

Formalisation of Ground Resolution and CDCL in Isabelle/HOL

Mathias Fleury and Jasmin Blanchette

April 4, 2016

Contents

1	Rewrite systems and properties	2
1.1	Lifting of rewrite rules	2
1.2	Consistency preservation	4
1.3	Full Lifting	5
2	Transformation testing	6
2.1	Definition and first properties	6
2.2	Invariant conservation	8
2.2.1	Invariant while lifting of the rewriting relation	9
2.2.2	Invariant after all rewriting	10
3	Rewrite Rules	12
3.1	Elimination of the equivalences	12
3.2	Eliminate Implication	13
3.3	Eliminate all the True and False in the formula	15
3.4	PushNeg	21
3.5	Push inside	26
3.5.1	Only one type of connective in the formula (+ not)	35
3.5.2	Push Conjunction	38
3.5.3	Push Disjunction	39
4	The full transformations	40
4.1	Abstract Property characterizing that only some connective are inside the others	40
4.1.1	Definition	40
4.2	Conjunctive Normal Form	42
4.2.1	Full CNF transformation	42
4.3	Disjunctive Normal Form	43
4.3.1	Full DNF transform	44
5	More aggressive simplifications: Removing true and false at the beginning	44
5.1	Transformation	44
5.2	More invariants	46
5.3	The new CNF and DNF transformation	50

6 Link with Multiset Version	51
6.1 Transformation to Multiset	51
6.2 Equisatisfiability of the two Version	51

theory *Prop-Abstract-Transformation*
imports *Main Prop-Logic Wellfounded-More*

begin

This file is devoted to abstract properties of the transformations, like consistency preservation and lifting from terms to proposition.

1 Rewrite systems and properties

1.1 Lifting of rewrite rules

We can lift a rewrite relation r over a full formula: the relation r works on terms, while *propo-rew-step* works on formulas.

```
inductive propo-rew-step :: ('v propo  $\Rightarrow$  'v propo  $\Rightarrow$  bool)  $\Rightarrow$  'v propo  $\Rightarrow$  'v propo  $\Rightarrow$  bool
  for r :: 'v propo  $\Rightarrow$  'v propo  $\Rightarrow$  bool where
    global-rel: r  $\varphi$   $\psi \Longrightarrow$  propo-rew-step r  $\varphi$   $\psi$  |
    propo-rew-one-step-lift: propo-rew-step r  $\varphi$   $\varphi' \Longrightarrow$  wf-conn c ( $\psi s @ \varphi \# \psi s'$ )
       $\Longrightarrow$  propo-rew-step r (conn c ( $\psi s @ \varphi \# \psi s'$ )) (conn c ( $\psi s @ \varphi' \# \psi s'$ ))
```

Here is a more precise link between the lifting and the subformulas: if a rewriting takes place between φ and φ' , then there are two subformulas ψ in φ and ψ' in φ' , ψ' is the result of the rewriting of r on ψ .

This lemma is only a health condition:

```
lemma propo-rew-step-subformula-imp:
shows propo-rew-step r  $\varphi$   $\varphi' \Longrightarrow \exists \psi \psi'. \psi \preceq \varphi \wedge \psi' \preceq \varphi' \wedge r \psi \psi'$ 
  apply (induct rule: propo-rew-step.induct)
  using subformula.simps subformula-into-subformula apply blast
  using wf-conn-no-arity-change subformula-into-subformula wf-conn-no-arity-change-helper
  in-set-conv-decomp by metis
```

The converse is moreover true: if there is a ψ and ψ' , then every formula φ containing ψ , can be rewritten into a formula φ' , such that it contains ψ' .

```
lemma propo-rew-step-subformula-rec:
  fixes  $\psi \psi' \varphi :: 'v propo$ 
  shows  $\psi \preceq \varphi \Longrightarrow r \psi \psi' \Longrightarrow (\exists \varphi'. \psi' \preceq \varphi' \wedge propo-rew-step r \varphi \varphi')$ 
proof (induct  $\varphi$  rule: subformula.induct)
  case subformula-refl
  hence propo-rew-step r  $\psi \psi'$  using propo-rew-step.intros by auto
  moreover have  $\psi' \preceq \psi'$  using Prop-Logic.subformula-refl by auto
  ultimately show  $\exists \varphi'. \psi' \preceq \varphi' \wedge propo-rew-step r \varphi \varphi'$  by fastforce
next
  case (subformula-into-subformula  $\psi'' l c$ )
  note IH = this(4) and r = this(5) and  $\psi'' = this(1)$  and wf = this(2) and incl = this(3)
  then obtain  $\varphi'$  where *:  $\psi' \preceq \varphi' \wedge propo-rew-step r \psi'' \varphi'$  by metis
  moreover obtain  $\xi \xi' :: 'v propo list$  where
    l:  $l = \xi @ \psi'' \# \xi'$  using List.split-list  $\psi''$  by metis
  ultimately have propo-rew-step r (conn c l) (conn c ( $\xi @ \varphi' \# \xi'$ ))
    using propo-rew-step.intros(2) wf by metis
```

moreover have $\psi' \preceq \text{conn } c (\xi @ \varphi' \# \xi')$
using $\text{wf} * \text{wf-conn-no-arity-change Prop-Logic.subformula-into-subformula}$
by $(\text{metis (no-types) in-set-conv-decomp l wf-conn-no-arity-change-helper})$
ultimately show $\exists \varphi'. \psi' \preceq \varphi' \wedge \text{propo-rew-step } r (\text{conn } c l) \varphi'$ **by** metis
qed

lemma *propo-rew-step-subformula:*

$(\exists \psi \psi'. \psi \preceq \varphi \wedge r \psi \psi') \longleftrightarrow (\exists \varphi'. \text{propo-rew-step } r \varphi \varphi')$
using *propo-rew-step-subformula-imp propo-rew-step-subformula-rec* **by** metis+

lemma *consistency-decompose-into-list:*

assumes $\text{wf}: \text{wf-conn } c l$ **and** $\text{wf}': \text{wf-conn } c l'$
and $\text{same}: \forall n. (A \models l ! n \longleftrightarrow (A \models l' ! n))$
shows $(A \models \text{conn } c l) = (A \models \text{conn } c l')$

proof (*cases c rule: connective-cases-arity-2*)

case *nullary*

thus $(A \models \text{conn } c l) \longleftrightarrow (A \models \text{conn } c l')$ **using** wf wf' **by** auto

next

case *unary note c = this*

then obtain a **where** $l: l = [a]$ **using** *wf-conn-Not-decomp wf* **by** metis

obtain a' **where** $l': l' = [a']$ **using** *wf-conn-Not-decomp wf' c* **by** metis

have $A \models a \longleftrightarrow A \models a'$ **using** $l l'$ **by** $(\text{metis nth-Cons-0 same})$

thus $A \models \text{conn } c l \longleftrightarrow A \models \text{conn } c l'$ **using** $l l' c$ **by** auto

next

case *binary note c = this*

then obtain $a b$ **where** $l: l = [a, b]$

using *wf-conn-bin-list-length list-length2-decomp wf* **by** metis

obtain $a' b'$ **where** $l': l' = [a', b']$

using *wf-conn-bin-list-length list-length2-decomp wf' c* **by** metis

have $p: A \models a \longleftrightarrow A \models a' \wedge A \models b \longleftrightarrow A \models b'$

using $l l'$ **same** **by** $(\text{metis diff-Suc-1 nth-Cons' nat.distinct(2)})+$

show $A \models \text{conn } c l \longleftrightarrow A \models \text{conn } c l'$

using $\text{wf } c p$ **unfolding** *binary-connectives-def l l'* **by** auto

qed

Relation between *propo-rew-step* and the rewriting we have seen before: *propo-rew-step* $r \varphi \varphi'$ means that we rewrite ψ inside φ (ie at a path p) into ψ' .

lemma *propo-rew-step-rewrite:*

fixes $\varphi \varphi' :: 'v \text{ propo}$ **and** $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$

assumes *propo-rew-step* $r \varphi \varphi'$

shows $\exists \psi \psi' p. r \psi \psi' \wedge \text{path-to } p \varphi \psi \wedge \text{replace-at } p \varphi \psi' = \varphi'$

using *assms*

proof (*induct rule: propo-rew-step.induct*)

case (*global-rel* $\varphi \psi$)

moreover have *path-to* $[] \varphi \varphi$ **by** auto

moreover have *replace-at* $[] \varphi \psi = \psi$ **by** auto

ultimately show *?case* **by** metis

next

case (*propo-rew-one-step-lift* $\varphi \varphi' c \xi \xi'$) **note** $\text{rel} = \text{this}(1)$ **and** $\text{IH0} = \text{this}(2)$ **and** $\text{corr} = \text{this}(3)$

obtain $\psi \psi' p$ **where** $\text{IH}: r \psi \psi' \wedge \text{path-to } p \varphi \psi \wedge \text{replace-at } p \varphi \psi' = \varphi'$ **using** IH0 **by** metis

{

fix $x :: 'v$

assume $c = CT \vee c = CF \vee c = CVar x$

```

  hence False using corr by auto
  hence  $\exists \psi \psi' p. r \psi \psi' \wedge \text{path-to } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi$ 
     $\wedge \text{replace-at } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi' = \text{conn } c (\xi @ (\varphi' \# \xi'))$ 
    by fast
}
moreover {
  assume c: c = CNot
  hence empty:  $\xi = [] \ \xi' = []$  using corr by auto
  have path-to (L#p) (conn c ( $\xi @ (\varphi \# \xi')$ ))  $\psi$ 
    using c empty IH wf-conn-unary path-to-l by fastforce
  moreover have replace-at (L#p) (conn c ( $\xi @ (\varphi \# \xi')$ ))  $\psi' = \text{conn } c (\xi @ (\varphi' \# \xi'))$ 
    using c empty IH by auto
  ultimately have  $\exists \psi \psi' p. r \psi \psi' \wedge \text{path-to } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi$ 
     $\wedge \text{replace-at } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi' = \text{conn } c (\xi @ (\varphi' \# \xi'))$ 
    using IH by metis
}
moreover {
  assume c: c  $\in$  binary-connectives
  have length ( $\xi @ \varphi \# \xi'$ ) = 2 using wf-conn-bin-list-length corr c by metis
  hence length  $\xi + \text{length } \xi' = 1$  by auto
  hence ld: (length  $\xi = 1 \wedge \text{length } \xi' = 0$ )  $\vee$  (length  $\xi = 0 \wedge \text{length } \xi' = 1$ ) by arith
  obtain a b where ab: ( $\xi = [] \wedge \xi' = [b]$ )  $\vee$  ( $\xi = [a] \wedge \xi' = []$ )
    using ld by (case-tac  $\xi$ , case-tac  $\xi'$ , auto)
  {
    assume  $\varphi$ :  $\xi = [] \wedge \xi' = [b]$ 
    have path-to (L#p) (conn c ( $\xi @ (\varphi \# \xi')$ ))  $\psi$ 
      using  $\varphi \ c \ IH \ ab \ corr$  by (simp add: path-to-l)
    moreover have replace-at (L#p) (conn c ( $\xi @ (\varphi \# \xi')$ ))  $\psi' = \text{conn } c (\xi @ (\varphi' \# \xi'))$ 
      using c IH ab  $\varphi$  unfolding binary-connectives-def by auto
    ultimately have  $\exists \psi \psi' p. r \psi \psi' \wedge \text{path-to } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi$ 
       $\wedge \text{replace-at } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi' = \text{conn } c (\xi @ (\varphi' \# \xi'))$ 
      using IH by metis
  }
  moreover {
    assume  $\varphi$ :  $\xi = [a] \ \xi' = []$ 
    hence path-to (R#p) (conn c ( $\xi @ (\varphi \# \xi')$ ))  $\psi$ 
      using c IH corr path-to-r corr  $\varphi$  by (simp add: path-to-r)
    moreover have replace-at (R#p) (conn c ( $\xi @ (\varphi \# \xi')$ ))  $\psi' = \text{conn } c (\xi @ (\varphi' \# \xi'))$ 
      using c IH ab  $\varphi$  unfolding binary-connectives-def by auto
    ultimately have ?case using IH by metis
  }
  ultimately have ?case using ab by blast
}
ultimately show ?case using connective-cases-arity by blast
qed

```

1.2 Consistency preservation

We define *preserves-un-sat*: it means that a relation preserves consistency.

definition *preserves-un-sat* **where**

preserves-un-sat $r \longleftrightarrow (\forall \varphi \psi. r \varphi \psi \longrightarrow (\forall A. A \models \varphi \longleftrightarrow A \models \psi))$

lemma *propo-rew-step-preservers-val-explicit*:

propo-rew-step $r \varphi \psi \implies \text{preserves-un-sat } r \implies \text{propo-rew-step } r \varphi \psi \implies (\forall A. A \models \varphi \longleftrightarrow A \models \psi)$

```

unfolding preserves-un-sat-def
proof (induction rule: propo-rew-step.induct)
  case global-rel
  thus ?case by simp
next
  case (propo-rew-one-step-lift  $\varphi \varphi' c \xi \xi'$ ) note  $rel = this(1)$  and  $wf = this(2)$ 
    and  $IH = this(3)[OF\ this(4)\ this(1)]$  and  $consistent = this(4)$ 
  {
    fix A
    from IH have  $\forall n. (A \models (\xi @ \varphi \# \xi') ! n) = (A \models (\xi @ \varphi' \# \xi') ! n)$ 
      by (metis (mono-tags, hide-lams) list-update-length nth-Cons-0 nth-append-length-plus
        nth-list-update-neg)
    hence  $(A \models conn\ c\ (\xi @ \varphi \# \xi')) = (A \models conn\ c\ (\xi @ \varphi' \# \xi'))$ 
      by (meson consistency-decompose-into-list wf wf-conn-no-arity-change-helper
        wf-conn-no-arity-change)
  }
  thus  $\forall A. A \models conn\ c\ (\xi @ \varphi \# \xi') \longleftrightarrow A \models conn\ c\ (\xi @ \varphi' \# \xi')$  by auto
qed

```

```

lemma propo-rew-step-preservers-val':
  assumes preserves-un-sat r
  shows preserves-un-sat (propo-rew-step r)
  using assms by (simp add: preserves-un-sat-def propo-rew-step-preservers-val-explicit)

```

```

lemma preserves-un-sat-OO[intro]:
  preserves-un-sat f  $\implies$  preserves-un-sat g  $\implies$  preserves-un-sat (f OO g)
  unfolding preserves-un-sat-def by auto

```

```

lemma star-consistency-preservation-explicit:
  assumes (propo-rew-step r)**  $\varphi \psi$  and preserves-un-sat r
  shows  $\forall A. A \models \varphi \longleftrightarrow A \models \psi$ 
  using assms by (induct rule: rtranclp-induct)
    (auto simp add: propo-rew-step-preservers-val-explicit)

```

```

lemma star-consistency-preservation:
  preserves-un-sat r  $\implies$  preserves-un-sat (propo-rew-step r)**
  by (simp add: star-consistency-preservation-explicit preserves-un-sat-def)

```

1.3 Full Lifting

In the previous a relation was lifted to a formula, now we define the relation such it is applied as long as possible. The definition is thus simply: it can be derived and nothing more can be derived.

```

lemma full-ropo-rew-step-preservers-val[simp]:
  preserves-un-sat r  $\implies$  preserves-un-sat (full (propo-rew-step r))
  by (metis full-def preserves-un-sat-def star-consistency-preservation)

```

```

lemma full-propo-rew-step-subformula:
  full (propo-rew-step r)  $\varphi' \varphi \implies \neg(\exists \psi \psi'. \psi \preceq \varphi \wedge r \psi \psi')$ 
  unfolding full-def using propo-rew-step-subformula-rec by metis

```

2 Transformation testing

2.1 Definition and first properties

To prove correctness of our transformation, we create a *all-subformula-st* predicate. It tests recursively all subformulas. At each step, the actual formula is tested. The aim of this *test-symb* function is to test locally some properties of the formulas (i.e. at the level of the connective or at first level). This allows a clause description between the rewrite relation and the *test-symb*

definition *all-subformula-st* :: ('a propo \Rightarrow bool) \Rightarrow 'a propo \Rightarrow bool **where**
all-subformula-st test-symb $\varphi \equiv \forall \psi. \psi \preceq \varphi \longrightarrow \text{test-symb } \psi$

lemma *test-symb-imp-all-subformula-st[simp]*:
test-symb FT \Longrightarrow *all-subformula-st test-symb FT*
test-symb FF \Longrightarrow *all-subformula-st test-symb FF*
test-symb (FVar x) \Longrightarrow *all-subformula-st test-symb (FVar x)*
unfolding *all-subformula-st-def* **using** *subformula-leaf* **by** *metis+*

lemma *all-subformula-st-test-symb-true-phi*:
all-subformula-st test-symb $\varphi \Longrightarrow \text{test-symb } \varphi$
unfolding *all-subformula-st-def* **by** *auto*

lemma *all-subformula-st-decomp-imp*:
wf-conn c l $\Longrightarrow (\text{test-symb } (\text{conn } c \ l) \wedge (\forall \varphi \in \text{set } l. \text{all-subformula-st test-symb } \varphi))$
 \Longrightarrow *all-subformula-st test-symb (conn c l)*
unfolding *all-subformula-st-def* **by** *auto*

To ease the finding of proofs, we give some explicit theorem about the decomposition.

lemma *all-subformula-st-decomp-rec*:
all-subformula-st test-symb (conn c l) $\Longrightarrow \text{wf-conn } c \ l$
 $\Longrightarrow (\text{test-symb } (\text{conn } c \ l) \wedge (\forall \varphi \in \text{set } l. \text{all-subformula-st test-symb } \varphi))$
unfolding *all-subformula-st-def* **by** *auto*

lemma *all-subformula-st-decomp*:
fixes *c* :: 'v connective **and** *l* :: 'v propo list
assumes *wf-conn c l*
shows *all-subformula-st test-symb (conn c l)*
 $\longleftrightarrow (\text{test-symb } (\text{conn } c \ l) \wedge (\forall \varphi \in \text{set } l. \text{all-subformula-st test-symb } \varphi))$
using *assms all-subformula-st-decomp-rec all-subformula-st-decomp-imp* **by** *metis*

lemma *helper-fact*: *c* \in *binary-connectives* $\longleftrightarrow (c = COr \vee c = CAnd \vee c = CEq \vee c = CImp)$
unfolding *binary-connectives-def* **by** *auto*

lemma *all-subformula-st-decomp-explicit[simp]*:
fixes $\varphi \ \psi$:: 'v propo
shows *all-subformula-st test-symb (FAnd $\varphi \ \psi$)*
 $\longleftrightarrow (\text{test-symb } (FAnd \ \varphi \ \psi) \wedge \text{all-subformula-st test-symb } \varphi \wedge \text{all-subformula-st test-symb } \psi)$
and *all-subformula-st test-symb (FOr $\varphi \ \psi$)*
 $\longleftrightarrow (\text{test-symb } (FOr \ \varphi \ \psi) \wedge \text{all-subformula-st test-symb } \varphi \wedge \text{all-subformula-st test-symb } \psi)$
and *all-subformula-st test-symb (FNot φ)*
 $\longleftrightarrow (\text{test-symb } (FNot \ \varphi) \wedge \text{all-subformula-st test-symb } \varphi)$
and *all-subformula-st test-symb (FEq $\varphi \ \psi$)*
 $\longleftrightarrow (\text{test-symb } (FEq \ \varphi \ \psi) \wedge \text{all-subformula-st test-symb } \varphi \wedge \text{all-subformula-st test-symb } \psi)$
and *all-subformula-st test-symb (FImp $\varphi \ \psi$)*
 $\longleftrightarrow (\text{test-symb } (FImp \ \varphi \ \psi) \wedge \text{all-subformula-st test-symb } \varphi \wedge \text{all-subformula-st test-symb } \psi)$

proof –

have $\text{all-subformula-st test-symb } (F\text{And } \varphi \psi) \longleftrightarrow \text{all-subformula-st test-symb } (\text{conn } C\text{And } [\varphi, \psi])$
by *auto*
moreover have $\dots \longleftrightarrow \text{test-symb } (\text{conn } C\text{And } [\varphi, \psi]) \wedge (\forall \xi \in \text{set } [\varphi, \psi]. \text{all-subformula-st test-symb } \xi)$
using *all-subformula-st-decomp wf-conn-helper-facts(5)* **by** *metis*
finally show $\text{all-subformula-st test-symb } (F\text{And } \varphi \psi)$
 $\longleftrightarrow (\text{test-symb } (F\text{And } \varphi \psi) \wedge \text{all-subformula-st test-symb } \varphi \wedge \text{all-subformula-st test-symb } \psi)$
by *simp*

have $\text{all-subformula-st test-symb } (F\text{Or } \varphi \psi) \longleftrightarrow \text{all-subformula-st test-symb } (\text{conn } C\text{Or } [\varphi, \psi])$
by *auto*
moreover have $\dots \longleftrightarrow$
 $(\text{test-symb } (\text{conn } C\text{Or } [\varphi, \psi]) \wedge (\forall \xi \in \text{set } [\varphi, \psi]. \text{all-subformula-st test-symb } \xi))$
using *all-subformula-st-decomp wf-conn-helper-facts(6)* **by** *metis*
finally show $\text{all-subformula-st test-symb } (F\text{Or } \varphi \psi)$
 $\longleftrightarrow (\text{test-symb } (F\text{Or } \varphi \psi) \wedge \text{all-subformula-st test-symb } \varphi \wedge \text{all-subformula-st test-symb } \psi)$
by *simp*

have $\text{all-subformula-st test-symb } (F\text{Eq } \varphi \psi) \longleftrightarrow \text{all-subformula-st test-symb } (\text{conn } C\text{Eq } [\varphi, \psi])$
by *auto*
moreover have \dots
 $\longleftrightarrow (\text{test-symb } (\text{conn } C\text{Eq } [\varphi, \psi]) \wedge (\forall \xi \in \text{set } [\varphi, \psi]. \text{all-subformula-st test-symb } \xi))$
using *all-subformula-st-decomp wf-conn-helper-facts(8)* **by** *metis*
finally show $\text{all-subformula-st test-symb } (F\text{Eq } \varphi \psi)$
 $\longleftrightarrow (\text{test-symb } (F\text{Eq } \varphi \psi) \wedge \text{all-subformula-st test-symb } \varphi \wedge \text{all-subformula-st test-symb } \psi)$
by *simp*

have $\text{all-subformula-st test-symb } (F\text{Imp } \varphi \psi) \longleftrightarrow \text{all-subformula-st test-symb } (\text{conn } C\text{Imp } [\varphi, \psi])$
by *auto*
moreover have \dots
 $\longleftrightarrow (\text{test-symb } (\text{conn } C\text{Imp } [\varphi, \psi]) \wedge (\forall \xi \in \text{set } [\varphi, \psi]. \text{all-subformula-st test-symb } \xi))$
using *all-subformula-st-decomp wf-conn-helper-facts(7)* **by** *metis*
finally show $\text{all-subformula-st test-symb } (F\text{Imp } \varphi \psi)$
 $\longleftrightarrow (\text{test-symb } (F\text{Imp } \varphi \psi) \wedge \text{all-subformula-st test-symb } \varphi \wedge \text{all-subformula-st test-symb } \psi)$
by *simp*

have $\text{all-subformula-st test-symb } (F\text{Not } \varphi) \longleftrightarrow \text{all-subformula-st test-symb } (\text{conn } C\text{Not } [\varphi])$
by *auto*
moreover have $\dots = (\text{test-symb } (\text{conn } C\text{Not } [\varphi]) \wedge (\forall \xi \in \text{set } [\varphi]. \text{all-subformula-st test-symb } \xi))$
using *all-subformula-st-decomp wf-conn-helper-facts(1)* **by** *metis*
finally show $\text{all-subformula-st test-symb } (F\text{Not } \varphi)$
 $\longleftrightarrow (\text{test-symb } (F\text{Not } \varphi) \wedge \text{all-subformula-st test-symb } \varphi)$ **by** *simp*
qed

As *all-subformula-st* tests recursively, the function is true on every subformula.

lemma *subformula-all-subformula-st*:

$\psi \preceq \varphi \implies \text{all-subformula-st test-symb } \varphi \implies \text{all-subformula-st test-symb } \psi$
by (*induct rule: subformula.induct, auto simp add: all-subformula-st-decomp*)

The following theorem *no-test-symb-step-exists* shows the link between the *test-symb* function and the corresponding rewrite relation *r*: if we assume that if every time *test-symb* is true, then a *r* can be applied, finally as long as $\neg \text{all-subformula-st test-symb } \varphi$, then something can be rewritten in φ .

lemma *no-test-symb-step-exists*:

```

fixes r:: 'v propo  $\Rightarrow$  'v propo  $\Rightarrow$  bool and test-symb:: 'v propo  $\Rightarrow$  bool and x :: 'v
and  $\varphi$  :: 'v propo
assumes test-symb-false-nullary:  $\forall x. \text{test-symb } FF \wedge \text{test-symb } FT \wedge \text{test-symb } (FVar\ x)$ 
and  $\forall \varphi'. \varphi' \preceq \varphi \longrightarrow (\neg \text{test-symb } \varphi') \longrightarrow (\exists \psi. r\ \varphi'\ \psi)$  and
 $\neg \text{all-subformula-st test-symb } \varphi$ 
shows  $(\exists \psi\ \psi'. \psi \preceq \varphi \wedge r\ \psi\ \psi')$ 
using assms
proof (induct  $\varphi$  rule: propo-induct-arity)
case (nullary  $\varphi\ x$ )
thus  $\exists \psi\ \psi'. \psi \preceq \varphi \wedge r\ \psi\ \psi'$ 
using wf-conn-nullary test-symb-false-nullary by fastforce
next
case (unary  $\varphi$ ) note IH = this(1)[OF this(2)] and r = this(2) and nst = this(3) and subf =
this(4)
from r IH nst have H:  $\neg \text{all-subformula-st test-symb } \varphi \implies \exists \psi. \psi \preceq \varphi \wedge (\exists \psi'. r\ \psi\ \psi')$ 
by (metis subformula-in-subformula-not subformula-refl subformula-trans)
{
assume n:  $\neg \text{test-symb } (FNot\ \varphi)$ 
obtain  $\psi$  where  $r\ (FNot\ \varphi)\ \psi$  using subformula-refl r n nst by blast
moreover have  $FNot\ \varphi \preceq FNot\ \varphi$  using subformula-refl by auto
ultimately have  $\exists \psi\ \psi'. \psi \preceq FNot\ \varphi \wedge r\ \psi\ \psi'$  by metis
}
moreover {
assume n:  $\text{test-symb } (FNot\ \varphi)$ 
hence  $\neg \text{all-subformula-st test-symb } \varphi$ 
using all-subformula-st-decomp-explicit(3) nst subf by blast
hence  $\exists \psi\ \psi'. \psi \preceq FNot\ \varphi \wedge r\ \psi\ \psi'$ 
using H subformula-in-subformula-not subformula-refl subformula-trans by blast
}
ultimately show  $\exists \psi\ \psi'. \psi \preceq FNot\ \varphi \wedge r\ \psi\ \psi'$  by blast
next
case (binary  $\varphi\ \varphi1\ \varphi2$ )
note  $IH\ \varphi1-0 = \text{this}(1)[OF\ \text{this}(4)]$  and  $IH\ \varphi2-0 = \text{this}(2)[OF\ \text{this}(4)]$  and r = this(4)
and  $\varphi = \text{this}(3)$  and le = this(5) and nst = this(6)

obtain c :: 'v connective where
c:  $(c = CAnd \vee c = COr \vee c = CImp \vee c = CEq) \wedge \text{conn } c\ [\varphi1, \varphi2] = \varphi$ 
using  $\varphi$  by fastforce

hence corr:  $\text{wf-conn } c\ [\varphi1, \varphi2]$  using wf-conn.simps unfolding binary-connectives-def by auto
have inc:  $\varphi1 \preceq \varphi\ \varphi2 \preceq \varphi$  using binary-connectives-def c subformula-in-binary-conn by blast+
from r IH $\varphi1-0$  have IH $\varphi1$ :  $\neg \text{all-subformula-st test-symb } \varphi1 \implies \exists \psi\ \psi'. \psi \preceq \varphi1 \wedge r\ \psi\ \psi'$ 
using inc(1) subformula-trans le by blast
from r IH $\varphi2-0$  have IH $\varphi2$ :  $\neg \text{all-subformula-st test-symb } \varphi2 \implies \exists \psi. \psi \preceq \varphi2 \wedge (\exists \psi'. r\ \psi\ \psi')$ 
using inc(2) subformula-trans le by blast
have cases:  $\neg \text{test-symb } \varphi \vee \neg \text{all-subformula-st test-symb } \varphi1 \vee \neg \text{all-subformula-st test-symb } \varphi2$ 
using c nst by auto
show  $\exists \psi\ \psi'. \psi \preceq \varphi \wedge r\ \psi\ \psi'$ 
using IH $\varphi1$  IH $\varphi2$  subformula-trans inc subformula-refl cases le by blast
qed

```

2.2 Invariant conservation

If two rewrite relation are independant (or at least independant enough), then the property characterizing the first relation *all-subformula-st test-symb* remains true. The next show the

same property, with changes in the assumptions.

The assumption $\forall \varphi' \psi. \varphi' \preceq \Phi \longrightarrow r \varphi' \psi \longrightarrow \text{all-subformula-st test-symb } \varphi' \longrightarrow \text{all-subformula-st test-symb } \psi$ means that rewriting with r does not mess up the property we want to preserve locally.

The previous assumption is not enough to go from r to *propo-rew-step* r : we have to add the assumption that rewriting inside does not mess up the term: $\forall c \xi \varphi \xi' \varphi'. \varphi \preceq \Phi \longrightarrow \text{propo-rew-step } r \varphi \varphi' \longrightarrow \text{wf-conn } c (\xi @ \varphi \# \xi') \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi')) \longrightarrow \text{test-symb } \varphi' \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$

2.2.1 Invariant while lifting of the rewriting relation

The condition $\varphi \preceq \Phi$ (that will be used with $\Phi = \varphi$ most of the time) is here to ensure that the recursive conditions on Φ will moreover hold for the subterm we are rewriting. For example if there is no equivalence symbol in Φ , we do not have to care about equivalence symbols in the two previous assumptions.

lemma *propo-rew-step-inv-stay*:

```

fixes  $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$  and  $\text{test-symb} :: 'v \text{ propo} \Rightarrow \text{bool}$  and  $x :: 'v$ 
and  $\varphi \psi \Phi :: 'v \text{ propo}$ 
assumes  $H: \forall \varphi' \psi. \varphi' \preceq \Phi \longrightarrow r \varphi' \psi \longrightarrow \text{all-subformula-st test-symb } \varphi'$ 
 $\longrightarrow \text{all-subformula-st test-symb } \psi$ 
and  $H': \forall (c :: 'v \text{ connective}) \xi \varphi \xi' \varphi'. \varphi \preceq \Phi \longrightarrow \text{propo-rew-step } r \varphi \varphi'$ 
 $\longrightarrow \text{wf-conn } c (\xi @ \varphi \# \xi') \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi')) \longrightarrow \text{test-symb } \varphi'$ 
 $\longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$  and
 $\text{propo-rew-step } r \varphi \psi$  and
 $\varphi \preceq \Phi$  and
 $\text{all-subformula-st test-symb } \varphi$ 
shows  $\text{all-subformula-st test-symb } \psi$ 
using assms(3-5)
proof (induct rule: propo-rew-step.induct)
case global-rel
thus ?case using  $H$  by simp
next
case (propo-rew-one-step-lift  $\varphi \varphi' c \xi \xi'$ )
note  $\text{rel} = \text{this}(1)$  and  $\varphi = \text{this}(2)$  and  $\text{corr} = \text{this}(3)$  and  $\Phi = \text{this}(4)$  and  $\text{nst} = \text{this}(5)$ 
have  $\text{sq}: \varphi \preceq \Phi$ 
using  $\Phi \text{ corr subformula-into-subformula subformula-refl subformula-trans}$ 
by (metis in-set-conv-decomp)
from  $\text{corr}$  have  $\forall \psi. \psi \in \text{set } (\xi @ \varphi \# \xi') \longrightarrow \text{all-subformula-st test-symb } \psi$ 
using  $\text{all-subformula-st-decomp nst}$  by blast
hence *:  $\forall \psi. \psi \in \text{set } (\xi @ \varphi' \# \xi') \longrightarrow \text{all-subformula-st test-symb } \psi$  using  $\varphi \text{ sq}$  by fastforce
hence  $\text{test-symb } \varphi'$  using  $\text{all-subformula-st-test-symb-true-phi}$  by auto
moreover from  $\text{corr nst}$  have  $\text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi'))$ 
using  $\text{all-subformula-st-decomp}$  by blast
ultimately have  $\text{test-symb: test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$  using  $H' \text{ sq corr rel}$  by blast

have  $\text{wf-conn } c (\xi @ \varphi' \# \xi')$ 
by (metis wf-conn-no-arity-change-helper corr wf-conn-no-arity-change)
thus  $\text{all-subformula-st test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$ 
using *  $\text{test-symb}$  by (metis all-subformula-st-decomp)
qed

```

The need for $\varphi \preceq \Phi$ is not always necessary, hence we moreover have a version without inclusion.

lemma *propo-rew-step-inv-stay*:

fixes $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ **and** $\text{test-symb} :: 'v \text{ propo} \Rightarrow \text{bool}$ **and** $x :: 'v$

and $\varphi \psi :: 'v \text{ propo}$

assumes

$H: \forall \varphi' \psi. r \varphi' \psi \longrightarrow \text{all-subformula-st test-symb } \varphi' \longrightarrow \text{all-subformula-st test-symb } \psi$ **and**

$H': \forall (c :: 'v \text{ connective}) \xi \varphi \xi' \varphi'. \text{wf-conn } c (\xi @ \varphi \# \xi') \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi'))$
 $\longrightarrow \text{test-symb } \varphi' \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$ **and**

$\text{propo-rew-step } r \varphi \psi$ **and**

$\text{all-subformula-st test-symb } \varphi$

shows $\text{all-subformula-st test-symb } \psi$

using $\text{propo-rew-step-inv-stay}'[\text{of } \varphi \ r \ \text{test-symb } \varphi \ \psi]$ *assms subformula-refl* **by** *metis*

The lemmas can be lifted to $\text{propo-rew-step } r^\downarrow$ instead of propo-rew-step

2.2.2 Invariant after all rewriting

lemma *full-propo-rew-step-inv-stay-with-inc*:

fixes $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ **and** $\text{test-symb} :: 'v \text{ propo} \Rightarrow \text{bool}$ **and** $x :: 'v$

and $\varphi \psi :: 'v \text{ propo}$

assumes

$H: \forall \varphi \psi. \text{propo-rew-step } r \varphi \psi \longrightarrow \text{all-subformula-st test-symb } \varphi$
 $\longrightarrow \text{all-subformula-st test-symb } \psi$ **and**

$H': \forall (c :: 'v \text{ connective}) \xi \varphi \xi' \varphi'. \varphi \preceq \Phi \longrightarrow \text{propo-rew-step } r \varphi \varphi'$
 $\longrightarrow \text{wf-conn } c (\xi @ \varphi \# \xi') \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi')) \longrightarrow \text{test-symb } \varphi'$
 $\longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$ **and**

$\varphi \preceq \Phi$ **and**

$\text{full: full } (\text{propo-rew-step } r) \varphi \psi$ **and**

$\text{init: all-subformula-st test-symb } \varphi$

shows $\text{all-subformula-st test-symb } \psi$

using *assms unfolding full-def*

proof –

have $\text{rel: } (\text{propo-rew-step } r)^{**} \varphi \psi$

using *full unfolding full-def* **by** *auto*

thus $\text{all-subformula-st test-symb } \psi$

using *init*

proof (*induct rule: rtranclp-induct*)

case *base*

then show $\text{all-subformula-st test-symb } \varphi$ **by** *blast*

next

case (*step b c*) **note** $\text{star} = \text{this}(1)$ **and** $\text{IH} = \text{this}(3)$ **and** $\text{one} = \text{this}(2)$ **and** $\text{all} = \text{this}(4)$

then have $\text{all-subformula-st test-symb } b$ **by** *metis*

then show $\text{all-subformula-st test-symb } c$ **using** $\text{propo-rew-step-inv-stay}' H H' \text{ rel one}$ **by** *auto*

qed

qed

lemma *full-propo-rew-step-inv-stay'*:

fixes $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ **and** $\text{test-symb} :: 'v \text{ propo} \Rightarrow \text{bool}$ **and** $x :: 'v$

and $\varphi \psi :: 'v \text{ propo}$

assumes

$H: \forall \varphi \psi. \text{propo-rew-step } r \varphi \psi \longrightarrow \text{all-subformula-st test-symb } \varphi$
 $\longrightarrow \text{all-subformula-st test-symb } \psi$ **and**

$H': \forall (c :: 'v \text{ connective}) \xi \varphi \xi' \varphi'. \text{propo-rew-step } r \varphi \varphi' \longrightarrow \text{wf-conn } c (\xi @ \varphi \# \xi')$

$\longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi')) \longrightarrow \text{test-symb } \varphi' \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$ **and**

$\text{full: full } (\text{propo-rew-step } r) \varphi \psi$ **and**

$\text{init: all-subformula-st test-symb } \varphi$

shows $\text{all-subformula-st test-symb } \psi$

using *full-propo-rew-step-inv-stay-with-inc*[of *r test-symb* φ] *assms subformula-refl* **by** *metis*

lemma *full-propo-rew-step-inv-stay*:

fixes $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ **and** *test-symb* :: $'v \text{ propo} \Rightarrow \text{bool}$ **and** $x :: 'v$

and $\varphi \ \psi :: 'v \text{ propo}$

assumes

$H: \forall \varphi \ \psi. r \ \varphi \ \psi \longrightarrow \text{all-subformula-st test-symb } \varphi \longrightarrow \text{all-subformula-st test-symb } \psi$ **and**

$H': \forall (c :: 'v \text{ connective}) \ \xi \ \varphi \ \xi' \ \varphi'. \text{wf-conn } c \ (\xi @ \varphi \# \xi') \longrightarrow \text{test-symb } (\text{conn } c \ (\xi @ \varphi \# \xi'))$
 $\longrightarrow \text{test-symb } \varphi' \longrightarrow \text{test-symb } (\text{conn } c \ (\xi @ \varphi' \# \xi'))$ **and**

full: *full* (*propo-rew-step* *r*) $\varphi \ \psi$ **and**

init: *all-subformula-st test-symb* φ

shows *all-subformula-st test-symb* ψ

unfolding *full-def*

proof –

have *rel*: (*propo-rew-step* *r*)^{∗∗} $\varphi \ \psi$

using *full* **unfolding** *full-def* **by** *auto*

thus *all-subformula-st test-symb* ψ

using *init*

proof (*induct rule*: *rtranclp-induct*)

case *base*

thus *all-subformula-st test-symb* φ **by** *blast*

next

case (*step* *b c*)

note *star* = *this*(1) **and** *IH* = *this*(3) **and** *one* = *this*(2) **and** *all* = *this*(4)

hence *all-subformula-st test-symb* *b* **by** *metis*

thus *all-subformula-st test-symb* *c*

using *propo-rew-step-inv-stay subformula-refl* *H H' rel one* **by** *auto*

qed

qed

lemma *full-propo-rew-step-inv-stay-conn*:

fixes $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ **and** *test-symb* :: $'v \text{ propo} \Rightarrow \text{bool}$ **and** $x :: 'v$

and $\varphi \ \psi :: 'v \text{ propo}$

assumes

$H: \forall \varphi \ \psi. r \ \varphi \ \psi \longrightarrow \text{all-subformula-st test-symb } \varphi \longrightarrow \text{all-subformula-st test-symb } \psi$ **and**

$H': \forall (c :: 'v \text{ connective}) \ l \ l'. \text{wf-conn } c \ l \longrightarrow \text{wf-conn } c \ l'$

$\longrightarrow (\text{test-symb } (\text{conn } c \ l) \longleftrightarrow \text{test-symb } (\text{conn } c \ l'))$ **and**

full: *full* (*propo-rew-step* *r*) $\varphi \ \psi$ **and**

init: *all-subformula-st test-symb* φ

shows *all-subformula-st test-symb* ψ

proof –

have $\bigwedge (c :: 'v \text{ connective}) \ \xi \ \varphi \ \xi' \ \varphi'. \text{wf-conn } c \ (\xi @ \varphi \# \xi')$

$\implies \text{test-symb } (\text{conn } c \ (\xi @ \varphi \# \xi')) \implies \text{test-symb } \varphi' \implies \text{test-symb } (\text{conn } c \ (\xi @ \varphi' \# \xi'))$

using *H'* **by** (*metis wf-conn-no-arity-change-helper wf-conn-no-arity-change*)

thus *all-subformula-st test-symb* ψ

using *H full init full-propo-rew-step-inv-stay* **by** *blast*

qed

end

theory *Prop-Normalisation*

imports *Main Prop-Logic Prop-Abstract-Transformation ../lib/Multiset-More*

begin

Given the previous definition about abstract rewriting and theorem about them, we now have the detailed rule making the transformation into CNF/DNF.

3 Rewrite Rules

The idea of Christoph Weidenbach's book is to remove gradually the operators: first equivalencies, then implication, after that the unused true/false and finally the reorganizing the or/and. We will prove each transformation separately.

3.1 Elimination of the equivalences

The first transformation consists in removing every equivalence symbol.

inductive *elim-equiv* :: 'v propo \Rightarrow 'v propo \Rightarrow bool **where**
elim-equiv[simp]: *elim-equiv* (FEq φ ψ) (FAnd (FImp φ ψ) (FImp ψ φ))

lemma *elim-equiv-transformation-consistent*:
 $A \models \text{FEq } \varphi \ \psi \longleftrightarrow A \models \text{FAnd } (\text{FImp } \varphi \ \psi) \ (\text{FImp } \psi \ \varphi)$
by *auto*

lemma *elim-equiv-explicit*: *elim-equiv* $\varphi \ \psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$
by (*induct* rule: *elim-equiv.induct*, *auto*)

lemma *elim-equiv-consistent*: *preserves-un-sat elim-equiv*
unfolding *preserves-un-sat-def* **by** (*simp* add: *elim-equiv-explicit*)

lemma *elimEquiv-lifted-consistant*:
preserves-un-sat (*full* (*propo-rew-step elim-equiv*))
by (*simp* add: *elim-equiv-consistent*)

This function ensures that there is no equivalencies left in the formula tested by *no-equiv-symb*.

fun *no-equiv-symb* :: 'v propo \Rightarrow bool **where**
no-equiv-symb (FEq -) = False |
no-equiv-symb - = True

Given the definition of *no-equiv-symb*, it does not depend on the formula, but only on the connective used.

lemma *no-equiv-symb-conn-characterization*[simp]:
fixes *c* :: 'v connective **and** *l* :: 'v propo list
assumes *wf*: *wf-conn c l*
shows *no-equiv-symb* (*conn c l*) $\longleftrightarrow c \neq \text{CEq}$
by (*metis* *connective.distinct*(13,25,35,43) *wf no-equiv-symb.elims*(3) *no-equiv-symb.simps*(1)
wf-conn.cases wf-conn-list(6))

definition *no-equiv* **where** *no-equiv* = *all-subformula-st no-equiv-symb*

lemma *no-equiv-eq*[simp]:
fixes $\varphi \ \psi$:: 'v propo
shows
 $\neg \text{no-equiv } (\text{FEq } \varphi \ \psi)$
no-equiv FT
no-equiv FF
using *no-equiv-symb.simps*(1) *all-subformula-st-test-symb-true-phi* **unfolding** *no-equiv-def* **by** *auto*

The following lemma helps to reconstruct *no-equiv* expressions: this representation is easier to use than the set definition.

lemma *all-subformula-st-decomp-explicit-no-equiv*[iff]:

fixes $\varphi \psi :: 'v \text{ propo}$

shows

$\text{no-equiv } (F\text{Not } \varphi) \longleftrightarrow \text{no-equiv } \varphi$
 $\text{no-equiv } (F\text{And } \varphi \psi) \longleftrightarrow (\text{no-equiv } \varphi \wedge \text{no-equiv } \psi)$
 $\text{no-equiv } (F\text{Or } \varphi \psi) \longleftrightarrow (\text{no-equiv } \varphi \wedge \text{no-equiv } \psi)$
 $\text{no-equiv } (F\text{Imp } \varphi \psi) \longleftrightarrow (\text{no-equiv } \varphi \wedge \text{no-equiv } \psi)$
by (*auto simp: no-equiv-def*)

A theorem to show the link between the rewrite relation *elim-equiv* and the function *no-equiv-symb*. This theorem is one of the assumption we need to characterize the transformation.

lemma *no-equiv-elim-equiv-step*:

fixes $\varphi :: 'v \text{ propo}$

assumes *no-equiv*: $\neg \text{no-equiv } \varphi$

shows $\exists \psi \psi'. \psi \preceq \varphi \wedge \text{elim-equiv } \psi \psi'$

proof –

have *test-symb-false-nullary*:

$\forall x::'v. \text{no-equiv-symb } FF \wedge \text{no-equiv-symb } FT \wedge \text{no-equiv-symb } (F\text{Var } x)$

unfolding *no-equiv-def* **by** *auto*

moreover {

fix $c::'v \text{ connective}$ **and** $l::'v \text{ propo list}$ **and** $\psi::'v \text{ propo}$

assume *a1*: $\text{elim-equiv } (\text{conn } c \ l) \ \psi$

have $\bigwedge p \text{ pa}. \neg \text{elim-equiv } (p::'v \text{ propo}) \text{ pa} \vee \neg \text{no-equiv-symb } p$

using *elim-equiv.cases no-equiv-symb.simps(1)* **by** *blast*

then have $\text{elim-equiv } (\text{conn } c \ l) \ \psi \implies \neg \text{no-equiv-symb } (\text{conn } c \ l)$ **using** *a1* **by** *metis*

}

moreover have $H': \forall \psi. \neg \text{elim-equiv } FT \ \psi \vee \psi. \neg \text{elim-equiv } FF \ \psi \vee \psi \ x. \neg \text{elim-equiv } (F\text{Var } x) \ \psi$

using *elim-equiv.cases* **by** *auto*

moreover have $\bigwedge \varphi. \neg \text{no-equiv-symb } \varphi \implies \exists \psi. \text{elim-equiv } \varphi \ \psi$

by (*case-tac* φ , *auto simp: elim-equiv.simps*)

then have $\bigwedge \varphi'. \varphi' \preceq \varphi \implies \neg \text{no-equiv-symb } \varphi' \implies \exists \psi. \text{elim-equiv } \varphi' \ \psi$ **by** *force*

ultimately show *?thesis*

using *no-test-symb-step-exists no-equiv test-symb-false-nullary unfolding no-equiv-def* **by** *blast*

qed

Given all the previous theorem and the characterization, once we have rewritten everything, there is no equivalence symbol any more.

lemma *no-equiv-full-propo-rew-step-elim-equiv*:

$\text{full } (\text{propo-rew-step } \text{elim-equiv}) \ \varphi \ \psi \implies \text{no-equiv } \psi$

using *full-propo-rew-step-subformula no-equiv-elim-equiv-step* **by** *blast*

3.2 Eliminate Implication

After that, we can eliminate the implication symbols.

inductive *elim-imp* $:: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ **where**

[*simp*]: *elim-imp* (*FImp* $\varphi \ \psi$) (*FOr* (*FNot* φ) ψ)

lemma *elim-imp-transformation-consistent*:

$A \models F\text{Imp } \varphi \ \psi \longleftrightarrow A \models F\text{Or } (F\text{Not } \varphi) \ \psi$

by *auto*

lemma *elim-imp-explicit*: $\text{elim-imp } \varphi \ \psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$

by (*induct* $\varphi \ \psi$ *rule: elim-imp.induct, auto*)

lemma *elim-imp-consistent*: *preserves-un-sat elim-imp*

unfolding *preserves-un-sat-def* **by** (*simp add: elim-imp-explicit*)

lemma *elim-imp-lifted-consistant*:
preserves-un-sat (full (propo-rew-step elim-imp))
by (*simp add: elim-imp-consistent*)

fun *no-imp-symb* **where**
no-imp-symb (FImp -) = False |
no-imp-symb - = True

lemma *no-imp-symb-conn-characterization*:
wf-conn c l \implies no-imp-symb (conn c l) \longleftrightarrow c \neq CImp
by (*induction rule: wf-conn-induct*) *auto*

definition *no-imp* **where** *no-imp \equiv all-subformula-st no-imp-symb*
declare *no-imp-def[simp]*

lemma *no-imp-Imp[simp]*:
 \neg *no-imp (FImp φ ψ)*
no-imp FT
no-imp FF
unfolding *no-imp-def* **by** *auto*

lemma *all-subformula-st-decomp-explicit-imp[simp]*:
fixes $\varphi \psi :: 'v$ *propo*
shows
no-imp (FNot φ) \longleftrightarrow no-imp φ
no-imp (FAnd φ ψ) \longleftrightarrow (no-imp $\varphi \wedge$ no-imp ψ)
no-imp (FOr φ ψ) \longleftrightarrow (no-imp $\varphi \wedge$ no-imp ψ)
by *auto*

Invariant of the *elim-imp* transformation

lemma *elim-imp-no-equiv*:
elim-imp $\varphi \psi \implies$ no-equiv $\varphi \implies$ no-equiv ψ
by (*induct $\varphi \psi$ rule: elim-imp.induct, auto*)

lemma *elim-imp-inv*:
fixes $\varphi \psi :: 'v$ *propo*
assumes *full (propo-rew-step elim-imp) $\varphi \psi$ and no-equiv φ*
shows *no-equiv ψ*
using *full-propo-rew-step-inv-stay-conn[of elim-imp no-equiv-symb $\varphi \psi$] assms elim-imp-no-equiv*
no-equiv-symb-conn-characterization **unfolding** *no-equiv-def* **by** *metis*

lemma *no-no-imp-elim-imp-step-exists*:
fixes $\varphi :: 'v$ *propo*
assumes *no-equiv: \neg no-imp φ*
shows $\exists \psi \psi'. \psi \preceq \varphi \wedge$ *elim-imp $\psi \psi'$*

proof –

have *test-symb-false-nullary: $\forall x. \text{no-imp-symb } FF \wedge \text{no-imp-symb } FT \wedge \text{no-imp-symb } (FVar (x :: 'v))$*
by *auto*
moreover {
fix $c :: 'v$ *connective* **and** $l :: 'v$ *propo list* **and** $\psi :: 'v$ *propo*
have *H: elim-imp (conn c l) $\psi \implies \neg$ no-imp-symb (conn c l)*
by (*auto elim: elim-imp.cases*)
}

```

moreover
  have  $H': \forall \psi. \neg \text{elim-imp } FT \ \psi \ \forall \psi. \neg \text{elim-imp } FF \ \psi \ \forall \psi \ x. \neg \text{elim-imp } (FVar \ x) \ \psi$ 
    by (auto elim: elim-imp.cases)+
moreover
  have  $\bigwedge \varphi. \neg \text{no-imp-symb } \varphi \implies \exists \psi. \text{elim-imp } \varphi \ \psi$ 
    by (case-tac  $\varphi$ ) (force simp: elim-imp.simps)+
  then have  $(\bigwedge \varphi'. \varphi' \preceq \varphi \implies \neg \text{no-imp-symb } \varphi' \implies \exists \psi. \text{elim-imp } \varphi' \ \psi)$  by force
ultimately show ?thesis
  using no-test-symb-step-exists no-equiv test-symb-false-nullary unfolding no-imp-def by blast
qed

```

```

lemma no-imp-full-propo-rew-step-elim-imp: full (propo-rew-step elim-imp)  $\varphi \ \psi \implies \text{no-imp } \psi$ 
  using full-propo-rew-step-subformula no-no-imp-elim-imp-step-exists by blast

```

3.3 Eliminate all the True and False in the formula

Contrary to the book, we have to give the transformation and the “commutative” transformation. The latter is implicit in the book.

inductive elimTB **where**

ElimTB1: elimTB (FAnd φ FT) φ |

ElimTB1': elimTB (FAnd FT φ) φ |

ElimTB2: elimTB (FAnd φ FF) FF |

ElimTB2': elimTB (FAnd FF φ) FF |

ElimTB3: elimTB (FOr φ FT) FT |

ElimTB3': elimTB (FOr FT φ) FT |

ElimTB4: elimTB (FOr φ FF) φ |

ElimTB4': elimTB (FOr FF φ) φ |

ElimTB5: elimTB (FNot FT) FF |

ElimTB6: elimTB (FNot FF) FT

lemma elimTB-consistent: preserves-un-sat elimTB

proof –

```

{
  fix  $\varphi \ \psi:: 'b \text{ propo}$ 
  have  $\text{elimTB } \varphi \ \psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$  by (induction rule: elimTB.inducts) auto
}
then show ?thesis using preserves-un-sat-def by auto
qed

```

inductive no-T-F-symb :: $'v \text{ propo} \Rightarrow \text{bool}$ **where**

no-T-F-symb-comp: $c \neq CF \implies c \neq CT \implies \text{wf-conn } c \ l \implies (\forall \varphi \in \text{set } l. \varphi \neq FT \wedge \varphi \neq FF)$
 $\implies \text{no-T-F-symb } (\text{conn } c \ l)$

lemma wf-conn-no-T-F-symb-iff[simp]:

$\text{wf-conn } c \ \psi s \implies$

$\text{no-T-F-symb } (\text{conn } c \ \psi s) \longleftrightarrow (c \neq CF \wedge c \neq CT \wedge (\forall \psi \in \text{set } \psi s. \psi \neq FF \wedge \psi \neq FT))$

unfolding no-T-F-symb.simps **apply** (cases c)

using wf-conn-list(1) **apply** fastforce

using wf-conn-list(2) **apply** fastforce

```

    using wf-conn-list(3) apply fastforce
    apply (metis (no-types, hide-lams) conn-inj connective.distinct(5,17))
    using conn-inj apply blast+
done

```

lemma *wf-conn-no-T-F-symb-iff-explicit*[simp]:

```

no-T-F-symb (FAnd  $\varphi$   $\psi$ )  $\longleftrightarrow (\forall \chi \in \text{set } [\varphi, \psi]. \chi \neq FF \wedge \chi \neq FT)$ 
no-T-F-symb (FOr  $\varphi$   $\psi$ )  $\longleftrightarrow (\forall \chi \in \text{set } [\varphi, \psi]. \chi \neq FF \wedge \chi \neq FT)$ 
no-T-F-symb (FEq  $\varphi$   $\psi$ )  $\longleftrightarrow (\forall \chi \in \text{set } [\varphi, \psi]. \chi \neq FF \wedge \chi \neq FT)$ 
no-T-F-symb (FImp  $\varphi$   $\psi$ )  $\longleftrightarrow (\forall \chi \in \text{set } [\varphi, \psi]. \chi \neq FF \wedge \chi \neq FT)$ 
  apply (metis conn.simps(36) conn.simps(37) conn.simps(5) propo.distinct(19)
    wf-conn-helper-facts(5) wf-conn-no-T-F-symb-iff)
  apply (metis conn.simps(36) conn.simps(37) conn.simps(6) propo.distinct(22)
    wf-conn-helper-facts(6) wf-conn-no-T-F-symb-iff)
  using wf-conn-no-T-F-symb-iff apply fastforce
by (metis conn.simps(36) conn.simps(37) conn.simps(7) propo.distinct(23) wf-conn-helper-facts(7)
  wf-conn-no-T-F-symb-iff)

```

lemma *no-T-F-symb-false*[simp]:

```

fixes c :: 'v connective
shows
   $\neg$ no-T-F-symb (FT :: 'v propo)
   $\neg$ no-T-F-symb (FF :: 'v propo)
  by (metis (no-types) conn.simps(1,2) wf-conn-no-T-F-symb-iff wf-conn-nullary)+

```

lemma *no-T-F-symb-bool*[simp]:

```

fixes x :: 'v
shows no-T-F-symb (FVar x)
using no-T-F-symb-comp wf-conn-nullary by (metis connective.distinct(3, 15) conn.simps(3)
  empty-iff list.set(1))

```

lemma *no-T-F-symb-fnot-imp*:

```

 $\neg$ no-T-F-symb (FNot  $\varphi$ )  $\implies \varphi = FT \vee \varphi = FF$ 
proof (rule ccontr)
  assume n:  $\neg$  no-T-F-symb (FNot  $\varphi$ )
  assume  $\neg (\varphi = FT \vee \varphi = FF)$ 
  then have  $\forall \varphi' \in \text{set } [\varphi]. \varphi' \neq FT \wedge \varphi' \neq FF$  by auto
  moreover have wf-conn CNot  $[\varphi]$  by simp
  ultimately have no-T-F-symb (FNot  $\varphi$ )
    using no-T-F-symb.intros by (metis conn.simps(4) connective.distinct(5,17))
  then show False using n by blast

```

qed

lemma *no-T-F-symb-fnot*[simp]:

```

no-T-F-symb (FNot  $\varphi$ )  $\longleftrightarrow \neg(\varphi = FT \vee \varphi = FF)$ 
using no-T-F-symb.simps no-T-F-symb-fnot-imp by (metis conn-inj-not(2) list.set-intros(1))

```

Actually it is not possible to remove every *FT* and *FF*: if the formula is equal to true or false, we can not remove it.

inductive *no-T-F-symb-except-toplevel* **where**

```

no-T-F-symb-except-toplevel-true[simp]: no-T-F-symb-except-toplevel FT |
no-T-F-symb-except-toplevel-false[simp]: no-T-F-symb-except-toplevel FF |
noTrue-no-T-F-symb-except-toplevel[simp]: no-T-F-symb  $\varphi \implies$  no-T-F-symb-except-toplevel  $\varphi$ 

```


lemma *no-T-F-symb-except-toplevel-bool*:

fixes $x :: 'v$

shows *no-T-F-symb-except-toplevel* (*FVar* x)

by *simp*

lemma *no-T-F-symb-except-toplevel-not-decom*:

$\varphi \neq FT \implies \varphi \neq FF \implies$ *no-T-F-symb-except-toplevel* (*FNot* φ)

by *simp*

lemma *no-T-F-symb-except-toplevel-bin-decom*:

fixes $\varphi \psi :: 'v$ *propo*

assumes $\varphi \neq FT$ **and** $\varphi \neq FF$ **and** $\psi \neq FT$ **and** $\psi \neq FF$

and $c :: \text{binary-connectives}$

shows *no-T-F-symb-except-toplevel* (*conn* c $[\varphi, \psi]$)

by (*metis* (*no-types*, *lifting*) *assms* c *conn.simps*(4) *list.discI* *noTrue-no-T-F-symb-except-toplevel* *wf-conn-no-T-F-symb-iff* *no-T-F-symb-fnot* *set.ConsD* *wf-conn-binary* *wf-conn-helper-facts*(1) *wf-conn-list-decomp*(1,2))

lemma *no-T-F-symb-except-toplevel-if-is-a-true-false*:

fixes $l :: 'v$ *propo list* **and** $c :: 'v$ *connective*

assumes *corr*: *wf-conn* c l

and $FT \in \text{set } l \vee FF \in \text{set } l$

shows \neg *no-T-F-symb-except-toplevel* (*conn* c l)

by (*metis* *assms* *empty-iff* *no-T-F-symb-except-toplevel.simps* *wf-conn-no-T-F-symb-iff* *set-empty* *wf-conn-list*(1,2))

lemma *no-T-F-symb-except-top-level-false-example*[*simp*]:

fixes $\varphi \psi :: 'v$ *propo*

assumes $\varphi = FT \vee \psi = FT \vee \varphi = FF \vee \psi = FF$

shows

\neg *no-T-F-symb-except-toplevel* (*FAnd* $\varphi \psi$)

\neg *no-T-F-symb-except-toplevel* (*FOr* $\varphi \psi$)

\neg *no-T-F-symb-except-toplevel* (*FImp* $\varphi \psi$)

\neg *no-T-F-symb-except-toplevel* (*FEq* $\varphi \psi$)

using *assms* *no-T-F-symb-except-toplevel-if-is-a-true-false* **unfolding** *binary-connectives-def*

by (*metis* (*no-types*) *conn.simps*(5-8) *insert-iff* *list.simps*(14-15) *wf-conn-helper-facts*(5-8))+

lemma *no-T-F-symb-except-top-level-false-not*[*simp*]:

fixes $\varphi \psi :: 'v$ *propo*

assumes $\varphi = FT \vee \varphi = FF$

shows

\neg *no-T-F-symb-except-toplevel* (*FNot* φ)

by (*simp* *add*: *assms* *no-T-F-symb-except-toplevel.simps*)

This is the local extension of *no-T-F-symb-except-toplevel*.

definition *no-T-F-except-top-level* **where**

no-T-F-except-top-level \equiv *all-subformula-st* *no-T-F-symb-except-toplevel*

This is another property we will use. While this version might seem to be the one we want to prove, it is not since *FT* can not be reduced.

definition *no-T-F* **where**

no-T-F \equiv *all-subformula-st* *no-T-F-symb*

lemma *no-T-F-except-top-level-false*:
fixes $l :: 'v \text{ propo list}$ **and** $c :: 'v \text{ connective}$
assumes *wf-conn* $c \ l$
and $FT \in \text{set } l \vee FF \in \text{set } l$
shows $\neg \text{no-T-F-except-top-level } (\text{conn } c \ l)$
by (*simp add: all-subformula-st-decomp assms no-T-F-except-top-level-def*
no-T-F-symb-except-toplevel-if-is-a-true-false)

lemma *no-T-F-except-top-level-false-example*[*simp*]:
fixes $\varphi \ \psi :: 'v \text{ propo}$
assumes $\varphi = FT \vee \psi = FT \vee \varphi = FF \vee \psi = FF$
shows
 $\neg \text{no-T-F-except-top-level } (F\text{And } \varphi \ \psi)$
 $\neg \text{no-T-F-except-top-level } (F\text{Or } \varphi \ \psi)$
 $\neg \text{no-T-F-except-top-level } (F\text{Eq } \varphi \ \psi)$
 $\neg \text{no-T-F-except-top-level } (F\text{Imp } \varphi \ \psi)$
by (*metis all-subformula-st-test-symb-true-phi assms no-T-F-except-top-level-def*
no-T-F-symb-except-top-level-false-example)+

lemma *no-T-F-symb-except-toplevel-no-T-F-symb*:
 $\text{no-T-F-symb-except-toplevel } \varphi \implies \varphi \neq FF \implies \varphi \neq FT \implies \text{no-T-F-symb } \varphi$
by (*induct rule: no-T-F-symb-except-toplevel.induct, auto*)

The two following lemmas give the precise link between the two definitions.

lemma *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb*:
 $\text{no-T-F-except-top-level } \varphi \implies \varphi \neq FF \implies \varphi \neq FT \implies \text{no-T-F } \varphi$
unfolding *no-T-F-except-top-level-def no-T-F-def* **apply** (*induct* φ)
using *no-T-F-symb-fnot* **by** *fastforce*+

lemma *no-T-F-no-T-F-except-top-level*:
 $\text{no-T-F } \varphi \implies \text{no-T-F-except-top-level } \varphi$
unfolding *no-T-F-except-top-level-def no-T-F-def*
unfolding *all-subformula-st-def* **by** *auto*

lemma *no-T-F-except-top-level-simp*[*simp*]: $\text{no-T-F-except-top-level } FF \ \text{no-T-F-except-top-level } FT$
unfolding *no-T-F-except-top-level-def* **by** *auto*

lemma *no-T-F-no-T-F-except-top-level'*[*simp*]:
 $\text{no-T-F-except-top-level } \varphi \longleftrightarrow (\varphi = FF \vee \varphi = FT \vee \text{no-T-F } \varphi)$
using *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb no-T-F-no-T-F-except-top-level*
by *auto*

lemma *no-T-F-bin-decomp*[*simp*]:
assumes $c: c \in \text{binary-connectives}$
shows $\text{no-T-F } (\text{conn } c \ [\varphi, \psi]) \longleftrightarrow (\text{no-T-F } \varphi \wedge \text{no-T-F } \psi)$

proof –

have *wf: wf-conn* $c \ [\varphi, \psi]$ **using** c **by** *auto*
then have $\text{no-T-F } (\text{conn } c \ [\varphi, \psi]) \longleftrightarrow (\text{no-T-F-symb } (\text{conn } c \ [\varphi, \psi]) \wedge \text{no-T-F } \varphi \wedge \text{no-T-F } \psi)$
by (*simp add: all-subformula-st-decomp no-T-F-def*)
then show $\text{no-T-F } (\text{conn } c \ [\varphi, \psi]) \longleftrightarrow (\text{no-T-F } \varphi \wedge \text{no-T-F } \psi)$
using c *wf all-subformula-st-decomp list.discI no-T-F-def no-T-F-symb-except-toplevel-bin-decom*
no-T-F-symb-except-toplevel-no-T-F-symb no-T-F-symb-false(1,2) wf-conn-helper-facts(2,3)
wf-conn-list(1,2) **by** *metis*

qed

```

lemma no-T-F-bin-decomp-expanded[simp]:
  assumes c:  $c = CAnd \vee c = COr \vee c = CEq \vee c = CImp$ 
  shows no-T-F (conn c [ $\varphi$ ,  $\psi$ ])  $\longleftrightarrow$  (no-T-F  $\varphi \wedge$  no-T-F  $\psi$ )
  using no-T-F-bin-decomp assms unfolding binary-connectives-def by blast

lemma no-T-F-comp-expanded-explicit[simp]:
  fixes  $\varphi \psi :: 'v \text{ propo}$ 
  shows
    no-T-F (FAnd  $\varphi \psi$ )  $\longleftrightarrow$  (no-T-F  $\varphi \wedge$  no-T-F  $\psi$ )
    no-T-F (FOr  $\varphi \psi$ )  $\longleftrightarrow$  (no-T-F  $\varphi \wedge$  no-T-F  $\psi$ )
    no-T-F (FEq  $\varphi \psi$ )  $\longleftrightarrow$  (no-T-F  $\varphi \wedge$  no-T-F  $\psi$ )
    no-T-F (FImp  $\varphi \psi$ )  $\longleftrightarrow$  (no-T-F  $\varphi \wedge$  no-T-F  $\psi$ )
  using assms conn.simps(5-8) no-T-F-bin-decomp-expanded by (metis (no-types))+

lemma no-T-F-comp-not[simp]:
  fixes  $\varphi \psi :: 'v \text{ propo}$ 
  shows no-T-F (FNot  $\varphi$ )  $\longleftrightarrow$  no-T-F  $\varphi$ 
  by (metis all-subformula-st-decomp-explicit(3) all-subformula-st-test-symb-true-phi no-T-F-def
    no-T-F-symb-false(1,2) no-T-F-symb-fnot-imp)

lemma no-T-F-decomp:
  fixes  $\varphi \psi :: 'v \text{ propo}$ 
  assumes  $\varphi$ : no-T-F (FAnd  $\varphi \psi$ )  $\vee$  no-T-F (FOr  $\varphi \psi$ )  $\vee$  no-T-F (FEq  $\varphi \psi$ )  $\vee$  no-T-F (FImp  $\varphi \psi$ )
  shows no-T-F  $\psi$  and no-T-F  $\varphi$ 
  using assms by auto

lemma no-T-F-decomp-not:
  fixes  $\varphi :: 'v \text{ propo}$ 
  assumes  $\varphi$ : no-T-F (FNot  $\varphi$ )
  shows no-T-F  $\varphi$ 
  using assms by auto

lemma no-T-F-symb-except-toplevel-step-exists:
  fixes  $\varphi \psi :: 'v \text{ propo}$ 
  assumes no-equiv  $\varphi$  and no-imp  $\varphi$ 
  shows  $\psi \preceq \varphi \implies \neg$  no-T-F-symb-except-toplevel  $\psi \implies \exists \psi'. \text{elimTB } \psi \psi'$ 
proof (induct  $\psi$  rule: propo-induct-arity)
  case (nullary  $\varphi' x$ )
  then have False using no-T-F-symb-except-toplevel-true no-T-F-symb-except-toplevel-false by auto
  then show ?case by blast
next
  case (unary  $\psi$ )
  then have  $\psi = FF \vee \psi = FT$  using no-T-F-symb-except-toplevel-not-decom by blast
  then show ?case using ElimTB5 ElimTB6 by blast
next
  case (binary  $\varphi' \psi_1 \psi_2$ )
  note IH1 = this(1) and IH2 = this(2) and  $\varphi' = \text{this}(3)$  and  $F\varphi = \text{this}(4)$  and  $n = \text{this}(5)$ 
  {
    assume  $\varphi' = FImp \psi_1 \psi_2 \vee \varphi' = FEq \psi_1 \psi_2$ 
    then have False using  $n$  F $\varphi$  subformula-all-subformula-st assms
      by (metis (no-types) no-equiv-eq(1) no-equiv-def no-imp-imp(1) no-imp-def)
    then have ?case by blast
  }
moreover {

```

```

    assume  $\varphi'$ :  $\varphi' = FAnd \ \psi1 \ \psi2 \vee \varphi' = FOr \ \psi1 \ \psi2$ 
    then have  $\psi1 = FT \vee \psi2 = FT \vee \psi1 = FF \vee \psi2 = FF$ 
      using no-T-F-symb-except-toplevel-bin-decom conn.simps(5,6) n unfolding binary-connectives-def
      by fastforce+
    then have ?case using elimTB.intros  $\varphi'$  by blast
  }
  ultimately show ?case using  $\varphi'$  by blast
qed

```

lemma *no-T-F-except-top-level-rew*:

```

  fixes  $\varphi :: 'v \text{ propo}$ 
  assumes noTB:  $\neg \text{no-T-F-except-top-level } \varphi$  and no-equiv: no-equiv  $\varphi$  and no-imp: no-imp  $\varphi$ 
  shows  $\exists \psi \ \psi'. \ \psi \preceq \varphi \wedge \text{elimTB } \psi \ \psi'$ 
proof -
  have test-symb-false-nullary:  $\forall x. \text{no-T-F-symb-except-toplevel } (FF :: 'v \text{ propo})$ 
     $\wedge \text{no-T-F-symb-except-toplevel } FT \wedge \text{no-T-F-symb-except-toplevel } (FVar \ (x :: 'v))$  by auto
  moreover {
    fix  $c :: 'v \text{ connective}$  and  $l :: 'v \text{ propo list}$  and  $\psi :: 'v \text{ propo}$ 
    have H: elimTB (conn  $c \ l$ )  $\psi \implies \neg \text{no-T-F-symb-except-toplevel } (\text{conn } c \ l)$ 
      by (cases (conn  $c \ l$ ) rule: elimTB.cases, auto)
  }
  moreover {
    fix  $x :: 'v$ 
    have H': no-T-F-except-top-level FT no-T-F-except-top-level FF
      no-T-F-except-top-level (FVar  $x$ )
      by (auto simp: no-T-F-except-top-level-def test-symb-false-nullary)
  }
  moreover {
    fix  $\psi$ 
    have  $\psi \preceq \varphi \implies \neg \text{no-T-F-symb-except-toplevel } \psi \implies \exists \psi'. \text{elimTB } \psi \ \psi'$ 
      using no-T-F-symb-except-toplevel-step-exists no-equiv no-imp by auto
  }
  ultimately show ?thesis
    using no-test-symb-step-exists noTB unfolding no-T-F-except-top-level-def by blast
qed

```

lemma *elimTB-inv*:

```

  fixes  $\varphi \ \psi :: 'v \text{ propo}$ 
  assumes full (propo-rew-step elimTB)  $\varphi \ \psi$ 
  and no-equiv  $\varphi$  and no-imp  $\varphi$ 
  shows no-equiv  $\psi$  and no-imp  $\psi$ 
proof -
  {
    fix  $\varphi \ \psi :: 'v \text{ propo}$ 
    have H: elimTB  $\varphi \ \psi \implies \text{no-equiv } \varphi \implies \text{no-equiv } \psi$ 
      by (induct  $\varphi \ \psi$  rule: elimTB.induct, auto)
  }
  then show no-equiv  $\psi$ 
    using full-propo-rew-step-inv-stay-conn[of elimTB no-equiv-symb  $\varphi \ \psi$ ]
      no-equiv-symb-conn-characterization assms unfolding no-equiv-def by metis
next
  {
    fix  $\varphi \ \psi :: 'v \text{ propo}$ 
    have H: elimTB  $\varphi \ \psi \implies \text{no-imp } \varphi \implies \text{no-imp } \psi$ 
      by (induct  $\varphi \ \psi$  rule: elimTB.induct, auto)
  }

```

```

}
then show no-imp  $\psi$ 
  using full-propo-rew-step-inv-stay-conn[of elimTB no-imp-symb  $\varphi$   $\psi$ ] assms
  no-imp-symb-conn-characterization unfolding no-imp-def by metis
qed

```

lemma *elimTB-full-propo-rew-step*:
 fixes $\varphi \psi :: 'v$ propo
 assumes no-equiv φ and no-imp φ and full (propo-rew-step elimTB) $\varphi \psi$
 shows no-T-F-except-top-level ψ
 using full-propo-rew-step-subformula no-T-F-except-top-level-rew assms elimTB-inv by fastforce

3.4 PushNeg

Push the negation inside the formula, until the litteral.

inductive *pushNeg* **where**
PushNeg1[simp]: *pushNeg* (FNot (FAnd $\varphi \psi$)) (FOr (FNot φ) (FNot ψ)) |
PushNeg2[simp]: *pushNeg* (FNot (FOr $\varphi \psi$)) (FAnd (FNot φ) (FNot ψ)) |
PushNeg3[simp]: *pushNeg* (FNot (FNot φ)) φ

lemma *pushNeg-transformation-consistent*:
 $A \models \text{FNot } (FAnd \varphi \psi) \longleftrightarrow A \models (FOr (FNot \varphi) (FNot \psi))$
 $A \models \text{FNot } (FOr \varphi \psi) \longleftrightarrow A \models (FAnd (FNot \varphi) (FNot \psi))$
 $A \models \text{FNot } (FNot \varphi) \longleftrightarrow A \models \varphi$
 by auto

lemma *pushNeg-explicit*: *pushNeg* $\varphi \psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$
 by (induct $\varphi \psi$ rule: *pushNeg.induct*, auto)

lemma *pushNeg-consistent*: *preserves-un-sat pushNeg*
 unfolding *preserves-un-sat-def* by (simp add: *pushNeg-explicit*)

lemma *pushNeg-lifted-consistant*:
preserves-un-sat (full (propo-rew-step *pushNeg*))
 by (simp add: *pushNeg-consistent*)

fun *simple* **where**
simple FT = True |
simple FF = True |
simple (FVar $-$) = True |
simple $-$ = False

lemma *simple-decomp*:
 $\text{simple } \varphi \longleftrightarrow (\varphi = FT \vee \varphi = FF \vee (\exists x. \varphi = FVar x))$
 by (cases φ) auto

lemma *subformula-conn-decomp-simple*:
 fixes $\varphi \psi :: 'v$ propo
 assumes s: *simple* ψ
 shows $\varphi \preceq \text{FNot } \psi \longleftrightarrow (\varphi = \text{FNot } \psi \vee \varphi = \psi)$
proof –
 have $\varphi \preceq \text{conn } CNot [\psi] \longleftrightarrow (\varphi = \text{conn } CNot [\psi] \vee (\exists \psi \in \text{set } [\psi]. \varphi \preceq \psi))$

using *subformula-conn-decomp wf-conn-helper-facts(1)* **by** *metis*
then show $\varphi \preceq FNot\ \psi \longleftrightarrow (\varphi = FNot\ \psi \vee \varphi = \psi)$ **using** *s* **by** *(auto simp: simple-decomp)*
qed

lemma *subformula-conn-decomp-explicit[simp]*:
fixes $\varphi :: 'v\ propo$ **and** $x :: 'v$
shows
 $\varphi \preceq FNot\ FT \longleftrightarrow (\varphi = FNot\ FT \vee \varphi = FT)$
 $\varphi \preceq FNot\ FF \longleftrightarrow (\varphi = FNot\ FF \vee \varphi = FF)$
 $\varphi \preceq FNot\ (FVar\ x) \longleftrightarrow (\varphi = FNot\ (FVar\ x) \vee \varphi = FVar\ x)$
by *(auto simp: subformula-conn-decomp-simple)*

fun *simple-not-symb* **where**
simple-not-symb (*FNot* φ) = (*simple* φ) |
simple-not-symb - = *True*

definition *simple-not* **where**
simple-not = *all-subformula-st simple-not-symb*
declare *simple-not-def[simp]*

lemma *simple-not-Not[simp]*:
 \neg *simple-not* (*FNot* (*FAnd* $\varphi\ \psi$))
 \neg *simple-not* (*FNot* (*FOr* $\varphi\ \psi$))
by *auto*

lemma *simple-not-step-exists*:
fixes $\varphi\ \psi :: 'v\ propo$
assumes *no-equiv* φ **and** *no-imp* φ
shows $\psi \preceq \varphi \implies \neg$ *simple-not-symb* $\psi \implies \exists \psi'. \text{pushNeg}\ \psi\ \psi'$
apply (*induct* ψ , *auto*)
apply (*rename-tac* ψ , *case-tac* ψ , *auto intro: pushNeg.intros*)
by (*metis assms(1,2) no-imp-Imp(1) no-equiv-eq(1) no-imp-def no-equiv-def*
subformula-in-subformula-not subformula-all-subformula-st)**+**

lemma *simple-not-rew*:
fixes $\varphi :: 'v\ propo$
assumes *noTB*: \neg *simple-not* φ **and** *no-equiv*: *no-equiv* φ **and** *no-imp*: *no-imp* φ
shows $\exists \psi\ \psi'. \psi \preceq \varphi \wedge \text{pushNeg}\ \psi\ \psi'$
proof –
have $\forall x. \text{simple-not-symb}\ (FF :: 'v\ propo) \wedge \text{simple-not-symb}\ FT \wedge \text{simple-not-symb}\ (FVar\ (x :: 'v))$
by *auto*
moreover {
fix $c :: 'v\ connective$ **and** $l :: 'v\ propo\ list$ **and** $\psi :: 'v\ propo$
have $H: \text{pushNeg}\ (\text{conn}\ c\ l)\ \psi \implies \neg \text{simple-not-symb}\ (\text{conn}\ c\ l)$
by (*cases* (*conn* $c\ l$) *rule: pushNeg.cases*) *auto*
}
moreover {
fix $x :: 'v$
have $H': \text{simple-not}\ FT\ \text{simple-not}\ FF\ \text{simple-not}\ (FVar\ x)$
by *simp-all*
}
moreover {
fix $\psi :: 'v\ propo$
have $\psi \preceq \varphi \implies \neg$ *simple-not-symb* $\psi \implies \exists \psi'. \text{pushNeg}\ \psi\ \psi'$

```

    using simple-not-step-exists no-equiv no-imp by blast
  }
  ultimately show ?thesis using no-test-symb-step-exists noTB unfolding simple-not-def by blast
qed

```

lemma *no-T-F-except-top-level-pushNeg1*:

```

  no-T-F-except-top-level (FNot (FAnd  $\varphi$   $\psi$ ))  $\implies$  no-T-F-except-top-level (FOr (FNot  $\varphi$ ) (FNot  $\psi$ ))
  using no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb no-T-F-comp-not no-T-F-decomp(1)
    no-T-F-decomp(2) no-T-F-no-T-F-except-top-level by (metis no-T-F-comp-expanded-explicit(2)
      propo.distinct(5,17))

```

lemma *no-T-F-except-top-level-pushNeg2*:

```

  no-T-F-except-top-level (FNot (FOr  $\varphi$   $\psi$ ))  $\implies$  no-T-F-except-top-level (FAnd (FNot  $\varphi$ ) (FNot  $\psi$ ))
  by auto

```

lemma *no-T-F-symb-pushNeg*:

```

  no-T-F-symb (FOr (FNot  $\varphi'$ ) (FNot  $\psi'$ ))
  no-T-F-symb (FAnd (FNot  $\varphi'$ ) (FNot  $\psi'$ ))
  no-T-F-symb (FNot (FNot  $\varphi'$ ))
  by auto

```

lemma *propo-rew-step-pushNeg-no-T-F-symb*:

```

  propo-rew-step pushNeg  $\varphi$   $\psi \implies$  no-T-F-except-top-level  $\varphi \implies$  no-T-F-symb  $\varphi \implies$  no-T-F-symb  $\psi$ 
  apply (induct rule: propo-rew-step.induct)
  apply (cases rule: pushNeg.cases)
  apply simp-all
  apply (metis no-T-F-symb-pushNeg(1))
  apply (metis no-T-F-symb-pushNeg(2))
  apply (simp, metis all-subformula-st-test-symb-true-phi no-T-F-def)

```

proof –

```

  fix  $\varphi \varphi'$ :: 'a propo and  $c$ :: 'a connective and  $\xi \xi'$ :: 'a propo list
  assume rel: propo-rew-step pushNeg  $\varphi \varphi'$ 
  and IH: no-T-F  $\varphi \implies$  no-T-F-symb  $\varphi \implies$  no-T-F-symb  $\varphi'$ 
  and wf: wf-conn  $c$  ( $\xi @ \varphi \# \xi'$ )
  and n: conn  $c$  ( $\xi @ \varphi \# \xi'$ ) = FF  $\vee$  conn  $c$  ( $\xi @ \varphi \# \xi'$ ) = FT  $\vee$  no-T-F (conn  $c$  ( $\xi @ \varphi \# \xi'$ ))
  and x:  $c \neq CF \wedge c \neq CT \wedge \varphi \neq FF \wedge \varphi \neq FT \wedge (\forall \psi \in \text{set } \xi \cup \text{set } \xi'. \psi \neq FF \wedge \psi \neq FT)$ 
  then have  $c \neq CF \wedge c \neq CT \wedge \text{wf-conn } c$  ( $\xi @ \varphi' \# \xi'$ )
    using wf-conn-no-arity-change-helper wf-conn-no-arity-change by metis
  moreover have  $n'$ : no-T-F (conn  $c$  ( $\xi @ \varphi \# \xi'$ )) using n by (simp add: wf wf-conn-list(1,2))
  moreover
  {
    have no-T-F  $\varphi$ 
      by (metis Un-iff all-subformula-st-decomp list.set-intros(1) n' wf no-T-F-def set-append)
    moreover then have no-T-F-symb  $\varphi$ 
      by (simp add: all-subformula-st-test-symb-true-phi no-T-F-def)
    ultimately have  $\varphi' \neq FF \wedge \varphi' \neq FT$ 
      using IH no-T-F-symb-false(1) no-T-F-symb-false(2) by blast
    then have  $\forall \psi \in \text{set } (\xi @ \varphi' \# \xi'). \psi \neq FF \wedge \psi \neq FT$  using x by auto
  }
  ultimately show no-T-F-symb (conn  $c$  ( $\xi @ \varphi' \# \xi'$ )) by (simp add: x)
qed

```

lemma *propo-rew-step-pushNeg-no-T-F*:

```

  propo-rew-step pushNeg  $\varphi \psi \implies$  no-T-F  $\varphi \implies$  no-T-F  $\psi$ 

```

proof (induct rule: propo-rew-step.induct)

```

case global-rel
then show ?case
  by (metis (no-types, lifting) no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb
      no-T-F-def no-T-F-except-top-level-pushNeg1 no-T-F-except-top-level-pushNeg2
      no-T-F-no-T-F-except-top-level all-subformula-st-decomp-explicit(3) pushNeg.simps
      simple.simps(1,2,5,6))
next
case (prop-rew-one-step-lift  $\varphi$   $\varphi'$   $c$   $\xi$   $\xi'$ )
note rel = this(1) and IH = this(2) and wf = this(3) and no-T-F = this(4)
moreover have wf': wf-conn  $c$  ( $\xi$  @  $\varphi'$  #  $\xi'$ )
  using wf-conn-no-arity-change wf-conn-no-arity-change-helper wf by metis
ultimately show no-T-F (conn  $c$  ( $\xi$  @  $\varphi'$  #  $\xi'$ ))
  using all-subformula-st-test-symb-true-phi
  by (fastforce simp: no-T-F-def all-subformula-st-decomp wf wf')
qed

```

```

lemma pushNeg-inv:
  fixes  $\varphi$   $\psi$  :: 'v propo
  assumes full (prop-rew-step pushNeg)  $\varphi$   $\psi$ 
  and no-equiv  $\varphi$  and no-imp  $\varphi$  and no-T-F-except-top-level  $\varphi$ 
  shows no-equiv  $\psi$  and no-imp  $\psi$  and no-T-F-except-top-level  $\psi$ 
proof -
  {
    fix  $\varphi$   $\psi$  :: 'v propo
    assume rel: prop-rew-step pushNeg  $\varphi$   $\psi$ 
    and no: no-T-F-except-top-level  $\varphi$ 
    then have no-T-F-except-top-level  $\psi$ 
    proof -
      {
        assume  $\varphi = FT \vee \varphi = FF$ 
        from rel this have False
        apply (induct rule: prop-rew-step.induct)
        using pushNeg.cases apply blast
        using wf-conn-list(1) wf-conn-list(2) by auto
        then have no-T-F-except-top-level  $\psi$  by blast
      }
      moreover {
        assume  $\varphi \neq FT \wedge \varphi \neq FF$ 
        then have no-T-F  $\varphi$ 
        by (metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
        then have no-T-F  $\psi$ 
        using prop-rew-step-pushNeg-no-T-F rel by auto
        then have no-T-F-except-top-level  $\psi$  by (simp add: no-T-F-no-T-F-except-top-level)
      }
      ultimately show no-T-F-except-top-level  $\psi$  by metis
    qed
  }
  moreover {
    fix  $c$  :: 'v connective and  $\xi$   $\xi'$  :: 'v propo list and  $\zeta$   $\zeta'$  :: 'v propo
    assume rel: prop-rew-step pushNeg  $\zeta$   $\zeta'$ 
    and incl:  $\zeta \preceq \varphi$ 
    and corr: wf-conn  $c$  ( $\xi$  @  $\zeta$  #  $\xi'$ )
    and no-T-F: no-T-F-symb-except-toplevel (conn  $c$  ( $\xi$  @  $\zeta$  #  $\xi'$ ))
    and n: no-T-F-symb-except-toplevel  $\zeta'$ 
  }

```



```

have no-T-F-symb-except-toplevel (conn c (ξ @ ζ' # ξ'))
proof
  have p: no-T-F-symb (conn c (ξ @ ζ # ξ'))
    using corr wf-conn-list(1) wf-conn-list(2) no-T-F-symb-except-toplevel-no-T-F-symb no-T-F
    by blast
  have l: ∀ φ ∈ set (ξ @ ζ # ξ'). φ ≠ FT ∧ φ ≠ FF
    using corr wf-conn-no-T-F-symb-iff p by blast
  from rel incl have ζ' ≠ FT ∧ ζ' ≠ FF
    apply (induction ζ ζ' rule: propo-rew-step.induct)
    apply (cases rule: pushNeg.cases, auto)
    by (metis assms(4) no-T-F-symb-except-top-level-false-not no-T-F-except-top-level-def
      all-subformula-st-test-symb-true-phi subformula-in-subformula-not
      subformula-all-subformula-st append-is-Nil-conv list.distinct(1)
      wf-conn-no-arity-change-helper wf-conn-list(1,2) wf-conn-no-arity-change)+
  then have ∀ φ ∈ set (ξ @ ζ' # ξ'). φ ≠ FT ∧ φ ≠ FF using l by auto
  moreover have c ≠ CT ∧ c ≠ CF using corr by auto
  ultimately show no-T-F-symb (conn c (ξ @ ζ' # ξ'))
    by (metis corr no-T-F-symb-comp wf-conn-no-arity-change wf-conn-no-arity-change-helper)
qed
}
ultimately show no-T-F-except-top-level ψ
  using full-propo-rew-step-inv-stay-with-inc[of pushNeg no-T-F-symb-except-toplevel φ] assms
  subformula-refl unfolding no-T-F-except-top-level-def full-unfold by metis
next
{
  fix φ ψ :: 'v propo
  have H: pushNeg φ ψ ⇒ no-equiv φ ⇒ no-equiv ψ
    by (induct φ ψ rule: pushNeg.induct, auto)
}
then show no-equiv ψ
  using full-propo-rew-step-inv-stay-conn[of pushNeg no-equiv-symb φ ψ]
  no-equiv-symb-conn-characterization assms unfolding no-equiv-def full-unfold by metis
next
{
  fix φ ψ :: 'v propo
  have H: pushNeg φ ψ ⇒ no-imp φ ⇒ no-imp ψ
    by (induct φ ψ rule: pushNeg.induct, auto)
}
then show no-imp ψ
  using full-propo-rew-step-inv-stay-conn[of pushNeg no-imp-symb φ ψ] assms
  no-imp-symb-conn-characterization unfolding no-imp-def full-unfold by metis
qed

```

lemma *pushNeg-full-propo-rew-step:*

```

fixes φ ψ :: 'v propo
assumes
  no-equiv φ and
  no-imp φ and
  full (propo-rew-step pushNeg) φ ψ and
  no-T-F-except-top-level φ
shows simple-not ψ
using assms full-propo-rew-step-subformula pushNeg-inv(1,2) simple-not-rew by blast

```

3.5 Push inside

inductive *push-conn-inside* :: 'v connective \Rightarrow 'v connective \Rightarrow 'v propo \Rightarrow 'v propo \Rightarrow bool
for *c c'* :: 'v connective **where**
push-conn-inside-l[simp]: $c = CAnd \vee c = COr \Longrightarrow c' = CAnd \vee c' = COr$
 $\Longrightarrow \text{push-conn-inside } c \ c' \ (\text{conn } c \ [\text{conn } c' \ [\varphi 1, \varphi 2], \psi])$
 $(\text{conn } c' \ [\text{conn } c \ [\varphi 1, \psi], \text{conn } c \ [\varphi 2, \psi]]) \mid$
push-conn-inside-r[simp]: $c = CAnd \vee c = COr \Longrightarrow c' = CAnd \vee c' = COr$
 $\Longrightarrow \text{push-conn-inside } c \ c' \ (\text{conn } c \ [\psi, \text{conn } c' \ [\varphi 1, \varphi 2]])$
 $(\text{conn } c' \ [\text{conn } c \ [\psi, \varphi 1], \text{conn } c \ [\psi, \varphi 2]])$

lemma *push-conn-inside-explicit*: $\text{push-conn-inside } c \ c' \ \varphi \ \psi \Longrightarrow \forall A. A \models \varphi \longleftrightarrow A \models \psi$
by (induct $\varphi \ \psi$ rule: *push-conn-inside.induct*, auto)

lemma *push-conn-inside-consistent*: *preserves-un-sat* (*push-conn-inside* *c c'*)
unfolding *preserves-un-sat-def* **by** (*simp* add: *push-conn-inside-explicit*)

lemma *propo-rew-step-push-conn-inside*[simp]:
 $\neg \text{propo-rew-step } (\text{push-conn-inside } c \ c') \ FT \ \psi \ \neg \text{propo-rew-step } (\text{push-conn-inside } c \ c') \ FF \ \psi$
proof –
{
{
fix $\varphi \ \psi$
have $\text{push-conn-inside } c \ c' \ \varphi \ \psi \Longrightarrow \varphi = FT \vee \varphi = FF \Longrightarrow \text{False}$
by (induct rule: *push-conn-inside.induct*, auto)
} **note** $H = \text{this}$
fix φ
have $\text{propo-rew-step } (\text{push-conn-inside } c \ c') \ \varphi \ \psi \Longrightarrow \varphi = FT \vee \varphi = FF \Longrightarrow \text{False}$
apply (induct rule: *propo-rew-step.induct*, auto *simp*: *wf-conn-list*(1) *wf-conn-list*(2))
using H **by** *blast+*
}
then show
 $\neg \text{propo-rew-step } (\text{push-conn-inside } c \ c') \ FT \ \psi$
 $\neg \text{propo-rew-step } (\text{push-conn-inside } c \ c') \ FF \ \psi$ **by** *blast+*
qed

inductive *not-c-in-c'-symb* :: 'v connective \Rightarrow 'v connective \Rightarrow 'v propo \Rightarrow bool **for** *c c'* **where**
not-c-in-c'-symb-l[simp]: $\text{wf-conn } c \ [\text{conn } c' \ [\varphi, \varphi'], \psi] \Longrightarrow \text{wf-conn } c' \ [\varphi, \varphi']$
 $\Longrightarrow \text{not-c-in-c'-symb } c \ c' \ (\text{conn } c \ [\text{conn } c' \ [\varphi, \varphi'], \psi]) \mid$
not-c-in-c'-symb-r[simp]: $\text{wf-conn } c \ [\psi, \text{conn } c' \ [\varphi, \varphi']] \Longrightarrow \text{wf-conn } c' \ [\varphi, \varphi']$
 $\Longrightarrow \text{not-c-in-c'-symb } c \ c' \ (\text{conn } c \ [\psi, \text{conn } c' \ [\varphi, \varphi']])$

abbreviation *c-in-c'-symb* *c c' φ* $\equiv \neg \text{not-c-in-c'-symb } c \ c' \ \varphi$

lemma *c-in-c'-symb-simp*:
 $\text{not-c-in-c'-symb } c \ c' \ \xi \Longrightarrow \xi = FF \vee \xi = FT \vee \xi = FVar \ x \vee \xi = FNot \ FF \vee \xi = FNot \ FT$
 $\vee \xi = FNot \ (FVar \ x) \Longrightarrow \text{False}$
apply (induct rule: *not-c-in-c'-symb.induct*, auto *simp*: *wf-conn.simps* *wf-conn-list*(1–3))
using *conn-inj-not*(2) *wf-conn-binary* **unfolding** *binary-connectives-def* **by** *fastforce+*

lemma *c-in-c'-symb-simp'*[simp]:
 $\neg \text{not-c-in-c'-symb } c \ c' \ FF$
 $\neg \text{not-c-in-c'-symb } c \ c' \ FT$

$\neg \text{not-c-in-c'-symb } c \ c' \ (FVar \ x)$
 $\neg \text{not-c-in-c'-symb } c \ c' \ (FNot \ FF)$
 $\neg \text{not-c-in-c'-symb } c \ c' \ (FNot \ FT)$
 $\neg \text{not-c-in-c'-symb } c \ c' \ (FNot \ (FVar \ x))$
using $c\text{-in-c'-symb-simp}$ **by** metis+

definition $c\text{-in-c'-only}$ **where**

$c\text{-in-c'-only } c \ c' \equiv \text{all-subformula-st } (c\text{-in-c'-symb } c \ c')$

lemma $c\text{-in-c'-only-simp}[simp]:$

$c\text{-in-c'-only } c \ c' \ FF$
 $c\text{-in-c'-only } c \ c' \ FT$
 $c\text{-in-c'-only } c \ c' \ (FVar \ x)$
 $c\text{-in-c'-only } c \ c' \ (FNot \ FF)$
 $c\text{-in-c'-only } c \ c' \ (FNot \ FT)$
 $c\text{-in-c'-only } c \ c' \ (FNot \ (FVar \ x))$
unfolding $c\text{-in-c'-only-def}$ **by** auto

lemma $\text{not-c-in-c'-symb-commute}:$

$\text{not-c-in-c'-symb } c \ c' \ \xi \implies \text{wf-conn } c \ [\varphi, \psi] \implies \xi = \text{conn } c \ [\varphi, \psi]$
 $\implies \text{not-c-in-c'-symb } c \ c' \ (\text{conn } c \ [\psi, \varphi])$

proof (*induct rule: not-c-in-c'-symb.induct*)

case ($\text{not-c-in-c'-symb-r } \varphi' \ \varphi'' \ \psi'$) **note** $H = \text{this}$
then have $\psi: \psi = \text{conn } c' \ [\varphi'', \psi']$ **using** conn-inj **by** auto
have $\text{wf-conn } c \ [\text{conn } c' \ [\varphi'', \psi'], \varphi]$
using $H(1)$ $\text{wf-conn-no-arity-change length-Cons}$ **by** metis
then show $\text{not-c-in-c'-symb } c \ c' \ (\text{conn } c \ [\psi, \varphi])$
unfolding ψ **using** $\text{not-c-in-c'-symb.intros}(1)$ H **by** auto

next

case ($\text{not-c-in-c'-symb-l } \varphi' \ \varphi'' \ \psi'$) **note** $H = \text{this}$
then have $\varphi = \text{conn } c' \ [\varphi', \varphi'']$ **using** conn-inj **by** auto
moreover have $\text{wf-conn } c \ [\psi', \text{conn } c' \ [\varphi', \varphi'']]$
using $H(1)$ $\text{wf-conn-no-arity-change length-Cons}$ **by** metis
ultimately show $\text{not-c-in-c'-symb } c \ c' \ (\text{conn } c \ [\psi, \varphi])$
using $\text{not-c-in-c'-symb.intros}(2)$ conn-inj $\text{not-c-in-c'-symb-l.hyps}$
 $\text{not-c-in-c'-symb-l.prem}(1,2)$ **by** blast

qed

lemma $\text{not-c-in-c'-symb-commute}':$

$\text{wf-conn } c \ [\varphi, \psi] \implies c\text{-in-c'-symb } c \ c' \ (\text{conn } c \ [\varphi, \psi]) \longleftrightarrow c\text{-in-c'-symb } c \ c' \ (\text{conn } c \ [\psi, \varphi])$
using $\text{not-c-in-c'-symb-commute}$ $\text{wf-conn-no-arity-change}$ **by** (metis length-Cons)

lemma $\text{not-c-in-c'-comm}:$

assumes $\text{wf}: \text{wf-conn } c \ [\varphi, \psi]$
shows $c\text{-in-c'-only } c \ c' \ (\text{conn } c \ [\varphi, \psi]) \longleftrightarrow c\text{-in-c'-only } c \ c' \ (\text{conn } c \ [\psi, \varphi])$ (**is** $?A \longleftrightarrow ?B$)

proof –

have $?A \longleftrightarrow (c\text{-in-c'-symb } c \ c' \ (\text{conn } c \ [\varphi, \psi])$
 $\quad \wedge (\forall \xi \in \text{set } [\varphi, \psi]. \text{all-subformula-st } (c\text{-in-c'-symb } c \ c' \ \xi))$
using $\text{all-subformula-st-decomp wf}$ **unfolding** $c\text{-in-c'-only-def}$ **by** fastforce
also have $\dots \longleftrightarrow (c\text{-in-c'-symb } c \ c' \ (\text{conn } c \ [\psi, \varphi])$
 $\quad \wedge (\forall \xi \in \text{set } [\psi, \varphi]. \text{all-subformula-st } (c\text{-in-c'-symb } c \ c' \ \xi))$
using $\text{not-c-in-c'-symb-commute}' \ \text{wf}$ **by** auto
also
have $\text{wf-conn } c \ [\psi, \varphi]$ **using** $\text{wf-conn-no-arity-change wf}$ **by** (metis length-Cons)

then have ($c\text{-in-}c'\text{-symb } c \ c' \ (\text{conn } c \ [\psi, \varphi])$
 $\wedge (\forall \xi \in \text{set } [\psi, \varphi]. \text{all-subformula-st } (c\text{-in-}c'\text{-symb } c \ c') \ \xi))$
 $\longleftrightarrow ?B$
using *all-subformula-st-decomp* **unfolding** *c-in-c'-only-def* **by** *fastforce*
finally show *?thesis* .
qed

lemma *not-c-in-c'-simp[simp]*:
fixes $\varphi 1 \ \varphi 2 \ \psi :: 'v \text{ propo}$ **and** $x :: 'v$
shows
 $c\text{-in-}c'\text{-symb } c \ c' \ FT$
 $c\text{-in-}c'\text{-symb } c \ c' \ FF$
 $c\text{-in-}c'\text{-symb } c \ c' \ (FVar \ x)$
 $wf\text{-conn } c \ [\text{conn } c' \ [\varphi 1, \varphi 2], \psi] \implies wf\text{-conn } c' \ [\varphi 1, \varphi 2]$
 $\implies \neg c\text{-in-}c'\text{-only } c \ c' \ (\text{conn } c \ [\text{conn } c' \ [\varphi 1, \varphi 2], \psi])$
apply (*simp-all add: c-in-c'-only-def*)
using *all-subformula-st-test-symb-true-phi not-c-in-c'-symb-l* **by** *blast*

lemma *c-in-c'-symb-not[simp]*:
fixes $c \ c' :: 'v \text{ connective}$ **and** $\psi :: 'v \text{ propo}$
shows $c\text{-in-}c'\text{-symb } c \ c' \ (FNot \ \psi)$
proof –
{
fix $\xi :: 'v \text{ propo}$
have $not\text{-}c\text{-in-}c'\text{-symb } c \ c' \ (FNot \ \psi) \implies False$
apply (*induct FNot ψ rule: not-c-in-c'-symb.induct*)
using *conn-inj-not(2)* **by** *blast+*
}
then show *?thesis* **by** *auto*
qed

lemma *c-in-c'-symb-step-exists*:
fixes $\varphi :: 'v \text{ propo}$
assumes $c: c = CAnd \vee c = COr$ **and** $c': c' = CAnd \vee c' = COr$
shows $\psi \preceq \varphi \implies \neg c\text{-in-}c'\text{-symb } c \ c' \ \psi \implies \exists \psi'. \text{push-conn-inside } c \ c' \ \psi \ \psi'$
apply (*induct ψ rule: propo-induct-arity*)
apply *auto[2]*
proof –
fix $\psi 1 \ \psi 2 \ \varphi' :: 'v \text{ propo}$
assume *IH $\psi 1$: $\psi 1 \preceq \varphi \implies \neg c\text{-in-}c'\text{-symb } c \ c' \ \psi 1 \implies Ex (\text{push-conn-inside } c \ c' \ \psi 1)$*
and *IH $\psi 2$: $\psi 2 \preceq \varphi \implies \neg c\text{-in-}c'\text{-symb } c \ c' \ \psi 2 \implies Ex (\text{push-conn-inside } c \ c' \ \psi 2)$*
and $\varphi': \varphi' = FAnd \ \psi 1 \ \psi 2 \vee \varphi' = FOr \ \psi 1 \ \psi 2 \vee \varphi' = FImp \ \psi 1 \ \psi 2 \vee \varphi' = FEq \ \psi 1 \ \psi 2$
and *in φ : $\varphi' \preceq \varphi$ and $n0: \neg c\text{-in-}c'\text{-symb } c \ c' \ \varphi'$*
then have $n: not\text{-}c\text{-in-}c'\text{-symb } c \ c' \ \varphi'$ **by** *auto*
{
assume $\varphi': \varphi' = \text{conn } c \ [\psi 1, \psi 2]$
obtain $a \ b$ **where** $\psi 1 = \text{conn } c' \ [a, b] \vee \psi 2 = \text{conn } c' \ [a, b]$
using $n \ \varphi'$ **apply** (*induct rule: not-c-in-c'-symb.induct*)
using c **by** *force+*
then have $Ex (\text{push-conn-inside } c \ c' \ \varphi')$
unfolding φ' **apply** *auto*
using *push-conn-inside.intros(1) c c'* **apply** *blast*
using *push-conn-inside.intros(2) c c'* **by** *blast*
}
moreover {

```

    assume  $\varphi'$ :  $\varphi' \neq \text{conn } c [\psi 1, \psi 2]$ 
    have  $\forall \varphi \ c \ ca. \exists \varphi 1 \ \psi 1 \ \psi 2 \ \psi 1' \ \psi 2' \ \varphi 2'. \text{conn } (c::'v \text{ connective}) [\varphi 1, \text{conn } ca [\psi 1, \psi 2]] = \varphi$ 
       $\vee \text{conn } c [\text{conn } ca [\psi 1', \psi 2'], \varphi 2'] = \varphi \vee c\text{-in-}c'\text{-symb } c \ ca \ \varphi$ 
    by (metis not-c-in-c'-symb.cases)
    then have  $Ex \ (\text{push-conn-inside } c \ c' \ \varphi')$ 
    by (metis (no-types)  $c \ c' \ n \ \text{push-conn-inside-l} \ \text{push-conn-inside-r}$ )
  }
  ultimately show  $Ex \ (\text{push-conn-inside } c \ c' \ \varphi')$  by blast
qed

```

lemma *c-in-c'-symb-rew*:

```

  fixes  $\varphi :: 'v \text{ propo}$ 
  assumes noTB:  $\neg c\text{-in-}c'\text{-only } c \ c' \ \varphi$ 
  and  $c: c = CAnd \vee c = COr$  and  $c': c' = CAnd \vee c' = COr$ 
  shows  $\exists \psi \ \psi'. \psi \preceq \varphi \wedge \text{push-conn-inside } c \ c' \ \psi \ \psi'$ 
proof -
  have test-symb-false-nullary:
     $\forall x. c\text{-in-}c'\text{-symb } c \ c' \ (FF:: 'v \text{ propo}) \wedge c\text{-in-}c'\text{-symb } c \ c' \ FT$ 
     $\wedge c\text{-in-}c'\text{-symb } c \ c' \ (FVar \ (x:: 'v))$ 
  by auto
  moreover {
    fix  $x :: 'v$ 
    have  $H': c\text{-in-}c'\text{-symb } c \ c' \ FT \ c\text{-in-}c'\text{-symb } c \ c' \ FF \ c\text{-in-}c'\text{-symb } c \ c' \ (FVar \ x)$ 
    by simp+
  }
  moreover {
    fix  $\psi :: 'v \text{ propo}$ 
    have  $\psi \preceq \varphi \implies \neg c\text{-in-}c'\text{-symb } c \ c' \ \psi \implies \exists \psi'. \text{push-conn-inside } c \ c' \ \psi \ \psi'$ 
    by (auto simp: assms(2)  $c' \ c\text{-in-}c'\text{-symb-step-exists}$ )
  }
  ultimately show ?thesis using noTB no-test-symb-step-exists[of  $c\text{-in-}c'\text{-symb } c \ c'$ ]
    unfolding c-in-c'-only-def by metis
qed

```

lemma *push-conn-insidec-in-c'-symb-no-T-F*:

```

  fixes  $\varphi \ \psi :: 'v \text{ propo}$ 
  shows propo-rew-step ( $\text{push-conn-inside } c \ c' \ \varphi \ \psi \implies \text{no-T-F } \varphi \implies \text{no-T-F } \psi$ )
proof (induct rule: propo-rew-step.induct)
  case (global-rel  $\varphi \ \psi$ )
  then show no-T-F  $\psi$ 
    by (cases rule: push-conn-inside.cases, auto)
next
  case (propo-rew-one-step-lift  $\varphi \ \varphi' \ c \ \xi \ \xi'$ )
  note  $rel = \text{this}(1)$  and  $IH = \text{this}(2)$  and  $wf = \text{this}(3)$  and  $\text{no-T-F} = \text{this}(4)$ 
  have no-T-F  $\varphi$ 
    using  $wf \ \text{no-T-F} \ \text{no-T-F-def} \ \text{subformula-into-subformula} \ \text{subformula-all-subformula-st}$ 
    subformula-refl by (metis (no-types) in-set-conv-decomp)
  then have  $\varphi': \text{no-T-F } \varphi'$  using IH by blast

  have  $\forall \zeta \in \text{set } (\xi @ \varphi \ \# \ \xi'). \text{no-T-F } \zeta$  by (metis  $wf \ \text{no-T-F} \ \text{no-T-F-def} \ \text{all-subformula-st-decomp}$ )
  then have  $n: \forall \zeta \in \text{set } (\xi @ \varphi' \ \# \ \xi'). \text{no-T-F } \zeta$  using  $\varphi'$  by auto
  then have  $n': \forall \zeta \in \text{set } (\xi @ \varphi' \ \# \ \xi'). \zeta \neq FF \wedge \zeta \neq FT$ 
    using  $\varphi'$  by (metis no-T-F-symb-false(1) no-T-F-symb-false(2) no-T-F-def
      all-subformula-st-test-symb-true-phi)

```

```

have wf': wf-conn c (ξ @ φ' # ξ')
  using wf wf-conn-no-arity-change by (metis wf-conn-no-arity-change-helper)
{
  fix x :: 'v
  assume c = CT ∨ c = CF ∨ c = CVar x
  then have False using wf by auto
  then have no-T-F (conn c (ξ @ φ' # ξ')) by blast
}
moreover {
  assume c: c = CNot
  then have ξ = [] ξ' = [] using wf by auto
  then have no-T-F (conn c (ξ @ φ' # ξ'))
    using c by (metis φ' conn.simps(4) no-T-F-symb-false(1,2) no-T-F-symb-fnot no-T-F-def
      all-subformula-st-decomp-explicit(3) all-subformula-st-test-symb-true-phi self-append-conv2)
}
moreover {
  assume c: c ∈ binary-connectives
  then have no-T-F-symb (conn c (ξ @ φ' # ξ')) using wf' n' no-T-F-symb.simps by fastforce
  then have no-T-F (conn c (ξ @ φ' # ξ'))
    by (metis all-subformula-st-decomp-imp wf' n no-T-F-def)
}
ultimately show no-T-F (conn c (ξ @ φ' # ξ')) using connective-cases-arity by auto
qed

```

lemma *simple-propo-rew-step-push-conn-inside-inv*:

```

propo-rew-step (push-conn-inside c c') φ ψ ⇒ simple φ ⇒ simple ψ
  apply (induct rule: propo-rew-step.induct)
  apply (rename-tac φ, case-tac φ, auto simp: push-conn-inside.simps)[]
  by (metis append-is-Nil-conv list.distinct(1) simple.elims(2) wf-conn-list(1-3))

```

lemma *simple-propo-rew-step-inv-push-conn-inside-simple-not*:

```

fixes c c' :: 'v connective and φ ψ :: 'v propo
shows propo-rew-step (push-conn-inside c c') φ ψ ⇒ simple-not φ ⇒ simple-not ψ
proof (induct rule: propo-rew-step.induct)
  case (global-rel φ ψ)
  then show ?case by (cases φ, auto simp: push-conn-inside.simps)
next
  case (propo-rew-one-step-lift φ φ' ca ξ ξ') note rew = this(1) and IH = this(2) and wf = this(3)
  and simple = this(4)
  show ?case
    proof (cases ca rule: connective-cases-arity)
      case nullary
      then show ?thesis using propo-rew-one-step-lift by auto
    next
      case binary note ca = this
      obtain a b where ab: ξ @ φ' # ξ' = [a, b]
      using wf ca list-length2-decomp wf-conn-bin-list-length
      by (metis (no-types) wf-conn-no-arity-change-helper)
      have ∀ ζ ∈ set (ξ @ φ # ξ'). simple-not ζ
        by (metis wf all-subformula-st-decomp simple simple-not-def)
      then have ∀ ζ ∈ set (ξ @ φ' # ξ'). simple-not ζ using IH by simp
      moreover have simple-not-symb (conn ca (ξ @ φ' # ξ')) using ca

```

```

  by (metis ab conn.simps(5-8) helper-fact simple-not-symb.simps(5) simple-not-symb.simps(6)
      simple-not-symb.simps(7) simple-not-symb.simps(8))
  ultimately show ?thesis
    by (simp add: ab all-subformula-st-decomp ca)
next
  case unary
  then show ?thesis
    using rew simple-propo-rew-step-push-conn-inside-inv[OF rew] IH local.wf simple by auto
qed
qed

```

```

lemma propo-rew-step-push-conn-inside-simple-not:
  fixes  $\varphi \varphi' :: 'v$  propo and  $\xi \xi' :: 'v$  propo list and  $c :: 'v$  connective
  assumes
    propo-rew-step (push-conn-inside  $c \ c'$ )  $\varphi \ \varphi'$  and
    wf-conn  $c \ (\xi @ \varphi \# \xi')$  and
    simple-not-symb (conn  $c \ (\xi @ \varphi \# \xi')$ ) and
    simple-not-symb  $\varphi'$ 
  shows simple-not-symb (conn  $c \ (\xi @ \varphi' \# \xi')$ )
  using assms
proof (induction rule: propo-rew-step.induct)
print-cases
  case (global-rel)
  then show ?case
    by (metis conn.simps(12,17) list.discI push-conn-inside.cases simple-not-symb.elims(3)
        wf-conn-helper-facts(5) wf-conn-list(2) wf-conn-list(8) wf-conn-no-arity-change
        wf-conn-no-arity-change-helper)
next
  case (propo-rew-one-step-lift  $\varphi \ \varphi' \ c' \ \chi s \ \chi s'$ ) note tel = this(1) and wf = this(2) and
    IH = this(3) and wf' = this(4) and simple' = this(5) and simple = this(6)
  then show ?case
    proof (cases  $c'$  rule: connective-cases-arity)
    case nullary
    then show ?thesis using wf simple simple' by auto
    next
    case binary note  $c = \text{this}(1)$ 
    have corr': wf-conn  $c \ (\xi @ \text{conn } c' \ (\chi s @ \varphi' \# \chi s') \# \xi')$ 
      using wf wf-conn-no-arity-change
    by (metis wf' wf-conn-no-arity-change-helper)
    then show ?thesis
      using  $c$  propo-rew-one-step-lift wf
    by (metis conn.simps(17) connective.distinct(37) propo-rew-step-subformula-imp
        push-conn-inside.cases simple-not-symb.elims(3) wf-conn.simps wf-conn-list(2,8))
    next
    case unary
    then have empty:  $\chi s = [] \ \chi s' = []$  using wf by auto
    then show ?thesis using simple unary simple' wf wf'
    by (metis connective.distinct(37) connective.distinct(39) propo-rew-step-subformula-imp
        push-conn-inside.cases simple-not-symb.elims(3) tel wf-conn-list(8)
        wf-conn-no-arity-change wf-conn-no-arity-change-helper)
    qed
  qed
qed

```

```

lemma push-conn-inside-not-true-false:
  push-conn-inside  $c \ c' \ \varphi \ \psi \implies \psi \neq FT \wedge \psi \neq FF$ 
```

by (induct rule: push-conn-inside.induct, auto)

lemma *push-conn-inside-inv*:

fixes $\varphi \psi :: 'v \text{ propo}$

assumes *full* (propo-rew-step (push-conn-inside $c \ c'$) $\varphi \ \psi$)

and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ **and** *simple-not* φ

shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ **and** *simple-not* ψ

proof –

```
{
  {
    fix  $\varphi \psi :: 'v \text{ propo}$ 
    have  $H$ : push-conn-inside  $c \ c' \ \varphi \ \psi \implies$  all-subformula-st simple-not-symb  $\varphi$ 
       $\implies$  all-subformula-st simple-not-symb  $\psi$ 
      by (induct  $\varphi \ \psi$  rule: push-conn-inside.induct, auto)
  } note  $H = \text{this}$ 
```

fix $\varphi \psi :: 'v \text{ propo}$

have H : propo-rew-step (push-conn-inside $c \ c'$) $\varphi \ \psi \implies$ all-subformula-st simple-not-symb φ
 \implies all-subformula-st simple-not-symb ψ

apply (induct $\varphi \ \psi$ rule: propo-rew-step.induct)

using H **apply** *simp*

proof (rename-tac $\varphi \ \varphi'$ *ca* $\psi \ \psi'$, case-tac *ca* rule: connective-cases-arity)

fix $\varphi \ \varphi' :: 'v \text{ propo}$ **and** $c :: 'v \text{ connective}$ **and** $\xi \ \xi' :: 'v \text{ propo list}$

and $x :: 'v$

assume *wf-conn* $c \ (\xi @ \varphi \ \# \ \xi')$

and $c = CT \vee c = CF \vee c = CVar \ x$

then have $\xi @ \varphi \ \# \ \xi' = []$ **by** *auto*

then have *False* **by** *auto*

then show all-subformula-st simple-not-symb (conn $c \ (\xi @ \varphi' \ \# \ \xi')$) **by** *blast*

next

fix $\varphi \ \varphi' :: 'v \text{ propo}$ **and** $ca :: 'v \text{ connective}$ **and** $\xi \ \xi' :: 'v \text{ propo list}$

and $x :: 'v$

assume *rel*: propo-rew-step (push-conn-inside $c \ c'$) $\varphi \ \varphi'$

and $\varphi\text{-}\varphi'$: all-subformula-st simple-not-symb $\varphi \implies$ all-subformula-st simple-not-symb φ'

and *corr*: *wf-conn* $ca \ (\xi @ \varphi \ \# \ \xi')$

and n : all-subformula-st simple-not-symb (conn $ca \ (\xi @ \varphi \ \# \ \xi')$)

and c : $ca = CNot$

have *empty*: $\xi = [] \ \xi' = []$ **using** $c \ \text{corr}$ **by** *auto*

then have *simple-not*: all-subformula-st simple-not-symb (*FNot* φ) **using** $\text{corr} \ c \ n$ **by** *auto*

then have *simple* φ

using all-subformula-st-test-symb-true-phi *simple-not-symb.simps*(1) **by** *blast*

then have *simple* φ'

using *rel* *simple-propo-rew-step-push-conn-inside-inv* **by** *blast*

then show all-subformula-st simple-not-symb (conn $ca \ (\xi @ \varphi' \ \# \ \xi')$) **using** $c \ \text{empty}$

by (*metis* *simple-not* $\varphi\text{-}\varphi'$ *append-Nil* *conn.simps*(4) *all-subformula-st-decomp-explicit*(3) *simple-not-symb.simps*(1))

next

fix $\varphi \ \varphi' :: 'v \text{ propo}$ **and** $ca :: 'v \text{ connective}$ **and** $\xi \ \xi' :: 'v \text{ propo list}$

and $x :: 'v$

assume *rel*: propo-rew-step (push-conn-inside $c \ c'$) $\varphi \ \varphi'$

and $n\varphi$: all-subformula-st simple-not-symb $\varphi \implies$ all-subformula-st simple-not-symb φ'

and *corr*: *wf-conn* $ca \ (\xi @ \varphi \ \# \ \xi')$

and n : all-subformula-st simple-not-symb (conn $ca \ (\xi @ \varphi \ \# \ \xi')$)

and c : $ca \in \text{binary-connectives}$


```

have all-subformula-st simple-not-symb  $\varphi$ 
  using  $n$   $c$  corr all-subformula-st-decomp by fastforce
then have  $\varphi'$ : all-subformula-st simple-not-symb  $\varphi'$  using  $n\varphi$  by blast
obtain  $a$   $b$  where  $ab$ :  $[a, b] = (\xi @ \varphi \# \xi')$ 
  using corr c list-length2-decomp wf-conn-bin-list-length by metis
then have  $\xi @ \varphi' \# \xi' = [a, \varphi'] \vee (\xi @ \varphi' \# \xi') = [\varphi', b]$ 
  using  $ab$  by (metis (no-types, hide-lams) append-Cons append-Nil append-Nil2
    append-is-Nil-conv butlast.simps(2) butlast-append list.sel(3) tl-append2)
moreover
{
  fix  $\chi :: 'v$  propo
  have  $wf'$ : wf-conn  $ca$   $[a, b]$ 
    using  $ab$  corr by presburger
  have all-subformula-st simple-not-symb (conn  $ca$   $[a, b]$ )
    using  $ab$   $n$  by presburger
  then have all-subformula-st simple-not-symb  $\chi \vee \chi \notin \text{set } (\xi @ \varphi' \# \xi')$ 
    using  $wf'$  by (metis (no-types)  $\varphi'$  all-subformula-st-decomp calculation insert-iff
      list.set(2))
}
then have  $\forall \varphi. \varphi \in \text{set } (\xi @ \varphi' \# \xi') \longrightarrow \text{all-subformula-st simple-not-symb } \varphi$ 
  by (metis (no-types))

moreover have simple-not-symb (conn  $ca$   $(\xi @ \varphi' \# \xi')$ )
  using  $ab$  conn-inj-not(1) corr wf-conn-list-decomp(4) wf-conn-no-arity-change
    not-Cons-self2 self-append-conv2 simple-not-symb.elims(3) by (metis (no-types) c
    calculation(1) wf-conn-binary)
moreover have wf-conn  $ca$   $(\xi @ \varphi' \# \xi')$  using  $c$  calculation(1) by auto
ultimately show all-subformula-st simple-not-symb (conn  $ca$   $(\xi @ \varphi' \# \xi')$ )
  by (metis all-subformula-st-decomp-imp)
qed
}
moreover {
  fix  $ca :: 'v$  connective and  $\xi \xi' :: 'v$  propo list and  $\varphi \varphi' :: 'v$  propo
  have propo-rew-step (push-conn-inside  $c$   $c'$ )  $\varphi \varphi' \Longrightarrow wf\text{-conn } ca (\xi @ \varphi \# \xi')$ 
     $\Longrightarrow \text{simple-not-symb } (\text{conn } ca (\xi @ \varphi \# \xi')) \Longrightarrow \text{simple-not-symb } \varphi'$ 
     $\Longrightarrow \text{simple-not-symb } (\text{conn } ca (\xi @ \varphi' \# \xi'))$ 
  by (metis append-self-conv2 conn.simps(4) conn-inj-not(1) simple-not-symb.elims(3)
    simple-not-symb.simps(1) simple-propo-rew-step-push-conn-inside-inv
    wf-conn-no-arity-change-helper wf-conn-list-decomp(4) wf-conn-no-arity-change)
}
ultimately show simple-not  $\psi$ 
  using full-propo-rew-step-inv-stay'[of push-conn-inside c c' simple-not-symb] assms
  unfolding no-T-F-except-top-level-def simple-not-def full-unfold by metis
next
{
  fix  $\varphi \psi :: 'v$  propo
  have  $H$ : propo-rew-step (push-conn-inside  $c$   $c'$ )  $\varphi \psi \Longrightarrow \text{no-T-F-except-top-level } \varphi$ 
     $\Longrightarrow \text{no-T-F-except-top-level } \psi$ 
  proof –
    assume  $rel$ : propo-rew-step (push-conn-inside  $c$   $c'$ )  $\varphi \psi$ 
    and no-T-F-except-top-level  $\varphi$ 
    then have no-T-F  $\varphi \vee \varphi = FF \vee \varphi = FT$ 
      by (metis no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
    moreover {

```

```

    assume  $\varphi = FF \vee \varphi = FT$ 
    then have False using rel propo-rew-step-push-conn-inside by blast
    then have no-T-F-except-top-level  $\psi$  by blast
  }
  moreover {
    assume no-T-F  $\varphi \wedge \varphi \neq FF \wedge \varphi \neq FT$ 
    then have no-T-F  $\psi$  using rel push-conn-insidec-in-c'-symb-no-T-F by blast
    then have no-T-F-except-top-level  $\psi$  using no-T-F-no-T-F-except-top-level by blast
  }
  ultimately show no-T-F-except-top-level  $\psi$  by blast
qed
}
moreover {
  fix ca :: 'v connective and  $\xi \xi' :: 'v$  propo list and  $\varphi \varphi' :: 'v$  propo
  assume rel: propo-rew-step (push-conn-inside c c')  $\varphi \varphi'$ 
  assume corr: wf-conn ca ( $\xi @ \varphi \# \xi'$ )
  then have c: ca  $\neq CT \wedge ca \neq CF$  by auto
  assume no-T-F: no-T-F-symb-except-toplevel (conn ca ( $\xi @ \varphi \# \xi'$ ))
  have no-T-F-symb-except-toplevel (conn ca ( $\xi @ \varphi' \# \xi'$ ))
  proof
    have c: ca  $\neq CT \wedge ca \neq CF$  using corr by auto
    have  $\zeta: \forall \zeta \in \text{set } (\xi @ \varphi \# \xi'). \zeta \neq FT \wedge \zeta \neq FF$ 
      using corr no-T-F no-T-F-symb-except-toplevel-if-is-a-true-false by blast
    then have  $\varphi \neq FT \wedge \varphi \neq FF$  by auto
    from rel this have  $\varphi' \neq FT \wedge \varphi' \neq FF$ 
    apply (induct rule: propo-rew-step.induct)
    by (metis append-is-Nil-conv conn.simps(2) conn-inj list.distinct(1)
      wf-conn-helper-facts(3) wf-conn-list(1) wf-conn-no-arity-change
      wf-conn-no-arity-change-helper push-conn-inside-not-true-false)
    then have  $\forall \zeta \in \text{set } (\xi @ \varphi' \# \xi'). \zeta \neq FT \wedge \zeta \neq FF$  using  $\zeta$  by auto
    moreover have wf-conn ca ( $\xi @ \varphi' \# \xi'$ )
      using corr wf-conn-no-arity-change by (metis wf-conn-no-arity-change-helper)
    ultimately show no-T-F-symb (conn ca ( $\xi @ \varphi' \# \xi'$ )) using no-T-F-symb.intros c by metis
  qed
}
ultimately show no-T-F-except-top-level  $\psi$ 
using full-propo-rew-step-inv-stay'[of push-conn-inside c c' no-T-F-symb-except-toplevel]
assms unfolding no-T-F-except-top-level-def full-unfold by metis

next
{
  fix  $\varphi \psi :: 'v$  propo
  have H: push-conn-inside c c'  $\varphi \psi \implies \text{no-equiv } \varphi \implies \text{no-equiv } \psi$ 
    by (induct  $\varphi \psi$  rule: push-conn-inside.induct, auto)
}
then show no-equiv  $\psi$ 
using full-propo-rew-step-inv-stay-conn[of push-conn-inside c c' no-equiv-symb] assms
no-equiv-symb-conn-characterization unfolding no-equiv-def by metis

next
{
  fix  $\varphi \psi :: 'v$  propo
  have H: push-conn-inside c c'  $\varphi \psi \implies \text{no-imp } \varphi \implies \text{no-imp } \psi$ 
    by (induct  $\varphi \psi$  rule: push-conn-inside.induct, auto)
}

```

then show *no-imp* ψ
using *full-propo-rew-step-inv-stay-conn*[*of push-conn-inside* c c' *no-imp-symb*] *assms*
no-imp-symb-conn-characterization **unfolding** *no-imp-def* **by** *metis*
qed

lemma *push-conn-inside-full-propo-rew-step*:
fixes $\varphi \psi :: 'v \text{ propo}$
assumes
no-equiv φ **and**
no-imp φ **and**
full (*propo-rew-step* (*push-conn-inside* c c')) $\varphi \psi$ **and**
no-T-F-except-top-level φ **and**
simple-not φ **and**
 $c = CAnd \vee c = COr$ **and**
 $c' = CAnd \vee c' = COr$
shows *c-in-c'-only* c $c' \psi$
using *c-in-c'-symb-rew* *assms* *full-propo-rew-step-subformula* **by** *blast*

3.5.1 Only one type of connective in the formula (+ not)

inductive *only-c-inside-symb* $:: 'v \text{ connective} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ **for** $c :: 'v \text{ connective}$ **where**
simple-only-c-inside[*simp*]: *simple* $\varphi \Longrightarrow \text{only-c-inside-symb } c \varphi$ |
simple-cnot-only-c-inside[*simp*]: *simple* $\varphi \Longrightarrow \text{only-c-inside-symb } c (FNot \varphi)$ |
only-c-inside-into-only-c-inside: *wf-conn* $c \ l \Longrightarrow \text{only-c-inside-symb } c (\text{conn } c \ l)$

lemma *only-c-inside-symb-simp*[*simp*]:
only-c-inside-symb $c \ FF$ *only-c-inside-symb* $c \ FT$ *only-c-inside-symb* $c \ (FVar \ x)$ **by** *auto*

definition *only-c-inside* **where** *only-c-inside* $c = \text{all-subformula-st } (\text{only-c-inside-symb } c)$

lemma *only-c-inside-symb-decomp*:
only-c-inside-symb $c \ \psi \longleftrightarrow (\text{simple } \psi$
 $\vee (\exists \varphi'. \psi = FNot \varphi' \wedge \text{simple } \varphi')$
 $\vee (\exists l. \psi = \text{conn } c \ l \wedge \text{wf-conn } c \ l))$
by (*auto simp: only-c-inside-symb.intros(3)*) (*induct rule: only-c-inside-symb.induct, auto*)

lemma *only-c-inside-symb-decomp-not*[*simp*]:
fixes $c :: 'v \text{ connective}$
assumes $c: c \neq CNot$
shows *only-c-inside-symb* $c \ (FNot \ \psi) \longleftrightarrow \text{simple } \psi$
apply (*auto simp: only-c-inside-symb.intros(3)*)
by (*induct FNot* ψ *rule: only-c-inside-symb.induct, auto simp: wf-conn-list(8) c*)

lemma *only-c-inside-decomp-not*[*simp*]:
assumes $c: c \neq CNot$
shows *only-c-inside* $c \ (FNot \ \psi) \longleftrightarrow \text{simple } \psi$
by (*metis* (*no-types, hide-lams*) *all-subformula-st-def all-subformula-st-test-symb-true-phi c*
only-c-inside-def only-c-inside-symb-decomp-not simple-only-c-inside
subformula-conn-decomp-simple)

lemma *only-c-inside-decomp*:
only-c-inside $c \ \varphi \longleftrightarrow$

$(\forall \psi. \psi \preceq \varphi \longrightarrow (\text{simple } \psi \vee (\exists \varphi'. \psi = \text{FNot } \varphi' \wedge \text{simple } \varphi') \vee (\exists l. \psi = \text{conn } c \ l \wedge \text{wf-conn } c \ l)))$

unfolding *only-c-inside-def* **by** (*auto simp: all-subformula-st-def only-c-inside-symb-decomp*)

lemma *only-c-inside-c-c'-false*:

fixes $c \ c' :: 'v \text{ connective}$ **and** $l :: 'v \text{ propo list}$ **and** $\varphi :: 'v \text{ propo}$
assumes $cc': c \neq c'$ **and** $c: c = \text{CAnd} \vee c = \text{COr}$ **and** $c': c' = \text{CAnd} \vee c' = \text{COr}$
and *only*: *only-c-inside* $c \ \varphi$ **and** *incl*: $\text{conn } c' \ l \preceq \varphi$ **and** *wf*: $\text{wf-conn } c' \ l$
shows *False*

proof –

let $? \psi = \text{conn } c' \ l$
have $\text{simple } ? \psi \vee (\exists \varphi'. ? \psi = \text{FNot } \varphi' \wedge \text{simple } \varphi') \vee (\exists l. ? \psi = \text{conn } c \ l \wedge \text{wf-conn } c \ l)$
using *only-c-inside-decomp only incl* **by** *blast*
moreover **have** $\neg \text{simple } ? \psi$
using *wf simple-decomp* **by** (*metis c' connective.distinct(19) connective.distinct(7,9,21,29,31) wf-conn-list(1-3)*)
moreover
 $\{$
 $\quad \text{fix } \varphi'$
 $\quad \text{have } ? \psi \neq \text{FNot } \varphi' \text{ using } c' \text{ conn-inj-not}(1) \text{ wf by blast}$
 $\}$
ultimately obtain $l :: 'v \text{ propo list}$ **where** $? \psi = \text{conn } c \ l \wedge \text{wf-conn } c \ l$ **by** *metis*
then **have** $c = c'$ **using** *conn-inj wf* **by** *metis*
then **show** *False* **using** cc' **by** *auto*

qed

lemma *only-c-inside-implies-c-in-c'-symb*:

assumes $\delta: c \neq c'$ **and** $c: c = \text{CAnd} \vee c = \text{COr}$ **and** $c': c' = \text{CAnd} \vee c' = \text{COr}$
shows *only-c-inside* $c \ \varphi \implies \text{c-in-c'-symb } c \ c' \ \varphi$
apply (*rule ccontr*)
apply (*cases rule: not-c-in-c'-symb.cases, auto*)
by (*metis $\delta \ c \ c'$ connective.distinct(37,39) list.distinct(1) only-c-inside-c-c'-false subformula-in-binary-conn(1,2) wf-conn.simps*) $+$

lemma *c-in-c'-symb-decomp-level1*:

fixes $l :: 'v \text{ propo list}$ **and** $c \ c' \ ca :: 'v \text{ connective}$
shows $\text{wf-conn } ca \ l \implies ca \neq c \implies \text{c-in-c'-symb } c \ c' \ (\text{conn } ca \ l)$

proof –

have $\text{not-c-in-c'-symb } c \ c' \ (\text{conn } ca \ l) \implies \text{wf-conn } ca \ l \implies ca = c$
by (*induct conn ca l rule: not-c-in-c'-symb.induct, auto simp: conn-inj*)
then **show** $\text{wf-conn } ca \ l \implies ca \neq c \implies \text{c-in-c'-symb } c \ c' \ (\text{conn } ca \ l)$ **by** *blast*

qed

lemma *only-c-inside-implies-c-in-c'-only*:

assumes $\delta: c \neq c'$ **and** $c: c = \text{CAnd} \vee c = \text{COr}$ **and** $c': c' = \text{CAnd} \vee c' = \text{COr}$
shows *only-c-inside* $c \ \varphi \implies \text{c-in-c'-only } c \ c' \ \varphi$
unfolding *c-in-c'-only-def all-subformula-st-def*
using *only-c-inside-implies-c-in-c'-symb*
by (*metis all-subformula-st-def assms(1) c c' only-c-inside-def subformula-trans*)

lemma *c-in-c'-symb-c-implies-only-c-inside*:

assumes $\delta: c = \text{CAnd} \vee c = \text{COr}$ $c' = \text{CAnd} \vee c' = \text{COr}$ $c \neq c'$ **and** *wf*: $\text{wf-conn } c \ [\varphi, \psi]$
and *inv*: *no-equiv* ($\text{conn } c \ l$) *no-imp* ($\text{conn } c \ l$) *simple-not* ($\text{conn } c \ l$)

```

  shows wf-conn c l  $\implies$  c-in-c'-only c c' (conn c l)  $\implies$  ( $\forall \psi \in \text{set } l. \text{ only-c-inside } c \psi$ )
using inv
proof (induct conn c l arbitrary: l rule: propo-induct-arity)
  case (nullary x)
  then show ?case by (auto simp: wf-conn-list assms)
next
  case (unary  $\varphi$  la)
  then have c = CNot  $\wedge$  la = [ $\varphi$ ] by (metis (no-types) wf-conn-list(8))
  then show ?case using assms(2) assms(1) by blast
next
  case (binary  $\varphi_1$   $\varphi_2$ )
  note IH $\varphi_1$  = this(1) and IH $\varphi_2$  = this(2) and  $\varphi$  = this(3) and only = this(5) and wf = this(4)
  and no-equiv = this(6) and no-imp = this(7) and simple-not = this(8)
  then have l: l = [ $\varphi_1$ ,  $\varphi_2$ ] by (meson wf-conn-list(4-7))
  let ? $\varphi$  = conn c l

  obtain c1 l1 c2 l2 where  $\varphi_1$ :  $\varphi_1$  = conn c1 l1 and wf $\varphi_1$ : wf-conn c1 l1
  and  $\varphi_2$ :  $\varphi_2$  = conn c2 l2 and wf $\varphi_2$ : wf-conn c2 l2 using exists-c-conn by metis
  then have c-in-only $\varphi_1$ : c-in-c'-only c c' (conn c1 l1) and c-in-c'-only c c' (conn c2 l2)
  using only l unfolding c-in-c'-only-def using assms(1) by auto
  have inc $\varphi_1$ :  $\varphi_1 \preceq ?\varphi$  and inc $\varphi_2$ :  $\varphi_2 \preceq ?\varphi$ 
  using  $\varphi_1$   $\varphi_2$   $\varphi$  local.wf by (metis conn.simps(5-8) helper-fact subformula-in-binary-conn(1,2))+

  have c1-eq: c1  $\neq$  CEq and c2-eq: c2  $\neq$  CEq
  unfolding no-equiv-def using inc $\varphi_1$  inc $\varphi_2$  by (metis  $\varphi_1$   $\varphi_2$  wf $\varphi_1$  wf $\varphi_2$  assms(1) no-equiv
  no-equiv-eq(1) no-equiv-symb.elims(3) no-equiv-symb-conn-characterization wf-conn-list(4,5)
  no-equiv-def subformula-all-subformula-st)+
  have c1-imp: c1  $\neq$  CImp and c2-imp: c2  $\neq$  CImp
  using no-imp by (metis  $\varphi_1$   $\varphi_2$  all-subformula-st-decomp-explicit-imp(2,3) assms(1)
  conn.simps(5,6) l no-imp-Imp(1) no-imp-symb.elims(3) no-imp-symb-conn-characterization
  wf $\varphi_1$  wf $\varphi_2$  all-subformula-st-decomp no-imp-symb-conn-characterization)+
  have c1c: c1  $\neq$  c'
  proof
    assume c1c: c1 = c'
    then obtain  $\xi_1$   $\xi_2$  where l1: l1 = [ $\xi_1$ ,  $\xi_2$ ]
    by (metis assms(2) connective.distinct(37,39) helper-fact wf $\varphi_1$  wf-conn.simps
    wf-conn-list-decomp(1-3))
    have c-in-c'-only c c' (conn c [conn c' l1,  $\varphi_2$ ]) using c1c l only  $\varphi_1$  by auto
    moreover have not-c-in-c'-symb c c' (conn c [conn c' l1,  $\varphi_2$ ])
    using l1  $\varphi_1$  c1c l local.wf not-c-in-c'-symb-l wf $\varphi_1$  by blast
    ultimately show False using  $\varphi_1$  c1c l l1 local.wf not-c-in-c'-simp(4) wf $\varphi_1$  by blast
  qed
  then have ( $\varphi_1$  = conn c l1  $\wedge$  wf-conn c l1)  $\vee$  ( $\exists \psi_1. \varphi_1$  = FNot  $\psi_1$ )  $\vee$  simple  $\varphi_1$ 
  by (metis  $\varphi_1$  assms(1-3) c1-eq c1-imp simple.elims(3) wf $\varphi_1$  wf-conn-list(4) wf-conn-list(5-7))
  moreover {
    assume  $\varphi_1$  = conn c l1  $\wedge$  wf-conn c l1
    then have only-c-inside c  $\varphi_1$ 
    by (metis IH $\varphi_1$   $\varphi_1$  all-subformula-st-decomp-imp inc $\varphi_1$  no-equiv no-equiv-def no-imp no-imp-def
    c-in-only $\varphi_1$  only-c-inside-def only-c-inside-into-only-c-inside simple-not simple-not-def
    subformula-all-subformula-st)
  }
  moreover {
    assume  $\exists \psi_1. \varphi_1$  = FNot  $\psi_1$ 
    then obtain  $\psi_1$  where  $\varphi_1$  = FNot  $\psi_1$  by metis
    then have only-c-inside c  $\varphi_1$ 

```

```

    by (metis all-subformula-st-def assms(1) connective.distinct(37,39) incφ1
        only-c-inside-decomp-not simple-not simple-not-def simple-not-symb.simps(1))
  }
  moreover {
    assume simple φ1
    then have only-c-inside c φ1
      by (metis all-subformula-st-decomp-explicit(3) assms(1) connective.distinct(37,39)
          only-c-inside-decomp-not only-c-inside-def)
  }
  ultimately have only-c-insideφ1: only-c-inside c φ1 by metis

  have c-in-onlyφ2: c-in-c'-only c c' (conn c2 l2)
    using only l φ2 wfφ2 assms unfolding c-in-c'-only-def by auto
  have c2c: c2 ≠ c'
  proof
    assume c2c: c2 = c'
    then obtain ξ1 ξ2 where l2: l2 = [ξ1, ξ2]
      by (metis assms(2) wfφ2 wf-conn.simps connective.distinct(7,9,19,21,29,31,37,39))
    then have c-in-c'-symb c c' (conn c [φ1, conn c' l2])
      using c2c l only φ2 all-subformula-st-test-symb-true-phi unfolding c-in-c'-only-def by auto
    moreover have not-c-in-c'-symb c c' (conn c [φ1, conn c' l2])
      using assms(1) c2c l2 not-c-in-c'-symb-r wfφ2 wf-conn-helper-facts(5,6) by metis
    ultimately show False by auto
  qed
  then have (φ2 = conn c l2 ∧ wf-conn c l2) ∨ (∃ψ2. φ2 = FNot ψ2) ∨ simple φ2
    using c2-eq by (metis φ2 assms(1-3) c2-eq c2-imp simple.elims(3) wfφ2 wf-conn-list(4-7))
  moreover {
    assume φ2 = conn c l2 ∧ wf-conn c l2
    then have only-c-inside c φ2
      by (metis IHφ2 φ2 all-subformula-st-decomp incφ2 no-equiv no-equiv-def no-imp no-imp-def
          c-in-onlyφ2 only-c-inside-def only-c-inside-into-only-c-inside simple-not simple-not-def
          subformula-all-subformula-st)
  }
  moreover {
    assume ∃ψ2. φ2 = FNot ψ2
    then obtain ψ2 where φ2 = FNot ψ2 by metis
    then have only-c-inside c φ2
      by (metis all-subformula-st-def assms(1-3) connective.distinct(38,40) incφ2
          only-c-inside-decomp-not simple-not simple-not-def simple-not-symb.simps(1))
  }
  moreover {
    assume simple φ2
    then have only-c-inside c φ2
      by (metis all-subformula-st-decomp-explicit(3) assms(1) connective.distinct(37,39)
          only-c-inside-decomp-not only-c-inside-def)
  }
  ultimately have only-c-insideφ2: only-c-inside c φ2 by metis
  show ?case using l only-c-insideφ1 only-c-insideφ2 by auto
qed

```

3.5.2 Push Conjunction

definition *pushConj* **where** *pushConj* = *push-conn-inside CAnd COr*

lemma *pushConj-consistent*: *preserves-un-sat pushConj*

unfolding *pushConj-def* **by** (*simp add: push-conn-inside-consistent*)

definition *and-in-or-symb* **where** *and-in-or-symb* = *c-in-c'-symb* *CAnd* *COr*

definition *and-in-or-only* **where**

and-in-or-only = *all-subformula-st* (*c-in-c'-symb* *CAnd* *COr*)

lemma *pushConj-inv*:

fixes $\varphi \psi :: 'v \text{ propo}$

assumes *full* (*propo-rew-step* *pushConj*) $\varphi \psi$

and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ **and** *simple-not* φ

shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ **and** *simple-not* ψ

using *push-conn-inside-inv* *assms* **unfolding** *pushConj-def* **by** *metis+*

lemma *pushConj-full-propo-rew-step*:

fixes $\varphi \psi :: 'v \text{ propo}$

assumes

no-equiv φ **and**

no-imp φ **and**

full (*propo-rew-step* *pushConj*) $\varphi \psi$ **and**

no-T-F-except-top-level φ **and**

simple-not φ

shows *and-in-or-only* ψ

using *assms* *push-conn-inside-full-propo-rew-step*

unfolding *pushConj-def* *and-in-or-only-def* *c-in-c'-only-def* **by** (*metis* (*no-types*))

3.5.3 Push Disjunction

definition *pushDisj* **where** *pushDisj* = *push-conn-inside* *COr* *CAnd*

lemma *pushDisj-consistent*: *preserves-un-sat* *pushDisj*

unfolding *pushDisj-def* **by** (*simp* *add*: *push-conn-inside-consistent*)

definition *or-in-and-symb* **where** *or-in-and-symb* = *c-in-c'-symb* *COr* *CAnd*

definition *or-in-and-only* **where**

or-in-and-only = *all-subformula-st* (*c-in-c'-symb* *COr* *CAnd*)

lemma *not-or-in-and-only-or-and[simp]*:

$\sim \text{or-in-and-only } (FOr (FAnd \psi1 \psi2) \varphi')$

unfolding *or-in-and-only-def*

by (*metis* *all-subformula-st-test-symb-true-phi* *conn.simps*(5–6) *not-c-in-c'-symb-l*

wf-conn-helper-facts(5) *wf-conn-helper-facts*(6))

lemma *pushDisj-inv*:

fixes $\varphi \psi :: 'v \text{ propo}$

assumes *full* (*propo-rew-step* *pushDisj*) $\varphi \psi$

and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ **and** *simple-not* φ

shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ **and** *simple-not* ψ

using *push-conn-inside-inv* *assms* **unfolding** *pushDisj-def* **by** *metis+*

lemma *pushDisj-full-propo-rew-step*:

fixes $\varphi \psi :: 'v \text{ propo}$

assumes

no-equiv φ **and**

no-imp φ **and**
full (*propo-rew-step pushDisj*) φ ψ **and**
no-T-F-except-top-level φ **and**
simple-not φ
shows *or-in-and-only* ψ
using *assms push-conn-inside-full-propo-rew-step*
unfolding *pushDisj-def or-in-and-only-def c-in-c'-only-def* **by** (*metis (no-types)*)

4 The full transformations

4.1 Abstract Property characterizing that only some connective are inside the others

4.1.1 Definition

The normal is a super group of groups

inductive *grouped-by* :: 'a connective \Rightarrow 'a propo \Rightarrow bool **for** c **where**
simple-is-grouped[*simp*]: *simple* $\varphi \Rightarrow$ *grouped-by* c φ |
simple-not-is-grouped[*simp*]: *simple* $\varphi \Rightarrow$ *grouped-by* c (*FNot* φ) |
connected-is-group[*simp*]: *grouped-by* c $\varphi \Rightarrow$ *grouped-by* c $\psi \Rightarrow$ *wf-conn* c [φ , ψ]
 \Rightarrow *grouped-by* c (*conn* c [φ , ψ])

lemma *simple-clause*[*simp*]:
grouped-by c *FT*
grouped-by c *FF*
grouped-by c (*FVar* x)
grouped-by c (*FNot* *FT*)
grouped-by c (*FNot* *FF*)
grouped-by c (*FNot* (*FVar* x))
by *simp+*

lemma *only-c-inside-symb-c-eq-c'*:
only-c-inside-symb c (*conn* c' [$\varphi 1$, $\varphi 2$]) \Rightarrow $c' = CAnd \vee c' = COr \Rightarrow$ *wf-conn* c' [$\varphi 1$, $\varphi 2$]
 \Rightarrow $c' = c$
by (*induct* *conn* c' [$\varphi 1$, $\varphi 2$] *rule: only-c-inside-symb.induct, auto simp: conn-inj*)

lemma *only-c-inside-c-eq-c'*:
only-c-inside c (*conn* c' [$\varphi 1$, $\varphi 2$]) \Rightarrow $c' = CAnd \vee c' = COr \Rightarrow$ *wf-conn* c' [$\varphi 1$, $\varphi 2$] \Rightarrow $c = c'$
unfolding *only-c-inside-def all-subformula-st-def* **using** *only-c-inside-symb-c-eq-c'* *subformula-refl*
by *blast*

lemma *only-c-inside-imp-grouped-by*:
assumes $c: c \neq CNot$ **and** $c': c' = CAnd \vee c' = COr$
shows *only-c-inside* c $\varphi \Rightarrow$ *grouped-by* c φ (**is** ?*O* $\varphi \Rightarrow$?*G* φ)
proof (*induct* φ *rule: propo-induct-arity*)
case (*nullary* φ x)
then show ?*G* φ **by** *auto*
next
case (*unary* ψ)
then show ?*G* (*FNot* ψ) **by** (*auto simp: c*)
next
case (*binary* φ $\varphi 1$ $\varphi 2$)
note *IH* $\varphi 1 = this(1)$ **and** *IH* $\varphi 2 = this(2)$ **and** $\varphi = this(3)$ **and** *only* = *this(4)*
have $\varphi\text{-conn}: \varphi = \text{conn } c$ [$\varphi 1$, $\varphi 2$] **and** *wf*: *wf-conn* c [$\varphi 1$, $\varphi 2$]


```

proof –
  obtain  $c''\ l''$  where  $\varphi\text{-}c'': \varphi = \text{conn } c''\ l''$  and  $\text{wf}: \text{wf-conn } c''\ l''$ 
    using exists-c-conn by metis
  then have  $l'': l'' = [\varphi 1, \varphi 2]$  using  $\varphi$  by (metis wf-conn-list(4–7))
  have only-c-inside-symb  $c$  ( $\text{conn } c''\ [\varphi 1, \varphi 2]$ )
    using only all-subformula-st-test-symb-true-phi
    unfolding only-c-inside-def  $\varphi\text{-}c''\ l''$  by metis
  then have  $c = c''$ 
    by (metis  $\varphi\ \varphi\text{-}c''\ \text{conn-inj}\ \text{conn-inj-not}(2)\ l''\ \text{list.distinct}(1)\ \text{list.inject}\ \text{wf}$ 
      only-c-inside-symb.cases simple.simps(5–8))
  then show  $\varphi = \text{conn } c\ [\varphi 1, \varphi 2]$  and  $\text{wf-conn } c\ [\varphi 1, \varphi 2]$  using  $\varphi\text{-}c''\ \text{wf } l''$  by auto
qed
have grouped-by  $c\ \varphi 1$  using  $\text{wf } IH\ \varphi 1\ IH\ \varphi 2\ \varphi\text{-conn}\ \text{only } \varphi$  unfolding only-c-inside-def by auto
moreover have grouped-by  $c\ \varphi 2$ 
  using  $\text{wf } \varphi\ IH\ \varphi 1\ IH\ \varphi 2\ \varphi\text{-conn}\ \text{only}$  unfolding only-c-inside-def by auto
ultimately show  $?G\ \varphi$  using  $\varphi\text{-conn}\ \text{connected-is-group}\ \text{local.wf}$  by blast
qed

```

lemma *grouped-by-false*:

```

grouped-by  $c\ (\text{conn } c'\ [\varphi, \psi]) \implies c \neq c' \implies \text{wf-conn } c'\ [\varphi, \psi] \implies \text{False}$ 
apply (induct conn  $c'\ [\varphi, \psi]$  rule: grouped-by.induct)
apply (auto simp: simple-decomp wf-conn-list, auto simp: conn-inj)
by (metis list.distinct(1) list.sel(3) wf-conn-list(8))+

```

Then the CNF form is a conjunction of clauses: every clause is in CNF form and two formulas in CNF form can be related by an and.

inductive *super-grouped-by*:: '*a* *connective* \Rightarrow '*a* *connective* \Rightarrow '*a* *propo* \Rightarrow *bool* **for** $c\ c'$ **where**
grouped-is-super-grouped[*simp*]: *grouped-by* $c\ \varphi \implies \text{super-grouped-by } c\ c'\ \varphi \mid$
connected-is-super-group: *super-grouped-by* $c\ c'\ \varphi \implies \text{super-grouped-by } c\ c'\ \psi \implies \text{wf-conn } c\ [\varphi, \psi]$
 $\implies \text{super-grouped-by } c\ c'\ (\text{conn } c'\ [\varphi, \psi])$

lemma *simple-cnf*[*simp*]:

```

super-grouped-by  $c\ c'\ FT$ 
super-grouped-by  $c\ c'\ FF$ 
super-grouped-by  $c\ c'\ (FVar\ x)$ 
super-grouped-by  $c\ c'\ (FNot\ FT)$ 
super-grouped-by  $c\ c'\ (FNot\ FF)$ 
super-grouped-by  $c\ c'\ (FNot\ (FVar\ x))$ 
by auto

```

lemma *c-in-c'-only-super-grouped-by*:

```

assumes  $c: c = CAnd \vee c = COr$  and  $c': c' = CAnd \vee c' = COr$  and  $cc': c \neq c'$ 
shows  $\text{no-equiv } \varphi \implies \text{no-imp } \varphi \implies \text{simple-not } \varphi \implies \text{c-in-c'-only } c\ c'\ \varphi$ 
 $\implies \text{super-grouped-by } c\ c'\ \varphi$ 
(is  $?NE\ \varphi \implies ?NI\ \varphi \implies ?SN\ \varphi \implies ?C\ \varphi \implies ?S\ \varphi)$ 

```

proof (*induct* φ *rule: propo-induct-arity*)

```

case (nullary  $\varphi\ x$ )
then show  $?S\ \varphi$  by auto

```

next

```

case (unary  $\varphi$ )
then have simple-not-symb ( $FNot\ \varphi$ )
  using all-subformula-st-test-symb-true-phi unfolding simple-not-def by blast
then have  $\varphi = FT \vee \varphi = FF \vee (\exists\ x. \varphi = FVar\ x)$  by (cases  $\varphi$ , auto)
then show  $?S\ (FNot\ \varphi)$  by auto

```

```

next
case (binary  $\varphi$   $\varphi1$   $\varphi2$ )
note  $IH\varphi1 = this(1)$  and  $IH\varphi2 = this(2)$  and  $no-equiv = this(4)$  and  $no-imp = this(5)$ 
and  $simpleN = this(6)$  and  $c-in-c'-only = this(7)$  and  $\varphi' = this(3)$ 
{
  assume  $\varphi = FImp\ \varphi1\ \varphi2 \vee \varphi = FEq\ \varphi1\ \varphi2$ 
  then have False using no-equiv no-imp by auto
  then have  $?S\ \varphi$  by auto
}
moreover {
  assume  $\varphi: \varphi = conn\ c'\ [\varphi1, \varphi2] \wedge wf-conn\ c'\ [\varphi1, \varphi2]$ 
  have c-in-c'-only:  $c-in-c'-only\ c\ c'\ \varphi1 \wedge c-in-c'-only\ c\ c'\ \varphi2 \wedge c-in-c'-symb\ c\ c'\ \varphi$ 
  using c-in-c'-only  $\varphi'$  unfolding c-in-c'-only-def by auto
  have super-grouped-by  $c\ c'\ \varphi1$  using  $\varphi\ c'\ no-equiv\ no-imp\ simpleN\ IH\varphi1\ c-in-c'-only$  by auto
  moreover have super-grouped-by  $c\ c'\ \varphi2$ 
  using  $\varphi\ c'\ no-equiv\ no-imp\ simpleN\ IH\varphi2\ c-in-c'-only$  by auto
  ultimately have  $?S\ \varphi$ 
  using super-grouped-by.intros(2)  $\varphi$  by (metis c wf-conn-helper-facts(5,6))
}
moreover {
  assume  $\varphi: \varphi = conn\ c\ [\varphi1, \varphi2] \wedge wf-conn\ c\ [\varphi1, \varphi2]$ 
  then have only-c-inside  $c\ \varphi1 \wedge only-c-inside\ c\ \varphi2$ 
  using c-in-c'-symb-c-implies-only-c-inside  $c\ c'\ c-in-c'-only\ list.set-intros(1)$ 
  wf-conn-helper-facts(5,6) no-equiv no-imp simpleN last-ConsL last-ConsR last-in-set
  list.distinct(1) by (metis (no-types, hide-lams) cc')
  then have only-c-inside  $c\ (conn\ c\ [\varphi1, \varphi2])$ 
  unfolding only-c-inside-def using  $\varphi$ 
  by (simp add: only-c-inside-into-only-c-inside all-subformula-st-decomp)
  then have grouped-by  $c\ \varphi$  using  $\varphi\ only-c-inside-imp-grouped-by\ c$  by blast
  then have  $?S\ \varphi$  using super-grouped-by.intros(1) by metis
}
ultimately show  $?S\ \varphi$  by (metis  $\varphi'\ c\ c'\ cc'\ conn.simps(5,6)\ wf-conn-helper-facts(5,6)$ )
qed

```

4.2 Conjunctive Normal Form

definition *is-conj-with-TF* where $is-conj-with-TF == super-grouped-by\ COr\ CAnd$

lemma *or-in-and-only-conjunction-in-disj*:

shows $no-equiv\ \varphi \implies no-imp\ \varphi \implies simple-not\ \varphi \implies or-in-and-only\ \varphi \implies is-conj-with-TF\ \varphi$
using *c-in-c'-only-super-grouped-by*
unfolding *is-conj-with-TF-def or-in-and-only-def c-in-c'-only-def*
by (*simp add: c-in-c'-only-def c-in-c'-only-super-grouped-by*)

definition *is-cnf* where

$is-cnf\ \varphi \equiv is-conj-with-TF\ \varphi \wedge no-T-F-except-top-level\ \varphi$

4.2.1 Full CNF transformation

The full CNF transformation consists simply in chaining all the transformation defined before.

definition *cnf-rew* where $cnf-rew =$

(*full* (*propo-rew-step elim-equiv*)) *OO*
(*full* (*propo-rew-step elim-imp*)) *OO*
(*full* (*propo-rew-step elimTB*)) *OO*
(*full* (*propo-rew-step pushNeg*)) *OO*

(full (propo-rew-step pushDisj))

lemma *cnf-rew-consistent: preserves-un-sat cnf-rew*
by (simp add: cnf-rew-def elimEquiv-lifted-consistant elim-imp-lifted-consistant elimTB-consistent
preserves-un-sat-OO pushDisj-consistent pushNeg-lifted-consistant)

lemma *cnf-rew-is-cnf: cnf-rew φ $\varphi' \implies$ is-cnf φ'*

apply (unfold cnf-rew-def OO-def)

apply auto

proof –

fix φ φEq φImp φTB φNeg \varphiDisj :: 'v propo

assume *Eq: full (propo-rew-step elim-equiv) φ φEq*

then have *no-equiv: no-equiv φEq using no-equiv-full-propo-rew-step-elim-equiv* **by** blast

assume *Imp: full (propo-rew-step elim-imp) φEq φImp*

then have *no-imp: no-imp φImp using no-imp-full-propo-rew-step-elim-imp* **by** blast

have *no-imp-inv: no-equiv φImp using no-equiv Imp elim-imp-inv* **by** blast

assume *TB: full (propo-rew-step elimTB) φImp φTB*

then have *noTB: no-T-F-except-top-level φTB*

using *no-imp-inv no-imp elimTB-full-propo-rew-step* **by** blast

have *noTB-inv: no-equiv φTB no-imp φTB using elimTB-inv TB no-imp no-imp-inv* **by** blast+

assume *Neg: full (propo-rew-step pushNeg) φTB φNeg*

then have *noNeg: simple-not φNeg*

using *noTB-inv noTB pushNeg-full-propo-rew-step* **by** blast

have *noNeg-inv: no-equiv φNeg no-imp φNeg no-T-F-except-top-level φNeg*

using *pushNeg-inv Neg noTB noTB-inv* **by** blast+

assume *Disj: full (propo-rew-step pushDisj) φNeg \varphiDisj*

then have *no-Disj: or-in-and-only \varphiDisj*

using *noNeg-inv noNeg pushDisj-full-propo-rew-step* **by** blast

have *noDisj-inv: no-equiv \varphiDisj no-imp \varphiDisj no-T-F-except-top-level \varphiDisj*

simple-not \varphiDisj

using *pushDisj-inv Disj noNeg noNeg-inv* **by** blast+

moreover have *is-conj-with-TF \varphiDisj*

using *or-in-and-only-conjunction-in-disj noDisj-inv no-Disj* **by** blast

ultimately show *is-cnf \varphiDisj unfolding is-cnf-def* **by** blast

qed

4.3 Disjunctive Normal Form

definition *is-disj-with-TF* **where** *is-disj-with-TF* \equiv *super-grouped-by CAnd COr*

lemma *and-in-or-only-conjunction-in-disj:*

shows *no-equiv $\varphi \implies$ no-imp $\varphi \implies$ simple-not $\varphi \implies$ and-in-or-only $\varphi \implies$ is-disj-with-TF φ*

using *c-in-c'-only-super-grouped-by*

unfolding *is-disj-with-TF-def and-in-or-only-def c-in-c'-only-def*

by (simp add: c-in-c'-only-def c-in-c'-only-super-grouped-by)

definition *is-dnf* :: 'a propo \Rightarrow bool **where**

is-dnf $\varphi \iff$ is-disj-with-TF $\varphi \wedge$ no-T-F-except-top-level φ

4.3.1 Full DNF transform

The full DNF transformation consists simply in chaining all the transformation defined before.

definition *dnf-rew* **where** *dnf-rew* \equiv
 (full (propo-rew-step elim-equiv)) OO
 (full (propo-rew-step elim-imp)) OO
 (full (propo-rew-step elimTB)) OO
 (full (propo-rew-step pushNeg)) OO
 (full (propo-rew-step pushConj))

lemma *dnf-rew-consistent: preserves-un-sat dnf-rew*
by (simp add: dnf-rew-def elimEquiv-lifted-consistant elim-imp-lifted-consistant elimTB-consistent
 preserves-un-sat-OO pushConj-consistent pushNeg-lifted-consistant)

theorem *dnf-transformation-correction:*
dnf-rew φ $\varphi' \implies$ is-dnf φ'
apply (unfold dnf-rew-def OO-def)
by (meson and-in-or-only-conjunction-in-disj elimTB-full-propo-rew-step elimTB-inv(1,2)
 elim-imp-inv is-dnf-def no-equiv-full-propo-rew-step-elim-equiv
 no-imp-full-propo-rew-step-elim-imp pushConj-full-propo-rew-step pushConj-inv(1-4)
 pushNeg-full-propo-rew-step pushNeg-inv(1-3))

5 More aggressive simplifications: Removing true and false at the beginning

5.1 Transformation

We should remove *FT* and *FF* at the beginning and not in the middle of the algorithm. To do this, we have to use more rules (one for each connective):

inductive *elimTBFull* **where**
ElimTBFull1[simp]: elimTBFull (FAnd φ FT) φ |
ElimTBFull1'[simp]: elimTBFull (FAnd FT φ) φ |

ElimTBFull2[simp]: elimTBFull (FAnd φ FF) FF |
ElimTBFull2'[simp]: elimTBFull (FAnd FF φ) FF |

ElimTBFull3[simp]: elimTBFull (FOr φ FT) FT |
ElimTBFull3'[simp]: elimTBFull (FOr FT φ) FT |

ElimTBFull4[simp]: elimTBFull (FOr φ FF) φ |
ElimTBFull4'[simp]: elimTBFull (FOr FF φ) φ |

ElimTBFull5[simp]: elimTBFull (FNot FT) FF |
ElimTBFull5'[simp]: elimTBFull (FNot FF) FT |

ElimTBFull6-l[simp]: elimTBFull (FImp FT φ) φ |
ElimTBFull6-l'[simp]: elimTBFull (FImp FF φ) FT |
ElimTBFull6-r[simp]: elimTBFull (FImp φ FT) FT |
ElimTBFull6-r'[simp]: elimTBFull (FImp φ FF) (FNot φ) |

ElimTBFull7-l[simp]: elimTBFull (FEq FT φ) φ |
ElimTBFull7-l'[simp]: elimTBFull (FEq FF φ) (FNot φ) |
ElimTBFull7-r[simp]: elimTBFull (FEq φ FT) φ |
ElimTBFull7-r'[simp]: elimTBFull (FEq φ FF) (FNot φ)

The transformation is still consistent.

lemma *elimTBFull-consistent: preserves-un-sat elimTBFull*

proof –

```
{
  fix  $\varphi \psi :: 'b \text{ propo}$ 
  have  $\text{elimTBFull } \varphi \psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$ 
    by (induct-tac rule: elimTBFull.inducts, auto)
}
```

then show *?thesis* **using** *preserves-un-sat-def* **by** *auto*
qed

Contrary to the theorem $\llbracket \text{no-equiv } ?\varphi; \text{no-imp } ?\varphi; ?\psi \preceq ?\varphi; \neg \text{no-T-F-symb-except-toplevel } ?\psi \rrbracket \implies \exists \psi'. \text{elimTB } ?\psi \psi'$, we do not need the assumption *no-equiv* φ and *no-imp* φ , since our transformation is more general.

lemma *no-T-F-symb-except-toplevel-step-exists'*:

fixes $\varphi :: 'v \text{ propo}$

shows $\psi \preceq \varphi \implies \neg \text{no-T-F-symb-except-toplevel } \psi \implies \exists \psi'. \text{elimTBFull } \psi \psi'$

proof (*induct* ψ *rule: propo-induct-arity*)

case (*nullary* φ')

then have *False* **using** *no-T-F-symb-except-toplevel-true no-T-F-symb-except-toplevel-false* **by** *auto*

then show *Ex* (*elimTBFull* φ') **by** *blast*

next

case (*unary* ψ)

then have $\psi = FF \vee \psi = FT$ **using** *no-T-F-symb-except-toplevel-not-decom* **by** *blast*

then show *Ex* (*elimTBFull* (*FNot* ψ)) **using** *ElimTBFull5 ElimTBFull5'* **by** *blast*

next

case (*binary* $\varphi' \psi1 \psi2$)

then have $\psi1 = FT \vee \psi2 = FT \vee \psi1 = FF \vee \psi2 = FF$

by (*metis binary-connectives-def conn.simps(5-8) insertI1 insert-commute*
no-T-F-symb-except-toplevel-bin-decom binary.hyps(3))

then show *Ex* (*elimTBFull* φ') **using** *elimTBFull.intros binary.hyps(3)* **by** *blast*

qed

The same applies here. We do not need the assumption, but the deep link between $\neg \text{no-T-F-except-top-level}$ φ and the existence of a rewriting step, still exists.

lemma *no-T-F-except-top-level-rew'*:

fixes $\varphi :: 'v \text{ propo}$

assumes *noTB*: $\neg \text{no-T-F-except-top-level } \varphi$

shows $\exists \psi \psi'. \psi \preceq \varphi \wedge \text{elimTBFull } \psi \psi'$

proof –

have *test-symb-false-nullary*:

$\forall x. \text{no-T-F-symb-except-toplevel } (FF :: 'v \text{ propo}) \wedge \text{no-T-F-symb-except-toplevel } FT$
 $\wedge \text{no-T-F-symb-except-toplevel } (FVar (x :: 'v))$

by *auto*

moreover {

fix $c :: 'v \text{ connective}$ **and** $l :: 'v \text{ propo list}$ **and** $\psi :: 'v \text{ propo}$

have $H: \text{elimTBFull } (\text{conn } c \ l) \ \psi \implies \neg \text{no-T-F-symb-except-toplevel } (\text{conn } c \ l)$

by (*cases* (*conn* $c \ l$) *rule: elimTBFull.cases*) *auto*

}

ultimately show *?thesis*

using *no-test-symb-step-exists*[*of no-T-F-symb-except-toplevel* φ *elimTBFull*] *noTB*

no-T-F-symb-except-toplevel-step-exists' **unfolding** *no-T-F-except-top-level-def* **by** *metis*

qed

lemma *elimTBFull-full-propo-rew-step*:
fixes $\varphi \psi :: 'v \text{ propo}$
assumes *full* (*propo-rew-step elimTBFull*) $\varphi \psi$
shows *no-T-F-except-top-level* ψ
using *full-propo-rew-step-subformula no-T-F-except-top-level-rew' assms* **by** *fastforce*

5.2 More invariants

As the aim is to use the transformation as the first transformation, we have to show some more invariants for *elim-equiv* and *elim-imp*. For the other transformation, we have already proven it.

lemma *propo-rew-step-ElimEquiv-no-T-F*: *propo-rew-step elim-equiv* $\varphi \psi \implies \text{no-T-F } \varphi \implies \text{no-T-F } \psi$

proof (*induct rule: propo-rew-step.induct*)

fix $\varphi' :: 'v \text{ propo}$ **and** $\psi' :: 'v \text{ propo}$
assume *a1*: *no-T-F* φ'
assume *a2*: *elim-equiv* $\varphi' \psi'$
have $\forall x0 \ x1. (\neg \text{elim-equiv } (x1 :: 'v \text{ propo}) \ x0 \vee (\exists v2 \ v3 \ v4 \ v5 \ v6 \ v7. x1 = \text{FEq } v2 \ v3$
 $\wedge x0 = \text{FAnd } (\text{FImp } v4 \ v5) (\text{FImp } v6 \ v7) \wedge v2 = v4 \wedge v4 = v7 \wedge v3 = v5 \wedge v3 = v6))$
 $= (\neg \text{elim-equiv } x1 \ x0 \vee (\exists v2 \ v3 \ v4 \ v5 \ v6 \ v7. x1 = \text{FEq } v2 \ v3$
 $\wedge x0 = \text{FAnd } (\text{FImp } v4 \ v5) (\text{FImp } v6 \ v7) \wedge v2 = v4 \wedge v4 = v7 \wedge v3 = v5 \wedge v3 = v6))$
by *meson*
then have $\forall p \ pa. \neg \text{elim-equiv } (p :: 'v \text{ propo}) \ pa \vee (\exists pb \ pc \ pd \ pe \ pf \ pg. p = \text{FEq } pb \ pc$
 $\wedge pa = \text{FAnd } (\text{FImp } pd \ pe) (\text{FImp } pf \ pg) \wedge pb = pd \wedge pd = pg \wedge pc = pe \wedge pc = pf)$
using *elim-equiv.cases* **by** *force*
then show *no-T-F* ψ' **using** *a1 a2* **by** *fastforce*

next

fix $\varphi \varphi' :: 'v \text{ propo}$ **and** $\xi \xi' :: 'v \text{ propo list}$ **and** $c :: 'v \text{ connective}$
assume *rel*: *propo-rew-step elim-equiv* $\varphi \varphi'$
and *IH*: *no-T-F* $\varphi \implies \text{no-T-F } \varphi'$
and *corr*: *wf-conn* $c (\xi @ \varphi \# \xi')$
and *no-T-F*: *no-T-F* (*conn* $c (\xi @ \varphi \# \xi')$)
{
assume *c*: $c = \text{CNot}$
then have *empty*: $\xi = [] \ \xi' = []$ **using** *corr* **by** *auto*
then have *no-T-F* φ **using** *no-T-F c no-T-F-decomp-not* **by** *auto*
then have *no-T-F* (*conn* $c (\xi @ \varphi' \# \xi')$) **using** *c empty no-T-F-comp-not IH* **by** *auto*
}

moreover **{**

assume *c*: $c \in \text{binary-connectives}$
obtain *a b* **where** $ab: \xi @ \varphi \# \xi' = [a, b]$
using *corr c list-length2-decomp wf-conn-bin-list-length* **by** *metis*
then have $\varphi: \varphi = a \vee \varphi = b$
by (*metis append.simps(1) append-is-Nil-conv list.distinct(1) list.sel(3) nth-Cons-0 tl-append2*)
have $\zeta: \forall \zeta \in \text{set } (\xi @ \varphi \# \xi'). \text{no-T-F } \zeta$
using *no-T-F unfolding no-T-F-def* **using** *corr all-subformula-st-decomp* **by** *blast*

then have $\varphi': \text{no-T-F } \varphi'$ **using** *ab IH* φ **by** *auto*
have $l': \xi @ \varphi' \# \xi' = [\varphi', b] \vee \xi @ \varphi' \# \xi' = [a, \varphi']$
by (*metis (no-types, hide-lams) ab append-Cons append-Nil append-Nil2 butlast.simps(2) butlast-append list.distinct(1) list.sel(3)*)
then have $\forall \zeta \in \text{set } (\xi @ \varphi' \# \xi'). \text{no-T-F } \zeta$ **using** $\zeta \varphi' ab$ **by** *fastforce*
moreover

```

have  $\forall \zeta \in \text{set } (\xi @ \varphi \# \xi'). \zeta \neq FT \wedge \zeta \neq FF$ 
  using  $\zeta \text{ corr no-}T\text{-}F \text{ no-}T\text{-}F\text{-except-top-level-false no-}T\text{-}F\text{-no-}T\text{-}F\text{-except-top-level}$  by blast
then have  $\text{no-}T\text{-}F\text{-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$ 
  by (metis  $\varphi' l'$  ab all-subformula-st-test-symb-true-phi  $c \text{ list.distinct}(1)$ 
     $\text{list.set-intros}(1,2)$   $\text{no-}T\text{-}F\text{-symb-except-toplevel-bin-decom}$ 
     $\text{no-}T\text{-}F\text{-symb-except-toplevel-no-}T\text{-}F\text{-symb no-}T\text{-}F\text{-symb-false}(1,2)$   $\text{no-}T\text{-}F\text{-def wf-conn-binary}$ 
     $\text{wf-conn-list}(1,2))$ 
ultimately have  $\text{no-}T\text{-}F (\text{conn } c (\xi @ \varphi' \# \xi'))$ 
  by (metis  $l'$  all-subformula-st-decomp-imp  $c \text{ no-}T\text{-}F\text{-def wf-conn-binary}$ )
}
moreover {
  fix  $x$ 
  assume  $c = CVar\ x \vee c = CF \vee c = CT$ 
  then have False using corr by auto
  then have  $\text{no-}T\text{-}F (\text{conn } c (\xi @ \varphi' \# \xi'))$  by auto
}
ultimately show  $\text{no-}T\text{-}F (\text{conn } c (\xi @ \varphi' \# \xi'))$  using corr wf-conn.cases by metis
qed

```

lemma *elim-equiv-inv'*:

```

fixes  $\varphi \psi :: 'v \text{ propo}$ 
assumes full (propo-rew-step elim-equiv)  $\varphi \psi$  and  $\text{no-}T\text{-}F\text{-except-top-level } \varphi$ 
shows  $\text{no-}T\text{-}F\text{-except-top-level } \psi$ 
proof –
{
  fix  $\varphi \psi :: 'v \text{ propo}$ 
  have  $\text{propo-rew-step elim-equiv } \varphi \psi \implies \text{no-}T\text{-}F\text{-except-top-level } \varphi$ 
     $\implies \text{no-}T\text{-}F\text{-except-top-level } \psi$ 
  proof –
    assume rel:  $\text{propo-rew-step elim-equiv } \varphi \psi$ 
    and no:  $\text{no-}T\text{-}F\text{-except-top-level } \varphi$ 
    {
      assume  $\varphi = FT \vee \varphi = FF$ 
      from rel this have False
      apply (induct rule:  $\text{propo-rew-step.induct}$ , auto simp:  $\text{wf-conn-list}(1,2)$ )
      using elim-equiv.simps by blast+
      then have  $\text{no-}T\text{-}F\text{-except-top-level } \psi$  by blast
    }
    moreover {
      assume  $\varphi \neq FT \wedge \varphi \neq FF$ 
      then have  $\text{no-}T\text{-}F \varphi$ 
      by (metis  $\text{no no-}T\text{-}F\text{-symb-except-toplevel-all-subformula-st-no-}T\text{-}F\text{-symb}$ )
      then have  $\text{no-}T\text{-}F \psi$  using propo-rew-step-ElimEquiv-no-}T\text{-}F \text{ rel} by blast
      then have  $\text{no-}T\text{-}F\text{-except-top-level } \psi$  by (simp add:  $\text{no-}T\text{-}F\text{-no-}T\text{-}F\text{-except-top-level}$ )
    }
    ultimately show  $\text{no-}T\text{-}F\text{-except-top-level } \psi$  by metis
  qed
}
moreover {
  fix  $c :: 'v \text{ connective}$  and  $\xi \xi' :: 'v \text{ propo list}$  and  $\zeta \zeta' :: 'v \text{ propo}$ 
  assume rel:  $\text{propo-rew-step elim-equiv } \zeta \zeta'$ 
  and incl:  $\zeta \preceq \varphi$ 
  and corr:  $\text{wf-conn } c (\xi @ \zeta \# \xi')$ 
  and no-T-F:  $\text{no-}T\text{-}F\text{-symb-except-toplevel } (\text{conn } c (\xi @ \zeta \# \xi'))$ 
  and n:  $\text{no-}T\text{-}F\text{-symb-except-toplevel } \zeta'$ 

```

```

have no-T-F-symb-except-toplevel (conn c (ξ @ ζ' # ξ'))
proof
  have p: no-T-F-symb (conn c (ξ @ ζ # ξ'))
    using corr wf-conn-list(1) wf-conn-list(2) no-T-F-symb-except-toplevel-no-T-F-symb no-T-F
    by blast
  have l: ∀ φ ∈ set (ξ @ ζ # ξ'). φ ≠ FT ∧ φ ≠ FF
    using corr wf-conn-no-T-F-symb-iff p by blast
  from rel incl have ζ' ≠ FT ∧ ζ' ≠ FF
    apply (induction ζ ζ' rule: propo-rew-step.induct)
    apply (cases rule: elim-equiv.cases, auto simp: elim-equiv.simps)
    by (metis append-is-Nil-conv list.distinct wf-conn-list(1,2) wf-conn-no-arity-change
        wf-conn-no-arity-change-helper)+
  then have ∀ φ ∈ set (ξ @ ζ' # ξ'). φ ≠ FT ∧ φ ≠ FF using l by auto
  moreover have c ≠ CT ∧ c ≠ CF using corr by auto
  ultimately show no-T-F-symb (conn c (ξ @ ζ' # ξ'))
    by (metis corr wf-conn-no-arity-change wf-conn-no-arity-change-helper no-T-F-symb-comp)
qed
}
ultimately show no-T-F-except-top-level ψ
  using full-propo-rew-step-inv-stay-with-inc[of elim-equiv no-T-F-symb-except-toplevel φ]
  assms subformula-refl unfolding no-T-F-except-top-level-def by metis
qed

lemma propo-rew-step-ElimImp-no-T-F: propo-rew-step elim-imp φ ψ ⇒ no-T-F φ ⇒ no-T-F ψ
proof (induct rule: propo-rew-step.induct)
  case (global-rel φ' ψ')
  then show no-T-F ψ'
    using elim-imp.cases no-T-F-comp-not no-T-F-decomp(1,2)
    by (metis no-T-F-comp-expanded-explicit(2))
next
  case (propo-rew-one-step-lift φ φ' c ξ ξ')
  note rel = this(1) and IH = this(2) and corr = this(3) and no-T-F = this(4)
  {
    assume c: c = CNot
    then have empty: ξ = [] ξ' = [] using corr by auto
    then have no-T-F φ using no-T-F c no-T-F-decomp-not by auto
    then have no-T-F (conn c (ξ @ φ' # ξ')) using c empty no-T-F-comp-not IH by auto
  }
  moreover {
    assume c: c ∈ binary-connectives
    then obtain a b where ab: ξ @ φ # ξ' = [a, b]
      using corr list-length2-decomp wf-conn-bin-list-length by metis
    then have φ: φ = a ∨ φ = b
      by (metis append-self-conv2 wf-conn-list-decomp(4) wf-conn-unary list.discI list.sel(3)
          nth-Cons-0 tl-append2)
    have ζ: ∀ ζ ∈ set (ξ @ φ # ξ'). no-T-F ζ using ab c propo-rew-one-step-lift.prem by auto

    then have φ': no-T-F φ'
      using ab IH φ corr no-T-F no-T-F-def all-subformula-st-decomp-explicit by auto
    have χ: ξ @ φ' # ξ' = [φ', b] ∨ ξ @ φ' # ξ' = [a, φ]
      by (metis (no-types, hide-lams) ab append-Cons append-Nil append-Nil2 butlast.simps(2)
          butlast-append list.distinct(1) list.sel(3))
    then have ∀ ζ ∈ set (ξ @ φ' # ξ'). no-T-F ζ using ζ φ' ab by fastforce
    moreover

```



```

  have no-T-F (last (ξ @ φ' # ξ')) by (simp add: calculation)
  then have no-T-F-symb (conn c (ξ @ φ' # ξ'))
    by (metis χ φ' ζ ab all-subformula-st-test-symb-true-phi c last.simps list.distinct(1)
        list.set-intros(1) no-T-F-bin-decomp no-T-F-def)
  ultimately have no-T-F (conn c (ξ @ φ' # ξ')) using c χ by fastforce
}
moreover {
  fix x
  assume c = CVar x ∨ c = CF ∨ c = CT
  then have False using corr by auto
  then have no-T-F (conn c (ξ @ φ' # ξ')) by auto
}
ultimately show no-T-F (conn c (ξ @ φ' # ξ')) using corr wf-conn.cases by blast
qed

```

lemma *elim-imp-inv'*:

```

  fixes φ ψ :: 'v propo
  assumes full (propo-rew-step elim-imp) φ ψ and no-T-F-except-top-level φ
  shows no-T-F-except-top-level ψ
proof -
  {
    {
      fix φ ψ :: 'v propo
      have H: elim-imp φ ψ ⇒ no-T-F-except-top-level φ ⇒ no-T-F-except-top-level ψ
        by (induct φ ψ rule: elim-imp.induct, auto)
    } note H = this
    fix φ ψ :: 'v propo
    have propo-rew-step elim-imp φ ψ ⇒ no-T-F-except-top-level φ ⇒ no-T-F-except-top-level ψ
    proof -
      assume rel: propo-rew-step elim-imp φ ψ
      and no: no-T-F-except-top-level φ
      {
        assume φ = FT ∨ φ = FF
        from rel this have False
        apply (induct rule: propo-rew-step.induct)
        by (cases rule: elim-imp.cases, auto simp: wf-conn-list(1,2))
        then have no-T-F-except-top-level ψ by blast
      }
      moreover {
        assume φ ≠ FT ∧ φ ≠ FF
        then have no-T-F φ
          by (metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
        then have no-T-F ψ
          using rel propo-rew-step-ElimImp-no-T-F by blast
        then have no-T-F-except-top-level ψ by (simp add: no-T-F-no-T-F-except-top-level)
      }
      ultimately show no-T-F-except-top-level ψ by metis
    qed
  }
}
moreover {
  fix c :: 'v connective and ξ ξ' :: 'v propo list and ζ ζ' :: 'v propo
  assume rel: propo-rew-step elim-imp ζ ζ'
  and incl: ζ ≤ φ
  and corr: wf-conn c (ξ @ ζ # ξ')

```

```

and no-T-F: no-T-F-symb-except-toplevel (conn c (ξ @ ζ # ξ'))
and n: no-T-F-symb-except-toplevel ζ'
have no-T-F-symb-except-toplevel (conn c (ξ @ ζ' # ξ'))
proof
  have p: no-T-F-symb (conn c (ξ @ ζ # ξ'))
    by (simp add: corr no-T-F no-T-F-symb-except-toplevel-no-T-F-symb wf-conn-list(1,2))

  have l: ∀ φ ∈ set (ξ @ ζ # ξ'). φ ≠ FT ∧ φ ≠ FF
    using corr wf-conn-no-T-F-symb-iff p by blast
  from rel incl have ζ' ≠ FT ∧ ζ' ≠ FF
    apply (induction ζ ζ' rule: propo-rew-step.induct)
    apply (cases rule: elim-imp.cases, auto)
    using wf-conn-list(1,2) wf-conn-no-arity-change wf-conn-no-arity-change-helper
    by (metis append-is-Nil-conv list.distinct(1))+
  then have ∀ φ ∈ set (ξ @ ζ' # ξ'). φ ≠ FT ∧ φ ≠ FF using l by auto
  moreover have c ≠ CT ∧ c ≠ CF using corr by auto
  ultimately show no-T-F-symb (conn c (ξ @ ζ' # ξ'))
    using corr wf-conn-no-arity-change no-T-F-symb-comp
    by (metis wf-conn-no-arity-change-helper)
qed
}
ultimately show no-T-F-except-top-level ψ
  using full-propo-rew-step-inv-stay-with-inc[of elim-imp no-T-F-symb-except-toplevel φ]
  assms subformula-refl unfolding no-T-F-except-top-level-def by metis
qed

```

5.3 The new CNF and DNF transformation

The transformation is the same as before, but the order is not the same.

definition *dnf-rew'* :: 'a propo ⇒ 'a propo ⇒ bool **where**

```

dnf-rew' =
  (full (propo-rew-step elimTBFull)) OO
  (full (propo-rew-step elim-equiv)) OO
  (full (propo-rew-step elim-imp)) OO
  (full (propo-rew-step pushNeg)) OO
  (full (propo-rew-step pushConj))

```

lemma *dnf-rew'-consistent: preserves-un-sat dnf-rew'*

```

by (simp add: dnf-rew'-def elimEquiv-lifted-consistant elim-imp-lifted-consistant
  elimTBFull-consistent preserves-un-sat-OO pushConj-consistent pushNeg-lifted-consistant)

```

theorem *cnf-transformation-correction:*

```

  dnf-rew' φ φ' ⇒ is-dnf φ'
unfolding dnf-rew'-def OO-def
by (meson and-in-or-only-conjunction-in-disj elimTBFull-full-propo-rew-step elim-equiv-inv'
  elim-imp-inv elim-imp-inv' is-dnf-def no-equiv-full-propo-rew-step-elim-equiv
  no-imp-full-propo-rew-step-elim-imp pushConj-full-propo-rew-step pushConj-inv(1-4)
  pushNeg-full-propo-rew-step pushNeg-inv(1-3))

```

Given all the lemmas before the CNF transformation is easy to prove:

definition *cnf-rew'* :: 'a propo ⇒ 'a propo ⇒ bool **where**

```

cnf-rew' =
  (full (propo-rew-step elimTBFull)) OO
  (full (propo-rew-step elim-equiv)) OO
  (full (propo-rew-step elim-imp)) OO

```

(full (propo-rew-step pushNeg)) OO
 (full (propo-rew-step pushDisj))

lemma *cnf-rew'-consistent: preserves-un-sat cnf-rew'*

by (simp add: cnf-rew'-def elimEquiv-lifted-consistant elim-imp-lifted-consistant
 elimTBFull-consistent preserves-un-sat-OO pushDisj-consistent pushNeg-lifted-consistant)

theorem *cnf'-transformation-correction:*

cnf-rew' φ φ' \implies is-cnf φ'

unfolding *cnf-rew'-def OO-def*

by (meson elimTBFull-full-propo-rew-step elim-equiv-inv' elim-imp-inv elim-imp-inv' is-cnf-def
 no-equiv-full-propo-rew-step-elim-equiv no-imp-full-propo-rew-step-elim-imp
 or-in-and-only-conjunction-in-disj pushDisj-full-propo-rew-step pushDisj-inv(1-4)
 pushNeg-full-propo-rew-step pushNeg-inv(1) pushNeg-inv(2) pushNeg-inv(3))

end

theory *Prop-Logic-Multiset*

imports ../lib/Multiset-More Prop-Normalisation Partial-Clausal-Logic

begin

6 Link with Multiset Version

6.1 Transformation to Multiset

fun *mset-of-conj* :: 'a propo \Rightarrow 'a literal multiset **where**

mset-of-conj (FOr φ ψ) = *mset-of-conj* φ + *mset-of-conj* ψ |

mset-of-conj (FVar v) = {# Pos v #} |

mset-of-conj (FNot (FVar v)) = {# Neg v #} |

mset-of-conj FF = {#}

fun *mset-of-formula* :: 'a propo \Rightarrow 'a literal multiset set **where**

mset-of-formula (FAnd φ ψ) = *mset-of-formula* φ \cup *mset-of-formula* ψ |

mset-of-formula (FOr φ ψ) = {*mset-of-conj* (FOr φ ψ)} |

mset-of-formula (FVar ψ) = {*mset-of-conj* (FVar ψ)} |

mset-of-formula (FNot ψ) = {*mset-of-conj* (FNot ψ)} |

mset-of-formula FF = {{#}} |

mset-of-formula FT = {}

6.2 Equisatisfiability of the two Version

lemma *is-conj-with-TF-FNot:*

is-conj-with-TF (FNot φ) \longleftrightarrow ($\exists v. \varphi = \text{FVar } v \vee \varphi = \text{FF} \vee \varphi = \text{FT}$)

unfolding *is-conj-with-TF-def* **apply** (rule iffI)

apply (induction FNot φ rule: super-grouped-by.induct)

apply (induction FNot φ rule: grouped-by.induct)

apply simp

apply (cases φ ; simp)

apply auto

done

lemma *grouped-by-COr-FNot:*

grouped-by COr (FNot φ) \longleftrightarrow ($\exists v. \varphi = \text{FVar } v \vee \varphi = \text{FF} \vee \varphi = \text{FT}$)

unfolding *is-conj-with-TF-def* **apply** (rule iffI)

apply (induction FNot φ rule: grouped-by.induct)

apply simp

```

    apply (cases  $\varphi$ ; simp)
  apply auto
done

lemma
  shows  $\text{no-T-F-FF}[simp]: \neg \text{no-T-F FF}$  and
     $\text{no-T-F-FT}[simp]: \neg \text{no-T-F FT}$ 
  unfolding  $\text{no-T-F-def all-subformula-st-def}$  by auto

lemma grouped-by-CAnd-FAnd:
  grouped-by CAnd ( $\text{FAnd } \varphi 1 \varphi 2$ )  $\longleftrightarrow$  grouped-by CAnd  $\varphi 1 \wedge$  grouped-by CAnd  $\varphi 2$ 
  apply (rule iffI)
  apply (induction  $\text{FAnd } \varphi 1 \varphi 2$  rule: grouped-by.induct)
  using connected-is-group[of CAnd  $\varphi 1 \varphi 2$ ] by auto

lemma grouped-by-COr-FOr:
  grouped-by COr ( $\text{FOr } \varphi 1 \varphi 2$ )  $\longleftrightarrow$  grouped-by COr  $\varphi 1 \wedge$  grouped-by COr  $\varphi 2$ 
  apply (rule iffI)
  apply (induction  $\text{FOr } \varphi 1 \varphi 2$  rule: grouped-by.induct)
  using connected-is-group[of COr  $\varphi 1 \varphi 2$ ] by auto

lemma grouped-by-COr-FAnd[simp]:  $\neg$  grouped-by COr ( $\text{FAnd } \varphi 1 \varphi 2$ )
  apply clarify
  apply (induction  $\text{FAnd } \varphi 1 \varphi 2$  rule: grouped-by.induct)
  apply auto
done

lemma grouped-by-COr-FEq[simp]:  $\neg$  grouped-by COr ( $\text{FEq } \varphi 1 \varphi 2$ )
  apply clarify
  apply (induction  $\text{FEq } \varphi 1 \varphi 2$  rule: grouped-by.induct)
  apply auto
done

lemma [simp]:  $\neg$ grouped-by COr ( $\text{FImp } \varphi \psi$ )
  apply clarify
  by (induction  $\text{FImp } \varphi \psi$  rule: grouped-by.induct) simp-all

lemma [simp]:  $\neg$  is-conj-with-TF ( $\text{FImp } \varphi \psi$ )
  unfolding is-conj-with-TF-def apply clarify
  by (induction  $\text{FImp } \varphi \psi$  rule: super-grouped-by.induct) simp-all

lemma [simp]:  $\neg$ grouped-by COr ( $\text{FEq } \varphi \psi$ )
  apply clarify
  by (induction  $\text{FEq } \varphi \psi$  rule: grouped-by.induct) simp-all

lemma [simp]:  $\neg$  is-conj-with-TF ( $\text{FEq } \varphi \psi$ )
  unfolding is-conj-with-TF-def apply clarify
  by (induction  $\text{FEq } \varphi \psi$  rule: super-grouped-by.induct) simp-all

lemma is-conj-with-TF-Fand:
  is-conj-with-TF ( $\text{FAnd } \varphi 1 \varphi 2$ )  $\implies$  is-conj-with-TF  $\varphi 1 \wedge$  is-conj-with-TF  $\varphi 2$ 
  unfolding is-conj-with-TF-def
  apply (induction  $\text{FAnd } \varphi 1 \varphi 2$  rule: super-grouped-by.induct)
  apply (auto simp: grouped-by-CAnd-FAnd intro: grouped-is-super-grouped)[]

```

```

apply auto[]
done

```

lemma *is-conj-with-TF-FOr*:

```

is-conj-with-TF (FOr  $\varphi 1$   $\varphi 2$ )  $\implies$  grouped-by COr  $\varphi 1 \wedge$  grouped-by COr  $\varphi 2$ 
unfolding is-conj-with-TF-def
apply (induction FOr  $\varphi 1$   $\varphi 2$  rule: super-grouped-by.induct)
  apply (auto simp: grouped-by-COr-FOr)[]
apply auto[]
done

```

lemma *grouped-by-COr-mset-of-formula*:

```

grouped-by COr  $\varphi \implies$  mset-of-formula  $\varphi =$  (if  $\varphi = FT$  then  $\{\}$  else  $\{\text{mset-of-conj } \varphi\}$ )
by (induction  $\varphi$ ) (auto simp add: grouped-by-COr-FNot)

```

When a formula is in CNF form, then there is equisatisfiability between the multiset version and the CNF form. Remark that the definition for the entailment are slightly different: $op \models$ uses a function assigning *True* or *False*, while $op \models s$ uses a set where being in the list means entailment of a literal.

theorem

```

fixes  $\varphi :: 'v \text{ propo}$ 
assumes is-cnf  $\varphi$ 
shows eval A  $\varphi \longleftrightarrow$  Partial-Clausal-Logic.true-cls ( $\{\text{Pos } v | v. A \ v\} \cup \{\text{Neg } v | v. \neg A \ v\}$ )
  (mset-of-formula  $\varphi$ )
using assms
proof (induction  $\varphi$ )
  case FF
  then show ?case by auto
next
  case FT
  then show ?case by auto
next
  case (FVar  $v$ )
  then show ?case by auto
next
  case (FAnd  $\varphi \ \psi$ )
  then show ?case
  unfolding is-cnf-def by (auto simp: is-conj-with-TF-FNot dest: is-conj-with-TF-Fand
    dest!: is-conj-with-TF-FOr)
next
  case (FOr  $\varphi \ \psi$ )
  then have [simp]: mset-of-formula  $\varphi = \{\text{mset-of-conj } \varphi\}$  mset-of-formula  $\psi = \{\text{mset-of-conj } \psi\}$ 
    unfolding is-cnf-def by (auto dest!: is-conj-with-TF-FOr simp: grouped-by-COr-mset-of-formula
      split: if-splits)
  have is-conj-with-TF  $\varphi$  is-conj-with-TF  $\psi$ 
    using FOr(3) unfolding is-cnf-def no-T-F-def
    by (metis grouped-is-super-grouped is-conj-with-TF-FOr is-conj-with-TF-def) +
  then show ?case using FOr
    unfolding is-cnf-def by simp
next
  case (FImp  $\varphi \ \psi$ )
  then show ?case
    unfolding is-cnf-def by auto
next
  case (FEq  $\varphi \ \psi$ )

```

```

    then show ?case
      unfolding is-cn $f$ -def by auto
next
case (FNot  $\varphi$ )
then show ?case
  unfolding is-cn $f$ -def by (auto simp: is-conj-with-TF-FNot)
qed
end

```