*)

lemma *in-get-all-marked-decomposition-trail-update-trail[simp]*:
assumes $H: (L \# M1, M2) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$
shows $\text{trail } (\text{reduce-trail-to } M1 \ S) = M1$
proof –
obtain $K \ \text{mark}$ **where**
 $L: L = \text{Marked } K \ \text{mark}$
using H **by** (*cases* L) (*auto dest!*: *in-get-all-marked-decomposition-marked-or-empty*)
obtain c **where**
 $\text{tr-}S: \text{trail } S = c \ @ \ M2 \ @ \ L \ \# \ M1$
using H **by** *auto*
show *?thesis*
by (*rule* *reduce-trail-to-trail-tl-trail-decomp*[*of* - $c \ @ \ M2 \ K \ \text{mark}$])
 (*auto simp*: *tr-S* L)
qed

lemma *raw-conflicting-cons-trail[simp]*:
assumes *undefined-lit* ($\text{trail } S$) (*lit-of* L)
shows
 $\text{raw-conflicting } (\text{cons-trail } L \ S) = \text{None} \longleftrightarrow \text{raw-conflicting } S = \text{None}$
using *assms* *conflicting-cons-trail*[*of* $S \ L$] *map-option-is-None* **by** *fastforce*+

lemma *raw-conflicting-add-init-cls[simp]*:
 $\text{no-dup } (\text{trail } S) \implies$
 $\text{raw-conflicting } (\text{add-init-cls } C \ S) = \text{None} \longleftrightarrow \text{raw-conflicting } S = \text{None}$
using *map-option-is-None* *conflicting-add-init-cls*[*of* $S \ C$] **by** *fastforce*+

lemma *raw-conflicting-add-learned-cls[simp]*:
 $\text{no-dup } (\text{trail } S) \implies$
 $\text{raw-conflicting } (\text{add-learned-cls } C \ S) = \text{None} \longleftrightarrow \text{raw-conflicting } S = \text{None}$
using *map-option-is-None* *conflicting-add-learned-cls*[*of* $S \ C$] **by** *fastforce*+

lemma *raw-conflicting-update-backtrack-lvl[simp]*:
 $\text{raw-conflicting } (\text{update-backtrack-lvl } k \ S) = \text{None} \longleftrightarrow \text{raw-conflicting } S = \text{None}$
using *map-option-is-None* *conflicting-update-backtrack-lvl*[*of* $k \ S$] **by** *fastforce*+

end — end of *stateW* locale

19.2 CDCL Rules

Because of the strategy we will later use, we distinguish propagate, conflict from the other rules

locale *conflict-driven-clause-learning*_W =

*state*_W

— functions for clauses:

mset-cls insert-cls remove-lit

mset-clss union-clss in-clss insert-clss remove-from-clss

— functions for the conflicting clause:

mset-ccls union-ccls insert-ccls remove-clit

— conversion

ccls-of-cls cls-of-ccls

— functions for the state:

— access functions:

trail hd-trail trail raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting

— changing state:

*cons-trail tl-trail add-init-cls add-learned-cls remove-cls update-backtrack-lvl
update-conflicting*

— get state:

init-state

restart-state

for

mset-cls :: *'cls* ⇒ *'v clause* **and**

insert-cls :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**

remove-lit :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**

mset-clss :: *'clss* ⇒ *'v clauses* **and**

union-clss :: *'clss* ⇒ *'clss* ⇒ *'clss* **and**

in-clss :: *'cls* ⇒ *'clss* ⇒ *bool* **and**

insert-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**

remove-from-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**

mset-ccls :: *'ccls* ⇒ *'v clause* **and**

union-ccls :: *'ccls* ⇒ *'ccls* ⇒ *'ccls* **and**

insert-ccls :: *'v literal* ⇒ *'ccls* ⇒ *'ccls* **and**

remove-clit :: *'v literal* ⇒ *'ccls* ⇒ *'ccls* **and**

ccls-of-cls :: *'cls* ⇒ *'ccls* **and**

cls-of-ccls :: *'ccls* ⇒ *'cls* **and**

trail :: *'st* ⇒ (*'v, nat, 'v clause*) *marked-lits* **and**

hd-trail :: *'st* ⇒ (*'v, nat, 'cls*) *marked-lit* **and**

raw-init-clss :: *'st* ⇒ *'clss* **and**

raw-learned-clss :: *'st* ⇒ *'clss* **and**

backtrack-lvl :: *'st* ⇒ *nat* **and**

raw-conflicting :: *'st* ⇒ *'ccls option* **and**

cons-trail :: (*'v, nat, 'cls*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**

tl-trail :: *'st* ⇒ *'st* **and**

add-init-cls :: *'cls* ⇒ *'st* ⇒ *'st* **and**

add-learned-cls :: *'cls* ⇒ *'st* ⇒ *'st* **and**

remove-cls :: 'cls \Rightarrow 'st \Rightarrow 'st **and**
update-backtrack-lvl :: nat \Rightarrow 'st \Rightarrow 'st **and**
update-conflicting :: 'ccls option \Rightarrow 'st \Rightarrow 'st **and**

init-state :: 'clss \Rightarrow 'st **and**
restart-state :: 'st \Rightarrow 'st

begin

inductive *propagate* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

propagate-rule: *conflicting S* = None \Rightarrow

E ! \in ! *raw-clauses S* \Rightarrow

L \in # *mset-cls E* \Rightarrow

trail S \models as *CNot* (*mset-cls* (*remove-lit L E*)) \Rightarrow

undefined-lit (*trail S*) *L* \Rightarrow

T \sim *cons-trail* (*Propagated L E*) *S* \Rightarrow

propagate S T

inductive-cases *propagateE*: *propagate S T*

inductive *conflict* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

conflict-rule:

conflicting S = None \Rightarrow

D ! \in ! *raw-clauses S* \Rightarrow

trail S \models as *CNot* (*mset-cls D*) \Rightarrow

T \sim *update-conflicting* (*Some* (*ccls-of-cls D*)) *S* \Rightarrow

conflict S T

inductive-cases *conflictE*: *conflict S T*

inductive *backtrack* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

backtrack-rule:

raw-conflicting S = *Some D* \Rightarrow

L \in # *mset-ccls D* \Rightarrow

(*Marked K* (*i*+1) # *M1*, *M2*) \in *set* (*get-all-marked-decomposition* (*trail S*)) \Rightarrow

get-level (*trail S*) *L* = *backtrack-lvl S* \Rightarrow

get-level (*trail S*) *L* = *get-maximum-level* (*trail S*) (*mset-ccls D*) \Rightarrow

get-maximum-level (*trail S*) (*mset-ccls* (*remove-clit L D*)) \equiv *i* \Rightarrow

T \sim *cons-trail* (*Propagated L* (*cls-of-ccls D*))

(*reduce-trail-to M1*

(*add-learned-cls* (*cls-of-ccls D*)

(*update-backtrack-lvl i*

(*update-conflicting None S*)))) \Rightarrow

backtrack S T

inductive-cases *backtrackE*: *backtrack S T*

thm *backtrackE*

inductive *decide* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

decide-rule:

conflicting S = None \Rightarrow

undefined-lit (*trail S*) *L* \Rightarrow

atm-of L \in *atms-of-mm* (*init-clss S*) \Rightarrow

T \sim *cons-trail* (*Marked L* (*backtrack-lvl S* + 1)) (*incr-lvl S*) \Rightarrow

decide S T

inductive-cases *decideE*: *decide S T*

inductive *skip* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

skip-rule:

trail S = Propagated L C' # M \Rightarrow
raw-conflicting S = Some E \Rightarrow
 $-L \notin \# \text{ mset-ccls } E \Rightarrow$
 $\text{mset-ccls } E \neq \{\#\} \Rightarrow$
 $T \sim \text{tl-trail } S \Rightarrow$
skip S T

inductive-cases *skipE*: *skip S T*

get-maximum-level (Propagated L (C + {\#L\#}) # M) D = k \vee k = 0 (that was in a previous version of the book) is equivalent to *get-maximum-level (Propagated L (C + {\#L\#}) # M) D = k*, when the structural invariants holds.

inductive *resolve* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

resolve-rule: *trail S $\neq [] \Rightarrow$*

hd-raw-trail S = Propagated L E \Rightarrow
 $L \in \# \text{ mset-clc } E \Rightarrow$
raw-conflicting S = Some D' \Rightarrow
 $-L \in \# \text{ mset-ccls } D' \Rightarrow$
get-maximum-level (trail S) (mset-ccls (remove-clit (-L) D')) = backtrack-lvl S \Rightarrow
 $T \sim \text{update-conflicting (Some (union-ccls (remove-clit (-L) D') (ccls-of-clc (remove-lit L E))))$
(tl-trail S) \Rightarrow
resolve S T

inductive-cases *resolveE*: *resolve S T*

inductive *restart* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

restart: *state S = (M, N, U, k, None) $\Rightarrow \neg M \models_{asm} \text{clauses } S$*
 $\Rightarrow T \sim \text{restart-state } S$
 $\Rightarrow \text{restart } S T$

inductive-cases *restartE*: *restart S T*

We add the condition $C \notin \# \text{ init-clss } S$, to maintain consistency even without the strategy.

inductive *forget* :: 'st \Rightarrow 'st \Rightarrow bool **where**

forget-rule:

conflicting S = None \Rightarrow
 $C ! \in ! \text{ raw-learned-clss } S \Rightarrow$
 $\neg(\text{trail } S) \models_{asm} \text{clauses } S \Rightarrow$
 $\text{mset-clc } C \notin \text{set (get-all-mark-of-propagated (trail S))} \Rightarrow$
 $\text{mset-clc } C \notin \# \text{ init-clss } S \Rightarrow$
 $T \sim \text{remove-clc } C S \Rightarrow$
forget S T

inductive-cases *forgetE*: *forget S T*

inductive *cdcl_W-rf* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

restart: *restart S T $\Rightarrow \text{cdcl}_W\text{-rf } S T$ |*

forget: *forget S T $\Rightarrow \text{cdcl}_W\text{-rf } S T$ |*

inductive *cdcl_W-bj* :: 'st \Rightarrow 'st \Rightarrow bool **where**

skip: *skip S S' $\Rightarrow \text{cdcl}_W\text{-bj } S S'$ |*

resolve: $\text{resolve } S \ S' \Longrightarrow \text{cdcl}_W\text{-bj } S \ S' \mid$
backtrack: $\text{backtrack } S \ S' \Longrightarrow \text{cdcl}_W\text{-bj } S \ S'$

inductive-cases $\text{cdcl}_W\text{-bjE}$: $\text{cdcl}_W\text{-bj } S \ T$

inductive $\text{cdcl}_W\text{-o}$: $'st \Rightarrow 'st \Rightarrow \text{bool}$ **for** $S :: 'st$ **where**
decide: $\text{decide } S \ S' \Longrightarrow \text{cdcl}_W\text{-o } S \ S' \mid$
bj: $\text{cdcl}_W\text{-bj } S \ S' \Longrightarrow \text{cdcl}_W\text{-o } S \ S'$

inductive cdcl_W : $'st \Rightarrow 'st \Rightarrow \text{bool}$ **for** $S :: 'st$ **where**
propagate: $\text{propagate } S \ S' \Longrightarrow \text{cdcl}_W \ S \ S' \mid$
conflict: $\text{conflict } S \ S' \Longrightarrow \text{cdcl}_W \ S \ S' \mid$
other: $\text{cdcl}_W\text{-o } S \ S' \Longrightarrow \text{cdcl}_W \ S \ S' \mid$
rf: $\text{cdcl}_W\text{-rf } S \ S' \Longrightarrow \text{cdcl}_W \ S \ S'$

lemma *rtranclp-propagate-is-rtranclp-cdcl_W*:
 $\text{propagate}^{**} \ S \ S' \Longrightarrow \text{cdcl}_W^{**} \ S \ S'$
apply (*induction rule*: *rtranclp-induct*)
apply *simp*
apply (*frule* *propagate*)
using *rtranclp-trans*[*of cdcl_W*] **by** *blast*

lemma *cdcl_W-all-rules-induct*[*consumes 1*, *case-names propagate conflict forget restart decide skip resolve backtrack*]:

fixes $S :: 'st$
assumes
 cdcl_W : $\text{cdcl}_W \ S \ S'$ **and**
propagate: $\bigwedge T. \text{propagate } S \ T \Longrightarrow P \ S \ T$ **and**
conflict: $\bigwedge T. \text{conflict } S \ T \Longrightarrow P \ S \ T$ **and**
forget: $\bigwedge T. \text{forget } S \ T \Longrightarrow P \ S \ T$ **and**
restart: $\bigwedge T. \text{restart } S \ T \Longrightarrow P \ S \ T$ **and**
decide: $\bigwedge T. \text{decide } S \ T \Longrightarrow P \ S \ T$ **and**
skip: $\bigwedge T. \text{skip } S \ T \Longrightarrow P \ S \ T$ **and**
resolve: $\bigwedge T. \text{resolve } S \ T \Longrightarrow P \ S \ T$ **and**
backtrack: $\bigwedge T. \text{backtrack } S \ T \Longrightarrow P \ S \ T$
shows $P \ S \ S'$
using *assms*(1)
proof (*induct S' rule*: *cdcl_W.induct*)
case (*propagate S'*) **note** *propagate = this*(1)
then show ?*case* **using** *assms*(2) **by** *auto*
next
case (*conflict S'*)
then show ?*case* **using** *assms*(3) **by** *auto*
next
case (*other S'*)
then show ?*case*
proof (*induct rule*: *cdcl_W-o.induct*)
case (*decide U*)
then show ?*case* **using** *assms*(6) **by** *auto*
next
case (*bj S'*)
then show ?*case* **using** *assms*(7–9) **by** (*induction rule*: *cdcl_W-bj.induct*) *auto*
qed
next
case (*rf S'*)

then show *?case*
by (*induct rule: cdcl_W-rf.induct*) (*fast dest: forget restart*)+
qed

lemma *cdcl_W-all-induct*[*consumes 1, case-names propagate conflict forget restart decide skip resolve backtrack*]:
fixes *S :: 'st*
assumes
cdcl_W: cdcl_W S S' and
propagateH: $\bigwedge C L T. \text{conflicting } S = \text{None} \implies$
 $C \in \text{raw-clauses } S \implies$
 $L \in \# \text{ mset-cls } C \implies$
 $\text{trail } S \models_{\text{as}} \text{CNot } (\text{remove1-mset } L (\text{mset-cls } C)) \implies$
 $\text{undefined-lit } (\text{trail } S) L \implies$
 $T \sim \text{cons-trail } (\text{Propagated } L C) S \implies$
P S T and
conflictH: $\bigwedge D T. \text{conflicting } S = \text{None} \implies$
 $D \in \text{raw-clauses } S \implies$
 $\text{trail } S \models_{\text{as}} \text{CNot } (\text{mset-cls } D) \implies$
 $T \sim \text{update-conflicting } (\text{Some } (\text{ccls-of-cls } D)) S \implies$
P S T and
forgetH: $\bigwedge C U T. \text{conflicting } S = \text{None} \implies$
 $C \in \text{raw-learned-clss } S \implies$
 $\neg(\text{trail } S) \models_{\text{asm}} \text{clauses } S \implies$
 $\text{mset-cls } C \notin \text{set } (\text{get-all-mark-of-propagated } (\text{trail } S)) \implies$
 $\text{mset-cls } C \notin \# \text{ init-clss } S \implies$
 $T \sim \text{remove-cls } C S \implies$
P S T and
restartH: $\bigwedge T. \neg \text{trail } S \models_{\text{asm}} \text{clauses } S \implies$
 $\text{conflicting } S = \text{None} \implies$
 $T \sim \text{restart-state } S \implies$
P S T and
decideH: $\bigwedge L T. \text{conflicting } S = \text{None} \implies$
 $\text{undefined-lit } (\text{trail } S) L \implies$
 $\text{atm-of } L \in \text{atms-of-mm } (\text{init-clss } S) \implies$
 $T \sim \text{cons-trail } (\text{Marked } L (\text{backtrack-lvl } S + 1)) (\text{incr-lvl } S) \implies$
P S T and
skipH: $\bigwedge L C' M E T.$
 $\text{trail } S = \text{Propagated } L C' \# M \implies$
 $\text{raw-conflicting } S = \text{Some } E \implies$
 $-L \notin \# \text{ mset-ccls } E \implies \text{mset-ccls } E \neq \{\#\} \implies$
 $T \sim \text{tl-trail } S \implies$
P S T and
resolveH: $\bigwedge L E M D T.$
 $\text{trail } S = \text{Propagated } L (\text{mset-cls } E) \# M \implies$
 $L \in \# \text{ mset-cls } E \implies$
 $\text{hd-raw-trail } S = \text{Propagated } L E \implies$
 $\text{raw-conflicting } S = \text{Some } D \implies$
 $-L \in \# \text{ mset-ccls } D \implies$
 $\text{get-maximum-level } (\text{trail } S) (\text{mset-ccls } (\text{remove-clit } (-L) D)) = \text{backtrack-lvl } S \implies$
 $T \sim \text{update-conflicting}$
 $(\text{Some } (\text{union-ccls } (\text{remove-clit } (-L) D) (\text{ccls-of-cls } (\text{remove-lit } L E)))) (\text{tl-trail } S) \implies$
P S T and
backtrackH: $\bigwedge L D K i M1 M2 T.$
 $\text{raw-conflicting } S = \text{Some } D \implies$

```

    L ∈ # mset-ccls D ⇒
    (Marked K (i+1) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) ⇒
    get-level (trail S) L = backtrack-lvl S ⇒
    get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) ⇒
    get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≡ i ⇒
    T ~ cons-trail (Propagated L (cls-of-ccls D))
      (reduce-trail-to M1
        (add-learned-cls (cls-of-ccls D)
          (update-backtrack-lvl i
            (update-conflicting None S)))) ⇒
    P S T
  shows P S S'
  using cdclW
proof (induct S S' rule: cdclW-all-rules-induct)
  case (propagate S')
  then show ?case
    by (auto elim!: propagateE intro!: propagateH)
  next
  case (conflict S')
  then show ?case
    by (auto elim!: conflictE intro!: conflictH)
  next
  case (restart S')
  then show ?case
    by (auto elim!: restartE intro!: restartH)
  next
  case (decide T)
  then show ?case
    by (auto elim!: decideE intro!: decideH)
  next
  case (backtrack S')
  then show ?case by (auto elim!: backtrackE intro!: backtrackH
    simp del: state-simp simp add: state-eq-def)
  next
  case (forget S')
  then show ?case by (auto elim!: forgetE intro!: forgetH)
  next
  case (skip S')
  then show ?case by (auto elim!: skipE intro!: skipH)
  next
  case (resolve S')
  then show ?case
    using hd-raw-trail[of S] by (cases trail S) (auto elim!: resolveE intro!: resolveH)
qed

```

```

lemma cdclW-o-induct[consumes 1, case-names decide skip resolve backtrack]:
  fixes S :: 'st
  assumes cdclW: cdclW-o S T and
    decideH:  $\bigwedge L$  T. conflicting S = None ⇒ undefined-lit (trail S) L
    ⇒ atm-of L ∈ atms-of-mm (init-cls S)
    ⇒ T ~ cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S)
    ⇒ P S T and
    skipH:  $\bigwedge L$  C' M E T.
    trail S = Propagated L C' # M ⇒
    raw-conflicting S = Some E ⇒

```



```

    -L ∉ # mset-ccls E ⇒ mset-ccls E ≠ {#} ⇒
    T ∼ tl-trail S ⇒
    P S T and
  resolveH: ∧ L E M D T.
    trail S = Propagated L (mset-cls E) # M ⇒
    L ∈ # mset-cls E ⇒
    hd-raw-trail S = Propagated L E ⇒
    raw-conflicting S = Some D ⇒
    -L ∈ # mset-ccls D ⇒
    get-maximum-level (trail S) (mset-ccls (remove-clit (-L) D)) = backtrack-lvl S ⇒
    T ∼ update-conflicting
      (Some (union-ccls (remove-clit (-L) D) (ccls-of-cls (remove-lit L E)))) (tl-trail S) ⇒
    P S T and
  backtrackH: ∧ L D K i M1 M2 T.
    raw-conflicting S = Some D ⇒
    L ∈ # mset-ccls D ⇒
    (Marked K (i+1) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) ⇒
    get-level (trail S) L = backtrack-lvl S ⇒
    get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) ⇒
    get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≡ i ⇒
    T ∼ cons-trail (Propagated L (cls-of-ccls D))
      (reduce-trail-to M1
        (add-learned-cls (cls-of-ccls D)
          (update-backtrack-lvl i
            (update-conflicting None S)))) ⇒
    P S T
  shows P S T
  using cdclW apply (induct T rule: cdclW-o.induct)
  using assms(2) apply (auto elim: decideE)[1]
  apply (elim cdclW-bjE skipE resolveE backtrackE)
  apply (frule skipH; simp)
  using hd-raw-trail[of S] apply (cases trail S; auto elim!: resolveE intro!: resolveH)
  apply (frule backtrackH; simp-all del: state-simp add: state-eq-def)
  done

thm cdclW-o.induct
lemma cdclW-o-all-rules-induct[consumes 1, case-names decide backtrack skip resolve]:
  fixes S T :: 'st
  assumes
    cdclW-o S T and
    ∧ T. decide S T ⇒ P S T and
    ∧ T. backtrack S T ⇒ P S T and
    ∧ T. skip S T ⇒ P S T and
    ∧ T. resolve S T ⇒ P S T
  shows P S T
  using assms by (induct T rule: cdclW-o.induct) (auto simp: cdclW-bj.simps)

lemma cdclW-o-rule-cases[consumes 1, case-names decide backtrack skip resolve]:
  fixes S T :: 'st
  assumes
    cdclW-o S T and
    decide S T ⇒ P and
    backtrack S T ⇒ P and
    skip S T ⇒ P and
    resolve S T ⇒ P

```

shows P
 using *assms* by (auto simp: $cdcl_W$ -o.simps $cdcl_W$ -bj.simps)

19.3 Invariants

19.3.1 Properties of the trail

We here establish that: * the marks are exactly 1..k where k is the level * the consistency of the trail * the fact that there is no duplicate in the trail.

lemma *backtrack-lit-skipped*:

assumes

L : *get-level* (*trail* S) L = *backtrack-lvl* S **and**

$M1$: (*Marked* K ($i + 1$) # $M1$, $M2$) \in *set* (*get-all-marked-decomposition* (*trail* S)) **and**

no-dup: *no-dup* (*trail* S) **and**

bt-l: *backtrack-lvl* S = *length* (*get-all-levels-of-marked* (*trail* S)) **and**

order: *get-all-levels-of-marked* (*trail* S)

= *rev* [$1..<(1 + \text{length } (\text{get-all-levels-of-marked } (\text{trail } S)))$]

shows *atm-of* $L \notin \text{atm-of } \text{'lits-of-l } M1$

proof (*rule ccontr*)

let $?M$ = *trail* S

assume $L\text{-in-}M1$: $\neg \text{atm-of } L \notin \text{atm-of } \text{'lits-of-l } M1$

obtain c **where**

Mc : *trail* S = $c @ M2 @ \text{Marked } K (i + 1) \# M1$

using $M1$ **by** *blast*

have *atm-of* $L \notin \text{atm-of } \text{'lits-of-l } c$

using $L\text{-in-}M1$ *no-dup* **unfolding** Mc *lits-of-def* **by** *force*

have $g\text{-}M\text{-eq-}g\text{-}M1$: *get-level* $?M$ L = *get-level* $M1$ L

using $L\text{-in-}M1$ **unfolding** Mc **by** *auto*

have g : *get-all-levels-of-marked* $M1$ = *rev* [$1..<\text{Suc } i$]

using *order* **unfolding** Mc **by** (*auto* simp *del*: *upt-simps* simp: *rev-swap*[*symmetric*]

dest: *append-cons-eq-upt-length-i*)

then have Max (*set* ($0 \# \text{get-all-levels-of-marked } (\text{rev } M1))) < \text{Suc } i$ **by** *auto*

then have *get-level* $M1$ $L < \text{Suc } i$

using *get-rev-level-less-max-get-all-levels-of-marked*[*of rev M1 0 L*] **by** *linarith*

moreover have $\text{Suc } i \leq \text{backtrack-lvl } S$ **using** *bt-l* **by** (*simp* *add*: Mc g)

ultimately show *False* **using** L $g\text{-}M\text{-eq-}g\text{-}M1$ **by** *auto*

qed

lemma *cdcl_W-distinctinv-1*:

assumes

$cdcl_W$ S S' **and**

no-dup (*trail* S) **and**

backtrack-lvl S = *length* (*get-all-levels-of-marked* (*trail* S)) **and**

get-all-levels-of-marked (*trail* S) = *rev* [$1..<1 + \text{length } (\text{get-all-levels-of-marked } (\text{trail } S))$]

shows *no-dup* (*trail* S')

using *assms*

proof (*induct rule*: *cdcl_W-all-induct*)

case (*backtrack* L D K i $M1$ $M2$ T) **note** $decomp = \text{this}(3)$ **and** $L = \text{this}(4)$ **and** $T = \text{this}(7)$ **and**

$n\text{-}d = \text{this}(8)$

obtain c **where** Mc : *trail* S = $c @ M2 @ \text{Marked } K (i + 1) \# M1$

using $decomp$ **by** *auto*

have *no-dup* ($M2 @ \text{Marked } K (i + 1) \# M1$)

using Mc $n\text{-}d$ **by** *fastforce*

moreover have *atm-of* $L \notin (\lambda l. \text{atm-of } (\text{lit-of } l)) \text{'set } M1$

using *backtrack-lit-skipped*[*of S L K i M1 M2*] L $decomp$ *backtrack.prem*s

by (fastforce simp: lits-of-def)
 moreover then have undefined-lit M1 L
 by (simp add: defined-lit-map)
 ultimately show ?case using decomp T n-d by simp
 qed (auto simp: defined-lit-map)

lemma *cdcl_W-consistent-inv-2*:

assumes
 cdcl_W S S' and
 no-dup (trail S) and
 backtrack-lvl S = length (get-all-levels-of-marked (trail S)) and
 get-all-levels-of-marked (trail S) = rev [1.. $1 + \text{length (get-all-levels-of-marked (trail S))}$]
 shows consistent-interp (lits-of-l (trail S'))
 using *cdcl_W-distinctinv-1* [OF assms] distinct-consistent-interp by fast

lemma *cdcl_W-o-bt*:

assumes
 cdcl_W-o S S' and
 backtrack-lvl S = length (get-all-levels-of-marked (trail S)) and
 get-all-levels-of-marked (trail S) =
 rev ([1.. $1 + \text{length (get-all-levels-of-marked (trail S))}$]) and
 n-d[simp]: no-dup (trail S)
 shows backtrack-lvl S' = length (get-all-levels-of-marked (trail S'))
 using assms

proof (induct rule: *cdcl_W-o-induct*)

case (backtrack L D K i M1 M2 T) note decomp = this(3) and T = this(7) and level = this(9)
 have [simp]: trail (reduce-trail-to M1 S) = M1
 using decomp by auto
 obtain c where M: trail S = c @ M2 @ Marked K (i + 1) # M1 using decomp by auto
 have rev (get-all-levels-of-marked (trail S))
 = [1.. $1 + (\text{length (get-all-levels-of-marked (trail S))})$]
 using level by (auto simp: rev-swap[symmetric])
 moreover have atm-of L $\notin (\lambda l. \text{atm-of (lit-of l)})$ ‘set M1
 using backtrack-lit-skipped[of S L K i M1 M2] backtrack(4,8,9) decomp
 by (fastforce simp add: lits-of-def)
 moreover then have undefined-lit M1 L
 by (simp add: defined-lit-map)
 moreover then have no-dup (trail T)
 using T decomp n-d by (auto simp: defined-lit-map M)
 ultimately show ?case
 using T n-d unfolding M by (auto dest!: append-cons-eq-upt-length simp del: upt-simps)
 qed auto

lemma *cdcl_W-rf-bt*:

assumes
 cdcl_W-rf S S' and
 backtrack-lvl S = length (get-all-levels-of-marked (trail S)) and
 get-all-levels-of-marked (trail S) = rev [1.. $1 + \text{length (get-all-levels-of-marked (trail S))}$]
 shows backtrack-lvl S' = length (get-all-levels-of-marked (trail S'))
 using assms by (induct rule: *cdcl_W-rf.induct*) (auto elim: restartE forgetE)

lemma *cdcl_W-bt*:

assumes
 cdcl_W S S' and
 backtrack-lvl S = length (get-all-levels-of-marked (trail S)) and

```

  get-all-levels-of-marked (trail S)
  = rev ([1.. $(1 + \text{length } (\text{get-all-levels-of-marked } (\text{trail } S)))$ ]) and
  no-dup (trail S)
shows backtrack-lvl S' = length (get-all-levels-of-marked (trail S'))
using assms by (induct rule: cdclW.induct) (auto simp add: cdclW-o-bt cdclW-rf-bt
  elim: conflictE propagateE)

lemma cdclW-bt-level':
assumes
  cdclW S S' and
  backtrack-lvl S = length (get-all-levels-of-marked (trail S)) and
  get-all-levels-of-marked (trail S)
  = rev ([1.. $(1 + \text{length } (\text{get-all-levels-of-marked } (\text{trail } S)))$ ]) and
  n-d: no-dup (trail S)
shows get-all-levels-of-marked (trail S')
  = rev [1.. $(1 + \text{length } (\text{get-all-levels-of-marked } (\text{trail } S'))$ )]
using assms
proof (induct rule: cdclW-all-induct)
case (decide L T) note undef = this(2) and T = this(4)
let ?k = backtrack-lvl S
let ?M = trail S
let ?M' = Marked L (?k + 1) # trail S
have H: get-all-levels-of-marked ?M = rev [Suc 0.. $(1 + \text{length } (\text{get-all-levels-of-marked } ?M))$ ]
  using decide.prem by simp
have k: ?k = length (get-all-levels-of-marked ?M)
  using decide.prem by auto
have get-all-levels-of-marked ?M' = Suc ?k # get-all-levels-of-marked ?M by simp
then have get-all-levels-of-marked ?M' = Suc ?k #
  rev [Suc 0.. $(1 + \text{length } (\text{get-all-levels-of-marked } ?M))$ ]
  using H by auto
moreover have ... = rev [Suc 0.. $(1 + \text{length } (\text{get-all-levels-of-marked } ?M))$ ]
  unfolding k by simp
finally show ?case using T undef by (auto simp add: defined-lit-map)
next
case (backtrack L D K i M1 M2 T) note decomp = this(3) and confli = this(1) and T = this(7)
and
  all-marked = this(9) and bt-lvl = this(8)
have atm-of L  $\notin$  atm-of ' lits-of-l M1
  using backtrack-lit-skipped[of S L K i M1 M2] backtrack(4,8-10) decomp
  by (fastforce simp add: lits-of-def)
moreover then have undefined-lit M1 L
  by (auto simp: defined-lit-map lits-of-def)
then have [simp]: trail T = Propagated L (mset-ccls D) # M1
  using T decomp n-d by auto
obtain c where M: trail S = c @ M2 @ Marked K (i + 1) # M1 using decomp by auto
have get-all-levels-of-marked (rev (trail S))
  = [Suc 0.. $(2 + \text{length } (\text{get-all-levels-of-marked } c) + (\text{length } (\text{get-all-levels-of-marked } M2) + \text{length } (\text{get-all-levels-of-marked } M1)))$ ]
  using all-marked bt-lvl unfolding M by (auto simp: rev-swap[symmetric] simp del: upt-simps)
then show ?case
  using T by (auto simp: rev-swap M simp del: upt-simps dest!: append-cons-eq-upt(1))
qed auto

```

We write $1 + \text{length } (\text{get-all-levels-of-marked } (\text{trail } S))$ instead of $\text{backtrack-lvl } S$ to avoid non termination of rewriting.

definition $cdcl_W$ -M-level-inv :: 'st \Rightarrow bool **where**
 $cdcl_W$ -M-level-inv $S \longleftrightarrow$
 consistent-interp (lits-of-l (trail S))
 \wedge no-dup (trail S)
 \wedge backtrack-lvl $S = \text{length } (\text{get-all-levels-of-marked } (\text{trail } S))$
 \wedge get-all-levels-of-marked (trail S)
 $= \text{rev } [1..<1+\text{length } (\text{get-all-levels-of-marked } (\text{trail } S))]$

lemma $cdcl_W$ -M-level-inv-decomp:
assumes $cdcl_W$ -M-level-inv S
shows
 consistent-interp (lits-of-l (trail S)) **and**
 no-dup (trail S)
using assms **unfolding** $cdcl_W$ -M-level-inv-def **by** fastforce+

lemma $cdcl_W$ -consistent-inv:
fixes $S S' :: 'st$
assumes
 $cdcl_W S S'$ **and**
 $cdcl_W$ -M-level-inv S
shows $cdcl_W$ -M-level-inv S'
using assms $cdcl_W$ -consistent-inv-2 $cdcl_W$ -distinctinv-1 $cdcl_W$ -bt $cdcl_W$ -bt-level'
unfolding $cdcl_W$ -M-level-inv-def **by** meson+

lemma rtrancpl- $cdcl_W$ -consistent-inv:
assumes
 $cdcl_W^{**} S S'$ **and**
 $cdcl_W$ -M-level-inv S
shows $cdcl_W$ -M-level-inv S'
using assms **by** (induct rule: rtrancpl-induct) (auto intro: $cdcl_W$ -consistent-inv)

lemma trancpl- $cdcl_W$ -consistent-inv:
assumes
 $cdcl_W^{++} S S'$ **and**
 $cdcl_W$ -M-level-inv S
shows $cdcl_W$ -M-level-inv S'
using assms **by** (induct rule: trancpl-induct)
(auto intro: $cdcl_W$ -consistent-inv)

lemma $cdcl_W$ -M-level-inv-S0- $cdcl_W$ [simp]:
 $cdcl_W$ -M-level-inv (init-state N)
unfolding $cdcl_W$ -M-level-inv-def **by** auto

lemma $cdcl_W$ -M-level-inv-get-level-le-backtrack-lvl:
assumes inv: $cdcl_W$ -M-level-inv S
shows get-level (trail S) $L \leq$ backtrack-lvl S
proof –
 have get-all-levels-of-marked (trail S) = rev $[1..<1 + \text{backtrack-lvl } S]$
 using inv **unfolding** $cdcl_W$ -M-level-inv-def **by** auto
 then show ?thesis
 using get-rev-level-less-max-get-all-levels-of-marked[of rev (trail S) 0 L]
 by (auto simp: Max-n-upt)
qed

lemma backtrack-ex-decomp:

```

assumes
  M-l: cdclW-M-level-inv S and
  i-S: i < backtrack-lvl S
shows  $\exists K\ M1\ M2. (\text{Marked } K\ (i + 1) \# M1, M2) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$ 
proof –
  let ?M = trail S
  have
    g: get-all-levels-of-marked (trail S) = rev [Suc 0..Suc (backtrack-lvl S)]
    using M-l unfolding cdclW-M-level-inv-def by simp-all
  then have i+1 ∈ set (get-all-levels-of-marked (trail S))
    using i-S by auto

  then obtain c K c' where tr-S: trail S = c @ Marked K (i + 1) # c'
    using in-get-all-levels-of-marked-iff-decomp[of i+1 trail S] by auto

  obtain M1 M2 where (Marked K (i + 1) # M1, M2) ∈ set (get-all-marked-decomposition (trail S))
    using Marked-cons-in-get-all-marked-decomposition-append-Marked-cons unfolding tr-S by fast
  then show ?thesis by blast
qed

```

19.3.2 Better-Suited Induction Principle

We generalise the induction principle defined previously: the induction case for *backtrack* now includes the assumption that *undefined-lit M1 L*. This helps the simplifier and thus the automation.

lemma *backtrack-induction-lev*[*consumes 1, case-names M-devel-inv backtrack*]:

```

assumes
  bt: backtrack S T and
  inv: cdclW-M-level-inv S and
  backtrackH:  $\bigwedge K\ i\ M1\ M2\ L\ D\ T.$ 
    raw-conflicting S = Some D  $\implies$ 
     $L \in \# \text{mset-ccls } D \implies$ 
     $(\text{Marked } K\ (\text{Suc } i) \# M1, M2) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S)) \implies$ 
    get-level (trail S) L = backtrack-lvl S  $\implies$ 
    get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D)  $\implies$ 
    get-maximum-level (trail S) (remove1-mset L (mset-ccls D))  $\equiv i$   $\implies$ 
    undefined-lit M1 L  $\implies$ 
     $T \sim \text{cons-trail } (\text{Propagated } L\ (\text{cls-of-ccls } D))$ 
    (reduce-trail-to M1
      (add-learned-cls (cls-of-ccls D)
        (update-backtrack-lvl i
          (update-conflicting None S))))  $\implies$ 
     $P\ S\ T$ 
shows  $P\ S\ T$ 
proof –
  obtain K i M1 M2 L D where
    decomp:  $(\text{Marked } K\ (\text{Suc } i) \# M1, M2) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$  and
    L: get-level (trail S) L = backtrack-lvl S and
    conf: raw-conflicting S = Some D and
    LD:  $L \in \# \text{mset-ccls } D$  and
    lev-L: get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) and
    lev-D: get-maximum-level (trail S) (remove1-mset L (mset-ccls D))  $\equiv i$  and
    T:  $T \sim \text{cons-trail } (\text{Propagated } L\ (\text{cls-of-ccls } D))$ 
    (reduce-trail-to M1
      (add-learned-cls (cls-of-ccls D)

```

```

      (update-backtrack-lvl i
       (update-conflicting None S))))
using bt by (elim backtrackE) metis

have atm-of L  $\notin$  atm-of ' lits-of-l M1
  using backtrack-lit-skipped[of S L K i M1 M2] L decomp bt confl lev-L lev-D inv
  unfolding cdclW-M-level-inv-def by force
then have undefined-lit M1 L
  by (auto simp: defined-lit-map lits-of-def)
then show ?thesis
  using backtrackH[OF confl LD decomp L lev-L lev-D - T] by simp
qed

lemmas backtrack-induction-lev2 = backtrack-induction-lev[consumes 2, case-names backtrack]

lemma cdclW-all-induct-lev-full:
  fixes S :: 'st
  assumes
    cdclW: cdclW S S' and
    inv[simp]: cdclW-M-level-inv S and
    propagateH:  $\bigwedge C L T. \text{conflicting } S = \text{None} \implies$ 
      C ! $\in$ ! raw-clauses S  $\implies$ 
      L  $\in$  # mset-cls C  $\implies$ 
      trail S  $\models_{\text{as}}$  CNot (remove1-mset L (mset-cls C))  $\implies$ 
      undefined-lit (trail S) L  $\implies$ 
      T  $\sim$  cons-trail (Propagated L C) S  $\implies$ 
      P S T and
    conflictH:  $\bigwedge D T. \text{conflicting } S = \text{None} \implies$ 
      D ! $\in$ ! raw-clauses S  $\implies$ 
      trail S  $\models_{\text{as}}$  CNot (mset-cls D)  $\implies$ 
      T  $\sim$  update-conflicting (Some (ccls-of-cls D)) S  $\implies$ 
      P S T and
    forgetH:  $\bigwedge C T. \text{conflicting } S = \text{None} \implies$ 
      C ! $\in$ ! raw-learned-clss S  $\implies$ 
       $\neg$ (trail S)  $\models_{\text{asm}}$  clauses S  $\implies$ 
      mset-cls C  $\notin$  set (get-all-mark-of-propagated (trail S))  $\implies$ 
      mset-cls C  $\notin$  # init-clss S  $\implies$ 
      T  $\sim$  remove-cls C S  $\implies$ 
      P S T and
    restartH:  $\bigwedge T. \neg \text{trail } S \models_{\text{asm}}$  clauses S  $\implies$ 
      conflicting S = None  $\implies$ 
      T  $\sim$  restart-state S  $\implies$ 
      P S T and
    decideH:  $\bigwedge L T. \text{conflicting } S = \text{None} \implies$ 
      undefined-lit (trail S) L  $\implies$ 
      atm-of L  $\in$  atms-of-mm (init-clss S)  $\implies$ 
      T  $\sim$  cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S)  $\implies$ 
      P S T and
    skipH:  $\bigwedge L C' M E T. \text{trail } S = \text{Propagated } L C' \# M \implies$ 
      raw-conflicting S = Some E  $\implies$ 
       $\neg L \notin$  # mset-ccls E  $\implies$  mset-ccls E  $\neq$  {#}  $\implies$ 
      T  $\sim$  tl-trail S  $\implies$ 
      P S T and
    resolveH:  $\bigwedge L E M D T.$ 

```

```

trail S = Propagated L (mset-cls E) # M ==>
L ∈# mset-cls E ==>
hd-raw-trail S = Propagated L E ==>
raw-conflicting S = Some D ==>
-L ∈# mset-ccls D ==>
get-maximum-level (trail S) (mset-ccls (remove-clit (-L) D)) = backtrack-lvl S ==>
T ~ update-conflicting
  (Some (union-ccls (remove-clit (-L) D) (ccls-of-cls (remove-lit L E)))) (tl-trail S) ==>
P S T and
backtrackH: ⋀ K i M1 M2 L D T.
  raw-conflicting S = Some D ==>
  L ∈# mset-ccls D ==>
  (Marked K (Suc i) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) ==>
  get-level (trail S) L = backtrack-lvl S ==>
  get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) ==>
  get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≡ i ==>
  undefined-lit M1 L ==>
  T ~ cons-trail (Propagated L (cls-of-ccls D))
    (reduce-trail-to M1
      (add-learned-cls (cls-of-ccls D)
        (update-backtrack-lvl i
          (update-conflicting None S)))) ==>
P S T
shows P S S'
using cdclW
proof (induct S' rule: cdclW-all-rules-induct)
case (propagate S')
then show ?case
  by (auto elim!: propagateE intro!: propagateH)
next
case (conflict S')
then show ?case
  by (auto elim!: conflictE intro!: conflictH)
next
case (restart S')
then show ?case
  by (auto elim!: restartE intro!: restartH)
next
case (decide T)
then show ?case
  by (auto elim!: decideE intro!: decideH)
next
case (backtrack S')
then show ?case
  apply (induction rule: backtrack-induction-lev)
  apply (rule inv)
  by (rule backtrackH;
      fastforce simp del: state-simp simp add: state-eq-def dest!: HOL.meta-eq-to-obj-eq)
next
case (forget S')
then show ?case by (auto elim!: forgetE intro!: forgetH)
next
case (skip S')
then show ?case by (auto elim!: skipE intro!: skipH)
next

```



```

case (resolve S')
then show ?case
  using hd-raw-trail[of S] by (cases trail S) (auto elim!: resolveE intro!: resolveH)
qed

lemmas cdclW-all-induct-lev2 = cdclW-all-induct-lev-full[consumes 2, case-names propagate conflict
forget restart decide skip resolve backtrack]

lemmas cdclW-all-induct-lev = cdclW-all-induct-lev-full[consumes 1, case-names lev-inv propagate
conflict forget restart decide skip resolve backtrack]

thm cdclW-o-induct
lemma cdclW-o-induct-lev[consumes 1, case-names M-lev decide skip resolve backtrack]:
fixes S :: 'st
assumes
  cdclW: cdclW-o S T and
  inv[simp]: cdclW-M-level-inv S and
  decideH:  $\bigwedge L T. \text{conflicting } S = \text{None} \implies$ 
    undefined-lit (trail S) L  $\implies$ 
    atm-of L  $\in$  atms-of-mm (init-clss S)  $\implies$ 
    T  $\sim$  cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S)  $\implies$ 
    P S T and
  skipH:  $\bigwedge L C' M E T. \text{trail } S = \text{Propagated } L C' \# M \implies$ 
    raw-conflicting S = Some E  $\implies$ 
     $-L \notin \# \text{mset-ccls } E \implies \text{mset-ccls } E \neq \{\#\} \implies$ 
    T  $\sim$  tl-trail S  $\implies$ 
    P S T and
  resolveH:  $\bigwedge L E M D T. \text{trail } S = \text{Propagated } L (\text{mset-clc } E) \# M \implies$ 
    L  $\in \# \text{mset-clc } E \implies$ 
    hd-raw-trail S = Propagated L E  $\implies$ 
    raw-conflicting S = Some D  $\implies$ 
     $-L \in \# \text{mset-ccls } D \implies$ 
    get-maximum-level (trail S) (mset-ccls (remove-clit (-L) D)) = backtrack-lvl S  $\implies$ 
    T  $\sim$  update-conflicting
      (Some (union-ccls (remove-clit (-L) D) (ccls-of-clc (remove-lit L E)))) (tl-trail S)  $\implies$ 
    P S T and
  backtrackH:  $\bigwedge K i M1 M2 L D T. \text{raw-conflicting } S = \text{Some } D \implies$ 
    L  $\in \# \text{mset-ccls } D \implies$ 
    (Marked K (Suc i)  $\#$  M1, M2)  $\in$  set (get-all-marked-decomposition (trail S))  $\implies$ 
    get-level (trail S) L = backtrack-lvl S  $\implies$ 
    get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D)  $\implies$ 
    get-maximum-level (trail S) (remove1-mset L (mset-ccls D))  $\equiv$  i  $\implies$ 
    undefined-lit M1 L  $\implies$ 
    T  $\sim$  cons-trail (Propagated L (cls-of-ccls D))
      (reduce-trail-to M1
        (add-learned-clc (cls-of-ccls D)
          (update-backtrack-lvl i
            (update-conflicting None S))))  $\implies$ 
    P S T
shows P S T
using cdclW
proof (induct S T rule: cdclW-o-all-rules-induct)

```

```

case (decide T)
then show ?case
  by (auto elim!: decideE intro!: decideH)
next
case (backtrack S')
then show ?case
  apply (induction rule: backtrack-induction-lev)
  apply (rule inv)
  by (rule backtrackH;
    fastforce simp del: state-simp simp add: state-eq-def dest!: HOL.meta-eq-to-obj-eq)
next
case (skip S')
then show ?case by (auto elim!: skipE intro!: skipH)
next
case (resolve S')
then show ?case
  using hd-raw-trail[of S] by (cases trail S) (auto elim!: resolveE intro!: resolveH)
qed

lemmas cdclW-o-induct-lev2 = cdclW-o-induct-lev[consumes 2, case-names decide skip resolve backtrack]

```

19.3.3 Compatibility with $op \sim$

```

lemma propagate-state-eq-compatible:
assumes
  propa: propagate S T and
  SS':  $S \sim S'$  and
  TT':  $T \sim T'$ 
shows propagate S' T'
proof –
obtain C L where
  conf: conflicting S = None and
  C:  $C \notin \text{raw-clauses } S$  and
  L:  $L \in \# \text{mset-cls } C$  and
  tr:  $\text{trail } S \models_{\text{as}} C \text{Not } (\text{remove1-mset } L (\text{mset-cls } C))$  and
  undef: undefined-lit (trail S) L and
  T:  $T \sim \text{cons-trail } (\text{Propagated } L \ C) \ S$ 
using propa by (elim propagateE) auto

obtain C' where
  CC':  $\text{mset-cls } C' = \text{mset-cls } C$  and
  C':  $C' \notin \text{raw-clauses } S'$ 
using SS' C
in-mset-clss-exists-preimage[of mset-cls C raw-learned-clss S]
in-mset-clss-exists-preimage[of mset-cls C raw-init-clss S]
apply –
apply (frule in-clss-mset-clss)
by (auto simp: state-eq-def raw-clauses-def simp del: state-simp dest: in-clss-mset-clss)

show ?thesis
apply (rule propagate-rule[of - C'])
using state-eq-sym[of S S'] SS' conf C' CC' L tr undef TT' T
by (auto simp: state-eq-def simp del: state-simp)
qed

```

lemma *conflict-state-eq-compatible*:

assumes

conf: *conflict* *S* *T* **and**

TT': $T \sim T'$ **and**

SS': $S \sim S'$

shows *conflict* *S'* *T'*

proof –

obtain *D* **where**

conf: *conflicting* *S* = *None* **and**

D: $D \notin \text{raw-clauses } S$ **and**

tr: $\text{trail } S \models_{\text{as}} \text{CNot } (\text{mset-cls } D)$ **and**

T: $T \sim \text{update-conflicting } (\text{Some } (\text{ccls-of-cls } D)) \text{ } S$

using *conf* **by** (*elim conflictE*) *auto*

obtain *D'* **where**

DD': $\text{mset-cls } D' = \text{mset-cls } D$ **and**

D': $D' \notin \text{raw-clauses } S'$

using *D* *SS'* *in-mset-clss-exists-preimage* **by** *fastforce*

show *?thesis*

apply (*rule conflict-rule*[*of* - *D'*])

using *state-eq-sym*[*of* *S* *S'*] *SS'* *conf* *D'* *DD'* *tr* *TT'* *T*

by (*auto simp: state-eq-def simp del: state-simp*)

qed

lemma *backtrack-levE*[*consumes 2*]:

$\text{backtrack } S \text{ } S' \implies \text{cdcl}_W\text{-M-level-inv } S \implies$

$(\bigwedge K \ i \ M1 \ M2 \ L \ D.$

$\text{raw-conflicting } S = \text{Some } D \implies$

$L \in \# \text{ mset-ccls } D \implies$

$(\text{Marked } K \ (\text{Suc } i) \ \# \ M1, \ M2) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S)) \implies$

$\text{get-level } (\text{trail } S) \ L = \text{backtrack-lvl } S \implies$

$\text{get-level } (\text{trail } S) \ L = \text{get-maximum-level } (\text{trail } S) \ (\text{mset-ccls } D) \implies$

$\text{get-maximum-level } (\text{trail } S) \ (\text{remove1-mset } L \ (\text{mset-ccls } D)) \equiv i \implies$

$\text{undefined-lit } M1 \ L \implies$

$S' \sim \text{cons-trail } (\text{Propagated } L \ (\text{cls-of-ccls } D))$

$(\text{reduce-trail-to } M1$

$(\text{add-learned-cls } (\text{cls-of-ccls } D)$

$(\text{update-backtrack-lvl } i$

$(\text{update-conflicting } \text{None } S)))) \implies P) \implies$

P

using *assms* **by** (*induction rule: backtrack-induction-lev2*) *metis*

thm *allI*

lemma *backtrack-state-eq-compatible*:

assumes

bt: *backtrack* *S* *T* **and**

SS': $S \sim S'$ **and**

TT': $T \sim T'$ **and**

inv: $\text{cdcl}_W\text{-M-level-inv } S$

shows *backtrack* *S'* *T'*

proof –

obtain *D* *L* *K* *i* *M1* *M2* **where**

conf: *raw-conflicting* *S* = *Some* *D* **and**

L: $L \in \# \text{ mset-ccls } D$ **and**

```

    decomp: (Marked  $K$  (Suc  $i$ ) #  $M1$ ,  $M2$ )  $\in$  set (get-all-marked-decomposition (trail  $S$ )) and
    lev: get-level (trail  $S$ )  $L$  = backtrack-lvl  $S$  and
    max: get-level (trail  $S$ )  $L$  = get-maximum-level (trail  $S$ ) (mset-ccls  $D$ ) and
    max-D: get-maximum-level (trail  $S$ ) (remove1-mset  $L$  (mset-ccls  $D$ ))  $\equiv i$  and
    undef: undefined-lit  $M1$   $L$  and
    T:  $T \sim$  cons-trail (Propagated  $L$  (cls-of-ccls  $D$ ))
      (reduce-trail-to  $M1$ 
        (add-learned-cls (cls-of-ccls  $D$ )
          (update-backtrack-lvl  $i$ 
            (update-conflicting None  $S$ ))))
  using bt inv by (elim backtrack-levE)metis
  obtain  $D'$  where
     $D'$ : raw-conflicting  $S' =$  Some  $D'$ 
    using  $SS'$  conf by (cases raw-conflicting  $S'$ ) auto
  have [simp]: mset-ccls  $D =$  mset-ccls  $D'$ 
    using  $SS'$   $D'$  conf by (auto simp: state-eq-def simp del: state-simp)[]

  have  $T'$ :  $T' \sim$  cons-trail (Propagated  $L$  (cls-of-ccls  $D'$ ))
    (reduce-trail-to  $M1$  (add-learned-cls (cls-of-ccls  $D'$ )
      (update-backtrack-lvl  $i$  (update-conflicting None  $S'$ ))))
    using  $TT'$  unfolding state-eq-def
    using decomp undef inv  $SS'$   $T$  by (auto simp add: cdclW-M-level-inv-def)

  show ?thesis
    apply (rule backtrack-rule[of -  $D'$ ])
      apply (rule  $D'$ )
        using state-eq-sym[of  $S$   $S'$ ]  $TT'$   $SS'$   $D'$  conf  $L$  decomp lev max max-D undef  $T$ 
        apply (auto simp: state-eq-def simp del: state-simp)[]
        using decomp  $SS'$  lev  $SS'$  max-D max  $T'$  by (auto simp: state-eq-def simp del: state-simp)
  qed

```

lemma decide-state-eq-compatible:

```

  assumes
    decide  $S$   $T$  and
     $S \sim S'$  and
     $T \sim T'$ 
  shows decide  $S'$   $T'$ 
  using assms apply (elim decideE)
  by (rule decide-rule) (auto simp: state-eq-def raw-clauses-def simp del: state-simp)

```

lemma skip-state-eq-compatible:

```

  assumes
    skip: skip  $S$   $T$  and
     $SS'$ :  $S \sim S'$  and
     $TT'$ :  $T \sim T'$ 
  shows skip  $S'$   $T'$ 
  proof -
    obtain  $L$   $C'$   $M$   $E$  where
      tr: trail  $S =$  Propagated  $L$   $C'$  #  $M$  and
      raw: raw-conflicting  $S =$  Some  $E$  and
      L:  $-L \notin$  mset-ccls  $E$  and
      E: mset-ccls  $E \neq \{\#\}$  and
      T:  $T \sim$  tl-trail  $S$ 
    using skip by (elim skipE) simp
  qed

```

obtain E' **where** E' : *raw-conflicting* $S' = \text{Some } E'$
using SS' *raw* **by** (*cases* *raw-conflicting* S') (*auto simp: state-eq-def simp del: state-simp*)
show *?thesis*
apply (*rule skip-rule*)
using tr *raw* $L E T SS'$ **apply** (*auto simp: simp del:*)[]
using E' **apply** *simp*
using $E' SS' L$ *raw* E **apply** (*auto simp: state-eq-def simp del: state-simp*)[2]
using $T TT' SS'$ **by** (*auto simp: state-eq-def simp del: state-simp*)
qed

lemma *resolve-state-eq-compatible:*

assumes
res: resolve $S T$ **and**
 TT' : $T \sim T'$ **and**
 SS' : $S \sim S'$
shows *resolve* $S' T'$
proof –
obtain $E D L$ **where**
 tr : *trail* $S \neq []$ **and**
 hd : *hd-raw-trail* $S = \text{Propagated } L E$ **and**
 L : $L \in \# \text{ mset-cls } E$ **and**
 raw : *raw-conflicting* $S = \text{Some } D$ **and**
 LD : $-L \in \# \text{ mset-ccls } D$ **and**
 i : *get-maximum-level* (*trail* S) (*mset-ccls* (*remove-clit* ($-L$) D)) = *backtrack-lvl* S **and**
 T : $T \sim \text{update-conflicting} (\text{Some} (\text{union-ccls} (\text{remove-clit} (-L) D) (\text{ccls-of-cls} (\text{remove-lit } L E)))) (\text{tl-trail } S)$
using *assms* **by** (*elim resolveE*) *simp*

obtain E' **where**
 E' : *hd-raw-trail* $S' = \text{Propagated } L E'$
using SS' hd **by** (*metis* (*trail* $S \neq []$) *hd-raw-trail is-proped-def marked-lit.disc*(3) *marked-lit.inject*(2) *mmset-of-mlit.elims state-eq-trail*)
have [*simp*]: *mset-cls* $E = \text{mset-cls } E'$
using *hd-raw-trail*[*of* S] tr *hd-raw-trail*[*of* S'] $tr SS' hd E'$
by (*metis* *marked-lit.inject*(2) *mmset-of-mlit.simps*(1) *state-eq-trail*)
obtain D' **where**
 D' : *raw-conflicting* $S' = \text{Some } D'$
using SS' *raw* **by** *fastforce*
have [*simp*]: *mset-ccls* $D = \text{mset-ccls } D'$
using $D' SS'$ *raw* *state-simp*(5) **by** *fastforce*
have $T'T$: $T' \sim T$
using TT' *state-eq-sym* **by** *auto*
show *?thesis*
apply (*rule resolve-rule*)
using $tr SS'$ **apply** *simp*
using E' **apply** *simp*
using L **apply** *simp*
using D' **apply** *simp*
using $D' SS'$ *raw* LD **apply** (*auto simp add: state-eq-def simp del: state-simp*)[]
using $D' SS'$ *raw* LD **apply** (*auto simp add: state-eq-def simp del: state-simp*)[]
using *raw* $SS' i$ **apply** (*auto simp add: state-eq-def simp del: state-simp*)[]
using $T T'T SS'$ **by** (*auto simp: state-eq-def simp del: state-simp*)
qed

lemma *forget-state-eq-compatible:*

```

assumes
  forget: forget  $S$   $T$  and
  SS':  $S \sim S'$  and
  TT':  $T \sim T'$ 
shows forget  $S'$   $T'$ 
proof –
  obtain  $C$  where
    conf: conflicting  $S = \text{None}$  and
    C !∈! raw-learned-clss  $S$  and
    tr:  $\neg(\text{trail } S) \models_{asm} \text{clauses } S$  and
    C1: mset-cls  $C \notin \text{set } (\text{get-all-mark-of-propagated } (\text{trail } S))$  and
    C2: mset-cls  $C \notin \# \text{init-clss } S$  and
    T:  $T \sim \text{remove-cls } C$   $S$ 
    using forget by (elim forgetE) simp

  obtain  $C'$  where
    C':  $C' !\in! \text{raw-learned-clss } S'$  and
    [simp]: mset-cls  $C' = \text{mset-cls } C$ 
    using  $\langle C !\in! \text{raw-learned-clss } S \rangle$  SS' in-mset-clss-exists-preimage by fastforce
  show ?thesis
    apply (rule forget-rule)
      using SS' conf apply simp
      using C' apply simp
      using SS' tr apply simp
      using SS' C1 apply simp
      using SS' C2 apply simp
      using T TT' SS' by (auto simp: state-eq-def simp del: state-simp)
  qed

lemma cdclW-state-eq-compatible:
assumes
  cdclW  $S$   $T$  and  $\neg \text{restart } S$   $T$  and
   $S \sim S'$ 
   $T \sim T'$  and
  cdclW-M-level-inv  $S$ 
shows cdclW  $S'$   $T'$ 
using assms by (meson backtrack backtrack-state-eq-compatible bj cdclW.simps cdclW-o-rule-cases
  cdclW-rf.cases conflict-state-eq-compatible decide decide-state-eq-compatible forget
  forget-state-eq-compatible propagate-state-eq-compatible resolve resolve-state-eq-compatible
  skip skip-state-eq-compatible state-eq-ref)

lemma cdclW-bj-state-eq-compatible:
assumes
  cdclW-bj  $S$   $T$  and cdclW-M-level-inv  $S$ 
   $T \sim T'$ 
shows cdclW-bj  $S$   $T'$ 
using assms by (meson backtrack backtrack-state-eq-compatible cdclW-bjE resolve
  resolve-state-eq-compatible skip skip-state-eq-compatible state-eq-ref)

lemma tranclp-cdclW-bj-state-eq-compatible:
assumes
  cdclW-bj++  $S$   $T$  and inv: cdclW-M-level-inv  $S$  and
   $S \sim S'$  and
   $T \sim T'$ 
shows cdclW-bj++  $S'$   $T'$ 

```

```

using assms
proof (induction arbitrary: S' T')
  case base
  then show ?case
    unfolding trancpl-unfold-end by (meson backtrack-state-eq-compatible cdclW-bj.simps
      resolve-state-eq-compatible rtrancpl-unfold skip-state-eq-compatible)
  next
  case (step T U) note IH = this(3)[OF this(4-5)]
  have cdclW++ S T
    using trancpl-mono[of cdclW-bj cdclW] step.hyps(1) cdclW.other cdclW-o.bj by blast
  then have cdclW-M-level-inv T
    using inv trancpl-cdclW-consistent-inv by blast
  then have cdclW-bj++ T T'
    using  $\langle U \sim T' \rangle$  cdclW-bj-state-eq-compatible[of T U]  $\langle cdcl_W\text{-}bj\ T\ U \rangle$  by auto
  then show ?case
    using IH[of T] by auto
qed

```

19.3.4 Conservation of some Properties

```

lemma cdclW-o-no-more-init-clss:
  assumes
    cdclW-o S S' and
    inv: cdclW-M-level-inv S
  shows init-clss S = init-clss S'
  using assms by (induct rule: cdclW-o-induct-lev2) (auto simp: inv cdclW-M-level-inv-decomp)

```

```

lemma trancpl-cdclW-o-no-more-init-clss:
  assumes
    cdclW-o++ S S' and
    inv: cdclW-M-level-inv S
  shows init-clss S = init-clss S'
  using assms apply (induct rule: trancpl.induct)
  by (auto dest: cdclW-o-no-more-init-clss
    dest!: trancpl-cdclW-consistent-inv dest: trancpl-mono-explicit[of cdclW-o - - cdclW]
    simp: other)

```

```

lemma rtrancpl-cdclW-o-no-more-init-clss:
  assumes
    cdclW-o** S S' and
    inv: cdclW-M-level-inv S
  shows init-clss S = init-clss S'
  using assms unfolding rtrancpl-unfold by (auto intro: trancpl-cdclW-o-no-more-init-clss)

```

```

lemma cdclW-init-clss:
  assumes
    cdclW S T and
    inv: cdclW-M-level-inv S
  shows init-clss S = init-clss T
  using assms by (induct rule: cdclW-all-induct-lev2)
  (auto simp: inv cdclW-M-level-inv-decomp not-in-iff)

```

```

lemma rtrancpl-cdclW-init-clss:
  cdclW** S T  $\implies$  cdclW-M-level-inv S  $\implies$  init-clss S = init-clss T
  by (induct rule: rtrancpl-induct) (auto dest: cdclW-init-clss rtrancpl-cdclW-consistent-inv)

```

lemma *trancpl-cdcl_W-init-clss*:
 $cdcl_W^{++} S T \implies cdcl_W\text{-}M\text{-level-inv } S \implies \text{init-clss } S = \text{init-clss } T$
using *rtrancpl-cdcl_W-init-clss*[of *S T*] **unfolding** *rtrancpl-unfold* **by** *auto*

19.3.5 Learned Clause

This invariant shows that:

- the learned clauses are entailed by the initial set of clauses.
- the conflicting clause is entailed by the initial set of clauses.
- the marks are entailed by the clauses. A more precise version would be to show that either these marked are learned or are in the set of clauses

definition *cdcl_W-learned-clause* (*S*:: 'st) \longleftrightarrow
 $(\text{init-clss } S \models_{psm} \text{learned-clss } S$
 $\wedge (\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{init-clss } S \models_{pm} T)$
 $\wedge \text{set } (\text{get-all-mark-of-propagated } (\text{trail } S)) \subseteq \text{set-mset } (\text{clauses } S))$

lemma *cdcl_W-learned-clause-S0-cdcl_W[simp]*:
 $cdcl_W\text{-learned-clause } (\text{init-state } N)$
unfolding *cdcl_W-learned-clause-def* **by** *auto*

lemma *cdcl_W-learned-clss*:

assumes

$cdcl_W S S'$ **and**

learned: *cdcl_W-learned-clause* *S* **and**

lev-inv: *cdcl_W-M-level-inv* *S*

shows *cdcl_W-learned-clause* *S'*

using *assms*(1) *lev-inv* *learned*

proof (*induct rule*: *cdcl_W-all-induct-lev2*)

case (*backtrack* *K i M1 M2 L D T*) **note** *decomp* = *this*(3) **and** *confl* = *this*(1) **and** *undef* = *this*(7)
and *T* = *this*(8)

show ?*case*

using *decomp confl learned undef T* **unfolding** *cdcl_W-learned-clause-def*

by (*auto dest*!:: *get-all-marked-decomposition-exists-prepend*

simp: *raw-clauses-def lev-inv cdcl_W-M-level-inv-decomp dest*: *true-clss-clss-left-right*)

next

case (*resolve* *L C M D*) **note** *trail* = *this*(1) **and** *CL* = *this*(2) **and** *confl* = *this*(4) **and** *DL* = *this*(5)
and *lwl* = *this*(6) **and** *T* = *this*(7)

moreover

have $\text{init-clss } S \models_{psm} \text{learned-clss } S$

using *learned trail* **unfolding** *cdcl_W-learned-clause-def raw-clauses-def* **by** *auto*

then have $\text{init-clss } S \models_{pm} \text{mset-clss } C + \{\#L\# \}$

using *trail learned* **unfolding** *cdcl_W-learned-clause-def raw-clauses-def*

by (*auto dest*: *true-clss-clss-in-imp-true-clss-clss*)

moreover have $\text{remove1-mset } (- L) (\text{mset-clss } D) + \{\#- L\# \} = \text{mset-clss } D$

using *DL* **by** (*auto simp*: *multiset-eq-iff*)

moreover have $\text{remove1-mset } L (\text{mset-clss } C) + \{\#L\# \} = \text{mset-clss } C$

using *CL* **by** (*auto simp*: *multiset-eq-iff*)

ultimately show ?*case*

using *learned T*

by (*auto dest*: *mk-disjoint-insert*)


```

    simp add: cdclW-learned-clause-def raw-clauses-def
    intro!: true-clss-clss-union-mset-true-clss-clss-or-not-true-clss-clss-or[of - L])
next
case (restart T)
then show ?case
using learned learned-clss-restart-state[of T]
by (auto
    simp: raw-clauses-def state-eq-def cdclW-learned-clause-def
    simp del: state-simp
    dest: true-clss-clssm-subsetE)
next
case propagate
then show ?case using learned by (auto simp: cdclW-learned-clause-def)
next
case conflict
then show ?case using learned
by (fastforce simp: cdclW-learned-clause-def raw-clauses-def
    true-clss-clss-in-imp-true-clss-clss)
next
case (forget U)
then show ?case using learned
by (auto simp: cdclW-learned-clause-def raw-clauses-def split: if-split-asm)
qed (auto simp: cdclW-learned-clause-def raw-clauses-def)

lemma rtrancplp-cdclW-learned-clss:
  assumes
    cdclW** S S' and
    cdclW-M-level-inv S
    cdclW-learned-clause S
  shows cdclW-learned-clause S'
  using assms by induction (auto dest: cdclW-learned-clss intro: rtrancplp-cdclW-consistent-inv)

```

19.3.6 No alien atom in the state

This invariant means that all the literals are in the set of clauses.

definition *no-strange-atm* $S' \longleftrightarrow$ (
 $(\forall T. \text{conflicting } S' = \text{Some } T \longrightarrow \text{atms-of } T \subseteq \text{atms-of-mm } (\text{init-clss } S'))$
 $\wedge (\forall L \text{ mark. Propagated } L \text{ mark} \in \text{set } (\text{trail } S') \longrightarrow \text{atms-of } (\text{mark}) \subseteq \text{atms-of-mm } (\text{init-clss } S'))$
 $\wedge \text{atms-of-mm } (\text{learned-clss } S') \subseteq \text{atms-of-mm } (\text{init-clss } S')$
 $\wedge \text{atm-of } ' (\text{lits-of-l } (\text{trail } S')) \subseteq \text{atms-of-mm } (\text{init-clss } S'))$

lemma *no-strange-atm-decomp*:
 assumes *no-strange-atm* S
 shows *conflicting* S = *Some* T $\implies \text{atms-of } T \subseteq \text{atms-of-mm } (\text{init-clss } S)$
 and $(\forall L \text{ mark. Propagated } L \text{ mark} \in \text{set } (\text{trail } S) \longrightarrow \text{atms-of } (\text{mark}) \subseteq \text{atms-of-mm } (\text{init-clss } S))$
 and $\text{atms-of-mm } (\text{learned-clss } S) \subseteq \text{atms-of-mm } (\text{init-clss } S)$
 and $\text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-mm } (\text{init-clss } S)$
 using assms **unfolding** *no-strange-atm-def* **by** blast+

lemma *no-strange-atm-S0* [simp]: *no-strange-atm* (init-state N)
unfolding *no-strange-atm-def* **by** auto

lemma *in-atms-of-implies-atm-of-on-atms-of-ms*:

$C + \{\#L\# \} \in \# A \implies x \in \text{atms-of } C \implies x \in \text{atms-of-mm } A$
using *multi-member-split* **by** *fastforce*

lemma *propagate-no-strange-atm-inv*:

assumes

propagate S T **and**

alien: *no-strange-atm* S

shows *no-strange-atm* T

using *assms*(1)

proof (*induction*)

case (*propagate-rule* C L T) **note** *confl* = *this*(1) **and** C = *this*(2) **and** $C-L$ = *this*(3) **and**
 tr = *this*(4) **and** *undef* = *this*(5) **and** T = *this*(6)

have *atm-CL*: *atms-of* (*mset-cls* C) \subseteq *atms-of-mm* (*init-clss* S)

using C *alien* **unfolding** *no-strange-atm-def*

by (*auto simp*: *raw-clauses-def* *atms-of-ms-def* *dest!*:*in-clss-mset-clss*)

show ?*case*

unfolding *no-strange-atm-def*

proof (*intro conjI allI impI*, *goal-cases*)

case 1

then show ?*case*

using *confl* T *undef* **by** *auto*

next

case (2 L' *mark'*)

then show ?*case*

using $C-L$ T *alien* *undef* *atm-CL*

unfolding *no-strange-atm-def* *raw-clauses-def* **apply** *auto* **by** *blast*

next

case (3)

show ?*case* **using** T *alien* *undef* **unfolding** *no-strange-atm-def* **by** *auto*

next

case (4)

show ?*case*

using T *alien* *undef* $C-L$ *atm-CL* **unfolding** *no-strange-atm-def* **by** (*auto simp*: *atms-of-def*)

qed

qed

lemma *in-atms-of-remove1-mset-in-atms-of*:

$x \in \text{atms-of} (\text{remove1-mset } L \ C) \implies x \in \text{atms-of } C$

using *in-diffD* **unfolding** *atms-of-def* **by** *fastforce*

lemma *cdcl_W-no-strange-atm-explicit*:

assumes

cdcl_W S S' **and**

lev: *cdcl_W-M-level-inv* S **and**

conf: $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{atms-of } T \subseteq \text{atms-of-mm} (\text{init-clss } S)$ **and**

marked: $\forall L \text{ mark}. \text{Propagated } L \text{ mark} \in \text{set } (\text{trail } S)$

$\longrightarrow \text{atms-of mark} \subseteq \text{atms-of-mm} (\text{init-clss } S)$ **and**

learned: $\text{atms-of-mm} (\text{learned-clss } S) \subseteq \text{atms-of-mm} (\text{init-clss } S)$ **and**

trail: *atm-of* ' (*lits-of-l* (*trail* S))) \subseteq *atms-of-mm* (*init-clss* S)

shows

$(\forall T. \text{conflicting } S' = \text{Some } T \longrightarrow \text{atms-of } T \subseteq \text{atms-of-mm} (\text{init-clss } S')) \wedge$

$(\forall L \text{ mark}. \text{Propagated } L \text{ mark} \in \text{set } (\text{trail } S'))$

$\longrightarrow \text{atms-of } (\text{mark}) \subseteq \text{atms-of-mm} (\text{init-clss } S')) \wedge$

$\text{atms-of-mm} (\text{learned-clss } S') \subseteq \text{atms-of-mm} (\text{init-clss } S') \wedge$

```

    atm-of ' (lits-of-l (trail S'))  $\subseteq$  atms-of-mm (init-clss S')
    (is ?C S'  $\wedge$  ?M S'  $\wedge$  ?U S'  $\wedge$  ?V S')
  using assms(1,2)
proof (induct rule: cdelW-all-induct-lev2)
  case (propagate C L T) note confl = this(1) and C-L = this(2) and tr = this(3) and undef =
  this(4)
  and T = this(5)
  show ?case
    using propagate-rule[OF propagate.hyps(1-3) - propagate.hyps(5,6), simplified]
    propagate.hyps(4) propagate-no-strange-atm-inv[of S T]
    conf marked learned trail unfolding no-strange-atm-def by presburger
next
  case (decide L)
  then show ?case using learned marked conf trail unfolding raw-clauses-def by auto
next
  case (skip L C M D)
  then show ?case using learned marked conf trail by auto
next
  case (conflict D T) note D-S = this(2) and T = this(4)
  have D: atm-of ' set-mset (mset-cls D)  $\subseteq$   $\bigcup$  (atms-of ' (set-mset (clauses S)))
    using D-S by (auto simp add: atms-of-def atms-of-ms-def)
  moreover {
    fix xa :: 'v literal
    assume a1: atm-of ' set-mset (mset-cls D)  $\subseteq$  ( $\bigcup$   $x \in$  set-mset (init-clss S). atms-of x)
       $\cup$  ( $\bigcup$   $x \in$  set-mset (learned-clss S). atms-of x)
    assume a2:
      ( $\bigcup$   $x \in$  set-mset (learned-clss S). atms-of x)  $\subseteq$  ( $\bigcup$   $x \in$  set-mset (init-clss S). atms-of x)
    assume xa  $\in$  # mset-cls D
    then have atm-of xa  $\in$  UNION (set-mset (init-clss S)) atms-of
      using a2 a1 by (metis (no-types) Un-iff atm-of-lit-in-atms-of atms-of-def subset-Un-eq)
    then have  $\exists m \in$  set-mset (init-clss S). atm-of xa  $\in$  atms-of m
      by blast
  } note H = this
  ultimately show ?case using conflict.premis T learned marked conf trail
    unfolding atms-of-def atms-of-ms-def raw-clauses-def
    by (auto simp add: H)
next
  case (restart T)
  then show ?case using learned marked conf trail by auto
next
  case (forget C T) note confl = this(1) and C = this(4) and C-le = this(5) and
    T = this(6)
  have H:  $\bigwedge L$  mark. Propagated L mark  $\in$  set (trail S)  $\implies$  atms-of mark  $\subseteq$  atms-of-mm (init-clss S)
    using marked by simp
  show ?case unfolding raw-clauses-def apply (intro conjI)
    using conf confl T trail C unfolding raw-clauses-def apply (auto dest!: H)[]
    using T trail C C-le apply (auto dest!: H)[]
    using T learned C-le atms-of-ms-remove-subset[of set-mset (learned-clss S)] apply auto[]
    using T trail C-le apply (auto simp: raw-clauses-def lits-of-def)[]
  done
next
  case (backtrack K i M1 M2 L D T) note confl = this(1) and LD = this(2) and decomp = this(3)
  and
    undef = this(7) and T = this(8)
  have ?C T

```

```

    using conf T decomp undef lev by (auto simp: cdclW-M-level-inv-decomp)
  moreover have set M1 ⊆ set (trail S)
    using decomp by auto
  then have M: ?M T
    using marked conf undef confl T decomp lev
    by (auto simp: image-subset-iff raw-clauses-def cdclW-M-level-inv-decomp)
  moreover have ?U T
    using learned decomp conf confl T undef lev unfolding raw-clauses-def
    by (auto simp: cdclW-M-level-inv-decomp)
  moreover have ?V T
    using M conf confl trail T undef decomp lev LD
    by (auto simp: cdclW-M-level-inv-decomp atms-of-def
      dest!: get-all-marked-decomposition-exists-prepend)
  ultimately show ?case by blast
next
case (resolve L C M D T) note trail-S = this(1) and confl = this(4) and T = this(7)
let ?T = update-conflicting (Some ((remove-clit (-L) D) !⊔ ccls-of-cls ((remove-lit L C))))
  (tl-trail S)
have ?C ?T
  using confl trail-S conf marked by (auto dest!: in-atms-of-remove1-mset-in-atms-of)
moreover have ?M ?T
  using confl trail-S conf marked by auto
moreover have ?U ?T
  using trail learned by auto
moreover have ?V ?T
  using confl trail-S trail by auto
ultimately show ?case using T by simp
qed

```

lemma *cdcl_W-no-strange-atm-inv*:
 assumes *cdcl_W S S'* and *no-strange-atm S* and *cdcl_W-M-level-inv S*
 shows *no-strange-atm S'*
 using *cdcl_W-no-strange-atm-explicit*[*OF assms(1)*] *assms(2,3)* unfolding *no-strange-atm-def* by fast

lemma *rtranclp-cdcl_W-no-strange-atm-inv*:
 assumes *cdcl_W** S S'* and *no-strange-atm S* and *cdcl_W-M-level-inv S*
 shows *no-strange-atm S'*
 using *assms* by induction (auto intro: *cdcl_W-no-strange-atm-inv rtranclp-cdcl_W-consistent-inv*)

19.3.7 No duplicates all around

This invariant shows that there is no duplicate (no literal appearing twice in the formula). The last part could be proven using the previous invariant moreover.

definition *distinct-cdcl_W-state* (*S::'st*)
 $\longleftrightarrow ((\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{distinct-mset } T)$
 $\wedge \text{distinct-mset-mset } (\text{learned-clss } S)$
 $\wedge \text{distinct-mset-mset } (\text{init-clss } S)$
 $\wedge (\forall L \text{ mark. } (\text{Propagated } L \text{ mark} \in \text{set } (\text{trail } S) \longrightarrow \text{distinct-mset } (\text{mark}))))$

lemma *distinct-cdcl_W-state-decomp*:
 assumes *distinct-cdcl_W-state* (*S::'st*)
 shows $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{distinct-mset } T$
 and *distinct-mset-mset* (*learned-clss S*)
 and *distinct-mset-mset* (*init-clss S*)
 and $\forall L \text{ mark. } (\text{Propagated } L \text{ mark} \in \text{set } (\text{trail } S) \longrightarrow \text{distinct-mset } (\text{mark}))$

```

using assms unfolding distinct-cdclW-state-def by blast+

lemma distinct-cdclW-state-decomp-2:
  assumes distinct-cdclW-state (S::'st)
  shows conflicting S = Some T  $\implies$  distinct-mset T
  using assms unfolding distinct-cdclW-state-def by auto

lemma distinct-cdclW-state-S0-cdclW[simp]:
  distinct-mset-mset (mset-clss N)  $\implies$  distinct-cdclW-state (init-state N)
  unfolding distinct-cdclW-state-def by auto

lemma distinct-cdclW-state-inv:
  assumes
    cdclW S S' and
    lev-inv: cdclW-M-level-inv S and
    distinct-cdclW-state S
  shows distinct-cdclW-state S'
  using assms(1,2,2,3)
proof (induct rule: cdclW-all-induct-lev2)
  case (backtrack L D K i M1 M2)
  then show ?case
    using lev-inv unfolding distinct-cdclW-state-def
    by (auto dest: get-all-marked-decomposition-incl simp: cdclW-M-level-inv-decomp)
next
  case restart
  then show ?case
    unfolding distinct-cdclW-state-def distinct-mset-set-def raw-clauses-def
    using learned-clss-restart-state[of S] by auto
next
  case resolve
  then show ?case
    by (auto simp add: distinct-cdclW-state-def distinct-mset-set-def raw-clauses-def
      distinct-mset-single-add
      intro!: distinct-mset-union-mset)
qed (auto simp: distinct-cdclW-state-def distinct-mset-set-def raw-clauses-def
  dest!: in-clss-mset-clss in-diffD)

lemma rtanclp-distinct-cdclW-state-inv:
  assumes
    cdclW** S S' and
    cdclW-M-level-inv S and
    distinct-cdclW-state S
  shows distinct-cdclW-state S'
  using assms apply (induct rule: rtanclp-induct)
  using distinct-cdclW-state-inv rtanclp-cdclW-consistent-inv by blast+

```

19.3.8 Conflicts and co

This invariant shows that each mark contains a contradiction only related to the previously defined variable.

abbreviation *every-mark-is-a-conflict :: 'st \Rightarrow bool* **where**
every-mark-is-a-conflict S \equiv
 $\forall L \text{ mark } a \text{ b. } a @ \text{Propagated } L \text{ mark} \# b = (\text{trail } S)$
 $\longrightarrow (b \models_{as} CNot (\text{mark} - \{\#L\}) \wedge L \in \# \text{ mark})$

definition $cdcl_W$ -conflicting $S \equiv$
 $(\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{as} \text{CNot } T)$
 $\wedge \text{every-mark-is-a-conflict } S$

lemma *backtrack-atms-of-D-in-M1*:

fixes $M1 :: ('v, \text{nat}, 'v \text{ clause}) \text{ marked-lits}$

assumes

$\text{inv}: cdcl_W\text{-}M\text{-level-inv } S \text{ and}$

$\text{undef}: \text{undefined-lit } M1 \ L \text{ and}$

$i: \text{get-maximum-level } (\text{trail } S) \ (\text{mset-ccls } (\text{remove-clit } L \ D)) \equiv i \text{ and}$

$\text{decomp}: (\text{Marked } K \ (\text{Suc } i) \ \# \ M1, \ M2)$

$\in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S)) \text{ and}$

$S\text{-lvl}: \text{backtrack-lvl } S = \text{get-maximum-level } (\text{trail } S) \ (\text{mset-ccls } D) \text{ and}$

$S\text{-confl}: \text{raw-conflicting } S = \text{Some } D \text{ and}$

$\text{undef}: \text{undefined-lit } M1 \ L \text{ and}$

$T: T \sim \text{cons-trail } (\text{Propagated } L \ (\text{cls-of-ccls } D))$

$(\text{reduce-trail-to } M1$

$(\text{add-learned-cls } (\text{cls-of-ccls } D)$

$(\text{update-backtrack-lvl } i$

$(\text{update-conflicting } \text{None } S)))) \text{ and}$

$\text{confl}: \forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{as} \text{CNot } T$

shows $\text{atms-of } (\text{mset-ccls } (\text{remove-clit } L \ D)) \subseteq \text{atm-of ' lits-of-l } (tl \ (\text{trail } T))$

proof (rule ccontr)

let $?k = \text{get-maximum-level } (\text{trail } S) \ (\text{mset-ccls } D)$

let $?D = \text{mset-ccls } D$

let $?D' = \text{mset-ccls } (\text{remove-clit } L \ D)$

have $\text{trail } S \models_{as} \text{CNot } ?D$ **using** $\text{confl } S\text{-confl}$ **by** *auto*

then have $\text{vars-of-D}: \text{atms-of } ?D \subseteq \text{atm-of ' lits-of-l } (\text{trail } S)$ **unfolding** atms-of-def

by (meson image-subsetI true-annots-CNot-all-atms-defined)

obtain $M0$ **where** $M: \text{trail } S = M0 \ @ \ M2 \ @ \ \text{Marked } K \ (\text{Suc } i) \ \# \ M1$

using decomp **by** *auto*

have $\text{max}: ?k = \text{length } (\text{get-all-levels-of-marked } (M0 \ @ \ M2 \ @ \ \text{Marked } K \ (\text{Suc } i) \ \# \ M1))$

using inv **unfolding** $cdcl_W\text{-}M\text{-level-inv-def } S\text{-lvl } M$ **by** *simp*

assume $a: \neg ?thesis$

then obtain L' **where**

$L': L' \in \text{atms-of } ?D' \text{ and}$

$L'\text{-notin-M1}: L' \notin \text{atm-of ' lits-of-l } M1$

using $T \text{ undef decomp inv}$ **by** (auto simp: $cdcl_W\text{-}M\text{-level-inv-decomp}$)

then have $L'\text{-in}: L' \in \text{atm-of ' lits-of-l } (M0 \ @ \ M2 \ @ \ \text{Marked } K \ (i + 1) \ \# \ [])$

using vars-of-D **unfolding** M **by** (auto dest: $\text{in-atms-of-remove1-mset-in-atms-of}$)

then obtain L'' **where**

$L'' \in \# \ ?D' \text{ and}$

$L'': L' = \text{atm-of } L''$

using $L' \ L'\text{-notin-M1}$ **unfolding** atms-of-def **by** *auto*

have $\text{lev-}L''$:

$\text{get-level } (\text{trail } S) \ L'' = \text{get-rev-level } (\text{Marked } K \ (\text{Suc } i) \ \# \ \text{rev } M2 \ @ \ \text{rev } M0) \ (\text{Suc } i) \ L''$

using $L'\text{-notin-M1 } L'' \ M$ **by** (auto simp del: $\text{get-rev-level.simps}$)

have $\text{get-all-levels-of-marked } (\text{trail } S) = \text{rev } [1..<1+?k]$

using $\text{inv } S\text{-lvl}$ **unfolding** $cdcl_W\text{-}M\text{-level-inv-def}$ **by** *auto*

then have $\text{get-all-levels-of-marked } (M0 \ @ \ M2) = \text{rev } [\text{Suc } (\text{Suc } i)..<\text{Suc } ?k]$

unfolding M **by** (auto simp: $\text{rev-swap[symmetric]}$ dest!: $\text{append-cons-eq-upt-length-i-end}$)

then have $M: \text{get-all-levels-of-marked } M0 \ @ \ \text{get-all-levels-of-marked } M2$

$= \text{rev } [\text{Suc } (\text{Suc } i) .. < \text{Suc } (\text{length } (\text{get-all-levels-of-marked } (M0 @ M2 @ \text{Marked } K (\text{Suc } i) \# M1)))]$
unfolding *max* **unfolding** *M* **by** *simp*
have *get-rev-level* (*Marked K (Suc i) # rev (M0 @ M2)*) (*Suc i*) *L''*
 $\geq \text{Min } (\text{set } ((\text{Suc } i) \# \text{get-all-levels-of-marked } (\text{Marked } K (\text{Suc } i) \# \text{rev } (M0 @ M2))))$
using *get-rev-level-ge-min-get-all-levels-of-marked*[*of L''*
 $\text{rev } (M0 @ M2 @ [\text{Marked } K (\text{Suc } i)]) \text{ Suc } i] \text{ L'-in}$
unfolding *L''* **by** (*fastforce simp add: lits-of-def*)
also have *Min* (*set ((Suc i) # get-all-levels-of-marked (Marked K (Suc i) # rev (M0 @ M2)))*)
 $= \text{Min } (\text{set } ((\text{Suc } i) \# \text{get-all-levels-of-marked } (\text{rev } (M0 @ M2))))$ **by** *auto*
also have $\dots = \text{Min } (\text{set } ((\text{Suc } i) \# \text{get-all-levels-of-marked } M0 @ \text{get-all-levels-of-marked } M2))$
by (*simp add: Un-commute*)
also have $\dots = \text{Min } (\text{set } ((\text{Suc } i) \# [\text{Suc } (\text{Suc } i) .. < 2 + \text{length } (\text{get-all-levels-of-marked } M0)$
 $+ (\text{length } (\text{get-all-levels-of-marked } M2) + \text{length } (\text{get-all-levels-of-marked } M1))]))$
unfolding *M* **by** (*auto simp add: Un-commute*)
also have $\dots = \text{Suc } i$ **by** (*auto intro: Min-eqI*)
finally have *get-rev-level* (*Marked K (Suc i) # rev (M0 @ M2)*) (*Suc i*) *L''* $\geq \text{Suc } i$.
then have *get-level* (*trail S*) *L''* $\geq i + 1$
using *lev-L''* **by** *simp*
then have *get-maximum-level* (*trail S*) $?D' \geq i + 1$
using *get-maximum-level-ge-get-level*[*OF (L'' ∈ # ?D')*, *of trail S*] **by** *auto*
then show *False* **using** *i* **by** *auto*
qed

lemma *distinct-atms-of-incl-not-in-other*:

assumes
 $a1: \text{no-dup } (M @ M')$ **and**
 $a2: \text{atms-of } D \subseteq \text{atm-of ' lits-of-l } M'$ **and**
 $a3: x \in \text{atms-of } D$
shows $x \notin \text{atm-of ' lits-of-l } M$
proof –
have *ff1*: $\bigwedge l \text{ ms. } \text{undefined-lit } ms \ l \vee \text{atm-of } l$
 $\in \text{set } (\text{map } (\lambda m. \text{atm-of } (\text{lit-of } (m :: ('a, 'b, 'c) \text{ marked-lit}))) \text{ ms})$
by (*simp add: defined-lit-map*)
have *ff2*: $\bigwedge a. a \notin \text{atms-of } D \vee a \in \text{atm-of ' lits-of-l } M'$
using *a2* **by** (*meson subsetCE*)
have *ff3*: $\bigwedge a. a \notin \text{set } (\text{map } (\lambda m. \text{atm-of } (\text{lit-of } m)) M')$
 $\vee a \notin \text{set } (\text{map } (\lambda m. \text{atm-of } (\text{lit-of } m)) M)$
using *a1* **by** (*metis (lifting) IntI distinct-append empty-iff map-append*)
have $\forall L \ a \ f. \exists l. ((a :: 'a) \notin f \ ' L \vee (l :: 'a \text{ literal}) \in L) \wedge (a \notin f \ ' L \vee f \ l = a)$
by *blast*
then show $x \notin \text{atm-of ' lits-of-l } M$
using *ff3 ff2 ff1 a3* **by** (*metis (no-types) Marked-Propagated-in-iff-in-lits-of-l*)
qed

lemma *cdcl_W-propagate-is-conclusion*:

assumes
 $\text{cdcl}_W \ S \ S'$ **and**
 $\text{inv: cdcl}_W\text{-M-level-inv } S$ **and**
 $\text{decomp: all-decomposition-implies-m } (\text{init-clss } S) (\text{get-all-marked-decomposition } (\text{trail } S))$ **and**
 $\text{learned: cdcl}_W\text{-learned-clause } S$ **and**
 $\text{conf1: } \forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{as} CNot \ T$ **and**
 $\text{alien: no-strange-atm } S$
shows $\text{all-decomposition-implies-m } (\text{init-clss } S') (\text{get-all-marked-decomposition } (\text{trail } S'))$
using *assms(1,2)*

```

proof (induct rule: cdclW-all-induct-lev2)
  case restart
  then show ?case by auto
next
  case forget
  then show ?case using decomp by auto
next
  case conflict
  then show ?case using decomp by auto
next
  case (resolve L C M D) note tr = this(1) and T = this(7)
  let ?decomp = get-all-marked-decomposition M
  have M: set ?decomp = insert (hd ?decomp) (set (tl ?decomp))
    by (cases ?decomp) auto
  show ?case
    using decomp tr T unfolding all-decomposition-implies-def
    by (cases hd (get-all-marked-decomposition M))
      (auto simp: M)
next
  case (skip L C' M D) note tr = this(1) and T = this(5)
  have M: set (get-all-marked-decomposition M)
    = insert (hd (get-all-marked-decomposition M)) (set (tl (get-all-marked-decomposition M))))
    by (cases get-all-marked-decomposition M) auto
  show ?case
    using decomp tr T unfolding all-decomposition-implies-def
    by (cases hd (get-all-marked-decomposition M))
      (auto simp add: M)
next
  case decide note S = this(1) and undef = this(2) and T = this(4)
  show ?case using decomp T undef unfolding S all-decomposition-implies-def by auto
next
  case (propagate C L T) note propa = this(2) and L = this(3) and undef = this(5) and T = this(6)
  obtain a y where ay: hd (get-all-marked-decomposition (trail S)) = (a, y)
    by (cases hd (get-all-marked-decomposition (trail S))))
  then have M: trail S = y @ a using get-all-marked-decomposition-decomp by blast
  have M': set (get-all-marked-decomposition (trail S))
    = insert (a, y) (set (tl (get-all-marked-decomposition (trail S))))
    using ay by (cases get-all-marked-decomposition (trail S)) auto
  have unmark-l a  $\cup$  set-mset (init-clss S)  $\models_{ps}$  unmark-l y
    using decomp ay unfolding all-decomposition-implies-def
    by (cases get-all-marked-decomposition (trail S)) fastforce +
  then have a-Un-N-M: unmark-l a  $\cup$  set-mset (init-clss S)
     $\models_{ps}$  unmark-l (trail S)
    unfolding M by (auto simp add: all-in-true-clss-clss image-Un)

  have unmark-l a  $\cup$  set-mset (init-clss S)  $\models_p$  {#L#} (is ?I  $\models_p$  -)
  proof (rule true-clss-clss-plus-CNot)
    show ?I  $\models_p$  remove1-mset L (mset-clss C) + {#L#}
    apply (rule true-clss-clss-in-imp-true-clss-clss[of -
      set-mset (init-clss S)  $\cup$  set-mset (learned-clss S)])
    using learned propa L by (auto simp: raw-clauses-def cdclW-learned-clause-def
      true-annot-CNot-diff)
  next
  have unmark-l (trail S)  $\models_{ps}$  CNot (remove1-mset L (mset-clss C))
    using  $\langle \langle \text{trail } S \rangle \models_{as} \text{CNot } (\text{remove1-mset } L \text{ (mset-clss } C)) \rangle \text{ true-annots-true-clss-clss}$ 

```



```

    by blast
  then show ?I  $\models_{ps}$  CNot (remove1-mset L (mset-cls C))
    using a-Un-N-M true-clss-clss-left-right true-clss-clss-union-l-r by blast
qed
moreover have  $\bigwedge aa\ b.$ 
   $\forall (Ls, seen) \in set\ (get\_all\_marked\_decomposition\ (y\ @\ a)).$ 
   $unmark-l\ Ls \cup set\_mset\ (init\_clss\ S) \models_{ps} unmark-l\ seen$ 
 $\implies (aa, b) \in set\ (tl\ (get\_all\_marked\_decomposition\ (y\ @\ a)))$ 
 $\implies unmark-l\ aa \cup set\_mset\ (init\_clss\ S) \models_{ps} unmark-l\ b$ 
  by (metis (no-types, lifting) case-prod-conv get-all-marked-decomposition-never-empty-sym
    list.collapse list.set-intros(2))

ultimately show ?case
  using decomp T undef unfolding ay all-decomposition-implies-def
  using M  $\langle unmark-l\ a \cup set\_mset\ (init\_clss\ S) \models_{ps} unmark-l\ y \rangle$ 
  ay by auto
next
  case (backtrack K i M1 M2 L D T) note conf = this(1) and LD = this(2) and decomp' = this(3)
and
  lev-L = this(4) and undef = this(7) and T = this(8)
  let ?D = mset-ccls D
  let ?D' = mset-ccls (remove-clit L D)
  have  $\forall l \in set\ M2. \neg is\_marked\ l$ 
    using get-all-marked-decomposition-snd-not-marked decomp' by blast
  obtain M0 where M: trail S = M0 @ M2 @ Marked K (i + 1) # M1
    using decomp' by auto
  show ?case unfolding all-decomposition-implies-def
  proof
    fix x
    assume  $x \in set\ (get\_all\_marked\_decomposition\ (trail\ T))$ 
    then have x:  $x \in set\ (get\_all\_marked\_decomposition\ (Propagated\ L\ ?D\ \# M1))$ 
      using T decomp' undef inv by (simp add: cdclW-M-level-inv-decomp)
    let ?m = get-all-marked-decomposition (Propagated L ?D # M1)
    let ?hd = hd ?m
    let ?tl = tl ?m
    consider
      (hd)  $x = ?hd$ 
      | (tl)  $x \in set\ ?tl$ 
    using x by (cases ?m) auto
  then show case x of (Ls, seen)  $\Rightarrow unmark-l\ Ls \cup set\_mset\ (init\_clss\ T)$ 
     $\models_{ps} unmark-l\ seen$ 
  proof cases
    case tl
    then have  $x \in set\ (get\_all\_marked\_decomposition\ (trail\ S))$ 
      using tl-get-all-marked-decomposition-skip-some[of x] by (simp add: list.set-sel(2) M)
    then show ?thesis
      using decomp learned decomp confl alien inv T undef M
      unfolding all-decomposition-implies-def cdclW-M-level-inv-def
      by auto
  next
    case hd
    obtain M1' M1'' where M1:  $hd\ (get\_all\_marked\_decomposition\ M1) = (M1', M1'')$ 
      by (cases hd (get-all-marked-decomposition M1))
    then have x':  $x = (M1', Propagated\ L\ ?D\ \# M1'')$ 
      using  $\langle x = ?hd \rangle$  by auto

```

```

have (M1', M1'') ∈ set (get-all-marked-decomposition (trail S))
  using M1[symmetric] hd-get-all-marked-decomposition-skip-some[OF M1[symmetric],
    of M0 @ M2 - i+1] unfolding M by fastforce
then have 1: unmark-l M1' ∪ set-mset (init-clss S) ⊨ps unmark-l M1''
  using decomp unfolding all-decomposition-implies-def by auto

moreover
  have vars-of-D: atms-of ?D' ⊆ atm-of ' lits-of-l M1
    using backtrack-atms-of-D-in-M1[of S M1 L D i K M2 T] backtrack.hyps inv conf confl
    by (auto simp: cdclW-M-level-inv-decomp)
  have no-dup (trail S) using inv by (auto simp: cdclW-M-level-inv-decomp)
  then have vars-in-M1:
    ∀ x ∈ atms-of ?D'. x ∉ atm-of ' lits-of-l (M0 @ M2 @ Marked K (i + 1) # [])
    using vars-of-D distinct-atms-of-incl-not-in-other[of
      M0 @ M2 @ Marked K (i + 1) # [] M1] unfolding M by auto
  have trail S ⊨as CNot (remove1-mset L (mset-ccls D))
    using conf confl LD unfolding M true-annots-true-clss-def-iff-negation-in-model
    by (auto dest!: Multiset.in-diffD)
  then have M1 ⊨as CNot ?D'
    using vars-in-M1 true-annots-remove-if-notin-vars[of M0 @ M2 @ Marked K (i + 1) # []
      M1 CNot ?D'] conf confl unfolding M lits-of-def by simp
  have M1 = M1'' @ M1' by (simp add: M1 get-all-marked-decomposition-decomp)
  have TT: unmark-l M1' ∪ set-mset (init-clss S) ⊨ps CNot ?D'
    using true-annots-true-clss-clss[OF ⟨M1 ⊨as CNot ?D'⟩] true-clss-clss-left-right[OF 1]
    unfolding ⟨M1 = M1'' @ M1'⟩ by (auto simp add: inf-sup-aci(5,7))
  have init-clss S ⊨pm ?D' + {#L#}
    using conf learned confl LD unfolding cdclW-learned-clause-def by auto
  then have T': unmark-l M1' ∪ set-mset (init-clss S) ⊨p ?D' + {#L#} by auto
  have atms-of (?D' + {#L#}) ⊆ atms-of-mm (clauses S)
    using alien conf LD unfolding no-strange-atm-def raw-clauses-def by auto
  then have unmark-l M1' ∪ set-mset (init-clss S) ⊨p {#L#}
    using true-clss-clss-plus-CNot[OF T' TT] by auto

ultimately show ?thesis
  using T' T decomp' undef inv unfolding x' by (simp add: cdclW-M-level-inv-decomp)
qed
qed
qed

lemma cdclW-propagate-is-false:
  assumes
    cdclW S S' and
    lev: cdclW-M-level-inv S and
    learned: cdclW-learned-clause S and
    decomp: all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S)) and
    confl: ∀ T. conflicting S = Some T ⟶ trail S ⊨as CNot T and
    alien: no-strange-atm S and
    mark-conflict: every-mark-is-a-conflict S
  shows every-mark-is-a-conflict S'
  using assms(1,2)
proof (induct rule: cdclW-all-induct-lev2)
  case (propagate C L T) note LC = this(3) and confl = this(4) and undef = this(5) and T =
  this(6)
  show ?case
    proof (intro allI impI)

```

```

    fix  $L'$  mark  $a$   $b$ 
    assume  $a @ \text{Propagated } L' \text{ mark } \# b = \text{trail } T$ 
    then consider
      (hd)  $a = []$  and  $L = L'$  and  $\text{mark} = \text{mset-cls } C$  and  $b = \text{trail } S$ 
      | (tl)  $tl \ a @ \text{Propagated } L' \text{ mark } \# b = \text{trail } S$ 
      using  $T \text{ undef by (cases } a) \text{ fastforce+}$ 
    then show  $b \models_{as} CNot (\text{mark} - \{\#L'\# \}) \wedge L' \in \# \text{ mark}$ 
      using  $\text{mark-confli confl } LC \text{ by cases auto}$ 
    qed
  next
  case (decide  $L$ ) note  $\text{undef[simp]} = \text{this}(2)$  and  $T = \text{this}(4)$ 
  have  $\bigwedge a \ La \ \text{mark } b. a @ \text{Propagated } La \ \text{mark } \# b = \text{Marked } L (\text{backtrack-lvl } S+1) \# \text{trail } S$ 
     $\implies tl \ a @ \text{Propagated } La \ \text{mark } \# b = \text{trail } S \text{ by (case-tac } a) \text{ auto}$ 
  then show ?case using  $\text{mark-confli } T \text{ unfolding decide.hyps}(1) \text{ by fastforce}$ 
next
  case (skip  $L \ C' \ M \ D \ T$ ) note  $tr = \text{this}(1)$  and  $T = \text{this}(5)$ 
  show ?case
    proof (intro allI impI)
      fix  $L'$  mark  $a$   $b$ 
      assume  $a @ \text{Propagated } L' \text{ mark } \# b = \text{trail } T$ 
      then have  $a @ \text{Propagated } L' \text{ mark } \# b = M$  using  $tr \ T \text{ by simp}$ 
      then have  $(\text{Propagated } L \ C' \# a) @ \text{Propagated } L' \text{ mark } \# b = \text{Propagated } L \ C' \# M$  by auto
      moreover have  $\forall La \ \text{mark } a \ b. a @ \text{Propagated } La \ \text{mark } \# b = \text{Propagated } L \ C' \# M$ 
         $\longrightarrow b \models_{as} CNot (\text{mark} - \{\#La\# \}) \wedge La \in \# \text{ mark}$ 
        using  $\text{mark-confli unfolding skip.hyps}(1) \text{ by simp}$ 
      ultimately show  $b \models_{as} CNot (\text{mark} - \{\#L'\# \}) \wedge L' \in \# \text{ mark}$  by blast
    qed
  next
  case (conflict  $D$ )
  then show ?case using  $\text{mark-confli by simp}$ 
next
  case (resolve  $L \ C \ M \ D \ T$ ) note  $tr-S = \text{this}(1)$  and  $T = \text{this}(7)$ 
  show ?case unfolding  $\text{resolve.hyps}(1)$ 
    proof (intro allI impI)
      fix  $L'$  mark  $a$   $b$ 
      assume  $a @ \text{Propagated } L' \text{ mark } \# b = \text{trail } T$ 
      then have  $(\text{Propagated } L (\text{mset-cls } (L !++ C)) \# a) @ \text{Propagated } L' \text{ mark } \# b$ 
         $= \text{Propagated } L (\text{mset-cls } (L !++ C)) \# M$ 
        using  $T \ tr-S \text{ by auto}$ 
      then show  $b \models_{as} CNot (\text{mark} - \{\#L'\# \}) \wedge L' \in \# \text{ mark}$ 
        using  $\text{mark-confli unfolding tr-S by (metis Cons-eq-appendI list.sel}(3))$ 
    qed
  next
  case restart
  then show ?case by auto
next
  case forget
  then show ?case using  $\text{mark-confli by auto}$ 
next
  case (backtrack  $K \ i \ M1 \ M2 \ L \ D \ T$ ) note  $\text{conf} = \text{this}(1)$  and  $LD = \text{this}(2)$  and  $\text{decomp} = \text{this}(3)$ 
  and
     $\text{undef} = \text{this}(7)$  and  $T = \text{this}(8)$ 
  have  $\forall l \in \text{set } M2. \neg \text{is-marked } l$ 
    using  $\text{get-all-marked-decomposition-snd-not-marked decomp by blast}$ 
  obtain  $M0$  where  $M: \text{trail } S = M0 @ M2 @ \text{Marked } K (i + 1) \# M1$ 

```

```

using decomp by auto
have [simp]: trail (reduce-trail-to M1 (add-learned-cls (cls-of-ccls (insert-ccls L D))
```

$$(update-backtrack-lvl\ i\ (update-conflicting\ None\ S)))) = M1$$

```

using decomp lev by (auto simp: cdclW-M-level-inv-decomp)
let ?D = mset-ccls D
let ?D' = mset-ccls (remove-clit L D)
show ?case
proof (intro allI impI)
  fix La :: 'v literal and mark :: 'v literal multiset and
    a b :: ('v, nat, 'v literal multiset) marked-lit list
  assume a @ Propagated La mark # b = trail T
  then consider
    (hd-tr) a = [] and
      (Propagated La mark :: ('v, nat, 'v literal multiset) marked-lit)
        = Propagated L ?D and
        b = M1
    | (tl-tr) tl a @ Propagated La mark # b = M1
  using M T decomp undef lev by (cases a) (auto simp: cdclW-M-level-inv-def)
then show b  $\models_{as}$  CNot (mark - {#La#})  $\wedge$  La  $\in \#$  mark
proof cases
  case hd-tr note A = this(1) and P = this(2) and b = this(3)
  have trail S  $\models_{as}$  CNot ?D using conf confl by auto
  then have vars-of-D: atms-of ?D  $\subseteq$  atm-of ' lits-of-l (trail S)
    unfolding atms-of-def
    by (meson image-subsetI true-annots-CNot-all-atms-defined)
  have vars-of-D: atms-of ?D'  $\subseteq$  atm-of ' lits-of-l M1
    using backtrack-atms-of-D-in-M1 [of S M1 L D i K M2 T] T backtrack lev confl
    by (auto simp: cdclW-M-level-inv-decomp)
  have no-dup (trail S) using lev by (auto simp: cdclW-M-level-inv-decomp)
  then have  $\forall x \in atms-of\ ?D'.\ x \notin atm-of\ 'lits-of-l\ (M0\ @\ M2\ @\ Marked\ K\ (i + 1)\ \# \ [])$ 
    using vars-of-D distinct-atms-of-incl-not-in-other [of
      M0 @ M2 @ Marked K (i + 1) # [] M1] unfolding M by auto
  then have M1  $\models_{as}$  CNot ?D'
    using true-annots-remove-if-notin-vars [of M0 @ M2 @ Marked K (i + 1) # []
      M1 CNot ?D'] (trail S  $\models_{as}$  CNot ?D) unfolding M lits-of-def
    by (simp add: true-annot-CNot-diff)
  then show b  $\models_{as}$  CNot (mark - {#La#})  $\wedge$  La  $\in \#$  mark
    using P LD b by auto
next
  case tl-tr
  then obtain c' where c' @ Propagated La mark # b = trail S
    unfolding M by auto
  then show b  $\models_{as}$  CNot (mark - {#La#})  $\wedge$  La  $\in \#$  mark
    using mark-confl by auto
qed
qed
qed

```

lemma *cdcl_W-conflicting-is-false:*

assumes

cdcl_W S S' **and**

M-lev: cdcl_W-M-level-inv S **and**

confl-inv: $\forall T. conflicting\ S = Some\ T \longrightarrow trail\ S \models_{as}\ CNot\ T$ **and**

marked-confl: $\forall L\ mark\ a\ b. a\ @\ Propagated\ L\ mark\ \# \ b = (trail\ S)$

$\longrightarrow (b \models_{as}\ CNot\ (mark - \{ \#L\ \})) \wedge L \in \# \ mark)$ **and**

dist: *distinct-cdcl_W-state S*
shows $\forall T. \text{conflicting } S' = \text{Some } T \longrightarrow \text{trail } S' \models_{\text{as}} \text{CNot } T$
using *assms(1,2)*
proof (*induct rule: cdcl_W-all-induct-lev2*)
 case (*skip L C' M D T*) **note** *tr-S = this(1)* **and** *confl = this(2)* **and** *L-D = this(3)* **and** *T = this(5)*
 let *?D = mset-ccls D*
 have *D: Propagated L C' # M \models_{as} CNot (mset-ccls D)* **using** *assms skip* **by** *auto*
 moreover
 have *L \notin # ?D*
 proof (*rule ccontr*)
 assume $\neg ?thesis$
 then have $- L \in \text{lits-of-l } M$
 using *in-CNot-implies-uminus(2)[of L ?D Propagated L C' # M]*
 $\langle \text{Propagated } L \ C' \ # \ M \models_{\text{as}} \text{CNot } ?D \rangle$ **by** *simp*
 then show *False*
 by (*metis (no-types, hide-lams) M-lev cdcl_W-M-level-inv-decomp(1) consistent-interp-def*
 image-insert insert-iff list.set(2) lits-of-def marked-lit.sel(2) tr-S)
 qed
 ultimately show *?case*
 using *tr-S confl L-D T unfolding cdcl_W-M-level-inv-def*
 by (*auto intro: true-annots-CNot-lit-of-notin-skip*)
next
 case (*resolve L C M D T*) **note** *tr = this(1)* **and** *LC = this(2)* **and** *confl = this(4)* **and** *LD = this(5)*
 and *T = this(7)*
 let *?C = remove1-mset L (mset-cls C)*
 let *?D = remove1-mset (-L) (mset-ccls D)*
 show *?case*
 proof (*intro allI impI*)
 fix *T'*
 have *tl (trail S) \models_{as} CNot ?C* **using** *tr marked-confl* **by** *fastforce*
 moreover
 have *distinct-mset (?D + {#- L#})* **using** *confl dist LD*
 unfolding *distinct-cdcl_W-state-def* **by** *auto*
 then have $-L \notin \# ?D$ **unfolding** *distinct-mset-def*
 by (*meson $\langle \text{distinct-mset } (?D + \{ \# - L \# \}) \rangle$ distinct-mset-single-add*)
 have *M \models_{as} CNot ?D*
 proof $-$
 have *Propagated L (?C + {#L#}) # M \models_{as} CNot ?D \cup CNot {#- L#}*
 using *confl tr confl-inv LC* **by** (*metis CNot-plus LD insert-DiffM2 option.simps(9)*)
 then show *?thesis*
 using *M-lev $\langle - L \notin \# ?D \rangle$ tr true-annots-lit-of-notin-skip*
 unfolding *cdcl_W-M-level-inv-def* **by** *force*
 qed
 moreover assume *conflicting T = Some T'*
 ultimately
 show *trail T \models_{as} CNot T'*
 using *tr T* **by** *auto*
 qed
qed (*auto simp: M-lev cdcl_W-M-level-inv-decomp*)

lemma *cdcl_W-conflicting-decomp*:
assumes *cdcl_W-conflicting S*
shows $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{\text{as}} \text{CNot } T$

and $\forall L \text{ mark } a \ b. \ a \ @ \text{ Propagated } L \text{ mark } \# \ b = (\text{trail } S)$
 $\rightarrow (b \models_{as} CNot \ (\text{mark} - \{ \#L\# \}) \wedge L \in \# \text{ mark})$
using *assms* **unfolding** *cdcl_W-conflicting-def* **by** *blast+*

lemma *cdcl_W-conflicting-decomp2*:
assumes *cdcl_W-conflicting* *S* **and** *conflicting* *S* = *Some T*
shows *trail S* \models_{as} *CNot T*
using *assms* **unfolding** *cdcl_W-conflicting-def* **by** *blast+*

lemma *cdcl_W-conflicting-S0-cdcl_W[simp]*:
cdcl_W-conflicting (*init-state N*)
unfolding *cdcl_W-conflicting-def* **by** *auto*

19.3.9 Putting all the invariants together

lemma *cdcl_W-all-inv*:
assumes
cdcl_W: cdcl_W S S' and
1: all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S)) and
2: cdcl_W-learned-clause S and
4: cdcl_W-M-level-inv S and
5: no-strange-atm S and
7: distinct-cdcl_W-state S and
8: cdcl_W-conflicting S
shows
all-decomposition-implies-m (init-clss S') (get-all-marked-decomposition (trail S')) and
cdcl_W-learned-clause S' and
cdcl_W-M-level-inv S' and
no-strange-atm S' and
distinct-cdcl_W-state S' and
cdcl_W-conflicting S'
proof –
show *S1: all-decomposition-implies-m (init-clss S') (get-all-marked-decomposition (trail S'))*
using *cdcl_W-propagate-is-conclusion[OF cdcl_W 4 1 2 - 5] 8* **unfolding** *cdcl_W-conflicting-def*
by *blast*
show *S2: cdcl_W-learned-clause S' using cdcl_W-learned-clss[OF cdcl_W 2 4] .*
show *S4: cdcl_W-M-level-inv S' using cdcl_W-consistent-inv[OF cdcl_W 4] .*
show *S5: no-strange-atm S' using cdcl_W-no-strange-atm-inv[OF cdcl_W 5 4] .*
show *S7: distinct-cdcl_W-state S' using distinct-cdcl_W-state-inv[OF cdcl_W 4 7] .*
show *S8: cdcl_W-conflicting S'*
using *cdcl_W-conflicting-is-false[OF cdcl_W 4 - - 7] 8 cdcl_W-propagate-is-false[OF cdcl_W 4 2 1 - 5]*
unfolding *cdcl_W-conflicting-def* **by** *fast*
qed

lemma *rtrancp-cdcl_W-all-inv*:
assumes
cdcl_W: rtrancp cdcl_W S S' and
1: all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S)) and
2: cdcl_W-learned-clause S and
4: cdcl_W-M-level-inv S and
5: no-strange-atm S and
7: distinct-cdcl_W-state S and
8: cdcl_W-conflicting S
shows
all-decomposition-implies-m (init-clss S') (get-all-marked-decomposition (trail S')) and

```

    cdclW-learned-clause  $S'$  and
    cdclW-M-level-inv  $S'$  and
    no-strange-atm  $S'$  and
    distinct-cdclW-state  $S'$  and
    cdclW-conflicting  $S'$ 
  using assms
proof (induct rule: rtrancpl-induct)
  case base
    case 1 then show ?case by blast
    case 2 then show ?case by blast
    case 3 then show ?case by blast
    case 4 then show ?case by blast
    case 5 then show ?case by blast
    case 6 then show ?case by blast
  next
  case (step  $S' S''$ ) note  $H = \text{this}$ 
    case 1 with  $H(3-7)[OF \text{this}(1-6)]$  show ?case using cdclW-all-inv[ $OF H(2)$ ]
       $H$  by presburger
    case 2 with  $H(3-7)[OF \text{this}(1-6)]$  show ?case using cdclW-all-inv[ $OF H(2)$ ]
       $H$  by presburger
    case 3 with  $H(3-7)[OF \text{this}(1-6)]$  show ?case using cdclW-all-inv[ $OF H(2)$ ]
       $H$  by presburger
    case 4 with  $H(3-7)[OF \text{this}(1-6)]$  show ?case using cdclW-all-inv[ $OF H(2)$ ]
       $H$  by presburger
    case 5 with  $H(3-7)[OF \text{this}(1-6)]$  show ?case using cdclW-all-inv[ $OF H(2)$ ]
       $H$  by presburger
    case 6 with  $H(3-7)[OF \text{this}(1-6)]$  show ?case using cdclW-all-inv[ $OF H(2)$ ]
       $H$  by presburger
  qed

lemma all-invariant-S0-cdclW:
  assumes distinct-mset-mset (mset-clss  $N$ )
  shows
    all-decomposition-implies-m (init-clss (init-state  $N$ ))
      (get-all-marked-decomposition (trail (init-state  $N$ ))) and
    cdclW-learned-clause (init-state  $N$ ) and
     $\forall T. \text{conflicting (init-state } N) = \text{Some } T \longrightarrow (\text{trail (init-state } N)) \models_{as} CNot T$  and
    no-strange-atm (init-state  $N$ ) and
    consistent-interp (lits-of-l (trail (init-state  $N$ ))) and
     $\forall L \text{ mark } a \ b. a @ \text{Propagated } L \text{ mark } \# \ b = \text{trail (init-state } N) \longrightarrow$ 
       $(b \models_{as} CNot (\text{mark} - \{\#L\}) \wedge L \in \# \text{ mark})$  and
    distinct-cdclW-state (init-state  $N$ )
  using assms by auto

lemma cdclW-only-propagated-vars-unsat:
  assumes
    marked:  $\forall x \in \text{set } M. \neg \text{is-marked } x$  and
    DN:  $D \in \# \text{ clauses } S$  and
    D:  $M \models_{as} CNot D$  and
    inv: all-decomposition-implies-m  $N$  (get-all-marked-decomposition  $M$ ) and
    state: state  $S = (M, N, U, k, C)$  and
    learned-cl: cdclW-learned-clause  $S$  and
    atm-incl: no-strange-atm  $S$ 
  shows unsatisfiable (set-mset  $N$ )

```

```

proof (rule ccontr)
  assume  $\neg$  unsatisfiable (set-mset N)
  then obtain I where
    I: I  $\models_s$  set-mset N and
    cons: consistent-interp I and
    tot: total-over-m I (set-mset N)
    unfolding satisfiable-def by auto
  have atms-of-mm N  $\cup$  atms-of-mm U = atms-of-mm N
    using atm-incl state unfolding total-over-m-def no-strange-atm-def
    by (auto simp add: raw-clauses-def)
  then have total-over-m I (set-mset N) using tot unfolding total-over-m-def by auto
  moreover then have total-over-m I (set-mset (learned-clss S))
    using atm-incl state unfolding no-strange-atm-def total-over-m-def total-over-set-def
    by auto
  moreover have N  $\models_{psm}$  U using learned-cl state unfolding cdclW-learned-clause-def by auto
  ultimately have I-D: I  $\models$  D
    using I DN cons state unfolding true-clss-clss-def true-clss-def Ball-def
    by (auto simp add: raw-clauses-def)

  have l0: {unmark L | L. is-marked L  $\wedge$  L  $\in$  set M} = {} using marked by auto
  have atms-of-ms (set-mset N  $\cup$  unmark-l M) = atms-of-mm N
    using atm-incl state unfolding no-strange-atm-def by auto
  then have total-over-m I (set-mset N  $\cup$  unmark-l M)
    using tot unfolding total-over-m-def by auto
  then have I  $\models_s$  unmark-l M
    using all-decomposition-implies-propagated-lits-are-implied[OF inv] cons I
    unfolding true-clss-clss-def l0 by auto
  then have IM: I  $\models_s$  unmark-l M by auto
  {
    fix K
    assume K  $\in \#$  D
    then have  $\neg K \in$  lits-of-l M
      using D unfolding true-annots-def Ball-def CNot-def true-annot-def true-clss-def true-lit-def
      Bex-def by force
    then have  $\neg K \in$  I using IM true-clss-singleton-lit-of-implies-incl lits-of-def by fastforce }
  then have  $\neg$  I  $\models$  D using cons unfolding true-clss-def true-lit-def consistent-interp-def by auto
  then show False using I-D by blast
}
qed

```

We have actually a much stronger theorem, namely *all-decomposition-implies ?N* (*get-all-marked-decomposition ?M*) \implies *?N* \cup {unmark L | L. is-marked L \wedge L \in set ?M} \models_{ps} unmark-l ?M, that show that the only choices we made are marked in the formula

```

lemma
  assumes all-decomposition-implies-m N (get-all-marked-decomposition M)
  and  $\forall m \in$  set M.  $\neg$ is-marked m
  shows set-mset N  $\models_{ps}$  unmark-l M
proof -
  have T: {unmark L | L. is-marked L  $\wedge$  L  $\in$  set M} = {} using assms(2) by auto
  then show ?thesis
    using all-decomposition-implies-propagated-lits-are-implied[OF assms(1)] unfolding T by simp
qed

```

```

lemma conflict-with-false-implies-unsat:
  assumes

```



```

  cdclW: cdclW S S' and
  lev: cdclW-M-level-inv S and
  [simp]: conflicting S' = Some {#} and
  learned: cdclW-learned-clause S
shows unsatisfiable (set-mset (init-clss S))
using assms
proof -
  have cdclW-learned-clause S' using cdclW-learned-clss cdclW learned lev by auto
  then have init-clss S' ⊨pm {#} using assms(3) unfolding cdclW-learned-clause-def by auto
  then have init-clss S ⊨pm {#}
    using cdclW-init-clss[OF assms(1) lev] by auto
  then show ?thesis unfolding satisfiable-def true-clss-clss-def by auto
qed

```

```

lemma conflict-with-false-implies-terminated:
  assumes cdclW S S'
  and conflicting S = Some {#}
  shows False
  using assms by (induct rule: cdclW-all-induct) auto

```

19.3.10 No tautology is learned

This is a simple consequence of all we have shown previously. It is not strictly necessary, but helps finding a better bound on the number of learned clauses.

```

lemma learned-clss-are-not-tautologies:
  assumes
    cdclW S S' and
    lev: cdclW-M-level-inv S and
    conflicting: cdclW-conflicting S and
    no-tauto: ∀ s ∈ # learned-clss S. ¬tautology s
  shows ∀ s ∈ # learned-clss S'. ¬tautology s
  using assms
proof (induct rule: cdclW-all-induct-lev2)
  case (backtrack K i M1 M2 L D T) note confl = this(1)
  have consistent-interp (lits-of-l (trail S)) using lev by (auto simp: cdclW-M-level-inv-decomp)
  moreover
    have trail S ⊨as CNot (mset-ccls D)
      using conflicting confl unfolding cdclW-conflicting-def by auto
    then have lits-of-l (trail S) ⊨s CNot (mset-ccls D)
      using true-annots-true-clss by blast
    ultimately have ¬tautology (mset-ccls D) using consistent-CNot-not-tautology by blast
  then show ?case using backtrack no-tauto lev
    by (auto simp: cdclW-M-level-inv-decomp split: if-split-asm)
next
  case restart
  then show ?case using learned-clss-restart-state state-eq-learned-clss no-tauto
    by (metis (no-types, lifting) set-mset-mono subsetCE)
qed (auto dest!: in-diffD)

```

```

definition final-cdclW-state (S:: 'st)
  ⟷ (trail S ⊨asm init-clss S
    ∨ ((∀ L ∈ set (trail S). ¬is-marked L) ∧
       (∃ C ∈ # init-clss S. trail S ⊨as CNot C)))

```

```

definition termination-cdclW-state (S:: 'st)

```

$$\begin{aligned} \longleftrightarrow & (trail\ S \models_{asm} init_clss\ S \\ & \vee ((\forall L \in atm_of_mm\ (init_clss\ S). L \in atm_of\ 'a\ lits_of_l\ (trail\ S)) \\ & \wedge (\exists C \in \# \text{ init_clss } S. trail\ S \models_{as} CNot\ C))) \end{aligned}$$

19.4 CDCL Strong Completeness

fun *mapi* :: ('a \Rightarrow nat \Rightarrow 'b) \Rightarrow nat \Rightarrow 'a list \Rightarrow 'b list **where**
mapi - - [] = [] |
mapi f n (x # xs) = f x n # *mapi* f (n - 1) xs

lemma *mark-not-in-set-mapi[simp]*: $L \notin set\ M \implies Marked\ L\ k \notin set\ (mapi\ Marked\ i\ M)$
by (induct M arbitrary: i) auto

lemma *propagated-not-in-set-mapi[simp]*: $L \notin set\ M \implies Propagated\ L\ k \notin set\ (mapi\ Marked\ i\ M)$
by (induct M arbitrary: i) auto

lemma *image-set-mapi*:
 $f\ 'set\ (mapi\ g\ i\ M) = set\ (mapi\ (\lambda x\ i. f\ (g\ x\ i))\ i\ M)$
by (induction M arbitrary: i) auto

lemma *mapi-map-convert*:
 $\forall x\ i\ j. f\ x\ i = f\ x\ j \implies mapi\ f\ i\ M = map\ (\lambda x. f\ x\ 0)\ M$
by (induction M arbitrary: i) auto

lemma *defined-lit-mapi*: $defined_lit\ (mapi\ Marked\ i\ M)\ L \longleftrightarrow atm_of\ L \in atm_of\ 'set\ M$
by (induction M) (auto simp: defined-lit-map image-set-mapi mapi-map-convert)

lemma *cdcl_W-can-do-step*:
assumes
consistent-interp (set M) **and**
distinct M **and**
 $atm_of\ 'set\ M \subseteq atm_of_mm\ (mset_clss\ N)$
shows $\exists S. rtranclp\ cdcl_W\ (init_state\ N)\ S$
 $\wedge state\ S = (mapi\ Marked\ (length\ M)\ M, mset_clss\ N, \{\#\}, length\ M, None)$
using *assms*
proof (induct M)
case Nil
then show ?case **apply** - **by** (rule exI[of - init-state N]) auto
next
case (Cons L M) **note** IH = this(1)
have *consistent-interp* (set M) **and** *distinct* M **and** $atm_of\ 'set\ M \subseteq atm_of_mm\ (mset_clss\ N)$
using Cons.prem(1-3) **unfolding** *consistent-interp-def* **by** auto
then obtain S **where**
st: $cdcl_W^{**}\ (init_state\ N)\ S$ **and**
S: $state\ S = (mapi\ Marked\ (length\ M)\ M, mset_clss\ N, \{\#\}, length\ M, None)$
using IH **by** blast
let ?S₀ = *incr-lvl* (cons-trail (Marked L (length M + 1)) S)
have *undefined-lit* (mapi Marked (length M) M) L
using Cons.prem(1,2) **unfolding** *defined-lit-def* *consistent-interp-def* **by** fastforce
moreover have *init-clss* S = *mset-clss* N
using S **by** blast
moreover have $atm_of\ L \in atm_of_mm\ (mset_clss\ N)$ **using** Cons.prem(3) **by** auto
moreover have *undef*: *undefined-lit* (trail S) L
using S $\langle distinct\ (L\ \# M) \rangle$ *calculation*(1) **by** (auto simp: *defined-lit-mapi* *defined-lit-map*)
ultimately have $cdcl_W\ S\ ?S_0$
using $cdcl_W.other[OF\ cdcl_W-o.decide[OF\ decide-rule[of\ S\ L\ ?S_0]]]$ S

```

    by (auto simp: state-eq-def simp del: state-simp)
  then have  $cdcl_W^{**}$  (init-state  $N$ )  $?S_0$ 
    using  $st$  by auto
  then show  $?case$ 
    using  $S$  undef by (auto intro!: exI[of -  $?S_0$ ] del: simp del: )
qed

```

lemma $cdcl_W$ -strong-completeness:

assumes

MN : $set\ M \models_{sm} mset-clss\ N$ **and**
 $cons$: consistent-interp ($set\ M$) **and**
 $dist$: distinct M **and**
 atm : $atm-of\ ' (set\ M) \subseteq atms-of-mm\ (mset-clss\ N)$

obtains S **where**

$state\ S = (mapi\ Marked\ (length\ M)\ M, mset-clss\ N, \{\#\}, length\ M, None)$ **and**
 $rtranclp\ cdcl_W\ (init-state\ N)\ S$ **and**
 $final-cdcl_W$ -state S

proof –

obtain S **where**

st : $rtranclp\ cdcl_W\ (init-state\ N)\ S$ **and**
 S : $state\ S = (mapi\ Marked\ (length\ M)\ M, mset-clss\ N, \{\#\}, length\ M, None)$
using $cdcl_W$ -can-do-step[$OF\ cons\ dist\ atm$] **by** auto
have $lits-of-l\ (mapi\ Marked\ (length\ M)\ M) = set\ M$
by (induct M , auto)

then have $mapi\ Marked\ (length\ M)\ M \models_{asm} mset-clss\ N$ **using** MN true-annots-true-cls **by** metis

then have $final-cdcl_W$ -state S

using S **unfolding** $final-cdcl_W$ -state-def **by** auto

then show $?thesis$ **using** that $st\ S$ **by** blast

qed

19.5 Higher level strategy

The rules described previously do not lead to a conclusive state. We have to add a strategy.

19.5.1 Definition

lemma $tranclp$ -conflict:

$tranclp\ conflict\ S\ S' \implies conflict\ S\ S'$

apply (induct rule: $tranclp$.induct)

apply simp

by (metis conflictE conflicting-update-conflicting option.distinct(1) option.simps(8,9)
state-eq-conflicting)

lemma $tranclp$ -conflict-iff[iff]:

$full1\ conflict\ S\ S' \longleftrightarrow conflict\ S\ S'$

proof –

have $tranclp\ conflict\ S\ S' \implies conflict\ S\ S'$ **by** (meson $tranclp$ -conflict $rtranclpD$)

then show $?thesis$ **unfolding** $full1$ -def

by (metis conflict.simps conflicting-update-conflicting option.distinct(1) option.simps(9)
state-eq-conflicting $tranclp$.intros(1))

qed

inductive $cdcl_W$ -cp :: $'st \Rightarrow 'st \Rightarrow bool$ **where**

$conflict'[intro]$: $conflict\ S\ S' \implies cdcl_W$ -cp $S\ S'$ |

$propagate'$: $propagate\ S\ S' \implies cdcl_W$ -cp $S\ S'$

```

lemma rtrancp-cdclW-cp-rtrancp-cdclW:
  cdclW-cp** S T  $\implies$  cdclW** S T
  by (induction rule: rtrancp-induct) (auto simp: cdclW-cp.simps dest: cdclW.intros)

lemma cdclW-cp-state-eq-compatible:
  assumes
    cdclW-cp S T and
    S  $\sim$  S' and
    T  $\sim$  T'
  shows cdclW-cp S' T'
  using assms
  apply (induction)
    using conflict-state-eq-compatible apply auto[1]
  using propagate' propagate-state-eq-compatible by auto

lemma trancp-cdclW-cp-state-eq-compatible:
  assumes
    cdclW-cp++ S T and
    S  $\sim$  S' and
    T  $\sim$  T'
  shows cdclW-cp++ S' T'
  using assms
proof induction
  case base
  then show ?case
    using cdclW-cp-state-eq-compatible by blast
next
  case (step U V)
  obtain ss :: 'st where
    cdclW-cp S ss  $\wedge$  cdclW-cp** ss U
  by (metis (no-types) step(1) trancpD)
  then show ?case
    by (meson cdclW-cp-state-eq-compatible rtrancp.rtrancp-into-rtrancp rtrancp-into-trancp2
      state-eq-ref step(2) step(4) step(5))
qed

lemma option-full-cdclW-cp:
  conflicting S  $\neq$  None  $\implies$  full cdclW-cp S S
  unfolding full-def rtrancp-unfold trancp-unfold
  by (auto simp add: cdclW-cp.simps elim: conflictE propagateE)

lemma skip-unique:
  skip S T  $\implies$  skip S T'  $\implies$  T  $\sim$  T'
  by (fastforce simp: state-eq-def simp del: state-simp elim: skipE)

lemma resolve-unique:
  resolve S T  $\implies$  resolve S T'  $\implies$  T  $\sim$  T'
  by (fastforce simp: state-eq-def simp del: state-simp elim: resolveE)

lemma cdclW-cp-no-more-clauses:
  assumes cdclW-cp S S'
  shows clauses S = clauses S'
  using assms by (induct rule: cdclW-cp.induct) (auto elim!: conflictE propagateE)

```

lemma *trancpl-cdcl_W-cp-no-more-clauses*:
assumes *cdcl_W-cp⁺⁺ S S'*
shows *clauses S = clauses S'*
using *assms* **by** (*induct rule: trancpl.induct*) (*auto dest: cdcl_W-cp-no-more-clauses*)

lemma *rtrancpl-cdcl_W-cp-no-more-clauses*:
assumes *cdcl_W-cp^{**} S S'*
shows *clauses S = clauses S'*
using *assms* **by** (*induct rule: rtrancpl.induct*) (*fastforce dest: cdcl_W-cp-no-more-clauses*)⁺

lemma *no-conflict-after-conflict*:
conflict S T \implies \neg conflict T U
by (*metis None-eq-map-option-iff conflictE conflicting-update-conflicting option.distinct(1)*
state-simp(5))

lemma *no-propagate-after-conflict*:
conflict S T \implies \neg propagate T U
by (*metis conflictE conflicting-update-conflicting map-option-is-None option.distinct(1)*
propagate.cases state-eq-conflicting)

lemma *trancpl-cdcl_W-cp-propagate-with-conflict-or-not*:
assumes *cdcl_W-cp⁺⁺ S U*
shows (*propagate⁺⁺ S U \wedge conflicting U = None*)
 \vee ($\exists T D. \text{propagate}^{**} S T \wedge \text{conflict } T U \wedge \text{conflicting } U = \text{Some } D$)
proof –
have *propagate⁺⁺ S U \vee ($\exists T. \text{propagate}^{**} S T \wedge \text{conflict } T U$)*
using *assms* **by** *induction*
(*force simp: cdcl_W-cp.simps trancpl-into-rtrancpl dest: no-conflict-after-conflict*
no-propagate-after-conflict)⁺
moreover
have *propagate⁺⁺ S U \implies conflicting U = None*
unfolding *trancpl-unfold-end* **by** (*auto elim!: propagateE*)
moreover
have $\bigwedge T. \text{conflict } T U \implies \exists D. \text{conflicting } U = \text{Some } D$
by (*auto elim!: conflictE simp: state-eq-def simp del: state-simp*)
ultimately show *?thesis* **by** *meson*
qed

lemma *cdcl_W-cp-conflicting-not-empty[simp]*: *conflicting S = Some D \implies \neg cdcl_W-cp S S'*
proof
assume *cdcl_W-cp S S' and conflicting S = Some D*
then show *False* **by** (*induct rule: cdcl_W-cp.induct*)
(*auto elim: conflictE propagateE simp: state-eq-def simp del: state-simp*)
qed

lemma *no-step-cdcl_W-cp-no-conflict-no-propagate*:
assumes *no-step cdcl_W-cp S*
shows *no-step conflict S and no-step propagate S*
using *assms* *conflict'* **apply** *blast*
by (*meson assms conflict' propagate'*)

CDCL with the reasonable strategy: we fully propagate the conflict and propagate, then we apply any other possible rule *cdcl_W-o S S'* and re-apply conflict and propagate *full cdcl_W-cp S' S''*

inductive *cdcl_W-stgy* :: '*st* \Rightarrow '*st* \Rightarrow bool **for** *S* :: '*st* **where**

conflict': $full1\ cdcl_W\text{-}cp\ S\ S' \implies cdcl_W\text{-}stgy\ S\ S' \mid$
other': $cdcl_W\text{-}o\ S\ S' \implies no\text{-}step\ cdcl_W\text{-}cp\ S \implies full\ cdcl_W\text{-}cp\ S'\ S'' \implies cdcl_W\text{-}stgy\ S\ S''$

19.5.2 Invariants

These are the same invariants as before, but lifted

lemma *cdcl_W-cp-learned-clause-inv*:

assumes *cdcl_W-cp* $S\ S'$

shows *learned-clss* $S = \text{learned-clss } S'$

using *assms* **by** (*induct* rule: *cdcl_W-cp.induct*) (*fastforce* *elim*: *conflictE* *propagateE*)+

lemma *rtrancpl-cdcl_W-cp-learned-clause-inv*:

assumes *cdcl_W-cp*** $S\ S'$

shows *learned-clss* $S = \text{learned-clss } S'$

using *assms* **by** (*induct* rule: *rtrancpl-induct*) (*fastforce* *dest*: *cdcl_W-cp-learned-clause-inv*)+

lemma *trancpl-cdcl_W-cp-learned-clause-inv*:

assumes *cdcl_W-cp⁺⁺* $S\ S'$

shows *learned-clss* $S = \text{learned-clss } S'$

using *assms* **by** (*simp* *add*: *rtrancpl-cdcl_W-cp-learned-clause-inv* *trancpl-into-rtrancpl*)

lemma *cdcl_W-cp-backtrack-lvl*:

assumes *cdcl_W-cp* $S\ S'$

shows *backtrack-lvl* $S = \text{backtrack-lvl } S'$

using *assms* **by** (*induct* rule: *cdcl_W-cp.induct*) (*fastforce* *elim*: *conflictE* *propagateE*)+

lemma *rtrancpl-cdcl_W-cp-backtrack-lvl*:

assumes *cdcl_W-cp*** $S\ S'$

shows *backtrack-lvl* $S = \text{backtrack-lvl } S'$

using *assms* **by** (*induct* rule: *rtrancpl-induct*) (*fastforce* *dest*: *cdcl_W-cp-backtrack-lvl*)+

lemma *cdcl_W-cp-consistent-inv*:

assumes *cdcl_W-cp* $S\ S'$

and *cdcl_W-M-level-inv* S

shows *cdcl_W-M-level-inv* S'

using *assms*

proof (*induct* rule: *cdcl_W-cp.induct*)

case (*conflict'*)

then show ?*case* **using** *cdcl_W-consistent-inv* *cdcl_W.conflict* **by** *blast*

next

case (*propagate'* $S\ S'$)

have *cdcl_W* $S\ S'$

using *propagate'.hyps*(1) *propagate* **by** *blast*

then show *cdcl_W-M-level-inv* S'

using *propagate'.prems*(1) *cdcl_W-consistent-inv* *propagate* **by** *blast*

qed

lemma *full1-cdcl_W-cp-consistent-inv*:

assumes *full1* *cdcl_W-cp* $S\ S'$

and *cdcl_W-M-level-inv* S

shows *cdcl_W-M-level-inv* S'

using *assms* **unfolding** *full1-def*

by (*metis* *rtrancpl-cdcl_W-cp-rtrancpl-cdcl_W* *rtrancpl-unfold* *trancpl-cdcl_W-consistent-inv*)

lemma *rtrancpl-cdcl_W-cp-consistent-inv*:

assumes *rtrancp cdcl_W-cp S S'*
and *cdcl_W-M-level-inv S*
shows *cdcl_W-M-level-inv S'*
using *assms unfolding full1-def*
by (*induction rule: rtrancp-induct*) (*blast intro: cdcl_W-cp-consistent-inv*)+

lemma *cdcl_W-stgy-consistent-inv*:
assumes *cdcl_W-stgy S S'*
and *cdcl_W-M-level-inv S*
shows *cdcl_W-M-level-inv S'*
using *assms apply (induct rule: cdcl_W-stgy.induct)*
unfolding *full-unfold* **by** (*blast intro: cdcl_W-consistent-inv full1-cdcl_W-cp-consistent-inv cdcl_W.other*)+

lemma *rtrancp-cdcl_W-stgy-consistent-inv*:
assumes *cdcl_W-stgy** S S'*
and *cdcl_W-M-level-inv S*
shows *cdcl_W-M-level-inv S'*
using *assms by induction (auto dest!: cdcl_W-stgy-consistent-inv)*

lemma *cdcl_W-cp-no-more-init-clss*:
assumes *cdcl_W-cp S S'*
shows *init-clss S = init-clss S'*
using *assms by (induct rule: cdcl_W-cp.induct) (auto elim: conflictE propagateE)*

lemma *trancp-cdcl_W-cp-no-more-init-clss*:
assumes *cdcl_W-cp⁺⁺ S S'*
shows *init-clss S = init-clss S'*
using *assms by (induct rule: trancp.induct) (auto dest: cdcl_W-cp-no-more-init-clss)*

lemma *cdcl_W-stgy-no-more-init-clss*:
assumes *cdcl_W-stgy S S' and cdcl_W-M-level-inv S*
shows *init-clss S = init-clss S'*
using *assms*
apply (*induct rule: cdcl_W-stgy.induct*)
unfolding *full1-def full-def* **apply** (*blast dest: trancp-cdcl_W-cp-no-more-init-clss trancp-cdcl_W-o-no-more-init-clss*)
by (*metis cdcl_W-o-no-more-init-clss rtrancp-unfold trancp-cdcl_W-cp-no-more-init-clss*)

lemma *rtrancp-cdcl_W-stgy-no-more-init-clss*:
assumes *cdcl_W-stgy** S S' and cdcl_W-M-level-inv S*
shows *init-clss S = init-clss S'*
using *assms*
apply (*induct rule: rtrancp-induct, simp*)
using *cdcl_W-stgy-no-more-init-clss* **by** (*simp add: rtrancp-cdcl_W-stgy-consistent-inv*)

lemma *cdcl_W-cp-dropWhile-trail'*:
assumes *cdcl_W-cp S S'*
obtains *M where trail S' = M @ trail S and (∀ l ∈ set M. ¬is-marked l)*
using *assms by induction (fastforce elim: conflictE propagateE)*+

lemma *rtrancp-cdcl_W-cp-dropWhile-trail'*:
assumes *cdcl_W-cp** S S'*
obtains *M :: ('v, nat, 'v clause) marked-lit list where trail S' = M @ trail S and ∀ l ∈ set M. ¬is-marked l*

using *assms* **by induction** (*fastforce dest!: cdcl_W-cp-dropWhile-trail'*)**+**

lemma *cdcl_W-cp-dropWhile-trail*:

assumes *cdcl_W-cp S S'*

shows $\exists M. \text{trail } S' = M @ \text{trail } S \wedge (\forall l \in \text{set } M. \neg \text{is-marked } l)$

using *assms* **by induction** (*fastforce elim: conflictE propagateE*)**+**

lemma *rtrancp-cdcl_W-cp-dropWhile-trail*:

assumes *cdcl_W-cp** S S'*

shows $\exists M. \text{trail } S' = M @ \text{trail } S \wedge (\forall l \in \text{set } M. \neg \text{is-marked } l)$

using *assms* **by induction** (*fastforce dest: cdcl_W-cp-dropWhile-trail*)**+**

This theorem can be seen as a termination theorem for *cdcl_W-cp*.

lemma *length-model-le-vars*:

assumes

no-strange-atm S **and**

no-d: no-dup (trail S) **and**

finite (atms-of-mm (init-clss S))

shows $\text{length (trail } S) \leq \text{card (atms-of-mm (init-clss } S))$

proof –

obtain *M N U k D* **where** *S: state S = (M, N, U, k, D)* **by** (*cases state S, auto*)

have *finite (atm-of 'lits-of-l (trail S))*

using *assms(1,3) unfolding S* **by** (*auto simp add: finite-subset*)

have $\text{length (trail } S) = \text{card (atm-of 'lits-of-l (trail } S))$

using *no-dup-length-eq-card-atm-of-lits-of-l no-d* **by** *blast*

then show *?thesis* **using** *assms(1) unfolding no-strange-atm-def*

by (*auto simp add: assms(3) card-mono*)

qed

lemma *cdcl_W-cp-decreasing-measure*:

assumes

cdcl_W: cdcl_W-cp S T **and**

M-lev: cdcl_W-M-level-inv S **and**

alien: no-strange-atm S

shows $(\lambda S. \text{card (atms-of-mm (init-clss } S)) - \text{length (trail } S)$

$+ (\text{if conflicting } S = \text{None then } 1 \text{ else } 0)) S$

$> (\lambda S. \text{card (atms-of-mm (init-clss } S)) - \text{length (trail } S)$

$+ (\text{if conflicting } S = \text{None then } 1 \text{ else } 0)) T$

using *assms*

proof –

have $\text{length (trail } T) \leq \text{card (atms-of-mm (init-clss } T))$

apply (*rule length-model-le-vars*)

using *cdcl_W-no-strange-atm-inv alien M-lev* **apply** (*meson cdcl_W cdcl_W.simps cdcl_W-cp.cases*)

using *M-lev cdcl_W cdcl_W-cp-consistent-inv cdcl_W-M-level-inv-def* **apply** *blast*

using *cdcl_W* **by** (*auto simp: cdcl_W-cp.simps*)

with *assms*

show *?thesis* **by induction** (*auto elim!: conflictE propagateE*

simp del: state-simp simp: state-eq-def)**+**

qed

lemma *cdcl_W-cp-wf*: $\text{wf } \{(b, a). (\text{cdcl}_W\text{-M-level-inv } a \wedge \text{no-strange-atm } a) \wedge \text{cdcl}_W\text{-cp } a \ b\}$

apply (*rule wf-wf-if-measure'[of less-than - -*

$(\lambda S. \text{card (atms-of-mm (init-clss } S)) - \text{length (trail } S)$

$+ (\text{if conflicting } S = \text{None then } 1 \text{ else } 0))]$)


```

    apply simp
using cdclW-cp-decreasing-measure unfolding less-than-iff by blast

lemma rtrancpl-cdclW-all-struct-inv-cdclW-cp-iff-rtrancpl-cdclW-cp:
  assumes
    lev: cdclW-M-level-inv S and
    alien: no-strange-atm S
  shows (λa b. (cdclW-M-level-inv a ∧ no-strange-atm a) ∧ cdclW-cp a b)** S T
    ⟷ cdclW-cp** S T
  (is ?I S T ⟷ ?C S T)
proof
  assume
    ?I S T
  then show ?C S T by induction auto
next
  assume
    ?C S T
  then show ?I S T
  proof induction
    case base
    then show ?case by simp
  next
    case (step T U) note st = this(1) and cp = this(2) and IH = this(3)
    have cdclW** S T
      by (metis rtrancpl-unfold cdclW-cp-conflicting-not-empty cp st
        rtrancpl-propagate-is-rtrancpl-cdclW trancpl-cdclW-cp-propagate-with-conflict-or-not)
    then have
      cdclW-M-level-inv T and
      no-strange-atm T
      using ⟨cdclW** S T⟩ apply (simp add: asms(1) rtrancpl-cdclW-consistent-inv)
      using ⟨cdclW** S T⟩ alien rtrancpl-cdclW-no-strange-atm-inv lev by blast
    then have (λa b. (cdclW-M-level-inv a ∧ no-strange-atm a)
      ∧ cdclW-cp a b)** T U
      using cp by auto
    then show ?case using IH by auto
  qed
qed

lemma cdclW-cp-normalized-element:
  assumes
    lev: cdclW-M-level-inv S and
    no-strange-atm S
  obtains T where full cdclW-cp S T
proof -
  let ?inv = λa. (cdclW-M-level-inv a ∧ no-strange-atm a)
  obtain T where T: full (λa b. ?inv a ∧ cdclW-cp a b) S T
  using cdclW-cp-wf wf-exists-normal-form[of λa b. ?inv a ∧ cdclW-cp a b]
  unfolding full-def by blast
  then have cdclW-cp** S T
    using rtrancpl-cdclW-all-struct-inv-cdclW-cp-iff-rtrancpl-cdclW-cp asms unfolding full-def
    by blast
  moreover
    then have cdclW** S T
      using rtrancpl-cdclW-cp-rtrancpl-cdclW by blast
  then have

```

```

    cdclW-M-level-inv T and
    no-strange-atm T
    using ⟨cdclW** S T⟩ apply (simp add: assms(1) rtranclp-cdclW-consistent-inv)
    using ⟨cdclW** S T⟩ assms(2) rtranclp-cdclW-no-strange-atm-inv lev by blast
  then have no-step cdclW-cp T
    using T unfolding full-def by auto
  ultimately show thesis using that unfolding full-def by blast
qed

lemma always-exists-full-cdclW-cp-step:
  assumes no-strange-atm S
  shows ∃ S''. full cdclW-cp S S''
  using assms
proof (induct card (atms-of-mm (init-clss S) - atm-of 'lits-of-l (trail S)) arbitrary: S)
  case 0 note card = this(1) and alien = this(2)
  then have atm: atms-of-mm (init-clss S) = atm-of 'lits-of-l (trail S)
    unfolding no-strange-atm-def by auto
  { assume a: ∃ S'. conflict S S'
    then obtain S' where S': conflict S S' by metis
    then have ∀ S''. ¬cdclW-cp S' S''
      by (auto simp: cdclW-cp.simps elim!: conflictE propagateE
        simp del: state-simp simp: state-eq-def)
    then have ?case using a S' cdclW-cp.conflict' unfolding full-def by blast
  }
  moreover {
    assume a: ∃ S'. propagate S S'
    then obtain S' where propagate S S' by blast
    then obtain E L where
      S: conflicting S = None and
      E: E !∈ raw-clauses S and
      LE: L ∈ # mset-cls E and
      tr: trail S ⊨as CNot (mset-cls (remove-lit L E)) and
      undef: undefined-lit (trail S) L and
      S': S' ∼ cons-trail (Propagated L E) S
    by (elim propagateE) simp
    have atms-of-mm (learned-clss S) ⊆ atms-of-mm (init-clss S)
      using alien S unfolding no-strange-atm-def by auto
    then have atm-of L ∈ atms-of-mm (init-clss S)
      using E LE S undef unfolding raw-clauses-def by (force simp: in-implies-atm-of-on-atms-of-ms)
    then have False using undef S unfolding atm unfolding lits-of-def
      by (auto simp add: defined-lit-map)
  }
  ultimately show ?case unfolding full-def by (metis cdclW-cp.cases rtranclp.rtrancl-refl)
next
  case (Suc n) note IH = this(1) and card = this(2) and alien = this(3)
  { assume a: ∃ S'. conflict S S'
    then obtain S' where S': conflict S S' by metis
    then have ∀ S''. ¬cdclW-cp S' S''
      by (auto simp: cdclW-cp.simps elim!: conflictE propagateE
        simp del: state-simp simp: state-eq-def)
    then have ?case unfolding full-def Ex-def using S' cdclW-cp.conflict' by blast
  }
  moreover {
    assume a: ∃ S'. propagate S S'
    then obtain S' where propagate: propagate S S' by blast
  }

```

then obtain $E \ L$ **where**
 S : *conflicting* $S = \text{None}$ **and**
 E : $E \notin \text{raw-clauses } S$ **and**
 LE : $L \in \# \text{ mset-cls } E$ **and**
 tr : $\text{trail } S \models_{as} CNot (\text{mset-cls } (\text{remove-lit } L \ E))$ **and**
 $undef$: *undefined-lit* $(\text{trail } S) \ L$ **and**
 S' : $S' \sim \text{cons-trail } (\text{Propagated } L \ E) \ S$
by $(\text{elim propagate } E)$ *simp*
then have $\text{atm-of } L \notin \text{atm-of ' lits-of-l } (\text{trail } S)$
unfolding *lits-of-def* **by** $(\text{auto simp add: defined-lit-map})$
moreover
have *no-strange-atm* S' **using** *alien propagate propagate-no-strange-atm-inv* **by** *blast*
then have $\text{atm-of } L \in \text{atms-of-mm } (\text{init-clss } S)$
using $S' \ LE \ E \ undef$ **unfolding** *no-strange-atm-def*
by $(\text{auto simp: raw-clauses-def in-implies-atm-of-on-atms-of-ms})$
then have $\bigwedge A. \{\text{atm-of } L\} \subseteq \text{atms-of-mm } (\text{init-clss } S) - A \vee \text{atm-of } L \in A$ **by** *force*
moreover have $\text{Suc } n - \text{card } \{\text{atm-of } L\} = n$ **by** *simp*
moreover have $\text{card } (\text{atms-of-mm } (\text{init-clss } S) - \text{atm-of ' lits-of-l } (\text{trail } S)) = \text{Suc } n$
using $\text{card } S \ S'$ **by** *simp*
ultimately
have $\text{card } (\text{atms-of-mm } (\text{init-clss } S) - \text{atm-of ' insert } L (\text{lits-of-l } (\text{trail } S))) = n$
by $(\text{metis } (\text{no-types}) \text{Diff-insert card-Diff-subset finite.emptyI finite.insertI image-insert})$
then have $n = \text{card } (\text{atms-of-mm } (\text{init-clss } S') - \text{atm-of ' lits-of-l } (\text{trail } S'))$
using $\text{card } S \ S' \ undef$ **by** *simp*
then have $a1: \text{Ex } (\text{full cdcl}_W\text{-cp } S')$ **using** $IH \ \langle \text{no-strange-atm } S' \rangle$ **by** *blast*
have *?case*
proof –
obtain $S'' :: 'st$ **where**
 $ff1: \text{cdcl}_W\text{-cp}^{**} \ S' \ S'' \wedge \text{no-step } \text{cdcl}_W\text{-cp } S''$
using $a1$ **unfolding** *full-def* **by** *blast*
have $\text{cdcl}_W\text{-cp}^{**} \ S \ S''$
using $ff1 \ \text{cdcl}_W\text{-cp.intros}(2)[OF \ \text{propagate}]$
by $(\text{metis } (\text{no-types}) \text{converse-rtranclp-into-rtranclp})$
then have $\exists S''. \text{cdcl}_W\text{-cp}^{**} \ S \ S'' \wedge (\forall S'''. \neg \text{cdcl}_W\text{-cp } S'' \ S''')$
using $ff1$ **by** *blast*
then show *?thesis* **unfolding** *full-def*
by *meson*
qed
}
ultimately show *?case* **unfolding** *full-def* **by** $(\text{metis } \text{cdcl}_W\text{-cp.cases } rtranclp.rtrancl-refl)$
qed

19.5.3 Literal of highest level in conflicting clauses

One important property of the cdcl_W with strategy is that, whenever a conflict takes place, there is at least a literal of level k involved (except if we have derived the false clause). The reason is that we apply conflicts before a decision is taken.

abbreviation *no-clause-is-false* $:: 'st \Rightarrow \text{bool}$ **where**

no-clause-is-false \equiv

$\lambda S. (\text{conflicting } S = \text{None} \longrightarrow (\forall D \in \# \text{ clauses } S. \neg \text{trail } S \models_{as} CNot \ D))$

abbreviation *conflict-is-false-with-level* $:: 'st \Rightarrow \text{bool}$ **where**

conflict-is-false-with-level $S \equiv \forall D. \text{conflicting } S = \text{Some } D \longrightarrow D \neq \{\#\}$

$\longrightarrow (\exists L \in \# \ D. \text{get-level } (\text{trail } S) \ L = \text{backtrack-lvl } S)$

```

lemma not-conflict-not-any-negated-init-clss:
  assumes  $\forall S'. \neg \text{conflict } S S'$ 
  shows no-clause-is-false S
proof (clarify)
  fix D
  assume  $D \in \# \text{ local.clauses } S$  and raw-conflicting  $S = \text{None}$  and  $\text{trail } S \models_{as} \text{CNot } D$ 
  moreover then obtain D' where
    mset-cls  $D' = D$  and
     $D' \notin \text{ raw-clauses } S$ 
  using in-mset-clss-exists-preimage unfolding raw-clauses-def by blast
  ultimately show False
  using conflict-rule[of S D' update-conflicting (Some (ccls-of-cls D')) S] assms
  by auto
qed

lemma full-cdclW-cp-not-any-negated-init-clss:
  assumes full cdclW-cp S S'
  shows no-clause-is-false S'
  using assms not-conflict-not-any-negated-init-clss unfolding full-def by auto

lemma full1-cdclW-cp-not-any-negated-init-clss:
  assumes full1 cdclW-cp S S'
  shows no-clause-is-false S'
  using assms not-conflict-not-any-negated-init-clss unfolding full1-def by auto

lemma cdclW-stgy-not-non-negated-init-clss:
  assumes cdclW-stgy S S'
  shows no-clause-is-false S'
  using assms apply (induct rule: cdclW-stgy.induct)
  using full1-cdclW-cp-not-any-negated-init-clss full-cdclW-cp-not-any-negated-init-clss by metis+

lemma rtrancp-cdclW-stgy-not-non-negated-init-clss:
  assumes cdclW-stgy** S S' and no-clause-is-false S
  shows no-clause-is-false S'
  using assms by (induct rule: rtrancp-induct) (auto simp: cdclW-stgy-not-non-negated-init-clss)

lemma cdclW-stgy-conflict-ex-lit-of-max-level:
  assumes cdclW-cp S S'
  and no-clause-is-false S
  and cdclW-M-level-inv S
  shows conflict-is-false-with-level S'
  using assms
proof (induct rule: cdclW-cp.induct)
  case conflict'
  then show ?case by (auto elim: conflictE)
next
  case propagate'
  then show ?case by (auto elim: propagateE)
qed

lemma no-chained-conflict:
  assumes conflict S S'
  and conflict S' S''
  shows False
  using assms unfolding conflict.simps

```

```

by (metis conflicting-update-conflicting option.distinct(1) option.simps(9) state-eq-conflicting)

lemma rtrancpl-cdclW-cp-propa-or-propa-conf:
  assumes cdclW-cp** S U
  shows propagate** S U ∨ (∃ T. propagate** S T ∧ conflict T U)
  using assms
proof induction
  case base
  then show ?case by auto
next
  case (step U V) note SU = this(1) and UV = this(2) and IH = this(3)
  consider (confl) T where propagate** S T and conflict T U
    | (propa) propagate** S U using IH by auto
  then show ?case
  proof cases
    case confl
    then have False using UV by (auto elim: conflictE)
    then show ?thesis by fast
  next
    case propa
    also have conflict U V ∨ propagate U V using UV by (auto simp add: cdclW-cp.simps)
    ultimately show ?thesis by force
  qed
qed

lemma rtrancpl-cdclW-co-conflict-ex-lit-of-max-level:
  assumes full: full cdclW-cp S U
  and cls-f: no-clause-is-false S
  and conflict-is-false-with-level S
  and lev: cdclW-M-level-inv S
  shows conflict-is-false-with-level U
proof (intro allI impI)
  fix D
  assume
    confl: conflicting U = Some D and
    D: D ≠ {#}
  consider (CT) conflicting S = None | (SD) D' where conflicting S = Some D'
  by (cases conflicting S) auto
  then show ∃ L ∈ #D. get-level (trail U) L = backtrack-lvl U
  proof cases
    case SD
    then have S = U
      by (metis (no-types) assms(1) cdclW-cp-conflicting-not-empty full-def rtrancplD trancplD)
    then show ?thesis using assms(3) confl D by blast
  next
    case CT
    have init-clss U = init-clss S and learned-clss U = learned-clss S
      using full unfolding full-def
      apply (metis (no-types) rtrancplD trancpl-cdclW-cp-no-more-init-clss)
      by (metis (mono-tags, lifting) full full-def rtrancpl-cdclW-cp-learned-clause-inv)
    obtain T where propagate** S T and TU: conflict T U
    proof -
      have f5: U ≠ S
        using confl CT by force
      then have cdclW-cp++ S U

```

```

    by (metis full full-def rtrancpD)
  have  $\bigwedge p \text{ pa. } \neg \text{propagate } p \text{ pa} \vee \text{conflicting } \text{pa} =$ 
    (None::'v clause option)
  by (auto elim: propagateE)
  then show ?thesis
    using f5 that trancp-cdclW-cp-propagate-with-conflict-or-not[OF  $\langle \text{cdcl}_W\text{-cp}^{++} S U \rangle$ ]
    full confl CT unfolding full-def by auto
qed
obtain D' where
  raw-conflicting T = None and
  D': D' ! $\in$ ! raw-clauses T and
  tr: trail T  $\models_{\text{as}}$  CNot (mset-cls D') and
  U: U  $\sim$  update-conflicting (Some (ccls-of-cls D')) T
  using TU by (auto elim!: conflictE)
have init-clss T = init-clss S and learned-clss T = learned-clss S
  using U  $\langle \text{init-clss } U = \text{init-clss } S \rangle \langle \text{learned-clss } U = \text{learned-clss } S \rangle$  by auto
then have D  $\in \#$  clauses S
  using confl U D' by (auto simp: raw-clauses-def)
then have  $\neg \text{trail } S \models_{\text{as}}$  CNot D
  using cls-f CT by simp

moreover
  obtain M where tr-U: trail U = M @ trail S and nm:  $\forall m \in \text{set } M. \neg \text{is-marked } m$ 
    by (metis (mono-tags, lifting) assms(1) full-def rtrancp-cdclW-cp-dropWhile-trail)
  have trail U  $\models_{\text{as}}$  CNot D
    using tr confl U by (auto elim!: conflictE)
ultimately obtain L where L  $\in \#$  D and  $\neg L \in \text{lits-of-l } M$ 
  unfolding tr-U CNot-def true-annot-def Ball-def true-annot-def true-cls-def by force

moreover have inv-U: cdclW-M-level-inv U
  by (metis cdclW-stgy.conflict' cdclW-stgy-consistent-inv full full-unfold lev)
moreover
  have backtrack-lvl U = backtrack-lvl S
    using full unfolding full-def by (auto dest: rtrancp-cdclW-cp-backtrack-lvl)

moreover
  have no-dup (trail U)
    using inv-U unfolding cdclW-M-level-inv-def by auto
  { fix x :: ('v, nat, 'v clause) marked-lit and
    xb :: ('v, nat, 'v clause) marked-lit
    assume a1: atm-of L = atm-of (lit-of xb)
    moreover assume a2:  $\neg L = \text{lit-of } x$ 
    moreover assume a3:  $(\lambda l. \text{atm-of } (\text{lit-of } l)) \text{ ' set } M$ 
       $\cap (\lambda l. \text{atm-of } (\text{lit-of } l)) \text{ ' set } (\text{trail } S) = \{\}$ 
    moreover assume a4:  $x \in \text{set } M$ 
    moreover assume a5:  $xb \in \text{set } (\text{trail } S)$ 
    moreover have atm-of ( $\neg L$ ) = atm-of L
      by auto
    ultimately have False
      by auto
  }
  then have LS: atm-of L  $\notin$  atm-of ' lits-of-l (trail S)
    using  $\langle \neg L \in \text{lits-of-l } M \rangle \langle \text{no-dup } (\text{trail } U) \rangle$  unfolding tr-U lits-of-def by auto
ultimately have get-level (trail U) L = backtrack-lvl U
  proof (cases get-all-levels-of-marked (trail S)  $\neq []$ , goal-cases)

```

case 2 **note** $LD = \text{this}(1)$ **and** $LM = \text{this}(2)$ **and** $\text{inv-}U = \text{this}(3)$ **and** $US = \text{this}(4)$ **and**
 $LS = \text{this}(5)$ **and** $ne = \text{this}(6)$
have $\text{backtrack-lvl } S = 0$
using $\text{lev } ne$ **unfolding** $\text{cdcl}_W\text{-}M\text{-level-inv-def}$ **by** auto
moreover **have** $\text{get-rev-level } (\text{rev } M) \ 0 \ L = 0$
using nm **by** auto
ultimately **show** $?thesis$ **using** $LS \ ne \ US$ **unfolding** $tr\text{-}U$
by $(\text{simp add: get-all-levels-of-marked-nil-iff-not-is-marked lits-of-def})$
next
case 1 **note** $LD = \text{this}(1)$ **and** $LM = \text{this}(2)$ **and** $\text{inv-}U = \text{this}(3)$ **and** $US = \text{this}(4)$ **and**
 $LS = \text{this}(5)$ **and** $ne = \text{this}(6)$

have $\text{hd } (\text{get-all-levels-of-marked } (\text{trail } S)) = \text{backtrack-lvl } S$
using $ne \ lev$ **unfolding** $\text{cdcl}_W\text{-}M\text{-level-inv-def}$
by $(\text{cases get-all-levels-of-marked } (\text{trail } S)) \ \text{auto}$
moreover **have** $\text{atm-of } L \in \text{atm-of } ' \text{ lits-of-l } M$
using $\langle -L \in \text{lits-of-l } M \rangle$ **by** $(\text{simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set lits-of-def})$
ultimately **show** $?thesis$
using $nm \ ne \ \text{get-level-skip-beginning-hd-get-all-levels-of-marked}[OF \ LS, \ \text{of } M]$
 $\text{get-level-skip-in-all-not-marked}[\text{of rev } M \ L \ \text{backtrack-lvl } S]$
unfolding $\text{lits-of-def } US \ tr\text{-}U$
by auto
qed
then **show** $\exists L \in \#D. \ \text{get-level } (\text{trail } U) \ L = \text{backtrack-lvl } U$
using $\langle L \in \# \ D \rangle$ **by** blast
qed
qed

19.5.4 Literal of highest level in marked literals

definition $\text{mark-is-false-with-level} :: 'st \Rightarrow \text{bool}$ **where**

$\text{mark-is-false-with-level } S' \equiv$

$\forall D \ M1 \ M2 \ L. \ M1 \ @ \ \text{Propagated } L \ D \ \# \ M2 = \text{trail } S' \longrightarrow D - \{\#L\# \} \neq \{\#\}$
 $\longrightarrow (\exists L. L \in \# \ D \wedge \text{get-level } (\text{trail } S') \ L = \text{get-maximum-possible-level } M1)$

definition $\text{no-more-propagation-to-do} :: 'st \Rightarrow \text{bool}$ **where**

$\text{no-more-propagation-to-do } S \equiv$

$\forall D \ M \ M' \ L. \ D + \{\#L\# \} \in \# \ \text{clauses } S \longrightarrow \text{trail } S = M' \ @ \ M \longrightarrow M \models_{as} CNot \ D$
 $\longrightarrow \text{undefined-lit } M \ L \longrightarrow \text{get-maximum-possible-level } M < \text{backtrack-lvl } S$
 $\longrightarrow (\exists L. L \in \# \ D \wedge \text{get-level } (\text{trail } S) \ L = \text{get-maximum-possible-level } M)$

lemma $\text{propagate-no-more-propagation-to-do}$:

assumes $\text{propagate: propagate } S \ S'$

and $H: \text{no-more-propagation-to-do } S$

and $\text{lev-inv: cdcl}_W\text{-}M\text{-level-inv } S$

shows $\text{no-more-propagation-to-do } S'$

using assms

proof –

obtain $E \ L$ **where**

$S: \text{conflicting } S = \text{None}$ **and**

$E: E \ !\in! \ \text{raw-clauses } S$ **and**

$LE: L \in \# \ \text{mset-cls } E$ **and**

$tr: \text{trail } S \models_{as} CNot \ (\text{mset-cls } (\text{remove-lit } L \ E))$ **and**

$\text{undefL: undefined-lit } (\text{trail } S) \ L$ **and**

$S': S' \sim \text{cons-trail } (\text{Propagated } L \ E) \ S$

```

using propagate by (elim propagateE) simp
let ?M' = Propagated L (mset-cls E) # trail S
show ?thesis unfolding no-more-propagation-to-do-def
proof (intro allI impI)
  fix D M1 M2 L'
  assume
    D-L: D + {#L'#} ∈# clauses S' and
    trail S' = M2 @ M1 and
    get-max: get-maximum-possible-level M1 < backtrack-lvl S' and
    M1 ⊢as CNot D and
    undef: undefined-lit M1 L'
  have tl M2 @ M1 = trail S ∨ (M2 = [] ∧ M1 = Propagated L (mset-cls E) # trail S)
    using ⟨trail S' = M2 @ M1⟩ S' S undefL lev-inv
    by (cases M2) (auto simp: cdclW-M-level-inv-decomp)
  moreover {
    assume tl M2 @ M1 = trail S
    moreover have D + {#L'#} ∈# clauses S
      using D-L S S' undefL unfolding raw-clauses-def by auto
    moreover have get-maximum-possible-level M1 < backtrack-lvl S
      using get-max S S' undefL by auto
    ultimately obtain L' where L' ∈# D and
      get-level (trail S) L' = get-maximum-possible-level M1
      using H ⟨M1 ⊢as CNot D⟩ undef unfolding no-more-propagation-to-do-def by metis
    moreover
      { have cdclW-M-level-inv S'
        using cdclW-consistent-inv lev-inv cdclW.propagate[OF propagate] by blast
        then have no-dup ?M' using S' undefL unfolding cdclW-M-level-inv-def by auto
        moreover
          have atm-of L' ∈ atm-of ' (lits-of-l M1)
            using ⟨L' ∈# D⟩ ⟨M1 ⊢as CNot D⟩ by (metis atm-of-uminus image-eqI
              in-CNot-implies-uminus(2))
          then have atm-of L' ∈ atm-of ' (lits-of-l (trail S))
            using ⟨tl M2 @ M1 = trail S⟩[symmetric] S undefL by auto
          ultimately have atm-of L ≠ atm-of L' unfolding lits-of-def by auto
        }
    ultimately have ∃ L' ∈# D. get-level (trail S') L' = get-maximum-possible-level M1
      using S S' undefL by auto
  }
  moreover {
    assume M2 = [] and M1: M1 = Propagated L (mset-cls E) # trail S
    have cdclW-M-level-inv S'
      using cdclW-consistent-inv[OF lev-inv] cdclW.propagate[OF propagate] by blast
    then have get-all-levels-of-marked (trail S') = rev [Suc 0..<(Suc 0+backtrack-lvl S)]
      using S' undefL unfolding cdclW-M-level-inv-def by auto
    then have get-maximum-possible-level M1 = backtrack-lvl S'
      using get-maximum-possible-level-max-get-all-levels-of-marked[of M1] S' M1 undefL
      by (auto intro: Max-eqI)
    then have False using get-max by auto
  }
  ultimately show ∃ L. L ∈# D ∧ get-level (trail S') L = get-maximum-possible-level M1
    by fast
qed
qed

```

lemma conflict-no-more-propagation-to-do:


```

assumes
  conflict: conflict S S' and
  H: no-more-propagation-to-do S and
  M: cdclW-M-level-inv S
shows no-more-propagation-to-do S'
using assms unfolding no-more-propagation-to-do-def by (force elim!: conflictE)

lemma cdclW-cp-no-more-propagation-to-do:
assumes
  conflict: cdclW-cp S S' and
  H: no-more-propagation-to-do S and
  M: cdclW-M-level-inv S
shows no-more-propagation-to-do S'
using assms
proof (induct rule: cdclW-cp.induct)
case (conflict' S S')
then show ?case using conflict-no-more-propagation-to-do[of S S'] by blast
next
case (propagate' S S') note S = this
show 1: no-more-propagation-to-do S'
  using propagate-no-more-propagation-to-do[of S S'] S by blast
qed

lemma cdclW-then-exists-cdclW-stgy-step:
assumes
  o: cdclW-o S S' and
  alien: no-strange-atm S and
  lev: cdclW-M-level-inv S
shows  $\exists S'. \text{cdcl}_W\text{-stgy } S S'$ 
proof –
obtain S'' where full cdclW-cp S' S''
  using always-exists-full-cdclW-cp-step alien cdclW-no-strange-atm-inv cdclW-o-no-more-init-clss
  o other lev by (meson cdclW-consistent-inv)
then show ?thesis
  using assms by (metis always-exists-full-cdclW-cp-step cdclW-stgy.conflict' full-unfold other')
qed

lemma backtrack-no-decomp:
assumes
  S: raw-conflicting S = Some E and
  LE:  $L \in \# \text{mset-ccls } E$  and
  L: get-level (trail S) L = backtrack-lvl S and
  D: get-maximum-level (trail S) (remove1-mset L (mset-ccls E)) < backtrack-lvl S and
  bt: backtrack-lvl S = get-maximum-level (trail S) (mset-ccls E) and
  M-L: cdclW-M-level-inv S
shows  $\exists S'. \text{cdcl}_W\text{-o } S S'$ 
proof –
have L-D: get-level (trail S) L = get-maximum-level (trail S) (mset-ccls E)
  using L D bt by (simp add: get-maximum-level-plus)
let ?i = get-maximum-level (trail S) (remove1-mset L (mset-ccls E))
obtain K M1 M2 where
  K: (Marked K (?i + 1)  $\#$  M1, M2)  $\in$  set (get-all-marked-decomposition (trail S))
  using backtrack-ex-decomp[OF M-L, of ?i] D S by auto
show ?thesis using backtrack-rule[OF S LE K L] bt L bj cdclW-bj.simps by auto
qed

```

lemma *cdcl_W-stgy-final-state-conclusive*:

assumes

termi: $\forall S'. \neg \text{cdcl}_W\text{-stgy } S S'$ **and**

decomp: *all-decomposition-implies-m* (*init-clss* *S*) (*get-all-marked-decomposition* (*trail* *S*)) **and**

learned: *cdcl_W-learned-clause* *S* **and**

level-inv: *cdcl_W-M-level-inv* *S* **and**

alien: *no-strange-atm* *S* **and**

no-dup: *distinct-cdcl_W-state* *S* **and**

confl: *cdcl_W-conflicting* *S* **and**

confl-k: *conflict-is-false-with-level* *S*

shows (*conflicting* *S* = *Some* {#} \wedge *unsatisfiable* (*set-mset* (*init-clss* *S*)))

\vee (*conflicting* *S* = *None* \wedge *trail* *S* \models_{as} *set-mset* (*init-clss* *S*))

proof –

let *?M* = *trail* *S*

let *?N* = *init-clss* *S*

let *?k* = *backtrack-lvl* *S*

let *?U* = *learned-clss* *S*

consider

(*None*) *raw-conflicting* *S* = *None*

| (*Some-Empty*) *E* **where** *raw-conflicting* *S* = *Some* *E* **and** *mset-ccls* *E* = {#}

| (*Some*) *E'* **where** *raw-conflicting* *S* = *Some* *E'* **and**

conflicting *S* = *Some* (*mset-ccls* *E'*) **and** *mset-ccls* *E'* \neq {#}

by (*cases* *conflicting* *S*, *simp*) *auto*

then show *?thesis*

proof *cases*

case (*Some-Empty* *E*)

then have *conflicting* *S* = *Some* {#} **by** *auto*

then have *unsatisfiable* (*set-mset* (*init-clss* *S*))

using *assms*(3) **unfolding** *cdcl_W-learned-clause-def* *true-clss-clss-def*

by (*metis* (*no-types*, *lifting*) *Un-insert-right* *atms-of-empty* *satisfiable-def*

sup-bot.right-neutral *total-over-m-insert* *total-over-set-empty* *true-clss-empty*)

then show *?thesis* **using** *Some-Empty* **by** *auto*

next

case *None*

{ **assume** $\neg ?M \models_{\text{asm}} ?N$

have *atm-of* ' (*lits-of-l* *?M*) = *atms-of-mm* *?N* (**is** *?A* = *?B*)

proof

show *?A* \subseteq *?B* **using** *alien* **unfolding** *no-strange-atm-def* **by** *auto*

show *?B* \subseteq *?A*

proof (*rule ccontr*)

assume $\neg ?B \subseteq ?A$

then obtain *l* **where** *l* \in *?B* **and** *l* \notin *?A* **by** *auto*

then have *undefined-lit* *?M* (*Pos* *l*)

using (*l* \notin *?A*) **unfolding** *lits-of-def* **by** (*auto simp add: defined-lit-map*)

moreover have *conflicting* *S* = *None*

using *None* **by** *auto*

ultimately have $\exists S'. \text{cdcl}_W\text{-o } S S'$

using *cdcl_W-o.decide* *decide-rule* (*l* \in *?B*) *no-strange-atm-def*

by (*metis* *literal.sel*(1) *state-eq-def*)

then show *False*

using *termi* *cdcl_W-then-exists-cdcl_W-stgy-step*[*OF* - *alien*] *level-inv* **by** *blast*

qed

qed

obtain *D* **where** $\neg ?M \models_a D$ **and** *D* $\in \#$ *?N*

```

    using  $\langle \neg ?M \models_{asm} ?N \rangle$  unfolding lits-of-def true-annots-def Ball-def by auto
have atms-of  $D \subseteq \text{atm-of } \langle \text{lits-of-l } ?M \rangle$ 
    using  $\langle D \in \# ?N \rangle$  unfolding  $\langle \text{atm-of } \langle \text{lits-of-l } ?M \rangle = \text{atms-of-mm } ?N \rangle$  atms-of-ms-def
    by (auto simp add: atms-of-def)
then have  $a1: \text{atm-of } \langle \text{set-mset } D \subseteq \text{atm-of } \langle \text{lits-of-l } (\text{trail } S) \rangle$ 
    by (auto simp add: atms-of-def lits-of-def)
have total-over-m  $(\text{lits-of-l } ?M) \{D\}$ 
    using  $\langle \text{atms-of } D \subseteq \text{atm-of } \langle \text{lits-of-l } ?M \rangle$ 
    atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set by (fastforce simp: total-over-set-def)
then have  $?M \models_{as} \text{CNot } D$ 
    using total-not-true-cls-true-clss-CNot  $\langle \neg \text{trail } S \models_a D \rangle$  true-annot-def
    true-annots-true-cls by fastforce
then have False
proof -
  obtain  $S'$  where
     $f2: \text{full cdcl}_W\text{-cp } S S'$ 
    by (meson alien always-exists-full-cdcl_W-cp-step level-inv)
  then have  $S' = S$ 
    using cdcl_W-stgy.conflict'[of S] by (metis (no-types) full-unfold termi)
  then show  $?thesis$ 
    using  $f2 \langle D \in \# \text{init-clss } S \rangle \text{None } \langle \text{trail } S \models_{as} \text{CNot } D \rangle$ 
    raw-clauses-def full-cdcl_W-cp-not-any-negated-init-clss by auto
qed
}
then have  $?M \models_{asm} ?N$  by blast
then show  $?thesis$ 
  using None by auto
next
case (Some E') note  $\text{raw-conf} = \text{this}(1)$  and  $LD = \text{this}(2)$  and  $\text{nempty} = \text{this}(3)$ 
then obtain  $L D$  where
   $E'[simp]: \text{mset-ccls } E' = D + \{\#L\# \}$  and
  lev-L: get-level  $?M L = ?k$ 
  by (metis (mono-tags) confl-k insert-DiffM2)
let  $?D = D + \{\#L\# \}$ 
have  $?D \neq \{\# \}$  by auto
have  $?M \models_{as} \text{CNot } ?D$  using confl LD unfolding cdcl_W-conflicting-def by auto
then have  $?M \neq []$  unfolding true-annots-def Ball-def true-annot-def true-cls-def by force
have  $M: ?M = \text{hd } ?M \# \text{tl } ?M$  using  $\langle ?M \neq [] \rangle$  list.collapse by fastforce
have g-a-l: get-all-levels-of-marked  $?M = \text{rev } [1..<1 + ?k]$ 
  using level-inv lev-L M unfolding cdcl_W-M-level-inv-def by auto
have g-k: get-maximum-level  $(\text{trail } S) D \leq ?k$ 
  using get-maximum-possible-level-ge-get-maximum-level[of ?M]
  get-maximum-possible-level-max-get-all-levels-of-marked[of ?M]
  by (auto simp add: Max-n-upt g-a-l)
{
  assume marked: is-marked  $(\text{hd } ?M)$ 
  then obtain  $k'$  where  $k': k' + 1 = ?k$ 
    using level-inv M unfolding cdcl_W-M-level-inv-def
    by (cases hd (trail S); cases trail S) auto
  obtain  $L' l'$  where  $L': \text{hd } ?M = \text{Marked } L' l'$  using marked by (cases hd ?M) auto
  have marked-hd-tl: get-all-levels-of-marked  $(\text{hd } (\text{trail } S) \# \text{tl } (\text{trail } S))$ 
     $= \text{rev } [1..<1 + \text{length } (\text{get-all-levels-of-marked } ?M)]$ 
    using level-inv lev-L M unfolding cdcl_W-M-level-inv-def M[symmetric]
    by blast
  then have  $l'\text{-tl}: l' \# \text{get-all-levels-of-marked } (\text{tl } ?M)$ 

```

= rev [1.. $1 + \text{length } (\text{get-all-levels-of-marked } ?M)$] **unfolding** L' **by** *simp*
moreover have ... = length (get-all-levels-of-marked ?M)
 # rev [1.. $\text{length } (\text{get-all-levels-of-marked } ?M)$]
using $M \text{ Suc-le-mono calculation by } (\text{fastforce simp add: upt.simps}(2))$
finally have
 $l'\text{-cons: } l' \# \text{get-all-levels-of-marked } (\text{tl } (\text{trail } S)) =$
 length (get-all-levels-of-marked (trail S))
 # rev [1.. $\text{length } (\text{get-all-levels-of-marked } (\text{trail } S))]$ **and**
 $l' = ?k$ **and**
 $g\text{-r: } \text{get-all-levels-of-marked } (\text{tl } (\text{trail } S))$
 = rev [1.. $\text{length } (\text{get-all-levels-of-marked } (\text{trail } S))]$
using $\text{level-inv lev-L } M$ **unfolding** $\text{cdcl}_W\text{-M-level-inv-def}$ **by** *auto*

have *: $\bigwedge \text{list. no-dup list} \implies$
 - $L \in \text{lits-of-l list} \implies \text{atm-of } L \in \text{atm-of ' lits-of-l list}$
by (metis atm-of-uminus imageI)
have $L'\text{-L: } L' = -L$
proof (rule ccontr)
assume $\neg ?thesis$
moreover have $-L \in \text{lits-of-l } ?M$ **using** *confl LD* **unfolding** $\text{cdcl}_W\text{-conflicting-def}$ **by** *auto*
ultimately have $\text{get-level } (\text{hd } (\text{trail } S)) \# \text{tl } (\text{trail } S)) L = \text{get-level } (\text{tl } ?M) L$
using $\text{cdcl}_W\text{-M-level-inv-decomp}(1)[\text{OF level-inv}] L' M \text{ atm-of-eq-atm-of}$
unfolding *lits-of-def consistent-interp-def*
by (metis (mono-tags, hide-lams) marked-lit.sel(1) get-level-skip-beginning image-eqI
 list.set-intros(1))
moreover
have length (get-all-levels-of-marked (trail S)) = $?k$
using $\text{level-inv unfolding cdcl}_W\text{-M-level-inv-def}$ **by** *auto*
then have $\text{Max } (\text{set } (0 \# \text{get-all-levels-of-marked } (\text{tl } (\text{trail } S)))) = ?k - 1$
unfolding $g\text{-r by } (\text{auto simp add: Max-n-upt})$
then have $\text{get-level } (\text{tl } ?M) L < ?k$
using $\text{get-maximum-possible-level-ge-get-level}[\text{of tl } ?M L]$
by (metis One-nat-def add.right-neutral add-Suc-right diff-add-inverse2
 get-maximum-possible-level-max-get-all-levels-of-marked k' le-imp-less-Suc
 list.simps(15))
finally show *False* **using** $\text{lev-L } M$ **by** *auto*

qed
have $L: \text{hd } ?M = \text{Marked } (-L) ?k$ **using** $\langle l' = ?k \rangle L'\text{-L } L'$ **by** *auto*

have $\text{get-maximum-level } (\text{trail } S) D < ?k$
proof (rule ccontr)
assume $\neg ?thesis$
then have $\text{get-maximum-level } (\text{trail } S) D = ?k$ **using** $M g\text{-k unfolding } L$ **by** *auto*
then obtain L'' **where** $L'' \in \# D$ **and** $L\text{-k: } \text{get-level } ?M L'' = ?k$
using $\text{get-maximum-level-exists-lit}[\text{of } ?k ?M D]$ **unfolding** $k'[\text{symmetric}]$ **by** *auto*
have $L \neq L''$ **using** $\text{no-dup } \langle L'' \in \# D \rangle$
unfolding $\text{distinct-cdcl}_W\text{-state-def } LD$
by (metis E' add.right-neutral add-diff-cancel-right'
 distinct-mem-diff-mset union-commute union-single-eq-member)
have $L'' = -L$
proof (rule ccontr)
assume $\neg ?thesis$
then have $\text{get-level } ?M L'' = \text{get-level } (\text{tl } ?M) L''$
using $M \langle L \neq L'' \rangle \text{get-level-skip-beginning}[\text{of } L'' \text{hd } ?M \text{tl } ?M]$ **unfolding** L
by (auto simp: atm-of-eq-atm-of)

```

then show False
  by (metis L-k Max-n-upt One-nat-def Suc-n-not-le-n ⟨l' = backtrack-lvl S⟩
      add-Suc-right add-implies-diff g-r
      get-all-levels-of-marked-rev-eq-rev-get-all-levels-of-marked list.set(2)
      get-rev-level-less-max-get-all-levels-of-marked k' l'-cons list.sel(1)
      rev-rev-ident semiring-normalization-rules(6) set-upt)
qed
then have taut: tautology (D + {#L#})
  using ⟨L'' ∈ # D⟩ by (metis add.commute mset-leD mset-le-add-left multi-member-this
      tautology-minus)
have consistent-interp (lits-of-l ?M)
  using level-inv unfolding cdclW-M-level-inv-def by auto
then have ¬?M ⊨as CNot ?D
  using taut by (metis ⟨L'' = - L⟩ ⟨L'' ∈ # D⟩ add.commute consistent-interp-def
      diff-union-cancelR in-CNot-implies-uminus(2) in-diffD multi-member-this)
moreover have ?M ⊨as CNot ?D
  using confl no-dup LD unfolding cdclW-conflicting-def by auto
ultimately show False by blast
qed note H = this
have get-maximum-level (trail S) D < get-maximum-level (trail S) (D + {#L#})
  using H by (auto simp: get-maximum-level-plus lev-L max-def)
moreover have backtrack-lvl S = get-maximum-level (trail S) (D + {#L#})
  using H by (auto simp: get-maximum-level-plus lev-L max-def)
ultimately have False
  using backtrack-no-decomp[OF raw-conf - lev-L] level-inv termi
      cdclW-then-exists-cdclW-stgy-step[of S] alien unfolding E'
  by (auto simp add: lev-L max-def)
} note not-is-marked = this

moreover {
  let ?D = D + {#L#}
  have ?D ≠ {#} by auto
  have ?M ⊨as CNot ?D using confl LD unfolding cdclW-conflicting-def by auto
  then have ?M ≠ [] unfolding true-annots-def Ball-def true-annot-def true-cls-def by force
  assume nm: ¬is-marked (hd ?M)
  then obtain L' C where L'C: hd-raw-trail S = Propagated L' C
    by (metis ⟨trail S ≠ []⟩ hd-raw-trail is-marked-def mset-of-mlit.elims)
  then have hd ?M = Propagated L' (mset-cls C)
    using ⟨trail S ≠ []⟩ hd-raw-trail mset-of-mlit.simps(1) by fastforce
  then have M: ?M = Propagated L' (mset-cls C) # tl ?M
    using ⟨?M ≠ []⟩ list.collapse by fastforce
  then obtain C' where C': mset-cls C = C' + {#L'#}
    using confl unfolding cdclW-conflicting-def by (metis append-Nil diff-single-eq-union)
  { assume -L' ∉ # ?D
    then have Ex (skip S)
      using skip-rule[OF M raw-conf] unfolding E' by auto
    then have False
      using cdclW-then-exists-cdclW-stgy-step[of S] alien level-inv termi
      by (auto dest: cdclW-o.intros cdclW-bj.intros)
  }
}
moreover {
  assume L'D: -L' ∈ # ?D
  then obtain D' where D': ?D = D' + {#-L'#} by (metis insert-DiffM2)
  have g-r: get-all-levels-of-marked (Propagated L' (mset-cls C) # tl (trail S))
    = rev [Suc 0..<Suc (length (get-all-levels-of-marked (trail S)))]

```

```

    using level-inv M unfolding cdclW-M-level-inv-def by auto
  have Max (insert 0
    (set (get-all-levels-of-marked (Propagated L' (mset-cls C) # tl (trail S)))) = ?k
    using level-inv M unfolding g-r cdclW-M-level-inv-def set-rev
    by (auto simp add:Max-n-upt)
  then have get-maximum-level (trail S) D' ≤ ?k
    using get-maximum-possible-level-ge-get-maximum-level[of
      Propagated L' (mset-cls C) # tl ?M] M
    unfolding get-maximum-possible-level-max-get-all-levels-of-marked by auto
  then have get-maximum-level (trail S) D' = ?k
    ∨ get-maximum-level (trail S) D' < ?k
    using le-neq-implies-less by blast
  moreover {
    assume g-D'-k: get-maximum-level (trail S) D' = ?k
    then have f1: get-maximum-level (trail S) D' = backtrack-lvl S
      using M by auto
    then have Ex (cdclW-o S)
      using f1 resolve-rule[of S L' C , OF ⟨trail S ≠ []⟩ - - raw-conf] raw-conf g-D'-k
      L'C L'D unfolding C' D' E'
      by (fastforce simp add: D' intro: cdclW-o.intros cdclW-bj.intros)
    then have False
      by (meson alien cdclW-then-exists-cdclW-stgy-step termi level-inv)
  }
  moreover {
    assume a1: get-maximum-level (trail S) D' < ?k
    then have f3: get-maximum-level (trail S) D' < get-level (trail S) (-L')
      using a1 lev-L by (metis D' get-maximum-level-ge-get-level insert-noteq-member
        not-less)
    moreover have backtrack-lvl S = get-level (trail S) L'
      apply (subst M)
      unfolding rev.simps
      apply (subst get-rev-level-can-skip-correctly-ordered)
      using level-inv unfolding cdclW-M-level-inv-def
      apply (subst (asm) (2) M) apply (simp add: cdclW-M-level-inv-decomp)
      using level-inv unfolding cdclW-M-level-inv-def
      apply (subst (asm) (2) M) apply (auto simp: cdclW-M-level-inv-decomp lits-of-def)[]
      using level-inv unfolding cdclW-M-level-inv-def
      apply (subst (asm) (4) M) apply (auto simp add: cdclW-M-level-inv-decomp)[]
      using level-inv unfolding cdclW-M-level-inv-def
      apply (subst (asm) (4) M) by (auto simp add: cdclW-M-level-inv-decomp)[]
    moreover
      then have get-level (trail S) L' = get-maximum-level (trail S) (D' + {#- L'#})
        using a1 by (auto simp add: get-maximum-level-plus max-def)
    ultimately have False
      using M backtrack-no-decomp[of S - -L', OF raw-conf]
      cdclW-then-exists-cdclW-stgy-step L'D level-inv termi alien
      unfolding D' E' by auto
  }
  ultimately have False by blast
}
ultimately have False by blast
}
ultimately show ?thesis by blast
qed

```

lemma *cdcl_W-cp-tranclp-cdcl_W*:
 $cdcl_W\text{-}cp\ S\ S' \implies cdcl_W^{++}\ S\ S'$
apply (*induct rule*: *cdcl_W-cp.induct*)
by (*meson* *cdcl_W.conflict cdcl_W.propagate tranclp.r-into-trancl tranclp.trancl-into-trancl*)**+**

lemma *tranclp-cdcl_W-cp-tranclp-cdcl_W*:
 $cdcl_W\text{-}cp^{++}\ S\ S' \implies cdcl_W^{++}\ S\ S'$
apply (*induct rule*: *tranclp.induct*)
apply (*simp add*: *cdcl_W-cp-tranclp-cdcl_W*)
by (*meson* *cdcl_W-cp-tranclp-cdcl_W tranclp-trans*)

lemma *cdcl_W-stgy-tranclp-cdcl_W*:
 $cdcl_W\text{-}stgy\ S\ S' \implies cdcl_W^{++}\ S\ S'$
proof (*induct rule*: *cdcl_W-stgy.induct*)
case *conflict'*
then show *?case*
unfolding *full1-def* **by** (*simp add*: *tranclp-cdcl_W-cp-tranclp-cdcl_W*)
next
case (*other' S' S''*)
then have $S' = S'' \vee cdcl_W\text{-}cp^{++}\ S'\ S''$
by (*simp add*: *rtranclp-unfold full-def*)
then show *?case*
using *other'* **by** (*meson* *cdcl_W.other tranclp.r-into-trancl tranclp-cdcl_W-cp-tranclp-cdcl_W tranclp-trans*)
qed

lemma *tranclp-cdcl_W-stgy-tranclp-cdcl_W*:
 $cdcl_W\text{-}stgy^{++}\ S\ S' \implies cdcl_W^{++}\ S\ S'$
apply (*induct rule*: *tranclp.induct*)
using *cdcl_W-stgy-tranclp-cdcl_W* **apply** *blast*
by (*meson* *cdcl_W-stgy-tranclp-cdcl_W tranclp-trans*)

lemma *rtranclp-cdcl_W-stgy-rtranclp-cdcl_W*:
 $cdcl_W\text{-}stgy^{**}\ S\ S' \implies cdcl_W^{**}\ S\ S'$
using *rtranclp-unfold*[*of cdcl_W-stgy S S'*] *tranclp-cdcl_W-stgy-tranclp-cdcl_W*[*of S S'*] **by** *auto*

lemma *not-empty-get-maximum-level-exists-lit*:
assumes $n: D \neq \{\#\}$
and *max*: *get-maximum-level M D = n*
shows $\exists L \in \#D. \text{get-level } M\ L = n$
proof –
have *f*: *finite* (*insert* 0 (($\lambda L. \text{get-level } M\ L$) ‘ *set-mset D*)) **by** *auto*
then have $n \in ((\lambda L. \text{get-level } M\ L) \text{ ‘ set-mset } D)$
using *n max get-maximum-level-exists-lit-of-max-level image-iff*
unfolding *get-maximum-level-def* **by** *force*
then show $\exists L \in \#D. \text{get-level } M\ L = n$ **by** *auto*
qed

lemma *cdcl_W-o-conflict-is-false-with-level-inv*:
assumes
cdcl_W-o S S' and
lev: *cdcl_W-M-level-inv S and*
confl-inv: *conflict-is-false-with-level S and*
n-d: *distinct-cdcl_W-state S and*

conflicting: $cdcl_W$ -conflicting S
shows *conflict-is-false-with-level* S'
using *assms*(1,2)
proof (*induct rule*: $cdcl_W$ -o-induct-lev2)
 case (*resolve* L C M D T) **note** $tr-S = this(1)$ **and** $confl = this(4)$ **and** $LD = this(5)$ **and** $T = this(7)$
 have uL -not- D : $-L \notin \#$ *remove1-mset* $(-L)$ (*mset-ccls* D)
 using n - d *confl* **unfolding** *distinct-cdcl_W-state-def* *distinct-mset-def*
 by (*metis* *distinct-cdcl_W-state-def* *distinct-mem-diff-mset* *multi-member-last* n - d *option.simps*(9))
 moreover **have** L -not- D : $L \notin \#$ *remove1-mset* $(-L)$ (*mset-ccls* D)
 proof (*rule ccontr*)
 assume $\neg ?thesis$
 then **have** $L \in \#$ *mset-ccls* D
 by (*auto simp: in-remove1-mset-neg*)
 moreover **have** *Propagated* L (*mset-cls* C) $\#$ $M \models_{as}$ *CNot* (*mset-ccls* D)
 using *conflicting confl tr-S* **unfolding** *cdcl_W-conflicting-def* **by** *auto*
 ultimately **have** $-L \in$ *lits-of-l* (*Propagated* L (*mset-cls* C) $\#$ M)
 using *in-CNot-implies-uminus*(2) **by** *blast*
 moreover **have** *no-dup* (*Propagated* L (*mset-cls* C) $\#$ M)
 using *lev tr-S* **unfolding** *cdcl_W-M-level-inv-def* **by** *auto*
 ultimately **show** *False* **unfolding** *lits-of-def* **by** (*metis* *consistent-interp-def* *image-eqI* *list.set-intros*(1) *lits-of-def* *marked-lit.sel*(2) *distinct-consistent-interp*)
 qed
 ultimately
 have g - D : *get-maximum-level* (*Propagated* L (*mset-cls* C) $\#$ M) (*remove1-mset* $(-L)$ (*mset-ccls* D))
 $=$ *get-maximum-level* M (*remove1-mset* $(-L)$ (*mset-ccls* D))
 proof –
 have $\forall a f L. ((a::'v) \in f \text{ ' } L) = (\exists l. (l::'v \text{ literal}) \in L \wedge a = f l)$
 by *blast*
 then **show** *?thesis*
 using *get-maximum-level-skip-first*[*of* L *remove1-mset* $(-L)$ (*mset-ccls* D) *mset-cls* C M]
 unfolding *atms-of-def*
 by (*metis* (*no-types*) uL -not- D L -not- D *atm-of-eq-atm-of*)
 qed
 have lev - L [*simp*]: *get-level* M $L = 0$
 apply (*rule atm-of-notin-get-rev-level-eq-0*)
 using lev **unfolding** *cdcl_W-M-level-inv-def* $tr-S$ **by** (*auto simp: lits-of-def*)

 have D : *get-maximum-level* M (*remove1-mset* $(-L)$ (*mset-ccls* D)) $=$ *backtrack-lvl* S
 using *resolve.hyps*(6) LD **unfolding** $tr-S$ **by** (*auto simp: get-maximum-level-plus max-def g-D*)
 have *get-all-levels-of-marked* $M = rev$ [*Suc* 0..*Suc* (*backtrack-lvl* S)]
 using lev **unfolding** $tr-S$ *cdcl_W-M-level-inv-def* **by** *auto*
 then **have** *get-maximum-level* M (*remove1-mset* L (*mset-cls* C)) \leq *backtrack-lvl* S
 using *get-maximum-possible-level-ge-get-maximum-level*[*of* M]
 get-maximum-possible-level-max-get-all-levels-of-marked[*of* M] **by** (*auto simp: Max-n-upt*)
 then **have**
 get-maximum-level M (*remove1-mset* $(-L)$ (*mset-ccls* D) $\# \cup$ *remove1-mset* L (*mset-cls* C)) $=$
 backtrack-lvl S
 by (*auto simp: get-maximum-level-union-mset get-maximum-level-plus max-def D*)
 then **show** *?case*
 using $tr-S$ *not-empty-get-maximum-level-exists-lit*[*of*
 remove1-mset $(-L)$ (*mset-ccls* D) $\# \cup$ *remove1-mset* L (*mset-cls* C) M] T
 by *auto*
next


```

case (skip  $L$   $C'$   $M$   $D$   $T$ ) note  $tr-S = this(1)$  and  $D = this(2)$  and  $T = this(5)$ 
then obtain  $La$  where
   $La \in \#$  mset-ccls  $D$  and
  get-level (Propagated  $L$   $C'$   $\#$   $M$ )  $La = backtrack-lvl$   $S$ 
  using skip confl-inv by auto
moreover
  have atm-of  $La \neq atm-of$   $L$ 
  proof (rule ccontr)
    assume  $\neg ?thesis$ 
    then have  $La: La = L$  using  $\langle La \in \#$  mset-ccls  $D \rangle \langle \neg L \notin \#$  mset-ccls  $D \rangle$ 
      by (auto simp add: atm-of-eq-atm-of)
    have Propagated  $L$   $C'$   $\#$   $M \models_{as} CNot$  (mset-ccls  $D$ )
      using conflicting tr-S D unfolding cdclW-conflicting-def by auto
    then have  $\neg L \in lits-of-l$   $M$ 
      using  $\langle La \in \#$  mset-ccls  $D \rangle$  in-CNot-implies-uminus(2)[of L mset-ccls D
        Propagated L C' # M] unfolding  $La$ 
      by auto
    then show False using lev tr-S unfolding cdclW-M-level-inv-def consistent-interp-def by auto
  qed
  then have get-level (Propagated  $L$   $C'$   $\#$   $M$ )  $La = get-level$   $M$   $La$  by auto
ultimately show ?case using  $D$   $tr-S$   $T$  by auto
next
case backtrack
then show ?case
  by (auto split: if-split-asm simp: cdclW-M-level-inv-decomp lev)
qed auto

```

19.5.5 Strong completeness

```

lemma cdclW-cp-propagate-confl:
  assumes cdclW-cp  $S$   $T$ 
  shows propagate**  $S$   $T \vee (\exists S'. propagate** S S' \wedge conflict S' T)$ 
  using assms by induction blast+

lemma rtrancp-cdclW-cp-propagate-confl:
  assumes cdclW-cp**  $S$   $T$ 
  shows propagate**  $S$   $T \vee (\exists S'. propagate** S S' \wedge conflict S' T)$ 
  by (simp add: assms rtrancp-cdclW-cp-propa-or-propa-confl)

lemma propagate-high-levelE:
  assumes propagate  $S$   $T$ 
  obtains  $M' N' U k L C$  where
    state  $S = (M', N', U, k, None)$  and
    state  $T = (Propagated L (C + \{\#L\}) \# M', N', U, k, None)$  and
     $C + \{\#L\} \in \#$  local.clauses  $S$  and
     $M' \models_{as} CNot$   $C$  and
    undefined-lit (trail  $S$ )  $L$ 
proof –
  obtain  $E L$  where
    conf: conflicting  $S = None$  and
     $E: E ! \in !$  raw-clauses  $S$  and
     $LE: L \in \#$  mset-cls  $E$  and
     $tr: trail$   $S \models_{as} CNot$  (mset-cls (remove-lit  $L$   $E$ )) and
    undef: undefined-lit (trail  $S$ )  $L$  and
     $T: T \sim cons-trail$  (Propagated  $L$   $E$ )  $S$ 
  using assms by (elim propagateE) simp

```

```

obtain  $M\ N\ U\ k$  where
   $S$ : state  $S = (M, N, U, k, \text{None})$ 
  using conf by auto
show thesis
  using that[of  $M\ N\ U\ k\ L\ \text{remove1-mset}\ L\ (\text{mset-cl}\ E)$ ]  $S\ T\ LE\ E\ tr\ \text{undef}$ 
  by auto
qed

lemma cdclW-cp-propagate-completeness:
  assumes  $MN$ :  $\text{set } M \models_s \text{set-mset } N$  and
  cons: consistent-interp ( $\text{set } M$ ) and
  tot: total-over-m ( $\text{set } M$ ) ( $\text{set-mset } N$ ) and
  lits-of-l ( $\text{trail } S$ )  $\subseteq \text{set } M$  and
  init-clss  $S = N$  and
  propagate**  $S\ S'$  and
  learned-clss  $S = \{\#\}$ 
  shows  $\text{length } (\text{trail } S) \leq \text{length } (\text{trail } S') \wedge \text{lits-of-l } (\text{trail } S') \subseteq \text{set } M$ 
  using assms(6,4,5,7)
proof (induction rule: rtranclp-induct)
  case base
  then show ?case by auto
next
  case (step  $Y\ Z$ )
  note  $st = \text{this}(1)$  and  $\text{propa} = \text{this}(2)$  and  $IH = \text{this}(3)$  and  $\text{lits}' = \text{this}(4)$  and  $NS = \text{this}(5)$  and
     $\text{learned} = \text{this}(6)$ 
  then have  $\text{len}: \text{length } (\text{trail } S) \leq \text{length } (\text{trail } Y)$  and  $LM: \text{lits-of-l } (\text{trail } Y) \subseteq \text{set } M$ 
    by blast+

obtain  $M'\ N'\ U\ k\ C\ L$  where
   $Y$ : state  $Y = (M', N', U, k, \text{None})$  and
   $Z$ : state  $Z = (\text{Propagated } L\ (C + \{\#L\})\ \# M', N', U, k, \text{None})$  and
   $C: C + \{\#L\} \in \# \text{ clauses } Y$  and
   $M'-C: M' \models_{as} C\text{Not } C$  and
  undefined-lit ( $\text{trail } Y$ )  $L$ 
  using  $\text{propa}$  by (auto elim: propagate-high-levelE)
have init-clss  $S = \text{init-clss } Y$ 
  using  $st$  by induction (auto elim: propagateE)
then have [simp]:  $N' = N$  using  $NS\ Y\ Z$  by simp
have learned-clss  $Y = \{\#\}$ 
  using  $st$  learned by induction (auto elim: propagateE)
then have [simp]:  $U = \{\#\}$  using  $Y$  by auto
have  $\text{set } M \models_s C\text{Not } C$ 
  using  $M'-C\ LM\ Y$  unfolding true-annots-def Ball-def true-annot-def true-clss-def true-cl-def
  by force
moreover
  have  $\text{set } M \models C + \{\#L\}$ 
  using  $MN\ C\ \text{learned } Y\ NS\ \langle \text{init-clss } S = \text{init-clss } Y \rangle \langle \text{learned-clss } Y = \{\#\} \rangle$ 
  unfolding true-clss-def raw-clauses-def by fastforce
ultimately have  $L \in \text{set } M$  by (simp add: cons consistent-CNot-not)
then show ?case using  $LM\ \text{len}\ Y\ Z$  by auto
qed

lemma
  assumes propagate**  $S\ X$ 
  shows

```

rtrancp-propagate-init-clss: $\text{init-clss } X = \text{init-clss } S$ and
rtrancp-propagate-learned-clss: $\text{learned-clss } X = \text{learned-clss } S$
using *assms* **by** (*induction rule*: *rtrancp-induct*) (*auto elim*: *propagateE*)

lemma *completeness-is-a-full1-propagation*:

fixes $S :: 'st$ **and** $M :: 'v$ *literal list*
assumes MN : $\text{set } M \models_s \text{set-mset } N$
and *cons*: *consistent-interp* ($\text{set } M$)
and *tot*: *total-over-m* ($\text{set } M$) ($\text{set-mset } N$)
and *alien*: *no-strange-atm* S
and *learned*: $\text{learned-clss } S = \{\#\}$
and $\text{clsS}[\text{simp}]$: $\text{init-clss } S = N$
and *lits*: $\text{lits-of-l } (\text{trail } S) \subseteq \text{set } M$
shows $\exists S'. \text{propagate}^{**} S S' \wedge \text{full cdcl}_W\text{-cp } S S'$

proof –

obtain S' **where** *full*: $\text{full cdcl}_W\text{-cp } S S'$
using *always-exists-full-cdcl_W-cp-step alien* **by** *blast*
then consider (*propa*) $\text{propagate}^{**} S S'$
 $|$ (*confl*) $\exists X. \text{propagate}^{**} S X \wedge \text{conflict } X S'$
using *rtrancp-cdcl_W-cp-propagate-confl* **unfolding** *full-def* **by** *blast*
then show *?thesis*

proof *cases*

case *propa* **then show** *?thesis* **using** *full* **by** *blast*

next

case *confl*

then obtain X **where**

X : $\text{propagate}^{**} S X$ **and**

$X\text{conf}$: $\text{conflict } X S'$

by *blast*

have clsX : $\text{init-clss } X = \text{init-clss } S$

using X **by** (*blast dest*: *rtrancp-propagate-init-clss*)

have learnedX : $\text{learned-clss } X = \{\#\}$

using X **learned by** (*auto dest*: *rtrancp-propagate-learned-clss*)

obtain E **where**

E : $E \in \# \text{ init-clss } X + \text{learned-clss } X$ **and**

$\text{Not-}E$: $\text{trail } X \models_{as} C\text{Not } E$

using $X\text{conf}$ **by** (*auto simp add*: *raw-clauses-def elim!*: *conflictE*)

have $\text{lits-of-l } (\text{trail } X) \subseteq \text{set } M$

using *cdcl_W-cp-propagate-completeness*[*OF assms(1–3) lits - X learned*] **learned by** *auto*

then have MNE : $\text{set } M \models_s C\text{Not } E$

using $\text{Not-}E$

by (*fastforce simp add*: *true-annots-def true-annot-def true-clss-def true-clss-def*)

have $\neg \text{set } M \models_s \text{set-mset } N$

using E *consistent-CNot-not*[*OF cons MNE*]

unfolding $\text{learnedX true-clss-def}$ **unfolding** clsX clsS **by** *auto*

then show *?thesis* **using** MN **by** *blast*

qed

qed

See also $\text{cdcl}_W\text{-cp}^{**} ?S ?S' \implies \exists M. \text{trail } ?S' = M @ \text{trail } ?S \wedge (\forall l \in \text{set } M. \neg \text{is-marked } l)$

lemma *rtrancp-propagate-is-trail-append*:

$\text{propagate}^{**} S T \implies \exists c. \text{trail } T = c @ \text{trail } S$

by (*induction rule*: *rtrancp-induct*) (*auto elim*: *propagateE*)

lemma *rtrancp-propagate-is-update-trail*:

$\text{propagate}^{**} S T \implies \text{cdcl}_W\text{-}M\text{-level-inv } S \implies$
 $\text{init-clss } S = \text{init-clss } T \wedge \text{learned-clss } S = \text{learned-clss } T \wedge \text{backtrack-lvl } S = \text{backtrack-lvl } T$
 $\wedge \text{conflicting } S = \text{conflicting } T$
proof (induction rule: *rtrancpl-induct*)
case *base*
then show ?*case unfolding state-eq-def* **by** (*auto simp: cdcl_W-M-level-inv-decomp*)
next
case (*step T U*) **note** *IH = this(3)[OF this(4)]*
moreover have *cdcl_W-M-level-inv U*
using *rtrancpl-cdcl_W-consistent-inv* $\langle \text{propagate}^{**} S T \rangle \langle \text{propagate } T U \rangle$
rtrancpl-mono[*of propagate cdcl_W*] *cdcl_W-cp-consistent-inv propagate'*
*rtrancpl-propagate-is-rtrancpl-cdcl_W step.prem*s **by** *blast*
then have *no-dup* (*trail U*) **unfolding** *cdcl_W-M-level-inv-def* **by** *auto*
ultimately show ?*case using* $\langle \text{propagate } T U \rangle$ **unfolding** *state-eq-def*
by (*fastforce simp: elim: propagateE*)
qed

lemma *cdcl_W-stgy-strong-completeness-n*:

assumes
MN: set M \models_s set-mset (mset-clss N) and
cons: consistent-interp (set M) and
tot: total-over-m (set M) (set-mset (mset-clss N)) and
atm-incl: atm-of ' (set M) \subseteq atms-of-mm (mset-clss N) and
distM: distinct M and
length: n \leq length M
shows
 $\exists M' k S. \text{length } M' \geq n \wedge$
 $\text{lits-of-l } M' \subseteq \text{set } M \wedge$
 $\text{no-dup } M' \wedge$
 $\text{state } S = (M', \text{mset-clss } N, \{\#\}, k, \text{None}) \wedge$
 $\text{cdcl}_W\text{-stgy}^{**} (\text{init-state } N) S$
using *length*
proof (induction *n*)
case 0
have *state (init-state N) = ([], mset-clss N, {\#}, 0, None)*
by (*auto simp: state-eq-def simp del: state-simp*)
moreover have
 $0 \leq \text{length } []$ **and**
 $\text{lits-of-l } [] \subseteq \text{set } M$ **and**
 $\text{cdcl}_W\text{-stgy}^{**} (\text{init-state } N) (\text{init-state } N)$
and *no-dup []*
by (*auto simp: state-eq-def simp del: state-simp*)
ultimately show ?*case using state-eq-sym* **by** *blast*
next
case (*Suc n*) **note** *IH = this(1) and n = this(2)*
then obtain *M' k S* **where**
 $\text{l-M': length } M' \geq n$ **and**
 $M': \text{lits-of-l } M' \subseteq \text{set } M$ **and**
 $n\text{-d}[simp]: \text{no-dup } M'$ **and**
 $S: \text{state } S = (M', \text{mset-clss } N, \{\#\}, k, \text{None})$ **and**
 $st: \text{cdcl}_W\text{-stgy}^{**} (\text{init-state } N) S$
by *auto*
have
 $M: \text{cdcl}_W\text{-M-level-inv } S$ **and**
 $\text{alien: no-strange-atm } S$

```

    using cdclW-M-level-inv-S0-cdclW rtrancpl-cdclW-stgy-consistent-inv st apply blast
    using cdclW-M-level-inv-S0-cdclW no-strange-atm-S0 rtrancpl-cdclW-no-strange-atm-inv
    rtrancpl-cdclW-stgy-rtrancpl-cdclW st by blast

{ assume no-step: ¬no-step propagate S
  obtain S' where S': propagate** S S' and full: full cdclW-cp S S'
    using completeness-is-a-full1-propagation[OF assms(1-3), of S] alien M' S
    by (auto simp: comp-def)
  have lev: cdclW-M-level-inv S'
    using M S' rtrancpl-cdclW-consistent-inv rtrancpl-propagate-is-rtrancpl-cdclW by blast
  then have n-d'[simp]: no-dup (trail S')
    unfolding cdclW-M-level-inv-def by auto
  have length (trail S) ≤ length (trail S') ∧ lits-of-l (trail S') ⊆ set M
    using S' full cdclW-cp-propagate-completeness[OF assms(1-3), of S] M' S
    by (auto simp: comp-def)
  moreover
    have full: full1 cdclW-cp S S'
      using full no-step no-step-cdclW-cp-no-conflict-no-propagate(2) unfolding full1-def full-def
      rtrancpl-unfold by blast
    then have cdclW-stgy S S' by (simp add: cdclW-stgy.conflict')
  moreover
    have propa: propagate++ S S' using S' full unfolding full1-def by (metis rtrancplD trancplD)
    have trail S = M'
      using S by (auto simp: comp-def rev-map)
    with propa have length (trail S') > n
      using l-M' propa by (induction rule: trancpl.induct) (auto elim: propagateE)
  moreover
    have stS': cdclW-stgy** (init-state N) S'
      using st cdclW-stgy.conflict'[OF full] by auto
    then have init-clss S' = mset-clss N
      using stS' rtrancpl-cdclW-stgy-no-more-init-clss by fastforce
  moreover
    have
      [simp]: learned-clss S' = {#} and
      [simp]: init-clss S' = init-clss S and
      [simp]: conflicting S' = None
      using trancpl-into-rtrancpl[OF ⟨propagate++ S S'⟩] S
      rtrancpl-propagate-is-update-trail[of S S'] S M unfolding state-eq-def
      by (auto simp: comp-def)
    have S-S': state S' = (trail S', mset-clss N, {#}, backtrack-lvl S', None)
      using S by auto
    have cdclW-stgy** (init-state N) S'
      apply (rule rtrancpl.rtrancpl-into-rtrancpl)
      using st apply simp
      using ⟨cdclW-stgy S S'⟩ by simp
    ultimately have ?case
      apply -
      apply (rule exI[of - trail S'], rule exI[of - backtrack-lvl S'], rule exI[of - S'])
      using S-S' by (auto simp: state-eq-def simp del: state-simp)
  }
moreover {
  assume no-step: no-step propagate S
  have ?case
    proof (cases length M' ≥ Suc n)
      case True

```

```

then show ?thesis using l-M' M' st M alien S n-d by blast
next
case False
then have n': length M' = n using l-M' by auto
have no-conf!: no-step conflict S
proof -
  { fix D
    assume D ∈# mset-cls N and M' ⊨as CNot D
    then have set M ⊨ D using MN unfolding true-cls-def by auto
    moreover have set M ⊨s CNot D
      using ⟨M' ⊨as CNot D⟩ M'
      by (metis le-iff-sup true-annots-true-cls true-cls-union-increase)
    ultimately have False using cons consistent-CNot-not by blast
  }
then show ?thesis
  using S by (auto simp: true-cls-def comp-def rev-map
    raw-clauses-def dest!: in-cls-mset-cls elim!: conflictE)
qed
have lenM: length M = card (set M) using distM by (induction M) auto
have no-dup M' using S M unfolding cdclW-M-level-inv-def by auto
then have card (lits-of-l M') = length M'
  by (induction M') (auto simp add: lits-of-def card-insert-if)
then have lits-of-l M' ⊆ set M
  using n M' n' lenM by auto
then obtain m where m: m ∈ set M and undef-m: m ∉ lits-of-l M' by auto
moreover have undef: undefined-lit M' m
  using M' Marked-Propagated-in-iff-in-lits-of-l calculation(1,2) cons
    consistent-interp-def by (metis (no-types, lifting) subset-eq)
moreover have atm-of m ∈ atms-of-mm (init-cls S)
  using atm-incl calculation S by auto
ultimately
  have dec: decide S (cons-trail (Marked m (k+1)) (incr-lvl S))
    using decide-rule[of S -
      cons-trail (Marked m (k + 1)) (incr-lvl S)] S
    by auto
let ?S' = cons-trail (Marked m (k+1)) (incr-lvl S)
have lits-of-l (trail ?S') ⊆ set M using m M' S undef by auto
moreover have no-strange-atm ?S'
  using alien dec M by (meson cdclW-no-strange-atm-inv decide other)
ultimately obtain S'' where S'': propagate** ?S' S'' and full: full cdclW-cp ?S' S''
  using completeness-is-a-full1-propagation[OF assms(1-3), of ?S'] S undef
  by auto
have cdclW-M-level-inv ?S'
  using M dec rtranclp-mono[of decide cdclW] by (meson cdclW-consistent-inv decide other)
then have lev'': cdclW-M-level-inv S''
  using S'' rtranclp-cdclW-consistent-inv rtranclp-propagate-is-rtranclp-cdclW by blast
then have n-d'': no-dup (trail S'')
  unfolding cdclW-M-level-inv-def by auto
have length (trail ?S') ≤ length (trail S'') ∧ lits-of-l (trail S'') ⊆ set M
  using S'' full cdclW-cp-propagate-completeness[OF assms(1-3), of ?S' S''] m M' S undef
  by simp
then have Suc n ≤ length (trail S'') ∧ lits-of-l (trail S'') ⊆ set M
  using l-M' S undef by auto
moreover
  have cdclW-M-level-inv (cons-trail (Marked m (Suc (backtrack-lvl S))))

```

```

      (update-backtrack-lvl (Suc (backtrack-lvl S)) S))
    using S ‹cdclW-M-level-inv (cons-trail (Marked m (k + 1)) (incr-lvl S))› by auto
  then have S'':
    state S'' = (trail S'', mset-cls N, {#}, backtrack-lvl S'', None)
    using rtrancplp-propagate-is-update-trail[OF S''] S undef n-d'' lev''
    by auto
  then have cdclW-stgy** (init-state N) S''
    using cdclW-stgy.intros(2)[OF decide[OF dec] - full] no-step no-confl st
    by (auto simp: cdclW-cp.simps)
  ultimately show ?thesis using S'' n-d'' by blast
qed
}
ultimately show ?case by blast
qed

```

lemma *cdcl_W-stgy-strong-completeness:*

assumes

MN: set M ⊨_s set-mset (mset-cls N) and

cons: consistent-interp (set M) and

tot: total-over-m (set M) (set-mset (mset-cls N)) and

atm-incl: atm-of ' (set M) ⊆ atms-of-mm (mset-cls N) and

distM: distinct M

shows

∃ M' k S.

lits-of-l M' = set M ∧

state S = (M', mset-cls N, {#}, k, None) ∧

*cdcl_W-stgy** (init-state N) S ∧*

final-cdcl_W-state S

proof –

from *cdcl_W-stgy-strong-completeness-n*[OF *assms*, of length M]

obtain M' k T **where**

l: length M ≤ length M' and

M'-M: lits-of-l M' ⊆ set M and

no-dup: no-dup M' and

T: state T = (M', mset-cls N, {#}, k, None) and

*st: cdcl_W-stgy** (init-state N) T*

by *auto*

have *card (set M) = length M* **using** *distM* **by** (*simp add: distinct-card*)

moreover

have *cdcl_W-M-level-inv T*

using *rtrancplp-cdcl_W-stgy-consistent-inv*[OF *st*] *T* **by** *auto*

then have *card (set ((map (λl. atm-of (lit-of l)) M')) = length M'*

using *distinct-card no-dup* **by** *fastforce*

moreover have *card (lits-of-l M') = card (set ((map (λl. atm-of (lit-of l)) M'))*

using *no-dup unfolding lits-of-def apply (induction M')* **by** (*auto simp add: card-insert-if*)

ultimately have *card (set M) ≤ card (lits-of-l M')* **using** *l unfolding lits-of-def* **by** *auto*

then have *set M = lits-of-l M'*

using *M'-M card-seteq* **by** *blast*

moreover

then have *M' ⊨_{asm} mset-cls N*

using *MN unfolding true-annots-def Ball-def true-annot-def true-cls-def* **by** *auto*

then have *final-cdcl_W-state T*

using *T no-dup unfolding final-cdcl_W-state-def* **by** *auto*

ultimately show ?thesis **using** *st T* **by** *blast*

qed

19.5.6 No conflict with only variables of level less than backtrack level

This invariant is stronger than the previous argument in the sense that it is a property about all possible conflicts.

definition *no-smaller-conflict* ($S :: 'st$) \equiv
 $(\forall M K i M' D. M' @ \text{Marked } K i \# M = \text{trail } S \longrightarrow D \in \# \text{ clauses } S$
 $\longrightarrow \neg M \models_{as} \text{CNot } D)$

lemma *no-smaller-conflict-init-sate*[simp]:
no-smaller-conflict (init-state N) **unfolding** *no-smaller-conflict-def* **by** *auto*

lemma *cdcl_W-o-no-smaller-conflict-inv*:

fixes $S S' :: 'st$
assumes
cdcl_W-o $S S'$ **and**
lev: *cdcl_W-M-level-inv* S **and**
max-lev: *conflict-is-false-with-level* S **and**
smaller: *no-smaller-conflict* S **and**
no-f: *no-clause-is-false* S
shows *no-smaller-conflict* S'
using *assms*(1,2) **unfolding** *no-smaller-conflict-def*
proof (induct rule: *cdcl_W-o-induct-lev2*)
case (decide $L T$) **note** *conflict* = *this*(1) **and** *undef* = *this*(2) **and** $T = \text{this}(4)$
have [simp]: *clauses* $T = \text{clauses } S$
using T *undef* **by** *auto*
show ?case
proof (intro allI impI)
fix $M'' K i M' Da$
assume $M'' @ \text{Marked } K i \# M' = \text{trail } T$
and $Da \in \# \text{ local.clauses } T$
then have $tl M'' @ \text{Marked } K i \# M' = \text{trail } S$
 $\vee (M'' = [] \wedge \text{Marked } K i \# M' = \text{Marked } L (\text{backtrack-lvl } S + 1) \# \text{trail } S)$
using T *undef* **by** (cases M'') *auto*
moreover {
assume $tl M'' @ \text{Marked } K i \# M' = \text{trail } S$
then have $\neg M' \models_{as} \text{CNot } Da$
using $D T$ *undef* *no-f* *conflict* *smaller* **unfolding** *no-smaller-conflict-def* *smaller* **by** *fastforce*
}
moreover {
assume $\text{Marked } K i \# M' = \text{Marked } L (\text{backtrack-lvl } S + 1) \# \text{trail } S$
then have $\neg M' \models_{as} \text{CNot } Da$ **using** *no-f* D *conflict* T **by** *auto*
}
ultimately show $\neg M' \models_{as} \text{CNot } Da$ **by** *fast*
qed
next
case *resolve*
then show ?case **using** *smaller* *no-f* *max-lev* **unfolding** *no-smaller-conflict-def* **by** *auto*
next
case *skip*
then show ?case **using** *smaller* *no-f* *max-lev* **unfolding** *no-smaller-conflict-def* **by** *auto*
next
case (backtrack $K i M1 M2 L D T$) **note** *conflict* = *this*(1) **and** $LD = \text{this}(2)$ **and** *decomp* = *this*(3)
and
 $\text{undef} = \text{this}(7)$ **and** $T = \text{this}(8)$
obtain c **where** $M: \text{trail } S = c @ M2 @ \text{Marked } K (i+1) \# M1$


```

using decomp by auto

show ?case
proof (intro allI impI)
  fix M ia K' M' Da

  assume M' @ Marked K' ia # M = trail T
  then have tl M' @ Marked K' ia # M = M1
    using T decomp undef lev by (cases M') (auto simp: cdclW-M-level-inv-decomp)
  let ?S' = (cons-trail (Propagated L (cls-of-ccls D))
    (reduce-trail-to M1 (add-learned-cls (cls-of-ccls D))
    (update-backtrack-lvl i (update-conflicting None S))))
  assume D: Da ∈ # clauses T
  moreover {
    assume Da ∈ # clauses S
    then have ¬M ⊨as CNot Da using ⟨tl M' @ Marked K' ia # M = M1⟩ M confl undef smaller
      unfolding no-smaller-confl-def by auto
  }
  moreover {
    assume Da: Da = mset-ccls D
    have ¬M ⊨as CNot Da
    proof (rule ccontr)
      assume ¬ ?thesis
      then have −L ∈ lits-of-l M
        using LD unfolding Da by (simp add: in-CNot-implies-uminus(2))
      then have −L ∈ lits-of-l (Propagated L (mset-ccls D) # M1)
        using UnI2 ⟨tl M' @ Marked K' ia # M = M1⟩
        by auto
      moreover
        have backtrack S ?S'
          using backtrack-rule[of S] backtrack.hyps
          by (force simp: state-eq-def simp del: state-simp)
        then have cdclW-M-level-inv ?S'
          using cdclW-consistent-inv[OF - lev] other[OF bj] by (auto intro: cdclW-bj.intros)
        then have no-dup (Propagated L (mset-ccls D) # M1)
          using decomp undef lev unfolding cdclW-M-level-inv-def by auto
        ultimately show False
          using undef by (auto simp: Marked-Propagated-in-iff-in-lits-of-l)
    qed
  }
  ultimately show ¬M ⊨as CNot Da
    using T undef decomp lev unfolding cdclW-M-level-inv-def by fastforce
qed
qed

lemma conflict-no-smaller-confl-inv:
  assumes conflict S S'
  and no-smaller-confl S
  shows no-smaller-confl S'
  using assms unfolding no-smaller-confl-def by (fastforce elim: conflictE)

lemma propagate-no-smaller-confl-inv:
  assumes propagate: propagate S S'
  and n-l: no-smaller-confl S
  shows no-smaller-confl S'

```

```

unfolding no-smaller-conflict-def
proof (intro allI impI)
  fix  $M' K i M'' D$ 
  assume  $M': M'' @ \text{Marked } K i \# M' = \text{trail } S'$ 
  and  $D \in \# \text{ clauses } S'$ 
  obtain  $M N U k C L$  where
     $S$ : state  $S = (M, N, U, k, \text{None})$  and
     $S'$ : state  $S' = (\text{Propagated } L (C + \{\#L\}) \# M, N, U, k, \text{None})$  and
     $C + \{\#L\} \in \# \text{ clauses } S$  and
     $M \models_{\text{as}} C \text{Not } C$  and
    undefined-lit  $M L$ 
  using propagate by (auto elim: propagate-high-levelE)
  have  $\text{tl } M'' @ \text{Marked } K i \# M' = \text{trail } S$  using  $M' S S'$ 
  by (metis Pair-inject list.inject list.sel(3) marked-lit.distinct(1) self-append-conv2
    tl-append2)
  then have  $\neg M' \models_{\text{as}} C \text{Not } D$ 
  using  $\langle D \in \# \text{ clauses } S' \rangle n-l S S'$  raw-clauses-def unfolding no-smaller-conflict-def by auto
  then show  $\neg M' \models_{\text{as}} C \text{Not } D$  by auto
qed

```

```

lemma cdclW-cp-no-smaller-conflict-inv:
  assumes propagate: cdclW-cp  $S S'$ 
  and n-l: no-smaller-conflict  $S$ 
  shows no-smaller-conflict  $S'$ 
  using assms
proof (induct rule: cdclW-cp.induct)
  case (conflict'  $S S'$ )
  then show ?case using conflict-no-smaller-conflict-inv[of  $S S'$ ] by blast
next
  case (propagate'  $S S'$ )
  then show ?case using propagate-no-smaller-conflict-inv[of  $S S'$ ] by fastforce
qed

```

```

lemma rtrancp-cdclW-cp-no-smaller-conflict-inv:
  assumes propagate: cdclW-cp**  $S S'$ 
  and n-l: no-smaller-conflict  $S$ 
  shows no-smaller-conflict  $S'$ 
  using assms
proof (induct rule: rtrancp.induct)
  case base
  then show ?case by simp
next
  case (step  $S' S''$ )
  then show ?case using cdclW-cp-no-smaller-conflict-inv[of  $S' S''$ ] by fast
qed

```

```

lemma trancp-cdclW-cp-no-smaller-conflict-inv:
  assumes propagate: cdclW-cp++  $S S'$ 
  and n-l: no-smaller-conflict  $S$ 
  shows no-smaller-conflict  $S'$ 
  using assms
proof (induct rule: trancp.induct)
  case (r-into-tranc  $S S'$ )
  then show ?case using cdclW-cp-no-smaller-conflict-inv[of  $S S'$ ] by blast
next

```

case (tranc1-into-tranc1 S S' S'')
 then show ?case using cdcl_W-cp-no-smaller-conf1-inv[of S' S''] by fast
 qed

lemma full-cdcl_W-cp-no-smaller-conf1-inv:
 assumes full cdcl_W-cp S S'
 and n-l: no-smaller-conf1 S
 shows no-smaller-conf1 S'
 using assms unfolding full-def
 using rtrancp-cdcl_W-cp-no-smaller-conf1-inv[of S S'] by blast

lemma full1-cdcl_W-cp-no-smaller-conf1-inv:
 assumes full1 cdcl_W-cp S S'
 and n-l: no-smaller-conf1 S
 shows no-smaller-conf1 S'
 using assms unfolding full1-def
 using trancp-cdcl_W-cp-no-smaller-conf1-inv[of S S'] by blast

lemma cdcl_W-stgy-no-smaller-conf1-inv:
 assumes cdcl_W-stgy S S'
 and n-l: no-smaller-conf1 S
 and conflict-is-false-with-level S
 and cdcl_W-M-level-inv S
 shows no-smaller-conf1 S'
 using assms

proof (induct rule: cdcl_W-stgy.induct)
 case (conflict' S')
 then show ?case using full1-cdcl_W-cp-no-smaller-conf1-inv[of S S'] by blast
 next
 case (other' S' S'')
 have no-smaller-conf1 S'
 using cdcl_W-o-no-smaller-conf1-inv[OF other'.hyps(1) other'.prems(3,2,1)]
 not-conflict-not-any-negated-init-clss other'.hyps(2) cdcl_W-cp.simps by auto
 then show ?case using full-cdcl_W-cp-no-smaller-conf1-inv[of S' S''] other'.hyps by blast
 qed

lemma is-conflicting-exists-conflict:
 assumes $\neg(\forall D \in \# \text{init-clss } S' + \text{learned-clss } S'. \neg \text{trail } S' \models_{\text{as}} \text{CNot } D)$
 and conflicting S' = None
 shows $\exists S''. \text{conflict } S' S''$
 using assms raw-clauses-def not-conflict-not-any-negated-init-clss by fastforce

lemma cdcl_W-o-conflict-is-no-clause-is-false:
 fixes S S' :: 'st
 assumes
 cdcl_W-o S S' and
 lev: cdcl_W-M-level-inv S and
 max-lev: conflict-is-false-with-level S and
 no-f: no-clause-is-false S and
 no-l: no-smaller-conf1 S
 shows no-clause-is-false S'
 $\vee (\text{conflicting } S' = \text{None}$
 $\longrightarrow (\forall D \in \# \text{clauses } S'. \text{trail } S' \models_{\text{as}} \text{CNot } D$
 $\longrightarrow (\exists L. L \in \# D \wedge \text{get-level } (\text{trail } S') L = \text{backtrack-lvl } S'))$
 using assms(1,2)

```

proof (induct rule: cdclW-o-induct-lev2)
  case (decide L T) note  $S = \text{this}(1)$  and  $\text{undef} = \text{this}(2)$  and  $T = \text{this}(4)$ 
  show ?case
  proof (rule HOL.disjI2, clarify)
    fix D
    assume D:  $D \in \# \text{ clauses } T$  and M-D:  $\text{trail } T \models_{as} \text{CNot } D$ 
    let ?M = trail S
    let ?M' = trail T
    let ?k = backtrack-lvl S
    have  $\neg ?M \models_{as} \text{CNot } D$ 
      using no-f D S T undef by auto
    have  $\neg L \in \# D$ 
    proof (rule ccontr)
      assume  $\neg ?thesis$ 
      have ?M  $\models_{as} \text{CNot } D$ 
      unfolding true-annots-def Ball-def true-annot-def CNot-def true-cls-def
      proof (intro allI impI)
        fix x
        assume  $x: x \in \{\{\# - L\# \} \mid L. L \in \# D\}$ 

        then obtain L' where  $L': x = \{\# - L'\# \}$   $L' \in \# D$  by auto
        obtain L'' where  $L'' \in \# x$  and  $\text{lits-of-l } (\text{Marked } L (\text{?k} + 1) \# ?M) \models_l L''$ 
          using M-D x T undef unfolding true-annots-def Ball-def true-annot-def CNot-def
            true-cls-def Bex-def by auto
        show  $\exists L \in \# x. \text{lits-of-l } ?M \models_l L$  unfolding Bex-def
          using L'(1) L'(2)  $\langle - L \notin \# D \rangle \langle L'' \in \# x \rangle$ 
             $\langle \text{lits-of-l } (\text{Marked } L (\text{backtrack-lvl } S + 1) \# \text{trail } S) \models_l L' \rangle$  by auto
        qed
        then show False using  $\langle \neg ?M \models_{as} \text{CNot } D \rangle$  by auto
      qed
    have atm-of L  $\notin \text{atm-of } \langle \text{lits-of-l } ?M \rangle$ 
      using undef defined-lit-map unfolding lits-of-def by fastforce
    then have get-level (Marked L (?k + 1) # ?M) (-L) = ?k + 1 by simp
    then show  $\exists La. La \in \# D \wedge \text{get-level } ?M' La = \text{backtrack-lvl } T$ 
      using  $\langle -L \in \# D \rangle$  T undef by auto
    qed
  next
  case resolve
  then show ?case by auto
  next
  case skip
  then show ?case by auto
  next
  case (backtrack K i M1 M2 L D T) note decomp = this(3) and undef = this(7) and  $T = \text{this}(8)$ 
  show ?case
  proof (rule HOL.disjI2, clarify)
    fix Da
    assume Da:  $Da \in \# \text{ clauses } T$ 
    and M-D:  $\text{trail } T \models_{as} \text{CNot } Da$ 
    obtain c where M:  $\text{trail } S = c @ M2 @ \text{Marked } K (i + 1) \# M1$ 
      using decomp by auto
    have tr-T:  $\text{trail } T = \text{Propagated } L (\text{mset-ccls } D) \# M1$ 
      using T decomp undef lev by (auto simp: cdclW-M-level-inv-decomp)
    have backtrack S T
      using backtrack-rule[of S] backtrack.hyps T

```

```

    by (force simp del: state-simp simp: state-eq-def)
  then have lev': cdclW-M-level-inv T
    using cdclW-consistent-inv lev other cdclW-bj.backtrack cdclW-o.bj by blast
  then have - L ∉ lits-of-l M1
    using lev cdclW-M-level-inv-def Marked-Propagated-in-iff-in-lits-of-l undef by blast
  { assume Da ∈# clauses S
    then have ¬M1 ⊨as CNot Da using no-l M unfolding no-smaller-conflict-def by auto
  }
  moreover {
    assume Da: Da = mset-ccls D
    have ¬M1 ⊨as CNot Da using ⟨- L ∉ lits-of-l M1⟩ unfolding Da
      using backtrack.hyps(2) in-CNot-implies-uminus(2) by auto
  }
  ultimately have ¬M1 ⊨as CNot Da
    using Da T undef decomp lev by (fastforce simp: cdclW-M-level-inv-decomp)
  then have -L ∈# Da
    using M-D ⟨- L ∉ lits-of-l M1⟩ T unfolding tr-T true-annots-true-clss true-clss-def
    by (auto simp: uminus-lit-swap)
  have g-M1: get-all-levels-of-marked M1 = rev [1..i+1]
    using lev lev' T decomp undef unfolding cdclW-M-level-inv-def by auto
  have no-dup (Propagated L (mset-ccls D) # M1)
    using lev lev' T decomp undef unfolding cdclW-M-level-inv-def by auto
  then have L: atm-of L ∉ atm-of 'lits-of-l M1' unfolding lits-of-def by auto
  have get-level (Propagated L (mset-ccls D) # M1) (-L) = i
    using get-level-get-rev-level-get-all-levels-of-marked[OF L,
      of [Propagated L (mset-ccls D)]]
    by (simp add: g-M1 split: if-splits)
  then show ∃ La. La ∈# Da ∧ get-level (trail T) La = backtrack-lvl T
    using ⟨-L ∈# Da⟩ T decomp undef lev by (auto simp: cdclW-M-level-inv-def)
qed
qed

```

lemma full1-cdcl_W-cp-exists-conflict-decompose:

```

assumes
  confl: ∃ D ∈ #clauses S. trail S ⊨as CNot D and
  full: full cdclW-cp S U and
  no-conf: conflicting S = None and
  lev: cdclW-M-level-inv S
shows ∃ T. propagate** S T ∧ conflict T U
proof -
  consider (propa) propagate** S U
    | (confl) T where propagate** S T and conflict T U
  using full unfolding full-def by (blast dest: rtranclp-cdclW-cp-propa-or-propa-conf)
  then show ?thesis
  proof cases
    case confl
    then show ?thesis by blast
  next
    case propa
    then have conflicting U = None and
      [simp]: learned-clss U = learned-clss S and
      [simp]: init-clss U = init-clss S
      using no-conf rtranclp-propagate-is-update-trail lev by auto
    moreover
      obtain D where D: D ∈ #clauses U and

```

```

    trS: trail S  $\models_{as}$  CNot D
    using confl raw-clauses-def by auto
    obtain M where M: trail U = M @ trail S
    using full rtrancp-cdclW-cp-dropWhile-trail unfolding full-def by meson
    have tr-U: trail U  $\models_{as}$  CNot D
    apply (rule true-annots-mono)
    using trS unfolding M by simp-all
    have  $\exists V. \text{conflict } U \ V$ 
    using  $\langle \text{conflicting } U = \text{None} \rangle \ D$  raw-clauses-def not-conflict-not-any-negated-init-clss tr-U
    by meson
    then have False using full cdclW-cp.conflict' unfolding full-def by blast
    then show ?thesis by fast
qed
qed

```

lemma full1-cdcl_W-cp-exists-conflict-full1-decompose:

```

assumes
  confl:  $\exists D \in \# \text{clauses } S. \text{trail } S \models_{as} \text{CNot } D$  and
  full: full cdclW-cp S U and
  no-confl: conflicting S = None and
  lev: cdclW-M-level-inv S
shows  $\exists T \ D. \text{propagate}^{**} \ S \ T \wedge \text{conflict } T \ U$ 
 $\wedge \text{trail } T \models_{as} \text{CNot } D \wedge \text{conflicting } U = \text{Some } D \wedge D \in \# \text{clauses } S$ 
proof -
  obtain T where propa: propagate** S T and conf: conflict T U
  using full1-cdclW-cp-exists-conflict-decompose[OF assms] by blast
  have p: learned-clss T = learned-clss S init-clss T = init-clss S
  using propa lev rtrancp-propagate-is-update-trail by auto
  have c: learned-clss U = learned-clss T init-clss U = init-clss T
  using conf by (auto elim: conflictE)
  obtain D where trail T  $\models_{as}$  CNot D  $\wedge$  conflicting U = Some D  $\wedge$  D  $\in \#$  clauses S
  using conf p c by (fastforce simp: raw-clauses-def elim!: conflictE)
  then show ?thesis
  using propa conf by blast
qed

```

lemma cdcl_W-stgy-no-smaller-conf:

```

assumes
  cdclW-stgy S S' and
  n-l: no-smaller-conf S and
  conflict-is-false-with-level S and
  cdclW-M-level-inv S and
  no-clause-is-false S and
  distinct-cdclW-state S and
  cdclW-conflicting S
shows no-smaller-conf S'
using assms
proof (induct rule: cdclW-stgy.induct)
  case (conflict' S')
  show no-smaller-conf S'
  using conflict'.hyps conflict'.prems(1) full1-cdclW-cp-no-smaller-conf-inv by blast
next
  case (other' S' S'')
  have lev': cdclW-M-level-inv S'
  using cdclW-consistent-inv other other'.hyps(1) other'.prems(3) by blast

```

```

show no-smaller-confl  $S''$ 
  using cdclW-stgy-no-smaller-confl-inv [OF cdclW-stgy.other' [OF other'.hyps(1-3)]]
  other'.prems(1-3) by blast
qed

lemma cdclW-stgy-ex-lit-of-max-level:
assumes
  cdclW-stgy  $S$   $S'$  and
  n-l: no-smaller-confl  $S$  and
  conflict-is-false-with-level  $S$  and
  cdclW-M-level-inv  $S$  and
  no-clause-is-false  $S$  and
  distinct-cdclW-state  $S$  and
  cdclW-conflicting  $S$ 
shows conflict-is-false-with-level  $S'$ 
using assms
proof (induct rule: cdclW-stgy.induct)
case (conflict'  $S'$ )
have no-smaller-confl  $S'$ 
  using conflict'.hyps conflict'.prems(1) full1-cdclW-cp-no-smaller-confl-inv by blast
moreover have conflict-is-false-with-level  $S'$ 
  using conflict'.hyps conflict'.prems(2-4)
  rtrancpl-cdclW-co-conflict-ex-lit-of-max-level[of  $S$   $S'$ ]
  unfolding full-def full1-def rtrancpl-unfold by presburger
then show ?case by blast
next
case (other'  $S'$   $S''$ )
have lev': cdclW-M-level-inv  $S'$ 
  using cdclW-consistent-inv other other'.hyps(1) other'.prems(3) by blast
moreover
have no-clause-is-false  $S'$ 
   $\vee$  (conflicting  $S' = \text{None}$   $\longrightarrow$  ( $\forall D \in \# \text{clauses } S'. \text{trail } S' \models_{\text{as}} C\text{Not } D$ 
     $\longrightarrow$  ( $\exists L. L \in \# D \wedge \text{get-level}(\text{trail } S') L = \text{backtrack-lvl } S')$ ))
  using cdclW-o-conflict-is-no-clause-is-false[of  $S$   $S'$ ] other'.hyps(1) other'.prems(1-4) by fast
moreover {
  assume no-clause-is-false  $S'$ 
  {
    assume conflicting  $S' = \text{None}$ 
    then have conflict-is-false-with-level  $S'$  by auto
    moreover have full cdclW-cp  $S'$   $S''$ 
      by (metis (no-types) other'.hyps(3))
    ultimately have conflict-is-false-with-level  $S''$ 
      using rtrancpl-cdclW-co-conflict-ex-lit-of-max-level[of  $S'$   $S''$ ] lev' no-clause-is-false  $S'$ 
      by blast
  }
}
moreover
{
  assume c: conflicting  $S' \neq \text{None}$ 
  have conflicting  $S \neq \text{None}$  using other'.hyps(1) c
  by (induct rule: cdclW-o-induct) auto
  then have conflict-is-false-with-level  $S'$ 
    using cdclW-o-conflict-is-false-with-level-inv[OF other'.hyps(1)]
    other'.prems(3,5,6,2) by blast
  moreover have cdclW-cp**  $S'$   $S''$  using other'.hyps(3) unfolding full-def by auto
  then have  $S' = S''$  using c
}

```

```

    by (induct rule: rtranclp-induct)
      (fastforce intro: option.exhaust)+
    ultimately have conflict-is-false-with-level S'' by auto
  }
  ultimately have conflict-is-false-with-level S'' by blast
}
moreover {
  assume
    confl: conflicting S' = None and
    D-L:  $\forall D \in \# \text{ clauses } S'. \text{ trail } S' \models_{\text{as}} \text{CNot } D$ 
       $\longrightarrow (\exists L. L \in \# D \wedge \text{get-level } (\text{trail } S') L = \text{backtrack-lvl } S')$ 
  { assume  $\forall D \in \# \text{ clauses } S'. \neg \text{ trail } S' \models_{\text{as}} \text{CNot } D$ 
    then have no-clause-is-false S' using confl by simp
    then have conflict-is-false-with-level S'' using calculation(3) by presburger
  }
  moreover {
    assume  $\neg(\forall D \in \# \text{ clauses } S'. \neg \text{ trail } S' \models_{\text{as}} \text{CNot } D)$ 
    then obtain T D where
      propagate** S' T and
      conflict T S'' and
      D:  $D \in \# \text{ clauses } S'$  and
      trail S''  $\models_{\text{as}} \text{CNot } D$  and
      conflicting S'' = Some D
    using full1-cdclW-cp-exists-conflict-full1-decompose[OF - - confl]
      other'(3) lev' by (metis (mono-tags, lifting) conflictE state-eq-trail
        trail-update-conflicting)
    obtain M where M: trail S'' = M @ trail S' and nm:  $\forall m \in \text{set } M. \neg \text{is-marked } m$ 
    using rtranclp-cdclW-cp-dropWhile-trail other'(3) unfolding full-def by meson
    have btS: backtrack-lvl S'' = backtrack-lvl S'
    using other'.hypos(3) unfolding full-def by (metis rtranclp-cdclW-cp-backtrack-lvl)
    have inv: cdclW-M-level-inv S''
    by (metis (no-types) cdclW-stgy.conflict' cdclW-stgy-consistent-inv full-unfold lev'
      other'.hypos(3))
    then have nd: no-dup (trail S'')
    by (metis (no-types) cdclW-M-level-inv-decomp(2))
    have conflict-is-false-with-level S''
    proof cases
      assume trail S'  $\models_{\text{as}} \text{CNot } D$ 
      moreover then obtain L where
        L  $\in \# D$  and
        lev-L: get-level (trail S') L = backtrack-lvl S'
        using D-L D by blast
      moreover
        have LS':  $-L \in \text{lits-of-l } (\text{trail } S')$ 
        using  $\langle \text{trail } S' \models_{\text{as}} \text{CNot } D \rangle \langle L \in \# D \rangle \text{in-CNot-implies-uminus}(2)$  by blast
      { fix x :: ('v, nat, 'v clause) marked-lit and
        xb :: ('v, nat, 'v clause) marked-lit
        assume a1:  $x \in \text{set } (\text{trail } S')$  and
        a2:  $xb \in \text{set } M$  and
        a3:  $(\lambda l. \text{atm-of } (\text{lit-of } l)) \text{ 'set } M \cap (\lambda l. \text{atm-of } (\text{lit-of } l)) \text{ 'set } (\text{trail } S') = \{\}$  and
        a4:  $-L = \text{lit-of } x$  and
        a5:  $\text{atm-of } L = \text{atm-of } (\text{lit-of } xb)$ 
        moreover have atm-of (lit-of x) = atm-of L
        using a4 by (metis (no-types) atm-of-uminus)
      }
    }
  }
}

```



```

    ultimately have False
      using a5 a3 a2 a1 by auto
  }
  then have atm-of L  $\notin$  atm-of ‘lits-of-l M’
    using nd LS' unfolding M by (auto simp add: lits-of-def)
  then have get-level (trail S'') L = get-level (trail S') L
    unfolding M by (simp add: lits-of-def)
  ultimately show ?thesis using btS ‘conflicting S'' = Some D’ by auto
next
assume  $\neg \text{trail } S' \models_{as} CNot\ D$ 
then obtain L where L  $\in \# D$  and LM:  $\neg L \in \text{lits-of-l } M$ 
  using ‘trail S''  $\models_{as} CNot\ D$ ’ unfolding M
  by (auto simp add: true-cls-def M true-annots-def true-annot-def
    split: if-split-asm)
{ fix x :: ‘v, nat, ‘v clause’ marked-lit and
  xb :: ‘v, nat, ‘v clause’ marked-lit
  assume a1: xb  $\in$  set (trail S') and
    a2: x  $\in$  set M and
    a3: atm-of L = atm-of (lit-of xb) and
    a4:  $\neg L = \text{lit-of } x$  and
    a5:  $(\lambda l. \text{atm-of (lit-of } l)) \text{ ‘set } M \cap (\lambda l. \text{atm-of (lit-of } l)) \text{ ‘set (trail S')}$ 
      = {}
  moreover have atm-of (lit-of xb) = atm-of ( $\neg L$ )
    using a3 by simp
  ultimately have False
    by auto }
then have LS': atm-of L  $\notin$  atm-of ‘lits-of-l (trail S')’
  using nd ‘L  $\in \# D$ ’ LM unfolding M by (auto simp add: lits-of-def)
show ?thesis
proof cases
  assume ne: get-all-levels-of-marked (trail S') = []
  have backtrack-lvl S'' = 0
    using inv ne nm unfolding cdclW-M-level-inv-def M
    by (simp add: get-all-levels-of-marked-nil-iff-not-is-marked)
  moreover
  have a1: get-level M L = 0
    using nm by auto
  then have get-level (M @ trail S') L = 0
    by (metis LS' get-all-levels-of-marked-nil-iff-not-is-marked
      get-level-skip-beginning-not-marked lits-of-def ne)
  ultimately show ?thesis using ‘conflicting S'' = Some D’ ‘L  $\in \# D$ ’ unfolding M
    by auto
next
assume ne: get-all-levels-of-marked (trail S')  $\neq$  []
have hd (get-all-levels-of-marked (trail S')) = backtrack-lvl S'
  using ne lev' M nm unfolding cdclW-M-level-inv-def
  by (cases get-all-levels-of-marked (trail S'))
  (simp-all add: get-all-levels-of-marked-nil-iff-not-is-marked[symmetric])
moreover have atm-of L  $\in$  atm-of ‘lits-of-l M’
  using ‘ $\neg L \in \text{lits-of-l } M$ ’
  by (simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set lits-of-def)
ultimately show ?thesis
  using nm ne ‘L  $\in \# D$ ’ ‘conflicting S'' = Some D’
    get-level-skip-beginning-hd-get-all-levels-of-marked[OF LS', of M]
    get-level-skip-in-all-not-marked[of rev M L backtrack-lvl S']

```

```

      unfolding lits-of-def btS M
    by auto
  qed
}
ultimately have conflict-is-false-with-level S'' by blast
}
moreover
{
  assume conflicting S' ≠ None
  have no-clause-is-false S' using ⟨conflicting S' ≠ None⟩ by auto
  then have conflict-is-false-with-level S'' using calculation(3) by presburger
}
ultimately show ?case by fast
qed

```

lemma *rtranclp-cdcl_W-stgy-no-smaller-confl-inv*:

```

assumes
  cdclW-stgy** S S' and
  n-l: no-smaller-confl S and
  cls-false: conflict-is-false-with-level S and
  lev: cdclW-M-level-inv S and
  no-f: no-clause-is-false S and
  dist: distinct-cdclW-state S and
  conflicting: cdclW-conflicting S and
  decomp: all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S)) and
  learned: cdclW-learned-clause S and
  alien: no-strange-atm S
shows no-smaller-confl S' ∧ conflict-is-false-with-level S'
using assms(1)
proof (induct rule: rtranclp-induct)
  case base
  then show ?case using n-l cls-false by auto
next
  case (step S' S'') note st = this(1) and cdcl = this(2) and IH = this(3)
  have no-smaller-confl S' and conflict-is-false-with-level S'
    using IH by blast+
  moreover have cdclW-M-level-inv S'
    using st lev rtranclp-cdclW-stgy-rtranclp-cdclW
    by (blast intro: rtranclp-cdclW-consistent-inv)+
  moreover have no-clause-is-false S'
    using st no-f rtranclp-cdclW-stgy-not-non-negated-init-clss by presburger
  moreover have distinct-cdclW-state S'
    using rtanclp-distinct-cdclW-state-inv[of S S'] lev rtranclp-cdclW-stgy-rtranclp-cdclW[OF st]
    dist by auto
  moreover have cdclW-conflicting S'
    using rtranclp-cdclW-all-inv(6)[of S S'] st alien conflicting decomp dist learned lev
    rtranclp-cdclW-stgy-rtranclp-cdclW by blast
  ultimately show ?case
    using cdclW-stgy-no-smaller-confl[OF cdcl] cdclW-stgy-ex-lit-of-max-level[OF cdcl] by fast
qed

```

19.5.7 Final States are Conclusive

lemma *full-cdcl_W-stgy-final-state-conclusive-non-false*:

fixes S' :: 'st

assumes *full*: *full cdcl_W-stgy (init-state N) S'*
and *no-d*: *distinct-mset-mset (mset-clss N)*
and *no-empty*: $\forall D \in \#mset-clss\ N. D \neq \{\#\}$
shows (*conflicting S' = Some {#} \wedge unsatisfiable (set-mset (init-clss S'))*)
 \vee (*conflicting S' = None \wedge trail S' \models_{asm} init-clss S'*)
proof –
let ?S = *init-state N*
have
termi: $\forall S''. \neg cdcl_W\text{-stgy } S' S''$ **and**
step: *cdcl_W-stgy^{*} ?S S' using full unfolding full-def by auto*
moreover have
learned: *cdcl_W-learned-clause S' and*
level-inv: *cdcl_W-M-level-inv S' and*
alien: *no-strange-atm S' and*
no-dup: *distinct-cdcl_W-state S' and*
confl: *cdcl_W-conflicting S' and*
decomp: *all-decomposition-implies-m (init-clss S') (get-all-marked-decomposition (trail S'))*
using *no-d tranclp-cdcl_W-stgy-tranclp-cdcl_W[of ?S S'] step rtranclp-cdcl_W-all-inv(1-6)[of ?S S']*
unfolding *rtranclp-unfold by auto*
moreover
have $\forall D \in \#mset-clss\ N. \neg [] \models_{as} CNot\ D$ **using** *no-empty by auto*
then have *confl-k: conflict-is-false-with-level S'*
using *rtranclp-cdcl_W-stgy-no-smaller-confl-inv[OF step] no-d by auto*
show *?thesis*
using *cdcl_W-stgy-final-state-conclusive[OF termi decomp learned level-inv alien no-dup confl*
confl-k] .
qed

lemma *conflict-is-full1-cdcl_W-cp*:
assumes *cp: conflict S S'*
shows *full1 cdcl_W-cp S S'*
proof –
have *cdcl_W-cp S S' and conflicting S' \neq None*
using *cp cdcl_W-cp.intros by (auto elim!: conflictE simp: state-eq-def simp del: state-simp)*
then have *cdcl_W-cp⁺⁺ S S' by blast*
moreover have *no-step cdcl_W-cp S'*
using $\langle \text{conflicting } S' \neq \text{None} \rangle$ **by** (*metis cdcl_W-cp-conflicting-not-empty*
option.exhaust)
ultimately show *full1 cdcl_W-cp S S' unfolding full1-def by blast+*
qed

lemma *cdcl_W-cp-fst-empty-conflicting-false*:
assumes
cdcl_W-cp S S' and
trail S = [] and
conflicting S \neq None
shows *False*
using *assms by (induct rule: cdcl_W-cp.induct) (auto elim: propagateE conflictE)*

lemma *cdcl_W-o-fst-empty-conflicting-false*:
assumes *cdcl_W-o S S'*
and *trail S = []*
and *conflicting S \neq None*
shows *False*

```

using assms by (induct rule: cdclW-o-induct) auto

lemma cdclW-stgy-fst-empty-conflicting-false:
  assumes cdclW-stgy S S'
  and trail S = []
  and conflicting S ≠ None
  shows False
  using assms apply (induct rule: cdclW-stgy.induct)
  using trancplD cdclW-cp-fst-empty-conflicting-false unfolding full1-def apply metis
  using cdclW-o-fst-empty-conflicting-false by blast
thm cdclW-cp.induct[split-format(complete)]

lemma cdclW-cp-conflicting-is-false:
  cdclW-cp S S' ⇒ conflicting S = Some {#} ⇒ False
  by (induction rule: cdclW-cp.induct) (auto elim: propagateE conflictE)

lemma rtrancpl-cdclW-cp-conflicting-is-false:
  cdclW-cp++ S S' ⇒ conflicting S = Some {#} ⇒ False
  apply (induction rule: trancpl.induct)
  by (auto dest: cdclW-cp-conflicting-is-false)

lemma cdclW-o-conflicting-is-false:
  cdclW-o S S' ⇒ conflicting S = Some {#} ⇒ False
  by (induction rule: cdclW-o-induct) auto

lemma cdclW-stgy-conflicting-is-false:
  cdclW-stgy S S' ⇒ conflicting S = Some {#} ⇒ False
  apply (induction rule: cdclW-stgy.induct)
  unfolding full1-def apply (metis (no-types) cdclW-cp-conflicting-not-empty trancplD)
  unfolding full-def by (metis conflict-with-false-implies-terminated other)

lemma rtrancpl-cdclW-stgy-conflicting-is-false:
  cdclW-stgy* S S' ⇒ conflicting S = Some {#} ⇒ S' = S
  apply (induction rule: rtrancpl.induct)
  apply simp
  using cdclW-stgy-conflicting-is-false by blast

lemma full-cdclW-init-clss-with-false-normal-form:
  assumes
     $\forall m \in \text{set } M. \neg \text{is-marked } m$  and
    E = Some D and
    state S = (M, N, U, 0, E)
    full cdclW-stgy S S' and
    all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S))
    cdclW-learned-clause S
    cdclW-M-level-inv S
    no-strange-atm S
    distinct-cdclW-state S
    cdclW-conflicting S
  shows  $\exists M''. \text{state } S' = (M'', N, U, 0, \text{Some } \{ \# \})$ 
  using assms(10,9,8,7,6,5,4,3,2,1)
proof (induction M arbitrary: E D S)
  case Nil
  then show ?case
    using rtrancpl-cdclW-stgy-conflicting-is-false unfolding full-def cdclW-conflicting-def

```

```

    by fastforce
next
case (Cons L M) note IH = this(1) and full = this(8) and E = this(10) and inv = this(2-7) and
  S = this(9) and nm = this(11)
obtain K p where K: L = Propagated K p
  using nm by (cases L) auto
have every-mark-is-a-conflict S using inv unfolding cdclW-conflicting-def by auto
then have MpK: M ⊨as CNot ( p - {#K#} ) and Kp: K ∈# p
  using S unfolding K by fastforce+
then have p: p = (p - {#K#}) + {#K#}
  by (auto simp add: multiset-eq-iff)
then have K': L = Propagated K ((p - {#K#}) + {#K#})
  using K by auto
obtain p' where
  p': hd-raw-trail S = Propagated K p' and
  pp': mset-cls p' = p
  using hd-raw-trail[of S] S K by (cases hd-raw-trail S) auto
obtain raw-D where
  raw-D: raw-conflicting S = Some raw-D
  using S E by (cases raw-conflicting S) auto
then have raw-DD: mset-ccls raw-D = D
  using S E by auto
consider (D) D = {#} | (D') D ≠ {#} by blast
then show ?case
  proof cases
    case D
    then show ?thesis
      using full rtrancp-cdclW-stgy-conflicting-is-false S unfolding full-def E D by auto
  next
    case D'
    then have no-p: no-step propagate S and no-c: no-step conflict S
      using S E by (auto elim: propagateE conflictE)
    then have no-step cdclW-cp S by (auto simp: cdclW-cp.simps)
    have res-skip: ∃ T. (resolve S T ∧ no-step skip S ∧ full cdclW-cp T T)
      ∨ (skip S T ∧ no-step resolve S ∧ full cdclW-cp T T)
    proof cases
      assume ¬lit-of L ∉# D
      then obtain T where sk: skip S T
        using S D' K skip-rule unfolding E by fastforce
      then have res: no-step resolve S
        using ⟨¬lit-of L ∉# D⟩ S D' K hd-raw-trail[of S] unfolding E
        by (auto elim!: skipE resolveE)
      have full cdclW-cp T T
        using sk by (auto intro!: option-full-cdclW-cp elim: skipE)
      then show ?thesis
        using sk res by blast
    next
      assume LD: ¬¬lit-of L ∉# D
      then have D: Some D = Some ((D - {#¬lit-of L#}) + {#¬lit-of L#})
        by (auto simp add: multiset-eq-iff)

      have ∧L. get-level M L = 0
        by (simp add: nm)
      then have get-maximum-level (Propagated K (p - {#K#} + {#K#}) # M) (D - {#¬
K#}) = 0

```

```

using LD get-maximum-level-exists-lit-of-max-level
proof –
  obtain  $L'$  where get-level ( $L \# M$ )  $L' = \text{get-maximum-level } (L \# M) \ D$ 
    using LD get-maximum-level-exists-lit-of-max-level[of  $D \ L \# M$ ] by fastforce
  then show ?thesis by (metis (mono-tags)  $K'$  get-level-skip-all-not-marked
    get-maximum-level-exists-lit nm not-gr0)
  qed
then obtain  $T$  where sk: resolve  $S \ T$ 
  using resolve-rule[of  $S \ K \ p' \ \text{raw-}D$ ]  $S \ p' \ \langle K \in \# \ p \rangle \ \text{raw-}D \ LD$ 
  unfolding  $K' \ D \ E \ pp' \ \text{raw-}DD$  by auto
then have res: no-step skip  $S$ 
  using LD  $S \ D' \ K \ \text{hd-raw-trail}$ [of  $S$ ] unfolding  $E$ 
  by (auto elim!: skipE resolveE)
have full cdclW-cp  $T \ T$ 
  using sk by (auto simp: option-full-cdclW-cp elim: resolveE)
then show ?thesis
  using sk res by blast
qed
then have step-s:  $\exists T. \text{cdcl}_W\text{-stgy } S \ T$ 
  using (no-step cdclW-cp S) other' by (meson bj resolve skip)
have get-all-marked-decomposition ( $L \ \# \ M$ ) =  $[(\ [], \ L \# M)]$ 
  using nm unfolding K apply (induction M rule: marked-lit-list-induct, simp)
  by (rename-tac L l xs, case-tac hd (get-all-marked-decomposition xs), auto)+
then have no-b: no-step backtrack  $S$ 
  using nm S by (auto elim: backtrackE)
have no-d: no-step decide  $S$ 
  using  $S \ E$  by (auto elim: decideE)

have full-S-S: full cdclW-cp S S
  using  $S \ E$  by (auto simp add: option-full-cdclW-cp)
then have no-f: no-step (full1 cdclW-cp) S
  unfolding full-def full1-def rtranclp-unfold by (meson tranclpD)
obtain  $T$  where
  s: cdclW-stgy S T and st: cdclW-stgy** T S'
  using full step-s full unfolding full-def by (metis rtranclp-unfold tranclpD)
have resolve S T  $\vee$  skip S T
  using s no-b no-d res-skip full-S-S cdclW-cp-state-eq-compatible resolve-unique
  skip-unique unfolding cdclW-stgy.simps cdclW-o.simps full-unfold
  full1-def by (blast dest!: tranclpD elim!: cdclW-bj.cases)+
then obtain  $D'$  where  $T: \text{state } T = (M, N, U, 0, \text{Some } D')$ 
  using  $S \ E$  by (auto elim!: skipE resolveE simp: state-eq-def simp del: state-simp)

have st-c: cdclW** S T
  using  $E \ T \ \text{rtranclp-cdcl}_W\text{-stgy-rtranclp-cdcl}_W \ s$  by blast
have cdclW-conflicting T
  using rtranclp-cdclW-all-inv(6)[OF st-c inv(6,5,4,3,2,1)] .
show ?thesis
  apply (rule IH[of  $T$ ])
    using rtranclp-cdclW-all-inv(6)[OF st-c inv(6,5,4,3,2,1)] apply blast
    using rtranclp-cdclW-all-inv(5)[OF st-c inv(6,5,4,3,2,1)] apply blast
    using rtranclp-cdclW-all-inv(4)[OF st-c inv(6,5,4,3,2,1)] apply blast
    using rtranclp-cdclW-all-inv(3)[OF st-c inv(6,5,4,3,2,1)] apply blast
    using rtranclp-cdclW-all-inv(2)[OF st-c inv(6,5,4,3,2,1)] apply blast
    using rtranclp-cdclW-all-inv(1)[OF st-c inv(6,5,4,3,2,1)] apply blast
  apply (metis full-def st full)

```

```

    using T E apply blast
    apply auto[]
    using nm by simp
qed
qed

lemma full-cdclW-stgy-final-state-conclusive-is-one-false:
  fixes S' :: 'st
  assumes full: full cdclW-stgy (init-state N) S'
  and no-d: distinct-mset-mset (mset-clss N)
  and empty: {#} ∈ # (mset-clss N)
  shows conflicting S' = Some {#} ∧ unsatisfiable (set-mset (init-clss S'))
proof -
  let ?S = init-state N
  have cdclW-stgy** ?S S' and no-step cdclW-stgy S' using full unfolding full-def by auto
  then have plus-or-eq: cdclW-stgy++ ?S S' ∨ S' = ?S unfolding rtranclp-unfold by auto
  have ∃ S''. conflict ?S S''
    using empty not-conflict-not-any-negated-init-clss[of init-state N] by auto

  then have cdclW-stgy: ∃ S'. cdclW-stgy ?S S'
    using cdclW-cp.conflict'[of ?S] conflict-is-full1-cdclW-cp cdclW-stgy.intros(1) by metis
  have S' ≠ ?S using (no-step cdclW-stgy S') cdclW-stgy by blast

  then obtain St:: 'st where St: cdclW-stgy ?S St and cdclW-stgy** St S'
    using plus-or-eq by (metis (no-types) (cdclW-stgy** ?S S') converse-rtranclpE)
  have st: cdclW** ?S St
    by (simp add: rtranclp-unfold (cdclW-stgy ?S St) cdclW-stgy-tranclp-cdclW)

  have ∃ T. conflict ?S T
    using empty not-conflict-not-any-negated-init-clss[of ?S] by force
  then have fullSt: full1 cdclW-cp ?S St
    using St unfolding cdclW-stgy.simps by blast
  then have bt: backtrack-lvl St = (0::nat)
    using rtranclp-cdclW-cp-backtrack-lvl unfolding full1-def
    by (fastforce dest!: tranclp-into-rtranclp)
  have cls-St: init-clss St = mset-clss N
    using fullSt cdclW-stgy-no-more-init-clss[OF St] by auto
  have conflicting St ≠ None
  proof (rule ccontr)
    assume conf: ¬ ?thesis
    obtain E where
      ES: E !∈! raw-init-clss St and
      E: mset-clss E = {#}
      using empty cls-St by (metis in-mset-clss-exists-preimage)
    then have ∃ T. conflict St T
      using empty cls-St conflict-rule[of St E] ES conf unfolding E
      by (auto simp: raw-clauses-def dest: in-mset-clss-exists-preimage)
    then show False using fullSt unfolding full1-def by blast
  qed

  have 1: ∀ m ∈ set (trail St). ¬ is-marked m
    using fullSt unfolding full1-def by (auto dest!: tranclp-into-rtranclp
      rtranclp-cdclW-cp-dropWhile-trail)
  have 2: full cdclW-stgy St S'

```

```

    using ⟨cdclW-stgy** St S'⟩ ⟨no-step cdclW-stgy S'⟩ bt unfolding full-def by auto
have 3: all-decomposition-implies-m
    (init-clss St)
    (get-all-marked-decomposition
     (trail St))
    using rtrancpl-cdclW-all-inv(1)[OF st] no-d bt by simp
have 4: cdclW-learned-clause St
    using rtrancpl-cdclW-all-inv(2)[OF st] no-d bt bt by simp
have 5: cdclW-M-level-inv St
    using rtrancpl-cdclW-all-inv(3)[OF st] no-d bt by simp
have 6: no-strange-atm St
    using rtrancpl-cdclW-all-inv(4)[OF st] no-d bt by simp
have 7: distinct-cdclW-state St
    using rtrancpl-cdclW-all-inv(5)[OF st] no-d bt by simp
have 8: cdclW-conflicting St
    using rtrancpl-cdclW-all-inv(6)[OF st] no-d bt by simp
have init-clss S' = init-clss St and conflicting S' = Some {#}
    using ⟨conflicting St ≠ None⟩ full-cdclW-init-clss-with-false-normal-form[OF 1, of - - St]
    2 3 4 5 6 7 8 st apply (metis ⟨cdclW-stgy** St S'⟩ rtrancpl-cdclW-stgy-no-more-init-clss)
    using ⟨conflicting St ≠ None⟩ full-cdclW-init-clss-with-false-normal-form[OF 1, of - - St - -
    S'] 2 3 4 5 6 7 8 by (metis bt option.exhaust prod.inject)

moreover have init-clss S' = mset-clss N
    using ⟨cdclW-stgy** (init-state N) S'⟩ rtrancpl-cdclW-stgy-no-more-init-clss by fastforce
moreover have unsatisfiable (set-mset (mset-clss N))
    by (meson empty satisfiable-def true-clss-empty true-clss-def)
ultimately show ?thesis by auto
qed

```

lemma *full-cdcl_W-stgy-final-state-conclusive:*

```

    fixes S' :: 'st
    assumes full: full cdclW-stgy (init-state N) S' and no-d: distinct-mset-mset (mset-clss N)
    shows (conflicting S' = Some {#} ∧ unsatisfiable (set-mset (init-clss S')))
    ∨ (conflicting S' = None ∧ trail S' ⊨asm init-clss S')
    using assms full-cdclW-stgy-final-state-conclusive-is-one-false
    full-cdclW-stgy-final-state-conclusive-non-false by blast

```

lemma *full-cdcl_W-stgy-final-state-conclusive-from-init-state:*

```

    fixes S' :: 'st
    assumes full: full cdclW-stgy (init-state N) S'
    and no-d: distinct-mset-mset (mset-clss N)
    shows (conflicting S' = Some {#} ∧ unsatisfiable (set-mset (mset-clss N)))
    ∨ (conflicting S' = None ∧ trail S' ⊨asm (mset-clss N) ∧ satisfiable (set-mset (mset-clss N)))
proof -
    have N: init-clss S' = (mset-clss N)
    using full unfolding full-def by (auto dest: rtrancpl-cdclW-stgy-no-more-init-clss)
    consider
    (confl) conflicting S' = Some {#} and unsatisfiable (set-mset (init-clss S'))
    | (sat) conflicting S' = None and trail S' ⊨asm init-clss S'
    using full-cdclW-stgy-final-state-conclusive[OF assms] by auto
    then show ?thesis
    proof cases
    case confl
    then show ?thesis by (auto simp: N)

```



```

next
  case sat
  have  $cdcl_W$ -M-level-inv (init-state  $N$ ) by auto
  then have  $cdcl_W$ -M-level-inv  $S'$ 
    using full rtranclp-cdclW-stgy-consistent-inv unfolding full-def by blast
  then have consistent-interp (lits-of-l (trail  $S'$ )) unfolding  $cdcl_W$ -M-level-inv-def by blast
  moreover have lits-of-l (trail  $S'$ )  $\models_s$  set-mset (init-clss  $S'$ )
    using sat(2) by (auto simp add: true-annots-def true-annot-def true-clss-def)
  ultimately have satisfiable (set-mset (init-clss  $S'$ )) by simp
  then show ?thesis using sat unfolding  $N$  by blast
qed
qed
end
end
theory CDCL-W-Termination
imports CDCL-W
begin

context conflict-driven-clause-learningW
begin

```

19.6 Termination

The condition that no learned clause is a tautology is overkill (in the sense that the no-duplicate condition is enough), but we can reuse *simple-clss*.

The invariant contains all the structural invariants that holds,

definition $cdcl_W$ -all-struct-inv **where**

```

 $cdcl_W$ -all-struct-inv  $S \longleftrightarrow$ 
  no-strange-atm  $S \wedge$ 
   $cdcl_W$ -M-level-inv  $S \wedge$ 
  ( $\forall s \in \# \text{learned-clss } S. \neg \text{tautology } s$ )  $\wedge$ 
  distinct-cdclW-state  $S \wedge$ 
   $cdcl_W$ -conflicting  $S \wedge$ 
  all-decomposition-implies-m (init-clss  $S$ ) (get-all-marked-decomposition (trail  $S$ ))  $\wedge$ 
   $cdcl_W$ -learned-clause  $S$ 

```

lemma $cdcl_W$ -all-struct-inv-inv:

assumes $cdcl_W$ S S' **and** $cdcl_W$ -all-struct-inv S

shows $cdcl_W$ -all-struct-inv S'

unfolding $cdcl_W$ -all-struct-inv-def

proof (intro HOL.conjI)

show no-strange-atm S'

using $cdcl_W$ -all-inv[OF assms(1)] assms(2) **unfolding** $cdcl_W$ -all-struct-inv-def **by** auto

show $cdcl_W$ -M-level-inv S'

using $cdcl_W$ -all-inv[OF assms(1)] assms(2) **unfolding** $cdcl_W$ -all-struct-inv-def **by** fast

show distinct-cdcl_W-state S'

using $cdcl_W$ -all-inv[OF assms(1)] assms(2) **unfolding** $cdcl_W$ -all-struct-inv-def **by** fast

show $cdcl_W$ -conflicting S'

using $cdcl_W$ -all-inv[OF assms(1)] assms(2) **unfolding** $cdcl_W$ -all-struct-inv-def **by** fast

show all-decomposition-implies-m (init-clss S') (get-all-marked-decomposition (trail S'))

using $cdcl_W$ -all-inv[OF assms(1)] assms(2) **unfolding** $cdcl_W$ -all-struct-inv-def **by** fast

show $cdcl_W$ -learned-clause S'

using $cdcl_W$ -all-inv[OF assms(1)] assms(2) **unfolding** $cdcl_W$ -all-struct-inv-def **by** fast

show $\forall s \in \# \text{learned-clss } S'. \neg \text{tautology } s$

using *assms*(1)[*THEN learned-clss-are-not-tautologies*] *assms*(2)
unfolding *cdcl_W-all-struct-inv-def* **by** *fast*
qed

lemma *rtrancp-cdcl_W-all-struct-inv-inv*:
assumes *cdcl_W** S S'* **and** *cdcl_W-all-struct-inv S*
shows *cdcl_W-all-struct-inv S'*
using *assms* **by** *induction* (*auto intro: cdcl_W-all-struct-inv-inv*)

lemma *cdcl_W-stgy-cdcl_W-all-struct-inv*:
cdcl_W-stgy S T \implies cdcl_W-all-struct-inv S \implies cdcl_W-all-struct-inv T
by (*meson cdcl_W-stgy-trancp-cdcl_W rtrancp-cdcl_W-all-struct-inv-inv rtrancp-unfold*)

lemma *rtrancp-cdcl_W-stgy-cdcl_W-all-struct-inv*:
*cdcl_W-stgy** S T \implies cdcl_W-all-struct-inv S \implies cdcl_W-all-struct-inv T*
by (*induction rule: rtrancp-induct*) (*auto intro: cdcl_W-stgy-cdcl_W-all-struct-inv*)

19.7 No Relearning of a clause

lemma *cdcl_W-o-new-clause-learned-is-backtrack-step*:
assumes *learned: D \in # learned-clss T* **and**
new: D \notin # learned-clss S **and**
cdcl_W: cdcl_W-o S T **and**
lev: cdcl_W-M-level-inv S
shows *backtrack S T \wedge conflicting S = Some D*
using *cdcl_W lev learned new*
proof (*induction rule: cdcl_W-o-induct-lev2*)
case (*backtrack K i M1 M2 L C T*) **note** *decomp = this(3)* **and** *undef = this(6)* **and** *andef = this(7)*
and
T = this(8) **and** *D-T = this(9)* **and** *D-S = this(10)*
then have *D = mset-ccls C*
using *not-gr0 lev* **by** (*auto simp: cdcl_W-M-level-inv-decomp*)
then show *?case*
using *T backtrack.hyps(1-5) backtrack.intros[OF backtrack.hyps(1,2)] backtrack.hyps(3-6)*
by *auto*
qed *auto*

lemma *cdcl_W-cp-new-clause-learned-has-backtrack-step*:
assumes *learned: D \in # learned-clss T* **and**
new: D \notin # learned-clss S **and**
cdcl_W: cdcl_W-stgy S T **and**
lev: cdcl_W-M-level-inv S
shows $\exists S'. \text{backtrack } S S' \wedge \text{cdcl}_W\text{-stgy}^{**} S' T \wedge \text{conflicting } S = \text{Some } D$
using *cdcl_W learned new*
proof (*induction rule: cdcl_W-stgy.induct*)
case (*conflict' S'*)
then show *?case*
unfolding *full1-def* **by** (*metis (mono-tags, lifting) rtrancp-cdcl_W-cp-learned-clause-inv*
trancp-into-rtrancp)
next
case (*other' S' S''*)
then have *D \in # learned-clss S'*
unfolding *full-def* **by** (*auto dest: rtrancp-cdcl_W-cp-learned-clause-inv*)
then show *?case*
using *cdcl_W-o-new-clause-learned-is-backtrack-step[OF - $\langle D \notin \# \text{learned-clss } S \rangle \langle \text{cdcl}_W\text{-o } S S' \rangle$*
 $\langle \text{full cdcl}_W\text{-cp } S' S'' \rangle \text{lev}$] **by** (*metis cdcl_W-stgy.conflict' full-unfold r-into-rtrancp*)

$rtrancplp.rtrancpl-refl)$
qed

lemma *rtrancplp-cdcl_W-cp-new-clause-learned-has-backtrack-step:*
assumes *learned: $D \in \#$ learned-clss T and*
new: $D \notin \#$ learned-clss S and
*cdcl_W: $cdcl_W-stgy^{**} S T$ and*
lev: $cdcl_W-M-level-inv S$
shows $\exists S' S''. cdcl_W-stgy^{**} S S' \wedge backtrack S' S'' \wedge conflicting S' = Some D \wedge$
 $cdcl_W-stgy^{**} S'' T$
using *cdcl_W learned new*
proof (*induction rule: rtrancplp-induct*)
case *base*
then show *?case by blast*
next
case (*step $T U$ note $st = this(1)$ and $o = this(2)$ and $IH = this(3)$ and*
 $D-U = this(4)$ and $D-S = this(5)$)
show *?case*
proof (*cases $D \in \#$ learned-clss T*)
case *True*
then obtain *$S' S''$ where*
 *st' : $cdcl_W-stgy^{**} S S'$ and*
 bt : $backtrack S' S''$ and
 $confl$: $conflicting S' = Some D$ and
 *st'' : $cdcl_W-stgy^{**} S'' T$*
using *$IH D-S$ by metis*
have $cdcl_W-stgy^{++} S'' U$
using *$st'' o$ by force*
then show *?thesis*
by (*meson bt confl rtrancplp-unfold st'*)
next
case *False*
have *$cdcl_W-M-level-inv T$*
using *lev rtrancplp-cdcl_W-stgy-consistent-inv st by blast*
then obtain *S' where*
 bt : $backtrack T S'$ and
 *st' : $cdcl_W-stgy^{**} S' U$ and*
 $confl$: $conflicting T = Some D$
using *$cdcl_W-cp-new-clause-learned-has-backtrack-step[OF D-U False o]$*
by *metis*
then have *$cdcl_W-stgy^{**} S T$ and*
 $backtrack T S'$ and
 $conflicting T = Some D$ and
 *$cdcl_W-stgy^{**} S' U$*
using *$o st$ by auto*
then show *?thesis by blast*
qed

qed

lemma *propagate-no-more-Marked-lit:*
assumes *propagate $S S'$*
shows *$Marked K i \in set (trail S) \longleftrightarrow Marked K i \in set (trail S')$*
using *assms by (auto elim: propagateE)*

lemma *conflict-no-more-Marked-lit:*

```

assumes conflict  $S S'$ 
shows  $\text{Marked } K i \in \text{set } (\text{trail } S) \longleftrightarrow \text{Marked } K i \in \text{set } (\text{trail } S')$ 
using assms by (auto elim: conflictE)

lemma cdclW-cp-no-more-Marked-lit:
assumes cdclW-cp  $S S'$ 
shows  $\text{Marked } K i \in \text{set } (\text{trail } S) \longleftrightarrow \text{Marked } K i \in \text{set } (\text{trail } S')$ 
using assms apply (induct rule: cdclW-cp.induct)
using conflict-no-more-Marked-lit propagate-no-more-Marked-lit by auto

lemma rtrancpl-cdclW-cp-no-more-Marked-lit:
assumes cdclW-cp**  $S S'$ 
shows  $\text{Marked } K i \in \text{set } (\text{trail } S) \longleftrightarrow \text{Marked } K i \in \text{set } (\text{trail } S')$ 
using assms apply (induct rule: rtrancpl-induct)
using cdclW-cp-no-more-Marked-lit by blast+

lemma cdclW-o-no-more-Marked-lit:
assumes cdclW-o  $S S'$  and lev: cdclW-M-level-inv  $S$  and  $\neg \text{decide } S S'$ 
shows  $\text{Marked } K i \in \text{set } (\text{trail } S') \longrightarrow \text{Marked } K i \in \text{set } (\text{trail } S)$ 
using assms
proof (induct rule: cdclW-o-induct-lev2)
  case backtrack note decomp = this(3) and undef = this(7) and  $T = \text{this}(8)$ 
  then show ?case using lev by (auto simp: cdclW-M-level-inv-decomp)
next
  case (decide  $L T$ )
  then show ?case using decide-rule[OF decide.hyps] by blast
qed auto

lemma cdclW-new-marked-at-beginning-is-decide:
assumes cdclW-stgy  $S S'$  and
lev: cdclW-M-level-inv  $S$  and
trail  $S' = M' @ \text{Marked } L i \# M$  and
trail  $S = M$ 
shows  $\exists T. \text{decide } S T \wedge \text{no-step } \text{cdcl}_W\text{-cp } S$ 
using assms
proof (induct rule: cdclW-stgy.induct)
  case (conflict'  $S'$ ) note st = this(1) and no-dup = this(2) and  $S' = \text{this}(3)$  and  $S = \text{this}(4)$ 
  have cdclW-M-level-inv  $S'$ 
    using full1-cdclW-cp-consistent-inv no-dup st by blast
  then have  $\text{Marked } L i \in \text{set } (\text{trail } S')$  and  $\text{Marked } L i \notin \text{set } (\text{trail } S)$ 
    using no-dup unfolding S S' cdclW-M-level-inv-def by (auto simp add: rev-image-eqI)
  then have False
    using st rtrancpl-cdclW-cp-no-more-Marked-lit[of S S']
    unfolding full1-def rtrancpl-unfold by blast
  then show ?case by fast
next
  case (other'  $T U$ ) note o = this(1) and ns = this(2) and st = this(3) and no-dup = this(4) and
     $S' = \text{this}(5)$  and  $S = \text{this}(6)$ 
  have cdclW-M-level-inv  $U$ 
    by (metis (full-types) lev cdclW.simps cdclW-consistent-inv full-def o
      other'.hyps(3) rtrancpl-cdclW-cp-consistent-inv)
  then have  $\text{Marked } L i \in \text{set } (\text{trail } U)$  and  $\text{Marked } L i \notin \text{set } (\text{trail } S)$ 
    using no-dup unfolding S S' cdclW-M-level-inv-def by (auto simp add: rev-image-eqI)
  then have  $\text{Marked } L i \in \text{set } (\text{trail } T)$ 
    using st rtrancpl-cdclW-cp-no-more-Marked-lit unfolding full-def by blast

```

then show *?case*
using *cdcl_W-o-no-more-Marked-lit[OF o] ⟨Marked L i ∉ set (trail S)⟩ ns lev* **by meson**
qed

lemma *cdcl_W-o-is-decide*:

assumes *cdcl_W-o S T* **and** *lev: cdcl_W-M-level-inv S*
trail T = drop (length M₀) M' @ Marked L i # H @ M **and**
 $\neg (\exists M'. \text{trail } S = M' @ \text{Marked } L \ i \ # \ H @ M)$
shows *decide S T*
using *assms*

proof (*induction rule:cdcl_W-o-induct-lev2*)

case (*backtrack K i M1 M2 L D T*)
then obtain *c* **where** *trail S = c @ M2 @ Marked K (Suc i) # M1*
by auto
show *?case*
using *backtrack lev*
apply (*cases drop (length M₀) M'*)
apply (*auto simp: cdcl_W-M-level-inv-decomp*)
using *⟨trail S = c @ M2 @ Marked K (Suc i) # M1⟩*
by (*auto simp: cdcl_W-M-level-inv-decomp*)

next

case *decide*
show *?case* **using** *decide-rule[of S] decide(1-4)* **by auto**

qed auto

lemma *rtranclp-cdcl_W-new-marked-at-beginning-is-decide*:

assumes *cdcl_W-stgy** R U* **and**
trail U = M' @ Marked L i # H @ M **and**
trail R = M **and**
cdcl_W-M-level-inv R
shows
 $\exists S \ T \ T'. \text{cdcl}_W\text{-stgy}^{**} \ R \ S \wedge \text{decide } S \ T \wedge \text{cdcl}_W\text{-stgy}^{**} \ T \ U \wedge \text{cdcl}_W\text{-stgy}^{**} \ S \ U \wedge$
 $\text{no-step cdcl}_W\text{-cp } S \wedge \text{trail } T = \text{Marked } L \ i \ # \ H @ M \wedge \text{trail } S = H @ M \wedge \text{cdcl}_W\text{-stgy } S \ T' \wedge$
 $\text{cdcl}_W\text{-stgy}^{**} \ T' \ U$
using *assms*

proof (*induct arbitrary: M H M' i rule: rtranclp-induct*)

case *base*
then show *?case* **by auto**

next

case (*step T U*) **note** *st = this(1)* **and** *IH = this(3)* **and** *s = this(2)* **and**
U = this(4) **and** *S = this(5)* **and** *lev = this(6)*

show *?case*

proof (*cases* $\exists M'. \text{trail } T = M' @ \text{Marked } L \ i \ # \ H @ M$)

case *False*

with *s* **show** *?thesis* **using** *U s st S*

proof *induction*

case (*conflict' W*) **note** *cp = this(1)* **and** *nd = this(2)* **and** *W = this(3)*
then obtain *M₀* **where** *trail W = M₀ @ trail T* **and** *nmarked: $\forall l \in \text{set } M_0. \neg \text{is-marked } l$*
using *rtranclp-cdcl_W-cp-dropWhile-trail unfolding full1-def rtranclp-unfold* **by meson**
then have *MV: $M' @ \text{Marked } L \ i \ # \ H @ M = M_0 @ \text{trail } T$* **unfolding** *W* **by simp**
then have *V: trail T = drop (length M₀) (M' @ Marked L i # H @ M)*
by auto
have *takeWhile (Not o is-marked) M' = M₀ @ takeWhile (Not o is-marked) (trail T)*
using *arg-cong[OF MV, of takeWhile (Not o is-marked)] nmarked*
by (*simp add: takeWhile-tail*)

```

from arg-cong[OF this, of length] have  $\text{length } M_0 \leq \text{length } M'$ 
  unfolding length-append by (metis (no-types, lifting) Nat.le-trans le-add1
    length-takeWhile-le)
  then have False using nd V by auto
  then show ?case by fast
next
  case (other' T' U) note  $o = \text{this}(1)$  and  $ns = \text{this}(2)$  and  $cp = \text{this}(3)$  and  $nd = \text{this}(4)$ 
    and  $U = \text{this}(5)$  and  $st = \text{this}(6)$ 
  obtain  $M_0$  where  $\text{trail } U = M_0 @ \text{trail } T'$  and  $n\text{marked}: \forall l \in \text{set } M_0. \neg \text{is-marked } l$ 
    using rtranclp-cdclW-cp-dropWhile-trail cp unfolding full-def by meson
  then have  $MV: M' @ \text{Marked } L \ i \ \# \ H @ M = M_0 @ \text{trail } T'$  unfolding U by simp
  then have  $V: \text{trail } T' = \text{drop } (\text{length } M_0) (M' @ \text{Marked } L \ i \ \# \ H @ M)$ 
    by auto
  have  $\text{takeWhile } (\text{Not } o \text{ is-marked}) \ M' = M_0 @ \text{takeWhile } (\text{Not } o \text{ is-marked}) (\text{trail } T')$ 
    using arg-cong[OF MV, of takeWhile (Not o is-marked)] nmarked
    by (simp add: takeWhile-tail)
  from arg-cong[OF this, of length] have  $\text{length } M_0 \leq \text{length } M'$ 
    unfolding length-append by (metis (no-types, lifting) Nat.le-trans le-add1
      length-takeWhile-le)
  then have  $\text{tr-}T': \text{trail } T' = \text{drop } (\text{length } M_0) \ M' @ \text{Marked } L \ i \ \# \ H @ M$  using V by auto
  then have  $LT': \text{Marked } L \ i \in \text{set } (\text{trail } T')$  by auto
  moreover
    have cdclW-M-level-inv T
      using lev rtranclp-cdclW-stgy-consistent-inv step.hyps(1) by blast
    then have decide T T' using o nd tr-T' cdclW-o-is-decide by metis
  ultimately have decide T T' using cdclW-o-no-more-Marked-lit[OF o] by blast
  then have  $1: \text{cdcl}_W\text{-stgy}^{**} \ R \ T$  and  $2: \text{decide } T \ T'$  and  $3: \text{cdcl}_W\text{-stgy}^{**} \ T' \ U$ 
    using st other'.prems(4)
    by (metis cdclW-stgy.conflict' cp full-unfold r-into-rtranclp rtranclp.rtrancl-refl) +
  have [simp]:  $\text{drop } (\text{length } M_0) \ M' = []$ 
    using  $\langle \text{decide } T \ T' \rangle \langle \text{Marked } L \ i \in \text{set } (\text{trail } T') \rangle \ nd \ \text{tr-}T'$ 
    by (auto simp add: Cons-eq-append-conv elim: decideE)
  have  $T': \text{drop } (\text{length } M_0) \ M' @ \text{Marked } L \ i \ \# \ H @ M = \text{Marked } L \ i \ \# \ \text{trail } T$ 
    using  $\langle \text{decide } T \ T' \rangle \langle \text{Marked } L \ i \in \text{set } (\text{trail } T') \rangle \ nd \ \text{tr-}T'$ 
    by (auto elim: decideE)
  have  $\text{trail } T' = \text{Marked } L \ i \ \# \ \text{trail } T$ 
    using  $\langle \text{decide } T \ T' \rangle \langle \text{Marked } L \ i \in \text{set } (\text{trail } T') \rangle \ \text{tr-}T'$ 
    by (auto elim: decideE)
  then have  $5: \text{trail } T' = \text{Marked } L \ i \ \# \ H @ M$ 
    using append.simps(1) list.sel(3) local.other'(5) tl-append2 by (simp add: tr-T')
  have  $6: \text{trail } T = H @ M$ 
    by (metis (no-types)  $\langle \text{trail } T' = \text{Marked } L \ i \ \# \ \text{trail } T \rangle$ 
       $\langle \text{trail } T' = \text{drop } (\text{length } M_0) \ M' @ \text{Marked } L \ i \ \# \ H @ M \rangle \ \text{append-Nil list.sel(3) nd}$ 
      tl-append2)
  have  $7: \text{cdcl}_W\text{-stgy}^{**} \ T \ U$  using other'.prems(4) st by auto
  have  $8: \text{cdcl}_W\text{-stgy} \ T \ U \ \text{cdcl}_W\text{-stgy}^{**} \ U \ U$ 
    using cdclW-stgy.other'[OF other'(1-3)] by simp-all
  show ?case apply (rule exI[of - T], rule exI[of - T'], rule exI[of - U])
    using ns 1 2 3 5 6 7 8 by fast
  qed
next
  case True
  then obtain  $M'$  where  $T: \text{trail } T = M' @ \text{Marked } L \ i \ \# \ H @ M$  by metis
  from IH[OF this S lev] obtain  $S' \ S'' \ S'''$  where
     $1: \text{cdcl}_W\text{-stgy}^{**} \ R \ S'$  and

```

2: *decide* $S' S''$ **and**
 3: $cdcl_W\text{-stgy}^{**} S'' T$ **and**
 4: *no-step* $cdcl_W\text{-cp} S'$ **and**
 6: $trail S'' = \text{Marked } L i \# H @ M$ **and**
 7: $trail S' = H @ M$ **and**
 8: $cdcl_W\text{-stgy}^{**} S' T$ **and**
 9: $cdcl_W\text{-stgy} S' S'''$ **and**
 10: $cdcl_W\text{-stgy}^{**} S''' T$
by *blast*
have $cdcl_W\text{-stgy}^{**} S'' U$ **using** $s \langle cdcl_W\text{-stgy}^{**} S'' T \rangle$ **by** *auto*
moreover have $cdcl_W\text{-stgy}^{**} S' U$ **using** $8 s$ **by** *auto*
moreover have $cdcl_W\text{-stgy}^{**} S''' U$ **using** $10 s$ **by** *auto*
ultimately show *?thesis* **apply** – **apply** (*rule* $exI[of - S']$, *rule* $exI[of - S'']$)
using $1\ 2\ 4\ 6\ 7\ 8\ 9$ **by** *blast*
qed
qed

lemma *rtrancpl-cdcl_W-new-marked-at-beginning-is-decide'*:
assumes $cdcl_W\text{-stgy}^{**} R U$ **and**
 $trail U = M' @ \text{Marked } L i \# H @ M$ **and**
 $trail R = M$ **and**
 $cdcl_W\text{-M-level-inv } R$
shows $\exists y y'. cdcl_W\text{-stgy}^{**} R y \wedge cdcl_W\text{-stgy } y y' \wedge \neg (\exists c. trail y = c @ \text{Marked } L i \# H @ M)$
 $\wedge (\lambda a b. cdcl_W\text{-stgy } a b \wedge (\exists c. trail a = c @ \text{Marked } L i \# H @ M))^{**} y' U$
proof –
fix T'
obtain $S' T T'$ **where**
 $st: cdcl_W\text{-stgy}^{**} R S'$ **and**
 $decide S' T$ **and**
 $TU: cdcl_W\text{-stgy}^{**} T U$ **and**
 $no\text{-step } cdcl_W\text{-cp } S'$ **and**
 $trT: trail T = \text{Marked } L i \# H @ M$ **and**
 $trS': trail S' = H @ M$ **and**
 $S'U: cdcl_W\text{-stgy}^{**} S' U$ **and**
 $S'T': cdcl_W\text{-stgy } S' T'$ **and**
 $T'U: cdcl_W\text{-stgy}^{**} T' U$
using *rtrancpl-cdcl_W-new-marked-at-beginning-is-decide*[*OF* *assms*] **by** *blast*
have $n: \neg (\exists c. trail S' = c @ \text{Marked } L i \# H @ M)$ **using** trS' **by** *auto*
show *?thesis*
using *rtrancpl-trans*[*OF* st] *rtrancpl-exists-last-with-prop*[*of* $cdcl_W\text{-stgy } S' T' -$
 $\lambda a -. \neg (\exists c. trail a = c @ \text{Marked } L i \# H @ M), OF S'T' T'U n$]
by *meson*
qed

lemma *beginning-not-marked-invert*:
assumes $A: M @ A = M' @ \text{Marked } K i \# H$ **and**
 $nm: \forall m \in set\ M. \neg is\text{-marked } m$
shows $\exists M. A = M @ \text{Marked } K i \# H$
proof –
have $A = drop\ (length\ M)\ (M' @ \text{Marked } K i \# H)$
using *arg-cong*[*OF* A , *of* $drop\ (length\ M)$] **by** *auto*
moreover have $drop\ (length\ M)\ (M' @ \text{Marked } K i \# H) = drop\ (length\ M)\ M' @ \text{Marked } K i \# H$
using nm **by** (*metis* (*no-types*, *lifting*) $A\ drop\ Cons'\ drop\ append\ marked\ lit.disc(1)\ not\ gr0$
 $nth\ append\ nth\ append\ length\ nth\ mem\ zero\ less\ diff$)
finally show *?thesis* **by** *fast*

qed

lemma *cdcl_W-stgy-trail-has-new-marked-is-decide-step*:

assumes *cdcl_W-stgy* *S T*

$\neg (\exists c. \text{trail } S = c @ \text{Marked } L i \# H @ M)$ **and**

$(\lambda a b. \text{cdcl}_W\text{-stgy } a b \wedge (\exists c. \text{trail } a = c @ \text{Marked } L i \# H @ M))^* T U$ **and**

$\exists M'. \text{trail } U = M' @ \text{Marked } L i \# H @ M$ **and**

lev: *cdcl_W-M-level-inv* *S*

shows $\exists S'. \text{decide } S S' \wedge \text{full } \text{cdcl}_W\text{-cp } S' T \wedge \text{no-step } \text{cdcl}_W\text{-cp } S$

using *assms*(3,1,2,4,5)

proof *induction*

case (*step* *T U*)

then show ?*case* **by** *fastforce*

next

case *base*

then show ?*case*

proof (*induction rule*: *cdcl_W-stgy.induct*)

case (*conflict'* *T*) **note** *cp* = *this*(1) **and** *nd* = *this*(2) **and** *M'* = *this*(3) **and** *no-dup* = *this*(3)

then obtain *M'* **where** *M'*: *trail* *T* = *M'* @ *Marked* *L i* # *H* @ *M* **by** *metis*

obtain *M''* **where** *M''*: *trail* *T* = *M''* @ *trail* *S* **and** *nm*: $\forall m \in \text{set } M''. \neg \text{is-marked } m$

using *cp* **unfolding** *full1-def*

by (*metis* *rtranclp-cdcl_W-cp-dropWhile-trail'* *tranclp-into-rtranclp*)

have *False*

using *beginning-not-marked-invert*[*of* *M''* *trail* *S* *M'* *L i* # *H* @ *M*] *M'* *nm* *nd* **unfolding** *M''*

by *fast*

then show ?*case* **by** *fast*

next

case (*other'* *T U'*) **note** *o* = *this*(1) **and** *ns* = *this*(2) **and** *cp* = *this*(3) **and** *nd* = *this*(4)

and *trU'* = *this*(5)

have *cdcl_W-cp*** *T U'* **using** *cp* **unfolding** *full-def* **by** *blast*

from *rtranclp-cdcl_W-cp-dropWhile-trail*[*OF* *this*]

have $\exists M'. \text{trail } T = M' @ \text{Marked } L i \# H @ M$

using *trU'* *beginning-not-marked-invert*[*of* - *trail* *T* - *L i* # *H* @ *M*] **by** *metis*

then obtain *M'* **where** *M'*: *trail* *T* = *M'* @ *Marked* *L i* # *H* @ *M*

by *auto*

with *o* *lev* *nd* *cp* *ns*

show ?*case*

proof (*induction rule*: *cdcl_W-o-induct-lev2*)

case (*decide* *L*) **note** *dec* = *this*(1) **and** *cp* = *this*(5) **and** *ns* = *this*(4)

then have *decide* *S* (*cons-trail* (*Marked* *L* (*backtrack-lvl* *S* + 1)) (*incr-lvl* *S*))

using *decide.hyps* *decide.intros*[*of* *S*] **by** *force*

then show ?*case* **using** *cp* *decide.premis* **by** (*meson* *decide-state-eq-compatible* *ns* *state-eq-ref* *state-eq-sym*)

next

case (*backtrack* *K j M1 M2 L' D T*) **note** *decomp* = *this*(3) **and** *undef* = *this*(7) **and**

T = *this*(8) **and** *trT* = *this*(12)

obtain *MS3* **where** *MS3*: *trail* *S* = *MS3* @ *M2* @ *Marked* *K* (*Suc* *j*) # *M1*

using *get-all-marked-decomposition-exists-prepend*[*OF* *decomp*] **by** *metis*

have *tl* (*M'* @ *Marked* *L i* # *H* @ *M*) = *tl* *M'* @ *Marked* *L i* # *H* @ *M*

using *lev* *trT* *T* *lev* *undef* *decomp* **by** (*cases* *M'*) (*auto simp*: *cdcl_W-M-level-inv-decomp*)

then have *M''*: *M1* = *tl* *M'* @ *Marked* *L i* # *H* @ *M*

using *arg-cong*[*OF* *trT*[*simplified*], *of* *tl*] *T* *decomp* *undef* *lev*

by (*simp add*: *cdcl_W-M-level-inv-decomp*)

have *False* **using** *nd* *MS3* *T* *undef* *decomp* **unfolding** *M''* **by** *auto*

then show ?*case* **by** *fast*

qed auto
 qed
 qed

lemma *rtrancpl-cdcl_W-stgy-with-trail-end-has-trail-end:*

assumes $(\lambda a b. \text{cdcl}_W\text{-stgy } a b \wedge (\exists c. \text{trail } a = c @ \text{Marked } L i \# H @ M))^{**} T U$ **and**
 $\exists M'. \text{trail } U = M' @ \text{Marked } L i \# H @ M$
shows $\exists M'. \text{trail } T = M' @ \text{Marked } L i \# H @ M$
using *assms* **by** (*induction rule: rtrancpl-induct*) *auto*

lemma *remove1-mset-eq-remove1-mset-same:*

remove1-mset $L D = \text{remove1-mset } L' D \implies L \in \# D \implies L = L'$
by (*metis diff-single-trivial insert-DiffM multi-drop-mem-not-eq single-eq-single union-right-cancel*)

lemma *cdcl_W-o-cannot-learn:*

assumes
cdcl_W-o $y z$ **and**
lev: *cdcl_W-M-level-inv* y **and**
trM: *trail* $y = c @ \text{Marked } Kh i \# H$ **and**
DL: $D \notin \# \text{learned-clss } y$ **and**
LD: $L \in \# D$ **and**
DH: *atms-of* (*remove1-mset* $L D$) \subseteq *atm-of* '*lits-of-l* H **and**
LH: *atm-of* $L \notin$ *atm-of* '*lits-of-l* H **and**
learned: $\forall T. \text{conflicting } y = \text{Some } T \longrightarrow \text{trail } y \models_{\text{as}} \text{CNot } T$ **and**
 z : *trail* $z = c' @ \text{Marked } Kh i \# H$

shows $D \notin \# \text{learned-clss } z$
using *assms*(1–2) *trM DL DH LH learned z*

proof (*induction rule: cdcl_W-o-induct-lev2*)

case (*backtrack* $K j M1 M2 L' D' T$) **note** *confl* = *this*(1) **and** *LD'* = *this*(2) **and** *decomp* = *this*(3)
and *levL* = *this*(4) **and** *levD* = *this*(5) **and** $j = \text{this}(6)$ **and** *undef* = *this*(7) **and** $T = \text{this}(8)$ **and**
 $z = \text{this}(14)$

obtain $M3$ **where** *M3*: *trail* $y = M3 @ M2 @ \text{Marked } K (\text{Suc } j) \# M1$

using *decomp get-all-marked-decomposition-exists-prepend* **by** *metis*

have M : *trail* $y = c @ \text{Marked } Kh i \# H$ **using** *trM* **by** *simp*

have H : *get-all-levels-of-marked* (*trail* y) = *rev* [*1..<1* + *backtrack-lvl* y]

using *lev unfolding cdcl_W-M-level-inv-def* **by** *auto*

have $c' @ \text{Marked } Kh i \# H = \text{Propagated } L' (\text{mset-ccls } D') \# \text{trail } (\text{reduce-trail-to } M1 y)$

using z *decomp undef T lev* **by** (*force simp: cdcl_W-M-level-inv-def*)

then obtain d **where** d : $M1 = d @ \text{Marked } Kh i \# H$

by (*metis (no-types) decomp in-get-all-marked-decomposition-trail-update-trail list.inject list.sel(3) marked-lit.distinct(1) self-append-conv2 tl-append2*)

have $i \in \text{set } (\text{get-all-levels-of-marked } (M3 @ M2 @ \text{Marked } K (\text{Suc } j) \# d @ \text{Marked } Kh i \# H))$
by *auto*

then have $i > 0$ **unfolding** $H[\text{unfolded } M3 d]$ **by** *auto*

show *?case*

proof

assume $D \in \# \text{learned-clss } T$

then have DLD' : $D = \text{mset-ccls } D'$

using *DL T neq0-conv undef decomp lev* **by** (*fastforce simp: cdcl_W-M-level-inv-def*)

have $L\text{-cKh}$: *atm-of* $L \in$ *atm-of* '*lits-of-l* ($c @ [\text{Marked } Kh i]$)

using *LH learned M DLD'[symmetric] confl LD' LD*

apply (*auto simp add: image-iff dest!: in-CNot-implies-uminus*)

```

  apply (metis atm-of-uminus)+ done
have get-all-levels-of-marked (M3 @ M2 @ Marked K (j + 1) # M1)
  = rev [1.. $1 + \text{backtrack-lvl } y$ ]
  using lev unfolding cdcW-M-level-inv-def M3 by auto
from arg-cong[OF this, of  $\lambda a. (\text{Suc } j) \in \text{set } a$ ] have backtrack-lvl  $y \geq j$  by auto

have DD'[simp]: remove1-mset L D = mset-ccls D' - {#L'#}
proof (rule ccontr)
  assume DD':  $\neg ?thesis$ 
  then have L'  $\in \#$  remove1-mset L D using DLD' LD by (metis LD' in-remove1-mset-neq)
  then have get-level (trail y) L'  $\leq$  get-maximum-level (trail y) (remove1-mset L D)
    using get-maximum-level-ge-get-level by blast
  moreover {
    have get-maximum-level (trail y) (remove1-mset L D) =
      get-maximum-level H (remove1-mset L D)
      using DH unfolding M by (simp add: get-maximum-level-skip-beginning)
    moreover
      have get-all-levels-of-marked (trail y) = rev [1.. $1 + \text{backtrack-lvl } y$ ]
        using lev unfolding cdcW-M-level-inv-def by auto
      then have get-all-levels-of-marked H = rev [1.. $i$ ]
        unfolding M by (auto dest: append-cons-eq-upt-length-i
          simp add: rev-swap[symmetric])
      then have get-maximum-possible-level H < i
        using get-maximum-possible-level-max-get-all-levels-of-marked[of H] <i > 0> by auto
      ultimately have get-maximum-level (trail y) (remove1-mset L D) < i
        by (metis (full-types) dual-order.strict-trans nat-neq-iff not-le
          get-maximum-possible-level-ge-get-maximum-level) }
  moreover
    have L  $\in \#$  remove1-mset L' (mset-ccls D')
      using DLD'[symmetric] DD' LD by (metis in-remove1-mset-neq)
    then have get-maximum-level (trail y) (remove1-mset L' (mset-ccls D'))  $\geq$ 
      get-level (trail y) L
      using get-maximum-level-ge-get-level by blast
  moreover {
    have get-all-levels-of-marked (c @ [Marked Kh i]) = rev [i.. $\text{backtrack-lvl } y + 1$ ]
      using append-cons-eq-upt-length-i-end[of rev (get-all-levels-of-marked H) i
        rev (get-all-levels-of-marked c) Suc 0 Suc (backtrack-lvl y)] H
      unfolding M apply (auto simp add: rev-swap[symmetric])
      by (metis (no-types, hide-lams) Nil-is-append-conv Suc-le-eq less-Suc-eq list.sel(1)
        rev.simps(2) rev-rev-ident upt-Suc upt-rec)
    have get-level (trail y) L = get-level (c @ [Marked Kh i]) L
      using L-cKh LH unfolding M by simp
    have get-level (c @ [Marked Kh i]) L  $\geq i$ 
      using L-cKh levL
      <get-all-levels-of-marked (c @ [Marked Kh i]) = rev [i.. $\text{backtrack-lvl } y + 1$ ]>
      calculation(1,2) by auto
    then have get-level (trail y) L  $\geq i$ 
      using M <get-level (trail y) L = get-level (c @ [Marked Kh i]) L> by auto }
  moreover
    have get-maximum-level (trail y) (remove1-mset L' (mset-ccls D'))
      < get-level (trail y) L
      using <j  $\leq$  backtrack-lvl y> levL j calculation(1-4) by linarith
    ultimately show False using backtrack.hyps(4) by linarith
qed
then have LL': L = L'

```

using $LD\ LD'$ *remove1-mset-eq-remove1-mset-same* **unfolding** $DLD'[symmetric]$ **by** *fast*
have $nd: no\text{-}dup\ (trail\ y)$ **using** *lev* **unfolding** $cdcl_W\text{-}M\text{-}level\text{-}inv\text{-}def$ **by** *auto*

{ **assume** $D: remove1\text{-}mset\ L\ (mset\text{-}ccls\ D') = \{\#\}$
then have $j: j = 0$ **using** $levD\ j$ **by** (*simp add: LL'*)
have $\forall m \in set\ M1. \neg is\text{-}marked\ m$
using H **unfolding** $M3\ j$
by (*auto simp add: rev-swap[symmetric] get-all-levels-of-marked-no-marked*
dest!: append-cons-eq-upt-length-i)
then have *False* **using** d **by** *auto*
}

moreover {
assume $D[simp]: remove1\text{-}mset\ L\ (mset\text{-}ccls\ D') \neq \{\#\}$
have $i \leq j$
using H **unfolding** $M3\ d$ **by** (*auto simp add: rev-swap[symmetric]*
dest: upt-decomp-lt)
have $j > 0$ **apply** (*rule ccontr*)
using $H\ \langle i > 0 \rangle$ **unfolding** $M3\ d$
by (*auto simp add: rev-swap[symmetric] dest!: upt-decomp-lt*)
obtain L'' **where**
 $L'' \in \# remove1\text{-}mset\ L\ (mset\text{-}ccls\ D')$ **and**
 $L''D': get\text{-}level\ (trail\ y)\ L'' = get\text{-}maximum\text{-}level\ (trail\ y)$
 $(remove1\text{-}mset\ L\ (mset\text{-}ccls\ D'))$
using *get-maximum-level-exists-lit-of-max-level[OF D, of trail y]* **by** *auto*
have $L''M: atm\text{-}of\ L'' \in atm\text{-}of\ \text{' lits-of-l } (trail\ y)$
using *get-rev-level-ge-0-atm-of-in[of 0 rev (trail y) L'']* $\langle j > 0 \rangle\ levD\ L''D'$
 $\langle j \leq backtrack\text{-}lvl\ y \rangle\ levL$ **by** (*simp add: LL' j*)
then have $L'' \in lits\text{-}of\text{-}l\ (Marked\ Kh\ i\ \# d)$
proof –
{
assume $L''H: atm\text{-}of\ L'' \in atm\text{-}of\ \text{' lits-of-l } H$
have $get\text{-}all\text{-}levels\text{-}of\text{-}marked\ H = rev\ [1..<i]$
using H **unfolding** M
by (*auto simp add: rev-swap[symmetric] dest!: append-cons-eq-upt-length-i*)
moreover have $get\text{-}level\ (trail\ y)\ L'' = get\text{-}level\ H\ L''$
using $L''H$ **unfolding** M **by** *simp*
ultimately have *False*
using $levD\ \langle j > 0 \rangle\ get\text{-}rev\text{-}level\text{-}in\text{-}levels\text{-}of\text{-}marked[of\ rev\ H\ 0\ L'']$ $\langle i \leq j \rangle$
unfolding $L''D'[symmetric]\ nd$
by (*metis L''D' LL' Max-n-upt Nat.le-trans One-nat-def Suc-pred* $\langle 0 < i \rangle$
get-all-levels-of-marked-rev-eq-rev-get-all-levels-of-marked
get-rev-level-less-max-get-all-levels-of-marked j lessI list.simps(15)
not-less rev-rev-ident set-upt)
}
moreover
have $atm\text{-}of\ L'' \in atm\text{-}of\ \text{' lits-of-l } H$
using $DD'\ DH\ \langle L'' \in \# remove1\text{-}mset\ L\ (mset\text{-}ccls\ D') \rangle\ atm\text{-}of\text{-}lit\text{-}in\text{-}atms\text{-}of\ LL'\ LD$
 LD' **by** *fastforce*
ultimately show *?thesis*
using $DD'\ DH\ \langle L'' \in \# remove1\text{-}mset\ L\ (mset\text{-}ccls\ D') \rangle\ atm\text{-}of\text{-}lit\text{-}in\text{-}atms\text{-}of$
by *auto*
qed
moreover
have $atm\text{-}of\ L'' \in atms\text{-}of\ (remove1\text{-}mset\ L\ (mset\text{-}ccls\ D'))$
using $\langle L'' \in \# remove1\text{-}mset\ L\ (mset\text{-}ccls\ D') \rangle$ **by** (*auto simp: atms-of-def*)

```

    then have  $\text{atm-of } L'' \in \text{atm-of ' lits-of-l } H$ 
      using  $DH$  unfolding  $DD'$  unfolding  $LL'$  by blast
    ultimately have  $\text{False}$ 
      using  $nd$  unfolding  $M3$  d  $LL'$  by (auto simp: lits-of-def)
  }
  ultimately show  $\text{False}$  by blast
qed
qed auto

```

lemma $\text{cdcl}_W\text{-stgy-with-trail-end-has-not-been-learned}$:

```

assumes
   $\text{cdcl}_W\text{-stgy } y \ z$  and
   $\text{cdcl}_W\text{-M-level-inv } y$  and
   $\text{trail } y = c @ \text{Marked } Kh \ i \ \# \ H$  and
   $D \notin \# \text{ learned-clss } y$  and
   $LD: L \in \# \ D$  and
   $DH: \text{atms-of } (\text{remove1-mset } L \ D) \subseteq \text{atm-of ' lits-of-l } H$  and
   $LH: \text{atm-of } L \notin \text{atm-of ' lits-of-l } H$  and
   $\forall T. \text{conflicting } y = \text{Some } T \longrightarrow \text{trail } y \models_{as} CNot \ T$  and
   $\text{trail } z = c' @ \text{Marked } Kh \ i \ \# \ H$ 
shows  $D \notin \# \text{ learned-clss } z$ 
using  $\text{assms}$ 
proof induction
  case  $\text{conflict'}$ 
  then show ?case
    unfolding  $\text{full1-def}$  using  $\text{trancpl-cdcl}_W\text{-cp-learned-clause-inv}$  by auto
next
  case ( $\text{other' } T \ U$ ) note  $o = \text{this}(1)$  and  $cp = \text{this}(3)$  and  $lev = \text{this}(4)$  and  $\text{trY} = \text{this}(5)$  and
     $\text{notin} = \text{this}(6)$  and  $LD = \text{this}(7)$  and  $DH = \text{this}(8)$  and  $LH = \text{this}(9)$  and  $\text{confl} = \text{this}(10)$  and
     $\text{trU} = \text{this}(11)$ 
  obtain  $c'$  where  $c': \text{trail } T = c' @ \text{Marked } Kh \ i \ \# \ H$ 
    using  $cp$  beginning-not-marked-invert[of - trail  $T$   $c'$   $Kh \ i \ H$ ]
      rtrancpl-cdcl $_W$ -cp-dropWhile-trail[of  $T \ U$ ] unfolding  $\text{trU full-def}$  by fastforce
  show ?case
    using  $\text{cdcl}_W\text{-o-cannot-learn}[OF \ o \ lev \ \text{trY} \ \text{notin} \ LD \ DH \ LH \ \text{confl} \ c']$ 
      rtrancpl-cdcl $_W$ -cp-learned-clause-inv  $cp$  unfolding  $\text{full-def}$  by auto
qed

```

lemma $\text{rtrancpl-cdcl}_W\text{-stgy-with-trail-end-has-not-been-learned}$:

```

assumes
  ( $\lambda a \ b. \text{cdcl}_W\text{-stgy } a \ b \wedge (\exists c. \text{trail } a = c @ \text{Marked } K \ i \ \# \ H @ [])$ )**  $S \ z$  and
   $\text{cdcl}_W\text{-all-struct-inv } S$  and
   $\text{trail } S = c @ \text{Marked } K \ i \ \# \ H$  and
   $D \notin \# \text{ learned-clss } S$  and
   $LD: L \in \# \ D$  and
   $DH: \text{atms-of } (\text{remove1-mset } L \ D) \subseteq \text{atm-of ' lits-of-l } H$  and
   $LH: \text{atm-of } L \notin \text{atm-of ' lits-of-l } H$  and
   $\exists c'. \text{trail } z = c' @ \text{Marked } K \ i \ \# \ H$ 
shows  $D \notin \# \text{ learned-clss } z$ 
using  $\text{assms}(1-4, 8)$ 
proof (induction rule:  $\text{rtrancpl-induct}$ )
  case base
  then show ?case by auto[1]
next

```

```

case (step  $T\ U$ ) note  $st = \text{this}(1)$  and  $s = \text{this}(2)$  and  $IH = \text{this}(3)[OF\ \text{this}(4-6)]$ 
  and  $lev = \text{this}(4)$  and  $trS = \text{this}(5)$  and  $DL-S = \text{this}(6)$  and  $trU = \text{this}(7)$ 
obtain  $c$  where  $c: \text{trail } T = c @ \text{Marked } K\ i \# H$  using  $s$  by auto
obtain  $c'$  where  $c': \text{trail } U = c' @ \text{Marked } K\ i \# H$  using  $trU$  by blast
have  $cdcl_W^{**}\ S\ T$ 
proof –
  have  $\forall p\ pa. \exists s\ sa. \forall sb\ sc\ sd\ se. (\neg p^{**}\ (sb::'st)\ sc \vee p\ s\ sa \vee pa^{**}\ sb\ sc)$ 
     $\wedge (\neg pa\ s\ sa \vee \neg p^{**}\ sd\ se \vee pa^{**}\ sd\ se)$ 
    by (metis (no-types) mono-rtrancpl)
  then have  $cdcl_W\text{-stgy}^{**}\ S\ T$ 
    using  $st$  by blast
  then show ?thesis
    using rtrancpl-cdcl_W-stgy-rtrancpl-cdcl_W by blast
qed
then have  $lev': cdcl_W\text{-all-struct-inv}\ T$ 
  using rtrancpl-cdcl_W-all-struct-inv-inv[of  $S\ T$ ]  $lev$  by auto
then have  $confl': \forall Ta. \text{conflicting } T = \text{Some } Ta \longrightarrow \text{trail } T \models_{as} CNot\ Ta$ 
  unfolding cdcl_W-all-struct-inv-def cdcl_W-conflicting-def by blast
show ?case
  apply (rule cdcl_W-stgy-with-trail-end-has-not-been-learned[OF - -  $c - LD\ DH\ LH\ confl'\ c$ ])
  using  $s\ lev'\ IH\ c$  unfolding cdcl_W-all-struct-inv-def by blast
qed

lemma cdcl_W-stgy-new-learned-clause:
assumes  $cdcl_W\text{-stgy}\ S\ T$  and
   $lev: cdcl_W\text{-M-level-inv}\ S$  and
   $E \notin \# \text{learned-clss}\ S$  and
   $E \in \# \text{learned-clss}\ T$ 
shows  $\exists S'. \text{backtrack } S\ S' \wedge \text{conflicting } S = \text{Some } E \wedge \text{full } cdcl_W\text{-cp}\ S'\ T$ 
using assms
proof induction
case conflict'
  then show ?case unfolding full1-def by (auto dest: rtrancpl-cdcl_W-cp-learned-clause-inv)
next
case (other'  $T\ U$ ) note  $o = \text{this}(1)$  and  $cp = \text{this}(3)$  and  $\text{not-yet} = \text{this}(5)$  and  $\text{learned} = \text{this}(6)$ 
have  $E \in \# \text{learned-clss}\ T$ 
  using  $\text{learned}\ cp\ \text{rtrancpl-cdcl_W-cp-learned-clause-inv}$  unfolding full-def by auto
then have  $\text{backtrack } S\ T$  and  $\text{conflicting } S = \text{Some } E$ 
  using cdcl_W-o-new-clause-learned-is-backtrack-step[OF -  $\text{not-yet } o$ ]  $lev$  by blast
then show ?case using  $cp$  by blast
qed

lemma cdcl_W-stgy-no-relearned-clause:
assumes
   $invR: cdcl_W\text{-all-struct-inv}\ R$  and
   $st': cdcl_W\text{-stgy}^{**}\ R\ S$  and
   $bt: \text{backtrack } S\ T$  and
   $confl: \text{raw-conflicting } S = \text{Some } E$  and
   $\text{already-learned}: \text{mset-ccls } E \in \# \text{clauses } S$  and
   $R: \text{trail } R = []$ 
shows False
proof –
  have  $M\text{-lev}: cdcl_W\text{-M-level-inv}\ R$ 
  using  $invR$  unfolding cdcl_W-all-struct-inv-def by auto
  have  $cdcl_W\text{-M-level-inv}\ S$ 

```

using $M\text{-lev assms}(2)$ $\text{rtrancp-cdcl}_W\text{-stgy-consistent-inv}$ **by** blast
with bt **obtain** $L\ M1\ M2\text{-loc}\ K\ i$ **where**
 $T: T \sim \text{cons-trail } (\text{Propagated } L\ (\text{cls-of-ccls } E))$
 $(\text{reduce-trail-to } M1\ (\text{add-learned-cls } (\text{cls-of-ccls } E))$
 $(\text{update-backtrack-lvl } i\ (\text{update-conflicting } \text{None } S))))$
and
 $\text{decomp}: (\text{Marked } K\ (\text{Suc } i) \# M1, M2\text{-loc}) \in$
 $\text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$ **and**
 $LD: L \in \# \text{mset-ccls } E$ **and**
 $k: \text{get-level } (\text{trail } S)\ L = \text{backtrack-lvl } S$ **and**
 $\text{level}: \text{get-level } (\text{trail } S)\ L = \text{get-maximum-level } (\text{trail } S)\ (\text{mset-ccls } E)$ **and**
 $\text{confl-S}: \text{raw-conflicting } S = \text{Some } E$ **and**
 $i: i = \text{get-maximum-level } (\text{trail } S)\ (\text{remove1-mset } L\ (\text{mset-ccls } E))$ **and**
 $\text{undef}: \text{undefined-lit } M1\ L$
using confl **by** $(\text{induction rule: backtrack-induction-lev2})$ fastforce
obtain $M2$ **where**
 $M: \text{trail } S = M2 @ \text{Marked } K\ (\text{Suc } i) \# M1$
using $\text{get-all-marked-decomposition-exists-prepend}[OF\ \text{decomp}]$ **unfolding** i **by** $(\text{metis append-assoc})$
let $?E = \text{mset-ccls } E$
let $?E' = \text{remove1-mset } L\ ?E$
have $\text{invS}: \text{cdcl}_W\text{-all-struct-inv } S$
using invR $\text{rtrancp-cdcl}_W\text{-all-struct-inv-inv}$ $\text{rtrancp-cdcl}_W\text{-stgy-rtrancp-cdcl}_W\ st'$ **by** blast
then have $\text{conf}: \text{cdcl}_W\text{-conflicting } S$ **unfolding** $\text{cdcl}_W\text{-all-struct-inv-def}$ **by** blast
then have $\text{trail } S \models_{as} \text{CNot } ?E$ **unfolding** $\text{cdcl}_W\text{-conflicting-def}$ confl-S **by** auto
then have $MD: \text{trail } S \models_{as} \text{CNot } ?E$ **by** auto
then have $MD': \text{trail } S \models_{as} \text{CNot } ?E'$ **using** $\text{true-annot-CNot-diff}$ **by** blast
have $\text{lev}': \text{cdcl}_W\text{-M-level-inv } S$ **using** invS **unfolding** $\text{cdcl}_W\text{-all-struct-inv-def}$ **by** blast

have $\text{get-lvls-M}: \text{get-all-levels-of-marked } (\text{trail } S) = \text{rev } [1..<\text{Suc } (\text{backtrack-lvl } S)]$
using lev' **unfolding** $\text{cdcl}_W\text{-M-level-inv-def}$ **by** auto

have $\text{lev}: \text{cdcl}_W\text{-M-level-inv } R$ **using** invR **unfolding** $\text{cdcl}_W\text{-all-struct-inv-def}$ **by** blast
then have $\text{vars-of-D}: \text{atms-of } ?E' \subseteq \text{atm-of ' lits-of-l } M1$
using $\text{backtrack-atms-of-D-in-M1}[OF\ \text{lev}'\ \text{undef} - \text{decomp} - - - T]$ $\text{confl-S}\ \text{conf}\ T\ \text{decomp}\ k\ \text{level}$
 $\text{lev}'\ i\ \text{undef}$ **unfolding** $\text{cdcl}_W\text{-conflicting-def}$ **by** $(\text{auto simp: cdcl}_W\text{-M-level-inv-decomp})$
have $\text{no-dup } (\text{trail } S)$ **using** lev' **by** $(\text{auto simp: cdcl}_W\text{-M-level-inv-decomp})$
have $\text{vars-in-M1}: \forall x \in \text{atms-of } ?E'. x \notin \text{atm-of ' lits-of-l } (M2 @ [\text{Marked } K\ (i + 1)])$
unfolding Set.Ball-def **apply** (intro impI allI)
apply $(\text{rule vars-of-D distinct-atms-of-incl-not-in-other}[of\ M2 @ \text{Marked } K\ (i + 1) \# []\ M1\ ?E'])$
using $\langle \text{no-dup } (\text{trail } S) \rangle\ M\ \text{vars-of-D}$ **by** simp-all
have $M1\text{-D}: M1 \models_{as} \text{CNot } ?E'$
using $\text{vars-in-M1}\ \text{true-annot-remove-if-notin-vars}[of\ M2 @ \text{Marked } K\ (i + 1) \# []\ M1\ \text{CNot } ?E']$
 $MD'\ M$ **by** simp

have $\text{get-lvls-M}: \text{get-all-levels-of-marked } (\text{trail } S) = \text{rev } [1..<\text{Suc } (\text{backtrack-lvl } S)]$
using lev' **unfolding** $\text{cdcl}_W\text{-M-level-inv-def}$ **by** auto
then have $\text{backtrack-lvl } S > 0$ **unfolding** M **by** $(\text{auto split: if-split-asm simp add: upt.simps}(2))$

obtain $M1'\ K'\ Ls$ **where**
 $M': \text{trail } S = Ls @ \text{Marked } K'\ (\text{backtrack-lvl } S) \# M1'$ **and**
 $Ls: \forall l \in \text{set } Ls. \neg \text{is-marked } l$ **and**
 $\text{set } M1 \subseteq \text{set } M1'$
proof –

```

let ?Ls = takeWhile (Not o is-marked) (trail S)
have MLs: trail S = ?Ls @ dropWhile (Not o is-marked) (trail S)
  by auto
have dropWhile (Not o is-marked) (trail S) ≠ [] unfolding M by auto
moreover
  from hd-dropWhile[OF this] have is-marked(hd (dropWhile (Not o is-marked) (trail S)))
    by simp
ultimately
  obtain K' K'k where
    K'k: dropWhile (Not o is-marked) (trail S)
      = Marked K' K'k # tl (dropWhile (Not o is-marked) (trail S))
  by (cases dropWhile (Not o is-marked) (trail S);
      cases hd (dropWhile (Not o is-marked) (trail S)))
    simp-all
moreover have ∀ l ∈ set ?Ls. ¬is-marked l using set-takeWhileD by force
moreover
  have get-all-levels-of-marked (trail S)
    = K'k # get-all-levels-of-marked(tl (dropWhile (Not o is-marked) (trail S)))
  apply (subst MLs, subst K'k)
  using calculation(2) by (auto simp add: get-all-levels-of-marked-no-marked)
  then have K'k = backtrack-lvl S
  using calculation(2) by (auto split: if-split-asm simp add: get-lvls-M upt.simps(2))
moreover have set M1 ⊆ set (tl (dropWhile (Not o is-marked) (trail S)))
  unfolding M by (induction M2) auto
ultimately show ?thesis using that MLs by metis
qed

have get-lvls-M: get-all-levels-of-marked (trail S) = rev [1..W-M-level-inv-def by auto
then have backtrack-lvl S > 0 unfolding M by (auto split: if-split-asm simp add: upt.simps(2) i)

have M1'-D: M1' ⊨as CNot ?E' using M1-D (set M1 ⊆ set M1') by (auto intro: true-annots-mono)
have -L ∈ lits-of-l (trail S) using conf confl-S LD unfolding cdclW-conflicting-def
  by (auto simp: in-CNot-implies-uminus)
have lvls-M1': get-all-levels-of-marked M1' = rev [1..W-stgy** R Y and
  YZ: cdclW-stgy Y Z and
  nt: ¬ (∃ c. trail Y = c @ Marked K' (backtrack-lvl S) # M1' @ []) and
  Z: (λa b. cdclW-stgy a b ∧ (∃ c. trail a = c @ Marked K' (backtrack-lvl S) # M1' @ []))** Z S
  using rtrancpl-cdclW-new-marked-at-beginning-is-decide[OF st' - lev, of Ls K'
    backtrack-lvl S M1' []] unfolding R M' by auto
have [simp]: cdclW-M-level-inv Y
  using RY lev rtrancpl-cdclW-stgy-consistent-inv by blast

```

```

obtain  $M'$  where  $trZ$ :  $trail\ Z = M' @ Marked\ K' (backtrack-lvl\ S) \# M1'$ 
  using  $rtrancplp-cdcl_W-stgy-with-trail-end-has-trail-end[OF\ Z]$   $M'$  by auto
have  $no-dup\ (trail\ Y)$ 
  using  $RY\ lev\ rtrancplp-cdcl_W-stgy-consistent-inv$  unfolding  $cdcl_W-M-level-inv-def$  by blast
then obtain  $Y'$  where
   $dec$ :  $decide\ Y\ Y'$  and
   $Y'Z$ :  $full\ cdcl_W-cp\ Y'\ Z$  and
   $no-step\ cdcl_W-cp\ Y$ 
  using  $cdcl_W-stgy-trail-has-new-marked-is-decide-step[OF\ YZ\ nt\ Z]$   $M'$  by auto
have  $trY$ :  $trail\ Y = M1'$ 
proof –
  obtain  $M'$  where  $M$ :  $trail\ Z = M' @ Marked\ K' (backtrack-lvl\ S) \# M1'$ 
    using  $rtrancplp-cdcl_W-stgy-with-trail-end-has-trail-end[OF\ Z]$   $M'$  by auto
  obtain  $M''$  where  $M''$ :  $trail\ Z = M'' @ trail\ Y'$  and  $\forall m \in set\ M''. \neg is-marked\ m$ 
    using  $Y'Z\ rtrancplp-cdcl_W-cp-dropWhile-trail'$  unfolding  $full-def$  by blast
  obtain  $M'''$  where  $trail\ Y' = M''' @ Marked\ K' (backtrack-lvl\ S) \# M1'$ 
    using  $M''$  unfolding  $M$ 
    by (metis (no-types, lifting)  $\langle \forall m \in set\ M''. \neg is-marked\ m \rangle$  beginning-not-marked-invert)
  then show  $?thesis$  using  $dec\ nt$  by (induction  $M'''$ ) (auto elim: decideE)
qed
have  $Y-CT$ :  $conflicting\ Y = None$  using  $\langle decide\ Y\ Y' \rangle$  by (auto elim: decideE)
have  $cdcl_W^{**}\ R\ Y$  by (simp add: RY rtrancplp-cdcl_W-stgy-rtrancplp-cdcl_W)
then have  $init-clss\ Y = init-clss\ R$  using  $rtrancplp-cdcl_W-init-clss[of\ R\ Y]$   $M-lev$  by auto
{ assume  $DL$ :  $mset-ccls\ E \notin \# clauses\ Y$ 
  have  $atm-of\ L \notin atm-of\ 'lits-of-l\ M1$ 
    apply (rule backtrack-lit-skipped[of\ S])
    using  $decomp\ i\ k\ lev'$  unfolding  $cdcl_W-M-level-inv-def$  by auto
  then have  $LM1$ :  $undefined-lit\ M1\ L$ 
    by (metis Marked-Propagated-in-iff-in-lits-of-l atm-of-uminus image-eqI)
  have  $L-trY$ :  $undefined-lit\ (trail\ Y)\ L$ 
    using  $L-notin\ \langle no-dup\ (trail\ S) \rangle$  unfolding  $defined-lit-map\ trY\ M'$ 
    by (auto simp add: image-iff lits-of-def)
  obtain  $E'$  where
     $E'$ :  $E' ! \in ! raw-clauses\ Y$  and
     $EE'$ :  $mset-clss\ E' = mset-ccls\ E$ 
    using  $DL\ in-mset-clss-exists-preimage$  by blast
  have  $Ex$  (propagate  $Y$ )
    using  $propagate-rule[of\ Y\ E'\ L]\ DL\ M1'-D\ L-trY\ Y-CT\ trY\ LD\ E'$ 
    by (auto simp: EE')
  then have  $False$  using  $\langle no-step\ cdcl_W-cp\ Y \rangle$  propagate' by blast
}
moreover {
  assume  $DL$ :  $mset-ccls\ E \notin \# clauses\ Y$ 
  have  $lY-lZ$ :  $learned-clss\ Y = learned-clss\ Z$ 
    using  $dec\ Y'Z\ rtrancplp-cdcl_W-cp-learned-clause-inv[of\ Y'\ Z]$  unfolding  $full-def$ 
    by (auto elim: decideE)
  have  $invZ$ :  $cdcl_W-all-struct-inv\ Z$ 
    by (meson  $RY\ YZ\ invR\ r-into-rtrancplp\ rtrancplp-cdcl_W-all-struct-inv-inv$ 
       $rtrancplp-cdcl_W-stgy-rtrancplp-cdcl_W$ )
  have  $n$ :  $mset-ccls\ E \notin \# learned-clss\ Z$ 
    using  $DL\ lY-lZ\ YZ$  unfolding  $raw-clauses-def$  by auto
  have  $?E \notin \# learned-clss\ S$ 
    apply (rule  $rtrancplp-cdcl_W-stgy-with-trail-end-has-not-been-learned[OF\ Z\ invZ\ trZ]$ )
    apply (simp add: n)
    using  $LD$  apply simp

```



```

    apply (metis (no-types, lifting) ⟨set M1 ⊆ set M1⟩ image-mono order-trans
      vars-of-D lits-of-def)
    using L-notin ⟨no-dup (trail S)⟩ unfolding M' by (auto simp add: image-iff lits-of-def)
  then have False
    using already-learned DL confl st' M-lev rtranclp-cdclW-stgy-no-more-init-clss[of R S]
    unfolding M'
    by (simp add: ⟨init-clss Y = init-clss R⟩ raw-clauses-def confl-S
      rtranclp-cdclW-stgy-no-more-init-clss)
  }
  ultimately show False by blast
qed

```

lemma *rtranclp-cdcl_W-stgy-distinct-mset-clauses:*

```

  assumes
    invR: cdclW-all-struct-inv R and
    st: cdclW-stgy** R S and
    dist: distinct-mset (clauses R) and
    R: trail R = []
  shows distinct-mset (clauses S)
  using st
proof (induction)
  case base
  then show ?case using dist by simp
next
  case (step S T) note st = this(1) and s = this(2) and IH = this(3)
  from s show ?case
  proof (cases rule: cdclW-stgy.cases)
    case conflict'
    then show ?thesis
      using IH unfolding full1-def by (auto dest: tranclp-cdclW-cp-no-more-clauses)
  next
    case (other' S') note o = this(1) and full = this(3)
    have [simp]: clauses T = clauses S'
      using full unfolding full-def by (auto dest: rtranclp-cdclW-cp-no-more-clauses)
    show ?thesis
      using o IH
    proof (cases rule: cdclW-o-rule-cases)
      case backtrack
      moreover
        have cdclW-all-struct-inv S
          using invR rtranclp-cdclW-stgy-cdclW-all-struct-inv st by blast
        then have cdclW-M-level-inv S
          unfolding cdclW-all-struct-inv-def by auto
      ultimately obtain E where
        conflicting S = Some E and
        cls-S': clauses S' = {#E#} + clauses S
        using ⟨cdclW-M-level-inv S⟩
        by (induction rule: backtrack-induction-lev2) (auto simp: cdclW-M-level-inv-decomp)
      then have E ∉ # clauses S
        using cdclW-stgy-no-relearned-clause R invR local.backtrack st by blast
      then show ?thesis using IH by (simp add: distinct-mset-add-single cls-S')
    qed (auto elim: decideE skipE resolveE)
  qed
qed

```

```

lemma cdclW-stgy-distinct-mset-clauses:
  assumes
    st: cdclW-stgy** (init-state N) S and
    no-duplicate-clause: distinct-mset (mset-clss N) and
    no-duplicate-in-clause: distinct-mset-mset (mset-clss N)
  shows distinct-mset (clauses S)
  using rtrancp-cdclW-stgy-distinct-mset-clauses[OF - st] assms
  by (auto simp: cdclW-all-struct-inv-def distinct-cdclW-state-def)

```

19.8 Decrease of a measure

```

fun cdclW-measure where
cdclW-measure S =
  [( $3::nat$ ) ^ (card (atms-of-mm (init-clss S))) - card (set-mset (learned-clss S)),
   if conflicting S = None then 1 else 0,
   if conflicting S = None then card (atms-of-mm (init-clss S)) - length (trail S)
   else length (trail S)
  ]

```

```

lemma length-model-le-vars-all-inv:
  assumes cdclW-all-struct-inv S
  shows length (trail S) ≤ card (atms-of-mm (init-clss S))
  using assms length-model-le-vars[of S] unfolding cdclW-all-struct-inv-def
  by (auto simp: cdclW-M-level-inv-decomp)
end

```

```

context conflict-driven-clause-learningW
begin

```

```

lemma learned-clss-less-upper-bound:
  fixes S :: 'st
  assumes
    distinct-cdclW-state S and
     $\forall s \in \# \text{learned-clss } S. \neg \text{tautology } s$ 
  shows card(set-mset (learned-clss S)) ≤  $3 \wedge$  card (atms-of-mm (learned-clss S))
proof -
  have set-mset (learned-clss S) ⊆ simple-clss (atms-of-mm (learned-clss S))
  apply (rule simplified-in-simple-clss)
  using assms unfolding distinct-cdclW-state-def by auto
  then have card(set-mset (learned-clss S))
    ≤ card (simple-clss (atms-of-mm (learned-clss S)))
  by (simp add: simple-clss-finite card-mono)
  then show ?thesis
  by (meson atms-of-ms-finite simple-clss-card finite-set-mset order-trans)
qed

```

```

lemma cdclW-measure-decreasing:
  fixes S :: 'st
  assumes
    cdclW S S' and
    no-restart:
       $\neg(\text{learned-clss } S \subseteq \# \text{learned-clss } S' \wedge [] = \text{trail } S' \wedge \text{conflicting } S' = \text{None})$ 
    and
    no-forget: learned-clss S ⊆ # learned-clss S' and
    no-relearn:  $\bigwedge S'. \text{backtrack } S S' \implies \forall T. \text{conflicting } S = \text{Some } T \longrightarrow T \notin \# \text{learned-clss } S$ 

```

and
alien: *no-strange-atm S* **and**
M-level: *cdcl_W-M-level-inv S* **and**
no-taut: $\forall s \in \# \text{ learned-clss } S. \neg \text{tautology } s$ **and**
no-dup: *distinct-cdcl_W-state S* **and**
conf: *cdcl_W-conflicting S*
shows (*cdcl_W-measure S'*, *cdcl_W-measure S*) $\in \text{lexn } \{(a, b). a < b\} \ 3$
using *assms(1) M-level assms(2,3)*
proof (*induct rule: cdcl_W-all-induct-lev2*)
case (*propagate C L*) **note** *conf = this(1)* **and** *undef = this(5)* **and** *T = this(6)*
have *propa*: *propagate S (cons-trail (Propagated L C) S)*
using *propagate-rule[OF propagate.hyps(1,2)] propagate.hyps* **by** *auto*
then have *no-dup'*: *no-dup (Propagated L (mset-cls C) # trail S)*
using *M-level cdcl_W-M-level-inv-decomp(2) undef defined-lit-map* **by** *auto*

let *?N = init-clss S*
have *no-strange-atm (cons-trail (Propagated L C) S)*
using *alien cdcl_W.propagate cdcl_W-no-strange-atm-inv propa M-level* **by** *blast*
then have *atm-of ' lits-of-l (Propagated L (mset-cls C) # trail S)*
 \subseteq *atms-of-mm (init-clss S)*
using *undef unfolding no-strange-atm-def* **by** *auto*
then have *card (atm-of ' lits-of-l (Propagated L (mset-cls C) # trail S))*
 \leq *card (atms-of-mm (init-clss S))*
by (*meson atms-of-ms-finite card-mono finite-set-mset*)
then have *length (Propagated L (mset-cls C) # trail S) \leq card (atms-of-mm ?N)*
using *no-dup-length-eq-card-atm-of-lits-of-l no-dup'* **by** *fastforce*
then have *H*: *card (atms-of-mm (init-clss S)) - length (trail S)*
 $=$ *Suc (card (atms-of-mm (init-clss S)) - Suc (length (trail S)))*
by *simp*
show *?case using conf T undef* **by** (*auto simp: H lexn3-conv*)
next
case (*decide L*) **note** *conf = this(1)* **and** *undef = this(2)* **and** *T = this(4)*
moreover
have *dec*: *decide S (cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S))*
using *decide-rule decide.hyps* **by** *force*
then have *cdcl_W:cdcl_W S (cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S))*
using *cdcl_W.simps cdcl_W-o.intros* **by** *blast*
moreover
have *lev*: *cdcl_W-M-level-inv (cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S))*
using *cdcl_W M-level cdcl_W-consistent-inv[OF cdcl_W]* **by** *auto*
then have *no-dup*: *no-dup (Marked L (backtrack-lvl S + 1) # trail S)*
using *undef unfolding cdcl_W-M-level-inv-def* **by** *auto*
have *no-strange-atm (cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S))*
using *M-level alien calculation(4) cdcl_W-no-strange-atm-inv* **by** *blast*
then have *length (Marked L ((backtrack-lvl S) + 1) # (trail S))*
 \leq *card (atms-of-mm (init-clss S))*
using *no-dup undef*
length-model-le-vars[of cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S)]
by *fastforce*
ultimately show *?case using conf* **by** (*simp add: lexn3-conv*)
next
case (*skip L C' M D*) **note** *tr = this(1)* **and** *conf = this(2)* **and** *T = this(5)*
show *?case using conf T* **by** (*simp add: tr lexn3-conv*)
next
case *conflict*

```

    then show ?case by (simp add: lexn3-conv)
next
  case resolve
  then show ?case using finite by (simp add: lexn3-conv)
next
  case (backtrack K i M1 M2 L D T) note conf = this(1) and decomp = this(3) and undef = this(7)
and
  T = this(8) and lev = this(9)
let ?S' = T
have bt: backtrack S ?S'
  using backtrack.hyps backtrack.intros[of S D L K i] by auto
have mset-ccls D  $\notin$  # learned-clss S
  using no-relearn conf bt by auto
then have card-T:
  card (set-mset ({#mset-ccls D#} + learned-clss S)) = Suc (card (set-mset (learned-clss S)))
  by simp
have distinct-cdclW-state ?S'
  using bt M-level distinct-cdclW-state-inv no-dup other cdclW-o.intros cdclW-bj.intros by blast
moreover have  $\forall s \in \# \text{learned-clss } ?S'. \neg \text{tautology } s$ 
  using learned-clss-are-not-tautologies[OF cdclW.other[OF cdclW-o.bj[OF
    cdclW-bj.backtrack[OF bt]]]] M-level no-taut confl by auto
ultimately have card (set-mset (learned-clss T))  $\leq 3 \wedge$  card (atms-of-mm (learned-clss T))
  by (auto simp: learned-clss-less-upper-bound)
then have H: card (set-mset ({#mset-ccls D#} + learned-clss S))
   $\leq 3 \wedge$  card (atms-of-mm ({#mset-ccls D#} + learned-clss S))
  using T undef decomp M-level by (simp add: cdclW-M-level-inv-decomp)
moreover
  have atms-of-mm ({#mset-ccls D#} + learned-clss S)  $\subseteq$  atms-of-mm (init-clss S)
  using alien conf unfolding no-strange-atm-def by auto
  then have card-f: card (atms-of-mm ({#mset-ccls D#} + learned-clss S))
     $\leq$  card (atms-of-mm (init-clss S))
    by (meson atms-of-ms-finite card-mono finite-set-mset)
  then have (3::nat)  $\wedge$  card (atms-of-mm ({#mset-ccls D#} + learned-clss S))
     $\leq 3 \wedge$  card (atms-of-mm (init-clss S)) by simp
ultimately have (3::nat)  $\wedge$  card (atms-of-mm (init-clss S))
   $\geq$  card (set-mset ({#mset-ccls D#} + learned-clss S))
  using le-trans by blast
then show ?case using decomp undef diff-less-mono2 card-T T M-level
  by (auto simp: cdclW-M-level-inv-decomp lexn3-conv)
next
  case restart
  then show ?case using alien by (auto simp: state-eq-def simp del: state-simp)
next
  case (forget C T) note no-forget = this(8)
  then have mset-cl C  $\in$  # learned-clss S and mset-cl C  $\notin$  # learned-clss T
    using forget.hyps by auto
  then show ?case using no-forget by (auto simp add: mset-leD)
qed

lemma propagate-measure-decreasing:
  fixes S :: 'st
  assumes propagate S S' and cdclW-all-struct-inv S
  shows (cdclW-measure S', cdclW-measure S)  $\in$  lexn {(a, b). a < b} 3
  apply (rule cdclW-measure-decreasing)
  using assms(1) propagate apply blast

```

```

    using assms(1) apply (auto simp add: propagate.simps)[3]
    using assms(2) apply (auto simp add: cdclW-all-struct-inv-def)
done

lemma conflict-measure-decreasing:
  fixes S :: 'st
  assumes conflict S S' and cdclW-all-struct-inv S
  shows (cdclW-measure S', cdclW-measure S) ∈ lexn {(a, b). a < b} 3
  apply (rule cdclW-measure-decreasing)
  using assms(1) conflict apply blast
    using assms(1) apply (auto simp: state-eq-def simp del: state-simp elim!: conflictE)[3]
    using assms(2) apply (auto simp add: cdclW-all-struct-inv-def elim: conflictE)
done

lemma decide-measure-decreasing:
  fixes S :: 'st
  assumes decide S S' and cdclW-all-struct-inv S
  shows (cdclW-measure S', cdclW-measure S) ∈ lexn {(a, b). a < b} 3
  apply (rule cdclW-measure-decreasing)
  using assms(1) decide other apply blast
    using assms(1) apply (auto simp: state-eq-def simp del: state-simp elim!: decideE)[3]
    using assms(2) apply (auto simp add: cdclW-all-struct-inv-def elim: decideE)
done

lemma trans-le:
  trans {(a, (b::nat)). a < b}
  unfolding trans-def by auto

lemma cdclW-cp-measure-decreasing:
  fixes S :: 'st
  assumes cdclW-cp S S' and cdclW-all-struct-inv S
  shows (cdclW-measure S', cdclW-measure S) ∈ lexn {(a, b). a < b} 3
  using assms
proof induction
  case conflict'
  then show ?case using conflict-measure-decreasing by blast
next
  case propagate'
  then show ?case using propagate-measure-decreasing by blast
qed

lemma tranclp-cdclW-cp-measure-decreasing:
  fixes S :: 'st
  assumes cdclW-cp++ S S' and cdclW-all-struct-inv S
  shows (cdclW-measure S', cdclW-measure S) ∈ lexn {(a, b). a < b} 3
  using assms
proof induction
  case base
  then show ?case using cdclW-cp-measure-decreasing by blast
next
  case (step T U) note st = this(1) and step = this(2) and IH = this(3) and inv = this(4)
  then have (cdclW-measure T, cdclW-measure S) ∈ lexn {a. case a of (a, b) ⇒ a < b} 3 by blast

  moreover have (cdclW-measure U, cdclW-measure T) ∈ lexn {a. case a of (a, b) ⇒ a < b} 3
    using cdclW-cp-measure-decreasing[OF step] rtranclp-cdclW-all-struct-inv-inv inv

```

```

  tranclp-cdclW-cp-tranclp-cdclW[OF st]
  unfolding trans-def rtranclp-unfold
  by blast
ultimately show ?case using lern-transI[OF trans-le] unfolding trans-def by blast
qed

lemma cdclW-stgy-step-decreasing:
  fixes R S T :: 'st
  assumes cdclW-stgy S T and
  cdclW-stgy** R S
  trail R = [] and
  cdclW-all-struct-inv R
  shows (cdclW-measure T, cdclW-measure S) ∈ lern {(a, b). a < b} 3
proof -
  have cdclW-all-struct-inv S
  using assms
  by (metis rtranclp-unfold rtranclp-cdclW-all-struct-inv-inv tranclp-cdclW-stgy-tranclp-cdclW)
with assms show ?thesis
proof induction
  case (conflict' V) note cp = this(1) and inv = this(5)
  show ?case
    using tranclp-cdclW-cp-measure-decreasing[OF HOL.conjunct1[OF cp[unfolded full1-def]] inv]
    .
next
  case (other' T U) note st = this(1) and H = this(4,5,6,7) and cp = this(3)
  have cdclW-all-struct-inv T
  using cdclW-all-struct-inv-inv other other'.hyps(1) other'.prems(4) by blast
from tranclp-cdclW-cp-measure-decreasing[OF - this]
have le-or-eq: (cdclW-measure U, cdclW-measure T) ∈ lern {a. case a of (a, b) ⇒ a < b} 3 ∨
  cdclW-measure U = cdclW-measure T
  using cp unfolding full-def rtranclp-unfold by blast
moreover
  have cdclW-M-level-inv S
  using cdclW-all-struct-inv-def other'.prems(4) by blast
with st have (cdclW-measure T, cdclW-measure S) ∈ lern {a. case a of (a, b) ⇒ a < b} 3
proof (induction rule:cdclW-o-induct-lev2)
  case (decide T)
  then show ?case using decide-measure-decreasing H decide.intros[OF decide.hyps] by blast
next
  case (backtrack K i M1 M2 L D T) note conf = this(1) and decomp = this(3) and
  undef = this(7) and T = this(8)
  have bt: backtrack S T
  apply (rule backtrack-rule)
  using backtrack.hyps by auto
  then have no-relearn: ∀ T. conflicting S = Some T ⟶ T ∉ # learned-clss S
  using cdclW-stgy-no-relearned-clause[of R S T] H conf
  unfolding cdclW-all-struct-inv-def raw-clauses-def by auto
  have inv: cdclW-all-struct-inv S
  using ⟨cdclW-all-struct-inv S⟩ by blast
  show ?case
  apply (rule cdclW-measure-decreasing)
  using bt cdclW-bj.backtrack cdclW-o.bj other apply simp
  using bt T undef decomp inv unfolding cdclW-all-struct-inv-def
  cdclW-M-level-inv-def apply auto[]
  using bt T undef decomp inv unfolding cdclW-all-struct-inv-def

```

```

      cdclW-M-level-inv-def apply auto[]
    using bt no-relearn apply auto[]
    using inv unfolding cdclW-all-struct-inv-def apply simp
    using inv unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def apply simp
    using inv unfolding cdclW-all-struct-inv-def apply simp
    using inv unfolding cdclW-all-struct-inv-def apply simp
    using inv unfolding cdclW-all-struct-inv-def by simp
  next
    case skip
    then show ?case by (auto simp: leW3-conv)
  next
    case resolve
    then show ?case by (auto simp: leW3-conv)
  qed
ultimately show ?case
  by (metis (full-types) leW-transI transD trans-le)
qed
qed

```

lemma tranclp-cdcl_W-stgy-decreasing:

```

fixes R S T :: 'st
assumes cdclW-stgy++ R S
trail R = [] and
cdclW-all-struct-inv R
shows (cdclW-measure S, cdclW-measure R) ∈ leW {(a, b). a < b} 3
using assms
apply induction
  using cdclW-stgy-step-decreasing[of R - R] apply blast
using cdclW-stgy-step-decreasing[of - - R] tranclp-into-rtranclp[of cdclW-stgy R]
leW-transI[OF trans-le, of 3] unfolding trans-def by blast

```

lemma tranclp-cdcl_W-stgy-S0-decreasing:

```

fixes R S T :: 'st
assumes
  pl: cdclW-stgy++ (init-state N) S and
  no-dup: distinct-mset-mset (mset-cls N)
shows (cdclW-measure S, cdclW-measure (init-state N)) ∈ leW {(a, b). a < b} 3
proof -
  have cdclW-all-struct-inv (init-state N)
    using no-dup unfolding cdclW-all-struct-inv-def by auto
  then show ?thesis using pl tranclp-cdclW-stgy-decreasing init-state-trail by blast
qed

```

lemma wf-tranclp-cdcl_W-stgy:

```

wf {(S::'st, init-state N)|
  S N. distinct-mset-mset (mset-cls N) ∧ cdclW-stgy++ (init-state N) S}
apply (rule wf-wf-if-measure'-notation2[of leW {(a, b). a < b} 3 - - cdclW-measure])
apply (simp add: wf wf-leW)
using tranclp-cdclW-stgy-S0-decreasing by blast

```

lemma cdcl_W-cp-wf-all-inv:

```

wf {(S', S). cdclW-all-struct-inv S ∧ cdclW-cp S S'}
(is wf ?R)
proof (rule wf-bounded-measure[of -
  λS. card (atms-of-mm (init-cls S))+1

```

```

    λS. length (trail S) + (if conflicting S = None then 0 else 1)], goal-cases)
  case (1 S S')
  then have cdclW-all-struct-inv S and cdclW-cp S S' by auto
  moreover then have cdclW-all-struct-inv S'
    using cdclW-cp.simps cdclW-all-struct-inv-inv conflict cdclW.intros cdclW-all-struct-inv-inv
    by blast+
  ultimately show ?case
    by (auto simp:cdclW-cp.simps state-eq-def simp del: state-simp elim!: conflictE propagateE
        dest: length-model-le-vars-all-inv)
qed

end

end

theory DPLL-CDCL-W-Implementation
imports Partial-Annotated-Clausal-Logic
begin

```

20 Simple Implementation of the DPLL and CDCL

20.1 Common Rules

20.1.1 Propagation

The following theorem holds:

lemma *lits-of-l-unfold*[iff]:
 $(\forall c \in \text{set } C. -c \in \text{lits-of-l } Ms) \longleftrightarrow Ms \models_{\text{as}} \text{CNot } (\text{mset } C)$
unfolding *true-annots-def Ball-def true-annot-def CNot-def* **by** auto

The right-hand version is written at a high-level, but only the left-hand side is executable.

definition *is-unit-clause* :: 'a literal list \Rightarrow ('a, 'b, 'c) marked-lit list \Rightarrow 'a literal option
where

is-unit-clause l M =
 (case List.filter (λa. atm-of a \notin atm-of ' lits-of-l M) l of
 a # [] \Rightarrow if M \models_{as} CNot (mset l - {#a#}) then Some a else None
 | - \Rightarrow None)

definition *is-unit-clause-code* :: 'a literal list \Rightarrow ('a, 'b, 'c) marked-lit list
 \Rightarrow 'a literal option **where**

is-unit-clause-code l M =
 (case List.filter (λa. atm-of a \notin atm-of ' lits-of-l M) l of
 a # [] \Rightarrow if ($\forall c \in \text{set } (\text{remove1 } a \text{ l}). -c \in \text{lits-of-l } M$) then Some a else None
 | - \Rightarrow None)

lemma *is-unit-clause-is-unit-clause-code*[code]:

is-unit-clause l M = *is-unit-clause-code* l M

proof -

have 1: $\bigwedge a. (\forall c \in \text{set } (\text{remove1 } a \text{ l}). -c \in \text{lits-of-l } M) \longleftrightarrow M \models_{\text{as}} \text{CNot } (\text{mset } l - \{\#a\})$
using *lits-of-l-unfold*[of remove1 - l, of - M] **by** simp
thus ?thesis
unfolding *is-unit-clause-code-def is-unit-clause-def* 1 **by** blast

qed

lemma *is-unit-clause-some-undef*:

assumes *is-unit-clause* l M = Some a

shows *undefined-lit* M a

proof –

have (case $[a \leftarrow l . \text{atm-of } a \notin \text{atm-of ' lits-of-l } M]$ of $[] \Rightarrow \text{None}$
 $| [a] \Rightarrow \text{if } M \models_{\text{as}} \text{CNot } (\text{mset } l - \{\#a\# \}) \text{ then } \text{Some } a \text{ else } \text{None}$
 $| a \# ab \# xa \Rightarrow \text{Map.empty } xa) = \text{Some } a$
using **assms** **unfolding** *is-unit-clause-def* .
hence $a \in \text{set } [a \leftarrow l . \text{atm-of } a \notin \text{atm-of ' lits-of-l } M]$
apply (cases $[a \leftarrow l . \text{atm-of } a \notin \text{atm-of ' lits-of-l } M]$)
apply *simp*
apply (rename-tac aa list; case-tac list) **by** (auto split: if-split-asm)
hence $\text{atm-of } a \notin \text{atm-of ' lits-of-l } M$ **by** auto
thus ?thesis
by (simp add: Marked-Propagated-in-iff-in-lits-of-l
atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)

qed

lemma *is-unit-clause-some-CNot*: $\text{is-unit-clause } l \ M = \text{Some } a \implies M \models_{\text{as}} \text{CNot } (\text{mset } l - \{\#a\# \})$
unfolding *is-unit-clause-def*

proof –

assume (case $[a \leftarrow l . \text{atm-of } a \notin \text{atm-of ' lits-of-l } M]$ of $[] \Rightarrow \text{None}$
 $| [a] \Rightarrow \text{if } M \models_{\text{as}} \text{CNot } (\text{mset } l - \{\#a\# \}) \text{ then } \text{Some } a \text{ else } \text{None}$
 $| a \# ab \# xa \Rightarrow \text{Map.empty } xa) = \text{Some } a$
thus ?thesis
apply (cases $[a \leftarrow l . \text{atm-of } a \notin \text{atm-of ' lits-of-l } M]$, *simp*)
apply *simp*
apply (rename-tac aa list; case-tac list) **by** (auto split: if-split-asm)

qed

lemma *is-unit-clause-some-in*: $\text{is-unit-clause } l \ M = \text{Some } a \implies a \in \text{set } l$
unfolding *is-unit-clause-def*

proof –

assume (case $[a \leftarrow l . \text{atm-of } a \notin \text{atm-of ' lits-of-l } M]$ of $[] \Rightarrow \text{None}$
 $| [a] \Rightarrow \text{if } M \models_{\text{as}} \text{CNot } (\text{mset } l - \{\#a\# \}) \text{ then } \text{Some } a \text{ else } \text{None}$
 $| a \# ab \# xa \Rightarrow \text{Map.empty } xa) = \text{Some } a$
thus $a \in \text{set } l$
by (cases $[a \leftarrow l . \text{atm-of } a \notin \text{atm-of ' lits-of-l } M]$)
(fastforce dest: filter-eq-ConsD split: if-split-asm split: list.splits)+

qed

lemma *is-unit-clause-nil*[*simp*]: $\text{is-unit-clause } [] \ M = \text{None}$
unfolding *is-unit-clause-def* **by** auto

20.1.2 Unit propagation for all clauses

Finding the first clause to propagate

fun *find-first-unit-clause* :: 'a literal list list \Rightarrow ('a, 'b, 'c) marked-lit list
 \Rightarrow ('a literal \times 'a literal list) option **where**
find-first-unit-clause (a # l) M =
(case *is-unit-clause* a M of
None \Rightarrow *find-first-unit-clause* l M
| Some L \Rightarrow Some (L, a)) |
find-first-unit-clause [] - = None

lemma *find-first-unit-clause-some*:
find-first-unit-clause l M = Some (a, c)

$\Rightarrow c \in \text{set } l \wedge M \models_{\text{as}} \text{CNot } (\text{mset } c - \{\#a\# \}) \wedge \text{undefined-lit } M a \wedge a \in \text{set } c$
apply (induction l)
apply simp
by (auto split: option.splits dest: is-unit-clause-some-in is-unit-clause-some-CNot
is-unit-clause-some-undef)

lemma propagate-is-unit-clause-not-None:

assumes dist: distinct c **and**
 $M: M \models_{\text{as}} \text{CNot } (\text{mset } c - \{\#a\# \})$ **and**
undef: undefined-lit M a **and**
ac: $a \in \text{set } c$
shows is-unit-clause c M \neq None

proof -

have $[a \leftarrow c . \text{atm-of } a \notin \text{atm-of ' lits-of-l } M] = [a]$
using assms
proof (induction c)
case Nil **thus** ?case **by** simp
next
case (Cons ac c)
show ?case
proof (cases a = ac)
case True
thus ?thesis **using** Cons
by (auto simp del: lits-of-l-unfold
simp add: lits-of-l-unfold[symmetric] Marked-Propagated-in-iff-in-lits-of-l
atm-of-eq-atm-of atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)
next
case False
hence T: $\text{mset } c + \{\#ac\# \} - \{\#a\# \} = \text{mset } c - \{\#a\# \} + \{\#ac\# \}$
by (auto simp add: multiset-eq-iff)
show ?thesis **using** False Cons
by (auto simp add: T atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)
qed
qed
thus ?thesis
using M **unfolding** is-unit-clause-def **by** auto
qed

lemma find-first-unit-clause-none:

distinct c $\Rightarrow c \in \text{set } l \Rightarrow M \models_{\text{as}} \text{CNot } (\text{mset } c - \{\#a\# \}) \Rightarrow \text{undefined-lit } M a \Rightarrow a \in \text{set } c$
 $\Rightarrow \text{find-first-unit-clause } l M \neq \text{None}$
by (induction l)
(auto split: option.split simp add: propagate-is-unit-clause-not-None)

20.1.3 Decide

fun find-first-unused-var :: 'a literal list list \Rightarrow 'a literal set \Rightarrow 'a literal option **where**

find-first-unused-var (a # l) M =
(case List.find ($\lambda \text{lit. lit} \notin M \wedge \neg \text{lit} \in M$) a of
None \Rightarrow find-first-unused-var l M
| Some a \Rightarrow Some a) |
find-first-unused-var [] - = None

lemma find-none[iff]:

List.find ($\lambda \text{lit. lit} \notin M \wedge \neg \text{lit} \in M$) a = None \longleftrightarrow atm-of ' set a \subseteq atm-of ' M
apply (induct a)

```

using atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
  by (force simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)+

lemma find-some: List.find ( $\lambda lit. lit \notin M \wedge \neg lit \notin M$ )  $a = Some\ b \implies b \in set\ a \wedge b \notin M \wedge \neg b \notin M$ 
unfolding find-Some-iff by (metis nth-mem)

lemma find-first-unused-var-None[iff]:
  find-first-unused-var  $l\ M = None \iff (\forall a \in set\ l. atm-of\ 'set\ a \subseteq atm-of\ 'M)$ 
by (induct l)
  (auto split: option.splits dest!: find-some
    simp add: image-subset-iff atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)

lemma find-first-unused-var-Some-not-all-incl:
  assumes find-first-unused-var  $l\ M = Some\ c$ 
  shows  $\neg(\forall a \in set\ l. atm-of\ 'set\ a \subseteq atm-of\ 'M)$ 
proof -
  have find-first-unused-var  $l\ M \neq None$ 
  using assms by (cases find-first-unused-var  $l\ M$ ) auto
  thus  $\neg(\forall a \in set\ l. atm-of\ 'set\ a \subseteq atm-of\ 'M)$  by auto
qed

lemma find-first-unused-var-Some:
  find-first-unused-var  $l\ M = Some\ a \implies (\exists m \in set\ l. a \in set\ m \wedge a \notin M \wedge \neg a \notin M)$ 
by (induct l) (auto split: option.splits dest: find-some)

lemma find-first-unused-var-undefined:
  find-first-unused-var  $l\ (lits-of-l\ Ms) = Some\ a \implies undefined-lit\ Ms\ a$ 
using find-first-unused-var-Some[of  $l\ lits-of-l\ Ms\ a$ ] Marked-Propagated-in-iff-in-lits-of-l
by blast

end
theory DPLL-W-Implementation
imports DPLL-CDCL-W-Implementation DPLL-W  $\sim\sim$  /src/HOL/Library/Code-Target-Numeral
begin

```

20.2 Simple Implementation of DPLL

20.2.1 Combining the propagate and decide: a DPLL step

definition $DPLL\text{-}step :: int\ dpll_W\text{-}marked\text{-}lits \times int\ literal\ list\ list$
 $\Rightarrow int\ dpll_W\text{-}marked\text{-}lits \times int\ literal\ list\ list$ **where**

```

DPLL-step = ( $\lambda(Ms, N).$ 
  (case find-first-unit-clause  $N\ Ms$  of
    Some  $(L, -) \Rightarrow (Propagated\ L\ () \# Ms, N)$ 
  | -  $\Rightarrow$ 
    if  $\exists C \in set\ N. (\forall c \in set\ C. \neg c \in lits-of-l\ Ms)$ 
    then
      (case backtrack-split  $Ms$  of
        ( $\neg, L \# M$ )  $\Rightarrow (Propagated\ (\neg\ (lit-of\ L))\ () \# M, N)$ 
      | ( $\neg, -$ )  $\Rightarrow (Ms, N)$ 
      )
    else
      (case find-first-unused-var  $N\ (lits-of-l\ Ms)$  of
        Some  $a \Rightarrow (Marked\ a\ () \# Ms, N)$ 
      | None  $\Rightarrow (Ms, N))))$ 

```

Example of propagation:

```
value DPLL-step ([Marked (Neg 1) ()], [[Pos (1::int), Neg 2]])
```

We define the conversion function between the states as defined in *Prop-DPLL* (with multisets) and here (with lists).

```
abbreviation toS ≡ λ(Ms::(int, unit, unit) marked-lit list)
  (N:: int literal list list). (Ms, mset (map mset N))
abbreviation toS' ≡ λ(Ms::(int, unit, unit) marked-lit list,
  N:: int literal list list). (Ms, mset (map mset N))
```

Proof of correctness of *DPLL-step*

lemma *DPLL-step-is-a-dpll_W-step*:

```
assumes step: (Ms', N') = DPLL-step (Ms, N)
and neq: (Ms, N) ≠ (Ms', N')
shows dpllW (toS Ms N) (toS Ms' N')
```

proof –

```
let ?S = (Ms, mset (map mset N))
{ fix L E
  assume unit: find-first-unit-clause N Ms = Some (L, E)
  hence Ms'N: (Ms', N') = (Propagated L () # Ms, N)
    using step unfolding DPLL-step-def by auto
  obtain C where
    C: C ∈ set N and
    Ms: Ms ⊨as CNot (mset C - {#L#}) and
    undef: undefined-lit Ms L and
    L ∈ set C using find-first-unit-clause-some[OF unit] by metis
  have dpllW (Ms, mset (map mset N))
    (Propagated L () # fst (Ms, mset (map mset N)), snd (Ms, mset (map mset N)))
    apply (rule dpllW.propagate)
    using Ms undef C ⟨L ∈ set C⟩ by (auto simp add: C)
  hence ?thesis using Ms'N by auto
}
```

moreover

```
{ assume unit: find-first-unit-clause N Ms = None
  assume exC: ∃ C ∈ set N. Ms ⊨as CNot (mset C)
  then obtain C where C: C ∈ set N and Ms: Ms ⊨as CNot (mset C) by auto
  then obtain L M M' where bt: backtrack-split Ms = (M', L # M)
    using step exC neq unfolding DPLL-step-def prod.case unit
    by (cases backtrack-split Ms, rename-tac b, case-tac b) auto
  hence is-marked L using backtrack-split-snd-hd-marked[of Ms] by auto
  have 1: dpllW (Ms, mset (map mset N))
    (Propagated (– lit-of L) () # M, snd (Ms, mset (map mset N)))
    apply (rule dpllW.backtrack[OF - ⟨is-marked L⟩, of ])
    using C Ms bt by auto
  moreover have (Ms', N') = (Propagated (– (lit-of L)) () # M, N)
    using step exC unfolding DPLL-step-def bt prod.case unit by auto
  ultimately have ?thesis by auto
}
```

moreover

```
{ assume unit: find-first-unit-clause N Ms = None
  assume exC: ¬ (∃ C ∈ set N. Ms ⊨as CNot (mset C))
  obtain L where unused: find-first-unused-var N (lits-of-l Ms) = Some L
    using step exC neq unfolding DPLL-step-def prod.case unit
    by (cases find-first-unused-var N (lits-of-l Ms)) auto
  have dpllW (Ms, mset (map mset N))
```

```

      (Marked L () # fst (Ms, mset (map mset N)), snd (Ms, mset (map mset N)))
    apply (rule dpllW.decided[of ?S L])
    using find-first-unused-var-Some[OF unused]
    by (auto simp add: Marked-Propagated-in-iff-in-lits-of-l atms-of-ms-def)
  moreover have (Ms', N') = (Marked L () # Ms, N)
    using step exC unfolding DPLL-step-def unused prod.case unit by auto
  ultimately have ?thesis by auto
}
ultimately show ?thesis by (cases find-first-unit-clause N Ms) auto
qed

lemma DPLL-step-stuck-final-state:
  assumes step: (Ms, N) = DPLL-step (Ms, N)
  shows conclusive-dpllW-state (toS Ms N)
proof -
  have unit: find-first-unit-clause N Ms = None
    using step unfolding DPLL-step-def by (auto split: option.splits)

  { assume n:  $\exists C \in \text{set } N. Ms \models_{as} CNot (mset C)$ 
    hence Ms: (Ms, N) = (case backtrack-split Ms of (x, [])  $\Rightarrow$  (Ms, N)
      | (x, L # M)  $\Rightarrow$  (Propagated (- lit-of L) () # M, N))
      using step unfolding DPLL-step-def by (simp add: unit)
  }

  have snd (backtrack-split Ms) = []
  proof (cases backtrack-split Ms, cases snd (backtrack-split Ms))
    fix a b
    assume backtrack-split Ms = (a, b) and snd (backtrack-split Ms) = []
    thus snd (backtrack-split Ms) = [] by blast
  next
    fix a b aa list
    assume
      bt: backtrack-split Ms = (a, b) and
      bt': snd (backtrack-split Ms) = aa # list
    hence Ms: Ms = Propagated (- lit-of aa) () # list using Ms by auto
    have is-marked aa using backtrack-split-snd-hd-marked[of Ms] bt bt' by auto
    moreover have fst (backtrack-split Ms) @ aa # list = Ms
      using backtrack-split-list-eq[of Ms] bt' by auto
    ultimately have False unfolding Ms by auto
    thus snd (backtrack-split Ms) = [] by blast
  qed

  hence ?thesis
    using n backtrack-snd-empty-not-marked[of Ms] unfolding conclusive-dpllW-state-def
    by (cases backtrack-split Ms) auto
}
moreover {
  assume n:  $\neg (\exists C \in \text{set } N. Ms \models_{as} CNot (mset C))$ 
  hence find-first-unused-var N (lits-of-l Ms) = None
    using step unfolding DPLL-step-def by (simp add: unit split: option.splits)
  hence a:  $\forall a \in \text{set } N. \text{atm-of } ' \text{ set } a \subseteq \text{atm-of } ' (\text{lits-of-l } Ms)$  by auto
  have fst (toS Ms N)  $\models_{asm}$  snd (toS Ms N) unfolding true-annots-def CNot-def Ball-def
  proof clarify
    fix x
    assume x:  $x \in \text{set-mset } (\text{clauses } (\text{toS } Ms N))$ 
    hence  $\neg Ms \models_{as} CNot x$  using n unfolding true-annots-def CNot-def Ball-def by auto
  }
}

```

```

    moreover have total-over-m (lits-of-l Ms) {x}
      using a x image-iff in-mono atms-of-s-def
      unfolding total-over-m-def total-over-set-def lits-of-def by fastforce
    ultimately show fst (toS Ms N)  $\models$  a x
      using total-not-CNot[of lits-of-l Ms x] by (simp add: true-annot-def true-annots-true-cls)
  qed
  hence ?thesis unfolding conclusive-dpllW-state-def by blast
}
ultimately show ?thesis by blast
qed

```

20.2.2 Adding invariants

Invariant tested in the function `function DPLL-ci :: int dpllW-marked-lits \Rightarrow int literal list list`

```

 $\Rightarrow$  int dpllW-marked-lits  $\times$  int literal list list where
DPLL-ci Ms N =
  (if  $\neg$ dpllW-all-inv (Ms, mset (map mset N))
   then (Ms, N)
   else
    let (Ms', N') = DPLL-step (Ms, N) in
    if (Ms', N') = (Ms, N) then (Ms, N) else DPLL-ci Ms' N)
  by fast+
termination
proof (relation {(S', S). (toS' S', toS' S)  $\in$  {(S', S). dpllW-all-inv S  $\wedge$  dpllW S S'}})
  show wf {(S', S). (toS' S', toS' S)  $\in$  {(S', S). dpllW-all-inv S  $\wedge$  dpllW S S'}}
    using wf-if-measure-f[OF dpllW-wf, of toS'] by auto
next
  fix Ms :: int dpllW-marked-lits and N x xa y
  assume  $\neg \neg$  dpllW-all-inv (toS Ms N)
  and step: x = DPLL-step (Ms, N)
  and x: (xa, y) = x
  and (xa, y)  $\neq$  (Ms, N)
  thus ((xa, N), Ms, N)  $\in$  {(S', S). (toS' S', toS' S)  $\in$  {(S', S). dpllW-all-inv S  $\wedge$  dpllW S S'}}
    using DPLL-step-is-a-dpllW-step dpllW-same-clauses split-conv by fastforce
qed

```

No invariant tested `function (domintros) DPLL-part :: int dpllW-marked-lits \Rightarrow int literal list list`

```

 $\Rightarrow$ 
  int dpllW-marked-lits  $\times$  int literal list list where
DPLL-part Ms N =
  (let (Ms', N') = DPLL-step (Ms, N) in
   if (Ms', N') = (Ms, N) then (Ms, N) else DPLL-part Ms' N)
  by fast+

```

lemma `snd-DPLL-step[simp]:`
`snd (DPLL-step (Ms, N)) = N`
unfolding `DPLL-step-def` **by** (auto split: if-split option.splits prod.splits list.splits)

lemma `dpllW-all-inv-implicS-2-eq3-and-dom:`
assumes `dpllW-all-inv (Ms, mset (map mset N))`
shows `DPLL-ci Ms N = DPLL-part Ms N \wedge DPLL-part-dom (Ms, N)`
using `assms`
proof (induct rule: `DPLL-ci.induct`)
case (1 Ms N)
have `snd (DPLL-step (Ms, N)) = N` **by** auto

then obtain Ms' where $Ms': DPLL\text{-}step (Ms, N) = (Ms', N)$ by $(cases\ DPLL\text{-}step (Ms, N))\ auto$
have $inv': dpll_W\text{-}all\text{-}inv (toS\ Ms'\ N)$ by $(metis\ (mono\text{-}tags)\ 1.prem\ DPLL\text{-}step\text{-}is\text{-}a\text{-}dpll_W\text{-}step\ Ms'\ dpll_W\text{-}all\text{-}inv\ old.prod.inject)$
{ assume $(Ms', N) \neq (Ms, N)$
hence $DPLL\text{-}ci\ Ms'\ N = DPLL\text{-}part\ Ms'\ N \wedge DPLL\text{-}part\text{-}dom (Ms', N)$ using $1(1)[of - Ms'\ N]$
 Ms'
 $1(2)\ inv'$ by $auto$
hence $DPLL\text{-}part\text{-}dom (Ms, N)$ using $DPLL\text{-}part.domintros\ Ms'$ by $fastforce$
moreover have $DPLL\text{-}ci\ Ms\ N = DPLL\text{-}part\ Ms\ N$ using $1.prem\ DPLL\text{-}part.psims\ Ms'$
 $\langle DPLL\text{-}ci\ Ms'\ N = DPLL\text{-}part\ Ms'\ N \wedge DPLL\text{-}part\text{-}dom (Ms', N) \rangle \langle DPLL\text{-}part\text{-}dom (Ms, N) \rangle$ by
 $auto$
ultimately have $?case$ by $blast$
}
moreover {
assume $(Ms', N) = (Ms, N)$
hence $?case$ using $DPLL\text{-}part.domintros\ DPLL\text{-}part.psims\ Ms'$ by $fastforce$
}
ultimately show $?case$ by $blast$
qed

lemma $DPLL\text{-}ci\text{-}dpll_W\text{-}rtranclp$:

assumes $DPLL\text{-}ci\ Ms\ N = (Ms', N')$
shows $dpll_W^{} (toS\ Ms\ N) (toS\ Ms'\ N')$**
using $assms$

proof (*induct $Ms\ N$ arbitrary: $Ms'\ N'$ rule: $DPLL\text{-}ci.induct$*)

case $(1\ Ms\ N\ Ms'\ N')$ note $IH = this(1)$ and $step = this(2)$

obtain $S_1\ S_2$ where $S: (S_1, S_2) = DPLL\text{-}step (Ms, N)$ by $(cases\ DPLL\text{-}step (Ms, N))\ auto$

{ assume $\neg dpll_W\text{-}all\text{-}inv (toS\ Ms\ N)$
hence $(Ms, N) = (Ms', N)$ using $step$ by $auto$
hence $?case$ by $auto$
}

moreover

{ assume $dpll_W\text{-}all\text{-}inv (toS\ Ms\ N)$
and $(S_1, S_2) = (Ms, N)$
hence $?case$ using $S\ step$ by $auto$
}

moreover

{ assume $dpll_W\text{-}all\text{-}inv (toS\ Ms\ N)$
and $(S_1, S_2) \neq (Ms, N)$
moreover obtain $S_1'\ S_2'$ where $DPLL\text{-}ci\ S_1\ N = (S_1', S_2')$ by $(cases\ DPLL\text{-}ci\ S_1\ N)\ auto$
moreover have $DPLL\text{-}ci\ Ms\ N = DPLL\text{-}ci\ S_1\ N$ using $DPLL\text{-}ci.sims[of\ Ms\ N]\ calculation$
proof –

have $(case\ (S_1, S_2)\ of\ (ms, lss) \Rightarrow$
if $(ms, lss) = (Ms, N)$ then (Ms, N) else $DPLL\text{-}ci\ ms\ N = DPLL\text{-}ci\ Ms\ N$
using $S\ DPLL\text{-}ci.sims[of\ Ms\ N]\ calculation$ by $presburger$
hence $(if\ (S_1, S_2) = (Ms, N)\ then\ (Ms, N)\ else\ DPLL\text{-}ci\ S_1\ N) = DPLL\text{-}ci\ Ms\ N$
by $fastforce$
thus $?thesis$
using $calculation(2)$ by $presburger$

qed

ultimately have $dpll_W^{} (toS\ S_1'\ N) (toS\ Ms'\ N)$ using $IH[of\ (S_1, S_2)\ S_1\ S_2]\ S\ step$ by $simp$**

moreover have $dpll_W (toS\ Ms\ N) (toS\ S_1\ N)$

by $(metis\ DPLL\text{-}step\text{-}is\text{-}a\text{-}dpll_W\text{-}step\ S\ \langle (S_1, S_2) \neq (Ms, N) \rangle\ prod.sel(2)\ snd\text{-}DPLL\text{-}step)$

```

ultimately have ?case by (metis (mono-tags, hide-lams) IH S ⟨(S1, S2) ≠ (Ms, N)⟩
  ⟨DPLL-ci Ms N = DPLL-ci S1 N⟩ ⟨dpllW-all-inv (toS Ms N)⟩ converse-rtrancpl-into-rtrancpl
  local.step)
}
ultimately show ?case by blast
qed

```

lemma *dpll_W-all-inv-dpll_W-trancpl-irrefl*:

```

assumes dpllW-all-inv (Ms, N)
and dpllW++ (Ms, N) (Ms, N)
shows False

```

proof –

```

have 1: wf {(S', S). dpllW-all-inv S ∧ dpllW++ S S'} using dpllW-wf-trancpl by auto
have ((Ms, N), (Ms, N)) ∈ {(S', S). dpllW-all-inv S ∧ dpllW++ S S'} using assms by auto
thus False using wf-not-refl[OF 1] by blast

```

qed

lemma *DPLL-ci-final-state*:

```

assumes step: DPLL-ci Ms N = (Ms, N)
and inv: dpllW-all-inv (toS Ms N)
shows conclusive-dpllW-state (toS Ms N)

```

proof –

```

have st: dpllW** (toS Ms N) (toS Ms N) using DPLL-ci-dpllW-rtrancpl[OF step] .

```

```

have DPLL-step (Ms, N) = (Ms, N)

```

proof (rule ccontr)

```

obtain Ms' N' where Ms'N: (Ms', N') = DPLL-step (Ms, N)

```

```

by (cases DPLL-step (Ms, N)) auto

```

```

assume ¬ ?thesis

```

```

hence DPLL-ci Ms' N = (Ms, N) using step inv st Ms'N[symmetric] by fastforce

```

```

hence dpllW++ (toS Ms N) (toS Ms N)

```

```

by (metis DPLL-ci-dpllW-rtrancpl DPLL-step-is-a-dpllW-step Ms'N ⟨DPLL-step (Ms, N) ≠ (Ms,
N)⟩

```

```

  prod.sel(2) rtrancpl-into-trancpl2 snd-DPLL-step)

```

```

thus False using dpllW-all-inv-dpllW-trancpl-irrefl inv by auto

```

qed

```

thus ?thesis using DPLL-step-stuck-final-state[of Ms N] by simp

```

qed

lemma *DPLL-step-obtains*:

```

obtains Ms' where (Ms', N) = DPLL-step (Ms, N)

```

```

unfolding DPLL-step-def by (metis (no-types, lifting) DPLL-step-def prod.collapse snd-DPLL-step)

```

lemma *DPLL-ci-obtains*:

```

obtains Ms' where (Ms', N) = DPLL-ci Ms N

```

proof (induct rule: *DPLL-ci.induct*)

```

case (1 Ms N) note IH = this(1) and that = this(2)

```

```

obtain S where SN: (S, N) = DPLL-step (Ms, N) using DPLL-step-obtains by metis

```

```

{ assume ¬ dpllW-all-inv (toS Ms N)

```

```

  hence ?case using that by auto

```

```

}

```

moreover {

```

  assume n: (S, N) ≠ (Ms, N)

```

```

  and inv: dpllW-all-inv (toS Ms N)

```

```

  have ∃ ms. DPLL-step (Ms, N) = (ms, N)

```

```

  by (metis ⟨∧thesis⟩. (∧S. (S, N) = DPLL-step (Ms, N) ⇒ thesis) ⇒ thesis)

```


hence ?thesis
 using IH that by fastforce
 }
 moreover {
 assume n: (S, N) = (Ms, N)
 hence ?case using SN that by fastforce
 }
 ultimately show ?case by blast
 qed

lemma DPLL-ci-no-more-step:

assumes step: DPLL-ci Ms N = (Ms', N')
 shows DPLL-ci Ms' N' = (Ms', N')
 using assms

proof (induct arbitrary: Ms' N' rule: DPLL-ci.induct)

case (1 Ms N Ms' N') note IH = this(1) and step = this(2)

obtain S₁ where S: (S₁, N) = DPLL-step (Ms, N) using DPLL-step-obtains by auto

{ assume ¬dpll_W-all-inv (toS Ms N)

hence ?case using step by auto

}

moreover {

assume dpll_W-all-inv (toS Ms N)

and (S₁, N) = (Ms, N)

hence ?case using S step by auto

}

moreover

{ assume inv: dpll_W-all-inv (toS Ms N)

assume n: (S₁, N) ≠ (Ms, N)

obtain S₁' where SS: (S₁', N) = DPLL-ci S₁ N using DPLL-ci-obtains by blast

moreover have DPLL-ci Ms N = DPLL-ci S₁ N

proof –

have (case (S₁, N) of (ms, lss) ⇒ if (ms, lss) = (Ms, N) then (Ms, N) else DPLL-ci ms N)
 = DPLL-ci Ms N

using S DPLL-ci.simps[of Ms N] calculation inv by presburger

hence (if (S₁, N) = (Ms, N) then (Ms, N) else DPLL-ci S₁ N) = DPLL-ci Ms N

by fastforce

thus ?thesis

using calculation n by presburger

qed

moreover

have DPLL-ci S₁' N = (S₁', N) using step IH[OF - - S n SS[symmetric]] inv by blast

ultimately have ?case using step by fastforce

}

ultimately show ?case by blast

qed

lemma DPLL-part-dpll_W-all-inv-final:

fixes M Ms': (int, unit, unit) marked-lit list and

N :: int literal list list

assumes inv: dpll_W-all-inv (Ms, mset (map mset N))

and MsN: DPLL-part Ms N = (Ms', N)

shows conclusive-dpll_W-state (toS Ms' N) ∧ dpll_W** (toS Ms N) (toS Ms' N)

proof –

have 2: $DPLL\text{-}ci\ Ms\ N = DPLL\text{-}part\ Ms\ N$ **using** $inv\ dpll_W\text{-}all\text{-}inv\text{-}implyS\text{-}2\text{-}eq3\text{-}and\text{-}dom$ **by** *blast*
hence $star: dpll_W^{**}\ (toS\ Ms\ N)\ (toS\ Ms'\ N)$ **unfolding** MsN **using** $DPLL\text{-}ci\text{-}dpll_W\text{-}rtranclp$ **by** *blast*
hence $inv': dpll_W\text{-}all\text{-}inv\ (toS\ Ms'\ N)$ **using** $inv\ rtranclp\text{-}dpll_W\text{-}all\text{-}inv$ **by** *blast*
show $?thesis$ **using** $star\ DPLL\text{-}ci\text{-}final\text{-}state[OF\ DPLL\text{-}ci\text{-}no\text{-}more\text{-}step\ inv']\ 2$ **unfolding** MsN **by** *blast*
qed

Embedding the invariant into the type

Defining the type `typedef dpllW-state =`

`{(M::(int, unit, unit) marked-lit list, N::int literal list list).`

`dpllW-all-inv (toS M N)}`

`morphisms rough-state-of state-of`

proof

show $([], []) \in \{(M, N). dpll_W\text{-}all\text{-}inv\ (toS\ M\ N)\}$ **by** (*auto simp add: dpll_W-all-inv-def*)

qed

lemma

$DPLL\text{-}part\text{-}dom\ ([],\ N)$

using $assms\ dpll_W\text{-}all\text{-}inv\text{-}implyS\text{-}2\text{-}eq3\text{-}and\text{-}dom[of\ [],\ N]$ **by** (*simp add: dpll_W-all-inv-def*)

Some type classes **instantiation** $dpll_W\text{-}state :: equal$

begin

definition $equal\text{-}dpll_W\text{-}state :: dpll_W\text{-}state \Rightarrow dpll_W\text{-}state \Rightarrow bool$ **where**

$equal\text{-}dpll_W\text{-}state\ S\ S' = (rough\text{-}state\text{-}of\ S = rough\text{-}state\text{-}of\ S')$

instance

by *standard (simp add: rough-state-of-inject equal-dpll_W-state-def)*

end

DPLL **definition** $DPLL\text{-}step' :: dpll_W\text{-}state \Rightarrow dpll_W\text{-}state$ **where**

$DPLL\text{-}step'\ S = state\text{-}of\ (DPLL\text{-}step\ (rough\text{-}state\text{-}of\ S))$

declare $rough\text{-}state\text{-}of\text{-}inverse[simp]$

lemma $DPLL\text{-}step\text{-}dpll_W\text{-}conc\text{-}inv:$

$DPLL\text{-}step\ (rough\text{-}state\text{-}of\ S) \in \{(M, N). dpll_W\text{-}all\text{-}inv\ (toS\ M\ N)\}$

by (*smt DPLL-ci.simps DPLL-ci-dpll_W-rtranclp case-prodE case-prodI2 rough-state-of mem-Collect-eq old.prod.case prod.sel(2) rtranclp-dpll_W-all-inv snd-DPLL-step*)

lemma $rough\text{-}state\text{-}of\text{-}DPLL\text{-}step'\text{-}DPLL\text{-}step[simp]:$

$rough\text{-}state\text{-}of\ (DPLL\text{-}step'\ S) = DPLL\text{-}step\ (rough\text{-}state\text{-}of\ S)$

using $DPLL\text{-}step\text{-}dpll_W\text{-}conc\text{-}inv\ DPLL\text{-}step'\text{-}def\ state\text{-}of\text{-}inverse$ **by** *auto*

function $DPLL\text{-}tot :: dpll_W\text{-}state \Rightarrow dpll_W\text{-}state$ **where**

$DPLL\text{-}tot\ S =$

$(let\ S' = DPLL\text{-}step'\ S\ in$

$if\ S' = S\ then\ S\ else\ DPLL\text{-}tot\ S')$

by *fast+*

termination

proof (*relation* $\{(T', T).$

$(rough\text{-}state\text{-}of\ T', rough\text{-}state\text{-}of\ T)$

$\in \{(S', S). (toS'\ S', toS'\ S)$

$\in \{(S', S). dpll_W\text{-}all\text{-}inv\ S \wedge dpll_W\ S\ S'\}\}$)

show *wf* $\{(b, a).$

```

      (rough-state-of b, rough-state-of a)
      ∈ {(b, a). (toS' b, toS' a)}
      ∈ {(b, a). dpllW-all-inv a ∧ dpllW a b}}
    using wf-if-measure-f[OF wf-if-measure-f[OF dpllW-wf, of toS'], of rough-state-of] .
next
fix S x
assume x: x = DPLL-step' S
and x ≠ S
have dpllW-all-inv (case rough-state-of S of (Ms, N) ⇒ (Ms, mset (map mset N)))
  by (metis (no-types, lifting) case-prodE mem-Collect-eq old.prod.case rough-state-of)
moreover have dpllW (case rough-state-of S of (Ms, N) ⇒ (Ms, mset (map mset N)))
  (case rough-state-of (DPLL-step' S) of (Ms, N) ⇒ (Ms, mset (map mset N)))
proof -
  obtain Ms N where Ms: (Ms, N) = rough-state-of S by (cases rough-state-of S) auto
  have dpllW-all-inv (toS' (Ms, N)) using calculation unfolding Ms by blast
  moreover obtain Ms' N' where Ms': (Ms', N') = rough-state-of (DPLL-step' S)
    by (cases rough-state-of (DPLL-step' S)) auto
  ultimately have dpllW-all-inv (toS' (Ms', N')) unfolding Ms'
    by (metis (no-types, lifting) case-prod-unfold mem-Collect-eq rough-state-of)

  have dpllW (toS Ms N) (toS Ms' N')
    apply (rule DPLL-step-is-a-dpllW-step[of Ms' N' Ms N])
    unfolding Ms Ms' using ⟨x ≠ S⟩ rough-state-of-inject x by fastforce+
    thus ?thesis unfolding Ms[symmetric] Ms'[symmetric] by auto
qed
ultimately show (x, S) ∈ {(T', T). (rough-state-of T', rough-state-of T)}
  ∈ {(S', S). (toS' S', toS' S) ∈ {(S', S). dpllW-all-inv S ∧ dpllW S S'}}}
  by (auto simp add: x)
qed

lemma [code]:
DPLL-tot S =
  (let S' = DPLL-step' S in
   if S' = S then S else DPLL-tot S') by auto

lemma DPLL-tot-DPLL-step-DPLL-tot[simp]: DPLL-tot (DPLL-step' S) = DPLL-tot S
  apply (cases DPLL-step' S = S)
  apply simp
  unfolding DPLL-tot.simps[of S] by (simp del: DPLL-tot.simps)

lemma DOPLL-step'-DPLL-tot[simp]:
DPLL-step' (DPLL-tot S) = DPLL-tot S
  by (rule DPLL-tot.induct[of λS. DPLL-step' (DPLL-tot S) = DPLL-tot S S])
  (metis (full-types) DPLL-tot.simps)

lemma DPLL-tot-final-state:
  assumes DPLL-tot S = S
  shows conclusive-dpllW-state (toS' (rough-state-of S))
proof -
  have DPLL-step' S = S using assms[symmetric] DOPLL-step'-DPLL-tot by metis
  hence DPLL-step (rough-state-of S) = (rough-state-of S)
    unfolding DPLL-step'-def using DPLL-step-dpllW-conc-inv rough-state-of-inverse
    by (metis rough-state-of-DPLL-step'-DPLL-step)

```

```

thus ?thesis
  by (metis (mono-tags, lifting) DPLL-step-stuck-final-state old.prod.exhaust split-conv)
qed

lemma DPLL-tot-star:
  assumes rough-state-of (DPLL-tot S) = S'
  shows dpllW** (toS' (rough-state-of S)) (toS' S')
  using assms
proof (induction arbitrary: S' rule: DPLL-tot.induct)
  case (1 S S')
  let ?x = DPLL-step' S
  { assume ?x = S
    then have ?case using 1(2) by simp
  }
  moreover {
    assume S: ?x ≠ S
    have ?case
    apply (cases DPLL-step' S = S)
    using S apply blast
    by (smt 1.IH 1.prem DPLL-step-is-a-dpllW-step DPLL-tot.simps case-prodE2
      rough-state-of-DPLL-step'-DPLL-step rtranclp.rtrancl-into-rtrancl rtranclp.rtrancl-refl
      rtranclp-idemp split-conv)
  }
  ultimately show ?case by auto
qed

```

```

lemma rough-state-of-rough-state-of-nil[simp]:
  rough-state-of (state-of ([], N)) = ([], N)
  apply (rule DPLL-W-Implementation.dpllW-state.state-of-inverse)
  unfolding dpllW-all-inv-def by auto

```

Theorem of correctness

```

lemma DPLL-tot-correct:
  assumes rough-state-of (DPLL-tot (state-of ([], N))) = (M, N')
  and (M', N'') = toS' (M, N')
  shows M' ⊨asm N'' ↔ satisfiable (set-mset N'')
proof –
  have dpllW** (toS' ([], N)) (toS' (M, N')) using DPLL-tot-star[OF assms(1)] by auto
  moreover have conclusive-dpllW-state (toS' (M, N'))
    using DPLL-tot-final-state by (metis (mono-tags, lifting) DPLL-step'-DPLL-tot DPLL-tot.simps
      assms(1))
  ultimately show ?thesis using dpllW-conclusive-state-correct by (smt DPLL-ci.simps
    DPLL-ci-dpllW-rtranclp assms(2) dpllW-all-inv-def prod.case prod.sel(1) prod.sel(2)
    rtranclp-dpllW-inv(3) rtranclp-dpllW-inv-starting-from-0)
qed

```

20.2.3 Code export

A conversion to DPLL-W-Implementation.dpll_W-state **definition** Con :: (int, unit, unit) marked-lit list × int literal list list
 \Rightarrow dpll_W-state **where**
 Con xs = state-of (if dpll_W-all-inv (toS (fst xs) (snd xs)) then xs else ([], []))

```

lemma [code abstype]:
  Con (rough-state-of S) = S
  using rough-state-of[of S] unfolding Con-def by auto

```

declare *rough-state-of-DPLL-step'-DPLL-step*[code abstract]

lemma *Con-DPLL-step-rough-state-of-state-of*[simp]:

Con (DPLL-step (rough-state-of s)) = state-of (DPLL-step (rough-state-of s))

unfolding *Con-def* **by** (*metis* (*mono-tags*, *lifting*) *DPLL-step-dpll_W-conc-inv mem-Collect-eq prod.case-eq-if*)

A slightly different version of *DPLL-tot* where the returned boolean indicates the result.

definition *DPLL-tot-rep* **where**

DPLL-tot-rep *S* =

(*let* (*M*, *N*) = (*rough-state-of* (*DPLL-tot* *S*)) *in* ($\forall A \in \text{set } N. (\exists a \in \text{set } A. a \in \text{lits-of-l } (M)), M$))

One version of the generated SML code is here, but not included in the generated document. The only differences are:

- export *'a literal* from the SML Module *Clausal-Logic*;
- export the constructor *Con* from *DPLL-W-Implementation*;
- export the *int* constructor from *Arith*.

All these allows to test on the code on some examples.

end

theory *CDCL-W-Implementation*

imports *DPLL-CDCL-W-Implementation CDCL-W-Termination*

begin

notation *image-mset* (**infixr** *'#* 90)

type-synonym *'a cdcl_W-mark* = *'a literal list*

type-synonym *cdcl_W-marked-level* = *nat*

type-synonym *'v cdcl_W-marked-lit* = (*'v*, *cdcl_W-marked-level*, *'v cdcl_W-mark*) *marked-lit*

type-synonym *'v cdcl_W-marked-lits* = (*'v*, *cdcl_W-marked-level*, *'v cdcl_W-mark*) *marked-lits*

type-synonym *'v cdcl_W-state* =

'v cdcl_W-marked-lits \times *'v literal list list* \times *'v literal list list* \times *nat* \times

'v literal list option

abbreviation *raw-trail* :: *'a* \times *'b* \times *'c* \times *'d* \times *'e* \Rightarrow *'a* **where**

raw-trail $\equiv (\lambda(M, -). M)$

abbreviation *raw-cons-trail* :: *'a* \Rightarrow *'a list* \times *'b* \times *'c* \times *'d* \times *'e* \Rightarrow *'a list* \times *'b* \times *'c* \times *'d* \times *'e*

where

raw-cons-trail $\equiv (\lambda L (M, S). (L \# M, S))$

abbreviation *raw-tl-trail* :: *'a list* \times *'b* \times *'c* \times *'d* \times *'e* \Rightarrow *'a list* \times *'b* \times *'c* \times *'d* \times *'e* **where**

raw-tl-trail $\equiv (\lambda(M, S). (tl M, S))$

abbreviation *raw-init-clss* :: *'a* \times *'b* \times *'c* \times *'d* \times *'e* \Rightarrow *'b* **where**

raw-init-clss $\equiv \lambda(M, N, -). N$

abbreviation *raw-learned-clss* :: *'a* \times *'b* \times *'c* \times *'d* \times *'e* \Rightarrow *'c* **where**

raw-learned-clss $\equiv \lambda(M, N, U, -). U$

abbreviation *raw-backtrack-lvl* :: *'a* \times *'b* \times *'c* \times *'d* \times *'e* \Rightarrow *'d* **where**

raw-backtrack-lvl $\equiv \lambda(M, N, U, k, -). k$

abbreviation *raw-update-backtrack-lvl* $:: 'd \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e$
where

raw-update-backtrack-lvl $\equiv \lambda k (M, N, U, -, S). (M, N, U, k, S)$

abbreviation *raw-conflicting* $:: 'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'e$ **where**

raw-conflicting $\equiv \lambda(M, N, U, k, D). D$

abbreviation *raw-update-conflicting* $:: 'e \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e$
where

raw-update-conflicting $\equiv \lambda S (M, N, U, k, -). (M, N, U, k, S)$

abbreviation *raw-add-learned-cl* **where**

raw-add-learned-cl $\equiv \lambda C (M, N, U, S). (M, N, \{\#C\# \} + U, S)$

abbreviation *raw-remove-cl* **where**

raw-remove-cl $\equiv \lambda C (M, N, U, S). (M, \text{removeAll-mset } C \ N, \text{removeAll-mset } C \ U, S)$

type-synonym *'v cdcl_W-state-inv-st* = (*'v*, *nat*, *'v literal list*) *marked-lit list* \times
'v literal list list \times *'v literal list list* \times *nat* \times *'v literal list option*

abbreviation *raw-S0-cdcl_W* *N* $\equiv (([], N, [], 0, \text{None}) :: 'v \text{ cdcl}_W\text{-state-inv-st})$

fun *mmset-of-mlit'* $:: ('v, \text{nat}, 'v \text{ literal list}) \text{ marked-lit} \Rightarrow ('v, \text{nat}, 'v \text{ clause}) \text{ marked-lit}$
where

mmset-of-mlit' (*Propagated L C*) = *Propagated L (mset C)* |

mmset-of-mlit' (*Marked L i*) = *Marked L i*

lemma *lit-of-mmset-of-mlit'*[*simp*]:

lit-of (mmset-of-mlit' xa) = *lit-of xa*

by (*induction xa*) *auto*

abbreviation *trail* **where**

trail S $\equiv \text{map } \text{mmset-of-mlit}' (\text{raw-trail } S)$

abbreviation *clauses-of-l* **where**

clauses-of-l $\equiv \lambda L. \text{mset } (\text{map } \text{mset } L)$

global-interpretation *state_W-ops*

mset $:: 'v \text{ literal list} \Rightarrow 'v \text{ clause}$

op $\# \text{remove1}$

clauses-of-l op @ $\lambda L \ C. L \in \text{set } C \text{ op} \# \lambda C. \text{remove1-cond } (\lambda L. \text{mset } L = \text{mset } C)$

mset $\lambda xs \ ys. \text{case-prod append } (\text{fold } (\lambda x \ (ys, zs). (\text{remove1 } x \ ys, x \# \ zs)) \ xs \ (ys, []))$

op $\# \text{remove1}$

id id

$\lambda(M, -). \text{map } \text{mmset-of-mlit}' \ M \ \lambda(M, -). \text{hd } M$

$\lambda(-, N, -). N$

$\lambda(-, -, U, -). U$

$\lambda(-, -, -, k, -). k$

$\lambda(-, -, -, -, C). C$

$\lambda L (M, S). (L \# M, S)$
 $\lambda(M, S). (tl\ M, S)$
 $\lambda C (M, N, S). (M, C \# N, S)$
 $\lambda C (M, N, U, S). (M, N, C \# U, S)$
 $\lambda C (M, N, U, S). (M, filter\ (\lambda L. mset\ L \neq mset\ C)\ N, filter\ (\lambda L. mset\ L \neq mset\ C)\ U, S)$
 $\lambda(k::nat)\ (M, N, U, -, D). (M, N, U, k, D)$
 $\lambda D (M, N, U, k, -). (M, N, U, k, D)$
 $\lambda N. ([], N, [], 0, None)$
 $\lambda(-, N, U, -). ([], N, U, 0, None)$
apply *unfold-locales* **by** (*auto simp: hd-map comp-def map-tl ac-simps union-mset-list mset-map-mset-remove1-cond ex-mset*)

lemma *mmset-of-mlit'-mmset-of-mlit*: *mmset-of-mlit' l = mmset-of-mlit l*
apply (*induct l*)
apply *auto*
done

lemma *clauses-of-l-filter-removeAll*:
clauses-of-l [L ← a . mset L ≠ mset C] = mset (removeAll (mset C) (map mset a))
by (*induct a*) *auto*

interpretation *state_w*
mset::'v literal list ⇒ 'v clause
op # remove1

clauses-of-l op @ λL C. L ∈ set C op # λC. remove1-cond (λL. mset L = mset C)

mset λxs ys. case-prod append (fold (λx (ys, zs). (remove1 x ys, x # zs)) xs (ys, []))
op # remove1

id id

$\lambda(M, -). map\ mmset-of-mlit'\ M\ \lambda(M, -). hd\ M$
 $\lambda(-, N, -). N$
 $\lambda(-, -, U, -). U$
 $\lambda(-, -, -, k, -). k$
 $\lambda(-, -, -, -, C). C$

$\lambda L (M, S). (L \# M, S)$
 $\lambda(M, S). (tl\ M, S)$
 $\lambda C (M, N, S). (M, C \# N, S)$
 $\lambda C (M, N, U, S). (M, N, C \# U, S)$
 $\lambda C (M, N, U, S). (M, filter\ (\lambda L. mset\ L \neq mset\ C)\ N, filter\ (\lambda L. mset\ L \neq mset\ C)\ U, S)$
 $\lambda(k::nat)\ (M, N, U, -, D). (M, N, U, k, D)$
 $\lambda D (M, N, U, k, -). (M, N, U, k, D)$
 $\lambda N. ([], N, [], 0, None)$
 $\lambda(-, N, U, -). ([], N, U, 0, None)$
apply *unfold-locales*
apply (*rename-tac S, case-tac S*)
by (*auto simp: hd-map comp-def map-tl ac-simps clauses-of-l-filter-removeAll mmset-of-mlit'-mmset-of-mlit*)

global-interpretation *conflict-driven-clause-learning_w*
mset::'v literal list ⇒ 'v clause

op # remove1

clauses-of-l op @ λL C. L ∈ set C op # λC. remove1-cond (λL. mset L = mset C)

mset λxs ys. case-prod append (fold (λx (ys, zs). (remove1 x ys, x # zs)) xs (ys, []))
op # remove1

id id

λ(M, -). map mmset-of-mlit' M λ(M, -). hd M

λ(-, N, -). N

λ(-, -, U, -). U

λ(-, -, -, k, -). k

λ(-, -, -, -, C). C

λL (M, S). (L # M, S)

λ(M, S). (tl M, S)

λC (M, N, S). (M, C # N, S)

λC (M, N, U, S). (M, N, C # U, S)

λC (M, N, U, S). (M, filter (λL. mset L ≠ mset C) N, filter (λL. mset L ≠ mset C) U, S)

λ(k::nat) (M, N, U, -, D). (M, N, U, k, D)

λD (M, N, U, k, -). (M, N, U, k, D)

λN. ([], N, [], 0, None)

λ(-, N, U, -). ([], N, U, 0, None)

by *intro-locales*

declare *state-simp[simp del] raw-clauses-def[simp] state-eq-def[simp]*

notation *state-eq* (**infix** \sim 50)

term *reduce-trail-to*

lemma *reduce-trail-to-map[simp]:*

reduce-trail-to (map f M1) = reduce-trail-to M1

by (*rule ext*) (*auto intro: reduce-trail-to-length*)

20.3 CDCL Implementation

20.3.1 Definition of the rules

Types **lemma** *true-clss-remdups[simp]:*

I ⊨_s (mset ∘ remdups) ' N ⟷ I ⊨_s mset ' N

by (*simp add: true-clss-def*)

lemma *satisfiable-mset-remdups[simp]:*

satisfiable ((mset ∘ remdups) ' N) ⟷ satisfiable (mset ' N)

unfolding *satisfiable-carac[symmetric]* **by** *simp*

We need some functions to convert between our abstract state *nat cdcl_W-state* and the concrete state *'v cdcl_W-state-inv-st*.

abbreviation *convertC* :: *'a list option ⇒ 'a multiset option* **where**

convertC ≡ *map-option mset*

lemma *convert-Propagated[elim!]:*

mmset-of-mlit' z = Propagated L C ⟹ (∃ C'. z = Propagated L C' ∧ C = mset C')

by (*cases z*) *auto*

lemma *get-rev-level-map-convert:*

get-rev-level (map mmset-of-mlit' M) n x = *get-rev-level* M n x
by (induction M arbitrary: n rule: marked-lit-list-induct) auto

lemma *get-level-map-convert*[simp]:
get-level (map mmset-of-mlit' M) = *get-level* M
using *get-rev-level-map-convert*[of rev M] **by** (simp add: rev-map)

lemma *get-rev-level-map-mmsetof-mlit*[simp]:
get-rev-level (map mmset-of-mlit M) = *get-rev-level* M
by (induction M rule: marked-lit-list-induct) (auto intro!: ext)

lemma *get-level-map-mmsetof-mlit*[simp]:
get-level (map mmset-of-mlit M) = *get-level* M
using *get-rev-level-map-mmsetof-mlit*[of rev M] **unfolding** rev-map **by** simp

lemma *get-maximum-level-map-convert*[simp]:
get-maximum-level (map mmset-of-mlit' M) D = *get-maximum-level* M D
by (induction D) (auto simp add: get-maximum-level-plus)

lemma *get-all-levels-of-marked-map-convert*[simp]:
get-all-levels-of-marked (map mmset-of-mlit' M) = (*get-all-levels-of-marked* M)
by (induction M rule: marked-lit-list-induct) auto

lemma *reduce-trail-to-empty-trail*[simp]:
reduce-trail-to F ([], aa, ab, ac, b) = ([], aa, ab, ac, b)
using *reduce-trail-to.simps* **by** auto

lemma *raw-trail-reduce-trail-to-length-le*:
assumes length F > length (raw-trail S)
shows raw-trail (*reduce-trail-to* F S) = []
using *assms* *trail-reduce-trail-to-length-le*[of S F]
by (cases S, cases *reduce-trail-to* F S) auto

lemma *reduce-trail-to*:
reduce-trail-to F S =
 ((if length (raw-trail S) ≥ length F
 then drop (length (raw-trail S) - length F) (raw-trail S)
 else []), raw-init-clss S, raw-learned-clss S, raw-backtrack-lvl S, raw-conflicting S)
 (is ?S = -)

proof (induction F S rule: *reduce-trail-to.induct*)
case (1 F S) **note** IH = this
show ?case
proof (cases raw-trail S)
case Nil
then show ?thesis **using** IH **by** (cases S) auto
next
case (Cons L M)
then show ?thesis
apply (cases Suc (length M) > length F)
prefer 2 **using** IH *reduce-trail-to-length-ne*[of S F] **apply** (cases S) **apply** auto[]
apply (subgoal-tac Suc (length M) - length F = Suc (length M - length F))
using *reduce-trail-to-length-ne*[of S F] IH **by** (cases S) (auto simp add:)
qed
qed

Definition an abstract type

```

typedef 'v cdclW-state-inv = {S::'v cdclW-state-inv-st. cdclW-all-struct-inv S}
morphisms rough-state-of state-of
proof
  show ([], [], [], 0, None) ∈ {S. cdclW-all-struct-inv S}
  by (auto simp add: cdclW-all-struct-inv-def)
qed

instantiation cdclW-state-inv :: (type) equal
begin
definition equal-cdclW-state-inv :: 'v cdclW-state-inv ⇒ 'v cdclW-state-inv ⇒ bool where
  equal-cdclW-state-inv S S' = (rough-state-of S = rough-state-of S')
instance
  by standard (simp add: rough-state-of-inject equal-cdclW-state-inv-def)
end

lemma lits-of-map-convert[simp]: lits-of-l (map mmset-of-mlit' M) = lits-of-l M
  by (induction M rule: marked-lit-list-induct) simp-all

lemma undefined-lit-map-convert[iff]:
  undefined-lit (map mmset-of-mlit' M) L ⟷ undefined-lit M L
  by (auto simp add: defined-lit-map image-image mmset-of-mlit'-mmset-of-mlit)

lemma true-annot-map-convert[simp]: map mmset-of-mlit' M ⊨a N ⟷ M ⊨a N
  by (induction M rule: marked-lit-list-induct) (simp-all add: true-annot-def
    mmset-of-mlit'-mmset-of-mlit lits-of-def)

lemma true-annots-map-convert[simp]: map mmset-of-mlit' M ⊨as N ⟷ M ⊨as N
  unfolding true-annots-def by auto

lemmas propagateE
lemma find-first-unit-clause-some-is-propagate:
  assumes H: find-first-unit-clause (N @ U) M = Some (L, C)
  shows propagate (M, N, U, k, None) (Propagated L C # M, N, U, k, None)
  using assms
  by (auto dest!: find-first-unit-clause-some intro!: propagate-rule)

```

20.3.2 The Transitions

Propagate **definition** do-propagate-step **where**

```

do-propagate-step S =
  (case S of
    (M, N, U, k, None) ⇒
      (case find-first-unit-clause (N @ U) M of
        Some (L, C) ⇒ (Propagated L C # M, N, U, k, None)
        | None ⇒ (M, N, U, k, None))
  | S ⇒ S)

```

```

lemma do-propagate-step:
  do-propagate-step S ≠ S ⟹ propagate S (do-propagate-step S)
apply (cases S, cases conflicting S)
using find-first-unit-clause-some-is-propagate[of raw-init-clss S raw-learned-clss S]
by (auto simp add: do-propagate-step-def split: option.splits)

```

```

lemma do-propagate-step-option[simp]:
  conflicting S ≠ None ⟹ do-propagate-step S = S
unfolding do-propagate-step-def by (cases S, cases conflicting S) auto

```

thm *prod-cases*

lemma *do-propagate-step-no-step*:

assumes *dist*: $\forall c \in \text{set } (\text{raw-clauses } S). \text{ distinct } c$ **and**

prop-step: $\text{do-propagate-step } S = S$

shows *no-step propagate S*

proof (*standard, standard*)

fix *T*

assume *propagate S T*

then obtain *C L* **where**

toSS: *conflicting S = None* **and**

C: $C \in \text{set } (\text{raw-clauses } S)$ **and**

L: $L \in \text{set } C$ **and**

MC: $\text{raw-trail } S \models_{\text{as}} C \text{Not } (\text{mset } (\text{remove1 } L \ C))$ **and**

T: $T \sim \text{raw-cons-trail } (\text{Propagated } L \ C) \ S$ **and**

undef: *undefined-lit (raw-trail S) L*

apply (*cases S rule: prod-cases5*)

by (*elim propagateE*) *simp*

let *?M* = *raw-trail S*

let *?N* = *raw-init-clss S*

let *?U* = *raw-learned-clss S*

let *?k* = *raw-backtrack-lvl S*

let *?D* = *None*

have *S*: $S = (?M, ?N, ?U, ?k, ?D)$

using *toSS* **by** (*cases S, cases conflicting S*) *simp-all*

have *find-first-unit-clause (?N @ ?U) ?M* $\neq \text{None}$

apply (*rule dist find-first-unit-clause-none[of C ?N @ ?U ?M L, OF -]*)

using *C dist apply auto[]*

using *C apply auto[1]*

using *MC apply auto[1]*

using *undef apply auto[1]*

using *L by auto*

then show *False* **using** *prop-step S unfolding do-propagate-step-def* **by** (*cases S*) *auto*

qed

Conflict **fun** *find-conflict* **where**

find-conflict M [] = *None* |

find-conflict M (N # Ns) = (*if* ($\forall c \in \text{set } N. -c \in \text{lits-of-l } M$) *then Some N* *else find-conflict M Ns*)

lemma *find-conflict-Some*:

find-conflict M Ns = *Some N* $\implies N \in \text{set } Ns \wedge M \models_{\text{as}} C \text{Not } (\text{mset } N)$

by (*induction Ns rule: find-conflict.induct*)

(*auto split: if-split-asm*)

lemma *find-conflict-None*:

find-conflict M Ns = *None* $\longleftrightarrow (\forall N \in \text{set } Ns. \neg M \models_{\text{as}} C \text{Not } (\text{mset } N))$

by (*induction Ns*) *auto*

lemma *find-conflict-None-no-confl*:

find-conflict M (N @ U) = *None* $\longleftrightarrow \text{no-step conflict } (M, N, U, k, \text{None})$

by (*auto simp add: find-conflict-None conflict.simps*)

definition *do-conflict-step* **where**

do-conflict-step S =

```

(case S of
  (M, N, U, k, None) ⇒
    (case find-conflict M (N @ U) of
      Some a ⇒ (M, N, U, k, Some a)
      | None ⇒ (M, N, U, k, None))
| S ⇒ S)

```

lemma *do-conflict-step*:
do-conflict-step S ≠ S ⇒ conflict S (do-conflict-step S)
apply (cases S, cases conflicting S)
unfolding *conflict.simps do-conflict-step-def*
by (auto dest!: *find-conflict-Some split: option.splits simp: state-eq-def*)

lemma *do-conflict-step-no-step*:
do-conflict-step S = S ⇒ no-step conflict S
apply (cases S, cases conflicting S)
unfolding *do-conflict-step-def*
using *find-conflict-None-no-conf*[of raw-trail S raw-init-clss S raw-learned-clss S
raw-backtrack-lvl S]
by (auto split: *option.split elim: conflictE*)

lemma *do-conflict-step-option[simp]*:
conflicting S ≠ None ⇒ do-conflict-step S = S
unfolding *do-conflict-step-def* **by** (cases S, cases conflicting S) auto

lemma *do-conflict-step-conflicting[dest]*:
do-conflict-step S ≠ S ⇒ conflicting (do-conflict-step S) ≠ None
unfolding *do-conflict-step-def* **by** (cases S, cases conflicting S) (auto split: *option.splits*)

definition *do-cp-step* **where**
do-cp-step S =
(*do-propagate-step o do-conflict-step*) S

lemma *cp-step-is-cdcl_W-cp*:
assumes *H: do-cp-step S ≠ S*
shows *cdcl_W-cp S (do-cp-step S)*
proof –
show *?thesis*
proof (cases *do-conflict-step S ≠ S*)
case *True*
then have *do-propagate-step (do-conflict-step S) = do-conflict-step S*
by *auto*
then show *?thesis*
by (auto simp add: *do-conflict-step do-conflict-step-conflicting do-cp-step-def True*)
next
case *False*
then have *confl[simp]: do-conflict-step S = S* **by** *simp*
show *?thesis*
proof (cases *do-propagate-step S = S*)
case *True*
then show *?thesis*
using *H* **by** (*simp add: do-cp-step-def*)
next
case *False*
let *?S = S*

```

    let ?T = (do-propagate-step S)
    let ?U = (do-conflict-step (do-propagate-step S))
    have propa: propagate S ?T using False do-propagate-step by blast
    moreover have ns: no-step conflict S using confl do-conflict-step-no-step by blast
    ultimately show ?thesis
      using cdclW-cp.intros(2)[of ?S ?T] confl unfolding do-cp-step-def by auto
  qed
qed
qed

lemma do-cp-step-eq-no-prop-no-confl:
  do-cp-step S = S  $\implies$  do-conflict-step S = S  $\wedge$  do-propagate-step S = S
  by (cases S, cases raw-conflicting S)
  (auto simp add: do-conflict-step-def do-propagate-step-def do-cp-step-def split: option.splits)

lemma no-cdclW-cp-iff-no-propagate-no-conflict:
  no-step cdclW-cp S  $\longleftrightarrow$  no-step propagate S  $\wedge$  no-step conflict S
  by (auto simp: cdclW-cp.simps)

lemma do-cp-step-eq-no-step:
  assumes
    H: do-cp-step S = S and
     $\forall c \in \text{set } (\text{raw-init-clss } S @ \text{raw-learned-clss } S). \text{ distinct } c$ 
  shows no-step cdclW-cp S
  unfolding no-cdclW-cp-iff-no-propagate-no-conflict
  using assms apply (cases S, cases conflicting S)
  using do-propagate-step-no-step[of S]
  by (auto dest!: do-cp-step-eq-no-prop-no-confl[simplified] do-conflict-step-no-step
    split: option.splits)

lemma cdclW-cp-cdclW-st: cdclW-cp S S'  $\implies$  cdclW** S S'
  by (simp add: cdclW-cp-tranclp-cdclW tranclp-into-rtranclp)

lemma cdclW-all-struct-inv-rough-state[simp]: cdclW-all-struct-inv (rough-state-of S)
  using rough-state-of by auto

lemma [simp]: cdclW-all-struct-inv S  $\implies$  rough-state-of (state-of S) = S
  by (simp add: state-of-inverse)

lemma rough-state-of-state-of-do-cp-step[simp]:
  rough-state-of (state-of (do-cp-step (rough-state-of S))) = do-cp-step (rough-state-of S)
proof -
  have cdclW-all-struct-inv (do-cp-step (rough-state-of S))
  apply (cases do-cp-step (rough-state-of S) = (rough-state-of S))
  apply simp
  using cp-step-is-cdclW-cp[of rough-state-of S] cdclW-all-struct-inv-rough-state[of S]
  cdclW-cp-cdclW-st rtranclp-cdclW-all-struct-inv-inv by blast
  then show ?thesis by auto
qed

Skip fun do-skip-step :: 'v cdclW-state-inv-st  $\Rightarrow$  'v cdclW-state-inv-st where
do-skip-step (Propagated L C # Ls, N, U, k, Some D) =
  (if  $\neg L \in \text{set } D \wedge D \neq []$ 
  then (Ls, N, U, k, Some D)
  else (Propagated L C # Ls, N, U, k, Some D)) |

```

do-skip-step $S = S$

lemma *do-skip-step*:

do-skip-step $S \neq S \implies \text{skip } S$ (*do-skip-step* *induct*)
apply (*induction* S *rule*: *do-skip-step.induct*)
by (*auto simp add*: *skip.simps*)

lemma *do-skip-step-no*:

do-skip-step $S = S \implies \text{no-step skip } S$
by (*induction* S *rule*: *do-skip-step.induct*)
(*auto simp add*: *other split*: *if-split-asm elim*!: *skipE*)

lemma *do-skip-step-trail-is-None*[*iff*]:

do-skip-step $S = (a, b, c, d, \text{None}) \longleftrightarrow S = (a, b, c, d, \text{None})$
by (*cases* S *rule*: *do-skip-step.cases*) *auto*

Resolve fun *maximum-level-code*:: '*a* literal list \Rightarrow ('*a*, nat, '*a* literal list) marked-lit list \Rightarrow nat
where

maximum-level-code [] = 0 |
maximum-level-code ($L \# Ls$) $M = \max$ (*get-level* M L) (*maximum-level-code* Ls M)

lemma *maximum-level-code-eq-get-maximum-level*[*code, simp*]:

maximum-level-code D $M = \text{get-maximum-level } M$ (*mset* D)
by (*induction* D) (*auto simp add*: *get-maximum-level-plus*)

fun *do-resolve-step* :: '*v* *cdcl_W*-state-inv-st \Rightarrow '*v* *cdcl_W*-state-inv-st **where**

do-resolve-step (*Propagated* L $C \# Ls$, N , U , k , *Some* D) =
(*if* $-L \in \text{set } D \wedge \text{maximum-level-code } (\text{remove1 } (-L) D) (\text{Propagated } L \ C \ \# \ Ls) = k$
then (Ls , N , U , k , *Some* (*remdups* (*remove1* L C @ *remove1* $(-L) D$)))
else (*Propagated* L $C \# Ls$, N , U , k , *Some* D)) |
do-resolve-step $S = S$

lemma *do-resolve-step*:

cdcl_W-all-struct-inv $S \implies \text{do-resolve-step } S \neq S$
 $\implies \text{resolve } S$ (*do-resolve-step* S)

proof (*induction* S *rule*: *do-resolve-step.induct*)

case ($1 \ L \ C \ M \ N \ U \ k \ D$)

then have

LD: $-L \in \text{set } D$ **and**

M: *maximum-level-code* (*remove1* $(-L) D$) (*Propagated* L $C \# M$) = k

by (*cases mset* $D - \{\#-L\} = \{\#\}$,

auto dest!: *get-maximum-level-exists-lit-of-max-level*[*of* - *Propagated* L $C \# M$]

split: *if-split-asm*) +

have *every-mark-is-a-conflict* (*Propagated* L $C \# M$, N , U , k , *Some* D)

using $1(1)$ **unfolding** *cdcl_W*-all-struct-inv-def *cdcl_W*-conflicting-def **by** *fast*

then have *LC*: $L \in \text{set } C$ **by** *fastforce*

then obtain C' **where** C : *mset* $C = C' + \{\#L\}$

by (*metis add.commute in-multiset-in-set insert-DiffM*)

obtain D' **where** D : *mset* $D = D' + \{\#-L\}$

using $\langle -L \in \text{set } D \rangle$ **by** (*metis add.commute in-multiset-in-set insert-DiffM*)

have $D'L$: $D' + \{\#-L\} - \{\#-L\} = D'$ **by** (*auto simp add*: *multiset-eq-iff*)

have *CL*: *mset* $C - \{\#L\} + \{\#L\} = \text{mset } C$ **using** $\langle L \in \text{set } C \rangle$ **by** (*auto simp add*: *multiset-eq-iff*)

have *max*: *get-maximum-level* (*Propagated* L ($C' + \{\#L\}$) $\# \text{map mmset-of-mlit}' M$) $D' = k$

using $M[\text{simplified}]$ **unfolding** *maximum-level-code-eq-get-maximum-level* $C[\text{symmetric}]$ *CL*

```

  by (metis D D'L get-maximum-level-map-convert list.simps(9) mset-of-multiset'.simps(1))
have distinct-mset (mset C) and distinct-mset (mset D)
  using ⟨cdclW-all-struct-inv (Propagated L C # M, N, U, k, Some D)⟩
  unfolding cdclW-all-struct-inv-def distinct-cdclW-state-def
  by auto
then have conf: (mset C - {#L#}) # ∪ (mset D - {#- L#}) =
  remdups-mset (mset C - {#L#} + (mset D - {#- L#}))
  by (auto simp: distinct-mset-remdups-union-mset)
show ?case
  apply (rule resolve-rule)
  using LC LD max M conf C D by (auto simp: subset-mset.sup.commute)
qed auto

```

```

lemma do-resolve-step-no:
  do-resolve-step S = S ⟹ no-step resolve S
  apply (cases S; cases (raw-trail S); cases raw-conflicting S)
  by (auto
    elim!: resolveE split: if-split-asm
    dest!: union-single-eq-member
    simp del: in-multiset-in-set get-maximum-level-map-convert
    simp: get-maximum-level-map-convert[symmetric] do-resolve-step)

```

```

lemma rough-state-of-state-of-resolve[simp]:
  cdclW-all-struct-inv S ⟹ rough-state-of (state-of (do-resolve-step S)) = do-resolve-step S
  apply (rule state-of-inverse)
  apply (cases do-resolve-step S = S)
  apply simp
  by (blast dest: other resolve bj do-resolve-step cdclW-all-struct-inv-inv)

```

```

lemma do-resolve-step-trail-is-None[iff]:
  do-resolve-step S = (a, b, c, d, None) ⟷ S = (a, b, c, d, None)
  by (cases S rule: do-resolve-step.cases) auto

```

Backjumping **fun** find-level-decomp **where**

```

find-level-decomp M [] D k = None |
find-level-decomp M (L # Ls) D k =
  (case (get-level M L, maximum-level-code (D @ Ls) M) of
    (i, j) ⇒ if i = k ∧ j < i then Some (L, j) else find-level-decomp M Ls (L#D) k
  )

```

```

lemma find-level-decomp-some:
  assumes find-level-decomp M Ls D k = Some (L, j)
  shows L ∈ set Ls ∧ get-maximum-level M (mset (remove1 L (Ls @ D))) = j ∧ get-level M L = k
  using assms

```

proof (induction Ls arbitrary: D)

```

  case Nil
  then show ?case by simp

```

next

```

  case (Cons L' Ls) note IH = this(1) and H = this(2)

```

```

def find ≡ (if get-level M L' ≠ k ∨ ¬ get-maximum-level M (mset D + mset Ls) < get-level M L'
  then find-level-decomp M Ls (L' # D) k
  else Some (L', get-maximum-level M (mset D + mset Ls)))
have a1: ⋀D. find-level-decomp M Ls D k = Some (L, j) ⟹
  L ∈ set Ls ∧ get-maximum-level M (mset Ls + mset D - {#L#}) = j ∧ get-level M L = k

```

```

    using IH by simp
  have a2: find = Some (L, j)
    using H unfolding find-def by (auto split: if-split-asm)
  { assume Some (L', get-maximum-level M (mset D + mset Ls)) ≠ find
    then have f3: L ∈ set Ls and get-maximum-level M (mset Ls + mset (L' # D) - {#L#}) = j
      using a1 IH a2 unfolding find-def by meson+
    moreover then have mset Ls + mset D - {#L#} + {#L'#} = {#L'#} + mset D + (mset Ls
- {#L#})
      by (auto simp: ac-simps multiset-eq-iff Suc-leI)
    ultimately have f4: get-maximum-level M (mset Ls + mset D - {#L#} + {#L'#}) = j
      by (metis add.commute diff-union-single-conv in-multiset-in-set mset.simps(2))
  } note f4 = this
  have {#L'#} + (mset Ls + mset D) = mset Ls + (mset D + {#L'#})
    by (auto simp: ac-simps)
  then have
    (L = L' → get-maximum-level M (mset Ls + mset D) = j ∧ get-level M L' = k) and
    (L ≠ L' → L ∈ set Ls ∧ get-maximum-level M (mset Ls + mset D - {#L#} + {#L'#}) = j ∧
      get-level M L = k)
    using f4 a2 a1[of L' # D] unfolding find-def by (metis (no-types) add-diff-cancel-left'
      mset.simps(2) option.inject prod.inject union-commute)+
  then show ?case by simp
qed

```

lemma find-level-decomp-none:

```

  assumes find-level-decomp M Ls E k = None and mset (L # D) = mset (Ls @ E)
  shows ¬(L ∈ set Ls ∧ get-maximum-level M (mset D) < k ∧ k = get-level M L)
  using assms

```

proof (induction Ls arbitrary: E L D)

```

  case Nil
  then show ?case by simp
next
  case (Cons L' Ls) note IH = this(1) and find-none = this(2) and LD = this(3)
  have mset D + {#L'#} = mset E + (mset Ls + {#L'#}) ⇒ mset D = mset E + mset Ls
    by (metis add-right-imp-eq union-assoc)
  then show ?case
    using find-none IH[of L' # E L D] LD by (auto simp add: ac-simps split: if-split-asm)
qed

```

fun bt-cut where

```

bt-cut i (Propagated - - # Ls) = bt-cut i Ls |
bt-cut i (Marked K k # Ls) = (if k = Suc i then Some (Marked K k # Ls) else bt-cut i Ls) |
bt-cut i [] = None

```

lemma bt-cut-some-decomp:

```

  bt-cut i M = Some M' ⇒ ∃ K M2 M1. M = M2 @ M' ∧ M' = Marked K (i+1) # M1
  by (induction i M rule: bt-cut.induct) (auto split: if-split-asm)

```

lemma bt-cut-not-none: M = M2 @ Marked K (Suc i) # M' ⇒ bt-cut i M ≠ None

by (induction M2 arbitrary: M rule: marked-lit-list-induct) auto

lemma get-all-marked-decomposition-ex:

```

  ∃ N. (Marked K (Suc i) # M', N) ∈ set (get-all-marked-decomposition (M2 @ Marked K (Suc i) #
M'))

```

apply (induction M2 rule: marked-lit-list-induct)

apply auto[2]

by (rename-tac L m xs , case-tac get-all-marked-decomposition (xs @ Marked K (Suc i) # M'))
 auto

lemma *bt-cut-in-get-all-marked-decomposition*:

$bt-cut\ i\ M = Some\ M' \implies \exists M2. (M', M2) \in set\ (get-all-marked-decomposition\ M)$
by (auto dest!: bt-cut-some-decomp simp add: get-all-marked-decomposition-ex)

fun *do-backtrack-step* **where**

do-backtrack-step ($M, N, U, k, Some\ D$) =
 (case *find-level-decomp* $M\ D$ [] k of
 None $\Rightarrow (M, N, U, k, Some\ D)$
 | *Some* (L, j) \Rightarrow
 (case *bt-cut* $j\ M$ of
 Some (Marked - - # Ls) $\Rightarrow (Propagated\ L\ D\ \#\ Ls, N, D\ \#\ U, j, None)$
 | - $\Rightarrow (M, N, U, k, Some\ D)$)
)
do-backtrack-step $S = S$

lemma *get-all-marked-decomposition-map-convert*:

(*get-all-marked-decomposition* (map *mmset-of-mlit'* M)) =
 map ($\lambda(a, b). (map\ mmset-of-mlit'\ a, map\ mmset-of-mlit'\ b)$) (*get-all-marked-decomposition* M)
apply (*induction* M rule: marked-lit-list-induct)
apply *simp*
by (rename-tac $L\ l\ xs$, case-tac get-all-marked-decomposition xs ; auto)+

lemma *do-backtrack-step*:

assumes
 db : *do-backtrack-step* $S \neq S$ **and**
 inv : *cdcl_W-all-struct-inv* S
shows *backtrack* S (*do-backtrack-step* S)
proof (cases S , cases *raw-conflicting* S , goal-cases)
 case (1 $M\ N\ U\ k\ E$)
 then show ?case **using** db **by** auto
next
 case (2 $M\ N\ U\ k\ E\ C$) **note** $S = this(1)$ **and** $confl = this(2)$
 have $E: E = Some\ C$ **using** $S\ confl$ **by** auto

obtain $L\ j$ **where** fd : *find-level-decomp* $M\ C$ [] $k = Some\ (L, j)$
using db **unfolding** $S\ E$ **by** (cases C) (auto split: if-split-asm option.splits)
have
 $L \in set\ C$ **and**
 j : *get-maximum-level* M (*mset* (*remove1* $L\ C$)) = j **and**
 $levL$: *get-level* $M\ L = k$
using *find-level-decomp-some*[OF fd] **by** auto
obtain C' **where** C : *mset* $C = mset\ C' + \{\#L\#\}$
using ($L \in set\ C$) **by** (*metis* *add commute ex-mset in-multiset-in-set insert-DiffM*)
obtain M_2 **where** M_2 : *bt-cut* $j\ M = Some\ M_2$
using $db\ fd$ **unfolding** $S\ E$ **by** (auto split: option.splits)
obtain $M1\ K$ **where** $M1$: $M_2 = Marked\ K\ (Suc\ j)\ \#\ M1$
using *bt-cut-some-decomp*[OF M_2] **by** (cases M_2) auto
obtain c **where** c : $M = c\ @\ Marked\ K\ (Suc\ j)\ \#\ M1$
using *bt-cut-in-get-all-marked-decomposition*[OF M_2]
unfolding $M1$ **by** *fastforce*
have *get-all-levels-of-marked* (map *mmset-of-mlit'* M) = *rev* [$1..<Suc\ k$]
using inv **unfolding** *cdcl_W-all-struct-inv-def* *cdcl_W-M-level-inv-def* S **by** auto

```

from arg-cong[OF this, of  $\lambda a. \text{Suc } j \in \text{set } a$ ] have  $j \leq k$  unfolding c by auto
have max-l-j: maximum-level-code  $C' M = j$ 
  using db fd  $M_2 C$  unfolding S E by (auto
    split: option.splits list.splits marked-lit.splits
    dest!: find-level-decomp-some)[1]
have get-maximum-level  $M (\text{mset } C) \geq k$ 
  using  $\langle L \in \text{set } C \rangle \text{levL } \text{get-maximum-level-ge-get-level}$  by (metis set-mset-mset)
moreover have get-maximum-level  $M (\text{mset } C) \leq k$ 
  using get-maximum-level-exists-lit-of-max-level[of mset C M] inv
    cdclW-M-level-inv-get-level-le-backtrack-lvl[of S]
  unfolding  $C \text{cdcl}_W\text{-all-struct-inv-def } S$  by (auto dest: sym[of get-level - -])
ultimately have get-maximum-level  $M (\text{mset } C) = k$  by auto

obtain  $M_2$  where  $M_2: (M_2, M_2) \in \text{set } (\text{get-all-marked-decomposition } M)$ 
  using bt-cut-in-get-all-marked-decomposition[OF M2] by metis
have decomp:
  (Marked K (Suc (get-maximum-level  $M (\text{remove1-mset } L (\text{mset } C))$ )))  $\#$  (map mmset-of-mlit'  $M_1$ ),
  (map mmset-of-mlit'  $M_2$ ))  $\in$ 
  set (get-all-marked-decomposition (map mmset-of-mlit'  $M$ ))
  using imageI[of - -  $\lambda(a, b). (\text{map mmset-of-mlit'} a, \text{map mmset-of-mlit'} b), \text{OF } M_2]$  j
  unfolding S E M1 by (auto simp add: get-all-marked-decomposition-map-convert)
have red: (reduce-trail-to (map mmset-of-mlit'  $M_1$ ))
  ( $M, N, C \# U, \text{get-maximum-level } M (\text{remove1-mset } L (\text{mset } C)), \text{None}$ )
  = ( $M_1, N, C \# U, \text{get-maximum-level } M (\text{remove1-mset } L (\text{mset } C)), \text{None}$ )
  using  $M_2 M_1$  by (auto simp: reduce-trail-to)
show ?case
  apply (rule backtrack-rule)
  using  $M_2 \text{fd confl } \langle L \in \text{set } C \rangle j \text{decomp levL } \langle \text{get-maximum-level } M (\text{mset } C) = k \rangle$ 
  unfolding S E M1 apply (auto simp: mset-map)[6]
  unfolding CDCL-W-Implementation.state-eq-def
  using  $M_2 \text{fd confl } \langle L \in \text{set } C \rangle j \text{decomp levL } \langle \text{get-maximum-level } M (\text{mset } C) = k \rangle \text{red}$ 
  unfolding S E M1
  by auto
qed

lemma map-eq-list-length:
  map f L = L'  $\implies$  length L = length L'
  by auto

lemma map-mmset-of-mlit-eq-cons:
  assumes map mmset-of-mlit'  $M = a @ c$ 
  obtains  $a' c'$  where
     $M = a' @ c'$  and
     $a = \text{map mmset-of-mlit'} a'$  and
     $c = \text{map mmset-of-mlit'} c'$ 
  using that[of take (length a) M drop (length a) M]
  assms by (metis append-eq-conv-conj append-take-drop-id drop-map take-map)

lemma do-backtrack-step-no:
  assumes
    db: do-backtrack-step  $S = S$  and
    inv: cdclW-all-struct-inv  $S$ 
  shows no-step backtrack S
proof (rule ccontr, cases S, cases conflicting S, goal-cases)
  case 1

```

```

then show ?case using db by (auto split: option.splits elim: backtrackE)
next
case (2 M N U k E C) note bt = this(1) and S = this(2) and confl = this(3)
obtain K j M1 M2 L D where
  CE: raw-conflicting S = Some D and
  LD: L ∈ # mset D and
  decomp: (Marked K (Suc j) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) and
  levL: get-level (raw-trail S) L = raw-backtrack-lvl S and
  k: get-level (raw-trail S) L = get-maximum-level (raw-trail S) (mset D) and
  j: get-maximum-level (raw-trail S) (remove1-mset L (mset D)) ≡ j and
  undef: undefined-lit M1 L
using bt apply clarsimp
apply (elim backtrack-levE)
  using inv unfolding cdclW-all-struct-inv-def apply fast
apply (cases S)
by (auto simp add: get-all-marked-decomposition-map-convert)

obtain c where c: trail S = c @ M2 @ Marked K (Suc j) # M1
  using decomp by blast
have get-all-levels-of-marked (trail S) = rev [1.. $\text{Suc } k$ ]
  using inv unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def S by auto
from arg-cong[OF this, of  $\lambda a. \text{Suc } j \in \text{set } a$ ] have  $k > j$ 
  unfolding c by (auto simp: get-all-marked-decomposition-map-convert)
have [simp]: L ∈ set D
  using LD by auto
have CD: C = mset D
  using CE confl by auto
obtain D' where
  E: E = Some D and
  DD': mset D = {#L#} + mset D'
  using that[of remove1 L D]
  using S CE confl LD by (auto simp add: insert-DiffM)
have find-level-decomp M D []  $k \neq \text{None}$ 
  apply rule
  apply (drule find-level-decomp-none[of - - - L D'])
  using DD'  $\langle k > j \rangle$  mset-eq-setD S levL unfolding k[symmetric] j[symmetric]
  by (auto simp: ac-simps)
then obtain L' j' where fd-some: find-level-decomp M D []  $k = \text{Some } (L', j')$ 
  by (cases find-level-decomp M D [] k) auto
have L': L' = L
  proof (rule ccontr)
    assume  $\neg$  ?thesis
    then have L' ∈ # mset (remove1 L D)
      by (metis fd-some find-level-decomp-some in-set-remove1 set-mset-mset)
    then have get-level M L' ≤ get-maximum-level M (mset (remove1 L D))
      using get-maximum-level-ge-get-level by blast
    then show False using  $\langle k > j \rangle$  j find-level-decomp-some[OF fd-some] S DD' by auto
  qed
then have j': j' = j using find-level-decomp-some[OF fd-some] j S DD' by auto

obtain c' M1' where cM: M = c' @ Marked K (Suc j) # M1'
  apply (rule map-mmset-of-mlit-eq-cons[of M c @ M2 Marked K (Suc j) # M1])
  using c S apply simp
  apply (rule map-mmset-of-mlit-eq-cons[of - [Marked K (Suc j)] M1])
  apply auto[]

```

```

  apply (rename-tac a b' aa b, case-tac aa)
  apply auto[]
  apply (rename-tac a b' aa b, case-tac aa)
  by auto
have btc-none: bt-cut j M ≠ None
  apply (rule bt-cut-not-none[of M ])
  using cM by simp
show ?case using db unfolding S E
  by (auto split: option.splits list.splits marked-lit.splits
      simp add: fd-some L' j' btc-none
      dest: bt-cut-some-decomp)
qed

```

```

lemma rough-state-of-state-of-backtrack[simp]:
  assumes inv: cdclW-all-struct-inv S
  shows rough-state-of (state-of (do-backtrack-step S)) = do-backtrack-step S
proof (rule state-of-inverse)
  have f2: backtrack S (do-backtrack-step S) ∨ do-backtrack-step S = S
    using do-backtrack-step inv by blast
  have ∧p. ¬ cdclW-o S p ∨ cdclW-all-struct-inv p
    using inv cdclW-all-struct-inv-inv other by blast
  then have do-backtrack-step S = S ∨ cdclW-all-struct-inv (do-backtrack-step S)
    using f2 inv cdclW-o.intros cdclW-bj.intros by blast
  then show do-backtrack-step S ∈ {S. cdclW-all-struct-inv S}
    using inv by fastforce
qed

```

```

Decide fun do-decide-step where
do-decide-step (M, N, U, k, None) =
  (case find-first-unused-var N (lits-of-l M) of
    None ⇒ (M, N, U, k, None)
  | Some L ⇒ (Marked L (Suc k) # M, N, U, k+1, None)) |
do-decide-step S = S

```

```

lemma do-decide-step:
  fixes S :: 'v cdclW-state-inv-st
  assumes do-decide-step S ≠ S
  shows decide S (do-decide-step S)
  using assms
  apply (cases S, cases conflicting S)
  defer
  apply (auto split: option.splits simp add: decide.simps Marked-Propagated-in-iff-in-lits-of-l
      dest: find-first-unused-var-undefined find-first-unused-var-Some
      intro:)[1]
proof -
  fix a :: ('v, nat, 'v literal list) marked-lit list and
    b :: 'v literal list list and c :: 'v literal list list and
    d :: nat and e :: 'v literal list option
  {
    fix a :: ('v, nat, 'v literal list) marked-lit list and
      b :: 'v literal list list and c :: 'v literal list list and
      d :: nat and x2 :: 'v literal and m :: 'v literal list
    assume a1: m ∈ set b
    assume x2 ∈ set m
    then have f2: atm-of x2 ∈ atms-of (mset m)

```

```

    by simp
  have  $\bigwedge f. (f\ m::'v\ clause) \in f\ 'set\ b$ 
    using a1 by blast
  then have  $\bigwedge f. (atms-of\ (f\ m)::'v\ set) \subseteq atms-of-ms\ (f\ 'set\ b)$ 
    by simp
  then have  $\bigwedge n\ f. (n::'v) \in atms-of-ms\ (f\ 'set\ b) \vee n \notin atms-of\ (f\ m)$ 
    by (meson contra-subsetD)
  then have  $atm-of\ x2 \in atms-of-ms\ (mset\ 'set\ b)$ 
    using f2 by blast
} note H = this
{
  fix m :: 'v literal list and x2
  have  $m \in set\ b \implies x2 \in set\ m \implies x2 \notin lits-of-l\ a \implies \neg x2 \notin lits-of-l\ a \implies$ 
     $\exists aa \in set\ b. \neg atm-of\ 'set\ aa \subseteq atm-of\ 'lits-of-l\ a$ 
    by (meson atm-of-in-atm-of-set-in-uminus contra-subsetD rev-image-eqI)
} note H' = this

assume do-decide-step S  $\neq$  S and
  S = (a, b, c, d, e) and
  conflicting S = None
then show decide S (do-decide-step S)
  using H H' by (auto split: option.splits simp: lits-of-def decide.simps
    Marked-Propagated-in-iff-in-lits-of-l
    dest!: find-first-unused-var-Some)
qed

lemma mmset-of-mlit'-eq-Marked[iff]: mmset-of-mlit' z = Marked x k  $\longleftrightarrow$  z = Marked x k
  by (cases z) auto

lemma do-decide-step-no:
  do-decide-step S = S  $\implies$  no-step decide S
  apply (cases S, cases conflicting S)

  apply (auto simp: atms-of-ms-mset-unfold Marked-Propagated-in-iff-in-lits-of-l lits-of-def
    dest!: atm-of-in-atm-of-set-in-uminus
    elim!: decideE
    split: option.splits)+
  using atm-of-eq-atm-of by blast

lemma rough-state-of-state-of-do-decide-step[simp]:
  cdclW-all-struct-inv S  $\implies$  rough-state-of (state-of (do-decide-step S)) = do-decide-step S
proof (subst state-of-inverse, goal-cases)
  case 1
  then show ?case
    by (cases do-decide-step S = S)
      (auto dest: do-decide-step decide other intro: cdclW-all-struct-inv-inv)
qed simp

lemma rough-state-of-state-of-do-skip-step[simp]:
  cdclW-all-struct-inv S  $\implies$  rough-state-of (state-of (do-skip-step S)) = do-skip-step S
  apply (subst state-of-inverse, cases do-skip-step S = S)
  apply simp
  by (blast dest: other skip bj do-skip-step cdclW-all-struct-inv-inv)+

```

20.3.3 Code generation

Type definition There are two invariants: one while applying conflict and propagate and one for the other rules

declare *rough-state-of-inverse*[*simp add*]

definition *Con* **where**

Con xs = state-of (if cdcl_W-all-struct-inv xs then xs else ([], [], [], 0, None))

lemma [*code abstype*]:

Con (rough-state-of S) = S

using *rough-state-of*[*of S*] **unfolding** *Con-def* **by** *simp*

definition *do-cp-step'* **where**

do-cp-step' S = state-of (do-cp-step (rough-state-of S))

typedef *'v cdcl_W-state-inv-from-init-state* = $\{S :: 'v \text{ cdcl}_W\text{-state-inv-st. cdcl}_W\text{-all-struct-inv } S \wedge \text{cdcl}_W\text{-stgy}^{**}(\text{raw-S0-cdcl}_W(\text{raw-init-clss } S)) S\}$

morphisms *rough-state-from-init-state-of state-from-init-state-of*

proof

show $([], [], [], 0, \text{None}) \in \{S. \text{cdcl}_W\text{-all-struct-inv } S \wedge \text{cdcl}_W\text{-stgy}^{**}(\text{raw-S0-cdcl}_W(\text{raw-init-clss } S)) S\}$

by (*auto simp add: cdcl_W-all-struct-inv-def*)

qed

instantiation *cdcl_W-state-inv-from-init-state* :: (*type*) *equal*

begin

definition *equal-cdcl_W-state-inv-from-init-state* :: *'v cdcl_W-state-inv-from-init-state* \Rightarrow

'v cdcl_W-state-inv-from-init-state \Rightarrow *bool* **where**

equal-cdcl_W-state-inv-from-init-state S S' \longleftrightarrow

(rough-state-from-init-state-of S = rough-state-from-init-state-of S')

instance

by *standard (simp add: rough-state-from-init-state-of-inject equal-cdcl_W-state-inv-from-init-state-def)*

end

definition *ConI* **where**

ConI S = state-from-init-state-of (if cdcl_W-all-struct-inv S

*$\wedge \text{cdcl}_W\text{-stgy}^{**}(\text{raw-S0-cdcl}_W(\text{raw-init-clss } S)) S$ then S else ([], [], [], 0, None))*

lemma [*code abstype*]:

ConI (rough-state-from-init-state-of S) = S

using *rough-state-from-init-state-of*[*of S*] **unfolding** *ConI-def*

by (*simp add: rough-state-from-init-state-of-inverse*)

definition *id-of-I-to* :: *'v cdcl_W-state-inv-from-init-state* \Rightarrow *'v cdcl_W-state-inv* **where**

id-of-I-to S = state-of (rough-state-from-init-state-of S)

lemma [*code abstract*]:

rough-state-of (id-of-I-to S) = rough-state-from-init-state-of S

unfolding *id-of-I-to-def* **using** *rough-state-from-init-state-of*[*of S*] **by** *auto*

Conflict and Propagate function *do-full1-cp-step* :: *'v cdcl_W-state-inv* \Rightarrow *'v cdcl_W-state-inv* **where**

do-full1-cp-step S =

(let S' = do-cp-step' S in

if $S = S'$ then S else $\text{do-full1-cp-step } S'$)
by *auto*
termination
proof (*relation* $\{(T', T). (\text{rough-state-of } T', \text{rough-state-of } T) \in \{(S', S). (S', S) \in \{(S', S). \text{cdcl}_W\text{-all-struct-inv } S \wedge \text{cdcl}_W\text{-cp } S S'\}\}, \text{goal-cases})\}$, *goal-cases*)
 case 1
show ?*case*
 using *wf-if-measure-f*[*OF wf-if-measure-f*[*OF cdcl_W-cp-wf-all-inv, of*], *of rough-state-of*] .
next
 case (2 $S' S$)
then show ?*case*
 unfolding *do-cp-step'-def*
 apply *simp*
 by (*metis cp-step-is-cdcl_W-cp rough-state-of-inverse*)
qed

lemma *do-full1-cp-step-fix-point-of-do-full1-cp-step*:
 $\text{do-cp-step}(\text{rough-state-of } (\text{do-full1-cp-step } S)) = \text{rough-state-of } (\text{do-full1-cp-step } S)$
by (*rule do-full1-cp-step.induct*[*of* $\lambda S. \text{do-cp-step}(\text{rough-state-of } (\text{do-full1-cp-step } S)) = \text{rough-state-of } (\text{do-full1-cp-step } S)$])
 (*metis (full-types) do-full1-cp-step.elims rough-state-of-state-of-do-cp-step do-cp-step'-def*)

lemma *in-clauses-rough-state-of-is-distinct*:
 $c \in \text{set } (\text{raw-init-clss } (\text{rough-state-of } S) @ \text{raw-learned-clss } (\text{rough-state-of } S)) \implies \text{distinct } c$
apply (*cases rough-state-of* S)
using *rough-state-of*[*of* S] **by** (*auto simp add: distinct-mset-set-distinct cdcl_W-all-struct-inv-def distinct-cdcl_W-state-def*)

lemma *do-full1-cp-step-full*:
 $\text{full } \text{cdcl}_W\text{-cp } (\text{rough-state-of } S)$
 $(\text{rough-state-of } (\text{do-full1-cp-step } S))$
unfolding *full-def*
proof (*rule conjI, induction S rule: do-full1-cp-step.induct*)
 case (1 S)
then have *f1*:
 $\text{cdcl}_W\text{-cp}^{**} ((\text{do-cp-step } (\text{rough-state-of } S))) ($
 $\text{rough-state-of } (\text{do-full1-cp-step } (\text{state-of } (\text{do-cp-step } (\text{rough-state-of } S)))))$
 $\vee \text{state-of } (\text{do-cp-step } (\text{rough-state-of } S)) = S$
using *rough-state-of-state-of-do-cp-step*[*of* S] **unfolding** *do-cp-step'-def* **by** *fastforce*
have *f2*: $\bigwedge c. (\text{if } c = \text{state-of } (\text{do-cp-step } (\text{rough-state-of } c))$
 $\text{then } c \text{ else } \text{do-full1-cp-step } (\text{state-of } (\text{do-cp-step } (\text{rough-state-of } c))))$
 $= \text{do-full1-cp-step } c$
by (*metis (full-types) do-cp-step'-def do-full1-cp-step.simps*)
have *f3*: $\neg \text{cdcl}_W\text{-cp } (\text{rough-state-of } S) (\text{do-cp-step } (\text{rough-state-of } S))$
 $\vee \text{state-of } (\text{do-cp-step } (\text{rough-state-of } S)) = S$
 $\vee \text{cdcl}_W\text{-cp}^{++} (\text{rough-state-of } S)$
 $(\text{rough-state-of } (\text{do-full1-cp-step } (\text{state-of } (\text{do-cp-step } (\text{rough-state-of } S)))))$
using *f1* **by** (*meson rtranclp-into-tranclp2*)
{ assume $\text{do-full1-cp-step } S \neq S$
then have $\text{do-cp-step } (\text{rough-state-of } S) = \text{rough-state-of } S$
 $\longrightarrow \text{cdcl}_W\text{-cp}^{**} (\text{rough-state-of } S) (\text{rough-state-of } (\text{do-full1-cp-step } S))$
 $\vee \text{do-cp-step } (\text{rough-state-of } S) \neq \text{rough-state-of } S$
 $\wedge \text{state-of } (\text{do-cp-step } (\text{rough-state-of } S)) \neq S$
using *f2 f1* **by** (*metis (no-types)*)
then have $\text{do-cp-step } (\text{rough-state-of } S) \neq \text{rough-state-of } S$

```

    ∧ state-of (do-cp-step (rough-state-of S)) ≠ S
  ∨ cdclW-cp** (rough-state-of S) (rough-state-of (do-full1-cp-step S))
  by (metis rough-state-of-state-of-do-cp-step)
then have cdclW-cp** (rough-state-of S) (rough-state-of (do-full1-cp-step S))
  using f3 f2 by (metis (no-types) cp-step-is-cdclW-cp tranclp-into-rtranclp) }
then show ?case
  by fastforce
next
show no-step cdclW-cp (rough-state-of (do-full1-cp-step S))
  apply (rule do-cp-step-eq-no-step[OF do-full1-cp-step-fix-point-of-do-full1-cp-step[of S]])
  using in-clauses-rough-state-of-is-distinct unfolding do-cp-step'-def by blast
qed

```

lemma [code abstract]:
 rough-state-of (do-cp-step' S) = do-cp-step (rough-state-of S)
 unfolding do-cp-step'-def by auto

The other rules fun do-other-step where

```

do-other-step S =
  (let T = do-skip-step S in
    if T ≠ S
    then T
    else
      (let U = do-resolve-step T in
        if U ≠ T
        then U else
          (let V = do-backtrack-step U in
            if V ≠ U then V else do-decide-step V)))

```

lemma do-other-step:
 assumes inv: cdcl_W-all-struct-inv S and
 st: do-other-step S ≠ S
 shows cdcl_W-o S (do-other-step S)
 using st inv by (auto split: if-split-asm
 simp add: Let-def
 intro: do-skip-step do-resolve-step do-backtrack-step do-decide-step
 cdcl_W-o.intros cdcl_W-bj.intros)

lemma do-other-step-no:
 assumes inv: cdcl_W-all-struct-inv S and
 st: do-other-step S = S
 shows no-step cdcl_W-o S
 using st inv by (auto split: if-split-asm elim: cdcl_W-bjE
 simp add: Let-def cdcl_W-bj.simps elim!: cdcl_W-o.cases
 dest!: do-skip-step-no do-resolve-step-no do-backtrack-step-no do-decide-step-no)

lemma rough-state-of-state-of-do-other-step[simp]:
 rough-state-of (state-of (do-other-step (rough-state-of S))) = do-other-step (rough-state-of S)
proof (cases do-other-step (rough-state-of S) = rough-state-of S)
 case True
 then show ?thesis by simp
next
 case False
 have cdcl_W-o (rough-state-of S) (do-other-step (rough-state-of S))
 by (metis False cdcl_W-all-struct-inv-rough-state do-other-step[of rough-state-of S])

then have $cdcl_W\text{-all-struct-inv}$ ($do\text{-other-step}$ ($rough\text{-state-of}$ S))
using $cdcl_W\text{-all-struct-inv-inv}$ $cdcl_W\text{-all-struct-inv-rough-state}$ $other$ **by** $blast$
then show $?thesis$
by ($simp$ add : $CollectI$ $state\text{-of-inverse}$)
qed

definition $do\text{-other-step}'$ **where**

$do\text{-other-step}' S =$
 $state\text{-of}$ ($do\text{-other-step}$ ($rough\text{-state-of}$ S))

lemma $rough\text{-state-of-}do\text{-other-step}'$ [*code abstract*]:

$rough\text{-state-of}$ ($do\text{-other-step}' S$) = $do\text{-other-step}$ ($rough\text{-state-of}$ S)

apply ($cases$ $do\text{-other-step}$ ($rough\text{-state-of}$ S) = $rough\text{-state-of}$ S)

unfolding $do\text{-other-step}'\text{-def}$ **apply** $simp$

using $do\text{-other-step}$ [*of* $rough\text{-state-of}$ S] **by** ($auto$ $intro$: $cdcl_W\text{-all-struct-inv-inv}$
 $cdcl_W\text{-all-struct-inv-rough-state}$ $other$ $state\text{-of-inverse}$)

definition $do\text{-cdcl}_W\text{-stgy-step}$ **where**

$do\text{-cdcl}_W\text{-stgy-step}$ $S =$
 (let $T = do\text{-full1-cp-step}$ S *in*
 if $T \neq S$
 then T
 else
 (let $U = (do\text{-other-step}' T)$ *in*
 ($do\text{-full1-cp-step}$ U)))

definition $do\text{-cdcl}_W\text{-stgy-step}'$ **where**

$do\text{-cdcl}_W\text{-stgy-step}' S = state\text{-from-init-state-of}$ ($rough\text{-state-of}$ ($do\text{-cdcl}_W\text{-stgy-step}$ ($id\text{-of-I-to}$ S)))

lemma $toS\text{-do-full1-cp-step-not-eq}$: $do\text{-full1-cp-step}$ $S \neq S \implies$

$rough\text{-state-of}$ $S \neq rough\text{-state-of}$ ($do\text{-full1-cp-step}$ S)

proof –

assume $a1$: $do\text{-full1-cp-step}$ $S \neq S$

then have $S \neq do\text{-cp-step}' S$

by $fastforce$

then show $?thesis$

by ($metis$ ($no\text{-types}$) $do\text{-cp-step}'\text{-def}$ $do\text{-full1-cp-step-fix-point-of-}do\text{-full1-cp-step}$
 $rough\text{-state-of-inverse}$)

qed

$do\text{-full1-cp-step}$ should not be unfolded anymore:

declare $do\text{-full1-cp-step.simps}$ [$simp$ del]

Correction of the transformation **lemma** $do\text{-cdcl}_W\text{-stgy-step}$:

assumes $do\text{-cdcl}_W\text{-stgy-step}$ $S \neq S$

shows $cdcl_W\text{-stgy}$ ($rough\text{-state-of}$ S) ($rough\text{-state-of}$ ($do\text{-cdcl}_W\text{-stgy-step}$ S))

proof ($cases$ $do\text{-full1-cp-step}$ $S = S$)

case $False$

then show $?thesis$

using $assms$ $do\text{-full1-cp-step-full}$ [*of* S] **unfolding** $full\text{-unfold}$ $do\text{-cdcl}_W\text{-stgy-step-def}$

by ($auto$ $intro$!: $cdcl_W\text{-stgy.intros}$ $dest$: $toS\text{-do-full1-cp-step-not-eq}$)

next

case $True$

have $cdcl_W\text{-o}$ ($rough\text{-state-of}$ S) ($rough\text{-state-of}$ ($do\text{-other-step}' S$))

by (smt $True$ $assms$ $cdcl_W\text{-all-struct-inv-rough-state}$ $do\text{-cdcl}_W\text{-stgy-step-def}$ $do\text{-other-step}$)

```

    rough-state-of-do-other-step' rough-state-of-inverse)
moreover
  have
    np: no-step propagate (rough-state-of S) and
    nc: no-step conflict (rough-state-of S)
    apply (metis True cdclW-cp.simps do-cp-step-eq-no-step
      do-full1-cp-step-fix-point-of-do-full1-cp-step in-clauses-rough-state-of-is-distinct)
    by (metis True do-conflict-step-no-step do-cp-step-eq-no-prop-no-confl
      do-full1-cp-step-fix-point-of-do-full1-cp-step)
  then have no-step cdclW-cp (rough-state-of S)
    by (simp add: cdclW-cp.simps)
moreover have full cdclW-cp (rough-state-of (do-other-step' S))
  (rough-state-of (do-full1-cp-step (do-other-step' S)))
  using do-full1-cp-step-full by auto
ultimately show ?thesis
  using assms True unfolding do-cdclW-stgy-step-def
  by (auto intro!: cdclW-stgy.other' dest: toS-do-full1-cp-step-not-eq)
qed

lemma do-skip-step-trail-changed-or-conflict:
  assumes d: do-other-step S ≠ S
  and inv: cdclW-all-struct-inv S
  shows trail S ≠ trail (do-other-step S)
proof -
  have M:  $\bigwedge M K M1 c. M = c @ K \# M1 \implies \text{Suc}(\text{length } M1) \leq \text{length } M$ 
    by auto
  have cdclW-M-level-inv S
    using inv unfolding cdclW-all-struct-inv-def by auto
  have cdclW-o S (do-other-step S) using do-other-step[OF inv d] .
  then show ?thesis
    using ⟨cdclW-M-level-inv S⟩
  proof (induction do-other-step S rule: cdclW-o-induct-lev2)
    case decide
    then show ?thesis
      apply (cases S)
      apply (auto dest!: find-first-unused-var-Some
        simp: split: option.splits)
      by (meson atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set contra-subsetD)
  next
  case (skip)
  then show ?case
    by (cases S; cases do-other-step S) force
  next
  case (resolve)
  then show ?case
    by (cases S, cases do-other-step S) force
  next
  case (backtrack K i M1 M2 L D) note decomp = this(1) and confl-S = this(3) and undef =
    this(6)
    and U = this(7)
  then show ?case
    apply (cases do-other-step S)
    apply (auto split: if-split-asm simp: Let-def)
    apply (cases S rule: do-skip-step.cases; auto split: if-split-asm)
    apply (cases S rule: do-skip-step.cases; auto split: if-split-asm)

```

```

    apply (cases S rule: do-backtrack-step.cases;
      auto split: if-split-asm option.splits list.splits marked-lit.splits
      dest!: bt-cut-some-decomp simp: Let-def)
  using d apply (cases S rule: do-decide-step.cases; auto split: option.splits)[]
done
qed
qed

```

lemma *do-full1-cp-step-induct*:
 $(\bigwedge S. (S \neq \text{do-cp-step}' S \implies P (\text{do-cp-step}' S)) \implies P S) \implies P a0$
using *do-full1-cp-step.induct* **by** *metis*

lemma *do-cp-step-neq-trail-increase*:
 $\exists c. \text{raw-trail } (\text{do-cp-step } S) = c @ \text{raw-trail } S \wedge (\forall m \in \text{set } c. \neg \text{is-marked } m)$
by (cases S, cases *raw-conflicting* S)
 (auto simp add: do-cp-step-def do-conflict-step-def do-propagate-step-def split: option.splits)

lemma *do-full1-cp-step-neq-trail-increase*:
 $\exists c. \text{raw-trail } (\text{rough-state-of } (\text{do-full1-cp-step } S)) = c @ \text{raw-trail } (\text{rough-state-of } S)$
 $\wedge (\forall m \in \text{set } c. \neg \text{is-marked } m)$
apply (induction rule: do-full1-cp-step-induct)
apply (rename-tac S, case-tac $\text{do-cp-step}' S = S$)
apply (simp add: do-full1-cp-step.simps)
by (smt Un-iff append-assoc do-cp-step'-def do-cp-step-neq-trail-increase do-full1-cp-step.simps
 rough-state-of-state-of-do-cp-step set-append)

lemma *do-cp-step-conflicting*:
 $\text{conflicting } (\text{rough-state-of } S) \neq \text{None} \implies \text{do-cp-step}' S = S$
unfolding *do-cp-step'-def* *do-cp-step-def* **by** *simp*

lemma *do-full1-cp-step-conflicting*:
 $\text{conflicting } (\text{rough-state-of } S) \neq \text{None} \implies \text{do-full1-cp-step } S = S$
unfolding *do-cp-step'-def* *do-cp-step-def*
apply (induction rule: do-full1-cp-step-induct)
by (rename-tac S, case-tac $S \neq \text{do-cp-step}' S$)
 (auto simp add: do-full1-cp-step.simps do-cp-step-conflicting)

lemma *do-decide-step-not-conflicting-one-more-decide*:
assumes
conflicting S = None **and**
do-decide-step S \neq S
shows Suc (length (filter is-marked (raw-trail S)))
 = length (filter is-marked (raw-trail (do-decide-step S)))
using *assms* **unfolding** *do-other-step'-def*
by (cases S) (force simp: Let-def split: if-split-asm option.splits
 dest!: find-first-unused-var-Some-not-all-incl)

lemma *do-decide-step-not-conflicting-one-more-decide-bt*:
assumes *conflicting* S \neq None **and**
do-decide-step S \neq S
shows length (filter is-marked (raw-trail S)) <
 length (filter is-marked (raw-trail (do-decide-step S)))
using *assms* **unfolding** *do-other-step'-def* **by** (cases S, cases *conflicting* S)
 (auto simp add: Let-def split: if-split-asm option.splits)

lemma *do-other-step-not-conflicting-one-more-decide-bt*:

assumes

conflicting (*rough-state-of* *S*) \neq *None* **and**

conflicting (*rough-state-of* (*do-other-step'* *S*)) = *None* **and**

do-other-step' *S* \neq *S*

shows *length* (*filter is-marked* (*raw-trail* (*rough-state-of* *S*)))

$>$ *length* (*filter is-marked* (*raw-trail* (*rough-state-of* (*do-other-step'* *S*))))

proof (*cases* *S*, *goal-cases*)

case (*1 y*) **note** *S* = *this*(*1*) **and** *inv* = *this*(*2*)

obtain *M N U k E* **where** *y*: *y* = (*M*, *N*, *U*, *k*, *Some E*)

using *assms*(*1*) *S inv* **by** (*cases y, cases conflicting y*) *auto*

have *M*: *rough-state-of* (*state-of* (*M*, *N*, *U*, *k*, *Some E*)) = (*M*, *N*, *U*, *k*, *Some E*)

using *inv y* **by** (*auto simp add: state-of-inverse*)

have *bt*: *do-other-step'* *S* = *state-of* (*do-backtrack-step* (*rough-state-of* *S*))

proof (*cases rough-state-of S rule: do-decide-step.cases*)

case *1*

then show *?thesis*

using *assms*(*1,2*) **by** *auto*]

next

case (*2 v vb vd vf vh*)

have *f3*: $\bigwedge c.$ (*if* *do-skip-step* (*rough-state-of* *c*) \neq *rough-state-of* *c*

then *do-skip-step* (*rough-state-of* *c*)

else if *do-resolve-step* (*do-skip-step* (*rough-state-of* *c*)) \neq *do-skip-step* (*rough-state-of* *c*)

then *do-resolve-step* (*do-skip-step* (*rough-state-of* *c*))

else if *do-backtrack-step* (*do-resolve-step* (*do-skip-step* (*rough-state-of* *c*)))

\neq *do-resolve-step* (*do-skip-step* (*rough-state-of* *c*))

then *do-backtrack-step* (*do-resolve-step* (*do-skip-step* (*rough-state-of* *c*)))

else *do-decide-step* (*do-backtrack-step* (*do-resolve-step*

(*do-skip-step* (*rough-state-of* *c*))))

= *rough-state-of* (*do-other-step'* *c*)

by (*simp add: rough-state-of-do-other-step'*)

have

(*raw-trail* (*rough-state-of* (*do-other-step'* *S*)),

raw-init-clss (*rough-state-of* (*do-other-step'* *S*)),

raw-learned-clss (*rough-state-of* (*do-other-step'* *S*)),

raw-backtrack-lvl (*rough-state-of* (*do-other-step'* *S*)), *None*)

= *rough-state-of* (*do-other-step'* *S*)

using *assms*(*2*) **by** (*cases do-other-step' S*) *auto*

then show *?thesis*

using *f3 2* **by** (*metis* (*no-types*) *do-decide-step.simps*(*2*) *do-resolve-step-trail-is-None*

do-skip-step-trail-is-None *rough-state-of-inverse*)

qed

show *?case*

using *assms*(*2*) *S* **unfolding** *bt y inv*

apply *simp*

by (*auto simp add: M bt-cut-not-none*

split: option.splits

dest!: *bt-cut-some-decomp*)

qed

lemma *do-other-step-not-conflicting-one-more-decide*:

assumes *conflicting* (*rough-state-of* *S*) = *None* **and**

do-other-step' *S* \neq *S*

shows *1* + *length* (*filter is-marked* (*raw-trail* (*rough-state-of* *S*)))

$= \text{length } (\text{filter is-marked } (\text{raw-trail } (\text{rough-state-of } (\text{do-other-step}' S))))$
proof (cases S , goal-cases)
case (1 y) **note** $S = \text{this}(1)$ **and** $\text{inv} = \text{this}(2)$
obtain $M N U k$ **where** $y: y = (M, N, U, k, \text{None})$ **using** $\text{assms}(1)$ S inv **by** (cases y) *auto*
have M : $\text{rough-state-of } (\text{state-of } (M, N, U, k, \text{None})) = (M, N, U, k, \text{None})$
using $\text{inv } y$ **by** (*auto simp add: state-of-inverse*)
have $\text{state-of } (\text{do-decide-step } (M, N, U, k, \text{None})) \neq \text{state-of } (M, N, U, k, \text{None})$
using $\text{assms}(2)$ **unfolding** $\text{do-other-step}'\text{-def } y \text{ inv } S$ **by** (*auto simp add: M*)
then have $f4$: $\text{do-skip-step } (\text{rough-state-of } S) = \text{rough-state-of } S$
unfolding $S M y$ **by** (*metis (full-types) do-skip-step.simps(4)*)
have $f5$: $\text{do-resolve-step } (\text{rough-state-of } S) = \text{rough-state-of } S$
unfolding $S M y$ **by** (*metis (no-types) do-resolve-step.simps(4)*)
have $f6$: $\text{do-backtrack-step } (\text{rough-state-of } S) = \text{rough-state-of } S$
unfolding $S M y$ **by** (*metis (no-types) do-backtrack-step.simps(2)*)
have $\text{do-other-step } (\text{rough-state-of } S) \neq \text{rough-state-of } S$
using $\text{assms}(2)$ **unfolding** $S M y$ $\text{do-other-step}'\text{-def}$ **by** (*metis (no-types)*)
then show ?case
using $f6 f5 f4$ **by** (*simp add: assms(1) do-decide-step-not-conflicting-one-more-decide do-other-step'-def*)
qed

lemma $\text{rough-state-of-state-of-do-skip-step-rough-state-of}[simp]$:
 $\text{rough-state-of } (\text{state-of } (\text{do-skip-step } (\text{rough-state-of } S))) = \text{do-skip-step } (\text{rough-state-of } S)$
by (*smt do-other-step.simps rough-state-of-inverse rough-state-of-state-of-do-other-step*)

lemma $\text{conflicting-do-resolve-step-iff}[iff]$:
 $\text{conflicting } (\text{do-resolve-step } S) = \text{None} \longleftrightarrow \text{conflicting } S = \text{None}$
by (cases S *rule: do-resolve-step.cases*)
(auto simp add: Let-def split: option.splits)

lemma $\text{conflicting-do-skip-step-iff}[iff]$:
 $\text{conflicting } (\text{do-skip-step } S) = \text{None} \longleftrightarrow \text{conflicting } S = \text{None}$
by (cases S *rule: do-skip-step.cases*)
(auto simp add: Let-def split: option.splits)

lemma $\text{conflicting-do-decide-step-iff}[iff]$:
 $\text{conflicting } (\text{do-decide-step } S) = \text{None} \longleftrightarrow \text{conflicting } S = \text{None}$
by (cases S *rule: do-decide-step.cases*)
(auto simp add: Let-def split: option.splits)

lemma $\text{conflicting-do-backtrack-step-imp}[simp]$:
 $\text{do-backtrack-step } S \neq S \implies \text{conflicting } (\text{do-backtrack-step } S) = \text{None}$
by (cases S *rule: do-backtrack-step.cases*)
(auto simp add: Let-def split: list.splits option.splits marked-lit.splits)

lemma $\text{do-skip-step-eq-iff-trail-eq}$:
 $\text{do-skip-step } S = S \longleftrightarrow \text{trail } (\text{do-skip-step } S) = \text{trail } S$
by (cases S *rule: do-skip-step.cases*) *auto*

lemma $\text{do-decide-step-eq-iff-trail-eq}$:
 $\text{do-decide-step } S = S \longleftrightarrow \text{trail } (\text{do-decide-step } S) = \text{trail } S$
by (cases S *rule: do-decide-step.cases*) (*auto split: option.split*)

lemma $\text{do-backtrack-step-eq-iff-trail-eq}$:
 $\text{do-backtrack-step } S = S \longleftrightarrow \text{raw-trail } (\text{do-backtrack-step } S) = \text{raw-trail } S$

by (cases S rule: do-backtrack-step.cases)
 (auto split: option.split list.splits marked-lit.splits
 dest!: bt-cut-in-get-all-marked-decomposition)

lemma do-resolve-step-eq-iff-trail-eq:
 do-resolve-step $S = S \longleftrightarrow \text{trail } (\text{do-resolve-step } S) = \text{trail } S$
 by (cases S rule: do-resolve-step.cases) auto

lemma do-other-step-eq-iff-trail-eq:
 do-other-step $S = S \longleftrightarrow \text{raw-trail } (\text{do-other-step } S) = \text{raw-trail } S$

apply
 (auto simp add: Let-def do-skip-step-eq-iff-trail-eq
 do-decide-step-eq-iff-trail-eq do-backtrack-step-eq-iff-trail-eq
 do-resolve-step-eq-iff-trail-eq
)
apply (simp add: do-resolve-step-eq-iff-trail-eq[symmetric]
 do-skip-step-eq-iff-trail-eq[symmetric])
apply (simp add: do-skip-step-eq-iff-trail-eq[symmetric]
 do-decide-step-eq-iff-trail-eq do-backtrack-step-eq-iff-trail-eq[symmetric]
 do-resolve-step-eq-iff-trail-eq[symmetric]
)
done

lemma do-full1-cp-step-do-other-step'-normal-form[dest!]:

assumes H : do-full1-cp-step (do-other-step' S) = S
shows do-other-step' $S = S \wedge \text{do-full1-cp-step } S = S$

proof –

let $?T = \text{do-other-step}' S$
 { **assume** confl : conflicting (rough-state-of $?T$) $\neq \text{None}$
then have tr : trail (rough-state-of (do-full1-cp-step $?T$)) = trail (rough-state-of $?T$)
using do-full1-cp-step-conflicting **by** fastforce
have raw-trail (rough-state-of (do-full1-cp-step (do-other-step' S))) =
 raw-trail (rough-state-of S)
using arg-cong[OF H , of $\lambda S. \text{raw-trail } (\text{rough-state-of } S)$] .
then have raw-trail (rough-state-of (do-other-step' S)) = raw-trail (rough-state-of S)
using confl **by** (auto simp add: do-full1-cp-step-conflicting)
then have do-other-step' $S = S$
by (simp add: do-other-step-eq-iff-trail-eq[symmetric] do-other-step'-def
 del: do-other-step.simps)

}

moreover {

assume eq[simp]: do-other-step' $S = S$
obtain c **where** c : raw-trail (rough-state-of (do-full1-cp-step S)) =
 $c @ \text{raw-trail } (\text{rough-state-of } S)$
using do-full1-cp-step-neq-trail-increase **by** auto

moreover have raw-trail (rough-state-of (do-full1-cp-step S)) = raw-trail (rough-state-of S)
using arg-cong[OF H , of $\lambda S. \text{raw-trail } (\text{rough-state-of } S)$] **by** simp

finally have $c = []$ **by** blast

then have do-full1-cp-step $S = S$ **using** assms **by** auto

}

moreover {

assume confl : conflicting (rough-state-of $?T$) = None **and** neg : do-other-step' $S \neq S$
obtain c **where**

```

  c: raw-trail (rough-state-of (do-full1-cp-step ?T)) = c @ raw-trail (rough-state-of ?T) and
  nm:  $\forall m \in \text{set } c. \neg \text{is-marked } m$ 
  using do-full1-cp-step-neq-trail-increase by auto
  have length (filter is-marked (raw-trail (rough-state-of (do-full1-cp-step ?T))))
    = length (filter is-marked (raw-trail (rough-state-of ?T)))
  using nm unfolding c by force
  moreover have length (filter is-marked (raw-trail (rough-state-of S)))
     $\neq$  length (filter is-marked (raw-trail (rough-state-of ?T)))
  using do-other-step-not-conflicting-one-more-decide[OF - neq]
  do-other-step-not-conflicting-one-more-decide-bt[of S, OF - confl neq]
  by linarith
  finally have False unfolding H by blast
}
ultimately show ?thesis by blast
qed

```

```

lemma do-cdclW-stgy-step-no:
  assumes S: do-cdclW-stgy-step S = S
  shows no-step cdclW-stgy (rough-state-of S)
proof -
  {
    fix S'
    assume full1 cdclW-cp (rough-state-of S) S'
    then have False
      using do-full1-cp-step-full[of S] unfolding full-def S rtranclp-unfold full1-def
      by (smt assms do-cdclW-stgy-step-def tranclpD)
  }
  moreover {
    fix S' S''
    assume cdclW-o (rough-state-of S) S' and
      no-step propagate (rough-state-of S) and
      no-step conflict (rough-state-of S) and
      full cdclW-cp S' S''
    then have False
      using assms unfolding do-cdclW-stgy-step-def
      by (smt cdclW-all-struct-inv-rough-state do-full1-cp-step-do-other-step'-normal-form
        do-other-step-no rough-state-of-do-other-step')
  }
  ultimately show ?thesis using assms by (force simp: cdclW-cp.simps cdclW-stgy.simps)
qed

```

```

lemma toS-rough-state-of-state-of-rough-state-from-init-state-of[simp]:
  rough-state-of (state-of (rough-state-from-init-state-of S))
    = rough-state-from-init-state-of S
  using rough-state-from-init-state-of[of S] by (auto simp add: state-of-inverse)

```

```

lemma cdclW-cp-is-rtranclp-cdclW: cdclW-cp S T  $\implies$  cdclW** S T
  apply (induction rule: cdclW-cp.induct)
  using conflict apply blast
  using propagate by blast

```

```

lemma rtranclp-cdclW-cp-is-rtranclp-cdclW: cdclW-cp** S T  $\implies$  cdclW** S T
  apply (induction rule: rtranclp-induct)
  apply simp
  by (fastforce dest!: cdclW-cp-is-rtranclp-cdclW)

```

lemma *cdcl_W-stgy-is-rtrancpl-cdcl_W*:
*cdcl_W-stgy S T \implies cdcl_W** S T*
apply (*induction rule: cdcl_W-stgy.induct*)
using *cdcl_W-stgy.conflict' rtrancpl-cdcl_W-stgy-rtrancpl-cdcl_W* **apply** *blast*
unfolding *full-def* **by** (*fastforce dest!:other rtrancpl-cdcl_W-cp-is-rtrancpl-cdcl_W*)

lemma *cdcl_W-stgy-init-clss: cdcl_W-stgy S T \implies cdcl_W-M-level-inv S \implies init-clss S = init-clss T*
using *rtrancpl-cdcl_W-init-clss cdcl_W-stgy-is-rtrancpl-cdcl_W* **by** *fast*

lemma *clauses-toS-rough-state-of-do-cdcl_W-stgy-step[simp]*:
init-clss (rough-state-of (do-cdcl_W-stgy-step (state-of (rough-state-from-init-state-of S))))
= init-clss (rough-state-from-init-state-of S) (is - = init-clss ?S)

proof (*cases do-cdcl_W-stgy-step (state-of ?S) = state-of ?S*)
case *True*
then show *?thesis* **by** *simp*

next
case *False*
have $\bigwedge c. \text{cdcl}_W\text{-M-level-inv (rough-state-of c)}$
using *cdcl_W-all-struct-inv-def cdcl_W-all-struct-inv-rough-state* **by** *blast*
then have $\bigwedge c. \text{init-clss (rough-state-of c) = init-clss (rough-state-of (do-cdcl}_W\text{-stgy-step c))}$
 $\vee \text{do-cdcl}_W\text{-stgy-step c = c}$
using *cdcl_W-stgy-no-more-init-clss do-cdcl_W-stgy-step* **by** *blast*
then show *?thesis*
using *False* **by** *force*

qed

lemma *raw-init-clss-do-cp-step[simp]*:
raw-init-clss (do-cp-step S) = raw-init-clss S
by (*cases S*) (*auto simp: do-cp-step-def do-propagate-step-def do-conflict-step-def*
split: option.splits)

lemma *raw-init-clss-do-cp-step'[simp]*:
raw-init-clss (rough-state-of (do-cp-step' S)) = raw-init-clss (rough-state-of S)
by (*simp add: do-cp-step'-def*)

lemma *raw-init-clss-rough-state-of-do-full1-cp-step[simp]*:
raw-init-clss (rough-state-of (do-full1-cp-step S))
= raw-init-clss (rough-state-of S)
apply (*rule do-full1-cp-step.induct[of $\lambda S.$*
raw-init-clss (rough-state-of (do-full1-cp-step S))
= raw-init-clss (rough-state-of S)])
by (*metis (mono-tags, lifting) do-full1-cp-step.simps raw-init-clss-do-cp-step'*)

lemma *raw-init-clss-do-skip-def[simp]*:
raw-init-clss (do-skip-step S) = raw-init-clss S
by (*cases S rule: do-skip-step.cases*) (*auto simp: do-other-step'-def Let-def*
split: option.splits)

lemma *raw-init-clss-do-resolve-def[simp]*:
raw-init-clss (do-resolve-step S) = raw-init-clss S
by (*cases S rule: do-resolve-step.cases*) (*auto simp: do-other-step'-def Let-def*
split: option.splits)

lemma *raw-init-clss-do-backtrack-def[simp]*:
raw-init-clss (do-backtrack-step S) = raw-init-clss S

by (cases S rule: do-backtrack-step.cases) (auto simp: do-other-step'-def Let-def
split: option.splits list.splits marked-lit.splits)

lemma raw-init-clss-do-decide-def[simp]:
raw-init-clss (do-decide-step S) = raw-init-clss S
by (cases S rule: do-decide-step.cases) (auto simp: do-other-step'-def Let-def
split: option.splits)

lemma raw-init-clss-rough-state-of-do-other-step'[simp]:
raw-init-clss (rough-state-of (do-other-step' S))
= raw-init-clss (rough-state-of S)
by (cases S) (auto simp: do-other-step'-def Let-def do-skip-step.cases
split: option.splits)

lemma [simp]:
raw-init-clss (rough-state-of (do-cdcl_W-stgy-step (state-of (rough-state-from-init-state-of S))))
=
raw-init-clss (rough-state-from-init-state-of S)
unfolding do-cdcl_W-stgy-step-def **by** (auto simp: Let-def)

lemma rough-state-from-init-state-of-do-cdcl_W-stgy-step'[code abstract]:
rough-state-from-init-state-of (do-cdcl_W-stgy-step' S) =
rough-state-of (do-cdcl_W-stgy-step (id-of-I-to S))

proof –
let ? S = (rough-state-from-init-state-of S)
have cdcl_W-stgy** (raw-S0-cdcl_W (raw-init-clss (rough-state-from-init-state-of S)))
(rough-state-from-init-state-of S)
using rough-state-from-init-state-of[of S] **by** auto
moreover have cdcl_W-stgy**
(rough-state-from-init-state-of S)
(rough-state-of (do-cdcl_W-stgy-step
(state-of (rough-state-from-init-state-of S))))
using do-cdcl_W-stgy-step[of state-of ? S]
by (cases do-cdcl_W-stgy-step (state-of ? S) = state-of ? S) auto
ultimately show ?thesis
unfolding do-cdcl_W-stgy-step'-def id-of-I-to-def
by (auto intro: state-from-init-state-of-inverse)
qed

All rules together **function** do-all-cdcl_W-stgy **where**

do-all-cdcl_W-stgy S =
(let T = do-cdcl_W-stgy-step' S in
if T = S then S else do-all-cdcl_W-stgy T)

by fast+

termination

proof (relation {(T , S)}.
(cdcl_W-measure (rough-state-from-init-state-of T),
cdcl_W-measure (rough-state-from-init-state-of S))
∈ le_{rn} {(a , b). $a < b$ } 3}, goal-cases)
case 1
show ?case **by** (rule wf-if-measure-f) (auto intro!: wf-le_{rn} wf-less)
next
case (2 S T) **note** T = this(1) **and** ST = this(2)
let ? S = rough-state-from-init-state-of S

```

have S: cdclW-stgy** (raw-S0-cdclW (raw-init-clss ?S)) ?S
  using rough-state-from-init-state-of[of S] by auto
moreover have cdclW-stgy (rough-state-from-init-state-of S)
  (rough-state-from-init-state-of T)

proof -
  have  $\bigwedge c.$  rough-state-of (state-of (rough-state-from-init-state-of c)) =
    rough-state-from-init-state-of c
  using rough-state-from-init-state-of by force
  then have do-cdclW-stgy-step (state-of (rough-state-from-init-state-of S))
     $\neq$  state-of (rough-state-from-init-state-of S)
  using ST T rough-state-from-init-state-of-inverse
  unfolding id-of-I-to-def do-cdclW-stgy-step'-def
  by fastforce
  from do-cdclW-stgy-step[OF this] show ?thesis
    by (simp add: T id-of-I-to-def rough-state-from-init-state-of-do-cdclW-stgy-step')
qed
moreover
  have cdclW-all-struct-inv (rough-state-from-init-state-of S)
    using rough-state-from-init-state-of[of S] by auto
  then have cdclW-all-struct-inv (raw-S0-cdclW (raw-init-clss (rough-state-from-init-state-of S)))
    by (cases rough-state-from-init-state-of S)
      (auto simp add: cdclW-all-struct-inv-def distinct-cdclW-state-def)
  ultimately show ?case
    by (auto intro!: cdclW-stgy-step-decreasing[of - - raw-S0-cdclW (raw-init-clss ?S)]
      simp del: cdclW-measure.simps)
qed

thm do-all-cdclW-stgy.induct
lemma do-all-cdclW-stgy-induct:
  ( $\bigwedge S.$  (do-cdclW-stgy-step' S  $\neq$  S  $\implies$  P (do-cdclW-stgy-step' S))  $\implies$  P S)  $\implies$  P a0
  using do-all-cdclW-stgy.induct by metis

lemma [simp]: raw-init-clss (rough-state-from-init-state-of (do-all-cdclW-stgy S)) =
  raw-init-clss (rough-state-from-init-state-of S)
apply (induction rule: do-all-cdclW-stgy-induct)
by (smt do-all-cdclW-stgy.simps do-cdclW-stgy-step-def id-of-I-to-def
  raw-init-clss-rough-state-of-do-full1-cp-step raw-init-clss-rough-state-of-do-other-step'
  rough-state-from-init-state-of-do-cdclW-stgy-step'
  toS-rough-state-of-state-of-rough-state-from-init-state-of)

lemma no-step-cdclW-stgy-cdclW-all:
  fixes S :: 'a cdclW-state-inv-from-init-state
  shows no-step cdclW-stgy (rough-state-from-init-state-of (do-all-cdclW-stgy S))
  apply (induction S rule: do-all-cdclW-stgy-induct)
  apply (rename-tac S, case-tac do-cdclW-stgy-step' S  $\neq$  S)
proof -
  fix Sa :: 'a cdclW-state-inv-from-init-state
  assume a1:  $\neg$  do-cdclW-stgy-step' Sa  $\neq$  Sa
  { fix pp
    have (if True then Sa else do-all-cdclW-stgy Sa) = do-all-cdclW-stgy Sa
      using a1 by auto
    then have  $\neg$  cdclW-stgy (rough-state-from-init-state-of (do-all-cdclW-stgy Sa)) pp
      using a1 by (smt do-cdclW-stgy-step-no id-of-I-to-def
        rough-state-from-init-state-of-do-cdclW-stgy-step' rough-state-of-inverse) }

```

```

then show no-step cdclW-stgy (rough-state-from-init-state-of (do-all-cdclW-stgy Sa))
  by fastforce
next
fix Sa :: 'a cdclW-state-inv-from-init-state
assume a1: do-cdclW-stgy-step' Sa ≠ Sa
  ⇒ no-step cdclW-stgy (rough-state-from-init-state-of
    (do-all-cdclW-stgy (do-cdclW-stgy-step' Sa)))
assume a2: do-cdclW-stgy-step' Sa ≠ Sa
have do-all-cdclW-stgy Sa = do-all-cdclW-stgy (do-cdclW-stgy-step' Sa)
  by (metis (full-types) do-all-cdclW-stgy.simps)
then show no-step cdclW-stgy (rough-state-from-init-state-of (do-all-cdclW-stgy Sa))
  using a2 a1 by presburger
qed

lemma do-all-cdclW-stgy-is-rtranclp-cdclW-stgy:
  cdclW-stgy** (rough-state-from-init-state-of S)
    (rough-state-from-init-state-of (do-all-cdclW-stgy S))
proof (induction S rule: do-all-cdclW-stgy-induct)
case (1 S) note IH = this(1)
show ?case
  proof (cases do-cdclW-stgy-step' S = S)
    case True
      then show ?thesis by simp
    next
      case False
        have f2: do-cdclW-stgy-step (id-of-I-to S) = id-of-I-to S →
          rough-state-from-init-state-of (do-cdclW-stgy-step' S)
          = rough-state-of (state-of (rough-state-from-init-state-of S))
          unfolding rough-state-from-init-state-of-do-cdclW-stgy-step'
            id-of-I-to-def by presburger
        have f3: do-all-cdclW-stgy S = do-all-cdclW-stgy (do-cdclW-stgy-step' S)
          by (metis (full-types) do-all-cdclW-stgy.simps)
        have cdclW-stgy (rough-state-from-init-state-of S)
          (rough-state-from-init-state-of (do-cdclW-stgy-step' S))
          = cdclW-stgy (rough-state-of (id-of-I-to S))
            (rough-state-of (do-cdclW-stgy-step (id-of-I-to S)))
          unfolding id-of-I-to-def rough-state-from-init-state-of-do-cdclW-stgy-step'
            toS-rough-state-of-state-of-rough-state-from-init-state-of by presburger
        then show ?thesis
          using f3 f2 IH do-cdclW-stgy-step
          by (smt False toS-rough-state-of-state-of-rough-state-from-init-state-of tranclp.intros(1)
            tranclp-into-rtranclp transitive-closurep-trans'(2))
  qed
qed

```

Final theorem:

```

lemma consistent-interp-mmset-of-mlit[simp]:
  consistent-interp (lit-of ' mmset-of-mlit' ' set M') ↔
  consistent-interp (lit-of ' set M')
by (auto simp: image-image)

```

lemma DPLL-tot-correct:

```

assumes
  r: rough-state-from-init-state-of (do-all-cdclW-stgy (state-from-init-state-of
    ([], map remdups N, [], 0, None))) = S and

```

```

  S: (M', N', U', k, E) = S
shows (E ≠ Some [] ∧ satisfiable (set (map mset N)))
  ∨ (E = Some [] ∧ unsatisfiable (set (map mset N)))
proof –
  let ?N = map remdups N
  have inv: cdclW-all-struct-inv ([], map remdups N, [], 0, None)
    unfolding cdclW-all-struct-inv-def distinct-cdclW-state-def distinct-mset-set-def by auto
  then have S0: rough-state-of (state-of ([], map remdups N, [], 0, None))
    = ([], map remdups N, [], 0, None) by simp
  have 1: full cdclW-stgy ([], ?N, [], 0, None) S
    unfolding full-def apply rule
    using do-all-cdclW-stgy-is-rtrancpl-cdclW-stgy[of
      state-from-init-state-of ([], map remdups N, [], 0, None)] inv
    by (auto simp del: do-all-cdclW-stgy.simps simp: state-from-init-state-of-inverse
      r[symmetric] no-step-cdclW-stgy-cdclW-all)+
  moreover have 2: finite (set (map mset ?N)) by auto
  moreover have 3: distinct-mset-set (set (map mset ?N))
    unfolding distinct-mset-set-def by auto
  moreover
    have cdclW-all-struct-inv S
      by (metis (no-types) cdclW-all-struct-inv-rough-state r
        toS-rough-state-of-state-of-rough-state-from-init-state-of)
    then have cons: consistent-interp (lits-of-l M')
      unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def S[symmetric]
      by (auto simp: lits-of-def)
  moreover
    have [simp]:
      rough-state-from-init-state-of (state-from-init-state-of (raw-S0-cdclW (map remdups N)))
      = raw-S0-cdclW (map remdups N)
    apply (rule cdclW-state-inv-from-init-state.state-from-init-state-of-inverse)
    using 3 by (auto simp: cdclW-all-struct-inv-def distinct-cdclW-state-def
      image-image comp-def)
    have raw-init-clss ([], ?N, [], 0, None) = raw-init-clss S
      using arg-cong[OF r, of raw-init-clss] unfolding S[symmetric]
      by (simp del: do-all-cdclW-stgy.simps)
    then have N': N' = map remdups N
      using S[symmetric] by auto
  have conflicting S = Some {#} ∧ unsatisfiable (set-mset (init-clss S)) ∨
    conflicting S = None ∧ (case S of (M, uu-) ⇒ map mmset-of-mlit' M) ⊨asm init-clss S
  apply (rule full-cdclW-stgy-final-state-conclusive)
    using 1 apply simp
    using 2 apply simp
    using 3 by simp
  then have (E ≠ Some [] ∧ satisfiable (set (map mset ?N)))
    ∨ (E = Some [] ∧ unsatisfiable (set (map mset ?N)))
    using cons unfolding S[symmetric] N' apply (auto simp: comp-def)
    by (simp add: true-annots-true-clss)
  then show ?thesis by auto
qed

```

The Code The SML code is skipped in the documentation, but stays to ensure that some version of the exported code is working. The only difference between the generated code and the one used here is the export of the constructor ConI.

```

end
theory CDCL-W-Merge
imports CDCL-W-Termination
begin

```

21 Link between Weidenbach's and NOT's CDCL

21.1 Inclusion of the states

```

context conflict-driven-clause-learningW
begin
declare  $cdcl_W.intros[intro]$   $cdcl_W-bj.intros[intro]$   $cdcl_W-o.intros[intro]$ 

lemma backtrack-no-cdclW-bj:
  assumes  $cdcl: cdcl_W-bj\ T\ U$  and  $inv: cdcl_W-M-level-inv\ V$ 
  shows  $\neg backtrack\ V\ T$ 
  using  $cdcl\ inv$ 
  apply (induction rule:  $cdcl_W-bj.induct$ )
  apply (elim skipE, force elim!: backtrack-levE[OF - inv] simp:  $cdcl_W-M-level-inv-def$ )
  apply (elim resolveE, force elim!: backtrack-levE[OF - inv] simp:  $cdcl_W-M-level-inv-def$ )
  apply standard
  apply (elim backtrack-levE[OF - inv], elim backtrackE)
  apply (force simp del: state-simp simp add: state-eq-def  $cdcl_W-M-level-inv-decomp$ )
  done

inductive skip-or-resolve :: 'st  $\Rightarrow$  'st  $\Rightarrow$  bool where
  s-or-r-skip[ $intro$ ]:  $skip\ S\ T \Longrightarrow skip-or-resolve\ S\ T$  |
  s-or-r-resolve[ $intro$ ]:  $resolve\ S\ T \Longrightarrow skip-or-resolve\ S\ T$ 

lemma rtrancpl-cdclW-bj-skip-or-resolve-backtrack:
  assumes  $cdcl_W-bj^{**}\ S\ U$  and  $inv: cdcl_W-M-level-inv\ S$ 
  shows  $skip-or-resolve^{**}\ S\ U \vee (\exists T. skip-or-resolve^{**}\ S\ T \wedge backtrack\ T\ U)$ 
  using  $assms$ 
proof (induction)
  case base
  then show ?case by simp
next
  case (step  $U\ V$ ) note  $st = this(1)$  and  $bj = this(2)$  and  $IH = this(3)[OF\ this(4)]$ 
  consider
    ( $SU$ )  $S = U$ 
  | ( $SUp$ )  $cdcl_W-bj^{++}\ S\ U$ 
  using  $st$  unfolding rtrancpl-unfold by blast
  then show ?case
  proof cases
    case  $SUp$ 
    have  $\bigwedge T. skip-or-resolve^{**}\ S\ T \Longrightarrow cdcl_W^{**}\ S\ T$ 
    using mono-rtrancpl[of skip-or-resolve  $cdcl_W$ ]
    by (blast intro: skip-or-resolve.cases)
    then have  $skip-or-resolve^{**}\ S\ U$ 
    using  $bj\ IH\ inv$  backtrack-no-cdclW-bj rtrancpl-cdclW-consistent-inv[OF - inv] by meson
    then show ?thesis
    using  $bj$  by (auto simp:  $cdcl_W-bj.simps$  dest!: skip-or-resolve.intros)
  next
    case  $SU$ 

```

```

    then show ?thesis
    using bj by (auto simp: cdclW-bj.simps dest!: skip-or-resolve.intros)
qed
qed

```

```

lemma rtrancpl-skip-or-resolve-rtrancpl-cdclW:
  skip-or-resolve** S T  $\implies$  cdclW** S T
  by (induction rule: rtrancpl-induct)
  (auto dest!: cdclW-bj.intros cdclW.intros cdclW-o.intros simp: skip-or-resolve.simps)

```

```

definition backjump-l-cond :: 'v clause  $\Rightarrow$  'v clause  $\Rightarrow$  'v literal  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  bool where
backjump-l-cond  $\equiv$   $\lambda C\ C'\ L'\ S\ T.$  True

```

```

definition invNOT :: 'st  $\Rightarrow$  bool where
invNOT  $\equiv$   $\lambda S.$  no-dup (trail S)

```

```

declare invNOT-def[simp]
end

```

```

context conflict-driven-clause-learningW
begin

```

21.2 More lemmas conflict-propagate and backjumping

21.2.1 Termination

```

lemma cdclW-cp-normalized-element-all-inv:
  assumes inv: cdclW-all-struct-inv S
  obtains T where full cdclW-cp S T
  using assms cdclW-cp-normalized-element unfolding cdclW-all-struct-inv-def by blast
thm backtrackE

```

```

lemma cdclW-bj-measure:
  assumes cdclW-bj S T and cdclW-M-level-inv S
  shows length (trail S) + (if conflicting S = None then 0 else 1)
    > length (trail T) + (if conflicting T = None then 0 else 1)
  using assms by (induction rule: cdclW-bj.induct)
  (force dest:arg-cong[of - - length]
   intro: get-all-marked-decomposition-exists-prepend
   elim!: backtrack-levE skipE resolveE
   simp: cdclW-M-level-inv-def)+

```

```

lemma wf-cdclW-bj:
  wf {(b,a). cdclW-bj a b  $\wedge$  cdclW-M-level-inv a}
  apply (rule wfP-if-measure[of  $\lambda$ -. True
    -  $\lambda T.$  length (trail T) + (if conflicting T = None then 0 else 1), simplified])
  using cdclW-bj-measure by simp

```

```

lemma cdclW-bj-exists-normal-form:
  assumes lev: cdclW-M-level-inv S
  shows  $\exists T.$  full cdclW-bj S T
proof -
  obtain T where T: full ( $\lambda a\ b.$  cdclW-bj a b  $\wedge$  cdclW-M-level-inv a) S T
  using wf-exists-normal-form-full[OF wf-cdclW-bj] by auto
  then have cdclW-bj** S T
  by (auto dest: rtrancpl-and-rtrancpl-left simp: full-def)

```

moreover
then have $cdcl_W^{**} S T$
using $mono-rtrancpl[of\ cdcl_W-bj\ cdcl_W]$ **by** $blast$
then have $cdcl_W-M-level-inv\ T$
using $rtrancpl-cdcl_W-consistent-inv\ lev$ **by** $auto$
ultimately show $?thesis$ **using** T **unfolding** $full-def$ **by** $auto$
qed
lemma $rtrancpl-skip-state-decomp$:
assumes $skip^{**} S T$ **and** $no-dup\ (trail\ S)$
shows
 $\exists M. trail\ S = M @ trail\ T \wedge (\forall m \in set\ M. \neg is-marked\ m)$
 $init-clss\ S = init-clss\ T$
 $learned-clss\ S = learned-clss\ T$
 $backtrack-lvl\ S = backtrack-lvl\ T$
 $conflicting\ S = conflicting\ T$
using $assms$ **by** $(induction\ rule:\ rtrancpl-induct)$
 $(auto\ simp\ del:\ state-simp\ simp:\ state-eq-def\ elim!:\ skipE)$

21.2.2 More backjumping

Backjumping after skipping or jump directly **lemma** $rtrancpl-skip-backtrack-backtrack$:

assumes
 $skip^{**} S T$ **and**
 $backtrack\ T W$ **and**
 $cdcl_W-all-struct-inv\ S$
shows $backtrack\ S W$
using $assms$
proof $induction$
case $base$
then show $?case$ **by** $simp$
next
case $(step\ T\ V)$ **note** $st = this(1)$ **and** $skip = this(2)$ **and** $IH = this(3)$ **and** $bt = this(4)$ **and**
 $inv = this(5)$
have $skip^{**} S V$
using $st\ skip$ **by** $auto$
then have $cdcl_W-all-struct-inv\ V$
using $rtrancpl-mono[of\ skip\ cdcl_W]$ $assms(3)$ $rtrancpl-cdcl_W-all-struct-inv-inv\ mono-rtrancpl$
by $(auto\ dest!:\ bj\ other\ cdcl_W-bj.skip)$
then have $cdcl_W-M-level-inv\ V$
unfolding $cdcl_W-all-struct-inv-def$ **by** $auto$
then obtain $K\ i\ M1\ M2\ L\ D$ **where**
 $conf:\ raw-conflicting\ V = Some\ D$ **and**
 $LD:\ L \in \# mset-ccls\ D$ **and**
 $decomp:\ (Marked\ K\ (Suc\ i) \# M1, M2) \in set\ (get-all-marked-decomposition\ (trail\ V))$ **and**
 $lev-L:\ get-level\ (trail\ V)\ L = backtrack-lvl\ V$ **and**
 $max:\ get-level\ (trail\ V)\ L = get-maximum-level\ (trail\ V)\ (mset-ccls\ D)$ **and**
 $max-D:\ get-maximum-level\ (trail\ V)\ (remove1-mset\ L\ (mset-ccls\ D)) \equiv i$ **and**
 $undef:\ undefined-lit\ M1\ L$ **and**
 $W:\ W \sim cons-trail\ (Propagated\ L\ (cls-of-ccls\ D))$
 $(reduce-trail-to\ M1$
 $(add-learned-cls\ (cls-of-ccls\ D)$
 $(update-backtrack-lvl\ i$
 $(update-conflicting\ None\ V))))$
using $bt\ inv$ **by** $(elim\ backtrack-levE)\ metis+$
obtain $L'\ C'\ M\ E$ **where**
 $tr:\ trail\ T = Propagated\ L'\ C'\ \# M$ **and**

```

raw: raw-conflicting  $T = \text{Some } E$  and
LE:  $-L' \notin \# \text{ mset-ccls } E$  and
E:  $\text{mset-ccls } E \neq \{\#\}$  and
V:  $V \sim \text{tl-trail } T$ 
using skip by (elim skipE) metis
let ?M = Propagated  $L' C' \# \text{ trail } V$ 
have tr-M: trail  $T = ?M$ 
using tr V by auto
have MT:  $M = \text{tl } (\text{trail } T)$  and MV:  $M = \text{trail } V$ 
using tr V by auto
have DE[simp]:  $\text{mset-ccls } D = \text{mset-ccls } E$ 
using V conf raw by (auto simp add: state-eq-def simp del: state-simp)
have cdclW** S T using bj cdclW-bj.skip mono-rtranclp[of skip cdclW S T] other st by meson
then have inv': cdclW-all-struct-inv T
using rtranclp-cdclW-all-struct-inv-inv inv by blast
have M-lev: cdclW-M-level-inv T using inv' unfolding cdclW-all-struct-inv-def by auto
then have n-d': no-dup ?M
using tr-M unfolding cdclW-M-level-inv-def by auto
let ?k = backtrack-lvl T
have [simp]:
backtrack-lvl V = ?k
using V by simp
have ?k > 0
using decomp M-lev V tr unfolding cdclW-M-level-inv-def by auto
then have atm-of L ∈ atm-of ' lits-of-l (trail V)
using lev-L get-rev-level-ge-0-atm-of-in[of 0 rev (trail V) L] by auto
then have L-L': atm-of L ≠ atm-of L'
using n-d' unfolding lits-of-def by auto
have L'-M: atm-of L' ∉ atm-of ' lits-of-l (trail V)
using n-d' unfolding lits-of-def by auto
have ?M ⊨as CNot (mset-ccls D)
using inv' raw unfolding cdclW-conflicting-def cdclW-all-struct-inv-def tr-M by auto
then have L' ∉ # mset-ccls (remove-clit L D)
using L-L' L'-M ⟨Propagated L' C' # trail V ⊨as CNot (mset-ccls D)⟩
unfolding true-annots-true-clss true-clss-def
by (auto simp: uminus-lit-swap atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set dest!: in-diffD)
have [simp]: trail (reduce-trail-to M1 T) = M1
using decomp undef tr W V by auto
have skip** S V
using st skip by auto
have no-dup (trail S)
using inv unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def by auto
then have [simp]: init-clss S = init-clss V and [simp]: learned-clss S = learned-clss V
using rtranclp-skip-state-decomp[OF ⟨skip** S V⟩] V
by (auto simp del: state-simp simp: state-eq-def)
then have
W-S:  $W \sim \text{cons-trail } (\text{Propagated } L \text{ (cls-of-ccls } E)) \text{ (reduce-trail-to } M1$ 
 $\text{ (add-learned-clss (cls-of-ccls } E) \text{ (update-backtrack-lvl } i \text{ (update-conflicting None } T))))$ 
using W V undef M-lev decomp tr
by (auto simp del: state-simp simp: state-eq-def cdclW-M-level-inv-def)

obtain M2' where
decomp':  $(\text{Marked } K \text{ (} i+1 \text{)} \# M1, M2') \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } T))$ 
using decomp V unfolding tr-M by (cases hd (get-all-marked-decomposition (trail V)),
cases get-all-marked-decomposition (trail V)) auto

```



```

moreover
  from  $L-L'$  have  $\text{get-level } ?M \ L = ?k$ 
    using  $\text{lev-L } V$  by ( $\text{auto split: if-split-asm}$ )
moreover
  have  $\text{atm-of } L' \notin \text{atms-of } (\text{mset-ccls } D)$ 
    by ( $\text{metis DE LE } L-L' \langle L' \notin \# \text{mset-ccls } (\text{remove-clit } L \ D) \rangle \text{ in-remove1-mset-neq remove-clit}$ 
       $\text{atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set atms-of-def}$ )
  then have  $\text{get-level } ?M \ L = \text{get-maximum-level } ?M \ (\text{mset-ccls } D)$ 
    using  $\text{calculation(2) lev-L max}$  by  $\text{auto}$ 
moreover
  have  $\text{atm-of } L' \notin \text{atms-of } (\text{mset-ccls } (\text{remove-clit } L \ D))$ 
    by ( $\text{metis DE LE } L-L' \langle L' \notin \# \text{mset-ccls } (\text{remove-clit } L \ D) \rangle \text{ in-remove1-mset-neq remove-clit}$ 
       $\text{atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set atms-of-def in-remove1-mset-neq remove-clit}$ 
       $\text{in-atms-of-remove1-mset-in-atms-of}$ )
  have  $i = \text{get-maximum-level } ?M \ (\text{mset-ccls } (\text{remove-clit } L \ D))$ 
    using  $\text{max-D } \langle \text{atm-of } L' \notin \text{atms-of } (\text{mset-ccls } (\text{remove-clit } L \ D)) \rangle$  by  $\text{auto}$ 

ultimately have  $\text{backtrack } T \ W$ 
  apply –
  apply ( $\text{rule backtrack-rule[of } T - L \ K \ i \ M1 \ M2' \ W, \ OF \ \text{raw}]$ )
  unfolding  $\text{tr-M[symmetric]}$ 
    using  $LD$  apply  $\text{simp}$ 
    apply  $\text{simp}$ 
    apply  $\text{simp}$ 
    apply  $\text{simp}$ 
    apply  $\text{auto}[]$ 
  using  $W-S$  by  $\text{auto}$ 
then show  $?thesis$  using  $IH \ inv$  by  $\text{blast}$ 
qed

lemma  $\text{fst-get-all-marked-decomposition-prepend-not-marked:}$ 
  assumes  $\forall m \in \text{set } MS. \neg \text{is-marked } m$ 
  shows  $\text{set } (\text{map fst } (\text{get-all-marked-decomposition } M))$ 
     $= \text{set } (\text{map fst } (\text{get-all-marked-decomposition } (MS @ M)))$ 
  using  $\text{assms apply (induction MS rule: marked-lit-list-induct)}$ 
  apply  $\text{auto[2]}$ 
  by ( $\text{rename-tac } L \ m \ xs; \text{case-tac get-all-marked-decomposition } (xs @ M)) \text{ simp-all}$ 

See also  $\llbracket \text{skip}^{**} \ ?S \ ?T; \text{backtrack } ?T \ ?W; \text{cdcl}_W\text{-all-struct-inv } ?S \rrbracket \implies \text{backtrack } ?S \ ?W$ 

lemma  $\text{rtrancplp-skip-backtrack-backtrack-end:}$ 
  assumes
     $\text{skip: skip}^{**} \ S \ T$  and
     $\text{bt: backtrack } S \ W$  and
     $\text{inv: cdcl}_W\text{-all-struct-inv } S$ 
  shows  $\text{backtrack } T \ W$ 
  using  $\text{assms}$ 
proof –
  have  $M\text{-lev: cdcl}_W\text{-M-level-inv } S$ 
    using  $\text{bt inv unfolding cdcl}_W\text{-all-struct-inv-def}$  by ( $\text{auto elim!: backtrack-levE}$ )
  then obtain  $K \ i \ M1 \ M2 \ L \ D$  where
     $\text{raw-S: raw-conflicting } S = \text{Some } D$  and
     $LD: L \in \# \text{mset-ccls } D$  and
     $\text{decomp: (Marked } K \ (\text{Suc } i) \ \# \ M1, \ M2) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$  and
     $\text{lev-l: get-level } (\text{trail } S) \ L = \text{backtrack-lvl } S$  and
     $\text{lev-l-D: get-level } (\text{trail } S) \ L = \text{get-maximum-level } (\text{trail } S) \ (\text{mset-ccls } D)$  and

```

i: *get-maximum-level* (*trail S*) (*remove1-mset L (mset-ccls D)*) $\equiv i$ **and**
undef: *undefined-lit M1 L* **and**
W: $W \sim \text{cons-trail } (\text{Propagated } L \text{ (cls-of-ccls } D))$
 (*reduce-trail-to M1*
 (*add-learned-cls (cls-of-ccls D)*
 (*update-backtrack-lvl i*
 (*update-conflicting None S*))))

using *bt* **by** (*elim backtrack-levE*)
 (*simp-all add: cdcl_W-M-level-inv-decomp state-eq-def del: state-simp*)
let *?D* = *remove1-mset L (mset-ccls D)*

have [*simp*]: *no-dup (trail S)*
 using *M-lev* **by** (*auto simp: cdcl_W-M-level-inv-decomp*)
have *cdcl_W-all-struct-inv T*
 using *mono-rtrancpl[of skip cdcl_W]* **by** (*smt bj cdcl_W-bj.skip inv local.skip other*
 rtrancpl-cdcl_W-all-struct-inv-inv)
then have [*simp*]: *no-dup (trail T)*
 unfolding *cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def* **by** *auto*

obtain *MS M_T* **where** *M*: *trail S = MS @ M_T* **and** *M_T*: *M_T = trail T* **and** *nm*: $\forall m \in \text{set } MS. \neg \text{is-marked } m$
 using *rtrancpl-skip-state-decomp(1)[OF skip] raw-S M-lev* **by** *auto*
have *T*: *state T = (M_T, init-cls S, learned-cls S, backtrack-lvl S, Some (mset-ccls D))*
 using *M_T rtrancpl-skip-state-decomp[of S T] skip raw-S*
 by (*auto simp del: state-simp simp: state-eq-def*)

have *cdcl_W-all-struct-inv T*
 apply (*rule rtrancpl-cdcl_W-all-struct-inv-inv[OF - inv]*)
 using *bj cdcl_W-bj.skip local.skip other rtrancpl-mono[of skip cdcl_W]* **by** *blast*
then have *M_T \models_{as} CNot (mset-ccls D)*
 unfolding *cdcl_W-all-struct-inv-def cdcl_W-conflicting-def* **using** *T* **by** *blast*
then have $\forall L \in \# \text{mset-ccls } D. \text{atm-of } L \in \text{atm-of ' lits-of-l } M_T$
 by (*meson atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set*
 true-annots-true-cls-def-iff-negation-in-model)
moreover have *no-dup (trail S)*
 using *inv unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def* **by** *auto*
ultimately have $\forall L \in \# \text{mset-ccls } D. \text{atm-of } L \notin \text{atm-of ' lits-of-l } MS$
 unfolding *M* **unfolding** *lits-of-def* **by** *auto*
then have *H*: $\bigwedge L. L \in \# \text{mset-ccls } D \implies \text{get-level } (\text{trail } S) \text{ } L = \text{get-level } M_T \text{ } L$
 unfolding *M* **by** (*fastforce simp: lits-of-def*)
have [*simp*]: *get-maximum-level (trail S) (mset-ccls D) = get-maximum-level M_T (mset-ccls D)*
 using $\langle M_T \models_{as} \text{CNot } (\text{mset-ccls } D) \rangle$ *M nm* **by** (*metis true-annots-CNot-all-atms-defined*
 get-maximum-level-skip-un-marked-not-present)

have *lev-l'*: *get-level M_T L = backtrack-lvl S*
 using *lev-l LD* **by** (*auto simp: H*)
have [*simp*]: *trail (reduce-trail-to M1 T) = M1*
 using *T decomp M nm* **by** (*smt M_T append-assoc beginning-not-marked-invert*
 get-all-marked-decomposition-exists-prepend reduce-trail-to-trail-tl-trail-decomp)
have *W*: $W \sim \text{cons-trail } (\text{Propagated } L \text{ (cls-of-ccls } D))$ (*reduce-trail-to M1*
 (*add-learned-cls (cls-of-ccls D)* (*update-backtrack-lvl i* (*update-conflicting None T*))))
 using *W T i decomp undef* **by** (*auto simp del: state-simp simp: state-eq-def*)
have *lev-l-D'*: *get-level M_T L = get-maximum-level M_T (mset-ccls D)*
 using *lev-l-D LD* **by** (*auto simp: H*)

```

have [simp]: get-maximum-level (trail S) ?D = get-maximum-level MT ?D
  by (smt H get-maximum-level-exists-lit get-maximum-level-ge-get-level in-diffD le-antisym
    not-gr0 not-less)
then have i': i = get-maximum-level MT ?D
  using i by auto
have Marked K (i + 1) # M1 ∈ set (map fst (get-all-marked-decomposition (trail S)))
  using Set.imageI[OF decomp, of fst] by auto
then have Marked K (i + 1) # M1 ∈ set (map fst (get-all-marked-decomposition MT))
  using fst-get-all-marked-decomposition-prepend-not-marked[OF nm] unfolding M by auto
then obtain M2' where decomp':(Marked K (i+1) # M1, M2') ∈ set (get-all-marked-decomposition
MT)
  by auto
then show backtrack T W
  using T decomp' lev-l' lev-l-D' i' W LD undef
  by (force intro!: backtrack.intros simp del: state-simp simp: state-eq-def)
qed

```

```

lemma cdclW-bj-decomp-resolve-skip-and-bj:
  assumes cdclW-bj** S T and inv: cdclW-M-level-inv S
  shows (skip-or-resolve** S T
    ∨ (∃ U. skip-or-resolve** S U ∧ backtrack U T))
  using assms
proof induction
  case base
  then show ?case by simp
next
  case (step T U) note st = this(1) and bj = this(2) and IH = this(3)
  have IH: skip-or-resolve** S T
  proof -
    { assume (∃ U. skip-or-resolve** S U ∧ backtrack U T)
      then obtain V where
        bt: backtrack V T and
        skip-or-resolve** S V
        by blast
      have cdclW** S V
        using (skip-or-resolve** S V) rtranclp-skip-or-resolve-rtranclp-cdclW by blast
      then have cdclW-M-level-inv V and cdclW-M-level-inv S
        using rtranclp-cdclW-consistent-inv inv by blast+
      with bj bt have False using backtrack-no-cdclW-bj by simp
    }
    then show ?thesis using IH inv by blast
  qed
show ?case
  using bj
proof (cases rule: cdclW-bj.cases)
  case backtrack
  then show ?thesis using IH by blast
qed (metis (no-types, lifting) IH rtranclp.simps skip-or-resolve.simps)+
qed

```

```

lemma resolve-skip-deterministic:
  resolve S T ⇒ skip S U ⇒ False
  by (auto elim!: skipE resolveE dest: hd-raw-trail)

```

```

lemma backtrack-unique:

```

assumes
bt-T: *backtrack S T* **and**
bt-U: *backtrack S U* **and**
inv: *cdcl_W-all-struct-inv S*
shows $T \sim U$
proof –
have *lev*: *cdcl_W-M-level-inv S*
using *inv* **unfolding** *cdcl_W-all-struct-inv-def* **by** *auto*
then obtain *K i M1 M2 L D* **where**
raw-S: *raw-conflicting S = Some D* **and**
LD: $L \in \# \text{ mset-ccls } D$ **and**
decomp: $(\text{Marked } K (\text{Suc } i) \# M1, M2) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$ **and**
lev-l: *get-level (trail S) L = backtrack-lvl S* **and**
lev-l-D: *get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D)* **and**
i: *get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) \equiv i* **and**
undef: *undefined-lit M1 L* **and**
T: $T \sim \text{cons-trail } (\text{Propagated } L (\text{cls-of-ccls } D))$
 $(\text{reduce-trail-to } M1$
 $(\text{add-learned-cls } (\text{cls-of-ccls } D)$
 $(\text{update-backtrack-lvl } i$
 $(\text{update-conflicting } \text{None } S))))$
using *bt-T* **by** $(\text{elim backtrack-levE}) (\text{force simp: cdcl}_W\text{-M-level-inv-def})+$

obtain *K' i' M1' M2' L' D'* **where**
raw-S': *raw-conflicting S = Some D'* **and**
LD': $L' \in \# \text{ mset-ccls } D'$ **and**
decomp': $(\text{Marked } K' (\text{Suc } i') \# M1', M2') \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$ **and**
lev-l: *get-level (trail S) L' = backtrack-lvl S* **and**
lev-l-D: *get-level (trail S) L' = get-maximum-level (trail S) (mset-ccls D')* **and**
i': *get-maximum-level (trail S) (remove1-mset L' (mset-ccls D')) \equiv i'* **and**
undef': *undefined-lit M1' L'* **and**
U: $U \sim \text{cons-trail } (\text{Propagated } L' (\text{cls-of-ccls } D'))$
 $(\text{reduce-trail-to } M1'$
 $(\text{add-learned-cls } (\text{cls-of-ccls } D')$
 $(\text{update-backtrack-lvl } i'$
 $(\text{update-conflicting } \text{None } S))))$
using *bt-U lev* **by** $(\text{elim backtrack-levE}) (\text{force simp: cdcl}_W\text{-M-level-inv-def})+$
obtain *c* **where** *M*: *trail S = c @ M2 @ Marked K (i + 1) # M1*
using *decomp* **by** *auto*
obtain *c'* **where** *M'*: *trail S = c' @ M2' @ Marked K' (i' + 1) # M1'*
using *decomp'* **by** *auto*
have *marked*: *get-all-levels-of-marked (trail S) = rev [1.. $1 + \text{backtrack-lvl } S$]*
using *inv* **unfolding** *cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def* **by** *auto*
then have $i < \text{backtrack-lvl } S$
unfolding *M* **by** $(\text{force simp add: rev-swap[symmetric] dest!: arg-cong[of - - set]})$

have $[\text{simp}]: L' = L$
proof (rule ccontr)
assume $\neg ?thesis$
then have $L' \in \# \text{ remove1-mset } L (\text{mset-ccls } D)$
using *raw-S raw-S' LD LD'* **by** $(\text{simp add: in-remove1-mset-neq})$
then have *get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) \geq backtrack-lvl S*
using $(\text{get-level } (\text{trail } S) L' = \text{backtrack-lvl } S)$ *get-maximum-level-ge-get-level*
by *metis*

```

    then show False using i' i < i < backtrack-lvl S by auto
  qed
then have [simp]: mset-ccls D' = mset-ccls D
  using raw-S raw-S' by auto
have [simp]: i' = i
  using i i' by auto

```

Automation in a step later...

```

have H:  $\bigwedge a A B. \text{insert } a A = B \implies a : B$ 
  by blast
have get-all-levels-of-marked (c @ M2) = rev [i+2..<1+backtrack-lvl S] and
  get-all-levels-of-marked (c' @ M2') = rev [i+2..<1+backtrack-lvl S]
  using marked unfolding M
  using marked unfolding M'
  unfolding rev-swap[symmetric] by (auto dest: append-cons-eq-upt-length-i-end)
from arg-cong[OF this(1), of set] arg-cong[OF this(2), of set]
have
  dropWhile ( $\lambda L. \neg \text{is-marked } L \vee \text{level-of } L \neq \text{Suc } i$ ) (c @ M2) = [] and
  dropWhile ( $\lambda L. \neg \text{is-marked } L \vee \text{level-of } L \neq \text{Suc } i$ ) (c' @ M2') = []
  unfolding dropWhile-eq-Nil-conv Ball-def
  by (intro allI; rename-tac x; case-tac x; auto dest!: H simp add: in-set-conv-decomp)+

then have [simp]: M1' = M1
  using arg-cong[OF M, of dropWhile ( $\lambda L. \neg \text{is-marked } L \vee \text{level-of } L \neq \text{Suc } i$ )]
  unfolding M' by auto
show ?thesis using T U undef inv decomp by (auto simp del: state-simp simp: state-eq-def
  cdclW-all-struct-inv-def cdclW-M-level-inv-decomp)
qed

```

lemma *if-can-apply-backtrack-no-more-resolve:*

```

assumes
  skip: skip** S U and
  bt: backtrack S T and
  inv: cdclW-all-struct-inv S
shows  $\neg \text{resolve } U V$ 
proof (rule ccontr)
  assume resolve:  $\neg \neg \text{resolve } U V$ 

```

```

obtain L E D where
  U: trail U  $\neq []$  and
  tr-U: hd-raw-trail U = Propagated L E and
  LE: L  $\in \#$  mset-cls E and
  raw-U: raw-conflicting U = Some D and
  LD:  $-L \in \#$  mset-ccls D and
  get-maximum-level (trail U) (mset-ccls (remove-clit (-L) D)) = backtrack-lvl U and
  V: V  $\sim$  update-conflicting (Some (union-ccls (remove-clit (-L) D)
    (ccls-of-cls (remove-lit L E))))
    (tl-trail U)
  using resolve by (auto elim!: resolveE)
have cdclW-all-struct-inv U
  using mono-rtrancp[of skip cdclW] by (meson bj cdclW-bj.skip inv local.skip other
    rtrancp-cdclW-all-struct-inv-inv)
then have [iff]: no-dup (trail S) cdclW-M-level-inv S and [iff]: no-dup (trail U)
  using inv unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def by blast+
then have

```

S : $\text{init-clss } U = \text{init-clss } S$
 $\text{learned-clss } U = \text{learned-clss } S$
 $\text{backtrack-lvl } U = \text{backtrack-lvl } S$
 $\text{conflicting } S = \text{Some } (\text{mset-ccls } D)$
using $\text{rtrancplp-skip-state-decomp}[OF \text{ skip}] \ U \text{ raw-}U$
by $(\text{auto simp del: state-simp simp: state-eq-def})$
obtain M_0 **where**
 $\text{tr-}S$: $\text{trail } S = M_0 @ \text{trail } U$ **and**
 nm : $\forall m \in \text{set } M_0. \neg \text{is-marked } m$
using $\text{rtrancplp-skip-state-decomp}[OF \text{ skip}]$ **by** blast

obtain $K' \ i' \ M1' \ M2' \ L' \ D'$ **where**
 $\text{raw-}S'$: $\text{raw-conflicting } S = \text{Some } D'$ **and**
 LD' : $L' \in \# \text{ mset-ccls } D'$ **and**
 decomp' : $(\text{Marked } K' (\text{Suc } i') \# M1', M2') \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$ **and**
 lev-l : $\text{get-level } (\text{trail } S) \ L' = \text{backtrack-lvl } S$ **and**
 lev-l-D : $\text{get-level } (\text{trail } S) \ L' = \text{get-maximum-level } (\text{trail } S) (\text{mset-ccls } D')$ **and**
 i' : $\text{get-maximum-level } (\text{trail } S) (\text{remove1-mset } L' (\text{mset-ccls } D')) \equiv i'$ **and**
 undef' : $\text{undefined-lit } M1' \ L'$ **and**
 R : $T \sim \text{cons-trail } (\text{Propagated } L' (\text{cls-of-ccls } D'))$
 $(\text{reduce-trail-to } M1'$
 $(\text{add-learned-cls } (\text{cls-of-ccls } D')$
 $(\text{update-backtrack-lvl } i'$
 $(\text{update-conflicting } \text{None } S))))$
using bt **by** $(\text{elim backtrack-levE}) (\text{fastforce simp: } S \text{ state-eq-def simp del: state-simp}) +$
obtain c **where** M : $\text{trail } S = c @ M2' @ \text{Marked } K' (i' + 1) \# M1'$
using $\text{get-all-marked-decomposition-exists-prepend}[OF \text{ decomp}']$ **by** auto
have marked : $\text{get-all-levels-of-marked } (\text{trail } S) = \text{rev } [1..<1+\text{backtrack-lvl } S]$
using $\text{inv unfolding cdcl}_W\text{-all-struct-inv-def cdcl}_W\text{-M-level-inv-def}$ **by** auto
then have $i' < \text{backtrack-lvl } S$
unfolding M **by** $(\text{force simp add: rev-swap[symmetric] dest!: arg-cong[of - - set]})$

have U : $\text{trail } U = \text{Propagated } L (\text{mset-cls } E) \# \text{trail } V$
using $\text{tr-}S \text{ hd-raw-trail}[OF \ U] \ U \ S \ V \ \text{tr-}U$ **by** $(\text{auto simp: lits-of-def})$
have $DD'[simp]$: $\text{mset-ccls } D' = \text{mset-ccls } D$
using $\text{raw-}U \text{ raw-}S' \ S$ **by** auto
have $[simp]$: $L' = -L$
proof (rule ccontr)
assume $\neg ?thesis$
then have $-L \in \# \text{ remove1-mset } L' (\text{mset-ccls } D')$
using $DD' \ LD' \ LD$ **by** $(\text{simp add: in-remove1-mset-neq})$
moreover
have M' : $\text{trail } S = M_0 @ \text{Propagated } L (\text{mset-cls } E) \# \text{trail } V$
using $\text{tr-}S \text{ unfolding } U$ **by** auto
have $\text{no-dup } (\text{trail } S)$
using $\text{inv } U \text{ unfolding cdcl}_W\text{-all-struct-inv-def cdcl}_W\text{-M-level-inv-def}$ **by** auto
then have atm-L-notin-M : $\text{atm-of } L \notin \text{atm-of } ' (\text{lits-of-l } (\text{trail } V))$
using $M' \ U \ S$ **by** $(\text{auto simp: lits-of-def})$
have $\text{get-all-levels-of-marked } (\text{trail } S) = \text{rev } [1..<1+\text{backtrack-lvl } S]$
using $\text{inv } U \text{ unfolding cdcl}_W\text{-all-struct-inv-def cdcl}_W\text{-M-level-inv-def}$ **by** auto
then have $\text{get-all-levels-of-marked } (\text{trail } U) = \text{rev } [1..<1+\text{backtrack-lvl } S]$
using $\text{nm } M' \ U$ **by** $(\text{simp add: get-all-levels-of-marked-no-marked})$
then have get-lev-L :
 $\text{get-level}(\text{Propagated } L (\text{mset-cls } E) \# \text{trail } V) \ L = \text{backtrack-lvl } S$
using $\text{get-level-get-rev-level-get-all-levels-of-marked}[OF \text{ atm-L-notin-M},$

```

    of [Propagated L (mset-cls E)] U by auto
  have atm-of L  $\notin$  atm-of ' (lits-of-l (rev M0))
    using ⟨no-dup (trail S)⟩ M' by (auto simp: lits-of-def)
  then have get-level (trail S) L = backtrack-lvl S
    by (metis M' get-lev-L get-rev-level-notin-end rev-append)
ultimately
  have get-maximum-level (trail S) (remove1-mset L' (mset-ccls D'))  $\geq$  backtrack-lvl S
    by (metis get-maximum-level-ge-get-level get-rev-level-uminus)
  then show False
    using ⟨i' < backtrack-lvl S⟩ i' by auto
qed
have cdclW** S U
  using bj cdclW-bj.skip local.skip mono-rtrancpl[of skip cdclW S U] other by meson
then have cdclW-all-struct-inv U
  using inv rtrancpl-cdclW-all-struct-inv-inv by blast
then have Propagated L (mset-cls E) # trail V  $\models$ as CNot (mset-ccls D')
  using cdclW-all-struct-inv-def cdclW-conflicting-def raw-U U by auto
then have  $\forall L' \in \#$  (remove1-mset L' (mset-ccls D')) . atm-of L'  $\in$  atm-of ' lits-of-l (Propagated L
(mset-cls E) # trail U)
  using U atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set in-CNot-implies-uminus(2)
  by (fastforce dest: in-diffD)
then have get-maximum-level (trail S) (remove1-mset L' (mset-ccls D')) = backtrack-lvl S
  using get-maximum-level-skip-un-marked-not-present[of remove1-mset L' (mset-ccls D')
    trail U M0] tr-S nm U
  ⟨get-maximum-level (trail U) (mset-ccls (remove-clit (- L) D)) = backtrack-lvl U⟩
  by (auto simp: S)
then show False
  using i' ⟨i' < backtrack-lvl S⟩ by auto
qed

```

lemma *if-can-apply-resolve-no-more-backtrack:*

```

assumes
  skip: skip** S U and
  resolve: resolve S T and
  inv: cdclW-all-struct-inv S
shows  $\neg$ backtrack U V
using assms
by (meson if-can-apply-backtrack-no-more-resolve rtrancpl.rtrancpl-refl
  rtrancpl-skip-backtrack-backtrack)

```

lemma *if-can-apply-backtrack-skip-or-resolve-is-skip:*

```

assumes
  bt: backtrack S T and
  skip: skip-or-resolve** S U and
  inv: cdclW-all-struct-inv S
shows skip** S U
using assms(2,3,1)
by induction (simp-all add: if-can-apply-backtrack-no-more-resolve skip-or-resolve.simps)

```

lemma *cdcl_W-bj-bj-decomp:*

```

assumes cdclW-bj** S W and cdclW-all-struct-inv S
shows
  ( $\exists T U V. (\lambda S T. \text{skip-or-resolve } S T \wedge \text{no-step backtrack } S)$ ** S T
 $\wedge (\lambda T U. \text{resolve } T U \wedge \text{no-step backtrack } T)$  T U
 $\wedge \text{skip** } U V \wedge \text{backtrack } V W$ )

```

```

    ∨ (∃ T U. (λS T. skip-or-resolve S T ∧ no-step backtrack S)** S T
      ∧ (λT U. resolve T U ∧ no-step backtrack T) T U ∧ skip** U W)
    ∨ (∃ T. skip** S T ∧ backtrack T W)
    ∨ skip** S W (is ?RB S W ∨ ?R S W ∨ ?SB S W ∨ ?S S W)
  using assms
proof induction
  case base
  then show ?case by simp
next
  case (step W X) note st = this(1) and bj = this(2) and IH = this(3)[OF this(4)] and inv = this(4)

  have ¬?RB S W and ¬?SB S W
  proof (clarify, goal-cases)
    case (1 T U V)
    have skip-or-resolve** S T
    using 1(1) by (auto dest!: rtrancpl-and-rtrancpl-left)
    then show False
    by (metis (no-types, lifting) 1(2) 1(4) 1(5) backtrack-no-cdclW-bj
      cdclW-all-struct-inv-def cdclW-all-struct-inv-inv cdclW-o.bj local.bj other
      resolve rtrancpl-cdclW-all-struct-inv-inv rtrancpl-skip-backtrack-backtrack
      rtrancpl-skip-or-resolve-rtrancpl-cdclW step.prems)
  next
    case 2
    then show ?case by (meson assms(2) cdclW-all-struct-inv-def backtrack-no-cdclW-bj
      local.bj rtrancpl-skip-backtrack-backtrack)
  qed
  then have IH: ?R S W ∨ ?S S W using IH by blast

  have cdclW** S W using mono-rtrancpl[of cdclW-bj cdclW] st by blast
  then have inv-W: cdclW-all-struct-inv W by (simp add: rtrancpl-cdclW-all-struct-inv-inv
    step.prems)
  consider
    (BT) X' where backtrack W X'
  | (skip) no-step backtrack W and skip W X
  | (resolve) no-step backtrack W and resolve W X
  using bj cdclW-bj.cases by meson
  then show ?case
  proof cases
    case (BT X')
    then consider
      (bt) backtrack W X
    | (sk) skip W X
    using bj if-can-apply-backtrack-no-more-resolve[of W W X' X] inv-W cdclW-bj.cases by fast
  then show ?thesis
  proof cases
    case bt
    then show ?thesis using IH by auto
  next
    case sk
    then show ?thesis using IH by (meson rtrancpl-trans r-into-rtrancpl)
  qed
next
  case skip
  then show ?thesis using IH by (meson rtrancpl.rtrancpl-into-rtrancpl)
next

```



```

case resolve note no-bt = this(1) and res = this(2)
consider
  (RS) T U where
    ( $\lambda S T. \text{skip-or-resolve } S T \wedge \text{no-step backtrack } S$ )** S T and
    resolve T U and
    no-step backtrack T and
    skip** U W
  | (S) skip** S W
using IH by auto
then show ?thesis
proof cases
  case (RS T U)
  have cdclW** S T
    using RS(1) cdclW-bj.resolve cdclW-o.bj other skip
    mono-rtrancpl[of ( $\lambda S T. \text{skip-or-resolve } S T \wedge \text{no-step backtrack } S$ ) cdclW S T]
    by (meson skip-or-resolve.cases)
  then have cdclW-all-struct-inv U
    by (meson RS(2) cdclW-all-struct-inv-inv cdclW-bj.resolve cdclW-o.bj other
      rtrancpl-cdclW-all-struct-inv-inv step.prems)
  { fix U'
    assume skip** U U' and skip** U' W
    have cdclW-all-struct-inv U'
      using  $\langle \text{cdcl}_W\text{-all-struct-inv } U \rangle \langle \text{skip}^{**} U U' \rangle \text{rtrancpl-cdcl}_W\text{-all-struct-inv-inv}$ 
        cdclW-o.bj rtrancpl-mono[of skip cdclW] other skip by blast
    then have no-step backtrack U'
      using if-can-apply-backtrack-no-more-resolve[OF  $\langle \text{skip}^{**} U' W \rangle$ ] res by blast
  }
with  $\langle \text{skip}^{**} U W \rangle$ 
have ( $\lambda S T. \text{skip-or-resolve } S T \wedge \text{no-step backtrack } S$ )** U W
  proof induction
    case base
    then show ?case by simp
  next
  case (step V W) note st = this(1) and skip = this(2) and IH = this(3) and H = this(4)
  have  $\bigwedge U'. \text{skip}^{**} U' V \implies \text{skip}^{**} U' W$ 
    using skip by auto
  then have ( $\lambda S T. \text{skip-or-resolve } S T \wedge \text{no-step backtrack } S$ )** U V
    using IH H by blast
  moreover have ( $\lambda S T. \text{skip-or-resolve } S T \wedge \text{no-step backtrack } S$ )** V W
    by (simp add: local.skip r-into-rtrancpl st step.prems skip-or-resolve.intros)
  ultimately show ?case by simp
qed
then show ?thesis
proof –
  have f1:  $\forall p pa pb pc. \neg p (pa) pb \vee \neg p^{**} pb pc \vee p^{**} pa pc$ 
    by (meson converse-rtrancpl-into-rtrancpl)
  have skip-or-resolve T U  $\wedge$  no-step backtrack T
    using RS(2) RS(3) by force
  then have ( $\lambda p pa. \text{skip-or-resolve } p pa \wedge \text{no-step backtrack } p$ )** T W
  proof –
    have ( $\exists vr19 vr16 vr17 vr18. vr19 (vr16::'st) vr17 \wedge vr19^{**} vr17 vr18$ 
       $\wedge \neg vr19^{**} vr16 vr18$ )
       $\vee \neg (\text{skip-or-resolve } T U \wedge \text{no-step backtrack } T)$ 
       $\vee \neg (\lambda uu uua. \text{skip-or-resolve } uu uua \wedge \text{no-step backtrack } uu)$ ** U W

```

```

      ∨ (λuu uua. skip-or-resolve uu uua ∧ no-step backtrack uu)** T W
    by force
  then show ?thesis
    by (metis (no-types) (λS T. skip-or-resolve S T ∧ no-step backtrack S)** U W)
      (skip-or-resolve T U ∧ no-step backtrack T) f1)
  qed
  then have (λp pa. skip-or-resolve p pa ∧ no-step backtrack p)** S W
    using RS(1) by force
  then show ?thesis
    using no-bt res by blast
  qed
next
case S
{ fix U'
  assume skip** S U' and skip** U' W
  then have cdclW** S U'
    using mono-rtrancpl[of skip cdclW S U'] by (simp add: cdclW-o.bj other skip)
  then have cdclW-all-struct-inv U'
    by (metis (no-types, hide-lams) (cdclW-all-struct-inv S)
      rtrancpl-cdclW-all-struct-inv-inv)
  then have no-step backtrack U'
    using if-can-apply-backtrack-no-more-resolve[OF (skip** U' W)] res by blast
}
with S
have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** S W
  proof induction
    case base
    then show ?case by simp
  next
  case (step V W) note st = this(1) and skip = this(2) and IH = this(3) and H = this(4)
  have ∧U'. skip** U' V ⇒ skip** U' W
    using skip by auto
  then have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** S V
    using IH H by blast
  moreover have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** V W
    by (simp add: local.skip r-into-rtrancpl st step.prem skip-or-resolve.intros)
  ultimately show ?case by simp
  qed
  then show ?thesis using res no-bt by blast
  qed
qed

```

The case distinction is needed, since $T \sim V$ does not imply that $R^{**} T V$.

lemma *cdcl_W-bj-strongly-confluent*:

```

  assumes
    cdclW-bj** S V and
    cdclW-bj** S T and
    n-s: no-step cdclW-bj V and
    inv: cdclW-all-struct-inv S
  shows T ~ V ∨ cdclW-bj** T V
  using assms(2)
proof induction
  case base

```

```

then show ?case by (simp add: assms(1))
next
case (step T U) note st = this(1) and s-o-r = this(2) and IH = this(3)
have cdclW** S T
  using st mono-rtrancpl[of cdclW-bj cdclW] other by blast
then have lev-T: cdclW-M-level-inv T
  using inv rtrancpl-cdclW-consistent-inv[of S T]
  unfolding cdclW-all-struct-inv-def by auto

consider
  (TV) T ~ V
  | (bj-TV) cdclW-bj** T V
using IH by blast
then show ?case
proof cases
case TV
have no-step cdclW-bj T
  using ⟨cdclW-M-level-inv T⟩ n-s cdclW-bj-state-eq-compatible[of T - V] TV
  by (meson backtrack-state-eq-compatible cdclW-bj.simps resolve-state-eq-compatible
    skip-state-eq-compatible state-eq-ref)
then show ?thesis
  using s-o-r by auto
next
case bj-TV
then obtain U' where
  T-U': cdclW-bj T U' and
  cdclW-bj** U' V
  using IH n-s s-o-r by (metis rtrancpl-unfold trancplD)
have cdclW** S T
  by (metis (no-types, hide-lams) bj mono-rtrancpl[of cdclW-bj cdclW] other st)
then have inv-T: cdclW-all-struct-inv T
  by (metis (no-types, hide-lams) inv rtrancpl-cdclW-all-struct-inv-inv)

have lev-U: cdclW-M-level-inv U
  using s-o-r cdclW-consistent-inv lev-T other by blast
show ?thesis
  using s-o-r
  proof cases
  case backtrack
  then obtain V0 where skip** T V0 and backtrack V0 V
    using IH if-can-apply-backtrack-skip-or-resolve-is-skip[OF backtrack - inv-T]
    cdclW-bj-decomp-resolve-skip-and-bj
    by (meson bj-TV cdclW-bj.backtrack inv-T lev-T n-s
      rtrancpl-skip-backtrack-backtrack-end)
  then have cdclW-bj** T V0 and cdclW-bj V0 V
    using rtrancpl-mono[of skip cdclW-bj] by blast+
  then show ?thesis
    using ⟨backtrack V0 V⟩ ⟨skip** T V0⟩ backtrack-unique inv-T local.backtrack
    rtrancpl-skip-backtrack-backtrack by auto
  next
  case resolve
  then have U ~ U'
    by (meson T-U' cdclW-bj.simps if-can-apply-backtrack-no-more-resolve inv-T
      resolve-skip-deterministic resolve-unique rtrancpl.rtrancl-refl)
  then show ?thesis

```

```

    using ⟨cdclW-bj** U' V⟩ unfolding rtrancpl-unfold
    by (meson T-U' bj cdclW-consistent-inv lev-T other state-eq-ref state-eq-sym
        trancpl-cdclW-bj-state-eq-compatible)
  next
  case skip
  consider
    (sk) skip T U'
  | (bt) backtrack T U'
  using T-U' by (meson cdclW-bj.cases local.skip resolve-skip-deterministic)
  then show ?thesis
  proof cases
    case sk
    then show ?thesis
    using ⟨cdclW-bj** U' V⟩ unfolding rtrancpl-unfold
    by (meson T-U' bj cdclW-all-inv(3) cdclW-all-struct-inv-def inv-T local.skip other
        trancpl-cdclW-bj-state-eq-compatible skip-unique state-eq-ref)
  next
  case bt
  have skip++ T U
    using local.skip by blast
  have cdclW-bj U U'
    by (meson ⟨skip++ T U⟩ backtrack bt inv-T rtrancpl-skip-backtrack-backtrack-end
        trancpl-into-rtrancpl)
  then have cdclW-bj++ U V
    using ⟨cdclW-bj** U' V⟩ by auto
  then show ?thesis
    by (meson trancpl-into-rtrancpl)
  qed
qed
qed
qed

```

lemma *cdcl_W-bj-unique-normal-form*:

```

  assumes
    ST: cdclW-bj** S T and SU: cdclW-bj** S U and
    n-s-U: no-step cdclW-bj U and
    n-s-T: no-step cdclW-bj T and
    inv: cdclW-all-struct-inv S
  shows T ~ U
proof -
  have T ~ U ∨ cdclW-bj** T U
    using ST SU cdclW-bj-strongly-confluent inv n-s-U by blast
  then show ?thesis
    by (metis (no-types) n-s-T rtrancpl-unfold state-eq-ref trancpl-unfold-begin)
qed

```

lemma *full-cdcl_W-bj-unique-normal-form*:

```

  assumes full cdclW-bj S T and full cdclW-bj S U and
    inv: cdclW-all-struct-inv S
  shows T ~ U
    using cdclW-bj-unique-normal-form assms unfolding full-def by blast

```

21.3 CDCL FW

inductive *cdcl_W-merge-restart* :: 'st ⇒ 'st ⇒ bool **where**

fw-r-propagate: $\text{propagate } S \ S' \implies \text{cdcl}_W\text{-merge-restart } S \ S' \mid$
fw-r-conflict: $\text{conflict } S \ T \implies \text{full } \text{cdcl}_W\text{-bj } T \ U \implies \text{cdcl}_W\text{-merge-restart } S \ U \mid$
fw-r-decide: $\text{decide } S \ S' \implies \text{cdcl}_W\text{-merge-restart } S \ S' \mid$
fw-r-rf: $\text{cdcl}_W\text{-rf } S \ S' \implies \text{cdcl}_W\text{-merge-restart } S \ S'$

lemma *rtrancpl-cdcl_W-bj-rtrancpl-cdcl_W*:
 $\text{cdcl}_W\text{-bj}^{**} \ S \ T \implies \text{cdcl}_W^{**} \ S \ T$
using *mono-rtrancpl*[of *cdcl_W-bj cdcl_W*] **by** *blast*

lemma *cdcl_W-merge-restart-cdcl_W*:
assumes *cdcl_W-merge-restart* *S T*
shows $\text{cdcl}_W^{**} \ S \ T$
using *assms*

proof *induction*

case (*fw-r-conflict* *S T U*) **note** *confl = this(1)* **and** *bj = this(2)*
have $\text{cdcl}_W \ S \ T$ **using** *confl* **by** (*simp add: cdcl_W.intros r-into-rtrancpl*)
moreover
have $\text{cdcl}_W\text{-bj}^{**} \ T \ U$ **using** *bj* **unfolding** *full-def* **by** *auto*
then have $\text{cdcl}_W^{**} \ T \ U$ **using** *rtrancpl-cdcl_W-bj-rtrancpl-cdcl_W* **by** *blast*
ultimately show *?case* **by** *auto*

qed (*simp-all add: cdcl_W-o.intros cdcl_W.intros r-into-rtrancpl*)

lemma *cdcl_W-merge-restart-conflicting-true-or-no-step*:
assumes *cdcl_W-merge-restart* *S T*
shows $\text{conflicting } T = \text{None} \vee \text{no-step } \text{cdcl}_W \ T$
using *assms*

proof *induction*

case (*fw-r-conflict* *S T U*) **note** *confl = this(1)* **and** *n-s = this(2)*
{ fix *D V*
assume $\text{cdcl}_W \ U \ V$ **and** $\text{conflicting } U = \text{Some } D$
then have *False*
using *n-s* **unfolding** *full-def*
by (*induction rule: cdcl_W-all-rules-induct*)
(auto dest!: cdcl_W-bj.intros elim: decideE propagateE conflictE forgetE restartE)
}
then show *?case* **by** (*cases conflicting U*) *fastforce+*

qed (*auto simp add: cdcl_W-rf.simps elim: propagateE decideE restartE forgetE*)

inductive *cdcl_W-merge* :: *'st \Rightarrow 'st \Rightarrow bool* **where**
fw-propagate: $\text{propagate } S \ S' \implies \text{cdcl}_W\text{-merge } S \ S' \mid$
fw-conflict: $\text{conflict } S \ T \implies \text{full } \text{cdcl}_W\text{-bj } T \ U \implies \text{cdcl}_W\text{-merge } S \ U \mid$
fw-decide: $\text{decide } S \ S' \implies \text{cdcl}_W\text{-merge } S \ S' \mid$
fw-forget: $\text{forget } S \ S' \implies \text{cdcl}_W\text{-merge } S \ S'$

lemma *cdcl_W-merge-cdcl_W-merge-restart*:
 $\text{cdcl}_W\text{-merge } S \ T \implies \text{cdcl}_W\text{-merge-restart } S \ T$
by (*meson cdcl_W-merge.cases cdcl_W-merge-restart.simps forget*)

lemma *rtrancpl-cdcl_W-merge-trancpl-cdcl_W-merge-restart*:
 $\text{cdcl}_W\text{-merge}^{**} \ S \ T \implies \text{cdcl}_W\text{-merge-restart}^{**} \ S \ T$
using *rtrancpl-mono*[of *cdcl_W-merge cdcl_W-merge-restart*] *cdcl_W-merge-cdcl_W-merge-restart* **by** *blast*

lemma *cdcl_W-merge-rtrancpl-cdcl_W*:
 $\text{cdcl}_W\text{-merge } S \ T \implies \text{cdcl}_W^{**} \ S \ T$
using *cdcl_W-merge-cdcl_W-merge-restart cdcl_W-merge-restart-cdcl_W* **by** *blast*

```

lemma rtrancp-cdclW-merge-rtrancp-cdclW:
  cdclW-merge** S T  $\implies$  cdclW** S T
  using rtrancp-mono[of cdclW-merge cdclW**] cdclW-merge-rtrancp-cdclW by auto

lemmas rulesE =
  skipE resolveE backtrackE propagateE conflictE decideE restartE forgetE

lemma cdclW-all-struct-inv-trancp-cdclW-merge-trancp-cdclW-merge-cdclW-all-struct-inv:
  assumes
    inv: cdclW-all-struct-inv b
    cdclW-merge++ b a
  shows  $(\lambda S T. \text{cdcl}_W\text{-all-struct-inv } S \wedge \text{cdcl}_W\text{-merge } S T)^{++} b a$ 
  using assms(2)
proof induction
  case base
  then show ?case using inv by auto
next
  case (step c d) note st = this(1) and fw = this(2) and IH = this(3)
  have cdclW-all-struct-inv c
    using trancp-into-rtrancp[OF st] cdclW-merge-rtrancp-cdclW
    assms(1) rtrancp-cdclW-all-struct-inv-inv rtrancp-mono[of cdclW-merge cdclW**] by fastforce
  then have  $(\lambda S T. \text{cdcl}_W\text{-all-struct-inv } S \wedge \text{cdcl}_W\text{-merge } S T)^{++} c d$ 
    using fw by auto
  then show ?case using IH by auto
qed

lemma backtrack-is-full1-cdclW-bj:
  assumes bt: backtrack S T and inv: cdclW-M-level-inv S
  shows full1 cdclW-bj S T
  using bt inv backtrack-no-cdclW-bj unfolding full1-def by blast

lemma rtranc-cdclW-conflicting-true-cdclW-merge-restart:
  assumes cdclW** S V and inv: cdclW-M-level-inv S and conflicting S = None
  shows  $(\text{cdcl}_W\text{-merge-restart}^{**} S V \wedge \text{conflicting } V = \text{None})$ 
   $\vee (\exists T U. \text{cdcl}_W\text{-merge-restart}^{**} S T \wedge \text{conflicting } V \neq \text{None} \wedge \text{conflict } T U \wedge \text{cdcl}_W\text{-bj}^{**} U V)$ 
  using assms
proof induction
  case base
  then show ?case by simp
next
  case (step U V) note st = this(1) and cdclW = this(2) and IH = this(3)[OF this(4-)] and
    confl[simp] = this(5) and inv = this(4)
  from cdclW
  show ?case
  proof (cases)
    case propagate
    moreover then have conflicting U = None and conflicting V = None
      by (auto elim: propagateE)
    ultimately show ?thesis using IH cdclW-merge-restart.fw-r-propagate[of U V] by auto
  next
    case conflict
    moreover then have conflicting U = None and conflicting V  $\neq$  None
      by (auto elim!: conflictE simp del: state-simp simp: state-eq-def)
    ultimately show ?thesis using IH by auto

```

```

next
case other
then show ?thesis
proof cases
case decide
then show ?thesis using IH cdclW-merge-restart.fw-r-decide[of U V] by (auto elim: decideE)
next
case bj
moreover {
assume skip-or-resolve U V
have f1: cdclW-bj++ U V
by (simp add: local.bj tranclp.r-into-trancl)
obtain T T' :: 'st where
f2: cdclW-merge-restart** S U
  ∨ cdclW-merge-restart** S T ∧ conflicting U ≠ None
  ∧ conflict T T' ∧ cdclW-bj** T' U
using IH confl by blast
have conflicting V ≠ None ∧ conflicting U ≠ None
using ⟨skip-or-resolve U V⟩
by (auto simp: skip-or-resolve.simps state-eq-def elim!: skipE resolveE
simp del: state-simp)
then have ?thesis
by (metis (full-types) IH f1 rtranclp-trans tranclp-into-rtranclp)
}
moreover {
assume backtrack U V
then have conflicting U ≠ None by (auto elim: backtrackE)
then obtain T T' where
cdclW-merge-restart** S T and
conflicting U ≠ None and
conflict T T' and
cdclW-bj** T' U
using IH confl by meson
have invU: cdclW-M-level-inv U
using inv rtranclp-cdclW-consistent-inv step.hyps(1) by blast
then have conflicting V = None
using ⟨backtrack U V⟩ inv by (auto elim: backtrack-levE
simp: cdclW-M-level-inv-decomp)
have full cdclW-bj T' V
apply (rule rtranclp-fullI[of cdclW-bj T' U V])
using ⟨cdclW-bj** T' U⟩ apply fast
using ⟨backtrack U V⟩ backtrack-is-full1-cdclW-bj invU unfolding full1-def full-def
by blast
then have ?thesis
using cdclW-merge-restart.fw-r-conflict[of T T' V] ⟨conflict T T'⟩
⟨cdclW-merge-restart** S T⟩ ⟨conflicting V = None⟩ by auto
}
ultimately show ?thesis by (auto simp: cdclW-bj.simps)
qed
next
case rf
moreover then have conflicting U = None and conflicting V = None
by (auto simp: cdclW-rf.simps elim: restartE forgetE)
ultimately show ?thesis using IH cdclW-merge-restart.fw-r-rf[of U V] by auto
qed

```

qed

lemma *no-step-cdcl_W-no-step-cdcl_W-merge-restart*: *no-step cdcl_W S \implies no-step cdcl_W-merge-restart S*

by (auto simp: cdcl_W.simps cdcl_W-merge-restart.simps cdcl_W-o.simps cdcl_W-bj.simps)

lemma *no-step-cdcl_W-merge-restart-no-step-cdcl_W*:

assumes

conflicting S = None and

cdcl_W-M-level-inv S and

no-step cdcl_W-merge-restart S

shows *no-step cdcl_W S*

proof –

{ fix *S'*

assume *conflict S S'*

then have *cdcl_W S S'* using *cdcl_W.conflict* by auto

then have *cdcl_W-M-level-inv S'*

using *assms(2) cdcl_W-consistent-inv* by blast

then obtain *S''* where *full cdcl_W-bj S' S''*

using *cdcl_W-bj-exists-normal-form[of S']* by auto

then have *False*

using *⟨conflict S S'⟩ assms(3) fw-r-conflict* by blast

}

then show *?thesis*

using *assms unfolding cdcl_W.simps cdcl_W-merge-restart.simps cdcl_W-o.simps cdcl_W-bj.simps*

by (auto elim: *skipE resolveE backtrackE conflictE decideE restartE*)

qed

lemma *cdcl_W-merge-restart-no-step-cdcl_W-bj*:

assumes

cdcl_W-merge-restart S T

shows *no-step cdcl_W-bj T*

using *assms*

by (*induction rule: cdcl_W-merge-restart.induct*)

(*force simp: cdcl_W-bj.simps cdcl_W-rf.simps cdcl_W-merge-restart.simps full-def*

elim!: rulesE)+

lemma *rtrancpl-cdcl_W-merge-restart-no-step-cdcl_W-bj*:

assumes

*cdcl_W-merge-restart** S T* and

conflicting S = None

shows *no-step cdcl_W-bj T*

using *assms unfolding rtrancpl-unfold*

apply (*elim disjE*)

apply (*force simp: cdcl_W-bj.simps cdcl_W-rf.simps elim!: rulesE*)

by (auto simp: *trancpl-unfold-end simp: cdcl_W-merge-restart-no-step-cdcl_W-bj*)

If *conflicting S \neq None*, we cannot say anything.

Remark that this theorem does not say anything about well-foundedness: even if you know that one relation is well-founded, it only states that the normal forms are shared.

lemma *conflicting-true-full-cdcl_W-iff-full-cdcl_W-merge*:

assumes *conf!: conflicting S = None* and *lev: cdcl_W-M-level-inv S*

shows *full cdcl_W S V \longleftrightarrow full cdcl_W-merge-restart S V*

proof

assume *full: full cdcl_W-merge-restart S V*


```

then have st: cdclW** S V
  using rtrancp-mono[of cdclW-merge-restart cdclW**] cdclW-merge-restart-cdclW
  unfolding full-def by auto

have n-s: no-step cdclW-merge-restart V
  using full unfolding full-def by auto
have n-s-bj: no-step cdclW-bj V
  using rtrancp-cdclW-merge-restart-no-step-cdclW-bj confl full unfolding full-def by auto
have  $\bigwedge S'. \text{conflict } V S' \implies \text{cdcl}_W\text{-M-level-inv } S'$ 
  using cdclW.conflict cdclW-consistent-inv lev rtrancp-cdclW-consistent-inv st by blast
then have  $\bigwedge S'. \text{conflict } V S' \implies \text{False}$ 
  using n-s n-s-bj cdclW-bj-exists-normal-form cdclW-merge-restart.simps by meson
then have n-s-cdclW: no-step cdclW V
  using n-s n-s-bj by (auto simp: cdclW.simps cdclW-o.simps cdclW-merge-restart.simps)
then show full cdclW S V using st unfolding full-def by auto
next
assume full: full cdclW S V
have no-step cdclW-merge-restart V
  using full no-step-cdclW-no-step-cdclW-merge-restart unfolding full-def by blast
moreover
consider
  (fw) cdclW-merge-restart** S V and conflicting V = None
| (bj) T U where
  cdclW-merge-restart** S T and
  conflicting V  $\neq$  None and
  conflict T U and
  cdclW-bj** U V
  using full rtrancp-cdclW-conflicting-true-cdclW-merge-restart confl lev unfolding full-def
  by meson
then have cdclW-merge-restart** S V
proof cases
  case fw
  then show ?thesis by fast
next
  case (bj T U)
  have no-step cdclW-bj V
    using full unfolding full-def by (meson cdclW-o.bj other)
  then have full cdclW-bj U V
    using  $\langle \text{cdcl}_W\text{-bj}^{**} U V \rangle$  unfolding full-def by auto
  then have cdclW-merge-restart T V
    using  $\langle \text{conflict } T U \rangle$  cdclW-merge-restart.fw-r-conflict by blast
  then show ?thesis using  $\langle \text{cdcl}_W\text{-merge-restart}^{**} S T \rangle$  by auto
qed
ultimately show full cdclW-merge-restart S V unfolding full-def by fast
qed

lemma init-state-true-full-cdclW-iff-full-cdclW-merge:
  shows full cdclW (init-state N) V  $\longleftrightarrow$  full cdclW-merge-restart (init-state N) V
  by (rule conflicting-true-full-cdclW-iff-full-cdclW-merge) auto

```

21.4 FW with strategy

21.4.1 The intermediate step

inductive cdcl_W-s' :: 'st \Rightarrow 'st \Rightarrow bool where
 conflict': full1 cdcl_W-cp S S' \implies cdcl_W-s' S S' |

decide': $\text{decide } S \ S' \implies \text{no-step } \text{cdcl}_W\text{-cp } S \implies \text{full } \text{cdcl}_W\text{-cp } S' \ S'' \implies \text{cdcl}_W\text{-s'} \ S \ S'' \mid$
bj': $\text{full1 } \text{cdcl}_W\text{-bj } S \ S' \implies \text{no-step } \text{cdcl}_W\text{-cp } S \implies \text{full } \text{cdcl}_W\text{-cp } S' \ S'' \implies \text{cdcl}_W\text{-s'} \ S \ S''$

inductive-cases $\text{cdcl}_W\text{-s'}E$: $\text{cdcl}_W\text{-s'} \ S \ T$

lemma *rtrancpl-cdcl_W-bj-full1-cdclp-cdcl_W-stgy*:

$\text{cdcl}_W\text{-bj}^{**} \ S \ S' \implies \text{full } \text{cdcl}_W\text{-cp } S' \ S'' \implies \text{cdcl}_W\text{-stgy}^{**} \ S \ S''$

proof (*induction rule: converse-rtrancpl-induct*)

case *base*

then show *?case by (metis cdcl_W-stgy.conflict' full-unfold rtrancpl.simps)*

next

case (*step* $T \ U$) **note** $st = \text{this}(2)$ **and** $bj = \text{this}(1)$ **and** $IH = \text{this}(3)[OF \ \text{this}(4)]$

have *no-step cdcl_W-cp T*

using *bj by (auto simp add: cdcl_W-bj.simps cdcl_W-cp.simps elim!: rulesE)*

consider

(*U*) $U = S'$

| (*U'*) U' **where** *cdcl_W-bj U U'* **and** *cdcl_W-bj** U' S'*

using *st by (metis converse-rtrancplE)*

then show *?case*

proof *cases*

case *U*

then show *?thesis*

using (*no-step cdcl_W-cp T*) *cdcl_W-o.bj local.bj other' step.prem*s **by** (*meson r-into-rtrancpl*)

next

case U' **note** $U' = \text{this}(1)$

have *no-step cdcl_W-cp U*

using U' **by** (*fastforce simp: cdcl_W-cp.simps cdcl_W-bj.simps elim: rulesE*)

then have *full cdcl_W-cp U U*

by (*simp add: full-unfold*)

then have *cdcl_W-stgy T U*

using (*no-step cdcl_W-cp T*) *cdcl_W-stgy.simps local.bj cdcl_W-o.bj* **by** *meson*

then show *?thesis using IH by auto*

qed

qed

lemma *cdcl_W-s'-is-rtrancpl-cdcl_W-stgy*:

$\text{cdcl}_W\text{-s'} \ S \ T \implies \text{cdcl}_W\text{-stgy}^{**} \ S \ T$

apply (*induction rule: cdcl_W-s'.induct*)

apply (*auto intro: cdcl_W-stgy.intros*)[]

apply (*meson decide other' r-into-rtrancpl*)

by (*metis full1-def rtrancpl-cdcl_W-bj-full1-cdclp-cdcl_W-stgy trancpl-into-rtrancpl*)

lemma *cdcl_W-cp-cdcl_W-bj-bissimulation*:

assumes

full cdcl_W-cp T U **and**

*cdcl_W-bj** T T'* **and**

cdcl_W-all-struct-inv T **and**

no-step cdcl_W-bj T'

shows *full cdcl_W-cp T' U*

$\vee (\exists U' \ U''. \text{full } \text{cdcl}_W\text{-cp } T' \ U'' \wedge \text{full1 } \text{cdcl}_W\text{-bj } U \ U' \wedge \text{full } \text{cdcl}_W\text{-cp } U' \ U''$
 $\wedge \text{cdcl}_W\text{-s}^{**} \ U \ U'')$

using *assms(2,1,3,4)*

proof (*induction rule: rtrancpl-induct*)

case *base*

then show *?case by blast*

next

```

case (step T' T'') note st = this(1) and bj = this(2) and IH = this(3)[OF this(4,5)] and
  full = this(4) and inv = this(5)
have cdclW-bj** T T''
  using local.bj st by auto
then have cdclW** T T''
  using rtrancp-cdclW-bj-rtrancp-cdclW by blast
then have inv-T'': cdclW-all-struct-inv T''
  using inv rtrancp-cdclW-all-struct-inv-inv by blast
have cdclW-bj++ T T''
  using local.bj st by auto
have full1 cdclW-bj T T''
  by (metis ⟨cdclW-bj++ T T'⟩ full1-def step.prem(3))
then have T = U
proof -
  obtain Z where cdclW-bj T Z
    using ⟨cdclW-bj++ T T'⟩ by (blast dest: trancpD)
  { assume cdclW-cp++ T U
    then obtain Z' where cdclW-cp T Z'
      by (meson trancpD)
    then have False
      using ⟨cdclW-bj T Z⟩ by (fastforce simp: cdclW-bj.simps cdclW-cp.simps
        elim: rulesE)
  }
  then show ?thesis
    using full unfolding full-def rtrancp-unfold by blast
qed
obtain U'' where full cdclW-cp T'' U''
  using cdclW-cp-normalized-element-all-inv inv-T'' by blast
moreover then have cdclW-stgy** U U''
  by (metis ⟨T = U⟩ ⟨cdclW-bj++ T T'⟩ rtrancp-cdclW-bj-full1-cdclp-cdclW-stgy rtrancp-unfold)
moreover have cdclW-s** U U''
proof -
  obtain ss :: 'st ⇒ 'st where
    f1: ∀ x2. (∃ v3. cdclW-cp x2 v3) = cdclW-cp x2 (ss x2)
    by moura
  have ¬ cdclW-cp U (ss U)
    by (meson full full-def)
  then show ?thesis
    using f1 by (metis (no-types) ⟨T = U⟩ ⟨full1 cdclW-bj T T'⟩ bj' calculation(1)
      r-into-rtrancp)
qed
ultimately show ?case
  using ⟨full1 cdclW-bj T T'⟩ ⟨full cdclW-cp T'' U''⟩ unfolding ⟨T = U⟩ by blast
qed

```

lemma cdcl_W-cp-cdcl_W-bj-bissimulation':

```

assumes
  full cdclW-cp T U and
  cdclW-bj** T T' and
  cdclW-all-struct-inv T and
  no-step cdclW-bj T'
shows full cdclW-cp T' U
  ∨ (∃ U'. full1 cdclW-bj U U' ∧ (∀ U''. full cdclW-cp U' U'' ⟶ full cdclW-cp T' U''
    ∧ cdclW-s/* U U'))

```

```

using assms(2,1,3,4)
proof (induction rule: rtrancplp-induct)
  case base
  then show ?case by blast
next
case (step T' T'') note st = this(1) and bj = this(2) and IH = this(3)[OF this(4,5)] and
  full = this(4) and inv = this(5)
have cdclW** T T''
  by (metis local.bj rtrancplp.simps rtrancplp-cdclW-bj-rtrancplp-cdclW st)
then have inv-T'': cdclW-all-struct-inv T''
  using inv rtrancplp-cdclW-all-struct-inv-inv by blast
have cdclW-bj++ T T''
  using local.bj st by auto
have full1 cdclW-bj T T''
  by (metis <cdclW-bj++ T T''> full1-def step.prem(3))
then have T = U
proof -
  obtain Z where cdclW-bj T Z
  using <cdclW-bj++ T T''> by (blast dest: trancplpD)
  { assume cdclW-cp++ T U
    then obtain Z' where cdclW-cp T Z'
    by (meson trancplpD)
    then have False
    using <cdclW-bj T Z> by (fastforce simp: cdclW-bj.simps cdclW-cp.simps elim: rulesE)
  }
  then show ?thesis
  using full unfolding full-def rtrancplp-unfold by blast
qed
{ fix U''
  assume full cdclW-cp T'' U''
  moreover then have cdclW-stgy** U U''
  by (metis <T = U> <cdclW-bj++ T T''> rtrancplp-cdclW-bj-full1-cdclp-cdclW-stgy rtrancplp-unfold)
  moreover have cdclW-s'** U U''
  proof -
    obtain ss :: 'st ⇒ 'st where
      f1: ∀ x2. (∃ v3. cdclW-cp x2 v3) = cdclW-cp x2 (ss x2)
    by mouna
    have ¬ cdclW-cp U (ss U)
    by (meson assms(1) full-def)
    then show ?thesis
    using f1 by (metis (no-types) <T = U> <full1 cdclW-bj T T''> bj' calculation(1)
      r-into-rtrancplp)
  qed
  ultimately have full1 cdclW-bj U T'' and cdclW-s'** T'' U''
  using <full1 cdclW-bj T T''> <full cdclW-cp T'' U''> unfolding <T = U>
  apply blast
  by (metis <full cdclW-cp T'' U''> cdclW-s'.simps full-unfold rtrancplp.simps)
}
then show ?case
  using <full1 cdclW-bj T T''> full bj' unfolding <T = U> full-def by (metis r-into-rtrancplp)
qed

lemma cdclW-stgy-cdclW-s'-connected:
  assumes cdclW-stgy S U and cdclW-all-struct-inv S
  shows cdclW-s' S U

```

```

     $\vee (\exists U'. \text{full1 } \text{cdcl}_W\text{-bj } U \ U' \wedge (\forall U''. \text{full } \text{cdcl}_W\text{-cp } U' \ U'' \longrightarrow \text{cdcl}_W\text{-s}' S \ U''))$ 
  using assms
proof (induction rule: cdclW-stgy.induct)
  case (conflict' T)
  then have cdclW-s' S T
    using cdclW-s'.conflict' by blast
  then show ?case
    by blast
next
case (other' T U) note o = this(1) and n-s = this(2) and full = this(3) and inv = this(4)
show ?case
  using o
  proof cases
    case decide
    then show ?thesis using cdclW-s'.simps full n-s by blast
  next
  case bj
  have inv-T: cdclW-all-struct-inv T
    using cdclW-all-struct-inv-inv o other other'.prems by blast
  consider
    (cp) full cdclW-cp T U and no-step cdclW-bj T
  | (fbj) T' where full1 cdclW-bj T T'
  apply (cases no-step cdclW-bj T)
  using full apply blast
  using cdclW-bj-exists-normal-form[of T] inv-T unfolding cdclW-all-struct-inv-def
  by (metis full-unfold)
  then show ?thesis
  proof cases
    case cp
    then show ?thesis
    proof -
      obtain ss :: 'st  $\Rightarrow$  'st where
        f1:  $\forall s \ sa \ sb. (\neg \text{full1 } \text{cdcl}_W\text{-bj } s \ sa \vee \text{cdcl}_W\text{-cp } s \ (ss \ s) \vee \neg \text{full } \text{cdcl}_W\text{-cp } sa \ sb)$ 
         $\vee \text{cdcl}_W\text{-s}' s \ sb$ 
      using bj' by moura
      have full1 cdclW-bj S T
        by (simp add: cp(2) full1-def local.bj tranclp.r-into-trancl)
      then show ?thesis
        using f1 full n-s by blast
    qed
  next
  case (fbj U')
  then have full1 cdclW-bj S U'
    using bj unfolding full1-def by auto
  moreover have no-step cdclW-cp S
    using n-s by blast
  moreover have T = U
    using full fbj unfolding full1-def full-def rtranclp-unfold
    by (force dest!: tranclpD simp:cdclW-bj.simps elim: rulesE)
  ultimately show ?thesis using cdclW-s'.bj'[of S U] using fbj by blast
  qed
  qed
  qed

```

lemma *cdcl_W-stgy-cdcl_W-s'-connected'*:

```

assumes  $cdcl_W\text{-stgy } S \ U$  and  $cdcl_W\text{-all-struct-inv } S$ 
shows  $cdcl_W\text{-s}' S \ U$ 
   $\vee (\exists U' \ U''. \ cdcl_W\text{-s}' S \ U'' \wedge full1 \ cdcl_W\text{-bj } U \ U' \wedge full \ cdcl_W\text{-cp } U' \ U'')$ 
using assms
proof (induction rule: cdcl_W-stgy.induct)
  case (conflict'  $T$ )
  then have  $cdcl_W\text{-s}' S \ T$ 
    using  $cdcl_W\text{-s}'.conflict'$  by blast
  then show ?case
    by blast
next
  case (other'  $T \ U$ ) note  $o = this(1)$  and  $n\text{-s} = this(2)$  and  $full = this(3)$  and  $inv = this(4)$ 
  show ?case
    using  $o$ 
  proof cases
    case decide
    then show ?thesis using  $cdcl_W\text{-s}'.simps \ full \ n\text{-s}$  by blast
  next
  case bj
  have  $cdcl_W\text{-all-struct-inv } T$ 
    using  $cdcl_W\text{-all-struct-inv-inv } o \ other \ other'.prems$  by blast
  then obtain  $T'$  where  $T': full \ cdcl_W\text{-bj } T \ T'$ 
    using  $cdcl_W\text{-bj-exists-normal-form}$  unfolding  $full\text{-def } cdcl_W\text{-all-struct-inv-def}$  by metis
  then have  $full \ cdcl_W\text{-bj } S \ T'$ 
  proof –
    have  $f1: cdcl_W\text{-bj}^{**} T \ T' \wedge no\text{-step } cdcl_W\text{-bj } T'$ 
      by (metis (no-types)  $T' \ full\text{-def}$ )
    then have  $cdcl_W\text{-bj}^{**} S \ T'$ 
      by (meson converse-rtranclp-into-rtranclp local.bj)
    then show ?thesis
      using  $f1$  by (simp add: full-def)
  qed
  have  $cdcl_W\text{-bj}^{**} T \ T'$ 
    using  $T'$  unfolding  $full\text{-def}$  by simp
  have  $cdcl_W\text{-all-struct-inv } T$ 
    using  $cdcl_W\text{-all-struct-inv-inv } o \ other \ other'.prems$  by blast
  then consider
    ( $T'U$ )  $full \ cdcl_W\text{-cp } T' \ U$ 
  | ( $U$ )  $U' \ U''$  where
     $full \ cdcl_W\text{-cp } T' \ U''$  and
     $full1 \ cdcl_W\text{-bj } U \ U'$  and
     $full \ cdcl_W\text{-cp } U' \ U''$  and
     $cdcl_W\text{-s}^{**} U \ U''$ 
    using  $cdcl_W\text{-cp-cdcl_W-bj-bissimulation}[OF \ full \ \langle cdcl_W\text{-bj}^{**} T \ T' \rangle] \ T'$  unfolding  $full\text{-def}$ 
    by blast
  then show ?thesis by (metis  $T' \ cdcl_W\text{-s}'.simps \ full\text{-fullI local.bj } n\text{-s}$ )
qed
qed

```

```

lemma  $cdcl_W\text{-stgy-cdcl_W-s}'\text{-no-step}$ :
assumes  $cdcl_W\text{-stgy } S \ U$  and  $cdcl_W\text{-all-struct-inv } S$  and  $no\text{-step } cdcl_W\text{-bj } U$ 
shows  $cdcl_W\text{-s}' S \ U$ 
using  $cdcl_W\text{-stgy-cdcl_W-s}'\text{-connected}[OF \ assms(1,2)] \ assms(3)$ 
by (metis (no-types, lifting)  $full1\text{-def } tranclpD$ )

```

```

lemma rtranclp-cdclW-stgy-connected-to-rtranclp-cdclW-s':
  assumes cdclW-stgy** S U and inv: cdclW-M-level-inv S
  shows cdclW-s'** S U  $\vee (\exists T. \text{cdcl}_W\text{-s}'^{**} S T \wedge \text{cdcl}_W\text{-bj}^{++} T U \wedge \text{conflicting } U \neq \text{None})$ 
  using assms(1)
proof induction
  case base
  then show ?case by simp
next
  case (step T V) note st = this(1) and o = this(2) and IH = this(3)
  from o show ?case
  proof cases
    case conflict'
    then have f2: cdclW-s' T V
    using cdclW-s'.conflict' by blast
    obtain ss :: 'st where
      f3: S = T  $\vee$  cdclW-stgy** S ss  $\wedge$  cdclW-stgy ss T
      by (metis (full-types) rtranclp.simps st)
    obtain ssa :: 'st where
      ssa: cdclW-cp T ssa
      using conflict' by (metis (no-types) full1-def tranclpD)
    have  $\forall s. \neg \text{full } \text{cdcl}_W\text{-cp } s \ T$ 
    by (meson ssa full-def)
    then have S = T
    by (metis (full-types) f3 ssa cdclW-stgy.cases full1-def)
    then show ?thesis
    using f2 by blast
  next
  case (other' U) note o = this(1) and n-s = this(2) and full = this(3)
  then show ?thesis
  using o
  proof (cases rule: cdclW-o-rule-cases)
    case decide
    then have cdclW-s'** S T
    using IH by (auto elim: rulesE)
    then show ?thesis
    by (meson decide decide' full n-s rtranclp.rtrancl-into-rtrancl)
  next
  case backtrack
  consider
    (s') cdclW-s'** S T
    | (bj) S' where cdclW-s'** S S' and cdclW-bj++ S' T and conflicting T  $\neq$  None
    using IH by blast
  then show ?thesis
  proof cases
    case s'
    moreover
    have cdclW-M-level-inv T
    using inv local.step(1) rtranclp-cdclW-stgy-consistent-inv by auto
    then have full1 cdclW-bj T U
    using backtrack-is-full1-cdclW-bj backtrack by blast
    then have cdclW-s' T V
    using full bj' n-s by blast
    ultimately show ?thesis by auto
  next
  case (bj S') note S-S' = this(1) and bj-T = this(2)

```

```

have no-step cdclW-cp S'
  using bj-T by (fastforce simp: cdclW-cp.simps cdclW-bj.simps dest!: tranclpD
    elim: rulesE)
moreover
  have cdclW-M-level-inv T
    using inv local.step(1) rtranclp-cdclW-stgy-consistent-inv by auto
  then have full1 cdclW-bj T U
    using backtrack-is-full1-cdclW-bj backtrack by blast
  then have full1 cdclW-bj S' U
    using bj-T unfolding full1-def by fastforce
  ultimately have cdclW-s' S' V using full by (simp add: bj')
  then show ?thesis using S-S' by auto
qed
next
case skip
then have [simp]: U = V
  using full converse-rtranclpE unfolding full-def by (fastforce elim: rulesE)
then have confl-V: conflicting V ≠ None
  using skip by (auto elim!: rulesE simp del: state-simp simp: state-eq-def)
consider
  (s') cdclW-s'^** S T
  | (bj) S' where cdclW-s'^** S S' and cdclW-bj++ S' T and conflicting T ≠ None
  using IH by blast
then show ?thesis
proof cases
  case s'
  show ?thesis using s' confl-V skip by force
next
  case (bj S') note S-S' = this(1) and bj-T = this(2)
  have cdclW-bj++ S' V
    using skip bj-T by (metis ⟨U = V⟩ cdclW-bj.skip tranclp.simps)
  then show ?thesis using S-S' confl-V by auto
qed
next
case resolve
then have [simp]: U = V
  using full unfolding full-def rtranclp-unfold
  by (auto elim!: rulesE dest!: tranclpD
    simp del: state-simp simp: state-eq-def cdclW-cp.simps)
have confl-V: conflicting V ≠ None
  using resolve by (auto elim!: rulesE simp del: state-simp simp: state-eq-def)

consider
  (s') cdclW-s'^** S T
  | (bj) S' where cdclW-s'^** S S' and cdclW-bj++ S' T and conflicting T ≠ None
  using IH by blast
then show ?thesis
proof cases
  case s'
  have cdclW-bj++ T V
    using resolve by force
  then show ?thesis using s' confl-V by auto
next
  case (bj S') note S-S' = this(1) and bj-T = this(2)
  have cdclW-bj++ S' V

```



```

      using resolve bj-T by (metis ⟨U = V⟩ cdclW-bj.resolve tranclp.simps)
    then show ?thesis using confl-V S-S' by auto
  qed
qed
qed
qed

lemma n-step-cdclW-stgy-iff-no-step-cdclW-cl-cdclW-o:
  assumes inv: cdclW-all-struct-inv S
  shows no-step cdclW-s' S  $\longleftrightarrow$  no-step cdclW-cp S  $\wedge$  no-step cdclW-o S (is ?S' S  $\longleftrightarrow$  ?C S  $\wedge$  ?O S)
proof
  assume ?C S  $\wedge$  ?O S
  then show ?S' S
    by (auto simp: cdclW-s'.simps full1-def tranclp-unfold-begin)
next
  assume n-s: ?S' S
  have ?C S
  proof (rule ccontr)
    assume  $\neg$  ?thesis
    then obtain S' where cdclW-cp S S'
      by auto
    then obtain T where full1 cdclW-cp S T
      using cdclW-cp-normalized-element-all-inv inv by (metis (no-types, lifting) full-unfold)
    then show False using n-s cdclW-s'.conflict' by blast
  qed
  moreover have ?O S
  proof (rule ccontr)
    assume  $\neg$  ?thesis
    then obtain S' where cdclW-o S S'
      by auto
    then obtain T where full1 cdclW-cp S' T
      using cdclW-cp-normalized-element-all-inv inv
      by (meson cdclW-all-struct-inv-def n-s
        cdclW-stgy-cdclW-s'-connected' cdclW-then-exists-cdclW-stgy-step )
    then show False using n-s by (meson ⟨cdclW-o S S'⟩ cdclW-all-struct-inv-def
      cdclW-stgy-cdclW-s'-connected' cdclW-then-exists-cdclW-stgy-step inv)
  qed
  ultimately show ?C S  $\wedge$  ?O S by auto
qed

lemma cdclW-s'-tranclp-cdclW:
  cdclW-s' S S'  $\implies$  cdclW++ S S'
proof (induct rule: cdclW-s'.induct)
  case conflict'
  then show ?case
    by (simp add: full1-def tranclp-cdclW-cp-tranclp-cdclW)
next
  case decide'
  then show ?case
    using cdclW-stgy.simps cdclW-stgy-tranclp-cdclW by (meson cdclW-o.simps)
next
  case (bj' Sa S'a S'') note a2 = this(1) and a1 = this(2) and n-s = this(3)
  obtain ss :: 'st  $\Rightarrow$  'st  $\Rightarrow$  ('st  $\Rightarrow$  'st  $\Rightarrow$  bool)  $\Rightarrow$  'st where
     $\forall x0\ x1\ x2. (\exists v3. x2\ x1\ v3 \wedge x2^{**}\ v3\ x0) = (x2\ x1\ (ss\ x0\ x1\ x2) \wedge x2^{**}\ (ss\ x0\ x1\ x2)\ x0)$ 
    by moura

```

then have $f3: \forall p\ s\ sa. \neg p^{++}\ s\ sa \vee p\ s\ (ss\ sa\ s\ p) \wedge p^{**}\ (ss\ sa\ s\ p)\ sa$
by (*metis* (*full-types*) *trancplD*)
have $cdcl_W\text{-}bj^{++}\ Sa\ S'a \wedge no\text{-}step\ cdcl_W\text{-}bj\ S'a$
using *a2* **by** (*simp* *add: full1-def*)
then have $cdcl_W\text{-}bj\ Sa\ (ss\ S'a\ Sa\ cdcl_W\text{-}bj) \wedge cdcl_W\text{-}bj^{**}\ (ss\ S'a\ Sa\ cdcl_W\text{-}bj)\ S'a$
using *f3* **by** *auto*
then show $cdcl_W^{++}\ Sa\ S''$
using *a1* *n-s* **by** (*meson* *bj* *other* *rtrancpl-cdcl_W-bj-full1-cdclp-cdcl_W-stgy*
rtrancpl-cdcl_W-stgy-rtrancpl-cdcl_W *rtrancpl-into-trancpl2*)
qed

lemma *trancpl-cdcl_W-s'-trancpl-cdcl_W*:
 $cdcl_W\text{-}s'^{++}\ S\ S' \implies cdcl_W^{++}\ S\ S'$
apply (*induct* *rule: trancpl.induct*)
using *cdcl_W-s'-trancpl-cdcl_W* **apply** *blast*
by (*meson* *cdcl_W-s'-trancpl-cdcl_W* *trancpl-trans*)

lemma *rtrancpl-cdcl_W-s'-rtrancpl-cdcl_W*:
 $cdcl_W\text{-}s'^{**}\ S\ S' \implies cdcl_W^{**}\ S\ S'$
using *rtrancpl-unfold*[*of* *cdcl_W-s' S S'*] *trancpl-cdcl_W-s'-trancpl-cdcl_W*[*of* *S S'*] **by** *auto*

lemma *full-cdcl_W-stgy-iff-full-cdcl_W-s'*:
assumes *inv: cdcl_W-all-struct-inv S*
shows $full\ cdcl_W\text{-}stgy\ S\ T \longleftrightarrow full\ cdcl_W\text{-}s'\ S\ T$ (*is* $?S \longleftrightarrow ?S'$)

proof
assume $?S'$
then have $cdcl_W^{**}\ S\ T$
using *rtrancpl-cdcl_W-s'-rtrancpl-cdcl_W*[*of* *S T*] **unfolding** *full-def* **by** *blast*
then have *inv'*: $cdcl_W\text{-}all\text{-}struct\text{-}inv\ T$
using *rtrancpl-cdcl_W-all-struct-inv-inv* *inv* **by** *blast*
have $cdcl_W\text{-}stgy^{**}\ S\ T$
using $\langle ?S' \rangle$ **unfolding** *full-def*
using *cdcl_W-s'-is-rtrancpl-cdcl_W-stgy* *rtrancpl-mono*[*of* *cdcl_W-s' cdcl_W-stgy^{**}*] **by** *auto*
then show $?S$
using $\langle ?S' \rangle$ *inv'* *cdcl_W-stgy-cdcl_W-s'-connected'* **unfolding** *full-def* **by** *blast*

next
assume $?S$
then have *inv-T: cdcl_W-all-struct-inv T*
by (*metis* *assms* *full-def* *rtrancpl-cdcl_W-all-struct-inv-inv* *rtrancpl-cdcl_W-stgy-rtrancpl-cdcl_W*)

consider
 $(s')\ cdcl_W\text{-}s'^{**}\ S\ T$
 $| (st)\ S'$ **where** $cdcl_W\text{-}s'^{**}\ S\ S'$ **and** $cdcl_W\text{-}bj^{++}\ S'\ T$ **and** *conflicting* $T \neq None$
using *rtrancpl-cdcl_W-stgy-connected-to-rtrancpl-cdcl_W-s'*[*of* *S T*] *inv* $\langle ?S \rangle$
unfolding *full-def* *cdcl_W-all-struct-inv-def*
by *blast*
then show $?S'$
proof *cases*
case s'
have *no-step* $cdcl_W\text{-}s'\ T$
using $\langle full\ cdcl_W\text{-}stgy\ S\ T \rangle$ **unfolding** *full-def*
by (*meson* *cdcl_W-all-struct-inv-def* *cdcl_W-s'E* *cdcl_W-stgy.conflict'*
cdcl_W-then-exists-cdcl_W-stgy-step *inv-T* *n-step-cdcl_W-stgy-iff-no-step-cdcl_W-cl-cdcl_W-o*)
then show *?thesis*
using s' **unfolding** *full-def* **by** *blast*

```

next
  case (st S')
  have full cdclW-cp T T
    using option-full-cdclW-cp st(3) by blast
  moreover
    have n-s: no-step cdclW-bj T
      by (metis ⟨full cdclW-stgy S T⟩ bj inv-T cdclW-all-struct-inv-def
        cdclW-then-exists-cdclW-stgy-step full-def)
    then have full1 cdclW-bj S' T
      using st(2) unfolding full1-def by blast
  moreover have no-step cdclW-cp S'
    using st(2) by (fastforce dest!: tranclpD simp: cdclW-cp.simps cdclW-bj.simps
      elim: rulesE)
  ultimately have cdclW-s' S' T
    using cdclW-s'.bj'[of S' T T] by blast
  then have cdclW-sfs* S T
    using st(1) by auto
  moreover have no-step cdclW-s' T
    using inv-T ⟨full cdclW-cp T T⟩ ⟨full cdclW-stgy S T⟩ unfolding full-def
    by (metis cdclW-all-struct-inv-def cdclW-then-exists-cdclW-stgy-step
      n-step-cdclW-stgy-iff-no-step-cdclW-cl-cdclW-o)
  ultimately show ?thesis
    unfolding full-def by blast
qed
qed

lemma conflict-step-cdclW-stgy-step:
  assumes
    conflict S T
    cdclW-all-struct-inv S
  shows ∃ T. cdclW-stgy S T
proof -
  obtain U where full cdclW-cp S U
    using cdclW-cp-normalized-element-all-inv assms by blast
  then have full1 cdclW-cp S U
    by (metis cdclW-cp.conflict' assms(1) full-unfold)
  then show ?thesis using cdclW-stgy.conflict' by blast
qed

lemma decide-step-cdclW-stgy-step:
  assumes
    decide S T
    cdclW-all-struct-inv S
  shows ∃ T. cdclW-stgy S T
proof -
  obtain U where full cdclW-cp T U
    using cdclW-cp-normalized-element-all-inv by (meson assms(1) assms(2) cdclW-all-struct-inv-inv
      cdclW-cp-normalized-element-all-inv decide other)
  then show ?thesis
    by (metis assms cdclW-cp-normalized-element-all-inv cdclW-stgy.conflict' decide full-unfold
      other')
qed

lemma rtranclp-cdclW-cp-conflicting-Some:
  cdclW-cp** S T ⟹ conflicting S = Some D ⟹ S = T

```

```

using rtrancpD trancpD by fastforce

inductive cdclW-merge-cp :: 'st ⇒ 'st ⇒ bool where
  conflict': conflict S T ⇒ full cdclW-bj T U ⇒ cdclW-merge-cp S U |
  propagate': propagate++ S S' ⇒ cdclW-merge-cp S S'

lemma cdclW-merge-restart-cases[consumes 1, case-names conflict propagate]:
  assumes
    cdclW-merge-cp S U and
    ∧ T. conflict S T ⇒ full cdclW-bj T U ⇒ P and
    propagate++ S U ⇒ P
  shows P
  using assms unfolding cdclW-merge-cp.simps by auto

lemma cdclW-merge-cp-trancp-cdclW-merge:
  cdclW-merge-cp S T ⇒ cdclW-merge++ S T
  apply (induction rule: cdclW-merge-cp.induct)
  using cdclW-merge.simps apply auto[1]
  using trancp-mono[of propagate cdclW-merge] fw-propagate by blast

lemma rtrancp-cdclW-merge-cp-rtrancp-cdclW:
  cdclW-merge-cp** S T ⇒ cdclW** S T
  apply (induction rule: rtrancp-induct)
  apply simp
  unfolding cdclW-merge-cp.simps by (meson cdclW-merge-restart-cdclW fw-r-conflict
    rtrancp-propagate-is-rtrancp-cdclW rtrancp-trans trancp-into-rtrancp)

lemma full1-cdclW-bj-no-step-cdclW-bj:
  full1 cdclW-bj S T ⇒ no-step cdclW-cp S
  unfolding full1-def by (metis rtrancp-unfold cdclW-cp-conflicting-not-empty option.exhaust
    rtrancp-cdclW-merge-restart-no-step-cdclW-bj trancpD)

inductive cdclW-s'-without-decide where
  conflict'-without-decide[intro]: full1 cdclW-cp S S' ⇒ cdclW-s'-without-decide S S' |
  bj'-without-decide[intro]: full1 cdclW-bj S S' ⇒ no-step cdclW-cp S ⇒ full cdclW-cp S' S''
    ⇒ cdclW-s'-without-decide S S''

lemma rtrancp-cdclW-s'-without-decide-rtrancp-cdclW:
  cdclW-s'-without-decide** S T ⇒ cdclW** S T
  apply (induction rule: rtrancp-induct)
  apply simp
  by (meson cdclW-s'.simps cdclW-s'-trancp-cdclW cdclW-s'-without-decide.simps
    rtrancp-trancp-trancp trancp-into-rtrancp)

lemma rtrancp-cdclW-s'-without-decide-rtrancp-cdclW-s':
  cdclW-s'-without-decide** S T ⇒ cdclW-s'*** S T
  proof (induction rule: rtrancp-induct)
  case base
  then show ?case by simp
next
  case (step y z) note a2 = this(2) and a1 = this(3)
  have cdclW-s' y z
  using a2 by (metis (no-types) bj' cdclW-s'.conflict' cdclW-s'-without-decide.cases)
  then show cdclW-s'*** S z
  using a1 by (meson r-into-rtrancp rtrancp-trans)

```

qed

lemma *rtrancpl-cdcl_W-merge-cp-is-rtrancpl-cdcl_W-s'-without-decide:*

assumes

*cdcl_W-merge-cp** S V*

conflicting S = None

shows

*(cdcl_W-s'-without-decide** S V)*

$\vee (\exists T. \text{cdcl}_W\text{-s'-without-decide}^{**} S T \wedge \text{propagate}^{++} T V)$

$\vee (\exists T U. \text{cdcl}_W\text{-s'-without-decide}^{**} S T \wedge \text{full1 cdcl}_W\text{-bj } T U \wedge \text{propagate}^{**} U V)$

using *assms*

proof (*induction rule: rtrancpl-induct*)

case *base*

then show *?case by simp*

next

case (*step U V*) **note** *st = this(1) and cp = this(2) and IH = this(3)[OF this(4)]*

from *cp show ?case*

proof (*cases rule: cdcl_W-merge-restart-cases*)

case *propagate*

then show *?thesis using IH by (meson rtrancpl-trancpl-trancpl trancpl-into-rtrancpl)*

next

case (*conflict U'*) **note** *confl = this(1) and bj = this(2)*

have *full1-U-U': full1 cdcl_W-cp U U'*

by (*simp add: conflict-is-full1-cdcl_W-cp local.conflict(1)*)

consider

*(s') cdcl_W-s'-without-decide** S U*

| (*propa*) *T' where cdcl_W-s'-without-decide** S T' and propagate⁺⁺ T' U*

| (*bj-prop*) *T' T'' where*

*cdcl_W-s'-without-decide** S T' and*

full1 cdcl_W-bj T' T'' and

*propagate** T'' U*

using *IH by blast*

then show *?thesis*

proof *cases*

case *s'*

have *cdcl_W-s'-without-decide U U'*

using *full1-U-U' conflict'-without-decide by blast*

then have *cdcl_W-s'-without-decide** S U'*

using *⟨cdcl_W-s'-without-decide** S U⟩ by auto*

moreover have *U' = V ∨ full1 cdcl_W-bj U' V*

using *bj by (meson full-unfold)*

ultimately show *?thesis by blast*

next

case *propa* **note** *s' = this(1) and T'-U = this(2)*

have *full1 cdcl_W-cp T' U'*

using *rtrancpl-mono[of propagate cdcl_W-cp] T'-U cdcl_W-cp.propagate' full1-U-U'*

rtrancpl-full1I[of cdcl_W-cp T'] by (metis (full-types) predicate2D predicate2I

trancpl-into-rtrancpl)

have *cdcl_W-s'-without-decide** S U'*

using *⟨full1 cdcl_W-cp T' U'⟩ conflict'-without-decide s' by force*

have *full1 cdcl_W-bj U' V ∨ V = U' using bj unfolding full-unfold by blast*

then show *?thesis*

using *⟨cdcl_W-s'-without-decide** S U'⟩ by blast*

next

case *bj-prop* **note** *s' = this(1) and bj-T' = this(2) and T''-U = this(3)*

```

have no-step cdclW-cp T'
  using bj-T' full1-cdclW-bj-no-step-cdclW-bj by blast
moreover have full1 cdclW-cp T'' U'
  using rtrancp-mono[of propagate cdclW-cp] T''-U cdclW-cp.propagate' full1-U-U'
  rtrancp-full1I[of cdclW-cp T''] by blast
ultimately have cdclW-s'-without-decide T' U'
  using bj'-without-decide[of T' T'' U'] bj-T' by (simp add: full-unfold)
then have cdclW-s'-without-decide** S U'
  using s' rtrancp.intros(2)[of - S T' U'] by blast
then show ?thesis
  using local.bj unfolding full-unfold by blast
qed
qed
qed

lemma rtrancp-cdclW-s'-without-decide-is-rtrancp-cdclW-merge-cp:
  assumes
    cdclW-s'-without-decide** S V and
    confl: conflicting S = None
  shows
    (cdclW-merge-cp** S V ∧ conflicting V = None)
    ∨ (cdclW-merge-cp** S V ∧ conflicting V ≠ None ∧ no-step cdclW-cp V ∧ no-step cdclW-bj V)
    ∨ (∃ T. cdclW-merge-cp** S T ∧ conflict T V)
  using assms(1)
proof (induction)
  case base
  then show ?case using confl by auto
next
  case (step U V) note st = this(1) and s = this(2) and IH = this(3)
  from s show ?case
  proof (cases rule: cdclW-s'-without-decide.cases)
    case conflict'-without-decide
    then have rt: cdclW-cp++ U V unfolding full1-def by fast
    then have conflicting U = None
      using trancp-cdclW-cp-propagate-with-conflict-or-not[of U V]
      conflict by (auto dest!: trancpD simp: rtrancp-unfold elim: rulesE)
    then have cdclW-merge-cp** S U using IH by (auto elim: rulesE
      simp del: state-simp simp: state-eq-def)
    consider
      (propa) propagate++ U V
      | (confl') conflict U V
      | (propa-confl') U' where propagate++ U U' conflict U' V
    using trancp-cdclW-cp-propagate-with-conflict-or-not[OF rt] unfolding rtrancp-unfold
    by fastforce
  then show ?thesis
  proof cases
    case propa
    then have cdclW-merge-cp U V
      by (auto intro: cdclW-merge-cp.intros)
    moreover have conflicting V = None
      using propa unfolding trancp-unfold-end by (auto elim: rulesE)
    ultimately show ?thesis using ⟨cdclW-merge-cp** S U⟩ by (auto elim!: rulesE
      simp del: state-simp simp: state-eq-def)
  next
    case confl'
  end

```

```

    then show ?thesis using ⟨cdclW-merge-cp** S U⟩ by auto
next
  case propa-confl' note propa = this(1) and confl' = this(2)
  then have cdclW-merge-cp U U' by (auto intro: cdclW-merge-cp.intros)
  then have cdclW-merge-cp** S U' using ⟨cdclW-merge-cp** S U⟩ by auto
  then show ?thesis using ⟨cdclW-merge-cp** S U⟩ confl' by auto
qed
next
case (bj'-without-decide U') note full-bj = this(1) and cp = this(3)
then have conflicting U ≠ None
  using full-bj unfolding full1-def by (fastforce dest!: tranclpD simp: cdclW-bj.simps
    elim: rulesE)
with IH obtain T where
  S-T: cdclW-merge-cp** S T and T-U: conflict T U
  using full-bj unfolding full1-def by (blast dest: tranclpD)
then have cdclW-merge-cp T U'
  using cdclW-merge-cp.conflict'[of T U U'] full-bj by (simp add: full-unfold)
then have S-U': cdclW-merge-cp** S U' using S-T by auto
consider
  (n-s) U' = V
  | (propa) propagate++ U' V
  | (confl') conflict U' V
  | (propa-confl') U'' where propagate++ U' U'' conflict U'' V
  using tranclp-cdclW-cp-propagate-with-conflict-or-not cp
  unfolding rtranclp-unfold full-def by metis
then show ?thesis
proof cases
case propa
  then have cdclW-merge-cp U' V by (blast intro: cdclW-merge-cp.intros)
  moreover have conflicting V = None
    using propa unfolding tranclp-unfold-end by (auto elim: rulesE)
  ultimately show ?thesis using S-U' by (auto elim: rulesE
    simp del: state-simp simp: state-eq-def)
next
case confl'
  then show ?thesis using S-U' by auto
next
case propa-confl' note propa = this(1) and confl = this(2)
  have cdclW-merge-cp U' U'' using propa by (blast intro: cdclW-merge-cp.intros)
  then show ?thesis using S-U' confl by (meson rtranclp.rtrancl-into-rtrancl)
next
case n-s
  then show ?thesis
    using S-U' apply (cases conflicting V = None)
    using full-bj apply simp
    by (metis cp full-def full-unfold full-bj)
qed
qed
qed

```

lemma *no-step-cdcl_W-s'-no-ste-cdcl_W-merge-cp:*
assumes
cdcl_W-all-struct-inv S
conflicting S = None
no-step cdcl_W-s' S

shows *no-step cdcl_W-merge-cp S*
using *assms apply (auto simp: cdcl_W-s'.simps cdcl_W-merge-cp.simps)*
using *conflict-is-full1-cdcl_W-cp apply blast*
using *cdcl_W-cp-normalized-element-all-inv cdcl_W-cp.propagate' by (metis cdcl_W-cp.propagate' full-unfold tranclpD)*

The *no-step decide S* is needed, since *cdcl_W-merge-cp* is *cdcl_W-s'* without *decide*.

lemma *conflicting-true-no-step-cdcl_W-merge-cp-no-step-s'-without-decide:*

assumes
confl: conflicting S = None and
inv: cdcl_W-M-level-inv S and
n-s: no-step cdcl_W-merge-cp S
shows *no-step cdcl_W-s'-without-decide S*
proof (rule *ccontr*)
assume \neg *no-step cdcl_W-s'-without-decide S*
then obtain *T* **where**
cdcl_W: cdcl_W-s'-without-decide S T
by *auto*
then have *inv-T: cdcl_W-M-level-inv T*
using *rtranclp-cdcl_W-s'-without-decide-rtranclp-cdcl_W[of S T]*
rtranclp-cdcl_W-consistent-inv inv **by** *blast*
from *cdcl_W* **show** *False*
proof *cases*
case *conflict'-without-decide*
have *no-step propagate S*
using *n-s* **by** (*blast intro: cdcl_W-merge-cp.intros*)
then have *conflict S T*
using *local.conflict' tranclp-cdcl_W-cp-propagate-with-conflict-or-not[of S T]*
local.conflict'-without-decide unfolding full1-def rtranclp-unfold
by (*metis tranclp-unfold-begin*)
moreover
then obtain *T'* **where** *full cdcl_W-bj T T'*
using *cdcl_W-bj-exists-normal-form inv-T* **by** *blast*
ultimately show *False* **using** *cdcl_W-merge-cp.conflict' n-s* **by** *meson*
next
case (*bj'-without-decide S'*)
then show *?thesis*
using *confl unfolding full1-def* **by** (*fastforce simp: cdcl_W-bj.simps dest: tranclpD*
elim: rulesE)
qed
qed

lemma *conflicting-true-no-step-s'-without-decide-no-step-cdcl_W-merge-cp:*

assumes
inv: cdcl_W-all-struct-inv S and
n-s: no-step cdcl_W-s'-without-decide S
shows *no-step cdcl_W-merge-cp S*
proof (rule *ccontr*)
assume \neg *?thesis*
then obtain *T* **where** *cdcl_W-merge-cp S T*
by *auto*
then show *False*
proof *cases*
case (*conflict' S'*)
then show *False* **using** *n-s conflict'-without-decide conflict-is-full1-cdcl_W-cp* **by** *blast*


```

next
  case propagate'
  moreover
    have cdclW-all-struct-inv T
      using inv by (meson local.propagate' rtrancp-cdclW-all-struct-inv-inv
        rtrancp-propagate-is-rtrancp-cdclW trancp-into-rtrancp)
    then obtain U where full cdclW-cp T U
      using cdclW-cp-normalized-element-all-inv by auto
    ultimately have full1 cdclW-cp S U
      using trancp-full-full1I[of cdclW-cp S T U] cdclW-cp.propagate'
      trancp-mono[of propagate cdclW-cp] by blast
    then show False using conflict'-without-decide n-s by blast
qed
qed

```

lemma *no-step-cdcl_W-merge-cp-no-step-cdcl_W-cp:*
no-step cdcl_W-merge-cp S \implies cdcl_W-M-level-inv S \implies no-step cdcl_W-cp S
using *cdcl_W-bj-exists-normal-form cdcl_W-consistent-inv[OF cdcl_W.conflict, of S]*
by (metis cdcl_W-cp.cases cdcl_W-merge-cp.simps trancp.intros(1))

lemma *conflicting-not-true-rtrancp-cdcl_W-merge-cp-no-step-cdcl_W-bj:*
assumes
conflicting S = None and
*cdcl_W-merge-cp** S T*
shows *no-step cdcl_W-bj T*
using *assms(2,1) by (induction)*
(fastforce simp: cdcl_W-merge-cp.simps full-def trancp-unfold-end cdcl_W-bj.simps
elim: rulesE)+

lemma *conflicting-true-full-cdcl_W-merge-cp-iff-full-cdcl_W-s'-without-decode:*
assumes
confl: conflicting S = None and
inv: cdcl_W-all-struct-inv S
shows
full cdcl_W-merge-cp S V \longleftrightarrow full cdcl_W-s'-without-decode S V (is ?fw \longleftrightarrow ?s')

proof
assume ?fw
then have *st: cdcl_W-merge-cp** S V and n-s: no-step cdcl_W-merge-cp V*
unfolding *full-def by blast+*
have *inv-V: cdcl_W-all-struct-inv V*
using *rtrancp-cdcl_W-merge-cp-rtrancp-cdcl_W[of S V] <?fw> unfolding full-def*
by (simp add: inv rtrancp-cdcl_W-all-struct-inv-inv)
consider
 (s') cdcl_W-s'-without-decode** S V
 | (propa) T **where** *cdcl_W-s'-without-decode** S T and propagate⁺⁺ T V*
 | (bj) T U **where** *cdcl_W-s'-without-decode** S T and full1 cdcl_W-bj T U and propagate** U V*
using *rtrancp-cdcl_W-merge-cp-is-rtrancp-cdcl_W-s'-without-decode confl st n-s by metis*
then have *cdcl_W-s'-without-decode** S V*
proof cases
 case s'
 then show ?thesis .
next
 case propa **note** *s' = this(1) and propa = this(2)*
 have *no-step cdcl_W-cp V*
 using *no-step-cdcl_W-merge-cp-no-step-cdcl_W-cp n-s inv-V*

```

    unfolding cdclW-all-struct-inv-def by blast
  then have full1 cdclW-cp T V
    using propa tranclp-mono[of propagate cdclW-cp] cdclW-cp.propagate' unfolding full1-def
    by blast
  then have cdclW-s'-without-decide T V
    using conflict'-without-decide by blast
  then show ?thesis using s' by auto
next
case bj note s' = this(1) and bj = this(2) and propa = this(3)
have no-step cdclW-cp V
  using no-step-cdclW-merge-cp-no-step-cdclW-cp n-s inv-V
  unfolding cdclW-all-struct-inv-def by blast
then have full cdclW-cp U V
  using propa rtranclp-mono[of propagate cdclW-cp] cdclW-cp.propagate' unfolding full-def
  by blast
moreover have no-step cdclW-cp T
  using bj unfolding full1-def by (fastforce dest!: tranclpD simp:cdclW-bj.simps elim: rulesE)
ultimately have cdclW-s'-without-decide T V
  using bj'-without-decide[of T U V] bj by blast
then show ?thesis using s' by auto
qed
moreover have no-step cdclW-s'-without-decide V
proof (cases conflicting V = None)
case False
{ fix ss :: 'st
  have ff1:  $\forall s \text{ sa. } \neg \text{cdcl}_W\text{-s}' s \text{ sa} \vee \text{full1 cdcl}_W\text{-cp s sa}$ 
     $\vee (\exists sb. \text{decide s sb} \wedge \text{no-step cdcl}_W\text{-cp s} \wedge \text{full cdcl}_W\text{-cp sb sa})$ 
     $\vee (\exists sb. \text{full1 cdcl}_W\text{-bj s sb} \wedge \text{no-step cdcl}_W\text{-cp s} \wedge \text{full cdcl}_W\text{-cp sb sa})$ 
    by (metis cdclW-s'.cases)
  have ff2:  $(\forall p \text{ s sa. } \neg \text{full1 p (s::'st) sa} \vee p^{++} \text{ s sa} \wedge \text{no-step p sa})$ 
     $\wedge (\forall p \text{ s sa. } (\neg p^{++} (s::'st) sa \vee (\exists s. p \text{ s sa})) \vee \text{full1 p s sa})$ 
    by (meson full1-def)
  obtain ssa :: ('st  $\Rightarrow$  'st  $\Rightarrow$  bool)  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  'st where
    ff3:  $\forall p \text{ s sa. } \neg p^{++} \text{ s sa} \vee p \text{ s (ssa p s sa)} \wedge p^{**} (ssa p s sa) \text{ sa}$ 
    by (metis (no-types) tranclpD)
  then have a3:  $\neg \text{cdcl}_W\text{-cp}^{++} V \text{ ss}$ 
    using False by (metis option-full-cdclW-cp full-def)
  have  $\bigwedge s. \neg \text{cdcl}_W\text{-bj}^{++} V s$ 
    using ff3 False by (metis confl st
      conflicting-not-true-rtranclp-cdclW-merge-cp-no-step-cdclW-bj)
  then have  $\neg \text{cdcl}_W\text{-s}'\text{-without-decide } V \text{ ss}$ 
    using ff1 a3 ff2 by (metis cdclW-s'-without-decide.cases)
}
}
then show ?thesis
  by fastforce
next
case True
then show ?thesis
  using conflicting-true-no-step-cdclW-merge-cp-no-step-s'-without-decide n-s inv-V
  unfolding cdclW-all-struct-inv-def by simp
qed
ultimately show ?s' unfolding full-def by blast
next
assume s': ?s'
then have st: cdclW-s'-without-decide** S V and n-s: no-step cdclW-s'-without-decide V

```

```

  unfolding full-def by auto
then have cdclW** S V
  using rtrancp-cdclW-s'-without-decide-rtrancp-cdclW st by blast
then have inv-V: cdclW-all-struct-inv V using inv rtrancp-cdclW-all-struct-inv-inv by blast
then have n-s-cp-V: no-step cdclW-cp V
  using cdclW-cp-normalized-element-all-inv[of V] full-fullI[of cdclW-cp V] n-s
  conflict'-without-decide conflicting-true-no-step-s'-without-decide-no-step-cdclW-merge-cp
  no-step-cdclW-merge-cp-no-step-cdclW-cp
  unfolding cdclW-all-struct-inv-def by presburger
have n-s-bj: no-step cdclW-bj V
proof (rule ccontr)
  assume ¬ ?thesis
  then obtain W where W: cdclW-bj V W by blast
  have cdclW-all-struct-inv W
    using W cdclW.simps cdclW-all-struct-inv-inv inv-V by blast
  then obtain W' where full1 cdclW-bj V W'
    using cdclW-bj-exists-normal-form[of W] full-fullI[of cdclW-bj V W] W
    unfolding cdclW-all-struct-inv-def
    by blast
  moreover
    then have cdclW++ V W'
      using trancp-mono[of cdclW-bj cdclW] cdclW.other cdclW-o.bj unfolding full1-def by blast
    then have cdclW-all-struct-inv W'
      by (meson inv-V rtrancp-cdclW-all-struct-inv-inv trancp-into-rtrancp)
    then obtain X where full cdclW-cp W' X
      using cdclW-cp-normalized-element-all-inv by blast
    ultimately show False
      using bj'-without-decide n-s-cp-V n-s by blast
qed
from s' consider
  (cp-true) cdclW-merge-cp** S V and conflicting V = None
| (cp-false) cdclW-merge-cp** S V and conflicting V ≠ None and no-step cdclW-cp V and
  no-step cdclW-bj V
| (cp-conf) T where cdclW-merge-cp** S T conflict T V
using rtrancp-cdclW-s'-without-decide-is-rtrancp-cdclW-merge-cp[of S V] confl
unfolding full-def by meson
then have cdclW-merge-cp** S V
proof cases
  case cp-conf note S-T = this(1) and conf-V = this(2)
  have full cdclW-bj V V
    using conf-V n-s-bj unfolding full-def by fast
  then have cdclW-merge-cp T V
    using cdclW-merge-cp.conflict' conf-V by auto
  then show ?thesis using S-T by auto
qed fast+
moreover
  then have cdclW** S V using rtrancp-cdclW-merge-cp-rtrancp-cdclW by blast
  then have cdclW-all-struct-inv V
    using inv rtrancp-cdclW-all-struct-inv-inv by blast
  then have no-step cdclW-merge-cp V
    using conflicting-true-no-step-s'-without-decide-no-step-cdclW-merge-cp s'
    unfolding full-def by blast
  ultimately show ?fw unfolding full-def by auto
qed

```

lemma *conflicting-true-full1-cdcl_W-merge-cp-iff-full1-cdcl_W-s'-without-decode:*
assumes
confl: conflicting S = None and
inv: cdcl_W-all-struct-inv S
shows
full1 cdcl_W-merge-cp S V \longleftrightarrow full1 cdcl_W-s'-without-decide S V
proof –
have *full cdcl_W-merge-cp S V = full cdcl_W-s'-without-decide S V*
using *confl conflicting-true-full-cdcl_W-merge-cp-iff-full-cdcl_W-s'-without-decode inv*
by *simp*
then show *?thesis unfolding full-unfold full1-def tranclp-unfold-begin by blast*
qed

lemma *conflicting-true-full1-cdcl_W-merge-cp-imp-full1-cdcl_W-s'-without-decode:*
assumes
fw: full1 cdcl_W-merge-cp S V and
inv: cdcl_W-all-struct-inv S
shows
full1 cdcl_W-s'-without-decide S V
proof –
have *conflicting S = None*
using *fw unfolding full1-def by (auto dest!: tranclpD simp: cdcl_W-merge-cp.simps elim: rulesE)*
then show *?thesis*
using *conflicting-true-full1-cdcl_W-merge-cp-iff-full1-cdcl_W-s'-without-decode fw inv by simp*
qed

inductive *cdcl_W-merge-stgy where*
fw-s-cp[intro]: full1 cdcl_W-merge-cp S T \implies cdcl_W-merge-stgy S T |
fw-s-decide[intro]: decide S T \implies no-step cdcl_W-merge-cp S \implies full cdcl_W-merge-cp T U
 \implies *cdcl_W-merge-stgy S U*

lemma *cdcl_W-merge-stgy-tranclp-cdcl_W-merge:*
assumes *fw: cdcl_W-merge-stgy S T*
shows *cdcl_W-merge⁺⁺ S T*
proof –
{ fix S T
assume *full1 cdcl_W-merge-cp S T*
then have *cdcl_W-merge⁺⁺ S T*
using *tranclp-mono[of cdcl_W-merge-cp cdcl_W-merge⁺⁺] cdcl_W-merge-cp-tranclp-cdcl_W-merge*
unfolding *full1-def*
by *auto*
} note *full1-cdcl_W-merge-cp-cdcl_W-merge = this*
show *?thesis*
using *fw*
apply *(induction rule: cdcl_W-merge-stgy.induct)*
using *full1-cdcl_W-merge-cp-cdcl_W-merge apply simp*
unfolding *full-unfold by (auto dest!: full1-cdcl_W-merge-cp-cdcl_W-merge fw-decide)*
qed

lemma *rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W-merge:*
assumes *fw: cdcl_W-merge-stgy^{**} S T*
shows *cdcl_W-merge^{**} S T*
using *fw cdcl_W-merge-stgy-tranclp-cdcl_W-merge rtranclp-mono[of cdcl_W-merge-stgy cdcl_W-merge⁺⁺]*
unfolding *tranclp-rtranclp-rtranclp by blast*

lemma *cdcl_W-merge-stgy-rtrancpl-cdcl_W*:
cdcl_W-merge-stgy *S T* \implies *cdcl_W** S T*
apply (*induction rule*: *cdcl_W-merge-stgy.induct*)
using *rtrancpl-cdcl_W-merge-cp-rtrancpl-cdcl_W* **unfolding** *full1-def*
apply (*simp add*: *trancpl-into-rtrancpl*)
using *rtrancpl-cdcl_W-merge-cp-rtrancpl-cdcl_W* *cdcl_W-o.decide* *cdcl_W.other* **unfolding** *full-def*
by (*meson r-into-rtrancpl rtrancpl-trans*)

lemma *rtrancpl-cdcl_W-merge-stgy-rtrancpl-cdcl_W*:
*cdcl_W-merge-stgy** S T* \implies *cdcl_W** S T*
using *rtrancpl-mono*[*of cdcl_W-merge-stgy cdcl_W***] *cdcl_W-merge-stgy-rtrancpl-cdcl_W* **by** *auto*

lemma *cdcl_W-merge-stgy-cases*[*consumes 1, case-names fw-s-cp fw-s-decide*]:
assumes
cdcl_W-merge-stgy S U
full1 cdcl_W-merge-cp S U \implies P
 $\bigwedge T. \text{decide } S \ T \implies \text{no-step } cdcl_W\text{-merge-cp } S \implies \text{full } cdcl_W\text{-merge-cp } T \ U \implies P$
shows *P*
using *assms* **by** (*auto simp*: *cdcl_W-merge-stgy.simps*)

inductive *cdcl_W-s'-w* :: '*st* \Rightarrow '*st* \Rightarrow *bool* **where**
conflict': *full1 cdcl_W-s'-without-decide S S' \implies cdcl_W-s'-w S S' |*
decide': *decide S S' \implies no-step cdcl_W-s'-without-decide S \implies full cdcl_W-s'-without-decide S' S''*
 \implies *cdcl_W-s'-w S S''*

lemma *cdcl_W-s'-w-rtrancpl-cdcl_W*:
cdcl_W-s'-w S T \implies *cdcl_W** S T*
apply (*induction rule*: *cdcl_W-s'-w.induct*)
using *rtrancpl-cdcl_W-s'-without-decide-rtrancpl-cdcl_W* **unfolding** *full1-def*
apply (*simp add*: *trancpl-into-rtrancpl*)
using *rtrancpl-cdcl_W-s'-without-decide-rtrancpl-cdcl_W* **unfolding** *full-def*
by (*meson decide other rtrancpl-into-trancpl2 trancpl-into-rtrancpl*)

lemma *rtrancpl-cdcl_W-s'-w-rtrancpl-cdcl_W*:
*cdcl_W-s'-w** S T* \implies *cdcl_W** S T*
using *rtrancpl-mono*[*of cdcl_W-s'-w cdcl_W***] *cdcl_W-s'-w-rtrancpl-cdcl_W* **by** *auto*

lemma *no-step-cdcl_W-cp-no-step-cdcl_W-s'-without-decide*:
assumes *no-step cdcl_W-cp S* **and** *conflicting S = None* **and** *inv*: *cdcl_W-M-level-inv S*
shows *no-step cdcl_W-s'-without-decide S*
by (*metis assms cdcl_W-cp.conflict' cdcl_W-cp.propagate' cdcl_W-merge-restart-cases trancplD*
conflicting-true-no-step-cdcl_W-merge-cp-no-step-s'-without-decide)

lemma *no-step-cdcl_W-cp-no-step-cdcl_W-merge-restart*:
assumes *no-step cdcl_W-cp S* **and** *conflicting S = None*
shows *no-step cdcl_W-merge-cp S*
by (*metis assms*(1) *cdcl_W-cp.conflict' cdcl_W-cp.propagate' cdcl_W-merge-restart-cases trancplD*)

lemma *after-cdcl_W-s'-without-decide-no-step-cdcl_W-cp*:
assumes *cdcl_W-s'-without-decide S T*
shows *no-step cdcl_W-cp T*
using *assms* **by** (*induction rule*: *cdcl_W-s'-without-decide.induct*) (*auto simp*: *full1-def full-def*)

lemma *no-step-cdcl_W-s'-without-decide-no-step-cdcl_W-cp*:
cdcl_W-all-struct-inv S \implies *no-step cdcl_W-s'-without-decide S* \implies *no-step cdcl_W-cp S*
by (*simp add*: *conflicting-true-no-step-s'-without-decide-no-step-cdcl_W-merge-cp*)

no-step-cdcl_W-merge-cp-no-step-cdcl_W-cp cdcl_W-all-struct-inv-def)

lemma *after-cdcl_W-s'-w-no-step-cdcl_W-cp*:
assumes *cdcl_W-s'-w S T and cdcl_W-all-struct-inv S*
shows *no-step cdcl_W-cp T*
using *assms*
proof (*induction rule: cdcl_W-s'-w.induct*)
case *conflict'*
then show *?case*
by (*auto simp: full1-def tranclp-unfold-end after-cdcl_W-s'-without-decide-no-step-cdcl_W-cp*)
next
case (*decide' S T U*)
moreover
then have *cdcl_W** S U*
using *rtranclp-cdcl_W-s'-without-decide-rtranclp-cdcl_W[of T U] cdcl_W.other[of S T]*
cdcl_W-o.decide unfolding full-def by auto
then have *cdcl_W-all-struct-inv U*
using *decide'.prems rtranclp-cdcl_W-all-struct-inv-inv by blast*
ultimately show *?case*
using *no-step-cdcl_W-s'-without-decide-no-step-cdcl_W-cp unfolding full-def by blast*
qed

lemma *rtranclp-cdcl_W-s'-w-no-step-cdcl_W-cp-or-eq*:
assumes *cdcl_W-s'-w** S T and cdcl_W-all-struct-inv S*
shows *S = T ∨ no-step cdcl_W-cp T*
using *assms*
proof (*induction rule: rtranclp-induct*)
case *base*
then show *?case by simp*
next
case (*step T U*)
moreover have *cdcl_W-all-struct-inv T*
using *rtranclp-cdcl_W-s'-w-rtranclp-cdcl_W[of S U] assms(2) rtranclp-cdcl_W-all-struct-inv-inv*
rtranclp-cdcl_W-s'-w-rtranclp-cdcl_W step.hyps(1) by blast
ultimately show *?case using after-cdcl_W-s'-w-no-step-cdcl_W-cp by fast*
qed

lemma *rtranclp-cdcl_W-merge-stgy'-no-step-cdcl_W-cp-or-eq*:
assumes *cdcl_W-merge-stgy** S T and inv: cdcl_W-all-struct-inv S*
shows *S = T ∨ no-step cdcl_W-cp T*
using *assms*
proof (*induction rule: rtranclp-induct*)
case *base*
then show *?case by simp*
next
case (*step T U*)
moreover have *cdcl_W-all-struct-inv T*
using *rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W[of S U] assms(2) rtranclp-cdcl_W-all-struct-inv-inv*
rtranclp-cdcl_W-s'-w-rtranclp-cdcl_W step.hyps(1)
by (*meson rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W*)
ultimately show *?case*
using *after-cdcl_W-s'-w-no-step-cdcl_W-cp inv unfolding cdcl_W-all-struct-inv-def*
by (*metis cdcl_W-all-struct-inv-def cdcl_W-merge-stgy.simps full1-def full-def*
no-step-cdcl_W-merge-cp-no-step-cdcl_W-cp rtranclp-cdcl_W-all-struct-inv-inv
rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W tranclp.intros(1) tranclp-into-rtranclp)

qed

lemma *no-step-cdcl_W-s'-without-decide-no-step-cdcl_W-bj*:
assumes *no-step cdcl_W-s'-without-decide S and inv: cdcl_W-all-struct-inv S*
shows *no-step cdcl_W-bj S*
proof (*rule ccontr*)
assume $\neg ?thesis$
then obtain *T where S-T: cdcl_W-bj S T*
by *auto*
have *cdcl_W-all-struct-inv T*
using *S-T cdcl_W-all-struct-inv-inv inv other* **by** *blast*
then obtain *T' where full1 cdcl_W-bj S T'*
using *cdcl_W-bj-exists-normal-form[of T] full-full1 S-T* **unfolding** *cdcl_W-all-struct-inv-def*
by *metis*
moreover
then have *cdcl_W** S T'*
using *rtrancpl-mono[of cdcl_W-bj cdcl_W] cdcl_W.other cdcl_W-o.bj trancpl-into-rtrancpl[of cdcl_W-bj]*
unfolding *full1-def* **by** *blast*
then have *cdcl_W-all-struct-inv T'*
using *inv rtrancpl-cdcl_W-all-struct-inv-inv* **by** *blast*
then obtain *U where full cdcl_W-cp T' U*
using *cdcl_W-cp-normalized-element-all-inv* **by** *blast*
moreover have *no-step cdcl_W-cp S*
using *S-T* **by** (*auto simp: cdcl_W-bj.simps elim: rulesE*)
ultimately show *False*
using *assms cdcl_W-s'-without-decide.intros(2)[of S T' U]* **by** *fast*
qed

lemma *cdcl_W-s'-w-no-step-cdcl_W-bj*:
assumes *cdcl_W-s'-w S T and cdcl_W-all-struct-inv S*
shows *no-step cdcl_W-bj T*
using *assms* **apply** *induction*
using *rtrancpl-cdcl_W-s'-without-decide-rtrancpl-cdcl_W rtrancpl-cdcl_W-all-struct-inv-inv*
no-step-cdcl_W-s'-without-decide-no-step-cdcl_W-bj **unfolding** *full1-def*
apply (*meson trancpl-into-rtrancpl*)
using *rtrancpl-cdcl_W-s'-without-decide-rtrancpl-cdcl_W rtrancpl-cdcl_W-all-struct-inv-inv*
no-step-cdcl_W-s'-without-decide-no-step-cdcl_W-bj **unfolding** *full-def*
by (*meson cdcl_W-merge-restart-cdcl_W fw-r-decide*)

lemma *rtrancpl-cdcl_W-s'-w-no-step-cdcl_W-bj-or-eq*:
assumes *cdcl_W-s'-w** S T and cdcl_W-all-struct-inv S*
shows *S = T \vee no-step cdcl_W-bj T*
using *assms* **apply** *induction*
apply *simp*
using *rtrancpl-cdcl_W-s'-w-rtrancpl-cdcl_W rtrancpl-cdcl_W-all-struct-inv-inv*
cdcl_W-s'-w-no-step-cdcl_W-bj **by** *meson*

lemma *rtrancpl-cdcl_W-s'-no-step-cdcl_W-s'-without-decide-decomp-into-cdcl_W-merge*:
assumes
*cdcl_W-s'*** R V and*
conflicting R = None and
inv: cdcl_W-all-struct-inv R
shows (*cdcl_W-merge-stgy** R V \wedge conflicting V = None*)
 \vee (*cdcl_W-merge-stgy** R V \wedge conflicting V \neq None \wedge no-step cdcl_W-bj V*)
 \vee ($\exists S T U. \text{cdcl}_W\text{-merge-stgy}^{***} R S \wedge \text{no-step cdcl}_W\text{-merge-cp } S \wedge \text{decide } S T$)

```

     $\wedge \text{cdcl}_W\text{-merge-cp}^{**} T U \wedge \text{conflict } U V$ 
 $\vee (\exists S T. \text{cdcl}_W\text{-merge-stgy}^{**} R S \wedge \text{no-step cdcl}_W\text{-merge-cp } S \wedge \text{decide } S T$ 
 $\wedge \text{cdcl}_W\text{-merge-cp}^{**} T V$ 
 $\wedge \text{conflicting } V = \text{None})$ 
 $\vee (\text{cdcl}_W\text{-merge-cp}^{**} R V \wedge \text{conflicting } V = \text{None})$ 
 $\vee (\exists U. \text{cdcl}_W\text{-merge-cp}^{**} R U \wedge \text{conflict } U V)$ 
using assms(1,2)
proof induction
  case base
  then show ?case by simp
next
  case (step  $V W$ ) note  $st = \text{this}(1)$  and  $s' = \text{this}(2)$  and  $IH = \text{this}(3)[\text{OF } \text{this}(4)]$  and
 $n\text{-s-}R = \text{this}(4)$ 
  from  $s'$ 
  show ?case
  proof cases
    case conflict'
    consider
      ( $s'$ )  $\text{cdcl}_W\text{-merge-stgy}^{**} R V$ 
      | (dec-conf)  $S T U$  where  $\text{cdcl}_W\text{-merge-stgy}^{**} R S$  and  $\text{no-step cdcl}_W\text{-merge-cp } S$  and
 $\text{decide } S T$  and  $\text{cdcl}_W\text{-merge-cp}^{**} T U$  and  $\text{conflict } U V$ 
      | (dec)  $S T$  where  $\text{cdcl}_W\text{-merge-stgy}^{**} R S$  and  $\text{no-step cdcl}_W\text{-merge-cp } S$  and  $\text{decide } S T$ 
and  $\text{cdcl}_W\text{-merge-cp}^{**} T V$  and  $\text{conflicting } V = \text{None}$ 
      | (cp)  $\text{cdcl}_W\text{-merge-cp}^{**} R V$ 
      | (cp-conf)  $U$  where  $\text{cdcl}_W\text{-merge-cp}^{**} R U$  and  $\text{conflict } U V$ 
    using  $IH$  by meson
  then show ?thesis
  proof cases
  next
    case  $s'$ 
    then have  $R = V$ 
    by (metis full1-def inv local.conflict' tranclp-unfold-begin
 $\text{rtranclp-cdcl}_W\text{-merge-stgy}'\text{-no-step-cdcl}_W\text{-cp-or-eq}$ )
    consider
      ( $V\text{-}W$ )  $V = W$ 
      | (propa)  $\text{propagate}^{++} V W$  and  $\text{conflicting } W = \text{None}$ 
      | (propa-conf)  $V'$  where  $\text{propagate}^{**} V V'$  and  $\text{conflict } V' W$ 
    using tranclp-cdclW-cp-propagate-with-conflict-or-not[of  $V W$ ] conflict'
    unfolding full-unfold full1-def by meson
  then show ?thesis
  proof cases
    case  $V\text{-}W$ 
    then show ?thesis using  $\langle R = V \rangle$   $n\text{-s-}R$  by simp
  next
    case propa
    then show ?thesis using  $\langle R = V \rangle$  by (auto intro: cdclW-merge-cp.intros)
  next
    case propa-conf
    moreover
      then have  $\text{cdcl}_W\text{-merge-cp}^{**} V V'$ 
      by (metis rtranclp-unfold cdclW-merge-cp.propagate' r-into-rtranclp)
    ultimately show ?thesis using  $s' \langle R = V \rangle$  by blast
  qed
next
  case dec-conf note  $- = \text{this}(5)$ 

```



```

then have False using conflict' unfolding full1-def by (auto dest!: tranclpD elim: rulesE)
then show ?thesis by fast
next
case dec note T-V = this(4)
consider
  (propa) propagate++ V W and conflicting W = None
  | (propa-conf) V' where propagate** V V' and conflict V' W
  using tranclp-cdclW-cp-propagate-with-conflict-or-not[of V W] conflict'
  unfolding full1-def by meson
then show ?thesis
proof cases
case propa
then show ?thesis
  by (meson T-V cdclW-merge-cp.propagate' dec rtranclp.rtrancl-into-rtrancl)
next
case propa-conf
then have cdclW-merge-cp** T V'
  using T-V by (metis rtranclp-unfold cdclW-merge-cp.propagate' rtranclp.simps)
then show ?thesis using dec propa-conf(2) by metis
qed
next
case cp
consider
  (propa) propagate++ V W and conflicting W = None
  | (propa-conf) V' where propagate** V V' and conflict V' W
  using tranclp-cdclW-cp-propagate-with-conflict-or-not[of V W] conflict'
  unfolding full1-def by meson
then show ?thesis
proof cases
case propa
then show ?thesis by (meson cdclW-merge-cp.propagate' cp
  rtranclp.rtrancl-into-rtrancl)
next
case propa-conf
then show ?thesis
  using propa-conf(2) cp
  by (metis (full-types) cdclW-merge-cp.propagate' rtranclp.rtrancl-into-rtrancl
    rtranclp-unfold)
qed
next
case cp-conf
then show ?thesis using conflict' unfolding full1-def by (fastforce dest!: tranclpD
  elim!: rulesE)
qed
next
case (decide' V')
then have conf-V: conflicting V = None
  by (auto elim: rulesE)
consider
  (s') cdclW-merge-stgy** R V
  | (dec-conf) S T U where cdclW-merge-stgy** R S and no-step cdclW-merge-cp S and
    decide S T and cdclW-merge-cp** T U and conflict U V
  | (dec) S T where cdclW-merge-stgy** R S and no-step cdclW-merge-cp S and decide S T
    and cdclW-merge-cp** T V and conflicting V = None
  | (cp) cdclW-merge-cp** R V

```

```

| (cp-conflict) U where cdclW-merge-cp** R U and conflict U V
using IH by meson
then show ?thesis
proof cases
  case s'
  have confl-V': conflicting V' = None using decide'(1) by (auto elim: rulesE)
  have full: full1 cdclW-cp V' W  $\vee$  (V' = W  $\wedge$  no-step cdclW-cp W)
    using decide'(3) unfolding full-unfold by blast
  consider
    (V'-W) V' = W
  | (propa) propagate++ V' W and conflicting W = None
  | (propa-conflict) V'' where propagate** V' V'' and conflict V'' W
  using tranclp-cdclW-cp-propagate-with-conflict-or-not[of V W] decide'
    ⟨full1 cdclW-cp V' W  $\vee$  V' = W  $\wedge$  no-step cdclW-cp W⟩ unfolding full1-def
  by (metis tranclp-cdclW-cp-propagate-with-conflict-or-not)
then show ?thesis
proof cases
  case V'-W
  then show ?thesis
    using confl-V' local.decide'(1,2) s' conf-V
    no-step-cdclW-cp-no-step-cdclW-merge-restart[of V]
    by auto
  next
  case propa
  then show ?thesis using local.decide'(1,2) s' by (metis cdclW-merge-cp.simps conf-V
    no-step-cdclW-cp-no-step-cdclW-merge-restart r-into-rtranclp)
  next
  case propa-conflict
  then have cdclW-merge-cp** V' V''
    by (metis rtranclp-unfold cdclW-merge-cp.propagate' r-into-rtranclp)
  then show ?thesis
    using local.decide'(1,2) propa-conflict(2) s' conf-V
    no-step-cdclW-cp-no-step-cdclW-merge-restart
    by metis
  qed
next
case (dec) note s' = this(1) and dec = this(2) and cp = this(3) and ns-cp-T = this(4)
have full cdclW-merge-cp T V
  unfolding full-def by (simp add: conf-V local.decide'(2)
    no-step-cdclW-cp-no-step-cdclW-merge-restart ns-cp-T)
moreover have no-step cdclW-merge-cp V
  by (simp add: conf-V local.decide'(2) no-step-cdclW-cp-no-step-cdclW-merge-restart)
moreover have no-step cdclW-merge-cp S
  by (metis dec)
ultimately have cdclW-merge-stgy S V
  using cp by blast
then have cdclW-merge-stgy** R V using s' by auto
consider
  (V'-W) V' = W
  | (propa) propagate++ V' W and conflicting W = None
  | (propa-conflict) V'' where propagate** V' V'' and conflict V'' W
  using tranclp-cdclW-cp-propagate-with-conflict-or-not[of V' W] decide'
  unfolding full-unfold full1-def by meson
then show ?thesis
proof cases

```

```

    case  $V'-W$ 
    moreover have conflicting  $V' = \text{None}$ 
      using decide'(1) by (auto elim: rulesE)
    ultimately show ?thesis
      using  $\langle \text{cdcl}_W\text{-merge-stgy}^{**} R V \rangle$  decide'  $\langle \text{no-step cdcl}_W\text{-merge-cp } V \rangle$  by blast
  next
    case propa
    moreover then have  $\text{cdcl}_W\text{-merge-cp } V' W$  by (blast intro:  $\text{cdcl}_W\text{-merge-cp.intros}$ )
    ultimately show ?thesis
      using  $\langle \text{cdcl}_W\text{-merge-stgy}^{**} R V \rangle$  decide'  $\langle \text{no-step cdcl}_W\text{-merge-cp } V \rangle$ 
      by (meson r-into-rtrancpl)
  next
    case propa-conf
    moreover then have  $\text{cdcl}_W\text{-merge-cp}^{**} V' V''$ 
      by (metis  $\text{cdcl}_W\text{-merge-cp.propagate' rtrancpl-unfold trancpl-unfold-end}$ )
    ultimately show ?thesis using  $\langle \text{cdcl}_W\text{-merge-stgy}^{**} R V \rangle$  decide'
       $\langle \text{no-step cdcl}_W\text{-merge-cp } V \rangle$  by (meson r-into-rtrancpl)
  qed
next
  case cp
  have  $\text{no-step cdcl}_W\text{-merge-cp } V$ 
    using conf-V local.decide'(2)  $\text{no-step-cdcl}_W\text{-cp-no-step-cdcl}_W\text{-merge-restart}$  by auto
  then have  $\text{full cdcl}_W\text{-merge-cp } R V$ 
    unfolding full-def using cp by fast
  then have  $\text{cdcl}_W\text{-merge-stgy}^{**} R V$ 
    unfolding full-unfold by auto
  have  $\text{full1 cdcl}_W\text{-cp } V' W \vee (V' = W \wedge \text{no-step cdcl}_W\text{-cp } W)$ 
    using decide'(3) unfolding full-unfold by blast

consider
  ( $V'-W$ )  $V' = W$ 
  | (propa)  $\text{propagate}^{++} V' W$  and conflicting  $W = \text{None}$ 
  | (propa-conf)  $V''$  where  $\text{propagate}^{**} V' V''$  and conflict  $V'' W$ 
  using  $\text{trancpl-cdcl}_W\text{-cp-propagate-with-conflict-or-not}$ [of  $V' W$ ] decide'
  unfolding full-unfold full1-def by meson
then show ?thesis

proof cases
  case  $V'-W$ 
  moreover have conflicting  $V' = \text{None}$ 
    using decide'(1) by (auto elim: rulesE)
  ultimately show ?thesis
    using  $\langle \text{cdcl}_W\text{-merge-stgy}^{**} R V \rangle$  decide'  $\langle \text{no-step cdcl}_W\text{-merge-cp } V \rangle$  by blast
next
  case propa
  moreover then have  $\text{cdcl}_W\text{-merge-cp } V' W$ 
    by (blast intro:  $\text{cdcl}_W\text{-merge-cp.intros}$ )
  ultimately show ?thesis using  $\langle \text{cdcl}_W\text{-merge-stgy}^{**} R V \rangle$  decide'
     $\langle \text{no-step cdcl}_W\text{-merge-cp } V \rangle$  by (meson r-into-rtrancpl)
next
  case propa-conf
  moreover then have  $\text{cdcl}_W\text{-merge-cp}^{**} V' V''$ 
    by (metis  $\text{cdcl}_W\text{-merge-cp.propagate' rtrancpl-unfold trancpl-unfold-end}$ )
  ultimately show ?thesis using  $\langle \text{cdcl}_W\text{-merge-stgy}^{**} R V \rangle$  decide'
     $\langle \text{no-step cdcl}_W\text{-merge-cp } V \rangle$  by (meson r-into-rtrancpl)

```

```

    qed
  next
    case (dec-conf)
    show ?thesis using conf-V dec-conf(5) by (auto elim!: rulesE
      simp del: state-simp simp: state-eq-def)
  next
    case cp-conf
    then show ?thesis using decide' apply - by (intro HOL.disjI2) (fastforce elim: rulesE
      simp del: state-simp simp: state-eq-def)
  qed
next
case (bj' V')
then have ¬no-step cdclW-bj V
  by (auto dest: tranclpD simp: full1-def)
then consider
  (s') cdclW-merge-stgy** R V and conflicting V = None
| (dec-conf) S T U where cdclW-merge-stgy** R S and no-step cdclW-merge-cp S and
  decide S T and cdclW-merge-cp** T U and conflict U V
| (dec) S T where cdclW-merge-stgy** R S and no-step cdclW-merge-cp S and decide S T
  and cdclW-merge-cp** T V and conflicting V = None
| (cp) cdclW-merge-cp** R V and conflicting V = None
| (cp-conf) U where cdclW-merge-cp** R U and conflict U V
using IH by meson
then show ?thesis
proof cases
  case s' note - = this(2)
  then have False
    using bj'(1) unfolding full1-def by (force dest!: tranclpD simp: cdclW-bj.simps
      elim: rulesE)
  then show ?thesis by fast
next
  case dec note - = this(5)
  then have False
    using bj'(1) unfolding full1-def by (force dest!: tranclpD simp: cdclW-bj.simps
      elim: rulesE)
  then show ?thesis by fast
next
  case dec-conf
  then have cdclW-merge-cp U V'
    using bj' cdclW-merge-cp.intros(1)[of U V V'] by (simp add: full-unfold)
  then have cdclW-merge-cp** T V'
    using dec-conf(4) by simp
  consider
    (V'-W) V' = W
  | (propa) propagate++ V' W and conflicting W = None
  | (propa-conf) V'' where propagate** V' V'' and conflict V'' W
  using tranclp-cdclW-cp-propagate-with-conflict-or-not[of V' W] bj'(3)
  unfolding full-unfold full1-def by meson
  then show ?thesis
proof cases
  case V'-W
  then have no-step cdclW-cp V'
    using bj'(3) unfolding full-def by auto
  then have no-step cdclW-merge-cp V'
    by (metis cdclW-cp.propagate' cdclW-merge-cp.cases tranclpD

```

```

    no-step-cdclW-cp-no-conflict-no-propagate(1) )
  then have full1 cdclW-merge-cp T V'
    unfolding full1-def using ⟨cdclW-merge-cp U V'⟩ dec-confl(4) by auto
  then have full cdclW-merge-cp T V'
    by (simp add: full-unfold)
  then have cdclW-merge-stgy S V'
    using dec-confl(3) cdclW-merge-stgy.fw-s-decide ⟨no-step cdclW-merge-cp S⟩ by blast
  then have cdclW-merge-stgy** R V'
    using ⟨cdclW-merge-stgy** R S⟩ by auto
  show ?thesis
  proof cases
    assume conflicting W = None
    then show ?thesis using ⟨cdclW-merge-stgy** R V'⟩ ⟨V' = W⟩ by auto
  next
    assume conflicting W ≠ None
    then show ?thesis
      using ⟨cdclW-merge-stgy** R V'⟩ ⟨V' = W⟩ by (metis ⟨cdclW-merge-cp U V'⟩
        conflictE conflicting-not-true-rtranclp-cdclW-merge-cp-no-step-cdclW-bj
        dec-confl(5) map-option-is-None r-into-rtranclp)
    qed
  next
    case propa
    moreover then have cdclW-merge-cp V' W by (blast intro: cdclW-merge-cp.intros)
    ultimately show ?thesis using decide' by (meson ⟨cdclW-merge-cp** T V'⟩ dec-confl(1-3)
      rtranclp.rtrancl-into-rtrancl)
  next
    case propa-confl
    moreover then have cdclW-merge-cp** V' V''
      by (metis cdclW-merge-cp.propagate' rtranclp-unfold tranclp-unfold-end)
    ultimately show ?thesis by (meson ⟨cdclW-merge-cp** T V'⟩ dec-confl(1-3) rtranclp-trans)
    qed
  next
    case cp note - = this(2)
    then show ?thesis using bj'(1) ⟨¬ no-step cdclW-bj V⟩
      conflicting-not-true-rtranclp-cdclW-merge-cp-no-step-cdclW-bj by auto
  next
    case cp-confl
    then have cdclW-merge-cp U V' by (simp add: cdclW-merge-cp.conflict' full-unfold
      local.bj'(1))
    consider
      (V'-W) V' = W
    | (propa) propagate++ V' W and conflicting W = None
    | (propa-confl) V'' where propagate** V' V'' and conflict V'' W
    using tranclp-cdclW-cp-propagate-with-conflict-or-not[of V' W] bj'
    unfolding full-unfold full1-def by meson
  then show ?thesis

  proof cases
    case V'-W
    show ?thesis
    proof cases
      assume conflicting V' = None
      then show ?thesis
        using V'-W ⟨cdclW-merge-cp U V'⟩ cp-confl(1) by force
    next

```

```

assume confl: conflicting  $V' \neq \text{None}$ 
then have no-step cdclW-merge-stgy  $V'$ 
  by (fastforce simp: cdclW-merge-stgy.simps full1-def full-def
    cdclW-merge-cp.simps dest!: trancplD elim: rulesE)
have no-step cdclW-merge-cp  $V'$ 
  using confl by (auto simp: full1-def full-def cdclW-merge-cp.simps
    dest!: trancplD elim: rulesE)
moreover have cdclW-merge-cp  $U\ W$ 
  using  $V'-W$   $\langle \text{cdcl}_W\text{-merge-cp } U\ V' \rangle$  by blast
ultimately have full1 cdclW-merge-cp  $R\ V'$ 
  using cp-confl(1)  $V'-W$  unfolding full1-def by auto
then have cdclW-merge-stgy  $R\ V'$ 
  by auto
moreover have no-step cdclW-merge-stgy  $V'$ 
  using confl  $\langle \text{no-step } \text{cdcl}_W\text{-merge-cp } V' \rangle$  by (auto simp: cdclW-merge-stgy.simps
    full1-def dest!: trancplD elim: rulesE)
ultimately have cdclW-merge-stgy**  $R\ V'$  by auto
{ fix ss :: 'st
  have cdclW-merge-cp  $U\ W$ 
    using  $V'-W$   $\langle \text{cdcl}_W\text{-merge-cp } U\ V' \rangle$  by blast
  then have  $\neg \text{cdcl}_W\text{-bj } W\ ss$ 
    by (meson conflicting-not-true-rtrancpl-cdclW-merge-cp-no-step-cdclW-bj
      cp-confl(1) rtrancpl.rtrancpl-into-rtrancpl step.prems)
  then have cdclW-merge-stgy**  $R\ W \wedge \text{conflicting } W = \text{None} \vee$ 
    cdclW-merge-stgy**  $R\ W \wedge \neg \text{cdcl}_W\text{-bj } W\ ss$ 
    using  $V'-W$   $\langle \text{cdcl}_W\text{-merge-stgy}^{**} R\ V' \rangle$  by presburger }
then show ?thesis
  by presburger
qed
next
case propa
moreover then have cdclW-merge-cp  $V'\ W$ 
  by (blast intro: cdclW-merge-cp.intros)
ultimately show ?thesis using  $\langle \text{cdcl}_W\text{-merge-cp } U\ V' \rangle$  cp-confl(1) by force
next
case propa-confl
moreover then have cdclW-merge-cp**  $V'\ V''$ 
  by (metis cdclW-merge-cp.propagate' rtrancpl-unfold trancpl-unfold-end)
ultimately show ?thesis
  using  $\langle \text{cdcl}_W\text{-merge-cp } U\ V' \rangle$  cp-confl(1) by (metis rtrancpl.rtrancpl-into-rtrancpl
    rtrancpl-trans)
qed
qed
qed
qed

```

lemma *decide-rtrancpl-cdcl_W-s'-rtrancpl-cdcl_W-s'*:

assumes

dec: *decide* $S\ T$ **and**

cdcl_W-s'^{**} $T\ U$ **and**

n-s-S: *no-step* *cdcl_W-cp* S **and**

no-step *cdcl_W-cp* U

shows *cdcl_W-s'*^{**} $S\ U$

using *assms*(2,4)

proof *induction*

```

case (step U V) note st = this(1) and s' = this(2) and IH = this(3) and n-s = this(4)
consider
  (TU) T = U
  | (s'-st) T' where cdclW-s' T T' and cdclW-s'** T' U
  using st[unfolded rtrancp-unfold] by (auto dest!: trancpD)
then show ?case
proof cases
  case TU
  then show ?thesis
  proof -
    assume a1: T = U
    then have f2: cdclW-s' T V
    using s' by force
    obtain ss :: 'st where
      ss: cdclW-s'** S T ∨ cdclW-cp T ss
    using a1 step.IH by blast-
    obtain ssa :: 'st ⇒ 'st where
      f3: ∀ s sa sb. (¬ decide s sa ∨ cdclW-cp s (ssa s) ∨ ¬ full cdclW-cp sa sb)
      ∨ cdclW-s' s sb
    using cdclW-s'.decide' by moura
    have ∀ s sa. ¬ cdclW-s' s sa ∨ full1 cdclW-cp s sa ∨
      (∃ sb. decide s sb ∧ no-step cdclW-cp s ∧ full cdclW-cp sb sa) ∨
      (∃ sb. full1 cdclW-bj s sb ∧ no-step cdclW-cp s ∧ full cdclW-cp sb sa)
    by (metis cdclW-s'E)
    then have ∃ s. cdclW-s'** S s ∧ cdclW-s' s V
    using f3 ss f2 by (metis dec full1-is-full n-s-S rtrancp-unfold)
    then show ?thesis
    by force
  qed
next
case (s'-st T') note s'-T' = this(1) and st = this(2)
have cdclW-s'** S T'
using s'-T'
proof cases
  case conflict'
  then have cdclW-s' S T'
    using dec cdclW-s'.decide' n-s-S by (simp add: full-unfold)
  then show ?thesis
    using st by auto
next
case (decide' T'')
then have cdclW-s' S T
  using dec cdclW-s'.decide' n-s-S by (simp add: full-unfold)
then show ?thesis using decide' s'-T' by auto
next
case bj'
then have False
  using dec unfolding full1-def by (fastforce dest!: trancpD simp: cdclW-bj.simps
    elim: rulesE)
then show ?thesis by fast
qed
then show ?thesis using s' st by auto
qed
next
case base

```

```

then have full cdclW-cp T T
  by (simp add: full-unfold)
then show ?case
  using cdclW-s'.simps dec n-s-S by auto
qed

lemma rtrancp-cdclW-merge-stgy-rtrancp-cdclW-s':
  assumes
    cdclW-merge-stgy** R V and
    inv: cdclW-all-struct-inv R
  shows cdclW-s'*** R V
  using assms(1)
proof induction
  case base
  then show ?case by simp
next
  case (step S T) note st = this(1) and fw = this(2) and IH = this(3)
  have cdclW-all-struct-inv S
    using inv rtrancp-cdclW-all-struct-inv-inv rtrancp-cdclW-merge-stgy-rtrancp-cdclW st by blast
  from fw show ?case
  proof (cases rule: cdclW-merge-stgy-cases)
    case fw-s-cp
    have  $\bigwedge s. \neg \text{full } \text{cdcl}_W\text{-merge-cp } s \ S$ 
      using fw-s-cp unfolding full-def full1-def by (metis trancp-unfold-begin)
    then have S = R
      using fw-s-cp unfolding full1-def by (metis cdclW-cp.conflict' cdclW-cp.propagate'
        cdclW-merge-cp.cases trancp-unfold-begin inv st
        rtrancp-cdclW-merge-stgy'-no-step-cdclW-cp-or-eq)
    then have full1 cdclW-s'-without-decide R T
      using inv local.fw-s-cp
      by (blast intro: conflicting-true-full1-cdclW-merge-cp-imp-full1-cdclW-s'-without-decode)
    then show ?thesis unfolding full1-def
      by (metis (no-types) rtrancp-cdclW-s'-without-decide-rtrancp-cdclW-s' rtrancp-unfold)
  next
    case (fw-s-decide S') note dec = this(1) and n-S = this(2) and full = this(3)
    moreover then have conflicting S' = None
      by (auto elim: rulesE)
    ultimately have full cdclW-s'-without-decide S' T
      by (meson  $\langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$  cdclW-merge-restart-cdclW fw-r-decide
        rtrancp-cdclW-all-struct-inv-inv
        conflicting-true-full-cdclW-merge-cp-iff-full-cdclW-s'-without-decode)
    then have a1: cdclW-s'*** S' T
      unfolding full-def by (metis (full-types) rtrancp-cdclW-s'-without-decide-rtrancp-cdclW-s')
    have cdclW-merge-stgy** S T
      using fw by blast
    then have cdclW-s'*** S T
      using decide-rtrancp-cdclW-s'-rtrancp-cdclW-s' a1 by (metis  $\langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$  dec
        n-S no-step-cdclW-merge-cp-no-step-cdclW-cp cdclW-all-struct-inv-def
        rtrancp-cdclW-merge-stgy'-no-step-cdclW-cp-or-eq)
    then show ?thesis using IH by auto
  qed
qed

```

```

lemma rtrancp-cdclW-merge-stgy-distinct-mset-clauses:
  assumes invR: cdclW-all-struct-inv R and

```



```

st: cdclW-merge-stgy** R S and
dist: distinct-mset (clauses R) and
R: trail R = []
shows distinct-mset (clauses S)
using rtrancpl-cdclW-stgy-distinct-mset-clauses[OF invR - dist R]
invR st rtrancpl-mono[of cdclW-s' cdclW-stgy**] cdclW-s'-is-rtrancpl-cdclW-stgy
by (auto dest!: cdclW-s'-is-rtrancpl-cdclW-stgy rtrancpl-cdclW-merge-stgy-rtrancpl-cdclW-s')

lemma no-step-cdclW-s'-no-step-cdclW-merge-stgy:
  assumes
    inv: cdclW-all-struct-inv R and s': no-step cdclW-s' R
  shows no-step cdclW-merge-stgy R
proof -
  { fix ss :: 'st
    obtain ssa :: 'st  $\Rightarrow$  'st  $\Rightarrow$  'st where
      ff1:  $\bigwedge s sa. \neg cdcl_W\text{-merge-stgy } s sa \vee full1\ cdcl_W\text{-merge-cp } s sa \vee decide\ s\ (ssa\ s\ sa)$ 
      using cdclW-merge-stgy.cases by moura
    obtain ssb :: ('st  $\Rightarrow$  'st  $\Rightarrow$  bool)  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  'st where
      ff2:  $\bigwedge p s sa. \neg p^{++}\ s\ sa \vee p\ s\ (ssb\ p\ s\ sa)$ 
      by (meson trancpl-unfold-begin)
    obtain ssc :: 'st  $\Rightarrow$  'st where
      ff3:  $\bigwedge s sa sb. (\neg cdcl_W\text{-all-struct-inv } s \vee \neg cdcl_W\text{-cp } s\ sa \vee cdcl_W\text{-s'}\ s\ (ssc\ s))$ 
         $\wedge (\neg cdcl_W\text{-all-struct-inv } s \vee \neg cdcl_W\text{-o } s\ sb \vee cdcl_W\text{-s'}\ s\ (ssc\ s))$ 
      using n-step-cdclW-stgy-iff-no-step-cdclW-cl-cdclW-o by moura
    then have ff4:  $\bigwedge s. \neg cdcl_W\text{-o } R\ s$ 
      using s' inv by blast
    have ff5:  $\bigwedge s. \neg cdcl_W\text{-cp}^{++}\ R\ s$ 
      using ff3 ff2 s' by (metis inv)
    have  $\bigwedge s. \neg cdcl_W\text{-bj}^{++}\ R\ s$ 
      using ff4 ff2 by (metis bj)
    then have  $\bigwedge s. \neg cdcl_W\text{-s'-without-decide } R\ s$ 
      using ff5 by (simp add: cdclW-s'-without-decide.simps full1-def)
    then have  $\neg cdcl_W\text{-s'-without-decide}^{++}\ R\ ss$ 
      using ff2 by blast
    then have  $\neg full1\ cdcl_W\text{-s'-without-decide } R\ ss$ 
      by (simp add: full1-def)
    then have  $\neg cdcl_W\text{-merge-stgy } R\ ss$ 
      using ff4 ff1 conflicting-true-full1-cdclW-merge-cp-imp-full1-cdclW-s'-without-decode inv
      by blast }
    then show ?thesis
      by fastforce
  }
qed
end

```

We will discharge the assumption later.

```

locale conflict-driven-clause-learningW-termination =
  conflict-driven-clause-learningW +
  assumes wf-cdclW-merge-inv: wf {(T, S). cdclW-all-struct-inv S  $\wedge$  cdclW-merge S T}
begin

```

```

lemma wf-trancpl-cdclW-merge: wf {(T, S). cdclW-all-struct-inv S  $\wedge$  cdclW-merge++ S T}
  using wf-trancpl[OF wf-cdclW-merge-inv]
  apply (rule wf-subset)
  by (auto simp: trancpl-set-trancpl
      cdclW-all-struct-inv-trancpl-cdclW-merge-trancpl-cdclW-merge-cdclW-all-struct-inv)

```

lemma *wf-cdcl_W-merge-cp*:
wf{(*T*, *S*). *cdcl_W-all-struct-inv S* \wedge *cdcl_W-merge-cp S T*}
using *wf-tranclp-cdcl_W-merge* **by** (*rule wf-subset*) (*auto simp: cdcl_W-merge-cp-tranclp-cdcl_W-merge*)

lemma *wf-cdcl_W-merge-stgy*:
wf{(*T*, *S*). *cdcl_W-all-struct-inv S* \wedge *cdcl_W-merge-stgy S T*}
using *wf-tranclp-cdcl_W-merge* **by** (*rule wf-subset*)
(*auto simp add: cdcl_W-merge-stgy-tranclp-cdcl_W-merge*)

lemma *cdcl_W-merge-cp-obtain-normal-form*:
assumes *inv: cdcl_W-all-struct-inv R*
obtains *S* **where** *full cdcl_W-merge-cp R S*

proof –

obtain *S* **where** *full* ($\lambda S T. \text{cdcl}_W\text{-all-struct-inv } S \wedge \text{cdcl}_W\text{-merge-cp } S T$) *R S*
using *wf-exists-normal-form-full*[*OF wf-cdcl_W-merge-cp*] **by** *blast*

then have

*st: (λS T. cdcl_W-all-struct-inv S \wedge cdcl_W-merge-cp S T)** R S* **and**

n-s: no-step ($\lambda S T. \text{cdcl}_W\text{-all-struct-inv } S \wedge \text{cdcl}_W\text{-merge-cp } S T$) *S*

unfolding *full-def* **by** *blast+*

have *cdcl_W-merge-cp** R S*

using *st* **by** *induction auto*

moreover

have *cdcl_W-all-struct-inv S*

using *st inv*

apply (*induction rule: rtranclp-induct*)

apply *simp*

by (*meson r-into-rtranclp rtranclp-cdcl_W-all-struct-inv-inv*
rtranclp-cdcl_W-merge-cp-rtranclp-cdcl_W)

then have *no-step cdcl_W-merge-cp S*

using *n-s* **by** *auto*

ultimately show *?thesis*

using *that unfolding full-def* **by** *blast*

qed

lemma *no-step-cdcl_W-merge-stgy-no-step-cdcl_W-s'*:

assumes

inv: cdcl_W-all-struct-inv R **and**

confl: conflicting R = None **and**

n-s: no-step cdcl_W-merge-stgy R

shows *no-step cdcl_W-s' R*

proof (*rule ccontr*)

assume $\neg ?thesis$

then obtain *S* **where** *cdcl_W-s' R S* **by** *auto*

then show *False*

proof *cases*

case *conflict'*

then obtain *S'* **where** *full1 cdcl_W-merge-cp R S'*

proof –

obtain *R' :: 'e* **where**

cdcl_W-merge-cp R R'

using *inv* **unfolding** *cdcl_W-all-struct-inv-def* **by** (*meson confl*

cdcl_W-s'-without-decide.simps conflict'

conflicting-true-no-step-cdcl_W-merge-cp-no-step-s'-without-decide)

then show *?thesis*

```

    using that by (metis cdclW-merge-cp-obtain-normal-form full-unfold inv)
  qed
  then show False using n-s by blast
next
case (decide' R')
then have cdclW-all-struct-inv R'
  using inv cdclW-all-struct-inv-inv cdclW.other cdclW-o.decide by meson
then obtain R'' where full cdclW-merge-cp R' R''
  using cdclW-merge-cp-obtain-normal-form by blast
moreover have no-step cdclW-merge-cp R
  by (simp add: confl local.decide'(2) no-step-cdclW-cp-no-step-cdclW-merge-restart)
ultimately show False using n-s cdclW-merge-stgy.intros local.decide'(1) by blast
next
case (bj' R')
then show False
  using confl no-step-cdclW-cp-no-step-cdclW-s'-without-decide inv
  unfolding cdclW-all-struct-inv-def by auto
qed
qed

lemma rtranclp-cdclW-merge-cp-no-step-cdclW-bj:
  assumes conflicting R = None and cdclW-merge-cp** R S
  shows no-step cdclW-bj S
  using assms conflicting-not-true-rtranclp-cdclW-merge-cp-no-step-cdclW-bj by auto

lemma rtranclp-cdclW-merge-stgy-no-step-cdclW-bj:
  assumes confl: conflicting R = None and cdclW-merge-stgy** R S
  shows no-step cdclW-bj S
  using assms(2)
proof induction
  case base
  then show ?case
    using confl by (auto simp: cdclW-bj.simps elim: rulesE)
next
case (step S T) note st = this(1) and fw = this(2) and IH = this(3)
have confl-S: conflicting S = None
  using fw apply cases
  by (auto simp: full1-def cdclW-merge-cp.simps dest!: tranclpD elim: rulesE)
from fw show ?case
  proof cases
    case fw-s-cp
    then show ?thesis
      using rtranclp-cdclW-merge-cp-no-step-cdclW-bj confl-S
      by (simp add: full1-def tranclp-into-rtranclp)
  next
    case (fw-s-decide S')
    moreover then have conflicting S' = None by (auto elim: rulesE)
    ultimately show ?thesis
      using conflicting-not-true-rtranclp-cdclW-merge-cp-no-step-cdclW-bj
      unfolding full-def by meson
  qed
qed
qed

end

```

```

end
theory CDCL-W-Restart
imports CDCL-W-Merge
begin

```

21.5 Adding Restarts

```

locale cdclW-restart =
  conflict-driven-clause-learningW
  — functions for clauses:
  mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss

  — functions for the conflicting clause:
  mset-ccls union-ccls insert-ccls remove-clit

  — conversion
  ccls-of-cls cls-of-ccls

  — functions for the state:
  — access functions:
  trail hd-raw-trail raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting
  — changing state:
  cons-trail tl-trail add-init-cls add-learned-cls remove-cls update-backtrack-lvl
  update-conflicting

  — get state:
  init-state
  restart-state
for
  mset-cls:: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and

  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and

  mset-ccls:: 'ccls  $\Rightarrow$  'v clause and
  union-ccls :: 'ccls  $\Rightarrow$  'ccls  $\Rightarrow$  'ccls and
  insert-ccls :: 'v literal  $\Rightarrow$  'ccls  $\Rightarrow$  'ccls and
  remove-clit :: 'v literal  $\Rightarrow$  'ccls  $\Rightarrow$  'ccls and

  ccls-of-cls :: 'cls  $\Rightarrow$  'ccls and
  cls-of-ccls :: 'ccls  $\Rightarrow$  'cls and

  trail :: 'st  $\Rightarrow$  ('v, nat, 'v clause) marked-lits and
  hd-raw-trail :: 'st  $\Rightarrow$  ('v, nat, 'cls) marked-lit and
  raw-init-clss :: 'st  $\Rightarrow$  'clss and
  raw-learned-clss :: 'st  $\Rightarrow$  'clss and
  backtrack-lvl :: 'st  $\Rightarrow$  nat and
  raw-conflicting :: 'st  $\Rightarrow$  'ccls option and

  cons-trail :: ('v, nat, 'cls) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and

```

```

tl-trail :: 'st ⇒ 'st and
add-init-cls :: 'cls ⇒ 'st ⇒ 'st and
add-learned-cls :: 'cls ⇒ 'st ⇒ 'st and
remove-cls :: 'cls ⇒ 'st ⇒ 'st and
update-backtrack-lvl :: nat ⇒ 'st ⇒ 'st and
update-conflicting :: 'ccls option ⇒ 'st ⇒ 'st and

init-state :: 'clss ⇒ 'st and
restart-state :: 'st ⇒ 'st +
fixes f :: nat ⇒ nat
assumes f: unbounded f
begin

```

The condition of the differences of cardinality has to be strict. Otherwise, you could be in a strange state, where nothing remains to do, but a restart is done. See the proof of well-foundedness.

inductive *cdcl_W-merge-with-restart* **where**

restart-step:

```

(cdeclW-merge-stgy  $\sim$  (card (set-mset (learned-clss T)) - card (set-mset (learned-clss S)))) S T
⇒ card (set-mset (learned-clss T)) - card (set-mset (learned-clss S)) > f n
⇒ restart T U ⇒ cdeclW-merge-with-restart (S, n) (U, Suc n) |

```

restart-full: full1 cdecl_W-merge-stgy S T ⇒ cdecl_W-merge-with-restart (S, n) (T, Suc n)

lemma cdecl_W-merge-with-restart S T ⇒ cdecl_W-merge-restart** (fst S) (fst T)

by (induction rule: cdecl_W-merge-with-restart.induct)

```

(auto dest!: relpowp-imp-rtranclp cdeclW-merge-stgy-tranclp-cdeclW-merge tranclp-into-rtranclp
  rtranclp-cdeclW-merge-stgy-rtranclp-cdeclW-merge rtranclp-cdeclW-merge-tranclp-cdeclW-merge-restart
  fw-r-rf cdeclW-rf.restart
  simp: full1-def)

```

lemma cdecl_W-merge-with-restart-rtranclp-cdecl_W:

cdecl_W-merge-with-restart S T ⇒ cdecl_W** (fst S) (fst T)

by (induction rule: cdecl_W-merge-with-restart.induct)

```

(auto dest!: relpowp-imp-rtranclp rtranclp-cdeclW-merge-stgy-rtranclp-cdeclW cdeclW.rf
  cdeclW-rf.restart tranclp-into-rtranclp simp: full1-def)

```

lemma cdecl_W-merge-with-restart-increasing-number:

cdecl_W-merge-with-restart S T ⇒ snd T = 1 + snd S

by (induction rule: cdecl_W-merge-with-restart.induct) auto

lemma full1 cdecl_W-merge-stgy S T ⇒ cdecl_W-merge-with-restart (S, n) (T, Suc n)

using restart-full **by** blast

lemma cdecl_W-all-struct-inv-learned-clss-bound:

assumes inv: cdecl_W-all-struct-inv S

shows set-mset (learned-clss S) ⊆ simple-clss (atms-of-mm (init-clss S))

proof

fix C

assume C: C ∈ set-mset (learned-clss S)

have distinct-mset C

```

  using C inv unfolding cdeclW-all-struct-inv-def distinct-cdeclW-state-def distinct-mset-set-def
  by auto

```

moreover **have** ¬tautology C

```

  using C inv unfolding cdeclW-all-struct-inv-def cdeclW-learned-clause-def by auto

```

moreover

```

have atms-of  $C \subseteq \text{atms-of-mm } (\text{learned-clss } S)$ 
  using C by auto
then have atms-of  $C \subseteq \text{atms-of-mm } (\text{init-clss } S)$ 
  using inv unfolding cdclW-all-struct-inv-def no-strange-atm-def by force
moreover have finite (atms-of-mm (init-clss S))
  using inv unfolding cdclW-all-struct-inv-def by auto
ultimately show  $C \in \text{simple-clss } (\text{atms-of-mm } (\text{init-clss } S))$ 
  using distinct-mset-not-tautology-implies-in-simple-clss simple-clss-mono
  by blast
qed

```

lemma *cdcl_W-merge-with-restart-init-clss*:

```

cdclW-merge-with-restart S T  $\implies \text{cdcl}_W\text{-M-level-inv } (\text{fst } S) \implies$ 
init-clss (fst S) = init-clss (fst T)
using cdclW-merge-with-restart-rtrancpl-cdclW rtrancpl-cdclW-init-clss by blast

```

lemma

```

wf {(T, S). cdclW-all-struct-inv (fst S)  $\wedge$  cdclW-merge-with-restart S T}
proof (rule ccontr)
  assume  $\neg ?thesis$ 
  then obtain g where
    g:  $\bigwedge i. \text{cdcl}_W\text{-merge-with-restart } (g\ i) (g\ (\text{Suc } i))$  and
    inv:  $\bigwedge i. \text{cdcl}_W\text{-all-struct-inv } (\text{fst } (g\ i))$ 
  unfolding wf-iff-no-infinite-down-chain by fast
  { fix i
    have init-clss (fst (g i)) = init-clss (fst (g 0))
      apply (induction i)
      apply simp
      using g inv unfolding cdclW-all-struct-inv-def by (metis cdclW-merge-with-restart-init-clss)
    } note init-g = this
  let ?S = g 0
  have finite (atms-of-mm (init-clss (fst ?S)))
    using inv unfolding cdclW-all-struct-inv-def by auto
  have snd-g:  $\bigwedge i. \text{snd } (g\ i) = i + \text{snd } (g\ 0)$ 
    apply (induct-tac i)
    apply simp
    by (metis Suc-eq-plus1-left add-Suc cdclW-merge-with-restart-increasing-number g)
  then have snd-g-0:  $\bigwedge i. i > 0 \implies \text{snd } (g\ i) = i + \text{snd } (g\ 0)$ 
    by blast
  have unbounded-f-g: unbounded ( $\lambda i. f\ (\text{snd } (g\ i))$ )
    using f unfolding bounded-def by (metis add commute f less-or-eq-imp-le snd-g
      not-bounded-nat-exists-larger not-le le-iff-add)

  obtain k where
    f-g-k:  $f\ (\text{snd } (g\ k)) > \text{card } (\text{simple-clss } (\text{atms-of-mm } (\text{init-clss } (\text{fst } ?S))))$  and
     $k > \text{card } (\text{simple-clss } (\text{atms-of-mm } (\text{init-clss } (\text{fst } ?S))))$ 
    using not-bounded-nat-exists-larger[OF unbounded-f-g] by blast

```

The following does not hold anymore with the non-strict version of cardinality in the definition.

```

{ fix i
  assume no-step cdclW-merge-stgy (fst (g i))
  with g[of i]
  have False
    proof (induction rule: cdclW-merge-with-restart.induct)
      case (restart-step T S n) note H = this(1) and c = this(2) and n-s = this(4)

```

```

    obtain  $S'$  where  $cdcl_W$ -merge-stgy  $S S'$ 
      using  $H c$  by (metis gr-implies-not0 relpowp-E2)
    then show False using  $n-s$  by auto
  next
    case (restart-full  $S T$ )
    then show False unfolding full1-def by (auto dest: tranclpD)
  qed
} note  $H = this$ 
obtain  $m T$  where
   $m: m = \text{card} (\text{set-mset} (\text{learned-clss } T)) - \text{card} (\text{set-mset} (\text{learned-clss} (\text{fst } (g k))))$  and
   $m > f (\text{snd } (g k))$  and
  restart  $T (\text{fst } (g (k+1)))$  and
   $cdcl_W$ -merge-stgy:  $(cdcl_W$ -merge-stgy  $\sim m$ )  $(\text{fst } (g k)) T$ 
  using  $g[\text{of } k] H[\text{of } \text{Suc } k]$  by (force simp:  $cdcl_W$ -merge-with-restart.simps full1-def)
have  $cdcl_W$ -merge-stgy**  $(\text{fst } (g k)) T$ 
  using  $cdcl_W$ -merge-stgy relpowp-imp-rtranclp by metis
then have  $cdcl_W$ -all-struct-inv  $T$ 
  using  $inv[\text{of } k] rtranclp$ - $cdcl_W$ -all-struct-inv-inv  $rtranclp$ - $cdcl_W$ -merge-stgy- $rtranclp$ - $cdcl_W$ 
  by blast
moreover have  $\text{card} (\text{set-mset} (\text{learned-clss } T)) - \text{card} (\text{set-mset} (\text{learned-clss} (\text{fst } (g k))))$ 
   $> \text{card} (\text{simple-clss} (\text{atms-of-mm} (\text{init-clss} (\text{fst } ?S))))$ 
  unfolding  $m[\text{symmetric}]$  using  $\langle m > f (\text{snd } (g k)) \rangle f-g-k$  by linarith
then have  $\text{card} (\text{set-mset} (\text{learned-clss } T))$ 
   $> \text{card} (\text{simple-clss} (\text{atms-of-mm} (\text{init-clss} (\text{fst } ?S))))$ 
  by linarith
moreover
  have  $\text{init-clss} (\text{fst } (g k)) = \text{init-clss } T$ 
    using  $\langle cdcl_W$ -merge-stgy**  $(\text{fst } (g k)) T \rangle rtranclp$ - $cdcl_W$ -merge-stgy- $rtranclp$ - $cdcl_W$ 
     $rtranclp$ - $cdcl_W$ -init-clss  $inv$  unfolding  $cdcl_W$ -all-struct-inv-def by blast
  then have  $\text{init-clss} (\text{fst } ?S) = \text{init-clss } T$ 
    using  $init-g[\text{of } k]$  by auto
ultimately show False
  using  $cdcl_W$ -all-struct-inv-learned-clss-bound
  by (simp add:  $\langle \text{finite} (\text{atms-of-mm} (\text{init-clss} (\text{fst } (g 0)))) \rangle \text{simple-clss-finite}$ 
     $\text{card-mono leD}$ )
qed

lemma  $cdcl_W$ -merge-with-restart-distinct-mset-clauses:
  assumes  $invR: cdcl_W$ -all-struct-inv  $(fst R)$  and
   $st: cdcl_W$ -merge-with-restart  $R S$  and
   $dist: \text{distinct-mset} (\text{clauses} (\text{fst } R))$  and
   $R: \text{trail } (\text{fst } R) = []$ 
  shows  $\text{distinct-mset} (\text{clauses} (\text{fst } S))$ 
  using  $assms(2,1,3,4)$ 
proof (induction)
  case (restart-full  $S T$ )
  then show ?case using  $rtranclp$ - $cdcl_W$ -merge-stgy-distinct-mset-clauses[ $of S T$ ] unfolding full1-def
    by (auto dest: tranclp-into-rtranclp)
next
  case (restart-step  $T S n U$ )
  then have  $\text{distinct-mset} (\text{clauses } T)$ 
    using  $rtranclp$ - $cdcl_W$ -merge-stgy-distinct-mset-clauses[ $of S T$ ] unfolding full1-def
    by (auto dest: relpowp-imp-rtranclp)
  then show ?case using  $\langle \text{restart } T U \rangle$  by (metis clauses-restart distinct-mset-union fstI
     $mset-le-exists-conv restart.cases state-eq-clauses$ )

```

qed

inductive *cdcl_W-with-restart* **where**

restart-step:

$(\text{cdcl}_W\text{-stgy} \sim (\text{card } (\text{set-mset } (\text{learned-clss } T)) - \text{card } (\text{set-mset } (\text{learned-clss } S)))) S T \implies$
 $\text{card } (\text{set-mset } (\text{learned-clss } T)) - \text{card } (\text{set-mset } (\text{learned-clss } S)) > f n \implies$
 $\text{restart } T U \implies$

cdcl_W-with-restart (*S*, *n*) (*U*, *Suc n*) |

restart-full: *full1 cdcl_W-stgy S T* \implies *cdcl_W-with-restart* (*S*, *n*) (*T*, *Suc n*)

lemma *cdcl_W-with-restart-rtranclp-cdcl_W*:

cdcl_W-with-restart S T \implies *cdcl_W*** (*fst S*) (*fst T*)

apply (*induction rule*: *cdcl_W-with-restart.induct*)

by (*auto dest!*: *relopw-imp-rtranclp tranclp-into-rtranclp fw-r-rf*

cdcl_W-rf.restart rtranclp-cdcl_W-stgy-rtranclp-cdcl_W cdcl_W-merge-restart-cdcl_W

simp: *full1-def*)

lemma *cdcl_W-with-restart-increasing-number*:

cdcl_W-with-restart S T \implies *snd T* = 1 + *snd S*

by (*induction rule*: *cdcl_W-with-restart.induct*) *auto*

lemma *full1 cdcl_W-stgy S T* \implies *cdcl_W-with-restart* (*S*, *n*) (*T*, *Suc n*)

using *restart-full* **by** *blast*

lemma *cdcl_W-with-restart-init-clss*:

cdcl_W-with-restart S T \implies *cdcl_W-M-level-inv* (*fst S*) \implies *init-clss* (*fst S*) = *init-clss* (*fst T*)

using *cdcl_W-with-restart-rtranclp-cdcl_W rtranclp-cdcl_W-init-clss* **by** *blast*

lemma

wf {(*T*, *S*). *cdcl_W-all-struct-inv* (*fst S*) \wedge *cdcl_W-with-restart S T*}

proof (*rule ccontr*)

assume \neg *?thesis*

then obtain *g* **where**

g: $\bigwedge i. \text{cdcl}_W\text{-with-restart } (g\ i) (g\ (\text{Suc } i))$ **and**

inv: $\bigwedge i. \text{cdcl}_W\text{-all-struct-inv } (\text{fst } (g\ i))$

unfolding *wf-iff-no-infinite-down-chain* **by** *fast*

{ **fix** *i*

have *init-clss* (*fst* (*g i*)) = *init-clss* (*fst* (*g 0*))

apply (*induction i*)

apply *simp*

using *g inv* **unfolding** *cdcl_W-all-struct-inv-def* **by** (*metis cdcl_W-with-restart-init-clss*)

} **note** *init-g* = *this*

let *?S* = *g 0*

have *finite* (*atms-of-mm* (*init-clss* (*fst ?S*)))

using *inv* **unfolding** *cdcl_W-all-struct-inv-def* **by** *auto*

have *snd-g*: $\bigwedge i. \text{snd } (g\ i) = i + \text{snd } (g\ 0)$

apply (*induct-tac i*)

apply *simp*

by (*metis Suc-eq-plus1-left add-Suc cdcl_W-with-restart-increasing-number g*)

then have *snd-g-0*: $\bigwedge i. i > 0 \implies \text{snd } (g\ i) = i + \text{snd } (g\ 0)$

by *blast*

have *unbounded-f-g*: *unbounded* ($\lambda i. f\ (\text{snd } (g\ i))$)

using *f* **unfolding** *bounded-def* **by** (*metis add.commute f less-or-eq-imp-le snd-g not-bounded-nat-exists-larger not-le le-iff-add*)

obtain k where

$f\text{-}g\text{-}k$: $f \text{ (snd } (g \ k)) > \text{card } (\text{simple-clss } (\text{atms-of-mm } (\text{init-clss } (\text{fst } ?S))))$ and
 $k > \text{card } (\text{simple-clss } (\text{atms-of-mm } (\text{init-clss } (\text{fst } ?S))))$
 using *not-bounded-nat-exists-larger*[*OF unbounded-f-g*] by *blast*

The following does not hold anymore with the non-strict version of cardinality in the definition.

```

{ fix i
  assume no-step cdclW-stgy (fst (g i))
  with g[of i]
  have False
  proof (induction rule: cdclW-with-restart.induct)
    case (restart-step T S n) note H = this(1) and c = this(2) and n-s = this(4)
    obtain S' where cdclW-stgy S S'
    using H c by (metis gr-implies-not0 relpowp-E2)
    then show False using n-s by auto
  next
    case (restart-full S T)
    then show False unfolding full1-def by (auto dest: tranclpD)
  qed
} note H = this
obtain m T where
  m: m = card (set-mset (learned-clss T)) - card (set-mset (learned-clss (fst (g k)))) and
  m > f (snd (g k)) and
  restart T (fst (g (k+1))) and
  cdclW-merge-stgy: (cdclW-stgy  $\sim$  m) (fst (g k)) T
  using g[of k] H[of Suc k] by (force simp: cdclW-with-restart.simps full1-def)
have cdclW-stgy** (fst (g k)) T
  using cdclW-merge-stgy relpowp-imp-rtranclp by metis
then have cdclW-all-struct-inv T
  using inv[of k] rtranclp-cdclW-all-struct-inv-inv rtranclp-cdclW-stgy-rtranclp-cdclW by blast
moreover have card (set-mset (learned-clss T)) - card (set-mset (learned-clss (fst (g k))))
  > card (simple-clss (atms-of-mm (init-clss (fst ?S))))
  unfolding m[symmetric] using m > f (snd (g k)) f-g-k by linarith
then have card (set-mset (learned-clss T))
  > card (simple-clss (atms-of-mm (init-clss (fst ?S))))
  by linarith
moreover
  have init-clss (fst (g k)) = init-clss T
    using cdclW-stgy** (fst (g k)) T rtranclp-cdclW-stgy-rtranclp-cdclW rtranclp-cdclW-init-clss
    inv unfolding cdclW-all-struct-inv-def
    by blast
  then have init-clss (fst ?S) = init-clss T
    using init-g[of k] by auto
ultimately show False
  using cdclW-all-struct-inv-learned-clss-bound
  by (simp add: finite (atms-of-mm (init-clss (fst (g 0)))) simple-clss-finite
    card-mono leD)
qed

```

lemma *cdcl_W-with-restart-distinct-mset-clauses*:

assumes *invR*: *cdcl_W-all-struct-inv* (fst *R*) and
st: *cdcl_W-with-restart* *R S* and
dist: *distinct-mset* (*clauses* (fst *R*)) and
R: *trail* (fst *R*) = []
shows *distinct-mset* (*clauses* (fst *S*))

```

using assms(2,1,3,4)
proof (induction)
  case (restart-full S T)
  then show ?case using rtrancpl-cdclW-stgy-distinct-mset-clauses[of S T] unfolding full1-def
    by (auto dest: trancpl-into-rtrancpl)
next
  case (restart-step T S n U)
  then have distinct-mset (clauses T) using rtrancpl-cdclW-stgy-distinct-mset-clauses[of S T]
    unfolding full1-def by (auto dest: relpowp-imp-rtrancpl)
  then show ?case using (restart T U) by (metis clauses-restart distinct-mset-union fstI
    mset-le-exists-conv restart.cases state-eq-clauses)
qed
end

```

```

locale luby-sequence =
  fixes ur :: nat
  assumes ur > 0
begin

```

```

lemma exists-luby-decomp:
  fixes i :: nat
  shows  $\exists k::nat. (2^{k-1} \leq i \wedge i < 2^k - 1) \vee i = 2^k - 1$ 
proof (induction i)
  case 0
  then show ?case
    by (rule exI[of - 0], simp)
next
  case (Suc n)
  then obtain k where  $2^{k-1} \leq n \wedge n < 2^k - 1 \vee n = 2^k - 1$ 
    by blast
  then consider
    (st-interv)  $2^{k-1} \leq n$  and  $n \leq 2^k - 2$ 
  | (end-interv)  $2^{k-1} \leq n$  and  $n = 2^k - 2$ 
  | (pow2)  $n = 2^k - 1$ 
  by linarith
  then show ?case
  proof cases
    case st-interv
    then show ?thesis apply - apply (rule exI[of - k])
      by (metis (no-types, lifting) One-nat-def Suc-diff-Suc Suc-lessI
         $(2^{k-1} \leq n \wedge n < 2^k - 1 \vee n = 2^k - 1)$  diff-self-eq-0
        dual-order.trans le-SucI le-imp-less-Suc numeral-2-eq-2 one-le-numeral
        one-le-power zero-less-numeral zero-less-power)
    next
    case end-interv
    then show ?thesis apply - apply (rule exI[of - k]) by auto
  next
    case pow2
    then show ?thesis apply - apply (rule exI[of - k+1]) by auto
  qed
qed

```

Luby sequences are defined by:

- $2^k - 1$, if $i = (2::'a)^k - (1::'a)$

- *luby-sequence-core* $(i - 2^k - 1 + 1)$, if $(2::'a)^k - 1 \leq i$ and $i \leq (2::'a)^k - (1::'a)$

Then the sequence is then scaled by a constant unit run (called *ur* here), strictly positive.

function *luby-sequence-core* :: *nat* \Rightarrow *nat* **where**

luby-sequence-core *i* =

(if $\exists k. i = 2^k - 1$

then $2^((\text{SOME } k. i = 2^k - 1) - 1)$

else *luby-sequence-core* $(i - 2^((\text{SOME } k. 2^{(k-1)} \leq i \wedge i < 2^k - 1) - 1) + 1))$)

by *auto*

termination

proof (*relation less-than*, *goal-cases*)

case 1

then show ?*case* **by** *auto*

next

case (2 *i*)

let ?*k* = (*SOME* *k*. $2^k - 1 \leq i \wedge i < 2^{k+1} - 1$)

have $2^{(?k - 1)} \leq i \wedge i < 2^{?k} - 1$

apply (*rule someI-ex*)

using 2 *exists-luby-decomp* **by** *blast*

then show ?*case*

proof –

have $\forall n \text{ na. } \neg (1::\text{nat}) \leq n \vee 1 \leq n \wedge \text{na}$

by (*meson one-le-power*)

then have *f1*: $(1::\text{nat}) \leq 2^{(?k - 1)}$

using *one-le-numeral* **by** *blast*

have *f2*: $i - 2^{(?k - 1)} + 2^{(?k - 1)} = i$

using $\langle 2^{(?k - 1)} \leq i \wedge i < 2^{?k} - 1 \rangle$ *le-add-diff-inverse2* **by** *blast*

have *f3*: $2^{?k} - 1 \neq \text{Suc } 0$

using *f1* $\langle 2^{(?k - 1)} \leq i \wedge i < 2^{?k} - 1 \rangle$ **by** *linarith*

have $2^{?k} - (1::\text{nat}) \neq 0$

using $\langle 2^{(?k - 1)} \leq i \wedge i < 2^{?k} - 1 \rangle$ *gr-implies-not0* **by** *blast*

then have *f4*: $2^{?k} \neq (1::\text{nat})$

by *linarith*

have *f5*: $\forall n \text{ na. if } \text{na} = 0 \text{ then } (n::\text{nat}) \wedge \text{na} = 1 \text{ else } n \wedge \text{na} = n * n \wedge (\text{na} - 1)$

by (*simp add: power-eq-if*)

then have ?*k* $\neq 0$

using *f4* **by** *meson*

then have $2^{(?k - 1)} \neq \text{Suc } 0$

using *f5 f3* **by** *presburger*

then have $\text{Suc } 0 < 2^{(?k - 1)}$

using *f1* **by** *linarith*

then show ?*thesis*

using *f2 less-than-iff* **by** *presburger*

qed

qed

function *natlog2* :: *nat* \Rightarrow *nat* **where**

natlog2 *n* = (if *n* = 0 then 0 else 1 + *natlog2* (*n* div 2))

using *not0-implies-Suc* **by** *auto*

termination **by** (*relation measure* ($\lambda n. n$)) *auto*

declare *natlog2.simps*[*simp del*]

declare *luby-sequence-core.simps*[*simp del*]

lemma *two-pover-n-eq-two-power-n'-eq*:
 assumes $H: (2::nat) \wedge (k::nat) - 1 = 2 \wedge k' - 1$
 shows $k' = k$
proof –
 have $(2::nat) \wedge (k::nat) = 2 \wedge k'$
 using H by (metis *One-nat-def Suc-pred zero-less-numeral zero-less-power*)
 then show *?thesis* by simp
qed

lemma *luby-sequence-core-two-power-minus-one*:
 luby-sequence-core $(2^k - 1) = 2^{(k-1)}$ (is $?L = ?K$)
proof –
 have *decomp*: $\exists ka. 2^k - 1 = 2^{ka} - 1$
 by auto
 have $?L = 2^{((SOME\ k'. (2::nat) \wedge k - 1 = 2^{k'} - 1) - 1)}$
 apply (subst luby-sequence-core.simps, subst *decomp*)
 by simp
 moreover have $(SOME\ k'. (2::nat) \wedge k - 1 = 2^{k'} - 1) = k$
 apply (rule some-equality)
 apply simp
 using *two-pover-n-eq-two-power-n'-eq* by blast
 ultimately show *?thesis* by presburger
qed

lemma *different-luby-decomposition-false*:
 assumes
 $H: 2 \wedge (k - Suc\ 0) \leq i$ and
 $k': i < 2 \wedge k' - Suc\ 0$ and
 $k-k': k > k'$
 shows *False*
proof –
 have $2 \wedge k' - Suc\ 0 < 2 \wedge (k - Suc\ 0)$
 using $k-k'$ less-eq-Suc-le by auto
 then show *?thesis*
 using $H\ k'$ by linarith
qed

lemma *luby-sequence-core-not-two-power-minus-one*:
 assumes
 $k-i: 2 \wedge (k - 1) \leq i$ and
 $i-k: i < 2^k - 1$
 shows luby-sequence-core $i = luby-sequence-core\ (i - 2 \wedge (k - 1) + 1)$
proof –
 have $H: \neg (\exists ka. i = 2^{ka} - 1)$
 proof (rule ccontr)
 assume $\neg ?thesis$
 then obtain $k': nat$ where $k': i = 2^{k'} - 1$ by blast
 have $(2::nat) \wedge k' - 1 < 2^k - 1$
 using $i-k$ unfolding k' .
 then have $(2::nat) \wedge k' < 2^k$
 by linarith
 then have $k' < k$
 by simp
 have $2 \wedge (k - 1) \leq 2 \wedge k' - (1::nat)$

```

    using k-i unfolding k' .
  then have  $(2::nat) \wedge (k-1) < 2 \wedge k'$ 
    by (metis Suc-diff-1 not-le not-less-eq zero-less-numeral zero-less-power)
  then have  $k-1 < k'$ 
    by simp

  show False using  $\langle k' < k \rangle \langle k-1 < k' \rangle$  by linarith
qed
have  $\bigwedge k k'. 2 \wedge (k - \text{Suc } 0) \leq i \implies i < 2 \wedge k - \text{Suc } 0 \implies 2 \wedge (k' - \text{Suc } 0) \leq i \implies$ 
 $i < 2 \wedge k' - \text{Suc } 0 \implies k = k'$ 
  by (meson different-luby-decomposition-false linorder-neqE-nat)
then have k:  $(\text{SOME } k. 2 \wedge (k - \text{Suc } 0) \leq i \wedge i < 2 \wedge k - \text{Suc } 0) = k$ 
  using k-i i-k by auto
show ?thesis
  apply (subst luby-sequence-core.simps[of i], subst H)
  by (simp add: k)
qed

```

```

lemma unbounded-luby-sequence-core: unbounded luby-sequence-core
  unfolding bounded-def
proof
  assume  $\exists b. \forall n. \text{luby-sequence-core } n \leq b$ 
  then obtain b where b:  $\bigwedge n. \text{luby-sequence-core } n \leq b$ 
    by metis
  have luby-sequence-core  $(2^{b+1} - 1) = 2^b$ 
    using luby-sequence-core-two-power-minus-one[of b+1] by simp
  moreover have  $(2::nat) \wedge b > b$ 
    by (induction b) auto
  ultimately show False using b[of  $2^{b+1} - 1$ ] by linarith
qed

```

```

abbreviation luby-sequence :: nat  $\Rightarrow$  nat where
luby-sequence n  $\equiv$  ur * luby-sequence-core n

```

```

lemma bounded-luby-sequence: unbounded luby-sequence
  using bounded-const-product[of ur] luby-sequence-axioms
  luby-sequence-def unbounded-luby-sequence-core by blast

```

```

lemma luby-sequence-core-0: luby-sequence-core 0 = 1
proof -
  have 0:  $(0::nat) = 2^0 - 1$ 
    by auto
  show ?thesis
    by (subst 0, subst luby-sequence-core-two-power-minus-one) simp
qed

```

```

lemma luby-sequence-core  $n \geq 1$ 
proof (induction n rule: nat-less-induct-case)
  case 0
  then show ?case by (simp add: luby-sequence-core-0)
next
  case (Suc n) note IH = this

```

```

consider
  (interv) k where  $2 \wedge (k - 1) \leq \text{Suc } n$  and  $\text{Suc } n < 2 \wedge k - 1$ 

```

```

| (pow2) k where Suc n = 2 ^ k - Suc 0
using exists-luby-decomp[of Suc n] by auto

then show ?case
proof cases
  case pow2
  show ?thesis
    using luby-sequence-core-two-power-minus-one pow2 by auto
  next
  case interv
  have n: Suc n - 2 ^ (k - 1) + 1 < Suc n
  by (metis Suc-1 Suc-eq-plus1 add.commute add-diff-cancel-left' add-less-mono1 gr0I
    interv(1) interv(2) le-add-diff-inverse2 less-Suc-eq not-le power-0 power-one-right
    power-strict-increasing-iff)
  show ?thesis
    apply (subst luby-sequence-core-not-two-power-minus-one[OF interv])
    using IH n by auto
qed
qed
end

locale luby-sequence-restart =
  luby-sequence ur +
  conflict-driven-clause-learningW — functions for clauses:
  mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss

  — functions for the conflicting clause:
  mset-ccls union-ccls insert-ccls remove-clit

  — conversion
  ccls-of-cls cls-of-ccls

  — functions for the state:
  — access functions:
  trail hd-raw-trail raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting
  — changing state:
  cons-trail tl-trail add-init-cls add-learned-cls remove-cls update-backtrack-lvl
  update-conflicting

  — get state:
  init-state
  restart-state
for
  ur :: nat and
  mset-cls:: 'cls ⇒ 'v clause and
  insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
  remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and

  mset-clss:: 'clss ⇒ 'v clauses and
  union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
  in-clss :: 'cls ⇒ 'clss ⇒ bool and
  insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
  remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and

```

```

mset-ccls :: 'ccls ⇒ 'v clause and
union-ccls :: 'ccls ⇒ 'ccls ⇒ 'ccls and
insert-ccls :: 'v literal ⇒ 'ccls ⇒ 'ccls and
remove-clit :: 'v literal ⇒ 'ccls ⇒ 'ccls and

ccls-of-cl :: 'cls ⇒ 'ccls and
cls-of-ccls :: 'ccls ⇒ 'cls and

trail :: 'st ⇒ ('v, nat, 'v clause) marked-lits and
hd-raw-trail :: 'st ⇒ ('v, nat, 'cls) marked-lit and
raw-init-clss :: 'st ⇒ 'clss and
raw-learned-clss :: 'st ⇒ 'clss and
backtrack-lvl :: 'st ⇒ nat and
raw-conflicting :: 'st ⇒ 'ccls option and

cons-trail :: ('v, nat, 'cls) marked-lit ⇒ 'st ⇒ 'st and
tl-trail :: 'st ⇒ 'st and
add-init-cl :: 'cls ⇒ 'st ⇒ 'st and
add-learned-cl :: 'cls ⇒ 'st ⇒ 'st and
remove-cl :: 'cls ⇒ 'st ⇒ 'st and
update-backtrack-lvl :: nat ⇒ 'st ⇒ 'st and
update-conflicting :: 'ccls option ⇒ 'st ⇒ 'st and

init-state :: 'clss ⇒ 'st and
restart-state :: 'st ⇒ 'st
begin

sublocale cdclW-restart - - - - - luby-sequence
  apply unfold-locales
  using bounded-luby-sequence by blast

end
end
theory CDCL-WNOT
imports CDCL-NOT CDCL-W-Termination CDCL-W-Merge
begin

```

22 Link between Weidenbach's and NOT's CDCL

22.1 Inclusion of the states

```

declare upt.simps(2)[simp del]

fun convert-marked-lit-from-W where
  convert-marked-lit-from-W (Propagated L -) = Propagated L () |
  convert-marked-lit-from-W (Marked L -) = Marked L ()

abbreviation convert-trail-from-W ::
  ('v, 'lvl, 'a) marked-lit list
  ⇒ ('v, unit, unit) marked-lit list where
  convert-trail-from-W ≡ map convert-marked-lit-from-W

lemma lits-of-l-convert-trail-from-W[simp]:
  lits-of-l (convert-trail-from-W M) = lits-of-l M
  by (induction rule: marked-lit-list-induct) simp-all

```

lemma *lit-of-convert-trail-from-W*[simp]:
lit-of (convert-marked-lit-from-W L) = *lit-of* L
by (cases L) *auto*

lemma *no-dup-convert-from-W*[simp]:
no-dup (convert-trail-from-W M) \longleftrightarrow *no-dup* M
by (auto simp: comp-def)

lemma *convert-trail-from-W-true-annots*[simp]:
convert-trail-from-W M \models_{as} C \longleftrightarrow M \models_{as} C
by (auto simp: true-annots-true-cls image-image lits-of-def)

lemma *defined-lit-convert-trail-from-W*[simp]:
defined-lit (convert-trail-from-W S) L \longleftrightarrow *defined-lit* S L
by (auto simp: defined-lit-map image-comp)

The values 0 and {#} are dummy values.

consts *dummy-cls* :: 'cls
fun *convert-marked-lit-from-NOT*
:: ('a, 'e, 'b) marked-lit \Rightarrow ('a, nat, 'cls) marked-lit **where**
convert-marked-lit-from-NOT (Propagated L -) = Propagated L *dummy-cls* |
convert-marked-lit-from-NOT (Marked L -) = Marked L 0

abbreviation *convert-trail-from-NOT* **where**
convert-trail-from-NOT \equiv map *convert-marked-lit-from-NOT*

lemma *undefined-lit-convert-trail-from-NOT*[simp]:
undefined-lit (convert-trail-from-NOT F) L \longleftrightarrow *undefined-lit* F L
by (induction F rule: marked-lit-list-induct) (auto simp: defined-lit-map)

lemma *lits-of-l-convert-trail-from-NOT*:
lits-of-l (convert-trail-from-NOT F) = *lits-of-l* F
by (induction F rule: marked-lit-list-induct) *auto*

lemma *convert-trail-from-W-from-NOT*[simp]:
convert-trail-from-W (convert-trail-from-NOT M) = M
by (induction rule: marked-lit-list-induct) *auto*

lemma *convert-trail-from-W-convert-lit-from-NOT*[simp]:
convert-marked-lit-from-W (convert-marked-lit-from-NOT L) = L
by (cases L) *auto*

abbreviation *trail*_{NOT} **where**
*trail*_{NOT} S \equiv convert-trail-from-W (fst S)

lemma *undefined-lit-convert-trail-from-W*[iff]:
undefined-lit (convert-trail-from-W M) L \longleftrightarrow *undefined-lit* M L
by (auto simp: defined-lit-map image-comp)

lemma *lit-of-convert-marked-lit-from-NOT*[iff]:
lit-of (convert-marked-lit-from-NOT L) = *lit-of* L
by (cases L) *auto*

sublocale *state*_W \subseteq *dpll-state-ops*


```

mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
λS. convert-trail-from-W (trail S)
raw-clauses
λL S. cons-trail (convert-marked-lit-from-NOT L) S
λS. tl-trail S
λC S. add-learned-cls C S
λC S. remove-cls C S
by unfold-locales

```

context $state_W$

begin

lemma *convert-marked-lit-from-W-convert-marked-lit-from-NOT*[simp]:

convert-marked-lit-from-W (mmset-of-mlit (convert-marked-lit-from-NOT L)) = L

by (cases L) auto

end

sublocale $state_W \subseteq dpll\text{-}state$

```

mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
λS. convert-trail-from-W (trail S)
raw-clauses
λL S. cons-trail (convert-marked-lit-from-NOT L) S
λS. tl-trail S
λC S. add-learned-cls C S
λC S. remove-cls C S
by unfold-locales (auto simp: map-tl o-def)

```

context $state_W$

begin

declare $state\text{-}simp_{NOT}[simp\ del]$

end

sublocale *conflict-driven-clause-learning*_W $\subseteq cdcl_{NOT}\text{-merge-bj-learn-ops}$

```

mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
λS. convert-trail-from-W (trail S)
raw-clauses
λL S. cons-trail (convert-marked-lit-from-NOT L) S
λS. tl-trail S
λC S. add-learned-cls C S
λC S. remove-cls C S
λ- -. True
λ- S. raw-conflicting S = None
λC C' L' S T. backjump-l-cond C C' L' S T
  ∧ distinct-mset (C' + {#L'#}) ∧ ¬tautology (C' + {#L'#})
by unfold-locales

```

thm $cdcl_{NOT}\text{-merge-bj-learn-proxy.}axioms$

sublocale *conflict-driven-clause-learning*_W $\subseteq cdcl_{NOT}\text{-merge-bj-learn-proxy}$

```

mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss

```

```

λS. convert-trail-from-W (trail S)
raw-clauses
λL S. cons-trail (convert-marked-lit-from-NOT L) S
λS. tl-trail S
λC S. add-learned-cls C S
λC S. remove-cls C S

λ- -. True
λ- S. raw-conflicting S = None
backjump-l-cond
invNOT
proof (unfold-locales, goal-cases)
  case 2
  then show ?case using cdclNOT-merged-bj-learn-no-dup-inv by (auto simp: comp-def)
next
  case (1 C' S C F' K F L)
  moreover
    let ?C' = remdups-mset C'
    have L ∉ # C'
      using ⟨F ⊨as CNot C'⟩ ⟨undefined-lit F L⟩ Marked-Propagated-in-iff-in-lits-of-l
      in-CNot-implies-uminus(2) by fast
    then have distinct-mset (?C' + {#L#})
      by (simp add: distinct-mset-single-add)
  moreover
    have no-dup F
      using ⟨invNOT S⟩ ⟨convert-trail-from-W (trail S) = F' @ Marked K () # F⟩
      unfolding invNOT-def
      by (smt comp-apply distinct.simps(2) distinct-append list.simps(9) map-append
        no-dup-convert-from-W)
    then have consistent-interp (lits-of-l F)
      using distinct-consistent-interp by blast
    then have ¬ tautology (C')
      using ⟨F ⊨as CNot C'⟩ consistent-CNot-not-tautology true-annots-true-cls by blast
    then have ¬ tautology (?C' + {#L#})
      using ⟨F ⊨as CNot C'⟩ ⟨undefined-lit F L⟩ by (metis CNot-remdups-mset
        Marked-Propagated-in-iff-in-lits-of-l add.commute in-CNot-uminus tautology-add-single
        tautology-remdups-mset true-annot-singleton true-annots-def)
  show ?case
  proof -
    have f2: no-dup (convert-trail-from-W (trail S))
      using ⟨invNOT S⟩ unfolding invNOT-def by (simp add: o-def)
    have f3: atm-of L ∈ atms-of-mm (clauses S)
      ∪ atm-of ' lits-of-l (convert-trail-from-W (trail S))
      using ⟨convert-trail-from-W (trail S) = F' @ Marked K () # F⟩
      ⟨atm-of L ∈ atms-of-mm (clauses S) ∪ atm-of ' lits-of-l (F' @ Marked K () # F)⟩ by auto
    have f4: clauses S ⊨pm remdups-mset C' + {#L#}
      by (metis (no-types) ⟨L ∉ # C'⟩ ⟨clauses S ⊨pm C' + {#L#}⟩ remdups-mset-singleton-sum(2)
        true-clss-cls-remdups-mset union-commute)
    have F ⊨as CNot (remdups-mset C')
      by (simp add: ⟨F ⊨as CNot C'⟩)
    obtain D where D: mset-cls D = remdups-mset C' + {#L#}
      using ex-mset-cls by blast
    have Ex (backjump-l S)
      apply standard
      apply (rule backjump-l.intros[OF - f2, of - - -])

```

```

    using f4 f3 f2 <¬ tautology (remdups-mset C' + {#L#})>
    calculation(2-5,9) <F ⊨as CNot (remdups-mset C')>
    state-eqNOT-ref D unfolding backjump-l-cond-def by blast+
then show ?thesis
    by blast
qed
qed

sublocale conflict-driven-clause-learningW ⊆ cdclNOT-merge-bj-learn-proxy2 - - - - -
  λS. convert-trail-from-W (trail S)
  raw-clauses
  λL S. cons-trail (convert-marked-lit-from-NOT L) S
  λS. tl-trail S
  λC S. add-learned-cls C S
  λC S. remove-cls C S
  λ-. True
  λ- S. raw-conflicting S = None backjump-l-cond invNOT
by unfold-locales

sublocale conflict-driven-clause-learningW ⊆ cdclNOT-merge-bj-learn - - - - -
  λS. convert-trail-from-W (trail S)
  raw-clauses
  λL S. cons-trail (convert-marked-lit-from-NOT L) S
  λS. tl-trail S
  λC S. add-learned-cls C S
  λC S. remove-cls C S
  backjump-l-cond
  λ- -. True
  λ- S. raw-conflicting S = None invNOT
apply unfold-locales
  using dpll-bj-no-dup apply (simp add: comp-def)
  using cdclNOT.simps cdclNOT-no-dup no-dup-convert-from-W unfolding invNOT-def by blast

context conflict-driven-clause-learningW
begin

Notations are lost while proving locale inclusion:
notation state-eqNOT (infix ~NOT 50)

```

22.2 Additional Lemmas between NOT and W states

```

lemma trailW-eq-reduce-trail-toNOT-eq:
  trail S = trail T ⇒ trail (reduce-trail-toNOT F S) = trail (reduce-trail-toNOT F T)
proof (induction F S arbitrary: T rule: reduce-trail-toNOT.induct)
  case (1 F S T) note IH = this(1) and tr = this(2)
  then have [] = convert-trail-from-W (trail S)
    ∨ length F = length (convert-trail-from-W (trail S))
    ∨ trail (reduce-trail-toNOT F (tl-trail S)) = trail (reduce-trail-toNOT F (tl-trail T))
  using IH by (metis (no-types) trail-tl-trail)
  then show trail (reduce-trail-toNOT F S) = trail (reduce-trail-toNOT F T)
    using tr by (metis (no-types) reduce-trail-toNOT.elims)
qed

```

```

lemma trail-reduce-trail-toNOT-add-learned-cls:
  no-dup (trail S) ⇒
  trail (reduce-trail-toNOT M (add-learned-cls D S)) = trail (reduce-trail-toNOT M S)

```

by (rule trail_W-eq-reduce-trail-to_{NOT}-eq) simp

lemma reduce-trail-to_{NOT}-reduce-trail-convert:

reduce-trail-to_{NOT} C S = reduce-trail-to (convert-trail-from-NOT C) S

apply (induction C S rule: reduce-trail-to_{NOT}.induct)

apply (subst reduce-trail-to_{NOT}.simps, subst reduce-trail-to.simps)

by auto

lemma reduce-trail-to-map[simp]:

reduce-trail-to (map f M) S = reduce-trail-to M S

by (rule reduce-trail-to-length) simp

lemma reduce-trail-to_{NOT}-map[simp]:

reduce-trail-to_{NOT} (map f M) S = reduce-trail-to_{NOT} M S

by (rule reduce-trail-to_{NOT}-length) simp

lemma skip-or-resolve-state-change:

assumes skip-or-resolve** S T

shows

∃ M. trail S = M @ trail T ∧ (∀ m ∈ set M. ¬is-marked m)

clauses S = clauses T

backtrack-lvl S = backtrack-lvl T

using assms

proof (induction rule: rtranclp-induct)

case base

case 1 show ?case **by** simp

case 2 show ?case **by** simp

case 3 show ?case **by** simp

next

case (step T U) **note** st = this(1) **and** s-o-r = this(2) **and** IH = this(3) **and** IH' = this(3-5)

case 2 show ?case **using** IH' s-o-r **by** (auto elim!: rulesE simp: skip-or-resolve.simps)

case 3 show ?case **using** IH' s-o-r **by** (auto elim!: rulesE simp: skip-or-resolve.simps)

case 1 show ?case

using s-o-r

proof cases

case s-or-r-skip

then show ?thesis **using** IH **by** (auto elim!: rulesE simp: skip-or-resolve.simps)

next

case s-or-r-resolve

then show ?thesis

using IH **by** (cases trail T) (auto elim!: rulesE simp: skip-or-resolve.simps dest!: hd-raw-trail)

qed

qed

22.3 More lemmas conflict-propagate and backjumping

22.4 CDCL FW

lemma cdcl_W-merge-is-cdcl_{NOT}-merged-bj-learn:

assumes

inv: cdcl_W-all-struct-inv S **and**

cdcl_W:cdcl_W-merge S T

shows cdcl_{NOT}-merged-bj-learn S T

∨ (no-step cdcl_W-merge T ∧ conflicting T ≠ None)

```

using cdclW inv
proof induction
case (fw-propagate S T) note propa = this(1)
then obtain M N U k L C where
  H: state S = (M, N, U, k, None) and
  CL: C + {#L#} ∈ # clauses S and
  M-C: M ⊨as CNot C and
  undef: undefined-lit (trail S) L and
  T: state T = (Propagated L (C + {#L#})) # M, N, U, k, None)
  by (auto elim: propagate-high-levelE)
have propagateNOT S T
  using H CL T undef M-C by (auto simp: state-eqNOT-def state-eq-def raw-clauses-def
    simp del: state-simp)
then show ?case
  using cdclNOT-merged-bj-learn.intros(2) by blast
next
case (fw-decide S T) note dec = this(1) and inv = this(2)
then obtain L where
  undef-L: undefined-lit (trail S) L and
  atm-L: atm-of L ∈ atms-of-mm (init-clss S) and
  T: T ∼ cons-trail (Marked L (Suc (backtrack-lvl S)))
    (update-backtrack-lvl (Suc (backtrack-lvl S)) S)
  by (auto elim: decideE)
have decideNOT S T
  apply (rule decideNOT.decideNOT)
  using undef-L apply simp
  using atm-L inv unfolding cdclW-all-struct-inv-def no-strange-atm-def raw-clauses-def
  apply auto[]
  using T undef-L unfolding state-eq-def state-eqNOT-def by (auto simp: raw-clauses-def)
then show ?case using cdclNOT-merged-bj-learn-decideNOT by blast
next
case (fw-forget S T) note rf = this(1) and inv = this(2)
then obtain C where
  S: conflicting S = None and
  C-le: C !∈! raw-learned-clss S and
  ¬(trail S) ⊨asm clauses S and
  mset-cls C ∉ set (get-all-mark-of-propagated (trail S)) and
  C-init: mset-cls C ∉ # init-clss S and
  T: T ∼ remove-cls C S
  by (auto elim: forgetE)
have init-clss S ⊨pm mset-cls C
  using inv C-le unfolding cdclW-all-struct-inv-def cdclW-learned-clause-def raw-clauses-def
  by (meson in-clss-mset-clss true-clss-clss-in-imp-true-clss-clss)
then have S-C: removeAll-mset (mset-cls C) (clauses S) ⊨pm mset-cls C
  using C-init C-le unfolding raw-clauses-def by (auto simp add: Un-Diff ac-simps)
have forgetNOT S T
  apply (rule forgetNOT.forgetNOT)
  using S-C apply blast
  using S apply simp
  using C-init C-le apply (simp add: raw-clauses-def)
  using T C-le C-init by (auto
    simp: state-eq-def Un-Diff state-eqNOT-def raw-clauses-def ac-simps
    simp del: state-simp)
then show ?case using cdclNOT-merged-bj-learn-forgetNOT by blast
next

```

```

case (fw-conflict S T U) note confl = this(1) and bj = this(2) and inv = this(3)
obtain CS CT where
  confl-T: raw-conflicting T = Some CT and
  CT: mset-ccls CT = mset-cls CS and
  CS: CS !∈! raw-clauses S and
  tr-S-CS: trail S ⊨as CNot (mset-cls CS)
  using confl by (elim conflictE) (auto simp del: state-simp simp: state-eq-def)
have cdclW-all-struct-inv T
  using cdclW.sims cdclW-all-struct-inv-inv confl inv by blast
then have cdclW-M-level-inv T
  unfolding cdclW-all-struct-inv-def by auto
then consider
  (no-bt) skip-or-resolve** T U
  | (bt) T' where skip-or-resolve** T T' and backtrack T' U
  using bj rtranclp-cdclW-bj-skip-or-resolve-backtrack unfolding full-def by meson
then show ?case
proof cases
  case no-bt
  then have conflicting U ≠ None
    using confl by (induction rule: rtranclp-induct)
    (auto simp del: state-simp simp: skip-or-resolve.sims state-eq-def elim!: rulesE)
  moreover then have no-step cdclW-merge U
    by (auto simp: cdclW-merge.sims elim: rulesE)
  ultimately show ?thesis by blast
next
  case bt note s-or-r = this(1) and bt = this(2)
  have cdclW** T T'
    using s-or-r mono-rtranclp[of skip-or-resolve cdclW] rtranclp-skip-or-resolve-rtranclp-cdclW
    by blast
  then have cdclW-M-level-inv T'
    using rtranclp-cdclW-consistent-inv (cdclW-M-level-inv T) by blast
  then obtain M1 M2 i D L K where
    confl-T': raw-conflicting T' = Some D and
    LD: L ∈# mset-ccls D and
    M1-M2: (Marked K (i+1) # M1, M2) ∈ set (get-all-marked-decomposition (trail T')) and
    get-level (trail T') L = backtrack-lvl T' and
    get-level (trail T') L = get-maximum-level (trail T') (mset-ccls D) and
    get-maximum-level (trail T') (mset-ccls (remove-clit L D)) = i and
    undef-L: undefined-lit M1 L and
    U: U ∼ cons-trail (Propagated L (cls-of-ccls D))
      (reduce-trail-to M1
        (add-learned-cls (cls-of-ccls D)
          (update-backtrack-lvl i
            (update-conflicting None T'))))))
    using bt by (auto elim: backtrack-levE)
  have [simp]: clauses S = clauses T
    using confl by (auto elim: rulesE)
  have [simp]: clauses T = clauses T'
    using s-or-r
  proof (induction)
    case base
    then show ?case by simp
  next
    case (step U V) note st = this(1) and s-o-r = this(2) and IH = this(3)
    have clauses U = clauses V

```

```

    using s-or-r by (auto simp: skip-or-resolve.simps elim: rulesE)
    then show ?case using IH by auto
qed
have inv-T: cdclW-all-struct-inv T
  by (meson cdclW-cp.simps confl inv r-into-rtrancplp rtrancplp-cdclW-all-struct-inv-inv
      rtrancplp-cdclW-cp-rtrancplp-cdclW)
have cdclW** T T'
  using rtrancplp-skip-or-resolve-rtrancplp-cdclW s-or-r by blast
have inv-T': cdclW-all-struct-inv T'
  using ⟨cdclW** T T'⟩ inv-T rtrancplp-cdclW-all-struct-inv-inv by blast
have inv-U: cdclW-all-struct-inv U
  using cdclW-merge-restart-cdclW confl fw-r-conflict inv local.bj
  rtrancplp-cdclW-all-struct-inv-inv by blast

have [simp]: init-clss S = init-clss T'
  using ⟨cdclW** T T'⟩ cdclW-init-clss confl cdclW-all-struct-inv-def conflict inv
  by (metis ⟨cdclW-M-level-inv T'⟩ rtrancplp-cdclW-init-clss)
then have atm-L: atm-of L ∈ atms-of-mm (clauses S)
  using inv-T' confl-T' LD unfolding cdclW-all-struct-inv-def no-strange-atm-def
  raw-clauses-def
  by (simp add: atms-of-def image-subset-iff)
obtain M where tr-T: trail T = M @ trail T'
  using s-or-r skip-or-resolve-state-change by meson
obtain M' where
  tr-T': trail T' = M' @ Marked K (i+1) # tl (trail U) and
  tr-U: trail U = Propagated L (mset-ccls D) # tl (trail U)
  using U M1-M2 undef-L inv-T' unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def
  by fastforce
def M'' ≡ M @ M'
have tr-T: trail S = M'' @ Marked K (i+1) # tl (trail U)
  using tr-T tr-T' confl unfolding M''-def by (auto elim: rulesE)
have init-clss T' + learned-clss S ⊨pm mset-ccls D
  using inv-T' confl-T' unfolding cdclW-all-struct-inv-def cdclW-learned-clause-def
  raw-clauses-def by simp
have reduce-trail-to (convert-trail-from-NOT (convert-trail-from-W M1)) S =
  reduce-trail-to M1 S
  by (rule reduce-trail-to-length) simp
moreover have trail (reduce-trail-to M1 S) = M1
  apply (rule reduce-trail-to-skip-beginning[of - M @ - @ M2 @ [Marked K (Suc i)]])
  using confl M1-M2 ⟨trail T = M @ trail T'⟩
  apply (auto dest!: get-all-marked-decomposition-exists-prepend
      elim!: conflictE)
  by (rule sym) auto
ultimately have [simp]: trail (reduce-trail-toNOT M1 S) = M1
  using M1-M2 confl by (subst reduce-trail-toNOT-reduce-trail-convert)
  (auto simp: comp-def elim: rulesE)
have every-mark-is-a-conflict U
  using inv-U unfolding cdclW-all-struct-inv-def cdclW-conflicting-def by simp
then have U-D: tl (trail U) ⊨as CNot (remove1-mset L (mset-ccls D))
  by (metis append-self-conv2 tr-U)
thm backjump-l[of - - - - L cls-of-ccls D - remove1-mset L (mset-ccls D)]
have backjump-l S U
  apply (rule backjump-l[of - - - - L cls-of-ccls D - remove1-mset L (mset-ccls D)])
  using tr-T apply simp
  using inv unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def

```

```

    apply (simp add: comp-def)
  using U M1-M2 confl undef-L M1-M2 inv-T' inv undef-L unfolding cdclW-all-struct-inv-def
    cdclW-M-level-inv-def apply (auto simp: state-eqNOT-def
      trail-reduce-trail-toNOT-add-learned-cls)[]
  using CS apply auto[]
  using tr-S-CS apply simp

  using U undef-L M1-M2 inv-T' inv unfolding cdclW-all-struct-inv-def
    cdclW-M-level-inv-def apply auto[]
  using undef-L atm-L apply (simp add: trail-reduce-trail-toNOT-add-learned-cls)
  using (init-cls T' + learned-cls S ⊨pm mset-ccls D) LD unfolding raw-clauses-def
    apply simp
  using LD apply simp
  apply (metis U-D convert-trail-from-W-true-annots)
  using inv-T' inv-U U confl-T' undef-L M1-M2 LD unfolding cdclW-all-struct-inv-def
    distinct-cdclW-state-def by (simp add: cdclW-M-level-inv-decomp backjump-l-cond-def)
  then show ?thesis using cdclNOT-merged-bj-learn-backjump-l by fast
qed
qed

```

abbreviation $cdcl_{NOT}\text{-restart}$ **where**
 $cdcl_{NOT}\text{-restart} \equiv restart\text{-ops}.cdcl_{NOT}\text{-raw-restart } cdcl_{NOT} \text{ restart}$

lemma $cdcl_W\text{-merge-restart-is-cdcl}_{NOT}\text{-merged-bj-learn-restart-no-step}$:

```

assumes
  inv: cdclW-all-struct-inv S and
  cdclW: cdclW-merge-restart S T
shows cdclNOT-restart** S T ∨ (no-step cdclW-merge T ∧ conflicting T ≠ None)
proof –
consider
  (fw) cdclW-merge S T
  | (fw-r) restart S T
using cdclW by (meson cdclW-merge-restart.simps cdclW-rf.cases fw-conflict fw-decide fw-forget
  fw-propagate)
then show ?thesis
proof cases
  case fw
  then have IH: cdclNOT-merged-bj-learn S T ∨ (no-step cdclW-merge T ∧ conflicting T ≠ None)
    using inv cdclW-merge-is-cdclNOT-merged-bj-learn by blast
  have invS: invNOT S
    using inv unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def by auto
  have ff2: cdclNOT++ S T ⟶ cdclNOT** S T
    by (meson tranclp-into-rtranclp)
  have ff3: no-dup (convert-trail-from-W (trail S))
    using invS by (simp add: comp-def)
  have cdclNOT ≤ cdclNOT-restart
    by (auto simp: restart-ops.cdclNOT-raw-restart.simps)
  then show ?thesis
    using ff3 ff2 IH cdclNOT-merged-bj-learn-is-tranclp-cdclNOT
      rtranclp-mono[of cdclNOT cdclNOT-restart] invS predicate2D by blast
next
  case fw-r
  then show ?thesis by (blast intro: restart-ops.cdclNOT-raw-restart.intros)
qed
qed

```


abbreviation $\mu_{FW} :: 'st \Rightarrow nat$ **where**

$\mu_{FW} S \equiv (if\ no_step\ cdcl_W\text{-merge}\ S\ then\ 0\ else\ 1 + \mu_{CDCL}\text{'-merged}\ (set\ mset\ (init\ clss\ S))\ S)$

lemma $cdcl_W\text{-merge-}\mu_{FW}\text{-decreasing}$:

assumes

$inv: cdcl_W\text{-all-struct-inv}\ S$ **and**

$fw: cdcl_W\text{-merge}\ S\ T$

shows $\mu_{FW} T < \mu_{FW} S$

proof –

let $?A = init\ clss\ S$

have $atm\text{-clauses}: atm\text{-of-mm}\ (clauses\ S) \subseteq atm\text{-of-mm}\ ?A$

using inv **unfolding** $cdcl_W\text{-all-struct-inv-def}\ no\text{-strange-atm-def}\ raw\text{-clauses-def}$ **by** $auto$

have $atm\text{-trail}: atm\text{-of}\ 'lits\text{-of-l}\ (trail\ S) \subseteq atm\text{-of-mm}\ ?A$

using inv **unfolding** $cdcl_W\text{-all-struct-inv-def}\ no\text{-strange-atm-def}\ raw\text{-clauses-def}$ **by** $auto$

have $n\text{-d}: no_dup\ (trail\ S)$

using inv **unfolding** $cdcl_W\text{-all-struct-inv-def}$ **by** $(auto\ simp: cdcl_W\text{-M-level-inv-decomp})$

have $[simp]: \neg no_step\ cdcl_W\text{-merge}\ S$

using fw **by** $auto$

have $[simp]: init\ clss\ S = init\ clss\ T$

using $cdcl_W\text{-merge-restart-cdcl}_W[of\ S\ T]\ inv\ rtranclp\text{-cdcl}_W\text{-init-clss}$

unfolding $cdcl_W\text{-all-struct-inv-def}$

by $(meson\ cdcl_W\text{-merge.simps}\ cdcl_W\text{-merge-restart.simps}\ cdcl_W\text{-rf.simps}\ fw)$

consider

$(merged)\ cdcl_{NOT}\text{-merged-bj-learn}\ S\ T$

| $(n\text{-s})\ no_step\ cdcl_W\text{-merge}\ T$

using $cdcl_W\text{-merge-is-cdcl}_{NOT}\text{-merged-bj-learn}\ inv\ fw$ **by** $blast$

then show $?thesis$

proof $cases$

case $merged$

then show $?thesis$

using $cdcl_{NOT}\text{-decreasing-measure}[OF\ -\ atm\text{-clauses},\ of\ T]\ atm\text{-trail}\ n\text{-d}$

by $(auto\ split: if_split\ simp: comp_def\ image\ image\ lits\text{-of-def})$

next

case $n\text{-s}$

then show $?thesis$ **by** $simp$

qed

qed

lemma $wf\text{-cdcl}_W\text{-merge}: wf\ \{(T, S). cdcl_W\text{-all-struct-inv}\ S \wedge cdcl_W\text{-merge}\ S\ T\}$

apply $(rule\ wfP\text{-if-measure}[of\ -\ \mu_{FW}])$

using $cdcl_W\text{-merge-}\mu_{FW}\text{-decreasing}$ **by** $blast$

sublocale $conflict\text{-driven-clause-learning}_W\text{-termination}$

by $unfold\text{-locales}\ (simp\ add: wf\text{-cdcl}_W\text{-merge})$

lemma $full\text{-cdcl}_W\text{-s'full-cdcl}_W\text{-merge-restart}$:

assumes

$conflicting\ R = None$ **and**

$inv: cdcl_W\text{-all-struct-inv}\ R$

shows $full\ cdcl_W\text{-s'}\ R\ V \longleftrightarrow full\ cdcl_W\text{-merge-stgy}\ R\ V$ **(is** $?s' \longleftrightarrow ?fw)$

proof

assume $?s'$

then have $cdcl_W\text{-s'}^{**}\ R\ V$ **unfolding** $full\text{-def}$ **by** $blast$

have $cdcl_W\text{-all-struct-inv}\ V$

```

using ⟨cdclW-s'** R V⟩ inv rtrancp-cdclW-all-struct-inv-inv rtrancp-cdclW-s'-rtrancp-cdclW
by blast
then have n-s: no-step cdclW-merge-stgy V
  using no-step-cdclW-s'-no-step-cdclW-merge-stgy by (meson ⟨full cdclW-s' R V⟩ full-def)
have n-s-bj: no-step cdclW-bj V
  by (metis ⟨cdclW-all-struct-inv V⟩ ⟨full cdclW-s' R V⟩ bj full-def
      n-step-cdclW-stgy-iff-no-step-cdclW-cl-cdclW-o)
have n-s-cp: no-step cdclW-merge-cp V
proof -
  { fix ss :: 'st
    obtain ssa :: 'st ⇒ 'st where
      ff1: ∀ s. ¬ cdclW-all-struct-inv s ∨ cdclW-s'-without-decide s (ssa s)
        ∨ no-step cdclW-merge-cp s
      using conflicting-true-no-step-s'-without-decide-no-step-cdclW-merge-cp by moura
    have (∀ p s sa. ¬ full p (s::'st) sa ∨ p** s sa ∧ no-step p sa) and
      (∀ p s sa. (¬ p** (s::'st) sa ∨ (∃ s. p sa s)) ∨ full p s sa)
      by (meson full-def)+
    then have ¬ cdclW-merge-cp V ss
      using ff1 by (metis (no-types) ⟨cdclW-all-struct-inv V⟩ ⟨full cdclW-s' R V⟩ cdclW-s'.sims
          cdclW-s'-without-decide.cases) }
    then show ?thesis
      by blast
  }
qed
consider
  (fw-no-confl) cdclW-merge-stgy** R V and conflicting V = None
| (fw-confl) cdclW-merge-stgy** R V and conflicting V ≠ None and no-step cdclW-bj V
| (fw-dec-confl) S T U where cdclW-merge-stgy** R S and no-step cdclW-merge-cp S and
  decide S T and cdclW-merge-cp** T U and conflict U V
| (fw-dec-no-confl) S T where cdclW-merge-stgy** R S and no-step cdclW-merge-cp S and
  decide S T and cdclW-merge-cp** T V and conflicting V = None
| (cp-no-confl) cdclW-merge-cp** R V and conflicting V = None
| (cp-confl) U where cdclW-merge-cp** R U and conflict U V
using rtrancp-cdclW-s'-no-step-cdclW-s'-without-decide-decomp-into-cdclW-merge[OF
  ⟨cdclW-s'** R V⟩ assms] by auto
then show ?fw
proof cases
  case fw-no-confl
  then show ?thesis using n-s unfolding full-def by blast
next
  case fw-confl
  then show ?thesis using n-s unfolding full-def by blast
next
  case fw-dec-confl
  have cdclW-merge-cp U V
  using n-s-bj by (metis cdclW-merge-cp.sims full-unfold fw-dec-confl(5))
  then have full1 cdclW-merge-cp T V
  unfolding full1-def by (metis fw-dec-confl(4) n-s-cp trancp-unfold-end)
  then have cdclW-merge-stgy S V using ⟨decide S T⟩ ⟨no-step cdclW-merge-cp S⟩ by auto
  then show ?thesis using n-s ⟨cdclW-merge-stgy** R S⟩ unfolding full-def by auto
next
  case fw-dec-no-confl
  then have full cdclW-merge-cp T V
  using n-s-cp unfolding full-def by blast
  then have cdclW-merge-stgy S V using ⟨decide S T⟩ ⟨no-step cdclW-merge-cp S⟩ by auto
  then show ?thesis using n-s ⟨cdclW-merge-stgy** R S⟩ unfolding full-def by auto

```

```

next
  case cp-no-confl
  then have full cdclW-merge-cp R V
    by (simp add: full-def n-s-cp)
  then have  $R = V \vee \text{cdcl}_W\text{-merge-stgy}^{++} R V$ 
    using fw-s-cp unfolding full-unfold fw-s-cp
    by (metis (no-types) rtranclp-unfold tranclp-unfold-end)
  then show ?thesis
    by (simp add: full-def n-s rtranclp-unfold)
next
  case cp-confl
  have full cdclW-bj V V
    using n-s-bj unfolding full-def by blast
  then have full1 cdclW-merge-cp R V
    unfolding full1-def by (meson cdclW-merge-cp.conflict' cp-confl(1,2) n-s-cp
      rtranclp-into-tranclp1)
  then show ?thesis using n-s unfolding full-def by auto
qed
next
  assume ?fw
  then have cdclW** R V using rtranclp-mono[of cdclW-merge-stgy cdclW**]
    cdclW-merge-stgy-rtranclp-cdclW unfolding full-def by auto
  then have inv': cdclW-all-struct-inv V using inv rtranclp-cdclW-all-struct-inv-inv by blast
  have cdclW-s'** R V
    using (?fw) by (simp add: full-def inv rtranclp-cdclW-merge-stgy-rtranclp-cdclW-s')
  moreover have no-step cdclW-s' V
  proof cases
    assume conflicting V = None
    then show ?thesis
      by (metis inv' (full cdclW-merge-stgy R V) full-def
        no-step-cdclW-merge-stgy-no-step-cdclW-s')
  next
    assume confl-V: conflicting V ≠ None
    then have no-step cdclW-bj V
    using rtranclp-cdclW-merge-stgy-no-step-cdclW-bj by (meson (full cdclW-merge-stgy R V)
      assms(1) full-def)
    then show ?thesis using confl-V by (fastforce simp: cdclW-s'.simps full1-def cdclW-cp.simps
      dest!: tranclpD elim: rulesE)
  qed
  ultimately show ?s' unfolding full-def by blast
qed

```

```

lemma full-cdclW-stgy-full-cdclW-merge:
  assumes
    conflicting R = None and
    inv: cdclW-all-struct-inv R
  shows full cdclW-stgy R V  $\longleftrightarrow$  full cdclW-merge-stgy R V
  by (simp add: assms(1) full-cdclW-s'-full-cdclW-merge-restart full-cdclW-stgy-iff-full-cdclW-s'
    inv)

```

```

lemma full-cdclW-merge-stgy-final-state-conclusive':
  fixes S' :: 'st
  assumes full: full cdclW-merge-stgy (init-state N) S'
  and no-d: distinct-mset-mset (mset-cls N)
  shows (conflicting S' = Some {#}  $\wedge$  unsatisfiable (set-mset (mset-cls N)))

```

```

     $\vee (\text{conflicting } S' = \text{None} \wedge \text{trail } S' \models_{\text{asm}} \text{mset-clss } N \wedge \text{satisfiable } (\text{set-mset } (\text{mset-clss } N)))$ 
proof –
  have cdclW-all-struct-inv (init-state N)
    using no-d unfolding cdclW-all-struct-inv-def by auto
  moreover have conflicting (init-state N) = None
    by auto
  ultimately show ?thesis
    using full full-cdclW-stgy-final-state-conclusive-from-init-state
      full-cdclW-stgy-full-cdclW-merge no-d by presburger
qed
end

end
theory CDCL-W-Incremental
imports CDCL-W-Termination
begin

```

23 Incremental SAT solving

```

context conflict-driven-clause-learningW
begin

```

This invariant holds all the invariant related to the strategy. See the structural invariant in *cdcl_W-all-struct-inv*

```

definition cdclW-stgy-invariant where
cdclW-stgy-invariant S  $\longleftrightarrow$ 
  conflict-is-false-with-level S
   $\wedge$  no-clause-is-false S
   $\wedge$  no-smaller-confl S
   $\wedge$  no-clause-is-false S

```

lemma *cdcl_W-stgy-cdcl_W-stgy-invariant*:

```

assumes
  cdclW: cdclW-stgy S T and
  inv-s: cdclW-stgy-invariant S and
  inv: cdclW-all-struct-inv S
shows
  cdclW-stgy-invariant T
unfolding cdclW-stgy-invariant-def cdclW-all-struct-inv-def apply (intro conjI)
apply (rule cdclW-stgy-ex-lit-of-max-level[of S])
using assms unfolding cdclW-stgy-invariant-def cdclW-all-struct-inv-def apply auto[7]
using cdclW cdclW-stgy-not-non-negated-init-clss apply simp
apply (rule cdclW-stgy-no-smaller-confl-inv)
using assms unfolding cdclW-stgy-invariant-def cdclW-all-struct-inv-def apply auto[4]
using cdclW cdclW-stgy-not-non-negated-init-clss by auto

```

lemma *rtrancpl-cdcl_W-stgy-cdcl_W-stgy-invariant*:

```

assumes
  cdclW: cdclW-stgy** S T and
  inv-s: cdclW-stgy-invariant S and
  inv: cdclW-all-struct-inv S
shows
  cdclW-stgy-invariant T
using assms apply (induction)
apply simp

```

using *cdcl_W-stgy-cdcl_W-stgy-invariant rtrancp-cdcl_W-all-struct-inv-inv*
rtrancp-cdcl_W-stgy-rtrancp-cdcl_W **by** *blast*

abbreviation *decr-bt-lvl* **where**

decr-bt-lvl S \equiv *update-backtrack-lvl (backtrack-lvl S - 1) S*

When we add a new clause, we reduce the trail until we get to the first literal included in *C*. Then we can mark the conflict.

fun *cut-trail-wrt-clause* **where**

cut-trail-wrt-clause C [] S = *S* |
cut-trail-wrt-clause C (Marked L - # M) S =
 (if $-L \in \# C$ then *S*
 else *cut-trail-wrt-clause C M (decr-bt-lvl (tl-trail S))*) |
cut-trail-wrt-clause C (Propagated L - # M) S =
 (if $-L \in \# C$ then *S*
 else *cut-trail-wrt-clause C M (tl-trail S)*)

definition *add-new-clause-and-update* :: '*ccls* \Rightarrow '*st* \Rightarrow '*st* **where**

add-new-clause-and-update C S =
 (if *trail S* \models *as CNot (mset-ccls C)*
 then *update-conflicting (Some C) (add-init-cls (cls-of-ccls C)*
 (*cut-trail-wrt-clause (mset-ccls C) (trail S) S*))
 else *add-init-cls (cls-of-ccls C) S*)

thm *cut-trail-wrt-clause.induct*

lemma *init-clss-cut-trail-wrt-clause[simp]*:

init-clss (cut-trail-wrt-clause C M S) = *init-clss S*
by (*induction rule: cut-trail-wrt-clause.induct*) *auto*

lemma *learned-clss-cut-trail-wrt-clause[simp]*:

learned-clss (cut-trail-wrt-clause C M S) = *learned-clss S*
by (*induction rule: cut-trail-wrt-clause.induct*) *auto*

lemma *conflicting-clss-cut-trail-wrt-clause[simp]*:

conflicting (cut-trail-wrt-clause C M S) = *conflicting S*
by (*induction rule: cut-trail-wrt-clause.induct*) *auto*

lemma *trail-cut-trail-wrt-clause*:

$\exists M. \text{trail } S = M @ \text{trail } (\text{cut-trail-wrt-clause } C (\text{trail } S) S)$

proof (*induction trail S arbitrary: S rule: marked-lit-list-induct*)

case *nil*

then show ?*case* **by** *simp*

next

case (*marked L l M*) **note** *IH* = *this(1)[of decr-bt-lvl (tl-trail S)]* **and** *M* = *this(2)[symmetric]*
then show ?*case* **using** *Cons-eq-appendI* **by** *fastforce+*

next

case (*proped L l M*) **note** *IH* = *this(1)[of tl-trail S]* **and** *M* = *this(2)[symmetric]*
then show ?*case* **using** *Cons-eq-appendI* **by** *fastforce+*

qed

lemma *n-dup-no-dup-trail-cut-trail-wrt-clause[simp]*:

assumes *n-d: no-dup (trail T)*
shows *no-dup (trail (cut-trail-wrt-clause C (trail T) T))*

proof —

obtain *M* **where**

```

    M: trail T = M @ trail (cut-trail-wrt-clause C (trail T) T)
    using trail-cut-trail-wrt-clause[of T C] by auto
  show ?thesis
    using n-d unfolding arg-cong[OF M, of no-dup] by auto
qed

lemma cut-trail-wrt-clause-backtrack-lvl-length-marked:
  assumes
    backtrack-lvl T = length (get-all-levels-of-marked (trail T))
  shows
    backtrack-lvl (cut-trail-wrt-clause C (trail T) T) =
      length (get-all-levels-of-marked (trail (cut-trail-wrt-clause C (trail T) T)))
  using assms
proof (induction trail T arbitrary:T rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
  case (marked L l M) note IH = this(1)[of decr-bt-lvl (tl-trail T)] and M = this(2)[symmetric]
    and bt = this(3)
  then show ?case by auto
next
  case (proped L l M) note IH = this(1)[of tl-trail T] and M = this(2)[symmetric] and bt = this(3)
  then show ?case by auto
qed

lemma cut-trail-wrt-clause-get-all-levels-of-marked:
  assumes get-all-levels-of-marked (trail T) = rev [Suc 0..  

    Suc (length (get-all-levels-of-marked (trail T)))]
  shows
    get-all-levels-of-marked (trail ((cut-trail-wrt-clause C (trail T) T))) = rev [Suc 0..  

    Suc (length (get-all-levels-of-marked (trail ((cut-trail-wrt-clause C (trail T) T)))))]
  using assms
proof (induction trail T arbitrary:T rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
  case (marked L l M) note IH = this(1)[of decr-bt-lvl (tl-trail T)] and M = this(2)[symmetric]
    and bt = this(3)
  then show ?case by (cases count C L = 0) auto
next
  case (proped L l M) note IH = this(1)[of tl-trail T] and M = this(2)[symmetric] and bt = this(3)
  then show ?case by (cases count C L = 0) auto
qed

lemma cut-trail-wrt-clause-CNot-trail:
  assumes trail T  $\models_{as}$  CNot C
  shows
    (trail ((cut-trail-wrt-clause C (trail T) T)))  $\models_{as}$  CNot C
  using assms
proof (induction trail T arbitrary:T rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
  case (marked L l M) note IH = this(1)[of decr-bt-lvl (tl-trail T)] and M = this(2)[symmetric]
    and bt = this(3)

```

```

show ?case
proof (cases count C (-L) = 0)
  case False
  then show ?thesis
    using IH M bt by (auto simp: true-annots-true-cls)
next
case True
obtain mma :: 'v literal multiset where
  f6: (mma ∈ {{#- l#} | l. l ∈# C} → M ⊨a mma) → M ⊨as {{#- l#} | l. l ∈# C}
  using true-annots-def by blast
have mma ∈ {{#- l#} | l. l ∈# C} → trail T ⊨a mma
  using CNot-def M bt by (metis (no-types) true-annots-def)
then have M ⊨as {{#- l#} | l. l ∈# C}
  using f6 True M bt by (force simp: count-eq-zero-iff)
then show ?thesis
  using IH true-annots-true-cls M by (auto simp: CNot-def)
qed
next
case (proped L l M) note IH = this(1)[of tl-trail T] and M = this(2)[symmetric] and bt = this(3)
show ?case
proof (cases count C (-L) = 0)
  case False
  then show ?thesis
    using IH M bt by (auto simp: true-annots-true-cls)
next
case True
obtain mma :: 'v literal multiset where
  f6: (mma ∈ {{#- l#} | l. l ∈# C} → M ⊨a mma) → M ⊨as {{#- l#} | l. l ∈# C}
  using true-annots-def by blast
have mma ∈ {{#- l#} | l. l ∈# C} → trail T ⊨a mma
  using CNot-def M bt by (metis (no-types) true-annots-def)
then have M ⊨as {{#- l#} | l. l ∈# C}
  using f6 True M bt by (force simp: count-eq-zero-iff)
then show ?thesis
  using IH true-annots-true-cls M by (auto simp: CNot-def)
qed
qed

lemma cut-trail-wrt-clause-hd-trail-in-or-empty-trail:
  ((∀ L ∈# C. -L ∉ lits-of-l (trail T)) ∧ trail (cut-trail-wrt-clause C (trail T) T) = [])
  ∨ (-lit-of (hd (trail (cut-trail-wrt-clause C (trail T) T))) ∈# C
    ∧ length (trail (cut-trail-wrt-clause C (trail T) T)) ≥ 1)
  using assms
proof (induction trail T arbitrary: T rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
case (marked L l M) note IH = this(1)[of decr-bt-lvl (tl-trail T)] and M = this(2)[symmetric]
  then show ?case by simp force
next
case (proped L l M) note IH = this(1)[of tl-trail T] and M = this(2)[symmetric]
  then show ?case by simp force
qed

```

We can fully run *cdcl_W*-s or add a clause. Remark that we use *cdcl_W*-s to avoid an explicit *skip*, *resolve*, and *backtrack* normalisation to get rid of the conflict *C* if possible.

inductive *incremental-cdcl_W* :: 'st ⇒ 'st ⇒ bool **for** *S* **where**

add-conflict:

trail S ⊨_{asm} *init-clss S* ⇒ *distinct-mset (mset-ccls C)* ⇒ *conflicting S = None* ⇒
trail S ⊨_{as} *CNot (mset-ccls C)* ⇒
full cdcl_W-stgy
 (*update-conflicting (Some C)*
 (*add-init-cls (cls-of-ccls C) (cut-trail-wrt-clause (mset-ccls C) (trail S) S)*)) *T* ⇒
incremental-cdcl_W S T |

add-no-conflict:

trail S ⊨_{asm} *init-clss S* ⇒ *distinct-mset (mset-ccls C)* ⇒ *conflicting S = None* ⇒
 ¬*trail S* ⊨_{as} *CNot (mset-ccls C)* ⇒
full cdcl_W-stgy (add-init-cls (cls-of-ccls C) S) T ⇒
incremental-cdcl_W S T

lemma *cdcl_W-all-struct-inv-add-new-clause-and-update-cdcl_W-all-struct-inv*:

assumes

inv-T: *cdcl_W-all-struct-inv T* **and**
tr-T-N[simp]: *trail T* ⊨_{asm} *N* **and**
tr-C[simp]: *trail T* ⊨_{as} *CNot (mset-ccls C)* **and**
 [*simp*]: *distinct-mset (mset-ccls C)*

shows *cdcl_W-all-struct-inv (add-new-clause-and-update C T) (is cdcl_W-all-struct-inv ?T')*

proof –

let *?T* = *update-conflicting (Some C)*
 (*add-init-cls (cls-of-ccls C) (cut-trail-wrt-clause (mset-ccls C) (trail T) T)*)

obtain *M* **where**

M: *trail T* = *M @ trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)*
using *trail-cut-trail-wrt-clause[of T mset-ccls C]* **by** *blast*

have *H[dest]*: $\bigwedge x. x \in \text{lits-of-l } (\text{trail } (\text{cut-trail-wrt-clause } (\text{mset-ccls } C) (\text{trail } T) T)) \Rightarrow$
 $x \in \text{lits-of-l } (\text{trail } T)$

using *inv-T arg-cong[OF M, of lits-of-l]* **by** *auto*

have *H'[dest]*: $\bigwedge x. x \in \text{set } (\text{trail } (\text{cut-trail-wrt-clause } (\text{mset-ccls } C) (\text{trail } T) T)) \Rightarrow$
 $x \in \text{set } (\text{trail } T)$

using *inv-T arg-cong[OF M, of set]* **by** *auto*

have *H-proped*: $\bigwedge x. x \in \text{set } (\text{get-all-mark-of-propagated } (\text{trail } (\text{cut-trail-wrt-clause } (\text{mset-ccls } C) (\text{trail } T) T))) \Rightarrow$
 $x \in \text{set } (\text{get-all-mark-of-propagated } (\text{trail } T))$

using *inv-T arg-cong[OF M, of get-all-mark-of-propagated]* **by** *auto*

have [*simp*]: *no-strange-atm ?T*

using *inv-T unfolding cdcl_W-all-struct-inv-def no-strange-atm-def add-new-clause-and-update-def*
cdcl_W-M-level-inv-def **by** (*auto 20 1*)

have *M-lev*: *cdcl_W-M-level-inv T*

using *inv-T unfolding cdcl_W-all-struct-inv-def* **by** *blast*

then have *no-dup (M @ trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))*

unfolding *cdcl_W-M-level-inv-def* **unfolding** *M[symmetric]* **by** *auto*

then have [*simp*]: *no-dup (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))*
by *auto*

have *consistent-interp (lits-of-l (M @ trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))*

using *M-lev unfolding cdcl_W-M-level-inv-def* **unfolding** *M[symmetric]* **by** *auto*

then have [*simp*]: *consistent-interp (lits-of-l (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))*

unfolding *consistent-interp-def* **by** *auto*

have [*simp*]: *cdcl_W-M-level-inv ?T*


```

using M-lev cut-trail-wrt-clause-get-all-levels-of-marked[of T mset-ccls C]
unfolding cdclW-M-level-inv-def by (auto dest: H H')
  simp: M-lev cdclW-M-level-inv-def cut-trail-wrt-clause-backtrack-lvl-length-marked)

have [simp]:  $\bigwedge s. s \in \# \text{ learned-clss } T \implies \neg \text{tautology } s$ 
  using inv-T unfolding cdclW-all-struct-inv-def by auto

have distinct-cdclW-state T
  using inv-T unfolding cdclW-all-struct-inv-def by auto
then have [simp]: distinct-cdclW-state ?T
  unfolding distinct-cdclW-state-def by auto

have cdclW-conflicting T
  using inv-T unfolding cdclW-all-struct-inv-def by auto
have trail ?T  $\models_{\text{as}}$  CNot (mset-ccls C)
  by (simp add: cut-trail-wrt-clause-CNot-trail)
then have [simp]: cdclW-conflicting ?T
  unfolding cdclW-conflicting-def apply simp
  by (metis M  $\langle$ cdclW-conflicting T $\rangle$  append-assoc cdclW-conflicting-decomp(2))

have
  decomp-T: all-decomposition-implies-m (init-clss T) (get-all-marked-decomposition (trail T))
  using inv-T unfolding cdclW-all-struct-inv-def by auto
have all-decomposition-implies-m (init-clss ?T)
  (get-all-marked-decomposition (trail ?T))
  unfolding all-decomposition-implies-def
  proof clarify
    fix a b
    assume  $(a, b) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } ?T))$ 
    from in-get-all-marked-decomposition-in-get-all-marked-decomposition-prepend[OF this, of M]
    obtain b' where
       $(a, b' @ b) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } T))$ 
      using M by auto
    then have unmark-l a  $\cup$  set-mset (init-clss T)  $\models_{\text{ps}}$  unmark-l (b' @ b)
      using decomp-T unfolding all-decomposition-implies-def by fastforce
    then have unmark-l a  $\cup$  set-mset (init-clss ?T)  $\models_{\text{ps}}$  unmark-l (b @ b')
      by (simp add: Un-commute)
    then show unmark-l a  $\cup$  set-mset (init-clss ?T)  $\models_{\text{ps}}$  unmark-l b
      by (auto simp: image-Un)
  qed

have [simp]: cdclW-learned-clause ?T
  using inv-T unfolding cdclW-all-struct-inv-def cdclW-learned-clause-def
  by (auto dest!: H-proped simp: raw-clauses-def)
show ?thesis
  using  $\langle \text{all-decomposition-implies-m } (\text{init-clss } ?T) \text{ (get-all-marked-decomposition (trail ?T))} \rangle$ 
  unfolding cdclW-all-struct-inv-def by (auto simp: add-new-clause-and-update-def)
qed

lemma cdclW-all-struct-inv-add-new-clause-and-update-cdclW-stgy-inv:
assumes
  inv-s: cdclW-stgy-invariant T and
  inv: cdclW-all-struct-inv T and
  tr-T-N[simp]: trail T  $\models_{\text{asm}}$  N and

```

$tr-C[simp]: \text{trail } T \models_{as} CNot \ (mset-ccls \ C) \text{ and}$
 $[simp]: \text{distinct-mset} \ (mset-ccls \ C)$
shows $cdcl_W\text{-stgy-invariant} \ (\text{add-new-clause-and-update} \ C \ T)$
 $(\text{is } cdcl_W\text{-stgy-invariant} \ ?T')$
proof –
have $cdcl_W\text{-all-struct-inv} \ ?T'$
using $cdcl_W\text{-all-struct-inv-add-new-clause-and-update-cdcl}_W\text{-all-struct-inv assms}$ **by** *blast*
then have
 $no\text{-dup-cut-}T[simp]: no\text{-dup} \ (\text{trail} \ (\text{cut-trail-wrt-clause} \ (mset-ccls \ C) \ (\text{trail} \ T) \ T)) \text{ and}$
 $n\text{-d}[simp]: no\text{-dup} \ (\text{trail} \ T)$
using $cdcl_W\text{-M-level-inv-decomp}(2) \ cdcl_W\text{-all-struct-inv-def inv}$
 $n\text{-dup-no-dup-trail-cut-trail-wrt-clause}$ **by** *blast+*
then have $\text{trail} \ (\text{add-new-clause-and-update} \ C \ T) \models_{as} CNot \ (mset-ccls \ C)$
by $(simp \text{ add: add-new-clause-and-update-def cut-trail-wrt-clause-}CNot\text{-trail}$
 $cdcl_W\text{-M-level-inv-def } cdcl_W\text{-all-struct-inv-def})$
obtain MT **where**
 $MT: \text{trail } T = MT \ @ \ \text{trail} \ (\text{cut-trail-wrt-clause} \ (mset-ccls \ C) \ (\text{trail} \ T) \ T)$
using $\text{trail-cut-trail-wrt-clause}$ **by** *blast*
consider
 $(false) \ \forall L \in \#mset-ccls \ C. - \ L \notin \text{lits-of-l} \ (\text{trail} \ T) \text{ and}$
 $\text{trail} \ (\text{cut-trail-wrt-clause} \ (mset-ccls \ C) \ (\text{trail} \ T) \ T) = []$
 $| \ (not\text{-false})$
 $- \text{lit-of} \ (\text{hd} \ (\text{trail} \ (\text{cut-trail-wrt-clause} \ (mset-ccls \ C) \ (\text{trail} \ T) \ T))) \in \#(mset-ccls \ C) \text{ and}$
 $1 \leq \text{length} \ (\text{trail} \ (\text{cut-trail-wrt-clause} \ (mset-ccls \ C) \ (\text{trail} \ T) \ T))$
using $\text{cut-trail-wrt-clause-hd-trail-in-or-empty-trail}[of \ mset-ccls \ C \ T]$ **by** *auto*
then show $?thesis$
proof *cases*
case *false* **note** $C = \text{this}(1)$ **and** $\text{empty-tr} = \text{this}(2)$
then have $[simp]: mset-ccls \ C = \{\#\}$
by $(simp \text{ add: in-}CNot\text{-implies-uminus}(2) \ \text{multiset-eqI})$
show $?thesis$
using $\text{empty-tr unfolding } cdcl_W\text{-stgy-invariant-def no-smaller-conflict-def}$
 $cdcl_W\text{-all-struct-inv-def}$ **by** $(\text{auto } simp: \text{add-new-clause-and-update-def})$
next
case *not-false* **note** $C = \text{this}(1)$ **and** $l = \text{this}(2)$
let $?L = - \text{lit-of} \ (\text{hd} \ (\text{trail} \ (\text{cut-trail-wrt-clause} \ (mset-ccls \ C) \ (\text{trail} \ T) \ T)))$
have $\text{get-all-levels-of-marked} \ (\text{trail} \ (\text{add-new-clause-and-update} \ C \ T)) =$
 $\text{rev} \ [1..<1 + \text{length} \ (\text{get-all-levels-of-marked} \ (\text{trail} \ (\text{add-new-clause-and-update} \ C \ T)))]$
using $\langle cdcl_W\text{-all-struct-inv} \ ?T' \rangle$ **unfolding** $cdcl_W\text{-all-struct-inv-def } cdcl_W\text{-M-level-inv-def}$
by *blast*
moreover
have $\text{backtrack-lvl} \ (\text{cut-trail-wrt-clause} \ (mset-ccls \ C) \ (\text{trail} \ T) \ T) =$
 $\text{length} \ (\text{get-all-levels-of-marked} \ (\text{trail} \ (\text{add-new-clause-and-update} \ C \ T)))$
using $\langle cdcl_W\text{-all-struct-inv} \ ?T' \rangle$ **unfolding** $cdcl_W\text{-all-struct-inv-def } cdcl_W\text{-M-level-inv-def}$
by $(\text{auto } simp: \text{add-new-clause-and-update-def})$
moreover
have $no\text{-dup} \ (\text{trail} \ (\text{cut-trail-wrt-clause} \ (mset-ccls \ C) \ (\text{trail} \ T) \ T))$
using $\langle cdcl_W\text{-all-struct-inv} \ ?T' \rangle$ **unfolding** $cdcl_W\text{-all-struct-inv-def } cdcl_W\text{-M-level-inv-def}$
by $(\text{auto } simp: \text{add-new-clause-and-update-def})$
then have $\text{atm-of} \ ?L \notin \text{atm-of} \ ' \text{lits-of-l}$
 $(\text{tl} \ (\text{trail} \ (\text{cut-trail-wrt-clause} \ (mset-ccls \ C) \ (\text{trail} \ T) \ T)))$
by $(\text{cases } \text{trail} \ (\text{cut-trail-wrt-clause} \ (mset-ccls \ C) \ (\text{trail} \ T) \ T))$
 $(\text{auto } simp: \text{lits-of-def})$
ultimately have $L: \text{get-level} \ (\text{trail} \ (\text{cut-trail-wrt-clause} \ (mset-ccls \ C) \ (\text{trail} \ T) \ T)) \ (-?L)$

```

= length (get-all-levels-of-marked (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
using get-level-get-rev-level-get-all-levels-of-marked[OF
  ‹atm-of ?L ∉ atm-of ‘ lits-of-l (tl (trail (cut-trail-wrt-clause (mset-ccls C)
    (trail T) T)))›,
  of [hd (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))]]

apply (cases trail (add-init-cls (cls-of-ccls C)
  (cut-trail-wrt-clause (mset-ccls C) (trail T) T));
  cases hd (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
using l by (auto split: if-split-asm
  simp: rev-swap[symmetric] add-new-clause-and-update-def)

have L': length (get-all-levels-of-marked (trail (cut-trail-wrt-clause (mset-ccls C)
  (trail T) T)))
= backtrack-lvl (cut-trail-wrt-clause (mset-ccls C) (trail T) T)
using ‹cdclW-all-struct-inv ?T'› unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def
by (auto simp: add-new-clause-and-update-def)

have [simp]: no-smaller-confl (update-conflicting (Some C)
  (add-init-cls (cls-of-ccls C) (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
unfolding no-smaller-confl-def
proof (clarify, goal-cases)
case (1 M K i M' D)
then consider
  (DC) D = mset-ccls C
  | (D-T) D ∈ # clauses T
by (auto simp: raw-clauses-def split: if-split-asm)
then show False
proof cases
case D-T
have no-smaller-confl T
using inv-s unfolding cdclW-stgy-invariant-def by auto
have (MT @ M') @ Marked K i # M = trail T
using MT 1(1) by auto
thus False using D-T ‹no-smaller-confl T› 1(3) unfolding no-smaller-confl-def by blast
next
case DC note -[simp] = this
then have atm-of (−?L) ∈ atm-of ‘ (lits-of-l M)
using 1(3) C in-CNot-implies-uminus(2) by blast
moreover
have lit-of (hd (M' @ Marked K i # [])) = −?L
using l 1(1)[symmetric] inv
by (cases trail (add-init-cls (cls-of-ccls C)
  (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
  (auto dest!: arg-cong[of - # - - hd] simp: hd-append cdclW-all-struct-inv-def
    cdclW-M-level-inv-def)
from arg-cong[OF this, of atm-of]
have atm-of (−?L) ∈ atm-of ‘ (lits-of-l (M' @ Marked K i # []))
by (cases (M' @ Marked K i # [])) auto
moreover have no-dup (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))
using ‹cdclW-all-struct-inv ?T'› unfolding cdclW-all-struct-inv-def
  cdclW-M-level-inv-def by (auto simp: add-new-clause-and-update-def)
ultimately show False
unfolding 1(1)[symmetric, simplified] by (auto simp: lits-of-def)
qed

```

```

    qed
    show ?thesis using L L' C
      unfolding cdclW-stgy-invariant-def
      unfolding cdclW-all-struct-inv-def by (auto simp: add-new-clause-and-update-def)
    qed
  qed

lemma full-cdclW-stgy-inv-normal-form:
  assumes
    full: full cdclW-stgy S T and
    inv-s: cdclW-stgy-invariant S and
    inv: cdclW-all-struct-inv S
  shows conflicting T = Some {#}  $\wedge$  unsatisfiable (set-mset (init-clss S))
     $\vee$  conflicting T = None  $\wedge$  trail T  $\models_{asm}$  init-clss S  $\wedge$  satisfiable (set-mset (init-clss S))
proof -
  have no-step cdclW-stgy T
    using full unfolding full-def by blast
  moreover have cdclW-all-struct-inv T and inv-s: cdclW-stgy-invariant T
    apply (metis rtrancpl-cdclW-stgy-rtrancpl-cdclW full full-def inv
      rtrancpl-cdclW-all-struct-inv-inv)
    by (metis full full-def inv inv-s rtrancpl-cdclW-stgy-cdclW-stgy-invariant)
  ultimately have conflicting T = Some {#}  $\wedge$  unsatisfiable (set-mset (init-clss T))
     $\vee$  conflicting T = None  $\wedge$  trail T  $\models_{asm}$  init-clss T
    using cdclW-stgy-final-state-conclusive[of T] full
    unfolding cdclW-all-struct-inv-def cdclW-stgy-invariant-def full-def by fast
  moreover have consistent-interp (lits-of-l (trail T))
    using  $\langle$ cdclW-all-struct-inv T $\rangle$  unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def
    by auto
  moreover have init-clss S = init-clss T
    using inv unfolding cdclW-all-struct-inv-def
    by (metis rtrancpl-cdclW-stgy-no-more-init-clss full full-def)
  ultimately show ?thesis
    by (metis satisfiable-carac' true-annot-def true-annots-def true-clss-def)
qed

```

```

lemma incremental-cdclW-inv:
  assumes
    inc: incremental-cdclW S T and
    inv: cdclW-all-struct-inv S and
    s-inv: cdclW-stgy-invariant S
  shows
    cdclW-all-struct-inv T and
    cdclW-stgy-invariant T
  using inc
proof (induction)
  case (add-confl C T)
  let ?T = (update-conflicting (Some C) (add-init-cls (cls-of-ccls C)
    (cut-trail-wrt-clause (mset-ccls C) (trail S) S)))
  have cdclW-all-struct-inv ?T and inv-s-T: cdclW-stgy-invariant ?T
    using add-confl.hyps(1,2,4) add-new-clause-and-update-def
    cdclW-all-struct-inv-add-new-clause-and-update-cdclW-all-struct-inv inv apply auto[1]
    using add-confl.hyps(1,2,4) add-new-clause-and-update-def
    cdclW-all-struct-inv-add-new-clause-and-update-cdclW-stgy-inv inv s-inv by auto
  case 1 show ?case
    by (metis add-confl.hyps(1,2,4,5) add-new-clause-and-update-def)

```

```

    cdclW-all-struct-inv-add-new-clause-and-update-cdclW-all-struct-inv
    rtrancpl-cdclW-all-struct-inv-inv rtrancpl-cdclW-stgy-rtrancpl-cdclW full-def inv)

case 2 show ?case
  by (metis inv-s-T add-confl.hyps(1,2,4,5) add-new-clause-and-update-def
      cdclW-all-struct-inv-add-new-clause-and-update-cdclW-all-struct-inv full-def inv
      rtrancpl-cdclW-stgy-cdclW-stgy-invariant)
next
case (add-no-confl C T)
case 1
have cdclW-all-struct-inv (add-init-cls (cls-of-ccls C) S)
  using inv <distinct-mset (mset-ccls C)> unfolding cdclW-all-struct-inv-def no-strange-atm-def
  cdclW-M-level-inv-def distinct-cdclW-state-def cdclW-conflicting-def cdclW-learned-clause-def
  by (auto 9 1 simp: all-decomposition-implies-insert-single raw-clauses-def)

then show ?case
  using add-no-confl(5) unfolding full-def by (auto intro: rtrancpl-cdclW-stgy-cdclW-all-struct-inv)
case 2
have nc:  $\forall M. (\exists K i M'. \text{trail } S = M' @ \text{Marked } K i \# M) \longrightarrow \neg M \models_{\text{as}} \text{CNot } (\text{mset-ccls } C)$ 
  using < $\neg \text{trail } S \models_{\text{as}} \text{CNot } (\text{mset-ccls } C)$ >
  by (auto simp: true-annots-true-cls-def-iff-negation-in-model)

have cdclW-stgy-invariant (add-init-cls (cls-of-ccls C) S)
  using s-inv < $\neg \text{trail } S \models_{\text{as}} \text{CNot } (\text{mset-ccls } C)$ > inv unfolding cdclW-stgy-invariant-def
  no-smaller-confl-def eq-commute[of - trail -] cdclW-M-level-inv-def cdclW-all-struct-inv-def
  by (auto simp: raw-clauses-def nc)
then show ?case
  by (metis <cdclW-all-struct-inv (add-init-cls (cls-of-ccls C) S)> add-no-confl.hyps(5) full-def
      rtrancpl-cdclW-stgy-cdclW-stgy-invariant)
qed

lemma rtrancpl-incremental-cdclW-inv:
  assumes
    inc: incremental-cdclW** S T and
    inv: cdclW-all-struct-inv S and
    s-inv: cdclW-stgy-invariant S
  shows
    cdclW-all-struct-inv T and
    cdclW-stgy-invariant T
  using inc apply induction
  using inv apply simp
  using s-inv apply simp
  using incremental-cdclW-inv by blast+

lemma incremental-conclusive-state:
  assumes
    inc: incremental-cdclW S T and
    inv: cdclW-all-struct-inv S and
    s-inv: cdclW-stgy-invariant S
  shows conflicting T = Some {#}  $\wedge$  unsatisfiable (set-mset (init-cls T))
     $\vee$  conflicting T = None  $\wedge$  trail T  $\models_{\text{asm}} \text{init-cls } T \wedge$  satisfiable (set-mset (init-cls T))
  using inc
proof induction
print-cases
case (add-confl C T) note tr = this(1) and dist = this(2) and conf = this(3) and C = this(4) and

```

```

full = this(5)

have full cdclW-stgy T T
  using full unfolding full-def by auto
then show ?case
  using full C conf dist tr
  by (metis full-cdclW-stgy-inv-normal-form incremental-cdclW.simps incremental-cdclW-inv(1)
      incremental-cdclW-inv(2) inv s-inv)
next
case (add-no-conf C T) note tr = this(1) and dist = this(2) and conf = this(3) and C = this(4)
  and full = this(5)

have full cdclW-stgy T T
  using full unfolding full-def by auto
then show ?case
  by (meson C conf dist full full-cdclW-stgy-inv-normal-form incremental-cdclW.add-no-conf
      incremental-cdclW-inv(1) incremental-cdclW-inv(2) inv s-inv tr)
qed

lemma tranclp-incremental-correct:
  assumes
    inc: incremental-cdclW++ S T and
    inv: cdclW-all-struct-inv S and
    s-inv: cdclW-stgy-invariant S
  shows conflicting T = Some {#} ∧ unsatisfiable (set-mset (init-cls T))
    ∨ conflicting T = None ∧ trail T ⊨asm init-cls T ∧ satisfiable (set-mset (init-cls T))
  using inc apply induction
  using assms incremental-conclusive-state apply blast
  by (meson incremental-conclusive-state inv rtranclp-incremental-cdclW-inv s-inv
      tranclp-into-rtranclp)

end

end

```

24 2-Watched-Literal

```

theory CDCL-Two-Watched-Literals
imports CDCL-WNOT
begin

```

We will directly on the two-watched literals datastructure with lists: it could be also seen as a state over some abstract clause representation we would later refine as lists. However, as we need a way to select element from a clause, working on lists is better.

24.1 Datastructure and Access Functions

Only the 2-watched literals have to be verified here: the backtrack level and the trail that appear in the state are not related to the 2-watched algorithm.

```

datatype 'v twl-clause =
  TWL-Clause (watched: 'v literal list) (unwatched: 'v literal list)

datatype 'v twl-state =
  TWL-State (raw-trail: ('v, nat, 'v twl-clause) marked-lit list)

```

(*raw-init-clss*: 'v twl-clause list)
 (*raw-learned-clss*: 'v twl-clause list) (*backtrack-lvl*: nat)
 (*raw-conflicting*: 'v literal list option)

fun *mmset-of-mlit'* :: ('v, nat, 'v twl-clause) marked-lit \Rightarrow ('v, nat, 'v clause) marked-lit
where
mmset-of-mlit' (*Propagated L C*) = *Propagated L* (*mset* (*watched C* @ *unwatched C*)) |
mmset-of-mlit' (*Marked L i*) = *Marked L i*

lemma *lit-of-mmset-of-mlit'[simp]*: *lit-of* (*mmset-of-mlit' x*) = *lit-of x*
by (*cases x*) *auto*

lemma *lits-of-mmset-of-mlit'[simp]*: *lits-of* (*mmset-of-mlit' S*) = *lits-of S*
by (*auto simp: lits-of-def image-image*)

abbreviation *trail* **where**
trail S \equiv *map mmset-of-mlit' (raw-trail S)*

abbreviation *clauses-of-l* **where**
clauses-of-l \equiv $\lambda L. \text{mset } (\text{map mset } L)$

definition *raw-clause* :: 'v twl-clause \Rightarrow 'v literal list **where**
raw-clause C \equiv *watched C* @ *unwatched C*

abbreviation *raw-clss* :: 'v twl-state \Rightarrow 'v clauses **where**
raw-clss S \equiv *clauses-of-l (map raw-clause (raw-init-clss S @ raw-learned-clss S))*

interpretation *raw-cl*
 $\lambda C. \text{mset } (\text{raw-clause } C)$
 $\lambda L \ C. \text{TWL-Clause } (\text{watched } C) \ (L \# \text{unwatched } C)$
 $\lambda L \ C. \text{TWL-Clause } [] \ (\text{remove1 } L \ (\text{raw-clause } C))$
apply (*unfold-locales*)
by (*auto simp:hd-map comp-def map-tl ac-simps*
mset-map-mset-remove1-cond ex-mset raw-clause-def
simp del:)

lemma *XXX*:
 $\text{mset } (\text{map } (\lambda x. \text{mset } (\text{unwatched } x) + \text{mset } (\text{watched } x))$
 $(\text{remove1-cond } (\lambda D. \text{mset } (\text{raw-clause } D) = \text{mset } (\text{raw-clause } a)) \ Cs)) =$
 $\text{remove1-mset } (\text{mset } (\text{raw-clause } a)) \ (\text{mset } (\text{map } (\lambda x. \text{mset } (\text{raw-clause } x)) \ Cs))$
apply (*induction Cs*)
apply *simp*
by (*auto simp: ac-simps remove1-mset-single-add raw-clause-def*)

interpretation *raw-clss*
 $\lambda C. \text{mset } (\text{raw-clause } C)$
 $\lambda L \ C. \text{TWL-Clause } (\text{watched } C) \ (L \# \text{unwatched } C)$
 $\lambda L \ C. \text{TWL-Clause } [] \ (\text{remove1 } L \ (\text{raw-clause } C))$
 $\lambda C. \text{clauses-of-l } (\text{map raw-clause } C) \ \text{op } @$
 $\lambda L \ C. L \in \text{set } C \ \text{op } \# \lambda C. \text{remove1-cond } (\lambda D. \text{mset } (\text{raw-clause } D) = \text{mset } (\text{raw-clause } C))$
apply (*unfold-locales*)
using *XXX* **by** (*auto simp:hd-map comp-def map-tl ac-simps raw-clause-def*
union-mset-list mset-map-mset-remove1-cond ex-mset
simp del:)

lemma *ex-mset-unwatched-watched*:

$\exists a. \text{mset } (\text{unwatched } a) + \text{mset } (\text{watched } a) = E$

proof –

obtain *e* **where** $\text{mset } e = E$

using *ex-mset* **by** *blast*

then have $\text{mset } (\text{unwatched } (\text{TWL-Clause } [] e)) + \text{mset } (\text{watched } (\text{TWL-Clause } [] e)) = E$

by *auto*

then show *?thesis* **by** *fast*

qed

thm *CDCL-Two-Watched-Literals.raw-cls-axioms*

interpretation *twl: state_W-ops*

$\lambda C. \text{mset } (\text{raw-clause } C)$

$\lambda L C. \text{TWL-Clause } (\text{watched } C) (L \# \text{unwatched } C)$

$\lambda L C. \text{TWL-Clause } [] (\text{remove1 } L (\text{raw-clause } C))$

$\lambda C. \text{clauses-of-l } (\text{map raw-clause } C) \text{ op } @$

$\lambda L C. L \in \text{set } C \text{ op } \# \lambda C. \text{remove1-cond } (\lambda D. \text{mset } (\text{raw-clause } D) = \text{mset } (\text{raw-clause } C))$

$\text{mset } \lambda xs \text{ ys. case-prod append } (\text{fold } (\lambda x (ys, zs). (\text{remove1 } x \text{ ys}, x \# zs)) \text{ xs } (ys, []))$

$\text{op } \# \text{remove1}$

$\text{raw-clause } \lambda C. \text{TWL-Clause } [] C$

$\text{trail } \lambda S. \text{hd } (\text{raw-trail } S)$

$\text{raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting}$

apply *unfold-locales* **apply** (*auto simp: hd-map comp-def map-tl ac-simps raw-clause-def*

union-mset-list mset-map-mset-remove1-cond ex-mset-unwatched-watched)

done

declare *CDCL-Two-Watched-Literals.twl.mset-ccls-ccls-of-cls[simp del]*

lemma *mmset-of-mlit'-mmset-of-mlit[simp]*:

$\text{twl.mmset-of-mlit } L = \text{mmset-of-mlit}' L$

by (*metis mmset-of-mlit'.simps(1) mmset-of-mlit'.simps(2) twl.mmset-of-mlit.elims raw-clause-def*)

definition

$\text{candidates-propagate} :: 'v \text{ twl-state} \Rightarrow ('v \text{ literal} \times 'v \text{ twl-clause}) \text{ set}$

where

$\text{candidates-propagate } S =$

$\{(L, C) \mid L C.$

$C \in \text{set } (\text{twl.raw-clauses } S) \wedge$

$\text{set } (\text{watched } C) - (\text{uminus } ' \text{ lits-of-l } (\text{trail } S)) = \{L\} \wedge$

$\text{undefined-lit } (\text{raw-trail } S) L\}$

definition $\text{candidates-conflict} :: 'v \text{ twl-state} \Rightarrow 'v \text{ twl-clause set}$ **where**

$\text{candidates-conflict } S =$

$\{C. C \in \text{set } (\text{twl.raw-clauses } S) \wedge$

$\text{set } (\text{watched } C) \subseteq \text{uminus } ' \text{ lits-of-l } (\text{raw-trail } S)\}$

primrec (*nonexhaustive*) $\text{index} :: 'a \text{ list} \Rightarrow 'a \Rightarrow \text{nat}$ **where**

$\text{index } (a \# l) c = (\text{if } a = c \text{ then } 0 \text{ else } 1 + \text{index } l c)$

lemma *index-nth*:

$a \in \text{set } l \implies l ! (\text{index } l a) = a$

by (induction l) auto

24.2 Invariants

We need the following property about updates: if there is a literal L with $-L$ in the trail, and L is not watched, then it stays unwatched; i.e., while updating with *rewatch* it does not get swap with a watched literal L' such that $-L'$ is in the trail.

primrec *watched-decided-most-recently* :: ('v, 'wl, 'mark) marked-lit list \Rightarrow 'v twl-clause \Rightarrow bool

where

watched-decided-most-recently M (TWL-Clause W UW) \longleftrightarrow
 $(\forall L' \in \text{set } W. \forall L \in \text{set } UW. \\ -L' \in \text{lits-of-l } M \longrightarrow -L \in \text{lits-of-l } M \longrightarrow L \notin \# \text{ mset } W \longrightarrow \\ \text{index } (\text{map lit-of } M) (-L') \leq \text{index } (\text{map lit-of } M) (-L))$

Here are the invariant strictly related to the 2-WL data structure.

primrec *wf-tw-cl* :: ('v, 'wl, 'mark) marked-lit list \Rightarrow 'v twl-clause \Rightarrow bool **where**
wf-tw-cl M (TWL-Clause W UW) \longleftrightarrow
 $\text{distinct } W \wedge \text{length } W \leq 2 \wedge (\text{length } W < 2 \longrightarrow \text{set } UW \subseteq \text{set } W) \wedge \\ (\forall L \in \text{set } W. -L \in \text{lits-of-l } M \longrightarrow (\forall L' \in \text{set } UW. L' \notin \text{set } W \longrightarrow -L' \in \text{lits-of-l } M)) \wedge \\ \text{watched-decided-most-recently } M$ (TWL-Clause W UW)

lemma *size-mset-2*: $\text{size } x1 = 2 \longleftrightarrow (\exists a \ b. x1 = \{\#a, b\})$

apply (cases $x1$)

apply *simp*

by (metis (no-types, hide-lams) *Suc-eq-plus1 one-add-one size-1-singleton-mset size-Diff-singleton size-Suc-Diff1 size-single union-single-eq-diff union-single-eq-member*)

lemma *distinct-mset-size-2*: $\text{distinct-mset } \{\#a, b\} \longleftrightarrow a \neq b$

unfolding *distinct-mset-def* **by** *auto*

lemma *wf-tw-cl-annotation-independant*:

assumes M : $\text{map lit-of } M = \text{map lit-of } M'$

shows *wf-tw-cl* M (TWL-Clause W UW) \longleftrightarrow *wf-tw-cl* M' (TWL-Clause W UW)

proof –

have $\text{lits-of-l } M = \text{lits-of-l } M'$

using *arg-cong[OF M, of set]* **by** (*simp add: lits-of-def*)

then show *?thesis*

by (*simp add: lits-of-def M*)

qed

lemma *wf-tw-cl-wf-tw-cl-tl*:

assumes *wf*: *wf-tw-cl* M C **and** *n-d*: *no-dup* M

shows *wf-tw-cl* (tl M) C

proof (cases M)

case *Nil*

then show *?thesis* **using** *wf*

by (cases C) (*simp add: wf-tw-cl.simps[of tl -]*)

next

case (*Cons* l M') **note** $M = \text{this}(1)$

obtain W UW **where** C : $C = \text{TWL-Clause } W$ UW

by (cases C)

{ fix L L'

assume

LW : $L \in \text{set } W$ **and**

```

    LM: - L ∈ lits-of-l M' and
    L'UW: L' ∈ set UW and
    L'∉ set W
  then have
    L'M: - L' ∈ lits-of-l M
    using wf by (auto simp: C M)
  have watched-decided-most-recently M C
    using wf by (auto simp: C)
  then have
    index (map lit-of M) (-L) ≤ index (map lit-of M) (-L')
    using LM L'M L'UW LW ⟨L'∉ set W⟩ C M unfolding lits-of-def
    by (fastforce simp: lits-of-def)
  then have - L' ∈ lits-of-l M'
    using ⟨L'∉ set W⟩ LW L'M by (auto simp: C M split: if-split-asm)
}
moreover
{
  fix L' L
  assume
    L' ∈ set W and
    L ∈ set UW and
    L'M: - L' ∈ lits-of-l M' and
    - L ∈ lits-of-l M' and
    L ∉ set W
  moreover
    have lit-of l ≠ - L'
    using n-d unfolding M
      by (metis (no-types) L'M M Marked-Propagated-in-iff-in-lits-of-l defined-lit-map
        distinct.simps(2) list.simps(9) set-map)
  moreover have watched-decided-most-recently M C
    using wf by (auto simp: C)
  ultimately have index (map lit-of M') (- L') ≤ index (map lit-of M') (- L)
    by (fastforce simp: M C split: if-split-asm)
}
moreover have distinct W and length W ≤ 2 and (length W < 2 ⟶ set UW ⊆ set W)
  using wf by (auto simp: C M)
ultimately show ?thesis by (auto simp add: M C)
qed

```

lemma wf-twl-cls-append:

```

  assumes
    n-d: no-dup (M' @ M) and
    wf: wf-twl-cls (M' @ M) C
  shows wf-twl-cls M C
  using wf n-d apply (induction M')
  apply simp
  using wf-twl-cls-wf-twl-cls-tl by fastforce

```

definition wf-twl-state :: 'v twl-state ⇒ bool **where**

```

wf-twl-state S ⟷
  (∀ C ∈ set (twl.raw-clauses S). wf-twl-cls (raw-trail S) C) ∧ no-dup (raw-trail S)

```

lemma wf-candidates-propagate-sound:

```

  assumes wf: wf-twl-state S and
  cand: (L, C) ∈ candidates-propagate S

```

shows $\text{raw-trail } S \models_{\text{as}} \text{CNot } (\text{mset } (\text{removeAll } L \text{ (raw-clause } C))) \wedge \text{undefined-lit } (\text{raw-trail } S) L$
(is $? \text{Not} \wedge ? \text{undef})$

proof

def $M \equiv \text{raw-trail } S$

def $N \equiv \text{raw-init-clss } S$

def $U \equiv \text{raw-learned-clss } S$

note $\text{MNU-defs } [\text{simp}] = M\text{-def } N\text{-def } U\text{-def}$

have cw :

$C \in \text{set } (N @ U)$

$\text{set } (\text{watched } C) - \text{uminus ' lits-of-l } M = \{L\}$

$\text{undefined-lit } M L$

using $\text{cand unfolding candidates-propagate-def MNU-defs twl.raw-clauses-def}$ **by** auto

obtain $W UW$ **where** $cw\text{-eq: } C = \text{TWL-Clause } W UW$

by $(\text{cases } C)$

have $l\text{-w: } L \in \text{set } W$

using $cw(2)$ $cw\text{-eq}$ **by** auto

have $wf\text{-c: } wf\text{-twl-clss } M C$

using $wf\text{-c}(1)$ **unfolding** $wf\text{-twl-state-def}$ **by** $(\text{simp add: twl.raw-clauses-def})$

have $w\text{-nw}$:

$\text{distinct } W$

$\text{length } W < 2 \implies \text{set } UW \subseteq \text{set } W$

$\bigwedge L L'. L \in \text{set } W \implies -L \in \text{lits-of-l } M \implies L' \in \text{set } UW \implies L' \notin \text{set } W \implies -L' \in \text{lits-of-l } M$

using $wf\text{-c}$ **unfolding** $cw\text{-eq}$ **by** $(\text{auto simp: image-image})$

have $\forall L' \in \text{set } (\text{raw-clause } C) - \{L\}. -L' \in \text{lits-of-l } M$

proof $(\text{cases } \text{length } W < 2)$

case True

moreover **have** $\text{size } W \neq 0$

using $cw(2)$ $cw\text{-eq}$ **by** auto

ultimately **have** $\text{size } W = 1$

by linarith

then **have** $w: W = [L]$

using $l\text{-w}$ **by** $(\text{auto simp: length-list-Suc-0})$

from True **have** $\text{set } UW \subseteq \text{set } W$

using $w\text{-nw}(2)$ **by** blast

then **show** $?thesis$

using $w\text{-nw}(1)$ $cw\text{-eq}$ **by** $(\text{auto simp: raw-clause-def})$

next

case sz2: False

show $?thesis$

proof

fix L'

assume l' : $L' \in \text{set } (\text{raw-clause } C) - \{L\}$

have $\text{ex-la: } \exists La. La \neq L \wedge La \in \text{set } W$

proof $(\text{cases } W)$

case $w: \text{Nil}$

thus $?thesis$

using $l\text{-w}$ **by** auto

next

```

    case lb: (Cons Lb W')
    show ?thesis
    proof (cases W')
      case Nil
      thus ?thesis
        using lb sz2 by simp
    next
      case lc: (Cons Lc W'')
      thus ?thesis
        by (metis distinct-length-2-or-more lb list.set-intros(1) list.set-intros(2) w-nw(1))
    qed
  qed
  then obtain La where la: La ≠ L La ∈ set W
    by blast
  then have La ∈ uminus ' lits-of-l M
    using cw(2)[unfolded cw-eq, simplified, folded M-def] ⟨La ∈ set W⟩ ⟨La ≠ L⟩ by auto
  then have nla: ¬La ∈ lits-of-l M
    by (auto simp: image-image)
  then show ¬L' ∈ lits-of-l M

  proof -
    have f1: L' ∈ set (raw-clause C)
      using l' by blast
    have f2: L' ∉ {L}
      using l' by fastforce
    have ∧l L. ¬ (l::'a literal) ∈ L ∨ l ∉ uminus ' L
      by force
    then show ?thesis
      using cw(1) cw-eq w-nw(3) raw-clause-def by (metis DiffI Un-iff cw(2) f1 f2 la(2) nla
        set-append twl-clause.sel(1) twl-clause.sel(2))
  qed
  qed
  then show ?Not
    unfolding true-annots-def by (auto simp: image-image Ball-def CNot-def)

  show ?undef
    using cw(3) unfolding M-def by blast
  qed

lemma wf-candidates-propagate-complete:
  assumes wf: wf-twll-state S and
    c-mem: C ∈ set (twl.raw-clauses S) and
    l-mem: L ∈ set (raw-clause C) and
    unsat: trail S ⊨as CNot (mset-set (set (raw-clause C) - {L})) and
    undef: undefined-lit (raw-trail S) L
  shows (L, C) ∈ candidates-propagate S
proof -
  def M ≡ raw-trail S
  def N ≡ raw-init-clss S
  def U ≡ raw-learned-clss S

  note MNU-defs [simp] = M-def N-def U-def

  obtain W UW where cw-eq: C = TWL-Clause W UW

```

```

by (cases C, blast)

have wf-c: wf-twl-cls M C
  using wf-c-mem unfolding wf-twl-state-def by simp

have w-nw:
  distinct W
  length W < 2  $\implies$  set UW  $\subseteq$  set W
   $\bigwedge L L'. L \in \text{set } W \implies -L \in \text{lits-of-l } M \implies L' \in \text{set } UW \implies L' \notin \text{set } W \implies -L' \in \text{lits-of-l } M$ 
  using wf-c unfolding cw-eq by (auto simp: image-image)

have unit-set: set W - (uminus ' lits-of-l M) = {L} (is ?W = ?L)
proof
  show ?W  $\subseteq$  {L}
  proof
    fix L'
    assume l': L'  $\in$  ?W
    hence l'-mem-w: L'  $\in$  set W
      by (simp add: in-diffD)
    have L'  $\notin$  uminus ' lits-of-l M
      using l' by blast
    then have  $\neg M \models_a \{\# - L' \#\}$ 
      by (auto simp: lits-of-def uminus-lit-swap image-image)
    moreover have L'  $\in$  set (raw-clause C)
      using c-mem cw-eq l'-mem-w by (auto simp: raw-clause-def)
    ultimately have L' = L
      using unsat[unfolded CNot-def true-annots-def, simplified]
      unfolding M-def by fastforce
    then show L'  $\in$  {L}
      by simp
  qed
next
  show {L}  $\subseteq$  ?W
  proof clarify
    have L  $\in$  set W
    proof (cases W)
      case Nil
      thus ?thesis
        using w-nw(2) cw-eq l-mem by (auto simp: raw-clause-def)
    next
      case (Cons La W')
      thus ?thesis
        proof (cases La = L)
          case True
          thus ?thesis
            using Cons by simp
        next
          case False
          have -La  $\in$  lits-of-l M
            using False Cons cw-eq unsat[unfolded CNot-def true-annots-def, simplified]
            by (fastforce simp: raw-clause-def)
          then show ?thesis
            using Cons cw-eq l-mem undef w-nw(3)
            by (auto simp: Marked-Propagated-in-iff-in-lits-of-l raw-clause-def)
        qed
    qed
  qed

```

```

qed
moreover have  $L \notin \# \text{ mset-set } (\text{uminus } ' \text{ lits-of-l } M)$ 
  using undef by (auto simp: Marked-Propagated-in-iff-in-lits-of-l image-image)
ultimately show  $L \in ?W$ 
  by simp
qed
qed

show ?thesis
  unfolding candidates-propagate-def using unit-set undef c-mem unfolding cw-eq M-def
  by (auto simp: image-image cw-eq intro!: exI[of - C])
qed

lemma wf-candidates-conflict-sound:
  assumes wf: wf-twl-state  $S$  and
    cand:  $C \in \text{candidates-conflict } S$ 
  shows  $\text{trail } S \models_{\text{as}} C \text{Not } (\text{mset } (\text{raw-clause } C)) \wedge C \in \text{set } (\text{twl.raw-clauses } S)$ 
proof
  def  $M \equiv \text{raw-trail } S$ 
  def  $N \equiv \text{raw-init-clss } S$ 
  def  $U \equiv \text{raw-learned-clss } S$ 

  note MNU-defs [simp] = M-def N-def U-def

  have cw:
     $C \in \text{set } (N @ U)$ 
     $\text{set } (\text{watched } C) \subseteq \text{uminus } ' \text{ lits-of-l } (\text{trail } S)$ 
    using cand[unfolded candidates-conflict-def, simplified] unfolding twl.raw-clauses-def by auto

  obtain  $W \text{ } UW$  where cw-eq:  $C = \text{TWL-Clause } W \text{ } UW$ 
    by (cases  $C$ , blast)

  have wf-c: wf-twl-cls  $M \text{ } C$ 
    using wf cw(1) unfolding wf-twl-state-def by (simp add: comp-def twl.raw-clauses-def)

  have w-nw:
    distinct  $W$ 
     $\text{length } W < 2 \implies \text{set } UW \subseteq \text{set } W$ 
     $\bigwedge L \text{ } L'. L \in \text{set } W \implies \neg L \in \text{lits-of-l } M \implies L' \in \text{set } UW \implies L' \notin \text{set } W \implies \neg L' \in \text{lits-of-l } M$ 
    using wf-c unfolding cw-eq by (auto simp: image-image)

  have  $\forall L \in \text{set } (\text{raw-clause } C). \neg L \in \text{lits-of-l } M$ 
proof (cases  $W$ )
  case Nil
  then have raw-clause  $C = []$ 
    using cw(1) cw-eq w-nw(2) by (auto simp: raw-clause-def)
  then show ?thesis
    by simp
next
  case (Cons  $La \text{ } W'$ ) note  $W' = \text{this}(1)$ 
  show ?thesis
proof
  fix  $L$ 
  assume  $l: L \in \text{set } (\text{raw-clause } C)$ 
  show  $\neg L \in \text{lits-of-l } M$ 

```

```

proof (cases  $L \in \text{set } W$ )
  case True
    thus ?thesis
      using  $\text{cw}(2)$   $\text{cw-eq}$  by fastforce
  next
    case False
    thus ?thesis
      using  $W'$   $\text{cw}(2)$   $\text{cw-eq}$   $l$   $w\text{-nw}(3)$  unfolding  $M\text{-def}$   $\text{raw-clause-def}$ 
      by (metis (no-types, lifting)  $\text{UnE}$   $\text{imageE}$   $\text{list.set-intros}(1)$ 
         $\text{lits-of-mmset-of-mlit}'$   $\text{rev-subsetD}$   $\text{set-append}$   $\text{set-map}$   $\text{twl-clause.sel}(1)$ 
         $\text{twl-clause.sel}(2)$   $\text{uminus-of-uminus-id}$ )
    qed
  qed
qed
then show  $\text{trail } S \models_{\text{as}} \text{CNot } (\text{mset } (\text{raw-clause } C))$ 
  unfolding  $\text{CNot-def}$   $\text{true-annots-def}$  by auto

show  $C \in \text{set } (\text{twl.raw-clauses } S)$ 
  using  $\text{cw}$  unfolding  $\text{twl.raw-clauses-def}$  by auto
qed

lemma wf-candidates-conflict-complete:
  assumes  $\text{wf}$ :  $\text{wf-tw-l-state } S$  and
     $c\text{-mem}$ :  $C \in \text{set } (\text{twl.raw-clauses } S)$  and
     $\text{unsat}$ :  $\text{trail } S \models_{\text{as}} \text{CNot } (\text{mset } (\text{raw-clause } C))$ 
  shows  $C \in \text{candidates-conflict } S$ 
proof –
  def  $M \equiv \text{raw-trail } S$ 
  def  $N \equiv \text{twl.init-clss } S$ 
  def  $U \equiv \text{twl.learned-clss } S$ 

  note  $MNU\text{-defs } [\text{simp}] = M\text{-def } N\text{-def } U\text{-def}$ 

  obtain  $W$   $UW$  where  $\text{cw-eq}$ :  $C = \text{TWL-Clause } W$   $UW$ 
  by (cases  $C$ , blast)

  have  $\text{wf-c}$ :  $\text{wf-tw-l-clss } M$   $C$ 
  using  $\text{wf}$   $c\text{-mem}$  unfolding  $\text{wf-tw-l-state-def}$  by simp

  have  $w\text{-nw}$ :
    distinct  $W$ 
     $\text{length } W < 2 \implies \text{set } UW \subseteq \text{set } W$ 
     $\bigwedge L \ L'. L \in \text{set } W \implies \neg L \in \text{lits-of-l } M \implies L' \in \text{set } UW \implies L' \notin \text{set } W \implies \neg L' \in \text{lits-of-l } M$ 
    using  $\text{wf-c}$  unfolding  $\text{cw-eq}$  by (auto simp: image-image)

  have  $\bigwedge L. L \in \text{set } (\text{raw-clause } C) \implies \neg L \in \text{lits-of-l } M$ 
  unfolding  $M\text{-def}$  using  $\text{unsat}[\text{unfolded } \text{CNot-def } \text{true-annots-def}, \text{simplified}]$  by auto
  then have  $\text{set } (\text{raw-clause } C) \subseteq \text{uminus } ' \text{lits-of-l } M$ 
  by (metis imageI subsetI uminus-of-uminus-id)
  then have  $\text{set } W \subseteq \text{uminus } ' \text{lits-of-l } M$ 
  using  $\text{cw-eq}$  by (auto simp:  $\text{raw-clause-def}$ )
  then have  $\text{subset}$ :  $\text{set } W \subseteq \text{uminus } ' \text{lits-of-l } M$ 
  by (simp add:  $w\text{-nw}(1)$ )

  have  $W = \text{watched } C$ 

```

```

    using cw-eq twl-clause.sel(1) by simp
  then show ?thesis
    using MNU-defs c-mem subset candidates-conflict-def by blast
qed

```

```

typedef 'v wf-twℓ = {S :: 'v twℓ-state. wf-twℓ-state S}
morphisms rough-state-of-twℓ twℓ-of-rough-state
proof -
  have TWℓ-State ([::('v, nat, 'v twℓ-clause) marked-lits)
    [ ] 0 None ∈ {S :: 'v twℓ-state. wf-twℓ-state S}
    by (auto simp: wf-twℓ-state-def twℓ.raw-clauses-def)
  then show ?thesis by auto
qed

```

```

lemma [code abstype]:
  twℓ-of-rough-state (rough-state-of-twℓ S) = S
  by (fact CDCL-Two-Watched-Literals.wf-twℓ.rough-state-of-twℓ-inverse)

```

```

lemma wf-twℓ-state-rough-state-of-twℓ[simp]: wf-twℓ-state (rough-state-of-twℓ S)
  using rough-state-of-twℓ by auto

```

abbreviation *candidates-conflict-twℓ* :: 'v wf-twℓ \Rightarrow 'v twℓ-clause set **where**
candidates-conflict-twℓ S \equiv *candidates-conflict* (rough-state-of-twℓ S)

abbreviation *candidates-propagate-twℓ* :: 'v wf-twℓ \Rightarrow ('v literal \times 'v twℓ-clause) set **where**
candidates-propagate-twℓ S \equiv *candidates-propagate* (rough-state-of-twℓ S)

abbreviation *raw-trail-twℓ* :: 'a wf-twℓ \Rightarrow ('a, nat, 'a twℓ-clause) marked-lit list **where**
raw-trail-twℓ S \equiv *raw-trail* (rough-state-of-twℓ S)

abbreviation *trail-twℓ* :: 'a wf-twℓ \Rightarrow ('a, nat, 'a literal multiset) marked-lit list **where**
trail-twℓ S \equiv *trail* (rough-state-of-twℓ S)

abbreviation *raw-clauses-twℓ* :: 'a wf-twℓ \Rightarrow 'a twℓ-clause list **where**
raw-clauses-twℓ S \equiv *twℓ.raw-clauses* (rough-state-of-twℓ S)

abbreviation *raw-init-clss-twℓ* :: 'a wf-twℓ \Rightarrow 'a twℓ-clause list **where**
raw-init-clss-twℓ S \equiv *raw-init-clss* (rough-state-of-twℓ S)

abbreviation *raw-learned-clss-twℓ* :: 'a wf-twℓ \Rightarrow 'a twℓ-clause list **where**
raw-learned-clss-twℓ S \equiv *raw-learned-clss* (rough-state-of-twℓ S)

abbreviation *backtrack-lvl-twℓ* **where**
backtrack-lvl-twℓ S \equiv *backtrack-lvl* (rough-state-of-twℓ S)

abbreviation *raw-conflicting-twℓ* **where**
raw-conflicting-twℓ S \equiv *raw-conflicting* (rough-state-of-twℓ S)

```

lemma wf-candidates-twℓ-conflict-complete:
  assumes
    c-mem: C ∈ set (raw-clauses-twℓ S) and
    unsat: trail-twℓ S  $\models$  as CNot (mset (raw-clause C))
  shows C ∈ candidates-conflict-twℓ S
  using c-mem unsat wf-candidates-conflict-complete wf-twℓ-state-rough-state-of-twℓ by blast

```


abbreviation *update-backtrack-lvl* **where**

update-backtrack-lvl k $S \equiv$
 $TWL\text{-}State\ (raw\text{-}trail\ S)\ (raw\text{-}init\text{-}clss\ S)\ (raw\text{-}learned\text{-}clss\ S)\ k\ (raw\text{-}conflicting\ S)$

abbreviation *update-conflicting* **where**

update-conflicting C $S \equiv$
 $TWL\text{-}State\ (raw\text{-}trail\ S)\ (raw\text{-}init\text{-}clss\ S)\ (raw\text{-}learned\text{-}clss\ S)\ (backtrack\text{-}lvl\ S)\ C$

24.3 Abstract 2-WL

definition *tl-trail* **where**

tl-trail $S =$
 $TWL\text{-}State\ (tl\ (raw\text{-}trail\ S))\ (raw\text{-}init\text{-}clss\ S)\ (raw\text{-}learned\text{-}clss\ S)\ (backtrack\text{-}lvl\ S)\ (raw\text{-}conflicting\ S)$

locale *abstract-tw* $=$

fixes

watch $:: 'v\ twl\text{-}state \Rightarrow 'v\ literal\ list \Rightarrow 'v\ twl\text{-}clause$ **and**

rewatch $:: 'v\ literal \Rightarrow 'v\ twl\text{-}state \Rightarrow$

$'v\ twl\text{-}clause \Rightarrow 'v\ twl\text{-}clause$ **and**

restart-learned $:: 'v\ twl\text{-}state \Rightarrow 'v\ twl\text{-}clause\ list$

assumes

clause-watch: $no\text{-}dup\ (raw\text{-}trail\ S) \Longrightarrow mset\ (raw\text{-}clause\ (watch\ S\ C)) = mset\ C$ **and**

wf-watch: $no\text{-}dup\ (raw\text{-}trail\ S) \Longrightarrow wf\text{-}twl\text{-}cls\ (raw\text{-}trail\ S)\ (watch\ S\ C)$ **and**

clause-rewatch: $mset\ (raw\text{-}clause\ (rewatch\ L'\ S\ C')) = mset\ (raw\text{-}clause\ C')$ **and**

wf-rewatch:

$no\text{-}dup\ (raw\text{-}trail\ S) \Longrightarrow undefined\text{-}lit\ (raw\text{-}trail\ S)\ (lit\text{-}of\ L) \Longrightarrow$

$wf\text{-}twl\text{-}cls\ (raw\text{-}trail\ S)\ C' \Longrightarrow$

$wf\text{-}twl\text{-}cls\ (L\ \# \ raw\text{-}trail\ S)\ (rewatch\ (lit\text{-}of\ L)\ S\ C')$

and

restart-learned: $mset\ (restart\text{-}learned\ S) \subseteq\# mset\ (raw\text{-}learned\text{-}clss\ S)$ — We need *mset* and not *set* to take care of duplicates.

begin

definition

cons-trail $:: ('v, nat, 'v\ twl\text{-}clause)\ marked\text{-}lit \Rightarrow 'v\ twl\text{-}state \Rightarrow 'v\ twl\text{-}state$

where

cons-trail $L\ S =$

$TWL\text{-}State\ (L\ \# \ raw\text{-}trail\ S)\ (map\ (rewatch\ (lit\text{-}of\ L)\ S)\ (raw\text{-}init\text{-}clss\ S))$

$(map\ (rewatch\ (lit\text{-}of\ L)\ S)\ (raw\text{-}learned\text{-}clss\ S))\ (backtrack\text{-}lvl\ S)\ (raw\text{-}conflicting\ S)$

definition

add-init-cl $:: 'v\ literal\ list \Rightarrow 'v\ twl\text{-}state \Rightarrow 'v\ twl\text{-}state$

where

add-init-cl $C\ S =$

$TWL\text{-}State\ (raw\text{-}trail\ S)\ (watch\ S\ C\ \# \ raw\text{-}init\text{-}clss\ S)\ (raw\text{-}learned\text{-}clss\ S)\ (backtrack\text{-}lvl\ S)\ (raw\text{-}conflicting\ S)$

definition

add-learned-cl $:: 'v\ literal\ list \Rightarrow 'v\ twl\text{-}state \Rightarrow 'v\ twl\text{-}state$

where

add-learned-cl $C\ S =$

$TWL\text{-}State\ (raw\text{-}trail\ S)\ (raw\text{-}init\text{-}clss\ S)\ (watch\ S\ C\ \# \ raw\text{-}learned\text{-}clss\ S)\ (backtrack\text{-}lvl\ S)\ (raw\text{-}conflicting\ S)$

definition

$remove_cls :: 'v \text{ literal list} \Rightarrow 'v \text{ twl-state} \Rightarrow 'v \text{ twl-state}$
where
 $remove_cls \ C \ S =$
 $TWL_State \ (raw_trail \ S)$
 $(removeAll_cond \ (\lambda D. \ mset \ (raw_clause \ D) = mset \ C) \ (raw_init_clss \ S))$
 $(removeAll_cond \ (\lambda D. \ mset \ (raw_clause \ D) = mset \ C) \ (raw_learned_clss \ S))$
 $(backtrack_lvl \ S)$
 $(raw_conflicting \ S)$

definition $init_state :: 'v \text{ literal list list} \Rightarrow 'v \text{ twl-state}$ **where**
 $init_state \ N = fold \ add_init_cls \ N \ (TWL_State \ [] \ [] \ 0 \ None)$

lemma *unchanged-fold-add-init-cls:*
 $raw_trail \ (fold \ add_init_cls \ Cs \ (TWL_State \ M \ N \ U \ k \ C)) = M$
 $raw_learned_clss \ (fold \ add_init_cls \ Cs \ (TWL_State \ M \ N \ U \ k \ C)) = U$
 $backtrack_lvl \ (fold \ add_init_cls \ Cs \ (TWL_State \ M \ N \ U \ k \ C)) = k$
 $raw_conflicting \ (fold \ add_init_cls \ Cs \ (TWL_State \ M \ N \ U \ k \ C)) = C$
by $(induct \ Cs \ arbitrary: N) \ (auto \ simp: add_init_cls_def)$

lemma *unchanged-init-state[simp]:*
 $raw_trail \ (init_state \ N) = []$
 $raw_learned_clss \ (init_state \ N) = []$
 $backtrack_lvl \ (init_state \ N) = 0$
 $raw_conflicting \ (init_state \ N) = None$
unfolding $init_state_def$ **by** $(rule \ unchanged_fold_add_init_cls)+$

lemma *clauses-init-fold-add-init:*
 $no_dup \ M \Longrightarrow$
 $twl.init_clss \ (fold \ add_init_cls \ Cs \ (TWL_State \ M \ N \ U \ k \ C)) =$
 $clauses_of_l \ Cs + clauses_of_l \ (map \ raw_clause \ N)$
by $(induct \ Cs \ arbitrary: N) \ (auto \ simp: add_init_cls_def \ clause_watch \ comp_def \ ac_simps)$

lemma *init-clss-init-state[simp]:* $twl.init_clss \ (init_state \ N) = clauses_of_l \ N$
unfolding $init_state_def$ **by** $(subst \ clauses_init_fold_add_init) \ simp_all$

definition $restart'$ **where**
 $restart' \ S = TWL_State \ [] \ (raw_init_clss \ S) \ (restart_learned \ S) \ 0 \ None$

end

24.4 Instanciation of the previous locale

definition $watch_nat :: 'v \text{ twl-state} \Rightarrow 'v \text{ literal list} \Rightarrow 'v \text{ twl-clause}$ **where**
 $watch_nat \ S \ C =$
 $(let$
 $\ C' = remdups \ C;$
 $\ neg_not_assigned = filter \ (\lambda L. \ -L \notin lits_of_l \ (raw_trail \ S)) \ C';$
 $\ neg_assigned_sorted_by_trail = filter \ (\lambda L. \ L \in set \ C) \ (map \ (\lambda L. \ -lit_of \ L) \ (raw_trail \ S));$
 $\ W = take \ 2 \ (neg_not_assigned \ @ \ neg_assigned_sorted_by_trail);$
 $\ UW = foldr \ remove1 \ W \ C$
 $\ in \ TWL_Clause \ W \ UW)$

lemma *list-cases2:*
fixes $l :: 'a \text{ list}$
assumes
 $l = [] \Longrightarrow P$ **and**

$\bigwedge x. l = [x] \implies P$ **and**
 $\bigwedge x\ y\ xs. l = x \# y \# xs \implies P$
shows P
by (*metis* *assms* *list.collapse*)

lemma *filter-in-list-prop-verifiedD*:
assumes $[L \leftarrow P \ . \ Q \ L] = l$
shows $\forall x \in \text{set } l. x \in \text{set } P \wedge Q \ x$
using *assms* **by** *auto*

lemma *no-dup-filter-diff*:
assumes $n\text{-d}: \text{no-dup } M$ **and** $H: [L \leftarrow \text{map } (\lambda L. - \text{lit-of } L) \ M. L \in \text{set } C] = l$
shows *distinct* l
unfolding $H[\text{symmetric}]$
apply (*rule distinct-filter*)
using $n\text{-d}$ **by** (*induction* M) *auto*

lemma *watch-nat-lists-disjointD*:
assumes
 $l: [L \leftarrow \text{remdups } C. - L \notin \text{lits-of-l } (\text{raw-trail } S)] = l$ **and**
 $l': [L \leftarrow \text{map } (\lambda L. - \text{lit-of } L) (\text{raw-trail } S) . L \in \text{set } C] = l'$
shows $\forall x \in \text{set } l. \forall y \in \text{set } l'. x \neq y$
by (*auto simp: l[symmetric] l'[symmetric] lits-of-def image-image*)

lemma *watch-nat-list-cases-witness*[*consumes 2, case-names nil-nil nil-single nil-other single-nil single-other other*]:

fixes

$C :: 'v$ *literal list* **and**

$S :: 'v$ *twl-state*

defines

$xs \equiv [L \leftarrow \text{remdups } C. - L \notin \text{lits-of-l } (\text{raw-trail } S)]$ **and**

$ys \equiv [L \leftarrow \text{map } (\lambda L. - \text{lit-of } L) (\text{raw-trail } S) . L \in \text{set } C]$

assumes

$n\text{-d}: \text{no-dup } (\text{raw-trail } S)$ **and**

nil-nil: $xs = [] \implies ys = [] \implies P$ **and**

nil-single:

$\bigwedge a. xs = [] \implies ys = [a] \implies a \in \text{set } C \implies P$ **and**

nil-other: $\bigwedge a\ b\ ys'. xs = [] \implies ys = a \# b \# ys' \implies a \neq b \implies P$ **and**

single-nil: $\bigwedge a. xs = [a] \implies ys = [] \implies P$ **and**

single-other: $\bigwedge a\ b\ ys'. xs = [a] \implies ys = b \# ys' \implies a \neq b \implies P$ **and**

other: $\bigwedge a\ b\ xs'. xs = a \# b \# xs' \implies a \neq b \implies P$

shows P

proof –

note $xs\text{-def}[simp]$ **and** $ys\text{-def}[simp]$

have *dist*: $\bigwedge P. \text{distinct } [L \leftarrow \text{remdups } C \ . \ P \ L]$

by *auto*

then have $H: \bigwedge a\ b\ P\ xs. [L \leftarrow \text{remdups } C \ . \ P \ L] = a \# b \# xs \implies a \neq b$

by (*metis distinct-length-2-or-more*)

show *?thesis*

apply (*cases* $[L \leftarrow \text{remdups } C. - L \notin \text{lits-of-l } (\text{raw-trail } S)]$

rule: list-cases2;

cases $[L \leftarrow \text{map } (\lambda L. - \text{lit-of } L) (\text{raw-trail } S) . L \in \text{set } C]$ *rule: list-cases2*)

using *nil-nil* **apply** *simp*

using *nil-single* **apply** (*force dest: filter-in-list-prop-verifiedD*)

using *nil-other no-dup-filter-diff*[*OF n-d, of C*]

apply *fastforce*
using *single-nil* **apply** *simp*
using *single-other* *xs-def* *ys-def* **apply** (*metis* *list.set-intros*(1) *watch-nat-lists-disjointD*)
using *single-other* **unfolding** *xs-def* *ys-def* **apply** (*metis* *list.set-intros*(1) *watch-nat-lists-disjointD*)
using *other* *xs-def* *ys-def* **by** (*metis* *H*)
qed

lemma *watch-nat-list-cases* [*consumes* 1, *case-names* *nil-nil* *nil-single* *nil-other* *single-nil* *single-other* *other*]:

fixes

C :: 'v literal list **and**

S :: 'v twl-state

defines

xs \equiv [*L* \leftarrow *remdups* *C* . $-$ *L* \notin *lits-of-l* (*raw-trail* *S*)] **and**

ys \equiv [*L* \leftarrow *map* (λL . $-$ *lit-of* *L*) (*raw-trail* *S*) . *L* \in *set* *C*]

assumes

n-d: *no-dup* (*raw-trail* *S*) **and**

nil-nil: *xs* = [] \implies *ys* = [] \implies *P* **and**

nil-single:

$\bigwedge a$. *xs* = [] \implies *ys* = [*a*] \implies *a* \in *set* *C* \implies *P* **and**

nil-other: $\bigwedge a$ *b* *ys'*. *xs* = [] \implies *ys* = *a* # *b* # *ys'* \implies *a* \neq *b* \implies *P* **and**

single-nil: $\bigwedge a$. *xs* = [*a*] \implies *ys* = [] \implies *P* **and**

single-other: $\bigwedge a$ *b* *ys'*. *xs* = [*a*] \implies *ys* = *b* # *ys'* \implies *a* \neq *b* \implies *P* **and**

other: $\bigwedge a$ *b* *xs'*. *xs* = *a* # *b* # *xs'* \implies *a* \neq *b* \implies *P*

shows *P*

using *watch-nat-list-cases-witness*[*OF* *n-d*, *of* *C* *P*]

nil-nil *nil-single* *nil-other* *single-nil* *single-other* *other*

unfolding *xs-def*[*symmetric*] *ys-def*[*symmetric*] **by** *auto*

lemma *watch-nat-lists-set-union-witness*:

fixes

C :: 'v literal list **and**

S :: 'v twl-state

defines

xs \equiv [*L* \leftarrow *remdups* *C* . $-$ *L* \notin *lits-of-l* (*raw-trail* *S*)] **and**

ys \equiv [*L* \leftarrow *map* (λL . $-$ *lit-of* *L*) (*raw-trail* *S*) . *L* \in *set* *C*]

assumes *n-d*: *no-dup* (*raw-trail* *S*)

shows *set* *C* = *set* *xs* \cup *set* *ys*

using *n-d* **unfolding** *xs-def* *ys-def* **by** (*auto* *simp*: *lits-of-def* *comp-def* *uminus-lit-swap*)

lemma *mset-intersection-inclusion*: $A + (B - A) = B \longleftrightarrow A \subseteq \# B$

apply (*rule* *iffI*)

apply (*metis* *mset-le-add-left*)

by (*auto* *simp*: *ac-simps* *multiset-eq-iff* *subsetq-mset-def*)

lemma *clause-watch-nat*:

assumes *no-dup* (*raw-trail* *S*)

shows *mset* (*raw-clause* (*watch-nat* *S* *C*)) = *mset* *C*

using *assms*

apply (*cases* *rule*: *watch-nat-list-cases*[*OF* *assms*(1), *of* *C*])

by (*auto* *dest*: *filter-in-list-prop-verifiedD* *simp*: *watch-nat-def* *multiset-eq-iff* *raw-clause-def*)

lemma *index-uminus-index-map-uminus*:

$-a \in \text{set } L \implies \text{index } L \ (-a) = \text{index } (\text{map } \text{uminus } L) \ (a::'a \text{ literal})$

```

by (induction L) auto

lemma index-filter:
  a ∈ set L ⇒ b ∈ set L ⇒ P a ⇒ P b ⇒
  index L a ≤ index L b ⇔ index (filter P L) a ≤ index (filter P L) b
by (induction L) auto

lemma foldr-remove1-W-Nil[simp]: foldr remove1 W [] = []
by (induct W) auto

lemma image-lit-of-mmset-of-mlit'[simp]:
  lit-of ' mmset-of-mlit' ' A = lit-of ' A
by (auto simp: image-image comp-def)

lemma distinct-filter-eq:
  assumes distinct xs
  shows [L ← xs. L = a] = (if a ∈ set xs then [a] else [])
using assms by (induction xs) auto

lemma no-dup-distinct-map-uminus-lit-of:
  no-dup xs ⇒ distinct (map (λL. - lit-of L) xs)
by (induction xs) auto

lemma wf-watch-witness:
  fixes C :: 'v literal list and
  S :: 'v twl-state
  defines
    ass: neg-not-assigned ≡ filter (λL. -L ∉ lits-of-l (raw-trail S)) (remdups C) and
    tr: neg-assigned-sorted-by-trail ≡ filter (λL. L ∈ set C) (map (λL. -lit-of L) (raw-trail S))
  defines
    W: W ≡ take 2 (neg-not-assigned @ neg-assigned-sorted-by-trail)
  assumes
    n-d[simp]: no-dup (raw-trail S)
  shows wf-twlc (raw-trail S) (TWL-Clause W (foldr remove1 W C))
  unfolding wf-twlc.simps
proof (intro conjI, goal-cases)
  case 1
  then show ?case using n-d W unfolding ass tr
  apply (cases rule: watch-nat-list-cases-witness[of S C, OF n-d])
  by (auto simp: distinct-mset-add-single)
next
  case 2
  then show ?case unfolding W by simp
next
  case 3
  show ?case using n-d
  proof (cases rule: watch-nat-list-cases-witness[of S C])
    case nil-nil
    then have set C = set [] ∪ set []
    using watch-nat-lists-set-union-witness n-d by metis
    then show ?thesis
    by simp
  next
    case (nil-single a)
    moreover have ∧x. set C = {a} ⇒ - a ∈ lits-of-l (trail S) ⇒ x ∈ set (remove1 a C) ⇒

```

```

     $x = a$ 
    using notin-set-remove1 by auto
  ultimately show ?thesis
    using watch-nat-lists-set-union-witness[of  $S$   $C$ ] 3 by (auto simp:  $W$  ass tr comp-def)
next
  case nil-other
  then show ?thesis
    using 3 by (auto simp:  $W$  ass tr)
next
  case (single-nil  $a$ )
  show ?thesis
    using watch-nat-lists-set-union-witness[of  $S$   $C$ ] 3
    by (fastforce simp add:  $W$  ass tr single-nil comp-def distinct-filter-eq
        no-dup-distinct-map-uminus-lit-of min-def)
next
  case single-other
  then show ?thesis
    using 3 by (auto simp:  $W$  ass tr)
next
  case other
  then show ?thesis
    using 3 by (auto simp:  $W$  ass tr)
qed
next
  case 4 note -[simp] = this
  show ?case
    using n-d apply (cases rule: watch-nat-list-cases-witness[of  $S$   $C$ ])
    apply (auto dest: filter-in-list-prop-verifiedD
        simp:  $W$  ass tr lits-of-def filter-empty-conv)[4]
    using watch-nat-lists-set-union-witness[of  $S$   $C$ ]
    by (force dest: filter-in-list-prop-verifiedD simp:  $W$  ass tr lits-of-def)+
next
  case 5
  from n-d show ?case
  proof (cases rule: watch-nat-list-cases-witness[of  $S$   $C$ ])
    case nil-nil
    then show ?thesis by (auto simp:  $W$  ass tr)
  next
    case nil-single
    then show ?thesis
      using watch-nat-lists-set-union-witness[of  $S$   $C$ ] tr by (fastforce simp:  $W$  ass)
  next
    case nil-other
    then show ?thesis
      unfolding watched-decided-most-recently.simps Ball-def
      apply (intro allI impI)
      apply (subst index-uminus-index-map-uminus,
        simp add: index-uminus-index-map-uminus lits-of-def o-def)
      apply (subst index-uminus-index-map-uminus,
        simp add: index-uminus-index-map-uminus lits-of-def o-def)

      apply (subst index-filter[of - -  $\lambda L. L \in \text{set } C$ ])
      by (auto dest: filter-in-list-prop-verifiedD
        simp: uminus-lit-swap lits-of-def o-def  $W$  ass tr dest: in-diffD)
  next

```

```

case single-nil
then show ?thesis
  using watch-nat-lists-set-union-witness[of S C] tr by (fastforce simp: W ass)
next
case single-other
then show ?thesis
  unfolding watched-decided-most-recently.simps Ball-def
  apply (clarify)
  apply (subst index-uminus-index-map-uminus,
    simp add: index-uminus-index-map-uminus lits-of-def image-image o-def)
  apply (subst index-uminus-index-map-uminus,
    simp add: index-uminus-index-map-uminus lits-of-def o-def)

  apply (subst index-filter[of - -  $\lambda L. L \in \text{set } C$ ])
  by (auto dest: filter-in-list-prop-verifiedD
    simp: W ass tr uminus-lit-swap lits-of-def o-def dest: in-diffD)
next
case other
then show ?thesis
  unfolding watched-decided-most-recently.simps
  apply clarify
  apply (subst index-uminus-index-map-uminus,
    simp add: index-uminus-index-map-uminus lits-of-def o-def)[1]
  apply (subst index-uminus-index-map-uminus,
    simp add: index-uminus-index-map-uminus lits-of-def o-def)[1]

  apply (subst index-filter[of - -  $\lambda L. L \in \text{set } C$ ])
  by (auto dest: filter-in-list-prop-verifiedD
    simp: index-uminus-index-map-uminus lits-of-def o-def uminus-lit-swap
    W ass tr)
qed
qed

```

lemma *wf-watch-nat: no-dup (raw-trail S) \implies wf-twl-cls (raw-trail S) (watch-nat S C)*
using *wf-watch-witness*[of *S C*] *watch-nat-def* **by** *metis*

definition

```

rewatch-nat ::
  'v literal  $\Rightarrow$  'v twl-state  $\Rightarrow$  'v twl-clause  $\Rightarrow$  'v twl-clause
where
  rewatch-nat L S C =
    (if  $\neg L \in \text{set } (\text{watched } C)$  then
      case filter ( $\lambda L'. L' \notin \text{set } (\text{watched } C) \wedge \neg L' \notin \text{insert } L (\text{lits-of-l } (\text{trail } S))$ )
        (unwatched C) of
        []  $\Rightarrow$  C
      | L' # -  $\Rightarrow$ 
        TWL-Clause (L' # remove1 (-L) (watched C)) (-L # remove1 L' (unwatched C))
    else
      C)

```

lemma *clause-rewatch-nat:*

```

fixes UW :: 'v literal list and
  S :: 'v twl-state and
  L :: 'v literal and C :: 'v twl-clause
shows mset (raw-clause (rewatch-nat L S C)) = mset (raw-clause C)

```

```

using List.set-remove1-subset[of  $-L$  watched  $C$ ]
apply (cases  $C$ )
by (auto simp: raw-clause-def rewatch-nat-def ac-simps multiset-eq-iff
    split: list.split
    dest: filter-in-list-prop-verifiedD)

lemma filter-sorted-list-of-multiset-Nil:
 $[x \leftarrow \text{sorted-list-of-multiset } M. p \ x] = [] \longleftrightarrow (\forall x \in \# M. \neg p \ x)$ 
by auto (metis empty-iff filter-set list.set(1) member-filter set-sorted-list-of-multiset)

lemma filter-sorted-list-of-multiset-ConsD:
 $[x \leftarrow \text{sorted-list-of-multiset } M. p \ x] = x \ \# \ xs \implies p \ x$ 
by (metis filter-set insert-iff list.set(2) member-filter)

lemma mset-minus-single-eq-empty:
 $a - \{\#b\} = \{\#\} \longleftrightarrow a = \{\#b\} \vee a = \{\#\}$ 
by (metis Multiset.diff-cancel add.right-neutral diff-single-eq-union
    diff-single-trivial zero-diff)

lemma size-mset-le-2-cases:
assumes  $size \ W \leq 2$ 
shows  $W = \{\#\} \vee (\exists a. W = \{\#a\}) \vee (\exists a \ b. W = \{\#a, b\})$ 
by (metis One-nat-def Suc-1 Suc-eq-plus1-left assms linorder-not-less nat-less-le
    not-less-eq-eq le-iff-add size-1-singleton-mset
    size-eq-0-iff-empty size-mset-2)

lemma filter-sorted-list-of-multiset-eqD:
assumes  $[x \leftarrow \text{sorted-list-of-multiset } A. p \ x] = x \ \# \ xs$  (is  $?comp = -$ )
shows  $x \in \# A$ 
proof  $-$ 
  have  $x \in \text{set } ?comp$ 
  using assms by simp
  then have  $x \in \text{set } (\text{sorted-list-of-multiset } A)$ 
  by simp
  then show  $x \in \# A$ 
  by simp
qed

lemma clause-rewatch-witness':
assumes
  wf: wf-twl-cls (raw-trail  $S$ )  $C$  and
  undef: undefined-lit (raw-trail  $S$ ) (lit-of  $L$ )
shows wf-twl-cls ( $L \ \# \ \text{raw-trail } S$ ) (rewatch-nat (lit-of  $L$ )  $S \ C$ )
proof (cases  $- \text{lit-of } L \in \text{set } (\text{watched } C)$ )
  case False
  then show ?thesis
  apply (cases  $C$ )
  using wf undef unfolding rewatch-nat-def
  by (auto simp: uminus-lit-swap Marked-Propagated-in-iff-in-lits-of-l comp-def)
next
  case falsified: True

  let  $?unwatched\text{-nonfalsified} =$ 
     $[L' \leftarrow \text{unwatched } C. L' \notin \text{set } (\text{watched } C) \wedge - L' \notin \text{insert } (\text{lit-of } L) (\text{lits-of-l } (\text{trail } S))]$ 
  obtain  $W \ UW$  where  $C: C = \text{TWL-Clause } W \ UW$ 

```



```

by (cases C)

show ?thesis
proof (cases ?unwatched-nonfalsified)
  case Nil
  show ?thesis
    using falsified Nil
  apply (simp only: wf-twl-cls.simps if-True list.cases C rewatch-nat-def)
  apply (intro conjI)
  proof goal-cases
    case 1
    then show ?case using wf C by simp
  next
    case 2
    then show ?case using wf C by simp
  next
    case 3
    then show ?case using wf C by simp
  next
    case 4
    have  $\bigwedge p l. \text{filter } p (\text{unwatched } C) \neq [] \vee l \notin \text{set } UW \vee \neg p l$ 
      unfolding C by (metis (no-types) filter-empty-conv twl-clause.sel(2))
    then show ?case
      using 4(2) C by auto
  next
    case 5
    then show ?case
      using wf by (fastforce simp add: C comp-def uminus-lit-swap)
  qed
next
case (Cons L' Ls)
show ?thesis
  unfolding rewatch-nat-def
  using falsified Cons
  apply (simp only: wf-twl-cls.simps if-True list.cases C)
  apply (intro conjI)
  proof goal-cases
    case 1
    have distinct (watched (TWL-Clause W UW))
      using wf unfolding C by auto
    moreover have  $L' \notin \text{set } (\text{remove1 } (\neg \text{lit-of } L) (\text{watched } (\text{TWL-Clause } W UW)))$ 
      using 1(2) not-gr0 by (fastforce dest: filter-in-list-prop-verifiedD in-diffD)
    ultimately show ?case
      by (auto simp: distinct-mset-single-add)
  next
    case 2
    have f2:  $[l \leftarrow \text{unwatched } (\text{TWL-Clause } W UW) . l \notin \text{set } (\text{watched } (\text{TWL-Clause } W UW)) \wedge \neg l \notin \text{insert } (\text{lit-of } L) (\text{lits-of-l } (\text{trail } S))] \neq []$ 
      using 2(2) by simp
    then have  $\neg \text{set } UW \subseteq \text{set } W$ 
      using 2 by (auto simp add: filter-empty-conv)
    then show ?case
      using wf C 2(1) by (auto simp: length-remove1)
  next
    case 3

```

```

have W: length W ≤ Suc 0 ↔ length W = 0 ∨ length W = Suc 0
  by linarith
show ?case
  using wf C 3 by (auto simp: length-remove1 W length-list-Suc-0 dest!: subset-singletonD)
next
case 4
have H: ∀ L ∈ set W. ¬ L ∈ lits-of-l (trail S) →
  (∀ L' ∈ set UW. L' ∉ set W → ¬ L' ∈ lits-of-l (trail S))
  using wf by (auto simp: C)
have W: length W ≤ 2 and W-UW: length W < 2 → set UW ⊆ set W
  using wf by (auto simp: C)
have distinct: distinct W
  using wf by (auto simp: C)
show ?case
  using 4
  unfolding C watched-decided-most-recently.simps Ball-def twl-clause.sel
  apply (intro allI impI)
  apply (rename-tac xW xUW)
  apply (case-tac ¬ lit-of L = xW; case-tac xW = xUW; case-tac L' = xW)
    apply (auto simp: uminus-lit-swap)[2]
    apply (force dest: filter-in-list-prop-verifiedD)
    using H distinct apply (fastforce)
    using distinct apply (fastforce)
    using distinct apply (fastforce)
    apply (force dest: filter-in-list-prop-verifiedD)
    using H by (auto simp: uminus-lit-swap)
next
case 5
have H: ∀ x. x ∈ set W → ¬ x ∈ lits-of-l (trail S) → (∀ x. x ∈ set UW → x ∉ set W
  → ¬ x ∈ lits-of-l (trail S))
  using wf by (auto simp: C)
show ?case
  unfolding C watched-decided-most-recently.simps Ball-def
  proof (intro allI impI conjI, goal-cases)
    case (1 xW x)
    show ?case
      proof (cases ¬ lit-of L = xW)
        case True
        then show ?thesis
          by (cases xW = x) (auto simp: uminus-lit-swap)
      next
        case False note LxW = this
        have f9: L' ∈ set [l ← unwatched C. l ∉ set (watched (TWL-Clause W UW))
          ∧ ¬ l ∉ lits-of-l (L # raw-trail S)]
          using 1(2) 5 C by auto
        moreover then have f11: ¬ xW ∈ lits-of-l (trail S)
          using 1(3) LxW by (auto simp: uminus-lit-swap)
        moreover then have xW ∉ set W
          using f9 1(2) H by (auto simp: C)
        ultimately have False
          using 1 by auto
        then show ?thesis
          by fast
      qed
    qed
  qed

```

```

      qed
    qed
  qed

```

```

interpretation twl: abstract-twl watch-nat rewatch-nat raw-learned-clss
  apply unfold-locales
  apply (rule clause-watch-nat; simp add: image-image comp-def)
  apply (rule wf-watch-nat; simp add: image-image comp-def)
  apply (rule clause-rewatch-nat)
  apply (rule clause-rewatch-witness'; simp add: image-image comp-def)
  apply (simp)
done

```

```

interpretation twl2: abstract-twl watch-nat rewatch-nat λ-. []
  apply unfold-locales
  apply (rule clause-watch-nat; simp add: image-image comp-def)
  apply (rule wf-watch-nat; simp add: image-image comp-def)
  apply (rule clause-rewatch-nat)
  apply (rule clause-rewatch-witness'; simp add: image-image comp-def)
  apply (simp)
done

```

```

end

```

25 Invariants for 2 Watched-Literals

```

theory CDCL-Two-Watched-Literals-Invariant
imports CDCL-Two-Watched-Literals DPLL-CDCL-W-Implementation
begin

```

25.1 Interpretation for *conflict-driven-clause-learning_W.cdcl_W*

We define here the 2-WL with the invariant and show the role of the candidates.

```

context abstract-twl
begin

```

25.1.1 Direct Interpretation

```

lemma mset-map-removeAll-cond:
  mset (map (λx. mset (raw-clause x))
    (removeAll-cond (λD. mset (raw-clause D) = mset (raw-clause C)) N))
  = mset (removeAll (mset (raw-clause C)) (map (λx. mset (raw-clause x)) N))
by (induction N) auto

```

```

lemma mset-raw-init-clss-init-state:
  mset (map (λx. mset (raw-clause x)) (raw-init-clss (init-state (map raw-clause N))))
  = mset (map (λx. mset (raw-clause x)) N)
by (metis (no-types, lifting) init-clss-init-state map-eq-conv map-map o-def)

```

```

interpretation rough-cdcl: stateW
  λC. mset (raw-clause C)

```

```

  λL C. TWL-Clause (watched C) (L # unwatched C)

```

$\lambda L C. \text{TWL-Clause } [] (\text{remove1 } L (\text{raw-clause } C))$
 $\lambda C. \text{clauses-of-l } (\text{map raw-clause } C) \text{ op } @$
 $\lambda L C. L \in \text{set } C \text{ op } \# \lambda C. \text{remove1-cond } (\lambda D. \text{mset } (\text{raw-clause } D) = \text{mset } (\text{raw-clause } C))$

$\text{mset } \lambda xs \text{ ys. case-prod append } (\text{fold } (\lambda x (ys, zs). (\text{remove1 } x \text{ ys}, x \# zs)) \text{ xs } (ys, []))$
 $\text{op } \# \text{remove1}$

$\text{raw-clause } \lambda C. \text{TWL-Clause } [] C$
 $\text{trail } \lambda S. \text{hd } (\text{raw-trail } S)$
 $\text{raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting}$
 $\text{cons-trail tl-trail } \lambda C. \text{add-init-cls } (\text{raw-clause } C) \lambda C. \text{add-learned-cls } (\text{raw-clause } C)$
 $\lambda C. \text{remove-cls } (\text{raw-clause } C)$
 $\text{update-backtrack-lvl}$
 $\text{update-conflicting } \lambda N. \text{init-state } (\text{map raw-clause } N) \text{ restart'}$
apply *unfold-locales*
apply (*case-tac raw-trail S*)
apply (*simp-all add: add-init-cls-def add-learned-cls-def clause-rewatch clause-watch*
cons-trail-def remove-cls-def restart'-def tl-trail-def map-tl comp-def
ac-simps mset-map-removeAll-cond mset-raw-init-clss-init-state)

apply (*auto simp: mset-map image-mset-subseteq-mono[OF restart-learned]*)
done

interpretation *rough-cdcl: conflict-driven-clause-learning_w*

$\lambda C. \text{mset } (\text{raw-clause } C)$

$\lambda L C. \text{TWL-Clause } (\text{watched } C) (L \# \text{unwatched } C)$
 $\lambda L C. \text{TWL-Clause } [] (\text{remove1 } L (\text{raw-clause } C))$
 $\lambda C. \text{clauses-of-l } (\text{map raw-clause } C) \text{ op } @$
 $\lambda L C. L \in \text{set } C \text{ op } \# \lambda C. \text{remove1-cond } (\lambda D. \text{mset } (\text{raw-clause } D) = \text{mset } (\text{raw-clause } C))$

$\text{mset } \lambda xs \text{ ys. case-prod append } (\text{fold } (\lambda x (ys, zs). (\text{remove1 } x \text{ ys}, x \# zs)) \text{ xs } (ys, []))$
 $\text{op } \# \text{remove1}$

$\lambda C. \text{raw-clause } C \lambda C. \text{TWL-Clause } [] C$
 $\text{trail } \lambda S. \text{hd } (\text{raw-trail } S)$
 $\text{raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting}$
 $\text{cons-trail tl-trail } \lambda C. \text{add-init-cls } (\text{raw-clause } C) \lambda C. \text{add-learned-cls } (\text{raw-clause } C)$
 $\lambda C. \text{remove-cls } (\text{raw-clause } C)$
 $\text{update-backtrack-lvl}$
 $\text{update-conflicting } \lambda N. \text{init-state } (\text{map raw-clause } N) \text{ restart'}$
by *unfold-locales*

declare *local.rough-cdcl.mset-ccls-ccls-of-cls[simp del]*

25.1.2 Opaque Type with Invariant

declare *rough-cdcl.state-simp[simp del]*

definition *cons-trail-tw_w* :: (*'v*, *nat*, *'v twl-clause*) *marked-lit* \Rightarrow *'v wf-tw_w* \Rightarrow *'v wf-tw_w*
where
 $\text{cons-trail-tw}_w L S \equiv \text{twl-of-rough-state } (\text{cons-trail } L (\text{rough-state-of-tw}_w S))$

lemma *wf-tw_w-state-cons-trail*:

assumes

undef: undefined-lit (raw-trail S) (lit-of L) **and**

wf : $wf\text{-}twl\text{-}state\ S$
shows $wf\text{-}twl\text{-}state\ (cons\text{-}trail\ L\ S)$
using $undef\ wf\ wf\text{-}rewatch[of\ S]$ **unfolding** $wf\text{-}twl\text{-}state\text{-}def\ Ball\text{-}def$
by $(auto\ simp: cons\text{-}trail\text{-}def\ defined\text{-}lit\text{-}map\ comp\text{-}def\ image\text{-}def\ twl.raw\text{-}clauses\text{-}def)$

lemma $rough\text{-}state\text{-}of\text{-}twl\text{-}cons\text{-}trail$:
 $undefined\text{-}lit\ (raw\text{-}trail\text{-}twl\ S)\ (lit\text{-}of\ L) \implies$
 $rough\text{-}state\text{-}of\text{-}twl\ (cons\text{-}trail\text{-}twl\ L\ S) = cons\text{-}trail\ L\ (rough\text{-}state\text{-}of\text{-}twl\ S)$
using $rough\text{-}state\text{-}of\text{-}twl\ twl\text{-}of\text{-}rough\text{-}state\text{-}inverse\ wf\text{-}twl\text{-}state\text{-}cons\text{-}trail$
unfolding $cons\text{-}trail\text{-}twl\text{-}def$ **by** $blast$

abbreviation $add\text{-}init\text{-}cls\text{-}twl$ **where**
 $add\text{-}init\text{-}cls\text{-}twl\ C\ S \equiv twl\text{-}of\text{-}rough\text{-}state\ (add\text{-}init\text{-}cls\ C\ (rough\text{-}state\text{-}of\text{-}twl\ S))$

lemma $wf\text{-}twl\text{-}add\text{-}init\text{-}cls$: $wf\text{-}twl\text{-}state\ S \implies wf\text{-}twl\text{-}state\ (add\text{-}init\text{-}cls\ L\ S)$
unfolding $wf\text{-}twl\text{-}state\text{-}def$ **by** $(auto\ simp: wf\text{-}watch\ add\text{-}init\text{-}cls\text{-}def\ comp\text{-}def\ twl.raw\text{-}clauses\text{-}def\ split: if\text{-}split\text{-}asm)$

lemma $rough\text{-}state\text{-}of\text{-}twl\text{-}add\text{-}init\text{-}cls$:
 $rough\text{-}state\text{-}of\text{-}twl\ (add\text{-}init\text{-}cls\text{-}twl\ L\ S) = add\text{-}init\text{-}cls\ L\ (rough\text{-}state\text{-}of\text{-}twl\ S)$
using $rough\text{-}state\text{-}of\text{-}twl\ twl\text{-}of\text{-}rough\text{-}state\text{-}inverse\ wf\text{-}twl\text{-}add\text{-}init\text{-}cls$ **by** $blast$

abbreviation $add\text{-}learned\text{-}cls\text{-}twl$ **where**
 $add\text{-}learned\text{-}cls\text{-}twl\ C\ S \equiv twl\text{-}of\text{-}rough\text{-}state\ (add\text{-}learned\text{-}cls\ C\ (rough\text{-}state\text{-}of\text{-}twl\ S))$

lemma $wf\text{-}twl\text{-}add\text{-}learned\text{-}cls$: $wf\text{-}twl\text{-}state\ S \implies wf\text{-}twl\text{-}state\ (add\text{-}learned\text{-}cls\ L\ S)$
unfolding $wf\text{-}twl\text{-}state\text{-}def$ **by** $(auto\ simp: wf\text{-}watch\ add\text{-}learned\text{-}cls\text{-}def\ twl.raw\text{-}clauses\text{-}def\ split: if\text{-}split\text{-}asm)$

lemma $rough\text{-}state\text{-}of\text{-}twl\text{-}add\text{-}learned\text{-}cls$:
 $rough\text{-}state\text{-}of\text{-}twl\ (add\text{-}learned\text{-}cls\text{-}twl\ L\ S) = add\text{-}learned\text{-}cls\ L\ (rough\text{-}state\text{-}of\text{-}twl\ S)$
using $rough\text{-}state\text{-}of\text{-}twl\ twl\text{-}of\text{-}rough\text{-}state\text{-}inverse\ wf\text{-}twl\text{-}add\text{-}learned\text{-}cls$ **by** $blast$

abbreviation $remove\text{-}cls\text{-}twl$ **where**
 $remove\text{-}cls\text{-}twl\ C\ S \equiv twl\text{-}of\text{-}rough\text{-}state\ (remove\text{-}cls\ C\ (rough\text{-}state\text{-}of\text{-}twl\ S))$

lemma $set\text{-}removeAll\text{-}condD$: $x \in set\ (removeAll\text{-}cond\ f\ xs) \implies x \in set\ xs$
by $(induction\ xs)\ (auto\ split: if\text{-}split\text{-}asm)$

lemma $wf\text{-}twl\text{-}remove\text{-}cls$: $wf\text{-}twl\text{-}state\ S \implies wf\text{-}twl\text{-}state\ (remove\text{-}cls\ L\ S)$
unfolding $wf\text{-}twl\text{-}state\text{-}def$ **by** $(auto\ simp: wf\text{-}watch\ remove\text{-}cls\text{-}def\ twl.raw\text{-}clauses\text{-}def\ comp\text{-}def\ split: if\text{-}split\text{-}asm\ dest: set\text{-}removeAll\text{-}condD)$

lemma $rough\text{-}state\text{-}of\text{-}twl\text{-}remove\text{-}cls$:
 $rough\text{-}state\text{-}of\text{-}twl\ (remove\text{-}cls\text{-}twl\ L\ S) = remove\text{-}cls\ L\ (rough\text{-}state\text{-}of\text{-}twl\ S)$
using $rough\text{-}state\text{-}of\text{-}twl\ twl\text{-}of\text{-}rough\text{-}state\text{-}inverse\ wf\text{-}twl\text{-}remove\text{-}cls$ **by** $blast$

abbreviation $init\text{-}state\text{-}twl$ **where**
 $init\text{-}state\text{-}twl\ N \equiv twl\text{-}of\text{-}rough\text{-}state\ (init\text{-}state\ N)$

lemma $wf\text{-}twl\text{-}state\text{-}wf\text{-}twl\text{-}state\text{-}fold\text{-}add\text{-}init\text{-}cls$:
assumes $wf\text{-}twl\text{-}state\ S$
shows $wf\text{-}twl\text{-}state\ (fold\ add\text{-}init\text{-}cls\ N\ S)$
using $assms$ **apply** $(induction\ N\ arbitrary: S)$
apply $(auto\ simp: wf\text{-}twl\text{-}state\text{-}def)\ []$

by (simp add: wf-twl-add-init-cl)

lemma *wf-twl-state-epsilon-state*[simp]:
wf-twl-state (TWL-State [] [] 0 None)
 by (auto simp: wf-twl-state-def twl.raw-clauses-def)

lemma *wf-twl-init-state*: *wf-twl-state* (init-state N)
 unfolding init-state-def by (auto intro!: wf-twl-state-wf-twl-state-fold-add-init-cl)

lemma *rough-state-of-twl-init-state*:
rough-state-of-twl (init-state-twl N) = init-state N
 by (simp add: twl-of-rough-state-inverse wf-twl-init-state)

abbreviation *tl-trail-twl* **where**
tl-trail-twl S \equiv *twl-of-rough-state* (tl-trail (rough-state-of-twl S))

lemma *wf-twl-state-tl-trail*: *wf-twl-state* S \implies *wf-twl-state* (tl-trail S)
 by (auto simp add: twl-of-rough-state-inverse wf-twl-init-state wf-twl-cl-wf-twl-cl-tl
 tl-trail-def wf-twl-state-def distinct-tl map-tl comp-def twl.raw-clauses-def)

lemma *rough-state-of-twl-tl-trail*:
rough-state-of-twl (tl-trail-twl S) = tl-trail (rough-state-of-twl S)
 using rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-tl-trail by blast

abbreviation *update-backtrack-lvl-twl* **where**
update-backtrack-lvl-twl k S \equiv *twl-of-rough-state* (update-backtrack-lvl k (rough-state-of-twl S))

lemma *wf-twl-state-update-backtrack-lvl*:
wf-twl-state S \implies *wf-twl-state* (update-backtrack-lvl k S)
 unfolding wf-twl-state-def by (auto simp: comp-def twl.raw-clauses-def)

lemma *rough-state-of-twl-update-backtrack-lvl*:
rough-state-of-twl (update-backtrack-lvl-twl k S) = update-backtrack-lvl k
 (rough-state-of-twl S)
 using rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-update-backtrack-lvl by fast

abbreviation *update-conflicting-twl* **where**
update-conflicting-twl k S \equiv *twl-of-rough-state* (update-conflicting k (rough-state-of-twl S))

lemma *wf-twl-state-update-conflicting*:
wf-twl-state S \implies *wf-twl-state* (update-conflicting k S)
 unfolding wf-twl-state-def by (auto simp: twl.raw-clauses-def comp-def)

lemma *rough-state-of-twl-update-conflicting*:
rough-state-of-twl (update-conflicting-twl k S) = update-conflicting k
 (rough-state-of-twl S)
 using rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-update-conflicting by fast

abbreviation *raw-clauses-twl* **where**
raw-clauses-twl S \equiv *twl.raw-clauses* (rough-state-of-twl S)

abbreviation *restart-twl* **where**
restart-twl S \equiv *twl-of-rough-state* (restart' (rough-state-of-twl S))

lemma *mset-union-mset-setD*:
mset A $\subseteq\#$ *mset* B \implies *set* A \subseteq *set* B

by auto

lemma *wf-wf-restart'*: *wf-twl-state* *S* \implies *wf-twl-state* (*restart'* *S*)
unfolding *restart'-def* *wf-twl-state-def* **apply** *standard*
apply *clarify*
apply (*rename-tac* *x*)
apply (*subgoal-tac* *wf-twl-cls* (*raw-trail* *S*) *x*)
apply (*case-tac* *x*)
using *restart-learned* **by** (*auto simp: twl.raw-clauses-def comp-def dest: mset-union-mset-setD*)

lemma *rough-state-of-twl-restart-twl*:
rough-state-of-twl (*restart-twl* *S*) = *restart'* (*rough-state-of-twl* *S*)
by (*simp add: twl-of-rough-state-inverse wf-wf-restart'*)

lemma *undefined-lit-trail-twl-raw-trail*[*iff*]:
undefined-lit (*trail-twl* *S*) *L* \longleftrightarrow *undefined-lit* (*raw-trail-twl* *S*) *L*
by (*auto simp: defined-lit-map image-image*)

sublocale *wf-twl: conflict-driven-clause-learning_w*
 $\lambda C. \text{mset } (\text{raw-clause } C)$

$\lambda L C. \text{TWL-Clause } (\text{watched } C) (L \# \text{unwatched } C)$
 $\lambda L C. \text{TWL-Clause } [] (\text{remove1 } L (\text{raw-clause } C))$
 $\lambda C. \text{clauses-of-l } (\text{map raw-clause } C) \text{ op } @$
 $\lambda L C. L \in \text{set } C \text{ op } \# \lambda C. \text{remove1-cond } (\lambda D. \text{mset } (\text{raw-clause } D) = \text{mset } (\text{raw-clause } C))$

$\text{mset } \lambda xs \text{ ys. case-prod append (fold } (\lambda x (ys, zs). (\text{remove1 } x \text{ ys}, x \# zs)) \text{ xs } (ys, []))$
 $\text{op } \# \text{remove1}$

$\lambda C. \text{raw-clause } C \lambda C. \text{TWL-Clause } [] C$
trail-twl $\lambda S. \text{hd } (\text{raw-trail-twl } S)$
raw-init-clss-twl
raw-learned-clss-twl
backtrack-lvl-twl
raw-conflicting-twl
cons-trail-twl
tl-trail-twl
 $\lambda C. \text{add-init-cls-twl } (\text{raw-clause } C)$
 $\lambda C. \text{add-learned-cls-twl } (\text{raw-clause } C)$
 $\lambda C. \text{remove-cls-twl } (\text{raw-clause } C)$
update-backtrack-lvl-twl
update-conflicting-twl
 $\lambda N. \text{init-state-twl } (\text{map raw-clause } N)$
restart-twl

apply *unfold-locales*

using *rough-cdcl.hd-raw-trail* **apply** *blast*
apply (*simp-all add: rough-state-of-twl-cons-trail rough-state-of-twl-tl-trail*
rough-state-of-twl-add-init-cls rough-state-of-twl-add-learned-cls
rough-state-of-twl-remove-cls rough-state-of-twl-update-backtrack-lvl
rough-state-of-twl-update-conflicting)[7]
using *rough-cdcl.init-clss-cons-trail rough-cdcl.init-clss-tl-trail*
rough-cdcl.init-clss-add-init-cls rough-cdcl.init-clss-remove-cls
rough-cdcl.init-clss-add-learned-cls
rough-cdcl.init-clss-update-backtrack-lvl
rough-cdcl.init-clss-update-conflicting

```

apply (auto simp add: rough-state-of-twl-cons-trail rough-state-of-twl-tl-trail
  rough-state-of-twl-add-init-cls rough-state-of-twl-add-learned-cls
  rough-state-of-twl-remove-cls rough-state-of-twl-update-backtrack-lvl
  rough-state-of-twl-update-conflicting comp-def)[7]
using rough-cdcl.learned-clss-cons-trail rough-cdcl.learned-clss-tl-trail
  rough-cdcl.learned-clss-add-init-cls rough-cdcl.learned-clss-remove-cls
  rough-cdcl.learned-clss-add-learned-cls
  rough-cdcl.learned-clss-update-backtrack-lvl
  rough-cdcl.learned-clss-update-conflicting
apply (auto simp add: rough-state-of-twl-cons-trail rough-state-of-twl-tl-trail
  rough-state-of-twl-add-init-cls rough-state-of-twl-add-learned-cls
  rough-state-of-twl-remove-cls rough-state-of-twl-update-backtrack-lvl
  rough-state-of-twl-update-conflicting comp-def)[7]
apply (auto simp add: rough-state-of-twl-cons-trail rough-state-of-twl-tl-trail
  rough-state-of-twl-add-init-cls rough-state-of-twl-add-learned-cls
  rough-state-of-twl-remove-cls rough-state-of-twl-update-backtrack-lvl
  rough-state-of-twl-update-conflicting comp-def)[14]
using init-clss-init-state apply (auto simp: rough-state-of-twl-init-state)[5]
using rough-cdcl.init-clss-restart-state rough-cdcl.learned-clss-restart-state
apply (auto simp: rough-state-of-twl-restart-twl)[5]
done

```

```

declare local.rough-cdcl.mset-ccls-ccls-of-cls[simp del]
abbreviation state-eq-twl (infix  $\sim$  TWL 51) where
  state-eq-twl  $S S' \equiv$  rough-cdcl.state-eq (rough-state-of-twl  $S$ ) (rough-state-of-twl  $S'$ )
notation wf-twl.state-eq (infix  $\sim$  51)
declare wf-twl.state-simp[simp del]

```

To avoid ambiguities:

```

no-notation state-eq-twl (infix  $\sim$  51)

```

```

inductive propagate-twl :: 'v wf-twl  $\Rightarrow$  'v wf-twl  $\Rightarrow$  bool where
  propagate-twl-rule: (L, C)  $\in$  candidates-propagate-twl  $S \Rightarrow$ 
     $S' \sim$  cons-trail-twl (Propagated L C)  $S \Rightarrow$ 
    raw-conflicting-twl  $S =$  None  $\Rightarrow$ 
    propagate-twl  $S S'$ 

```

```

inductive-cases propagate-twlE: propagate-twl  $S T$ 

```

```

lemma distinct-filter-eq-if:
  distinct C  $\Rightarrow$  length (filter (op = L) C) = (if L  $\in$  set C then 1 else 0)
by (induction C) auto

```

```

lemma distinct-mset-remove1-All:
  distinct-mset C  $\Rightarrow$  remove1-mset L C = removeAll-mset L C
by (auto simp: multiset-eq-iff distinct-mset-count-less-1)

```

```

lemma propagate-twl-iff-propagate:
  assumes inv: wf-twl.cdclw-all-struct-inv S
  shows wf-twl.propagate  $S T \longleftrightarrow$  propagate-twl  $S T$  (is ?P  $\longleftrightarrow$  ?T)

```

proof

```

assume ?P

```

```

then obtain L E where

```

```

  raw-conflicting-twl S = None and

```

```

  CL-Clauses: E  $\in$  set (wf-twl.raw-clauses S) and

```



```

  LE:  $L \in \# \text{ mset } (\text{raw-clause } E)$  and
  tr-CNot:  $\text{trail-twl } S \models_{\text{as}} \text{CNot } (\text{remove1-mset } L (\text{mset } (\text{raw-clause } E)))$  and
  undef-lot[simp]:  $\text{undefined-lit } (\text{trail-twl } S) \text{ } L$  and
   $T \sim \text{cons-trail-twl } (\text{Propagated } L \ E) \ S$ 
  by (blast elim: wf-twl.propagateE)
have distinct (raw-clause E)
  using inv CL-Clauses unfolding wf-twl.cdclW-all-struct-inv-def distinct-mset-set-def
  wf-twl.distinct-cdclW-state-def wf-twl.raw-clauses-def by auto
then have X:  $\text{remove1-mset } L (\text{mset } (\text{raw-clause } E)) = \text{mset-set } (\text{set } (\text{raw-clause } E) - \{L\})$ 
  by (auto simp: multiset-eq-iff raw-clause-def count-mset distinct-filter-eq-if)
have (L, E)  $\in \text{candidates-propagate-twl } S$ 
  apply (rule wf-candidates-propagate-complete)
    using rough-state-of-twl apply auto[]
    using CL-Clauses unfolding wf-twl.raw-clauses-def twl.raw-clauses-def
    apply auto[]
    using LE apply simp
    using tr-CNot X apply simp
    using undef-lot apply blast
  done
show ?T
  apply (rule propagate-twl-rule)
    apply (rule  $\langle (L, E) \in \text{candidates-propagate-twl } S \rangle$ )
    using  $\langle T \sim \text{cons-trail-twl } (\text{Propagated } L \ E) \ S \rangle$ 
    apply (auto simp:  $\langle \text{raw-conflicting-twl } S = \text{None} \rangle$  wf-twl.state-eq-def)
  done
next
assume ?T
then obtain L C where
  LC:  $(L, C) \in \text{candidates-propagate-twl } S$  and
  T:  $T \sim \text{cons-trail-twl } (\text{Propagated } L \ C) \ S$  and
  confl:  $\text{raw-conflicting-twl } S = \text{None}$ 
  by (auto elim: propagate-twlE)
have
  C'S:  $C \in \text{set } (\text{raw-clauses-twl } S)$  and
  L:  $\text{set } (\text{watched } C) - \text{uminus ' lits-of-l } (\text{trail-twl } S) = \{L\}$  and
  undef:  $\text{undefined-lit } (\text{trail-twl } S) \text{ } L$ 
  using LC unfolding candidates-propagate-def wf-twl.raw-clauses-def by auto
have dist: distinct (raw-clause C)
  using inv C'S unfolding wf-twl.cdclW-all-struct-inv-def wf-twl.distinct-cdclW-state-def
  distinct-mset-set-def twl.raw-clauses-def by fastforce
then have C-L-L:  $\text{mset-set } (\text{set } (\text{raw-clause } C) - \{L\}) = \text{mset } (\text{raw-clause } C) - \{\#L\# \}$ 
  by (metis distinct-mset-distinct distinct-mset-minus distinct-mset-set-mset-ident mset-remove1
    set-mset-mset set-remove1-eq)

show ?P
apply (rule wf-twl.propagate-rule[of S C L])
  using confl apply auto[]
  using C'S unfolding twl.raw-clauses-def apply (simp add: wf-twl.raw-clauses-def)
  using L unfolding candidates-propagate-def apply (auto simp: raw-clause-def)[]
  using wf-candidates-propagate-sound[OF - LC] rough-state-of-twl dist
  apply (simp add: distinct-mset-remove1-All true-annots-true-cl)
  using undef apply simp
using T undef by (smt wf-twl.backtrack-lvl-cons-trail confl wf-twl.init-clss-cons-trail
  wf-twl.learned-clss-cons-trail marked-lit.sel(2) wf-twl.raw-conflicting-cons-trail
  wf-twl.state-eq-def wf-twl.trail-cons-trail wf-twl.mset-of-mlit.simps(1))

```

wf-twl.mset-cls-cls-of-ccls)
qed

no-notation *twl.state-eq-twl* (**infix** \sim *TWL 51*)

inductive *conflict-twl* **where**

conflict-twl-rule:

$C \in \text{candidates-conflict-twl } S \implies$
 $S' \sim \text{update-conflicting-twl } (\text{Some } (\text{raw-clause } C)) \ S \implies$
 $\text{raw-conflicting-twl } S = \text{None} \implies$
 $\text{conflict-twl } S \ S'$

inductive-cases *conflict-twlE*: *conflict-twl* $S \ T$

lemma *conflict-twl-iff-conflict*:

shows $\text{wf-twl.conflict } S \ T \longleftrightarrow \text{conflict-twl } S \ T$ (**is** $?C \longleftrightarrow ?T$)

proof

assume $?C$

then obtain D **where**

$S: \text{raw-conflicting-twl } S = \text{None}$ **and**
 $D: D \in \text{set } (\text{wf-twl.raw-clauses } S)$ **and**
 $MD: \text{trail-twl } S \models_{\text{as}} \text{CNot } (\text{mset } (\text{raw-clause } D))$ **and**
 $T: T \sim \text{update-conflicting-twl } (\text{Some } (\text{raw-clause } D)) \ S$
by (*elim wf-twl.conflictE*)

have $D \in \text{candidates-conflict-twl } S$

apply (*rule wf-candidates-conflict-complete*)

apply *simp*

using D **apply** (*auto simp: wf-twl.raw-clauses-def twl.raw-clauses-def*)[]

using $MD \ S$ **by** *auto*

moreover have $T \sim \text{twl-of-rough-state } (\text{update-conflicting } (\text{Some } (\text{raw-clause } D)))$
(*rough-state-of-twl* S)

using T **unfolding** *rough-cdcl.state-eq-def wf-twl.state-eq-def* **by** *auto*

ultimately show $?T$

using S **by** (*auto intro: conflict-twlE*)

next

assume $?T$

then obtain C **where**

$C: C \in \text{candidates-conflict-twl } S$ **and**
 $T: T \sim \text{update-conflicting-twl } (\text{Some } (\text{raw-clause } C)) \ S$ **and**
 $\text{confl: raw-conflicting-twl } S = \text{None}$
by (*auto elim: conflict-twlE*)

have

$C \in \text{set } (\text{wf-twl.raw-clauses } S)$

using C **unfolding** *candidates-conflict-def wf-twl.raw-clauses-def twl.raw-clauses-def* **by** *auto*

moreover have $\text{trail-twl } S \models_{\text{as}} \text{CNot } (\text{mset } (\text{raw-clause } C))$

using *wf-candidates-conflict-sound[OF - C]* **by** *auto*

ultimately show $?C$ **apply** –

apply (*rule wf-twl.conflict.conflict-rule[of - C]*)

using *confl T unfolding rough-cdcl.state-eq-def* **by** (*auto simp del: map-map*)

qed

inductive *cdcl_W-twl* :: ' v *wf-twl* \Rightarrow ' v *wf-twl* \Rightarrow *bool* **for** $S ::$ ' v *wf-twl* **where**

propagate: *propagate-twl* $S \ S' \implies \text{cdcl}_W\text{-twl } S \ S' \mid$

conflict: *conflict-twl* $S \ S' \implies \text{cdcl}_W\text{-twl } S \ S' \mid$

other: $wf\text{-}twl.cdcl_W\text{-}o\ S\ S' \implies cdcl_W\text{-}twl\ S\ S'$
rf: $wf\text{-}twl.cdcl_W\text{-}rf\ S\ S' \implies cdcl_W\text{-}twl\ S\ S'$

lemma *cdcl_W-twl-iff-cdcl_W*:

assumes *wf-twl.cdcl_W-all-struct-inv S*

shows $cdcl_W\text{-}twl\ S\ T \longleftrightarrow wf\text{-}twl.cdcl_W\ S\ T$

by (*simp add: assms wf-twl.cdcl_W.simps cdcl_W-twl.simps conflict-twl-iff-conflict propagate-twl-iff-propagate del: map-map*)

lemma *rtrancpl-cdcl_W-twl-all-struct-inv-inv*:

assumes *cdcl_W-twl** S T and wf-twl.cdcl_W-all-struct-inv S*

shows *wf-twl.cdcl_W-all-struct-inv T*

using *assms by (induction rule: rtrancpl-induct)*

(*simp-all add: cdcl_W-twl-iff-cdcl_W wf-twl.cdcl_W-all-struct-inv-inv del: map-map*)

lemma *rtrancpl-cdcl_W-twl-iff-rtrancpl-cdcl_W*:

assumes *wf-twl.cdcl_W-all-struct-inv S*

shows $cdcl_W\text{-}twl^{**}\ S\ T \longleftrightarrow wf\text{-}twl.cdcl_W^{**}\ S\ T$ (**is** $?T \longleftrightarrow ?W$)

proof

assume *?W*

then show *?T*

proof (*induction rule: rtrancpl-induct*)

case *base*

then show *?case by simp*

next

case (*step T U*) **note** *st = this(1) and cdcl = this(2) and IH = this(3)*

have *cdcl_W-twl T U*

using *assms st cdcl wf-twl.rtrancpl-cdcl_W-all-struct-inv-inv cdcl_W-twl-iff-cdcl_W*

by *blast*

then show *?case using IH by auto*

qed

next

assume *?T*

then show *?W*

proof (*induction rule: rtrancpl-induct*)

case *base*

then show *?case by simp*

next

case (*step T U*) **note** *st = this(1) and cdcl = this(2) and IH = this(3)*

have *wf-twl.cdcl_W T U*

using *assms st cdcl rtrancpl-cdcl_W-twl-all-struct-inv-inv cdcl_W-twl-iff-cdcl_W*

by *blast*

then show *?case using IH by auto*

qed

qed

end

end

26 Implementation for 2 Watched-Literals

theory *CDCL-Two-Watched-Literals-Implementation*

imports *CDCL-Two-Watched-Literals-Invariant*

begin

The general idea is the following:

1. Build a “propagate” queue and a conflict clause.
2. While updating the data-structure: if you find a conflicting clause, update the conflict clause. Otherwise prepend the propagated clause.
3. While updating, when looking for conflicts and propagation, work with respect to the trail of the state and the propagate queue (and not only the trail of the state).
4. As long as the propagate queue is not empty, dequeue the first element, push it on the trail (with the *conflict-driven-clause-learning_W.propagate* rule), propagate, and update the data-structure.
5. if a conflict has been found such that it is entailed by the trail only (i.e. without the propagate queue), then apply the *conflict-driven-clause-learning_W.conflict* rule.

It is important to remember that a conflicting clause with respect to the trail and the queue might not be the earliest conflicting clause, meaning that the proof of non-redundancy should not work anymore.

However, once a conflict has been found, we can stop adding literals to the queue: we just have to finish updating the data-structure (both to keep the invariant and find a potentially better conflict). A conflict is better when it involves less literals, i.e. less propagations are needed before finding the conflict.

datatype *'v candidate* =
Prop-Or-Conf
 (*prop-queue*: (*'v literal* × *'v twl-clause*) *list*)
 (*conflict*: *'v twl-clause option*)

Morally instead of (*'v literal* × *'v twl-clause*) *list*, we should use (*'v, nat, 'v twl-clause*) *marked-lits* with only *Propagated*. However, we do not want to define the function for *Marked* too. The following function makes the conversion from the pair to the trail:

abbreviation *get-trail-of-cand where*
get-trail-of-cand C ≡ *map (case-prod Propagated) (prop-queue C)*

datatype *'v twl-state-cands* =
TWL-State-Cand (twl-state: 'v twl-state)
 (*cand*: *'v candidate*)

fun *find-earliest-conflict* :: (*'v, nat, 'v twl-clause*) *marked-lits* ⇒
'v twl-clause option ⇒ *'v twl-clause option* ⇒ *'v twl-clause option where*
find-earliest-conflict - None C = C |
find-earliest-conflict - C None = C |
find-earliest-conflict [] C - = C |
find-earliest-conflict (L # M) (Some C) (Some D) =
 (*case (M ⊨_a mset (raw-clause C), ¬M ⊨_a mset (raw-clause D)) of*
 (*True, True*) ⇒ *find-earliest-conflict M (Some C) (Some D)*
 | (*False, True*) ⇒ *Some D*
 | (*True, False*) ⇒ *Some C*
 | - ⇒ *Some C*)

lemma *find-earliest-conflict-cases:*
find-earliest-conflict M (Some C) (Some D) = Some C ∨

find-earliest-conflict M (*Some* C) (*Some* D) = *Some* D
by (*induction* M) (*auto split*: *bool.splits*)

While updating the clauses, there are several cases:

- L is not watched and there is nothing to do;
- there is a literal to be watched: there are swapped;
- there is no literal to be watched, the other literal is not assigned: the clause is a propagate or a conflict candidate;
- there is no literal to be watched, the other literal is $-L$: the clause is a tautology and nothing special is done;
- there is no literal to be watched, but the other literal is true: there is nothing to do;
- there is no literal to be watched, but the other literal is false: the clause is a conflict candidate.

The function returns a couple composed of a list of clauses and a candidate.

fun

rewatch-nat-cand-single-clause ::
'v literal \Rightarrow (*'v*, *nat*, *'v twl-clause*) *marked-lits* \Rightarrow *'v twl-clause* \Rightarrow
'v twl-clause list \times *'v candidate* \Rightarrow
'v twl-clause list \times *'v candidate*

where

rewatch-nat-cand-single-clause L M C (Cs , Ks) =
 (*if* $-L \in \text{set } (\text{watched } C)$ *then*
 case filter ($\lambda L'. L' \notin \text{set } (\text{watched } C) \wedge$
 $- L' \notin \text{insert } L (\text{lits-of-l } (M @ \text{get-trail-of-cand } Ks)))$ (*unwatched* C) *of*
 $\square \Rightarrow$
 (*case remove1* ($-L$) (*watched* C) *of* ($*$ *contains at most a single element* $*$)
 $\square \Rightarrow (C \# Cs, \text{Prop-Or-Conf } (\text{prop-queue } Ks)$
 (*find-earliest-conflict* (*get-trail-of-cand* $Ks @ M$) (*Some* C) (*conflict* Ks)))
 $| L' \# - \Rightarrow$
 if *undefined-lit* (*get-trail-of-cand* $Ks @ M$) $L' \wedge \text{atm-of } L \neq \text{atm-of } L'$
 then ($C \# Cs, \text{Prop-Or-Conf } ((L', C) \# \text{prop-queue } Ks) (\text{conflict } Ks))$
 else
 (*if* $-L' \in \text{lits-of-l } (\text{get-trail-of-cand } Ks @ M)$
 then ($C \# Cs, \text{Prop-Or-Conf } (\text{prop-queue } Ks)$
 (*find-earliest-conflict* (*get-trail-of-cand* $Ks @ M$) (*Some* C) (*conflict* Ks)))
 else ($C \# Cs, Ks$)))
 $| L' \# - \Rightarrow$
 (*TWL-Clause* ($L' \# \text{remove1 } (-L) (\text{watched } C)$) ($-L \# \text{remove1 } L' (\text{unwatched } C)$) $\# Cs, Ks$)
 else
 ($C \# Cs, Ks$))

declare *rewatch-nat-cand-single-clause.simps*[*simp del*]

fun *raw-cons-trail* **where**

raw-cons-trail L (*TWL-State* M N U k C) = *TWL-State* ($L \# M$) N U k C

lemma *length-raw-trail-raw-cons-trails*[*simp*]:

length (*raw-trail* (*raw-cons-trail* (*Propagated* L C') S)) = *Suc* (*length* (*raw-trail* S))

```

by (cases S) auto

lemma twl-raw-clauses-raw-cons-trail[simp]:
  twl.raw-clauses (raw-cons-trail (Propagated L C') S) = twl.raw-clauses S
by (cases S) (auto simp: twl.raw-clauses-def)

fun raw-cons-trail-pq where
  raw-cons-trail-pq L (TWL-State-Cand S Q) = TWL-State-Cand (raw-cons-trail L S) Q

fun update-conflicting-pq where
  update-conflicting-pq L (TWL-State-Cand S Q) = TWL-State-Cand (update-conflicting L S) Q

lemma raw-trail-raw-cons-trail[simp]:
  raw-trail (raw-cons-trail (Propagated L C') S) = Propagated L C' # raw-trail S
by (cases S) auto

fun  $\mu$ TWL where
   $\mu$ TWL (TWL-State-Cand U (Prop-Or-Conf Q D)) =
    card (set-mset (twl.clauses U)) -
    card {C  $\in$  set-mset (twl.clauses U).  $\neg$ raw-trail U @ get-trail-of-cand (Prop-Or-Conf Q D)  $\models_a$  C}

function do-propagate-or-conflict-step :: 'a twl-state-cands  $\Rightarrow$  'a twl-state-cands where
  do-propagate-or-conflict-step (TWL-State-Cand S (Prop-Or-Conf [] (Some D))) =
    (if trail S  $\models_{as}$  CNot (mset (raw-clause D))
     then (update-conflicting-pq (Some (raw-clause D)) (TWL-State-Cand S (Prop-Or-Conf [] None)))
     else TWL-State-Cand S (Prop-Or-Conf [] (Some D))) |
  do-propagate-or-conflict-step (TWL-State-Cand S (Prop-Or-Conf [] None)) =
    TWL-State-Cand S (Prop-Or-Conf [] None) |
  do-propagate-or-conflict-step (TWL-State-Cand S (Prop-Or-Conf (l @ [(L, C')]) (Some D))) =
    (if trail S  $\models_{as}$  CNot (mset (raw-clause D))
     then update-conflicting-pq (Some (raw-clause D)) (TWL-State-Cand S (Prop-Or-Conf l None))
     else do-propagate-or-conflict-step
        (raw-cons-trail-pq (Propagated L C') (TWL-State-Cand S (Prop-Or-Conf l (Some D))))) |
  do-propagate-or-conflict-step (TWL-State-Cand S (Prop-Or-Conf (l @ [(L, C')]) None)) =
    do-propagate-or-conflict-step
      (raw-cons-trail-pq (Propagated L C') (TWL-State-Cand S (Prop-Or-Conf l None)))
  apply (rename-tac P x)
  apply (case-tac x, case-tac cand x; case-tac conflict (cand x);
        case-tac prop-queue (cand x) rule: rev-cases; simp)
  by auto

termination
  apply (relation {(T, S).
    ([ $\mu$ TWL T, length (prop-queue (cand T))], [ $\mu$ TWL S, length (prop-queue (cand S))])
     $\in$  lern less-than 2})
proof (goal-cases)
  case 1
  show ?case by (auto simp: wf-lern intro!: wf-if-measure-f)
next
  case (2 S l L C')
  show ?case
    by (auto simp add: comp-def lern-n true-annot-def simp del: lern.sims(2))
next
  case (3 S l L C')
  show ?case
    by (auto simp add: comp-def lern-n true-annot-def simp del: lern.sims(2))

```

qed

lemma *CNot-mset-replicate[simp]:*
 $CNot (mset (replicate n L)) = (if\ n = 0\ then\ \{\}\ else\ \{\{\#-L\#\})$
 by (*induction n*) *auto*

end

theory *Prop-Superposition*
imports *Partial-Clausal-Logic ../lib/Herbrand-Interpretation*
begin

27 Superposition

no-notation *Herbrand-Interpretation.true-cls* (**infix** \models 50)

notation *Herbrand-Interpretation.true-cls* (**infix** \models_h 50)

no-notation *Herbrand-Interpretation.true-clss* (**infix** \models_s 50)

notation *Herbrand-Interpretation.true-clss* (**infix** \models_{hs} 50)

lemma *herbrand-interp-iff-partial-interp-cls:*
 $S \models_h C \longleftrightarrow \{Pos\ P|P. P \in S\} \cup \{Neg\ P|P. P \notin S\} \models C$
 unfolding *Herbrand-Interpretation.true-cls-def Partial-Clausal-Logic.true-cls-def*
 by *auto*

lemma *herbrand-consistent-interp:*
 $consistent_interp (\{Pos\ P|P. P \in S\} \cup \{Neg\ P|P. P \notin S\})$
 unfolding *consistent-interp-def* **by** *auto*

lemma *herbrand-total-over-set:*
 $total_over_set (\{Pos\ P|P. P \in S\} \cup \{Neg\ P|P. P \notin S\})\ T$
 unfolding *total-over-set-def* **by** *auto*

lemma *herbrand-total-over-m:*
 $total_over_m (\{Pos\ P|P. P \in S\} \cup \{Neg\ P|P. P \notin S\})\ T$
 unfolding *total-over-m-def* **by** (*auto simp add: herbrand-total-over-set*)

lemma *herbrand-interp-iff-partial-interp-clss:*
 $S \models_{hs} C \longleftrightarrow \{Pos\ P|P. P \in S\} \cup \{Neg\ P|P. P \notin S\} \models_s C$
 unfolding *true-clss-def Ball-def herbrand-interp-iff-partial-interp-cls*
 Partial-Clausal-Logic.true-clss-def **by** *auto*

definition *clss-lt* :: $'a::wellorder\ clauses \Rightarrow 'a\ clause \Rightarrow 'a\ clauses$ **where**
 $clss_lt\ N\ C = \{D \in N. D \# \subset \# C\}$

notation (*latex output*)
clss-lt ($-\hat{<}^{bsup}-\hat{<}^{esup}$)

locale *selection* =
 fixes $S :: 'a\ clause \Rightarrow 'a\ clause$
 assumes
 $S_selects_subsesteq: \bigwedge C. S\ C \leq \# C$ **and**
 $S_selects_neg_lits: \bigwedge C\ L. L \in \# S\ C \Longrightarrow is_neg\ L$

locale *ground-resolution-with-selection* =
 selection S **for** $S :: ('a :: wellorder)\ clause \Rightarrow 'a\ clause$

begin

context

fixes $N :: 'a \text{ clause set}$

begin

We do not create an equivalent of δ , but we directly defined N_C by inlining the definition.

function

$\text{production} :: 'a \text{ clause} \Rightarrow 'a \text{ interp}$

where

$\text{production } C =$

$\{A. C \in N \wedge C \neq \{\#\} \wedge \text{Max } (\text{set-mset } C) = \text{Pos } A \wedge \text{count } C (\text{Pos } A) \leq 1$
 $\wedge \neg (\bigcup D \in \{D. D \# \subset \# C\}. \text{production } D) \models_h C \wedge S C = \{\#\}\}$

by auto

termination by $(\text{relation } \{(D, C). D \# \subset \# C\}) (\text{auto simp: wf-less-multiset})$

declare $\text{production.simps}[simp \text{ del}]$

definition $\text{interp} :: 'a \text{ clause} \Rightarrow 'a \text{ interp}$ **where**

$\text{interp } C = (\bigcup D \in \{D. D \# \subset \# C\}. \text{production } D)$

lemma production-unfold :

$\text{production } C = \{A. C \in N \wedge C \neq \{\#\} \wedge \text{Max } (\text{set-mset } C) = \text{Pos } A \wedge \text{count } C (\text{Pos } A) \leq 1 \wedge \neg$
 $\text{interp } C \models_h C \wedge S C = \{\#\}\}$

unfolding interp-def **by** $(\text{rule } \text{production.simps})$

abbreviation $\text{productive } A \equiv (\text{production } A \neq \{\})$

abbreviation $\text{produces} :: 'a \text{ clause} \Rightarrow 'a \Rightarrow \text{bool}$ **where**

$\text{produces } C A \equiv \text{production } C = \{A\}$

lemma producesD :

$\text{produces } C A \Longrightarrow C \in N \wedge C \neq \{\#\} \wedge \text{Pos } A = \text{Max } (\text{set-mset } C) \wedge \text{count } C (\text{Pos } A) \leq 1 \wedge$
 $\neg \text{interp } C \models_h C \wedge S C = \{\#\}$

unfolding production-unfold **by auto**

lemma $\text{produces } C A \Longrightarrow \text{Pos } A \in \# C$

by $(\text{simp add: Max-in-lits producesD})$

lemma $\text{interp'-def-in-set}$:

$\text{interp } C = (\bigcup D \in \{D \in N. D \# \subset \# C\}. \text{production } D)$

unfolding interp-def **apply auto**

unfolding production-unfold **apply auto**

done

lemma $\text{production-iff-produces}$:

$\text{produces } D A \longleftrightarrow A \in \text{production } D$

unfolding production-unfold **by auto**

definition $\text{Interp} :: 'a \text{ clause} \Rightarrow 'a \text{ interp}$ **where**

$\text{Interp } C = \text{interp } C \cup \text{production } C$

lemma

assumes $\text{produces } C P$

shows $\text{Interp } C \models_h C$

unfolding *Interp-def* *assms* **using** *producesD*[*OF* *assms*]
by (*metis* *Max-in-lits* *Un-insert-right* *insertI1* *pos-literal-in-imp-true-cls*)

definition *INTERP* :: 'a *interp* **where**
INTERP = ($\bigcup D \in N.$ *production* *D*)

lemma *interp-subseteq-Interp*[*simp*]: *interp* *C* \subseteq *Interp* *C*
unfolding *Interp-def* **by** *simp*

lemma *Interp-as-UNION*: *Interp* *C* = ($\bigcup D \in \{D. D \# \subseteq \# C\}.$ *production* *D*)
unfolding *Interp-def* *interp-def* *le-multiset-def* **by** *fast*

lemma *productive-not-empty*: *productive* *C* $\implies C \neq \{\#\}$
unfolding *production-unfold* **by** *auto*

lemma *productive-imp-produces-Max-literal*: *productive* *C* \implies *produces* *C* (*atm-of* (*Max* (*set-mset* *C*)))
unfolding *production-unfold* **by** (*auto* *simp* *del*: *atm-of-Max-lit*)

lemma *productive-imp-produces-Max-atom*: *productive* *C* \implies *produces* *C* (*Max* (*atms-of* *C*))
unfolding *atms-of-def* *Max-atm-of-set-mset-commute*[*OF* *productive-not-empty*]
by (*rule* *productive-imp-produces-Max-literal*)

lemma *produces-imp-Max-literal*: *produces* *C* *A* $\implies A = \text{atm-of } (\text{Max } (\text{set-mset } C))$
by (*metis* *Max-singleton* *insert-not-empty* *productive-imp-produces-Max-literal*)

lemma *produces-imp-Max-atom*: *produces* *C* *A* $\implies A = \text{Max } (\text{atms-of } C)$
by (*metis* *Max-singleton* *insert-not-empty* *productive-imp-produces-Max-atom*)

lemma *produces-imp-Pos-in-lits*: *produces* *C* *A* $\implies \text{Pos } A \in \# C$
by (*auto* *intro*: *Max-in-lits* *dest*!: *producesD*)

lemma *productive-in-N*: *productive* *C* $\implies C \in N$
unfolding *production-unfold* **by** *auto*

lemma *produces-imp-atms-leq*: *produces* *C* *A* $\implies B \in \text{atms-of } C \implies B \leq A$
by (*metis* *Max-ge* *finite-atms-of* *insert-not-empty* *productive-imp-produces-Max-atom* *singleton-inject*)

lemma *produces-imp-neg-notin-lits*: *produces* *C* *A* $\implies \neg \text{Neg } A \in \# C$
by (*rule* *pos-Max-imp-neg-notin*) (*auto* *dest*: *producesD*)

lemma *less-eq-imp-interp-subseteq-interp*: *C* $\# \subseteq \# D \implies \text{interp } C \subseteq \text{interp } D$
unfolding *interp-def* **by** *auto* (*metis* *multiset-order.order.strict-trans2*)

lemma *less-eq-imp-interp-subseteq-Interp*: *C* $\# \subseteq \# D \implies \text{interp } C \subseteq \text{Interp } D$
unfolding *Interp-def* **using** *less-eq-imp-interp-subseteq-interp* **by** *blast*

lemma *less-imp-production-subseteq-interp*: *C* $\# \subset \# D \implies \text{production } C \subseteq \text{interp } D$
unfolding *interp-def* **by** *fast*

lemma *less-eq-imp-production-subseteq-Interp*: *C* $\# \subseteq \# D \implies \text{production } C \subseteq \text{Interp } D$
unfolding *Interp-def* **using** *less-imp-production-subseteq-interp*
by (*metis* *multiset-order.le-imp-less-or-eq* *le-supI1* *sup-ge2*)

lemma *less-imp-Interp-subseteq-interp*: $C \# \subseteq \# D \implies \text{Interp } C \subseteq \text{interp } D$
unfolding *Interp-def*
by (auto simp: less-eq-imp-interp-subseteq-interp less-imp-production-subseteq-interp)

lemma *less-eq-imp-Interp-subseteq-Interp*: $C \# \subseteq \# D \implies \text{Interp } C \subseteq \text{Interp } D$
using *less-imp-Interp-subseteq-interp*
unfolding *Interp-def* **by** (metis multiset-order.le-imp-less-or-eq le-supI2 subset-refl sup-commute)

lemma *false-Interp-to-true-interp-imp-less-multiset*: $A \notin \text{Interp } C \implies A \in \text{interp } D \implies C \# \subseteq \# D$
using *less-eq-imp-interp-subseteq-Interp* multiset-linorder.not-less **by** blast

lemma *false-interp-to-true-interp-imp-less-multiset*: $A \notin \text{interp } C \implies A \in \text{interp } D \implies C \# \subseteq \# D$
using *less-eq-imp-interp-subseteq-interp* multiset-linorder.not-less **by** blast

lemma *false-Interp-to-true-Interp-imp-less-multiset*: $A \notin \text{Interp } C \implies A \in \text{Interp } D \implies C \# \subseteq \# D$
using *less-eq-imp-Interp-subseteq-Interp* multiset-linorder.not-less **by** blast

lemma *false-interp-to-true-Interp-imp-le-multiset*: $A \notin \text{interp } C \implies A \in \text{Interp } D \implies C \# \subseteq \# D$
using *less-imp-Interp-subseteq-interp* multiset-linorder.not-less **by** blast

lemma *interp-subseteq-INTERP*: $\text{interp } C \subseteq \text{INTERP}$
unfolding *interp-def INTERP-def* **by** (auto simp: production-unfold)

lemma *production-subseteq-INTERP*: $\text{production } C \subseteq \text{INTERP}$
unfolding *INTERP-def* **using** *production-unfold* **by** blast

lemma *Interp-subseteq-INTERP*: $\text{Interp } C \subseteq \text{INTERP}$
unfolding *Interp-def* **by** (auto intro!: interp-subseteq-INTERP production-subseteq-INTERP)

This lemma corresponds to theorem 2.7.6 page 66 of CW.

lemma *produces-imp-in-interp*:
assumes *a-in-c*: $\text{Neg } A \in \# C$ **and** *d*: *produces* D A
shows $A \in \text{interp } C$
proof –
from *d* **have** $\text{Max } (\text{set-mset } D) = \text{Pos } A$
using *production-unfold* **by** blast
hence $D \# \subseteq \# \{\# \text{Neg } A \#\}$
by (auto intro: *Max-pos-neg-less-multiset*)
moreover **have** $\{\# \text{Neg } A \#\} \# \subseteq \# C$
by (rule *less-eq-imp-le-multiset*) (rule *mset-le-single[OF a-in-c]*)
ultimately **show** *?thesis*
using *d* **by** (blast dest: *less-eq-imp-interp-subseteq-interp less-imp-production-subseteq-interp*)
qed

lemma *neg-notin-Interp-not-produce*: $\text{Neg } A \in \# C \implies A \notin \text{Interp } D \implies C \# \subseteq \# D \implies \neg \text{produces } D'' A$
by (auto dest: *produces-imp-in-interp less-eq-imp-interp-subseteq-Interp*)

lemma *in-production-imp-produces*: $A \in \text{production } C \implies \text{produces } C A$
by (metis *insert-absorb productive-imp-produces-Max-atom singleton-insert-inj-eq'*)

lemma *not-produces-imp-notin-production*: $\neg \text{produces } C A \implies A \notin \text{production } C$
by (metis *in-production-imp-produces*)

lemma *not-produces-imp-notin-interp*: $(\bigwedge D. \neg \text{produces } D A) \implies A \notin \text{interp } C$

unfolding *interp-def* **by** (*fast intro!*: *in-production-imp-produces*)

The results below corresponds to Lemma 3.4.

Nitpicking: If $D = D'$ and D is productive, $I^D \subseteq I_{D'}$ does not hold.

lemma *true-Interp-imp-general*:

assumes

c-le-d: $C \# \subseteq \# D$ **and**

d-lt-d': $D \# \subset \# D'$ **and**

c-at-d: $\text{Interp } D \models_h C$ **and**

subs: $\text{interp } D' \subseteq (\bigcup C \in CC. \text{production } C)$

shows $(\bigcup C \in CC. \text{production } C) \models_h C$

proof (*cases* $\exists A. \text{Pos } A \in \# C \wedge A \in \text{Interp } D$)

case *True*

then obtain A **where** *a-in-c*: $\text{Pos } A \in \# C$ **and** *a-at-d*: $A \in \text{Interp } D$

by *blast*

from *a-at-d* **have** $A \in \text{interp } D'$

using *d-lt-d'* *less-imp-Interp-subseteq-interp* **by** *blast*

thus *?thesis*

using *subs a-in-c* **by** (*blast dest*: *contra-subsetD*)

next

case *False*

then obtain A **where** *a-in-c*: $\text{Neg } A \in \# C$ **and** $A \notin \text{Interp } D$

using *c-at-d* *unfolding true-cls-def* **by** *blast*

hence $\bigwedge D''. \neg \text{produces } D'' A$

using *c-le-d* *neg-notin-Interp-not-produce* **by** *simp*

thus *?thesis*

using *a-in-c* *subs not-produces-imp-notin-production* **by** *auto*

qed

lemma *true-Interp-imp-interp*: $C \# \subseteq \# D \implies D \# \subset \# D' \implies \text{Interp } D \models_h C \implies \text{interp } D' \models_h C$
using *interp-def true-Interp-imp-general* **by** *simp*

lemma *true-Interp-imp-Interp*: $C \# \subseteq \# D \implies D \# \subset \# D' \implies \text{Interp } D \models_h C \implies \text{Interp } D' \models_h C$
using *Interp-as-UNION interp-subseteq-Interp true-Interp-imp-general* **by** *simp*

lemma *true-Interp-imp-INTERP*: $C \# \subseteq \# D \implies \text{Interp } D \models_h C \implies \text{INTERP} \models_h C$
using *INTERP-def interp-subseteq-INTERP*
true-Interp-imp-general[OF - less-multiset-right-total]
by *simp*

lemma *true-interp-imp-general*:

assumes

c-le-d: $C \# \subseteq \# D$ **and**

d-lt-d': $D \# \subset \# D'$ **and**

c-at-d: $\text{interp } D \models_h C$ **and**

subs: $\text{interp } D' \subseteq (\bigcup C \in CC. \text{production } C)$

shows $(\bigcup C \in CC. \text{production } C) \models_h C$

proof (*cases* $\exists A. \text{Pos } A \in \# C \wedge A \in \text{interp } D$)

case *True*

then obtain A **where** *a-in-c*: $\text{Pos } A \in \# C$ **and** *a-at-d*: $A \in \text{interp } D$

by *blast*

from *a-at-d* **have** $A \in \text{interp } D'$

using *d-lt-d'* *less-eq-imp-interp-subseteq-interp[OF multiset-order.less-imp-le]* **by** *blast*

thus *?thesis*

using *subs a-in-c* **by** (*blast dest*: *contra-subsetD*)

```

next
case False
then obtain A where a-in-c: Neg A ∈# C and A ∉ interp D
  using c-at-d unfolding true-cls-def by blast
hence  $\bigwedge D''. \neg \text{produces } D'' A$ 
  using c-le-d by (auto dest: produces-imp-in-interp less-eq-imp-interp-subseteq-interp)
thus ?thesis
  using a-in-c subs not-produces-imp-notin-production by auto
qed

```

This lemma corresponds to theorem 2.7.6 page 66 of CW. Here the strict maximality is important

```

lemma true-interp-imp-interp: C #⊆# D ⇒ D #⊂# D' ⇒ interp D ⊨h C ⇒ interp D' ⊨h C
  using interp-def true-interp-imp-general by simp

```

```

lemma true-interp-imp-Interp: C #⊆# D ⇒ D #⊂# D' ⇒ interp D ⊨h C ⇒ Interp D' ⊨h C
  using Interp-as-UNION interp-subseteq-Interp[of D'] true-interp-imp-general by simp

```

```

lemma true-interp-imp-INTERP: C #⊆# D ⇒ interp D ⊨h C ⇒ INTERP ⊨h C
  using INTERP-def interp-subseteq-INTERP
  true-interp-imp-general[OF - less-multiset-right-total]
  by simp

```

```

lemma productive-imp-false-interp: productive C ⇒ ¬ interp C ⊨h C
  unfolding production-unfold by auto

```

This lemma corresponds to theorem 2.7.6 page 66 of CW. Here the strict maximality is important

```

lemma cls-gt-double-pos-no-production:
  assumes D: {#Pos P, Pos P#} #⊂# C
  shows ¬produces C P
proof -
  let ?D = {#Pos P, Pos P#}
  note D' = D[unfolded less-multiset_HO]
  consider
    (P) count C (Pos P) ≥ 2
  | (Q) Q where Q > Pos P and Q ∈# C
    using HOL.spec[OF HOL.conjunct2[OF D'], of Pos P] by (auto split: if-split-asm)
  thus ?thesis
  proof cases
    case Q
    have Q ∈ set-mset C
    using Q(2) by (auto split: if-split-asm)
    then have Max (set-mset C) > Pos P
    using Q(1) Max-gr-iff by blast
    thus ?thesis
    unfolding production-unfold by auto
  next
    case P
    thus ?thesis
    unfolding production-unfold by auto
  qed
qed

```

This lemma corresponds to theorem 2.7.6 page 66 of CW.

```

lemma
  assumes D: C + {#Neg P#} #⊂# D

```

shows *production* $D \neq \{P\}$
proof –
note $D' = D[\text{unfolded less-multiset}_{HO}]$
consider
 $(P) \text{ Neg } P \in \# D$
 $| (Q) Q \text{ where } Q > \text{Neg } P \text{ and } \text{count } D Q > \text{count } (C + \{\# \text{Neg } P \# \}) Q$
 using $HOL.\text{spec}[OF HOL.\text{conjunct2}[OF D], \text{ of Neg } P] \text{ count-greater-zero-iff}$ **by** *fastforce*
thus *?thesis*
proof *cases*
 case Q
 have $Q \in \text{set-mset } D$
 using $Q(2) \text{ gr-implies-not0}$ **by** *fastforce*
 then have $\text{Max } (\text{set-mset } D) > \text{Neg } P$
 using $Q(1) \text{ Max-gr-iff}$ **by** *blast*
 hence $\text{Max } (\text{set-mset } D) > \text{Pos } P$
 using $\text{less-trans}[\text{of Pos } P \text{ Neg } P \text{ Max } (\text{set-mset } D)]$ **by** *auto*
 thus *?thesis*
 unfolding *production-unfold* **by** *auto*
next
 case P
 hence $\text{Max } (\text{set-mset } D) > \text{Pos } P$
 by $(\text{meson } \text{Max-ge finite-set-mset le-less-trans linorder-not-le pos-less-neg})$
 thus *?thesis*
 unfolding *production-unfold* **by** *auto*
qed
qed

lemma *in-interp-is-produced:*

assumes $P \in \text{INTERP}$
shows $\exists D. D + \{\# \text{Pos } P \# \} \in N \wedge \text{produces } (D + \{\# \text{Pos } P \# \}) P$
using *assms unfolding INTERP-def UN-iff production-iff-produces Ball-def*
by $(\text{metis ground-resolution-with-selection.produces-imp-Pos-in-lits insert-DiffM2}$
 $\text{ground-resolution-with-selection-axioms not-produces-imp-notin-production})$

end
end

abbreviation $M\text{Max } M \equiv \text{Max } (\text{set-mset } M)$

27.1 We can now define the rules of the calculus

inductive *superposition-rules* :: $'a \text{ clause} \Rightarrow 'a \text{ clause} \Rightarrow 'a \text{ clause} \Rightarrow \text{bool}$ **where**
factoring: $\text{superposition-rules } (C + \{\# \text{Pos } P \# \} + \{\# \text{Pos } P \# \}) B (C + \{\# \text{Pos } P \# \}) |$
superposition-l: $\text{superposition-rules } (C_1 + \{\# \text{Pos } P \# \}) (C_2 + \{\# \text{Neg } P \# \}) (C_1 + C_2)$

inductive *superposition* :: $'a \text{ clauses} \Rightarrow 'a \text{ clauses} \Rightarrow \text{bool}$ **where**
superposition: $A \in N \Longrightarrow B \in N \Longrightarrow \text{superposition-rules } A B C$
 $\Longrightarrow \text{superposition } N (N \cup \{C\})$

definition *abstract-red* :: $'a::\text{wellorder clause} \Rightarrow 'a \text{ clauses} \Rightarrow \text{bool}$ **where**
abstract-red $C N = (\text{class-lt } N C \models_p C)$

lemma *less-multiset[iff]:* $M < N \longleftrightarrow M \# \subset \# N$
unfolding *less-multiset-def* **by** *auto*

lemma *less-eq-multiset[iff]*: $M \leq N \longleftrightarrow M \# \subseteq \# N$
unfolding *less-eq-multiset-def* **by** *auto*

lemma *herbrand-true-clss-true-clss-clss-herbrand-true-clss*:

assumes

$AB: A \models_{hs} B$ **and**

$BC: B \models_p C$

shows $A \models_h C$

proof –

let $?I = \{Pos\ P \mid P. P \in A\} \cup \{Neg\ P \mid P. P \notin A\}$

have $B: ?I \models_s B$ **using** AB

by (*auto simp add: herbrand-interp-iff-partial-interp-clss*)

have $IH: \bigwedge I. total-over-set\ I\ (atms-of\ C) \implies total-over-m\ I\ B \implies consistent-interp\ I$
 $\implies I \models_s B \implies I \models C$ **using** BC

by (*auto simp add: true-clss-clss-def*)

show *?thesis*

unfolding *herbrand-interp-iff-partial-interp-clss*

by (*auto intro: IH[of ?I] simp add: herbrand-total-over-set herbrand-total-over-m herbrand-consistent-interp B*)

qed

lemma *abstract-red-subset-mset-abstract-red*:

assumes

abstr: *abstract-red* $C\ N$ **and**

c-lt-d: $C \subseteq \# D$

shows *abstract-red* $D\ N$

proof –

have $\{D \in N. D \# \subset \# C\} \subseteq \{D' \in N. D' \# \subset \# D\}$

using *c-lt-d less-eq-imp-le-multiset* **by** *fastforce*

thus *?thesis*

using *abstr* **unfolding** *abstract-red-def clss-lt-def*

by (*metis (no-types, lifting) c-lt-d subset-mset.diff-add true-clss-clss-mono-r' true-clss-clss-subset*)

qed

lemma *true-clss-clss-extended*:

assumes

$A \models_p B$ **and**

tot: *total-over-m* $I\ (A)$ **and**

cons: *consistent-interp* I **and**

$I-A: I \models_s A$

shows $I \models B$

proof –

let $?I = I \cup \{Pos\ P \mid P. P \in atms-of\ B \wedge P \notin atms-of-s\ I\}$

have *consistent-interp* $?I$

using *cons* **unfolding** *consistent-interp-def atms-of-s-def atms-of-def*

apply (*auto 1 5 simp add: image-iff*)

by (*metis atm-of-uminus literal.sel(1)*)

moreover have *total-over-m* $?I\ (A \cup \{B\})$

proof –

obtain $aa :: 'a\ set \Rightarrow 'a\ literal\ set \Rightarrow 'a$ **where**

$f2: \forall x0\ x1. (\exists v2. v2 \in x0 \wedge Pos\ v2 \notin x1 \wedge Neg\ v2 \notin x1)$

$\longleftrightarrow (aa\ x0\ x1 \in x0 \wedge Pos\ (aa\ x0\ x1) \notin x1 \wedge Neg\ (aa\ x0\ x1) \notin x1)$

```

    by moura
  have  $\forall a. a \notin \text{atms-of-ms } A \vee \text{Pos } a \in I \vee \text{Neg } a \in I$ 
    using tot by (simp add: total-over-m-def total-over-set-def)
  hence aa ( $\text{atms-of-ms } A \cup \text{atms-of-ms } \{B\}$ ) ( $I \cup \{\text{Pos } a \mid a \in \text{atms-of } B \wedge a \notin \text{atms-of-s } I\}$ )
     $\notin \text{atms-of-ms } A \cup \text{atms-of-ms } \{B\} \vee \text{Pos } (aa (\text{atms-of-ms } A \cup \text{atms-of-ms } \{B\})$ 
      ( $I \cup \{\text{Pos } a \mid a \in \text{atms-of } B \wedge a \notin \text{atms-of-s } I\})) \in I$ 
       $\cup \{\text{Pos } a \mid a \in \text{atms-of } B \wedge a \notin \text{atms-of-s } I\}$ 
       $\vee \text{Neg } (aa (\text{atms-of-ms } A \cup \text{atms-of-ms } \{B\})$ 
        ( $I \cup \{\text{Pos } a \mid a \in \text{atms-of } B \wedge a \notin \text{atms-of-s } I\})) \in I$ 
         $\cup \{\text{Pos } a \mid a \in \text{atms-of } B \wedge a \notin \text{atms-of-s } I\}$ 
    by auto
  hence total-over-set ( $I \cup \{\text{Pos } a \mid a \in \text{atms-of } B \wedge a \notin \text{atms-of-s } I\}$ )
    ( $\text{atms-of-ms } A \cup \text{atms-of-ms } \{B\}$ )
    using f2 by (meson total-over-set-def)
  thus ?thesis
    by (simp add: total-over-m-def)
qed
moreover have  $?I \models_s A$ 
  using I-A by auto
ultimately have  $?I \models B$ 
  using  $\langle A \models_p B \rangle$  unfolding true-clss-clss-def by auto
thus ?thesis

oops
lemma
  assumes
    CP:  $\neg \text{clss-lt } N (\{\#C\# \} + \{\#E\# \}) \models_p \{\#C\# \} + \{\#Neg P\# \}$  and
    clss-lt  $N (\{\#C\# \} + \{\#E\# \}) \models_p \{\#E\# \} + \{\#Pos P\# \} \vee \text{clss-lt } N (\{\#C\# \} + \{\#E\# \}) \models_p$ 
 $\{\#C\# \} + \{\#Neg P\# \}$ 
  shows  $\text{clss-lt } N (\{\#C\# \} + \{\#E\# \}) \models_p \{\#E\# \} + \{\#Pos P\# \}$ 

oops

locale ground-ordered-resolution-with-redundancy =
  ground-resolution-with-selection +
  fixes redundant :: 'a::wellorder clause  $\Rightarrow$  'a clauses  $\Rightarrow$  bool
  assumes
    redundant-iff-abstract:  $\text{redundant } A N \longleftrightarrow \text{abstract-red } A N$ 
begin
definition saturated :: 'a clauses  $\Rightarrow$  bool where
  saturated  $N \longleftrightarrow (\forall A B C. A \in N \longrightarrow B \in N \longrightarrow \neg \text{redundant } A N \longrightarrow \neg \text{redundant } B N$ 
     $\longrightarrow \text{superposition-rules } A B C \longrightarrow \text{redundant } C N \vee C \in N)$ 

lemma
  assumes
    saturated: saturated  $N$  and
    finite: finite  $N$  and
    empty:  $\{\#\} \notin N$ 
  shows  $\text{INTERP } N \models_{hs} N$ 
proof (rule ccontr)
  let  $?N_{\mathcal{I}} = \text{INTERP } N$ 
  assume  $\neg ?thesis$ 
  hence not-empty:  $\{E \in N. \neg ?N_{\mathcal{I}} \models_h E\} \neq \{\}$ 
    unfolding true-clss-def Ball-def by auto
  def  $D \equiv \text{Min } \{E \in N. \neg ?N_{\mathcal{I}} \models_h E\}$ 
  have [simp]:  $D \in N$ 

```

```

unfolding D-def
  by (metis (mono-tags, lifting) Min-in not-empty finite mem-Collect-eq rev-finite-subset subsetI)
have not-d-interp:  $\neg ?N_{\mathcal{I}} \models_h D$ 
unfolding D-def
  by (metis (mono-tags, lifting) Min-in finite mem-Collect-eq not-empty rev-finite-subset subsetI)
have cls-not-D:  $\bigwedge E. E \in N \implies E \neq D \implies \neg ?N_{\mathcal{I}} \models_h E \implies D \leq E$ 
  using finite D-def by (auto simp del: less-eq-multiset)
obtain C L where D:  $D = C + \{\#L\#\}$  and LSD:  $L \in\# S D \vee (S D = \{\#\} \wedge \text{Max } (\text{set-mset } D)$ 
 $= L)$ 
proof (cases  $S D = \{\#\}$ )
  case False
  then obtain L where  $L \in\# S D$ 
  using Max-in-lits by blast
moreover
  hence  $L \in\# D$ 
  using S-selects-subseteq[of D] by auto
  hence  $D = (D - \{\#L\#\}) + \{\#L\#\}$ 
  by auto
  ultimately show ?thesis using that by blast
next
  let  $?L = \text{MMax } D$ 
  case True
  moreover
  have  $?L \in\# D$ 
  by (metis (no-types, lifting) Max-in-lits ( $D \in N$ ) empty)
  hence  $D = (D - \{\#?L\#\}) + \{\#?L\#\}$ 
  by auto
  ultimately show ?thesis using that by blast
qed
have red:  $\neg \text{redundant } D N$ 
proof (rule ccontr)
  assume red[simplified]:  $\sim\sim \text{redundant } D N$ 
  have  $\forall E < D. E \in N \longrightarrow ?N_{\mathcal{I}} \models_h E$ 
  using cls-not-D not-le by fastforce
  hence  $?N_{\mathcal{I}} \models_{hs} \text{clss-lt } N D$ 
  unfolding clss-lt-def true-clss-def Ball-def by blast
  thus False
  using red not-d-interp unfolding abstract-red-def redundant-iff-abstract
  using herbrand-true-clss-true-clss-clss-herbrand-true-clss by fast
qed

consider
  (L) P where  $L = \text{Pos } P$  and  $S D = \{\#\}$  and  $\text{Max } (\text{set-mset } D) = \text{Pos } P$ 
  | (Lneg) P where  $L = \text{Neg } P$ 
  using LSD S-selects-neg-lits[of L D] by (cases L) auto
thus False
proof cases
  case L note  $P = \text{this}(1)$  and  $S = \text{this}(2)$  and  $\text{max} = \text{this}(3)$ 
  have count  $D L > 1$ 
  proof (rule ccontr)
  assume  $\sim ?thesis$ 
  hence count:  $\text{count } D L = 1$ 
  unfolding D by (auto simp: not-in-iff)
  have  $\neg ?N_{\mathcal{I}} \models_h D$ 
  using not-d-interp true-interp-imp-INTERP ground-resolution-with-selection-axioms

```



```

    by blast
  hence produces  $N \ D \ P$ 
    using not-empty empty finite  $\langle D \in N \rangle$  count  $L$ 
      true-interp-imp-INTERP unfolding production-iff-produces unfolding production-unfold
    by (auto simp add: max not-empty)
  hence  $INTERP \ N \models_h D$ 
    unfolding  $D$ 
    by (metis pos-literal-in-imp-true-cls produces-imp-Pos-in-lits
      production-subseteq-INTERP singletonI subsetCE)
  thus False
    using not-d-interp by blast
qed
then have  $Pos \ P \in\# \ C$ 
  by (simp add:  $P \ D$ )
then obtain  $C'$  where  $C':D = C' + \{\#Pos \ P\# \} + \{\#Pos \ P\# \}$ 
  unfolding  $D$  by (metis (full-types)  $P$  insert-DiffM2)
have sup: superposition-rules  $D \ D \ (D - \{\#L\# \})$ 
  unfolding  $C' \ L$  by (auto simp add: superposition-rules.simps)
have  $C' + \{\#Pos \ P\# \} \# \subset \# \ C' + \{\#Pos \ P\# \} + \{\#Pos \ P\# \}$ 
  by auto
moreover have  $\neg ?N_{\mathcal{I}} \models_h (D - \{\#L\# \})$ 
  using not-d-interp unfolding  $C' \ L$  by auto
ultimately have  $C' + \{\#Pos \ P\# \} \notin N$ 
  by (metis (no-types, lifting)  $C' \ P$  add-diff-cancel-right' cls-not-D less-multiset
    multi-self-add-other-not-self not-le)
have  $D - \{\#L\# \} \# \subset \# \ D$ 
  unfolding  $C' \ L$  by auto
have  $c'-p-p: C' + \{\#Pos \ P\# \} + \{\#Pos \ P\# \} - \{\#Pos \ P\# \} = C' + \{\#Pos \ P\# \}$ 
  by auto
have redundant  $(C' + \{\#Pos \ P\# \}) \ N$ 
  using saturated red sup  $\langle D \in N \rangle \langle C' + \{\#Pos \ P\# \} \notin N \rangle$  unfolding saturated-def  $C' \ L \ c'-p-p$ 
  by blast
moreover have  $C' + \{\#Pos \ P\# \} \subseteq \# \ C' + \{\#Pos \ P\# \} + \{\#Pos \ P\# \}$ 
  by auto
ultimately show False
  using red unfolding  $C'$  redundant-iff-abstract by (blast dest:
    abstract-red-subset-mset-abstract-red)
next
case Lneg note  $L = this(1)$ 
have  $P \in ?N_{\mathcal{I}}$ 
  using not-d-interp unfolding  $D$  true-cls-def  $L$  by (auto split: if-split-asm)
then obtain  $E$  where
  DPN:  $E + \{\#Pos \ P\# \} \in N$  and
  prod: production  $N \ (E + \{\#Pos \ P\# \}) = \{P\}$ 
  using in-interp-is-produced by blast
have sup-EC: superposition-rules  $(E + \{\#Pos \ P\# \}) \ (C + \{\#Neg \ P\# \}) \ (E + C)$ 
  using superposition-l by fast
hence superposition  $N \ (N \cup \{E+C\})$ 
  using DPN  $\langle D \in N \rangle$  unfolding  $D \ L$  by (auto simp add: superposition.simps)
have
  PMax:  $Pos \ P = MMax \ (E + \{\#Pos \ P\# \})$  and
  count  $(E + \{\#Pos \ P\# \}) \ (Pos \ P) \leq 1$  and
   $S \ (E + \{\#Pos \ P\# \}) = \{\# \}$  and
   $\neg interp \ N \ (E + \{\#Pos \ P\# \}) \models_h E + \{\#Pos \ P\# \}$ 
  using prod unfolding production-unfold by auto

```

have $Neg P \notin \# E$
using *prod produces-imp-neg-notin-lits* **by** *force*
hence $\bigwedge y. y \in \# (E + \{\#Pos P\})$
 $\implies count (E + \{\#Pos P\}) (Neg P) < count (C + \{\#Neg P\}) (Neg P)$
using *count-greater-zero-iff* **by** *fastforce*
moreover have $\bigwedge y. y \in \# (E + \{\#Pos P\}) \implies y < Neg P$
using *PMax* **by** (*metis DPN Max-less-iff empty finite-set-mset pos-less-neg set-mset-eq-empty-iff*)
moreover have $E + \{\#Pos P\} \neq C + \{\#Neg P\}$
using *prod produces-imp-neg-notin-lits* **by** *force*
ultimately have $E + \{\#Pos P\} \# \subset \# C + \{\#Neg P\}$
unfolding *less-multiset_{HO}* **by** (*metis count-greater-zero-iff less-iff-Suc-add zero-less-Suc*)
have *ce-lt-d*: $C + E \# \subset \# D$
unfolding *D L* **by** (*simp add*: $\langle \bigwedge y. y \in \# E + \{\#Pos P\} \implies y < Neg P \rangle$ *ex-gt-imp-less-multiset*)
have $?N_{\mathcal{I}} \models_h E + \{\#Pos P\}$
using $\langle P \in ?N_{\mathcal{I}} \rangle$ **by** *blast*
have $?N_{\mathcal{I}} \models_h C + E \vee C + E \notin N$
using *ce-lt-d cls-not-D* **unfolding** *D-def* **by** *fastforce*
have $Pos P \notin \# C + E$
using *D* $\langle P \in \text{ground-resolution-with-selection.INTERP } S N \rangle$
 $\langle count (E + \{\#Pos P\}) (Pos P) \leq 1 \rangle$ *multi-member-skip not-d-interp*
by (*auto simp: not-in-iff*)
hence $\bigwedge y. y \in \# C + E$
 $\implies count (C + E) (Pos P) < count (E + \{\#Pos P\}) (Pos P)$
using *set-mset-def* **by** *fastforce*

have $\neg \text{redundant} (C + E) N$
proof (*rule ccontr*)
assume *red'*[*simplified*]: $\neg ?thesis$
have *abs*: *clss-lt* $N (C + E) \models_p C + E$
using *redundant-iff-abstract red'* **unfolding** *abstract-red-def* **by** *auto*
have *clss-lt* $N (C + E) \models_p E + \{\#Pos P\} \vee \text{clss-lt } N (C + E) \models_p C + \{\#Neg P\}$
proof *clarify*
assume *CP*: $\neg \text{clss-lt } N (C + E) \models_p C + \{\#Neg P\}$
{ fix *I*
assume
total-over-m $I (\text{clss-lt } N (C + E) \cup \{E + \{\#Pos P\}\})$ **and**
consistent-interp *I* **and**
 $I \models_s \text{clss-lt } N (C + E)$
hence $I \models C + E$
using *abs* **sorry**
moreover have $\neg I \models C + \{\#Neg P\}$
using *CP* **unfolding** *true-clss-cls-def*
sorry
ultimately have $I \models E + \{\#Pos P\}$ **by** *auto*
}
then show *clss-lt* $N (C + E) \models_p E + \{\#Pos P\}$
unfolding *true-clss-cls-def* **by** *auto*
qed

moreover have *clss-lt* $N (C + E) \subseteq \text{clss-lt } N (C + \{\#Neg P\})$
using *ce-lt-d mult-less-trans* **unfolding** *clss-lt-def D L* **by** *force*
ultimately have *redundant* $(C + \{\#Neg P\}) N \vee \text{clss-lt } N (C + E) \models_p E + \{\#Pos P\}$
unfolding *redundant-iff-abstract abstract-red-def* **using** *true-clss-cls-subset* **by** *blast*
show *False* **sorry**
qed

```

moreover have  $\neg$  redundant ( $E + \{\#Pos\ P\# \}$ )  $N$ 
  sorry
ultimately have  $CEN: C + E \in N$ 
  using  $\langle D \in N \rangle \langle E + \{\#Pos\ P\# \} \in N \rangle$  saturated sup-EC red unfolding saturated-def  $D\ L$ 
  by (metis union-commute)
have  $CED: C + E \neq D$ 
  using  $D\ ce-lt-d$  by auto
have interp:  $\neg\ INTERP\ N \models_h C + E$ 
sorry
show False
  using cls-not-D[OF CEN CED interp] ce-lt-d unfolding INTERP-def less-eq-multiset-def by
auto
  qed
qed

end

lemma tautology-is-redundant:
  assumes tautology  $C$ 
  shows abstract-red  $C\ N$ 
  using assms unfolding abstract-red-def true-clss-cls-def tautology-def by auto

lemma subsumed-is-redundant:
  assumes  $AB: A \subset\# B$ 
  and  $AN: A \in N$ 
  shows abstract-red  $B\ N$ 
proof –
  have  $A \in clss-lt\ N\ B$  using  $AN\ AB$  unfolding clss-lt-def
  by (auto dest: less-eq-imp-le-multiset simp add: multiset-order.dual-order.order-iff-strict)
  thus ?thesis
  using  $AB$  unfolding abstract-red-def true-clss-cls-def Partial-Clausal-Logic.true-clss-def
  by blast
qed

inductive redundant :: ' $a$  clause  $\Rightarrow$  ' $a$  clauses  $\Rightarrow$  bool where
  subsumption:  $A \in N \Longrightarrow A \subset\# B \Longrightarrow$  redundant  $B\ N$ 

lemma redundant-is-redundancy-criterion:
  fixes  $A :: 'a :: wellorder$  clause and  $N :: 'a :: wellorder$  clauses
  assumes redundant  $A\ N$ 
  shows abstract-red  $A\ N$ 
  using assms
proof (induction rule: redundant.induct)
  case (subsumption  $A\ B\ N$ )
  thus ?case
  using subsumed-is-redundant[of A N B] unfolding abstract-red-def clss-lt-def by auto
qed

lemma redundant-mono:
   $redundant\ A\ N \Longrightarrow A \subseteq\# B \Longrightarrow$  redundant  $B\ N$ 
  apply (induction rule: redundant.induct)
  by (meson subset-mset.less-le-trans subsumption)

locale truc =
  selection S for S :: nat clause  $\Rightarrow$  nat clause

```

begin

end

end