Formalisation of Ground Resolution and CDCL in Isabelle/HOL

Mathias Fleury and Jasmin Blanchette

February 17, 2016

Contents

theory *Wellfounded-More*
imports *Main*

begin

1 Transitions

This theory contains more facts about closure, the definition of full transformations, and well-foundedness.

1.1 More theorems about Closures

This is the equivalent of $?r \leq ?s \Longrightarrow ?r^{**} \leq ?s^{**}$ for *tranclp*

lemma *tranclp-mono-explicit*:
$r^{++}\ a\ b \Longrightarrow r \leq s \Longrightarrow s^{++}\ a\ b$
using *rtranclp-mono* **by** (*auto dest!*: *tranclpD intro*: *rtranclp-into-tranclp2*)

lemma *tranclp-mono*:
assumes *mono*: $r \leq s$
shows $r^{++} \leq s^{++}$
using *rtranclp-mono*[*OF mono*] *mono* **by** (*auto dest!*: *tranclpD intro*: *rtranclp-into-tranclp2*)

lemma *tranclp-idemp-rel*:
$R^{++++}\ a\ b \longleftrightarrow R^{++}\ a\ b$
apply (*rule iffI*)
prefer *2* **apply** *blast*
by (*induction rule*: *tranclp-induct*) *auto*

Equivalent of $?r^{****} = ?r^{**}$

lemma *trancl-idemp*: $(r^+)^+ = r^+$
by *simp*

lemmas *tranclp-idemp*[*simp*] = *trancl-idemp*[*to-pred*]

This theorem already exists as $?r^{**}\ ?a\ ?b \equiv ?a = ?b \vee ?r^{++}\ ?a\ ?b$ (and sledgehammer uses it), but it makes sense to duplicate it, because it is unclear how stable the lemmas in Nitpick are.

lemma *rtranclp-unfold*: *rtranclp r a b* $\longleftrightarrow$ ($a = b \vee$ *tranclp r a b*)
by (*meson rtranclp.simps rtranclpD tranclp-into-rtranclp*)

lemma *tranclp-unfold-end*: *tranclp r a b* $\longleftrightarrow$ ($\exists\, a'$. *rtranclp r a a'* $\wedge$ *r a' b*)
by (*metis rtranclp.rtrancl-refl rtranclp-into-tranclp1 tranclp.cases tranclp-into-rtranclp*)

lemma *tranclp-unfold-begin*: *tranclp r a b* $\longleftrightarrow$ ($\exists\, a'$. *r a a'* $\wedge$ *rtranclp r a' b*) by (*meson rtranclp-into-tranclp2 tranclpD*)

lemma *trancl-set-tranclp*: $(a,\ b) \in \{(b,a).\ P\ a\ b\}^+ \longleftrightarrow P^{++}\ b\ a$
apply (*rule iffI*)
apply (*induction rule*: *trancl-induct*; *simp*)
apply (*induction rule*: *tranclp-induct*; *auto simp*: *trancl-into-trancl2*)
done

lemma *tranclp-rtranclp-rtranclp-rel*: $R^{++**}\ a\ b \longleftrightarrow R^{**}\ a\ b$
by (*simp add*: *rtranclp-unfold*)

lemma *tranclp-rtranclp-rtranclp*[*simp*]: $R^{++**} = R^{**}$
by (*fastforce simp*: *rtranclp-unfold*)

lemma *rtranclp-exists-last-with-prop*:
assumes $R\ x\ z$
and $R^{**}\ z\ z'$ **and** $P\ x\ z$
shows $\exists\, y\ y'.\ R^{**}\ x\ y \land R\ y\ y' \land P\ y\ y' \land (\lambda a\ b.\ R\ a\ b \land \neg P\ a\ b)^{**}\ y'\ z'$
using *assms*(*2*,*1*,*3*)
proof (*induction arbitrary*:)
case *base*
then show *?case* **by** *auto*
next
case (*step* $z'\ z''$) **note** $z = this(2)$ **and** $IH = this(3)[OF\ this(4-5)]$
show *?case*
apply (*cases* $P\ z'\ z''$)
apply (*rule exI*[*of* - z'], *rule exI*[*of* - z''])
using *z assms*(*1*) *step.hyps*(*1*) *step.prems*(*2*) **apply** *auto*[*1*]
using *IH z rtranclp.rtrancl-into-rtrancl* **by** *fastforce*
qed

lemma *rtranclp-and-rtranclp-left*: $(\lambda\ a\ b.\ P\ a\ b \land Q\ a\ b)^{**}\ S\ T \Longrightarrow P^{**}\ S\ T$
by (*induction rule*: *rtranclp-induct*) *auto*

1.2 Full Transitions

We define here properties to define properties after all possible transitions.

abbreviation *no-step step* $S \equiv (\forall\, S'.\ \neg step\ S\ S')$

definition *full1* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ **where**
full1 transf $= (\lambda S\ S'.\ tranclp\ transf\ S\ S' \land (\forall\, S''.\ \neg\ transf\ S'\ S''))$

definition *full*:: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ **where**
full transf $= (\lambda S\ S'.\ rtranclp\ transf\ S\ S' \land (\forall\, S''.\ \neg\ transf\ S'\ S''))$

lemma *rtranclp-full1I*:
$R^{**}\ a\ b \Longrightarrow full1\ R\ b\ c \Longrightarrow full1\ R\ a\ c$
unfolding *full1-def* **by** *auto*

lemma *tranclp-full1I*:
$R^{++}\ a\ b \Longrightarrow full1\ R\ b\ c \Longrightarrow full1\ R\ a\ c$
unfolding *full1-def* **by** *auto*

lemma *rtranclp-fullI*:
$R^{**}\ a\ b \Longrightarrow full\ R\ b\ c \Longrightarrow full\ R\ a\ c$

unfolding *full-def* **by** *auto*

lemma *tranclp-full-full1I*:
R^{++} a b $\Longrightarrow$ *full* R b c $\Longrightarrow$ *full1* R a c
unfolding *full-def full1-def* **by** *auto*

lemma *full-fullI*:
R a b $\Longrightarrow$ *full* R b c $\Longrightarrow$ *full1* R a c
unfolding *full-def full1-def* **by** *auto*

lemma *full-unfold*:
full r S S' $\longleftrightarrow$ (($S = S' \wedge$ *no-step* r S') $\vee$ *full1* r S S')
unfolding *full-def full1-def* **by** (*auto simp add*: *rtranclp-unfold*)

lemma *full1-is-full*[*intro*]: *full1* R S T $\Longrightarrow$ *full* R S T
by (*simp add*: *full-unfold*)

lemma *not-full1-rtranclp-relation*: $\neg$*full1* R^{**} a b
by (*meson full1-def rtranclp.rtrancl-refl*)

lemma *not-full-rtranclp-relation*: $\neg$*full* R^{**} a b
by (*meson full-fullI not-full1-rtranclp-relation rtranclp.rtrancl-refl*)

lemma *full1-tranclp-relation-full*:
full1 R^{++} a b $\longleftrightarrow$ *full1* R a b
by (*metis converse-tranclpE full1-def reflclp-tranclp rtranclpD rtranclp-idemp rtranclp-reflclp tranclp.r-into-trancl tranclp-into-rtranclp*)

lemma *full-tranclp-relation-full*:
full R^{++} a b $\longleftrightarrow$ *full* R a b
by (*metis full-unfold full1-tranclp-relation-full tranclp.r-into-trancl tranclpD*)

lemma *rtranclp-full1-eq-or-full1*:
(*full1* R)** a b $\longleftrightarrow$ ($a = b \vee$ *full1* R a b)
proof $-$
have $\forall p\ a\ aa.\ \neg\ p^{**}\ (a{::}'a)\ aa \vee a = aa \vee (\exists ab.\ p^{**}\ a\ ab \wedge p\ ab\ aa)$
by (*metis rtranclp.cases*)
then obtain $aa :: ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a$ **where**
f1: $\forall p\ a\ ab.\ \neg\ p^{**}\ a\ ab \vee a = ab \vee p^{**}\ a\ (aa\ p\ a\ ab) \wedge p\ (aa\ p\ a\ ab)\ ab$
by *moura*
{ assume $a \neq b$
{ assume $\neg$ *full1* R a b $\wedge$ $a \neq b$
then have $a \neq b \wedge a \neq b \wedge \neg$ *full1* R (aa (*full1* R) a b) b $\vee$ $\neg$ (*full1* R)** a b $\wedge$ $a \neq b$
using *f1* **by** (*metis* (*no-types*) *full1-def full1-tranclp-relation-full*)
then have *?thesis*
using *f1* **by** *blast* **}**
then have *?thesis*
by *auto* **}**
then show *?thesis*
by *fastforce*
qed

lemma *tranclp-full1-full1*:
(*full1* R)$^{++}$ a b $\longleftrightarrow$ *full1* R a b
by (*metis full1-def rtranclp-full1-eq-or-full1 tranclp-unfold-begin*)

1.3 Well-Foundedness and Full Transitions

```
lemma wf-exists-normal-form:
  assumes wf:wf {(x, y). R y x}
  shows ∃ b. R** a b ∧ no-step R b
proof (rule ccontr)
  assume ¬ ?thesis
  then have H: ⋀b. ¬ R** a b ∨ ¬no-step R b
    by blast
  def F ≡ rec-nat a (λi b. SOME c. R b c)
  have [simp]: F 0 = a
    unfolding F-def by auto
  have [simp]: ⋀i. F (Suc i) = (SOME b. R (F i) b)
    using F-def by simp
  { fix i
    have ∀ j<i. R (F j) (F (Suc j))
      proof (induction i)
        case 0
        then show ?case by auto
      next
        case (Suc i)
        then have R** a (F i)
          by (induction i) auto
        then have R (F i) (SOME b. R (F i) b)
          using H by (simp add: someI-ex)
        then have ∀ j < Suc i. R (F j) (F (Suc j))
          using H Suc by (simp add: less-Suc-eq)
        then show ?case by fast
      qed
  }
  then have ∀ j. R (F j) (F (Suc j)) by blast
  then show False
    using wf unfolding wfP-def wf-iff-no-infinite-down-chain by blast
qed

lemma wf-exists-normal-form-full:
  assumes wf:wf {(x, y). R y x}
  shows ∃ b. full R a b
  using wf-exists-normal-form[OF assms] unfolding full-def by blast
```

1.4 More Well-Foundedness

A little list of theorems that could be useful, but are hidden:

- link between *wf* and infinite chains: $wf\ ?r = (\neg (\exists f.\ \forall i.\ (f\ (Suc\ i), f\ i) \in ?r))$, $[\![wf\ ?r; \bigwedge k.\ (?f\ (Suc\ k), ?f\ k) \notin ?r \Longrightarrow ?thesis]\!] \Longrightarrow ?thesis$

```
lemma wf-if-measure-in-wf:
  wf R ⟹ (⋀a b. (a, b) ∈ S ⟹ (ν a, ν b)∈R) ⟹ wf S
  by (metis in-inv-image wfE-min wfI-min wf-inv-image)

lemma wfP-if-measure: fixes f :: 'a ⇒ nat
shows (⋀x y. P x ⟹ g x y  ⟹ f y < f x) ⟹ wf {(y,x). P x ∧ g x y}
  apply(insert wf-measure[of f])
  apply(simp only: measure-def inv-image-def less-than-def less-eq)
```

apply(*erule wf-subset*)
apply *auto*
done

lemma *wf-if-measure-f*:
assumes *wf r*
shows *wf* $\{(b,\ a).\ (f\ b,\ f\ a) \in r\}$
using *assms* **by** (*metis inv-image-def wf-inv-image*)

lemma *wf-wf-if-measure'*:
assumes *wf r* **and** *H*: $(\bigwedge x\ y.\ P\ x \Longrightarrow g\ x\ y \Longrightarrow (f\ y,\ f\ x) \in r)$
shows *wf* $\{(y,x).\ P\ x \wedge g\ x\ y\}$
proof −
have *wf* $\{(b,\ a).\ (f\ b,\ f\ a) \in r\}$ **using** *assms(1) wf-if-measure-f* **by** *auto*
then have *wf* $\{(b,\ a).\ P\ a \wedge g\ a\ b \wedge (f\ b,\ f\ a) \in r\}$
using *wf-subset*[*of* - $\{(b,\ a).\ P\ a \wedge g\ a\ b \wedge (f\ b,\ f\ a) \in r\}$] **by** *auto*
moreover have $\{(b,\ a).\ P\ a \wedge g\ a\ b \wedge (f\ b,\ f\ a) \in r\} \subseteq \{(b,\ a).\ (f\ b,\ f\ a) \in r\}$ **by** *auto*
moreover have $\{(b,\ a).\ P\ a \wedge g\ a\ b \wedge (f\ b,\ f\ a) \in r\} = \{(b,\ a).\ P\ a \wedge g\ a\ b\}$ **using** *H* **by** *auto*
ultimately show *?thesis* **using** *wf-subset* **by** *simp*
qed

lemma *wf-lex-less*: *wf* (*lex* $\{(a,\ b).\ (a{::}nat) < b\}$)
proof −
have *m*: $\{(a,\ b).\ a < b\}$ = *measure id* **by** *auto*
show *?thesis* **apply** (*rule wf-lex*) **unfolding** *m* **by** *auto*
qed

lemma *wfP-if-measure2*: **fixes** $f :: {}'a \Rightarrow nat$
shows $(\bigwedge x\ y.\ P\ x\ y \Longrightarrow g\ x\ y \ \Longrightarrow f\ x < f\ y) \Longrightarrow$ *wf* $\{(x,y).\ P\ x\ y \wedge g\ x\ y\}$
apply(*insert wf-measure*[*of f*])
apply(*simp only*: *measure-def inv-image-def less-than-def less-eq*)
apply(*erule wf-subset*)
apply *auto*
done

lemma *lexord-on-finite-set-is-wf*:
assumes
P-finite: $\bigwedge U.\ P\ U \longrightarrow U \in A$ **and**
finite: *finite A* **and**
wf: *wf R* **and**
trans: *trans R*
shows *wf* $\{(T,\ S).\ (P\ S \wedge P\ T) \wedge (T,\ S) \in$ *lexord R*$\}$
proof (*rule wfP-if-measure2*)
fix *T S*
assume *P*: $P\ S \wedge P\ T$ **and**
s-le-t: $(T,\ S) \in$ *lexord R*
let $?f = \lambda S.\ \{U.\ (U,\ S) \in$ *lexord R* $\wedge\ P\ U \wedge P\ S\}$
have $?f\ T \subseteq\ ?f\ S$
using *s-le-t P lexord-trans trans* **by** *auto*
moreover have $T \in\ ?f\ S$
using *s-le-t P* **by** *auto*
moreover have $T \notin\ ?f\ T$
using *s-le-t* **by** (*auto simp add*: *lexord-irreflexive local.wf*)
ultimately have $\{U.\ (U,\ T) \in$ *lexord R* $\wedge\ P\ U \wedge P\ T\} \subset \{U.\ (U,\ S) \in$ *lexord R* $\wedge\ P\ U \wedge P\ S\}$
by *auto*

moreover have *finite* $\{U.\ (U, S) \in \mathit{lexord}\ R \wedge P\ U \wedge P\ S\}$
  using *finite* **by** (*metis* (*no-types*, *lifting*) *P-finite finite-subset mem-Collect-eq subsetI*)
ultimately show $\mathit{card}\ (?f\ T) < \mathit{card}\ (?f\ S)$ **by** (*simp add*: *psubset-card-mono*)
qed

lemma *wf-fst-wf-pair*:
  assumes $\mathit{wf}\ \{(M', M).\ R\ M'\ M\}$
  shows $\mathit{wf}\ \{((M', N'), (M, N)).\ R\ M'\ M\}$
proof –
  have $\mathit{wf}\ (\{(M', M).\ R\ M'\ M\}\ {<}{*}\mathit{lex}{*}{>}\ \{\})$
    using *assms* **by** *auto*
  then show *?thesis*
    by (*rule wf-subset*) *auto*
qed

lemma *wf-snd-wf-pair*:
  assumes $\mathit{wf}\ \{(M', M).\ R\ M'\ M\}$
  shows $\mathit{wf}\ \{((M', N'), (M, N)).\ R\ N'\ N\}$
proof –
  have *wf*: $\mathit{wf}\ \{((M', N'), (M, N)).\ R\ M'\ M\}$
    using *assms wf-fst-wf-pair* **by** *auto*
  then have *wf*: $\bigwedge P.\ (\forall x.\ (\forall y.\ (y, x) \in \{((M', N'), M, N).\ R\ M'\ M\} \longrightarrow P\ y) \longrightarrow P\ x) \Longrightarrow \mathit{All}\ P$
    unfolding *wf-def* **by** *auto*
  show *?thesis*
    unfolding *wf-def*
    proof (*intro allI impI*)
      fix $P :: {'c} \times {'a} \Rightarrow \mathit{bool}$ **and** $x :: {'c} \times {'a}$
      assume H: $\forall x.\ (\forall y.\ (y, x) \in \{((M', N'), M, y).\ R\ N'\ y\} \longrightarrow P\ y) \longrightarrow P\ x$
      obtain $a\ b$ **where** x: $x = (a, b)$ **by** (*cases* x)
      have P: $P\ x = (P \circ (\lambda(a, b).\ (b, a)))\ (b, a)$
        unfolding x **by** *auto*
      show $P\ x$
        using $\mathit{wf}[\mathit{of}\ P\ o\ (\lambda(a, b).\ (b, a))]$ **apply** *rule*
          using H **apply** *simp*
        unfolding P **by** *blast*
    qed
qed

lemma *wf-if-measure-f-notation2*:
  assumes $\mathit{wf}\ r$
  shows $\mathit{wf}\ \{(b, h\ a)|b\ a.\ (f\ b, f\ (h\ a)) \in r\}$
  apply (*rule wf-subset*)
  using *wf-if-measure-f*[*OF assms*, *of f*] **by** *auto*

lemma *wf-wf-if-measure'-notation2*:
assumes $\mathit{wf}\ r$ **and** H: $(\bigwedge x\ y.\ P\ x \Longrightarrow g\ x\ y \Longrightarrow (f\ y, f\ (h\ x)) \in r)$
shows $\mathit{wf}\ \{(y,h\ x)|\ y\ x.\ P\ x \wedge g\ x\ y\}$
proof –
  have $\mathit{wf}\ \{(b, h\ a)|b\ a.\ (f\ b, f\ (h\ a)) \in r\}$ **using** *assms*(*1*) *wf-if-measure-f-notation2* **by** *auto*
  then have $\mathit{wf}\ \{(b, h\ a)|b\ a.\ P\ a \wedge g\ a\ b \wedge (f\ b, f\ (h\ a)) \in r\}$
    using *wf-subset*[*of* - $\{(b, h\ a)|\ b\ a.\ P\ a \wedge g\ a\ b \wedge (f\ b, f\ (h\ a)) \in r\}$] **by** *auto*
  moreover have $\{(b, h\ a)|b\ a.\ P\ a \wedge g\ a\ b \wedge (f\ b, f\ (h\ a)) \in r\}$
    $\subseteq \{(b, h\ a)|b\ a.\ (f\ b, f\ (h\ a)) \in r\}$ **by** *auto*
  moreover have $\{(b, h\ a)|b\ a.\ P\ a \wedge g\ a\ b \wedge (f\ b, f\ (h\ a)) \in r\} = \{(b, h\ a)|b\ a.\ P\ a \wedge g\ a\ b\}$

using *H* **by** *auto*
ultimately show *?thesis* **using** *wf-subset* **by** *simp*
qed

end
theory *List-More*
imports *Main*
begin

2 Various Lemmas

Close to $(\bigwedge n.\ \forall m<n.\ ?P\ m \Longrightarrow ?P\ n) \Longrightarrow ?P\ ?n$, but with a separation between zero and non-zero, and case names.

thm *nat-less-induct*
lemma *nat-less-induct-case*[*case-names 0 Suc*]:
assumes
P 0 **and**
$\bigwedge n.\ (\forall m < Suc\ n.\ P\ m) \Longrightarrow P\ (Suc\ n)$
shows *P n*
apply (*induction rule*: *nat-less-induct*)
by (*case-tac n*) (*auto intro*: *assms*)

This is only proved in simple cases by auto. In assumptions, nothing happens, and $?P\ (if\ ?Q\ then\ ?x\ else\ ?y) = (\neg\ (?Q \land \neg\ ?P\ ?x \lor \neg\ ?Q \land \neg\ ?P\ ?y))$ can blow up goals (because of other if expression).

lemma *if-0-1-ge-0*[*simp*]:
$0 < (if\ P\ then\ a\ else\ (0{::}nat)) \longleftrightarrow P \land 0 < a$
by *auto*

Bounded function have not been defined in Isabelle.

definition *bounded* **where**
$bounded\ f \longleftrightarrow (\exists\, b.\ \forall\, n.\ f\ n \le b)$

abbreviation *unbounded* :: $('a \Rightarrow 'b{::}ord) \Rightarrow bool$ **where**
$unbounded\ f \equiv \neg\ bounded\ f$

lemma *not-bounded-nat-exists-larger*:
fixes $f :: nat \Rightarrow nat$
assumes *unbound*: *unbounded f*
shows $\exists\, n.\ f\ n > m \land n > n_0$
proof (*rule ccontr*)
assume *H*: ¬ *?thesis*
have $finite\ \{f\ n|n.\ n \le n_0\}$
by *auto*
have $\bigwedge n.\ f\ n \le Max\ (\{f\ n|n.\ n \le n_0\} \cup \{m\})$
apply (*case-tac* $n \le n_0$)
apply (*metis* (*mono-tags*, *lifting*) *Max-ge Un-insert-right* ‹*finite* $\{f\ n\ |n.\ n \le n_0\}$›
finite-insert insertCI mem-Collect-eq sup-bot.right-neutral)
by (*metis* (*no-types*, *lifting*) *H Max-less-iff Un-insert-right* ‹*finite* $\{f\ n\ |n.\ n \le n_0\}$›
finite-insert insertI1 insert-not-empty leI sup-bot.right-neutral)
then show *False*
using *unbound* **unfolding** *bounded-def* **by** *auto*
qed

lemma *bounded-const-product*:
  fixes k :: *nat* **and** f :: *nat* ⇒ *nat*
  assumes $k > 0$
  shows *bounded f* ⟷ *bounded* ($\lambda i.\ k * f\ i$)
  unfolding *bounded-def* **apply** (*rule iffI*)
   using *mult-le-mono2* **apply** *blast*
  by (*meson assms le-less-trans less-or-eq-imp-le nat-mult-less-cancel-disj split-div-lemma*)

This lemma is not used, but here to show that a property that can be expected from *bounded* holds.

lemma *bounded-finite-linorder*:
  fixes f :: $'a \Rightarrow {}'a$ ::{*finite*, *linorder*}
  shows *bounded f*
proof −
  have $\bigwedge x.\ f\ x \leq Max\ \{f\ x|x.\ True\}$
    by (*metis* (*mono-tags*) *Max-ge finite mem-Collect-eq*)
  then show *?thesis*
    unfolding *bounded-def* **by** *blast*
qed

3 More List

3.1 *upt*

The simplification rules are not very handy, because [*?i*..<*Suc ?j*] = (*if ?i* ≤ *?j then* [*?i*..<*?j*] @ [*?j*] *else* []) leads to a case distinction, that we do not want if the condition is not in the context.

lemma *upt-Suc-le-append*: $\neg i \leq j \Longrightarrow$ [*i*..<*Suc j*] = []
  by *auto*

lemmas *upt-simps*[*simp*] = *upt-Suc-append upt-Suc-le-append*

declare *upt.simps*(*2*)[*simp del*]

lemma
  assumes $i \leq n - m$
  shows *take i* [*m*..<*n*] = [*m*..<*m*+*i*]
  by (*metis Nat.le-diff-conv2 add.commute assms diff-is-0-eq' linear take-upt upt-conv-Nil*)

The counterpart for this lemma when $n - m < i$ is *length ?xs* ≤ *?n* ⟹ *take ?n ?xs* = *?xs*. It is close to *?i* + *?m* ≤ *?n* ⟹ *take ?m* [*?i*..<*?n*] = [*?i*..<*?i* + *?m*], but seems more general.

lemma *take-upt-bound-minus*[*simp*]:
  assumes $i \leq n - m$
  shows *take i* [*m*..<*n*] = [*m* ..<*m*+*i*]
  using *assms* **by** (*induction i*) *auto*

lemma *append-cons-eq-upt*:
  assumes *A* @ *B* = [*m*..<*n*]
  shows *A* = [*m* ..<*m*+*length A*] **and** *B* = [*m* + *length A*..<*n*]
proof −
  have *take* (*length A*) (*A* @ *B*) = *A* **by** *auto*
  moreover

have *length* $A \leq n - m$ **using** *assms linear calculation* **by** *fastforce*
then have *take* (*length* A) $[m..<n] = [m ..<m+length\ A]$ **by** *auto*
ultimately show $A = [m ..<m+length\ A]$ **using** *assms* **by** *auto*
show $B = [m + length\ A..<n]$ **using** *assms* **by** (*metis append-eq-conv-conj drop-upt*)
qed

The converse of $?A\ @\ ?B = [?m..<?n] \Longrightarrow ?A = [?m..<?m + length\ ?A]$
$?A\ @\ ?B = [?m..<?n] \Longrightarrow ?B = [?m + length\ ?A..<?n]$ does not hold, for example if B is empty and A is $[0::'a]$:

lemma $A\ @\ B = [m..< n] \longleftrightarrow A = [m ..<m+length\ A] \wedge B = [m + length\ A..<n]$

oops

A more restrictive version holds:

lemma $B \neq [] \Longrightarrow A\ @\ B = [m..< n] \longleftrightarrow A = [m ..<m+length\ A] \wedge B = [m + length\ A..<n]$
(**is** $?P \Longrightarrow ?A = ?B$)
proof
assume $?A$ **then show** $?B$ **by** (*auto simp add*: *append-cons-eq-upt*)
next
assume $?P$ **and** $?B$
then show $?A$ **using** *append-eq-conv-conj* **by** *fastforce*
qed

lemma *append-cons-eq-upt-length-i*:
assumes $A\ @\ i\ \#\ B = [m..<n]$
shows $A = [m ..<i]$
proof −
have $A = [m ..< m + length\ A]$ **using** *assms append-cons-eq-upt* **by** *auto*
have $(A\ @\ i\ \#\ B)\ !\ (length\ A) = i$ **by** *auto*
moreover have $n - m = length\ (A\ @\ i\ \#\ B)$
using *assms length-upt* **by** *presburger*
then have $[m..<n]\ !\ (length\ A) = m + length\ A$ **by** *simp*
ultimately have $i = m + length\ A$ **using** *assms* **by** *auto*
then show *?thesis* **using** ‹$A = [m ..< m + length\ A]$› **by** *auto*
qed

lemma *append-cons-eq-upt-length*:
assumes $A\ @\ i\ \#\ B = [m..<n]$
shows *length* $A = i - m$
using *assms*
proof (*induction A arbitrary*: *m*)
case *Nil*
then show *?case* **by** (*metis append-Nil diff-is-0-eq list.size*(*3*) *order-refl upt-eq-Cons-conv*)
next
case (*Cons a A*)
then have *A*: $A\ @\ i\ \#\ B = [m + 1..<n]$ **by** (*metis append-Cons upt-eq-Cons-conv*)
then have $m < i$ **by** (*metis Cons.prems append-cons-eq-upt-length-i upt-eq-Cons-conv*)
with *Cons.IH*[*OF A*] **show** *?case* **by** *auto*
qed

lemma *append-cons-eq-upt-length-i-end*:
assumes $A\ @\ i\ \#\ B = [m..<n]$
shows $B = [Suc\ i ..<n]$
proof −
have $B = [Suc\ m + length\ A..<n]$ **using** *assms append-cons-eq-upt*[*of A* @ [*i*] *B m n*] **by** *auto*

have $(A \ @ \ i \ \# \ B) \ ! \ (length \ A) = i$ **by** *auto*
moreover have $n - m = length \ (A \ @ \ i \ \# \ B)$
using *assms length-upt* **by** *auto*
then have $[m..<n]! \ (length \ A) = m + length \ A$ **by** *simp*
ultimately have $i = m + length \ A$ **using** *assms* **by** *auto*
then show *?thesis* **using** ‹$B = [Suc \ m + length \ A..<n]$› **by** *auto*
qed

lemma *Max-n-upt*: $Max \ (insert \ 0 \ \{Suc \ 0..<n\}) = n - Suc \ 0$
proof (*induct n*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*) **note** $IH = this$
have i: $insert \ 0 \ \{Suc \ 0..<Suc \ n\} = insert \ 0 \ \{Suc \ 0..< n\} \cup \{n\}$ **by** *auto*
show *?case* **using** *IH* **unfolding** *i* **by** *auto*
qed

lemma *upt-decomp-lt*:
assumes H: $xs \ @ \ i \ \# \ ys \ @ \ j \ \# \ zs = [m \ ..< n]$
shows $i < j$
proof −
have xs: $xs = [m \ ..< i]$ **and** ys: $ys = [Suc \ i \ ..< j]$ **and** zs: $zs = [Suc \ j \ ..< n]$
using *H* **by** (*auto dest*: *append-cons-eq-upt-length-i append-cons-eq-upt-length-i-end*)
show *?thesis*
by (*metis append-cons-eq-upt-length-i-end assms lessI less-trans self-append-conv2 upt-eq-Cons-conv upt-rec ys*)
qed

3.2 Lexicographic ordering

We are working a lot on lexicographic ordering over pairs.

lemma *list-length2-append-cons*:
$[c, \ d] = ys \ @ \ y \ \# \ ys' \longleftrightarrow (ys = [] \land y = c \land ys' = [d]) \lor (ys = [c] \land y = d \land ys' = [])$
by (*cases ys*; *cases ys'*) *auto*

lemma *lexn2-conv*:
$([a, \ b], \ [c, \ d]) \in lexn \ r \ 2 \longleftrightarrow (a, \ c) \in r \lor (a = c \land (b, \ d) \in r)$
unfolding *lexn-conv* **by** (*auto simp add*: *list-length2-append-cons*)

end
theory *Prop-Logic*

imports *Main*

begin

4 Logics

In this section we define the syntax of the formula and an abstraction over it to have simpler proofs. After that we define some properties like subformula and rewriting.

4.1 Definition and abstraction

The propositional logic is defined inductively. The type parameter is the type of the variables.

datatype *'v propo* =
  FT | *FF* | *FVar 'v* | *FNot 'v propo* | *FAnd 'v propo 'v propo* | *FOr 'v propo 'v propo*
  | *FImp 'v propo 'v propo* | *FEq 'v propo 'v propo*

We do not define any notation for the formula, to distinguish properly between the formulas and Isabelle's logic.

To ease the proofs, we will write the the formula on a homogeneous manner, namely a connecting argument and a list of arguments.

datatype *'v connective* = *CT* | *CF* | *CVar 'v* | *CNot* | *CAnd* | *COr* | *CImp* | *CEq*

abbreviation *nullary-connective* ≡ {*CF*} ∪ {*CT*} ∪ {*CVar x* | *x. True*}
definition *binary-connectives* ≡ {*CAnd*, *COr*, *CImp*, *CEq*}

We define our own induction principal: instead of distinguishing every constructor, we group them by arity.

lemma *propo-induct-arity*[*case-names nullary unary binary*]:
  fixes φ ψ :: *'v propo*
  assumes *nullary*: ($\bigwedge\varphi$ *x*. φ = *FF* ∨ φ = *FT* ∨ φ = *FVar x* ⟹ *P* φ)
  and *unary*: ($\bigwedge\psi$. *P* ψ ⟹ *P* (*FNot* ψ))
  and *binary*: ($\bigwedge\varphi$ ψ*1* ψ*2*. *P* ψ*1* ⟹ *P* ψ*2* ⟹ φ = *FAnd* ψ*1* ψ*2* ∨ φ = *FOr* ψ*1* ψ*2* ∨ φ = *FImp* ψ*1* ψ*2*
    ∨ φ = *FEq* ψ*1* ψ*2* ⟹ *P* φ)
  shows *P* ψ
  apply (*induct rule*: *propo.induct*)
  using *assms* **by** *metis+*

The function *conn* is the interpretation of our representation (connective and list of arguments). We define any thing that has no sense to be false

fun *conn* :: *'v connective* ⇒ *'v propo list* ⇒ *'v propo* **where**
conn CT [] = *FT* |
conn CF [] = *FF* |
conn (*CVar v*) [] = *FVar v* |
conn CNot [φ] = *FNot* φ |
conn CAnd (φ# [ψ]) = *FAnd* φ ψ |
conn COr (φ# [ψ]) = *FOr* φ ψ |
conn CImp (φ# [ψ]) = *FImp* φ ψ |
conn CEq (φ# [ψ]) = *FEq* φ ψ |
conn - - = *FF*

We will often use case distinction, based on the arity of the *'v connective*, thus we define our own splitting principle.

lemma *connective-cases-arity*:
  assumes *nullary*: $\bigwedge$*x*. *c* = *CT* ∨ *c* = *CF* ∨ *c* = *CVar x* ⟹ *P*
  and *binary*: *c* ∈ *binary-connectives* ⟹ *P*
  and *unary*: *c* = *CNot* ⟹ *P*
  shows *P*
  using *assms* **by** (*case-tac c*, *auto simp add*: *binary-connectives-def*)

lemma *connective-cases-arity-2*[*case-names nullary unary binary*]:

assumes *nullary*: $c \in$ *nullary-connective* $\Longrightarrow P$
and *unary*: $c = CNot \ \Longrightarrow P$
and *binary*: $c \in$ *binary-connectives* $\Longrightarrow P$
shows P
using *assms* **by** (*case-tac c*, *auto simp add*: *binary-connectives-def*)

Our previous definition is not necessary correct (connective and list of arguments) , so we define an inductive predicate.

inductive *wf-conn* :: *'v connective* $\Rightarrow$ *'v propo list* $\Rightarrow$ *bool* **for** c :: *'v connective* **where**
wf-conn-nullary[*simp*]: $(c = CT \lor c = CF \lor c \ = CVar\ v) \Longrightarrow$ *wf-conn* c [] |
wf-conn-unary[*simp*]: $c = CNot \Longrightarrow$ *wf-conn* c $[\psi]$ |
wf-conn-binary[*simp*]: $c \in$ *binary-connectives* $\Longrightarrow$ *wf-conn* c $(\psi \ \# \ \psi' \ \# \ [])$
thm *wf-conn.induct*
lemma *wf-conn-induct*[*consumes 1*, *case-names CT CF CVar CNot COr CAnd CImp CEq*]:
assumes *wf-conn c x* **and**
$(\bigwedge v.\ c = CT \Longrightarrow P\ [])$ **and**
$(\bigwedge v.\ c = CF \Longrightarrow P\ [])$ **and**
$(\bigwedge v.\ c = CVar\ v \Longrightarrow P\ [])$ **and**
$(\bigwedge \psi.\ c = CNot \Longrightarrow P\ [\psi])$ **and**
$(\bigwedge \psi\ \psi'.\ c = COr \Longrightarrow P\ [\psi, \psi'])$ **and**
$(\bigwedge \psi\ \psi'.\ c = CAnd \Longrightarrow P\ [\psi, \psi'])$ **and**
$(\bigwedge \psi\ \psi'.\ c = CImp \Longrightarrow P\ [\psi, \psi'])$ **and**
$(\bigwedge \psi\ \psi'.\ c = CEq \Longrightarrow P\ [\psi, \psi'])$
shows $P\ x$
using *assms* **by** *induction* (*auto simp add*: *binary-connectives-def*)

4.2 properties of the abstraction

First we can define simplification rules.

lemma *wf-conn-conn*[*simp*]:
wf-conn $CT\ l \Longrightarrow$ *conn* $CT\ l = FT$
wf-conn $CF\ l \Longrightarrow$ *conn* $CF\ l = FF$
wf-conn $(CVar\ x)\ l \Longrightarrow$ *conn* $(CVar\ x)\ l = FVar\ x$
apply (*simp-all add*: *wf-conn.simps*)
unfolding *binary-connectives-def* **by** *simp-all*

lemma *wf-conn-list-decomp*[*simp*]:
wf-conn $CT\ l \longleftrightarrow l = []$
wf-conn $CF\ l \longleftrightarrow l = []$
wf-conn $(CVar\ x)\ l \longleftrightarrow l = []$
wf-conn $CNot\ (\xi\ @\ \varphi\ \#\ \xi') \longleftrightarrow \xi = [] \land \xi' = []$
apply (*simp-all add*: *wf-conn.simps*)
unfolding *binary-connectives-def* **apply** *simp-all*
by (*metis append-Nil append-is-Nil-conv list.distinct*(*1*) *list.sel*(*3*) *tl-append2*)

lemma *wf-conn-list*:
wf-conn $c\ l \Longrightarrow$ *conn* $c\ l = FT \longleftrightarrow (c = CT \land l = [])$
wf-conn $c\ l \Longrightarrow$ *conn* $c\ l = FF \ \longleftrightarrow (c = CF \land l = [])$
wf-conn $c\ l \Longrightarrow$ *conn* $c\ l = FVar\ x \ \longleftrightarrow (c = CVar\ x \land l = [])$
wf-conn $c\ l \Longrightarrow$ *conn* $c\ l = FAnd\ a\ b \ \longleftrightarrow (c = CAnd \land l = a\ \#\ b\ \#\ [])$
wf-conn $c\ l \Longrightarrow$ *conn* $c\ l = FOr\ a\ b \longleftrightarrow (c = COr \land l = a\ \#\ b\ \#\ [])$
wf-conn $c\ l \Longrightarrow$ *conn* $c\ l = FEq\ a\ b \longleftrightarrow (c = CEq \land l = a\ \#\ b\ \#\ [])$
wf-conn $c\ l \Longrightarrow$ *conn* $c\ l = FImp\ a\ b \longleftrightarrow (c = CImp \land l = a\ \#\ b\ \#\ [])$

wf-conn c l ⟹ *conn c l* = *FNot a* ⟷ (*c* = *CNot* ∧ *l* = *a* # [])
  apply (*induct l rule*: *wf-conn.induct*)
  unfolding *binary-connectives-def* **by** *auto*

In the binary connective cases, we will often decompose the list of arguments (of length 2) into two elements.

lemma *list-length2-decomp*: *length l* = *2* ⟹ (∃ *a b*. *l* = *a* # *b* # [])
  apply (*induct l*, *auto*)
  by (*case-tac l*, *auto*)

wf-conn for binary operators means that there are two arguments.

lemma *wf-conn-bin-list-length*:
  fixes *l* :: *'v propo list*
  assumes *conn*: *c* ∈ *binary-connectives*
  shows *length l* = *2* ⟷ *wf-conn c l*
proof
  assume *length l* = *2*
  thus *wf-conn c l* **using** *wf-conn-binary list-length2-decomp* **using** *conn* **by** *metis*
next
  assume *wf-conn c l*
  thus *length l* = *2* (**is** *?P l*)
    proof (*cases rule*: *wf-conn.induct*)
      case *wf-conn-nullary*
      thus *?P* [] **using** *conn binary-connectives-def*
        using *connective.distinct*(*11*) *connective.distinct*(*13*) *connective.distinct*(*9*) **by** *blast*
    next
      fix *ψ* :: *'v propo*
      case *wf-conn-unary*
      thus *?P* [*ψ*] **using** *conn binary-connectives-def*
        using *connective.distinct* **by** *blast*
    next
      fix *ψ ψ'*:: *'v propo*
      show *?P* [*ψ*, *ψ'*] **by** *auto*
    qed
qed

lemma *wf-conn-not-list-length*[*iff*]:
  fixes *l* :: *'v propo list*
  shows *wf-conn CNot l* ⟷ *length l* = *1*
  apply *auto*
  apply (*metis append-Nil connective.distinct*(*5*,*17*,*27*) *length-Cons list.size*(*3*) *wf-conn.simps wf-conn-list-decomp*(*4*))
  by (*simp add*: *length-Suc-conv wf-conn.simps*)

Decomposing the Not into an element is moreover very useful.

lemma *wf-conn-Not-decomp*:
  fixes *l* :: *'v propo list* **and** *a* :: *'v*
  assumes *corr*: *wf-conn CNot l*
  shows ∃ *a*. *l* = [*a*]
  by (*metis* (*no-types*, *lifting*) *One-nat-def Suc-length-conv corr length-0-conv wf-conn-not-list-length*)

The *wf-conn* remains correct if the length of list does not change. This lemma is very useful when we do one rewriting step

lemma *wf-conn-no-arity-change*:

length l = *length* $l' \Longrightarrow$ *wf-conn* c $l \longleftrightarrow$ *wf-conn* c l'
proof −
  {
    fix l l'
    have *length* l = *length* $l' \Longrightarrow$ *wf-conn* c $l \Longrightarrow$ *wf-conn* c l'
      apply (*cases* c l *rule*: *wf-conn.induct*, *auto*)
      by (*metis wf-conn-bin-list-length*)
  }
  thus *length* l = *length* $l' \Longrightarrow$ *wf-conn* c l = *wf-conn* c l' **by** *metis*
qed

lemma *wf-conn-no-arity-change-helper*:
  length $(\xi\ @\ \varphi\ \#\ \xi')$ = *length* $(\xi\ @\ \varphi'\ \#\ \xi')$
  by *auto*

The injectivity of *conn* is useful to prove equality of the connectives and the lists.

lemma *conn-inj-not*:
  assumes *correct*: *wf-conn* c l
  and *conn*: *conn* c l = *FNot* ψ
  shows c = *CNot* **and** $l = [\psi]$
  apply (*cases* c l *rule*: *wf-conn.cases*)
  using *correct conn* **unfolding** *binary-connectives-def* **apply** *auto*
  apply (*cases* c l *rule*: *wf-conn.cases*)
  using *correct conn* **unfolding** *binary-connectives-def* **by** *auto*

lemma *conn-inj*:
  fixes c *ca* :: *'v connective* **and** l ψs :: *'v propo list*
  assumes *corr*: *wf-conn ca* l
  and *corr'*: *wf-conn* c ψs
  and *eq*: *conn ca* l = *conn* c ψs
  shows $ca = c \wedge \psi s = l$
  using *corr*
proof (*cases ca* l *rule*: *wf-conn.cases*)
  case (*wf-conn-nullary* v)
  thus $ca = c \wedge \psi s = l$ **using** *assms*
      by (*metis conn.simps*(*1*) *conn.simps*(*2*) *conn.simps*(*3*) *wf-conn-list*(*1*−*3*))
next
  case (*wf-conn-unary* ψ')
  hence ∗: *FNot* ψ' = *conn* c ψs **using** *conn-inj-not eq assms* **by** *auto*
  hence $c = ca$ **by** (*metis conn-inj-not*(*1*) *corr'* *wf-conn-unary*(*2*))
  moreover have $\psi s = l$ **using** ∗ *conn-inj-not*(*2*) *corr'* *wf-conn-unary*(*1*) **by** *force*
  ultimately show $ca = c \wedge \psi s = l$ **by** *auto*
next
  case (*wf-conn-binary* ψ' ψ'')
  thus $ca = c \wedge \psi s = l$
    using *eq corr'* **unfolding** *binary-connectives-def* **apply** (*case-tac ca*, *auto simp add*: *wf-conn-list*)
    using *wf-conn-list*(*4*−*7*) *corr'* **by** *metis*+
qed

4.3 Subformulas and properties

A characterization using sub-formulas is interesting for rewriting: we will define our relation on the sub-term level, and then lift the rewriting on the term-level. So the rewriting takes place on a subformula.

inductive *subformula* :: $'v$ *propo* $\Rightarrow$ $'v$ *propo* $\Rightarrow$ *bool* (**infix** $\preceq$ *45*) **for** φ **where**
subformula-refl[*simp*]: $\varphi \preceq \varphi$ |
subformula-into-subformula: $\psi \in$ *set* $l \Longrightarrow$ *wf-conn c l* $\Longrightarrow \varphi \preceq \psi \Longrightarrow \varphi \preceq$ *conn c l*

On the *subformula-into-subformula*, we can see why we use our *conn* representation: one case is enough to express the subformulas property instead of listing all the cases.

This is an example of a property related to subformulas.

lemma *subformula-in-subformula-not*:
shows *b*: *FNot* $\varphi \preceq \psi \Longrightarrow \varphi \preceq \psi$
  apply (*induct rule*: *subformula.induct*)
  using *subformula-into-subformula wf-conn-unary subformula-refl list.set-intros*(*1*) *subformula-refl*
    by (*fastforce intro*: *subformula-into-subformula*)+

lemma *subformula-in-binary-conn*:
  assumes *conn*: $c \in$ *binary-connectives*
  shows $f \preceq$ *conn c* [f, g]
  and $g \preceq$ *conn c* [f, g]
proof −
  have *a*: *wf-conn c* (f# [g]) **using** *conn wf-conn-binary binary-connectives-def* **by** *auto*
  moreover have *b*: $f \preceq f$ **using** *subformula-refl* **by** *auto*
  ultimately show $f \preceq$ *conn c* [f, g]
    by (*metis append-Nil in-set-conv-decomp subformula-into-subformula*)
next
  have *a*: *wf-conn c* ([f] @ [g]) **using** *conn wf-conn-binary binary-connectives-def* **by** *auto*
  moreover have *b*: $g \preceq g$ **using** *subformula-refl* **by** *auto*
  ultimately show $g \preceq$ *conn c* [f, g] **using** *subformula-into-subformula* **by** *force*
qed

lemma *subformula-trans*:
 $\psi \preceq \psi' \Longrightarrow \varphi \preceq \psi \Longrightarrow \varphi \preceq \psi'$
  apply (*induct* ψ' *rule*: *subformula.inducts*)
  by (*auto simp add*: *subformula-into-subformula*)

lemma *subformula-leaf*:
  fixes φ ψ :: $'v$ *propo*
  assumes *incl*: $\varphi \preceq \psi$
  and *simple*: $\psi = FT \vee \psi = FF \vee \psi = FVar\ x$
  shows $\varphi = \psi$
  using *incl simple*
  by (*induct rule*: *subformula.induct*, *auto simp add*: *wf-conn-list*)

lemma *subfurmula-not-incl-eq*:
  assumes $\varphi \preceq$ *conn c l*
  and *wf-conn c l*
  and $\forall \psi.\ \psi \in$ *set* $l \longrightarrow \neg\ \varphi \preceq \psi$
  shows $\varphi =$ *conn c l*
  using *assms* **apply** (*induction conn c l rule*: *subformula.induct*, *auto*)
  using *conn-inj* **by** *blast*

lemma *wf-subformula-conn-cases*:
  wf-conn c l $\Longrightarrow \varphi \preceq$ *conn c l* $\longleftrightarrow$ ($\varphi =$ *conn c l* $\vee$ ($\exists \psi.\ \psi \in$ *set* $l \wedge \varphi \preceq \psi$))
  apply *standard*

using *subfurmula-not-incl-eq* **apply** *metis*
by (*auto simp add*: *subformula-into-subformula*)

lemma *subformula-decomp-explicit*[*simp*]:
$\varphi \preceq FAnd\ \psi\ \psi' \longleftrightarrow (\varphi = FAnd\ \psi\ \psi' \vee \varphi \preceq \psi \vee \varphi \preceq \psi')$ (**is** *?P FAnd*)
$\varphi \preceq FOr\ \psi\ \psi' \longleftrightarrow (\varphi = FOr\ \psi\ \psi' \vee \varphi \preceq \psi \vee \varphi \preceq \psi')$
$\varphi \preceq FEq\ \psi\ \psi' \longleftrightarrow (\varphi = FEq\ \psi\ \psi' \vee \varphi \preceq \psi \vee \varphi \preceq \psi')$
$\varphi \preceq FImp\ \psi\ \psi' \longleftrightarrow (\varphi = FImp\ \psi\ \psi' \vee \varphi \preceq \psi \vee \varphi \preceq \psi')$
proof −
have *wf-conn CAnd* $[\psi, \psi']$ **by** (*simp add*: *binary-connectives-def*)
hence $\varphi \preceq conn\ CAnd\ [\psi, \psi'] \longleftrightarrow (\varphi = conn\ CAnd\ [\psi, \psi'] \vee (\exists\psi''.\ \psi'' \in set\ [\psi, \psi'] \wedge \varphi \preceq \psi''))$
using *wf-subformula-conn-cases* **by** *metis*
thus *?P FAnd* **by** *auto*
next
have *wf-conn COr* $[\psi, \psi']$ **by** (*simp add*: *binary-connectives-def*)
hence $\varphi \preceq conn\ COr\ [\psi, \psi'] \longleftrightarrow (\varphi = conn\ COr\ [\psi, \psi'] \vee (\exists\psi''.\ \psi'' \in set\ [\psi, \psi'] \wedge \varphi \preceq \psi''))$
using *wf-subformula-conn-cases* **by** *metis*
thus *?P FOr* **by** *auto*
next
have *wf-conn CEq* $[\psi, \psi']$ **by** (*simp add*: *binary-connectives-def*)
hence $\varphi \preceq conn\ CEq\ [\psi, \psi'] \longleftrightarrow (\varphi = conn\ CEq\ [\psi, \psi'] \vee (\exists\psi''.\ \psi'' \in set\ [\psi, \psi'] \wedge \varphi \preceq \psi''))$
using *wf-subformula-conn-cases* **by** *metis*
thus *?P FEq* **by** *auto*
next
have *wf-conn CImp* $[\psi, \psi']$ **by** (*simp add*: *binary-connectives-def*)
hence $\varphi \preceq conn\ CImp\ [\psi, \psi'] \longleftrightarrow (\varphi = conn\ CImp\ [\psi, \psi'] \vee (\exists\psi''.\ \psi'' \in set\ [\psi, \psi'] \wedge \varphi \preceq \psi''))$
using *wf-subformula-conn-cases* **by** *metis*
thus *?P FImp* **by** *auto*
qed

lemma *wf-conn-helper-facts*[*iff*]:
wf-conn CNot $[\varphi]$
wf-conn CT []
wf-conn CF []
wf-conn (*CVar x*) []
wf-conn CAnd $[\varphi, \psi]$
wf-conn COr $[\varphi, \psi]$
wf-conn CImp $[\varphi, \psi]$
wf-conn CEq $[\varphi, \psi]$
using *wf-conn.intros* **unfolding** *binary-connectives-def* **by** *fastforce+*

lemma *exists-c-conn*: $\exists\ c\ l.\ \varphi = conn\ c\ l \wedge wf\text{-}conn\ c\ l$
by (*cases* φ) *force+*

lemma *subformula-conn-decomp*[*simp*]:
$wf\text{-}conn\ c\ l \Longrightarrow \varphi \preceq conn\ c\ l \longleftrightarrow (\varphi = conn\ c\ l \vee (\exists\ \psi \in set\ l.\ \varphi \preceq \psi))$
apply *auto*
proof −
{
fix ξ
have $\varphi \preceq \xi \Longrightarrow \xi = conn\ c\ l \Longrightarrow wf\text{-}conn\ c\ l \Longrightarrow \forall x{::}'a\ propo \in set\ l.\ \neg\ \varphi \preceq x \Longrightarrow \varphi = conn\ c\ l$
apply (*induct rule*: *subformula.induct*)
apply *simp*
using *conn-inj* **by** *blast*

}
moreover assume *wf-conn c l* **and** $\varphi \preceq$ *conn c l* **and** $\forall x::'a$ *propo*$\in$*set l*. $\neg\ \varphi \preceq x$
ultimately show $\varphi =$ *conn c l* **by** *metis*
next
fix ψ
assume *wf-conn c l* **and** $\psi \in$ *set l* **and** $\varphi \preceq \psi$
thus $\varphi \preceq$ *conn c l* **using** *wf-subformula-conn-cases* **by** *blast*
qed

lemma *subformula-leaf-explicit*[*simp*]:
$\varphi \preceq FT \longleftrightarrow \varphi = FT$
$\varphi \preceq FF \longleftrightarrow \varphi = FF$
$\varphi \preceq FVar\ x \longleftrightarrow \varphi = FVar\ x$
apply *auto*
using *subformula-leaf* **by** *metis* +

The variables inside the formula gives precisely the variables that are needed for the formula.

primrec *vars-of-prop*:: *'v propo* $\Rightarrow$ *'v set* **where**
vars-of-prop FT = {} |
vars-of-prop FF = {} |
vars-of-prop (*FVar x*) = {*x*} |
vars-of-prop (*FNot* φ) = *vars-of-prop* φ |
vars-of-prop (*FAnd* φ ψ) = *vars-of-prop* φ $\cup$ *vars-of-prop* ψ |
vars-of-prop (*FOr* φ ψ) = *vars-of-prop* φ $\cup$ *vars-of-prop* ψ |
vars-of-prop (*FImp* φ ψ) = *vars-of-prop* φ $\cup$ *vars-of-prop* ψ |
vars-of-prop (*FEq* φ ψ) = *vars-of-prop* φ $\cup$ *vars-of-prop* ψ

lemma *vars-of-prop-incl-conn*:
fixes ξ ξ' :: *'v propo list* **and** ψ :: *'v propo* **and** *c* :: *'v connective*
assumes *corr*: *wf-conn c l* **and** *incl*: $\psi \in$ *set l*
shows *vars-of-prop* $\psi \subseteq$ *vars-of-prop* (*conn c l*)
proof (*cases c rule*: *connective-cases-arity-2*)
case *nullary*
hence *False* **using** *corr incl* **by** *auto*
thus *vars-of-prop* $\psi \subseteq$ *vars-of-prop* (*conn c l*) **by** *blast*
next
case *binary* **note** *c* = *this*
then obtain *a b* **where** *ab*: *l* = [*a*, *b*]
using *wf-conn-bin-list-length list-length2-decomp corr* **by** *metis*
hence $\psi = a \vee \psi = b$ **using** *incl* **by** *auto*
thus *vars-of-prop* $\psi \subseteq$ *vars-of-prop* (*conn c l*)
using *ab c* **unfolding** *binary-connectives-def* **by** *auto*
next
case *unary* **note** *c* = *this*
fix φ :: *'v propo*
have *l* = [ψ] **using** *corr c incl split-list* **by** *force*
thus *vars-of-prop* $\psi \subseteq$ *vars-of-prop* (*conn c l*) **using** *c* **by** *auto*
qed

The set of variables is compatible with the subformula order.

lemma *subformula-vars-of-prop*:
$\varphi \preceq \psi \Longrightarrow$ *vars-of-prop* $\varphi \subseteq$ *vars-of-prop* ψ
apply (*induct rule*: *subformula.induct*)
apply *simp*

using *vars-of-prop-incl-conn* **by** *blast*

4.4 Positions

Instead of 1 or 2 we use L or R

datatype *sign* = L | R

We use *nil* instead of ε.

fun *pos* :: *'v propo* ⇒ *sign list set* **where**
pos FF = {[]} |
pos FT = {[]} |
pos (*FVar x*) = {[]} |
pos (*FAnd* φ ψ) = {[]} ∪ { L # p | p. p∈ *pos* φ} ∪ { R # p | p. p∈ *pos* ψ} |
pos (*FOr* φ ψ) = {[]} ∪ { L # p | p. p∈ *pos* φ} ∪ { R # p | p. p∈ *pos* ψ} |
pos (*FEq* φ ψ) = {[]} ∪ { L # p | p. p∈ *pos* φ} ∪ { R # p | p. p∈ *pos* ψ} |
pos (*FImp* φ ψ) = {[]} ∪ { L # p | p. p∈ *pos* φ} ∪ { R # p | p. p∈ *pos* ψ} |
pos (*FNot* φ) = {[]} ∪ { L # p | p. p∈ *pos* φ}

lemma *finite-pos*: *finite* (*pos* φ)
  by (*induct* φ, *auto*)

lemma *finite-inj-comp-set*:
  fixes *s* :: *'v set*
  assumes *finite*: *finite s*
  and *inj*: *inj f*
  shows *card* ({*f p* |*p*. *p* ∈ *s*}) = *card s*
  using *finite*
proof (*induct s rule*: *finite-induct*)
  show *card* {*f p* |*p*. *p* ∈ {}} = *card* {} **by** *auto*
next
  fix *x* :: *'v* **and** *s*:: *'v set*
  assume *f*: *finite s* **and** *notin*: *x* ∉ *s*
  and *IH*: *card* {*f p* |*p*. *p* ∈ *s*} = *card s*
  have *f'*: *finite* {*f p* |*p*. *p* ∈ *insert x s*} **using** *f* **by** *auto*
  have *notin'*: *f x* ∉ {*f p* |*p*. *p* ∈ *s*} **using** *notin inj injD* **by** *fastforce*
  have {*f p* |*p*. *p* ∈ *insert x s*} = *insert* (*f x*) {*f p* |*p*. *p*∈ *s*} **by** *auto*
  hence *card* {*f p* |*p*. *p* ∈ *insert x s*} = *1* + *card* {*f p* |*p*. *p* ∈ *s*}
    using *finite card-insert-disjoint f' notin'* **by** *auto*
  moreover have ... = *card* (*insert x s*) **using** *notin f IH* **by** *auto*
  finally show *card* {*f p* |*p*. *p* ∈ *insert x s*} = *card* (*insert x s*) **.**
qed

lemma *cons-inject*:
  inj (*op* # *s*)
  by (*meson injI list.inject*)

lemma *finite-insert-nil-cons*:
  finite s ⟹ *card* (*insert* [] {*L* # *p* |*p*. *p* ∈ *s*}) = *1* + *card* {*L* # *p* |*p*. *p* ∈ *s*}
using *card-insert-disjoint* **by** *auto*

lemma *cord-not*[*simp*]:
  card (*pos* (*FNot* φ)) = *1* + *card* (*pos* φ)

by (*simp add*: *cons-inject finite-inj-comp-set finite-pos*)

lemma *card-seperate*:
  assumes *finite s1* **and** *finite s2*
  shows *card* ($\{L \# p\ |p.\ p \in s1\} \cup \{R \# p\ |p.\ p \in s2\}$) = *card* ($\{L \# p\ |p.\ p \in s1\}$)
    + *card*($\{R \# p\ |p.\ p \in s2\}$) (**is** *card* (*?L*$\cup$*?R*) = *card ?L* + *card ?R*)
proof −
  have *finite ?L* **using** *assms* **by** *auto*
  moreover have *finite ?R* **using** *assms* **by** *auto*
  moreover have *?L* $\cap$ *?R* = {} **by** *blast*
  ultimately show *?thesis* **using** *assms card-Un-disjoint* **by** *blast*
qed

definition *prop-size* **where** *prop-size* φ = *card* (*pos* φ)

lemma *prop-size-vars-of-prop*:
  fixes φ :: *'v propo*
  shows *card* (*vars-of-prop* φ) $\leq$ *prop-size* φ
  unfolding *prop-size-def* **apply** (*induct* φ, *auto simp add*: *cons-inject finite-inj-comp-set finite-pos*)
proof −
  fix $\varphi 1$ $\varphi 2$ :: *'v propo*
  assume *IH1*: *card* (*vars-of-prop* $\varphi 1$) $\leq$ *card* (*pos* $\varphi 1$)
  and *IH2*: *card* (*vars-of-prop* $\varphi 2$) $\leq$ *card* (*pos* $\varphi 2$)
  let *?L* = $\{L \# p\ |p.\ p \in pos\ \varphi 1\}$
  let *?R* = $\{R \# p\ |p.\ p \in pos\ \varphi 2\}$
  have *card* (*?L* $\cup$ *?R*) = *card ?L* + *card ?R*
    using *card-seperate finite-pos* **by** *blast*
  moreover have ... = *card* (*pos* $\varphi 1$) + *card* (*pos* $\varphi 2$)
    by (*simp add*: *cons-inject finite-inj-comp-set finite-pos*)
  moreover have ... $\geq$ *card* (*vars-of-prop* $\varphi 1$) + *card* (*vars-of-prop* $\varphi 2$) **using** *IH1 IH2* **by** *arith*
  hence ... $\geq$ *card* (*vars-of-prop* $\varphi 1$ $\cup$ *vars-of-prop* $\varphi 2$) **using** *card-Un-le le-trans* **by** *blast*
  ultimately
    show *card* (*vars-of-prop* $\varphi 1$ $\cup$ *vars-of-prop* $\varphi 2$) $\leq$ *Suc* (*card* (*?L* $\cup$ *?R*))
      card (*vars-of-prop* $\varphi 1$ $\cup$ *vars-of-prop* $\varphi 2$) $\leq$ *Suc* (*card* (*?L* $\cup$ *?R*))
      card (*vars-of-prop* $\varphi 1$ $\cup$ *vars-of-prop* $\varphi 2$) $\leq$ *Suc* (*card* (*?L* $\cup$ *?R*))
      card (*vars-of-prop* $\varphi 1$ $\cup$ *vars-of-prop* $\varphi 2$) $\leq$ *Suc* (*card* (*?L* $\cup$ *?R*))
    by *auto*
qed

value *pos* (*FImp* (*FAnd* (*FVar P*) (*FVar Q*)) (*FOr* (*FVar P*) (*FVar Q*)))

inductive *path-to* :: *sign list* $\Rightarrow$ *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **where**
path-to-refl[*intro*]: *path-to* [] φ φ |
path-to-l: *c*$\in$*binary-connectives* $\vee$ *c* = *CNot* $\Longrightarrow$ *wf-conn c* (φ#*l*) $\Longrightarrow$ *path-to p* φ φ'
  $\Longrightarrow$ *path-to* (*L*#*p*) (*conn c* (φ#*l*)) φ' |
path-to-r: *c*$\in$*binary-connectives* $\Longrightarrow$ *wf-conn c* (ψ#φ#[]) $\Longrightarrow$ *path-to p* φ φ'
  $\Longrightarrow$ *path-to* (*R*#*p*) (*conn c* (ψ#φ#[])) φ'

There is a deep link between subformulas and pathes: a (correct) path leads to a subformula and a subformula is associated to a given path.

lemma *path-to-subformula*:

path-to p φ φ' $\Longrightarrow$ *φ'* $\preceq$ *φ*
apply (*induct rule*: *path-to.induct*)
apply *simp*
apply (*metis list.set-intros*(*1*) *subformula-into-subformula*)
using *subformula-trans subformula-in-binary-conn*(*2*) **by** *metis*

lemma *subformula-path-exists*:
fixes *φ φ'*:: *'v propo*
shows *φ'* $\preceq$ *φ* $\Longrightarrow$ ∃ *p. path-to p φ φ'*
proof (*induct rule*: *subformula.induct*)
case *subformula-refl*
have *path-to* [] *φ' φ'* **by** *auto*
thus ∃ *p. path-to p φ' φ'* **by** *metis*
next
case (*subformula-into-subformula ψ l c*)
note *wf* = *this*(*2*) **and** *IH* = *this*(*4*) **and** *ψ* = *this*(*1*)
then obtain *p* **where** *p*: *path-to p ψ φ'* **by** *metis*
{
fix *x* :: *'v*
assume *c* = *CT* ∨ *c* = *CF* ∨ *c* = *CVar x*
hence *False* **using** *subformula-into-subformula* **by** *auto*
hence ∃ *p. path-to p* (*conn c l*) *φ'* **by** *blast*
}
moreover {
assume *c*: *c* = *CNot*
hence *l* = [*ψ*] **using** *wf ψ wf-conn-Not-decomp* **by** *fastforce*
hence *path-to* (*L* # *p*) (*conn c l*) *φ'* **by** (*metis c wf-conn-unary p path-to-l*)
hence ∃ *p. path-to p* (*conn c l*) *φ'* **by** *blast*
}
moreover {
assume *c*: *c*∈ *binary-connectives*
obtain *a b* **where** *ab*: [*a*, *b*] = *l* **using** *subformula-into-subformula c wf-conn-bin-list-length list-length2-decomp* **by** *metis*
hence *a* = *ψ* ∨ *b* = *ψ* **using** *ψ* **by** *auto*
hence *path-to* (*L* # *p*) (*conn c l*) *φ'* ∨ *path-to* (*R* # *p*) (*conn c l*) *φ'* **using** *c path-to-l path-to-r p ab* **by** (*metis wf-conn-binary*)
hence ∃ *p. path-to p* (*conn c l*) *φ'* **by** *blast*
}
ultimately show ∃ *p. path-to p* (*conn c l*) *φ'* **using** *connective-cases-arity* **by** *metis*
qed

fun *replace-at* :: *sign list* ⇒ *'v propo* ⇒ *'v propo* ⇒ *'v propo* **where**
replace-at [] - *ψ* = *ψ* |
replace-at (*L* # *l*) (*FAnd φ φ'*) *ψ* = *FAnd* (*replace-at l φ ψ*) *φ'*|
replace-at (*R* # *l*) (*FAnd φ φ'*) *ψ* = *FAnd φ* (*replace-at l φ' ψ*) |
replace-at (*L* # *l*) (*FOr φ φ'*) *ψ* = *FOr* (*replace-at l φ ψ*) *φ'* |
replace-at (*R* # *l*) (*FOr φ φ'*) *ψ* = *FOr φ* (*replace-at l φ' ψ*) |
replace-at (*L* # *l*) (*FEq φ φ'*) *ψ* = *FEq* (*replace-at l φ ψ*) *φ'*|
replace-at (*R* # *l*) (*FEq φ φ'*) *ψ* = *FEq φ* (*replace-at l φ' ψ*) |
replace-at (*L* # *l*) (*FImp φ φ'*) *ψ* = *FImp* (*replace-at l φ ψ*) *φ'*|
replace-at (*R* # *l*) (*FImp φ φ'*) *ψ* = *FImp φ* (*replace-at l φ' ψ*) |
replace-at (*L* # *l*) (*FNot φ*) *ψ* = *FNot* (*replace-at l φ ψ*)

5 Semantics over the syntax

Given the syntax defined above, we define a semantics, by defining an evaluation function *eval*. This function is the bridge between the logic as we define it here and the built-in logic of Isabelle.

```
fun eval :: ('v ⇒ bool) ⇒ 'v propo ⇒ bool (infix ⊨ 50) where
𝒜 ⊨ FT = True |
𝒜 ⊨ FF = False |
𝒜 ⊨ FVar v = (𝒜 v) |
𝒜 ⊨ FNot φ = (¬(𝒜⊨ φ)) |
𝒜 ⊨ FAnd φ1 φ2 = (𝒜⊨φ1 ∧ 𝒜⊨φ2) |
𝒜 ⊨ FOr φ1 φ2 = (𝒜⊨φ1 ∨ 𝒜⊨φ2) |
𝒜 ⊨ FImp φ1 φ2 = (𝒜⊨φ1 ⟶ 𝒜⊨φ2)  |
𝒜 ⊨ FEq φ1 φ2 = (𝒜⊨φ1 ⟷ 𝒜 ⊨φ2)

definition evalf (infix ⊨f 50) where
evalf φ ψ = (∀ A. A ⊨ φ ⟶ A ⊨ ψ)
```

The deduction rule is in the book. And the proof looks like to the one of the book.

```
lemma deduction-rule:
  (φ ⊨f ψ) ⟷ (∀ A. (A⊨ FImp φ ψ))
proof
  assume H: φ ⊨f ψ
  {
    fix A
```

"Suppose that φ entails ψ (assumption $\varphi \models\!f\ \psi$) and let A be an arbitrary $'v$-valuation. We need to show $A \models FImp\ \varphi\ \psi$. "

```
    {
```

If $A\ \varphi = (1{::}'b)$, then $A\ \varphi = (1{::}'b)$, because φ entails ψ, and therefore $A \models FImp\ \varphi\ \psi$.

```
      assume A ⊨ φ
      hence A ⊨ ψ using H unfolding evalf-def by metis
      hence A ⊨ FImp φ ψ by auto
    }
    moreover {
```

For otherwise, if $A\ \varphi = (0{::}'b)$, then $A \models FImp\ \varphi\ \psi$ holds by definition, independently of the value of $A \models \psi$.

```
      assume ¬ A ⊨ φ
      hence A ⊨ FImp φ ψ by auto
    }
```

In both cases $A \models FImp\ \varphi\ \psi$.

```
    ultimately have A⊨ FImp φ ψ by blast
  }
  thus ∀ A. A ⊨ FImp φ ψ by blast
next
  show ∀ A. A ⊨ FImp φ ψ ⟹ φ ⊨f ψ
    proof (rule ccontr)
      assume ¬φ⊨f ψ
      then obtain A where A ⊨ φ ∧ ¬A⊨ψ using evalf-def by metis
      hence ¬ A ⊨ FImp φ ψ by auto
      moreover assume ∀ A. A ⊨ FImp φ ψ
```

ultimately show *False* **by** *blast*
qed
qed

A shorter proof:

lemma $\varphi \models f\ \psi \longleftrightarrow (\forall A.\ A\models FImp\ \varphi\ \psi)$
by (*simp add*: *evalf-def*)

definition *same-over-set*:: $('v \Rightarrow bool) \Rightarrow('v \Rightarrow bool) \Rightarrow 'v\ set \Rightarrow bool$ **where**
same-over-set $A\ B\ S = (\forall c\in S.\ A\ c = B\ c)$

If two mapping A and B have the same value over the variables, then the same formula are satisfiable.

lemma *same-over-set-eval*:
assumes *same-over-set* $A\ B$ (*vars-of-prop* φ)
shows $A \models \varphi \longleftrightarrow B \models \varphi$
using *assms* **unfolding** *same-over-set-def* **by** (*induct* φ, *auto*)

end
theory *Prop-Abstract-Transformation*
imports *Main Prop-Logic Wellfounded-More*

begin

This file is devoted to abstract properties of the transformations, like consistency preservation and lifting from terms to proposition.

6 Rewrite systems and properties

6.1 Lifting of rewrite rules

We can lift a rewrite relation r over a full1 formula: the relation r works on terms, while *propo-rew-step* works on formulas.

inductive *propo-rew-step* :: $('v\ propo \Rightarrow 'v\ propo \Rightarrow bool) \Rightarrow 'v\ propo \Rightarrow 'v\ propo \Rightarrow bool$
for r :: $'v\ propo \Rightarrow 'v\ propo \Rightarrow bool$ **where**
global-rel: $r\ \varphi\ \psi \Longrightarrow$ *propo-rew-step* $r\ \varphi\ \psi$ |
propo-rew-one-step-lift: *propo-rew-step* $r\ \varphi\ \varphi' \Longrightarrow$ *wf-conn* c $(\psi s\ @\ \varphi\ \#\ \psi s')$
$\Longrightarrow$ *propo-rew-step* r (*conn* c $(\psi s\ @\ \varphi\ \#\ \psi s')$) (*conn* c $(\psi s\ @\ \varphi'\#\ \psi s')$)

Here is a more precise link between the lifting and the subformulas: if a rewriting takes place between φ and φ', then there are two subformulas ψ in φ and ψ' in φ', ψ' is the result of the rewriting of r on ψ.

This lemma is only a health condition:

lemma *propo-rew-step-subformula-imp*:
shows *propo-rew-step* $r\ \varphi\ \varphi' \Longrightarrow \exists\ \psi\ \psi'.\ \psi \preceq \varphi \wedge \psi' \preceq \varphi' \wedge r\ \psi\ \psi'$
apply (*induct rule*: *propo-rew-step.induct*)
using *subformula.simps subformula-into-subformula* **apply** *blast*
using *wf-conn-no-arity-change subformula-into-subformula wf-conn-no-arity-change-helper in-set-conv-decomp* **by** *metis*

The converse is moreover true: if there is a ψ and ψ', then every formula φ containing ψ, can be rewritten into a formula φ', such that it contains φ'.

lemma *propo-rew-step-subformula-rec*:
  fixes $\psi\ \psi'\ \varphi :: {'v}\ propo$
  shows $\psi \preceq \varphi \Longrightarrow r\ \psi\ \psi' \Longrightarrow (\exists\varphi'.\ \psi' \preceq \varphi' \wedge \textit{propo-rew-step}\ r\ \varphi\ \varphi')$
proof (*induct* φ *rule*: *subformula.induct*)
  case *subformula-refl*
  hence *propo-rew-step* $r\ \psi\ \psi'$ **using** *propo-rew-step.intros* **by** *auto*
  moreover have $\psi' \preceq \psi'$ **using** *Prop-Logic.subformula-refl* **by** *auto*
  ultimately show $\exists\varphi'.\ \psi' \preceq \varphi' \wedge \textit{propo-rew-step}\ r\ \psi\ \varphi'$ **by** *fastforce*
next
  case (*subformula-into-subformula* $\psi''\ l\ c$)
  note $IH = this(4)$ **and** $r = this(5)$ **and** $\psi'' = this(1)$ **and** $wf = this(2)$ **and** $incl = this(3)$
  then obtain φ' **where** $*$: $\psi' \preceq \varphi' \wedge \textit{propo-rew-step}\ r\ \psi''\ \varphi'$ **by** *metis*
  moreover obtain $\xi\ \xi' :: {'v}\ propo\ list$ **where**
    l: $l = \xi\ @\ \psi''\ \#\ \xi'$ **using** *List.split-list* ψ'' **by** *metis*
  ultimately have *propo-rew-step* r (*conn* $c\ l$) (*conn* c ($\xi\ @\ \varphi'\ \#\ \xi'$))
    using *propo-rew-step.intros*(2) *wf* **by** *metis*
  moreover have $\psi' \preceq$ *conn* c ($\xi\ @\ \varphi'\ \#\ \xi'$)
    using *wf* $*$ *wf-conn-no-arity-change Prop-Logic.subformula-into-subformula*
    by (*metis* (*no-types*) *in-set-conv-decomp* l *wf-conn-no-arity-change-helper*)
  ultimately show $\exists\varphi'.\ \psi' \preceq \varphi' \wedge \textit{propo-rew-step}\ r\ (\textit{conn}\ c\ l)\ \varphi'$ **by** *metis*
qed

lemma *propo-rew-step-subformula*:
  $(\exists\psi\ \psi'.\ \psi \preceq \varphi \wedge r\ \psi\ \psi') \longleftrightarrow (\exists\varphi'.\ \textit{propo-rew-step}\ r\ \varphi\ \varphi')$
  using *propo-rew-step-subformula-imp propo-rew-step-subformula-rec* **by** *metis+*

lemma *consistency-decompose-into-list*:
  assumes *wf*: *wf-conn* $c\ l$ **and** *wf'*: *wf-conn* $c\ l'$
  and *same*: $\forall n.\ (A \models l\ !\ n \longleftrightarrow (A \models l'\ !\ n))$
  shows $(A \models \textit{conn}\ c\ l) = (A \models \textit{conn}\ c\ l')$
proof (*cases* c *rule*: *connective-cases-arity-2*)
  case *nullary*
  thus $(A \models \textit{conn}\ c\ l) \longleftrightarrow (A \models \textit{conn}\ c\ l')$ **using** *wf wf'* **by** *auto*
next
  case *unary* **note** $c = this$
  then obtain a **where** l: $l = [a]$ **using** *wf-conn-Not-decomp wf* **by** *metis*
  obtain a' **where** l': $l' = [a']$ **using** *wf-conn-Not-decomp wf'* c **by** *metis*
  have $A \models a \longleftrightarrow A \models a'$ **using** $l\ l'$ **by** (*metis nth-Cons-0 same*)
  thus $A \models \textit{conn}\ c\ l \longleftrightarrow A \models \textit{conn}\ c\ l'$ **using** $l\ l'\ c$ **by** *auto*
next
  case *binary* **note** $c = this$
  then obtain $a\ b$ **where** l: $l = [a, b]$
    using *wf-conn-bin-list-length list-length2-decomp wf* **by** *metis*
  obtain $a'\ b'$ **where** l': $l' = [a', b']$
    using *wf-conn-bin-list-length list-length2-decomp wf'* c **by** *metis*

  have p: $A \models a \longleftrightarrow A \models a'\ A \models b \longleftrightarrow A \models b'$
    using $l\ l'$ *same* **by** (*metis diff-Suc-1 nth-Cons' nat.distinct*(2))+
  show $A \models \textit{conn}\ c\ l \longleftrightarrow A \models \textit{conn}\ c\ l'$
    using *wf* $c\ p$ **unfolding** *binary-connectives-def* $l\ l'$ **by** *auto*
qed

Relation between *propo-rew-step* and the rewriting we have seen before: *propo-rew-step* $r\ \varphi\ \varphi'$ means that we rewrite ψ inside φ (ie at a path p) into ψ'.

lemma *propo-rew-step-rewrite*:

```
  fixes φ φ' :: 'v propo and r :: 'v propo ⇒ 'v propo ⇒ bool
  assumes propo-rew-step r φ φ'
  shows ∃ψ ψ' p. r ψ ψ' ∧ path-to p φ ψ ∧ replace-at p φ ψ' = φ'
  using assms
proof (induct rule: propo-rew-step.induct)
  case(global-rel φ ψ)
  moreover have path-to [] φ φ by auto
  moreover have replace-at [] φ ψ = ψ by auto
  ultimately show ?case by metis
next
  case (propo-rew-one-step-lift φ φ' c ξ ξ') note rel = this(1) and IH0 = this(2) and corr = this(3)
  obtain ψ ψ' p where IH: r ψ ψ' ∧ path-to p φ ψ ∧ replace-at p φ ψ' = φ' using IH0 by metis

  {
    fix x :: 'v
    assume c = CT ∨ c = CF ∨ c = CVar x
    hence False using corr by auto
    hence ∃ψ ψ' p. r ψ ψ' ∧ path-to p (conn c (ξ@ (φ # ξ'))) ψ
                  ∧ replace-at p (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
      by fast
  }
  moreover {
    assume c: c = CNot
    hence empty: ξ =[] ξ'=[] using corr by auto
    have path-to (L#p) (conn c (ξ@ (φ # ξ'))) ψ
      using c empty IH wf-conn-unary path-to-l by fastforce
    moreover have replace-at (L#p) (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
      using c empty IH by auto
    ultimately have ∃ψ ψ' p. r ψ ψ' ∧ path-to p (conn c (ξ@ (φ # ξ'))) ψ
                        ∧ replace-at p (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
    using IH by metis
  }
  moreover {
    assume c: c ∈ binary-connectives
    have length (ξ@ φ # ξ') = 2 using wf-conn-bin-list-length corr c by metis
    hence length ξ + length ξ' = 1 by auto
    hence ld: (length ξ = 1 ∧ length ξ' = 0) ∨ (length ξ = 0 ∧ length ξ' = 1)  by arith
    obtain a b where ab: (ξ=[] ∧ ξ'=[b]) ∨ (ξ=[a] ∧ ξ'=[])
      using ld by (case-tac ξ, case-tac ξ', auto)
    {
      assume φ: ξ=[] ∧ ξ'=[b]
      have path-to (L#p) (conn c (ξ@ (φ # ξ'))) ψ
        using φ c IH ab  corr  by (simp add: path-to-l)
      moreover have replace-at (L#p) (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
        using c IH ab φ unfolding binary-connectives-def by auto
      ultimately have ∃ψ ψ' p. r ψ ψ' ∧ path-to p (conn c (ξ@ (φ # ξ'))) ψ
        ∧ replace-at p (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
        using IH by metis
    }
    moreover {
      assume φ: ξ=[a]  ξ'=[]
      hence path-to (R#p) (conn c (ξ@ (φ # ξ'))) ψ
        using c IH corr path-to-r corr φ  by (simp add: path-to-r)
      moreover have replace-at (R#p) (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
        using c IH ab φ unfolding binary-connectives-def by auto
```

ultimately have *?case* **using** *IH* **by** *metis*
}
ultimately have *?case* **using** *ab* **by** *blast*
}
ultimately show *?case* **using** *connective-cases-arity* **by** *blast*
qed

6.2 Consistency preservation

We define *preserves-un-sat*: it means that a relation preserves consistency.

definition *preserves-un-sat* **where**
preserves-un-sat r $\longleftrightarrow$ $(\forall \varphi\ \psi.\ r\ \varphi\ \psi \longrightarrow (\forall A.\ A \models \varphi \longleftrightarrow A \models \psi))$

lemma *propo-rew-step-preservers-val-explicit*:
propo-rew-step r $\varphi\ \psi \Longrightarrow$ *preserves-un-sat r* $\Longrightarrow$ *propo-rew-step r* $\varphi\ \psi \Longrightarrow (\forall A.\ A \models\varphi \longleftrightarrow A\models\psi)$
unfolding *preserves-un-sat-def*
proof (*induction rule*: *propo-rew-step.induct*)
case *global-rel*
thus *?case* **by** *simp*
next
case (*propo-rew-one-step-lift* $\varphi\ \varphi'\ c\ \xi\ \xi'$) **note** *rel* = *this*(*1*) **and** *wf* = *this*(*2*)
and *IH* = *this*(*3*)[*OF this*(*4*) *this*(*1*)] **and** *consistent* = *this*(*4*)
{
fix *A*
from *IH* **have** $\forall n.\ (A \models (\xi\ @\ \varphi\ \#\ \xi')\ !\ n) = (A \models (\xi\ @\ \varphi'\ \#\ \xi')\ !\ n)$
by (*metis* (*mono-tags*, *hide-lams*) *list-update-length nth-Cons-0 nth-append-length-plus nth-list-update-neq*)
hence $(A \models$ *conn c* $(\xi\ @\ \varphi\ \#\ \xi')) = (A \models$ *conn c* $(\xi\ @\ \varphi'\ \#\ \xi'))$
by (*meson consistency-decompose-into-list wf wf-conn-no-arity-change-helper wf-conn-no-arity-change*)
}
thus $\forall A.\ A \models$ *conn c* $(\xi\ @\ \varphi\ \#\ \xi') \longleftrightarrow A \models$ *conn c* $(\xi\ @\ \varphi'\ \#\ \xi')$ **by** *auto*
qed

lemma *propo-rew-step-preservers-val'*:
assumes *preserves-un-sat r*
shows *preserves-un-sat* (*propo-rew-step r*)
using *assms* **by** (*simp add*: *preserves-un-sat-def propo-rew-step-preservers-val-explicit*)

lemma *preserves-un-sat-OO*[*intro*]:
preserves-un-sat f $\Longrightarrow$ *preserves-un-sat g* $\Longrightarrow$ *preserves-un-sat* (*f OO g*)
unfolding *preserves-un-sat-def* **by** *auto*

lemma *star-consistency-preservation-explicit*:
assumes (*propo-rew-step r*)^** $\varphi\ \psi$ **and** *preserves-un-sat r*
shows $\forall A.\ A \models \varphi \longleftrightarrow A \models \psi$
using *assms* **by** (*induct rule*: *rtranclp-induct*)
(*auto simp add*: *propo-rew-step-preservers-val-explicit*)

lemma *star-consistency-preservation*:
preserves-un-sat r $\Longrightarrow$ *preserves-un-sat* (*propo-rew-step r*)^**

by (*simp add*: *star-consistency-preservation-explicit preserves-un-sat-def*)

6.3 Full Lifting

In the previous a relation was lifted to a formula, now we define the relation such it is applied as long as possible. The definition is thus simply: it can be derived and nothing more can be derived.

lemma *full-ropo-rew-step-preservers-val*[*simp*]:
preserves-un-sat r $\Longrightarrow$ *preserves-un-sat* (*full* (*propo-rew-step r*))
by (*metis full-def preserves-un-sat-def star-consistency-preservation*)

lemma *full-propo-rew-step-subformula*:
full (*propo-rew-step r*) φ' φ $\Longrightarrow \neg(\exists\ \psi\ \psi'.\ \psi \preceq \varphi \wedge r\ \psi\ \psi')$
unfolding *full-def* **using** *propo-rew-step-subformula-rec* **by** *metis*

7 Transformation testing

7.1 Definition and first properties

To prove correctness of our transformation, we create a *all-subformula-st* predicate. It tests recursively all subformulas. At each step, the actual formula is tested. The aim of this *test-symb* function is to test locally some properties of the formulas (i.e. at the level of the connective or at first level). This allows a clause description between the rewrite relation and the *test-symb*

definition *all-subformula-st* :: (*'a propo* $\Rightarrow$ *bool*) $\Rightarrow$ *'a propo* $\Rightarrow$ *bool* **where**
all-subformula-st test-symb $\varphi \equiv \forall \psi.\ \psi \preceq \varphi \longrightarrow$ *test-symb* ψ

lemma *test-symb-imp-all-subformula-st*[*simp*]:
test-symb FT $\Longrightarrow$ *all-subformula-st test-symb FT*
test-symb FF $\Longrightarrow$ *all-subformula-st test-symb FF*
test-symb (*FVar x*) $\Longrightarrow$ *all-subformula-st test-symb* (*FVar x*)
unfolding *all-subformula-st-def* **using** *subformula-leaf* **by** *metis+*

lemma *all-subformula-st-test-symb-true-phi*:
all-subformula-st test-symb $\varphi \Longrightarrow$ *test-symb* φ
unfolding *all-subformula-st-def* **by** *auto*

lemma *all-subformula-st-decomp-imp*:
wf-conn c l $\Longrightarrow$ (*test-symb* (*conn c l*) $\wedge$ ($\forall \varphi \in$ *set l*. *all-subformula-st test-symb* φ))
$\Longrightarrow$ *all-subformula-st test-symb* (*conn c l*)
unfolding *all-subformula-st-def* **by** *auto*

To ease the finding of proofs, we give some explicit theorem about the decomposition.

lemma *all-subformula-st-decomp-rec*:
all-subformula-st test-symb (*conn c l*) $\Longrightarrow$ *wf-conn c l*
$\Longrightarrow$ (*test-symb* (*conn c l*) $\wedge$ ($\forall \varphi \in$ *set l*. *all-subformula-st test-symb* φ))
unfolding *all-subformula-st-def* **by** *auto*

lemma *all-subformula-st-decomp*:
fixes *c* :: *'v connective* **and** *l* :: *'v propo list*
assumes *wf-conn c l*
shows *all-subformula-st test-symb* (*conn c l*)

$\longleftrightarrow$ (*test-symb* (*conn c l*) $\wedge$ ($\forall\varphi\in$ *set l*. *all-subformula-st test-symb* φ))
using *assms all-subformula-st-decomp-rec all-subformula-st-decomp-imp* **by** *metis*

lemma *helper-fact*: $c \in$ *binary-connectives* $\longleftrightarrow$ (c = *COr* $\vee$ c = *CAnd* $\vee$ c = *CEq* $\vee$ c = *CImp*)
unfolding *binary-connectives-def* **by** *auto*
lemma *all-subformula-st-decomp-explicit*[*simp*]:
fixes φ ψ :: *'v propo*
shows *all-subformula-st test-symb* (*FAnd* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FAnd* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
and *all-subformula-st test-symb* (*FOr* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FOr* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
and *all-subformula-st test-symb* (*FNot* φ)
$\longleftrightarrow$ (*test-symb* (*FNot* φ) $\wedge$ *all-subformula-st test-symb* φ)
and *all-subformula-st test-symb* (*FEq* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FEq* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
and *all-subformula-st test-symb* (*FImp* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FImp* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
proof –
have *all-subformula-st test-symb* (*FAnd* φ ψ) $\longleftrightarrow$ *all-subformula-st test-symb* (*conn CAnd* [φ, ψ])
by *auto*
moreover have ... $\longleftrightarrow$*test-symb* (*conn CAnd* [φ, ψ])$\wedge$($\forall\xi\in$ *set* [φ, ψ]. *all-subformula-st test-symb* ξ)
using *all-subformula-st-decomp wf-conn-helper-facts*(*5*) **by** *metis*
finally show *all-subformula-st test-symb* (*FAnd* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FAnd* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
by *simp*

have *all-subformula-st test-symb* (*FOr* φ ψ) $\longleftrightarrow$ *all-subformula-st test-symb* (*conn COr* [φ, ψ])
by *auto*
moreover have ... $\longleftrightarrow$
(*test-symb* (*conn COr* [φ, ψ]) $\wedge$ ($\forall\xi\in$ *set* [φ, ψ]. *all-subformula-st test-symb* ξ))
using *all-subformula-st-decomp wf-conn-helper-facts*(*6*) **by** *metis*
finally show *all-subformula-st test-symb* (*FOr* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FOr* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
by *simp*

have *all-subformula-st test-symb* (*FEq* φ ψ) $\longleftrightarrow$ *all-subformula-st test-symb* (*conn CEq* [φ, ψ])
by *auto*
moreover have ...
$\longleftrightarrow$ (*test-symb* (*conn CEq* [φ, ψ]) $\wedge$ ($\forall\xi\in$ *set* [φ, ψ]. *all-subformula-st test-symb* ξ))
using *all-subformula-st-decomp wf-conn-helper-facts*(*8*) **by** *metis*
finally show *all-subformula-st test-symb* (*FEq* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FEq* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
by *simp*

have *all-subformula-st test-symb* (*FImp* φ ψ) $\longleftrightarrow$ *all-subformula-st test-symb* (*conn CImp* [φ, ψ])
by *auto*
moreover have ...
$\longleftrightarrow$(*test-symb* (*conn CImp* [φ, ψ]) $\wedge$ ($\forall\xi\in$ *set* [φ, ψ]. *all-subformula-st test-symb* ξ))
using *all-subformula-st-decomp wf-conn-helper-facts*(*7*) **by** *metis*
finally show *all-subformula-st test-symb* (*FImp* φ ψ)
$\longleftrightarrow$ (*test-symb* (*FImp* φ ψ) $\wedge$ *all-subformula-st test-symb* φ $\wedge$ *all-subformula-st test-symb* ψ)
by *simp*

have *all-subformula-st test-symb* (*FNot* φ) $\longleftrightarrow$ *all-subformula-st test-symb* (*conn CNot* [φ])

by *auto*
moreover have ... $= $ (*test-symb* (*conn CNot* $[\varphi]$) $\wedge$ ($\forall \xi \in$ *set* $[\varphi]$. *all-subformula-st test-symb* ξ))
using *all-subformula-st-decomp wf-conn-helper-facts*(*1*) **by** *metis*
finally show *all-subformula-st test-symb* (*FNot* φ)
$\longleftrightarrow$ (*test-symb* (*FNot* φ) $\wedge$ *all-subformula-st test-symb* φ) **by** *simp*
qed

As *all-subformula-st* tests recursively, the function is true on every subformula.

lemma *subformula-all-subformula-st*:
$\psi \preceq \varphi \Longrightarrow$ *all-subformula-st test-symb* $\varphi \Longrightarrow$ *all-subformula-st test-symb* ψ
by (*induct rule*: *subformula.induct*, *auto simp add*: *all-subformula-st-decomp*)

The following theorem *no-test-symb-step-exists* shows the link between the *test-symb* function and the corresponding rewrite relation r: if we assume that if every time *test-symb* is true, then a r can be applied, finally as long as $\neg$ *all-subformula-st test-symb* φ, then something can be rewritten in φ.

lemma *no-test-symb-step-exists*:
fixes r:: *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **and** *test-symb*:: *'v propo* $\Rightarrow$ *bool* **and** x :: *'v*
and φ :: *'v propo*
assumes *test-symb-false-nullary*: $\forall x$. *test-symb FF* $\wedge$ *test-symb FT* $\wedge$ *test-symb* (*FVar* x)
and $\forall \varphi'. \varphi' \preceq \varphi \longrightarrow (\neg$*test-symb* $\varphi') \longrightarrow (\exists \psi.\ r\ \varphi'\ \psi)$ **and**
$\neg$ *all-subformula-st test-symb* φ
shows $(\exists \psi\ \psi'.\ \psi \preceq \varphi \wedge r\ \psi\ \psi')$
using *assms*
proof (*induct* φ *rule*: *propo-induct-arity*)
case (*nullary* φ x)
thus $\exists \psi\ \psi'.\ \psi \preceq \varphi \wedge r\ \psi\ \psi'$
using *wf-conn-nullary test-symb-false-nullary* **by** *fastforce*
next
case (*unary* φ) **note** *IH* = *this*(*1*)[*OF this*(*2*)] **and** r = *this*(*2*) **and** *nst* = *this*(*3*) **and** *subf* = *this*(*4*)
from *r IH nst* **have** *H*: $\neg$ *all-subformula-st test-symb* $\varphi \Longrightarrow \exists \psi.\ \psi \preceq \varphi \wedge (\exists \psi'.\ r\ \psi\ \psi')$
by (*metis subformula-in-subformula-not subformula-refl subformula-trans*)
{
assume n: $\neg$*test-symb* (*FNot* φ)
obtain ψ **where** r (*FNot* φ) ψ **using** *subformula-refl r n nst* **by** *blast*
moreover have *FNot* $\varphi \preceq$ *FNot* φ **using** *subformula-refl* **by** *auto*
ultimately have $\exists \psi\ \psi'.\ \psi \preceq$ *FNot* $\varphi \wedge r\ \psi\ \psi'$ **by** *metis*
}
moreover {
assume n: *test-symb* (*FNot* φ)
hence $\neg$ *all-subformula-st test-symb* φ
using *all-subformula-st-decomp-explicit*(*3*) *nst subf* **by** *blast*
hence $\exists \psi\ \psi'.\ \psi \preceq$ *FNot* $\varphi \wedge r\ \psi\ \psi'$
using *H subformula-in-subformula-not subformula-refl subformula-trans* **by** *blast*
}
ultimately show $\exists \psi\ \psi'.\ \psi \preceq$ *FNot* $\varphi \wedge r\ \psi\ \psi'$ **by** *blast*
next
case (*binary* φ $\varphi 1$ $\varphi 2$)
note *IH*φ*1-0* = *this*(*1*)[*OF this*(*4*)] **and** *IH*φ*2-0* = *this*(*2*)[*OF this*(*4*)] **and** r = *this*(*4*)
and φ = *this*(*3*) **and** *le* = *this*(*5*) **and** *nst* = *this*(*6*)

obtain c :: *'v connective* **where**
c: (c = *CAnd* $\vee$ c = *COr* $\vee$ c = *CImp* $\vee$ c = *CEq*) $\wedge$ *conn* c $[\varphi 1, \varphi 2] = \varphi$
using φ **by** *fastforce*

hence *corr*: *wf-conn c* $[\varphi 1, \varphi 2]$ **using** *wf-conn.simps* **unfolding** *binary-connectives-def* **by** *auto*
have *inc*: $\varphi 1 \preceq \varphi$ $\varphi 2 \preceq \varphi$ **using** *binary-connectives-def c subformula-in-binary-conn* **by** *blast+*
from *r* $IH\varphi 1\text{-}0$ **have** $IH\varphi 1$: $\neg$ *all-subformula-st test-symb* $\varphi 1 \Longrightarrow \exists \psi\ \psi'.\ \psi \preceq \varphi 1 \wedge r\ \psi\ \psi'$
using *inc*(*1*) *subformula-trans le* **by** *blast*
from *r* $IH\varphi 2\text{-}0$ **have** $IH\varphi 2$: $\neg$ *all-subformula-st test-symb* $\varphi 2 \Longrightarrow \exists \psi.\ \psi \preceq \varphi 2 \wedge (\exists \psi'.\ r\ \psi\ \psi')$
using *inc*(*2*) *subformula-trans le* **by** *blast*
have *cases*: $\neg$*test-symb* φ $\vee$ $\neg$*all-subformula-st test-symb* $\varphi 1$ $\vee$ $\neg$*all-subformula-st test-symb* $\varphi 2$
using *c nst* **by** *auto*
show $\exists \psi\ \psi'.\ \psi \preceq \varphi \wedge r\ \psi\ \psi'$
using $IH\varphi 1$ $IH\varphi 2$ *subformula-trans inc subformula-refl cases le* **by** *blast*
qed

7.2 Invariant conservation

If two rewrite relation are independant (or at least independant enough), then the property characterizing the first relation *all-subformula-st test-symb* remains true. The next show the same property, with changes in the assumptions.

The assumption $\forall \varphi'\ \psi.\ \varphi' \preceq \Phi \longrightarrow r\ \varphi'\ \psi \longrightarrow$ *all-subformula-st test-symb* $\varphi' \longrightarrow$ *all-subformula-st test-symb* ψ means that rewriting with r does not mess up the property we want to preserve locally.

The previous assumption is not enough to go from r to *propo-rew-step* r: we have to add the assumption that rewriting inside does not mess up the term: $\forall c\ \xi\ \varphi\ \xi'\ \varphi'.\ \varphi \preceq \Phi \longrightarrow$ *propo-rew-step* $r\ \varphi\ \varphi' \longrightarrow$ *wf-conn* $c\ (\xi\ @\ \varphi\ \#\ \xi') \longrightarrow$ *test-symb* $(conn\ c\ (\xi\ @\ \varphi\ \#\ \xi')) \longrightarrow$ *test-symb* $\varphi' \longrightarrow$ *test-symb* $(conn\ c\ (\xi\ @\ \varphi'\ \#\ \xi'))$

7.2.1 Invariant while lifting of the rewriting relation

The condition $\varphi \preceq \Phi$ (that will by used with $\Phi = \varphi$ most of the time) is here to ensure that the recursive conditions on Φ will moreover hold for the subterm we are rewriting. For example if there is no equivalence symbol in Φ, we do not have to care about equivalence symbols in the two previous assumptions.

lemma *propo-rew-step-inv-stay'*:
fixes *r*:: *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **and** *test-symb*:: *'v propo* $\Rightarrow$ *bool* **and** *x* :: *'v*
and $\varphi\ \psi\ \Phi$:: *'v propo*
assumes *H*: $\forall \varphi'\ \psi.\ \varphi' \preceq \Phi \longrightarrow r\ \varphi'\ \psi \longrightarrow$ *all-subformula-st test-symb* φ'
$\longrightarrow$ *all-subformula-st test-symb* ψ
and *H'*: $\forall (c$:: *'v connective*$)\ \xi\ \varphi\ \xi'\ \varphi'.\ \varphi \preceq \Phi \longrightarrow$ *propo-rew-step* $r\ \varphi\ \varphi'$
$\longrightarrow$ *wf-conn* $c\ (\xi\ @\ \varphi\ \#\ \xi') \longrightarrow$ *test-symb* $(conn\ c\ (\xi\ @\ \varphi\ \#\ \xi')) \longrightarrow$ *test-symb* φ'
$\longrightarrow$ *test-symb* $(conn\ c\ (\xi\ @\ \varphi'\ \#\ \xi'))$ **and**
propo-rew-step $r\ \varphi\ \psi$ **and**
$\varphi \preceq \Phi$ **and**
all-subformula-st test-symb φ
shows *all-subformula-st test-symb* ψ
using *assms*(*3*$-$*5*)
proof (*induct rule*: *propo-rew-step.induct*)
case *global-rel*
thus *?case* **using** *H* **by** *simp*
next
case (*propo-rew-one-step-lift* $\varphi\ \varphi'\ c\ \xi\ \xi'$)
note *rel* = *this*(*1*) **and** φ = *this*(*2*) **and** *corr* = *this*(*3*) **and** Φ = *this*(*4*) **and** *nst* = *this*(*5*)

have *sq*: $\varphi \preceq \Phi$
  using Φ *corr subformula-into-subformula subformula-refl subformula-trans*
  by (*metis in-set-conv-decomp*)
from *corr* **have** $\forall\ \psi.\ \psi \in$ *set* $(\xi\ @\ \varphi\ \#\ \xi') \longrightarrow$ *all-subformula-st test-symb* ψ
  using *all-subformula-st-decomp nst* **by** *blast*
hence $*$: $\forall\psi.\ \psi \in$ *set* $(\xi\ @\ \varphi'\ \#\ \xi') \longrightarrow$ *all-subformula-st test-symb* ψ **using** φ *sq* **by** *fastforce*
hence *test-symb* φ' **using** *all-subformula-st-test-symb-true-phi* **by** *auto*
moreover from *corr nst* **have** *test-symb* (*conn c* $(\xi\ @\ \varphi\ \#\ \xi')$)
  using *all-subformula-st-decomp* **by** *blast*
ultimately have *test-symb*: *test-symb* (*conn c* $(\xi\ @\ \varphi'\ \#\ \xi')$) **using** H' *sq corr rel* **by** *blast*

have *wf-conn c* $(\xi\ @\ \varphi'\ \#\ \xi')$
  by (*metis wf-conn-no-arity-change-helper corr wf-conn-no-arity-change*)
thus *all-subformula-st test-symb* (*conn c* $(\xi\ @\ \varphi'\ \#\ \xi')$)
  using $*$ *test-symb* **by** (*metis all-subformula-st-decomp*)
qed

The need for $\varphi \preceq \Phi$ is not always necessary, hence we moreover have a version without inclusion.

lemma *propo-rew-step-inv-stay*:
  fixes *r*:: *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **and** *test-symb*:: *'v propo* $\Rightarrow$ *bool* **and** *x* :: *'v*
  and $\varphi\ \psi$ :: *'v propo*
  assumes
    H: $\forall\varphi'\ \psi.\ r\ \varphi'\ \psi \longrightarrow$ *all-subformula-st test-symb* $\varphi' \longrightarrow$ *all-subformula-st test-symb* ψ **and**
    H': $\forall$(*c*:: *'v connective*) $\xi\ \varphi\ \xi'\ \varphi'$. *wf-conn c* $(\xi\ @\ \varphi\ \#\ \xi') \longrightarrow$ *test-symb* (*conn c* $(\xi\ @\ \varphi\ \#\ \xi')$)
      $\longrightarrow$ *test-symb* $\varphi' \longrightarrow$ *test-symb* (*conn c* $(\xi\ @\ \varphi'\ \#\ \xi')$) **and**
    propo-rew-step r $\varphi\ \psi$ **and**
    all-subformula-st test-symb φ
  shows *all-subformula-st test-symb* ψ
  using *propo-rew-step-inv-stay'*[*of* φ *r test-symb* $\varphi\ \psi$] *assms subformula-refl* **by** *metis*

The lemmas can be lifted to *full* (*propo-rew-step r*) instead of *propo-rew-step*

7.2.2 Invariant after all rewriting

lemma *full-propo-rew-step-inv-stay-with-inc*:
  fixes *r*:: *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **and** *test-symb*:: *'v propo* $\Rightarrow$ *bool* **and** *x* :: *'v*
  and $\varphi\ \psi$ :: *'v propo*
  assumes
    H: $\forall\ \varphi\ \psi$. *propo-rew-step r* $\varphi\ \psi \longrightarrow$ *all-subformula-st test-symb* φ
      $\longrightarrow$ *all-subformula-st test-symb* ψ **and**
    H': $\forall$(*c*:: *'v connective*) $\xi\ \varphi\ \xi'\ \varphi'$. $\varphi \preceq \Phi \longrightarrow$ *propo-rew-step r* $\varphi\ \varphi'$
      $\longrightarrow$ *wf-conn c* $(\xi\ @\ \varphi\ \#\ \xi') \longrightarrow$ *test-symb* (*conn c* $(\xi\ @\ \varphi\ \#\ \xi')$) $\longrightarrow$ *test-symb* φ'
      $\longrightarrow$ *test-symb* (*conn c* $(\xi\ @\ \varphi'\ \#\ \xi')$) **and**
      $\varphi \preceq \Phi$ **and**
    full: *full* (*propo-rew-step r*) $\varphi\ \psi$ **and**
    init: *all-subformula-st test-symb* φ
  shows *all-subformula-st test-symb* ψ
  using *assms* **unfolding** *full-def*
proof –
  have *rel*: (*propo-rew-step r*)** $\varphi\ \psi$
    using *full* **unfolding** *full-def* **by** *auto*
  thus *all-subformula-st test-symb* ψ
    using *init*
    proof (*induct rule*: *rtranclp-induct*)
      case *base*

then show *all-subformula-st test-symb* φ **by** *blast*
next
case (*step b c*) **note** *star* = *this*(*1*) **and** *IH* = *this*(*3*) **and** *one* = *this*(*2*) **and** *all* = *this*(*4*)
then have *all-subformula-st test-symb b* **by** *metis*
then show *all-subformula-st test-symb c* **using** *propo-rew-step-inv-stay'* *H H'* *rel one* **by** *auto*
qed
qed

lemma *full-propo-rew-step-inv-stay'*:
fixes r:: *'v propo* ⇒ *'v propo* ⇒ *bool* **and** *test-symb*:: *'v propo* ⇒ *bool* **and** x :: *'v*
and φ ψ :: *'v propo*
assumes
H: ∀ φ ψ. *propo-rew-step r* φ ψ ⟶ *all-subformula-st test-symb* φ
⟶ *all-subformula-st test-symb* ψ **and**
H': ∀ (*c*:: *'v connective*) ξ φ ξ' φ'. *propo-rew-step r* φ φ' ⟶ *wf-conn c* (ξ @ φ # ξ')
⟶ *test-symb* (*conn c* (ξ @ φ # ξ')) ⟶ *test-symb* φ' ⟶ *test-symb* (*conn c* (ξ @ φ' # ξ')) **and**
full: *full* (*propo-rew-step r*) φ ψ **and**
init: *all-subformula-st test-symb* φ
shows *all-subformula-st test-symb* ψ
using *full-propo-rew-step-inv-stay-with-inc*[*of r test-symb* φ] *assms subformula-refl* **by** *metis*

lemma *full-propo-rew-step-inv-stay*:
fixes r:: *'v propo* ⇒ *'v propo* ⇒ *bool* **and** *test-symb*:: *'v propo* ⇒ *bool* **and** x :: *'v*
and φ ψ :: *'v propo*
assumes
H: ∀ φ ψ. r φ ψ ⟶ *all-subformula-st test-symb* φ ⟶ *all-subformula-st test-symb* ψ **and**
H': ∀ (*c*:: *'v connective*) ξ φ ξ' φ'. *wf-conn c* (ξ @ φ # ξ') ⟶ *test-symb* (*conn c* (ξ @ φ # ξ'))
⟶ *test-symb* φ' ⟶ *test-symb* (*conn c* (ξ @ φ' # ξ')) **and**
full: *full* (*propo-rew-step r*) φ ψ **and**
init: *all-subformula-st test-symb* φ
shows *all-subformula-st test-symb* ψ
unfolding *full-def*
proof −
have *rel*: (*propo-rew-step r*)^** φ ψ
using *full* **unfolding** *full-def* **by** *auto*
thus *all-subformula-st test-symb* ψ
using *init*
proof (*induct rule*: *rtranclp-induct*)
case *base*
thus *all-subformula-st test-symb* φ **by** *blast*
next
case (*step b c*)
note *star* = *this*(*1*) **and** *IH* = *this*(*3*) **and** *one* = *this*(*2*) **and** *all* = *this*(*4*)
hence *all-subformula-st test-symb b* **by** *metis*
thus *all-subformula-st test-symb c*
using *propo-rew-step-inv-stay subformula-refl H H'* *rel one* **by** *auto*
qed
qed

lemma *full-propo-rew-step-inv-stay-conn*:
fixes r:: *'v propo* ⇒ *'v propo* ⇒ *bool* **and** *test-symb*:: *'v propo* ⇒ *bool* **and** x :: *'v*
and φ ψ :: *'v propo*
assumes
H: ∀ φ ψ. r φ ψ ⟶ *all-subformula-st test-symb* φ ⟶ *all-subformula-st test-symb* ψ **and**

H': $\forall (c{::}\ 'v\ connective)\ l\ l'.\ wf\text{-}conn\ c\ l \longrightarrow wf\text{-}conn\ c\ l'$
$\longrightarrow (test\text{-}symb\ (conn\ c\ l) \longleftrightarrow test\text{-}symb\ (conn\ c\ l'))$ **and**
full: $full\ (propo\text{-}rew\text{-}step\ r)\ \varphi\ \psi$ **and**
init: $all\text{-}subformula\text{-}st\ test\text{-}symb\ \varphi$
shows $all\text{-}subformula\text{-}st\ test\text{-}symb\ \psi$
proof −
have $\bigwedge(c{::}\ 'v\ connective)\ \xi\ \varphi\ \xi'\ \varphi'.\ wf\text{-}conn\ c\ (\xi\ @\ \varphi\ \#\ \xi')$
$\Longrightarrow test\text{-}symb\ (conn\ c\ (\xi\ @\ \varphi\ \#\ \xi')) \Longrightarrow test\text{-}symb\ \varphi' \Longrightarrow test\text{-}symb\ (conn\ c\ (\xi\ @\ \varphi'\ \#\ \xi'))$
using H' **by** (*metis wf-conn-no-arity-change-helper wf-conn-no-arity-change*)
thus *all-subformula-st test-symb* ψ
using H *full init full-propo-rew-step-inv-stay* **by** *blast*
qed

end
theory *Prop-Normalisation*
imports *Main Prop-Logic Prop-Abstract-Transformation*
begin

Given the previous definition about abstract rewriting and theorem about them, we now have the detailed rule making the transformation into CNF/DNF.

8 Rewrite Rules

The idea of Christoph Weidenbach's book is to remove gradually the operators: first equivalencies, then implication, after that the unused true/false and finally the reorganizing the or/and. We will prove each transformation seperately.

8.1 Elimination of the equivalences

The first transformation consists in removing every equivalence symbol.

inductive *elim-equiv* :: $'v\ propo \Rightarrow\ 'v\ propo \Rightarrow bool$ **where**
elim-equiv[*simp*]: $elim\text{-}equiv\ (FEq\ \varphi\ \psi)\ (FAnd\ (FImp\ \varphi\ \psi)\ \ (FImp\ \psi\ \varphi))$

lemma *elim-equiv-transformation-consistent*:
$A \models FEq\ \varphi\ \psi \longleftrightarrow A \models FAnd\ (FImp\ \varphi\ \psi)\ (FImp\ \psi\ \varphi)$
by *auto*

lemma *elim-equiv-explicit*: $elim\text{-}equiv\ \varphi\ \psi \Longrightarrow \forall A.\ A \models \varphi \longleftrightarrow A \models \psi$
by (*induct rule*: *elim-equiv.induct*, *auto*)

lemma *elim-equiv-consistent*: *preserves-un-sat elim-equiv*
unfolding *preserves-un-sat-def* **by** (*simp add*: *elim-equiv-explicit*)

lemma *elimEquv-lifted-consistant*:
preserves-un-sat (*full* (*propo-rew-step elim-equiv*))
by (*simp add*: *elim-equiv-consistent*)

This function ensures that there is no equivalencies left in the formula tested by *no-equiv-symb*.

fun *no-equiv-symb* :: $'v\ propo \Rightarrow bool$ **where**
no-equiv-symb (*FEq* - -) = *False* |
no-equiv-symb - = *True*

Given the definition of *no-equiv-symb*, it does not depend on the formula, but only on the connective used.

lemma *no-equiv-symb-conn-characterization*[*simp*]:
  fixes c :: $'v$ *connective* **and** l :: $'v$ *propo list*
  assumes *wf*: *wf-conn c l*
  shows *no-equiv-symb* (*conn c l*) $\longleftrightarrow$ $c \neq$ *CEq*
    by (*metis connective.distinct*(*13*,*25*,*35*,*43*) *wf no-equiv-symb.elims*(*3*) *no-equiv-symb.simps*(*1*)
      wf-conn.cases wf-conn-list(*6*))

definition *no-equiv* **where** *no-equiv* = *all-subformula-st no-equiv-symb*

lemma *no-equiv-eq*[*simp*]:
  fixes φ ψ :: $'v$ *propo*
  shows
    $\neg$*no-equiv* (*FEq* φ ψ)
    no-equiv FT
    no-equiv FF
  using *no-equiv-symb.simps*(*1*) *all-subformula-st-test-symb-true-phi* **unfolding** *no-equiv-def* **by** *auto*

The following lemma helps to reconstruct *no-equiv* expressions: this representation is easier to use than the set definition.

lemma *all-subformula-st-decomp-explicit-no-equiv*[*iff*]:
fixes φ ψ :: $'v$ *propo*
shows
  no-equiv (*FNot* φ) $\longleftrightarrow$ *no-equiv* φ
  no-equiv (*FAnd* φ ψ) $\longleftrightarrow$ (*no-equiv* φ $\wedge$ *no-equiv* ψ)
  no-equiv (*FOr* φ ψ) $\longleftrightarrow$ (*no-equiv* φ $\wedge$ *no-equiv* ψ)
  no-equiv (*FImp* φ ψ) $\longleftrightarrow$ (*no-equiv* φ $\wedge$ *no-equiv* ψ)
  by (*auto simp add*: *no-equiv-def*)

A theorem to show the link between the rewrite relation *elim-equiv* and the function *no-equiv-symb*. This theorem is one of the assumption we need to characterize the transformation.

lemma *no-equiv-elim-equiv-step*:
  fixes φ :: $'v$ *propo*
  assumes *no-equiv*: $\neg$ *no-equiv* φ
  shows $\exists\,\psi$ ψ'. $\psi \preceq \varphi$ $\wedge$ *elim-equiv* ψ ψ'
proof −
  have *test-symb-false-nullary*:
    $\forall\,x$::$'v$. *no-equiv-symb FF* $\wedge$ *no-equiv-symb FT* $\wedge$ *no-equiv-symb* (*FVar* x)
    unfolding *no-equiv-def* **by** *auto*
  moreover {
    fix c:: $'v$ *connective* **and** l :: $'v$ *propo list* **and** ψ :: $'v$ *propo*
      assume *a1*: *elim-equiv* (*conn c l*) ψ
      have $\bigwedge p$ *pa*. $\neg$ *elim-equiv* (p::$'v$ *propo*) *pa* $\vee$ $\neg$ *no-equiv-symb* p
        using *elim-equiv.cases no-equiv-symb.simps*(*1*) **by** *blast*
      hence *elim-equiv* (*conn c l*) $\psi \Longrightarrow \neg$*no-equiv-symb* (*conn c l*) **using** *a1* **by** *metis*
  }
  moreover have H': $\forall\psi$. $\neg$*elim-equiv FT* ψ $\forall\psi$. $\neg$*elim-equiv FF* ψ $\forall\psi$ x. $\neg$*elim-equiv* (*FVar* x) ψ
    using *elim-equiv.cases* **by** *auto*
  moreover have $\bigwedge\varphi$. $\neg$ *no-equiv-symb* $\varphi \Longrightarrow \exists\,\psi$. *elim-equiv* φ ψ
    by (*case-tac* φ, *auto simp add*: *elim-equiv.simps*)
  hence $\bigwedge\varphi'$. $\varphi' \preceq \varphi \Longrightarrow \neg$*no-equiv-symb* $\varphi' \Longrightarrow$ $\exists\,\psi$. *elim-equiv* φ' ψ **by** *force*
  ultimately show *?thesis*
    using *no-test-symb-step-exists no-equiv test-symb-false-nullary* **unfolding** *no-equiv-def* **by** *blast*
qed

Given all the previous theorem and the characterization, once we have rewritten everything, there is no equivalence symbol any more.

lemma *no-equiv-full-propo-rew-step-elim-equiv*:
full (*propo-rew-step elim-equiv*) φ ψ $\Longrightarrow$ *no-equiv* ψ
using *full-propo-rew-step-subformula no-equiv-elim-equiv-step* **by** *blast*

8.2 Eliminate Implication

After that, we can eliminate the implication symbols.

inductive *elim-imp* :: $'v$ *propo* $\Rightarrow$ $'v$ *propo* $\Rightarrow$ *bool* **where**
[*simp*]: *elim-imp* (*FImp* φ ψ) (*FOr* (*FNot* φ) ψ)

lemma *elim-imp-transformation-consistent*:
$A \models$ *FImp* φ ψ $\longleftrightarrow$ $A \models$ *FOr* (*FNot* φ) ψ
by *auto*

lemma *elim-imp-explicit*: *elim-imp* φ ψ $\Longrightarrow$ $\forall A.\ A \models \varphi \longleftrightarrow A \models \psi$
by (*induct* φ ψ *rule*: *elim-imp.induct*, *auto*)

lemma *elim-imp-consistent*: *preserves-un-sat elim-imp*
unfolding *preserves-un-sat-def* **by** (*simp add*: *elim-imp-explicit*)

lemma *elim-imp-lifted-consistant*:
preserves-un-sat (*full* (*propo-rew-step elim-imp*))
by (*simp add*: *elim-imp-consistent*)

fun *no-imp-symb* **where**
no-imp-symb (*FImp* - -) = *False* |
no-imp-symb - = *True*

lemma *no-imp-symb-conn-characterization*:
wf-conn c l $\Longrightarrow$ *no-imp-symb* (*conn c l*) $\longleftrightarrow$ $c \neq$ *CImp*
by (*induction rule*: *wf-conn-induct*) *auto*

definition *no-imp* **where** *no-imp* $\equiv$ *all-subformula-st no-imp-symb*
declare *no-imp-def*[*simp*]

lemma *no-imp-Imp*[*simp*]:
$\neg$*no-imp* (*FImp* φ ψ)
no-imp FT
no-imp FF
unfolding *no-imp-def* **by** *auto*

lemma *all-subformula-st-decomp-explicit-imp*[*simp*]:
fixes φ ψ :: $'v$ *propo*
shows
no-imp (*FNot* φ) $\longleftrightarrow$ *no-imp* φ
no-imp (*FAnd* φ ψ) $\longleftrightarrow$ (*no-imp* φ $\wedge$ *no-imp* ψ)
no-imp (*FOr* φ ψ) $\longleftrightarrow$ (*no-imp* φ $\wedge$ *no-imp* ψ)
by *auto*

Invariant of the *elim-imp* transformation

lemma *elim-imp-no-equiv*:

elim-imp φ ψ $\Longrightarrow$ *no-equiv* φ $\Longrightarrow$ *no-equiv* ψ
by (*induct* φ ψ *rule*: *elim-imp.induct*, *auto*)

lemma *elim-imp-inv*:
fixes φ ψ :: *'v propo*
assumes *full* (*propo-rew-step elim-imp*) φ ψ
and *no-equiv* φ
shows *no-equiv* ψ
using *full-propo-rew-step-inv-stay-conn*[*of elim-imp no-equiv-symb* φ ψ] *assms elim-imp-no-equiv no-equiv-symb-conn-characterization* **unfolding** *no-equiv-def* **by** *metis*

lemma *no-no-imp-elim-imp-step-exists*:
fixes φ :: *'v propo*
assumes *no-equiv*: $\neg$ *no-imp* φ
shows $\exists \psi\ \psi'.\ \psi \preceq \varphi \wedge$ *elim-imp* ψ ψ'
proof −
have *test-symb-false-nullary*: $\forall x.$ *no-imp-symb FF* $\wedge$ *no-imp-symb FT* $\wedge$ *no-imp-symb* (*FVar* (x:: *'v*))
by *auto*
moreover **{**
fix c:: *'v connective* **and** l :: *'v propo list* **and** ψ :: *'v propo*
have H: *elim-imp* (*conn c l*) ψ $\Longrightarrow$ $\neg$*no-imp-symb* (*conn c l*)
by (*auto elim*: *elim-imp.cases*)
}
moreover
have H': $\forall \psi.$ $\neg$*elim-imp FT* ψ $\forall \psi.$ $\neg$*elim-imp FF* ψ $\forall \psi\ x.$ $\neg$*elim-imp* (*FVar x*) ψ
by (*auto elim*: *elim-imp.cases*)+
moreover have $\bigwedge\varphi.$ $\neg$ *no-imp-symb* φ $\Longrightarrow$ $\exists \psi.$ *elim-imp* φ ψ
apply (*case-tac* φ) **using** *elim-imp.simps* **by** *force*+
hence ($\bigwedge\varphi'.\ \varphi' \preceq \varphi \Longrightarrow$ $\neg$*no-imp-symb* φ' $\Longrightarrow$ $\exists\ \psi.$ *elim-imp* φ' ψ) **by** *force*
ultimately show *?thesis*
using *no-test-symb-step-exists no-equiv test-symb-false-nullary* **unfolding** *no-imp-def* **by** *blast*
qed

lemma *no-imp-full-propo-rew-step-elim-imp*: *full* (*propo-rew-step elim-imp*) φ ψ $\Longrightarrow$ *no-imp* ψ
using *full-propo-rew-step-subformula no-no-imp-elim-imp-step-exists* **by** *blast*

8.3 Eliminate all the True and False in the formula

Contrary to the book, we have to give the transformation and the "commutative" transformation. The latter is implicit in the book.

inductive *elimTB* **where**
ElimTB1: *elimTB* (*FAnd* φ *FT*) φ |
ElimTB1': *elimTB* (*FAnd FT* φ) φ |

ElimTB2: *elimTB* (*FAnd* φ *FF*) *FF* |
ElimTB2': *elimTB* (*FAnd FF* φ) *FF* |

ElimTB3: *elimTB* (*FOr* φ *FT*) *FT* |
ElimTB3': *elimTB* (*FOr FT* φ) *FT* |

ElimTB4: *elimTB* (*FOr* φ *FF*) φ |
ElimTB4': *elimTB* (*FOr FF* φ) φ |

ElimTB5: *elimTB* (*FNot FT*) *FF* |

ElimTB6: *elimTB* (*FNot FF*) *FT*

lemma *elimTB-consistent*: *preserves-un-sat elimTB*
proof −
  {
    fix φ ψ:: *'b propo*
    have *elimTB* φ ψ $\Longrightarrow \forall A.\ A \models \varphi \longleftrightarrow A \models \psi$ **by** (*induct-tac rule*: *elimTB.inducts*) *auto*
  }
  thus *?thesis* **using** *preserves-un-sat-def* **by** *auto*
qed

inductive *no-T-F-symb* :: *'v propo* $\Rightarrow$ *bool* **where**
no-T-F-symb-comp: $c \neq CF \Longrightarrow c \neq CT \Longrightarrow$ *wf-conn c l* $\Longrightarrow (\forall \varphi \in set\ l.\ \varphi \neq FT \wedge \varphi \neq FF)$
  $\Longrightarrow$ *no-T-F-symb* (*conn c l*)

lemma *wf-conn-no-T-F-symb-iff*[*simp*]:
  wf-conn c $\psi s \Longrightarrow$ *no-T-F-symb* (*conn c* ψs) $\longleftrightarrow (c \neq CF \wedge c \neq CT \wedge (\forall \psi \in set\ \psi s.\ \psi \neq FF \wedge \psi \neq FT))$
  unfolding *no-T-F-symb.simps* **apply** (*cases c*)
          using *wf-conn-list*(*1*) **apply** *fastforce*
         using *wf-conn-list*(*2*) **apply** *fastforce*
        using *wf-conn-list*(*3*) **apply** *fastforce*
       apply (*metis* (*no-types*, *hide-lams*) *conn-inj connective.distinct*(*5*,*17*))
      using *conn-inj* **apply** *blast+*
  done

lemma *wf-conn-no-T-F-symb-iff-explicit*[*simp*]:
no-T-F-symb (*FAnd* φ ψ) $\longleftrightarrow (\forall \chi \in set\ [\varphi, \psi].\ \chi \neq FF \wedge \chi \neq FT)$
no-T-F-symb (*FOr* φ ψ) $\longleftrightarrow (\forall \chi \in set\ [\varphi, \psi].\ \chi \neq FF \wedge \chi \neq FT)$
no-T-F-symb (*FEq* φ ψ) $\longleftrightarrow (\forall \chi \in set\ [\varphi, \psi].\ \chi \neq FF \wedge \chi \neq FT)$
no-T-F-symb (*FImp* φ ψ) $\longleftrightarrow (\forall \chi \in set\ [\varphi, \psi].\ \chi \neq FF \wedge \chi \neq FT)$
     apply (*metis conn.simps*(*36*) *conn.simps*(*37*) *conn.simps*(*5*) *propo.distinct*(*19*)
       wf-conn-helper-facts(*5*) *wf-conn-no-T-F-symb-iff*)
    apply (*metis conn.simps*(*36*) *conn.simps*(*37*) *conn.simps*(*6*) *propo.distinct*(*22*)
      wf-conn-helper-facts(*6*) *wf-conn-no-T-F-symb-iff*)
   using *wf-conn-no-T-F-symb-iff* **apply** *fastforce*
  by (*metis conn.simps*(*36*) *conn.simps*(*37*) *conn.simps*(*7*) *propo.distinct*(*23*) *wf-conn-helper-facts*(*7*)
    wf-conn-no-T-F-symb-iff)

lemma *no-T-F-symb-false*[*simp*]:
  fixes *c* :: *'v connective*
  shows
    ¬*no-T-F-symb* (*FT* :: *'v propo*)
    ¬*no-T-F-symb* (*FF* :: *'v propo*)
    by (*metis* (*no-types*) *conn.simps*(*1*,*2*) *wf-conn-no-T-F-symb-iff wf-conn-nullary*)+

lemma *no-T-F-symb-bool*[*simp*]:
  fixes *x* :: *'v*
  shows *no-T-F-symb* (*FVar x*)
  using *no-T-F-symb-comp wf-conn-nullary* **by** (*metis connective.distinct*(*3*, *15*) *conn.simps*(*3*)
    empty-iff list.set(*1*))

lemma *no-T-F-symb-fnot-imp*:
$\neg$*no-T-F-symb* (*FNot* φ) $\Longrightarrow$ φ = *FT* $\vee$ φ = *FF*
proof (*rule ccontr*)
assume *n*: $\neg$ *no-T-F-symb* (*FNot* φ)
assume $\neg$ (φ = *FT* $\vee$ φ = *FF*)
hence $\forall\varphi' \in$ *set* [φ]. $\varphi'\neq$*FT* $\wedge$ $\varphi'\neq$*FF* **by** *auto*
moreover have *wf-conn CNot* [φ] **by** *simp*
ultimately have *no-T-F-symb* (*FNot* φ)
using *no-T-F-symb.intros* **by** (*metis conn.simps*(*4*) *connective.distinct*(*5*,*17*))
thus *False* **using** *n* **by** *blast*
qed

lemma *no-T-F-symb-fnot*[*simp*]:
no-T-F-symb (*FNot* φ) $\longleftrightarrow$ $\neg$(φ = *FT* $\vee$ φ = *FF*)
using *no-T-F-symb.simps no-T-F-symb-fnot-imp* **by** (*metis conn-inj-not*(*2*) *list.set-intros*(*1*))

Actually it is not possible to remover every *FT* and *FF*: if the formula is equal to true or false, we can not remove it.

inductive *no-T-F-symb-except-toplevel* **where**
no-T-F-symb-except-toplevel-true[*simp*]: *no-T-F-symb-except-toplevel FT* |
no-T-F-symb-except-toplevel-false[*simp*]: *no-T-F-symb-except-toplevel FF* |
noTrue-no-T-F-symb-except-toplevel[*simp*]: *no-T-F-symb* φ $\Longrightarrow$ *no-T-F-symb-except-toplevel* φ

lemma *no-T-F-symb-except-toplevel-bool*[*simp*]:
fixes *x* :: *'v*
shows *no-T-F-symb-except-toplevel* (*FVar x*)
by *simp*

lemma *no-T-F-symb-except-toplevel-not-decom*:
$\varphi \neq$ *FT* $\Longrightarrow$ $\varphi \neq$ *FF* $\Longrightarrow$ *no-T-F-symb-except-toplevel* (*FNot* φ)
by *simp*

lemma *no-T-F-symb-except-toplevel-bin-decom*:
fixes φ ψ :: *'v propo*
assumes $\varphi \neq$ *FT* **and** $\varphi \neq$ *FF* **and** $\psi \neq$ *FT* **and** $\psi \neq$ *FF*
and *c*: *c*$\in$ *binary-connectives*
shows *no-T-F-symb-except-toplevel* (*conn c* [φ, ψ])
by (*metis* (*no-types*, *lifting*) *assms c conn.simps*(*4*) *list.discI noTrue-no-T-F-symb-except-toplevel wf-conn-no-T-F-symb-iff no-T-F-symb-fnot set-ConsD wf-conn-binary wf-conn-helper-facts*(*1*) *wf-conn-list-decomp*(*1*,*2*))

lemma *no-T-F-symb-except-toplevel-if-is-a-true-false*:
fixes *l* :: *'v propo list* **and** *c* :: *'v connective*
assumes *corr*: *wf-conn c l*
and *FT* $\in$ *set l* $\vee$ *FF* $\in$ *set l*
shows $\neg$*no-T-F-symb-except-toplevel* (*conn c l*)
by (*metis assms empty-iff no-T-F-symb-except-toplevel.simps wf-conn-no-T-F-symb-iff set-empty wf-conn-list*(*1*,*2*))

lemma *no-T-F-symb-except-top-level-false-example*[*simp*]:

fixes φ ψ :: *'v propo*
assumes $\varphi = FT \lor \psi = FT \lor \varphi = FF \lor \psi = FF$
shows
$\neg$ *no-T-F-symb-except-toplevel* (*FAnd* φ ψ)
$\neg$ *no-T-F-symb-except-toplevel* (*FOr* φ ψ)
$\neg$ *no-T-F-symb-except-toplevel* (*FImp* φ ψ)
$\neg$ *no-T-F-symb-except-toplevel* (*FEq* φ ψ)
using *assms no-T-F-symb-except-toplevel-if-is-a-true-false* **unfolding** *binary-connectives-def*
by (*metis* (*no-types*) *conn.simps*(*5−8*) *insert-iff list.simps*(*14−15*) *wf-conn-helper-facts*(*5−8*))+

lemma *no-T-F-symb-except-top-level-false-not*[*simp*]:
fixes φ ψ :: *'v propo*
assumes $\varphi = FT \lor \varphi = FF$
shows
$\neg$ *no-T-F-symb-except-toplevel* (*FNot* φ)
by (*simp add*: *assms no-T-F-symb-except-toplevel.simps*)

This is the local extension of *no-T-F-symb-except-toplevel*.

definition *no-T-F-except-top-level* **where**
no-T-F-except-top-level $\equiv$ *all-subformula-st no-T-F-symb-except-toplevel*

This is another property we will use. While this version might seem to be the one we want to prove, it is not since FT can not be reduced.

definition *no-T-F* **where**
no-T-F $\equiv$ *all-subformula-st no-T-F-symb*

lemma *no-T-F-except-top-level-false*:
fixes *l* :: *'v propo list* **and** *c* :: *'v connective*
assumes *wf-conn c l*
and $FT \in set\ l \lor FF \in set\ l$
shows $\neg$*no-T-F-except-top-level* (*conn c l*)
by (*simp add*: *all-subformula-st-decomp assms no-T-F-except-top-level-def*
no-T-F-symb-except-toplevel-if-is-a-true-false)

lemma *no-T-F-except-top-level-false-example*[*simp*]:
fixes φ ψ :: *'v propo*
assumes $\varphi = FT \lor \psi = FT \lor \varphi = FF \lor \psi = FF$
shows
$\neg$*no-T-F-except-top-level* (*FAnd* φ ψ)
$\neg$*no-T-F-except-top-level* (*FOr* φ ψ)
$\neg$*no-T-F-except-top-level* (*FEq* φ ψ)
$\neg$*no-T-F-except-top-level* (*FImp* φ ψ)
by (*metis all-subformula-st-test-symb-true-phi assms no-T-F-except-top-level-def*
no-T-F-symb-except-top-level-false-example)+

lemma *no-T-F-symb-except-toplevel-no-T-F-symb*:
no-T-F-symb-except-toplevel $\varphi \Longrightarrow \varphi \neq FF \Longrightarrow \varphi \neq FT \Longrightarrow$ *no-T-F-symb* φ
by (*induct rule*: *no-T-F-symb-except-toplevel.induct*, *auto*)

The two following lemmas give the precise link between the two definitions.

lemma *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb*:
no-T-F-except-top-level $\varphi \Longrightarrow \varphi \neq FF \Longrightarrow \varphi \neq FT \Longrightarrow$ *no-T-F* φ
unfolding *no-T-F-except-top-level-def no-T-F-def* **apply** (*induct* φ)

using *no-T-F-symb-fnot* **by** *fastforce+*

lemma *no-T-F-no-T-F-except-top-level*:
no-T-F $\varphi \Longrightarrow$ *no-T-F-except-top-level* φ
unfolding *no-T-F-except-top-level-def no-T-F-def*
unfolding *all-subformula-st-def* **by** *auto*

lemma *no-T-F-except-top-level-simp*[*simp*]: *no-T-F-except-top-level FF no-T-F-except-top-level FT*
unfolding *no-T-F-except-top-level-def* **by** *auto*

lemma *no-T-F-no-T-F-except-top-level'*[*simp*]:
no-T-F-except-top-level $\varphi \longleftrightarrow (\varphi = FF \vee \varphi = FT \vee$ *no-T-F* $\varphi)$
apply *auto*
using *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb no-T-F-no-T-F-except-top-level*
by *blast+*

lemma *no-T-F-bin-decomp*[*simp*]:
assumes *c*: $c \in$ *binary-connectives*
shows *no-T-F* (*conn c* $[\varphi, \psi]$) $\longleftrightarrow$ (*no-T-F* $\varphi \wedge$ *no-T-F* ψ)
proof −
have *wf*: *wf-conn c* $[\varphi, \psi]$ **using** *c* **by** *auto*
hence *no-T-F* (*conn c* $[\varphi, \psi]$) $\longleftrightarrow$ (*no-T-F-symb* (*conn c* $[\varphi, \psi]$) $\wedge$ *no-T-F* $\varphi \wedge$ *no-T-F* ψ)
by (*simp add*: *all-subformula-st-decomp no-T-F-def*)
thus *no-T-F* (*conn c* $[\varphi, \psi]$) $\longleftrightarrow$ (*no-T-F* $\varphi \wedge$ *no-T-F* ψ)
using *c wf all-subformula-st-decomp list.discI no-T-F-def no-T-F-symb-except-toplevel-bin-decom no-T-F-symb-except-toplevel-no-T-F-symb no-T-F-symb-false*(*1*,*2*) *wf-conn-helper-facts*(*2*,*3*) *wf-conn-list*(*1*,*2*) **by** *metis*
qed

lemma *no-T-F-bin-decomp-expanded*[*simp*]:
assumes *c*: $c = CAnd \vee c = COr \vee c = CEq \vee c = CImp$
shows *no-T-F* (*conn c* $[\varphi, \psi]$) $\longleftrightarrow$ (*no-T-F* $\varphi \wedge$ *no-T-F* ψ)
using *no-T-F-bin-decomp assms* **unfolding** *binary-connectives-def* **by** *blast*

lemma *no-T-F-comp-expanded-explicit*[*simp*]:
fixes $\varphi\ \psi$ :: *'v propo*
shows
no-T-F (*FAnd* $\varphi\ \psi$) $\longleftrightarrow$ (*no-T-F* $\varphi \wedge$ *no-T-F* ψ)
no-T-F (*FOr* $\varphi\ \psi$) $\longleftrightarrow$ (*no-T-F* $\varphi \wedge$ *no-T-F* ψ)
no-T-F (*FEq* $\varphi\ \psi$) $\longleftrightarrow$ (*no-T-F* $\varphi \wedge$ *no-T-F* ψ)
no-T-F (*FImp* $\varphi\ \psi$) $\longleftrightarrow$ (*no-T-F* $\varphi \wedge$ *no-T-F* ψ)
using *assms conn.simps*(*5*−*8*) *no-T-F-bin-decomp-expanded* **by** (*metis* (*no-types*))+

lemma *no-T-F-comp-not*[*simp*]:
fixes $\varphi\ \psi$ :: *'v propo*
shows *no-T-F* (*FNot* φ) $\longleftrightarrow$ *no-T-F* φ
by (*metis all-subformula-st-decomp-explicit*(*3*) *all-subformula-st-test-symb-true-phi no-T-F-def no-T-F-symb-false*(*1*,*2*) *no-T-F-symb-fnot-imp*)

lemma *no-T-F-decomp*:
fixes $\varphi\ \psi$ :: *'v propo*
assumes φ: *no-T-F* (*FAnd* $\varphi\ \psi$) $\vee$ *no-T-F* (*FOr* $\varphi\ \psi$) $\vee$ *no-T-F* (*FEq* $\varphi\ \psi$) $\vee$ *no-T-F* (*FImp* $\varphi\ \psi$)
shows *no-T-F* ψ **and** *no-T-F* φ
using *assms* **by** *auto*

lemma *no-T-F-decomp-not*:
  fixes φ :: $'v$ *propo*
  assumes φ: *no-T-F* (*FNot* φ)
  shows *no-T-F* φ
  using *assms* **by** *auto*

lemma *no-T-F-symb-except-toplevel-step-exists*:
  fixes φ ψ :: $'v$ *propo*
  assumes *no-equiv* φ **and** *no-imp* φ
  shows $\psi \preceq \varphi \Longrightarrow \neg$ *no-T-F-symb-except-toplevel* $\psi \Longrightarrow \exists \psi'$. *elimTB* ψ ψ'
proof (*induct* ψ *rule*: *propo-induct-arity*)
  case (*nullary* φ' *x*)
  hence *False* **using** *no-T-F-symb-except-toplevel-true no-T-F-symb-except-toplevel-false* **by** *auto*
  thus *?case* **by** *blast*
next
  case (*unary* ψ)
  hence $\psi = FF \vee \psi = FT$ **using** *no-T-F-symb-except-toplevel-not-decom* **by** *blast*
  thus *?case* **using** *ElimTB5 ElimTB6* **by** *blast*
next
  case (*binary* φ' $\psi 1$ $\psi 2$)
  note *IH1* = *this*(*1*) **and** *IH2* = *this*(*2*) **and** φ' = *this*(*3*) **and** $F\varphi$ = *this*(*4*) **and** *n* = *this*(*5*)
  {
    assume $\varphi' = FImp\ \psi 1\ \psi 2 \vee \varphi' = FEq\ \psi 1\ \psi 2$
    hence *False* **using** *n* $F\varphi$ *subformula-all-subformula-st assms* **by** (*metis* (*no-types*) *no-equiv-eq*(*1*)
      no-equiv-def no-imp-Imp(*1*) *no-imp-def*)
    hence *?case* **by** *blast*
  }
  moreover {
    assume φ': $\varphi' = FAnd\ \psi 1\ \psi 2 \vee \varphi' = FOr\ \psi 1\ \psi 2$
    hence $\psi 1 = FT \vee \psi 2 = FT \vee \psi 1 = FF \vee \psi 2 = FF$
     using *no-T-F-symb-except-toplevel-bin-decom conn.simps*(*5*,*6*) *n* **unfolding** *binary-connectives-def*
      by *fastforce+*
    hence *?case* **using** *elimTB.intros* φ' **by** *blast*
  }
  ultimately show *?case* **using** φ' **by** *blast*
qed

lemma *no-T-F-except-top-level-rew*:
  fixes φ :: $'v$ *propo*
  assumes *noTB*: $\neg$ *no-T-F-except-top-level* φ **and** *no-equiv*: *no-equiv* φ **and** *no-imp*: *no-imp* φ
  shows $\exists \psi\ \psi'.\ \psi \preceq \varphi \wedge$ *elimTB* ψ ψ'
proof −
  have *test-symb-false-nullary*: $\forall x$. *no-T-F-symb-except-toplevel* (*FF*:: $'v$ *propo*)
    $\wedge$ *no-T-F-symb-except-toplevel FT* $\wedge$ *no-T-F-symb-except-toplevel* (*FVar* (*x*:: $'v$)) **by** *auto*
  moreover {
     fix *c*:: $'v$ *connective* **and** *l* :: $'v$ *propo list* **and** ψ :: $'v$ *propo*
     have *H*: *elimTB* (*conn c l*) $\psi \Longrightarrow \neg$*no-T-F-symb-except-toplevel* (*conn c l*)
       by (*case-tac* (*conn c l*) *rule*: *elimTB.cases*, *auto*)
  }
  moreover {
     fix *x* :: $'v$
     have *H'*: *no-T-F-except-top-level FT* *no-T-F-except-top-level FF*

no-T-F-except-top-level (*FVar x*)
by (*auto simp add*: *no-T-F-except-top-level-def test-symb-false-nullary*)
}
moreover {
fix ψ
have $\psi \preceq \varphi \Longrightarrow \neg$ *no-T-F-symb-except-toplevel* $\psi \Longrightarrow \exists \psi'.$ *elimTB* $\psi\ \psi'$
using *no-T-F-symb-except-toplevel-step-exists no-equiv no-imp* **by** *auto*
}
ultimately show *?thesis*
using *no-test-symb-step-exists noTB* **unfolding** *no-T-F-except-top-level-def* **by** *blast*
qed

lemma *elimTB-inv*:
fixes $\varphi\ \psi$ :: *'v propo*
assumes *full* (*propo-rew-step elimTB*) $\varphi\ \psi$
and *no-equiv* φ **and** *no-imp* φ
shows *no-equiv* ψ **and** *no-imp* ψ
proof –
{
fix $\varphi\ \psi$ :: *'v propo*
have *H*: *elimTB* $\varphi\ \psi \Longrightarrow$ *no-equiv* $\varphi \Longrightarrow$ *no-equiv* ψ
by (*induct* $\varphi\ \psi$ *rule*: *elimTB.induct*, *auto*)
}
thus *no-equiv* ψ
using *full-propo-rew-step-inv-stay-conn*[*of elimTB no-equiv-symb* $\varphi\ \psi$]
no-equiv-symb-conn-characterization assms **unfolding** *no-equiv-def* **by** *metis*
next
{
fix $\varphi\ \psi$ :: *'v propo*
have *H*: *elimTB* $\varphi\ \psi \Longrightarrow$ *no-imp* $\varphi \Longrightarrow$ *no-imp* ψ
by (*induct* $\varphi\ \psi$ *rule*: *elimTB.induct*, *auto*)
}
thus *no-imp* ψ
using *full-propo-rew-step-inv-stay-conn*[*of elimTB no-imp-symb* $\varphi\ \psi$] *assms*
no-imp-symb-conn-characterization **unfolding** *no-imp-def* **by** *metis*
qed

lemma *elimTB-full-propo-rew-step*:
fixes $\varphi\ \psi$ :: *'v propo*
assumes *no-equiv* φ **and** *no-imp* φ **and** *full* (*propo-rew-step elimTB*) $\varphi\ \psi$
shows *no-T-F-except-top-level* ψ
using *full-propo-rew-step-subformula no-T-F-except-top-level-rew assms elimTB-inv* **by** *fastforce*

8.4 PushNeg

Push the negation inside the formula, until the litteral.

inductive *pushNeg* **where**
PushNeg1[*simp*]: *pushNeg* (*FNot* (*FAnd* $\varphi\ \psi$)) (*FOr* (*FNot* φ) (*FNot* ψ)) |
PushNeg2[*simp*]: *pushNeg* (*FNot* (*FOr* $\varphi\ \psi$)) (*FAnd* (*FNot* φ) (*FNot* ψ)) |
PushNeg3[*simp*]: *pushNeg* (*FNot* (*FNot* φ)) φ

lemma *pushNeg-transformation-consistent*:
$A \models$ *FNot* (*FAnd* $\varphi\ \psi$) $\longleftrightarrow A \models$ (*FOr* (*FNot* φ) (*FNot* ψ))
$A \models$ *FNot* (*FOr* $\varphi\ \psi$) $\longleftrightarrow A \models$ (*FAnd* (*FNot* φ) (*FNot* ψ))

$A \models FNot\ (FNot\ \varphi) \quad \longleftrightarrow A \models \varphi$
  by *auto*

lemma *pushNeg-explicit*: $pushNeg\ \varphi\ \psi \Longrightarrow \forall A.\ A \models \varphi \longleftrightarrow A \models \psi$
  by (*induct* φ ψ *rule*: *pushNeg.induct*, *auto*)

lemma *pushNeg-consistent*: *preserves-un-sat pushNeg*
  unfolding *preserves-un-sat-def* **by** (*simp add*: *pushNeg-explicit*)

lemma *pushNeg-lifted-consistant*:
preserves-un-sat (*full* (*propo-rew-step pushNeg*))
  by (*simp add*: *pushNeg-consistent*)

fun *simple* **where**
simple FT = *True* |
simple FF = *True* |
simple (*FVar* -) = *True* |
simple - = *False*

lemma *simple-decomp*:
  $simple\ \varphi \longleftrightarrow (\varphi = FT \lor \varphi = FF \lor (\exists x.\ \varphi = FVar\ x))$
  by (*case-tac* φ, *auto*)

lemma *subformula-conn-decomp-simple*:
  fixes φ ψ :: *'v propo*
  assumes *s*: *simple* ψ
  shows $\varphi \preceq FNot\ \psi \longleftrightarrow (\varphi = FNot\ \psi \lor \varphi = \psi)$
proof −
  have $\varphi \preceq conn\ CNot\ [\psi] \longleftrightarrow (\varphi = conn\ CNot\ [\psi] \lor (\exists\ \psi \in set\ [\psi].\ \varphi \preceq \psi))$
    using *subformula-conn-decomp* *wf-conn-helper-facts*(*1*) **by** *metis*
  thus $\varphi \preceq FNot\ \psi \longleftrightarrow (\varphi = FNot\ \psi \lor \varphi = \psi)$ **using** *s* **by** (*auto simp add*: *simple-decomp*)
qed

lemma *subformula-conn-decomp-explicit*[*simp*]:
  fixes φ :: *'v propo* **and** *x* :: *'v*
  shows
    $\varphi \preceq FNot\ FT \longleftrightarrow (\varphi = FNot\ FT \lor \varphi = FT)$
    $\varphi \preceq FNot\ FF \longleftrightarrow (\varphi = FNot\ FF \lor \varphi = FF)$
    $\varphi \preceq FNot\ (FVar\ x) \longleftrightarrow (\varphi = FNot\ (FVar\ x) \lor \varphi = FVar\ x)$
  by (*auto simp add*: *subformula-conn-decomp-simple*)

fun *simple-not-symb* **where**
simple-not-symb (*FNot* φ) = (*simple* φ) |
simple-not-symb - = *True*

definition *simple-not* **where**
simple-not = *all-subformula-st simple-not-symb*
declare *simple-not-def*[*simp*]

lemma *simple-not-Not*[*simp*]:
  ¬ *simple-not* (*FNot* (*FAnd* φ ψ))
  ¬ *simple-not* (*FNot* (*FOr* φ ψ))

by *auto*

lemma *simple-not-step-exists*:
fixes φ ψ :: *'v propo*
assumes *no-equiv* φ **and** *no-imp* φ
shows $\psi \preceq \varphi \Longrightarrow \neg$ *simple-not-symb* $\psi \Longrightarrow \exists \psi'$. *pushNeg* ψ ψ'
apply (*induct* ψ, *auto*)
apply (*case-tac* ψ, *auto intro*: *pushNeg.intros*)
by (*metis assms*(*1*,*2*) *no-imp-Imp*(*1*) *no-equiv-eq*(*1*) *no-imp-def no-equiv-def subformula-in-subformula-not subformula-all-subformula-st*)+

lemma *simple-not-rew*:
fixes φ :: *'v propo*
assumes *noTB*: $\neg$ *simple-not* φ **and** *no-equiv*: *no-equiv* φ **and** *no-imp*: *no-imp* φ
shows $\exists \psi$ ψ'. $\psi \preceq \varphi \wedge$ *pushNeg* ψ ψ'
proof –
have $\forall x$. *simple-not-symb* (*FF*:: *'v propo*) $\wedge$ *simple-not-symb FT* $\wedge$ *simple-not-symb* (*FVar* (*x*:: *'v*))
by *auto*
moreover {
fix *c*:: *'v connective* **and** *l* :: *'v propo list* **and** ψ :: *'v propo*
have *H*: *pushNeg* (*conn c l*) $\psi \Longrightarrow \neg$*simple-not-symb* (*conn c l*)
by (*case-tac* (*conn c l*) *rule*: *pushNeg.cases*, *simp-all*)
}
moreover {
fix *x* :: *'v*
have *H'*: *simple-not FT simple-not FF simple-not* (*FVar x*)
by *simp-all*
}
moreover {
fix ψ :: *'v propo*
have $\psi \preceq \varphi \Longrightarrow \neg$ *simple-not-symb* $\psi \Longrightarrow \exists \psi'$. *pushNeg* ψ ψ'
using *simple-not-step-exists no-equiv no-imp* **by** *blast*
}
ultimately show *?thesis* **using** *no-test-symb-step-exists noTB* **unfolding** *simple-not-def* **by** *blast*
qed

lemma *no-T-F-except-top-level-pushNeg1*:
no-T-F-except-top-level (*FNot* (*FAnd* φ ψ)) $\Longrightarrow$ *no-T-F-except-top-level* (*FOr* (*FNot* φ) (*FNot* ψ))
using *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb no-T-F-comp-not no-T-F-decomp*(*1*) *no-T-F-decomp*(*2*) *no-T-F-no-T-F-except-top-level* **by** (*metis no-T-F-comp-expanded-explicit*(*2*) *propo.distinct*(*5*,*17*))

lemma *no-T-F-except-top-level-pushNeg2*:
no-T-F-except-top-level (*FNot* (*FOr* φ ψ)) $\Longrightarrow$ *no-T-F-except-top-level* (*FAnd* (*FNot* φ) (*FNot* ψ))
by *auto*

lemma *no-T-F-symb-pushNeg*:
no-T-F-symb (*FOr* (*FNot* φ') (*FNot* ψ'))
no-T-F-symb (*FAnd* (*FNot* φ') (*FNot* ψ'))
no-T-F-symb (*FNot* (*FNot* φ'))
by *auto*

lemma *propo-rew-step-pushNeg-no-T-F-symb*:
propo-rew-step pushNeg φ $\psi \Longrightarrow$ *no-T-F-except-top-level* $\varphi \Longrightarrow$ *no-T-F-symb* $\varphi \Longrightarrow$ *no-T-F-symb* ψ
apply (*induct rule*: *propo-rew-step.induct*)

apply (*cases rule*: *pushNeg.cases*)
apply *simp-all*
apply (*metis no-T-F-symb-pushNeg*(*1*))
apply (*metis no-T-F-symb-pushNeg*(*2*))
apply (*simp*, *metis all-subformula-st-test-symb-true-phi no-T-F-def*)
proof −
fix φ φ':: *'a propo* **and** *c*:: *'a connective* **and** ξ ξ':: *'a propo list*
assume *rel*: *propo-rew-step pushNeg* φ φ'
and *IH*: *no-T-F* $\varphi \Longrightarrow$ *no-T-F-symb* $\varphi \Longrightarrow$ *no-T-F-symb* φ'
and *wf*: *wf-conn c* (ξ @ φ # ξ')
and *n*: *conn c* (ξ @ φ # ξ') = *FF* $\vee$ *conn c* (ξ @ φ # ξ') = *FT* $\vee$ *no-T-F* (*conn c* (ξ @ φ # ξ'))
and *x*: $c \neq CF \wedge c \neq CT \wedge \varphi \neq FF \wedge \varphi \neq FT \wedge (\forall \psi \in$ *set* $\xi \cup$ *set* ξ'. $\psi \neq FF \wedge \psi \neq FT)$
hence $c \neq CF \wedge c \neq CF \wedge$ *wf-conn c* (ξ @ φ' # ξ')
using *wf-conn-no-arity-change-helper wf-conn-no-arity-change* **by** *metis*
moreover have *n'*: *no-T-F* (*conn c* (ξ @ φ # ξ')) **using** *n* **by** (*simp add*: *wf wf-conn-list*(*1*,*2*))
moreover
{
have *no-T-F* φ
by (*metis Un-iff all-subformula-st-decomp list.set-intros*(*1*) *n' wf no-T-F-def set-append*)
moreover hence *no-T-F-symb* φ
by (*simp add*: *all-subformula-st-test-symb-true-phi no-T-F-def*)
ultimately have $\varphi' \neq FF \wedge \varphi' \neq FT$
using *IH no-T-F-symb-false*(*1*) *no-T-F-symb-false*(*2*) **by** *blast*
hence $\forall \psi \in$ *set* (ξ @ φ' # ξ'). $\psi \neq FF \wedge \psi \neq FT$ **using** *x* **by** *auto*
}
ultimately show *no-T-F-symb* (*conn c* (ξ @ φ' # ξ')) **by** (*simp add*: *x*)
qed

lemma *propo-rew-step-pushNeg-no-T-F*:
propo-rew-step pushNeg φ $\psi \Longrightarrow$ *no-T-F* $\varphi \Longrightarrow$ *no-T-F* ψ
proof (*induct rule*: *propo-rew-step.induct*)
case *global-rel*
thus *?case*
by (*metis* (*no-types*, *lifting*) *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb no-T-F-def no-T-F-except-top-level-pushNeg1 no-T-F-except-top-level-pushNeg2 no-T-F-no-T-F-except-top-level all-subformula-st-decomp-explicit*(*3*) *pushNeg.simps simple.simps*(*1*,*2*,*5*,*6*))
next
case (*propo-rew-one-step-lift* φ φ' *c* ξ ξ')
note *rel* = *this*(*1*) **and** *IH* = *this*(*2*) **and** *wf* = *this*(*3*) **and** *no-T-F* = *this*(*4*)
moreover have *wf'*: *wf-conn c* (ξ @ φ' # ξ')
using *wf-conn-no-arity-change wf-conn-no-arity-change-helper wf* **by** *metis*
ultimately show *no-T-F* (*conn c* (ξ @ φ' # ξ')) **unfolding** *no-T-F-def*
apply(*simp add*: *all-subformula-st-decomp wf wf'*)
using *all-subformula-st-test-symb-true-phi no-T-F-symb-false*(*1*) *no-T-F-symb-false*(*2*) **by** *blast*
qed

lemma *pushNeg-inv*:
fixes φ ψ :: *'v propo*
assumes *full* (*propo-rew-step pushNeg*) φ ψ
and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ
shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ
proof −
{

fix φ ψ :: *'v propo*
assume *rel*: *propo-rew-step pushNeg* φ ψ
and *no*: *no-T-F-except-top-level* φ
hence *no-T-F-except-top-level* ψ
proof −
{
assume $\varphi = FT \vee \varphi = FF$
from *rel this* **have** *False*
apply (*induct rule*: *propo-rew-step.induct*)
using *pushNeg.cases* **apply** *blast*
using *wf-conn-list*(*1*) *wf-conn-list*(*2*) **by** *auto*
hence *no-T-F-except-top-level* ψ **by** *blast*
}
moreover {
assume $\varphi \neq FT \wedge \varphi \neq FF$
hence *no-T-F* φ **by** (*metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb*)
hence *no-T-F* ψ **using** *propo-rew-step-pushNeg-no-T-F rel* **by** *auto*
hence *no-T-F-except-top-level* ψ **by** (*simp add*: *no-T-F-no-T-F-except-top-level*)
}
ultimately show *no-T-F-except-top-level* ψ **by** *metis*
qed
}
moreover {
fix *c* :: *'v connective* **and** ξ ξ' :: *'v propo list* **and** ζ ζ' :: *'v propo*
assume *rel*: *propo-rew-step pushNeg* ζ ζ'
and *incl*: $\zeta \preceq \varphi$
and *corr*: *wf-conn c* (ξ @ ζ # ξ')
and *no-T-F*: *no-T-F-symb-except-toplevel* (*conn c* (ξ @ ζ # ξ'))
and *n*: *no-T-F-symb-except-toplevel* ζ'
have *no-T-F-symb-except-toplevel* (*conn c* (ξ @ ζ' # ξ'))
proof
have *p*: *no-T-F-symb* (*conn c* (ξ @ ζ # ξ'))
using *corr wf-conn-list*(*1*) *wf-conn-list*(*2*) *no-T-F-symb-except-toplevel-no-T-F-symb no-T-F*
by *blast*
have *l*: $\forall \varphi \in set$ (ξ @ ζ # ξ'). $\varphi \neq FT \wedge \varphi \neq FF$
using *corr wf-conn-no-T-F-symb-iff p* **by** *blast*
from *rel incl* **have** $\zeta' \neq FT \wedge \zeta' \neq FF$
apply (*induction* ζ ζ' *rule*: *propo-rew-step.induct*)
apply (*cases rule*: *pushNeg.cases*, *auto*)
by (*metis assms*(*4*) *no-T-F-symb-except-top-level-false-not no-T-F-except-top-level-def all-subformula-st-test-symb-true-phi subformula-in-subformula-not subformula-all-subformula-st append-is-Nil-conv list.distinct*(*1*) *wf-conn-no-arity-change-helper wf-conn-list*(*1*,*2*) *wf-conn-no-arity-change*)+
hence $\forall \varphi \in set$ (ξ @ ζ' # ξ'). $\varphi \neq FT \wedge \varphi \neq FF$ **using** *l* **by** *auto*
moreover have $c \neq CT \wedge c \neq CF$ **using** *corr* **by** *auto*
ultimately show *no-T-F-symb* (*conn c* (ξ @ ζ' # ξ'))
by (*metis corr no-T-F-symb-comp wf-conn-no-arity-change wf-conn-no-arity-change-helper*)
qed
}
ultimately show *no-T-F-except-top-level* ψ
using *full-propo-rew-step-inv-stay-with-inc*[*of pushNeg no-T-F-symb-except-toplevel* φ] *assms subformula-refl* **unfolding** *no-T-F-except-top-level-def full-unfold* **by** *metis*
next
{
fix φ ψ :: *'v propo*

have *H*: *pushNeg* φ ψ $\Longrightarrow$ *no-equiv* φ $\Longrightarrow$ *no-equiv* ψ
by (*induct* φ ψ *rule*: *pushNeg.induct*, *auto*)
}
thus *no-equiv* ψ
using *full-propo-rew-step-inv-stay-conn*[*of pushNeg no-equiv-symb* φ ψ]
no-equiv-symb-conn-characterization assms **unfolding** *no-equiv-def full-unfold* **by** *metis*
next
{
fix φ ψ :: *'v propo*
have *H*: *pushNeg* φ ψ $\Longrightarrow$ *no-imp* φ $\Longrightarrow$ *no-imp* ψ
by (*induct* φ ψ *rule*: *pushNeg.induct*, *auto*)
}
thus *no-imp* ψ
using *full-propo-rew-step-inv-stay-conn*[*of pushNeg no-imp-symb* φ ψ] *assms*
no-imp-symb-conn-characterization **unfolding** *no-imp-def full-unfold* **by** *metis*
qed

lemma *pushNeg-full-propo-rew-step*:
fixes φ ψ :: *'v propo*
assumes
no-equiv φ **and**
no-imp φ **and**
full (*propo-rew-step pushNeg*) φ ψ **and**
no-T-F-except-top-level φ
shows *simple-not* ψ
using *assms full-propo-rew-step-subformula pushNeg-inv*(*1*,*2*) *simple-not-rew* **by** *blast*

8.5 Push inside

inductive *push-conn-inside* :: *'v connective* $\Rightarrow$ *'v connective* $\Rightarrow$ *'v propo* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool*
for *c c'*:: *'v connective* **where**
push-conn-inside-l[*simp*]: *c* = *CAnd* $\vee$ *c* = *COr* $\Longrightarrow$ *c'* = *CAnd* $\vee$ *c'* = *COr*
$\Longrightarrow$ *push-conn-inside c c'* (*conn c* [*conn c'* [$\varphi 1$, $\varphi 2$], ψ])
(*conn c'* [*conn c* [$\varphi 1$, ψ], *conn c* [$\varphi 2$, ψ]]) |
push-conn-inside-r[*simp*]: *c* = *CAnd* $\vee$ *c* = *COr* $\Longrightarrow$ *c'* = *CAnd* $\vee$ *c'* = *COr*
$\Longrightarrow$ *push-conn-inside c c'* (*conn c* [ψ, *conn c'* [$\varphi 1$, $\varphi 2$]])
(*conn c'* [*conn c* [ψ, $\varphi 1$], *conn c* [ψ, $\varphi 2$]])

lemma *push-conn-inside-explicit*: *push-conn-inside c c'* φ ψ $\Longrightarrow$ $\forall$ *A*. *A*$\models\varphi$ $\longleftrightarrow$ *A*$\models\psi$
by (*induct* φ ψ *rule*: *push-conn-inside.induct*, *auto*)

lemma *push-conn-inside-consistent*: *preserves-un-sat* (*push-conn-inside c c'*)
unfolding *preserves-un-sat-def* **by** (*simp add*: *push-conn-inside-explicit*)

lemma *propo-rew-step-push-conn-inside*[*simp*]:
$\neg$*propo-rew-step* (*push-conn-inside c c'*) *FT* ψ $\neg$*propo-rew-step* (*push-conn-inside c c'*) *FF* ψ
proof –
{
{
fix φ ψ
have *push-conn-inside c c'* φ ψ $\Longrightarrow$ φ = *FT* $\vee$ φ = *FF* $\Longrightarrow$ *False*
by (*induct rule*: *push-conn-inside.induct*, *auto*)
} note *H* = *this*
fix φ

have *propo-rew-step* (*push-conn-inside* c c') φ $\psi \Longrightarrow \varphi = FT \vee \varphi = FF \Longrightarrow False$
apply (*induct rule*: *propo-rew-step.induct*, *auto simp add*: *wf-conn-list*(*1*) *wf-conn-list*(*2*))
using *H* **by** *blast+*
}
thus
$\neg$*propo-rew-step* (*push-conn-inside* c c') *FT* ψ
$\neg$*propo-rew-step* (*push-conn-inside* c c') *FF* ψ **by** *blast+*
qed

inductive *not-c-in-c'-symb*:: *'v connective* $\Rightarrow$ *'v connective* $\Rightarrow$ *'v propo* $\Rightarrow$ *bool* **for** c c' **where**
not-c-in-c'-symb-l[*simp*]: *wf-conn* c [*conn* c' [φ, φ'], ψ] $\Longrightarrow$ *wf-conn* c' [φ, φ']
$\Longrightarrow$ *not-c-in-c'-symb* c c' (*conn* c [*conn* c' [φ, φ'], ψ]) |
not-c-in-c'-symb-r[*simp*]: *wf-conn* c [ψ, *conn* c' [φ, φ']] $\Longrightarrow$ *wf-conn* c' [φ, φ']
$\Longrightarrow$ *not-c-in-c'-symb* c c' (*conn* c [ψ, *conn* c' [φ, φ']])

abbreviation *c-in-c'-symb* c c' $\varphi \equiv \neg$*not-c-in-c'-symb* c c' φ

lemma *c-in-c'-symb-simp*:
not-c-in-c'-symb c c' $\xi \Longrightarrow \xi = FF \vee \xi = FT \vee \xi = FVar\ x \vee \xi = FNot\ FF \vee \xi = FNot\ FT$
$\vee\ \xi = FNot\ (FVar\ x) \Longrightarrow False$
apply (*induct rule*: *not-c-in-c'-symb.induct*, *auto simp add*: *wf-conn.simps* *wf-conn-list*(*1*–*3*))
using *conn-inj-not*(*2*) *wf-conn-binary* **unfolding** *binary-connectives-def* **by** *fastforce+*

lemma *c-in-c'-symb-simp'*[*simp*]:
$\neg$*not-c-in-c'-symb* c c' *FF*
$\neg$*not-c-in-c'-symb* c c' *FT*
$\neg$*not-c-in-c'-symb* c c' (*FVar* x)
$\neg$*not-c-in-c'-symb* c c' (*FNot FF*)
$\neg$*not-c-in-c'-symb* c c' (*FNot FT*)
$\neg$*not-c-in-c'-symb* c c' (*FNot* (*FVar* x))
using *c-in-c'-symb-simp* **by** *metis+*

definition *c-in-c'-only* **where**
c-in-c'-only c c' $\equiv$ *all-subformula-st* (*c-in-c'-symb* c c')

lemma *c-in-c'-only-simp*[*simp*]:
c-in-c'-only c c' *FF*
c-in-c'-only c c' *FT*
c-in-c'-only c c' (*FVar* x)
c-in-c'-only c c' (*FNot FF*)
c-in-c'-only c c' (*FNot FT*)
c-in-c'-only c c' (*FNot* (*FVar* x))
unfolding *c-in-c'-only-def* **by** *auto*

lemma *not-c-in-c'-symb-commute*:
not-c-in-c'-symb c c' $\xi \Longrightarrow$ *wf-conn* c [φ, ψ] $\Longrightarrow \xi =$ *conn* c [φ, ψ]
$\Longrightarrow$ *not-c-in-c'-symb* c c' (*conn* c [ψ, φ])
proof (*induct rule*: *not-c-in-c'-symb.induct*)
case (*not-c-in-c'-symb-r* φ' φ'' ψ') **note** $H = this$
hence ψ: $\psi =$ *conn* c' [φ'', ψ'] **using** *conn-inj* **by** *auto*
have *wf-conn* c [*conn* c' [φ'', ψ'], φ]
using *H*(*1*) *wf-conn-no-arity-change* *length-Cons* **by** *metis*

thus *not-c-in-c′-symb c c′* (*conn c* [ψ, φ])
  unfolding ψ **using** *not-c-in-c′-symb.intros*(*1*) *H* **by** *auto*
next
  case (*not-c-in-c′-symb-l* φ' φ'' ψ') **note** *H* = *this*
  hence φ = *conn c′* [φ', φ''] **using** *conn-inj* **by** *auto*
  moreover have *wf-conn c* [ψ', *conn c′* [φ', φ'']]
    using *H*(*1*) *wf-conn-no-arity-change length-Cons* **by** *metis*
  ultimately show *not-c-in-c′-symb c c′* (*conn c* [ψ, φ])
    using *not-c-in-c′-symb.intros*(*2*) *conn-inj not-c-in-c′-symb-l.hyps*
      not-c-in-c′-symb-l.prems(*1*,*2*) **by** *blast*
qed

lemma *not-c-in-c′-symb-commute′*:
  wf-conn c [φ, ψ] $\Longrightarrow$ *c-in-c′-symb c c′* (*conn c* [φ, ψ]) $\longleftrightarrow$ *c-in-c′-symb c c′* (*conn c* [ψ, φ])
  using *not-c-in-c′-symb-commute wf-conn-no-arity-change* **by** (*metis length-Cons*)

lemma *not-c-in-c′-comm*:
  assumes *wf*: *wf-conn c* [φ, ψ]
  shows *c-in-c′-only c c′* (*conn c* [φ, ψ]) $\longleftrightarrow$ *c-in-c′-only c c′* (*conn c* [ψ, φ]) (**is** *?A* $\longleftrightarrow$ *?B*)
proof −
  have *?A* $\longleftrightarrow$ (*c-in-c′-symb c c′* (*conn c* [φ, ψ])
        $\wedge$ ($\forall\xi \in$ *set* [φ, ψ]. *all-subformula-st* (*c-in-c′-symb c c′*) ξ))
    using *all-subformula-st-decomp wf* **unfolding** *c-in-c′-only-def* **by** *fastforce*
  also have ... $\longleftrightarrow$ (*c-in-c′-symb c c′* (*conn c* [ψ, φ])
          $\wedge$ ($\forall\xi \in$ *set* [ψ, φ]. *all-subformula-st* (*c-in-c′-symb c c′*) ξ))
    using *not-c-in-c′-symb-commute′ wf* **by** *auto*
  also
    have *wf-conn c* [ψ, φ] **using** *wf-conn-no-arity-change wf* **by** (*metis length-Cons*)
    hence (*c-in-c′-symb c c′* (*conn c* [ψ, φ])
        $\wedge$ ($\forall\xi \in$ *set* [ψ, φ]. *all-subformula-st* (*c-in-c′-symb c c′*) ξ))
      $\longleftrightarrow$ *?B*
      using *all-subformula-st-decomp* **unfolding** *c-in-c′-only-def* **by** *fastforce*
  finally show *?thesis* **.**
qed

lemma *not-c-in-c′-simp*[*simp*]:
  fixes $\varphi 1$ $\varphi 2$ ψ :: *′v propo* **and** *x* :: *′v*
  shows
  c-in-c′-symb c c′ FT
  c-in-c′-symb c c′ FF
  c-in-c′-symb c c′ (*FVar x*)
  wf-conn c [*conn c′* [$\varphi 1$, $\varphi 2$], ψ] $\Longrightarrow$ *wf-conn c′* [$\varphi 1$, $\varphi 2$]
    $\Longrightarrow$ $\neg$ *c-in-c′-only c c′* (*conn c* [*conn c′* [$\varphi 1$, $\varphi 2$], ψ])
  apply (*simp-all add*: *c-in-c′-only-def*)
  using *all-subformula-st-test-symb-true-phi not-c-in-c′-symb-l* **by** *blast*

lemma *c-in-c′-symb-not*[*simp*]:
  fixes *c c′* :: *′v connective* **and** ψ :: *′v propo*
  shows *c-in-c′-symb c c′* (*FNot* ψ)
proof −
  {
    fix ξ :: *′v propo*
    have *not-c-in-c′-symb c c′* (*FNot* ψ) $\Longrightarrow$ *False*
      apply (*induct FNot* ψ *rule*: *not-c-in-c′-symb.induct*)
      using *conn-inj-not*(*2*) **by** *blast*+

}
thus *?thesis* **by** *auto*
qed

lemma *c-in-c′-symb-step-exists*:
fixes φ :: *′v propo*
assumes *c*: $c = CAnd \vee c = COr$ **and** *c′*: $c' = CAnd \vee c' = COr$
shows $\psi \preceq \varphi \Longrightarrow \neg$ *c-in-c′-symb c c′* $\psi \Longrightarrow \exists \psi'.$ *push-conn-inside c c′* ψ ψ'
apply (*induct* ψ *rule*: *propo-induct-arity*)
apply *auto*[*2*]
proof −
fix $\psi 1$ $\psi 2$ φ':: *′v propo*
assume *IH*ψ*1*: $\psi 1 \preceq \varphi \Longrightarrow \neg$ *c-in-c′-symb c c′* $\psi 1 \Longrightarrow$ *Ex* (*push-conn-inside c c′* $\psi 1$)
and *IH*ψ*2*: $\psi 1 \preceq \varphi \Longrightarrow \neg$ *c-in-c′-symb c c′* $\psi 1 \Longrightarrow$ *Ex* (*push-conn-inside c c′* $\psi 1$)
and φ': $\varphi' = FAnd\ \psi 1\ \psi 2 \vee \varphi' = FOr\ \psi 1\ \psi 2 \vee \varphi' = FImp\ \psi 1\ \psi 2 \vee \varphi' = FEq\ \psi 1\ \psi 2$
and *in*φ: $\varphi' \preceq \varphi$ **and** *n0*: $\neg$*c-in-c′-symb c c′* φ'
hence *n*: *not-c-in-c′-symb c c′* φ' **by** *auto*
{
assume φ': $\varphi' = conn\ c\ [\psi 1, \psi 2]$
obtain *a b* **where** $\psi 1 = conn\ c'\ [a, b] \vee \psi 2 = conn\ c'\ [a, b]$
using *n* φ' **apply** (*induct rule*: *not-c-in-c′-symb.induct*)
using *c* **by** *force+*
hence *Ex* (*push-conn-inside c c′* φ')
unfolding φ' **apply** *auto*
using *push-conn-inside.intros*(*1*) *c c′* **apply** *blast*
using *push-conn-inside.intros*(*2*) *c c′* **by** *blast*
}
moreover {
assume φ': $\varphi' \neq conn\ c\ [\psi 1, \psi 2]$
have $\forall \varphi\ c\ ca.\ \exists \varphi 1\ \psi 1\ \psi 2\ \psi 1'\ \psi 2'\ \varphi 2'.$ *conn* (*c*::*′v connective*) $[\varphi 1, conn\ ca\ [\psi 1, \psi 2]] = \varphi$
$\vee\ conn\ c\ [conn\ ca\ [\psi 1', \psi 2'], \varphi 2'] = \varphi \vee$ *c-in-c′-symb c ca* φ
by (*metis not-c-in-c′-symb.cases*)
hence *Ex* (*push-conn-inside c c′* φ')
by (*metis* (*no-types*) *c c′ n push-conn-inside-l push-conn-inside-r*)
}
ultimately show *Ex* (*push-conn-inside c c′* φ') **by** *blast*
qed

lemma *c-in-c′-symb-rew*:
fixes φ :: *′v propo*
assumes *noTB*: $\neg$*c-in-c′-only c c′* φ
and *c*: $c = CAnd \vee c = COr$ **and** *c′*: $c' = CAnd \vee c' = COr$
shows $\exists \psi\ \psi'.\ \psi \preceq \varphi \wedge$ *push-conn-inside c c′* ψ ψ'
proof −
have *test-symb-false-nullary*:
$\forall x.$ *c-in-c′-symb c c′* (*FF*:: *′v propo*) $\wedge$ *c-in-c′-symb c c′ FT*
$\wedge$ *c-in-c′-symb c c′* (*FVar* (*x*:: *′v*))
by *auto*
moreover {
fix *x* :: *′v*
have *H′*: *c-in-c′-symb c c′ FT c-in-c′-symb c c′ FF c-in-c′-symb c c′* (*FVar x*)
by *simp+*
}
moreover {

fix ψ :: *'v propo*
have $\psi \preceq \varphi \Longrightarrow \neg$ *c-in-c'-symb c c'* $\psi \Longrightarrow \exists\psi'$. *push-conn-inside c c'* $\psi\ \psi'$
by (*auto simp add*: *assms*(*2*) *c' c-in-c'-symb-step-exists*)
}
ultimately show *?thesis* **using** *noTB no-test-symb-step-exists*[*of c-in-c'-symb c c'*]
unfolding *c-in-c'-only-def* **by** *metis*
qed

lemma *push-conn-insidec-in-c'-symb-no-T-F*:
fixes $\varphi\ \psi$ :: *'v propo*
shows *propo-rew-step* (*push-conn-inside c c'*) $\varphi\ \psi \Longrightarrow$ *no-T-F* $\varphi \Longrightarrow$ *no-T-F* ψ
proof (*induct rule*: *propo-rew-step.induct*)
case (*global-rel* $\varphi\ \psi$)
thus *no-T-F* ψ
by (*cases rule*: *push-conn-inside.cases*, *auto*)
next
case (*propo-rew-one-step-lift* $\varphi\ \varphi'$ *c* $\xi\ \xi'$)
note *rel* = *this*(*1*) **and** *IH* = *this*(*2*) **and** *wf* = *this*(*3*) **and** *no-T-F* = *this*(*4*)
have *no-T-F* φ
using *wf no-T-F no-T-F-def subformula-into-subformula subformula-all-subformula-st*
subformula-refl **by** (*metis* (*no-types*) *in-set-conv-decomp*)
hence φ': *no-T-F* φ' **using** *IH* **by** *blast*

have $\forall\zeta \in$ *set* (ξ @ φ # ξ'). *no-T-F* ζ **by** (*metis wf no-T-F no-T-F-def all-subformula-st-decomp*)
hence *n*: $\forall\zeta \in$ *set* (ξ @ φ' # ξ'). *no-T-F* ζ **using** φ' **by** *auto*
hence *n'*: $\forall\zeta \in$ *set* (ξ @ φ' # ξ'). $\zeta \neq$ *FF* $\wedge\ \zeta \neq$ *FT*
using φ' **by** (*metis no-T-F-symb-false*(*1*) *no-T-F-symb-false*(*2*) *no-T-F-def*
all-subformula-st-test-symb-true-phi)

have *wf'*: *wf-conn c* (ξ @ φ' # ξ')
using *wf wf-conn-no-arity-change* **by** (*metis wf-conn-no-arity-change-helper*)
{
fix *x* :: *'v*
assume *c* = *CT* $\vee$ *c* = *CF* $\vee$ *c* = *CVar x*
hence *False* **using** *wf* **by** *auto*
hence *no-T-F* (*conn c* (ξ @ φ' # ξ')) **by** *blast*
}
moreover {
assume *c*: *c* = *CNot*
hence ξ = [] ξ' = [] **using** *wf* **by** *auto*
hence *no-T-F* (*conn c* (ξ @ φ' # ξ'))
using *c* **by** (*metis* φ' *conn.simps*(*4*) *no-T-F-symb-false*(*1*,*2*) *no-T-F-symb-fnot no-T-F-def*
all-subformula-st-decomp-explicit(*3*) *all-subformula-st-test-symb-true-phi self-append-conv2*)
}
moreover {
assume *c*: *c* $\in$ *binary-connectives*
hence *no-T-F-symb* (*conn c* (ξ @ φ' # ξ')) **using** *wf' n' no-T-F-symb.simps* **by** *fastforce*
hence *no-T-F* (*conn c* (ξ @ φ' # ξ')) **by** (*metis all-subformula-st-decomp-imp wf' n no-T-F-def*)
}
ultimately show *no-T-F* (*conn c* (ξ @ φ' # ξ')) **using** *connective-cases-arity* **by** *auto*
qed

lemma *simple-propo-rew-step-push-conn-inside-inv*:
propo-rew-step (*push-conn-inside c c'*) $\varphi\ \psi \Longrightarrow$ *simple* $\varphi \Longrightarrow$ *simple* ψ

```
  apply (induct rule: propo-rew-step.induct)
  apply (case-tac φ, auto simp add: push-conn-inside.simps)[1]
  by (metis append-is-Nil-conv list.distinct(1) simple.elims(2) wf-conn-list(1−3))


lemma simple-propo-rew-step-inv-push-conn-inside-simple-not:
  fixes c c′ :: ′v connective and φ ψ :: ′v propo
  shows propo-rew-step (push-conn-inside c c′) φ ψ ⟹ simple-not φ ⟹ simple-not ψ
proof (induct rule: propo-rew-step.induct)
  case (global-rel φ ψ)
  thus ?case by (case-tac φ, auto simp add: push-conn-inside.simps)
next
  case (propo-rew-one-step-lift φ φ′ ca ξ ξ′)
  thus ?case
    proof (case-tac ca rule: connective-cases-arity, auto)
      fix φ φ′:: ′v propo and  c :: ′v connective and  ξ ξ′ :: ′v propo list
      assume rel: propo-rew-step (push-conn-inside c c′) φ φ′
      assume simple φ
      thus simple φ′ using rel simple-propo-rew-step-push-conn-inside-inv by blast
    next
      fix φ φ′:: ′v propo and  ca :: ′v connective and  ξ ξ′ :: ′v propo list
      assume rel: propo-rew-step (push-conn-inside c c′) φ φ′
      and IH: all-subformula-st simple-not-symb φ ⟹ all-subformula-st simple-not-symb φ′
      and wf: wf-conn ca (ξ @ φ # ξ′)
      and simple-not: all-subformula-st simple-not-symb (conn ca (ξ @ φ # ξ′))
      and ca: ca ∈ binary-connectives

      obtain a b where ab: ξ @ φ′ # ξ′ = [a, b]
        using wf ca  list-length2-decomp wf-conn-bin-list-length
        by (metis (no-types) wf-conn-no-arity-change-helper)
      have ∀ζ ∈ set (ξ @ φ # ξ′). simple-not ζ
        by (metis wf all-subformula-st-decomp simple-not simple-not-def)
      hence ∀ζ ∈ set (ξ @ φ′ # ξ′). simple-not ζ by (simp add: IH)
      moreover have simple-not-symb (conn ca (ξ @ φ′ # ξ′)) using ca
        by (metis ab conn.simps(5−8) helper-fact simple-not-symb.simps(5) simple-not-symb.simps(6)
          simple-not-symb.simps(7) simple-not-symb.simps(8))
      ultimately show all-subformula-st simple-not-symb (conn ca (ξ @ φ′ # ξ′))
        by (simp add: ab all-subformula-st-decomp ca)
    qed
qed

lemma propo-rew-step-push-conn-inside-simple-not:
  fixes φ φ′ :: ′v propo and ξ ξ′ :: ′v propo list and c :: ′v connective
  shows propo-rew-step (push-conn-inside c c′) φ φ′ ⟹ wf-conn c (ξ @ φ # ξ′)
    ⟹ simple-not-symb (conn c (ξ @ φ # ξ′)) ⟹ simple-not-symb φ′
    ⟹ simple-not-symb (conn c (ξ @ φ′ # ξ′))
  apply (induct rule: propo-rew-step.induct)
  apply (metis (no-types, lifting) append-eq-append-conv2 append-self-conv conn.simps(4)
    conn-inj-not(1) global-rel simple-not-symb.elims(3) simple-not-symb.simps(1)
    simple-propo-rew-step-push-conn-inside-inv wf-conn-list-decomp(4) wf-conn-no-arity-change
    wf-conn-no-arity-change-helper)

proof (case-tac c rule: connective-cases-arity, auto)
  fix φ φ′:: ′v propo and  ca:: ′v connective and χs χs′ :: ′v propo list
  assume simple-not-symb (conn c (ξ @ conn ca (χs @ φ # χs′) # ξ′))
```

and *simple-not-symb* (*conn ca* (χs @ φ' # $\chi s'$))
and *corr*: *wf-conn c* (ξ @ *conn ca* (χs @ φ # $\chi s'$) # ξ')
and *c*: $c \in$ *binary-connectives*
have *corr'*: *wf-conn c* (ξ @ *conn ca* (χs @ φ' # $\chi s'$) # ξ')
using *corr wf-conn-no-arity-change* **by** (*metis wf-conn-no-arity-change-helper*)
obtain *a b* **where** ξ @ *conn ca* (χs @ φ' # $\chi s'$) # $\xi' = [a, b]$
using *corr' c list-length2-decomp wf-conn-bin-list-length* **by** *metis*
thus *simple-not-symb* (*conn c* (ξ @ *conn ca* (χs @ φ' # $\chi s'$) # ξ'))
using *c* **unfolding** *binary-connectives-def* **by** *auto*
next
fix φ φ':: *'v propo* **and** *ca*:: *'v connective* **and** χs $\chi s'$ :: *'v propo list*
assume *corr-ca*: *wf-conn ca* (χs @ φ # $\chi s'$)
and *simple-not*: *simple* (*conn ca* (χs @ φ # $\chi s'$))
hence *False*
proof (*case-tac ca rule*: *connective-cases-arity*)
fix *x* :: *'v*
assume *simple* (*conn ca* (χs @ φ # $\chi s'$)) **and** $ca = CT \lor ca = CF \lor ca = CVar\ x$
hence χs @ φ # $\chi s'$ = [] **using** *corr-ca* **by** *auto*
thus *False* **by** *auto*
next
assume *simple*: *simple* (*conn ca* (χs @ φ # $\chi s'$))
and *ca*: *ca*$\in$ *binary-connectives*
obtain *a b* **where** *ab*: χs @ φ # $\chi s' = [a, b]$
using *corr-ca ca list-length2-decomp wf-conn-bin-list-length*
by (*metis append-assoc length-Cons length-append length-append-singleton*)
thus *False* **using** *simple ca ab conn.simps*(*5*,*6*,*7*,*8*) **unfolding** *binary-connectives-def* **by** *auto*
next
assume *simple*: *simple* (*conn ca* (χs @ φ # $\chi s'$))
and *ca*: $ca = CNot$
hence *empty*: χs = [] $\chi s'$ = [] **using** *corr-ca* **by** *auto*
thus *False* **using** *simple ca conn.simps*(*4*) **by** *auto*
qed
thus *simple* (*conn ca* (χs @ φ' # $\chi s'$)) **by** *blast*
qed

lemma *push-conn-inside-not-true-false*:
push-conn-inside c c' φ $\psi \Longrightarrow \psi \neq FT \land \psi \neq FF$
by (*induct rule*: *push-conn-inside.induct*, *auto*)

lemma *push-conn-inside-inv*:
fixes φ ψ :: *'v propo*
assumes *full* (*propo-rew-step* (*push-conn-inside c c'*)) φ ψ
and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ **and** *simple-not* φ
shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ **and** *simple-not* ψ
proof –
{
{
fix φ ψ :: *'v propo*
have *H*: *push-conn-inside c c'* φ $\psi \Longrightarrow$ *all-subformula-st simple-not-symb* φ
$\Longrightarrow$ *all-subformula-st simple-not-symb* ψ
by (*induct* φ ψ *rule*: *push-conn-inside.induct*, *auto*)
} **note** $H = this$

fix φ ψ :: *'v propo*
have *H*: *propo-rew-step* (*push-conn-inside c c'*) φ $\psi \Longrightarrow$ *all-subformula-st simple-not-symb* φ

$\Longrightarrow$ *all-subformula-st simple-not-symb* ψ
apply (*induct* φ ψ *rule*: *propo-rew-step.induct*)
using *H* **apply** *simp*
proof (*case-tac ca rule*: *connective-cases-arity*)
  fix φ φ' :: *'v propo* **and** *c*:: *'v connective* **and** ξ ξ':: *'v propo list*
  and *x*:: *'v*
  assume *wf-conn c* (ξ @ φ # ξ')
  and $c = CT \lor c = CF \lor c = CVar\ x$
  hence ξ @ φ # ξ' = [] **by** *auto*
  hence *False* **by** *auto*
  thus *all-subformula-st simple-not-symb* (*conn c* (ξ @ φ' # ξ')) **by** *blast*
next
  fix φ φ' :: *'v propo* **and** *ca*:: *'v connective* **and** ξ ξ':: *'v propo list*
  and *x* :: *'v*
  assume *rel*: *propo-rew-step* (*push-conn-inside c c'*) φ φ'
  and φ-φ': *all-subformula-st simple-not-symb* $\varphi \Longrightarrow$ *all-subformula-st simple-not-symb* φ'
  and *corr*: *wf-conn ca* (ξ @ φ # ξ')
  and *n*: *all-subformula-st simple-not-symb* (*conn ca* (ξ @ φ # ξ'))
  and *c*: *ca* = *CNot*

  have *empty*: ξ = [] ξ' = [] **using** *c corr* **by** *auto*
  hence *simple-not*:*all-subformula-st simple-not-symb* (*FNot* φ) **using** *corr c n* **by** *auto*
  hence *simple* φ
    using *all-subformula-st-test-symb-true-phi simple-not-symb.simps*(*1*) **by** *blast*
  hence *simple* φ'
    using *rel simple-propo-rew-step-push-conn-inside-inv* **by** *blast*
  thus *all-subformula-st simple-not-symb* (*conn ca* (ξ @ φ' # ξ')) **using** *c empty*
    by (*metis simple-not* φ-φ' *append-Nil conn.simps*(*4*) *all-subformula-st-decomp-explicit*(*3*)
      simple-not-symb.simps(*1*))
next
  fix φ φ' :: *'v propo* **and** *ca* :: *'v connective* **and** ξ ξ' :: *'v propo list*
  and *x* :: *'v*
  assume *rel*: *propo-rew-step* (*push-conn-inside c c'*) φ φ'
  and *n*φ: *all-subformula-st simple-not-symb* $\varphi \Longrightarrow$ *all-subformula-st simple-not-symb* φ'
  and *corr*: *wf-conn ca* (ξ @ φ # ξ')
  and *n*: *all-subformula-st simple-not-symb* (*conn ca* (ξ @ φ # ξ'))
  and *c*: *ca* $\in$ *binary-connectives*

  have *all-subformula-st simple-not-symb* φ
    using *n c corr all-subformula-st-decomp* **by** *fastforce*
  hence φ': *all-subformula-st simple-not-symb* φ' **using** *n*φ **by** *blast*
  obtain *a b* **where** *ab*: [*a*, *b*] = (ξ @ φ # ξ')
    using *corr c list-length2-decomp wf-conn-bin-list-length* **by** *metis*
  hence ξ @ φ' # ξ' = [*a*, φ'] $\lor$ (ξ @ φ' # ξ') = [φ', *b*]
    using *ab* **by** (*metis* (*no-types*, *hide-lams*) *append-Cons append-Nil append-Nil2*
      append-is-Nil-conv butlast.simps(*2*) *butlast-append list.sel*(*3*) *tl-append2*)
  moreover
  {
    fix χ :: *'v propo*
    have *wf'*: *wf-conn ca* [*a*, *b*]
      using *ab corr* **by** *presburger*
    have *all-subformula-st simple-not-symb* (*conn ca* [*a*, *b*])
      using *ab n* **by** *presburger*
    hence *all-subformula-st simple-not-symb* $\chi \lor \chi \notin$ *set* (ξ @ φ' # ξ')
      using *wf'* **by** (*metis* (*no-types*) φ' *all-subformula-st-decomp calculation insert-iff*

list.set(*2*))
}
hence $\forall\varphi.\ \varphi \in set\ (\xi\ @\ \varphi'\ \#\ \xi') \longrightarrow$ *all-subformula-st simple-not-symb* φ
by (*metis* (*no-types*))

moreover have *simple-not-symb* (*conn ca* (ξ @ φ' # ξ'))
using *ab conn-inj-not*(*1*) *corr wf-conn-list-decomp*(*4*) *wf-conn-no-arity-change not-Cons-self2 self-append-conv2 simple-not-symb.elims*(*3*) **by** (*metis* (*no-types*) *c calculation*(*1*) *wf-conn-binary*)
moreover have *wf-conn ca* (ξ @ φ' # ξ') **using** *c calculation*(*1*) **by** *auto*
ultimately show *all-subformula-st simple-not-symb* (*conn ca* (ξ @ φ' # ξ'))
by (*metis all-subformula-st-decomp-imp*)
qed
}
moreover {
fix *ca* :: *'v connective* **and** ξ ξ' :: *'v propo list* **and** φ φ' :: *'v propo*
have *propo-rew-step* (*push-conn-inside c c'*) φ φ' $\Longrightarrow$ *wf-conn ca* (ξ @ φ # ξ')
$\Longrightarrow$ *simple-not-symb* (*conn ca* (ξ @ φ # ξ')) $\Longrightarrow$ *simple-not-symb* φ'
$\Longrightarrow$ *simple-not-symb* (*conn ca* (ξ @ φ' # ξ'))
by (*metis append-self-conv2 conn.simps*(*4*) *conn-inj-not*(*1*) *simple-not-symb.elims*(*3*) *simple-not-symb.simps*(*1*) *simple-propo-rew-step-push-conn-inside-inv wf-conn-no-arity-change-helper wf-conn-list-decomp*(*4*) *wf-conn-no-arity-change*)
}
ultimately show *simple-not* ψ
using *full-propo-rew-step-inv-stay'*[*of push-conn-inside c c' simple-not-symb*] *assms*
unfolding *no-T-F-except-top-level-def simple-not-def full-unfold* **by** *metis*
next
{
fix φ ψ :: *'v propo*
have *H*: *propo-rew-step* (*push-conn-inside c c'*) φ ψ $\Longrightarrow$ *no-T-F-except-top-level* φ
$\Longrightarrow$ *no-T-F-except-top-level* ψ
proof −
assume *rel*: *propo-rew-step* (*push-conn-inside c c'*) φ ψ
and *no-T-F-except-top-level* φ
hence *no-T-F* $\varphi \vee \varphi = FF \vee \varphi = FT$
by (*metis no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb*)
moreover {
assume $\varphi = FF \vee \varphi = FT$
hence *False* **using** *rel propo-rew-step-push-conn-inside* **by** *blast*
hence *no-T-F-except-top-level* ψ **by** *blast*
}
moreover {
assume *no-T-F* $\varphi \wedge \varphi \neq FF \wedge \varphi \neq FT$
hence *no-T-F* ψ **using** *rel push-conn-insidec-in-c'-symb-no-T-F* **by** *blast*
hence *no-T-F-except-top-level* ψ **using** *no-T-F-no-T-F-except-top-level* **by** *blast*
}
ultimately show *no-T-F-except-top-level* ψ **by** *blast*
qed
}
moreover {
fix *ca* :: *'v connective* **and** ξ ξ' :: *'v propo list* **and** φ φ' :: *'v propo*
assume *rel*: *propo-rew-step* (*push-conn-inside c c'*) φ φ'
assume *corr*: *wf-conn ca* (ξ @ φ # ξ')
hence *c*: $ca \neq CT \wedge ca \neq CF$ **by** *auto*
assume *no-T-F*: *no-T-F-symb-except-toplevel* (*conn ca* (ξ @ φ # ξ'))

have *no-T-F-symb-except-toplevel* (*conn ca* (ξ @ φ′ # ξ′))
proof
have *c*: *ca* ≠ *CT* ∧ *ca* ≠ *CF* **using** *corr* **by** *auto*
have ζ: ∀ζ∈ *set* (ξ @ φ # ξ′). ζ≠*FT* ∧ ζ ≠ *FF*
using *corr no-T-F no-T-F-symb-except-toplevel-if-is-a-true-false* **by** *blast*
hence φ ≠ *FT* ∧ φ ≠ *FF* **by** *auto*
from *rel this* **have** φ′ ≠ *FT* ∧ φ′ ≠ *FF*
apply (*induct rule*: *propo-rew-step.induct*)
by (*metis append-is-Nil-conv conn.simps*(*2*) *conn-inj list.distinct*(*1*)
wf-conn-helper-facts(*3*) *wf-conn-list*(*1*) *wf-conn-no-arity-change*
wf-conn-no-arity-change-helper push-conn-inside-not-true-false)+
hence ∀ζ ∈ *set* (ξ @ φ′ # ξ′). ζ ≠ *FT* ∧ ζ ≠ *FF* **using** ζ **by** *auto*
moreover have *wf-conn ca* (ξ @ φ′ # ξ′)
using *corr wf-conn-no-arity-change* **by** (*metis wf-conn-no-arity-change-helper*)
ultimately show *no-T-F-symb* (*conn ca* (ξ @ φ′ # ξ′)) **using** *no-T-F-symb.intros c* **by** *metis*
qed
}
ultimately show *no-T-F-except-top-level* ψ
using *full-propo-rew-step-inv-stay′*[*of push-conn-inside c c′ no-T-F-symb-except-toplevel*]
assms **unfolding** *no-T-F-except-top-level-def full-unfold* **by** *metis*

next
{
fix φ ψ :: ′*v propo*
have *H*: *push-conn-inside c c′* φ ψ ⟹ *no-equiv* φ ⟹ *no-equiv* ψ
by (*induct* φ ψ *rule*: *push-conn-inside.induct*, *auto*)
}
thus *no-equiv* ψ
using *full-propo-rew-step-inv-stay-conn*[*of push-conn-inside c c′ no-equiv-symb*] *assms*
no-equiv-symb-conn-characterization **unfolding** *no-equiv-def* **by** *metis*

next
{
fix φ ψ :: ′*v propo*
have *H*: *push-conn-inside c c′* φ ψ ⟹ *no-imp* φ ⟹ *no-imp* ψ
by (*induct* φ ψ *rule*: *push-conn-inside.induct*, *auto*)
}
thus *no-imp* ψ
using *full-propo-rew-step-inv-stay-conn*[*of push-conn-inside c c′ no-imp-symb*] *assms*
no-imp-symb-conn-characterization **unfolding** *no-imp-def* **by** *metis*
qed

lemma *push-conn-inside-full-propo-rew-step*:
fixes φ ψ :: ′*v propo*
assumes
no-equiv φ **and**
no-imp φ **and**
full (*propo-rew-step* (*push-conn-inside c c′*)) φ ψ **and**
no-T-F-except-top-level φ **and**
simple-not φ **and**
c = *CAnd* ∨ *c* = *COr* **and**
c′ = *CAnd* ∨ *c′* = *COr*
shows *c-in-c′-only c c′* ψ
using *c-in-c′-symb-rew assms full-propo-rew-step-subformula* **by** *blast*

8.5.1 Only one type of connective in the formula (+ not)

inductive *only-c-inside-symb* :: $'v$ *connective* $\Rightarrow$ $'v$ *propo* $\Rightarrow$ *bool* **for** *c*:: $'v$ *connective* **where**
simple-only-c-inside[*simp*]: *simple* $\varphi \Longrightarrow$ *only-c-inside-symb c* φ |
simple-cnot-only-c-inside[*simp*]: *simple* $\varphi \Longrightarrow$ *only-c-inside-symb c* (*FNot* φ) |
only-c-inside-into-only-c-inside: *wf-conn c l* $\Longrightarrow$ *only-c-inside-symb c* (*conn c l*)

lemma *only-c-inside-symb-simp*[*simp*]:
only-c-inside-symb c FF only-c-inside-symb c FT only-c-inside-symb c (*FVar x*) **by** *auto*

definition *only-c-inside* **where** *only-c-inside c* = *all-subformula-st* (*only-c-inside-symb c*)

lemma *only-c-inside-symb-decomp*:
only-c-inside-symb c $\psi \longleftrightarrow$ (*simple* ψ
$\lor$ ($\exists$ φ'. ψ = *FNot* φ' $\land$ *simple* φ')
$\lor$ ($\exists$ *l*. ψ = *conn c l* $\land$ *wf-conn c l*))
by (*auto simp add*: *only-c-inside-symb.intros*(*3*)) (*induct rule*: *only-c-inside-symb.induct*, *auto*)

lemma *only-c-inside-symb-decomp-not*[*simp*]:
fixes *c* :: $'v$ *connective*
assumes *c*: $c \neq$ *CNot*
shows *only-c-inside-symb c* (*FNot* ψ) $\longleftrightarrow$ *simple* ψ
apply (*auto simp add*: *only-c-inside-symb.intros*(*3*))
by (*induct FNot* ψ *rule*: *only-c-inside-symb.induct*, *auto simp add*: *wf-conn-list*(*8*) *c*)

lemma *only-c-inside-decomp-not*[*simp*]:
assumes *c*: $c \neq$ *CNot*
shows *only-c-inside c* (*FNot* ψ) $\longleftrightarrow$ *simple* ψ
by (*metis* (*no-types*, *hide-lams*) *all-subformula-st-def all-subformula-st-test-symb-true-phi c*
only-c-inside-def only-c-inside-symb-decomp-not simple-only-c-inside
subformula-conn-decomp-simple)

lemma *only-c-inside-decomp*:
only-c-inside c $\varphi \longleftrightarrow$
($\forall \psi$. $\psi \preceq \varphi \longrightarrow$ (*simple* ψ $\lor$ ($\exists$ φ'. ψ = *FNot* φ' $\land$ *simple* φ')
$\lor$ ($\exists$ *l*. ψ = *conn c l* $\land$ *wf-conn c l*)))
unfolding *only-c-inside-def* **by** (*auto simp add*: *all-subformula-st-def only-c-inside-symb-decomp*)

lemma *only-c-inside-c-c'-false*:
fixes *c c'* :: $'v$ *connective* **and** *l* :: $'v$ *propo list* **and** φ :: $'v$ *propo*
assumes *cc'*: $c \neq c'$ **and** *c*: *c* = *CAnd* $\lor$ *c* = *COr* **and** *c'*: *c'* = *CAnd* $\lor$ *c'* = *COr*
and *only*: *only-c-inside c* φ **and** *incl*: *conn c' l* $\preceq \varphi$ **and** *wf*: *wf-conn c' l*
shows *False*
proof −
let *?*ψ = *conn c' l*
have *simple ?*ψ $\lor$ ($\exists$ φ'. *?*ψ = *FNot* φ' $\land$ *simple* φ') $\lor$ ($\exists$ *l*. *?*ψ = *conn c l* $\land$ *wf-conn c l*)
using *only-c-inside-decomp only incl* **by** *blast*
moreover have $\neg$ *simple ?*ψ
using *wf simple-decomp* **by** (*metis c' connective.distinct*(*19*) *connective.distinct*(*7*,*9*,*21*,*29*,*31*)
wf-conn-list(*1*−*3*))
moreover
{
fix φ'

have *?ψ ≠ FNot φ′* **using** *c′ conn-inj-not(1) wf* **by** *blast*
}
ultimately obtain *l* :: *′v propo list* **where** *?ψ = conn c l ∧ wf-conn c l* **by** *metis*
hence *c = c′* **using** *conn-inj wf* **by** *metis*
thus *False* **using** *cc′* **by** *auto*
qed

lemma *only-c-inside-implies-c-in-c′-symb*:
assumes *δ*: *c ≠ c′* **and** *c*: *c = CAnd ∨ c = COr* **and** *c′*: *c′ = CAnd ∨ c′ = COr*
shows *only-c-inside c φ ⟹ c-in-c′-symb c c′ φ*
apply (*rule ccontr*)
apply (*cases rule*: *not-c-in-c′-symb.cases*, *auto*)
by (*metis δ c c′ connective.distinct(37,39) list.distinct(1) only-c-inside-c-c′-false subformula-in-binary-conn(1,2) wf-conn.simps*)+

lemma *c-in-c′-symb-decomp-level1*:
fixes *l* :: *′v propo list* **and** *c c′ ca* :: *′v connective*
shows *wf-conn ca l ⟹ ca ≠ c ⟹ c-in-c′-symb c c′ (conn ca l)*
proof −
have *not-c-in-c′-symb c c′ (conn ca l) ⟹ wf-conn ca l ⟹ ca = c*
by (*induct conn ca l rule*: *not-c-in-c′-symb.induct*, *auto simp add*: *conn-inj*)
thus *wf-conn ca l ⟹ ca ≠ c ⟹ c-in-c′-symb c c′ (conn ca l)* **by** *blast*
qed

lemma *only-c-inside-implies-c-in-c′-only*:
assumes *δ*: *c ≠ c′* **and** *c*: *c = CAnd ∨ c = COr* **and** *c′*: *c′ = CAnd ∨ c′ = COr*
shows *only-c-inside c φ ⟹ c-in-c′-only c c′ φ*
unfolding *c-in-c′-only-def all-subformula-st-def*
using *only-c-inside-implies-c-in-c′-symb*
by (*metis all-subformula-st-def assms(1) c c′ only-c-inside-def subformula-trans*)

lemma *c-in-c′-symb-c-implies-only-c-inside*:
assumes *δ*: *c = CAnd ∨ c = COr c′ = CAnd ∨ c′ = COr c ≠ c′* **and** *wf*: *wf-conn c [φ, ψ]*
and *inv*: *no-equiv (conn c l) no-imp (conn c l) simple-not (conn c l)*
shows *wf-conn c l ⟹ c-in-c′-only c c′ (conn c l) ⟹ (∀ψ∈ set l. only-c-inside c ψ)*
using *inv*
proof (*induct conn c l arbitrary*: *l rule*: *propo-induct-arity*)
case (*nullary x*)
thus *?case* **by** (*auto simp add*: *wf-conn-list assms*)
next
case (*unary φ la*)
hence *c = CNot ∧ la = [φ]* **by** (*metis (no-types) wf-conn-list(8)*)
thus *?case* **using** *assms(2) assms(1)* **by** *blast*
next
case (*binary φ1 φ2*)
note *IHφ1 = this(1)* **and** *IHφ2 = this(2)* **and** *φ = this(3)* **and** *only = this(5)* **and** *wf = this(4)*
and *no-equiv = this(6)* **and** *no-imp = this(7)* **and** *simple-not = this(8)*
hence *l*: *l = [φ1, φ2]* **by** (*meson wf-conn-list(4−7)*)
let *?φ = conn c l*

obtain *c1 l1 c2 l2* **where** *φ1*: *φ1 = conn c1 l1* **and** *wfφ1*: *wf-conn c1 l1*
and *φ2*: *φ2 = conn c2 l2* **and** *wfφ2*: *wf-conn c2 l2* **using** *exists-c-conn* **by** *metis*
hence *c-in-onlyφ1*: *c-in-c′-only c c′ (conn c1 l1)* **and** *c-in-c′-only c c′ (conn c2 l2)*

using *only l* **unfolding** *c-in-c′-only-def* **using** *assms(1)* **by** *auto*
have *incφ1*: *φ1* ≼ *?φ* **and** *incφ2*: *φ2* ≼ *?φ*
using *φ1 φ2 φ local.wf* **by** (*metis conn.simps(5−8) helper-fact subformula-in-binary-conn(1,2)*)+

have *c1-eq*: *c1* ≠ *CEq* **and** *c2-eq*: *c2* ≠ *CEq*
unfolding *no-equiv-def* **using** *incφ1 incφ2* **by** (*metis φ1 φ2 wfφ1 wfφ2 assms(1) no-equiv no-equiv-eq(1) no-equiv-symb.elims(3) no-equiv-symb-conn-characterization wf-conn-list(4,5) no-equiv-def subformula-all-subformula-st*)+
have *c1-imp*: *c1* ≠ *CImp* **and** *c2-imp*: *c2* ≠ *CImp*
using *no-imp* **by** (*metis φ1 φ2 all-subformula-st-decomp-explicit-imp(2,3) assms(1) conn.simps(5,6) l no-imp-Imp(1) no-imp-symb.elims(3) no-imp-symb-conn-characterization wfφ1 wfφ2 all-subformula-st-decomp no-imp-symb-conn-characterization*)+
have *c1c*: *c1* ≠ *c′*
proof
assume *c1c*: *c1* = *c′*
then obtain *ξ1 ξ2* **where** *l1*: *l1* = [*ξ1*, *ξ2*]
by (*metis assms(2) connective.distinct(37,39) helper-fact wfφ1 wf-conn.simps wf-conn-list-decomp(1−3)*)
have *c-in-c′-only c c′* (*conn c* [*conn c′ l1*, *φ2*]) **using** *c1c l only φ1* **by** *auto*
moreover have *not-c-in-c′-symb c c′* (*conn c* [*conn c′ l1*, *φ2*])
using *l1 φ1 c1c l local.wf not-c-in-c′-symb-l wfφ1* **by** *blast*
ultimately show *False* **using** *φ1 c1c l l1 local.wf not-c-in-c′-simp(4) wfφ1* **by** *blast*
qed
hence (*φ1* = *conn c l1* ∧ *wf-conn c l1*) ∨ (∃*ψ1*. *φ1* = *FNot ψ1*) ∨ *simple φ1*
by (*metis φ1 assms(1−3) c1-eq c1-imp simple.elims(3) wfφ1 wf-conn-list(4) wf-conn-list(5−7)*)
moreover {
assume *φ1* = *conn c l1* ∧ *wf-conn c l1*
hence *only-c-inside c φ1*
by (*metis IHφ1 φ1 all-subformula-st-decomp-imp incφ1 no-equiv no-equiv-def no-imp no-imp-def c-in-onlyφ1 only-c-inside-def only-c-inside-into-only-c-inside simple-not simple-not-def subformula-all-subformula-st*)
}
moreover {
assume ∃*ψ1*. *φ1* = *FNot ψ1*
then obtain *ψ1* **where** *φ1* = *FNot ψ1* **by** *metis*
hence *only-c-inside c φ1*
by (*metis all-subformula-st-def assms(1) connective.distinct(37,39) incφ1 only-c-inside-decomp-not simple-not simple-not-def simple-not-symb.simps(1)*)
}
moreover {
assume *simple φ1*
hence *only-c-inside c φ1*
by (*metis all-subformula-st-decomp-explicit(3) assms(1) connective.distinct(37,39) only-c-inside-decomp-not only-c-inside-def*)
}
ultimately have *only-c-insideφ1*: *only-c-inside c φ1* **by** *metis*

have *c-in-onlyφ2*: *c-in-c′-only c c′* (*conn c2 l2*)
using *only l φ2 wfφ2 assms* **unfolding** *c-in-c′-only-def* **by** *auto*
have *c2c*: *c2* ≠ *c′*
proof
assume *c2c*: *c2* = *c′*
then obtain *ξ1 ξ2* **where** *l2*: *l2* = [*ξ1*, *ξ2*]
by (*metis assms(2) wfφ2 wf-conn.simps connective.distinct(7,9,19,21,29,31,37,39)*)
hence *c-in-c′-symb c c′* (*conn c* [*φ1*, *conn c′ l2*])

using *c2c l only φ2 all-subformula-st-test-symb-true-phi* **unfolding** *c-in-c′-only-def* **by** *auto*
moreover have *not-c-in-c′-symb c c′* (*conn c* [*φ1*, *conn c′ l2*])
using *assms*(*1*) *c2c l2 not-c-in-c′-symb-r wfφ2 wf-conn-helper-facts*(*5*,*6*) **by** *metis*
ultimately show *False* **by** *auto*
qed
hence (*φ2* = *conn c l2* ∧ *wf-conn c l2*) ∨ (∃*ψ2*. *φ2* = *FNot ψ2*) ∨ *simple φ2*
using *c2-eq* **by** (*metis φ2 assms*(*1*−*3*) *c2-eq c2-imp simple.elims*(*3*) *wfφ2 wf-conn-list*(*4*−*7*))
moreover {
assume *φ2* = *conn c l2* ∧ *wf-conn c l2*
hence *only-c-inside c φ2*
by (*metis IHφ2 φ2 all-subformula-st-decomp incφ2 no-equiv no-equiv-def no-imp no-imp-def c-in-onlyφ2 only-c-inside-def only-c-inside-into-only-c-inside simple-not simple-not-def subformula-all-subformula-st*)
}
moreover {
assume ∃*ψ2*. *φ2* = *FNot ψ2*
then obtain *ψ2* **where** *φ2* = *FNot ψ2* **by** *metis*
hence *only-c-inside c φ2*
by (*metis all-subformula-st-def assms*(*1*−*3*) *connective.distinct*(*38*,*40*) *incφ2 only-c-inside-decomp-not simple-not simple-not-def simple-not-symb.simps*(*1*))
}
moreover {
assume *simple φ2*
hence *only-c-inside c φ2*
by (*metis all-subformula-st-decomp-explicit*(*3*) *assms*(*1*) *connective.distinct*(*37*,*39*) *only-c-inside-decomp-not only-c-inside-def*)
}
ultimately have *only-c-insideφ2*: *only-c-inside c φ2* **by** *metis*
show *?case* **using** *l only-c-insideφ1 only-c-insideφ2* **by** *auto*
qed

8.5.2 Push Conjunction

definition *pushConj* **where** *pushConj* = *push-conn-inside CAnd COr*

lemma *pushConj-consistent*: *preserves-un-sat pushConj*
unfolding *pushConj-def* **by** (*simp add*: *push-conn-inside-consistent*)

definition *and-in-or-symb* **where** *and-in-or-symb* = *c-in-c′-symb CAnd COr*

definition *and-in-or-only* **where**
and-in-or-only = *all-subformula-st* (*c-in-c′-symb CAnd COr*)

lemma *pushConj-inv*:
fixes *φ ψ* :: *′v propo*
assumes *full* (*propo-rew-step pushConj*) *φ ψ*
and *no-equiv φ* **and** *no-imp φ* **and** *no-T-F-except-top-level φ* **and** *simple-not φ*
shows *no-equiv ψ* **and** *no-imp ψ* **and** *no-T-F-except-top-level ψ* **and** *simple-not ψ*
using *push-conn-inside-inv assms* **unfolding** *pushConj-def* **by** *metis+*

lemma *pushConj-full-propo-rew-step*:
fixes *φ ψ* :: *′v propo*
assumes
no-equiv φ **and**
no-imp φ **and**

full (*propo-rew-step pushConj*) φ ψ **and**
no-T-F-except-top-level φ **and**
simple-not φ
shows *and-in-or-only* ψ
using *assms push-conn-inside-full-propo-rew-step*
unfolding *pushConj-def and-in-or-only-def c-in-c′-only-def* **by** (*metis* (*no-types*))

8.5.3 Push Disjunction

definition *pushDisj* **where** *pushDisj* = *push-conn-inside COr CAnd*

lemma *pushDisj-consistent*: *preserves-un-sat pushDisj*
unfolding *pushDisj-def* **by** (*simp add*: *push-conn-inside-consistent*)

definition *or-in-and-symb* **where** *or-in-and-symb* = *c-in-c′-symb COr CAnd*

definition *or-in-and-only* **where**
or-in-and-only = *all-subformula-st* (*c-in-c′-symb COr CAnd*)

lemma *not-or-in-and-only-or-and*[*simp*]:
~*or-in-and-only* (*FOr* (*FAnd* $\psi 1$ $\psi 2$) φ')
unfolding *or-in-and-only-def*
by (*metis all-subformula-st-test-symb-true-phi conn.simps*($5-6$) *not-c-in-c′-symb-l*
wf-conn-helper-facts(5) *wf-conn-helper-facts*(6))

lemma *pushDisj-inv*:
fixes φ ψ :: *′v propo*
assumes *full* (*propo-rew-step pushDisj*) φ ψ
and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ **and** *simple-not* φ
shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ **and** *simple-not* ψ
using *push-conn-inside-inv assms* **unfolding** *pushDisj-def* **by** *metis*+

lemma *pushDisj-full-propo-rew-step*:
fixes φ ψ :: *′v propo*
assumes
no-equiv φ **and**
no-imp φ **and**
full (*propo-rew-step pushDisj*) φ ψ **and**
no-T-F-except-top-level φ **and**
simple-not φ
shows *or-in-and-only* ψ
using *assms push-conn-inside-full-propo-rew-step*
unfolding *pushDisj-def or-in-and-only-def c-in-c′-only-def* **by** (*metis* (*no-types*))

9 The full transformations

9.1 Abstract Property characterizing that only some connective are inside the others

9.1.1 Definition

The normal is a super group of groups

inductive *grouped-by* :: *′a connective* $\Rightarrow$ *′a propo* $\Rightarrow$ *bool* **for** *c* **where**
simple-is-grouped[*simp*]: *simple* $\varphi \Longrightarrow$ *grouped-by* *c* φ |

simple-not-is-grouped[*simp*]: *simple* $\varphi \Longrightarrow$ *grouped-by c* (*FNot* φ) |
connected-is-group[*simp*]: *grouped-by c* $\varphi \Longrightarrow$ *grouped-by c* $\psi \Longrightarrow$ *wf-conn c* [φ, ψ]
  $\Longrightarrow$ *grouped-by c* (*conn c* [φ, ψ])

lemma *simple-clause*[*simp*]:
  grouped-by c FT
  grouped-by c FF
  grouped-by c (*FVar x*)
  grouped-by c (*FNot FT*)
  grouped-by c (*FNot FF*)
  grouped-by c (*FNot* (*FVar x*))
  by *simp+*

lemma *only-c-inside-symb-c-eq-c'*:
  only-c-inside-symb c (*conn c'* [φ*1*, φ*2*]) $\Longrightarrow$ $c' = CAnd \lor c' = COr \Longrightarrow$ *wf-conn c'* [φ*1*, φ*2*]
    $\Longrightarrow c' = c$
  by (*induct conn c'* [φ*1*, φ*2*] *rule*: *only-c-inside-symb.induct*, *auto simp add*: *conn-inj*)

lemma *only-c-inside-c-eq-c'*:
  only-c-inside c (*conn c'* [φ*1*, φ*2*]) $\Longrightarrow$ $c' = CAnd \lor c' = COr \Longrightarrow$ *wf-conn c'* [φ*1*, φ*2*] $\Longrightarrow c = c'$
  unfolding *only-c-inside-def all-subformula-st-def* **using** *only-c-inside-symb-c-eq-c' subformula-refl*
  by *blast*

lemma *only-c-inside-imp-grouped-by*:
  assumes *c*: $c \neq CNot$ **and** *c'*: $c' = CAnd \lor c' = COr$
  shows *only-c-inside c* $\varphi \Longrightarrow$ *grouped-by c* φ (**is** *?O* $\varphi \Longrightarrow$ *?G* φ)
proof (*induct* φ *rule*: *propo-induct-arity*)
  case (*nullary* φ *x*)
  thus *?G* φ **by** *auto*
next
  case (*unary* ψ)
  thus *?G* (*FNot* ψ) **by** (*auto simp add*: *c*)
next
  case (*binary* φ φ*1* φ*2*)
  note *IH*φ*1* = *this*(*1*) **and** *IH*φ*2* = *this*(*2*) **and** φ = *this*(*3*) **and** *only* = *this*(*4*)
  have φ*-conn*: φ = *conn c* [φ*1*, φ*2*] **and** *wf*: *wf-conn c* [φ*1*, φ*2*]
    proof −
      obtain *c'' l''* **where** φ*-c''*: φ = *conn c'' l''* **and** *wf*: *wf-conn c'' l''*
        using *exists-c-conn* **by** *metis*
      hence *l''*: *l''* = [φ*1*, φ*2*] **using** φ **by** (*metis wf-conn-list*(*4*−*7*))
      have *only-c-inside-symb c* (*conn c''* [φ*1*, φ*2*])
        using *only all-subformula-st-test-symb-true-phi*
        unfolding *only-c-inside-def* φ*-c'' l''* **by** *metis*
      hence $c = c''$
        by (*metis* φ φ*-c'' conn-inj conn-inj-not*(*2*) *l'' list.distinct*(*1*) *list.inject wf*
          only-c-inside-symb.cases simple.simps(*5*−*8*))
      thus φ = *conn c* [φ*1*, φ*2*] **and** *wf-conn c* [φ*1*, φ*2*] **using** φ*-c'' wf l''* **by** *auto*
    qed
  have *grouped-by c* φ*1* **using** *wf IH*φ*1 IH*φ*2* φ*-conn only* φ **unfolding** *only-c-inside-def* **by** *auto*
  moreover have *grouped-by c* φ*2*
    using *wf* φ *IH*φ*1 IH*φ*2* φ*-conn only* **unfolding** *only-c-inside-def* **by** *auto*
  ultimately show *?G* φ **using** φ*-conn connected-is-group local.wf* **by** *blast*
qed

lemma *grouped-by-false*:
grouped-by c (*conn c'* [φ, ψ]) $\Longrightarrow$ *c* $\neq$ *c'* $\Longrightarrow$ *wf-conn c'* [φ, ψ] $\Longrightarrow$ *False*
apply (*induct conn c'* [φ, ψ] *rule*: *grouped-by.induct*)
apply (*auto simp add*: *simple-decomp wf-conn-list*, *auto simp add*: *conn-inj*)
by (*metis list.distinct*(*1*) *list.sel*(*3*) *wf-conn-list*(*8*))+

Then the CNF form is a conjunction of clauses: every clause is in CNF form and two formulas in CNF form can be related by an and.

inductive *super-grouped-by*:: *'a connective* $\Rightarrow$ *'a connective* $\Rightarrow$ *'a propo* $\Rightarrow$ *bool* **for** *c c'* **where**
grouped-is-super-grouped[*simp*]: *grouped-by c* φ $\Longrightarrow$ *super-grouped-by c c'* φ |
connected-is-super-group: *super-grouped-by c c'* φ $\Longrightarrow$ *super-grouped-by c c'* ψ $\Longrightarrow$ *wf-conn c* [φ, ψ]
$\Longrightarrow$ *super-grouped-by c c'* (*conn c'* [φ, ψ])

lemma *simple-cnf*[*simp*]:
super-grouped-by c c' FT
super-grouped-by c c' FF
super-grouped-by c c' (*FVar x*)
super-grouped-by c c' (*FNot FT*)
super-grouped-by c c' (*FNot FF*)
super-grouped-by c c' (*FNot* (*FVar x*))
by *auto*

lemma *c-in-c'-only-super-grouped-by*:
assumes *c*: *c* = *CAnd* $\vee$ *c* = *COr* **and** *c'*: *c'* = *CAnd* $\vee$ *c'* = *COr* **and** *cc'*: *c* $\neq$ *c'*
shows *no-equiv* φ $\Longrightarrow$ *no-imp* φ $\Longrightarrow$ *simple-not* φ $\Longrightarrow$ *c-in-c'-only c c'* φ
$\Longrightarrow$ *super-grouped-by c c'* φ
(**is** *?NE* φ $\Longrightarrow$ *?NI* φ $\Longrightarrow$ *?SN* φ $\Longrightarrow$ *?C* φ $\Longrightarrow$ *?S* φ)
proof (*induct* φ *rule*: *propo-induct-arity*)
case (*nullary* φ *x*)
thus *?S* φ **by** *auto*
next
case (*unary* φ)
hence *simple-not-symb* (*FNot* φ)
using *all-subformula-st-test-symb-true-phi* **unfolding** *simple-not-def* **by** *blast*
hence φ = *FT* $\vee$ φ = *FF* $\vee$ ($\exists$ *x*. φ = *FVar x*) **by** (*case-tac* φ, *auto*)
thus *?S* (*FNot* φ) **by** *auto*
next
case (*binary* φ $\varphi 1$ $\varphi 2$)
note *IH*$\varphi 1$ = *this*(*1*) **and** *IH*$\varphi 2$ = *this*(*2*) **and** *no-equiv* = *this*(*4*) **and** *no-imp* = *this*(*5*)
and *simpleN* = *this*(*6*) **and** *c-in-c'-only* = *this*(*7*) **and** φ' = *this*(*3*)
{
assume φ = *FImp* $\varphi 1$ $\varphi 2$ $\vee$ φ = *FEq* $\varphi 1$ $\varphi 2$
hence *False* **using** *no-equiv no-imp* **by** *auto*
hence *?S* φ **by** *auto*
}
moreover {
assume φ: φ = *conn c'* [$\varphi 1$, $\varphi 2$] $\wedge$ *wf-conn c'* [$\varphi 1$, $\varphi 2$]
have *c-in-c'-only*: *c-in-c'-only c c'* $\varphi 1$ $\wedge$ *c-in-c'-only c c'* $\varphi 2$ $\wedge$ *c-in-c'-symb c c'* φ
using *c-in-c'-only* φ' **unfolding** *c-in-c'-only-def* **by** *auto*
have *super-grouped-by c c'* $\varphi 1$ **using** φ *c' no-equiv no-imp simpleN IH*$\varphi 1$ *c-in-c'-only* **by** *auto*
moreover have *super-grouped-by c c'* $\varphi 2$
using φ *c' no-equiv no-imp simpleN IH*$\varphi 2$ *c-in-c'-only* **by** *auto*
ultimately have *?S* φ

using *super-grouped-by.intros*(*2*) φ **by** (*metis c wf-conn-helper-facts*(*5*,*6*))
}
moreover {
assume φ: φ = *conn c* [φ*1*, φ*2*] ∧ *wf-conn c* [φ*1*, φ*2*]
hence *only-c-inside c* φ*1* ∧ *only-c-inside c* φ*2*
using *c-in-c′-symb-c-implies-only-c-inside c c′ c-in-c′-only list.set-intros*(*1*)
wf-conn-helper-facts(*5*,*6*) *no-equiv no-imp simpleN last-ConsL last-ConsR last-in-set*
list.distinct(*1*) **by** (*metis* (*no-types*, *hide-lams*) *cc′*)
hence *only-c-inside c* (*conn c* [φ*1*, φ*2*])
unfolding *only-c-inside-def* **using** φ
by (*simp add*: *only-c-inside-into-only-c-inside all-subformula-st-decomp*)
hence *grouped-by c* φ **using** φ *only-c-inside-imp-grouped-by c* **by** *blast*
hence *?S* φ **using** *super-grouped-by.intros*(*1*) **by** *metis*
}
ultimately show *?S* φ **by** (*metis* φ' *c c′ cc′ conn.simps*(*5*,*6*) *wf-conn-helper-facts*(*5*,*6*))
qed

9.2 Conjunctive Normal Form

definition *is-conj-with-TF* **where** *is-conj-with-TF* == *super-grouped-by COr CAnd*

lemma *or-in-and-only-conjunction-in-disj*:
shows *no-equiv* φ ⟹ *no-imp* φ ⟹ *simple-not* φ ⟹ *or-in-and-only* φ ⟹ *is-conj-with-TF* φ
using *c-in-c′-only-super-grouped-by*
unfolding *is-conj-with-TF-def or-in-and-only-def c-in-c′-only-def*
by (*simp add*: *c-in-c′-only-def c-in-c′-only-super-grouped-by*)

definition *is-cnf* **where** *is-cnf* φ == *is-conj-with-TF* φ ∧ *no-T-F-except-top-level* φ

9.2.1 Full CNF transformation

The full1 CNF transformation consists simply in chaining all the transformation defined before.

definition *cnf-rew* **where** *cnf-rew* =
(*full* (*propo-rew-step elim-equiv*)) *OO*
(*full* (*propo-rew-step elim-imp*)) *OO*
(*full* (*propo-rew-step elimTB*)) *OO*
(*full* (*propo-rew-step pushNeg*)) *OO*
(*full* (*propo-rew-step pushDisj*))

lemma *cnf-rew-consistent*: *preserves-un-sat cnf-rew*
by (*simp add*: *cnf-rew-def elimEquv-lifted-consistant elim-imp-lifted-consistant elimTB-consistent*
preserves-un-sat-OO pushDisj-consistent pushNeg-lifted-consistant)

lemma *cnf-rew-is-cnf*: *cnf-rew* φ φ' ⟹ *is-cnf* φ'
apply (*unfold cnf-rew-def OO-def*)
apply *auto*
proof −
fix φ φ*Eq* φ*Imp* φ*TB* φ*Neg* φ*Disj* :: *′v propo*
assume *Eq*: *full* (*propo-rew-step elim-equiv*) φ φ*Eq*
hence *no-equiv*: *no-equiv* φ*Eq* **using** *no-equiv-full-propo-rew-step-elim-equiv* **by** *blast*

assume *Imp*: *full* (*propo-rew-step elim-imp*) φ*Eq* φ*Imp*
hence *no-imp*: *no-imp* φ*Imp* **using** *no-imp-full-propo-rew-step-elim-imp* **by** *blast*
have *no-imp-inv*: *no-equiv* φ*Imp* **using** *no-equiv Imp elim-imp-inv* **by** *blast*

assume *TB*: *full* (*propo-rew-step elimTB*) *φImp φTB*
hence *noTB*: *no-T-F-except-top-level φTB*
using *no-imp-inv no-imp elimTB-full-propo-rew-step* **by** *blast*
have *noTB-inv*: *no-equiv φTB no-imp φTB* **using** *elimTB-inv TB no-imp no-imp-inv* **by** *blast+*

assume *Neg*: *full* (*propo-rew-step pushNeg*) *φTB φNeg*
hence *noNeg*: *simple-not φNeg*
using *noTB-inv noTB pushNeg-full-propo-rew-step* **by** *blast*
have *noNeg-inv*: *no-equiv φNeg no-imp φNeg no-T-F-except-top-level φNeg*
using *pushNeg-inv Neg noTB noTB-inv* **by** *blast+*

assume *Disj*: *full* (*propo-rew-step pushDisj*) *φNeg φDisj*
hence *no-Disj*: *or-in-and-only φDisj*
using *noNeg-inv noNeg pushDisj-full-propo-rew-step* **by** *blast*
have *noDisj-inv*: *no-equiv φDisj no-imp φDisj no-T-F-except-top-level φDisj*
simple-not φDisj
using *pushDisj-inv Disj noNeg noNeg-inv* **by** *blast+*

moreover have *is-conj-with-TF φDisj*
using *or-in-and-only-conjunction-in-disj noDisj-inv no-Disj* **by** *blast*
ultimately show *is-cnf φDisj* **unfolding** *is-cnf-def* **by** *blast*
qed

9.3 Disjunctive Normal Form

definition *is-disj-with-TF* **where** *is-disj-with-TF* ≡ *super-grouped-by CAnd COr*

lemma *and-in-or-only-conjunction-in-disj*:
shows *no-equiv φ* ⟹ *no-imp φ* ⟹ *simple-not φ* ⟹ *and-in-or-only φ* ⟹ *is-disj-with-TF φ*
using *c-in-c′-only-super-grouped-by*
unfolding *is-disj-with-TF-def and-in-or-only-def c-in-c′-only-def*
by (*simp add*: *c-in-c′-only-def c-in-c′-only-super-grouped-by*)

definition *is-dnf* :: *′a propo* ⇒ *bool* **where**
is-dnf φ ⟷ *is-disj-with-TF φ* ∧ *no-T-F-except-top-level φ*

9.3.1 Full DNF transform

The full1 DNF transformation consists simply in chaining all the transformation defined before.

definition *dnf-rew* **where** *dnf-rew* ≡
(*full* (*propo-rew-step elim-equiv*)) *OO*
(*full* (*propo-rew-step elim-imp*)) *OO*
(*full* (*propo-rew-step elimTB*)) *OO*
(*full* (*propo-rew-step pushNeg*)) *OO*
(*full* (*propo-rew-step pushConj*))

lemma *dnf-rew-consistent*: *preserves-un-sat dnf-rew*
by (*simp add*: *dnf-rew-def elimEquv-lifted-consistant elim-imp-lifted-consistant elimTB-consistent preserves-un-sat-OO pushConj-consistent pushNeg-lifted-consistant*)

theorem *dnf-transformation-correction*:
dnf-rew φ φ′ ⟹ *is-dnf φ′*
apply (*unfold dnf-rew-def OO-def*)
by (*meson and-in-or-only-conjunction-in-disj elimTB-full-propo-rew-step elimTB-inv*(*1*,*2*)

elim-imp-inv is-dnf-def no-equiv-full-propo-rew-step-elim-equiv no-imp-full-propo-rew-step-elim-imp pushConj-full-propo-rew-step pushConj-inv(*1*−*4*) *pushNeg-full-propo-rew-step pushNeg-inv*(*1*−*3*))

10 More aggressive simplifications: Removing true and false at the beginning

10.1 Transformation

We should remove *FT* and *FF* at the beginning and not in the middle of the algorithm. To do this, we have to use more rules (one for each connective):

inductive *elimTBFull* **where**
ElimTBFull1[*simp*]: *elimTBFull* (*FAnd* φ *FT*) φ |
ElimTBFull1'[*simp*]: *elimTBFull* (*FAnd FT* φ) φ |

ElimTBFull2[*simp*]: *elimTBFull* (*FAnd* φ *FF*) *FF* |
ElimTBFull2'[*simp*]: *elimTBFull* (*FAnd FF* φ) *FF* |

ElimTBFull3[*simp*]: *elimTBFull* (*FOr* φ *FT*) *FT* |
ElimTBFull3'[*simp*]: *elimTBFull* (*FOr FT* φ) *FT* |

ElimTBFull4[*simp*]: *elimTBFull* (*FOr* φ *FF*) φ |
ElimTBFull4'[*simp*]: *elimTBFull* (*FOr FF* φ) φ |

ElimTBFull5[*simp*]: *elimTBFull* (*FNot FT*) *FF* |
ElimTBFull5'[*simp*]: *elimTBFull* (*FNot FF*) *FT* |

ElimTBFull6-l[*simp*]: *elimTBFull* (*FImp FT* φ) φ |
ElimTBFull6-l'[*simp*]: *elimTBFull* (*FImp FF* φ) *FT* |
ElimTBFull6-r[*simp*]: *elimTBFull* (*FImp* φ *FT*) *FT* |
ElimTBFull6-r'[*simp*]: *elimTBFull* (*FImp* φ *FF*) (*FNot* φ) |

ElimTBFull7-l[*simp*]: *elimTBFull* (*FEq FT* φ) φ |
ElimTBFull7-l'[*simp*]: *elimTBFull* (*FEq FF* φ) (*FNot* φ) |
ElimTBFull7-r[*simp*]: *elimTBFull* (*FEq* φ *FT*) φ |
ElimTBFull7-r'[*simp*]: *elimTBFull* (*FEq* φ *FF*) (*FNot* φ)

The transformation is still consistent.

lemma *elimTBFull-consistent*: *preserves-un-sat elimTBFull*
proof −
  {
    fix φ ψ:: *'b propo*
    have *elimTBFull* φ $\psi \Longrightarrow \forall A.\ A \models \varphi \longleftrightarrow A \models \psi$
      by (*induct-tac rule*: *elimTBFull.inducts*, *auto*)
  }
  thus *?thesis* **using** *preserves-un-sat-def* **by** *auto*
qed

Contrary to the theorem ⟦*no-equiv* $?\varphi$; *no-imp* $?\varphi$; $?\psi \preceq ?\varphi$; $\neg$ *no-T-F-symb-except-toplevel* $?\psi$⟧ $\Longrightarrow \exists \psi'.$ *elimTB* $?\psi$ ψ', we do not need the assumption *no-equiv* φ and *no-imp* φ, since our transformation is more general.

lemma *no-T-F-symb-except-toplevel-step-exists'*:
  fixes φ :: *'v propo*

shows $\psi \preceq \varphi \Longrightarrow \neg$ *no-T-F-symb-except-toplevel* $\psi \Longrightarrow \exists \psi'.$ *elimTBFull* ψ ψ'
proof (*induct* ψ *rule*: *propo-induct-arity*)
case (*nullary* φ')
hence *False* **using** *no-T-F-symb-except-toplevel-true* *no-T-F-symb-except-toplevel-false* **by** *auto*
thus *Ex* (*elimTBFull* φ') **by** *blast*
next
case (*unary* ψ)
hence $\psi = FF \lor \psi = FT$ **using** *no-T-F-symb-except-toplevel-not-decom* **by** *blast*
thus *Ex* (*elimTBFull* (*FNot* ψ)) **using** *ElimTBFull5* *ElimTBFull5'* **by** *blast*
next
case (*binary* φ' $\psi 1$ $\psi 2$)
hence $\psi 1 = FT \lor \psi 2 = FT \lor \psi 1 = FF \lor \psi 2 = FF$
by (*metis binary-connectives-def conn.simps*(*5*−*8*) *insertI1 insert-commute*
no-T-F-symb-except-toplevel-bin-decom binary.hyps(*3*))
thus *Ex* (*elimTBFull* φ') **using** *elimTBFull.intros binary.hyps*(*3*) **by** *blast*
qed

The same applies here. We do not need the assumption, but the deep link between $\neg$ *no-T-F-except-top-level* φ and the existence of a rewriting step, still exists.

lemma *no-T-F-except-top-level-rew'*:
fixes φ :: *'v propo*
assumes *noTB*: $\neg$ *no-T-F-except-top-level* φ
shows $\exists \psi\ \psi'.\ \psi \preceq \varphi \land$ *elimTBFull* ψ ψ'
proof −
have *test-symb-false-nullary*:
$\forall x.$ *no-T-F-symb-except-toplevel* (*FF*:: *'v propo*) $\land$ *no-T-F-symb-except-toplevel FT*
$\land$ *no-T-F-symb-except-toplevel* (*FVar* (*x*:: *'v*))
by *auto*
moreover {
fix *c*:: *'v connective* **and** *l* :: *'v propo list* **and** ψ :: *'v propo*
have *H*: *elimTBFull* (*conn c l*) $\psi \Longrightarrow \neg$*no-T-F-symb-except-toplevel* (*conn c l*)
by (*case-tac* (*conn c l*) *rule*: *elimTBFull.cases*, *simp-all*)
}
ultimately show *?thesis*
using *no-test-symb-step-exists*[*of no-T-F-symb-except-toplevel* φ *elimTBFull*] *noTB*
no-T-F-symb-except-toplevel-step-exists' **unfolding** *no-T-F-except-top-level-def* **by** *metis*
qed

lemma *elimTBFull-full-propo-rew-step*:
fixes φ ψ :: *'v propo*
assumes *full* (*propo-rew-step elimTBFull*) φ ψ
shows *no-T-F-except-top-level* ψ
using *full-propo-rew-step-subformula no-T-F-except-top-level-rew' assms* **by** *fastforce*

10.2 More invariants

As the aim is to use the transformation as the first transformation, we have to show some more invariants for *elim-equiv* and *elim-imp*. For the other transformation, we have already proven it.

lemma *propo-rew-step-ElimEquiv-no-T-F*: *propo-rew-step elim-equiv* φ $\psi \Longrightarrow$ *no-T-F* $\varphi \Longrightarrow$ *no-T-F* ψ
proof (*induct rule*: *propo-rew-step.induct*)
fix φ' :: *'v propo* **and** ψ' :: *'v propo*

assume *a1*: *no-T-F* φ'
assume *a2*: *elim-equiv* φ' ψ'
have $\forall$ *x0 x1*. ($\neg$ *elim-equiv* (*x1* :: *'v propo*) *x0* $\vee$ ($\exists$ *v2 v3 v4 v5 v6 v7*. *x1* = *FEq v2 v3*
$\wedge$ *x0* = *FAnd* (*FImp v4 v5*) (*FImp v6 v7*) $\wedge$ *v2* = *v4* $\wedge$ *v4* = *v7* $\wedge$ *v3* = *v5* $\wedge$ *v3* = *v6*))
= ($\neg$ *elim-equiv x1 x0* $\vee$ ($\exists$ *v2 v3 v4 v5 v6 v7*. *x1* = *FEq v2 v3*
$\wedge$ *x0* = *FAnd* (*FImp v4 v5*) (*FImp v6 v7*) $\wedge$ *v2* = *v4* $\wedge$ *v4* = *v7* $\wedge$ *v3* = *v5* $\wedge$ *v3* = *v6*))
by *meson*
hence $\forall$ *p pa*. $\neg$ *elim-equiv* (*p* :: *'v propo*) *pa* $\vee$ ($\exists$ *pb pc pd pe pf pg*. *p* = *FEq pb pc*
$\wedge$ *pa* = *FAnd* (*FImp pd pe*) (*FImp pf pg*) $\wedge$ *pb* = *pd* $\wedge$ *pd* = *pg* $\wedge$ *pc* = *pe* $\wedge$ *pc* = *pf*)
using *elim-equiv.cases* **by** *force*
thus *no-T-F* ψ' **using** *a1 a2* **by** *fastforce*
next
fix φ φ' :: *'v propo* **and** ξ ξ' :: *'v propo list* **and** *c* :: *'v connective*
assume *rel*: *propo-rew-step elim-equiv* φ φ'
and *IH*: *no-T-F* $\varphi \Longrightarrow$ *no-T-F* φ'
and *corr*: *wf-conn c* (ξ @ φ # ξ')
and *no-T-F*: *no-T-F* (*conn c* (ξ @ φ # ξ'))
{
assume *c*: *c* = *CNot*
hence *empty*: ξ = [] ξ' = [] **using** *corr* **by** *auto*
hence *no-T-F* φ **using** *no-T-F c no-T-F-decomp-not* **by** *auto*
hence *no-T-F* (*conn c* (ξ @ φ' # ξ')) **using** *c empty no-T-F-comp-not IH* **by** *auto*
}
moreover {
assume *c*: *c* $\in$ *binary-connectives*
obtain *a b* **where** *ab*: ξ @ φ # ξ' = [*a*, *b*]
using *corr c list-length2-decomp wf-conn-bin-list-length* **by** *metis*
hence φ: φ = *a* $\vee$ φ = *b*
by (*metis append.simps*(*1*) *append-is-Nil-conv list.distinct*(*1*) *list.sel*(*3*) *nth-Cons-0*
tl-append2)
have ζ: $\forall\zeta\in$ *set* (ξ @ φ # ξ'). *no-T-F* ζ
using *no-T-F* **unfolding** *no-T-F-def* **using** *corr all-subformula-st-decomp* **by** *blast*

hence φ': *no-T-F* φ' **using** *ab IH* φ **by** *auto*
have *l'*: ξ @ φ' # ξ' = [φ', *b*] $\vee$ ξ @ φ' # ξ' = [*a*, φ']
by (*metis* (*no-types*, *hide-lams*) *ab append-Cons append-Nil append-Nil2 butlast.simps*(*2*)
butlast-append list.distinct(*1*) *list.sel*(*3*))
hence $\forall\zeta \in$ *set* (ξ @ φ' # ξ'). *no-T-F* ζ **using** ζ φ' *ab* **by** *fastforce*
moreover
have $\forall\zeta \in$ *set* (ξ @ φ # ξ'). $\zeta \neq$ *FT* $\wedge$ $\zeta \neq$ *FF*
using ζ *corr no-T-F no-T-F-except-top-level-false no-T-F-no-T-F-except-top-level* **by** *blast*
hence *no-T-F-symb* (*conn c* (ξ @ φ' # ξ'))
by (*metis* φ' *l' ab all-subformula-st-test-symb-true-phi c list.distinct*(*1*)
list.set-intros(*1*,*2*) *no-T-F-symb-except-toplevel-bin-decom*
no-T-F-symb-except-toplevel-no-T-F-symb no-T-F-symb-false(*1*,*2*) *no-T-F-def wf-conn-binary*
wf-conn-list(*1*,*2*))
ultimately have *no-T-F* (*conn c* (ξ @ φ' # ξ'))
by (*metis l' all-subformula-st-decomp-imp c no-T-F-def wf-conn-binary*)
}
moreover {
fix *x*
assume *c* = *CVar x* $\vee$ *c* = *CF* $\vee$ *c* = *CT*
hence *False* **using** *corr* **by** *auto*
hence *no-T-F* (*conn c* (ξ @ φ' # ξ')) **by** *auto*
}

ultimately show *no-T-F* (*conn c* (ξ @ φ' # ξ')) **using** *corr wf-conn.cases* **by** *metis*
qed

lemma *elim-equiv-inv'*:
fixes φ ψ :: *'v propo*
assumes *full* (*propo-rew-step elim-equiv*) φ ψ **and** *no-T-F-except-top-level* φ
shows*no-T-F-except-top-level* ψ
proof −
{
fix φ ψ :: *'v propo*
have *propo-rew-step elim-equiv* φ ψ $\Longrightarrow$ *no-T-F-except-top-level* φ
$\Longrightarrow$ *no-T-F-except-top-level* ψ
proof −
assume *rel*: *propo-rew-step elim-equiv* φ ψ
and *no*: *no-T-F-except-top-level* φ
{
assume $\varphi = FT \lor \varphi = FF$
from *rel this* **have** *False*
apply (*induct rule*: *propo-rew-step.induct*, *auto simp add*: *wf-conn-list*(*1*,*2*))
using *elim-equiv.simps* **by** *blast+*
hence *no-T-F-except-top-level* ψ **by** *blast*
}
moreover {
assume $\varphi \neq FT \land \varphi \neq FF$
hence *no-T-F* φ **by** (*metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb*)
hence *no-T-F* ψ **using** *propo-rew-step-ElimEquiv-no-T-F rel* **by** *blast*
hence *no-T-F-except-top-level* ψ **by** (*simp add*: *no-T-F-no-T-F-except-top-level*)
}
ultimately show *no-T-F-except-top-level* ψ **by** *metis*
qed
}
moreover {
fix *c* :: *'v connective* **and** ξ ξ' :: *'v propo list* **and** ζ ζ' :: *'v propo*
assume *rel*: *propo-rew-step elim-equiv* ζ ζ'
and *incl*: $\zeta \preceq \varphi$
and *corr*: *wf-conn c* (ξ @ ζ # ξ')
and *no-T-F*: *no-T-F-symb-except-toplevel* (*conn c* (ξ @ ζ # ξ'))
and *n*: *no-T-F-symb-except-toplevel* ζ'
have *no-T-F-symb-except-toplevel* (*conn c* (ξ @ ζ' # ξ'))
proof
have *p*: *no-T-F-symb* (*conn c* (ξ @ ζ # ξ'))
using *corr wf-conn-list*(*1*) *wf-conn-list*(*2*) *no-T-F-symb-except-toplevel-no-T-F-symb no-T-F*
by *blast*
have *l*: $\forall\varphi\in$*set* (ξ @ ζ # ξ'). $\varphi \neq FT \land \varphi \neq FF$
using *corr wf-conn-no-T-F-symb-iff p* **by** *blast*
from *rel incl* **have** $\zeta'\neq FT \land\zeta'\neq FF$
apply (*induction* ζ ζ' *rule*: *propo-rew-step.induct*)
apply (*cases rule*: *elim-equiv.cases*, *auto simp add*: *elim-equiv.simps*)
by (*metis append-is-Nil-conv list.distinct wf-conn-list*(*1*,*2*) *wf-conn-no-arity-change*
wf-conn-no-arity-change-helper)+
hence $\forall\varphi \in$ *set* (ξ @ ζ' # ξ'). $\varphi \neq FT \land \varphi \neq FF$ **using** *l* **by** *auto*
moreover have $c \neq CT \land c \neq CF$ **using** *corr* **by** *auto*
ultimately show *no-T-F-symb* (*conn c* (ξ @ ζ' # ξ'))
by (*metis corr wf-conn-no-arity-change wf-conn-no-arity-change-helper no-T-F-symb-comp*)
qed

}
ultimately show *no-T-F-except-top-level* ψ
using *full-propo-rew-step-inv-stay-with-inc*[*of elim-equiv no-T-F-symb-except-toplevel* φ]
assms subformula-refl **unfolding** *no-T-F-except-top-level-def* **by** *metis*
qed

lemma *propo-rew-step-ElimImp-no-T-F*: *propo-rew-step elim-imp* φ $\psi \Longrightarrow$ *no-T-F* $\varphi \Longrightarrow$ *no-T-F* ψ
proof (*induct rule*: *propo-rew-step.induct*)
case (*global-rel* φ' ψ')
thus *no-T-F* ψ'
using *elim-imp.cases no-T-F-comp-not no-T-F-decomp*(*1*,*2*)
by (*metis no-T-F-comp-expanded-explicit*(*2*))
next
case (*propo-rew-one-step-lift* φ φ' *c* ξ ξ')
note *rel* = *this*(*1*) **and** *IH* = *this*(*2*) **and** *corr* = *this*(*3*) **and** *no-T-F* = *this*(*4*)
{
assume *c*: *c* = *CNot*
hence *empty*: $\xi = []$ $\xi' = []$ **using** *corr* **by** *auto*
hence *no-T-F* φ **using** *no-T-F c no-T-F-decomp-not* **by** *auto*
hence *no-T-F* (*conn c* (ξ @ φ' # ξ')) **using** *c empty no-T-F-comp-not IH* **by** *auto*
}
moreover {
assume *c*: $c \in$ *binary-connectives*
then obtain *a b* **where** *ab*: ξ @ φ # $\xi' = [a, b]$
using *corr list-length2-decomp wf-conn-bin-list-length* **by** *metis*
hence φ: $\varphi = a \vee \varphi = b$
by (*metis append-self-conv2 wf-conn-list-decomp*(*4*) *wf-conn-unary list.discI list.sel*(*3*)
nth-Cons-0 tl-append2)
have ζ: $\forall \zeta \in$ *set* (ξ @ φ # ξ'). *no-T-F* ζ **using** *ab c propo-rew-one-step-lift.prems* **by** *auto*

hence φ': *no-T-F* φ'
using *ab IH* φ *corr no-T-F no-T-F-def all-subformula-st-decomp-explicit* **by** *auto*
have χ: ξ @ φ' # $\xi' = [\varphi', b] \vee \xi$ @ φ' # $\xi' = [a, \varphi']$
by (*metis* (*no-types*, *hide-lams*) *ab append-Cons append-Nil append-Nil2 butlast.simps*(*2*)
butlast-append list.distinct(*1*) *list.sel*(*3*))
hence $\forall \zeta \in$ *set* (ξ @ φ' # ξ'). *no-T-F* ζ **using** ζ φ' *ab* **by** *fastforce*
moreover
have *no-T-F* (*last* (ξ @ φ' # ξ')) **by** (*simp add*: *calculation*)
hence *no-T-F-symb* (*conn c* (ξ @ φ' # ξ'))
by (*metis* χ φ' ζ *ab all-subformula-st-test-symb-true-phi c last.simps list.distinct*(*1*)
list.set-intros(*1*) *no-T-F-bin-decomp no-T-F-def*)
ultimately have *no-T-F* (*conn c* (ξ @ φ' # ξ')) **using** *c* χ **by** *fastforce*
}
moreover {
fix *x*
assume *c* = *CVar x* $\vee$ *c* = *CF* $\vee$ *c* = *CT*
hence *False* **using** *corr* **by** *auto*
hence *no-T-F* (*conn c* (ξ @ φ' # ξ')) **by** *auto*
}
ultimately show *no-T-F* (*conn c* (ξ @ φ' # ξ')) **using** *corr wf-conn.cases* **by** *blast*
qed

lemma *elim-imp-inv'*:

fixes φ ψ :: *'v propo*
assumes *full* (*propo-rew-step elim-imp*) φ ψ **and** *no-T-F-except-top-level* φ
shows *no-T-F-except-top-level* ψ
proof −
{
{
fix φ ψ :: *'v propo*
have *H*: *elim-imp* φ ψ $\Longrightarrow$ *no-T-F-except-top-level* φ $\Longrightarrow$ *no-T-F-except-top-level* ψ
by (*induct* φ ψ *rule*: *elim-imp.induct*, *auto*)
} **note** *H* = *this*
fix φ ψ :: *'v propo*
have *propo-rew-step elim-imp* φ ψ $\Longrightarrow$ *no-T-F-except-top-level* φ $\Longrightarrow$ *no-T-F-except-top-level* ψ
proof −
assume *rel*: *propo-rew-step elim-imp* φ ψ
and *no*: *no-T-F-except-top-level* φ
{
assume $\varphi = FT \vee \varphi = FF$
from *rel this* **have** *False*
apply (*induct rule*: *propo-rew-step.induct*)
by (*cases rule*: *elim-imp.cases*, *auto simp add*: *wf-conn-list*(*1*,*2*))
hence *no-T-F-except-top-level* ψ **by** *blast*
}
moreover {
assume $\varphi \neq FT \wedge \varphi \neq FF$
hence *no-T-F* φ **by** (*metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb*)
hence *no-T-F* ψ **using** *rel propo-rew-step-ElimImp-no-T-F* **by** *blast*
hence *no-T-F-except-top-level* ψ **by** (*simp add*: *no-T-F-no-T-F-except-top-level*)
}
ultimately show *no-T-F-except-top-level* ψ **by** *metis*
qed
}
moreover {
fix *c* :: *'v connective* **and** ξ ξ' :: *'v propo list* **and** ζ ζ' :: *'v propo*
assume *rel*: *propo-rew-step elim-imp* ζ ζ'
and *incl*: $\zeta \preceq \varphi$
and *corr*: *wf-conn c* (ξ @ ζ # ξ')
and *no-T-F*: *no-T-F-symb-except-toplevel* (*conn c* (ξ @ ζ # ξ'))
and *n*: *no-T-F-symb-except-toplevel* ζ'
have *no-T-F-symb-except-toplevel* (*conn c* (ξ @ ζ' # ξ'))
proof
have *p*: *no-T-F-symb* (*conn c* (ξ @ ζ # ξ'))
by (*simp add*: *corr no-T-F no-T-F-symb-except-toplevel-no-T-F-symb wf-conn-list*(*1*,*2*))

have *l*: $\forall\varphi\in$*set* (ξ @ ζ # ξ'). $\varphi \neq FT \wedge \varphi \neq FF$
using *corr wf-conn-no-T-F-symb-iff p* **by** *blast*
from *rel incl* **have** $\zeta' \neq FT \wedge \zeta' \neq FF$
apply (*induction* ζ ζ' *rule*: *propo-rew-step.induct*)
apply (*cases rule*: *elim-imp.cases*, *auto*)
using *wf-conn-list*(*1*,*2*) *wf-conn-no-arity-change wf-conn-no-arity-change-helper*
by (*metis append-is-Nil-conv list.distinct*(*1*))+
hence $\forall\varphi\in$*set* (ξ @ ζ' # ξ'). $\varphi \neq FT \wedge \varphi \neq FF$ **using** *l* **by** *auto*
moreover have $c \neq CT \wedge c \neq CF$ **using** *corr* **by** *auto*
ultimately show *no-T-F-symb* (*conn c* (ξ @ ζ' # ξ'))
using *corr wf-conn-no-arity-change no-T-F-symb-comp*
by (*metis wf-conn-no-arity-change-helper*)

    qed
  }
  ultimately show *no-T-F-except-top-level* ψ
    using *full-propo-rew-step-inv-stay-with-inc*[*of elim-imp no-T-F-symb-except-toplevel* φ]
    assms subformula-refl **unfolding** *no-T-F-except-top-level-def* **by** *metis*
qed

10.3 The new CNF and DNF transformation

The transformation is the same as before, but the order is not the same.

definition *dnf-rew'* :: *'a propo* ⇒ *'a propo* ⇒ *bool* **where** *dnf-rew'* ≡
  (*full* (*propo-rew-step elimTBFull*)) *OO*
  (*full* (*propo-rew-step elim-equiv*)) *OO*
  (*full* (*propo-rew-step elim-imp*)) *OO*
  (*full* (*propo-rew-step pushNeg*)) *OO*
  (*full* (*propo-rew-step pushConj*))

lemma *dnf-rew'-consistent*: *preserves-un-sat dnf-rew'*
  by (*simp add*: *dnf-rew'-def elimEquv-lifted-consistant elim-imp-lifted-consistant*
    elimTBFull-consistent preserves-un-sat-OO pushConj-consistent pushNeg-lifted-consistant)

theorem *cnf-transformation-correction*:
    dnf-rew' φ φ' $\Longrightarrow$ *is-dnf* φ'
  unfolding *dnf-rew'-def OO-def*
  by (*meson and-in-or-only-conjunction-in-disj elimTBFull-full-propo-rew-step elim-equiv-inv'*
    elim-imp-inv elim-imp-inv' is-dnf-def no-equiv-full-propo-rew-step-elim-equiv
    no-imp-full-propo-rew-step-elim-imp pushConj-full-propo-rew-step pushConj-inv(*1*−*4*)
    pushNeg-full-propo-rew-step pushNeg-inv(*1*−*3*))

Given all the lemmas before the CNF transformation is easy to prove:

definition *cnf-rew'* :: *'a propo* ⇒ *'a propo* ⇒ *bool* **where** *cnf-rew'* ≡
  (*full* (*propo-rew-step elimTBFull*)) *OO*
  (*full* (*propo-rew-step elim-equiv*)) *OO*
  (*full* (*propo-rew-step elim-imp*)) *OO*
  (*full* (*propo-rew-step pushNeg*)) *OO*
  (*full* (*propo-rew-step pushDisj*))

lemma *cnf-rew'-consistent*: *preserves-un-sat cnf-rew'*
  by (*simp add*: *cnf-rew'-def elimEquv-lifted-consistant elim-imp-lifted-consistant*
    elimTBFull-consistent preserves-un-sat-OO pushDisj-consistent pushNeg-lifted-consistant)

theorem *cnf'-transformation-correction*:
  cnf-rew' φ φ' $\Longrightarrow$ *is-cnf* φ'
  unfolding *cnf-rew'-def OO-def*
  by (*meson elimTBFull-full-propo-rew-step elim-equiv-inv' elim-imp-inv elim-imp-inv' is-cnf-def*
    no-equiv-full-propo-rew-step-elim-equiv no-imp-full-propo-rew-step-elim-imp
    or-in-and-only-conjunction-in-disj pushDisj-full-propo-rew-step pushDisj-inv(*1*−*4*)
    pushNeg-full-propo-rew-step pushNeg-inv(*1*) *pushNeg-inv*(*2*) *pushNeg-inv*(*3*))

end

11 Partial Clausal Logic

theory *Partial-Clausal-Logic*

imports *../lib/Clausal-Logic List-More*
begin

11.1 Clauses

Clauses are (finite) multisets of literals.

type-synonym *'a clause* = *'a literal multiset*
type-synonym *'v clauses* = *'v clause set*

11.2 Partial Interpretations

type-synonym *'a interp* = *'a literal set*

definition *true-lit* :: *'a interp* ⇒ *'a literal* ⇒ *bool* (**infix** ⊨*l 50*) **where**
  $I \models l\ L \longleftrightarrow L \in I$

declare *true-lit-def*[*simp*]

11.2.1 Consistency

definition *consistent-interp* :: *'a literal set* ⇒ *bool* **where**
consistent-interp $I = (\forall L.\ \neg(L \in I \wedge -\ L \in I))$

lemma *consistent-interp-empty*[*simp*]:
  consistent-interp {} **unfolding** *consistent-interp-def* **by** *auto*

lemma *consistent-interp-single*[*simp*]:
  consistent-interp {*L*} **unfolding** *consistent-interp-def* **by** *auto*

lemma *consistent-interp-subset*:
  assumes
    $A \subseteq B$ **and**
    consistent-interp B
  shows *consistent-interp A*
  using *assms* **unfolding** *consistent-interp-def* **by** *auto*

lemma *consistent-interp-change-insert*:
  $a \notin A \Longrightarrow -a \notin A \Longrightarrow$ *consistent-interp* (*insert* (−*a*) *A*) ⟷ *consistent-interp* (*insert a A*)
  unfolding *consistent-interp-def* **by** *fastforce*

lemma *consistent-interp-insert-pos*[*simp*]:
  $a \notin A \Longrightarrow$ *consistent-interp* (*insert a A*) ⟷ *consistent-interp* $A \wedge -a \notin A$
  unfolding *consistent-interp-def* **by** *auto*

lemma *consistent-interp-insert-not-in*:
  consistent-interp $A \Longrightarrow a \notin A \Longrightarrow -a \notin A \Longrightarrow$ *consistent-interp* (*insert a A*)
  unfolding *consistent-interp-def* **by** *auto*

11.2.2 Atoms

definition *atms-of-ms* :: *'a literal multiset set* ⇒ *'a set* **where**
atms-of-ms $\psi s = \bigcup$(*atms-of* ‘ ψs)

lemma *atms-of-msultiset*[*simp*]:
  atms-of (*mset a*) = *atm-of* ‘ *set a*

by (*induct a*) *auto*

lemma *atms-of-ms-mset-unfold*:
atms-of-ms (*mset* ‘ *b*) = ($\bigcup$*x*∈*b*. *atm-of* ‘ *set x*)
unfolding *atms-of-ms-def* **by** *simp*

definition *atms-of-s* :: *'a literal set* ⇒ *'a set* **where**
atms-of-s C = *atm-of* ‘ *C*

lemma *atms-of-ms-emtpy-set*[*simp*]:
atms-of-ms {} = {}
unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-memtpy*[*simp*]:
atms-of-ms {{#}} = {}
unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-mono*:
A ⊆ *B* ⟹ *atms-of-ms A* ⊆ *atms-of-ms B*
unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-finite*[*simp*]:
finite ψs ⟹ *finite* (*atms-of-ms ψs*)
unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-union*[*simp*]:
atms-of-ms (*ψs* ∪ *χs*) = *atms-of-ms ψs* ∪ *atms-of-ms χs*
unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-insert*[*simp*]:
atms-of-ms (*insert ψs χs*) = *atms-of ψs* ∪ *atms-of-ms χs*
unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-singleton*[*simp*]: *atms-of-ms* {*L*} = *atms-of L*
unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-atms-of-ms-mono*[*simp*]:
A ∈ *ψ* ⟹ *atms-of A* ⊆ *atms-of-ms ψ*
unfolding *atms-of-ms-def* **by** *fastforce*

lemma *atms-of-ms-single-set-mset-atns-of*[*simp*]:
atms-of-ms (*single* ‘ *set-mset B*) = *atms-of B*
unfolding *atms-of-ms-def atms-of-def* **by** *auto*

lemma *atms-of-ms-remove-incl*:
shows *atms-of-ms* (*Set.remove a ψ*) ⊆ *atms-of-ms ψ*
unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-remove-subset*:
atms-of-ms (*φ* − *ψ*) ⊆ *atms-of-ms φ*
unfolding *atms-of-ms-def* **by** *auto*

lemma *finite-atms-of-ms-remove-subset*[*simp*]:
finite (*atms-of-ms A*) ⟹ *finite* (*atms-of-ms* (*A* − *C*))
using *atms-of-ms-remove-subset*[*of A C*] *finite-subset* **by** *blast*

lemma *atms-of-ms-empty-iff*:
atms-of-ms $A = \{\} \longleftrightarrow A = \{\{\#\}\} \vee A = \{\}$
apply (*rule iffI*)
apply (*metis* (*no-types*, *lifting*) *atms-empty-iff-empty atms-of-atms-of-ms-mono insert-absorb singleton-iff singleton-insert-inj-eq′ subsetI subset-empty*)
apply *auto*[]
done

lemma *in-implies-atm-of-on-atms-of-ms*:
assumes $L \in\# C$ **and** $C \in N$
shows *atm-of* $L \in$ *atms-of-ms* N
using *atms-of-atms-of-ms-mono*[*of* C N] *assms* **by** (*simp add*: *atm-of-lit-in-atms-of subset-iff*)

lemma *in-plus-implies-atm-of-on-atms-of-ms*:
assumes $C+\{\#L\#\} \in N$
shows *atm-of* $L \in$ *atms-of-ms* N
using *in-implies-atm-of-on-atms-of-ms*[*of* C $+\{\#L\#\}$] *assms* **by** *auto*

lemma *in-m-in-literals*:
assumes $\{\#A\#\} + D \in \psi s$
shows *atm-of* $A \in$ *atms-of-ms* ψs
using *assms* **by** (*auto dest*: *atms-of-atms-of-ms-mono*)

lemma *atms-of-s-union*[*simp*]:
atms-of-s $(Ia \cup Ib) =$ *atms-of-s* $Ia \cup$ *atms-of-s* Ib
unfolding *atms-of-s-def* **by** *auto*

lemma *atms-of-s-single*[*simp*]:
atms-of-s $\{L\} = \{$*atm-of* $L\}$
unfolding *atms-of-s-def* **by** *auto*

lemma *atms-of-s-insert*[*simp*]:
atms-of-s (*insert* L Ib) $= \{$*atm-of* $L\} \cup$ *atms-of-s* Ib
unfolding *atms-of-s-def* **by** *auto*

lemma *in-atms-of-s-decomp*[*iff*]:
$P \in$ *atms-of-s* $I \longleftrightarrow$ (*Pos* $P \in I \vee$ *Neg* $P \in I$) (**is** $?P \longleftrightarrow ?Q$)
proof
assume $?P$
then show $?Q$ **unfolding** *atms-of-s-def* **by** (*metis image-iff literal.exhaust-sel*)
next
assume $?Q$
then show $?P$ **unfolding** *atms-of-s-def* **by** *force*
qed

lemma *atm-of-in-atm-of-set-in-uminus*:
atm-of $L' \in$ *atm-of* ‘ $B \Longrightarrow L' \in B \vee - L' \in B$
using *atms-of-s-def* **by** (*cases* L') *fastforce+*

11.2.3 Totality

definition *total-over-set* :: ′*a interp* ⇒ ′*a set* ⇒ *bool* **where**
total-over-set I $S = (\forall l \in S.$ *Pos* $l \in I \vee$ *Neg* $l \in I)$

definition *total-over-m* :: *'a literal set* ⇒ *'a clause set* ⇒ *bool* **where**
total-over-m I ψs = *total-over-set I* (*atms-of-ms ψs*)

lemma *total-over-set-empty*[*simp*]:
total-over-set I {}
unfolding *total-over-set-def* **by** *auto*

lemma *total-over-m-empty*[*simp*]:
total-over-m I {}
unfolding *total-over-m-def* **by** *auto*

lemma *total-over-set-single*[*iff*]:
total-over-set I {*L*} ⟷ (*Pos L* ∈ *I* ∨ *Neg L* ∈ *I*)
unfolding *total-over-set-def* **by** *auto*

lemma *total-over-set-insert*[*iff*]:
total-over-set I (*insert L Ls*) ⟷ ((*Pos L* ∈ *I* ∨ *Neg L* ∈ *I*) ∧ *total-over-set I Ls*)
unfolding *total-over-set-def* **by** *auto*

lemma *total-over-set-union*[*iff*]:
total-over-set I (*Ls* ∪ *Ls'*) ⟷ (*total-over-set I Ls* ∧ *total-over-set I Ls'*)
unfolding *total-over-set-def* **by** *auto*

lemma *total-over-m-subset*:
A ⊆ *B* ⟹ *total-over-m I B* ⟹ *total-over-m I A*
using *atms-of-ms-mono*[*of A*] **unfolding** *total-over-m-def total-over-set-def* **by** *auto*

lemma *total-over-m-sum*[*iff*]:
shows *total-over-m I* {*C* + *D*} ⟷ (*total-over-m I* {*C*} ∧ *total-over-m I* {*D*})
using *assms* **unfolding** *total-over-m-def total-over-set-def* **by** *auto*

lemma *total-over-m-union*[*iff*]:
total-over-m I (*A* ∪ *B*) ⟷ (*total-over-m I A* ∧ *total-over-m I B*)
unfolding *total-over-m-def total-over-set-def* **by** *auto*

lemma *total-over-m-insert*[*iff*]:
total-over-m I (*insert a A*) ⟷ (*total-over-set I* (*atms-of a*) ∧ *total-over-m I A*)
unfolding *total-over-m-def total-over-set-def* **by** *fastforce*

lemma *total-over-m-extension*:
fixes *I* :: *'v literal set* **and** *A* :: *'v clauses*
assumes *total*: *total-over-m I A*
shows ∃*I'*. *total-over-m* (*I* ∪ *I'*) (*A*∪*B*)
∧ (∀*x*∈*I'*. *atm-of x* ∈ *atms-of-ms B* ∧ *atm-of x* ∉ *atms-of-ms A*)
proof −
let *?I'* = {*Pos v* |*v*. *v*∈ *atms-of-ms B* ∧ *v* ∉ *atms-of-ms A*}
have (∀*x*∈*?I'*. *atm-of x* ∈ *atms-of-ms B* ∧ *atm-of x* ∉ *atms-of-ms A*) **by** *auto*
moreover have *total-over-m* (*I* ∪ *?I'*) (*A*∪*B*)
using *total* **unfolding** *total-over-m-def total-over-set-def* **by** *auto*
ultimately show *?thesis* **by** *blast*
qed

lemma *total-over-m-consistent-extension*:
fixes *I* :: *'v literal set* **and** *A* :: *'v clauses*
assumes *total*: *total-over-m I A*

and *cons*: *consistent-interp I*
shows $\exists I'$. *total-over-m* $(I \cup I')$ $(A \cup B)$
$\wedge$ $(\forall x{\in}I'$. *atm-of* $x \in$ *atms-of-ms* $B \wedge$ *atm-of* $x \notin$ *atms-of-ms* $A) \wedge$ *consistent-interp* $(I \cup I')$
proof −
let $?I' = \{Pos\ v\ |v.\ v{\in}$ *atms-of-ms* $B \wedge v \notin$ *atms-of-ms* $A \wedge Pos\ v \notin I \wedge Neg\ v \notin I\}$
have $(\forall x{\in}?I'$. *atm-of* $x \in$ *atms-of-ms* $B \wedge$ *atm-of* $x \notin$ *atms-of-ms* $A)$ **by** *auto*
moreover have *total-over-m* $(I \cup ?I')$ $(A{\cup}B)$
using *total* **unfolding** *total-over-m-def total-over-set-def* **by** *auto*
moreover have *consistent-interp* $(I \cup ?I')$
using *cons* **unfolding** *consistent-interp-def* **by** (*intro allI*) (*case-tac L*, *auto*)
ultimately show *?thesis* **by** *blast*
qed

lemma *total-over-set-atms-of*[*simp*]:
total-over-set Ia (*atms-of-s Ia*)
unfolding *total-over-set-def atms-of-s-def* **by** (*metis image-iff literal.exhaust-sel*)

lemma *total-over-set-literal-defined*:
assumes $\{\#A\#\} + D \in \psi s$
and *total-over-set I* (*atms-of-ms* ψs)
shows $A \in I \vee -A \in I$
using *assms* **unfolding** *total-over-set-def* **by** (*metis* (*no-types*) *Neg-atm-of-iff in-m-in-literals literal.collapse*(*1*) *uminus-Neg uminus-Pos*)

lemma *tot-over-m-remove*:
assumes *total-over-m* $(I \cup \{L\})$ $\{\psi\}$
and L: $\neg\ L \in\#\ \psi\ -L \notin\#\ \psi$
shows *total-over-m I* $\{\psi\}$
unfolding *total-over-m-def total-over-set-def*
proof
fix l
assume l: $l \in$ *atms-of-ms* $\{\psi\}$
then have $Pos\ l \in I \vee Neg\ l \in I \vee l =$ *atm-of* L
using *assms* **unfolding** *total-over-m-def total-over-set-def* **by** *auto*
moreover have *atm-of* $L \notin$ *atms-of-ms* $\{\psi\}$
proof (*rule ccontr*)
assume $\neg$ *?thesis*
then have *atm-of* $L \in$ *atms-of* ψ **by** *auto*
then have *Pos* (*atm-of* L) $\in\#\ \psi \vee$ *Neg* (*atm-of* L) $\in\#\ \psi$
using *atm-imp-pos-or-neg-lit* **by** *metis*
then have $L \in\#\ \psi \vee -\ L \in\#\ \psi$ **by** (*case-tac L*) *auto*
then show *False* **using** L **by** *auto*
qed
ultimately show $Pos\ l \in I \vee Neg\ l \in I$ **using** l **by** *metis*
qed

lemma *total-union*:
assumes *total-over-m I* ψ
shows *total-over-m* $(I \cup I')$ ψ
using *assms* **unfolding** *total-over-m-def total-over-set-def* **by** *auto*

lemma *total-union-2*:
assumes *total-over-m I* ψ
and *total-over-m* I' ψ'
shows *total-over-m* $(I \cup I')$ $(\psi \cup \psi')$

using *assms* **unfolding** *total-over-m-def total-over-set-def* **by** *auto*

11.2.4 Interpretations

definition *true-cls* :: *'a interp* ⇒ *'a clause* ⇒ *bool* (**infix** ⊨ *50*) **where**
$I \models C \longleftrightarrow (\exists L \in\# C.\ I \models l\ L)$

lemma *true-cls-empty*[*iff*]: $\neg\ I \models \{\#\}$
unfolding *true-cls-def* **by** *auto*

lemma *true-cls-singleton*[*iff*]: $I \models \{\#L\#\} \longleftrightarrow I \models l\ L$
unfolding *true-cls-def* **by** (*auto split*:*split-if-asm*)

lemma *true-cls-union*[*iff*]: $I \models C + D \longleftrightarrow I \models C \vee I \models D$
unfolding *true-cls-def* **by** *auto*

lemma *true-cls-mono-set-mset*: *set-mset* $C \subseteq$ *set-mset* $D \Longrightarrow I \models C \Longrightarrow I \models D$
unfolding *true-cls-def subset-eq Bex-mset-def* **by** (*metis mem-set-mset-iff*)

lemma *true-cls-mono-leD*[*dest*]: $A \subseteq\# B \Longrightarrow I \models A \Longrightarrow I \models B$
unfolding *true-cls-def* **by** *auto*

lemma
assumes $I \models \psi$
shows *true-cls-union-increase*[*simp*]: $I \cup I' \models \psi$
and *true-cls-union-increase'*[*simp*]: $I' \cup I \models \psi$
using *assms* **unfolding** *true-cls-def* **by** *auto*

lemma *true-cls-mono-set-mset-l*:
assumes $A \models \psi$
and $A \subseteq B$
shows $B \models \psi$
using *assms* **unfolding** *true-cls-def* **by** *auto*

lemma *true-cls-replicate-mset*[*iff*]: $I \models$ *replicate-mset* $n\ L \longleftrightarrow n \neq 0 \wedge I \models l\ L$
by (*induct n*) *auto*

lemma *true-cls-empty-entails*[*iff*]: $\neg\ \{\} \models N$
by (*auto simp add*: *true-cls-def*)

lemma *true-cls-not-in-remove*:
assumes $L \notin\# \chi$
and $I \cup \{L\} \models \chi$
shows $I \models \chi$
using *assms* **unfolding** *true-cls-def* **by** *auto*

definition *true-clss* :: *'a interp* ⇒ *'a clauses* ⇒ *bool* (**infix** ⊨*s 50*) **where**
$I \models s\ CC \longleftrightarrow (\forall C \in CC.\ I \models C)$

lemma *true-clss-empty*[*simp*]: $I \models s\ \{\}$
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-singleton*[*iff*]: $I \models s\ \{C\} \longleftrightarrow I \models C$
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-empty-entails-empty*[*iff*]: $\{\} \models s\ N \longleftrightarrow N = \{\}$

unfolding *true-clss-def* **by** (*auto simp add*: *true-cls-def*)

lemma *true-cls-insert-l* [*simp*]:
$M \models A \Longrightarrow insert\ L\ M \models A$
unfolding *true-cls-def* **by** *auto*

lemma *true-clss-union*[*iff*]: $I \models s\ CC \cup DD \longleftrightarrow I \models s\ CC \wedge I \models s\ DD$
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-insert*[*iff*]: $I \models s\ insert\ C\ DD \longleftrightarrow I \models C \wedge I \models s\ DD$
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-mono*: $DD \subseteq CC \Longrightarrow I \models s\ CC \Longrightarrow I \models s\ DD$
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-union-increase*[*simp*]:
assumes $I \models s\ \psi$
shows $I \cup I' \models s\ \psi$
using *assms* **unfolding** *true-clss-def* **by** *auto*

lemma *true-clss-union-increase'*[*simp*]:
assumes $I' \models s\ \psi$
shows $I \cup I' \models s\ \psi$
using *assms* **by** (*auto simp add*: *true-clss-def*)

lemma *true-clss-commute-l*:
$(I \cup I' \models s\ \psi) \longleftrightarrow (I' \cup I \models s\ \psi)$
by (*simp add*: *Un-commute*)

lemma *model-remove*[*simp*]: $I \models s\ N \Longrightarrow I \models s\ Set.remove\ a\ N$
by (*simp add*: *true-clss-def*)

lemma *model-remove-minus*[*simp*]: $I \models s\ N \Longrightarrow I \models s\ N - A$
by (*simp add*: *true-clss-def*)

lemma *notin-vars-union-true-cls-true-cls*:
assumes $\forall x \in I'.\ atm\text{-}of\ x \notin atms\text{-}of\text{-}ms\ A$
and $atms\text{-}of\ L \subseteq atms\text{-}of\text{-}ms\ A$
and $I \cup I' \models L$
shows $I \models L$
using *assms* **unfolding** *true-cls-def true-lit-def Bex-mset-def*
by (*metis Un-iff atm-of-lit-in-atms-of contra-subsetD*)

lemma *notin-vars-union-true-clss-true-clss*:
assumes $\forall x \in I'.\ atm\text{-}of\ x \notin atms\text{-}of\text{-}ms\ A$
and $atms\text{-}of\text{-}ms\ L \subseteq atms\text{-}of\text{-}ms\ A$
and $I \cup I' \models s\ L$
shows $I \models s\ L$
using *assms* **unfolding** *true-clss-def true-lit-def Ball-def*
by (*meson atms-of-atms-of-ms-mono notin-vars-union-true-cls-true-cls subset-trans*)

11.2.5 Satisfiability

definition *satisfiable* :: *'a clause set* $\Rightarrow$ *bool* **where**
$satisfiable\ CC \equiv \exists I.\ (I \models s\ CC \wedge consistent\text{-}interp\ I \wedge total\text{-}over\text{-}m\ I\ CC)$

lemma *satisfiable-single*[*simp*]:
satisfiable $\{\{\#L\#\}\}$
unfolding *satisfiable-def* **by** *fastforce*

abbreviation *unsatisfiable* :: *'a clause set* $\Rightarrow$ *bool* **where**
unsatisfiable CC $\equiv \neg$ *satisfiable CC*

lemma *satisfiable-decreasing*:
assumes *satisfiable* $(\psi \cup \psi')$
shows *satisfiable* ψ
using *assms total-over-m-union* **unfolding** *satisfiable-def* **by** *blast*

lemma *satisfiable-def-min*:
satisfiable CC
$\longleftrightarrow (\exists I.\ I \models s\ CC \wedge$ *consistent-interp* $I \wedge$ *total-over-m* $I\ CC \wedge$ *atm-of*'I = *atms-of-ms* $CC)$
(**is** *?sat* $\longleftrightarrow$ *?B*)
proof
assume *?B* **then show** *?sat* **by** (*auto simp add*: *satisfiable-def*)
next
assume *?sat*
then obtain *I* **where**
I-CC: $I \models s\ CC$ **and**
cons: *consistent-interp I* **and**
tot: *total-over-m I CC*
unfolding *satisfiable-def* **by** *auto*
let *?I* = $\{P.\ P \in I \wedge$ *atm-of* $P \in$ *atms-of-ms* $CC\}$

have *I-CC*: *?I* $\models s\ CC$
using *I-CC in-implies-atm-of-on-atms-of-ms* **unfolding** *true-clss-def Ball-def true-cls-def Bex-mset-def true-lit-def*
by *blast*

moreover have *cons*: *consistent-interp ?I*
using *cons* **unfolding** *consistent-interp-def* **by** *auto*
moreover have *total-over-m ?I CC*
using *tot* **unfolding** *total-over-m-def total-over-set-def* **by** *auto*
moreover
have *atms-CC-incl*: *atms-of-ms CC* $\subseteq$ *atm-of*'I
using *tot* **unfolding** *total-over-m-def total-over-set-def atms-of-ms-def*
by (*auto simp add*: *atms-of-def atms-of-s-def*[*symmetric*])
have *atm-of* ' *?I* = *atms-of-ms CC*
using *atms-CC-incl* **unfolding** *atms-of-ms-def* **by** *force*
ultimately show *?B* **by** *auto*
qed

11.2.6 Entailment for Multisets of Clauses

definition *true-cls-mset* :: *'a interp* $\Rightarrow$ *'a clause multiset* $\Rightarrow$ *bool* (**infix** $\models m$ *50*) **where**
$I \models m\ CC \longleftrightarrow (\forall C \in\#\ CC.\ I \models C)$

lemma *true-cls-mset-empty*[*simp*]: $I \models m\ \{\#\}$
unfolding *true-cls-mset-def* **by** *auto*

lemma *true-cls-mset-singleton*[*iff*]: $I \models m\ \{\#C\#\} \longleftrightarrow I \models C$
unfolding *true-cls-mset-def* **by** (*auto split*: *split-if-asm*)

lemma *true-cls-mset-union*[*iff*]: $I \models m\ CC + DD \longleftrightarrow I \models m\ CC \land I \models m\ DD$
  unfolding *true-cls-mset-def* **by** *fastforce*

lemma *true-cls-mset-image-mset*[*iff*]: $I \models m$ *image-mset f A* $\longleftrightarrow (\forall x \in\# A.\ I \models f\ x)$
  unfolding *true-cls-mset-def* **by** *fastforce*

lemma *true-cls-mset-mono*: *set-mset DD* $\subseteq$ *set-mset CC* $\Longrightarrow I \models m\ CC \Longrightarrow I \models m\ DD$
  unfolding *true-cls-mset-def subset-iff* **by** *auto*

lemma *true-clss-set-mset*[*iff*]: $I \models s$ *set-mset CC* $\longleftrightarrow I \models m\ CC$
  unfolding *true-clss-def true-cls-mset-def* **by** *auto*

lemma *true-cls-mset-increasing-r*[*simp*]:
  $I \models m\ CC \Longrightarrow I \cup J \models m\ CC$
  unfolding *true-cls-mset-def* **by** *auto*

theorem *true-cls-remove-unused*:
  assumes $I \models \psi$
  shows $\{v \in I.$ *atm-of* $v \in$ *atms-of* $\psi\} \models \psi$
  using *assms* **unfolding** *true-cls-def atms-of-def* **by** *auto*

theorem *true-clss-remove-unused*:
  assumes $I \models s\ \psi$
  shows $\{v \in I.$ *atm-of* $v \in$ *atms-of-ms* $\psi\} \models s\ \psi$
  unfolding *true-clss-def atms-of-def Ball-def*
proof (*intro allI impI*)
  fix x
  assume $x \in \psi$
  then have $I \models x$
    using *assms* **unfolding** *true-clss-def atms-of-def Ball-def* **by** *auto*

  then have $\{v \in I.$ *atm-of* $v \in$ *atms-of* $x\} \models x$
    by (*simp only*: *true-cls-remove-unused*[*of I*])
  moreover have $\{v \in I.$ *atm-of* $v \in$ *atms-of* $x\} \subseteq \{v \in I.$ *atm-of* $v \in$ *atms-of-ms* $\psi\}$
    using ‹$x \in \psi$› **by** (*auto simp add*: *atms-of-ms-def*)
  ultimately show $\{v \in I.$ *atm-of* $v \in$ *atms-of-ms* $\psi\} \models x$
    using *true-cls-mono-set-mset-l* **by** *blast*
qed

A simple application of the previous theorem:

lemma *true-clss-union-decrease*:
  assumes II': $I \cup I' \models \psi$
  and H: $\forall v \in I'.$ *atm-of* $v \notin$ *atms-of* ψ
  shows $I \models \psi$
proof −
  let $?I = \{v \in I \cup I'.$ *atm-of* $v \in$ *atms-of* $\psi\}$
  have $?I \models \psi$ **using** *true-cls-remove-unused* II' **by** *blast*
  moreover have $?I \subseteq I$ **using** H **by** *auto*
  ultimately show *?thesis* **using** *true-cls-mono-set-mset-l* **by** *blast*
qed

lemma *multiset-not-empty*:
  assumes $M \neq \{\#\}$
  and $x \in\# M$
  shows $\exists A.\ x = Pos\ A \lor x = Neg\ A$

using *assms literal.exhaust-sel* **by** *blast*

lemma *atms-of-ms-empty*:
fixes ψ :: *'v clauses*
assumes *atms-of-ms* $\psi = \{\}$
shows $\psi = \{\} \vee \psi = \{\{\#\}\}$
using *assms* **by** (*auto simp add*: *atms-of-ms-def*)

lemma *consistent-interp-disjoint*:
assumes *consI*: *consistent-interp I*
and *disj*: *atms-of-s A* $\cap$ *atms-of-s I* $= \{\}$
and *consA*: *consistent-interp A*
shows *consistent-interp* $(A \cup I)$
proof (*rule ccontr*)
assume $\neg$ *?thesis*
moreover have $\bigwedge L.\ \neg (L \in A \wedge -L \in I)$
using *disj* **unfolding** *atms-of-s-def* **by** (*auto simp add*: *rev-image-eqI*)
ultimately show *False*
using *consA consI* **unfolding** *consistent-interp-def* **by** (*metis* (*full-types*) *Un-iff literal.exhaust-sel uminus-Neg uminus-Pos*)
qed

lemma *total-remove-unused*:
assumes *total-over-m I* ψ
shows *total-over-m* $\{v \in I.$ *atm-of v* $\in$ *atms-of-ms* $\psi\}$ ψ
using *assms* **unfolding** *total-over-m-def total-over-set-def*
by (*metis* (*lifting*) *literal.sel*(*1*,*2*) *mem-Collect-eq*)

lemma *true-cls-remove-hd-if-notin-vars*:
assumes *insert a M'* $\models$ *D*
and *atm-of a* $\notin$ *atms-of D*
shows *M'* $\models$ *D*
using *assms* **by** (*auto simp add*: *atm-of-lit-in-atms-of true-cls-def*)

lemma *total-over-set-atm-of*:
fixes *I* :: *'v interp* **and** *K* :: *'v set*
shows *total-over-set I K* $\longleftrightarrow$ $(\forall l \in K.\ l \in$ (*atm-of* ` *I*))
unfolding *total-over-set-def* **by** (*metis atms-of-s-def in-atms-of-s-decomp*)

11.2.7 Tautologies

definition *tautology* (ψ:: *'v clause*) $\equiv \forall I.$ *total-over-set I* (*atms-of* ψ) $\longrightarrow I \models \psi$

lemma *tautology-Pos-Neg*[*intro*]:
assumes *Pos p* $\in\#$ *A* **and** *Neg p* $\in\#$ *A*
shows *tautology A*
using *assms* **unfolding** *tautology-def total-over-set-def true-cls-def Bex-mset-def*
by (*meson atm-iff-pos-or-neg-lit true-lit-def*)

lemma *tautology-minus*[*simp*]:
assumes *L* $\in\#$ *A* **and** $-L \in\#$ *A*
shows *tautology A*
by (*metis assms literal.exhaust tautology-Pos-Neg uminus-Neg uminus-Pos*)

lemma *tautology-exists-Pos-Neg*:
assumes *tautology* ψ

shows $\exists p.$ *Pos p* $\in\#$ ψ $\wedge$ *Neg p* $\in\#$ ψ
proof (*rule ccontr*)
assume *p*: $\neg$ ($\exists p.$ *Pos p* $\in\#$ ψ $\wedge$ *Neg p* $\in\#$ ψ)
let *?I* = $\{-L \mid L.\ L \in\# \psi\}$
have *total-over-set ?I* (*atms-of* ψ)
unfolding *total-over-set-def* **using** *atm-imp-pos-or-neg-lit* **by** *force*
moreover have $\neg$ *?I* $\models \psi$
unfolding *true-cls-def true-lit-def Bex-mset-def* **apply** *clarify*
using *p* **by** (*case-tac L*) *fastforce+*
ultimately show *False* **using** *assms* **unfolding** *tautology-def* **by** *auto*
qed

lemma *tautology-decomp*:
tautology $\psi \longleftrightarrow (\exists p.$ *Pos p* $\in\#$ ψ $\wedge$ *Neg p* $\in\#$ $\psi)$
using *tautology-exists-Pos-Neg* **by** *auto*

lemma *tautology-false*[*simp*]: $\neg$*tautology* $\{\#\}$
unfolding *tautology-def* **by** *auto*

lemma *tautology-add-single*:
tautology ($\{\#a\#\}$ + *L*) $\longleftrightarrow$ *tautology L* $\vee$ $-a \in\#$ *L*
unfolding *tautology-decomp* **by** (*cases a*) *auto*

lemma *minus-interp-tautology*:
assumes $\{-L \mid L.\ L\in\# \chi\} \models \chi$
shows *tautology* χ
proof −
obtain *L* **where** $L \in\# \chi \wedge -L \in\# \chi$
using *assms* **unfolding** *true-cls-def* **by** *auto*
then show *?thesis* **using** *tautology-decomp literal.exhaust uminus-Neg uminus-Pos* **by** *metis*
qed

lemma *remove-literal-in-model-tautology*:
assumes $I \cup \{$*Pos P*$\} \models \varphi$
and $I \cup \{$*Neg P*$\} \models \varphi$
shows $I \models \varphi \vee$ *tautology* φ
using *assms* **unfolding** *true-cls-def* **by** *auto*

lemma *tautology-imp-tautology*:
fixes χ χ' :: *'v clause*
assumes $\forall I.$ *total-over-m I* $\{\chi\} \longrightarrow I \models \chi \longrightarrow I \models \chi'$ **and** *tautology* χ
shows *tautology* χ' **unfolding** *tautology-def*
proof (*intro allI HOL.impI*)
fix *I* ::*'v literal set*
assume *totI*: *total-over-set I* (*atms-of* χ')
let *?I'* = $\{$*Pos v* $|v.\ v\in$ *atms-of* $\chi \wedge v \notin$ *atms-of-s I*$\}$
have *totI'*: *total-over-m* ($I \cup$ *?I'*) $\{\chi\}$ **unfolding** *total-over-m-def total-over-set-def* **by** *auto*
then have χ: $I \cup$ *?I'* $\models \chi$ **using** *assms*(*2*) **unfolding** *total-over-m-def tautology-def* **by** *simp*
then have $I \cup$ (*?I'*− *I*) $\models \chi'$ **using** *assms*(*1*) *totI'* **by** *auto*
moreover have $\bigwedge L.\ L \in\# \chi' \Longrightarrow L \notin$ *?I'*
using *totI* **unfolding** *total-over-set-def* **by** (*auto dest*: *pos-lit-in-atms-of*)
ultimately show $I \models \chi'$ **unfolding** *true-cls-def* **by** *auto*
qed

11.2.8 Entailment for clauses and propositions

definition *true-cls-cls* :: *'a clause* ⇒ *'a clause* ⇒ *bool* (**infix** $\models f$ *49*) **where**
$\psi \models f\ \chi \longleftrightarrow (\forall I.\ \textit{total-over-m}\ I\ (\{\psi\} \cup \{\chi\}) \longrightarrow \textit{consistent-interp}\ I \longrightarrow I \models \psi \longrightarrow I \models \chi)$

definition *true-cls-clss* :: *'a clause* ⇒ *'a clauses* ⇒ *bool* (**infix** $\models fs$ *49*) **where**
$\psi \models fs\ \chi \longleftrightarrow (\forall I.\ \textit{total-over-m}\ I\ (\{\psi\} \cup \chi) \longrightarrow \textit{consistent-interp}\ I \longrightarrow I \models \psi \longrightarrow I \models s\ \chi)$

definition *true-clss-cls* :: *'a clauses* ⇒ *'a clause* ⇒ *bool* (**infix** $\models p$ *49*) **where**
$N \models p\ \chi \longleftrightarrow (\forall I.\ \textit{total-over-m}\ I\ (N \cup \{\chi\}) \longrightarrow \textit{consistent-interp}\ I \longrightarrow I \models s\ N \longrightarrow I \models \chi)$

definition *true-clss-clss* :: *'a clauses* ⇒ *'a clauses* ⇒ *bool* (**infix** $\models ps$ *49*) **where**
$N \models ps\ N' \longleftrightarrow (\forall I.\ \textit{total-over-m}\ I\ (N \cup N') \longrightarrow \textit{consistent-interp}\ I \longrightarrow I \models s\ N \longrightarrow I \models s\ N')$

lemma *true-cls-cls-refl*[*simp*]:
$A \models f\ A$
unfolding *true-cls-cls-def* **by** *auto*

lemma *true-cls-cls-insert-l*[*simp*]:
$a \models f\ C \Longrightarrow \textit{insert}\ a\ A \models p\ C$
unfolding *true-cls-cls-def true-clss-cls-def true-clss-def* **by** *fastforce*

lemma *true-cls-clss-empty*[*iff*]:
$N \models fs\ \{\}$
unfolding *true-cls-clss-def* **by** *auto*

lemma *true-prop-true-clause*[*iff*]:
$\{\varphi\} \models p\ \psi \longleftrightarrow \varphi \models f\ \psi$
unfolding *true-cls-cls-def true-clss-cls-def* **by** *auto*

lemma *true-clss-clss-true-clss-cls*[*iff*]:
$N \models ps\ \{\psi\} \longleftrightarrow N \models p\ \psi$
unfolding *true-clss-clss-def true-clss-cls-def* **by** *auto*

lemma *true-clss-clss-true-cls-clss*[*iff*]:
$\{\chi\} \models ps\ \psi \longleftrightarrow \chi \models fs\ \psi$
unfolding *true-clss-clss-def true-cls-clss-def* **by** *auto*

lemma *true-clss-clss-empty*[*simp*]:
$N \models ps\ \{\}$
unfolding *true-clss-clss-def* **by** *auto*

lemma *true-clss-cls-subset*:
$A \subseteq B \Longrightarrow A \models p\ CC \Longrightarrow B \models p\ CC$
unfolding *true-clss-cls-def total-over-m-union* **by** (*simp add*: *total-over-m-subset true-clss-mono*)

lemma *true-clss-cs-mono-l*[*simp*]:
$A \models p\ CC \Longrightarrow A \cup B\ \models p\ CC$
by (*auto intro*: *true-clss-cls-subset*)

lemma *true-clss-cs-mono-l2*[*simp*]:
$B \models p\ CC \Longrightarrow A \cup B\ \models p\ CC$
by (*auto intro*: *true-clss-cls-subset*)

lemma *true-clss-cls-mono-r*[*simp*]:
$A \models p\ CC \Longrightarrow A \models p\ CC + CC'$

unfolding *true-clss-cls-def total-over-m-union total-over-m-sum* **by** *blast*

lemma *true-clss-cls-mono-r'*[*simp*]:
$A \models p\ CC' \Longrightarrow A \models p\ CC + CC'$
unfolding *true-clss-cls-def total-over-m-union total-over-m-sum* **by** *blast*

lemma *true-clss-clss-union-l*[*simp*]:
$A \models ps\ CC \Longrightarrow A \cup B\ \models ps\ CC$
unfolding *true-clss-clss-def total-over-m-union* **by** *fastforce*

lemma *true-clss-clss-union-l-r*[*simp*]:
$B \models ps\ CC \Longrightarrow A \cup B \models ps\ CC$
unfolding *true-clss-clss-def total-over-m-union* **by** *fastforce*

lemma *true-clss-cls-in*[*simp*]:
$CC \in A \Longrightarrow A \models p\ CC$
unfolding *true-clss-cls-def true-clss-def total-over-m-union* **by** *fastforce*

lemma *true-clss-cls-insert-l*[*simp*]:
$A \models p\ C \Longrightarrow$ *insert a* $A \models p\ C$
unfolding *true-clss-cls-def true-clss-def* **using** *total-over-m-union*
by (*metis Un-iff insert-is-Un sup.commute*)

lemma *true-clss-clss-insert-l*[*simp*]:
$A \models ps\ C \Longrightarrow$ *insert a* $A \models ps\ C$
unfolding *true-clss-cls-def true-clss-clss-def true-clss-def* **by** *blast*

lemma *true-clss-clss-union-and*[*iff*]:
$A \models ps\ C \cup D \longleftrightarrow (A \models ps\ C \wedge A \models ps\ D)$
proof
  {
    fix $A\ C\ D$ :: *'a clauses*
    assume A: $A \models ps\ C \cup D$
    have $A \models ps\ C$
        unfolding *true-clss-clss-def true-clss-cls-def insert-def total-over-m-insert*
      proof (*intro allI impI*)
        fix I
        assume *totAC*: *total-over-m* I $(A \cup C)$
        and *cons*: *consistent-interp* I
        and I: $I \models s\ A$
        then have *tot*: *total-over-m* I A **and** *tot'*: *total-over-m* I C **by** *auto*
        obtain I' **where** *tot'*: *total-over-m* $(I \cup I')$ $(A \cup C \cup D)$
        and *cons'*: *consistent-interp* $(I \cup I')$
        and H: $\forall x \in I'$. *atm-of* $x \in$ *atms-of-ms* $D \wedge$ *atm-of* $x \notin$ *atms-of-ms* $(A \cup C)$
          using *total-over-m-consistent-extension*[*OF - cons*, *of* $A \cup C$] *tot tot'* **by** *blast*
        moreover have $I \cup I' \models s\ A$ **using** I **by** *simp*
        ultimately have $I \cup I' \models s\ C \cup D$ **using** A **unfolding** *true-clss-clss-def* **by** *auto*
        then have $I \cup I' \models s\ C \cup D$ **by** *auto*
        then show $I \models s\ C$ **using** *notin-vars-union-true-clss-true-clss*[*of* I'] H **by** *auto*
      qed
  } **note** H = *this*
  assume $A \models ps\ C \cup D$
  then show $A \models ps\ C \wedge A \models ps\ D$ **using** H[*of* A] *Un-commute*[*of* C D] **by** *metis*
next
  assume $A \models ps\ C \wedge A \models ps\ D$

then show $A \models ps\ C \cup D$
unfolding *true-clss-clss-def* **by** *auto*
qed

lemma *true-clss-clss-insert*[*iff*]:
$A \models ps$ *insert* $L\ Ls \longleftrightarrow (A \models p\ L \land A \models ps\ Ls)$
using *true-clss-clss-union-and*[*of* A $\{L\}$ *Ls*] **by** *auto*

lemma *true-clss-clss-subset*:
$A \subseteq B \Longrightarrow A \models ps\ CC \Longrightarrow B \models ps\ CC$
by (*metis subset-Un-eq true-clss-clss-union-l*)

lemma *union-trus-clss-clss*[*simp*]: $A \cup B \models ps\ B$
unfolding *true-clss-clss-def* **by** *auto*

lemma *true-clss-clss-remove*[*simp*]:
$A \models ps\ B \Longrightarrow A \models ps\ B - C$
by (*metis Un-Diff-Int true-clss-clss-union-and*)

lemma *true-clss-clss-subsetE*:
$N \models ps\ B \Longrightarrow A \subseteq B \Longrightarrow N \models ps\ A$
by (*metis sup.orderE true-clss-clss-union-and*)

lemma *true-clss-clss-in-imp-true-clss-cls*:
assumes $N \models ps\ U$
and $A \in U$
shows $N \models p\ A$
using *assms mk-disjoint-insert* **by** *fastforce*

lemma *all-in-true-clss-clss*: $\forall x \in B.\ x \in A \Longrightarrow A \models ps\ B$
unfolding *true-clss-clss-def true-clss-def* **by** *auto*

lemma *true-clss-clss-left-right*:
assumes $A \models ps\ B$
and $A \cup B \models ps\ M$
shows $A \models ps\ M \cup B$
using *assms* **unfolding** *true-clss-clss-def* **by** *auto*

lemma *true-clss-clss-generalise-true-clss-clss*:
$A \cup C \models ps\ D \Longrightarrow B \models ps\ C \Longrightarrow A \cup B \models ps\ D$
proof −
assume *a1*: $A \cup C \models ps\ D$
assume $B \models ps\ C$
then have *f2*: $\bigwedge M.\ M \cup B \models ps\ C$
by (*meson true-clss-clss-union-l-r*)
have $\bigwedge M.\ C \cup (M \cup A) \models ps\ D$
using *a1* **by** (*simp add*: *Un-commute sup-left-commute*)
then show *?thesis*
using *f2* **by** (*metis* (*no-types*) *Un-commute true-clss-clss-left-right true-clss-clss-union-and*)
qed

lemma *true-clss-cls-or-true-clss-cls-or-not-true-clss-cls-or*:
assumes *D*: $N \models p\ D + \{\#- L\#\}$
and *C*: $N \models p\ C + \{\#L\#\}$

shows $N \models p\ D + C$
unfolding *true-clss-cls-def*
proof (*intro allI impI*)
fix I
assume *tot*: *total-over-m* I $(N \cup \{D + C\})$
and *consistent-interp* I
and $I \models s\ N$
{
assume L: $L \in I \vee -L \in I$
then have *total-over-m* I $\{D + \{\#- L\#\}\}$
using *tot* **by** (*cases* L) *auto*
then have $I \models D + \{\#- L\#\}$ **using** D ‹$I \models s\ N$› *tot* ‹*consistent-interp* I›
unfolding *true-clss-cls-def* **by** *auto*
moreover
have *total-over-m* I $\{C + \{\#L\#\}\}$
using L *tot* **by** (*cases* L) *auto*
then have $I \models C + \{\#L\#\}$
using C ‹$I \models s\ N$› *tot* ‹*consistent-interp* I› **unfolding** *true-clss-cls-def* **by** *auto*
ultimately have $I \models D + C$ **using** ‹*consistent-interp* I› *consistent-interp-def* **by** *fastforce*
}
moreover {
assume L: $L \notin I \wedge -L \notin I$
let $?I' = I \cup \{L\}$
have *consistent-interp* $?I'$ **using** L ‹*consistent-interp* I› **by** *auto*
moreover have *total-over-m* $?I'$ $\{D + \{\#- L\#\}\}$
using *tot* **unfolding** *total-over-m-def total-over-set-def* **by** (*auto simp add*: *atms-of-def*)
moreover have *total-over-m* $?I'$ N **using** *tot* **using** *total-union* **by** *blast*
moreover have $?I' \models s\ N$ **using** ‹$I \models s\ N$› **using** *true-clss-union-increase* **by** *blast*
ultimately have $?I' \models D + \{\#- L\#\}$
using D **unfolding** *true-clss-cls-def* **by** *blast*
then have $?I' \models D$ **using** L **by** *auto*
moreover
have *total-over-set* I (*atms-of* $(D + C)$) **using** *tot* **by** *auto*
then have $L \notin\# D \wedge -L \notin\# D$
using L **unfolding** *total-over-set-def atms-of-def* **by** (*cases* L) *force+*
ultimately have $I \models D + C$ **unfolding** *true-cls-def* **by** *auto*
}
ultimately show $I \models D + C$ **by** *blast*
qed

lemma *atms-of-union-mset*[*simp*]:
atms-of $(A \#\cup B)$ = *atms-of* $A \cup$ *atms-of* B
unfolding *atms-of-def* **by** (*auto simp*: *max-def split*: *split-if-asm*)

lemma *true-cls-union-mset*[*iff*]: $I \models C \#\cup D \longleftrightarrow I \models C \vee I \models D$
unfolding *true-cls-def* **by** (*force simp*: *max-def Bex-mset-def split*: *split-if-asm*)

lemma *true-clss-cls-union-mset-true-clss-cls-or-not-true-clss-cls-or*:
assumes D: $N \models p\ D + \{\#- L\#\}$
and C: $N \models p\ C + \{\#L\#\}$
shows $N \models p\ D \#\cup C$
unfolding *true-clss-cls-def*
proof (*intro allI impI*)
fix I

assume *tot*: *total-over-m* I $(N \cup \{D \ \#\cup\ C\})$
and *consistent-interp* I
and $I \models s\ N$
{
assume L: $L \in I \lor -L \in I$
then have *total-over-m* I $\{D + \{\#-\ L\#\}\}$
using *tot* **by** (*cases* L) *auto*
then have $I \models D + \{\#-\ L\#\}$ **using** D ‹$I \models s\ N$› *tot* ‹*consistent-interp* I›
unfolding *true-clss-cls-def* **by** *auto*
moreover
have *total-over-m* I $\{C + \{\#L\#\}\}$
using L *tot* **by** (*cases* L) *auto*
then have $I \models C + \{\#L\#\}$
using C ‹$I \models s\ N$› *tot* ‹*consistent-interp* I› **unfolding** *true-clss-cls-def* **by** *auto*
ultimately have $I \models D \ \#\cup\ C$ **using** ‹*consistent-interp* I› **unfolding** *consistent-interp-def*
by *auto*
}
moreover {
assume L: $L \notin I \land -L \notin I$
let $?I' = I \cup \{L\}$
have *consistent-interp* $?I'$ **using** L ‹*consistent-interp* I› **by** *auto*
moreover have *total-over-m* $?I'$ $\{D + \{\#-\ L\#\}\}$
using *tot* **unfolding** *total-over-m-def total-over-set-def* **by** (*auto simp add*: *atms-of-def*)
moreover have *total-over-m* $?I'$ N **using** *tot* **using** *total-union* **by** *blast*
moreover have $?I' \models s\ N$ **using** ‹$I \models s\ N$› **using** *true-clss-union-increase* **by** *blast*
ultimately have $?I' \models D + \{\#-\ L\#\}$
using D **unfolding** *true-clss-cls-def* **by** *blast*
then have $?I' \models D$ **using** L **by** *auto*
moreover
have *total-over-set* I (*atms-of* $(D + C)$) **using** *tot* **by** *auto*
then have $L \notin\# D \land -L \notin\# D$
using L **unfolding** *total-over-set-def atms-of-def* **by** (*cases* L) *force+*
ultimately have $I \models D \ \ \#\cup\ C$ **unfolding** *true-cls-def* **by** *auto*
}
ultimately show $I \models D \ \#\cup\ C$ **by** *blast*
qed

lemma *satisfiable-carac*[*iff*]:
$(\exists I.\ consistent\text{-}interp\ I \land I \models s\ \varphi) \longleftrightarrow satisfiable\ \varphi$ (**is** $(\exists I.\ ?Q\ I) \longleftrightarrow ?S$)
proof
assume $?S$
then show $\exists I.\ ?Q\ I$ **unfolding** *satisfiable-def* **by** *auto*
next
assume $\exists I.\ ?Q\ I$
then obtain I **where** *cons*: *consistent-interp* I **and** I: $I \models s\ \varphi$ **by** *metis*
let $?I' = \{Pos\ v\ |v.\ v \notin atms\text{-}of\text{-}s\ I \land v \in atms\text{-}of\text{-}ms\ \varphi\}$
have *consistent-interp* $(I \cup ?I')$
using *cons* **unfolding** *consistent-interp-def* **by** (*intro allI*) (*case-tac* L, *auto*)
moreover have *total-over-m* $(I \cup ?I')$ φ
unfolding *total-over-m-def total-over-set-def* **by** *auto*
moreover have $I \cup ?I' \models s\ \varphi$
using I **unfolding** *Ball-def true-clss-def true-cls-def* **by** *auto*
ultimately show $?S$ **unfolding** *satisfiable-def* **by** *blast*
qed

lemma *satisfiable-carac*′[*simp*]: *consistent-interp* $I \Longrightarrow I \models s\ \varphi \Longrightarrow$ *satisfiable* φ
  using *satisfiable-carac* **by** *metis*

11.3 Subsumptions

lemma *subsumption-total-over-m*:
  assumes $A \subseteq\# B$
  shows *total-over-m* I $\{B\} \Longrightarrow$ *total-over-m* I $\{A\}$
  using *assms* **unfolding** *subset-mset-def total-over-m-def total-over-set-def*
  by (*auto simp add*: *mset-le-exists-conv*)

lemma *atm-of-eq-atm-of*:
  atm-of $L =$ *atm-of* $L' \longleftrightarrow (L = L' \vee L = -L')$
  by (*cases* L; *cases* L') *auto*

lemma *atms-of-replicate-mset-replicate-mset-uminus*[*simp*]:
  atms-of (D − *replicate-mset* (*count* D L) L − *replicate-mset* (*count* D ($-L$)) ($-L$))
    = *atms-of* D − {*atm-of* L}
  by (*auto split*: *split-if-asm simp add*: *atm-of-eq-atm-of atms-of-def*)

lemma *subsumption-chained*:
  assumes $\forall I.$ *total-over-m* I $\{D\} \longrightarrow I \models D \longrightarrow I \models \varphi$
  and $C \subseteq\# D$
  shows ($\forall I.$ *total-over-m* I $\{C\} \longrightarrow I \models C \longrightarrow I \models \varphi$) $\vee$ *tautology* φ
  using *assms*
proof (*induct card* {*Pos* v | v. $v \in$ *atms-of* $D \wedge v \notin$ *atms-of* C} *arbitrary*: D
    rule: *nat-less-induct-case*)
  case *0* **note** $n =$ *this*(*1*) **and** $H =$ *this*(*2*) **and** *incl* = *this*(*3*)
  then have *atms-of* $D \subseteq$ *atms-of* C **by** *auto*
  then have $\forall I.$ *total-over-m* I $\{C\} \longrightarrow$ *total-over-m* I $\{D\}$
    unfolding *total-over-m-def total-over-set-def* **by** *auto*
  moreover have $\forall I.\ I \models C \longrightarrow I \models D$ **using** *incl true-cls-mono-leD* **by** *blast*
  ultimately show *?case* **using** H **by** *auto*
next
  case (*Suc* n D) **note** *IH* = *this*(*1*) **and** *card* = *this*(*2*) **and** $H =$ *this*(*3*) **and** *incl* = *this*(*4*)
  let *?atms* = {*Pos* v |v. $v \in$ *atms-of* $D \wedge v \notin$ *atms-of* C}
  have *finite ?atms* **by** *auto*
  then obtain L **where** L: $L \in$ *?atms*
    using *card* **by** (*metis* (*no-types*, *lifting*) *Collect-empty-eq card-0-eq mem-Collect-eq*
      nat.simps(*3*))
  let *?D*′ = D − *replicate-mset* (*count* D L) L − *replicate-mset* (*count* D ($-L$)) ($-L$)
  have *atms-of-D*: *atms-of-ms* $\{D\} \subseteq$ *atms-of-ms* {*?D*′} $\cup$ {*atm-of* L} **by** *auto*

  {
    fix I
    assume *total-over-m* I {*?D*′}
    then have *tot*: *total-over-m* ($I \cup \{L\}$) $\{D\}$
      unfolding *total-over-m-def total-over-set-def* **using** *atms-of-D* **by** *auto*

    assume *IDL*: $I \models$ *?D*′
    then have $I \cup \{L\} \models D$ **unfolding** *true-cls-def* **by** *force*
    then have $I \cup \{L\} \models \varphi$ **using** H *tot* **by** *auto*

    moreover
      have *tot*′: *total-over-m* ($I \cup \{-L\}$) $\{D\}$
        using *tot* **unfolding** *total-over-m-def total-over-set-def* **by** *auto*

have $I \cup \{-L\} \models D$ **using** *IDL* **unfolding** *true-cls-def* **by** *force*
then have $I \cup \{-L\} \models \varphi$ **using** *H tot'* **by** *auto*
ultimately have $I \models \varphi \vee$ *tautology* φ
using *L remove-literal-in-model-tautology* **by** *force*
} **note** $H' = this$

have $L \notin\# C$ **and** $-L \notin\# C$ **using** *L atm-iff-pos-or-neg-lit* **by** *force+*
then have *C-in-D'*: $C \subseteq\#$ *?D'* **using** ‹$C \subseteq\# D$› **by** (*auto simp add*: *subseteq-mset-def*)
have *card* {*Pos v* |*v*. $v \in$ *atms-of ?D'* $\wedge v \notin$ *atms-of C*} <
card {*Pos v* |*v*. $v \in$ *atms-of D* $\wedge v \notin$ *atms-of C*}
using *L* **by** (*auto intro*!: *psubset-card-mono*)
then show *?case*
using *IH C-in-D' H'* **unfolding** *card*[*symmetric*] **by** *blast*
qed

11.4 Removing Duplicates

lemma *tautology-remdups-mset*[*iff*]:
tautology (*remdups-mset C*) $\longleftrightarrow$ *tautology C*
unfolding *tautology-decomp* **by** *auto*

lemma *atms-of-remdups-mset*[*simp*]: *atms-of* (*remdups-mset C*) = *atms-of C*
unfolding *atms-of-def* **by** *auto*

lemma *true-cls-remdups-mset*[*iff*]: $I \models$ *remdups-mset C* $\longleftrightarrow I \models C$
unfolding *true-cls-def* **by** *auto*

lemma *true-clss-cls-remdups-mset*[*iff*]: $A \models p$ *remdups-mset C* $\longleftrightarrow A \models p\ C$
unfolding *true-clss-cls-def total-over-m-def* **by** *auto*

11.5 Set of all Simple Clauses

A simple clause contains no duplicate and is not tautology.

function *build-all-simple-clss* :: *'v* :: *linorder set* $\Rightarrow$ *'v clause set* **where**
build-all-simple-clss vars =
(*if* ¬*finite vars* $\vee$ *vars*= {}
then {{#}}
else
let cls' = *build-all-simple-clss* (*vars* − {*Min vars*}) *in*
{{#*Pos* (*Min vars*)#} + χ |χ . $\chi \in$ *cls'*} $\cup$
{{#*Neg* (*Min vars*)#} + χ |χ. $\chi \in$ *cls'*} $\cup$
cls')
by *auto*
termination by (*relation measure card*) (*auto simp add*: *card-gt-0-iff*)

To avoid infinite simplifier loops:

declare *build-all-simple-clss.simps*[*simp del*]

lemma *build-all-simple-clss-simps-if*[*simp*]:
¬*finite vars* $\vee$ *vars* = {} $\Longrightarrow$ *build-all-simple-clss vars* = {{#}}
by (*simp add*: *build-all-simple-clss.simps*)

lemma *build-all-simple-clss-simps-else*[*simp*]:
fixes *vars*::*'v* ::*linorder set*
defines *cls* $\equiv$ *build-all-simple-clss* (*vars* − {*Min vars*})

shows
finite vars $\wedge$ *vars* $\neq$ {} $\Longrightarrow$ *build-all-simple-clss* (*vars*::'*v* ::*linorder set*) =
{{#*Pos* (*Min vars*)#} + χ |χ. $\chi \in$ *cls*}
$\cup$ {{#*Neg* (*Min vars*)#} + χ |χ. $\chi \in$ *cls*}
$\cup$ *cls*
using *build-all-simple-clss.simps*[*of vars*] **unfolding** *Let-def cls-def* **by** *metis*

lemma *build-all-simple-clss-finite*:
fixes *atms* :: '*v*::*linorder set*
shows *finite* (*build-all-simple-clss atms*)
proof (*induct card atms arbitrary*: *atms rule*: *nat-less-induct*)
case (*1 atms*) **note** *IH* = *this*
{
assume *atms* = {} $\vee$ $\neg$*finite atms*
then have *finite* (*build-all-simple-clss atms*) **by** *auto*
}
moreover {
assume *atms*: *atms* $\neq$ {} **and** *fin*: *finite atms*
then have *Min atms* $\in$ *atms* **using** *Min-in* **by** *auto*
then have *card* (*atms* − {*Min atms*}) < *card atms* **using** *fin atms* **by** (*meson card-Diff1-less*)
then have *finite* (*build-all-simple-clss* (*atms* − {*Min atms*})) **using** *IH* **by** *auto*
then have *finite* (*build-all-simple-clss atms*) **by** (*simp add*: *atms fin*)
}
ultimately show *finite* (*build-all-simple-clss atms*) **by** *blast*
qed

lemma *build-all-simple-clssE*:
assumes
x $\in$ *build-all-simple-clss atms* **and**
finite atms
shows *atms-of x* $\subseteq$ *atms* $\wedge$ $\neg$*tautology x* $\wedge$ *distinct-mset x*
using *assms*
proof (*induct card atms arbitrary*: *atms x*)
case (*0 atms*)
then show *?case* **by** *auto*
next
case (*Suc n*) **note** *IH* = *this*(*1*) **and** *card* = *this*(*2*) **and** *x* = *this*(*3*) **and** *finite* = *this*(*4*)
obtain *v* **where** *v* $\in$ *atms* **and** *v*: *v* = *Min atms*
using *Min-in card local.finite* **by** *fastforce*

let *?atms'* = *atms* − {*v*}
have *build-all-simple-clss atms*
= {{#*Pos v*#} + χ |χ. $\chi \in$ *build-all-simple-clss* (*?atms'*)}
$\cup$ {{#*Neg v*#} + χ |χ. $\chi \in$ *build-all-simple-clss* (*?atms'*)}
$\cup$ *build-all-simple-clss* (*?atms'*)
using *build-all-simple-clss-simps-else*[*of atms*] *finite* ‹*v* $\in$ *atms*› **unfolding** *v*
by (*metis emptyE*)
then consider
(*Pos*) χ φ **where** *x* = {#φ#} + χ **and** $\chi \in$ *build-all-simple-clss* (*?atms'*) **and**
φ = *Pos v* $\vee$ φ = *Neg v*
| (*In*) *x* $\in$ *build-all-simple-clss* (*?atms'*)
using *x* **by** *auto*
then show *?case*
proof *cases*
case *In*

```
      then show ?thesis using card finite IH[of ?atms'] ‹v ∈ atms› by fastforce
    next
      case Pos note x-χ = this(1) and χ = this(2) and φ = this(3)
      have
        atms-of χ ⊆ atms − {v} and
        ¬ tautology χ and
        distinct-mset χ
          using card finite IH[of ?atms' χ] ‹v ∈ atms› x-χ χ by auto
      moreover then have count χ (Neg v) = 0
        using ‹v ∈ atms› unfolding x-χ by (metis Diff-insert-absorb Set.set-insert
          atm-iff-pos-or-neg-lit gr0I subset-iff)
      moreover have count χ (Pos v) = 0
        using ‹atms-of χ ⊆ atms − {v}› by (meson Diff-iff atm-iff-pos-or-neg-lit
          contra-subsetD insertI1 not-gr0)
      ultimately show ?thesis
        using ‹v ∈ atms› φ unfolding x-χ
        by (auto simp add: tautology-add-single distinct-mset-add-single)
    qed
qed

lemma cls-in-build-all-simple-clss:
  shows {#} ∈ build-all-simple-clss s
  by (induct s rule: build-all-simple-clss.induct)
  (metis (no-types, lifting) UnCI build-all-simple-clss.simps insertI1)

lemma build-all-simple-clss-card:
  fixes atms :: 'v :: linorder set
  assumes finite atms
  shows card (build-all-simple-clss atms) ≤ 3 ^(card atms)
  using assms
proof (induct card atms arbitrary: atms rule: nat-less-induct)
  case (1 atms) note IH = this(1) and finite = this(2)
  {
    assume atms = {}
    then have card (build-all-simple-clss atms) ≤ 3 ^(card atms) by auto
  }
  moreover {
    let ?P = {{#Pos (Min atms)#} + χ |χ. χ ∈ build-all-simple-clss (atms − {Min atms})}
    let ?N = {{#Neg (Min atms)#} + χ |χ. χ ∈ build-all-simple-clss (atms − {Min atms})}
    let ?Z = build-all-simple-clss (atms − {Min atms})
    assume atms: atms ≠ {}
    then have min: Min atms ∈ atms using Min-in finite by auto
    then have card-atms-1: card atms ≥ 1 by (simp add: Suc-leI atms card-gt-0-iff local.finite)
    have card (build-all-simple-clss atms) = card (?P ∪ ?N ∪ ?Z) using atms finite by simp
    moreover
      have ⋀M Ma. card ((M::'v literal multiset set) ∪ Ma) ≤ card Ma + card M
          by (simp add: add.commute card-Un-le)
      then have card (?P ∪ ?N ∪ ?Z) ≤ card ?Z + (card ?P + card ?N)
        by (meson Nat.le-trans card-Un-le nat-add-left-cancel-le)
      then have card (?P ∪ ?N ∪ ?Z) ≤ card ?P + card ?N + card ?Z

        by presburger
    also
      have PZ: card ?P ≤ card ?Z
        by (simp add: Setcompr-eq-image build-all-simple-clss-finite card-image-le)
```

```
      have NZ: card ?N ≤ card ?Z
        by (simp add: Setcompr-eq-image build-all-simple-clss-finite card-image-le)
      have card ?P + card ?N + card ?Z ≤ card ?Z + card ?Z + card ?Z
        using PZ NZ by linarith
    finally have card (build-all-simple-clss atms) ≤ card ?Z + card ?Z + card ?Z .
    moreover
      have finite': finite (atms − {Min atms}) and
        card: card (atms − {Min atms}) = card atms − 1
        using finite min by auto
      have card-inf: card (atms − {Min atms}) < card atms
        using card ‹card atms ≥  1› min by auto
      then have card ?Z ≤ 3 ^ (card atms − 1) using IH finite' card by metis
    moreover
      have (3::nat) ^ (card atms − 1) + 3 ^ (card atms − 1) + 3 ^ (card atms − 1)
        = 3 ∗ 3 ^ (card atms − 1) by simp
      then have (3::nat) ^ (card atms − 1) + 3 ^ (card atms − 1) + 3 ^ (card atms − 1)
        = 3 ^ (card atms) by (metis card card-Suc-Diff1 local.finite min power-Suc)
    ultimately have card (build-all-simple-clss atms) ≤ 3 ^ (card atms) by linarith
  }
  ultimately show card (build-all-simple-clss atms) ≤ 3 ^ (card atms) by metis
qed

lemma build-all-simple-clss-mono-disj:
  assumes atms ∩ atms'= {} and finite atms and  finite atms'
  shows build-all-simple-clss atms ⊆ build-all-simple-clss (atms ∪ atms')
  using assms
proof (induct card (atms ∪ atms') arbitrary: atms atms')
  case (0 atms' atms)
  then show ?case by auto
next
  case (Suc n atms atms') note IH = this(1) and c = this(2) and disj = this(3) and finite = this(4)
    and finite' = this(5)
  let ?min = Min (atms ∪ atms')
  have m: ?min ∈ atms ∨ ?min ∈ atms' by (metis Min-in Un-iff c card-eq-0-iff nat.distinct(1))
  moreover {
    assume min: ?min ∈ atms'
    then have min': ?min ∉ atms using disj by auto
    then have atms = atms − {?min} by fastforce
    then have n = card (atms ∪ (atms' − {?min}))
      using c min finite finite' by (metis Min-in Un-Diff card-Diff-singleton-if diff-Suc-1
        finite-UnI sup-eq-bot-iff)
    moreover have atms ∩ (atms' − {?min}) = {} using disj by auto
    moreover have finite (atms' − {?min}) using finite' by auto
    ultimately have build-all-simple-clss atms ⊆ build-all-simple-clss (atms ∪ (atms' − {?min}))
      using IH[of atms atms' − {?min}] finite by metis
    moreover have atms ∪ (atms' − {?min}) = (atms ∪ atms') − {?min} using min min' by auto
    ultimately have ?case by (metis (no-types, lifting) build-all-simple-clss.simps c card-0-eq
      finite' finite-UnI le-supI2 local.finite nat.distinct(1))
  }
  moreover {
    let ?atms' = atms − {Min atms}
    assume min: ?min ∈ atms
    moreover have min': ?min ∉ atms' using disj min by auto
    moreover have atms' − {?min} = atms'
      using ‹?min ∉ atms'› by fastforce
```

ultimately have $n = card\ (atms - \{?min\} \cup atms')$
by (*metis Min-in Un-Diff c card-0-eq card-Diff-singleton-if diff-Suc-1 finite' finite-Un finite nat.distinct(1)*)
moreover have *finite* $(atms - \{?min\})$ **using** *finite* **by** *auto*
moreover have $(atms - \{?min\}) \cap atms' = \{\}$ **using** *disj* **by** *auto*
ultimately have *build-all-simple-clss* $(atms - \{?min\})$
$\subseteq$ *build-all-simple-clss* $((atms- \{?min\}) \cup atms')$
using *IH*[*of atms* $- \{?min\}$ *atms'*] *finite'* **by** *metis*
moreover have *build-all-simple-clss atms*
$= \{\{\#Pos\ (Min\ atms)\#\} + \chi\ |\chi.\ \chi \in$ *build-all-simple-clss* $(?atms')\}$
$\cup \{\{\#Neg\ (Min\ atms)\#\} + \chi\ |\chi.\ \chi \in$ *build-all-simple-clss* $(?atms')\}$
$\cup$ *build-all-simple-clss* $(?atms')$
using *build-all-simple-clss-simps-else*[*of atms*] *finite min* **by** (*metis emptyE*)
moreover
let $?mcls =$ *build-all-simple-clss* $(atms \cup atms' - \{?min\})$
have *build-all-simple-clss* $(atms \cup atms')$
$= \{\{\#Pos\ (?min)\#\} + \chi\ |\chi.\ \chi \in ?mcls\} \cup \{\{\#Neg\ (?min)\#\} + \chi\ |\chi.\ \chi \in ?mcls\} \cup ?mcls$
using *build-all-simple-clss-simps-else*[*of atms* $\cup$ *atms'*] *finite' min*
by (*metis c card-eq-0-iff nat.distinct(1)*)
moreover have $atms \cup atms' - \{?min\} = atms - \{?min\} \cup atms'$
using *min min'* **by** (*simp add*: *Un-Diff*)
moreover have *Min atms* $=$ *?min* **using** *min min'* **by** (*simp add*: *Min-eqI finite' local.finite*)
ultimately have *?case* **by** *auto*
}
ultimately show *?case* **by** *metis*
qed

lemma *build-all-simple-clss-mono*:
assumes *finite*: *finite atms'* **and** *incl*: $atms \subseteq atms'$
shows *build-all-simple-clss atms* $\subseteq$ *build-all-simple-clss atms'*
proof $-$
have $atms' = atms \cup (atms' - atms)$ **using** *incl* **by** *auto*
moreover have *finite* $(atms' - atms)$ **using** *finite* **by** *auto*
moreover have $atms \cap (atms' - atms) = \{\}$ **by** *auto*
ultimately show *?thesis*
using *rev-finite-subset*[*OF assms*] *build-all-simple-clss-mono-disj* **by** (*metis* (*no-types*))
qed

lemma *distinct-mset-not-tautology-implies-in-build-all-simple-clss*:
assumes *distinct-mset* χ **and** $\neg$*tautology* χ
shows $\chi \in$ *build-all-simple-clss* (*atms-of* χ)
using *assms*
proof (*induct card* (*atms-of* χ) *arbitrary*: χ)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*) **note** $IH = this(1)$ **and** $simp = this(3)$ **and** $c = this(2)$ **and** $no\text{-}dup = this(4)$
have *finite*: *finite* (*atms-of* χ) **by** *simp*

with *no-dup atm-iff-pos-or-neg-lit* **obtain** L **where**
$L\chi$: $L \in\# \chi$ **and**
L-min: *atm-of* $L =$ *Min* (*atms-of* χ) **and**
$mL\chi$: $\neg -L \in\# \chi$
by (*metis Min-in c card-0-eq literal.sel(1,2) nat.distinct(1) tautology-minus*)
then have χL: $\chi = (\chi - \{\#L\#\}) + \{\#L\#\}$ **by** *auto*

have $atm\chi$: $atms\text{-}of\ \chi = atms\text{-}of\ (\chi - \{\#L\#\}) \cup \{atm\text{-}of\ L\}$
  using *arg-cong*[*OF* χL, *of atms-of*] **by** *simp*

have $a\chi$: $atms\text{-}of\ (\chi - \{\#L\#\}) = (atms\text{-}of\ \chi) - \{atm\text{-}of\ L\}$
  proof (*standard*, *standard*)
    fix v
    assume a: $v \in atms\text{-}of\ (\chi - \{\#L\#\})$
    then obtain l **where** l: $v = atm\text{-}of\ l$ **and** l': $l \in\#\ \chi - \{\#L\#\}$
      unfolding *atms-of-def* **by** *auto*
    moreover {
      assume $v = atm\text{-}of\ L$
      then have $L \in\#\ \chi - \{\#L\#\} \vee -L \in\#\ \chi - \{\#L\#\}$
        using $l'\ l$ **by** (*auto simp add*: *atm-of-eq-atm-of*)
      moreover have $L \notin\#\ \chi - \{\#L\#\}$ **using** ‹ $L \in\#\ \chi$› *simp* **unfolding** *distinct-mset-def* **by** *auto*
      ultimately have *False* **using** $mL\chi$ **by** *auto*
    }
    ultimately show $v \in atms\text{-}of\ \chi - \{atm\text{-}of\ L\}$
      by (*auto dest*: *atm-of-lit-in-atms-of split*: *split-if-asm*)
  next
    show $atms\text{-}of\ \chi - \{atm\text{-}of\ L\} \subseteq atms\text{-}of\ (\chi - \{\#L\#\})$ **using** $atm\chi$ **by** *auto*
  qed

let $?s' = build\text{-}all\text{-}simple\text{-}clss\ (atms\text{-}of\ (\chi - \{\#L\#\}))$
have $card\ (atms\text{-}of\ (\chi - \{\#L\#\})) = n$
  using *c finite* $a\chi$ **by** (*simp add*: $L\chi$ *atm-of-lit-in-atms-of*)
moreover have $distinct\text{-}mset\ (\chi - \{\#L\#\})$ **using** *simp* **by** *auto*
moreover have $\neg tautology\ (\chi - \{\#L\#\})$
  by (*meson Multiset.diff-le-self mset-leD no-dup tautology-decomp*)
ultimately have χin: $\chi - \{\#L\#\} \in build\text{-}all\text{-}simple\text{-}clss\ (atms\text{-}of\ (\chi - \{\#L\#\}))$
  using *IH* **by** *simp*
have $\chi = \{\#L\#\} + (\chi - \{\#L\#\})$ **using** χL **by** (*simp add*: *add.commute*)
then show *?case*
  using χin *L-min* $a\chi$
  by (*cases L*)
    (*auto simp add*: *build-all-simple-clss.simps*[*of atms-of* χ] *Let-def*)
qed

lemma *simplified-in-build-all*:
  assumes *finite* ψ **and** *distinct-mset-set* ψ **and** $\forall \chi \in \psi.\ \neg tautology\ \chi$
  shows $\psi \subseteq build\text{-}all\text{-}simple\text{-}clss\ (atms\text{-}of\text{-}ms\ \psi)$
  using *assms*
proof (*induct rule*: *finite.induct*)
  case *emptyI*
  then show *?case* **by** *simp*
next
  case (*insertI* $\psi\ \chi$) **note** *finite* = *this*(*1*) **and** *IH* = *this*(*2*) **and** *simp* = *this*(*3*) **and** *tauto* = *this*(*4*)
  have *distinct-mset* χ **and** $\neg tautology\ \chi$
    using *simp tauto* **unfolding** *distinct-mset-set-def* **by** *auto*
  from *distinct-mset-not-tautology-implies-in-build-all-simple-clss*[*OF this*]
  have χ: $\chi \in build\text{-}all\text{-}simple\text{-}clss\ (atms\text{-}of\ \chi)$ **.**
  then have $\psi \subseteq build\text{-}all\text{-}simple\text{-}clss\ (atms\text{-}of\text{-}ms\ \psi)$ **using** *IH simp tauto* **by** *auto*
  moreover
    have $atms\text{-}of\text{-}ms\ \psi \subseteq atms\text{-}of\text{-}ms\ (insert\ \chi\ \psi)$ **unfolding** *atms-of-ms-def atms-of-def* **by** *force*
  ultimately
    have $\psi \subseteq build\text{-}all\text{-}simple\text{-}clss\ (atms\text{-}of\text{-}ms\ (insert\ \chi\ \psi))$

by (*meson atms-of-ms-finite build-all-simple-clss-mono dual-order.trans finite.insertI local.finite*)
moreover
have $\chi \in$ *build-all-simple-clss* (*atms-of-ms* (*insert* χ ψ))
using χ *finite build-all-simple-clss-mono*[*of atms-of-ms* (*insert* χ ψ)] **by** *auto*
ultimately show *?case* **by** *auto*
qed

11.6 Experiment: Expressing the Entailments as Locales

locale *entail* =
fixes *entail* :: $'a$ *set* $\Rightarrow$ $'b$ $\Rightarrow$ *bool* (**infix** $\models e$ *50*)
assumes *entail-insert*[*simp*]: $I \neq \{\} \Longrightarrow$ *insert L I* $\models e$ $x \longleftrightarrow \{L\} \models e$ $x \vee I \models e$ x
assumes *entail-union*[*simp*]: $I \models e$ $A \Longrightarrow I \cup I' \models e$ A
begin

definition *entails* :: $'a$ *set* $\Rightarrow$ $'b$ *set* $\Rightarrow$ *bool* (**infix** $\models es$ *50*) **where**
$I \models es$ $A \longleftrightarrow (\forall a \in A.\ I \models e$ $a)$

lemma *entails-empty*[*simp*]:
$I \models es$ $\{\}$
unfolding *entails-def* **by** *auto*

lemma *entails-single*[*iff*]:
$I \models es$ $\{a\} \longleftrightarrow I \models e$ a
unfolding *entails-def* **by** *auto*

lemma *entails-insert-l*[*simp*]:
$M \models es$ $A \Longrightarrow$ *insert L M* $\models es$ A
unfolding *entails-def* **by** (*metis Un-commute entail-union insert-is-Un*)

lemma *entails-union*[*iff*]: $I \models es$ $CC \cup DD \longleftrightarrow I \models es$ $CC \wedge I \models es$ DD
unfolding *entails-def* **by** *blast*

lemma *entails-insert*[*iff*]: $I \models es$ *insert C DD* $\longleftrightarrow I \models e$ $C \wedge I \models es$ DD
unfolding *entails-def* **by** *blast*

lemma *entails-insert-mono*: $DD \subseteq CC \Longrightarrow I \models es$ $CC \Longrightarrow I \models es$ DD
unfolding *entails-def* **by** *blast*

lemma *entails-union-increase*[*simp*]:
assumes $I \models es$ ψ
shows $I \cup I' \models es$ ψ
using *assms* **unfolding** *entails-def* **by** *auto*

lemma *true-clss-commute-l*:
$(I \cup I' \models es$ $\psi) \longleftrightarrow (I' \cup I \models es$ $\psi)$
by (*simp add*: *Un-commute*)

lemma *entails-remove*[*simp*]: $I \models es$ $N \Longrightarrow I \models es$ *Set.remove a N*
by (*simp add*: *entails-def*)

lemma *entails-remove-minus*[*simp*]: $I \models es$ $N \Longrightarrow I \models es$ $N - A$
by (*simp add*: *entails-def*)

end

interpretation *true-cls*: *entail true-cls*
  by *standard* (*auto simp add*: *true-cls-def*)

11.7 Entailment to be extended

definition *true-clss-ext* :: *'a literal set* ⇒ *'a literal multiset set* ⇒ *bool* (**infix** ⊨*sext 49*)
where
I ⊨*sext* $N \longleftrightarrow (\forall J.\ I \subseteq J \longrightarrow$ *consistent-interp* $J \longrightarrow$ *total-over-m* $J\ N \longrightarrow J$ ⊨*s* $N)$

lemma *true-clss-imp-true-cls-ext*:
  I⊨*s* $N \Longrightarrow I$ ⊨*sext* N
  unfolding *true-clss-ext-def* **by** (*metis sup.orderE true-clss-union-increase'*)

lemma *true-clss-ext-decrease-right-remove-r*:
  assumes I ⊨*sext* N
  shows I ⊨*sext* $N - \{C\}$
  unfolding *true-clss-ext-def*
proof (*intro allI impI*)
  fix J
  assume
    $I \subseteq J$ **and**
    cons: *consistent-interp* J **and**
    tot: *total-over-m* J $(N - \{C\})$
  let *?J* = $J \cup \{$*Pos* (*atm-of* P)|P. $P \in\#\ C \wedge$ *atm-of* $P \notin$ *atm-of* ‘ $J\}$
  have $I \subseteq$ *?J* **using** ‹$I \subseteq J$› **by** *auto*
  moreover have *consistent-interp ?J*
    using *cons* **unfolding** *consistent-interp-def* **apply** −
    apply (*rule allI*) **by** (*case-tac L*) (*fastforce simp add*: *image-iff*)+
  moreover
    have *ex-or-eq*: $\bigwedge l\ R\ J.\ \exists P.\ (l = P \vee l = -P) \wedge P \in\#\ C \wedge P \notin J \wedge -P \notin J$
      $\longleftrightarrow\ (l \in\#\ C \wedge l \notin J \wedge -l \notin J) \vee (-l \in\#\ C \wedge l \notin J \wedge -l \notin J)$
      by (*metis uminus-of-uminus-id*)
    have *total-over-m ?J N*

    using *tot* **unfolding** *total-over-m-def total-over-set-def atms-of-ms-def*
    apply (*auto simp add*:*atms-of-def*)
    apply (*case-tac* $a \in N - \{C\}$)
      apply *auto*[]
    using *atms-of-s-def atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set* **by** *fastforce*+
  ultimately have *?J* ⊨*s* N
    using *assms* **unfolding** *true-clss-ext-def* **by** *blast*
  then have *?J* ⊨*s* $N - \{C\}$ **by** *auto*
  have $\{v \in$ *?J*. *atm-of* $v \in$ *atms-of-ms* $(N - \{C\})\} \subseteq J$
    using *tot* **unfolding** *total-over-m-def total-over-set-def*
    by (*auto intro*!: *rev-image-eqI*)
  then show J ⊨*s* $N - \{C\}$
    using *true-clss-remove-unused*[*OF* ‹*?J* ⊨*s* $N - \{C\}$›] **unfolding** *true-clss-def*
    by (*meson true-cls-mono-set-mset-l*)
qed

lemma *consistent-true-clss-ext-satisfiable*:
  assumes *consistent-interp* I **and** I ⊨*sext* A
  shows *satisfiable* A
  by (*metis Un-empty-left assms satisfiable-carac subset-Un-eq sup.left-idem*
    total-over-m-consistent-extension total-over-m-empty true-clss-ext-def)

lemma *not-consistent-true-clss-ext*:
  assumes $\neg$*consistent-interp I*
  shows $I \models sext\ A$
  by (*meson assms consistent-interp-subset true-clss-ext-def*)
end
theory *Prop-Resolution*
imports *Partial-Clausal-Logic List-More Wellfounded-More*

begin

12 Resolution

12.1 Simplification Rules

inductive *simplify* :: *'v clauses* ⇒ *'v clauses* ⇒ *bool* **for** *N* :: *'v clause set* **where**
tautology-deletion:
  $(A + \{\#Pos\ P\#\} + \{\#Neg\ P\#\}) \in N \Longrightarrow simplify\ \ N\ (N - \{A + \{\#Pos\ P\#\} + \{\#Neg\ P\#\}\})$|
condensation:
  $(A + \{\#L\#\} + \{\#L\#\}) \in N \Longrightarrow simplify\ N\ (\ N - \{A + \{\#L\#\} + \{\#L\#\}\} \cup \{A + \{\#L\#\}\})$ |
subsumption:
  $A \in N \Longrightarrow A \subset\# B \Longrightarrow B \in N \Longrightarrow simplify\ N\ (N - \{B\})$

lemma *simplify-preserves-un-sat'*:
  fixes *N N'* :: *'v clauses*
  assumes *simplify N N'*
  and *total-over-m I N*
  shows $I \models s\ N' \longrightarrow I \models s\ N$
  using *assms*
proof (*induct rule*: *simplify.induct*)
  case (*tautology-deletion A P*)
  then have $I \models A + \{\#Pos\ P\#\} + \{\#Neg\ P\#\}$
    by (*metis total-over-m-def total-over-set-literal-defined true-cls-singleton true-cls-union true-lit-def uminus-Neg union-commute*)
  then show *?case* **by** (*metis Un-Diff-cancel2 true-clss-singleton true-clss-union*)
next
  case (*condensation A P*)
  then show *?case* **by** (*metis Diff-insert-absorb Set.set-insert insertE true-cls-union true-clss-def true-clss-singleton true-clss-union*)
next
  case (*subsumption A B*)
  have $A \neq B$ **using** *subsumption.hyps*(*2*) **by** *auto*
  then have $I \models s\ N - \{B\} \Longrightarrow I \models A$ **using** ‹$A \in N$› **by** (*simp add*: *true-clss-def*)
  moreover have $I \models A \Longrightarrow I \models B$ **using** ‹$A <\# B$› **by** *auto*
  ultimately show *?case* **by** (*metis insert-Diff-single true-clss-insert*)
qed

lemma *simplify-preserves-un-sat*:
  fixes *N N'* :: *'v clauses*
  assumes *simplify N N'*
  and *total-over-m I N*
  shows $I \models s\ N \longrightarrow I \models s\ N'$
  using *assms* **apply** (*induct rule*: *simplify.induct*)
  using *true-clss-def* **by** *fastforce+*

lemma *simplify-preserves-un-sat''*:
  fixes *N N'* :: *'v clauses*
  assumes *simplify N N'*
  and *total-over-m I N'*
  shows $I \models s\ N \longrightarrow I \models s\ N'$
  using *assms* **apply** (*induct rule*: *simplify.induct*)
  using *true-clss-def* **by** *fastforce+*

lemma *simplify-preserves-un-sat-eq*:
  fixes *N N'* :: *'v clauses*
  assumes *simplify N N'*
  and *total-over-m I N*
  shows $I \models s\ N \longleftrightarrow I \models s\ N'$
  using *simplify-preserves-un-sat simplify-preserves-un-sat' assms* **by** *blast*

lemma *simplify-preserves-finite*:
 assumes *simplify* ψ ψ'
 shows *finite* $\psi \longleftrightarrow$ *finite* ψ'
 using *assms* **by** (*induct rule*: *simplify.induct*, *auto simp add*: *remove-def*)

lemma *rtranclp-simplify-preserves-finite*:
 assumes *rtranclp simplify* ψ ψ'
 shows *finite* $\psi \longleftrightarrow$ *finite* ψ'
 using *assms* **by** (*induct rule*: *rtranclp-induct*) (*auto simp add*: *simplify-preserves-finite*)

lemma *simplify-atms-of-ms*:
  assumes *simplify* ψ ψ'
  shows *atms-of-ms* $\psi' \subseteq$ *atms-of-ms* ψ
  using *assms* **unfolding** *atms-of-ms-def*
proof (*induct rule*: *simplify.induct*)
  case (*tautology-deletion A P*)
  then show *?case* **by** *auto*
next
  case (*condensation A P*)
  moreover have $A + \{\#P\#\} + \{\#P\#\} \in \psi \Longrightarrow \exists x \in \psi.$ *atm-of* $P \in$ *atm-of* ‘ *set-mset x*
    by (*metis Un-iff atms-of-def atms-of-plus atms-of-singleton insert-iff*)
  ultimately show *?case* **by** (*auto simp add*: *atms-of-def*)
next
  case (*subsumption A P*)
  then show *?case* **by** *auto*
qed

lemma *rtranclp-simplify-atms-of-ms*:
  assumes *rtranclp simplify* ψ ψ'
  shows *atms-of-ms* $\psi' \subseteq$ *atms-of-ms* ψ
  using *assms* **apply** (*induct rule*: *rtranclp-induct*)
   apply (*fastforce intro*: *simplify-atms-of-ms*)
  using *simplify-atms-of-ms* **by** *blast*

lemma *factoring-imp-simplify*:
  assumes $\{\#L\#\} + \{\#L\#\} + C \in N$
  shows $\exists N'.$ *simplify N N'*
proof −
  have $C + \{\#L\#\} + \{\#L\#\} \in N$ **using** *assms* **by** (*simp add*: *add.commute union-lcomm*)
  from *condensation*[*OF this*] **show** *?thesis* **by** *blast*

qed

12.2 Unconstrained Resolution

type-synonym *'v uncon-state* = *'v clauses*
inductive *uncon-res* :: *'v uncon-state* ⇒ *'v uncon-state* ⇒ *bool* **where**
resolution:
$\{\#Pos\ p\#\} + C \in N \Longrightarrow \{\#Neg\ p\#\} + D \in N \Longrightarrow (\{\#Pos\ p\#\} + C, \{\#Neg\ p\#\} + D) \notin$ *already-used*
$\Longrightarrow$ *uncon-res* (N) $(N \cup \{C + D\})$ |
factoring: $\{\#L\#\} + \{\#L\#\} + C \in N \Longrightarrow$ *uncon-res* N $(N \cup \{C + \{\#L\#\}\})$

lemma *uncon-res-increasing*:
assumes *uncon-res* S S' **and** $\psi \in S$
shows $\psi \in S'$
using *assms* **by** (*induct rule*: *uncon-res.induct*) *auto*

lemma *rtranclp-uncon-inference-increasing*:
assumes *rtranclp uncon-res* S S' **and** $\psi \in S$
shows $\psi \in S'$
using *assms* **by** (*induct rule*: *rtranclp-induct*) (*auto simp add*: *uncon-res-increasing*)

12.2.1 Subsumption

definition *subsumes* :: *'a literal multiset* ⇒ *'a literal multiset* ⇒ *bool* **where**
subsumes χ χ' $\longleftrightarrow$
$(\forall I.$ *total-over-m* I $\{\chi'\} \longrightarrow$ *total-over-m* I $\{\chi\})$
$\wedge$ $(\forall I.$ *total-over-m* I $\{\chi\} \longrightarrow I \models \chi \longrightarrow I \models \chi')$

lemma *subsumes-refl*[*simp*]:
subsumes χ χ
unfolding *subsumes-def* **by** *auto*

lemma *subsumes-subsumption*:
assumes *subsumes* D χ
and $C \subset\# D$ **and** $\neg$*tautology* χ
shows *subsumes* C χ **unfolding** *subsumes-def*
using *assms subsumption-total-over-m subsumption-chained* **unfolding** *subsumes-def*
by (*blast intro*!: *subset-mset.less-imp-le*)

lemma *subsumes-tautology*:
assumes *subsumes* $(C + \{\#Pos\ P\#\} + \{\#Neg\ P\#\})$ χ
shows *tautology* χ
using *assms* **unfolding** *subsumes-def* **by** (*simp add*: *tautology-def*)

12.3 Inference Rule

type-synonym *'v state* = *'v clauses* × (*'v clause* × *'v clause*) *set*
inductive *inference-clause* :: *'v state* ⇒ *'v clause* × (*'v clause* × *'v clause*) *set* ⇒ *bool*
(**infix** $\Rightarrow_{\mathrm{Res}}$ *100*) **where**
resolution:
$\{\#Pos\ p\#\} + C \in N \Longrightarrow \{\#Neg\ p\#\} + D \in N \Longrightarrow (\{\#Pos\ p\#\} + C, \{\#Neg\ p\#\} + D) \notin$ *already-used*
$\Longrightarrow$ *inference-clause* $(N,$ *already-used*$)$ $(C + D,$ *already-used* $\cup \{(\{\#Pos\ p\#\} + C, \{\#Neg\ p\#\} + D)\})$ |

factoring: $\{\#L\#\} + \{\#L\#\} + C \in N \Longrightarrow$ *inference-clause* (N, *already-used*) ($C + \{\#L\#\}$, *already-used*)

inductive *inference* :: $'v$ *state* $\Rightarrow$ $'v$ *state* $\Rightarrow$ *bool* **where**
inference-step: *inference-clause* S (*clause*, *already-used*)
  $\Longrightarrow$ *inference* S (*fst* $S \cup \{$*clause*$\}$, *already-used*)

abbreviation *already-used-inv*
  :: $'a$ *literal multiset set* $\times$ ($'a$ *literal multiset* $\times$ $'a$ *literal multiset*) *set* $\Rightarrow$ *bool* **where**
already-used-inv state $\equiv$
  $(\forall (A, B) \in$ *snd state*. $\exists p$. *Pos* $p \in\#$ $A \wedge$ *Neg* $p \in\#$ $B \wedge$
      $((\exists \chi \in$ *fst state*. *subsumes* χ $((A - \{\#Pos\ p\#\}) + (B - \{\#Neg\ p\#\})))$
      $\vee$ *tautology* $((A - \{\#Pos\ p\#\}) + (B - \{\#Neg\ p\#\}))))$

lemma *inference-clause-preserves-already-used-inv*:
  assumes *inference-clause* S S'
  and *already-used-inv* S
  shows *already-used-inv* (*fst* $S \cup \{$*fst* $S'\}$, *snd* S')
  using *assms* **apply** (*induct rule*: *inference-clause.induct*)
  by *fastforce+*

lemma *inference-preserves-already-used-inv*:
  assumes *inference* S S'
  and *already-used-inv* S
  shows *already-used-inv* S'
  using *assms*
proof (*induct rule*: *inference.induct*)
  case (*inference-step* S *clause already-used*)
  then show *?case*
    using *inference-clause-preserves-already-used-inv*[*of* S (*clause*, *already-used*)] **by** *simp*
qed

lemma *rtranclp-inference-preserves-already-used-inv*:
  assumes *rtranclp inference* S S'
  and *already-used-inv* S
  shows *already-used-inv* S'
  using *assms* **apply** (*induct rule*: *rtranclp-induct*, *simp*)
  using *inference-preserves-already-used-inv* **unfolding** *tautology-def* **by** *fast*

lemma *subsumes-condensation*:
  assumes *subsumes* ($C + \{\#L\#\} + \{\#L\#\}$) D
  shows *subsumes* ($C + \{\#L\#\}$) D
  using *assms* **unfolding** *subsumes-def* **by** *simp*

lemma *simplify-preserves-already-used-inv*:
  assumes *simplify* N N'
  and *already-used-inv* (N, *already-used*)
  shows *already-used-inv* (N', *already-used*)
  using *assms*
proof (*induct rule*: *simplify.induct*)
  case (*condensation* C L)
  then show *?case*
    using *subsumes-condensation* **by** *simp fast*
next
  {

fix a:: $'a$ **and** A :: $'a$ *set* **and** P
have $(\exists x \in \mathit{Set.remove}\ a\ A.\ P\ x) \longleftrightarrow (\exists x \in A.\ x \neq a \wedge P\ x)$ **by** *auto*
} **note** *ex-member-remove* = *this*
{
fix a $a0$ :: $'v$ *clause* **and** A :: $'v$ *clauses* **and** y
assume $a \in A$ **and** $a0 \subset\# a$
then have $(\exists x \in A.\ \mathit{subsumes}\ x\ y) \longleftrightarrow (\mathit{subsumes}\ a\ y \vee (\exists x \in A.\ x \neq a \wedge \mathit{subsumes}\ x\ y))$
by *auto*
} **note** *tt2* = *this*
case (*subsumption A B*) **note** $A = \mathit{this}(1)$ **and** $AB = \mathit{this}(2)$ **and** $B = \mathit{this}(3)$ **and** $\mathit{inv} = \mathit{this}(4)$
show *?case*
proof (*standard*, *standard*)
fix x a b
assume x: $x \in \mathit{snd}\ (N - \{B\}, \mathit{already\text{-}used})$ **and** [*simp*]: $x = (a, b)$
obtain p **where** p: $\mathit{Pos}\ p \in\# a \wedge \mathit{Neg}\ p \in\# b$ **and**
q: $(\exists\chi\in N.\ \mathit{subsumes}\ \chi\ (a - \{\#\mathit{Pos}\ p\#\} + (b - \{\#\mathit{Neg}\ p\#\})))$
$\vee\ \mathit{tautology}\ (a - \{\#\mathit{Pos}\ p\#\} + (b - \{\#\mathit{Neg}\ p\#\}))$
using *inv x* **by** *fastforce*
consider (*taut*) $\mathit{tautology}\ (a - \{\#\mathit{Pos}\ p\#\} + (b - \{\#\mathit{Neg}\ p\#\}))$ |
(χ) χ **where** $\chi \in N$ $\mathit{subsumes}\ \chi\ (a - \{\#\mathit{Pos}\ p\#\} + (b - \{\#\mathit{Neg}\ p\#\}))$
$\neg\mathit{tautology}\ (a - \{\#\mathit{Pos}\ p\#\} + (b - \{\#\mathit{Neg}\ p\#\}))$
using q **by** *auto*
then show
$\exists p.\ \mathit{Pos}\ p \in\# a \wedge \mathit{Neg}\ p \in\# b$
$\wedge\ ((\exists\chi\in\mathit{fst}\ (N - \{B\}, \mathit{already\text{-}used}).\ \mathit{subsumes}\ \chi\ (a - \{\#\mathit{Pos}\ p\#\} + (b - \{\#\mathit{Neg}\ p\#\})))$
$\vee\ \mathit{tautology}\ (a - \{\#\mathit{Pos}\ p\#\} + (b - \{\#\mathit{Neg}\ p\#\})))$
proof *cases*
case *taut*
then show *?thesis* **using** p **by** *auto*
next
case χ **note** $H = \mathit{this}$
show *?thesis* **using** *p A AB B* *subsumes-subsumption*[*OF - AB H*(*3*)] *H*(*1*,*2*) **by** *auto*
qed
qed
next
case (*tautology-deletion C P*)
then show *?case* **apply** *clarify*
proof −
fix a b
assume $C + \{\#\mathit{Pos}\ P\#\} + \{\#\mathit{Neg}\ P\#\} \in N$
assume $\mathit{already\text{-}used\text{-}inv}\ (N, \mathit{already\text{-}used})$
and $(a, b) \in \mathit{snd}\ (N - \{C + \{\#\mathit{Pos}\ P\#\} + \{\#\mathit{Neg}\ P\#\}\}, \mathit{already\text{-}used})$
then obtain p **where**
$\mathit{Pos}\ p \in\# a \wedge \mathit{Neg}\ p \in\# b\ \wedge$
$((\exists\chi\in\mathit{fst}\ (N \cup \{C + \{\#\mathit{Pos}\ P\#\} + \{\#\mathit{Neg}\ P\#\}\}, \mathit{already\text{-}used}).$
$\mathit{subsumes}\ \chi\ (a - \{\#\mathit{Pos}\ p\#\} + (b - \{\#\mathit{Neg}\ p\#\})))$
$\vee\ \mathit{tautology}\ (a - \{\#\mathit{Pos}\ p\#\} + (b - \{\#\mathit{Neg}\ p\#\})))$
by *fastforce*
moreover have $\mathit{tautology}\ (C + \{\#\mathit{Pos}\ P\#\} + \{\#\mathit{Neg}\ P\#\})$ **by** *auto*
ultimately show
$\exists p.\ \mathit{Pos}\ p \in\# a \wedge \mathit{Neg}\ p \in\# b$
$\wedge\ ((\exists\chi\in\mathit{fst}\ (N - \{C + \{\#\mathit{Pos}\ P\#\} + \{\#\mathit{Neg}\ P\#\}\}, \mathit{already\text{-}used}).$
$\mathit{subsumes}\ \chi\ (a - \{\#\mathit{Pos}\ p\#\} + (b - \{\#\mathit{Neg}\ p\#\})))$
$\vee\ \mathit{tautology}\ (a - \{\#\mathit{Pos}\ p\#\} + (b - \{\#\mathit{Neg}\ p\#\})))$
by (*metis* (*no-types*) *Diff-iff Un-insert-right empty-iff fst-conv insertE subsumes-tautology*

sup-bot.right-neutral)
qed
qed

lemma
factoring-satisfiable: $I \models \{\#L\#\} + \{\#L\#\} + C \longleftrightarrow I \models \{\#L\#\} + C$ **and**
resolution-satisfiable:
consistent-interp $I \Longrightarrow I \models \{\#Pos\ p\#\} + C \Longrightarrow I \models \{\#Neg\ p\#\} + D \Longrightarrow I \models C + D$ **and**
factoring-same-vars: *atms-of* $(\{\#L\#\} + \{\#L\#\} + C) =$ *atms-of* $(\{\#L\#\} + C)$
unfolding *true-cls-def consistent-interp-def* **by** (*fastforce split*: *split-if-asm*)+

lemma *inference-increasing*:
assumes *inference* $S\ S'$ **and** $\psi \in$ *fst* S
shows $\psi \in$ *fst* S'
using *assms* **by** (*induct rule*: *inference.induct*, *auto*)

lemma *rtranclp-inference-increasing*:
assumes *rtranclp inference* $S\ S'$ **and** $\psi \in$ *fst* S
shows $\psi \in$ *fst* S'
using *assms* **by** (*induct rule*: *rtranclp-induct*, *auto simp add*: *inference-increasing*)

lemma *inference-clause-already-used-increasing*:
assumes *inference-clause* $S\ S'$
shows *snd* $S \subseteq$ *snd* S'
using *assms* **by** (*induct rule*:*inference-clause.induct*, *auto*)

lemma *inference-already-used-increasing*:
assumes *inference* $S\ S'$
shows *snd* $S \subseteq$ *snd* S'
using *assms* **apply** (*induct rule*:*inference.induct*)
using *inference-clause-already-used-increasing* **by** *fastforce*

lemma *inference-clause-preserves-un-sat*:
fixes $N\ N'$ :: *'v clauses*
assumes *inference-clause* $T\ T'$
and *total-over-m* I (*fst* T)
and *consistent*: *consistent-interp* I
shows $I \models s$ *fst* $T \longleftrightarrow I \models s$ *fst* $T \cup \{$*fst* $T'\}$
using *assms* **apply** (*induct rule*: *inference-clause.induct*)
unfolding *consistent-interp-def true-clss-def* **by** *auto force*+

lemma *inference-preserves-un-sat*:
fixes $N\ N'$ :: *'v clauses*
assumes *inference* $T\ T'$
and *total-over-m* I (*fst* T)
and *consistent*: *consistent-interp* I
shows $I \models s$ *fst* $T \longleftrightarrow I \models s$ *fst* T'
using *assms* **apply** (*induct rule*: *inference.induct*)
using *inference-clause-preserves-un-sat* **by** *fastforce*

lemma *inference-clause-preserves-atms-of-ms*:
assumes *inference-clause* $S\ S'$

shows *atms-of-ms* (*fst* (*fst* $S \cup \{$*fst* $S'\}$, *snd* S')) $\subseteq$ *atms-of-ms* (*fst* S)
using *assms* **apply** (*induct rule*: *inference-clause.induct*)
apply *auto*
apply (*metis Set.set-insert UnCI atms-of-ms-insert atms-of-plus*)
apply (*metis Set.set-insert UnCI atms-of-ms-insert atms-of-plus*)
apply (*simp add*: *in-m-in-literals union-assoc*)
unfolding *atms-of-ms-def* **using** *assms* **by** *fastforce*

lemma *inference-preserves-atms-of-ms*:
fixes N N' :: *'v clauses*
assumes *inference* T T'
shows *atms-of-ms* (*fst* T') $\subseteq$ *atms-of-ms* (*fst* T)
using *assms* **apply** (*induct rule*: *inference.induct*)
using *inference-clause-preserves-atms-of-ms* **by** *fastforce*

lemma *inference-preserves-total*:
fixes N N' :: *'v clauses*
assumes *inference* (N, *already-used*) (N', *already-used'*)
shows *total-over-m* I $N \Longrightarrow$ *total-over-m* I N'
using *assms inference-preserves-atms-of-ms* **unfolding** *total-over-m-def total-over-set-def*
by *fastforce*

lemma *rtranclp-inference-preserves-total*:
assumes *rtranclp inference* T T'
shows *total-over-m* I (*fst* T) $\Longrightarrow$ *total-over-m* I (*fst* T')
using *assms* **by** (*induct rule*: *rtranclp-induct*, *auto simp add*: *inference-preserves-total*)

lemma *rtranclp-inference-preserves-un-sat*:
assumes *rtranclp inference* N N'
and *total-over-m* I (*fst* N)
and *consistent*: *consistent-interp* I
shows $I \models s$ *fst* $N \longleftrightarrow I \models s$ *fst* N'
using *assms* **apply** (*induct rule*: *rtranclp-induct*)
apply (*simp add*: *inference-preserves-un-sat*)
using *inference-preserves-un-sat rtranclp-inference-preserves-total* **by** *blast*

lemma *inference-preserves-finite*:
assumes *inference* ψ ψ' **and** *finite* (*fst* ψ)
shows *finite* (*fst* ψ')
using *assms* **by** (*induct rule*: *inference.induct*, *auto simp add*: *simplify-preserves-finite*)

lemma *inference-clause-preserves-finite-snd*:
assumes *inference-clause* ψ ψ' **and** *finite* (*snd* ψ)
shows *finite* (*snd* ψ')
using *assms* **by** (*induct rule*: *inference-clause.induct*, *auto*)

lemma *inference-preserves-finite-snd*:
assumes *inference* ψ ψ' **and** *finite* (*snd* ψ)
shows *finite* (*snd* ψ')
using *assms inference-clause-preserves-finite-snd* **by** (*induct rule*: *inference.induct*, *fastforce*)

lemma *rtranclp-inference-preserves-finite*:
assumes *rtranclp inference* ψ ψ' **and** *finite* (*fst* ψ)
shows *finite* (*fst* ψ')
using *assms* **by** (*induct rule*: *rtranclp-induct*)
(*auto simp add*: *simplify-preserves-finite inference-preserves-finite*)

lemma *consistent-interp-insert*:
assumes *consistent-interp I*
and *atm-of P* $\notin$ *atm-of ‘ I*
shows *consistent-interp* (*insert P I*)
proof −
have *P*: *insert P I* $= I \cup \{P\}$ **by** *auto*
show *?thesis* **unfolding** *P*
apply (*rule consistent-interp-disjoint*)
using *assms* **by** (*auto simp add*: *atms-of-s-def*)
qed

lemma *simplify-clause-preserves-sat*:
assumes *simp*: *simplify* ψ ψ'
and *satisfiable* ψ'
shows *satisfiable* ψ
using *assms*
proof *induction*
case (*tautology-deletion A P*) **note** *AP* = *this*(*1*) **and** *sat* = *this*(*2*)
let *?A'* = *A* + {#*Pos P*#} + {#*Neg P*#}
let *?ψ'* = ψ − {*?A'*}
obtain *I* **where**
I: *I* ⊨*s ?ψ'* **and**
cons: *consistent-interp I* **and**
tot: *total-over-m I ?ψ'*
using *sat* **unfolding** *satisfiable-def* **by** *auto*
{ assume *Pos P* ∈ *I* ∨ *Neg P* ∈ *I*
then have *I* ⊨ *?A'* **by** *auto*
then have *I* ⊨*s* ψ **using** *I* **by** (*metis insert-Diff tautology-deletion.hyps true-clss-insert*)
then have *?case* **using** *cons tot* **by** *auto*
}
moreover {
assume *Pos*: *Pos P* ∉ *I* **and** *Neg*: *Neg P* ∉ *I*
then have *consistent-interp* (*I* ∪ {*Pos P*}) **using** *cons* **by** *simp*
moreover have *I'A*: *I* ∪ {*Pos P*} ⊨ *?A'* **by** *auto*
have {*Pos P*} ∪ *I* ⊨*s* ψ − {*A* + {#*Pos P*#} + {#*Neg P*#}}
using ‹*I* ⊨*s* ψ − {*A* + {#*Pos P*#} + {#*Neg P*#}}› *true-clss-union-increase'* **by** *blast*
then have *I* ∪ {*Pos P*} ⊨*s* ψ
by (*metis* (*no-types*) *Un-empty-right Un-insert-left Un-insert-right I'A insert-Diff*
sup-bot.left-neutral tautology-deletion.hyps true-clss-insert)
ultimately have *?case* **using** *satisfiable-carac'* **by** *blast*
}
ultimately show *?case* **by** *blast*
next
case (*condensation A L*) **note** *AL* = *this*(*1*) **and** *sat* = *this*(*2*)
have *f3*: *simplify* ψ (ψ − {*A* + {#*L*#} + {#*L*#}} ∪ {*A* + {#*L*#}})
using *AL simplify.condensation* **by** *blast*
obtain *LL* :: *'a literal multiset set* ⇒ *'a literal set* **where**
f4: *LL* (ψ − {*A* + {#*L*#} + {#*L*#}} ∪ {*A* + {#*L*#}}) ⊨*s* ψ − {*A* + {#*L*#} + {#*L*#}} ∪ {*A*
+ {#*L*#}}

$\wedge$ *consistent-interp* $(LL\ (\psi - \{A + \{\#L\#\} + \{\#L\#\}\} \cup \{A + \{\#L\#\}\}))$
$\wedge$ *total-over-m* $(LL\ (\psi - \{A + \{\#L\#\} + \{\#L\#\}\}$
$\cup \{A + \{\#L\#\}\}))\ (\psi - \{A + \{\#L\#\} + \{\#L\#\}\} \cup \{A + \{\#L\#\}\})$
using *sat* **by** (*meson satisfiable-def*)
have *f5*: *insert* $(A + \{\#L\#\} + \{\#L\#\})\ (\psi - \{A + \{\#L\#\} + \{\#L\#\}\}) = \psi$
using *AL* **by** *fastforce*
have *atms-of* $(A + \{\#L\#\} + \{\#L\#\})$ = *atms-of* $(\{\#L\#\} + A)$
by *simp*
then show *?case*
using *f5 f4 f3* **by** (*metis* (*no-types*) *add.commute satisfiable-def simplify-preserves-un-sat′*
total-over-m-insert total-over-m-union)
next
case (*subsumption A B*) **note** A = *this*(*1*) **and** AB = *this*(*2*) **and** B = *this*(*3*) **and** *sat* = *this*(*4*)
let $?\psi' = \psi - \{B\}$
obtain I **where** I: $I \models s\ ?\psi'$ **and** *cons*: *consistent-interp I* **and** *tot*: *total-over-m I* $?\psi'$
using *sat* **unfolding** *satisfiable-def* **by** *auto*
have $I \models A$ **using** *A I* **by** (*metis AB Diff-iff subset-mset.less-irrefl singletonD true-clss-def*)
then have $I \models B$ **using** *AB subset-mset.less-imp-le true-cls-mono-leD* **by** *blast*
then have $I \models s\ \psi$ **using** *I* **by** (*metis insert-Diff-single true-clss-insert*)
then show *?case* **using** *cons satisfiable-carac′* **by** *blast*
qed

lemma *simplify-preserves-unsat*:
assumes *inference* $\psi\ \psi'$
shows *satisfiable* (*fst* ψ') $\longrightarrow$ *satisfiable* (*fst* ψ)
using *assms* **apply** (*induct rule*: *inference.induct*)
using *satisfiable-decreasing* **by** (*metis fst-conv*)+

lemma *inference-preserves-unsat*:
assumes $inference^{**}\ S\ S'$
shows *satisfiable* (*fst* S') $\longrightarrow$ *satisfiable* (*fst* S)
using *assms* **apply** (*induct rule*: *rtranclp-induct*)
apply *simp-all*
using *simplify-preserves-unsat* **by** *blast*

datatype *′v sem-tree* = *Node ′v ′v sem-tree ′v sem-tree* | *Leaf*

fun *sem-tree-size* :: *′v sem-tree* ⇒ *nat* **where**
sem-tree-size Leaf = *0* |
sem-tree-size (*Node - ag ad*) = *1* + *sem-tree-size ag* + *sem-tree-size ad*

lemma *sem-tree-size*[*case-names bigger*]:
$(\bigwedge xs$:: *′v sem-tree*. $(\bigwedge ys$:: *′v sem-tree*. *sem-tree-size ys* < *sem-tree-size xs* $\Longrightarrow P\ ys) \Longrightarrow P\ xs)$
$\Longrightarrow P\ xs$
by (*fact Nat.measure-induct-rule*)

fun *partial-interps* :: *′v sem-tree* ⇒ *′v interp* ⇒ *′v clauses* ⇒ *bool* **where**
partial-interps Leaf I $\psi = (\exists\chi.\ \neg\ I \models \chi \wedge \chi \in \psi \wedge$ *total-over-m I* $\{\chi\})$ |
partial-interps (*Node v ag ad*) $I\ \psi \longleftrightarrow$
(*partial-interps ag* $(I \cup \{Pos\ v\})\ \psi\ \wedge$ *partial-interps ad* $(I\cup \{Neg\ v\})\ \psi)$

lemma *simplify-preserve-partial-leaf*:
simplify N N′ $\Longrightarrow$ *partial-interps Leaf I N* $\Longrightarrow$ *partial-interps Leaf I N′*

apply (*induct rule*: *simplify.induct*)
  using *union-lcomm* **apply** *auto*[*1*]
 apply (*simp*, *metis atms-of-plus total-over-set-union true-cls-union*)
apply *simp*
by (*metis atms-of-ms-singleton mset-le-exists-conv subset-mset-def true-cls-mono-leD*
  total-over-m-def total-over-m-sum)

lemma *simplify-preserve-partial-tree*:
  assumes *simplify N N′*
  and *partial-interps t I N*
  shows *partial-interps t I N′*
  using *assms* **apply** (*induct t arbitrary*: *I*, *simp*)
  using *simplify-preserve-partial-leaf* **by** *metis*

lemma *inference-preserve-partial-tree*:
  assumes *inference S S′*
  and *partial-interps t I* (*fst S*)
  shows *partial-interps t I* (*fst S′*)
  using *assms* **apply** (*induct t arbitrary*: *I*, *simp-all*)
  by (*meson inference-increasing*)

lemma *rtranclp-inference-preserve-partial-tree*:
  assumes *rtranclp inference N N′*
  and *partial-interps t I* (*fst N*)
  shows *partial-interps t I* (*fst N′*)
  using *assms* **apply** (*induct rule*: *rtranclp-induct*, *auto*)
  using *inference-preserve-partial-tree* **by** *force*

function *build-sem-tree* :: *′v* :: *linorder set* ⇒ *′v clauses* ⇒ *′v sem-tree* **where**
build-sem-tree atms ψ =
  (*if atms* = {} ∨ ¬ *finite atms*
  then Leaf
  else Node (*Min atms*) (*build-sem-tree* (*Set.remove* (*Min atms*) *atms*) ψ)
    (*build-sem-tree* (*Set.remove* (*Min atms*) *atms*) ψ))
by *auto*
termination
  apply (*relation measure* (λ(*A*, -). *card A*), *simp-all*)
  apply (*metis Min-in card-Diff1-less remove-def*)+
done
declare *build-sem-tree.induct*[*case-names tree*]

lemma *unsatisfiable-empty*[*simp*]:
  ¬*unsatisfiable* {}
   unfolding *satisfiable-def* **apply** *auto*
  using *consistent-interp-def* **unfolding** *total-over-m-def total-over-set-def atms-of-ms-def* **by** *blast*

lemma *partial-interps-build-sem-tree-atms-general*:
  fixes ψ :: *′v* :: *linorder clauses* **and** *p* :: *′v literal list*
  assumes *unsat*: *unsatisfiable* ψ **and** *finite* ψ **and** *consistent-interp I*
  and *finite atms*
  and *atms-of-ms* ψ = *atms* ∪ *atms-of-s I* **and** *atms* ∩ *atms-of-s I* = {}

shows *partial-interps* (*build-sem-tree atms* ψ) *I* ψ
using *assms*
proof (*induct arbitrary*: *I rule*: *build-sem-tree.induct*)
case (*1 atms* ψ *Ia*) **note** *IH1* = *this*(*1*) **and** *IH2* = *this*(*2*) **and** *unsat* = *this*(*3*) **and** *finite* = *this*(*4*)
and *cons* = *this*(*5*) **and** *f* = *this*(*6*) **and** *un* = *this*(*7*) **and** *disj* = *this*(*8*)
{
assume *atms*: *atms* = {}
then have *atmsIa*: *atms-of-ms* ψ = *atms-of-s Ia* **using** *un* **by** *auto*
then have *total-over-m Ia* ψ **unfolding** *total-over-m-def atmsIa* **by** *auto*
then have χ: $\exists\chi \in \psi.\ \neg$ *Ia* $\models \chi$
using *unsat cons* **unfolding** *true-clss-def satisfiable-def* **by** *auto*
then have *build-sem-tree atms* ψ = *Leaf* **using** *atms* **by** *auto*
moreover
have *tot*: $\bigwedge\chi.\ \chi \in \psi \Longrightarrow$ *total-over-m Ia* $\{\chi\}$
unfolding *total-over-m-def total-over-set-def atms-of-ms-def atms-of-s-def*
using *atmsIa atms-of-ms-def* **by** *fastforce*
have *partial-interps Leaf Ia* ψ
using χ *tot* **by** (*auto simp add*: *total-over-m-def total-over-set-def atms-of-ms-def*)

ultimately have *?case* **by** *metis*
}
moreover {
assume *atms*: *atms* $\neq$ {}
have *build-sem-tree atms* ψ = *Node* (*Min atms*) (*build-sem-tree* (*Set.remove* (*Min atms*) *atms*) ψ)
(*build-sem-tree* (*Set.remove* (*Min atms*) *atms*) ψ)
using *build-sem-tree.simps*[*of atms* ψ] *f atms* **by** *metis*

have *consistent-interp* (*Ia* $\cup$ {*Pos* (*Min atms*)}) **unfolding** *consistent-interp-def*
by (*metis Int-iff Min-in Un-iff atm-of-uminus atms cons consistent-interp-def disj empty-iff*
f in-atms-of-s-decomp insert-iff literal.distinct(*1*) *literal.exhaust-sel literal.sel*(*2*)
uminus-Neg uminus-Pos)
moreover have *atms-of-ms* ψ = *Set.remove* (*Min atms*) *atms* $\cup$ *atms-of-s* (*Ia* $\cup$ {*Pos* (*Min atms*)})
using *Min-in atms f un* **by** *fastforce*
moreover have *disj'*: *Set.remove* (*Min atms*) *atms* $\cap$ *atms-of-s* (*Ia* $\cup$ {*Pos* (*Min atms*)}) = {}
by *simp* (*metis disj disjoint-iff-not-equal member-remove*)
moreover have *finite* (*Set.remove* (*Min atms*) *atms*) **using** *f* **by** (*simp add*: *remove-def*)
ultimately have *subtree1*: *partial-interps* (*build-sem-tree* (*Set.remove* (*Min atms*) *atms*) ψ)
(*Ia* $\cup$ {*Pos* (*Min atms*)}) ψ
using *IH1*[*of Ia* $\cup$ {*Pos* (*Min* (*atms*))}] *atms f unsat finite* **by** *metis*

have *consistent-interp* (*Ia* $\cup$ {*Neg* (*Min atms*)}) **unfolding** *consistent-interp-def*
by (*metis Int-iff Min-in Un-iff atm-of-uminus atms cons consistent-interp-def disj empty-iff*
f in-atms-of-s-decomp insert-iff literal.distinct(*1*) *literal.exhaust-sel literal.sel*(*2*)
uminus-Neg)
moreover have *atms-of-ms* ψ = *Set.remove* (*Min atms*) *atms* $\cup$ *atms-of-s* (*Ia* $\cup$ {*Neg* (*Min atms*)})
using ‹*atms-of-ms* ψ = *Set.remove* (*Min atms*) *atms* $\cup$ *atms-of-s* (*Ia* $\cup$ {*Pos* (*Min atms*)})› **by**
blast

moreover have *disj'*: *Set.remove* (*Min atms*) *atms* $\cap$ *atms-of-s* (*Ia* $\cup$ {*Neg* (*Min atms*)}) = {}
using *disj* **by** *auto*
moreover have *finite* (*Set.remove* (*Min atms*) *atms*) **using** *f* **by** (*simp add*: *remove-def*)
ultimately have *subtree2*: *partial-interps* (*build-sem-tree* (*Set.remove* (*Min atms*) *atms*) ψ)
(*Ia* $\cup$ {*Neg* (*Min atms*)}) ψ
using *IH2*[*of Ia* $\cup$ {*Neg* (*Min* (*atms*))}] *atms f unsat finite* **by** *metis*

then have *?case*
using *IH1 subtree1 subtree2 f local.finite unsat atms* **by** *simp*
}
ultimately show *?case* **by** *metis*
qed

lemma *partial-interps-build-sem-tree-atms*:
fixes ψ :: *'v* :: *linorder clauses* **and** *p* :: *'v literal list*
assumes *unsat*: *unsatisfiable* ψ **and** *finite*: *finite* ψ
shows *partial-interps* (*build-sem-tree* (*atms-of-ms* ψ) ψ) {} ψ
proof −
have *consistent-interp* {} **unfolding** *consistent-interp-def* **by** *auto*
moreover have *atms-of-ms* ψ = *atms-of-ms* ψ ∪ *atms-of-s* {} **unfolding** *atms-of-s-def* **by** *auto*
moreover have *atms-of-ms* ψ ∩ *atms-of-s* {} = {} **unfolding** *atms-of-s-def* **by** *auto*
moreover have *finite* (*atms-of-ms* ψ) **unfolding** *atms-of-ms-def* **using** *finite* **by** *simp*
ultimately show *partial-interps* (*build-sem-tree* (*atms-of-ms* ψ) ψ) {} ψ
using *partial-interps-build-sem-tree-atms-general*[*of* ψ {} *atms-of-ms* ψ] *assms* **by** *metis*
qed

lemma *can-decrease-count*:
fixes ψ'' :: *'v clauses* × (*'v clause* × *'v clause* × *'v*) *set*
assumes *count* χ *L* = *n*
and *L* ∈# χ **and** χ ∈ *fst* ψ
shows $\exists\psi'\ \chi'$. *inference*** $\psi\ \psi'$ ∧ χ' ∈ *fst* ψ' ∧ (∀ *L*. *L* ∈# χ ⟷ *L* ∈# χ')
∧ *count* χ' *L* = *1*
∧ (∀ φ. φ ∈ *fst* ψ ⟶ φ ∈ *fst* ψ')
∧ (*I* ⊨ χ ⟷ *I* ⊨ χ')
∧ (∀ *I'*. *total-over-m I'* {χ} ⟶ *total-over-m I'* {χ'})
using *assms*
proof (*induct n arbitrary*: χ ψ)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n* χ)
note *IH* = *this*(*1*) **and** *count* = *this*(*2*) **and** *L* = *this*(*3*) **and** χ = *this*(*4*)
{
assume *n* = *0*
then have *inference*** ψ ψ
and χ ∈ *fst* ψ
and ∀ *L*. (*L* ∈# χ) ⟷ (*L* ∈# χ)
and *count* χ *L* = (*1*::*nat*)
and ∀ φ. φ ∈ *fst* ψ ⟶ φ ∈ *fst* ψ
by (*auto simp add*: *count L* χ)
then have *?case* **by** *metis*
}
moreover {
assume *n* > *0*
then have ∃ *C*. χ = *C* + {#*L*, *L*#}
by (*metis L One-nat-def add-diff-cancel-right' count-diff count-single diff-Suc-Suc diff-zero local.count multi-member-split union-assoc*)
then obtain *C* **where** *C*: χ = *C* + {#*L*, *L*#} **by** *metis*
let *?*χ' = *C* +{#*L*#}
let *?*ψ' = (*fst* ψ ∪ {*?*χ'}, *snd* ψ)
have φ: ∀ φ ∈ *fst* ψ. (φ ∈ *fst* ψ ∨ φ ≠ *?*χ') ⟷ φ ∈ *fst ?*ψ' **unfolding** *C* **by** *auto*

have *inf*: *inference* ψ *?*ψ'
  using *C factoring* χ *prod.collapse union-commute inference-step* **by** *metis*
moreover have *count*$'$: *count ?*χ' $L = n$ **using** *C count* **by** *auto*
moreover have $L\chi'$: L :# *?*χ' **by** *auto*
moreover have $\chi'\psi'$: *?*$\chi' \in$ *fst ?*ψ' **by** *auto*
ultimately obtain ψ'' **and** χ''
where
  inference** *?*ψ' ψ'' **and**
  α: $\chi'' \in$ *fst* ψ'' **and**
  $\forall$ *La*. (*La* $\in$# *?*χ') $\longleftrightarrow$ (*La* $\in$# χ'') **and**
  β: *count* χ'' L = (*1*::*nat*) **and**
  φ': $\forall \varphi$. $\varphi \in$ *fst ?*ψ' $\longrightarrow$ $\varphi \in$ *fst* ψ'' **and**
  $I\chi$: $I \models$ *?*χ' $\longleftrightarrow$ $I \models \chi''$ **and**
  tot: $\forall I'$. *total-over-m* I' {*?*χ'} $\longrightarrow$ *total-over-m* I' {χ''}
  using *IH*[*of ?*χ' *?*ψ'] *count*$'$ $L\chi'$ $\chi'\psi'$ **by** *blast*

then have *inference*** ψ ψ''
and $\forall$ *La*. (*La* $\in$# χ) $\longleftrightarrow$ (*La* $\in$# χ'')
using *inf* **unfolding** *C* **by** *auto*
moreover have $\forall \varphi$. $\varphi \in$ *fst* ψ $\longrightarrow$ $\varphi \in$ *fst* ψ'' **using** φ φ' **by** *metis*
moreover have $I \models \chi \longleftrightarrow I \models \chi''$ **using** $I\chi$ **unfolding** *true-cls-def C* **by** *auto*
moreover have $\forall I'$. *total-over-m* I' {χ} $\longrightarrow$ *total-over-m* I' {χ''}
  using *tot* **unfolding** *C total-over-m-def* **by** *auto*
ultimately have *?case* **using** φ φ' α β **by** *metis*
}
ultimately show *?case* **by** *auto*
qed

lemma *can-decrease-tree-size*:
fixes ψ :: *'v state* **and** *tree* :: *'v sem-tree*
assumes *finite* (*fst* ψ) **and** *already-used-inv* ψ
and *partial-interps tree I* (*fst* ψ)
shows $\exists$(*tree*$'$:: *'v sem-tree*) ψ'. *inference*** ψ ψ' $\wedge$ *partial-interps tree*$'$ I (*fst* ψ')
  $\wedge$ (*sem-tree-size tree*$'$ < *sem-tree-size tree* $\vee$ *sem-tree-size tree* = *0*)
using *assms*
proof (*induct arbitrary*: *I rule*: *sem-tree-size*)
case (*bigger xs I*) **note** *IH* = *this*(*1*) **and** *finite* = *this*(*2*) **and** *a-u-i* = *this*(*3*) **and** *part* = *this*(*4*)

{
  assume *sem-tree-size xs* = *0*
  then have *?case* **using** *part* **by** *blast*
}

moreover {
  assume *sn0*: *sem-tree-size xs* > *0*
  obtain *ag ad v* **where** *xs*: *xs* = *Node v ag ad* **using** *sn0* **by** (*case-tac xs*, *auto*)
  {
    assume *sem-tree-size ag* = *0* **and** *sem-tree-size ad* = *0*
    then have *ag*: *ag* = *Leaf* **and** *ad*: *ad* = *Leaf* **by** (*case-tac ag*, *auto*) (*case-tac ad*, *auto*)

    then obtain χ χ' **where**
      χ: $\neg$ $I \cup$ {*Pos v*} $\models \chi$ **and**
      totχ: *total-over-m* ($I \cup$ {*Pos v*}) {χ} **and**
      $\chi\psi$: $\chi \in$ *fst* ψ **and**
      χ': $\neg$ $I \cup$ {*Neg v*} $\models \chi'$ **and**

totχ′: *total-over-m* $(I \cup \{Neg\ v\})$ $\{\chi'\}$ **and**
χ′ψ: $\chi' \in fst\ \psi$
using *part* **unfolding** *xs* **by** *auto*
have *Posv*: $\neg Pos\ v \in\#\ \chi$ **using** χ **unfolding** *true-cls-def true-lit-def* **by** *auto*
have *Negv*: $\neg Neg\ v \in\#\ \chi'$ **using** χ' **unfolding** *true-cls-def true-lit-def* **by** *auto*
{
assume *Negχ*: $\neg Neg\ v \in\#\ \chi$
have $\neg\ I \models \chi$ **using** χ *Posv* **unfolding** *true-cls-def true-lit-def* **by** *auto*
moreover have *total-over-m I* $\{\chi\}$
using *Posv Negχ atm-imp-pos-or-neg-lit totχ* **unfolding** *total-over-m-def total-over-set-def*
by *fastforce*
ultimately have *partial-interps Leaf I* $(fst\ \psi)$
and *sem-tree-size Leaf* $<$ *sem-tree-size xs*
and $inference^{**}\ \psi\ \psi$
unfolding *xs* **by** (*auto simp add*: *χψ*)
}
moreover {
assume *Posχ*: $\neg Pos\ v \in\#\ \chi'$
then have *Iχ*: $\neg\ I \models \chi'$ **using** χ' *Posv* **unfolding** *true-cls-def true-lit-def* **by** *auto*
moreover have *total-over-m I* $\{\chi'\}$
using *Negv Posχ atm-imp-pos-or-neg-lit totχ′*
unfolding *total-over-m-def total-over-set-def* **by** *fastforce*
ultimately have *partial-interps Leaf I* $(fst\ \psi)$ **and**
sem-tree-size Leaf $<$ *sem-tree-size xs* **and**
$inference^{**}\ \psi\ \psi$
using *χ′ψ Iχ* **unfolding** *xs* **by** *auto*
}
moreover {
assume *neg*: $Neg\ v \in\#\ \chi$ **and** *pos*: $Pos\ v \in\#\ \chi'$
then obtain $\psi'\ \chi 2$ **where** *inf*: *rtranclp inference* $\psi\ \psi'$ **and** *χ2incl*: $\chi 2 \in fst\ \psi'$
and *χχ2-incl*: $\forall L.\ L :\#\ \chi \longleftrightarrow L :\#\ \chi 2$
and *countχ2*: $count\ \chi 2\ (Neg\ v) = 1$
and *φ*: $\forall \varphi{::}'v\ literal\ multiset.\ \varphi \in fst\ \psi \longrightarrow \varphi \in fst\ \psi'$
and *Iχ*: $I \models \chi \longleftrightarrow I \models \chi 2$
and *tot-impχ*: $\forall I'.\ total\text{-}over\text{-}m\ I'\ \{\chi\} \longrightarrow total\text{-}over\text{-}m\ I'\ \{\chi 2\}$
using *can-decrease-count*[*of* χ *Neg v count* χ $(Neg\ v)$ ψ *I*] *χψ χ′ψ* **by** *auto*

have $\chi' \in fst\ \psi'$ **by** (*simp add*: *χ′ψ φ*)
with *pos*
obtain $\psi''\ \chi 2'$ **where**
inf′: $inference^{**}\ \psi'\ \psi''$
and *χ2′-incl*: $\chi 2' \in fst\ \psi''$
and *χ′χ2-incl*: $\forall L{::}'v\ literal.\ (L \in\#\ \chi') = (L \in\#\ \chi 2')$
and *countχ2′*: $count\ \chi 2'\ (Pos\ v) = (1{::}nat)$
and *φ′*: $\forall \varphi{::}'v\ literal\ multiset.\ \varphi \in fst\ \psi' \longrightarrow \varphi \in fst\ \psi''$
and *Iχ′*: $I \models \chi' \longleftrightarrow I \models \chi 2'$
and *tot-impχ′*: $\forall I'.\ total\text{-}over\text{-}m\ I'\ \{\chi'\} \longrightarrow total\text{-}over\text{-}m\ I'\ \{\chi 2'\}$
using *can-decrease-count*[*of* χ' *Pos v count* χ' $(Pos\ v)$ ψ' *I*] **by** *auto*

obtain *C* **where** *χ2*: $\chi 2 = C + \{\#Neg\ v\#\}$ **and** *negC*: $Neg\ v \notin\#\ C$ **and** *posC*: $Pos\ v \notin\#\ C$
by (*metis* (*no-types*, *lifting*) *One-nat-def Posv Suc-inject Suc-pred χχ2-incl countχ2*
count-diff count-single gr0I insert-DiffM insert-DiffM2 multi-member-skip
old.nat.distinct(*2*))

obtain C' **where**

$\chi 2'$: $\chi 2' = C' + \{\#Pos\ v\#\}$ **and**
$posC'$: $Pos\ v \notin\# C'$ **and**
$negC'$: $Neg\ v \notin\# C'$
proof −
assume $a1$: $\bigwedge C'.\ [\![\chi 2' = C' + \{\#Pos\ v\#\};\ Pos\ v \notin\# C';\ Neg\ v \notin\# C']\!] \Longrightarrow thesis$
have $f2$: $\bigwedge n.\ (n::nat) - n = 0$
by *simp*
have $Neg\ v \notin\# \chi 2' - \{\#Pos\ v\#\}$
using *Negv* $\chi'\chi 2$-*incl* **by** *auto*
then show *?thesis*
using *f2 a1* **by** (*metis add.commute count*$\chi 2'$ *count-diff count-single insert-DiffM less-nat-zero-code zero-less-one*)
qed

have *already-used-inv* ψ'
using *rtranclp-inference-preserves-already-used-inv*[*of* ψ ψ'] *a-u-i inf* **by** *blast*
then have *a-u-i*-ψ'': *already-used-inv* ψ''
using *rtranclp-inference-preserves-already-used-inv a-u-i inf'* **unfolding** *tautology-def*
by *simp*

have *totC*: *total-over-m I* $\{C\}$
using *tot-imp*χ *tot*χ *tot-over-m-remove*[*of I Pos v C*] *negC posC* **unfolding** $\chi 2$
by (*metis total-over-m-sum uminus-Neg uminus-of-uminus-id*)
have *totC'*: *total-over-m I* $\{C'\}$
using *tot-imp*χ' *tot*χ' *total-over-m-sum tot-over-m-remove*[*of I Neg v C'*] *negC' posC'*
unfolding $\chi 2'$ **by** (*metis total-over-m-sum uminus-Neg*)
have $\neg\ I \models C + C'$
using χ $I\chi$ χ' $I\chi'$ **unfolding** $\chi 2$ $\chi 2'$ *true-cls-def Bex-mset-def*
by (*metis add-gr-0 count-union true-cls-singleton true-cls-union-increase*)
then have *part-I*-ψ''': *partial-interps Leaf I* $(fst\ \psi'' \cup \{C + C'\})$
using *totC totC'* **by** *simp*
(*metis* $\langle\neg\ I \models C + C'\rangle$ *atms-of-ms-singleton total-over-m-def total-over-m-sum*)
{
assume $(\{\#Pos\ v\#\} + C', \{\#Neg\ v\#\} + C) \notin snd\ \psi''$
then have *inf''*: *inference* ψ'' $(fst\ \psi'' \cup \{C + C'\},\ snd\ \psi'' \cup \{(\chi 2', \chi 2)\})$
using *add.commute* φ' $\chi 2incl$ $\langle\chi 2' \in fst\ \psi''\rangle$ **unfolding** $\chi 2$ $\chi 2'$
by (*metis prod.collapse inference-step resolution*)
have $inference^{**}$ ψ $(fst\ \psi'' \cup \{C + C'\},\ snd\ \psi'' \cup \{(\chi 2', \chi 2)\})$
using *inf inf' inf''* *rtranclp-trans* **by** *auto*
moreover have *sem-tree-size Leaf* $<$ *sem-tree-size xs* **unfolding** *xs* **by** *auto*
ultimately have *?case* **using** *part-I*-ψ''' **by** (*metis fst-conv*)
}
moreover {
assume a: $(\{\#Pos\ v\#\} + C', \{\#Neg\ v\#\} + C) \in snd\ \psi''$
then have $(\exists \chi \in fst\ \psi''.\ (\forall I.\ total\text{-}over\text{-}m\ I\ \{C{+}C'\} \longrightarrow total\text{-}over\text{-}m\ I\ \{\chi\})$
$\wedge\ (\forall I.\ total\text{-}over\text{-}m\ I\ \{\chi\} \longrightarrow I \models \chi \longrightarrow I \models C' + C))$
$\vee$ *tautology* $(C' + C)$
proof −
obtain p **where** p: $Pos\ p \in\# (\{\#Pos\ v\#\} + C')$ **and**
n: $Neg\ p \in\# (\{\#Neg\ v\#\} + C)$ **and**
decomp: $((\exists \chi {\in} fst\ \psi''.$
$(\forall I.\ total\text{-}over\text{-}m\ I\ \{((\{\#Pos\ v\#\} + C') - \{\#Pos\ p\#\}$
$+ ((\{\#Neg\ v\#\} + C) - \{\#Neg\ p\#\})\}$
$\longrightarrow total\text{-}over\text{-}m\ I\ \{\chi\})$
$\wedge\ (\forall I.\ total\text{-}over\text{-}m\ I\ \{\chi\} \longrightarrow I \models \chi$

$\longrightarrow I \models (\{\#Pos\ v\#\} + C') - \{\#Pos\ p\#\} + ((\{\#Neg\ v\#\} + C) - \{\#Neg\ p\#\}))$
)
$\vee$ *tautology* $((\{\#Pos\ v\#\} + C') - \{\#Pos\ p\#\} + ((\{\#Neg\ v\#\} + C) - \{\#Neg\ p\#\})))$
using *a* **by** (*blast intro*: *allE*[*OF a-u-i-ψ''*[*unfolded subsumes-def Ball-def*],
of $(\{\#Pos\ v\#\} + C', \{\#Neg\ v\#\} + C)$]])
{ assume $p \neq v$
then have *Pos* $p \in\#\ C' \wedge$ *Neg* $p \in\#\ C$ **using** *p n* **by** *force*
then have *?thesis* **by** (*metis add-gr-0 count-union tautology-Pos-Neg*)
}
moreover {
assume $p = v$
then have *?thesis* **using** *decomp* **by** (*metis add.commute add-diff-cancel-left'*)
}
ultimately show *?thesis* **by** *auto*
qed
moreover {
assume $\exists\chi \in$ *fst* ψ''. $(\forall I.$ *total-over-m* $I\ \{C{+}C'\} \longrightarrow$ *total-over-m* $I\ \{\chi\})$
$\wedge\ (\forall I.$ *total-over-m* $I\ \{\chi\} \longrightarrow I \models \chi \longrightarrow I \models C' + C)$
then obtain ϑ **where** ϑ: $\vartheta \in$ *fst* ψ'' **and**
tot-ϑ-CC': $\forall I.$ *total-over-m* $I\ \{C{+}C'\} \longrightarrow$ *total-over-m* $I\ \{\vartheta\}$ **and**
ϑ-inv: $\forall I.$ *total-over-m* $I\ \{\vartheta\} \longrightarrow I \models \vartheta \longrightarrow I \models C' + C$ **by** *blast*
have *partial-interps Leaf I* (*fst* ψ'')
using *tot-ϑ-CC' ϑ ϑ-inv totC totC'* ‹$\neg\ I \models C + C'$› *total-over-m-sum* **by** *fastforce*
moreover have *sem-tree-size Leaf* < *sem-tree-size xs* **unfolding** *xs* **by** *auto*
ultimately have *?case* **by** (*metis inf inf' rtranclp-trans*)
}
moreover {
assume *tautCC'*: *tautology* $(C' + C)$
have *total-over-m* $I\ \{C'{+}C\}$ **using** *totC totC' total-over-m-sum* **by** *auto*
then have $\neg$*tautology* $(C' + C)$
using ‹$\neg\ I \models C + C'$› **unfolding** *add.commute*[*of C C'*] *total-over-m-def*
unfolding *tautology-def* **by** *auto*
then have *False* **using** *tautCC'* **unfolding** *tautology-def* **by** *auto*
}
ultimately have *?case* **by** *auto*
}
ultimately have *?case* **by** *auto*
}
ultimately have *?case* **using** *part* **by** (*metis* (*no-types*) *sem-tree-size.simps*(*1*))
}
moreover {
assume *size-ag*: *sem-tree-size ag* > *0*
have *sem-tree-size ag* < *sem-tree-size xs* **unfolding** *xs* **by** *auto*
moreover have *partial-interps ag* $(I \cup \{Pos\ v\})$ (*fst* ψ)
and *partad*: *partial-interps ad* $(I \cup \{Neg\ v\})$ (*fst* ψ)
using *part partial-interps.simps*(*2*) **unfolding** *xs* **by** *metis+*
moreover have *sem-tree-size ag* < *sem-tree-size xs* $\longrightarrow$ *finite* (*fst* ψ) $\longrightarrow$ *already-used-inv* ψ
$\longrightarrow$ (*partial-interps ag* $(I \cup \{Pos\ v\})$ (*fst* ψ) $\longrightarrow$
$(\exists$ *tree'* ψ'. *inference*** $\psi\ \psi' \wedge$ *partial-interps tree'* $(I \cup \{Pos\ v\})$ (*fst* ψ')
$\wedge$ (*sem-tree-size tree'* < *sem-tree-size ag* $\vee$ *sem-tree-size ag* = *0*)))
using *IH* **by** *auto*
ultimately obtain ψ' :: *'v state* **and** *tree'* :: *'v sem-tree* **where**
inf: *inference*** $\psi\ \psi'$
and *part*: *partial-interps tree'* $(I \cup \{Pos\ v\})$ (*fst* ψ')
and *size*: *sem-tree-size tree'* < *sem-tree-size ag* $\vee$ *sem-tree-size ag* = *0*

using *finite part rtranclp.rtrancl-refl a-u-i* **by** *blast*

have *partial-interps ad* $(I \cup \{Neg\ v\})$ $(fst\ \psi')$
using *rtranclp-inference-preserve-partial-tree inf partad* **by** *metis*
then have *partial-interps* (*Node v tree′ ad*) *I* $(fst\ \psi')$ **using** *part* **by** *auto*
then have *?case* **using** *inf size size-ag part* **unfolding** *xs* **by** *fastforce*
}
moreover {
assume *size-ad*: *sem-tree-size ad* > 0
have *sem-tree-size ad* $<$ *sem-tree-size xs* **unfolding** *xs* **by** *auto*
moreover have *partag*: *partial-interps ag* $(I \cup \{Pos\ v\})$ $(fst\ \psi)$ **and**
partial-interps ad $(I \cup \{Neg\ v\})$ $(fst\ \psi)$
using *part partial-interps.simps*(*2*) **unfolding** *xs* **by** *metis+*
moreover have *sem-tree-size ad* $<$ *sem-tree-size xs* $\longrightarrow$ *finite* $(fst\ \psi)$ $\longrightarrow$ *already-used-inv* ψ
$\longrightarrow$ (*partial-interps ad* $(I \cup \{Neg\ v\})$ $(fst\ \psi)$
$\longrightarrow$ $(\exists$ *tree′* ψ'. *inference*** $\psi\ \psi'$ $\wedge$ *partial-interps tree′* $(I \cup \{Neg\ v\})$ $(fst\ \psi')$
$\wedge$ (*sem-tree-size tree′* $<$ *sem-tree-size ad* $\vee$ *sem-tree-size ad* $= 0$)))
using *IH* **by** *auto*
ultimately obtain ψ' :: *′v state* **and** *tree′* :: *′v sem-tree* **where**
inf: *inference*** $\psi\ \psi'$
and *part*: *partial-interps tree′* $(I \cup \{Neg\ v\})$ $(fst\ \psi')$
and *size*: *sem-tree-size tree′* $<$ *sem-tree-size ad* $\vee$ *sem-tree-size ad* $= 0$
using *finite part rtranclp.rtrancl-refl a-u-i* **by** *blast*

have *partial-interps ag* $(I \cup \{Pos\ v\})$ $(fst\ \psi')$
using *rtranclp-inference-preserve-partial-tree inf partag* **by** *metis*
then have *partial-interps* (*Node v ag tree′*) *I* $(fst\ \psi')$ **using** *part* **by** *auto*
then have *?case* **using** *inf size size-ad* **unfolding** *xs* **by** *fastforce*
}
ultimately have *?case* **by** *auto*
}
ultimately show *?case* **by** *auto*
qed

lemma *inference-completeness-inv*:
fixes ψ :: *′v ::linorder state*
assumes
unsat: ¬*satisfiable* $(fst\ \psi)$ **and**
finite: *finite* $(fst\ \psi)$ **and**
a-u-v: *already-used-inv* ψ
shows $\exists\psi'$. (*inference*** $\psi\ \psi'$ $\wedge$ $\{\#\} \in fst\ \psi'$)
proof −
obtain *tree* **where** *partial-interps tree* {} $(fst\ \psi)$
using *partial-interps-build-sem-tree-atms assms* **by** *metis*
then show *?thesis*
using *unsat finite a-u-v*
proof (*induct tree arbitrary*: ψ *rule*: *sem-tree-size*)
case (*bigger tree* ψ) **note** $H = this$
{
fix χ
assume *tree*: *tree* = *Leaf*
obtain χ **where** χ: ¬ {} $\models \chi$ **and** *tot*χ: *total-over-m* {} $\{\chi\}$ **and** $\chi\psi$: $\chi \in fst\ \psi$
using *H* **unfolding** *tree* **by** *auto*
moreover have $\{\#\} = \chi$
using *tot*χ **unfolding** *total-over-m-def total-over-set-def* **by** *fastforce*

moreover have *inference*** ψ ψ **by** *auto*
ultimately have *?case* **by** *metis*
}
moreover {
fix *v tree1 tree2*
assume *tree*: *tree* = *Node v tree1 tree2*
obtain
tree' ψ' **where** *inf*: *inference*** ψ ψ' **and**
part': *partial-interps tree'* {} (*fst* ψ') **and**
decrease: *sem-tree-size tree'* < *sem-tree-size tree* ∨ *sem-tree-size tree* = *0*
using *can-decrease-tree-size*[*of* ψ] *H*(*2*,*4*,*5*) **unfolding** *tautology-def* **by** *meson*
have *sem-tree-size tree'* < *sem-tree-size tree* **using** *decrease* **unfolding** *tree* **by** *auto*
moreover have *finite* (*fst* ψ') **using** *rtranclp-inference-preserves-finite inf H*(*4*) **by** *metis*
moreover have *unsatisfiable* (*fst* ψ')
using *inference-preserves-unsat inf bigger.prems*(*2*) **by** *blast*
moreover have *already-used-inv* ψ'
using *H*(*5*) *inf rtranclp-inference-preserves-already-used-inv*[*of* ψ ψ'] **by** *auto*
ultimately have *?case* **using** *inf rtranclp-trans part' H*(*1*) **by** *fastforce*
}
ultimately show *?case* **by** (*case-tac tree*, *auto*)
qed
qed

lemma *inference-completeness*:
fixes ψ :: *'v* ::*linorder state*
assumes *unsat*: ¬*satisfiable* (*fst* ψ)
and *finite*: *finite* (*fst* ψ)
and *snd* ψ = {}
shows ∃ψ'. (*rtranclp inference* ψ ψ' ∧ {#} ∈ *fst* ψ')
proof −
have *already-used-inv* ψ **unfolding** *assms* **by** *auto*
then show *?thesis* **using** *assms inference-completeness-inv* **by** *blast*
qed

lemma *inference-soundness*:
fixes ψ :: *'v* ::*linorder state*
assumes *rtranclp inference* ψ ψ' **and** {#} ∈ *fst* ψ'
shows *unsatisfiable* (*fst* ψ)
using *assms* **by** (*meson rtranclp-inference-preserves-un-sat satisfiable-def true-cls-empty true-clss-def*)

lemma *inference-soundness-and-completeness*:
fixes ψ :: *'v* ::*linorder state*
assumes *finite*: *finite* (*fst* ψ)
and *snd* ψ = {}
shows (∃ψ'. (*inference*** ψ ψ' ∧ {#} ∈ *fst* ψ')) ⟷ *unsatisfiable* (*fst* ψ)
using *assms inference-completeness inference-soundness* **by** *metis*

12.4 Lemma about the simplified state

abbreviation *simplified* ψ ≡ (*no-step simplify* ψ)

lemma *simplified-count*:
assumes *simp*: *simplified* ψ **and** χ: χ ∈ ψ
shows *count* χ *L* ≤ *1*
proof −

{
let $?\chi' = \chi - \{\#L,\ L\#\}$
assume $count\ \chi\ L \geq 2$
then have $f1$: $count\ (\chi - \{\#L,\ L\#\} + \{\#L,\ L\#\})\ L = count\ \chi\ L$
by *simp*
then have $L \in\#\ \chi - \{\#L\#\}$
by *simp*
then have χ': $?\chi' + \{\#L\#\} + \{\#L\#\} = \chi$
using *f1* **by** (*metis* (*no-types*) *diff-diff-add diff-single-eq-union union-assoc union-single-eq-member*)
have $\exists\,\psi'.\ simplify\ \psi\ \psi'$
by (*metis* (*no-types*, *hide-lams*) χ χ' *add.commute factoring-imp-simplify union-assoc*)
then have *False* **using** *simp* **by** *auto*
}
then show *?thesis* **by** *arith*
qed

lemma *simplified-no-both*:
assumes *simp*: *simplified* ψ **and** χ: $\chi \in \psi$
shows $\neg\ (L \in\#\ \chi \wedge -L \in\#\ \chi)$
proof (*rule ccontr*)
assume $\neg\ \neg\ (L \in\#\ \chi \wedge -\ L \in\#\ \chi)$
then have $L \in\#\ \chi \wedge -\ L \in\#\ \chi$ **by** *metis*
then obtain χ' **where** $\chi = \chi' + \{\#Pos\ (atm\text{-}of\ L)\#\} + \{\#Neg\ (atm\text{-}of\ L)\#\}$
by (*metis Neg-atm-of-iff Pos-atm-of-iff diff-union-swap insert-DiffM2 uminus-Neg uminus-Pos*)
then show *False* **using** χ *simp tautology-deletion* **by** *fastforce*
qed

lemma *simplified-not-tautology*:
assumes *simplified* $\{\psi\}$
shows $\sim$*tautology* ψ
proof (*rule ccontr*)
assume $\sim$ *?thesis*
then obtain p **where** $Pos\ p \in\#\ \psi \wedge Neg\ p \in\#\ \psi$ **using** *tautology-decomp* **by** *metis*
then obtain χ **where** $\psi = \chi + \{\#Pos\ p\#\} + \{\#Neg\ p\#\}$
by (*metis insert-noteq-member literal.distinct*(*1*) *multi-member-split*)
then have $\sim$ *simplified* $\{\psi\}$ **by** (*auto intro*: *tautology-deletion*)
then show *False* **using** *assms* **by** *auto*
qed

lemma *simplified-remove*:
assumes *simplified* $\{\psi\}$
shows *simplified* $\{\psi - \{\#l\#\}\}$
proof (*rule ccontr*)
assume *ns*: $\neg$ *simplified* $\{\psi - \{\#l\#\}\}$
{
assume $\neg\ l \in\#\ \psi$
then have $\psi - \{\#l\#\} = \psi$ **by** *simp*
then have *False* **using** *ns assms* **by** *auto*
}
moreover {
assume $l\psi$: $l \in\#\ \psi$
have A: $\bigwedge A.\ A \in \{\psi - \{\#l\#\}\} \longleftrightarrow A + \{\#l\#\} \in \{\psi\}$ **by** (*auto simp add*: $l\psi$)
obtain l' **where** l': *simplify* $\{\psi - \{\#l\#\}\}\ l'$ **using** *ns* **by** *metis*
then have $\exists\, l'.\ simplify\ \{\psi\}\ l'$

proof (*induction rule*: *simplify.induct*)
case (*tautology-deletion A P*)
have $\{\#Neg\ P\#\} + (\{\#Pos\ P\#\} + (A + \{\#l\#\})) \in \{\psi\}$
by (*metis* (*no-types*) *A add.commute tautology-deletion.hyps union-lcomm*)
then show *?thesis*
by (*metis simplify.tautology-deletion*[*of* $A+\{\#l\#\}$ *P* $\{\psi\}$] *add.commute*)
next
case (*condensation A L*)
have $A + \{\#L\#\} + \{\#L\#\} + \{\#l\#\} \in \{\psi\}$
using *A condensation.hyps* **by** *blast*
then have $\{\#L, L\#\} + (A + \{\#l\#\}) \in \{\psi\}$
by (*metis* (*no-types*) *union-assoc union-commute*)
then show *?case*
using *factoring-imp-simplify* **by** *blast*
next
case (*subsumption A B*)
then show *?case* **by** *blast*
qed
then have *False* **using** *assms*(*1*) **by** *blast*
}
ultimately show *False* **by** *auto*
qed

lemma *in-simplified-simplified*:
assumes *simp*: *simplified* ψ **and** *incl*: $\psi' \subseteq \psi$
shows *simplified* ψ'
proof (*rule ccontr*)
assume $\neg$ *?thesis*
then obtain ψ'' **where** *simplify* $\psi'\ \psi''$ **by** *metis*
then have $\exists\, l'.$ *simplify* $\psi\ l'$
proof (*induction rule*: *simplify.induct*)
case (*tautology-deletion A P*)
then show *?thesis* **using** *simplify.tautology-deletion*[*of A P* ψ] *incl* **by** *blast*
next
case (*condensation A L*)
then show *?case* **using** *simplify.condensation*[*of A L* ψ] *incl* **by** *blast*
next
case (*subsumption A B*)
then show *?case* **using** *simplify.subsumption*[*of A* ψ *B*] *incl* **by** *auto*
qed
then show *False* **using** *assms*(*1*) **by** *blast*
qed

lemma *simplified-in*:
assumes *simplified* ψ
and $N \in \psi$
shows *simplified* $\{N\}$
using *assms* **by** (*metis Set.set-insert empty-subsetI in-simplified-simplified insert-mono*)

lemma *subsumes-imp-formula*:
assumes $\psi \leq\# \varphi$
shows $\{\psi\} \models p\ \varphi$
unfolding *true-clss-cls-def* **apply** *auto*
using *assms true-cls-mono-leD* **by** *blast*

lemma *simplified-imp-distinct-mset-tauto*:
  assumes *simp*: *simplified* ψ'
  shows *distinct-mset-set* ψ' **and** $\forall \chi \in \psi'$. $\neg$*tautology* χ
proof –
  show $\forall \chi \in \psi'$. $\neg$*tautology* χ
    using *simp* **by** (*auto simp add*: *simplified-in simplified-not-tautology*)

  show *distinct-mset-set* ψ'
    proof (*rule ccontr*)
      assume $\neg$*?thesis*
      then obtain χ **where** $\chi \in \psi'$ **and** $\neg$*distinct-mset* χ **unfolding** *distinct-mset-set-def* **by** *auto*
      then obtain L **where** *count* χ $L \geq 2$
        unfolding *distinct-mset-def* **by** (*metis gr-implies-not0 le-antisym less-one not-le simp simplified-count*)
      then show *False* **by** (*metis Suc-1* ‹$\chi \in \psi'$› *not-less-eq-eq simp simplified-count*)
    qed
qed

lemma *simplified-no-more-full1-simplified*:
  assumes *simplified* ψ
  shows $\neg$*full1 simplify* ψ ψ'
  using *assms* **unfolding** *full1-def* **by** (*meson tranclpD*)

12.5 Resolution and Invariants

inductive *resolution* :: *'v state* $\Rightarrow$ *'v state* $\Rightarrow$ *bool* **where**
full1-simp: *full1 simplify* N N' $\Longrightarrow$ *resolution* (N, *already-used*) (N', *already-used*) |
inferring: *inference* (N, *already-used*) (N', *already-used'*) $\Longrightarrow$ *simplified* N
  $\Longrightarrow$ *full simplify* N' N'' $\Longrightarrow$ *resolution* (N, *already-used*) (N'', *already-used'*)

12.5.1 Invariants

lemma *resolution-finite*:
  assumes *resolution* ψ ψ' **and** *finite* (*fst* ψ)
  shows *finite* (*fst* ψ')
  using *assms* **by** (*induct rule*: *resolution.induct*)
    (*auto simp add*: *full1-def full-def rtranclp-simplify-preserves-finite*
      dest: *tranclp-into-rtranclp inference-preserves-finite*)

lemma *rtranclp-resolution-finite*:
  assumes *resolution*** ψ ψ' **and** *finite* (*fst* ψ)
  shows *finite* (*fst* ψ')
  using *assms* **by** (*induct rule*: *rtranclp-induct*, *auto simp add*: *resolution-finite*)

lemma *resolution-finite-snd*:
  assumes *resolution* ψ ψ' **and** *finite* (*snd* ψ)
  shows *finite* (*snd* ψ')
  using *assms* **apply** (*induct rule*: *resolution.induct*, *auto simp add*: *inference-preserves-finite-snd*)
  using *inference-preserves-finite-snd snd-conv* **by** *metis*

lemma *rtranclp-resolution-finite-snd*:
  assumes *resolution*** ψ ψ' **and** *finite* (*snd* ψ)
  shows *finite* (*snd* ψ')
  using *assms* **by** (*induct rule*: *rtranclp-induct*, *auto simp add*: *resolution-finite-snd*)

lemma *resolution-always-simplified*:
assumes *resolution* ψ ψ'
shows *simplified* (*fst* ψ')
using *assms* **by** (*induct rule*: *resolution.induct*)
(*auto simp add*: *full1-def full-def*)

lemma *tranclp-resolution-always-simplified*:
assumes *tranclp resolution* ψ ψ'
shows *simplified* (*fst* ψ')
using *assms* **by** (*induct rule*: *tranclp.induct*, *auto simp add*: *resolution-always-simplified*)

lemma *resolution-atms-of*:
assumes *resolution* ψ ψ' **and** *finite* (*fst* ψ)
shows *atms-of-ms* (*fst* ψ') $\subseteq$ *atms-of-ms* (*fst* ψ)
using *assms* **apply** (*induct rule*: *resolution.induct*)
apply(*simp add*: *rtranclp-simplify-atms-of-ms tranclp-into-rtranclp full1-def*)
by (*metis* (*no-types*, *lifting*) *contra-subsetD fst-conv full-def*
inference-preserves-atms-of-ms rtranclp-simplify-atms-of-ms subsetI)

lemma *rtranclp-resolution-atms-of*:
assumes *resolution*** ψ ψ' **and** *finite* (*fst* ψ)
shows *atms-of-ms* (*fst* ψ') $\subseteq$ *atms-of-ms* (*fst* ψ)
using *assms* **apply** (*induct rule*: *rtranclp-induct*)
using *resolution-atms-of rtranclp-resolution-finite* **by** *blast+*

lemma *resolution-include*:
assumes *res*: *resolution* ψ ψ' **and** *finite*: *finite* (*fst* ψ)
shows *fst* ψ' $\subseteq$ *build-all-simple-clss* (*atms-of-ms* (*fst* ψ))
proof −
have *finite'*: *finite* (*fst* ψ') **using** *local.finite res resolution-finite* **by** *blast*
have *simplified* (*fst* ψ') **using** *res finite' resolution-always-simplified* **by** *blast*
then have *fst* ψ' $\subseteq$ *build-all-simple-clss* (*atms-of-ms* (*fst* ψ'))
using *simplified-in-build-all finite' simplified-imp-distinct-mset-tauto*[*of fst* ψ'] **by** *auto*
moreover have *atms-of-ms* (*fst* ψ') $\subseteq$ *atms-of-ms* (*fst* ψ)
using *res finite resolution-atms-of*[*of* ψ ψ'] **by** *auto*
ultimately show *?thesis* **by** (*meson atms-of-ms-finite local.finite order.trans rev-finite-subset*
build-all-simple-clss-mono)
qed

lemma *rtranclp-resolution-include*:
assumes *res*: *tranclp resolution* ψ ψ' **and** *finite*: *finite* (*fst* ψ)
shows *fst* ψ' $\subseteq$ *build-all-simple-clss* (*atms-of-ms* (*fst* ψ))
using *assms* **apply** (*induct rule*: *tranclp.induct*)
apply (*simp add*: *resolution-include*)
by (*meson atms-of-ms-finite build-all-simple-clss-finite build-all-simple-clss-mono finite-subset*
resolution-include rtranclp-resolution-atms-of set-rev-mp subsetI tranclp-into-rtranclp)

abbreviation *already-used-all-simple*
:: (*'a literal multiset* $\times$ *'a literal multiset*) *set* $\Rightarrow$ *'a set* $\Rightarrow$ *bool* **where**
already-used-all-simple already-used vars $\equiv$
($\forall$ (A, B) $\in$ *already-used*. *simplified* $\{A\}$ $\wedge$ *simplified* $\{B\}$ $\wedge$ *atms-of* A $\subseteq$ *vars* $\wedge$ *atms-of* B $\subseteq$ *vars*)

lemma *already-used-all-simple-vars-incl*:
assumes *vars* $\subseteq$ *vars'*
shows *already-used-all-simple a vars* $\Longrightarrow$ *already-used-all-simple a vars'*

using *assms* by *fast*

lemma *inference-clause-preserves-already-used-all-simple*:
assumes *inference-clause S S'*
and *already-used-all-simple* (*snd S*) *vars*
and *simplified* (*fst S*)
and *atms-of-ms* (*fst S*) $\subseteq$ *vars*
shows *already-used-all-simple* (*snd* (*fst S* $\cup$ {*fst S'*}, *snd S'*)) *vars*
using *assms*
proof (*induct rule*: *inference-clause.induct*)
case (*factoring L C N already-used*)
then show *?case* **by** (*simp add*: *simplified-in factoring-imp-simplify*)
next
case (*resolution P C N D already-used*) **note** *H* = *this*
show *?case* **apply** *clarify*
proof −
fix *A B v*
assume (*A*, *B*) $\in$ *snd* (*fst* (*N*, *already-used*)
$\cup$ {*fst* (*C* + *D*, *already-used* $\cup$ {({#*Pos P*#} + *C*, {#*Neg P*#} + *D*)})},
snd (*C* + *D*, *already-used* $\cup$ {({#*Pos P*#} + *C*, {#*Neg P*#} + *D*)}))
then have (*A*, *B*) $\in$ *already-used* $\vee$ (*A*, *B*) = ({#*Pos P*#} + *C*, {#*Neg P*#} + *D*) **by** *auto*
moreover {
assume (*A*, *B*) $\in$ *already-used*
then have *simplified* {*A*} $\wedge$ *simplified* {*B*} $\wedge$ *atms-of A* $\subseteq$ *vars* $\wedge$ *atms-of B* $\subseteq$ *vars*
using *H*(*4*) **by** *auto*
}
moreover {
assume *eq*: (*A*, *B*) = ({#*Pos P*#} + *C*, {#*Neg P*#} + *D*)
then have *simplified* {*A*} **using** *simplified-in H*(*1*,*5*) **by** *auto*
moreover have *simplified* {*B*} **using** *eq simplified-in H*(*2*,*5*) **by** *auto*
moreover have *atms-of A* $\subseteq$ *atms-of-ms N*
using *eq H*(*1*) *atms-of-atms-of-ms-mono*[*of A N*] **by** *auto*
moreover have *atms-of B* $\subseteq$ *atms-of-ms N*
using *eq H*(*2*) *atms-of-atms-of-ms-mono*[*of B N*] **by** *auto*
ultimately have *simplified* {*A*} $\wedge$ *simplified* {*B*} $\wedge$ *atms-of A* $\subseteq$ *vars* $\wedge$ *atms-of B* $\subseteq$ *vars*
using *H*(*6*) **by** *auto*
}
ultimately show *simplified* {*A*} $\wedge$ *simplified* {*B*} $\wedge$ *atms-of A* $\subseteq$ *vars* $\wedge$ *atms-of B* $\subseteq$ *vars*
by *fast*
qed
qed

lemma *inference-preserves-already-used-all-simple*:
assumes *inference S S'*
and *already-used-all-simple* (*snd S*) *vars*
and *simplified* (*fst S*)
and *atms-of-ms* (*fst S*) $\subseteq$ *vars*
shows *already-used-all-simple* (*snd S'*) *vars*
using *assms*
proof (*induct rule*: *inference.induct*)
case (*inference-step S clause already-used*)
then show *?case*
using *inference-clause-preserves-already-used-all-simple*[*of S* (*clause*, *already-used*) *vars*]
by *auto*
qed

lemma *already-used-all-simple-inv*:
  assumes *resolution* S S'
  and *already-used-all-simple* (*snd* S) *vars*
  and *atms-of-ms* (*fst* S) $\subseteq$ *vars*
  shows *already-used-all-simple* (*snd* S') *vars*
  using *assms*
proof (*induct rule*: *resolution.induct*)
  case (*full1-simp* N N')
  then show *?case* **by** *simp*
next
  case (*inferring* N *already-used* N' *already-used'* N'')
  then show *already-used-all-simple* (*snd* (N'', *already-used'*)) *vars*
    using *inference-preserves-already-used-all-simple*[*of* (N, *already-used*)] **by** *simp*
qed

lemma *rtranclp-already-used-all-simple-inv*:
  assumes *resolution*** S S'
  and *already-used-all-simple* (*snd* S) *vars*
  and *atms-of-ms* (*fst* S) $\subseteq$ *vars*
  and *finite* (*fst* S)
  shows *already-used-all-simple* (*snd* S') *vars*
  using *assms*
proof (*induct rule*: *rtranclp-induct*)
  case *base*
  then show *?case* **by** *simp*
next
  case (*step* S' S'') **note** *infstar* = *this*(*1*) **and** *IH* = *this*(*3*) **and** *res* = *this*(*2*) **and**
    already = *this*(*4*) **and** *atms* = *this*(*5*) **and** *finite* = *this*(*6*)
  have *already-used-all-simple* (*snd* S') *vars* **using** *IH already atms finite* **by** *simp*
  moreover have *atms-of-ms* (*fst* S') $\subseteq$ *atms-of-ms* (*fst* S)
    by (*simp add*: *infstar local.finite rtranclp-resolution-atms-of*)
  then have *atms-of-ms* (*fst* S') $\subseteq$ *vars* **using** *atms* **by** *auto*
  ultimately show *?case*
    using *already-used-all-simple-inv*[*OF res*] **by** *simp*
qed

lemma *inference-clause-simplified-already-used-subset*:
  assumes *inference-clause* S S'
  and *simplified* (*fst* S)
  shows *snd* S $\subset$ *snd* S'
  using *assms* **apply** (*induct rule*: *inference-clause.induct*, *auto*)
  using *factoring-imp-simplify* **by** *blast*

lemma *inference-simplified-already-used-subset*:
  assumes *inference* S S'
  and *simplified* (*fst* S)
  shows *snd* S $\subset$ *snd* S'
  using *assms* **apply** (*induct rule*: *inference.induct*)
  by (*metis inference-clause-simplified-already-used-subset snd-conv*)

lemma *resolution-simplified-already-used-subset*:
  assumes *resolution* S S'
  and *simplified* (*fst* S)
  shows *snd* S $\subset$ *snd* S'

using *assms* **apply** (*induct rule*: *resolution.induct*, *simp-all add*: *full1-def*)
apply (*meson tranclpD*)
by (*metis inference-simplified-already-used-subset fst-conv snd-conv*)

lemma *tranclp-resolution-simplified-already-used-subset*:
assumes *tranclp resolution S S'*
and *simplified* (*fst S*)
shows *snd S* $\subset$ *snd S'*
using *assms* **apply** (*induct rule*: *tranclp.induct*)
using *resolution-simplified-already-used-subset* **apply** *metis*
by (*meson tranclp-resolution-always-simplified resolution-simplified-already-used-subset less-trans*)

abbreviation *already-used-top vars* $\equiv$ *build-all-simple-clss vars* $\times$ *build-all-simple-clss vars*

lemma *already-used-all-simple-in-already-used-top*:
assumes *already-used-all-simple s vars* **and** *finite vars*
shows *s* $\subseteq$ *already-used-top vars*
proof
fix *x*
assume *x-s*: $x \in s$
obtain *A B* **where** *x*: $x = (A, B)$ **by** (*case-tac x*, *auto*)
then have *simplified* $\{A\}$ **and** *atms-of A* $\subseteq$ *vars* **using** *assms*(*1*) *x-s* **by** *fastforce+*
then have *A*: $A \in$ *build-all-simple-clss vars*
using *build-all-simple-clss-mono*[*of vars atms-of A*] *x assms*(*2*)
simplified-imp-distinct-mset-tauto[*of* $\{A\}$]
distinct-mset-not-tautology-implies-in-build-all-simple-clss **by** *fast*
moreover have *simplified* $\{B\}$ **and** *atms-of B* $\subseteq$ *vars* **using** *assms*(*1*) *x-s x* **by** *fast+*
then have *B*: $B \in$ *build-all-simple-clss vars*
using *simplified-imp-distinct-mset-tauto*[*of* $\{B\}$]
distinct-mset-not-tautology-implies-in-build-all-simple-clss
build-all-simple-clss-mono[*of vars atms-of B*] *x assms*(*2*) **by** *fast*
ultimately show $x \in$ *build-all-simple-clss vars* $\times$ *build-all-simple-clss vars*
unfolding *x* **by** *auto*
qed

lemma *already-used-top-finite*:
assumes *finite vars*
shows *finite* (*already-used-top vars*)
using *build-all-simple-clss-finite assms* **by** *auto*

lemma *already-used-top-increasing*:
assumes *var* $\subseteq$ *var'* **and** *finite var'*
shows *already-used-top var* $\subseteq$ *already-used-top var'*
using *assms build-all-simple-clss-mono* **by** *auto*

lemma *already-used-all-simple-finite*:
fixes *s* :: ('*a*::*linorder literal multiset* $\times$ '*a literal multiset*) *set* **and** *vars* :: '*a set*
assumes *already-used-all-simple s vars* **and** *finite vars*
shows *finite s*
using *assms already-used-all-simple-in-already-used-top*[*OF assms*(*1*)]
rev-finite-subset[*OF already-used-top-finite*[*of vars*]] **by** *auto*

abbreviation *card-simple vars* $\psi \equiv$ *card* (*already-used-top vars* $-$ ψ)

lemma *resolution-card-simple-decreasing*:
  assumes *res*: *resolution* ψ ψ'
  and *a-u-s*: *already-used-all-simple* (*snd* ψ) *vars*
  and *finite-v*: *finite vars*
  and *finite-fst*: *finite* (*fst* ψ)
  and *finite-snd*: *finite* (*snd* ψ)
  and *simp*: *simplified* (*fst* ψ)
  and *atms-of-ms* (*fst* ψ) $\subseteq$ *vars*
  shows *card-simple vars* (*snd* ψ') < *card-simple vars* (*snd* ψ)
proof −
  let *?vars* = *vars*
  let *?top* = *build-all-simple-clss ?vars* × *build-all-simple-clss ?vars*
  have *1*: *card-simple vars* (*snd* ψ) = *card ?top* − *card* (*snd* ψ)
    using *card-Diff-subset finite-snd already-used-all-simple-in-already-used-top*[*OF a-u-s*]
    finite-v **by** *metis*
  have *a-u-s'*: *already-used-all-simple* (*snd* ψ') *vars*
    using *already-used-all-simple-inv res a-u-s assms*(*7*) **by** *blast*
  have *f*: *finite* (*snd* ψ') **using** *already-used-all-simple-finite a-u-s' finite-v* **by** *auto*
  have *2*: *card-simple vars* (*snd* ψ') = *card ?top* − *card* (*snd* ψ')
    using *card-Diff-subset*[*OF f*] *already-used-all-simple-in-already-used-top*[*OF a-u-s' finite-v*]
    by *auto*
  have *card* (*already-used-top vars*) ≥ *card* (*snd* ψ')
    using *already-used-all-simple-in-already-used-top*[*OF a-u-s' finite-v*]
    card-mono[*of already-used-top vars snd* ψ'] *already-used-top-finite*[*OF finite-v*] **by** *metis*
  then show *?thesis*
    using *psubset-card-mono*[*OF f resolution-simplified-already-used-subset*[*OF res simp*]]
    unfolding *1 2* **by** *linarith*
qed

lemma *tranclp-resolution-card-simple-decreasing*:
  assumes *tranclp resolution* ψ ψ' **and** *finite-fst*: *finite* (*fst* ψ)
  and *already-used-all-simple* (*snd* ψ) *vars*
  and *atms-of-ms* (*fst* ψ) $\subseteq$ *vars*
  and *finite-v*: *finite vars*
  and *finite-snd*: *finite* (*snd* ψ)
  and *simplified* (*fst* ψ)
  shows *card-simple vars* (*snd* ψ') < *card-simple vars* (*snd* ψ)
  using *assms*
proof (*induct rule*: *tranclp.induct*)
  case (*r-into-trancl* ψ ψ')
  then show *?case* **by** (*simp add*: *resolution-card-simple-decreasing*)
next
  case (*trancl-into-trancl* ψ ψ' ψ'') **note** *res* = *this*(*1*) **and** *res'* = *this*(*3*) **and** *a-u-s* = *this*(*5*) **and**
    atms = *this*(*6*) **and** *f-v* = *this*(*7*) **and** *f-fst* = *this*(*4*) **and** *H* = *this*
  then have *card-simple vars* (*snd* ψ') < *card-simple vars* (*snd* ψ) **by** *auto*
  moreover have *a-u-s'*: *already-used-all-simple* (*snd* ψ') *vars*
    using *rtranclp-already-used-all-simple-inv*[*OF tranclp-into-rtranclp*[*OF res*] *a-u-s atms f-fst*] **.**
  have *finite* (*fst* ψ')
    by (*meson build-all-simple-clss-finite rev-finite-subset rtranclp-resolution-include*
      trancl-into-trancl.hyps(*1*) *trancl-into-trancl.prems*(*1*))
  moreover have *finite* (*snd* ψ') **using** *already-used-all-simple-finite*[*OF a-u-s' f-v*] **.**
  moreover have *simplified* (*fst* ψ') **using** *res tranclp-resolution-always-simplified* **by** *blast*
  moreover have *atms-of-ms* (*fst* ψ') $\subseteq$ *vars*
    by (*meson atms f-fst order.trans res rtranclp-resolution-atms-of tranclp-into-rtranclp*)

ultimately show *?case*
  using *resolution-card-simple-decreasing*[*OF res′ a-u-s′ f-v*] *f-v*
  less-trans[*of card-simple vars* (*snd* ψ'') *card-simple vars* (*snd* ψ')
    card-simple vars (*snd* ψ)]
  by *blast*
qed

lemma *tranclp-resolution-card-simple-decreasing-2*:
  assumes *tranclp resolution* ψ ψ'
  and *finite-fst*: *finite* (*fst* ψ)
  and *empty-snd*: *snd* ψ = {}
  and *simplified* (*fst* ψ)
  shows *card-simple* (*atms-of-ms* (*fst* ψ)) (*snd* ψ') < *card-simple* (*atms-of-ms* (*fst* ψ)) (*snd* ψ)
proof −
  let *?vars* = (*atms-of-ms* (*fst* ψ))
  have *already-used-all-simple* (*snd* ψ) *?vars* **unfolding** *empty-snd* **by** *auto*
  moreover have *atms-of-ms* (*fst* ψ) ⊆ *?vars* **by** *auto*
  moreover have *finite-v*: *finite ?vars* **using** *finite-fst* **by** *auto*
  moreover have *finite-snd*: *finite* (*snd* ψ) **unfolding** *empty-snd* **by** *auto*
  ultimately show *?thesis*
    using *assms*(*1*,*2*,*4*) *tranclp-resolution-card-simple-decreasing*[*of* ψ ψ'] **by** *presburger*
qed

12.5.2 well-foundness if the relation

lemma *wf-simplified-resolution*:
  assumes *f-vars*: *finite vars*
  shows *wf* {(*y*:: ′*v*:: *linorder state*, *x*). (*atms-of-ms* (*fst x*) ⊆ *vars* ∧ *simplified* (*fst x*)
    ∧ *finite* (*snd x*) ∧ *finite* (*fst x*) ∧ *already-used-all-simple* (*snd x*) *vars*) ∧ *resolution x y*}
proof −
  {
    fix *a b* :: ′*v*::*linorder state*
    assume (*b*, *a*) ∈ {(*y*, *x*). (*atms-of-ms* (*fst x*) ⊆ *vars* ∧ *simplified* (*fst x*) ∧ *finite* (*snd x*)
      ∧ *finite* (*fst x*) ∧ *already-used-all-simple* (*snd x*) *vars*) ∧ *resolution x y*}
    then have
      atms-of-ms (*fst a*) ⊆ *vars* **and**
      simp: *simplified* (*fst a*) **and**
      finite (*snd a*) **and**
      finite (*fst a*) **and**
      a-u-v: *already-used-all-simple* (*snd a*) *vars* **and**
      res: *resolution a b* **by** *auto*
    have *finite* (*already-used-top vars*) **using** *f-vars already-used-top-finite* **by** *blast*
    moreover have *already-used-top vars* ⊆ *already-used-top vars* **by** *auto*
    moreover have *snd b* ⊆ *already-used-top vars*
      using *already-used-all-simple-in-already-used-top*[*of snd b vars*]
      a-u-v already-used-all-simple-inv[*OF res*] ‹*finite* (*fst a*)› ‹*atms-of-ms* (*fst a*) ⊆ *vars*› *f-vars*
      by *presburger*
    moreover have *snd a* ⊂ *snd b* **using** *resolution-simplified-already-used-subset*[*OF res simp*] **.**
    ultimately have *finite* (*already-used-top vars*) ∧ *already-used-top vars* ⊆ *already-used-top vars*
      ∧ *snd b* ⊆ *already-used-top vars* ∧ *snd a* ⊂ *snd b* **by** *metis*
  }
  then show *?thesis* **using** *wf-bounded-set*[*of* {(*y*:: ′*v*:: *linorder state*, *x*).
    (*atms-of-ms* (*fst x*) ⊆ *vars*
    ∧ *simplified* (*fst x*) ∧ *finite* (*snd x*) ∧ *finite* (*fst x*)∧ *already-used-all-simple* (*snd x*) *vars*)
    ∧ *resolution x y*} λ-. *already-used-top vars snd*] **by** *auto*

qed

lemma *wf-simplified-resolution'*:
  assumes *f-vars*: *finite vars*
  shows *wf* $\{(y{::}\ 'v{::}\ linorder\ state,\ x).\ (atms\text{-}of\text{-}ms\ (fst\ x) \subseteq vars \wedge \neg simplified\ (fst\ x)$
    $\wedge\ finite\ (snd\ x) \wedge finite\ (fst\ x) \wedge already\text{-}used\text{-}all\text{-}simple\ (snd\ x)\ vars) \wedge resolution\ x\ y\}$
  unfolding *wf-def*
   apply (*simp add*: *resolution-always-simplified*)
  by (*metis* (*mono-tags*, *hide-lams*) *fst-conv resolution-always-simplified*)

lemma *wf-resolution*:
  assumes *f-vars*: *finite vars*
  shows *wf* $(\{(y{::}\ 'v{::}\ linorder\ state,\ x).\ (atms\text{-}of\text{-}ms\ (fst\ x) \subseteq vars \wedge simplified\ (fst\ x)$
      $\wedge\ finite\ (snd\ x) \wedge finite\ (fst\ x) \wedge already\text{-}used\text{-}all\text{-}simple\ (snd\ x)\ vars) \wedge resolution\ x\ y\}$
    $\cup\ \{(y,\ x).\ (atms\text{-}of\text{-}ms\ (fst\ x) \subseteq vars \wedge \neg\ simplified\ (fst\ x) \wedge finite\ (snd\ x) \wedge finite\ (fst\ x)$
      $\wedge\ already\text{-}used\text{-}all\text{-}simple\ (snd\ x)\ vars) \wedge resolution\ x\ y\})$ (**is** *wf* (*?R* ∪ *?S*))
proof −
  have *Domain ?R Int Range ?S* = {} **using** *resolution-always-simplified* **by** *auto blast*
  then show *wf* (*?R* ∪ *?S*)
    using *wf-simplified-resolution*[*OF f-vars*] *wf-simplified-resolution'*[*OF f-vars*] *wf-Un*[*of ?R ?S*]
    by *fast*
qed

lemma *rtrancp-simplify-already-used-inv*:
  assumes $simplify^{**}\ S\ S'$
  and *already-used-inv* (*S*, *N*)
  shows *already-used-inv* (*S'*, *N*)
  using *assms* **apply** *induction*
  using *simplify-preserves-already-used-inv* **by** *fast+*

lemma *full1-simplify-already-used-inv*:
  assumes *full1 simplify S S'*
  and *already-used-inv* (*S*, *N*)
  shows *already-used-inv* (*S'*, *N*)
  using *assms tranclp-into-rtranclp*[*of simplify S S'*] *rtrancp-simplify-already-used-inv*
  unfolding *full1-def* **by** *fast*

lemma *full-simplify-already-used-inv*:
  assumes *full simplify S S'*
  and *already-used-inv* (*S*, *N*)
  shows *already-used-inv* (*S'*, *N*)
  using *assms rtrancp-simplify-already-used-inv* **unfolding** *full-def* **by** *fast*
lemma *resolution-already-used-inv*:
  assumes *resolution S S'*
  and *already-used-inv S*
  shows *already-used-inv S'*
  using *assms*
proof *induction*
  case (*full1-simp N N' already-used*)
  then show *?case* **using** *full1-simplify-already-used-inv* **by** *fast*
next
  case (*inferring N already-used N' already-used' N'''*) **note** *inf* = *this*(*1*) **and** *full* = *this*(*3*) **and** *a-u-v* = *this*(*4*)
  then show *?case*
    using *inference-preserves-already-used-inv*[*OF inf a-u-v*] *full-simplify-already-used-inv full*

by *fast*
qed

lemma *rtranclp-resolution-already-used-inv*:
assumes *resolution*** *S S'*
and *already-used-inv S*
shows *already-used-inv S'*
using *assms* **apply** *induction*
using *resolution-already-used-inv* **by** *fast+*

lemma *rtanclp-simplify-preserves-unsat*:
assumes *simplify*** ψ ψ'
shows *satisfiable* $\psi' \longrightarrow$ *satisfiable* ψ
using *assms* **apply** *induction*
using *simplify-clause-preserves-sat* **by** *blast+*

lemma *full1-simplify-preserves-unsat*:
assumes *full1 simplify* ψ ψ'
shows *satisfiable* $\psi' \longrightarrow$ *satisfiable* ψ
using *assms rtanclp-simplify-preserves-unsat*[*of* ψ ψ'] *tranclp-into-rtranclp*
unfolding *full1-def* **by** *metis*

lemma *full-simplify-preserves-unsat*:
assumes *full simplify* ψ ψ'
shows *satisfiable* $\psi' \longrightarrow$ *satisfiable* ψ
using *assms rtanclp-simplify-preserves-unsat*[*of* ψ ψ'] **unfolding** *full-def* **by** *metis*

lemma *resolution-preserves-unsat*:
assumes *resolution* ψ ψ'
shows *satisfiable* (*fst* ψ') $\longrightarrow$ *satisfiable* (*fst* ψ)
using *assms* **apply** (*induct rule*: *resolution.induct*)
using *full1-simplify-preserves-unsat* **apply** (*metis fst-conv*)
using *full-simplify-preserves-unsat simplify-preserves-unsat* **by** *fastforce*

lemma *rtranclp-resolution-preserves-unsat*:
assumes *resolution*** ψ ψ'
shows *satisfiable* (*fst* ψ') $\longrightarrow$ *satisfiable* (*fst* ψ)
using *assms* **apply** *induction*
using *resolution-preserves-unsat* **by** *fast+*

lemma *rtranclp-simplify-preserve-partial-tree*:
assumes *simplify*** *N N'*
and *partial-interps t I N*
shows *partial-interps t I N'*
using *assms* **apply** (*induction*, *simp*)
using *simplify-preserve-partial-tree* **by** *metis*

lemma *full1-simplify-preserve-partial-tree*:
assumes *full1 simplify N N'*
and *partial-interps t I N*
shows *partial-interps t I N'*
using *assms rtranclp-simplify-preserve-partial-tree*[*of N N' t I*] *tranclp-into-rtranclp*
unfolding *full1-def* **by** *fast*

lemma *full-simplify-preserve-partial-tree*:

assumes *full simplify* N N'
and *partial-interps* t I N
shows *partial-interps* t I N'
using *assms rtranclp-simplify-preserve-partial-tree*[*of* N N' t I] *tranclp-into-rtranclp*
unfolding *full-def* **by** *fast*

lemma *resolution-preserve-partial-tree*:
assumes *resolution* S S'
and *partial-interps* t I (*fst* S)
shows *partial-interps* t I (*fst* S')
using *assms* **apply** *induction*
using *full1-simplify-preserve-partial-tree fst-conv* **apply** *metis*
using *full-simplify-preserve-partial-tree inference-preserve-partial-tree* **by** *fastforce*

lemma *rtranclp-resolution-preserve-partial-tree*:
assumes $resolution^{**}$ S S'
and *partial-interps* t I (*fst* S)
shows *partial-interps* t I (*fst* S')
using *assms* **apply** *induction*
using *resolution-preserve-partial-tree* **by** *fast+*
thm *nat-less-induct nat.induct*

lemma *nat-ge-induct*[*case-names 0 Suc*]:
assumes $P\ 0$
and $(\bigwedge n.\ (\bigwedge m.\ m<Suc\ n \Longrightarrow P\ m) \Longrightarrow P\ (Suc\ n))$
shows $P\ n$
using *assms* **apply** (*induct rule*: *nat-less-induct*)
by (*case-tac n*) *auto*

lemma *wf-always-more-step-False*:
assumes *wf* R
shows $(\forall x.\ \exists z.\ (z, x) \in R) \Longrightarrow False$
using *assms* **unfolding** *wf-def* **by** (*meson Domain.DomainI assms wfE-min*)

lemma *finite-finite-mset-element-of-mset*[*simp*]:
assumes *finite* N
shows *finite* $\{f\ \varphi\ L\ |\varphi\ L.\ \varphi \in N \wedge L \in\#\ \varphi \wedge P\ \varphi\ L\}$
using *assms*
proof (*induction N rule*: *finite-induct*)
case *empty*
show *?case* **by** *auto*
next
case (*insert x N*) **note** *finite* = *this*(*1*) **and** *IH* = *this*(*3*)
have $\{f\ \varphi\ L\ |\varphi\ L.\ (\varphi = x \vee \varphi \in N) \wedge L \in\#\ \varphi \wedge P\ \varphi\ L\} \subseteq \{f\ x\ L\ |\ L.\ L \in\#\ x \wedge P\ x\ L\}$
$\cup \{f\ \varphi\ L\ |\varphi\ L.\ \varphi \in N \wedge L \in\#\ \varphi \wedge P\ \varphi\ L\}$ **by** *auto*
moreover have *finite* $\{f\ x\ L\ |\ L.\ L \in\#\ x\}$ **by** *auto*
ultimately show *?case* **using** *IH finite-subset* **by** *fastforce*
qed

value *card*
value *filter-mset*
value $\{\#count\ \varphi\ L\ |L \in\#\ \varphi.\ 2 \leq count\ \varphi\ L\#\}$
value $(\lambda\varphi.\ msetsum\ \{\#count\ \varphi\ L\ |L \in\#\ \varphi.\ 2 \leq count\ \varphi\ L\#\})$

syntax
  -comprehension1'-mset :: *'a* ⇒ *'b* ⇒ *'b multiset* ⇒ *'a multiset*
    (({#-/. - : *setof* -#}))
translations
  {#*e*. *x*: *setof M*#} == *CONST set-mset* (*CONST image-mset* (%*x*. *e*) *M*)
value {# *a*. *a* : *setof* {#*1*,*1*,*2*::*int*#}#} = {*1*,*2*}

definition *sum-count-ge-2* :: *'a multiset set* ⇒ *nat* (Ξ) **where**
sum-count-ge-2 ≡ *folding.F* (λφ. *op* +(*msetsum* {#*count* φ *L* |*L* ∈# φ. *2* ≤ *count* φ *L*#})) *0*

interpretation *sum-count-ge-2*:
  folding (λφ. *op* +(*msetsum* {#*count* φ *L* |*L* ∈# φ. *2* ≤ *count* φ *L*#})) *0*
rewrites
  folding.F (λφ. *op* +(*msetsum* {#*count* φ *L* |*L* ∈# φ. *2* ≤ *count* φ *L*#})) *0* = *sum-count-ge-2*
proof −
  show *folding* (λφ. *op* + (*msetsum* (*image-mset* (*count* φ) {# *L* :# φ. *2* ≤ *count* φ *L*#})))
    by *standard auto*
  then interpret *sum-count-ge-2*:
    folding (λφ. *op* +(*msetsum* {#*count* φ *L* |*L* ∈# φ. *2* ≤ *count* φ *L*#})) *0* **.**
  show *folding.F* (λφ. *op* + (*msetsum* (*image-mset* (*count* φ) {# *L* :# φ. *2* ≤ *count* φ *L*#}))) *0*
    = *sum-count-ge-2* **by** (*auto simp add*: *sum-count-ge-2-def*)
qed

lemma *finite-incl-le-setsum*:
 finite (*B*::*'a multiset set*) ⟹ *A* ⊆ *B* ⟹ Ξ *A* ≤ Ξ *B*
proof (*induction arbitrary*:*A rule*: *finite-induct*)
  case *empty*
  then show *?case* **by** *simp*
next
  case (*insert a F*) **note** *finite* = *this*(*1*) **and** *aF* = *this*(*2*) **and** *IH* = *this*(*3*) **and** *AF* = *this*(*4*)
  show *?case*
    proof (*cases a* ∈ *A*)
      assume *a* ∉ *A*
      then have *A* ⊆ *F* **using** *AF* **by** *auto*
      then show *?case* **using** *IH*[*of A*] **by** (*simp add*: *aF local.finite*)
    next
      assume *aA*: *a* ∈ *A*
      then have *A* − {*a*} ⊆ *F* **using** *AF* **by** *auto*
      then have Ξ (*A* − {*a*}) ≤ Ξ *F* **using** *IH* **by** *blast*
      then show *?case*
        proof −
          obtain *nn* :: *nat* ⇒ *nat* ⇒ *nat* **where**
            ∀ *x0 x1*. (∃ *v2*. *x0* = *x1* + *v2*) = (*x0* = *x1* + *nn x0 x1*)
            by *moura*
          then have Ξ *F* = Ξ (*A* − {*a*}) + *nn* (Ξ *F*) (Ξ (*A* − {*a*}))
            using *Nat.le-iff-add* ‹Ξ (*A* − {*a*}) ≤ Ξ *F*› **by** *presburger*
          then show *?thesis*
            by (*metis* (*no-types*) *Nat.le-iff-add aA aF add.assoc finite.insertI finite-subset*
              insert.prems local.finite sum-count-ge-2.insert sum-count-ge-2.remove)
        qed
    qed
qed

lemma *mset-condensation1*:

$\{\#\ La :\#\ A + \{\#L\#\}.\ 2 \leq count\ (A + \{\#L\#\})\ La\#\} = \{\#\ La :\#\ A.\ La \neq L \wedge\ 2 \leq count\ A\ La\#\}$
$\#\cup$ (*if count A L* $\geq$ *1 then replicate-mset* (*count A L* + *1*) *L else* {#})
by (*auto intro*: *multiset-eqI*)
lemma *mset-condensation2*:
$\{\#\ La :\#\ A + \{\#L\#\}+ \{\#L\#\}.\ 2 \leq count\ (A + \{\#L\#\}+ \{\#L\#\})\ La\#\} = \{\#\ La :\#\ A.\ La \neq L \wedge$
$2 \leq count\ A\ La\#\}\ \#\cup\ (replicate\text{-}mset\ (count\ A\ L + 2)\ L)$
by (*auto intro*: *multiset-eqI*)

lemma *msetsum-disjoint*:
assumes $A\ \#\cap\ B = \{\#\}$
shows $(\sum La \in\# A\ \#\cup\ B.\ f\ La) =$
$(\sum La \in\# A.\ f\ La) + (\sum La \in\# B.\ f\ La)$
by (*metis assms diff-zero empty-sup image-mset-union msetsum.union multiset-inter-commute multiset-union-diff-commute sup-subset-mset-def zero-diff*)

lemma *msetsum-linear*[*simp*]:
fixes *C D* :: *'a* ⇒ *'b*::{*comm-monoid-add*}
shows $(\sum x \in\# A.\ C\ x + D\ x) = (\sum x \in\# A.\ C\ x) + (\sum x \in\# A.\ D\ x)$
by (*induction A*) (*auto simp*: *ac-simps*)

lemma *msetsum-if-eq*[*simp*]: $(\sum x \in\# A.\ if\ L = x\ then\ 1\ else\ 0) = count\ A\ L$
by (*induction A*) *auto*

lemma *filter-equality-in-mset*:
filter-mset (*op* = *L*) *A* = *replicate-mset* (*count A L*) *L*
by (*auto simp*: *multiset-eq-iff*)

lemma *comprehension-mset-False*[*simp*]:
$\{\#\ L \in\#\ A.\ False\#\} = \{\#\}$
by (*auto simp*: *multiset-eq-iff*)

lemma *simplify-finite-measure-decrease*:
simplify N N' ⟹ *finite N* ⟹ *card N'* + Ξ *N'* < *card N* + Ξ *N*
proof (*induction rule*: *simplify.induct*)
case (*tautology-deletion A P*) **note** *an* = *this*(*1*) **and** *fin* = *this*(*2*)
let *?N'* = *N* − {*A* + {#*Pos P*#} + {#*Neg P*#}}
have *card ?N'* < *card N*
by (*meson card-Diff1-less tautology-deletion.hyps tautology-deletion.prems*)
moreover have *?N'* ⊆ *N* **by** *auto*
then have *sum-count-ge-2 ?N'* ≤ *sum-count-ge-2 N* **using** *finite-incl-le-setsum*[*OF fin*] **by** *blast*
ultimately show *?case* **by** *linarith*
next
case (*condensation A L*) **note** *AN* = *this*(*1*) **and** *fin* = *this*(*2*)
let *?C'* = *A* + {#*L*#}
let *?C* = *A* + {#*L*#} + {#*L*#}
let *?N'* = *N* − {*?C*} ∪ {*?C'*}
have *card ?N'* ≤ *card N*
using *AN* **by** (*metis* (*no-types*, *lifting*) *Diff-subset Un-empty-right Un-insert-right card.remove card-insert-if card-mono fin finite-Diff order-refl*)
moreover have Ξ {*?C'*} < Ξ {*?C*}

```
proof -
  have mset-decomp:
    {# La ∈# A. (L = La ⟶ Suc 0 ≤ count A La) ∧ (L ≠ La ⟶ 2 ≤ count A La)#}
    = {# La ∈# A. L ≠ La ∧ 2 ≤ count A La#} +
      {# La ∈# A. L = La ∧ Suc 0 ≤ count A L#}
       by (auto simp: multiset-eq-iff ac-simps)
  have mset-decomp2: {# La ∈# A. L ≠ La ⟶ 2 ≤ count A La#} =
    {# La ∈# A. L ≠ La ∧ 2 ≤ count A La#} + replicate-mset (count A L) L
    by (auto simp: multiset-eq-iff)
  show ?thesis
    by (auto simp: mset-decomp mset-decomp2 filter-equality-in-mset ac-simps)
 qed
have Ξ ?N' < Ξ N
 proof cases
  assume a1: ?C' ∈ N
  then show ?thesis
   proof -
    have f2: ⋀m M. insert (m::'a literal multiset) (M − {m}) = M ∪ {} ∨ m ∉ M
      using Un-empty-right insert-Diff by blast
    have f3: ⋀m M Ma. insert (m::'a literal multiset) M − insert m Ma = M − insert m Ma
      by simp
    then have f4: ⋀M m. M − {m::'a literal multiset} = M ∪ {} ∨ m ∈ M
      using Diff-insert-absorb Un-empty-right by fastforce
    have f5: insert (A + {#L#} + {#L#}) N = N
      using f3 f2 Un-empty-right condensation.hyps insert-iff by fastforce
    have ⋀m M. insert (m::'a literal multiset) M = M ∪ {} ∨ m ∉ M
      using f3 f2 Un-empty-right add.right-neutral insert-iff by fastforce
    then have Ξ (N − {A + {#L#} + {#L#}}) < Ξ N
      using f5 f4 by (metis Un-empty-right ‹Ξ {A + {#L#}} < Ξ {A + {#L#} + {#L#}}›
        add.right-neutral add-diff-cancel-left' add-gr-0 diff-less fin finite.emptyI not-le
        sum-count-ge-2.empty sum-count-ge-2.insert-remove trans-le-add2)
    then show ?thesis
      using f3 f2 a1 by (metis (no-types) Un-empty-right Un-insert-right condensation.hyps
        insert-iff multi-self-add-other-not-self)
   qed
 next
  assume ?C' ∉ N
  have mset-decomp:
    {# La ∈# A. (L = La ⟶ Suc 0 ≤ count A La) ∧ (L ≠ La ⟶ 2 ≤ count A La)#}
    = {# La ∈# A. L ≠ La ∧ 2 ≤ count A La#} +
      {# La ∈# A. L = La ∧ Suc 0 ≤ count A L#}
       by (auto simp: multiset-eq-iff ac-simps)
  have mset-decomp2: {# La ∈# A. L ≠ La ⟶ 2 ≤ count A La#} =
    {# La ∈# A. L ≠ La ∧ 2 ≤ count A La#} + replicate-mset (count A L) L
    by (auto simp: multiset-eq-iff)

  show ?thesis
    using ‹Ξ {A + {#L#}} < Ξ {A + {#L#} + {#L#}}› condensation.hyps fin
    sum-count-ge-2.remove[of - A + {#L#} + {#L#}] ‹?C' ∉ N›
    by (auto simp: mset-decomp mset-decomp2 filter-equality-in-mset)
 qed
ultimately show ?case by linarith
next
 case (subsumption A B) note AN = this(1) and AB = this(2) and BN = this(3) and fin = this(4)
 have card (N − {B}) < card N using BN by (meson card-Diff1-less subsumption.prems)
```

moreover have $\Xi\ (N - \{B\}) \leq \Xi\ N$
by (*simp add*: *Diff-subset finite-incl-le-setsum subsumption.prems*)
ultimately show *?case* **by** *linarith*
qed

lemma *simplify-terminates*:
wf $\{(N', N).$ *finite* $N \land$ *simplify* $N\ N'\}$
using *assms* **apply** (*rule wfP-if-measure*[*of finite simplify* $\lambda N.$ *card* $N + \Xi\ N$])
using *simplify-finite-measure-decrease* **by** *blast*

lemma *wf-terminates*:
assumes *wf r*
shows $\exists N'.(N', N) \in r^* \ \land (\forall N''.\ (N'', N') \notin r)$
proof −
let $?P = \lambda N.\ (\exists N'.(N', N) \in r^* \ \land (\forall N''.\ (N'', N') \notin r))$
have $(\forall x.\ (\forall y.\ (y, x) \in r \longrightarrow ?P\ y) \longrightarrow ?P\ x)$
proof *clarify*
fix *x*
assume *H*: $\forall y.\ (y, x) \in r \longrightarrow ?P\ y$
{ assume $\exists y.\ (y, x) \in r$
then obtain *y* **where** *y*: $(y, x) \in r$ **by** *blast*
then have *?P y* **using** *H* **by** *blast*
then have *?P x* **using** *y* **by** (*meson rtrancl.rtrancl-into-rtrancl*)
}
moreover {
assume $\neg(\exists y.\ (y, x) \in r)$
then have *?P x* **by** *auto*
}
ultimately show *?P x* **by** *blast*
qed
moreover have $(\forall x.\ (\forall y.\ (y, x) \in r \longrightarrow ?P\ y) \longrightarrow ?P\ x) \longrightarrow$ *All ?P*
using *assms* **unfolding** *wf-def* **by** (*rule allE*)
ultimately have *All ?P* **by** *blast*
then show *?P N* **by** *blast*
qed

lemma *rtranclp-simplify-terminates*:
assumes *fin*: *finite N*
shows $\exists N'.$ *simplify*** $N\ N' \land$ *simplified* N'
proof −
have *H*: $\{(N', N).$ *finite* $N \land$ *simplify* $N\ N'\} = \{(N', N).$ *simplify* $N\ N' \land$ *finite* $N\}$ **by** *auto*
then have *wf*: *wf* $\{(N', N).$ *simplify* $N\ N' \land$ *finite* $N\}$
using *simplify-terminates* **by** (*simp add*: *H*)
obtain N' **where** N': $(N', N) \in \{(b, a).$ *simplify* $a\ b \land$ *finite* $a\}^*$ **and**
more: $(\forall N''.\ (N'', N') \notin \{(b, a).$ *simplify* $a\ b \land$ *finite* $a\})$
using *Prop-Resolution.wf-terminates*[*OF wf*, *of N*] **by** *blast*
have *1*: *simplify*** $N\ N'$
using N' **by** (*induction rule*: *rtrancl.induct*) *auto*
then have *finite* N' **using** *fin rtranclp-simplify-preserves-finite* **by** *blast*
then have *2*: $\forall N''.\ \neg$*simplify* $N'\ N''$ **using** *more* **by** *auto*

show *?thesis* **using** *1 2* **by** *blast*
qed

lemma *finite-simplified-full1-simp*:
  assumes *finite N*
  shows *simplified N* $\vee$ ($\exists$ *N'*. *full1 simplify N N'*)
  using *rtranclp-simplify-terminates*[*OF assms*] **unfolding** *full1-def*
  by (*metis Nitpick.rtranclp-unfold*)

lemma *finite-simplified-full-simp*:
  assumes *finite N*
  shows $\exists$ *N'*. *full simplify N N'*
  using *rtranclp-simplify-terminates*[*OF assms*] **unfolding** *full-def* **by** *metis*

lemma *can-decrease-tree-size-resolution*:
  fixes ψ :: *'v state* **and** *tree* :: *'v sem-tree*
  assumes *finite* (*fst* ψ) **and** *already-used-inv* ψ
  and *partial-interps tree I* (*fst* ψ)
  and *simplified* (*fst* ψ)
  shows $\exists$(*tree'*:: *'v sem-tree*) ψ'. *resolution*** ψ ψ' $\wedge$ *partial-interps tree' I* (*fst* ψ')
    $\wedge$ (*sem-tree-size tree'* < *sem-tree-size tree* $\vee$ *sem-tree-size tree* = *0*)
  using *assms*
proof (*induct arbitrary*: *I rule*: *sem-tree-size*)
  case (*bigger xs I*) **note** *IH* = *this*(*1*) **and** *finite* = *this*(*2*) **and** *a-u-i* = *this*(*3*) **and** *part* = *this*(*4*)
    and *simp* = *this*(*5*)

  { assume *sem-tree-size xs* = *0*
    then have *?case* **using** *part* **by** *blast*
  }

  moreover {
    assume *sn0*: *sem-tree-size xs* > *0*
    obtain *ag ad v* **where** *xs*: *xs* = *Node v ag ad* **using** *sn0* **by** (*case-tac xs*, *auto*)
    {
      assume *sem-tree-size ag* = *0* $\wedge$ *sem-tree-size ad* = *0*
      then have *ag*: *ag* = *Leaf* **and** *ad*: *ad* = *Leaf* **by** (*case-tac ag*, *auto*, *case-tac ad*, *auto*)

      then obtain χ χ' **where**
        χ: $\neg$ *I* $\cup$ {*Pos v*} $\models$ χ **and**
        totχ: *total-over-m* (*I* $\cup$ {*Pos v*}) {χ} **and**
        $\chi\psi$: χ $\in$ *fst* ψ **and**
        χ': $\neg$ *I* $\cup$ {*Neg v*} $\models$ χ' **and**
        totχ': *total-over-m* (*I* $\cup$ {*Neg v*}) {χ'} **and** $\chi'\psi$: χ' $\in$ *fst* ψ
        using *part* **unfolding** *xs* **by** *auto*
      have *Posv*: *Pos v* $\notin\#$ χ **using** χ **unfolding** *true-cls-def true-lit-def* **by** *auto*
      have *Negv*: *Neg v* $\notin\#$ χ' **using** χ' **unfolding** *true-cls-def true-lit-def* **by** *auto*
      {
        assume *Neg*χ: $\neg$*Neg v* $\in\#$ χ
        then have $\neg$ *I* $\models$ χ **using** χ *Posv* **unfolding** *true-cls-def true-lit-def* **by** *auto*
        moreover have *total-over-m I* {χ}
          using *Posv Neg*χ *atm-imp-pos-or-neg-lit tot*χ **unfolding** *total-over-m-def total-over-set-def*
          by *fastforce*
        ultimately have *partial-interps Leaf I* (*fst* ψ)
        and *sem-tree-size Leaf* < *sem-tree-size xs*
        and *resolution*** ψ ψ
          unfolding *xs* **by** (*auto simp add*: $\chi\psi$)
      }

moreover {
  assume *Posχ*: $\neg$*Pos v* $\in\#$ χ'
  then have *Iχ*: $\neg$ *I* $\models$ χ' **using** χ' *Posv* **unfolding** *true-cls-def true-lit-def* **by** *auto*
  moreover have *total-over-m I* $\{\chi'\}$
    using *Negv Posχ atm-imp-pos-or-neg-lit totχ'*
    unfolding *total-over-m-def total-over-set-def* **by** *fastforce*
  ultimately have *partial-interps Leaf I* (*fst* ψ)
  and *sem-tree-size Leaf* $<$ *sem-tree-size xs*
  and *resolution*** ψ ψ **using** $\chi'\psi$ *Iχ* **unfolding** *xs* **by** *auto*
}
moreover {
  assume *neg*: *Neg v* $\in\#$ χ **and** *pos*: *Pos v* $\in\#$ χ'
  have *count* χ (*Neg v*) = *1*
    using *simplified-count*[*OF simp χψ*] *neg* **by** (*metis One-nat-def Suc-le-mono Suc-pred eq-iff le0*)
  have *count* χ' (*Pos v*) = *1*
    using *simplified-count*[*OF simp χ'ψ*] *pos* **by** (*metis One-nat-def Suc-le-mono Suc-pred eq-iff le0*)
  obtain *C* **where** *χC*: $\chi = C + \{\#Neg\ v\#\}$ **and** *negC*: *Neg v* $\notin\#$ *C* **and** *posC*: *Pos v* $\notin\#$ *C*
    proof −
      assume *a1*: $\bigwedge C.\ [\![\chi = C + \{\#Neg\ v\#\};\ Neg\ v \notin\#\ C;\ Pos\ v \notin\#\ C]\!] \Longrightarrow thesis$
      have *f2*: $\bigwedge n.\ (0::nat) + n = n$
        by *simp*
      obtain *mm* :: *'v literal multiset* $\Rightarrow$ *'v literal* $\Rightarrow$ *'v literal multiset* **where**
        f3: $\{\#Neg\ v\#\} + mm\ \chi\ (Neg\ v) = \chi$
        by (*metis* (*no-types*) ‹*count* χ (*Neg v*) = *1*› *add.commute multi-member-split zero-less-one*)
      then have *Pos v* $\notin\#$ *mm* χ (*Neg v*)
        using *f2* **by** (*metis* (*no-types*) *Posv* ‹*count* χ (*Neg v*) = *1*› *add.right-neutral add-left-cancel count-single count-union less-nat-zero-code*)
      then show *?thesis*
        using *f3 a1* **by** (*metis* (*no-types*) ‹*count* χ (*Neg v*) = *1*› *add.commute add.right-neutral add-left-cancel count-single count-union less-nat-zero-code*)
    qed
  obtain *C'* **where**
    χC': $\chi' = C' + \{\#Pos\ v\#\}$ **and**
    posC': *Pos v* $\notin\#$ *C'* **and**
    negC': *Neg v* $\notin\#$ *C'*
    by (*metis* (*no-types, hide-lams*) *Negv* ‹*count* χ' (*Pos v*) = *1*› *add-diff-cancel-right' cancel-comm-monoid-add-class.diff-cancel count-diff count-single less-nat-zero-code mset-leD mset-le-add-left multi-member-split zero-less-one*)

  have *totC*: *total-over-m I* $\{C\}$
    using *totχ tot-over-m-remove*[*of I Pos v C*] *negC posC* **unfolding** *χC*
    by (*metis total-over-m-sum uminus-Neg uminus-of-uminus-id*)
  have *totC'*: *total-over-m I* $\{C'\}$
    using *totχ' total-over-m-sum tot-over-m-remove*[*of I Neg v C'*] *negC' posC'*
    unfolding *χC'* **by** (*metis total-over-m-sum uminus-Neg*)
  have $\neg$ *I* $\models$ $C + C'$
    using χ χ' *χC χC'* **by** *auto*
  then have *part-I-ψ'''*: *partial-interps Leaf I* (*fst* $\psi \cup \{C + C'\}$)
    using *totC totC'* ‹$\neg$ *I* $\models$ $C + C'$› **by** (*metis Un-insert-right insertI1 partial-interps.simps*(*1*) *total-over-m-sum*)
  {
    assume ($\{\#Pos\ v\#\} + C'$, $\{\#Neg\ v\#\} + C$) $\notin$ *snd* ψ

then have *inf''*: *inference ψ (fst ψ ∪ {C + C'}, snd ψ ∪ {(χ', χ)})*
by (*metis χ'ψ χC χC' χψ add.commute inference-step prod.collapse resolution*)
obtain *N'* **where** *full*: *full simplify (fst ψ ∪ {C + C'}) N'*
by (*metis finite-simplified-full-simp fst-conv inf'' inference-preserves-finite local.finite*)
have *resolution ψ (N', snd ψ ∪ {(χ', χ)})*
using *resolution.intros(2)[OF - simp full, of snd ψ snd ψ ∪ {(χ', χ)}] inf''*
by (*metis surjective-pairing*)
moreover have *partial-interps Leaf I N'*
using *full-simplify-preserve-partial-tree[OF full part-I-ψ'']* **.**
moreover have *sem-tree-size Leaf < sem-tree-size xs* **unfolding** *xs* **by** *auto*
ultimately have *?case*
by (*metis (no-types) prod.sel(1) rtranclp.rtrancl-into-rtrancl rtranclp.rtrancl-refl*)
}
moreover {
assume *a*: *({#Pos v#} + C', {#Neg v#} + C) ∈ snd ψ*
then have *(∃χ ∈ fst ψ. (∀I. total-over-m I {C+C'} ⟶ total-over-m I {χ})*
∧ (∀I. total-over-m I {χ} ⟶ I ⊨ χ ⟶ I ⊨ C' + C)) ∨ tautology (C' + C)
proof −
obtain *p* **where** *p*: *Pos p ∈# ({#Pos v#} + C') ∧ Neg p ∈# ({#Neg v#} + C)*
∧((∃χ∈fst ψ. (∀I. total-over-m I {({#Pos v#} + C') − {#Pos p#} + (({#Neg v#} + C) − {#Neg p#})} ⟶ total-over-m I {χ}) ∧ (∀I. total-over-m I {χ} ⟶ I ⊨ χ ⟶ I ⊨ ({#Pos v#} + C') − {#Pos p#} + (({#Neg v#} + C) − {#Neg p#}))) ∨ tautology (({#Pos v#} + C') − {#Pos p#} + (({#Neg v#} + C) − {#Neg p#})))
using *a* **by** (*blast intro*: *allE[OF a-u-i[unfolded subsumes-def Ball-def],*
of ({#Pos v#} + C', {#Neg v#} + C)])
{ assume *p ≠ v*
then have *Pos p ∈# C' ∧ Neg p ∈# C* **using** *p* **by** *force*
then have *?thesis* **by** (*metis add-gr-0 count-union tautology-Pos-Neg*)
}
moreover {
assume *p = v*
then have *?thesis* **using** *p* **by** (*metis add.commute add-diff-cancel-left'*)
}
ultimately show *?thesis* **by** *auto*
qed
moreover {
assume *∃χ ∈ fst ψ. (∀I. total-over-m I {C+C'} ⟶ total-over-m I {χ})*
∧ (∀I. total-over-m I {χ} ⟶ I ⊨ χ ⟶ I ⊨ C' + C)
then obtain *ϑ* **where**
ϑ: *ϑ ∈ fst ψ* **and**
tot-ϑ-CC': *∀I. total-over-m I {C+C'} ⟶ total-over-m I {ϑ}* **and**
ϑ-inv: *∀I. total-over-m I {ϑ} ⟶ I ⊨ ϑ ⟶ I ⊨ C' + C* **by** *blast*
have *partial-interps Leaf I (fst ψ)*
using *tot-ϑ-CC' ϑ ϑ-inv totC totC'* ‹¬ *I ⊨ C + C'*› *total-over-m-sum* **by** *fastforce*
moreover have *sem-tree-size Leaf < sem-tree-size xs* **unfolding** *xs* **by** *auto*
ultimately have *?case* **by** *blast*
}
moreover {
assume *tautCC'*: *tautology (C' + C)*
have *total-over-m I {C'+C}* **using** *totC totC' total-over-m-sum* **by** *auto*
then have *¬tautology (C' + C)*
using ‹¬ *I ⊨ C + C'*› **unfolding** *add.commute[of C C'] total-over-m-def*
unfolding *tautology-def* **by** *auto*
then have *False* **using** *tautCC'* **unfolding** *tautology-def* **by** *auto*

}
ultimately have *?case* by *auto*
}
ultimately have *?case* by *auto*
}
ultimately have *?case* using *part* by (*metis* (*no-types*) *sem-tree-size.simps*(*1*))
}
moreover {
assume *size-ag*: *sem-tree-size ag* > *0*
have *sem-tree-size ag* < *sem-tree-size xs* unfolding *xs* by *auto*
moreover have *partial-interps ag* ($I \cup \{Pos\ v\}$) (*fst* ψ)
and *partad*: *partial-interps ad* ($I \cup \{Neg\ v\}$) (*fst* ψ)
using *part partial-interps.simps*(*2*) unfolding *xs* by *metis+*
moreover
have *sem-tree-size ag* < *sem-tree-size xs* $\Longrightarrow$ *finite* (*fst* ψ) $\Longrightarrow$ *already-used-inv* ψ
$\Longrightarrow$ *partial-interps ag* ($I \cup \{Pos\ v\}$) (*fst* ψ) $\Longrightarrow$ *simplified* (*fst* ψ)
$\Longrightarrow$ $\exists$ *tree'* ψ'. *resolution*** ψ ψ' $\wedge$ *partial-interps tree'* ($I \cup \{Pos\ v\}$) (*fst* ψ')
$\wedge$ (*sem-tree-size tree'* < *sem-tree-size ag* $\vee$ *sem-tree-size ag* = *0*)
using *IH*[*of ag* $I \cup \{Pos\ v\}$] by *auto*
ultimately obtain ψ' :: *'v state* and *tree'* :: *'v sem-tree* where
inf: *resolution*** ψ ψ'
and *part*: *partial-interps tree'* ($I \cup \{Pos\ v\}$) (*fst* ψ')
and *size*: *sem-tree-size tree'* < *sem-tree-size ag* $\vee$ *sem-tree-size ag* = *0*
using *finite part rtranclp.rtrancl-refl a-u-i simp* by *blast*

have *partial-interps ad* ($I \cup \{Neg\ v\}$) (*fst* ψ')
using *rtranclp-resolution-preserve-partial-tree inf partad* by *fast*
then have *partial-interps* (*Node v tree' ad*) *I* (*fst* ψ') using *part* by *auto*
then have *?case* using *inf size size-ag part* unfolding *xs* by *fastforce*
}
moreover {
assume *size-ad*: *sem-tree-size ad* > *0*
have *sem-tree-size ad* < *sem-tree-size xs* unfolding *xs* by *auto*
moreover
have
partag: *partial-interps ag* ($I \cup \{Pos\ v\}$) (*fst* ψ) and
partial-interps ad ($I \cup \{Neg\ v\}$) (*fst* ψ)
using *part partial-interps.simps*(*2*) unfolding *xs* by *metis+*
moreover have *sem-tree-size ad* < *sem-tree-size xs* $\longrightarrow$ *finite* (*fst* ψ) $\longrightarrow$ *already-used-inv* ψ
$\longrightarrow$ (*partial-interps ad* ($I \cup \{Neg\ v\}$) (*fst* ψ) $\longrightarrow$ *simplified* (*fst* ψ)
$\longrightarrow$ ($\exists$ *tree'* ψ'. *resolution*** ψ ψ' $\wedge$ *partial-interps tree'* ($I \cup \{Neg\ v\}$) (*fst* ψ')
$\wedge$ (*sem-tree-size tree'* < *sem-tree-size ad* $\vee$ *sem-tree-size ad* = *0*)))
using *IH* by *blast*
ultimately obtain ψ' :: *'v state* and *tree'* :: *'v sem-tree* where
inf: *resolution*** ψ ψ'
and *part*: *partial-interps tree'* ($I \cup \{Neg\ v\}$) (*fst* ψ')
and *size*: *sem-tree-size tree'* < *sem-tree-size ad* $\vee$ *sem-tree-size ad* = *0*
using *finite part rtranclp.rtrancl-refl a-u-i simp* by *blast*

have *partial-interps ag* ($I \cup \{Pos\ v\}$) (*fst* ψ')
using *rtranclp-resolution-preserve-partial-tree inf partag* by *fast*
then have *partial-interps* (*Node v ag tree'*) *I* (*fst* ψ') using *part* by *auto*
then have *?case* using *inf size size-ad* unfolding *xs* by *fastforce*
}
ultimately have *?case* by *auto*

```
  }
  ultimately show ?case by auto
qed

lemma resolution-completeness-inv:
  fixes ψ :: 'v ::linorder state
  assumes
    unsat: ¬satisfiable (fst ψ) and
    finite: finite (fst ψ) and
    a-u-v: already-used-inv ψ
  shows ∃ψ'. (resolution** ψ ψ' ∧ {#} ∈ fst ψ')
proof −
  obtain tree where partial-interps tree {} (fst ψ)
    using partial-interps-build-sem-tree-atms assms by metis
  then show ?thesis
    using unsat finite a-u-v
    proof (induct tree arbitrary: ψ rule: sem-tree-size)
      case (bigger tree ψ) note H = this
      {
        fix χ
        assume tree: tree = Leaf
        obtain χ where χ: ¬ {} ⊨ χ and totχ: total-over-m {} {χ} and χψ: χ ∈ fst ψ
          using H unfolding tree by auto
        moreover have {#} = χ
          using H atms-empty-iff-empty totχ
          unfolding true-cls-def total-over-m-def total-over-set-def by fastforce
        moreover have resolution** ψ ψ by auto
        ultimately have ?case by metis
      }
      moreover {
        fix v tree1 tree2
        assume tree: tree = Node v tree1 tree2
        obtain ψ0 where ψ0: resolution** ψ ψ0 and simp: simplified (fst ψ0)
          proof −
            { assume simplified (fst ψ)
              moreover have resolution** ψ ψ by auto
              ultimately have thesis using that by blast
            }
            moreover {
              assume ¬simplified (fst ψ)
              then have ∃ψ'. full1 simplify (fst ψ) ψ'
                by (metis Nitpick.rtranclp-unfold bigger.prems(3) full1-def
                  rtranclp-simplify-terminates)
              then obtain N where full1 simplify (fst ψ) N by metis
              then have resolution ψ (N, snd ψ)
                using resolution.intros(1)[of fst ψ N snd ψ] by auto
              moreover have simplified N
                using ‹full1 simplify (fst ψ) N› unfolding full1-def by blast
              ultimately have ?thesis using that by force
            }
            ultimately show ?thesis by auto
          qed


        have p: partial-interps tree {} (fst ψ0)
```

and *uns*: *unsatisfiable* (*fst* ψ_0)
and *f*: *finite* (*fst* ψ_0)
and *a-u-v*: *already-used-inv* ψ_0
using ψ_0 *bigger.prems*(*1*) *rtranclp-resolution-preserve-partial-tree* **apply** *blast*
using ψ_0 *bigger.prems*(*2*) *rtranclp-resolution-preserves-unsat* **apply** *blast*
using ψ_0 *bigger.prems*(*3*) *rtranclp-resolution-finite* **apply** *blast*
using *rtranclp-resolution-already-used-inv*[*OF* ψ_0 *bigger.prems*(*4*)] **by** *blast*
obtain *tree'* ψ' **where**
inf: *resolution*** ψ_0 ψ' **and**
part': *partial-interps tree'* {} (*fst* ψ') **and**
decrease: *sem-tree-size tree'* < *sem-tree-size tree* ∨ *sem-tree-size tree* = *0*
using *can-decrease-tree-size-resolution*[*OF f a-u-v p simp*] **unfolding** *tautology-def*
by *meson*
have *s*: *sem-tree-size tree'* < *sem-tree-size tree* **using** *decrease* **unfolding** *tree* **by** *auto*
have *fin*: *finite* (*fst* ψ')
using *f inf rtranclp-resolution-finite* **by** *blast*
have *unsat*: *unsatisfiable* (*fst* ψ')
using *rtranclp-resolution-preserves-unsat inf uns* **by** *metis*
have *a-u-i'*: *already-used-inv* ψ'
using *a-u-v inf rtranclp-resolution-already-used-inv*[*of* ψ_0 ψ'] **by** *auto*
have *?case*
using *inf rtranclp-trans*[*of resolution*] *H*(*1*)[*OF s part' unsat fin a-u-i'*] ψ_0 **by** *blast*
}
ultimately show *?case* **by** (*case-tac tree*, *auto*)
qed
qed

lemma *resolution-preserves-already-used-inv*:
assumes *resolution S S'*
and *already-used-inv S*
shows *already-used-inv S'*
using *assms*
apply (*induct rule*: *resolution.induct*)
apply (*rule full1-simplify-already-used-inv*; *simp*)
apply (*rule full-simplify-already-used-inv*, *simp*)
apply (*rule inference-preserves-already-used-inv*, *simp*)
apply *blast*
done

lemma *rtranclp-resolution-preserves-already-used-inv*:
assumes *resolution*** *S S'*
and *already-used-inv S*
shows *already-used-inv S'*
using *assms*
apply (*induct rule*: *rtranclp-induct*)
apply *simp*
using *resolution-preserves-already-used-inv* **by** *fast*

lemma *resolution-completeness*:
fixes ψ :: *'v* ::*linorder state*
assumes *unsat*: ¬*satisfiable* (*fst* ψ)
and *finite*: *finite* (*fst* ψ)
and *snd* ψ = {}
shows $\exists\psi'$. (*resolution*** ψ ψ' ∧ {#} ∈ *fst* ψ')
proof −

have *already-used-inv* ψ **unfolding** *assms* **by** *auto*
then show *?thesis* **using** *assms resolution-completeness-inv* **by** *blast*
qed

lemma *rtranclp-preserves-sat*:
assumes *simplify*** S S'
and *satisfiable* S
shows *satisfiable* S'
using *assms* **apply** *induction*
apply *simp*
by (*meson satisfiable-carac satisfiable-def simplify-preserves-un-sat-eq*)

lemma *resolution-preserves-sat*:
assumes *resolution* S S'
and *satisfiable* (*fst* S)
shows *satisfiable* (*fst* S')
using *assms* **apply** (*induction rule*: *resolution.induct*)
using *rtranclp-preserves-sat tranclp-into-rtranclp* **unfolding** *full1-def* **apply** *fastforce*
by (*metis fst-conv full-def inference-preserves-un-sat rtranclp-preserves-sat satisfiable-carac′ satisfiable-def*)

lemma *rtranclp-resolution-preserves-sat*:
assumes *resolution*** S S'
and *satisfiable* (*fst* S)
shows *satisfiable* (*fst* S')
using *assms* **apply** (*induction rule*: *rtranclp-induct*)
apply *simp*
using *resolution-preserves-sat* **by** *blast*

lemma *resolution-soundness*:
fixes ψ :: $'v$ ::*linorder state*
assumes *resolution*** ψ ψ' **and** $\{\#\} \in$ *fst* ψ'
shows *unsatisfiable* (*fst* ψ)
using *assms* **by** (*meson rtranclp-resolution-preserves-sat satisfiable-def true-cls-empty true-clss-def*)

lemma *resolution-soundness-and-completeness*:
fixes ψ :: $'v$ ::*linorder state*
assumes *finite*: *finite* (*fst* ψ)
and *snd*: *snd* $\psi = \{\}$
shows $(\exists \psi'.$ (*resolution*** ψ $\psi' \wedge \{\#\} \in$ *fst* ψ')) $\longleftrightarrow$ *unsatisfiable* (*fst* ψ)
using *assms resolution-completeness resolution-soundness* **by** *metis*

lemma *simplified-falsity*:
assumes *simp*: *simplified* ψ
and $\{\#\} \in \psi$
shows $\psi = \{\{\#\}\}$
proof (*rule ccontr*)
assume H: $\neg$ *?thesis*
then obtain χ **where** $\chi \in \psi$ **and** $\chi \neq \{\#\}$ **using** *assms*(*2*) **by** *blast*
then have $\{\#\} \subset\# \chi$ **by** (*simp add*: *mset-less-empty-nonempty*)
then have *simplify* ψ ($\psi - \{\chi\}$)
using *simplify.subsumption*[*OF assms*(*2*) ‹$\{\#\} \subset\# \chi$› ‹$\chi \in \psi$›] **by** *blast*
then show *False* **using** *simp* **by** *blast*
qed

lemma *simplify-falsity-in-preserved*:
assumes *simplify* χs $\chi s'$
and $\{\#\} \in \chi s$
shows $\{\#\} \in \chi s'$
using *assms*
by *induction auto*

lemma *rtranclp-simplify-falsity-in-preserved*:
assumes $simplify^{**}$ χs $\chi s'$
and $\{\#\} \in \chi s$
shows $\{\#\} \in \chi s'$
using *assms*
by *induction* (*auto intro*: *simplify-falsity-in-preserved*)

lemma *resolution-falsity-get-falsity-alone*:
assumes *finite* (*fst* ψ)
shows $(\exists \psi'. (resolution^{**}\ \psi\ \psi' \land \{\#\} \in fst\ \psi')) \longleftrightarrow (\exists\, a\text{-}u\text{-}v.\ resolution^{**}\ \psi\ (\{\{\#\}\},\ a\text{-}u\text{-}v))$
(**is** $?A \longleftrightarrow ?B$)
proof
assume *?B*
then show *?A* **by** *auto*
next
assume *?A*
then obtain χs *a-u-v* **where** χs: $resolution^{**}$ ψ (χs, *a-u-v*) **and** F: $\{\#\} \in \chi s$ **by** *auto*
{ assume *simplified* χs
then have *?B* **using** *simplified-falsity*[*OF - F*] χs **by** *blast*
}
moreover {
assume $\neg$ *simplified* χs
then obtain $\chi s'$ **where** *full1 simplify* χs $\chi s'$
by (*metis* χs *assms finite-simplified-full1-simp fst-conv rtranclp-resolution-finite*)
then have $\{\#\} \in \chi s'$
unfolding *full1-def* **by** (*meson F rtranclp-simplify-falsity-in-preserved*
tranclp-into-rtranclp)
then have *?B*
by (*metis* χs ‹*full1 simplify* χs $\chi s'$› *fst-conv full1-simp resolution-always-simplified*
rtranclp.rtrancl-into-rtrancl simplified-falsity)
}
ultimately show *?B* **by** *blast*
qed

lemma *resolution-soundness-and-completeness'*:
fixes ψ :: *'v* ::*linorder state*
assumes
finite: *finite* (*fst* ψ)**and**
snd: *snd* $\psi = \{\}$
shows $(\exists\, a\text{-}u\text{-}v.\ (resolution^{**}\ \psi\ (\{\{\#\}\},\ a\text{-}u\text{-}v))) \longleftrightarrow unsatisfiable\ (fst\ \psi)$
using *assms resolution-completeness resolution-soundness resolution-falsity-get-falsity-alone*
by *metis*

end

theory *Partial-Annotated-Clausal-Logic*

imports *Partial-Clausal-Logic*

begin

13 Partial Clausal Logic

We here define marked literals (that will be used in both DPLL and CDCL) and the entailment corresponding to it.

13.1 Marked Literals

13.1.1 Definition

datatype $('v, 'lvl, 'mark)$ *marked-lit* =
  is-marked: *Marked* (*lit-of*: *'v literal*) (*level-of*: *'lvl*) |
  is-proped: *Propagated* (*lit-of*: *'v literal*) (*mark-of*: *'mark*)

lemma *marked-lit-list-induct*[*case-names nil marked proped*]:
  assumes P [] **and**
  $\bigwedge L\ l\ xs.\ P\ xs \Longrightarrow P\ (Marked\ L\ l\ \#\ xs)$ **and**
  $\bigwedge L\ m\ xs.\ P\ xs \Longrightarrow P\ (Propagated\ L\ m\ \#\ xs)$
  shows $P\ xs$
  using *assms* **apply** (*induction xs*, *simp*)
  by (*case-tac a*) *auto*

lemma *is-marked-ex-Marked*:
  is-marked $L \Longrightarrow \exists K\ lvl.\ L = Marked\ K\ lvl$
  by (*cases L*) *auto*

type-synonym $('v, 'l, 'm)$ *marked-lits* = $('v, 'l, 'm)$ *marked-lit list*

definition *lits-of* :: $('a, 'b, 'c)$ *marked-lit list* $\Rightarrow$ *'a literal set* **where**
lits-of Ls = *lit-of* ‘ (*set Ls*)

lemma *lits-of-empty*[*simp*]:
  lits-of [] = {} **unfolding** *lits-of-def* **by** *auto*

lemma *lits-of-cons*[*simp*]:
  lits-of (L # *Ls*) = *insert* (*lit-of L*) (*lits-of Ls*)
  unfolding *lits-of-def* **by** *auto*

lemma *lits-of-append*[*simp*]:
  lits-of (l @ l') = *lits-of* l $\cup$ *lits-of* l'
  unfolding *lits-of-def* **by** *auto*

lemma *finite-lits-of-def*[*simp*]: *finite* (*lits-of L*)
  unfolding *lits-of-def* **by** *auto*

lemma *lits-of-rev*[*simp*]: *lits-of* (*rev M*) = *lits-of M*
  unfolding *lits-of-def* **by** *auto*

lemma *set-map-lit-of-lits-of*[*simp*]:
  set (*map lit-of T*) = *lits-of T*
  unfolding *lits-of-def* **by** *auto*

lemma *atms-of-ms-lambda-lit-of-is-atm-of-lit-of*[*simp*]:
atms-of-ms $((\lambda a.\ \{\#\mathit{lit\text{-}of}\ a\#\})$ ' *set* $M')$ = *atm-of* ' *lits-of* M'
unfolding *atms-of-ms-def lits-of-def* **by** *auto*

lemma *lits-of-empty-is-empty*[*iff*]:
lits-of $M = \{\} \longleftrightarrow M = []$
by (*induct M*) *auto*

13.1.2 Entailment

definition *true-annot* :: ('*a*, '*l*, '*m*) *marked-lits* ⇒ '*a clause* ⇒ *bool* (**infix** ⊨*a 49*) **where**
$I \models a\ C \longleftrightarrow$ (*lits-of I*) $\models C$

definition *true-annots* :: ('*a*, '*l*, '*m*) *marked-lits* ⇒ '*a clauses* ⇒ *bool* (**infix** ⊨*as 49*) **where**
$I \models as\ CC \longleftrightarrow (\forall\, C \in CC.\ I \models a\ C)$

lemma *true-annot-empty-model*[*simp*]:
$\neg[] \models a\ \psi$
unfolding *true-annot-def true-cls-def* **by** *simp*

lemma *true-annot-empty*[*simp*]:
$\neg I \models a\ \{\#\}$
unfolding *true-annot-def true-cls-def* **by** *simp*

lemma *empty-true-annots-def*[*iff*]:
$[] \models as\ \psi \longleftrightarrow \psi = \{\}$
unfolding *true-annots-def* **by** *auto*

lemma *true-annots-empty*[*simp*]:
$I \models as\ \{\}$
unfolding *true-annots-def* **by** *auto*

lemma *true-annots-single-true-annot*[*iff*]:
$I \models as\ \{C\} \longleftrightarrow I \models a\ C$
unfolding *true-annots-def* **by** *auto*

lemma *true-annot-insert-l*[*simp*]:
$M \models a\ A \Longrightarrow L\ \#\ M \models a\ A$
unfolding *true-annot-def* **by** *auto*

lemma *true-annots-insert-l* [*simp*]:
$M \models as\ A \Longrightarrow L\ \#\ M \models as\ A$
unfolding *true-annots-def* **by** *auto*

lemma *true-annots-union*[*iff*]:
$M \models as\ A \cup B \longleftrightarrow (M \models as\ A \wedge M \models as\ B)$
unfolding *true-annots-def* **by** *auto*

lemma *true-annots-insert*[*iff*]:
$M \models as$ *insert a A* $\longleftrightarrow (M \models a\ a \wedge M \models as\ A)$
unfolding *true-annots-def* **by** *auto*

Link between ⊨*as* and ⊨*s*:

lemma *true-annots-true-cls*:
$I \models as\ CC \longleftrightarrow$ (*lits-of I*) $\models s\ CC$
unfolding *true-annots-def Ball-def true-annot-def true-clss-def* **by** *auto*

lemma *in-lit-of-true-annot*:
  $a \in$ *lits-of* $M \longleftrightarrow M \models a$ $\{\#a\#\}$
  unfolding *true-annot-def lits-of-def* **by** *auto*

lemma *true-annot-lit-of-notin-skip*:
  L # $M \models a$ $A \Longrightarrow$ *lit-of* $L \notin\#$ $A \Longrightarrow M \models a$ A
  unfolding *true-annot-def true-cls-def* **by** *auto*

lemma *true-clss-singleton-lit-of-implies-incl*:
  $I \models s$ $(\lambda a.$ $\{\#$*lit-of* $a\#\})$ ' *set* $MLs \Longrightarrow$ *lits-of* $MLs \subseteq I$
  unfolding *true-clss-def lits-of-def* **by** *auto*

lemma *true-annot-true-clss-cls*:
  $MLs \models a$ $\psi \Longrightarrow$ *set* (*map* $(\lambda a.$ $\{\#$*lit-of* $a\#\})$ $MLs) \models p$ ψ
  unfolding *true-annot-def true-clss-cls-def true-cls-def*
  by (*auto dest*: *true-clss-singleton-lit-of-implies-incl*)

lemma *true-annots-true-clss-cls*:
  $MLs \models as$ $\psi \Longrightarrow$ *set* (*map* $(\lambda a.$ $\{\#$*lit-of* $a\#\})$ $MLs) \models ps$ ψ
  by (*auto*
    dest: *true-clss-singleton-lit-of-implies-incl*
    simp add: *true-clss-def true-annots-def true-annot-def lits-of-def true-cls-def*
    true-clss-clss-def)

lemma *true-annots-marked-true-cls*[*iff*]:
  map $(\lambda M.$ *Marked* M $a)$ $M \models as$ $N \longleftrightarrow$ *set* $M \models s$ N
proof −
  have $*$: *lits-of* (*map* $(\lambda M.$ *Marked* M $a)$ $M)$ = *set* M **unfolding** *lits-of-def* **by** *force*
  show *?thesis* **by** (*simp add*: *true-annots-true-cls* $*$)
qed

lemma *true-annot-singleton*[*iff*]: $M \models a$ $\{\#L\#\} \longleftrightarrow L \in$ *lits-of* M
  unfolding *true-annot-def lits-of-def* **by** *auto*

lemma *true-annots-true-clss-clss*:
  $A \models as$ $\Psi \Longrightarrow (\lambda a.$ $\{\#$*lit-of* $a\#\})$ ' *set* $A \models ps$ Ψ
  unfolding *true-clss-clss-def true-annots-def true-clss-def*
  by (*auto*
    dest!: *true-clss-singleton-lit-of-implies-incl*
    simp add: *lits-of-def true-annot-def true-cls-def*)

lemma *true-annot-commute*:
  M @ $M' \models a$ $D \longleftrightarrow M'$ @ $M \models a$ D
  unfolding *true-annot-def* **by** (*simp add*: *Un-commute*)

lemma *true-annots-commute*:
  M @ $M' \models as$ $D \longleftrightarrow M'$ @ $M \models as$ D
  unfolding *true-annots-def* **by** (*auto simp add*: *true-annot-commute*)

lemma *true-annot-mono*[*dest*]:
  set $I \subseteq$ *set* $I' \Longrightarrow I \models a$ $N \Longrightarrow I' \models a$ N
  using *true-cls-mono-set-mset-l* **unfolding** *true-annot-def lits-of-def*
  by (*metis* (*no-types*) *Un-commute Un-upper1 image-Un sup.orderE*)

lemma *true-annots-mono*:
  $set\ I \subseteq set\ I' \Longrightarrow I \models as\ N \Longrightarrow I' \models as\ N$
  unfolding *true-annots-def* **by** *auto*

13.1.3 Defined and undefined literals

definition *defined-lit* :: $('a, 'l, 'm)$ *marked-lit list* $\Rightarrow$ *'a literal* $\Rightarrow$ *bool*
  where
$defined\text{-}lit\ I\ L \longleftrightarrow (\exists l.\ Marked\ L\ l \in set\ I) \lor (\exists P.\ Propagated\ L\ P \in set\ I)$
  $\lor (\exists l.\ Marked\ (-L)\ l \in set\ I) \lor (\exists P.\ Propagated\ (-L)\ P \in set\ I)$

abbreviation *undefined-lit* :: $('a, 'l, 'm)$ *marked-lit list* $\Rightarrow$ *'a literal* $\Rightarrow$ *bool*
where $undefined\text{-}lit\ I\ L \equiv \neg defined\text{-}lit\ I\ L$

lemma *defined-lit-rev*[*simp*]:
  $defined\text{-}lit\ (rev\ M)\ L \longleftrightarrow defined\text{-}lit\ M\ L$
  unfolding *defined-lit-def* **by** *auto*

lemma *atm-imp-marked-or-proped*:
  assumes $x \in set\ I$
  shows
    $(\exists l.\ Marked\ (-\ lit\text{-}of\ x)\ l \in set\ I)$
    $\lor (\exists l.\ Marked\ (lit\text{-}of\ x)\ l \in set\ I)$
    $\lor (\exists l.\ Propagated\ (-\ lit\text{-}of\ x)\ l \in set\ I)$
    $\lor (\exists l.\ Propagated\ (lit\text{-}of\ x)\ l \in set\ I)$
  using *assms marked-lit.exhaust-sel* **by** *metis*

lemma *literal-is-lit-of-marked*:
  assumes $L = lit\text{-}of\ x$
  shows $(\exists l.\ x = Marked\ L\ l) \lor (\exists l'.\ x = Propagated\ L\ l')$
  using *assms* **by** (*case-tac x*) *auto*

lemma *true-annot-iff-marked-or-true-lit*:
  $defined\text{-}lit\ I\ L \longleftrightarrow ((lits\text{-}of\ I) \models l\ L \lor (lits\text{-}of\ I) \models l\ -L)$
  unfolding *defined-lit-def* **by** (*auto simp add*: *lits-of-def rev-image-eqI*
    dest!: *literal-is-lit-of-marked*)

lemma *consistent-interp* $(lits\text{-}of\ I) \Longrightarrow I \models as\ N \Longrightarrow satisfiable\ N$
  by (*simp add*: *true-annots-true-cls*)

lemma *defined-lit-map*:
  $defined\text{-}lit\ Ls\ L \longleftrightarrow atm\text{-}of\ L \in (\lambda l.\ atm\text{-}of\ (lit\text{-}of\ l))$ ‘ $set\ Ls$
 unfolding *defined-lit-def* **apply** (*rule iffI*)
   using *image-iff* **apply** *fastforce*
 by (*fastforce simp add*: *atm-of-eq-atm-of dest*: *atm-imp-marked-or-proped*)

lemma *defined-lit-uminus*[*iff*]:
  $defined\text{-}lit\ I\ (-L) \longleftrightarrow defined\text{-}lit\ I\ L$
  unfolding *defined-lit-def* **by** *auto*

lemma *Marked-Propagated-in-iff-in-lits-of*:
  $defined\text{-}lit\ I\ L \longleftrightarrow (L \in lits\text{-}of\ I \lor -L \in lits\text{-}of\ I)$
  unfolding *lits-of-def defined-lit-def*
  by (*auto simp add*: *rev-image-eqI*) (*case-tac x*, *auto*)+

lemma *consistent-add-undefined-lit-consistent*[*simp*]:
  assumes
    consistent-interp (*lits-of Ls*) **and**
    undefined-lit Ls L
  shows *consistent-interp* (*insert L* (*lits-of Ls*))
  using *assms* **unfolding** *consistent-interp-def* **by** (*auto simp*: *Marked-Propagated-in-iff-in-lits-of*)

lemma *decided-empty*[*simp*]:
  ¬*defined-lit* [] *L*
  unfolding *defined-lit-def* **by** *simp*

13.2 Backtracking

fun *backtrack-split* :: ('*v*, '*l*, '*m*) *marked-lits*
  ⇒ ('*v*, '*l*, '*m*) *marked-lits* × ('*v*, '*l*, '*m*) *marked-lits* **where**
backtrack-split [] = ([], []) |
backtrack-split (*Propagated L P* # *mlits*) = *apfst* ((*op* #) (*Propagated L P*)) (*backtrack-split mlits*) |
backtrack-split (*Marked L l* # *mlits*) = ([], *Marked L l* # *mlits*)

lemma *backtrack-split-fst-not-marked*: *a* ∈ *set* (*fst* (*backtrack-split l*)) ⟹ ¬*is-marked a*
  by (*induct l rule*: *marked-lit-list-induct*) *auto*

lemma *backtrack-split-snd-hd-marked*:
  snd (*backtrack-split l*) ≠ [] ⟹ *is-marked* (*hd* (*snd* (*backtrack-split l*)))
  by (*induct l rule*: *marked-lit-list-induct*) *auto*

lemma *backtrack-split-list-eq*[*simp*]:
  fst (*backtrack-split l*) @ (*snd* (*backtrack-split l*)) = *l*
  by (*induct l rule*: *marked-lit-list-induct*) *auto*

lemma *backtrack-snd-empty-not-marked*:
  backtrack-split M = (*M''*, []) ⟹ ∀ *l*∈*set M*. ¬ *is-marked l*
  by (*metis append-Nil2 backtrack-split-fst-not-marked backtrack-split-list-eq snd-conv*)

lemma *backtrack-split-some-is-marked-then-snd-has-hd*:
  ∃ *l*∈*set M*. *is-marked l* ⟹ ∃ *M' L' M''*. *backtrack-split M* = (*M''*, *L'* # *M'*)
  by (*metis backtrack-snd-empty-not-marked list.exhaust prod.collapse*)

Another characterisation of the result of *backtrack-split*. This view allows some simpler proofs, since *takeWhile* and *dropWhile* are highly automated:

lemma *backtrack-split-takeWhile-dropWhile*:
  backtrack-split M = (*takeWhile* (*Not o is-marked*) *M*, *dropWhile* (*Not o is-marked*) *M*)
proof (*induct M*)
  case *Nil* **show** *?case* **by** *simp*
next
  case (*Cons L M*) **thus** *?case* **by** (*cases L*) *auto*
qed

13.3 Decomposition with respect to the marked literals

The pattern *get-all-marked-decomposition* [] = [([], [])] is necessary otherwise, we can call the *hd* function in the other pattern.

fun *get-all-marked-decomposition* :: ('*a*, '*l*, '*m*) *marked-lits*
  ⇒ (('*a*, '*l*, '*m*) *marked-lits* × ('*a*, '*l*, '*m*) *marked-lits*) *list* **where**
get-all-marked-decomposition (*Marked L l* # *Ls*) =

(*Marked L l # Ls*, []) # *get-all-marked-decomposition Ls* |
get-all-marked-decomposition (*Propagated L P# Ls*) =
(*apsnd* ((*op* #) (*Propagated L P*)) (*hd* (*get-all-marked-decomposition Ls*)))
tl (*get-all-marked-decomposition Ls*) |
get-all-marked-decomposition [] = [([], [])]

value *get-all-marked-decomposition* [*Propagated A5 B5*, *Marked C4 D4*, *Propagated A3 B3*, *Propagated A2 B2*, *Marked C1 D1*, *Propagated A0 B0*]

lemma *get-all-marked-decomposition-never-empty*[*iff*]:
get-all-marked-decomposition M = [] ⟷ *False*
by (*induct M*, *simp*) (*case-tac a*, *auto*)

lemma *get-all-marked-decomposition-never-empty-sym*[*iff*]:
[] = *get-all-marked-decomposition M* ⟷ *False*
using *get-all-marked-decomposition-never-empty*[*of M*] **by** *presburger*

lemma *get-all-marked-decomposition-decomp*:
hd (*get-all-marked-decomposition S*) = (*a*, *c*) ⟹ *S* = *c* @ *a*
proof (*induct S arbitrary*: *a c*)
case *Nil*
thus *?case* **by** *simp*
next
case (*Cons x A*)
thus *?case* **by** (*cases x*; *cases hd* (*get-all-marked-decomposition A*)) *auto*
qed

lemma *get-all-marked-decomposition-backtrack-split*:
backtrack-split S = (*M*, *M′*) ⟷ *hd* (*get-all-marked-decomposition S*) = (*M′*, *M*)
proof (*induction S arbitrary*: *M M′*)
case *Nil*
thus *?case* **by** *auto*
next
case (*Cons a S*)
thus *?case* **using** *backtrack-split-takeWhile-dropWhile* **by** (*cases a*) *force+*
qed

lemma *get-all-marked-decomposition-nil-backtrack-split-snd-nil*:
get-all-marked-decomposition S = [([], *A*)] ⟹ *snd* (*backtrack-split S*) = []
by (*simp add*: *get-all-marked-decomposition-backtrack-split sndI*)

lemma *get-all-marked-decomposition-length-1-fst-empty-or-length-1*:
assumes *get-all-marked-decomposition M* = (*a*, *b*) # []
shows *a* = [] ∨ (*length a* = *1* ∧ *is-marked* (*hd a*) ∧ *hd a* ∈ *set M*)
using *assms*
proof (*induct M arbitrary*: *a b*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*Cons m M*)
show *?case*
proof (*cases m*)
case (*Marked l mark*)
thus *?thesis* **using** *Cons* **by** *simp*
next

case (*Propagated l mark*)
thus *?thesis* **using** *Cons* **by** (*cases get-all-marked-decomposition M*) *force+*
qed
qed

lemma *get-all-marked-decomposition-fst-empty-or-hd-in-M*:
assumes *get-all-marked-decomposition M* = (*a*, *b*) # *l*
shows *a* = [] ∨ (*is-marked* (*hd a*) ∧ *hd a* ∈ *set M*)
using *assms* **apply** (*induct M arbitrary*: *a b rule*: *marked-lit-list-induct*)
apply *auto*[*2*]
by (*metis UnCI backtrack-split-snd-hd-marked get-all-marked-decomposition-backtrack-split get-all-marked-decomposition-decomp hd-in-set list.sel*(*1*) *set-append snd-conv*)

lemma *get-all-marked-decomposition-snd-not-marked*:
assumes (*a*, *b*) ∈ *set* (*get-all-marked-decomposition M*)
and *L* ∈ *set b*
shows ¬*is-marked L*
using *assms* **apply** (*induct M arbitrary*: *a b rule*: *marked-lit-list-induct*, *simp*)
by (*case-tac get-all-marked-decomposition xs*; *fastforce*)+

lemma *tl-get-all-marked-decomposition-skip-some*:
assumes *x* ∈ *set* (*tl* (*get-all-marked-decomposition M1*))
shows *x* ∈ *set* (*tl* (*get-all-marked-decomposition* (*M0* @ *M1*)))
using *assms*
by (*induct M0 rule*: *marked-lit-list-induct*)
(*auto simp add*: *list.set-sel*(*2*))

lemma *hd-get-all-marked-decomposition-skip-some*:
assumes (*x*, *y*) = *hd* (*get-all-marked-decomposition M1*)
shows (*x*, *y*) ∈ *set* (*get-all-marked-decomposition* (*M0* @ *Marked K i* # *M1*))
using *assms*
proof (*induct M0*)
case *Nil*
thus *?case* **by** *auto*
next
case (*Cons L M0*)
hence *xy*: (*x*, *y*) ∈ *set* (*get-all-marked-decomposition* (*M0* @ *Marked K i* # *M1*)) **by** *blast*
show *?case*
proof (*cases L*)
case (*Marked l m*)
thus *?thesis* **using** *xy* **by** *auto*
next
case (*Propagated l m*)
thus *?thesis*
using *xy Cons.prems*
by (*cases get-all-marked-decomposition* (*M0* @ *Marked K i* # *M1*))
(*auto dest*!: *get-all-marked-decomposition-decomp*
arg-cong[*of get-all-marked-decomposition - - hd*])
qed
qed

lemma *get-all-marked-decomposition-snd-union*:
set M = $\bigcup$(*set* ‘ *snd* ‘ *set* (*get-all-marked-decomposition M*)) ∪ {*L* |*L*. *is-marked L* ∧ *L* ∈ *set M*}
(**is** *?M M* = *?U M* ∪ *?Ls M*)
proof (*induct M arbitrary*:)

case *Nil*
thus *?case* **by** *simp*
next
case (*Cons L M*)
show *?case*
proof (*cases L*)
case (*Marked a l*) **note** $L = this$
hence $L \in$ *?Ls* ($L\#M$) **by** *auto*
moreover have *?U* ($L\#M$) = *?U M* **unfolding** *L* **by** *auto*
moreover have *?M M* = *?U M* $\cup$ *?Ls M* **using** *Cons.hyps* **by** *auto*
ultimately show *?thesis* **by** *auto*
next
case (*Propagated a P*)
thus *?thesis* **using** *Cons.hyps* **by** (*cases* (*get-all-marked-decomposition M*)) *auto*
qed
qed

lemma *in-get-all-marked-decomposition-in-get-all-marked-decomposition-prepend*:
(a, b) $\in$ *set* (*get-all-marked-decomposition* M') $\Longrightarrow$
$\exists\, b'$. (a, b' @ b) $\in$ *set* (*get-all-marked-decomposition* (M @ M'))
apply (*induction M rule*: *marked-lit-list-induct*)
apply (*metis append-Nil*)
apply *auto*[]
by (*case-tac get-all-marked-decomposition* (*xs* @ M')) *auto*

lemma *get-all-marked-decomposition-remove-unmarked-length*:
assumes $\forall\, l \in$ *set* M'. $\neg$*is-marked l*
shows *length* (*get-all-marked-decomposition* (M' @ M''))
= *length* (*get-all-marked-decomposition* M'')
using *assms* **by** (*induct* M' *arbitrary*: M'' *rule*: *marked-lit-list-induct*) *auto*

lemma *get-all-marked-decomposition-not-is-marked-length*:
assumes $\forall\, l \in$ *set* M'. $\neg$*is-marked l*
shows *1* + *length* (*get-all-marked-decomposition* (*Propagated* ($-L$) *P* # *M*))
= *length* (*get-all-marked-decomposition* (M' @ *Marked L l* # *M*))
using *assms get-all-marked-decomposition-remove-unmarked-length* **by** *fastforce*

lemma *get-all-marked-decomposition-last-choice*:
assumes *tl* (*get-all-marked-decomposition* (M' @ *Marked L l* # *M*)) $\neq$ []
and $\forall\, l \in$ *set* M'. $\neg$*is-marked l*
and *hd* (*tl* (*get-all-marked-decomposition* (M' @ *Marked L l* # *M*))) = ($M0'$, *M0*)
shows *hd* (*get-all-marked-decomposition* (*Propagated* ($-L$) *P* # *M*)) = ($M0'$, *Propagated* ($-L$) *P* #
M0)
using *assms* **by** (*induct* M' *rule*: *marked-lit-list-induct*) *auto*

lemma *get-all-marked-decomposition-except-last-choice-equal*:
assumes $\forall\, l \in$ *set* M'. $\neg$*is-marked l*
shows *tl* (*get-all-marked-decomposition* (*Propagated* ($-L$) *P* # *M*))
= *tl* (*tl* (*get-all-marked-decomposition* (M' @ *Marked L l* # *M*)))
using *assms* **by** (*induct* M' *rule*: *marked-lit-list-induct*) *auto*

lemma *get-all-marked-decomposition-hd-hd*:
assumes *get-all-marked-decomposition Ls* = (M, C) # (*M0*, $M0'$) # *l*
shows *tl M* = $M0'$ @ *M0* $\wedge$ *is-marked* (*hd M*)
using *assms*

proof (*induct Ls arbitrary*: *M C M0 M0′ l*)
  case *Nil*
  thus *?case* **by** *simp*
next
  case (*Cons a Ls M C M0 M0′ l*) **note** *IH* = *this*(*1*) **and** *g* = *this*(*2*)
  { fix *L level*
    assume *a*: *a* = *Marked L level*
    have *Ls* = *M0′* @ *M0*
      using *g a* **by** (*force intro*: *get-all-marked-decomposition-decomp*)
    hence *tl M* = *M0′* @ *M0* ∧ *is-marked* (*hd M*) **using** *g a* **by** *auto*
  }
  moreover {
    fix *L P*
    assume *a*: *a* = *Propagated L P*
    have *tl M* = *M0′* @ *M0* ∧ *is-marked* (*hd M*)
      using *IH Cons.prems* **unfolding** *a* **by** (*cases get-all-marked-decomposition Ls*) *auto*
  }
  ultimately show *?case* **by** (*cases a*) *auto*
qed

lemma *get-all-marked-decomposition-exists-prepend*[*dest*]:
  assumes (*a*, *b*) ∈ *set* (*get-all-marked-decomposition M*)
  shows ∃ *c*. *M* = *c* @ *b* @ *a*
  using *assms* **apply** (*induct M rule*: *marked-lit-list-induct*)
    apply *simp*
  by (*case-tac get-all-marked-decomposition xs*;
    auto dest!: *arg-cong*[*of get-all-marked-decomposition - - hd*]
      get-all-marked-decomposition-decomp)+

lemma *get-all-marked-decomposition-incl*:
  assumes (*a*, *b*) ∈ *set* (*get-all-marked-decomposition M*)
  shows *set b* ⊆ *set M* **and** *set a* ⊆ *set M*
  using *assms get-all-marked-decomposition-exists-prepend* **by** *fastforce*+

lemma *get-all-marked-decomposition-exists-prepend′*:
  assumes (*a*, *b*) ∈ *set* (*get-all-marked-decomposition M*)
  obtains *c* **where** *M* = *c* @ *b* @ *a*
  using *assms* **apply** (*induct M rule*: *marked-lit-list-induct*)
    apply *auto*[*1*]
  by (*case-tac hd* (*get-all-marked-decomposition xs*),
    auto dest!: *get-all-marked-decomposition-decomp simp add*: *list.set-sel*(*2*))+

lemma *union-in-get-all-marked-decomposition-is-subset*:
  assumes (*a*, *b*) ∈ *set* (*get-all-marked-decomposition M*)
  shows *set a* ∪ *set b* ⊆ *set M*
  using *assms* **by** *force*

definition *all-decomposition-implies* :: *′a literal multiset set*
  ⇒ ((*′a*, *′l*, *′m*) *marked-lit list* × (*′a*, *′l*, *′m*) *marked-lit list*) *list* ⇒ *bool* **where**
 all-decomposition-implies N S
   ⟷ (∀ (*Ls*, *seen*) ∈ *set S*. (λ*a*. {#*lit-of a*#}) ‘ *set Ls* ∪ *N* ⊨*ps* (λ*a*. {#*lit-of a*#}) ‘ *set seen*)

lemma *all-decomposition-implies-empty*[*iff*]:
  all-decomposition-implies N [] **unfolding** *all-decomposition-implies-def* **by** *auto*

lemma *all-decomposition-implies-single*[*iff*]:
  all-decomposition-implies N [(*Ls*, *seen*)]
    $\longleftrightarrow$ (λa. {#*lit-of a*#}) ' *set Ls* $\cup$ *N* $\models$*ps* (λa. {#*lit-of a*#}) ' *set seen*
  unfolding *all-decomposition-implies-def* **by** *auto*

lemma *all-decomposition-implies-append*[*iff*]:
  all-decomposition-implies N (*S* @ *S'*)
    $\longleftrightarrow$ (*all-decomposition-implies N S* $\wedge$ *all-decomposition-implies N S'*)
  unfolding *all-decomposition-implies-def* **by** *auto*

lemma *all-decomposition-implies-cons-pair*[*iff*]:
  all-decomposition-implies N ((*Ls*, *seen*) # *S'*)
    $\longleftrightarrow$ (*all-decomposition-implies N* [(*Ls*, *seen*)] $\wedge$ *all-decomposition-implies N S'*)
  unfolding *all-decomposition-implies-def* **by** *auto*

lemma *all-decomposition-implies-cons-single*[*iff*]:
  all-decomposition-implies N (*l* # *S'*) $\longleftrightarrow$
    ((λa. {#*lit-of a*#}) ' *set* (*fst l*) $\cup$ *N* $\models$*ps* (λa. {#*lit-of a*#}) ' *set* (*snd l*) $\wedge$
      all-decomposition-implies N S')
  unfolding *all-decomposition-implies-def* **by** *auto*

lemma *all-decomposition-implies-trail-is-implied*:
  assumes *all-decomposition-implies N* (*get-all-marked-decomposition M*)
  shows *N* $\cup$ {{#*lit-of L*#} |*L*. *is-marked L* $\wedge$ *L* $\in$ *set M*}
    $\models$*ps* (λa. {#*lit-of a*#}) ' $\bigcup$(*set* ' *snd* ' *set* (*get-all-marked-decomposition M*))
using *assms*
proof (*induct length* (*get-all-marked-decomposition M*) *arbitrary*: *M*)
  case *0*
  thus *?case* **by** *auto*
next
  case (*Suc n*) **note** *IH* = *this*(*1*) **and** *length* = *this*(*2*)
  {
    assume *length* (*get-all-marked-decomposition M*) $\leq$ *1*
    then obtain *a b* **where** *g*: *get-all-marked-decomposition M* = (*a*, *b*) # []
      by (*case-tac get-all-marked-decomposition M*) *auto*
    moreover {
      assume *a* = []
      hence *?case* **using** *Suc.prems g* **by** *auto*
    }
    moreover {
      assume *l*: *length a* = *1* **and** *m*: *is-marked* (*hd a*) **and** *hd*: *hd a* $\in$ *set M*
      hence (λa. {#*lit-of a*#}) (*hd a*) $\in$ {{#*lit-of L*#} |*L*. *is-marked L* $\wedge$ *L* $\in$ *set M*} **by** *auto*
      hence *H*: (λa. {#*lit-of a*#}) ' *set a* $\cup$ *N* $\subseteq$ *N* $\cup$ {{#*lit-of L*#} |*L*. *is-marked L* $\wedge$ *L* $\in$ *set M*}
        using *l* **by** (*cases a*) *auto*
      have *f1*: (λm. {#*lit-of m*#}) ' *set a* $\cup$ *N* $\models$*ps* (λm. {#*lit-of m*#}) ' *set b*
        using *Suc.prems* **unfolding** *all-decomposition-implies-def g* **by** *simp*
      have *?case*
        unfolding *g* **apply** (*rule true-clss-clss-subset*) **using** *f1 H* **by** *auto*
    }
    ultimately have *?case* **using** *get-all-marked-decomposition-length-1-fst-empty-or-length-1* **by** *blast*
  }
  moreover {
    assume *length* (*get-all-marked-decomposition M*) > *1*
    then obtain *Ls0 seen0 M'* **where**

Ls0: *get-all-marked-decomposition M* = (*Ls0*, *seen0*) # *get-all-marked-decomposition M'* **and**
length': *length* (*get-all-marked-decomposition M'*) = *n* **and**
M'-in-M: *set M'* ⊆ *set M*
using *length* **apply** (*induct M*)
apply *simp*
by (*case-tac a*, *case-tac hd* (*get-all-marked-decomposition M*))
(*auto simp add*: *subset-insertI2*)
{
assume *n* = *0*
hence *get-all-marked-decomposition M'* = [] **using** *length'* **by** *auto*
hence *?case* **using** *Suc.prems* **unfolding** *all-decomposition-implies-def Ls0* **by** *auto*
}
moreover {
assume *n*: *n* > *0*
then obtain *Ls1 seen1 l* **where** *Ls1*: *get-all-marked-decomposition M'* = (*Ls1*, *seen1*) # *l*
using *length'* **by** (*induct M'*, *simp*) (*case-tac a*, *auto*)

have *all-decomposition-implies N* (*get-all-marked-decomposition M'*)
using *Suc.prems* **unfolding** *Ls0 all-decomposition-implies-def* **by** *auto*
hence *N*: *N* ∪ {{#*lit-of L*#} |*L*. *is-marked L* ∧ *L* ∈ *set M'*}
⊨*ps* (λ*a*. {#*lit-of a*#}) ‘ ⋃(*set* ‘ *snd* ‘ *set* (*get-all-marked-decomposition M'*))
using *IH length'* **by** *auto*

have *l*: *N* ∪ {{#*lit-of L*#} |*L*. *is-marked L* ∧ *L* ∈ *set M'*}
⊆ *N* ∪ {{#*lit-of L*#} |*L*. *is-marked L* ∧ *L* ∈ *set M*}
using *M'-in-M* **by** *auto*
hence Ψ*N*: *N* ∪ {{#*lit-of L*#} |*L*. *is-marked L* ∧ *L* ∈ *set M*}
⊨*ps* (λ*a*. {#*lit-of a*#}) ‘ ⋃(*set* ‘ *snd* ‘ *set* (*get-all-marked-decomposition M'*))
using *true-clss-clss-subset*[*OF l N*] **by** *auto*
have *is-marked* (*hd Ls0*) **and** *LS*: *tl Ls0* = *seen1* @ *Ls1*
using *get-all-marked-decomposition-hd-hd*[*of M*] **unfolding** *Ls0 Ls1* **by** *auto*

have *LSM*: *seen1* @ *Ls1* = *M'* **using** *get-all-marked-decomposition-decomp*[*of M'*] *Ls1* **by** *auto*
have *M'*: *set M'* = *Union* (*set* ‘ *snd* ‘ *set* (*get-all-marked-decomposition M'*))
∪ {*L* |*L*. *is-marked L* ∧ *L* ∈ *set M'*}
using *get-all-marked-decomposition-snd-union* **by** *auto*

{
assume *Ls0* ≠ []
hence *hd Ls0* ∈ *set M* **using** *get-all-marked-decomposition-fst-empty-or-hd-in-M Ls0* **by** *blast*
hence *N* ∪ {{#*lit-of L*#} |*L*. *is-marked L* ∧ *L* ∈ *set M*} ⊨*p* (λ*a*. {#*lit-of a*#}) (*hd Ls0*)
using ‹*is-marked* (*hd Ls0*)› **by** (*metis* (*mono-tags*, *lifting*) *UnCI mem-Collect-eq*
true-clss-cls-in)
} **note** *hd-Ls0* = *this*

have *l*: (λ*a*. {#*lit-of a*#}) ‘ (⋃(*set* ‘ *snd* ‘ *set* (*get-all-marked-decomposition M'*))
∪ {*L* |*L*. *is-marked L* ∧ *L* ∈ *set M'*})
= (λ*a*. {#*lit-of a*#}) ‘
⋃(*set* ‘ *snd* ‘ *set* (*get-all-marked-decomposition M'*))
∪ {{#*lit-of L*#} |*L*. *is-marked L* ∧ *L* ∈ *set M'*}
by *auto*
have *N* ∪ {{#*lit-of L*#} |*L*. *is-marked L* ∧ *L* ∈ *set M'*} ⊨*ps*
(λ*a*. {#*lit-of a*#}) ‘ (⋃(*set* ‘ *snd* ‘ *set* (*get-all-marked-decomposition M'*))
∪ {*L* |*L*. *is-marked L* ∧ *L* ∈ *set M'*})
unfolding *l* **using** *N* **by** (*auto simp add*: *all-in-true-clss-clss*)

hence $N \cup \{\{\#lit\text{-}of\ L\#\} \mid L.\ is\text{-}marked\ L \wedge L \in set\ M'\} \models ps\ (\lambda a.\ \{\#lit\text{-}of\ a\#\})\ `\ set\ (tl\ Ls0)$
using M' **unfolding** *LS LSM* **by** *auto*
hence t: $N \cup \{\{\#lit\text{-}of\ L\#\} \mid L.\ is\text{-}marked\ L \wedge L \in set\ M'\}$
$\models ps\ (\lambda a.\ \{\#lit\text{-}of\ a\#\})\ `\ set\ (tl\ Ls0)$
by (*blast intro*: *all-in-true-clss-clss*)
hence $N \cup \{\{\#lit\text{-}of\ L\#\} \mid L.\ is\text{-}marked\ L \wedge L \in set\ M\}$
$\models ps\ (\lambda a.\ \{\#lit\text{-}of\ a\#\})\ `\ set\ (tl\ Ls0)$
using *M'-in-M true-clss-clss-subset*[*OF - t*,
of $N \cup \{\{\#lit\text{-}of\ L\#\} \mid L.\ is\text{-}marked\ L \wedge L \in set\ M\}$]
by *auto*
hence $N \cup \{\{\#lit\text{-}of\ L\#\} \mid L.\ is\text{-}marked\ L \wedge L \in set\ M\} \models ps\ (\lambda a.\ \{\#lit\text{-}of\ a\#\})\ `\ set\ Ls0$
using *hd-Ls0* **by** (*case-tac Ls0*, *auto*)

moreover have $(\lambda a.\ \{\#lit\text{-}of\ a\#\})\ `\ set\ Ls0 \cup N \models ps\ (\lambda a.\ \{\#lit\text{-}of\ a\#\})\ `\ set\ seen0$
using *Suc.prems* **unfolding** *Ls0 all-decomposition-implies-def* **by** *simp*
moreover have $\bigwedge M\ Ma.\ (M::'a\ literal\ multiset\ set) \cup Ma \models ps\ M$
by (*simp add*: *all-in-true-clss-clss*)
ultimately have Ψ: $N \cup \{\{\#lit\text{-}of\ L\#\} \mid L.\ is\text{-}marked\ L \wedge L \in set\ M\} \models ps$
$(\lambda a.\ \{\#lit\text{-}of\ a\#\})\ `\ set\ seen0$
by (*meson true-clss-clss-left-right true-clss-clss-union-and true-clss-clss-union-l-r*)
have $(\lambda a.\ \{\#lit\text{-}of\ a\#\})`(set\ seen0$
$\cup (\bigcup x{\in}set\ (get\text{-}all\text{-}marked\text{-}decomposition\ M').\ set\ (snd\ x)))$
$= (\lambda a.\ \{\#lit\text{-}of\ a\#\})\ `\ set\ seen0$
$\cup (\lambda a.\ \{\#lit\text{-}of\ a\#\})\ `\ (\bigcup x{\in}set\ (get\text{-}all\text{-}marked\text{-}decomposition\ M').\ set\ (snd\ x))$
by *auto*

hence *?case* **unfolding** *Ls0* **using** $\Psi\ \Psi N$ **by** *simp*
}
ultimately have *?case* **by** *auto*
}
ultimately show *?case* **by** *arith*
qed

lemma *all-decomposition-implies-propagated-lits-are-implied*:
assumes *all-decomposition-implies N* (*get-all-marked-decomposition M*)
shows $N \cup \{\{\#lit\text{-}of\ L\#\} \mid L.\ is\text{-}marked\ L \wedge L \in set\ M\} \models ps\ (\lambda a.\ \{\#lit\text{-}of\ a\#\})\ `\ set\ M$
(**is** $?I \models ps\ ?A$)
proof –
have $?I \models ps\ (\lambda a.\ \{\#lit\text{-}of\ a\#\})\ `\ \{L \mid L.\ is\text{-}marked\ L \wedge L \in set\ M\}$
by (*auto intro*: *all-in-true-clss-clss*)
moreover have $?I \models ps\ (\lambda a.\ \{\#lit\text{-}of\ a\#\})\ `\ \bigcup(set\ `\ snd\ `\ set\ (get\text{-}all\text{-}marked\text{-}decomposition\ M))$
using *all-decomposition-implies-trail-is-implied assms* **by** *blast*
ultimately have $N \cup \{\{\#lit\text{-}of\ m\#\} \mid m.\ is\text{-}marked\ m \wedge m \in set\ M\}$
$\models ps\ (\lambda m.\ \{\#lit\text{-}of\ m\#\})\ `\ \bigcup(set\ `\ snd\ `\ set\ (get\text{-}all\text{-}marked\text{-}decomposition\ M))$
$\cup (\lambda m.\ \{\#lit\text{-}of\ m\#\})\ `\ \{m \mid m.\ is\text{-}marked\ m \wedge m \in set\ M\}$
by *blast*
thus *?thesis*
by (*metis* (*no-types*) *get-all-marked-decomposition-snd-union*[*of M*] *image-Un*)
qed

lemma *all-decomposition-implies-insert-single*:
all-decomposition-implies N M $\Longrightarrow$ *all-decomposition-implies* (*insert C N*) *M*
unfolding *all-decomposition-implies-def* **by** *auto*

13.4 Negation of Clauses

definition *CNot* :: *'v clause* ⇒ *'v clauses* **where**
CNot ψ = { {#−*L*#} | *L*. *L* ∈# ψ }

lemma *in-CNot-uminus*[*iff*]:
shows {#*L*#} ∈ *CNot* ψ ⟷ −*L* ∈# ψ
using *assms* **unfolding** *CNot-def* **by** *force*

lemma *CNot-singleton*[*simp*]: *CNot* {#*L*#} = {{#−*L*#}} **unfolding** *CNot-def* **by** *auto*
lemma *CNot-empty*[*simp*]: *CNot* {#} = {} **unfolding** *CNot-def* **by** *auto*
lemma *CNot-plus*[*simp*]: *CNot* (*A* + *B*) = *CNot A* ∪ *CNot B* **unfolding** *CNot-def* **by** *auto*

lemma *CNot-eq-empty*[*iff*]:
CNot D = {} ⟷ *D* = {#}
unfolding *CNot-def* **by** (*auto simp add*: *multiset-eqI*)

lemma *in-CNot-implies-uminus*:
assumes *L* ∈# *D*
and *M* ⊨*as CNot D*
shows *M* ⊨*a* {#−*L*#} **and** −*L* ∈ *lits-of M*
using *assms* **by** (*auto simp add*: *true-annots-def true-annot-def CNot-def*)

lemma *CNot-remdups-mset*[*simp*]:
CNot (*remdups-mset A*) = *CNot A*
unfolding *CNot-def* **by** *auto*

lemma *Ball-CNot-Ball-mset*[*simp*] :
(∀ *x*∈*CNot D*. *P x*) ⟷ (∀ *L*∈# *D*. *P* {#−*L*#})
unfolding *CNot-def* **by** *auto*

lemma *consistent-CNot-not*:
assumes *consistent-interp I*
shows *I* ⊨*s CNot* φ ⟹ ¬*I* ⊨ φ
using *assms* **unfolding** *consistent-interp-def true-clss-def true-cls-def* **by** *auto*

lemma *total-not-true-cls-true-clss-CNot*:
assumes *total-over-m I* {φ} **and** ¬*I* ⊨ φ
shows *I* ⊨*s CNot* φ
using *assms* **unfolding** *total-over-m-def total-over-set-def true-clss-def true-cls-def CNot-def*
apply *clarify*
by (*case-tac L*) (*force intro*: *pos-lit-in-atms-of neg-lit-in-atms-of*)+

lemma *total-not-CNot*:
assumes *total-over-m I* {φ} **and** ¬*I* ⊨*s CNot* φ
shows *I* ⊨ φ
using *assms total-not-true-cls-true-clss-CNot* **by** *auto*

lemma *atms-of-ms-CNot-atms-of*[*simp*]:
atms-of-ms (*CNot C*) = *atms-of C*
unfolding *atms-of-ms-def atms-of-def CNot-def* **by** *fastforce*

lemma *true-clss-clss-contradiction-true-clss-cls-false*:
C ∈ *D* ⟹ *D* ⊨*ps CNot C* ⟹ *D* ⊨*p* {#}
unfolding *true-clss-clss-def true-clss-cls-def total-over-m-def*
by (*metis Un-commute atms-of-empty atms-of-ms-CNot-atms-of atms-of-ms-insert atms-of-ms-union*

consistent-CNot-not insert-absorb sup-bot.left-neutral true-clss-def)

lemma *true-annots-CNot-all-atms-defined*:
assumes $M \models as\ CNot\ T$ **and** *a1*: $L \in\#\ T$
shows *atm-of* $L \in$ *atm-of* ' *lits-of* M
by (*metis assms atm-of-uminus image-eqI in-CNot-implies-uminus*(*1*) *true-annot-singleton*)

lemma *true-clss-clss-false-left-right*:
assumes $\{\{\#L\#\}\} \cup B \models p\ \{\#\}$
shows $B \models ps\ CNot\ \{\#L\#\}$
unfolding *true-clss-clss-def true-clss-cls-def*
proof (*intro allI impI*)
fix *I*
assume
tot: *total-over-m* I ($B \cup CNot\ \{\#L\#\}$) **and**
cons: *consistent-interp* I **and**
I: $I \models s\ B$
have *total-over-m* I ($\{\{\#L\#\}\} \cup B$) **using** *tot* **by** *auto*
hence $\neg I \models s$ *insert* $\{\#L\#\}\ B$
using *assms cons* **unfolding** *true-clss-cls-def* **by** *simp*
thus $I \models s\ CNot\ \{\#L\#\}$
using *tot I* **by** (*cases L*) *auto*
qed

lemma *true-annots-true-cls-def-iff-negation-in-model*:
$M \models as\ CNot\ C \longleftrightarrow (\forall L \in\#\ C.\ -L \in$ *lits-of* M)
unfolding *CNot-def true-annots-true-cls true-clss-def* **by** *auto*

lemma *consistent-CNot-not-tautology*:
consistent-interp $M \Longrightarrow M \models s\ CNot\ D \Longrightarrow \neg$*tautology* D
by (*metis atms-of-ms-CNot-atms-of consistent-CNot-not satisfiable-carac′ satisfiable-def tautology-def total-over-m-def*)

lemma *atms-of-ms-CNot-atms-of-ms*: *atms-of-ms* (*CNot CC*) = *atms-of-ms* $\{CC\}$
by *simp*

lemma *total-over-m-CNot-toal-over-m*[*simp*]:
total-over-m I (*CNot* C) = *total-over-set* I (*atms-of* C)
unfolding *total-over-m-def total-over-set-def* **by** *auto*

lemma *uminus-lit-swap*: $-(a::'a\ literal) = i \longleftrightarrow a = -i$
by *auto*

lemma *true-clss-cls-plus-CNot*:
assumes *CC-L*: $A \models p\ CC + \{\#L\#\}$
and *CNot-CC*: $A \models ps\ CNot\ CC$
shows $A \models p\ \{\#L\#\}$
unfolding *true-clss-clss-def true-clss-cls-def CNot-def total-over-m-def*
proof (*intro allI impI*)
fix *I*
assume *tot*: *total-over-set* I (*atms-of-ms* ($A \cup \{\{\#L\#\}\}$))
and *cons*: *consistent-interp* I
and *I*: $I \models s\ A$
let $?I = I \cup \{Pos\ P|P.\ P \in$ *atms-of* $CC \land P \notin$ *atm-of* ' $I\}$
have *cons′*: *consistent-interp* $?I$

using *cons* **unfolding** *consistent-interp-def*
by (*auto simp add*: *uminus-lit-swap atms-of-def rev-image-eqI*)
have *I'*: *?I* $\models s$ *A*
using *I true-clss-union-increase* **by** *blast*
have *tot-CNot*: *total-over-m ?I* ($A \cup$ *CNot CC*)
using *tot atms-of-s-def* **by** (*fastforce simp add*: *total-over-m-def total-over-set-def*)

hence *tot-I-A-CC-L*: *total-over-m ?I* ($A \cup \{CC + \{\#L\#\}\}$)
using *tot* **unfolding** *total-over-m-def total-over-set-atm-of* **by** *auto*
hence *?I* $\models CC + \{\#L\#\}$ **using** *CC-L cons' I'* **unfolding** *true-clss-cls-def* **by** *blast*
moreover
have *?I* $\models s$ *CNot CC* **using** *CNot-CC cons' I' tot-CNot* **unfolding** *true-clss-clss-def* **by** *auto*
hence $\neg A \models p$ *CC*
by (*metis* (*no-types*, *lifting*) *I' atms-of-ms-CNot-atms-of-ms atms-of-ms-union cons'*
consistent-CNot-not tot-CNot total-over-m-def true-clss-cls-def)
hence $\neg$*?I* $\models CC$ **using** ‹*?I* $\models s$ *CNot CC*› *cons' consistent-CNot-not* **by** *blast*
ultimately have *?I* $\models \{\#L\#\}$ **by** *blast*
thus $I \models \{\#L\#\}$
by (*metis* (*no-types*, *lifting*) *atms-of-ms-union cons' consistent-CNot-not tot total-not-CNot*
total-over-m-def total-over-set-union true-clss-union-increase)
qed

lemma *true-annots-CNot-lit-of-notin-skip*:
assumes *LM*: $L \# M \models as$ *CNot A* **and** *LA*: *lit-of L* $\notin\#$ *A* $-$*lit-of L* $\notin\#$ *A*
shows $M \models as$ *CNot A*
using *LM* **unfolding** *true-annots-def Ball-def*
proof (*intro allI impI*)
fix *l*
assume *H*: $\forall x.\ x \in$ *CNot A* $\longrightarrow L \# M \models a\ x$ **and** *l*: $l \in$ *CNot A*
hence $L \# M \models a\ l$ **by** *auto*
thus $M \models a\ l$ **using** *LA l* **by** (*cases L*) (*auto simp add*: *CNot-def*)
qed

lemma *true-clss-clss-union-false-true-clss-clss-cnot*:
$A \cup \{B\} \models ps \{\{\#\}\} \longleftrightarrow A \models ps$ *CNot B*
using *total-not-CNot consistent-CNot-not* **unfolding** *total-over-m-def true-clss-clss-def*
by *fastforce*

lemma *true-annot-remove-hd-if-notin-vars*:
assumes $a \# M' \models a\ D$
and *atm-of* (*lit-of a*) $\notin$ *atms-of D*
shows $M' \models a\ D$
using *assms true-cls-remove-hd-if-notin-vars* **unfolding** *true-annot-def* **by** *auto*

lemma *true-annot-remove-if-notin-vars*:
assumes $M\ @\ M' \models a\ D$
and $\forall x \in$*atms-of D*. $x \notin$ *atm-of* ‘ *lits-of M*
shows $M' \models a\ D$
using *assms* **apply** (*induct M*, *simp*)
using *true-annot-remove-hd-if-notin-vars* **by** *force+*

lemma *true-annots-remove-if-notin-vars*:
assumes $M\ @\ M' \models as\ D$
and $\forall x \in$*atms-of-ms D*. $x \notin$ *atm-of* ‘ *lits-of M*
shows $M' \models as\ D$ **unfolding** *true-annots-def*

using *assms true-annot-remove-if-notin-vars*[*of M M'*]
unfolding *true-annots-def atms-of-ms-def* **by** *force*

lemma *all-variables-defined-not-imply-cnot*:
assumes $\forall$ *s* $\in$ *atms-of-ms* $\{B\}$. *s* $\in$ *atm-of* ' *lits-of A*
and $\neg$ *A* $\models$*a B*
shows *A* $\models$*as CNot B*
unfolding *true-annot-def true-annots-def Ball-def CNot-def true-lit-def*
proof (*clarify*, *rule ccontr*)
fix *L*
assume *LB*: *L* $\in$# *B* **and** $\neg$ *lits-of A* $\models$*l* $-$ *L*
hence *atm-of L* $\in$ *atm-of* ' *lits-of A*
using *assms*(*1*) **by** (*simp add*: *atm-of-lit-in-atms-of lits-of-def*)
hence *L* $\in$ *lits-of A* $\vee$ $-L$ $\in$ *lits-of A*
using *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set* **by** *metis*
hence *L* $\in$ *lits-of A* **using** ‹ $\neg$ *lits-of A* $\models$*l* $-$ *L*› **by** *auto*
thus *False*
using *LB assms*(*2*) **unfolding** *true-annot-def true-lit-def true-cls-def Bex-mset-def*
by *blast*
qed

lemma *CNot-union-mset*[*simp*]:
CNot (*A* #$\cup$ *B*) = *CNot A* $\cup$ *CNot B*
unfolding *CNot-def* **by** *auto*

13.5 Other

abbreviation *no-dup L* $\equiv$ *distinct* (*map* (λ*l*. *atm-of* (*lit-of l*)) *L*)

lemma *no-dup-rev*[*simp*]:
no-dup (*rev M*) $\longleftrightarrow$ *no-dup M*
by (*auto simp*: *rev-map*[*symmetric*])

lemma *no-dup-length-eq-card-atm-of-lits-of*:
assumes *no-dup M*
shows *length M* = *card* (*atm-of* ' *lits-of M*)
using *assms* **unfolding** *lits-of-def* **by** (*induct M*) (*auto simp add*: *image-image*)

lemma *distinctconsistent-interp*:
no-dup M $\Longrightarrow$ *consistent-interp* (*lits-of M*)
proof (*induct M*)
case *Nil*
show *?case* **by** *auto*
next
case (*Cons L M*)
hence *a1*: *consistent-interp* (*lits-of M*) **by** *auto*
have *a2*: *atm-of* (*lit-of L*) $\notin$ (λ*l*. *atm-of* (*lit-of l*)) ' *set M* **using** *Cons.prems* **by** *auto*
have *undefined-lit M* (*lit-of L*)
using *a2 image-iff* **unfolding** *defined-lit-def* **by** *fastforce*
thus *?case*
using *a1* **by** *simp*
qed

lemma *distinct-get-all-marked-decomposition-no-dup*:
assumes (*a*, *b*) $\in$ *set* (*get-all-marked-decomposition M*)
and *no-dup M*

shows *no-dup* $(a @ b)$
using *assms* **by** *force*

lemma *true-annots-lit-of-notin-skip*:
assumes $L \# M \models as\ CNot\ A$
and $-lit\text{-}of\ L \notin\# A$
and *no-dup* $(L \# M)$
shows $M \models as\ CNot\ A$
proof $-$
have $\forall l \in\# A.\ -l \in lits\text{-}of\ (L \# M)$
using *assms*(1) *in-CNot-implies-uminus*(2) **by** *blast*
moreover
have *atm-of* (*lit-of* L) $\notin$ *atm-of* ‘ *lits-of* M
using *assms*(3) **unfolding** *lits-of-def* **by** *force*
hence $-$ *lit-of* $L \notin$ *lits-of* M **unfolding** *lits-of-def*
by (*metis* (*no-types*) *atm-of-uminus imageI*)
ultimately have $\forall\ l \in\# A.\ -l \in lits\text{-}of\ M$
using *assms*(2) **unfolding** *Ball-mset-def* **by** (*metis insertE lits-of-cons uminus-of-uminus-id*)
thus *?thesis* **by** (*auto simp add*: *true-annots-def*)
qed

type-synonym *'v clauses = 'v clause multiset*

abbreviation *true-annots-mset* (**infix** $\models$*asm 50*) **where**
$I \models asm\ C \equiv I \models as\ (set\text{-}mset\ C)$

abbreviation *true-clss-clss-m*:: *'a clauses* $\Rightarrow$ *'a clauses* $\Rightarrow$ *bool* (**infix** $\models$*psm 50*) **where**
$I \models psm\ C \equiv set\text{-}mset\ I \models ps\ (set\text{-}mset\ C)$

Analog of $[\![?N \models ps\ ?B;\ ?A \subseteq ?B]\!] \Longrightarrow ?N \models ps\ ?A$

lemma *true-clss-clssm-subsetE*: $N \models psm\ B \Longrightarrow A \subseteq\# B \Longrightarrow N \models psm\ A$
using *set-mset-mono true-clss-clss-subsetE* **by** *blast*

abbreviation *true-clss-cls-m*:: *'a clauses* $\Rightarrow$ *'a clause* $\Rightarrow$ *bool* (**infix** $\models$*pm 50*) **where**
$I \models pm\ C \equiv set\text{-}mset\ I \models p\ C$

abbreviation *distinct-mset-mset* :: *'a multiset multiset* $\Rightarrow$ *bool* **where**
distinct-mset-mset $\Sigma \equiv$ *distinct-mset-set* (*set-mset* Σ)

abbreviation *all-decomposition-implies-m* **where**
all-decomposition-implies-m $A\ B \equiv$ *all-decomposition-implies* (*set-mset* A) B

abbreviation *atms-of-msu* **where**
atms-of-msu $U \equiv$ *atms-of-ms* (*set-mset* U)

abbreviation *true-clss-m*:: *'a interp* $\Rightarrow$ *'a clauses* $\Rightarrow$ *bool* (**infix** $\models$*sm 50*) **where**
$I \models sm\ C \equiv I \models s\ set\text{-}mset\ C$

abbreviation *true-clss-ext-m* (**infix** $\models$*sextm 49*) **where**
$I \models sextm\ C \equiv I \models sext\ set\text{-}mset\ C$
end
theory *CDCL-NOT*
imports *Partial-Annotated-Clausal-Logic List-More Wellfounded-More Partial-Clausal-Logic*
begin

14 NOT's CDCL

sledgehammer-params[*verbose*, *prover=e spass z3 cvc4 verit remote-vampire*]

declare *set-mset-minus-replicate-mset*[*simp*]

14.1 Auxiliary Lemmas and Measure

lemma *no-dup-cannot-not-lit-and-uminus*:
  $no\text{-}dup\ M \Longrightarrow -\ lit\text{-}of\ xa = lit\text{-}of\ x \Longrightarrow x \in set\ M \Longrightarrow xa \notin set\ M$
  by (*metis atm-of-uminus distinct-map inj-on-eq-iff uminus-not-id'*)

lemma *true-clss-single-iff-incl*:
  $I \models s\ single\ `\ B \longleftrightarrow B \subseteq I$
  unfolding *true-clss-def* **by** *auto*

lemma *atms-of-ms-single-atm-of*[*simp*]:
  $atms\text{-}of\text{-}ms\ \{\{\#lit\text{-}of\ L\#\}\ |L.\ P\ L\} = atm\text{-}of\ `\ \{lit\text{-}of\ L\ |L.\ P\ L\}$
  unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-uminus-lit-atm-of-lit-of*:
  $atms\text{-}of\ \{\#-\ lit\text{-}of\ x.\ x \in\#\ A\#\} = atm\text{-}of\ `\ (lit\text{-}of\ `\ (set\text{-}mset\ A))$
  unfolding *atms-of-def* **by** (*auto simp add*: *Fun.image-comp*)

lemma *atms-of-ms-single-image-atm-of-lit-of*:
  $atms\text{-}of\text{-}ms\ ((\lambda x.\ \{\#lit\text{-}of\ x\#\})\ `\ A) = atm\text{-}of\ `\ (lit\text{-}of\ `\ A)$
  unfolding *atms-of-ms-def* **by** *auto*

This measure can also be seen as the increasing lexicographic order: it is an order on bounded sequences, when each element is bounded. The proof involves a measure like the one defined here (the same?).

definition μ_C :: *nat* ⇒ *nat* ⇒ *nat list* ⇒ *nat* **where**
$\mu_C\ s\ b\ M \equiv (\sum i{=}0..{<}length\ M.\ M!i * b\hat{}\ (s + i - length\ M))$

lemma μ_C*-nil*[*simp*]:
  $\mu_C\ s\ b\ [] = 0$
  unfolding μ_C*-def* **by** *auto*

lemma μ_C*-single*[*simp*]:
  $\mu_C\ s\ b\ [L] = L * b\ \hat{}\ (s - Suc\ 0)$
  unfolding μ_C*-def* **by** *auto*

lemma *set-sum-atLeastLessThan-add*:
  $(\sum i{=}k..{<}k{+}(b{::}nat).\ f\ i) = (\sum i{=}0..{<}b.\ f\ (k{+}\ i))$
  by (*induction b*) *auto*

lemma *set-sum-atLeastLessThan-Suc*:
  $(\sum i{=}1..{<}Suc\ j.\ f\ i) = (\sum i{=}0..{<}j.\ f\ (Suc\ i))$
  using *set-sum-atLeastLessThan-add*[*of - 1 j*] **by** *force*

lemma μ_C*-cons*:
  $\mu_C\ s\ b\ (L\ \#\ M) = L * b\ \hat{}\ (s - 1 - length\ M) + \mu_C\ s\ b\ M$
proof −
  have $\mu_C\ s\ b\ (L\ \#\ M) = (\sum i{=}0..{<}length\ (L\#M).\ (L\#M)!i * b\hat{}\ (s + i - length\ (L\#M)))$
    unfolding μ_C*-def* **by** *blast*

also have ... $= (\sum i{=}0..{<}1.\ (L\#M)!i * b\hat{}\ (s + i - \mathit{length}\ (L\#M)))$
$+ (\sum i{=}1..{<}\mathit{length}\ (L\#M).\ (L\#M)!i * b\hat{}\ (s + i - \mathit{length}\ (L\#M)))$
by (*rule setsum-add-nat-ivl*[*symmetric*]) *simp-all*
finally have $\mu_C\ s\ b\ (L\ \#\ M) = L * b\ \hat{}\ (s - 1 - \mathit{length}\ M)$
$+ (\sum i{=}1..{<}\mathit{length}\ (L\#M).\ (L\#M)!i * b\hat{}\ (s + i - \mathit{length}\ (L\#M)))$
by *auto*
moreover {
have $(\sum i{=}1..{<}\mathit{length}\ (L\#M).\ (L\#M)!i * b\hat{}\ (s + i - \mathit{length}\ (L\#M))) =$
$(\sum i{=}0..{<}\mathit{length}\ (M).\ (L\#M)!(\mathit{Suc}\ i) * b\hat{}\ (s + (\mathit{Suc}\ i) - \mathit{length}\ (L\#M)))$
unfolding *length-Cons set-sum-atLeastLessThan-Suc* **by** *blast*
also have ... $= (\sum i{=}0..{<}\mathit{length}\ (M).\ M!i * b\hat{}\ (s + i - \mathit{length}\ M))$
by *auto*
finally have $(\sum i{=}1..{<}\mathit{length}\ (L\#M).\ (L\#M)!i * b\hat{}\ (s + i - \mathit{length}\ (L\#M))) = \mu_C\ s\ b\ M$
unfolding μ_C*-def* **.**
}
ultimately show *?thesis* **by** *presburger*
qed

lemma μ_C*-append*:
assumes $s \geq \mathit{length}\ (M@M')$
shows $\mu_C\ s\ b\ (M@M') = \mu_C\ (s - \mathit{length}\ M')\ b\ M + \mu_C\ s\ b\ M'$
proof –
have $\mu_C\ s\ b\ (M@M') = (\sum i{=}0..{<}\mathit{length}\ (M@M').\ (M@M')!i * b\hat{}\ (s + i - \mathit{length}\ (M@M')))$
unfolding μ_C*-def* **by** *blast*
moreover then have ... $= (\sum i{=}0..{<}\ \mathit{length}\ M.\ (M@M')!i * b\hat{}\ (s + i - \mathit{length}\ (M@M')))$
$+ (\sum i{=}\mathit{length}\ M..{<}\mathit{length}\ (M@M').\ (M@M')!i * b\hat{}\ (s + i - \mathit{length}\ (M@M')))$
by (*auto intro*!: *setsum-add-nat-ivl*[*symmetric*])
moreover
have $\forall i \in \{0..{<}\ \mathit{length}\ M\}.\ (M@M')!i * b\hat{}\ (s + i - \mathit{length}\ (M@M')) = M\ !\ i * b\ \hat{}\ (s - \mathit{length}\ M' + i - \mathit{length}\ M)$
using ‹$s \geq \mathit{length}\ (M@M')$› **by** (*auto simp add*: *nth-append ac-simps*)
then have $\mu_C\ (s - \mathit{length}\ M')\ b\ M = (\sum i{=}0..{<}\ \mathit{length}\ M.\ (M@M')!i * b\hat{}\ (s + i - \mathit{length}\ (M@M')))$
unfolding μ_C*-def* **by** *auto*
ultimately have $\mu_C\ s\ b\ (M@M') = \mu_C\ (s - \mathit{length}\ M')\ b\ M$
$+ (\sum i{=}\mathit{length}\ M..{<}\mathit{length}\ (M@M').\ (M@M')!i * b\hat{}\ (s + i - \mathit{length}\ (M@M')))$
by *auto*
moreover {
have $(\sum i{=}\mathit{length}\ M..{<}\mathit{length}\ (M@M').\ (M@M')!i * b\hat{}\ (s + i - \mathit{length}\ (M@M'))) =$
$(\sum i{=}0..{<}\mathit{length}\ M'.\ M'!i * b\hat{}\ (s + i - \mathit{length}\ M'))$
unfolding *length-append set-sum-atLeastLessThan-add* **by** *auto*
then have $(\sum i{=}\mathit{length}\ M..{<}\mathit{length}\ (M@M').\ (M@M')!i * b\hat{}\ (s + i - \mathit{length}\ (M@M'))) = \mu_C\ s\ b\ M'$
unfolding μ_C*-def* **.**
}
ultimately show *?thesis* **by** *presburger*
qed

lemma μ_C*-cons-non-empty-inf*:
assumes *M-ge-1*: $\forall i \in \mathit{set}\ M.\ i \geq 1$ **and** *M*: $M \neq []$
shows $\mu_C\ s\ b\ M \geq b\ \hat{}\ (s - \mathit{length}\ M)$
using *assms* **by** (*cases M*) (*auto simp*: *mult-eq-if* μ_C*-cons*)

Duplicate of " /src/HOL/ex/NatSum.thy" (but generalized to $(0::'a) \leq k$)

lemma *sum-of-powers*: $0 \leq k \Longrightarrow (k - 1) * (\sum i{=}0..{<}n.\ k\hat{}i) = k\hat{}n - (1::\mathit{nat})$

apply (*cases* $k = 0$)
apply (*cases* n; *simp*)
by (*induct* n) (*auto simp*: *Nat.nat-distrib*)

In the degenerated cases, we only have the large inequality holds. In the other cases, the following strict inequality holds:

lemma μ_C*-bounded-non-degenerated*:
fixes b ::*nat*
assumes
$b > 0$ **and**
$M \neq []$ **and**
M-le: $\forall\, i < \mathit{length}\ M.\ M!i < b$ **and**
$s \geq \mathit{length}\ M$
shows $\mu_C\ s\ b\ M < b\hat{}s$
proof −
consider ($b1$) $b{=}\ 1$ | (b) $b{>}1$ **using** ‹$b{>}0$› **by** (*cases b*) *auto*
then show *?thesis*
proof *cases*
case $b1$
then have $\forall\, i < \mathit{length}\ M.\ M!i = 0$ **using** *M-le* **by** *auto*
then have $\mu_C\ s\ b\ M = 0$ **unfolding** μ_C*-def* **by** *auto*
then show *?thesis* **using** ‹$b > 0$› **by** *auto*
next
case b
have $\forall\ i \in \{0..{<}\mathit{length}\ M\}.\ M!i * b\hat{}\ (s\ {+}i - \mathit{length}\ M) \leq (b{-}1) * b\hat{}\ (s\ {+}i - \mathit{length}\ M)$
using *M-le* ‹$b > 1$› **by** *auto*
then have $\mu_C\ s\ b\ M \leq\ (\sum i{=}0..{<}\mathit{length}\ M.\ (b{-}1) * b\hat{}\ (s\ {+}i - \mathit{length}\ M))$
using ‹$M{\neq}[]$› ‹$b{>}0$› **unfolding** μ_C*-def* **by** (*auto intro*: *setsum-mono*)
also
have $\forall\ i \in \{0..{<}\mathit{length}\ M\}.\ (b{-}1) * b\hat{}\ (s\ {+}i - \mathit{length}\ M) = (b{-}1) * b\hat{}i * b\hat{}(s - \mathit{length}\ M)$
by (*metis Nat.add-diff-assoc2 add.commute assms*(*4*) *mult.assoc power-add*)
then have $(\sum i{=}0..{<}\mathit{length}\ M.\ (b{-}1) * b\hat{}\ (s\ {+}i - \mathit{length}\ M))$
$= (\sum i{=}0..{<}\mathit{length}\ M.\ (b{-}1){*}\ b\hat{}\ i * b\hat{}(s - \mathit{length}\ M))$
by (*auto simp add*: *ac-simps*)
also have $\ldots\ = (\sum i{=}0..{<}\mathit{length}\ M.\ b\hat{}\ i) * b\hat{}(s - \mathit{length}\ M) * (b{-}1)$
by (*simp add*: *setsum-left-distrib setsum-right-distrib ac-simps*)
finally have $\mu_C\ s\ b\ M \leq (\sum i{=}0..{<}\mathit{length}\ M.\ b\hat{}\ i) * (b{-}1) * b\hat{}(s - \mathit{length}\ M)$
by (*simp add*: *ac-simps*)

also
have $(\sum i{=}0..{<}\mathit{length}\ M.\ b\hat{}\ i){*}\ (b{-}1) = b\ \hat{}\ (\mathit{length}\ M) - 1$
using *sum-of-powers*[*of b length M*] ‹$b{>}1$›
by (*auto simp add*: *ac-simps*)
finally have $\mu_C\ s\ b\ M \leq (b\ \hat{}\ (\mathit{length}\ M) - 1) * b\ \hat{}\ (s - \mathit{length}\ M)$
by *auto*
also have $\ldots\ < b\ \hat{}\ (\mathit{length}\ M) * b\ \hat{}\ (s - \mathit{length}\ M)$
using ‹$b{>}1$› **by** *auto*
also have $\ldots\ = b\ \hat{}\, s$
by (*metis assms*(*4*) *le-add-diff-inverse power-add*)
finally show *?thesis* **unfolding** μ_C*-def* **by** (*auto simp add*: *ac-simps*)
qed
qed

In the degenerate case $b = (0{::}'a)$, the list M is empty (since the list cannot contain any element).

lemma μ_C*-bounded*:
  fixes *b* ::*nat*
  assumes
    M-le: $\forall\, i < length\ M.\ M!i < b$ **and**
    $s \geq length\ M$
    $b > 0$
  shows $\mu_C\ s\ b\ M < b\ \hat{}\ s$
proof −
  consider $(M0)\ M = []\ |\ (M)\ b > 0$ **and** $M \neq []$
    using *M-le* **by** (*cases b*, *cases M*) *auto*
  then show *?thesis*
    proof *cases*
      case *M0*
      then show *?thesis* **using** *M-le* ‹$b > 0$› **by** *auto*
    next
      case *M*
      show *?thesis* **using** μ_C*-bounded-non-degenerated*[*OF M assms*(*1*,*2*)] **by** *arith*
    qed
qed

When $b = 0$, we cannot show that the measure is empty, since $0^0 = 1$.

lemma μ_C*-base-0*:
  assumes $length\ M \leq s$
  shows $\mu_C\ s\ 0\ M \leq M!0$
proof −
  {
    assume $s = length\ M$
    moreover {
      fix *n*
      have $(\sum i{=}0..{<}n.\ M\ !\ i * (0{::}nat)\ \hat{}\ i) \leq M\ !\ 0$
        apply (*induction n rule*: *nat-induct*)
        by *simp* (*case-tac n*, *auto*)
    }
    ultimately have *?thesis* **unfolding** μ_C*-def* **by** *auto*
  }
  moreover
  {
    assume $length\ M < s$
    then have $\mu_C\ s\ 0\ M = 0$ **unfolding** μ_C*-def* **by** *auto***}**
  ultimately show *?thesis* **using** *assms* **unfolding** μ_C*-def* **by** *linarith*
qed

14.2 Initial definitions

14.2.1 The state

We define here an abstraction over operation on the state we are manipulating.

locale *dpll-state* =
  fixes
    trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
    clauses :: *'st* ⇒ *'v clauses* **and**
    prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
    tl-trail :: *'st* ⇒*'st* **and**
    add-cls$_{NOT}$ :: *'v clause* ⇒ *'st* ⇒ *'st* **and**
    remove-cls$_{NOT}$ :: *'v clause* ⇒ *'st* ⇒ *'st*

assumes
trail-prepend-trail[*simp*]:
$\bigwedge$*st L. undefined-lit* (*trail st*) (*lit-of L*) $\Longrightarrow$ *trail* (*prepend-trail L st*) = *L* # *trail st*
and
tl-trail[*simp*]: *trail* (*tl-trail S*) = *tl* (*trail S*) **and**
trail-add-cls$_{NOT}$[*simp*]: $\bigwedge$*st C. no-dup* (*trail st*) $\Longrightarrow$ *trail* (*add-cls*$_{NOT}$ *C st*) = *trail st* **and**
trail-remove-cls$_{NOT}$[*simp*]: $\bigwedge$*st C. trail* (*remove-cls*$_{NOT}$ *C st*) = *trail st* **and**

clauses-prepend-trail[*simp*]:
$\bigwedge$*st L. undefined-lit* (*trail st*) (*lit-of L*) $\Longrightarrow$ *clauses* (*prepend-trail L st*) = *clauses st*
and
clauses-tl-trail[*simp*]: $\bigwedge$*st. clauses* (*tl-trail st*) = *clauses st* **and**
clauses-add-cls$_{NOT}$[*simp*]:
$\bigwedge$*st C. no-dup* (*trail st*) $\Longrightarrow$ *clauses* (*add-cls*$_{NOT}$ *C st*) = {#*C*#} + *clauses st* **and**
clauses-remove-cls$_{NOT}$[*simp*]: $\bigwedge$*st C. clauses* (*remove-cls*$_{NOT}$ *C st*) = *remove-mset C* (*clauses st*)
begin

function *reduce-trail-to*$_{NOT}$ :: *'a list* ⇒ *'st* ⇒ *'st* **where**
reduce-trail-to$_{NOT}$ *F S* =
(*if length* (*trail S*) = *length F* ∨ *trail S* = [] *then S else reduce-trail-to*$_{NOT}$ *F* (*tl-trail S*))
by *fast+*
termination by (*relation measure* (λ(-, *S*). *length* (*trail S*))) *auto*
declare *reduce-trail-to*$_{NOT}$.*simps*[*simp del*]

lemma
shows
reduce-trail-to$_{NOT}$-*nil*[*simp*]: *trail S* = [] $\Longrightarrow$ *reduce-trail-to*$_{NOT}$ *F S* = *S* **and**
reduce-trail-to$_{NOT}$-*eq-length*[*simp*]: *length* (*trail S*) = *length F* $\Longrightarrow$ *reduce-trail-to*$_{NOT}$ *F S* = *S*
by (*auto simp*: *reduce-trail-to*$_{NOT}$.*simps*)

lemma *reduce-trail-to*$_{NOT}$-*length-ne*[*simp*]:
length (*trail S*) ≠ *length F* $\Longrightarrow$ *trail S* ≠ [] $\Longrightarrow$
reduce-trail-to$_{NOT}$ *F S* = *reduce-trail-to*$_{NOT}$ *F* (*tl-trail S*)
by (*auto simp*: *reduce-trail-to*$_{NOT}$.*simps*)

lemma *trail-reduce-trail-to*$_{NOT}$-*length-le*:
assumes *length F* > *length* (*trail S*)
shows *trail* (*reduce-trail-to*$_{NOT}$ *F S*) = []
using *assms* **by** (*induction F S rule*: *reduce-trail-to*$_{NOT}$.*induct*)
(*simp add*: *less-imp-diff-less reduce-trail-to*$_{NOT}$.*simps*)

lemma *trail-reduce-trail-to*$_{NOT}$-*nil*[*simp*]:
trail (*reduce-trail-to*$_{NOT}$ [] *S*) = []
by (*induction* [] *S rule*: *reduce-trail-to*$_{NOT}$.*induct*)
(*simp add*: *less-imp-diff-less reduce-trail-to*$_{NOT}$.*simps*)

lemma *clauses-reduce-trail-to*$_{NOT}$-*nil*:
clauses (*reduce-trail-to*$_{NOT}$ [] *S*) = *clauses S*
by (*induction* [] *S rule*: *reduce-trail-to*$_{NOT}$.*induct*)
(*simp add*: *less-imp-diff-less reduce-trail-to*$_{NOT}$.*simps*)

lemma *trail-reduce-trail-to*$_{NOT}$-*drop*:
trail (*reduce-trail-to*$_{NOT}$ *F S*) =
(*if length* (*trail S*) ≥ *length F*
then drop (*length* (*trail S*) − *length F*) (*trail S*)

else [])
apply (*induction F S rule*: *reduce-trail-to*$_{NOT}$*.induct*)
apply (*rename-tac F S*)
apply (*case-tac trail S*)
 apply *auto*[]
apply (*rename-tac list*)
apply (*case-tac Suc* (*length list*) > *length F*)
 prefer *2* **apply** *simp*
apply (*subgoal-tac Suc* (*length list*) − *length F* = *Suc* (*length list* − *length F*))
 apply *simp*
apply *simp*
done

lemma *reduce-trail-to*$_{NOT}$*-skip-beginning*:
assumes *trail S* = F' @ *F*
shows *trail* (*reduce-trail-to*$_{NOT}$ *F S*) = *F*
using *assms* **by** (*auto simp*: *trail-reduce-trail-to*$_{NOT}$*-drop*)

lemma *reduce-trail-to*$_{NOT}$*-clauses*[*simp*]:
clauses (*reduce-trail-to*$_{NOT}$ *F S*) = *clauses S*
by (*induction F S rule*: *reduce-trail-to*$_{NOT}$*.induct*)
(*simp add*: *less-imp-diff-less reduce-trail-to*$_{NOT}$*.simps*)

abbreviation *trail-weight* **where**
trail-weight S ≡ *map* ((λ*l*. *1* + *length l*) *o snd*) (*get-all-marked-decomposition* (*trail S*))

definition *state-eq*$_{NOT}$ :: *'st* ⇒ *'st* ⇒ *bool* (**infix** ∼ *50*) **where**
S ∼ *T* ⟷ *trail S* = *trail T* ∧ *clauses S* = *clauses T*

lemma *state-eq*$_{NOT}$*-ref*[*simp*]:
S ∼ *S*
unfolding *state-eq*$_{NOT}$*-def* **by** *auto*

lemma *state-eq*$_{NOT}$*-sym*:
S ∼ *T* ⟷ *T* ∼ *S*
unfolding *state-eq*$_{NOT}$*-def* **by** *auto*

lemma *state-eq*$_{NOT}$*-trans*:
S ∼ *T* ⟹ *T* ∼ *U* ⟹ *S* ∼ *U*
unfolding *state-eq*$_{NOT}$*-def* **by** *auto*

lemma
shows
state-eq$_{NOT}$*-trail*: *S* ∼ *T* ⟹ *trail S* = *trail T* **and**
state-eq$_{NOT}$*-clauses*: *S* ∼ *T* ⟹ *clauses S* = *clauses T*
unfolding *state-eq*$_{NOT}$*-def* **by** *auto*

lemmas *state-simp*$_{NOT}$[*simp*]= *state-eq*$_{NOT}$*-trail state-eq*$_{NOT}$*-clauses*

lemma *trail-eq-reduce-trail-to*$_{NOT}$*-eq*:
trail S = *trail T* ⟹ *trail* (*reduce-trail-to*$_{NOT}$ *F S*) = *trail* (*reduce-trail-to*$_{NOT}$ *F T*)
apply (*induction F S arbitrary*: *T rule*: *reduce-trail-to*$_{NOT}$*.induct*)
by (*metis tl-trail reduce-trail-to*$_{NOT}$*-eq-length reduce-trail-to*$_{NOT}$*-length-ne reduce-trail-to*$_{NOT}$*-nil*)

lemma *reduce-trail-to$_{NOT}$-state-eq$_{NOT}$-compatible*:
  assumes ST: $S \sim T$
  shows *reduce-trail-to$_{NOT}$* F $S \sim$ *reduce-trail-to$_{NOT}$* F T
proof −
  have *clauses* (*reduce-trail-to$_{NOT}$* F S) = *clauses* (*reduce-trail-to$_{NOT}$* F T)
    using ST **by** *auto*
  moreover have *trail* (*reduce-trail-to$_{NOT}$* F S) = *trail* (*reduce-trail-to$_{NOT}$* F T)
    using *trail-eq-reduce-trail-to$_{NOT}$-eq*[*of* S T F] ST **by** *auto*
  ultimately show *?thesis* **by** (*auto simp del*: *state-simp$_{NOT}$* *simp*: *state-eq$_{NOT}$-def*)
qed

lemma *trail-reduce-trail-to$_{NOT}$-add-cls$_{NOT}$*[*simp*]:
  no-dup (*trail* S) $\Longrightarrow$
    trail (*reduce-trail-to$_{NOT}$* F (*add-cls$_{NOT}$* C S)) = *trail* (*reduce-trail-to$_{NOT}$* F S)
  by (*rule trail-eq-reduce-trail-to$_{NOT}$-eq*) *simp*

lemma *reduce-trail-to$_{NOT}$-trail-tl-trail-decomp*[*simp*]:
  trail $S = F'$ @ *Marked* K () # $F \Longrightarrow$
    trail (*reduce-trail-to$_{NOT}$* F (*tl-trail* S)) = F
  apply (*rule reduce-trail-to$_{NOT}$-skip-beginning*[*of* - *tl* (F' @ *Marked* K () # [])])
  by (*cases* F') (*auto simp add*:*tl-append reduce-trail-to$_{NOT}$-skip-beginning*)

end

14.2.2 Definition of the operation

locale *propagate-ops* =
  dpll-state trail clauses prepend-trail tl-trail add-cls$_{NOT}$ remove-cls$_{NOT}$ **for**
    trail :: $'st \Rightarrow$ ($'v$, *unit*, *unit*) *marked-lits* **and**
    clauses :: $'st \Rightarrow$ $'v$ *clauses* **and**
    prepend-trail :: ($'v$, *unit*, *unit*) *marked-lit* $\Rightarrow$ $'st \Rightarrow$ $'st$ **and**
    tl-trail :: $'st \Rightarrow$ $'st$ **and**
    add-cls$_{NOT}$ *remove-cls$_{NOT}$*:: $'v$ *clause* $\Rightarrow$ $'st \Rightarrow$ $'st$ **and**
    propagate-cond :: ($'v$, *unit*, *unit*) *marked-lit* $\Rightarrow$ $'st \Rightarrow$ *bool*
begin
inductive *propagate$_{NOT}$* :: $'st \Rightarrow$ $'st \Rightarrow$ *bool* **where**
propagate$_{NOT}$[*intro*]: C + {#L#} $\in$# *clauses* $S \Longrightarrow$ *trail* S $\models$*as* *CNot* C
  $\Longrightarrow$ *undefined-lit* (*trail* S) L
  $\Longrightarrow$ *propagate-cond* (*Propagated* L ()) S
  $\Longrightarrow$ $T \sim$ *prepend-trail* (*Propagated* L ()) S
  $\Longrightarrow$ *propagate$_{NOT}$* S T
inductive-cases *propagateE*[*elim*]: *propagate$_{NOT}$* S T

end

locale *decide-ops* =
  dpll-state trail clauses prepend-trail tl-trail add-cls$_{NOT}$ remove-cls$_{NOT}$ **for**
    trail :: $'st \Rightarrow$ ($'v$, *unit*, *unit*) *marked-lits* **and**
    clauses :: $'st \Rightarrow$ $'v$ *clauses* **and**
    prepend-trail :: ($'v$, *unit*, *unit*) *marked-lit* $\Rightarrow$ $'st \Rightarrow$ $'st$ **and**
    tl-trail :: $'st \Rightarrow$ $'st$ **and**
    add-cls$_{NOT}$ *remove-cls$_{NOT}$*:: $'v$ *clause* $\Rightarrow$ $'st \Rightarrow$ $'st$
begin
inductive *decide$_{NOT}$* :: $'st \Rightarrow$ $'st \Rightarrow$ *bool* **where**
decide$_{NOT}$[*intro*]: *undefined-lit* (*trail* S) $L \Longrightarrow$ *atm-of* $L \in$ *atms-of-msu* (*clauses* S)
  $\Longrightarrow$ $T \sim$ *prepend-trail* (*Marked* L ()) S

$\Longrightarrow$ *decide$_{NOT}$ S T*

inductive-cases *decideE*[*elim*]: *decide$_{NOT}$ S S'*
end

locale *backjumping-ops* =
dpll-state trail clauses prepend-trail tl-trail add-cls$_{NOT}$ remove-cls$_{NOT}$
for
trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
clauses :: *'st* ⇒ *'v clauses* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-cls$_{NOT}$ remove-cls$_{NOT}$:: *'v clause* ⇒ *'st* ⇒ *'st* +
fixes
backjump-conds :: *'v clause* ⇒ *'v clause* ⇒ *'v literal* ⇒ *'st* ⇒ *'st* ⇒ *bool*
begin
inductive *backjump* **where**
trail S = *F'* @ *Marked K* ()# *F*
$\Longrightarrow$ *T* ∼ *prepend-trail* (*Propagated L* ()) (*reduce-trail-to$_{NOT}$ F S*)
$\Longrightarrow$ *C* ∈# *clauses S*
$\Longrightarrow$ *trail S* ⊨*as CNot C*
$\Longrightarrow$ *undefined-lit F L*
$\Longrightarrow$ *atm-of L* ∈ *atms-of-msu* (*clauses S*) ∪ *atm-of* ‘ (*lits-of* (*trail S*))
$\Longrightarrow$ *clauses S* ⊨*pm C'* + {#*L*#}
$\Longrightarrow$ *F* ⊨*as CNot C'*
$\Longrightarrow$ *backjump-conds C C' L S T*
$\Longrightarrow$ *backjump S T*
inductive-cases *backjumpE*: *backjump S T*
end

14.3 DPLL with backjumping

locale *dpll-with-backjumping-ops* =
dpll-state trail clauses prepend-trail tl-trail add-cls$_{NOT}$ remove-cls$_{NOT}$ +
propagate-ops trail clauses prepend-trail tl-trail add-cls$_{NOT}$ remove-cls$_{NOT}$ propagate-conds +
decide-ops trail clauses prepend-trail tl-trail add-cls$_{NOT}$ remove-cls$_{NOT}$ +
backjumping-ops trail clauses prepend-trail tl-trail add-cls$_{NOT}$ remove-cls$_{NOT}$ backjump-conds
for
trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
clauses :: *'st* ⇒ *'v clauses* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-cls$_{NOT}$ remove-cls$_{NOT}$:: *'v clause* ⇒ *'st* ⇒ *'st* **and**
propagate-conds :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *bool* **and**
inv :: *'st* ⇒ *bool* **and**
backjump-conds :: *'v clause* ⇒ *'v clause* ⇒ *'v literal* ⇒ *'st* ⇒ *'st* ⇒ *bool* +
assumes
bj-can-jump:
⋀*S C F' K F L*.
inv S $\Longrightarrow$
no-dup (*trail S*) $\Longrightarrow$
trail S = *F'* @ *Marked K* () # *F* $\Longrightarrow$
C ∈# *clauses S* $\Longrightarrow$
trail S ⊨*as CNot C* $\Longrightarrow$
undefined-lit F L $\Longrightarrow$
atm-of L ∈ *atms-of-msu* (*clauses S*) ∪ *atm-of* ‘ (*lits-of* (*F'* @ *Marked K* () # *F*)) $\Longrightarrow$

clauses $S \models pm\ C' + \{\#L\#\} \Longrightarrow$
$F \models as\ CNot\ C' \Longrightarrow$
$\neg$*no-step backjump* S
begin

We cannot add a like condition *atms-of* $C' \subseteq$ *atms-of-ms* N because to ensure that we can backjump even if the last decision variable has disappeared.

The part of the condition *atm-of* $L \in$ *atm-of* ' *lits-of* (F' @ *Marked* K () # F) is important, otherwise you are not sure that you can backtrack.

14.3.1 Definition

We define dpll with backjumping:

inductive *dpll-bj* :: $'st \Rightarrow 'st \Rightarrow bool$ **for** S :: $'st$ **where**
bj-decide$_{NOT}$: *decide*$_{NOT}$ $S\ S' \Longrightarrow$ *dpll-bj* $S\ S'$ |
bj-propagate$_{NOT}$: *propagate*$_{NOT}$ $S\ S' \Longrightarrow$ *dpll-bj* $S\ S'$ |
bj-backjump: *backjump* $S\ S' \Longrightarrow$ *dpll-bj* $S\ S'$

lemmas *dpll-bj-induct* = *dpll-bj.induct*[*split-format*(*complete*)]
thm *dpll-bj-induct*[*OF dpll-with-backjumping-ops-axioms*]
lemma *dpll-bj-all-induct*[*consumes 2*, *case-names decide*$_{NOT}$ *propagate*$_{NOT}$ *backjump*]:
  fixes $S\ T$ :: $'st$
  assumes
    dpll-bj $S\ T$ **and**
    inv S
    $\bigwedge L\ T.$ *undefined-lit* (*trail* S) $L \Longrightarrow$ *atm-of* $L \in$ *atms-of-msu* (*clauses* S)
      $\Longrightarrow T \sim$ *prepend-trail* (*Marked* L ()) S
      $\Longrightarrow P\ S\ T$ **and**
    $\bigwedge C\ L\ T.\ C + \{\#L\#\} \in\#$ *clauses* $S \Longrightarrow$ *trail* $S \models as\ CNot\ C \Longrightarrow$ *undefined-lit* (*trail* S) L
      $\Longrightarrow T \sim$ *prepend-trail* (*Propagated* L ()) S
      $\Longrightarrow P\ S\ T$ **and**
    $\bigwedge C\ F'\ K\ F\ L\ C'\ T.\ C \in\#$ *clauses* $S \Longrightarrow F'$ @ *Marked* K () # $F \models as\ CNot\ C$
      $\Longrightarrow$ *trail* $S = F'$ @ *Marked* K () # F
      $\Longrightarrow$ *undefined-lit* $F\ L$
      $\Longrightarrow$ *atm-of* $L \in$ *atms-of-msu* (*clauses* S) $\cup$ *atm-of* ' (*lits-of* (F' @ *Marked* K () # F))
      $\Longrightarrow$ *clauses* $S \models pm\ C' + \{\#L\#\}$
      $\Longrightarrow F \models as\ CNot\ C'$
      $\Longrightarrow T \sim$ *prepend-trail* (*Propagated* L ()) (*reduce-trail-to*$_{NOT}$ $F\ S$)
      $\Longrightarrow P\ S\ T$
  shows $P\ S\ T$
  apply (*induct T rule*: *dpll-bj-induct*[*OF local.dpll-with-backjumping-ops-axioms*])
    apply (*rule assms*(*1*))
   using *assms*(*3*) **apply** *blast*
  apply (*elim propagateE*) **using** *assms*(*4*) **apply** *blast*
  apply (*elim backjumpE*) **using** *assms*(*5*) ‹*inv S*› **by** *simp*

14.3.2 Basic properties

First, some better suited induction principle **lemma** *dpll-bj-clauses*:
  assumes *dpll-bj* $S\ T$ **and** *inv* S
  shows *clauses* S = *clauses* T
  using *assms* **by** (*induction rule*: *dpll-bj-all-induct*) *auto*

No duplicates in the trail **lemma** *dpll-bj-no-dup*:

assumes *dpll-bj S T* **and** *inv S*
and *no-dup* (*trail S*)
shows *no-dup* (*trail T*)
using *assms* **by** (*induction rule*: *dpll-bj-all-induct*)
(*auto simp add*: *defined-lit-map reduce-trail-to$_{NOT}$-skip-beginning*)

Valuations **lemma** *dpll-bj-sat-iff*:
assumes *dpll-bj S T* **and** *inv S*
shows $I \models sm$ *clauses S* $\longleftrightarrow$ $I \models sm$ *clauses T*
using *assms* **by** (*induction rule*: *dpll-bj-all-induct*) *auto*

Clauses **lemma** *dpll-bj-atms-of-ms-clauses-inv*:
assumes
dpll-bj S T **and**
inv S
shows *atms-of-msu* (*clauses S*) = *atms-of-msu* (*clauses T*)
using *assms* **by** (*induction rule*: *dpll-bj-all-induct*) *auto*

lemma *dpll-bj-atms-in-trail*:
assumes
dpll-bj S T **and**
inv S **and**
atm-of ‘ (*lits-of* (*trail S*)) $\subseteq$ *atms-of-msu* (*clauses S*)
shows *atm-of* ‘ (*lits-of* (*trail T*)) $\subseteq$ *atms-of-msu* (*clauses S*)
using *assms* **by** (*induction rule*: *dpll-bj-all-induct*)
(*auto simp*: *in-plus-implies-atm-of-on-atms-of-ms reduce-trail-to$_{NOT}$-skip-beginning*)

lemma *dpll-bj-atms-in-trail-in-set*:
assumes *dpll-bj S T***and**
inv S **and**
atms-of-msu (*clauses S*) $\subseteq$ *A* **and**
atm-of ‘ (*lits-of* (*trail S*)) $\subseteq$ *A*
shows *atm-of* ‘ (*lits-of* (*trail T*)) $\subseteq$ *A*
using *assms* **by** (*induction rule*: *dpll-bj-all-induct*)
(*auto simp*: *in-plus-implies-atm-of-on-atms-of-ms*)

lemma *dpll-bj-all-decomposition-implies-inv*:
assumes
dpll-bj S T **and**
inv: *inv S* **and**
decomp: *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
shows *all-decomposition-implies-m* (*clauses T*) (*get-all-marked-decomposition* (*trail T*))
using *assms*(*1*,*2*)
proof (*induction rule*:*dpll-bj-all-induct*)
case *decide$_{NOT}$*
then show *?case* **using** *decomp* **by** *auto*
next
case (*propagate$_{NOT}$ C L T*) **note** *propa* = *this*(*1*) **and** *undef* = *this*(*3*) **and** *T* = *this*(*4*)
let *?M′* = *trail* (*prepend-trail* (*Propagated L* ()) *S*)
let *?N* = *clauses S*
obtain *a y l* **where** *ay*: *get-all-marked-decomposition ?M′* = (*a*, *y*) # *l*
by (*cases get-all-marked-decomposition ?M′*) *fastforce+*
then have *M′*: *?M′* = *y* @ *a* **using** *get-all-marked-decomposition-decomp*[*of ?M′*] **by** *auto*
have *M*: *get-all-marked-decomposition* (*trail S*) = (*a*, *tl y*) # *l*
using *ay undef* **by** (*cases get-all-marked-decomposition* (*trail S*)) *auto*

have y_0: $y = (Propagated\ L\ ())\ \#\ (tl\ y)$
using *ay undef* **by** (*auto simp add*: *M*)
from *arg-cong*[*OF this*, *of set*] **have** *y*[*simp*]: *set y* = *insert* (*Propagated L* ()) (*set* (*tl y*))
by *simp*
have *tr-S*: *trail S* = *tl y* @ *a*
using *arg-cong*[*OF M′*, *of tl*] y_0 *M get-all-marked-decomposition-decomp* **by** *force*
have *a-Un-N-M*: (λ*a*. {#*lit-of a*#}) ‘ *set a* ∪ *set-mset ?N* ⊨*ps* (λ*a*. {#*lit-of a*#}) ‘ *set* (*tl y*)
using *decomp ay* **unfolding** *all-decomposition-implies-def* **by** (*simp add*: *M*)+

moreover have (λ*a*. {#*lit-of a*#}) ‘ *set a* ∪ *set-mset ?N* ⊨*p* {#*L*#} (**is** *?I* ⊨*p* -)
proof (*rule true-clss-cls-plus-CNot*)
show *?I* ⊨*p C* + {#*L*#}
using *propa* $propagate_{NOT}$.*prems* **by** (*auto dest*!: *true-clss-clss-in-imp-true-clss-cls*)
next
have (λ*m*. {#*lit-of m*#}) ‘ *set ?M′* ⊨*ps CNot C*
using ‹*trail S* ⊨*as CNot C*› *undef* **by** (*auto simp add*: *true-annots-true-clss-clss*)
have *a1*: (λ*m*. {#*lit-of m*#}) ‘ *set a* ∪ (λ*m*. {#*lit-of m*#}) ‘ *set* (*tl y*) ⊨*ps CNot C*
using $propagate_{NOT}$.*hyps*(*2*) *tr-S true-annots-true-clss-clss*
by (*force simp add*: *image-Un sup-commute*)
have *a2*: *set-mset* (*clauses S*)∪ (λ*a*. {#*lit-of a*#}) ‘ *set a*
⊨*ps* (λ*a*. {#*lit-of a*#}) ‘ *set* (*tl y*)
using *calculation* **by** (*auto simp add*: *sup-commute*)
show (λ*m*. {#*lit-of m*#}) ‘ *set a* ∪ *set-mset* (*clauses S*)⊨*ps CNot C*
proof −
have *set-mset* (*clauses S*) ∪ (λ*m*. {#*lit-of m*#}) ‘ *set a* ⊨*ps*
(λ*m*. {#*lit-of m*#}) ‘ *set a* ∪ (λ*m*. {#*lit-of m*#}) ‘ *set* (*tl y*)
using *a2 true-clss-clss-def* **by** *blast*
then show (λ*m*. {#*lit-of m*#}) ‘ *set a* ∪ *set-mset* (*clauses S*)⊨*ps CNot C*
using *a1* **unfolding** *sup-commute* **by** (*meson true-clss-clss-left-right*
true-clss-clss-union-and true-clss-clss-union-l-r)
qed
qed

ultimately have (λ*a*. {#*lit-of a*#}) ‘ *set a* ∪ *set-mset ?N* ⊨*ps* (λ*a*. {#*lit-of a*#}) ‘ *set ?M′*
unfolding *M′* **by** (*auto simp add*: *all-in-true-clss-clss image-Un*)

then show *?case*
using *decomp T M undef* **unfolding** *ay all-decomposition-implies-def* **by** (*auto simp add*: *ay*)
next
case (*backjump C F′ K F L D T*) **note** *confl* = *this*(*2*) **and** *tr* = *this*(*3*) **and** *undef* = *this*(*4*)
and *L* = *this*(*5*) **and** *N-C* = *this*(*6*) **and** *vars-D* = *this*(*5*) **and** *T* = *this*(*8*)
have *decomp*: *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition F*)
using *decomp* **unfolding** *tr all-decomposition-implies-def*
by (*metis* (*no-types*, *lifting*) *get-all-marked-decomposition.simps*(*1*)
get-all-marked-decomposition-never-empty hd-Cons-tl insert-iff list.sel(*3*) *list.set*(*2*)
tl-get-all-marked-decomposition-skip-some)

moreover have (λ*a*. {#*lit-of a*#}) ‘ *set* (*fst* (*hd* (*get-all-marked-decomposition F*)))
∪ *set-mset* (*clauses S*)
⊨*ps* (λ*a*. {#*lit-of a*#}) ‘ *set* (*snd* (*hd* (*get-all-marked-decomposition F*)))
by (*metis all-decomposition-implies-cons-single decomp get-all-marked-decomposition-never-empty*
hd-Cons-tl)
moreover
have *vars-of-D*: *atms-of D* ⊆ *atm-of* ‘ *lits-of F*
using ‹*F* ⊨*as CNot D*› **unfolding** *atms-of-def*

by (*meson image-subsetI mem-set-mset-iff true-annots-CNot-all-atms-defined*)

obtain *a b li* **where** *F*: *get-all-marked-decomposition F* = (*a*, *b*) # *li*
by (*cases get-all-marked-decomposition F*) *auto*
have *F* = *b* @ *a*
using *get-all-marked-decomposition-decomp*[*of F a b*] *F* **by** *auto*
have *a-N-b*:(λa. {#*lit-of a*#}) ' *set a* ∪ *set-mset* (*clauses S*) ⊨*ps* (λa. {#*lit-of a*#}) ' *set b*
using *decomp* **unfolding** *all-decomposition-implies-def* **by** (*auto simp add*: *F*)

have *F-D*:(λa. {#*lit-of a*#}) ' *set F* ⊨*ps CNot D*
using ‹*F* ⊨*as CNot D*› **by** (*simp add*: *true-annots-true-clss-clss*)
then have (λa. {#*lit-of a*#}) ' *set a* ∪ (λa. {#*lit-of a*#}) ' *set b* ⊨*ps CNot D*
unfolding ‹*F* = *b* @ *a*› **by** (*simp add*: *image-Un sup.commute*)
have *a-N-CNot-D*: (λa. {#*lit-of a*#}) ' *set a* ∪ *set-mset* (*clauses S*)
⊨*ps CNot D* ∪ (λa. {#*lit-of a*#}) ' *set b*
apply (*rule true-clss-clss-left-right*)
using *a-N-b F-D* **unfolding** ‹*F* = *b* @ *a*› **by** (*auto simp add*: *image-Un ac-simps*)

have *a-N-D-L*: (λa. {#*lit-of a*#}) ' *set a* ∪ *set-mset* (*clauses S*) ⊨*p D*+{#*L*#}
by (*simp add*: *N-C*)
have (λa. {#*lit-of a*#}) ' *set a* ∪ *set-mset* (*clauses S*) ⊨*p* {#*L*#}
using *a-N-D-L a-N-CNot-D* **by** (*blast intro*: *true-clss-cls-plus-CNot*)
then show *?case*
using *decomp T tr undef* **unfolding** *all-decomposition-implies-def* **by** (*auto simp add*: *F*)
qed

14.3.3 Termination

Using a proper measure **lemma** *length-get-all-marked-decomposition-append-Marked*:
length (*get-all-marked-decomposition* (*F'* @ *Marked K* () # *F*)) =
length (*get-all-marked-decomposition F'*)
+ *length* (*get-all-marked-decomposition* (*Marked K* () # *F*))
− *1*
by (*induction F' rule*: *marked-lit-list-induct*) *auto*

lemma *take-length-get-all-marked-decomposition-marked-sandwich*:
take (*length* (*get-all-marked-decomposition F*))
(*map* (*f o snd*) (*rev* (*get-all-marked-decomposition* (*F'* @ *Marked K* () # *F*))))
=
map (*f o snd*) (*rev* (*get-all-marked-decomposition F*))

proof (*induction F' rule*: *marked-lit-list-induct*)
case *nil*
then show *?case* **by** *auto*
next
case (*marked K*)
then show *?case* **by** (*simp add*: *length-get-all-marked-decomposition-append-Marked*)
next
case (*proped L m F'*) **note** *IH* = *this*(*1*)
obtain *a b l* **where** *F'*: *get-all-marked-decomposition* (*F'* @ *Marked K* () # *F*) = (*a*, *b*) # *l*
by (*cases get-all-marked-decomposition* (*F'* @ *Marked K* () # *F*)) *auto*
have *length* (*get-all-marked-decomposition F*) − *length l* = *0*
using *length-get-all-marked-decomposition-append-Marked*[*of F' K F*]
unfolding *F'* **by** (*cases get-all-marked-decomposition F'*) *auto*
then show *?case*
using *IH* **by** (*simp add*: *F'*)

qed

lemma *length-get-all-marked-decomposition-length*:
length (*get-all-marked-decomposition M*) $\leq$ *1* + *length M*
by (*induction M rule*: *marked-lit-list-induct*) *auto*

lemma *length-in-get-all-marked-decomposition-bounded*:
assumes *i*:*i* $\in$ *set* (*trail-weight S*)
shows *i* $\leq$ *Suc* (*length* (*trail S*))
proof −
obtain *a b* **where**
(*a*, *b*) $\in$ *set* (*get-all-marked-decomposition* (*trail S*)) **and**
ib: *i* = *Suc* (*length b*)
using *i* **by** *auto*
then obtain *c* **where** *trail S* = *c* @ *b* @ *a*
using *get-all-marked-decomposition-exists-prepend'* **by** *metis*
from *arg-cong*[*OF this*, *of length*] **show** *?thesis* **using** *i ib* **by** *auto*
qed

Well-foundedness The bounds are the following:

- *1* + *card* (*atms-of-ms A*): *card* (*atms-of-ms A*) is an upper bound on the length of the list. As *get-all-marked-decomposition* appends an possibly empty couple at the end, adding one is needed.
- *2* + *card* (*atms-of-ms A*): *card* (*atms-of-ms A*) is an upper bound on the number of elements, where adding one is necessary for the same reason as for the bound on the list, and one is needed to have a strict bound.

abbreviation *unassigned-lit* :: *'b literal multiset set* $\Rightarrow$ *'a list* $\Rightarrow$ *nat* **where**
unassigned-lit N M $\equiv$ *card* (*atms-of-ms N*) − *length M*
lemma *dpll-bj-trail-mes-increasing-prop*:
fixes *M* :: (*'v*, *unit*, *unit*) *marked-lits* **and** *N* :: *'v clauses*
assumes
dpll-bj S T **and**
inv S **and**
NA: *atms-of-msu* (*clauses S*) $\subseteq$ *atms-of-ms A* **and**
MA: *atm-of* ‘ *lits-of* (*trail S*) $\subseteq$ *atms-of-ms A* **and**
n-d: *no-dup* (*trail S*) **and**
finite: *finite A*
shows μ_C (*1*+*card* (*atms-of-ms A*)) (*2*+*card* (*atms-of-ms A*)) (*trail-weight T*)
\> μ_C (*1*+*card* (*atms-of-ms A*)) (*2*+*card* (*atms-of-ms A*)) (*trail-weight S*)
using *assms*(*1*,*2*)
proof (*induction rule*: *dpll-bj-all-induct*)
case ($propagate_{NOT}$ *C L*) **note** *CLN* = *this*(*1*) **and** *MC* =*this*(*2*) **and** *undef-L* = *this*(*3*) **and** *T* = *this*(*4*)
have *incl*: *atm-of* ‘ *lits-of* (*Propagated L* () # *trail S*) $\subseteq$ *atms-of-ms A*
using $propagate_{NOT}$*.hyps propagate-ops.*$propagate_{NOT}$ *dpll-bj-atms-in-trail-in-set bj-*$propagate_{NOT}$ *NA MA CLN* **by** (*auto simp*: *in-plus-implies-atm-of-on-atms-of-ms*)

have *no-dup*: *no-dup* (*Propagated L* () # *trail S*)
using *defined-lit-map n-d undef-L* **by** *auto*
obtain *a b l* **where** *M*: *get-all-marked-decomposition* (*trail S*) = (*a*, *b*) # *l*
by (*case-tac get-all-marked-decomposition* (*trail S*)) *auto*

have *b-le-M*: *length b* $\leq$ *length* (*trail S*)
  using *get-all-marked-decomposition-decomp*[*of trail S*] **by** (*simp add*: *M*)
have *finite* (*atms-of-ms A*) **using** *finite* **by** *simp*

then have *length* (*Propagated L* () # *trail S*) $\leq$ *card* (*atms-of-ms A*)
  using *incl finite* **unfolding** *no-dup-length-eq-card-atm-of-lits-of*[*OF no-dup*]
  by (*simp add*: *card-mono*)
then have *latm*: *unassigned-lit A b* = *Suc* (*unassigned-lit A* (*Propagated L d* # *b*))
  using *b-le-M* **by** *auto*
then show *?case* **using** *T undef-L* **by** (*auto simp*: *latm M* μ_C*-cons*)
next
case (*decide*$_{NOT}$ *L*) **note** *undef-L* = *this*(*1*) **and** *MC* = *this*(*2*) **and** *T* = *this*(*3*)
have *incl*: *atm-of* ' *lits-of* (*Marked L* () # (*trail S*)) $\subseteq$ *atms-of-ms A*
  using *dpll-bj-atms-in-trail-in-set bj-decide*$_{NOT}$ *decide*$_{NOT}$.*decide*$_{NOT}$[*OF decide*$_{NOT}$.*hyps*] *NA MA MC*
  by *auto*

have *no-dup*: *no-dup* (*Marked L* () # (*trail S*))
  using *defined-lit-map n-d undef-L* **by** *auto*
obtain *a b l* **where** *M*: *get-all-marked-decomposition* (*trail S*) = (*a*, *b*) # *l*
  by (*case-tac get-all-marked-decomposition* (*trail S*)) *auto*

then have *length* (*Marked L* () # (*trail S*)) $\leq$ *card* (*atms-of-ms A*)
  using *incl finite* **unfolding** *no-dup-length-eq-card-atm-of-lits-of*[*OF no-dup*]
  by (*simp add*: *card-mono*)
then have *latm*: *unassigned-lit A* (*trail S*) = *Suc* (*unassigned-lit A* (*Marked L lv* # (*trail S*)))
  by *force*
show *?case* **using** *T undef-L* **by** (*simp add*: *latm* μ_C*-cons*)
next
case (*backjump C F' K F L C' T*) **note** *undef-L* = *this*(*4*) **and** *MC* =*this*(*1*) **and** *tr-S* = *this*(*3*) **and**
  L = *this*(*5*) **and** *T* = *this*(*8*)
have *incl*: *atm-of* ' *lits-of* (*Propagated L* () # *F*) $\subseteq$ *atms-of-ms A*
  using *dpll-bj-atms-in-trail-in-set NA MA tr-S L* **by** *auto*

have *no-dup*: *no-dup* (*Propagated L* () # *F*)
  using *defined-lit-map n-d undef-L tr-S* **by** *auto*
obtain *a b l* **where** *M*: *get-all-marked-decomposition* (*trail S*) = (*a*, *b*) # *l*
  by (*cases get-all-marked-decomposition* (*trail S*)) *auto*
have *b-le-M*: *length b* $\leq$ *length* (*trail S*)
  using *get-all-marked-decomposition-decomp*[*of trail S*] **by** (*simp add*: *M*)
have *fin-atms-A*: *finite* (*atms-of-ms A*) **using** *finite* **by** *simp*

then have *F-le-A*: *length* (*Propagated L* () # *F*) $\leq$ *card* (*atms-of-ms A*)
  using *incl finite* **unfolding** *no-dup-length-eq-card-atm-of-lits-of*[*OF no-dup*]
  by (*simp add*: *card-mono*)
have *tr-S-le-A*: *length* (*trail S*) $\leq$ (*card* (*atms-of-ms A*))
  using *n-d MA* **by** (*metis fin-atms-A card-mono no-dup-length-eq-card-atm-of-lits-of*)
obtain *a b l* **where** *F*: *get-all-marked-decomposition F* = (*a*, *b*) # *l*
  by (*cases get-all-marked-decomposition F*) *auto*
then have *F* = *b* @ *a*
  using *get-all-marked-decomposition-decomp*[*of Propagated L* () # *F a*
    Propagated L () # *b*] **by** *simp*
then have *latm*: *unassigned-lit A b* = *Suc* (*unassigned-lit A* (*Propagated L* () # *b*))
  using *F-le-A* **by** *simp*

obtain *rem* **where**
rem:*map* (*λa. Suc* (*length* (*snd a*))) (*rev* (*get-all-marked-decomposition* (*F′* @ *Marked K* () # *F*)))
= *map* (*λa. Suc* (*length* (*snd a*))) (*rev* (*get-all-marked-decomposition F*)) @ *rem*
using *take-length-get-all-marked-decomposition-marked-sandwich*[*of F λa. Suc* (*length a*) *F′ K*]
unfolding *o-def* **by** (*metis append-take-drop-id*)
then have *rem*: *map* (*λa. Suc* (*length* (*snd a*)))
(*get-all-marked-decomposition* (*F′* @ *Marked K* () # *F*))
= *rev rem* @ *map* (*λa. Suc* (*length* (*snd a*))) ((*get-all-marked-decomposition F*))
by (*simp add*: *rev-map*[*symmetric*] *rev-swap*)
have *length* (*rev rem* @ *map* (*λa. Suc* (*length* (*snd a*))) (*get-all-marked-decomposition F*))
≤ *Suc* (*card* (*atms-of-ms A*))
using *arg-cong*[*OF rem*, *of length*] *tr-S-le-A*
length-get-all-marked-decomposition-length[*of F′* @ *Marked K* () # *F*] *tr-S* **by** *auto*
moreover
{ fix *i* :: *nat* **and** *xs* :: *′a list*
have *i* < *length xs* ⟹ *length xs* − *Suc i* < *length xs*
by *auto*
then have *H*: *i*<*length xs* ⟹ *rev xs* ! *i* ∈ *set xs*
using *rev-nth*[*of i xs*] **unfolding** *in-set-conv-nth* **by** (*force simp add*: *in-set-conv-nth*)
} note *H* = *this*
have ∀ *i*<*length rem*. *rev rem* ! *i* < *card* (*atms-of-ms A*) + *2*
using *tr-S-le-A length-in-get-all-marked-decomposition-bounded*[*of - S*] **unfolding** *tr-S*
by (*force simp add*: *o-def rem dest*!: *H intro*: *length-get-all-marked-decomposition-length*)
ultimately show *?case*
using μ_C*-bounded*[*of rev rem card* (*atms-of-ms A*)+*2 unassigned-lit A l*] *T undef-L*
by (*simp add*: *rem* μ_C*-append* μ_C*-cons F tr-S*)
qed

lemma *dpll-bj-trail-mes-decreasing-prop*:
assumes *dpll*: *dpll-bj S T* **and** *inv*: *inv S* **and**
N-A: *atms-of-msu* (*clauses S*) ⊆ *atms-of-ms A* **and**
M-A: *atm-of ‘ lits-of* (*trail S*) ⊆ *atms-of-ms A* **and**
nd: *no-dup* (*trail S*) **and**
fin-A: *finite A*
shows (*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*))
− μ_C (*1*+*card* (*atms-of-ms A*)) (*2*+*card* (*atms-of-ms A*)) (*trail-weight T*)
< (*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*))
− μ_C (*1*+*card* (*atms-of-ms A*)) (*2*+*card* (*atms-of-ms A*)) (*trail-weight S*)
proof −
let *?b* = *2*+*card* (*atms-of-ms A*)
let *?s* = *1*+*card* (*atms-of-ms A*)
let *?μ* = μ_C *?s ?b*
have *M′-A*: *atm-of ‘ lits-of* (*trail T*) ⊆ *atms-of-ms A*
by (*meson M-A N-A dpll dpll-bj-atms-in-trail-in-set inv*)
have *nd′*: *no-dup* (*trail T*)
using ‹*dpll-bj S T*› *dpll-bj-no-dup nd inv* **by** *blast*
{ fix *i* :: *nat* **and** *xs* :: *′a list*
have *i* < *length xs* ⟹ *length xs* − *Suc i* < *length xs*
by *auto*
then have *H*: *i*<*length xs* ⟹ *xs* ! *i* ∈ *set xs*
using *rev-nth*[*of i xs*] **unfolding** *in-set-conv-nth* **by** (*force simp add*: *in-set-conv-nth*)
} note *H* = *this*

have *l-M-A*: *length* (*trail S*) ≤ *card* (*atms-of-ms A*)
by (*simp add*: *fin-A M-A card-mono no-dup-length-eq-card-atm-of-lits-of nd*)

have *l-M′-A*: *length* (*trail T*) ≤ *card* (*atms-of-ms A*)
  by (*simp add*: *fin-A M′-A card-mono no-dup-length-eq-card-atm-of-lits-of nd′*)
have *l-trail-weight-M*: *length* (*trail-weight T*) ≤ *1*+*card* (*atms-of-ms A*)
  using *l-M′-A length-get-all-marked-decomposition-length*[*of trail T*] **by** *auto*
have *bounded-M*: ∀ *i*<*length* (*trail-weight T*). (*trail-weight T*)! *i* < *card* (*atms-of-ms A*) + *2*
  using *length-in-get-all-marked-decomposition-bounded*[*of - T*] *l-M′-A*
  by (*metis* (*no-types*, *lifting*) *Nat.le-trans One-nat-def Suc-1 add.right-neutral add-Suc-right le-imp-less-Suc less-eq-Suc-le nth-mem*)

from *dpll-bj-trail-mes-increasing-prop*[*OF dpll inv N-A M-A nd fin-A*]
have μ_C *?s ?b* (*trail-weight S*) < μ_C *?s ?b* (*trail-weight T*) **by** *simp*
moreover from μ_C*-bounded*[*OF bounded-M l-trail-weight-M*]
  have μ_C *?s ?b* (*trail-weight T*) ≤ *?b* ^ *?s* **by** *auto*
ultimately show *?thesis* **by** *linarith*
qed

lemma *wf-dpll-bj*:
  assumes *fin*: *finite A*
  shows *wf* {(*T*, *S*). *dpll-bj S T*
    ∧ *atms-of-msu* (*clauses S*) ⊆ *atms-of-ms A* ∧ *atm-of* ‘ *lits-of* (*trail S*) ⊆ *atms-of-ms A*
    ∧ *no-dup* (*trail S*) ∧ *inv S*}
  (**is** *wf ?A*)
proof (*rule wf-bounded-measure*[*of -*
    λ-. (*2* + *card* (*atms-of-ms A*))^(*1* + *card* (*atms-of-ms A*))
    λ*S*. μ_C (*1*+*card* (*atms-of-ms A*)) (*2*+*card* (*atms-of-ms A*)) (*trail-weight S*)])
  fix *a b* :: *′st*
  let *?b* = *2*+*card* (*atms-of-ms A*)
  let *?s* = *1*+*card* (*atms-of-ms A*)
  let *?μ* = μ_C *?s ?b*
  assume *ab*: (*b*, *a*) ∈ {(*T*, *S*). *dpll-bj S T*
    ∧ *atms-of-msu* (*clauses S*) ⊆ *atms-of-ms A* ∧ *atm-of* ‘ *lits-of* (*trail S*) ⊆ *atms-of-ms A*
    ∧ *no-dup* (*trail S*) ∧ *inv S*}

  have *fin-A*: *finite* (*atms-of-ms A*)
    using *fin* **by** *auto*
  have
    dpll-bj: *dpll-bj a b* **and**
    N-A: *atms-of-msu* (*clauses a*) ⊆ *atms-of-ms A* **and**
    M-A: *atm-of* ‘ *lits-of* (*trail a*) ⊆ *atms-of-ms A* **and**
    nd: *no-dup* (*trail a*) **and**
    inv: *inv a*
    using *ab* **by** *auto*

  have *M′-A*: *atm-of* ‘ *lits-of* (*trail b*) ⊆ *atms-of-ms A*
    by (*meson M-A N-A* ‹*dpll-bj a b*› *dpll-bj-atms-in-trail-in-set inv*)
  have *nd′*: *no-dup* (*trail b*)
    using ‹*dpll-bj a b*› *dpll-bj-no-dup nd inv* **by** *blast*
  { fix *i* :: *nat* **and** *xs* :: *′a list*
    have *i* < *length xs* ⟹ *length xs* − *Suc i* < *length xs*
      by *auto*
    then have *H*: *i*<*length xs* ⟹ *xs* ! *i* ∈ *set xs*
      using *rev-nth*[*of i xs*] **unfolding** *in-set-conv-nth* **by** (*force simp add*: *in-set-conv-nth*)
  } note *H* = *this*

  have *l-M-A*: *length* (*trail a*) ≤ *card* (*atms-of-ms A*)

by (*simp add*: *fin-A M-A card-mono no-dup-length-eq-card-atm-of-lits-of nd*)
have *l-M′-A*: *length* (*trail b*) $\leq$ *card* (*atms-of-ms A*)
by (*simp add*: *fin-A M′-A card-mono no-dup-length-eq-card-atm-of-lits-of nd′*)
have *l-trail-weight-M*: *length* (*trail-weight b*) $\leq$ *1+card* (*atms-of-ms A*)
using *l-M′-A length-get-all-marked-decomposition-length*[*of trail b*] **by** *auto*
have *bounded-M*: $\forall$ *i<length* (*trail-weight b*). (*trail-weight b*)! *i* < *card* (*atms-of-ms A*) + *2*
using *length-in-get-all-marked-decomposition-bounded*[*of - b*] *l-M′-A*
by (*metis* (*no-types*, *lifting*) *Nat.le-trans One-nat-def Suc-1 add.right-neutral add-Suc-right le-imp-less-Suc less-eq-Suc-le nth-mem*)

from *dpll-bj-trail-mes-increasing-prop*[*OF dpll-bj inv N-A M-A nd fin*]
have μ_C *?s ?b* (*trail-weight a*) < μ_C *?s ?b* (*trail-weight b*) **by** *simp*
moreover from μ_C*-bounded*[*OF bounded-M l-trail-weight-M*]
have μ_C *?s ?b* (*trail-weight b*) $\leq$ *?b* ^ *?s* **by** *auto*
ultimately show *?b* ^ *?s* $\leq$ *?b* ^ *?s* $\wedge$
μ_C *?s ?b* (*trail-weight b*) $\leq$ *?b* ^ *?s* $\wedge$
μ_C *?s ?b* (*trail-weight a*) < μ_C *?s ?b* (*trail-weight b*)
by *blast*
qed

14.3.4 Normal Forms

We prove that given a normal form of DPLL, with some invariants, the either N is satisfiable and the built valuation M is a model; or N is unsatisfiable.

Idea of the proof: We have to prove tat *satisfiable N*, $\neg$ *M* $\models$*as N* and there is no remaining step is incompatible.

1. The *decide* rules tells us that every variable in N has a value.
2. $\neg$ *M* $\models$*as N* tells us that there is conflict.
3. There is at least one decision in the trail (otherwise, M is a model of N).
4. Now if we build the clause with all the decision literals of the trail, we can apply the *backjump* rule.

The assumption are saying that we have a finite upper bound A for the literals, that we cannot do any step *no-step dpll-bj S*

theorem *dpll-backjump-final-state*:
fixes *A* :: *′v literal multiset set* **and** *S T* :: *′st*
assumes
atms-of-msu (*clauses S*) $\subseteq$ *atms-of-ms A* **and**
atm-of ‘ *lits-of* (*trail S*) $\subseteq$ *atms-of-ms A* **and**
no-dup (*trail S*) **and**
finite A **and**
inv: *inv S* **and**
n-s: *no-step dpll-bj S* **and**
decomp: *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
shows *unsatisfiable* (*set-mset* (*clauses S*))
$\vee$ (*trail S* $\models$*asm clauses S* $\wedge$ *satisfiable* (*set-mset* (*clauses S*)))
proof −
let *?N* = *set-mset* (*clauses S*)
let *?M* = *trail S*
consider

(*sat*) *satisfiable ?N* **and** *?M* $\models$*as ?N*
| (*sat'*) *satisfiable ?N* **and** $\neg$ *?M* $\models$*as ?N*
| (*unsat*) *unsatisfiable ?N*
by *auto*
then show *?thesis*
proof *cases*
case *sat'* **note** *sat* = *this*(*1*) **and** *M* = *this*(*2*)
obtain *C* **where** *C* $\in$ *?N* **and** $\neg$*?M* $\models$*a C* **using** *M* **unfolding** *true-annots-def* **by** *auto*
obtain *I* :: *'v literal set* **where**
I $\models$*s ?N* **and**
cons: *consistent-interp I* **and**
tot: *total-over-m I ?N* **and**
atm-I-N: *atm-of ` I* $\subseteq$ *atms-of-ms ?N*
using *sat* **unfolding** *satisfiable-def-min* **by** *auto*
let *?I* = *I* $\cup$ {*P*| *P*. *P* $\in$ *lits-of ?M* $\wedge$ *atm-of P* $\notin$ *atm-of ` I*}
let *?O* = {{#*lit-of L*#} |*L*. *is-marked L* $\wedge$ *L* $\in$ *set ?M* $\wedge$ *atm-of* (*lit-of L*) $\notin$ *atms-of-ms ?N*}
have *cons-I'*: *consistent-interp ?I*
using *cons* **using** ‹*no-dup ?M*› **unfolding** *consistent-interp-def*
by (*auto simp add*: *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set lits-of-def*
dest!: *no-dup-cannot-not-lit-and-uminus*)
have *tot-I'*: *total-over-m ?I* (*?N* $\cup$ (λ*a*. {#*lit-of a*#}) ` *set ?M*)
using *tot atms-of-s-def* **unfolding** *total-over-m-def total-over-set-def*
by *fastforce*
have {*P* |*P*. *P* $\in$ *lits-of ?M* $\wedge$ *atm-of P* $\notin$ *atm-of ` I*} $\models$*s ?O*
using ‹*I*$\models$*s ?N*› *atm-I-N* **by** (*auto simp add*: *atm-of-eq-atm-of true-clss-def lits-of-def*)
then have *I'-N*: *?I* $\models$*s ?N* $\cup$ *?O*
using ‹*I*$\models$*s ?N*› *true-clss-union-increase* **by** *force*
have *tot'*: *total-over-m ?I* (*?N*$\cup$*?O*)
using *atm-I-N tot* **unfolding** *total-over-m-def total-over-set-def*
by (*force simp*: *image-iff lits-of-def dest*!: *is-marked-ex-Marked*)

have *atms-N-M*: *atms-of-ms ?N* $\subseteq$ *atm-of ` lits-of ?M*
proof (*rule ccontr*)
assume $\neg$ *?thesis*
then obtain *l* :: *'v* **where**
l-N: *l* $\in$ *atms-of-ms ?N* **and**
l-M: *l* $\notin$ *atm-of ` lits-of ?M*
by *auto*
have *undefined-lit ?M* (*Pos l*)
using *l-M* **by** (*metis Marked-Propagated-in-iff-in-lits-of*
atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set literal.sel(*1*))
from *bj-decide*$_{NOT}$[*OF decide*$_{NOT}$[*OF this*]] **show** *False*
using *l-N n-s* **by** (*metis literal.sel*(*1*) *state-eq*$_{NOT}$*-ref*)
qed

have *?M* $\models$*as CNot C*
by (*metis* ‹*C* $\in$ *set-mset* (*clauses S*)› ‹$\neg$ *trail S* $\models$*a C*› *all-variables-defined-not-imply-cnot*
atms-N-M atms-of-atms-of-ms-mono atms-of-ms-CNot-atms-of atms-of-ms-CNot-atms-of-ms
subset-eq)
have $\exists$ *l* $\in$ *set ?M*. *is-marked l*
proof (*rule ccontr*)
let *?O* = {{#*lit-of L*#} |*L*. *is-marked L* $\wedge$ *L* $\in$ *set ?M* $\wedge$ *atm-of* (*lit-of L*) $\notin$ *atms-of-ms ?N*}
have ϑ[*iff*]: $\bigwedge$*I*. *total-over-m I* (*?N* $\cup$ *?O* $\cup$ (λ*a*. {#*lit-of a*#}) ` *set ?M*)
$\longleftrightarrow$ *total-over-m I* (*?N* $\cup$(λ*a*. {#*lit-of a*#}) ` *set ?M*)
unfolding *total-over-set-def total-over-m-def atms-of-ms-def* **by** *auto*

assume ¬ *?thesis*
then have [*simp*]:{{#*lit-of L*#} |*L. is-marked L* ∧ *L* ∈ *set ?M*}
= {{#*lit-of L*#} |*L. is-marked L* ∧ *L* ∈ *set ?M* ∧ *atm-of* (*lit-of L*) ∉ *atms-of-ms ?N*}
by *auto*
then have *?N* ∪ *?O* ⊨*ps* (λ*a*. {#*lit-of a*#}) ‘ *set ?M*
using *all-decomposition-implies-propagated-lits-are-implied*[*OF decomp*] **by** *auto*

then have *?I* ⊨*s* (λ*a*. {#*lit-of a*#}) ‘ *set ?M*
using *cons-I′ I′-N tot-I′* ‹*?I* ⊨*s ?N* ∪ *?O*› **unfolding** ϑ *true-clss-clss-def* **by** *blast*
then have *lits-of ?M* ⊆ *?I*
unfolding *true-clss-def lits-of-def* **by** *auto*
then have *?M* ⊨*as ?N*
using *I′-N* ‹*C* ∈ *?N*› ‹¬ *?M* ⊨*a C*› *cons-I′ atms-N-M*
by (*meson* ‹*trail S* ⊨*as CNot C*› *consistent-CNot-not rev-subsetD sup-ge1 true-annot-def true-annots-def true-cls-mono-set-mset-l true-clss-def*)
then show *False* **using** *M* **by** *fast*
qed
from *List.split-list-first-propE*[*OF this*] **obtain** *K* :: ′*v literal* **and**
F F′ :: (′*v*, *unit*, *unit*) *marked-lit list* **where**
M-K: *?M* = *F′* @ *Marked K* () # *F* **and**
nm: ∀*f*∈*set F′*. ¬*is-marked f*
unfolding *is-marked-def* **by** (*metis* (*full-types*) *old.unit.exhaust*)
let *?K* = *Marked K* ()::(′*v*, *unit*, *unit*) *marked-lit*
have *?K* ∈ *set ?M*
unfolding *M-K* **by** *auto*
let *?C* = *image-mset lit-of* {#*L*∈#*mset ?M*. *is-marked L* ∧ *L*≠*?K*#} :: ′*v literal multiset*
let *?C′* = *set-mset* (*image-mset* (λ*L*::′*v literal*. {#*L*#}) (*?C*+{#*lit-of ?K*#}))
have *?N* ∪ {{#*lit-of L*#} |*L. is-marked L* ∧ *L* ∈ *set ?M*} ⊨*ps* (λ*a*. {#*lit-of a*#}) ‘ *set ?M*
using *all-decomposition-implies-propagated-lits-are-implied*[*OF decomp*] **.**
moreover have *C′*: *?C′* = {{#*lit-of L*#} |*L. is-marked L* ∧ *L* ∈ *set ?M*}
unfolding *M-K* **apply** *standard*
apply *force*
using *IntI* **by** *auto*
ultimately have *N-C-M*: *?N* ∪ *?C′* ⊨*ps* (λ*a*. {#*lit-of a*#}) ‘ *set ?M*
by *auto*
have *N-M-False*: *?N* ∪ (λ*L*. {#*lit-of L*#}) ‘ (*set ?M*) ⊨*ps* {{#}}
using *M* ‹*?M* ⊨*as CNot C*› ‹*C*∈*?N*› **unfolding** *true-clss-clss-def true-annots-def Ball-def true-annot-def* **by** (*metis consistent-CNot-not sup.orderE sup-commute true-clss-def true-clss-singleton-lit-of-implies-incl true-clss-union true-clss-union-increase*)

have *undefined-lit F K* **using** ‹*no-dup ?M*› **unfolding** *M-K* **by** (*simp add*: *defined-lit-map*)
moreover
have *?N* ∪ *?C′* ⊨*ps* {{#}}
proof −
have *A*: *?N* ∪ *?C′* ∪ (λ*a*. {#*lit-of a*#}) ‘ *set ?M* =
?N ∪ (λ*a*. {#*lit-of a*#}) ‘ *set ?M*
unfolding *M-K* **by** *auto*
show *?thesis*
using *true-clss-clss-left-right*[*OF N-C-M*, *of* {{#}}] *N-M-False* **unfolding** *A* **by** *auto*
qed
have *?N* ⊨*p image-mset uminus ?C* + {#−*K*#}
unfolding *true-clss-cls-def true-clss-clss-def total-over-m-def*
proof (*intro allI impI*)
fix *I*
assume

tot: *total-over-set I* (*atms-of-ms* (*?N* ∪ {*image-mset uminus ?C*+ {#− *K*#}})) **and**
cons: *consistent-interp I* **and**
I ⊨*s ?N*
have (*K* ∈ *I* ∧ −*K* ∉ *I*) ∨ (−*K* ∈ *I* ∧ *K* ∉ *I*)
using *cons tot* **unfolding** *consistent-interp-def* **by** (*cases K*) *auto*
have *tot′*: *total-over-set I*
(*atm-of* ‘ *lit-of* ‘ (*set ?M* ∩ {*L*. *is-marked L* ∧ *L* ≠ *Marked K* ()}))
using *tot* **by** (*auto simp add*: *atms-of-uminus-lit-atm-of-lit-of*)
{ fix *x* :: (′*v*, *unit*, *unit*) *marked-lit*
assume
a3: *lit-of x* ∉ *I* **and**
a1: *x* ∈ *set ?M* **and**
a4: *is-marked x* **and**
a5: *x* ≠ *Marked K* ()
then have *Pos* (*atm-of* (*lit-of x*)) ∈ *I* ∨ *Neg* (*atm-of* (*lit-of x*)) ∈ *I*
using *a5 a4 tot′ a1* **unfolding** *total-over-set-def atms-of-s-def* **by** *blast*
moreover have *f6*: *Neg* (*atm-of* (*lit-of x*)) = − *Pos* (*atm-of* (*lit-of x*))
by *simp*
ultimately have − *lit-of x* ∈ *I*
using *f6 a3* **by** (*metis* (*no-types*) *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set*
literal.sel(*1*))
} note *H* = *this*

have ¬*I* ⊨*s ?C′*
using ‹*?N* ∪ *?C′* ⊨*ps* {{#}}› *tot cons* ‹*I* ⊨*s ?N*›
unfolding *true-clss-clss-def total-over-m-def*
by (*simp add*: *atms-of-uminus-lit-atm-of-lit-of atms-of-ms-single-image-atm-of-lit-of*)
then show *I* ⊨ *image-mset uminus ?C* + {#− *K*#}
unfolding *true-clss-def true-cls-def Bex-mset-def*
using ‹(*K* ∈ *I* ∧ −*K* ∉ *I*) ∨ (−*K* ∈ *I* ∧ *K* ∉ *I*)›
by (*auto dest*!: *H*)
qed
moreover have *F* ⊨*as CNot* (*image-mset uminus ?C*)
using *nm* **unfolding** *true-annots-def CNot-def M-K* **by** (*auto simp add*: *lits-of-def*)
ultimately have *False*
using *bj-can-jump*[*of S F′ K F C* −*K*
image-mset uminus (*image-mset lit-of* {# *L* :# *mset ?M*. *is-marked L* ∧ *L* ≠ *Marked K* ()#})]
‹*C*∈*?N*› *n-s* ‹*?M* ⊨*as CNot C*› *bj-backjump inv* ‹*no-dup* (*trail S*)› **unfolding** *M-K* **by** *auto*
then show *?thesis* **by** *fast*
qed *auto*
qed

end

locale *dpll-with-backjumping* =
dpll-with-backjumping-ops trail clauses prepend-trail tl-trail $add\text{-}cls_{NOT}$ $remove\text{-}cls_{NOT}$
propagate-conds inv backjump-conds
for
trail :: ′*st* ⇒ (′*v*, *unit*, *unit*) *marked-lits* **and**
clauses :: ′*st* ⇒ ′*v clauses* **and**
prepend-trail :: (′*v*, *unit*, *unit*) *marked-lit* ⇒ ′*st* ⇒ ′*st* **and** *tl-trail* :: ′*st* ⇒ ′*st* **and**
$add\text{-}cls_{NOT}$ $remove\text{-}cls_{NOT}$:: ′*v clause* ⇒ ′*st* ⇒ ′*st* **and**
propagate-conds :: (′*v*, *unit*, *unit*) *marked-lit* ⇒ ′*st* ⇒ *bool* **and**
inv :: ′*st* ⇒ *bool* **and**
backjump-conds :: ′*v clause* ⇒ ′*v clause* ⇒ ′*v literal* ⇒ ′*st* ⇒ ′*st* ⇒ *bool*

+
assumes *dpll-bj-inv*:$\bigwedge S\ T.$ *dpll-bj* $S\ T \Longrightarrow$ *inv* $S \Longrightarrow$ *inv* T
begin

lemma *rtranclp-dpll-bj-inv*:
assumes *dpll-bj*** $S\ T$ **and** *inv* S
shows *inv* T
using *assms* **by** (*induction rule*: *rtranclp-induct*)
(*auto simp add*: *dpll-bj-no-dup intro*: *dpll-bj-inv*)

lemma *rtranclp-dpll-bj-no-dup*:
assumes *dpll-bj*** $S\ T$ **and** *inv* S
and *no-dup* (*trail* S)
shows *no-dup* (*trail* T)
using *assms* **by** (*induction rule*: *rtranclp-induct*)
(*auto simp add*: *dpll-bj-no-dup dest*: *rtranclp-dpll-bj-inv dpll-bj-inv*)

lemma *rtranclp-dpll-bj-atms-of-ms-clauses-inv*:
assumes
dpll-bj** $S\ T$ **and** *inv* S
shows *atms-of-msu* (*clauses* S) = *atms-of-msu* (*clauses* T)
using *assms* **by** (*induction rule*: *rtranclp-induct*)
(*auto dest*: *rtranclp-dpll-bj-inv dpll-bj-atms-of-ms-clauses-inv*)

lemma *rtranclp-dpll-bj-atms-in-trail*:
assumes
dpll-bj** $S\ T$ **and**
inv S **and**
atm-of ‘ (*lits-of* (*trail* S)) $\subseteq$ *atms-of-msu* (*clauses* S)
shows *atm-of* ‘ (*lits-of* (*trail* T)) $\subseteq$ *atms-of-msu* (*clauses* T)
using *assms* **apply** (*induction rule*: *rtranclp-induct*)
using *dpll-bj-atms-in-trail dpll-bj-atms-of-ms-clauses-inv rtranclp-dpll-bj-inv* **by** *auto*

lemma *rtranclp-dpll-bj-sat-iff*:
assumes *dpll-bj*** $S\ T$ **and** *inv* S
shows $I \models sm$ *clauses* $S \longleftrightarrow I \models sm$ *clauses* T
using *assms* **by** (*induction rule*: *rtranclp-induct*)
(*auto dest*!: *dpll-bj-sat-iff simp*: *rtranclp-dpll-bj-inv*)

lemma *rtranclp-dpll-bj-atms-in-trail-in-set*:
assumes
dpll-bj** $S\ T$ **and**
inv S
atms-of-msu (*clauses* S) $\subseteq A$ **and**
atm-of ‘ (*lits-of* (*trail* S)) $\subseteq A$
shows *atm-of* ‘ (*lits-of* (*trail* T)) $\subseteq A$
using *assms*
by (*induction rule*: *rtranclp-induct*)
(*auto dest*: *rtranclp-dpll-bj-inv*
simp add: *dpll-bj-atms-in-trail-in-set rtranclp-dpll-bj-atms-of-ms-clauses-inv*
rtranclp-dpll-bj-inv)

lemma *rtranclp-dpll-bj-all-decomposition-implies-inv*:
assumes
dpll-bj** $S\ T$ **and**

inv S
all-decomposition-implies-m (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
shows *all-decomposition-implies-m* (*clauses T*) (*get-all-marked-decomposition* (*trail T*))
using *assms* **by** (*induction rule*: *rtranclp-induct*)
(*auto intro*: *dpll-bj-all-decomposition-implies-inv simp*: *rtranclp-dpll-bj-inv*)

lemma *rtranclp-dpll-bj-inv-incl-dpll-bj-inv-trancl*:
$\{(T, S).\ dpll\text{-}bj^{++}\ S\ T$
$\wedge$ *atms-of-msu* (*clauses S*) $\subseteq$ *atms-of-ms A* $\wedge$ *atm-of* ' *lits-of* (*trail S*) $\subseteq$ *atms-of-ms A*
$\wedge$ *no-dup* (*trail S*) $\wedge$ *inv S*$\}$
$\subseteq \{(T, S).$ *dpll-bj S T* $\wedge$ *atms-of-msu* (*clauses S*) $\subseteq$ *atms-of-ms A*
$\wedge$ *atm-of* ' *lits-of* (*trail S*) $\subseteq$ *atms-of-ms A* $\wedge$ *no-dup* (*trail S*) $\wedge$ *inv S*$\}^+$
(**is** $?A \subseteq ?B^+$)
proof *standard*
fix *x*
assume *x-A*: $x \in ?A$
obtain *S T*::*'st* **where**
x[*simp*]: $x = (T, S)$ **by** (*cases x*) *auto*
have
$dpll\text{-}bj^{++}$ *S T* **and**
atms-of-msu (*clauses S*) $\subseteq$ *atms-of-ms A* **and**
atm-of ' *lits-of* (*trail S*) $\subseteq$ *atms-of-ms A* **and**
no-dup (*trail S*) **and**
inv S
using *x-A* **by** *auto*
then show $x \in ?B^+$ **unfolding** *x*
proof (*induction rule*: *tranclp-induct*)
case *base*
then show *?case* **by** *auto*
next
case (*step T U*) **note** *step* = *this*(*1*) **and** *ST* = *this*(*2*) **and** *IH* = *this*(*3*)[*OF this*(*4*−*7*)]
and *N-A* = *this*(*4*) **and** *M-A* = *this*(*5*) **and** *nd* = *this*(*6*) **and** *inv* = *this*(*7*)

have [*simp*]: *atms-of-msu* (*clauses S*) = *atms-of-msu* (*clauses T*)
using *step rtranclp-dpll-bj-atms-of-ms-clauses-inv tranclp-into-rtranclp inv* **by** *fastforce*
have *no-dup* (*trail T*)
using *local.step nd rtranclp-dpll-bj-no-dup tranclp-into-rtranclp inv* **by** *fastforce*
moreover have *atm-of* ' (*lits-of* (*trail T*)) $\subseteq$ *atms-of-ms A*
by (*metis inv M-A N-A local.step rtranclp-dpll-bj-atms-in-trail-in-set*
tranclp-into-rtranclp)
moreover have *inv T*
using *inv local.step rtranclp-dpll-bj-inv tranclp-into-rtranclp* **by** *fastforce*
ultimately have $(U, T) \in ?B$ **using** *ST N-A M-A inv* **by** *auto*
then show *?case* **using** *IH* **by** (*rule trancl-into-trancl2*)
qed
qed

lemma *wf-tranclp-dpll-bj*:
assumes *fin*: *finite A*
shows *wf* $\{(T, S).\ dpll\text{-}bj^{++}\ S\ T$
$\wedge$ *atms-of-msu* (*clauses S*) $\subseteq$ *atms-of-ms A* $\wedge$ *atm-of* ' *lits-of* (*trail S*) $\subseteq$ *atms-of-ms A*
$\wedge$ *no-dup* (*trail S*) $\wedge$ *inv S*$\}$
using *wf-trancl*[*OF wf-dpll-bj*[*OF fin*]] *rtranclp-dpll-bj-inv-incl-dpll-bj-inv-trancl*
by (*rule wf-subset*)

lemma *dpll-bj-sat-ext-iff*:
  dpll-bj S $T \Longrightarrow$ *inv* $S \Longrightarrow I{\models}sextm$ *clauses* $S \longleftrightarrow I{\models}sextm$ *clauses* T
  by (*simp add*: *dpll-bj-clauses*)

lemma *rtranclp-dpll-bj-sat-ext-iff*:
  dpll-bj** S $T \Longrightarrow$ *inv* $S \Longrightarrow I{\models}sextm$ *clauses* $S \longleftrightarrow I{\models}sextm$ *clauses* T
  by (*induction rule*: *rtranclp-induct*) (*simp-all add*: *rtranclp-dpll-bj-inv dpll-bj-sat-ext-iff*)

theorem *full-dpll-backjump-final-state*:
  fixes A :: *'v literal multiset set* **and** S T :: *'st*
  assumes
    full: *full dpll-bj* S T **and**
    atms-S: *atms-of-msu* (*clauses* S) $\subseteq$ *atms-of-ms* A **and**
    atms-trail: *atm-of* ‘ *lits-of* (*trail* S) $\subseteq$ *atms-of-ms* A **and**
    n-d: *no-dup* (*trail* S) **and**
    finite A **and**
    inv: *inv* S **and**
    decomp: *all-decomposition-implies-m* (*clauses* S) (*get-all-marked-decomposition* (*trail* S))
  shows *unsatisfiable* (*set-mset* (*clauses* S))
  $\vee$ (*trail* T $\models$*asm clauses* S $\wedge$ *satisfiable* (*set-mset* (*clauses* S)))
proof –
  have *st*: *dpll-bj*** S T **and** *no-step dpll-bj* T
    using *full* **unfolding** *full-def* **by** *fast+*
  moreover have *atms-of-msu* (*clauses* T) $\subseteq$ *atms-of-ms* A
    using *atms-S inv rtranclp-dpll-bj-atms-of-ms-clauses-inv st* **by** *blast*
  moreover have *atm-of* ‘ *lits-of* (*trail* T) $\subseteq$ *atms-of-ms* A
     using *atms-S atms-trail inv rtranclp-dpll-bj-atms-in-trail-in-set st* **by** *auto*
  moreover have *no-dup* (*trail* T)
    using *n-d inv rtranclp-dpll-bj-no-dup st* **by** *blast*
  moreover have *inv*: *inv* T
    using *inv rtranclp-dpll-bj-inv st* **by** *blast*
  moreover
    have *decomp*: *all-decomposition-implies-m* (*clauses* T) (*get-all-marked-decomposition* (*trail* T))
      using ‹*inv* S› *decomp rtranclp-dpll-bj-all-decomposition-implies-inv st* **by** *blast*
  ultimately have *unsatisfiable* (*set-mset* (*clauses* T))
    $\vee$ (*trail* T $\models$*asm clauses* T $\wedge$ *satisfiable* (*set-mset* (*clauses* T)))
    using ‹*finite* A› *dpll-backjump-final-state* **by** *force*
  then show *?thesis*
    by (*meson* ‹*inv* S› *rtranclp-dpll-bj-sat-iff satisfiable-carac st true-annots-true-cls*)
qed

corollary *full-dpll-backjump-final-state-from-init-state*:
  fixes A :: *'v literal multiset set* **and** S T :: *'st*
  assumes
    full: *full dpll-bj* S T **and**
    trail S = [] **and**
    clauses S = N **and**
    inv S
  shows *unsatisfiable* (*set-mset* N) $\vee$ (*trail* T $\models$*asm* N $\wedge$ *satisfiable* (*set-mset* N))
  using *assms full-dpll-backjump-final-state*[*of* S T *set-mset* N] **by** *auto*

lemma *tranclp-dpll-bj-trail-mes-decreasing-prop*:
  assumes *dpll*: *dpll-bj*$^{++}$ S T **and** *inv*: *inv* S **and**
  N-A: *atms-of-msu* (*clauses* S) $\subseteq$ *atms-of-ms* A **and**
  M-A: *atm-of* ‘ *lits-of* (*trail* S) $\subseteq$ *atms-of-ms* A **and**

n-d: *no-dup* (*trail S*) **and**
fin-A: *finite A*
shows (*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*))
− μ_C (*1*+*card* (*atms-of-ms A*)) (*2*+*card* (*atms-of-ms A*)) (*trail-weight T*)
< (*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*))
− μ_C (*1*+*card* (*atms-of-ms A*)) (*2*+*card* (*atms-of-ms A*)) (*trail-weight S*)
using *dpll*
proof (*induction*)
case *base*
then show *?case*
using *N-A M-A n-d dpll-bj-trail-mes-decreasing-prop fin-A inv* **by** *blast*
next
case (*step T U*) **note** *st* = *this*(*1*) **and** *dpll* = *this*(*2*) **and** *IH* = *this*(*3*)
have *atms-of-msu* (*clauses S*) = *atms-of-msu* (*clauses T*)
using *rtranclp-dpll-bj-atms-of-ms-clauses-inv* **by** (*metis dpll-bj-clauses dpll-bj-inv inv st tranclpD*)
then have *N-A′*: *atms-of-msu* (*clauses T*) ⊆ *atms-of-ms A*
using *N-A* **by** *auto*
moreover have *M-A′*: *atm-of* ‘ *lits-of* (*trail T*) ⊆ *atms-of-ms A*
by (*meson M-A N-A inv rtranclp-dpll-bj-atms-in-trail-in-set st dpll tranclp.r-into-trancl tranclp-into-rtranclp tranclp-trans*)
moreover have *nd*: *no-dup* (*trail T*)
by (*metis inv n-d rtranclp-dpll-bj-no-dup st tranclp-into-rtranclp*)
moreover have *inv T*
by (*meson dpll dpll-bj-inv inv rtranclp-dpll-bj-inv st tranclp-into-rtranclp*)
ultimately show *?case*
using *IH dpll-bj-trail-mes-decreasing-prop*[*of T U A*] *dpll fin-A* **by** *linarith*
qed

end

14.4 CDCL

14.4.1 Learn and Forget

locale *learn-ops* =
dpll-state trail clauses prepend-trail tl-trail add-cls$_{NOT}$ *remove-cls*$_{NOT}$
for
trail :: *′st* ⇒ (*′v*, *unit*, *unit*) *marked-lits* **and**
clauses :: *′st* ⇒ *′v clauses* **and**
prepend-trail :: (*′v*, *unit*, *unit*) *marked-lit* ⇒ *′st* ⇒ *′st* **and** *tl-trail* :: *′st* ⇒ *′st* **and**
add-cls$_{NOT}$ *remove-cls*$_{NOT}$:: *′v clause* ⇒ *′st* ⇒ *′st* +
fixes
learn-cond :: *′v clause* ⇒ *′st* ⇒ *bool*

begin
inductive *learn* :: *′st* ⇒ *′st* ⇒ *bool* **where**
clauses S ⊨*pm C* ⟹ *atms-of C* ⊆ *atms-of-msu* (*clauses S*) ∪ *atm-of* ‘ (*lits-of* (*trail S*))
⟹ *learn-cond C S*
⟹ *T* ∼ *add-cls*$_{NOT}$ *C S*
⟹ *learn S T*
inductive-cases *learnE*: *learn S T*

lemma *learn-*μ_C*-stable*:
assumes *learn S T* **and** *no-dup* (*trail S*)
shows μ_C *A B* (*trail-weight S*) = μ_C *A B* (*trail-weight T*)

using *assms* **by** (*auto elim*: *learnE*)
end

locale *forget-ops* =
dpll-state trail clauses prepend-trail tl-trail $add\text{-}cls_{NOT}$ $remove\text{-}cls_{NOT}$
for
trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
clauses :: *'st* ⇒ *'v clauses* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and** *tl-trail* :: *'st* ⇒ *'st* **and**
$add\text{-}cls_{NOT}$ $remove\text{-}cls_{NOT}$:: *'v clause* ⇒ *'st* ⇒ *'st* +
fixes
forget-cond :: *'v clause* ⇒ *'st* ⇒ *bool*
begin
inductive $forget_{NOT}$ :: *'st* ⇒ *'st* ⇒ *bool* **where**
$forget_{NOT}$:*clauses S* − *replicate-mset* (*count* (*clauses S*) *C*) *C* ⊨*pm C*
⟹ *forget-cond C S*
⟹ *C* ∈# *clauses S*
⟹ *T* ∼ $remove\text{-}cls_{NOT}$ *C S*
⟹ $forget_{NOT}$ *S T*
inductive-cases *forgetE*: $forget_{NOT}$ *S T*

lemma *forget-*μ_C*-stable*:
assumes $forget_{NOT}$ *S T*
shows μ_C *A B* (*trail-weight S*) = μ_C *A B* (*trail-weight T*)
using *assms* **by** (*auto elim*!: *forgetE*)
end

locale *learn-and-forget*$_{NOT}$ =
learn-ops trail clauses prepend-trail tl-trail $add\text{-}cls_{NOT}$ $remove\text{-}cls_{NOT}$ *learn-cond* +
forget-ops trail clauses prepend-trail tl-trail $add\text{-}cls_{NOT}$ $remove\text{-}cls_{NOT}$ *forget-cond*
for
trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
clauses :: *'st* ⇒ *'v clauses* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
$add\text{-}cls_{NOT}$ $remove\text{-}cls_{NOT}$:: *'v clause* ⇒ *'st* ⇒ *'st* **and**
learn-cond forget-cond :: *'v clause* ⇒ *'st* ⇒ *bool*
begin
inductive *learn-and-forget*$_{NOT}$ :: *'st* ⇒ *'st* ⇒ *bool*
where
lf-learn: *learn S T* ⟹ *learn-and-forget*$_{NOT}$ *S T* |
lf-forget: $forget_{NOT}$ *S T* ⟹ *learn-and-forget*$_{NOT}$ *S T*
end

14.4.2 Definition of CDCL

locale *conflict-driven-clause-learning-ops* =
dpll-with-backjumping-ops trail clauses prepend-trail tl-trail $add\text{-}cls_{NOT}$ $remove\text{-}cls_{NOT}$
propagate-conds inv backjump-conds +
learn-and-forget$_{NOT}$ *trail clauses prepend-trail tl-trail* $add\text{-}cls_{NOT}$ $remove\text{-}cls_{NOT}$ *learn-cond*
forget-cond
for
trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
clauses :: *'st* ⇒ *'v clauses* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**

add-cls$_{NOT}$ *remove-cls*$_{NOT}$:: *'v clause* ⇒ *'st* ⇒ *'st* **and**
propagate-conds :: (*'v*, *unit*, *unit*) *marked-lit* ⇒*'st* ⇒ *bool* **and**
inv :: *'st* ⇒ *bool* **and**
backjump-conds :: *'v clause* ⇒ *'v clause* ⇒ *'v literal* ⇒ *'st* ⇒ *'st* ⇒ *bool* **and**
learn-cond forget-cond :: *'v clause* ⇒ *'st* ⇒ *bool*
begin

inductive *cdcl*$_{NOT}$ :: *'st* ⇒ *'st* ⇒ *bool* **for** *S* :: *'st* **where**
c-dpll-bj: *dpll-bj S S'* ⟹ *cdcl*$_{NOT}$ *S S'* |
c-learn: *learn S S'* ⟹ *cdcl*$_{NOT}$ *S S'* |
c-forget$_{NOT}$: *forget*$_{NOT}$ *S S'* ⟹ *cdcl*$_{NOT}$ *S S'*

lemma *cdcl*$_{NOT}$*-all-induct*[*consumes 1*, *case-names dpll-bj learn forget*$_{NOT}$]:
fixes *S T* :: *'st*
assumes *cdcl*$_{NOT}$ *S T* **and**
dpll: ⋀*T*. *dpll-bj S T* ⟹ *P S T* **and**
learning:
⋀*C T*. *clauses S* ⊨*pm C* ⟹
atms-of C ⊆ *atms-of-msu* (*clauses S*) ∪ *atm-of* ‘ (*lits-of* (*trail S*)) ⟹
T ∼ *add-cls*$_{NOT}$ *C S* ⟹
P S T **and**
forgetting: ⋀*C T*. *clauses S* − *replicate-mset* (*count* (*clauses S*) *C*) *C* ⊨*pm C* ⟹
C ∈# *clauses S* ⟹
T ∼ *remove-cls*$_{NOT}$ *C S* ⟹
P S T
shows *P S T*
using *assms*(*1*) **by** (*induction rule*: *cdcl*$_{NOT}$*.induct*)
(*auto intro*: *assms*(*2*, *3*, *4*) *elim*!: *learnE forgetE*)+

lemma *cdcl*$_{NOT}$*-no-dup*:
assumes
cdcl$_{NOT}$ *S T* **and**
inv S **and**
no-dup (*trail S*)
shows *no-dup* (*trail T*)
using *assms* **by** (*induction rule*: *cdcl*$_{NOT}$*-all-induct*) (*auto intro*: *dpll-bj-no-dup*)

Consistency of the trail **lemma** *cdcl*$_{NOT}$*-consistent*:
assumes
cdcl$_{NOT}$ *S T* **and**
inv S **and**
no-dup (*trail S*)
shows *consistent-interp* (*lits-of* (*trail T*))
using *cdcl*$_{NOT}$*-no-dup*[*OF assms*] *distinctconsistent-interp* **by** *fast*

The subtle problem here is that tautologies can be removed, meaning that some variable can disappear of the problem. It is also possible that some variable of the trail are not in the clauses anymore.

lemma *cdcl*$_{NOT}$*-atms-of-ms-clauses-decreasing*:
assumes *cdcl*$_{NOT}$ *S T***and** *inv S* **and** *no-dup* (*trail S*)
shows *atms-of-msu* (*clauses T*) ⊆ *atms-of-msu* (*clauses S*) ∪ *atm-of* ‘ (*lits-of* (*trail S*))
using *assms* **by** (*induction rule*: *cdcl*$_{NOT}$*-all-induct*)
(*auto dest*!: *dpll-bj-atms-of-ms-clauses-inv set-mp simp add*: *atms-of-ms-def Union-eq*)

lemma *cdcl*$_{NOT}$*-atms-in-trail*:

assumes *cdcl*$_{NOT}$ *S T***and** *inv S* **and** *no-dup* (*trail S*)
and *atm-of* ‘ (*lits-of* (*trail S*)) ⊆ *atms-of-msu* (*clauses S*)
shows *atm-of* ‘ (*lits-of* (*trail T*)) ⊆ *atms-of-msu* (*clauses S*)
using *assms* **by** (*induction rule*: *cdcl*$_{NOT}$*-all-induct*) (*auto simp add*: *dpll-bj-atms-in-trail*)

lemma *cdcl*$_{NOT}$*-atms-in-trail-in-set*:
assumes
cdcl$_{NOT}$ *S T* **and** *inv S* **and** *no-dup* (*trail S*) **and**
atms-of-msu (*clauses S*) ⊆ *A* **and**
atm-of ‘ (*lits-of* (*trail S*)) ⊆ *A*
shows *atm-of* ‘ (*lits-of* (*trail T*)) ⊆ *A*
using *assms*
by (*induction rule*: *cdcl*$_{NOT}$*-all-induct*)
(*simp-all add*: *dpll-bj-atms-in-trail-in-set dpll-bj-atms-of-ms-clauses-inv*)

lemma *cdcl*$_{NOT}$*-all-decomposition-implies*:
assumes *cdcl*$_{NOT}$ *S T* **and** *inv S* **and** *n-d*[*simp*]: *no-dup* (*trail S*) **and**
all-decomposition-implies-m (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
shows
all-decomposition-implies-m (*clauses T*) (*get-all-marked-decomposition* (*trail T*))
using *assms*(*1*,*2*,*4*)
proof (*induction rule*: *cdcl*$_{NOT}$*-all-induct*)
case *dpll-bj*
then show *?case*
using *dpll-bj-all-decomposition-implies-inv n-d* **by** *blast*
next
case *learn*
then show *?case* **by** (*auto simp add*: *all-decomposition-implies-def*)
next
case (*forget*$_{NOT}$ *C T*) **note** *cls-C* = *this*(*1*) **and** *C* = *this*(*2*) **and** *T* = *this*(*3*) **and** *iniv* = *this*(*4*)
and
decomp = *this*(*5*)
show *?case*
unfolding *all-decomposition-implies-def Ball-def*
proof (*intro allI*, *clarify*)
fix *a b*
assume (*a*, *b*) ∈ *set* (*get-all-marked-decomposition* (*trail T*))
then have (*λa*. {#*lit-of a*#}) ‘ *set a* ∪ *set-mset* (*clauses S*) ⊨*ps* (*λa*. {#*lit-of a*#}) ‘ *set b*
using *decomp T* **by** (*auto simp add*: *all-decomposition-implies-def*)
moreover
have *C* ∈ *set-mset* (*clauses S*)
by (*simp add*: *C*)
then have *set-mset* (*clauses T*) ⊨*ps set-mset* (*clauses S*)
by (*metis* (*no-types*) *T clauses-remove-cls*$_{NOT}$ *cls-C insert-Diff order-refl*
set-mset-minus-replicate-mset(*1*) *state-eq*$_{NOT}$*-clauses true-clss-clss-def*
true-clss-clss-insert)
ultimately show (*λa*. {#*lit-of a*#}) ‘ *set a* ∪ *set-mset* (*clauses T*)
⊨*ps* (*λa*. {#*lit-of a*#}) ‘ *set b*
using *true-clss-clss-generalise-true-clss-clss* **by** *blast*
qed
qed

Extension of models **lemma** *cdcl*$_{NOT}$*-bj-sat-ext-iff*:
assumes *cdcl*$_{NOT}$ *S T***and** *inv S* **and** *n-d*: *no-dup* (*trail S*)
shows *I*⊨*sextm clauses S* ⟷ *I*⊨*sextm clauses T*

```
  using assms
proof (induction rule:cdcl_NOT-all-induct)
  case dpll-bj
  then show ?case by (simp add: dpll-bj-clauses)
next
  case (learn C T) note T = this(3)
  { fix J
    assume
      I ⊨sextm clauses S and
      I ⊆ J and
      tot: total-over-m J (set-mset ({#C#} + (clauses S))) and
      cons: consistent-interp J
    then have J ⊨sm clauses S unfolding true-clss-ext-def by auto

    moreover
      with ‹clauses S⊨pm C› have J ⊨ C
        using tot cons unfolding true-clss-cls-def by auto
    ultimately have J ⊨sm {#C#} + clauses S by auto
  }
  then have H: I ⊨sextm (clauses S) ⟹ I ⊨sext insert C (set-mset (clauses S))
    unfolding true-clss-ext-def by auto
  show ?case
    apply standard
      using T n-d apply (auto simp add: H)[]
    using T n-d apply simp
    by (metis Diff-insert-absorb insert-subset subsetI subset-antisym
      true-clss-ext-decrease-right-remove-r)
next
  case (forget_NOT C T) note cls-C = this(1) and T = this(3)
  { fix J
    assume
      I ⊨sext set-mset (clauses S) − {C} and
      I ⊆ J and
      tot: total-over-m J (set-mset (clauses S)) and
      cons: consistent-interp J
    then have J ⊨s set-mset (clauses S) − {C}
      unfolding true-clss-ext-def by (meson Diff-subset total-over-m-subset)

    moreover
      with cls-C have J ⊨ C
        using tot cons unfolding true-clss-cls-def
        by (metis Un-commute forget_NOT.hyps(2) insert-Diff insert-is-Un mem-set-mset-iff order-refl
          set-mset-minus-replicate-mset(1))
    ultimately have J ⊨sm (clauses S) by (metis insert-Diff-single true-clss-insert)
  }
  then have H: I ⊨sext set-mset (clauses S) − {C} ⟹ I ⊨sextm (clauses S)
    unfolding true-clss-ext-def by blast
  show ?case using T by (auto simp: true-clss-ext-decrease-right-remove-r H)
qed

end — end of conflict-driven-clause-learning-ops
```

14.5 CDCL with invariant

```
locale conflict-driven-clause-learning =
  conflict-driven-clause-learning-ops +
```

assumes *cdcl*$_{NOT}$*-inv*: $\bigwedge S\ T.\ cdcl_{NOT}\ S\ T \Longrightarrow inv\ S \Longrightarrow inv\ T$
begin
sublocale *dpll-with-backjumping*
  apply *unfold-locales*
  using *cdcl*$_{NOT}$*.simps cdcl*$_{NOT}$*-inv* **by** *auto*

lemma *rtranclp-cdcl*$_{NOT}$*-inv*:
  $cdcl_{NOT}{}^{**}\ S\ T \Longrightarrow inv\ S \Longrightarrow inv\ T$
  by (*induction rule*: *rtranclp-induct*) (*auto simp add*: *cdcl*$_{NOT}$*-inv*)

lemma *rtranclp-cdcl*$_{NOT}$*-no-dup*:
  assumes $cdcl_{NOT}{}^{**}\ S\ T$ **and** *inv S*
  and *no-dup* (*trail S*)
  shows *no-dup* (*trail T*)
  using *assms* **by** (*induction rule*: *rtranclp-induct*) (*auto intro*: *cdcl*$_{NOT}$*-no-dup rtranclp-cdcl*$_{NOT}$*-inv*)

lemma *rtranclp-cdcl*$_{NOT}$*-trail-clauses-bound*:
  assumes
    cdcl: $cdcl_{NOT}{}^{**}\ S\ T$ **and**
    inv: *inv S* **and**
    n-d: *no-dup* (*trail S*) **and**
    atms-clauses-S: *atms-of-msu* (*clauses S*) $\subseteq A$ **and**
    atms-trail-S: *atm-of* ‘(*lits-of* (*trail S*)) $\subseteq A$
  shows *atm-of* ‘(*lits-of* (*trail T*)) $\subseteq A \land$ *atms-of-msu* (*clauses T*) $\subseteq A$
  using *cdcl*
proof (*induction rule*: *rtranclp-induct*)
  case *base*
  then show *?case* **using** *atms-clauses-S atms-trail-S* **by** *simp*
next
  case (*step T U*) **note** *st = this*(*1*) **and** *cdcl*$_{NOT}$ = *this*(*2*) **and** *IH = this*(*3*)
  have *inv T* **using** *inv st rtranclp-cdcl*$_{NOT}$*-inv* **by** *blast*
  have *no-dup* (*trail T*)
    using *rtranclp-cdcl*$_{NOT}$*-no-dup*[*of S T*] *st cdcl*$_{NOT}$ *inv n-d* **by** *blast*
  then have *atms-of-msu* (*clauses U*) $\subseteq A$
    using *cdcl*$_{NOT}$*-atms-of-ms-clauses-decreasing*[*OF cdcl*$_{NOT}$] *IH n-d* ‹*inv T*› **by** *auto*
  moreover
    have *atm-of* ‘(*lits-of* (*trail U*)) $\subseteq A$
      using *cdcl*$_{NOT}$*-atms-in-trail-in-set*[*OF cdcl*$_{NOT}$, *of A*] ‹*no-dup* (*trail T*)›
      by (*meson atms-trail-S atms-clauses-S IH* ‹*inv T*› *cdcl*$_{NOT}$)
  ultimately show *?case* **by** *fast*
qed

lemma *rtranclp-cdcl*$_{NOT}$*-all-decomposition-implies*:
  assumes $cdcl_{NOT}{}^{**}\ S\ T$ **and** *inv S* **and** *no-dup* (*trail S*) **and**
    all-decomposition-implies-m (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
  shows
    all-decomposition-implies-m (*clauses T*) (*get-all-marked-decomposition* (*trail T*))
  using *assms* **by** (*induction*)
  (*auto intro*: *rtranclp-cdcl*$_{NOT}$*-inv cdcl*$_{NOT}$*-all-decomposition-implies rtranclp-cdcl*$_{NOT}$*-no-dup*)

lemma *rtranclp-cdcl*$_{NOT}$*-bj-sat-ext-iff*:
  assumes $cdcl_{NOT}{}^{**}\ S\ T$**and** *inv S* **and** *no-dup* (*trail S*)
  shows $I \models$*sextm clauses S* $\longleftrightarrow I \models$*sextm clauses T*
  using *assms* **apply** (*induction rule*: *rtranclp-induct*)
  using *cdcl*$_{NOT}$*-bj-sat-ext-iff* **by** (*auto intro*: *rtranclp-cdcl*$_{NOT}$*-inv rtranclp-cdcl*$_{NOT}$*-no-dup*)

definition $cdcl_{NOT}$*-NOT-all-inv* **where**
$cdcl_{NOT}$*-NOT-all-inv A S* $\longleftrightarrow$ (*finite A* $\wedge$ *inv S* $\wedge$ *atms-of-msu* (*clauses S*) $\subseteq$ *atms-of-ms A*
$\wedge$ *atm-of* ‘ *lits-of* (*trail S*) $\subseteq$ *atms-of-ms A* $\wedge$ *no-dup* (*trail S*))

lemma $cdcl_{NOT}$*-NOT-all-inv*:
assumes $cdcl_{NOT}{}^{**}$ *S T* **and** $cdcl_{NOT}$*-NOT-all-inv A S*
shows $cdcl_{NOT}$*-NOT-all-inv A T*
using *assms* **unfolding** $cdcl_{NOT}$*-NOT-all-inv-def*
by (*simp add*: *rtranclp-*$cdcl_{NOT}$*-inv rtranclp-*$cdcl_{NOT}$*-no-dup rtranclp-*$cdcl_{NOT}$*-trail-clauses-bound*)

abbreviation *learn-or-forget* **where**
learn-or-forget S T $\equiv$ (λ*S T*. *learn S T* $\vee$ $forget_{NOT}$ *S T*) *S T*

lemma *rtranclp-learn-or-forget-*$cdcl_{NOT}$:
learn-or-forget** *S T* $\Longrightarrow$ $cdcl_{NOT}{}^{**}$ *S T*
using *rtranclp-mono*[*of learn-or-forget* $cdcl_{NOT}$] $cdcl_{NOT}$*.c-learn* $cdcl_{NOT}$*.c-forget*$_{NOT}$ **by** *blast*

lemma *learn-or-forget-dpll-*μ_C:
assumes
l-f: *learn-or-forget*** *S T* **and**
dpll: *dpll-bj T U* **and**
inv: $cdcl_{NOT}$*-NOT-all-inv A S*
shows (*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*))
− μ_C (*1*+*card* (*atms-of-ms A*)) (*2*+*card* (*atms-of-ms A*)) (*trail-weight U*)
< (*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*))
− μ_C (*1*+*card* (*atms-of-ms A*)) (*2*+*card* (*atms-of-ms A*)) (*trail-weight S*)
(**is** *?μ U* < *?μ S*)
proof −
have *?μ S* = *?μ T*
using *l-f*
proof (*induction*)
case *base*
then show *?case* **by** *simp*
next
case (*step T U*)
moreover then have *no-dup* (*trail T*)
using *rtranclp-*$cdcl_{NOT}$*-no-dup*[*of S T*] $cdcl_{NOT}$*-NOT-all-inv-def inv*
rtranclp-learn-or-forget-$cdcl_{NOT}$ **by** *auto*
ultimately show *?case*
using *forget-*μ_C*-stable learn-*μ_C*-stable inv* **unfolding** $cdcl_{NOT}$*-NOT-all-inv-def* **by** *presburger*
qed
moreover have $cdcl_{NOT}$*-NOT-all-inv A T*
using *rtranclp-learn-or-forget-*$cdcl_{NOT}$ $cdcl_{NOT}$*-NOT-all-inv l-f inv* **by** *blast*
ultimately show *?thesis*
using *dpll-bj-trail-mes-decreasing-prop*[*of T U A*, *OF dpll*] *finite*
unfolding $cdcl_{NOT}$*-NOT-all-inv-def* **by** *linarith*
qed

lemma *infinite-*$cdcl_{NOT}$*-exists-learn-and-forget-infinite-chain*:
assumes
$\bigwedge$*i*. $cdcl_{NOT}$ (*f i*) (*f*(*Suc i*)) **and**
inv: $cdcl_{NOT}$*-NOT-all-inv A* (*f 0*)
shows ∃*j*. ∀*i*≥*j*. *learn-or-forget* (*f i*) (*f* (*Suc i*))

```
  using assms
proof (induction (2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A))
    − μ_C (1+card (atms-of-ms A)) (2+card (atms-of-ms A)) (trail-weight (f 0))
    arbitrary: f
    rule: nat-less-induct-case)
  case (Suc n) note IH = this(1) and μ = this(2) and cdcl_NOT = this(3) and inv = this(4)
  consider
      (dpll-end) ∃ j. ∀ i≥j. learn-or-forget (f i) (f (Suc i))
    | (dpll-more) ¬(∃ j. ∀ i≥j. learn-or-forget (f i) (f (Suc i)))
    by blast
  then show ?case
    proof cases
      case dpll-end
      then show ?thesis by auto
    next
      case dpll-more
      then have j: ∃ i. ¬ learn (f i) (f (Suc i)) ∧ ¬forget_NOT (f i) (f (Suc i))
        by blast
      obtain i where
        ¬learn (f i) (f (Suc i)) ∧ ¬forget_NOT (f i) (f (Suc i)) and
        ∀ k<i. learn-or-forget (f k) (f (Suc k))
        proof −
          obtain i_0 where ¬ learn (f i_0) (f (Suc i_0)) ∧ ¬forget_NOT (f i_0) (f (Suc i_0))
            using j by auto
          then have {i. i≤i_0 ∧  ¬ learn (f i) (f (Suc i)) ∧ ¬forget_NOT (f i) (f (Suc i))} ≠ {}
            by auto
          let ?I = {i. i≤i_0 ∧  ¬ learn (f i) (f (Suc i)) ∧ ¬forget_NOT (f i) (f (Suc i))}
          let ?i = Min ?I
          have finite ?I
            by auto
          have ¬ learn (f ?i) (f (Suc ?i)) ∧ ¬forget_NOT (f ?i) (f (Suc ?i))
            using Min-in[OF ‹finite ?I› ‹?I ≠ {}›] by auto
          moreover have ∀ k<?i. learn-or-forget (f k) (f (Suc k))
            using Min.coboundedI[of {i. i ≤ i_0 ∧ ¬ learn (f i) (f (Suc i)) ∧ ¬ forget_NOT (f i)
              (f (Suc i))}, simplified]
            by (meson ‹¬ learn (f i_0) (f (Suc i_0)) ∧ ¬ forget_NOT (f i_0) (f (Suc i_0))› less-imp-le
              dual-order.trans not-le)
          ultimately show ?thesis using that by blast
        qed
      def g ≡ λn. f (n + Suc i)
      have dpll-bj (f i) (g 0)
        using ‹¬ learn (f i) (f (Suc i)) ∧ ¬ forget_NOT (f i) (f (Suc i))› cdcl_NOT cdcl_NOT.cases
        g-def by auto
      {
        fix j
        assume j ≤ i
        then have learn-or-forget** (f 0) (f j)
          apply (induction j)
           apply simp
          by (metis (no-types, lifting) Suc-leD Suc-le-lessD rtranclp.simps
            ‹∀ k<i. learn (f k) (f (Suc k)) ∨ forget_NOT (f k) (f (Suc k))›)
      }
      then have learn-or-forget** (f 0) (f i) by blast
      then have (2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A))
           − μ_C (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A)) (trail-weight (g 0))
```

< (*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*))
− μ_C (*1* + *card* (*atms-of-ms A*)) (*2* + *card* (*atms-of-ms A*)) (*trail-weight* (*f 0*))
using *learn-or-forget-dpll-*μ_C[*of f 0 f i g 0 A*] *inv* ‹*dpll-bj* (*f i*) (*g 0*)›
unfolding $cdcl_{NOT}$*-NOT-all-inv-def* **by** *linarith*

moreover have $cdcl_{NOT}$*-i*: ${cdcl_{NOT}}^{**}$ (*f 0*) (*g 0*)
using *rtranclp-learn-or-forget-*$cdcl_{NOT}$[*of f 0 f i*] ‹*learn-or-forget*** (*f 0*) (*f i*)›
$cdcl_{NOT}$[*of i*] **unfolding** *g-def* **by** *auto*
moreover have $\bigwedge$*i*. $cdcl_{NOT}$ (*g i*) (*g* (*Suc i*))
using $cdcl_{NOT}$ *g-def* **by** *auto*
moreover have $cdcl_{NOT}$*-NOT-all-inv A* (*g 0*)
using *inv* $cdcl_{NOT}$*-i rtranclp-*$cdcl_{NOT}$*-trail-clauses-bound g-def* $cdcl_{NOT}$*-NOT-all-inv* **by** *auto*
ultimately obtain *j* **where** *j*: $\bigwedge$*i*. *i*≥*j* ⟹ *learn-or-forget* (*g i*) (*g* (*Suc i*))
using *IH* **unfolding** *μ*[*symmetric*] **by** *presburger*
show *?thesis*
proof
{
fix *k*
assume *k* ≥ *j* + *Suc i*
then have *learn-or-forget* (*f k*) (*f* (*Suc k*))
using *j*[*of k*−*Suc i*] **unfolding** *g-def* **by** *auto*
}
then show ∀ *k*≥*j*+*Suc i*. *learn-or-forget* (*f k*) (*f* (*Suc k*))
by *auto*
qed
qed
next
case *0* **note** *H* = *this*(*1*) **and** $cdcl_{NOT}$ = *this*(*2*) **and** *inv* = *this*(*3*)
show *?case*
proof (*rule ccontr*)
assume ¬ *?case*
then have *j*: ∃ *i*. ¬ *learn* (*f i*) (*f* (*Suc i*)) ∧ ¬$forget_{NOT}$ (*f i*) (*f* (*Suc i*))
by *blast*
obtain *i* **where**
¬*learn* (*f i*) (*f* (*Suc i*)) ∧ ¬$forget_{NOT}$ (*f i*) (*f* (*Suc i*)) **and**
∀ *k*<*i*. *learn-or-forget* (*f k*) (*f* (*Suc k*))
proof −
obtain i_0 **where** ¬ *learn* (*f* i_0) (*f* (*Suc* i_0)) ∧ ¬$forget_{NOT}$ (*f* i_0) (*f* (*Suc* i_0))
using *j* **by** *auto*
then have {*i*. *i*≤i_0 ∧ ¬ *learn* (*f i*) (*f* (*Suc i*)) ∧ ¬$forget_{NOT}$ (*f i*) (*f* (*Suc i*))} ≠ {}
by *auto*
let *?I* = {*i*. *i*≤i_0 ∧ ¬ *learn* (*f i*) (*f* (*Suc i*)) ∧ ¬$forget_{NOT}$ (*f i*) (*f* (*Suc i*))}
let *?i* = *Min ?I*
have *finite ?I*
by *auto*
have ¬ *learn* (*f ?i*) (*f* (*Suc ?i*)) ∧ ¬$forget_{NOT}$ (*f ?i*) (*f* (*Suc ?i*))
using *Min-in*[*OF* ‹*finite ?I*› ‹*?I* ≠ {}›] **by** *auto*
moreover have ∀ *k*<*?i*. *learn-or-forget* (*f k*) (*f* (*Suc k*))
using *Min.coboundedI*[*of* {*i*. *i* ≤ i_0 ∧ ¬ *learn* (*f i*) (*f* (*Suc i*)) ∧ ¬ $forget_{NOT}$ (*f i*)
(*f* (*Suc i*))}, *simplified*]
by (*meson* ‹¬ *learn* (*f* i_0) (*f* (*Suc* i_0)) ∧ ¬ $forget_{NOT}$ (*f* i_0) (*f* (*Suc* i_0))› *less-imp-le*
dual-order.trans not-le)
ultimately show *?thesis* **using** *that* **by** *blast*
qed
have *dpll-bj* (*f i*) (*f* (*Suc i*))

using ‹¬ *learn* (*f i*) (*f* (*Suc i*)) ∧ ¬ *forget*$_{NOT}$ (*f i*) (*f* (*Suc i*))› *cdcl*$_{NOT}$ *cdcl*$_{NOT}$*.cases*
by *blast*
{
fix *j*
assume $j \le i$
then have *learn-or-forget*** (*f 0*) (*f j*)
apply (*induction j*)
apply *simp*
by (*metis* (*no-types*, *lifting*) *Suc-leD Suc-le-lessD rtranclp.simps*
‹∀ *k*<*i*. *learn* (*f k*) (*f* (*Suc k*)) ∨ *forget*$_{NOT}$ (*f k*) (*f* (*Suc k*))›)
}
then have *learn-or-forget*** (*f 0*) (*f i*) by *blast*

then show *False*
using *learn-or-forget-dpll-*μ_C[*of f 0 f i f* (*Suc i*) *A*] *inv 0*
‹*dpll-bj* (*f i*) (*f* (*Suc i*))› unfolding *cdcl*$_{NOT}$*-NOT-all-inv-def* by *linarith*
qed
qed

lemma *wf-cdcl*$_{NOT}$*-no-learn-and-forget-infinite-chain*:
assumes
no-infinite-lf: ⋀*f j*. ¬ (∀ *i*≥*j*. *learn-or-forget* (*f i*) (*f* (*Suc i*)))
shows *wf* {(*T*, *S*). *cdcl*$_{NOT}$ *S T* ∧ *cdcl*$_{NOT}$*-NOT-all-inv A S*} (**is** *wf* {(*T*, *S*). *cdcl*$_{NOT}$ *S T*
∧ *?inv S*})
unfolding *wf-iff-no-infinite-down-chain*
proof (*rule ccontr*)
assume ¬ ¬ (∃ *f*. ∀ *i*. (*f* (*Suc i*), *f i*) ∈ {(*T*, *S*). *cdcl*$_{NOT}$ *S T* ∧ *?inv S*})
then obtain *f* **where**
∀ *i*. *cdcl*$_{NOT}$ (*f i*) (*f* (*Suc i*)) ∧ *?inv* (*f i*)
by *fast*
then have ∃ *j*. ∀ *i*≥*j*. *learn-or-forget* (*f i*) (*f* (*Suc i*))
using *infinite-cdcl*$_{NOT}$*-exists-learn-and-forget-infinite-chain*[*of f*] **by** *meson*
then show *False* **using** *no-infinite-lf* **by** *blast*
qed

lemma *inv-and-tranclp-cdcl-*$_{NOT}$*-tranclp-cdcl*$_{NOT}$*-and-inv*:
cdcl$_{NOT}$$^{++}$ *S T* ∧ *cdcl*$_{NOT}$*-NOT-all-inv A S* ⟷ (λ*S T*. *cdcl*$_{NOT}$ *S T* ∧ *cdcl*$_{NOT}$*-NOT-all-inv A*
S)$^{++}$ *S T*
(**is** *?A* ∧ *?I* ⟷ *?B*)
proof
assume *?A* ∧ *?I*
then have *?A* **and** *?I* **by** *blast+*
then show *?B*
apply *induction*
apply (*simp add*: *tranclp.r-into-trancl*)
by (*metis* (*no-types*, *lifting*) *cdcl*$_{NOT}$*-NOT-all-inv tranclp.simps tranclp-into-rtranclp*)
next
assume *?B*
then have *?A* **by** *induction auto*
moreover have *?I* **using** ‹*?B*› *tranclpD* **by** *fastforce*
ultimately show *?A* ∧ *?I* **by** *blast*
qed

lemma *wf-tranclp-cdcl*$_{NOT}$*-no-learn-and-forget-infinite-chain*:
assumes

no-infinite-lf: $\bigwedge f\ j.\ \neg\ (\forall\ i{\geq}j.$ *learn-or-forget* $(f\ i)\ (f\ (Suc\ i)))$
shows *wf* $\{(T, S).\ cdcl_{NOT}^{++}\ S\ T \wedge cdcl_{NOT}$*-NOT-all-inv* $A\ S\}$
using *wf-trancl*[*OF wf-cdcl*$_{NOT}$*-no-learn-and-forget-infinite-chain*[*OF no-infinite-lf*]]
apply (*rule wf-subset*)
by (*auto simp*: *trancl-set-tranclp inv-and-tranclp-cdcl-*$_{NOT}$*-tranclp-cdcl*$_{NOT}$*-and-inv*)

lemma *cdcl*$_{NOT}$*-final-state*:
assumes
n-s: *no-step cdcl*$_{NOT}$ *S* **and**
inv: *cdcl*$_{NOT}$*-NOT-all-inv A S* **and**
decomp: *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
shows *unsatisfiable* (*set-mset* (*clauses S*))
$\vee$ (*trail S* $\models$*asm clauses S* $\wedge$ *satisfiable* (*set-mset* (*clauses S*)))
proof –
have *n-s'*: *no-step dpll-bj S*
using *n-s* **by** (*auto simp*: *cdcl*$_{NOT}$*.simps*)
show *?thesis*
apply (*rule dpll-backjump-final-state*[*of S A*])
using *inv decomp n-s'* **unfolding** *cdcl*$_{NOT}$*-NOT-all-inv-def* **by** *auto*
qed

lemma *full-cdcl*$_{NOT}$*-final-state*:
assumes
full: *full cdcl*$_{NOT}$ *S T* **and**
inv: *cdcl*$_{NOT}$*-NOT-all-inv A S* **and**
n-d: *no-dup* (*trail S*) **and**
decomp: *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
shows *unsatisfiable* (*set-mset* (*clauses T*))
$\vee$ (*trail T* $\models$*asm clauses T* $\wedge$ *satisfiable* (*set-mset* (*clauses T*)))
proof –
have *st*: $cdcl_{NOT}^{**}\ S\ T$ **and** *n-s*: *no-step cdcl*$_{NOT}$ *T*
using *full* **unfolding** *full-def* **by** *blast+*
have *n-s'*: *cdcl*$_{NOT}$*-NOT-all-inv A T*
using *cdcl*$_{NOT}$*-NOT-all-inv inv st* **by** *blast*
moreover have *all-decomposition-implies-m* (*clauses T*) (*get-all-marked-decomposition* (*trail T*))
using *cdcl*$_{NOT}$*-NOT-all-inv-def decomp inv rtranclp-cdcl*$_{NOT}$*-all-decomposition-implies st* **by** *auto*
ultimately show *?thesis*
using *cdcl*$_{NOT}$*-final-state n-s* **by** *blast*
qed

end — end of *conflict-driven-clause-learning*

14.6 Termination

14.6.1 Restricting learn and forget

locale *conflict-driven-clause-learning-learning-before-backjump-only-distinct-learnt* =
conflict-driven-clause-learning trail clauses prepend-trail tl-trail add-cls$_{NOT}$ *remove-cls*$_{NOT}$
propagate-conds inv backjump-conds
$\lambda C\ S.$ *distinct-mset C* $\wedge \neg$*tautology C* $\wedge$ *learn-restrictions C S* $\wedge$
$(\exists\ F\ K\ d\ F'\ C'\ L.$ *trail S* $= F'$ @ *Marked K* () # $F \wedge C = C' + \{\#L\#\} \wedge F \models$*as CNot* C'
$\wedge\ C' + \{\#L\#\} \notin\#$ *clauses S*)
$\lambda C\ S.\ \neg(\exists\ F'\ F\ K\ d\ L.$ *trail S* $= F'$ @ *Marked K* () # $F \wedge F \models$*as CNot* $(C - \{\#L\#\}))$
$\wedge$ *forget-restrictions C S*
for
trail :: *'st* $\Rightarrow$ (*'v::linorder*, *unit*, *unit*) *marked-lits* **and**

clauses :: *'st* ⇒ *'v clauses* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-cls$_{NOT}$ *remove-cls*$_{NOT}$:: *'v clause* ⇒ *'st* ⇒ *'st* **and**
propagate-conds :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *bool* **and**
inv :: *'st* ⇒ *bool* **and**
backjump-conds :: *'v clause* ⇒ *'v clause* ⇒ *'v literal* ⇒ *'st* ⇒ *'st* ⇒ *bool* **and**
learn-restrictions forget-restrictions :: *'v clause* ⇒ *'st* ⇒ *bool*
begin

lemma *cdcl*$_{NOT}$*-learn-all-induct*[*consumes 1*, *case-names dpll-bj learn forget*$_{NOT}$]:
fixes *S T* :: *'st*
assumes *cdcl*$_{NOT}$ *S T* **and**
dpll: ⋀*T*. *dpll-bj S T* ⟹ *P S T* **and**
learning:
⋀*C F K F' C' L T*. *clauses S* ⊨*pm C*
⟹ *atms-of C* ⊆ *atms-of-msu* (*clauses S*) ∪ *atm-of* ‘ (*lits-of* (*trail S*))
⟹ *distinct-mset C* ⟹ ¬ *tautology C* ⟹ *learn-restrictions C S*
⟹ *trail S* = *F'* @ *Marked K* () # *F* ⟹ *C* = *C'* + {#*L*#} ⟹ *F* ⊨*as CNot C'*
⟹ *C'* + {#*L*#} ∉# *clauses S* ⟹ *T* ∼ *add-cls*$_{NOT}$ *C S*
⟹ *P S T* **and**
forgetting: ⋀*C T*. *clauses S* − *replicate-mset* (*count* (*clauses S*) *C*) *C* ⊨*pm C*
⟹ *C* ∈# *clauses S*
⟹ ¬(∃*F' F K L*. *trail S* = *F'* @ *Marked K* () # *F* ∧ *F* ⊨*as CNot* (*C* − {#*L*#}))
⟹ *T* ∼ *remove-cls*$_{NOT}$ *C S*
⟹ *forget-restrictions C S* ⟹ *P S T*
shows *P S T*
using *assms*(*1*)
apply (*induction rule*: *cdcl*$_{NOT}$*.induct*)
apply (*auto dest*: *assms*(*2*) *simp add*: *learn-ops-axioms*)[]
apply (*auto elim*!: *learn-ops.learn.cases*[*OF learn-ops-axioms*] *dest*: *assms*(*3*))[]
apply (*auto elim*!: *forget-ops.forget*$_{NOT}$*.cases*[*OF forget-ops-axioms*] *dest*!: *assms*(*4*))
done

lemma *rtranclp-cdcl*$_{NOT}$*-inv*:
cdcl$_{NOT}$** *S T* ⟹ *inv S* ⟹ *inv T*
apply (*induction rule*: *rtranclp-induct*)
apply *simp*
using *cdcl*$_{NOT}$*-inv* **unfolding** *conflict-driven-clause-learning-def*
conflict-driven-clause-learning-axioms-def **by** *blast*

lemma *learn-always-simple-clauses*:
assumes
learn: *learn S T* **and**
n-d: *no-dup* (*trail S*)
shows *set-mset* (*clauses T* − *clauses S*)
⊆ *build-all-simple-clss* (*atms-of-msu* (*clauses S*) ∪ *atm-of* ‘ *lits-of* (*trail S*))
proof
fix *C* **assume** *C*: *C* ∈ *set-mset* (*clauses T* − *clauses S*)
have *distinct-mset C* ¬*tautology C* **using** *learn C n-d* **by** (*elim learnE*; *auto*)+
then have *C* ∈ *build-all-simple-clss* (*atms-of C*)
using *distinct-mset-not-tautology-implies-in-build-all-simple-clss* **by** *blast*
moreover have *atms-of C* ⊆ *atms-of-msu* (*clauses S*) ∪ *atm-of* ‘ *lits-of* (*trail S*)
using *learn C n-d* **by** (*elim learnE*) (*auto simp*: *atms-of-ms-def atms-of-def image-Un*
true-annots-CNot-all-atms-defined)

moreover have *finite* (*atms-of-msu* (*clauses* S) $\cup$ *atm-of* ‘ *lits-of* (*trail* S))
  by *auto*
ultimately show $C \in$ *build-all-simple-clss* (*atms-of-msu* (*clauses* S) $\cup$ *atm-of* ‘ *lits-of* (*trail* S))
  using *build-all-simple-clss-mono* **by** (*metis* (*no-types*) *insert-subset mk-disjoint-insert*)
qed

definition *conflicting-bj-clss* $S \equiv$
  $\{C+\{\#L\#\}|C\ L.\ C+\{\#L\#\} \in\#$ *clauses* $S \wedge$ *distinct-mset* $(C+\{\#L\#\}) \wedge \neg$*tautology* $(C+\{\#L\#\})$
  $\wedge$ ($\exists F'\ K\ F.$ *trail* $S = F'$ @ *Marked* K () # $F \wedge F \models$*as* *CNot* C)}

lemma *conflicting-bj-clss-remove-cls*$_{NOT}$[*simp*]:
  conflicting-bj-clss (*remove-cls*$_{NOT}$ C S) = *conflicting-bj-clss* $S - \{C\}$
  unfolding *conflicting-bj-clss-def* **by** *fastforce*

lemma *conflicting-bj-clss-add-cls*$_{NOT}$*-state-eq*:
  $T \sim$ *add-cls*$_{NOT}$ C' $S \Longrightarrow$ *no-dup* (*trail* S) $\Longrightarrow$ *conflicting-bj-clss* T
  = *conflicting-bj-clss* S
  $\cup$ (*if* $\exists C\ L.\ C' = C + \{\#L\#\} \wedge$ *distinct-mset* $(C+\{\#L\#\}) \wedge \neg$*tautology* $(C+\{\#L\#\})$
  $\wedge$ ($\exists F'\ K\ d\ F.$ *trail* $S = F'$ @ *Marked* K () # $F \wedge F \models$*as* *CNot* C)
  then $\{C'\}$ *else* {})
  unfolding *conflicting-bj-clss-def* **by** *auto metis+*

lemma *conflicting-bj-clss-add-cls*$_{NOT}$:
  no-dup (*trail* S) $\Longrightarrow$
  conflicting-bj-clss (*add-cls*$_{NOT}$ C' S)
  = *conflicting-bj-clss* S
  $\cup$ (*if* $\exists C\ L.\ C' = C + \{\#L\#\} \wedge$ *distinct-mset* $(C+\{\#L\#\}) \wedge \neg$*tautology* $(C+\{\#L\#\})$
  $\wedge$ ($\exists F'\ K\ d\ F.$ *trail* $S = F'$ @ *Marked* K () # $F \wedge F \models$*as* *CNot* C)
  then $\{C'\}$ *else* {})
  using *conflicting-bj-clss-add-cls*$_{NOT}$*-state-eq* **by** *auto*

lemma *conflicting-bj-clss-incl-clauses*:
  conflicting-bj-clss $S \subseteq$ *set-mset* (*clauses* S)
  unfolding *conflicting-bj-clss-def* **by** *auto*

lemma *finite-conflicting-bj-clss*[*simp*]:
  finite (*conflicting-bj-clss* S)
  using *conflicting-bj-clss-incl-clauses*[*of* S] *rev-finite-subset* **by** *blast*

lemma *learn-conflicting-increasing*:
  no-dup (*trail* S) $\Longrightarrow$ *learn* S T $\Longrightarrow$ *conflicting-bj-clss* $S \subseteq$ *conflicting-bj-clss* T
  apply (*elim learnE*)
  by (*subst conflicting-bj-clss-add-cls*$_{NOT}$*-state-eq*[*of* T]) *auto*

abbreviation *conflicting-bj-clss-yet* b $S \equiv$
  $3 \mathbin{\hat{}} b$ − *card* (*conflicting-bj-clss* S)

abbreviation μ_L :: *nat* $\Rightarrow$ *'st* $\Rightarrow$ *nat* $\times$ *nat* **where**
  μ_L b $S \equiv$ (*conflicting-bj-clss-yet* b S, *card* (*set-mset* (*clauses* S)))

lemma *do-not-forget-before-backtrack-rule-clause-learned-clause-untouched*:
  assumes *forget*$_{NOT}$ S T
  shows *conflicting-bj-clss* S = *conflicting-bj-clss* T
  using *assms* **apply** *induction*
  unfolding *conflicting-bj-clss-def*

by (*metis* (*no-types*, *lifting*) *Diff-insert-absorb Set.set-insert clauses-remove-cls*$_{NOT}$
diff-union-cancelR insert-iff mem-set-mset-iff order-refl set-mset-minus-replicate-mset(*1*)
state-eq$_{NOT}$*-clauses state-eq*$_{NOT}$*-trail trail-remove-cls*$_{NOT}$)

lemma *forget-*μ_L*-decrease*:
assumes *forget*$_{NOT}$: *forget*$_{NOT}$ *S T*
shows (μ_L *b T*, μ_L *b S*) $\in$ *less-than* <$*$*lex*$*$> *less-than*
proof −
have *card* (*set-mset* (*clauses T*)) $<$ *card* (*set-mset* (*clauses S*))
using *forget*$_{NOT}$ **apply** *induction*
by (*metis card-Diff1-less clauses-remove-cls*$_{NOT}$ *finite-set-mset mem-set-mset-iff order-refl*
set-mset-minus-replicate-mset(*1*) *state-eq*$_{NOT}$*-clauses*)
then show *?thesis*
unfolding *do-not-forget-before-backtrack-rule-clause-learned-clause-untouched*[*OF forget*$_{NOT}$]
by *auto*
qed

lemma *set-condition-or-split*:
$\{a.\ (a = b \vee Q\ a) \wedge S\ a\} = (\textit{if}\ S\ b\ \textit{then}\ \{b\}\ \textit{else}\ \{\}) \cup \{a.\ Q\ a \wedge S\ a\}$
by *auto*

lemma *set-insert-neq*:
$A \neq \textit{insert}\ a\ A \longleftrightarrow a \notin A$
by *auto*

lemma *learn-*μ_L*-decrease*:
assumes *learnST*: *learn S T* **and** *n-d*: *no-dup* (*trail S*) **and**
A: *atms-of-msu* (*clauses S*) $\cup$ *atm-of* ‘ *lits-of* (*trail S*) $\subseteq$ *A* **and**
fin-A: *finite A*
shows (μ_L (*card A*) *T*, μ_L (*card A*) *S*) $\in$ *less-than* <$*$*lex*$*$> *less-than*
proof −
have [*simp*]: (*atms-of-msu* (*clauses T*) $\cup$ *atm-of* ‘ *lits-of* (*trail T*))
= (*atms-of-msu* (*clauses S*) $\cup$ *atm-of* ‘ *lits-of* (*trail S*))
using *learnST n-d* **by** (*elim learnE*) *auto*

then have *card* (*atms-of-msu* (*clauses T*) $\cup$ *atm-of* ‘ *lits-of* (*trail T*))
= *card* (*atms-of-msu* (*clauses S*) $\cup$ *atm-of* ‘ *lits-of* (*trail S*))
by (*auto intro*!: *card-mono*)
then have *3*: (*3*::*nat*) $\hat{}$ *card* (*atms-of-msu* (*clauses T*) $\cup$ *atm-of* ‘ *lits-of* (*trail T*))
= *3* $\hat{}$ *card* (*atms-of-msu* (*clauses S*) $\cup$ *atm-of* ‘ *lits-of* (*trail S*))
by (*auto intro*: *power-mono*)
moreover have *conflicting-bj-clss S* $\subseteq$ *conflicting-bj-clss T*
using *learnST n-d* **by** (*simp add*: *learn-conflicting-increasing*)
moreover have *conflicting-bj-clss S* $\neq$ *conflicting-bj-clss T*
using *learnST*
proof (*elim learnE*, *goal-cases*)
case (*1 C*) **note** *clss-S* = *this*(*1*) **and** *atms-C* = *this*(*2*) **and** *inv* = *this*(*3*) **and** *T* = *this*(*4*)
then obtain *F K F' C' L* **where**
tr-S: *trail S* = *F'* @ *Marked K* () # *F* **and**
C: *C* = *C'* + {#*L*#} **and**
F: *F* $\models$*as CNot C'* **and**
C-S:*C'* + {#*L*#} $\notin$# *clauses S*
by *blast*
moreover have *distinct-mset C* $\neg$ *tautology C* **using** *inv* **by** *blast*+
ultimately have *C'* + {#*L*#} $\in$ *conflicting-bj-clss T*

using T *n-d* **unfolding** *conflicting-bj-clss-def* **by** *fastforce*
moreover have $C' + \{\#L\#\} \notin$ *conflicting-bj-clss* S
using *C-S* **unfolding** *conflicting-bj-clss-def* **by** *auto*
ultimately show *?case* **by** *blast*
qed
moreover have *fin-T*: *finite* (*conflicting-bj-clss* T)
using *learnST* **by** *induction* (*auto simp add*: *conflicting-bj-clss-add-cls*$_{NOT}$)
ultimately have *card* (*conflicting-bj-clss* T) $\geq$ *card* (*conflicting-bj-clss* S)
using *card-mono* **by** *blast*

moreover
have *fin'*: *finite* (*atms-of-msu* (*clauses* T) $\cup$ *atm-of* ‘ *lits-of* (*trail* T))
by *auto*
have *1*:*atms-of-ms* (*conflicting-bj-clss* T) $\subseteq$ *atms-of-msu* (*clauses* T)
unfolding *conflicting-bj-clss-def atms-of-ms-def* **by** *auto*
have *2*: $\bigwedge x.$ $x \in$ *conflicting-bj-clss* $T \Longrightarrow \neg$ *tautology* $x \wedge$ *distinct-mset* x
unfolding *conflicting-bj-clss-def* **by** *auto*
have T: *conflicting-bj-clss* T
$\subseteq$ *build-all-simple-clss* (*atms-of-msu* (*clauses* T) $\cup$ *atm-of* ‘ *lits-of* (*trail* T))
by *standard* (*meson 1 2 fin'* ‹*finite* (*conflicting-bj-clss* T)› *build-all-simple-clss-mono distinct-mset-set-def simplified-in-build-all subsetCE sup.coboundedI1*)
moreover
then have #: *3* ^ *card* (*atms-of-msu* (*clauses* T) $\cup$ *atm-of* ‘ *lits-of* (*trail* T))
$\geq$ *card* (*conflicting-bj-clss* T)
by (*meson Nat.le-trans build-all-simple-clss-card build-all-simple-clss-finite card-mono fin'*)
have *atms-of-msu* (*clauses* T) $\cup$ *atm-of* ‘ *lits-of* (*trail* T) $\subseteq A$
using *learnE*[*OF learnST*] A **by** *simp*
then have *3* ^ (*card* A) $\geq$ *card* (*conflicting-bj-clss* T)
using # *fin-A* **by** (*meson build-all-simple-clss-card build-all-simple-clss-finite build-all-simple-clss-mono calculation*(*2*) *card-mono dual-order.trans*)
ultimately show *?thesis*
using *psubset-card-mono*[*OF fin-T*]
unfolding *less-than-iff lex-prod-def* **by** *clarify*
(*meson* ‹*conflicting-bj-clss* $S \neq$ *conflicting-bj-clss* T›
‹*conflicting-bj-clss* $S \subseteq$ *conflicting-bj-clss* T›
diff-less-mono2 le-less-trans not-le psubsetI)
qed

We have to assume the following:

- *inv* S: the invariant holds in the inital state.
- A is a (finite *finite* A) superset of the literals in the trail *atm-of* ‘ *lits-of* (*trail* S) $\subseteq$ *atms-of-ms* A and in the clauses *atms-of-msu* (*clauses* S) $\subseteq$ *atms-of-ms* A. This can the the set of all the literals in the starting set of clauses.
- *no-dup* (*trail* S): no duplicate in the trail. This is invariant along the path.

definition μ_{CDCL} **where**
μ_{CDCL} A $T \equiv$ ((*2*+*card* (*atms-of-ms* A)) ^ (*1*+*card* (*atms-of-ms* A))
$-$ μ_C (*1*+*card* (*atms-of-ms* A)) (*2*+*card* (*atms-of-ms* A)) (*trail-weight* T),
conflicting-bj-clss-yet (*card* (*atms-of-ms* A)) T, *card* (*set-mset* (*clauses* T)))
lemma *cdcl*$_{NOT}$*-decreasing-measure*:
assumes
cdcl$_{NOT}$ S T **and**

inv: *inv S* **and**
atm-clss: *atms-of-msu* (*clauses S*) $\subseteq$ *atms-of-ms A* **and**
atm-lits: *atm-of* ' *lits-of* (*trail S*) $\subseteq$ *atms-of-ms A* **and**
n-d: *no-dup* (*trail S*) **and**
fin-A: *finite A*
shows (μ_{CDCL} *A T*, μ_{CDCL} *A S*)
$\in$ *less-than* <*lex*> (*less-than* <*lex*> *less-than*)
using *assms*(*1*)
proof *induction*
case (*c-dpll-bj T*)
from *dpll-bj-trail-mes-decreasing-prop*[*OF this*(*1*) *inv atm-clss atm-lits n-d fin-A*]
show *?case* **unfolding** μ_{CDCL}*-def*
by (*meson in-lex-prod less-than-iff*)
next
case (*c-learn T*) **note** *learn* = *this*(*1*)
then have *S*: *trail S* = *trail T*
using *inv atm-clss atm-lits n-d fin-A*
by (*elim learnE*) *auto*
show *?case*
using *learn*-μ_L*-decrease*[*OF learn - *] *atm-clss atm-lits fin-A n-d* **unfolding** *S* μ_{CDCL}*-def* **by** *auto*
next
case (*c-forget*$_{NOT}$ *T*) **note** *forget*$_{NOT}$ = *this*(*1*)
have *trail S* = *trail T* **using** *forget*$_{NOT}$ **by** *induction auto*
then show *?case*
using *forget*-μ_L*-decrease*[*OF forget*$_{NOT}$] **unfolding** μ_{CDCL}*-def* **by** *auto*
qed

lemma *wf-cdcl*$_{NOT}$*-restricted-learning*:
assumes *finite A*
shows *wf* {(*T*, *S*).
(*atms-of-msu* (*clauses S*) $\subseteq$ *atms-of-ms A* $\wedge$ *atm-of* ' *lits-of* (*trail S*) $\subseteq$ *atms-of-ms A*
$\wedge$ *no-dup* (*trail S*)
$\wedge$ *inv S*)
$\wedge$ *cdcl*$_{NOT}$ *S T* }
by (*rule wf-wf-if-measure'*[*of less-than* <*lex*> (*less-than* <*lex*> *less-than*)])
(*auto intro*: *cdcl*$_{NOT}$*-decreasing-measure*[*OF - - - - - assms*])

definition $\mu_C{}'$ :: *'v literal multiset set* $\Rightarrow$ *'st* $\Rightarrow$ *nat* **where**
$\mu_C{}'$ *A T* $\equiv$ μ_C (*1*+*card* (*atms-of-ms A*)) (*2*+*card* (*atms-of-ms A*)) (*trail-weight T*)

definition $\mu_{CDCL}{}'$ :: *'v literal multiset set* $\Rightarrow$ *'st* $\Rightarrow$ *nat* **where**
$\mu_{CDCL}{}'$ *A T* $\equiv$
((*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*)) $-$ $\mu_C{}'$ *A T*) $*$ (*1*+ *3*^*card* (*atms-of-ms A*)) $*$ *2*
+ *conflicting-bj-clss-yet* (*card* (*atms-of-ms A*)) *T* $*$ *2*
+ *card* (*set-mset* (*clauses T*))

lemma *cdcl*$_{NOT}$*-decreasing-measure'*:
assumes
cdcl$_{NOT}$ *S T* **and**
inv: *inv S* **and**
atms-clss: *atms-of-msu* (*clauses S*) $\subseteq$ *atms-of-ms A* **and**
atms-trail: *atm-of* ' *lits-of* (*trail S*) $\subseteq$ *atms-of-ms A* **and**
n-d: *no-dup* (*trail S*) **and**
fin-A: *finite A*

shows $\mu_{CDCL}{}'$ *A T* $<$ $\mu_{CDCL}{}'$ *A S*
using *assms*(*1*)
proof (*induction rule*: $cdcl_{NOT}$*-learn-all-induct*)
case (*dpll-bj T*)
then have (*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*)) − $\mu_C{}'$ *A T*
< (*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*)) − $\mu_C{}'$ *A S*
using *dpll-bj-trail-mes-decreasing-prop fin-A inv n-d atms-clss atms-trail*
unfolding $\mu_C{}'$*-def* **by** *blast*
then have *XX*: ((*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*)) − $\mu_C{}'$ *A T*) + *1*
≤ (*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*)) − $\mu_C{}'$ *A S*
by *auto*
from *mult-le-mono1*[*OF this*, *of* (*1* + *3* ^ *card* (*atms-of-ms A*))]
have ((*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*)) − $\mu_C{}'$ *A T*) ∗
(*1* + *3* ^ *card* (*atms-of-ms A*)) + (*1* + *3* ^ *card* (*atms-of-ms A*))
≤ ((*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*)) − $\mu_C{}'$ *A S*)
∗ (*1* + *3* ^ *card* (*atms-of-ms A*))
unfolding *Nat.add-mult-distrib*
by *presburger*
moreover
have *cl-T-S*: *clauses T* = *clauses S*
using *dpll-bj.hyps inv dpll-bj-clauses* **by** *auto*
have *conflicting-bj-clss-yet* (*card* (*atms-of-ms A*)) *S* < *1*+ *3* ^ *card* (*atms-of-ms A*)
by *simp*
ultimately have ((*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*)) − $\mu_C{}'$ *A T*)
∗ (*1* + *3* ^ *card* (*atms-of-ms A*)) + *conflicting-bj-clss-yet* (*card* (*atms-of-ms A*)) *T*
< ((*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*)) − $\mu_C{}'$ *A S*) ∗(*1* + *3* ^ *card* (*atms-of-ms A*))
by *linarith*
then have ((*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*)) − $\mu_C{}'$ *A T*)
∗ (*1* + *3* ^ *card* (*atms-of-ms A*))
+ *conflicting-bj-clss-yet* (*card* (*atms-of-ms A*)) *T*
< ((*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*)) − $\mu_C{}'$ *A S*)
∗ (*1* + *3* ^ *card* (*atms-of-ms A*))
+ *conflicting-bj-clss-yet* (*card* (*atms-of-ms A*)) *S*
by *linarith*
then have ((*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*)) − $\mu_C{}'$ *A T*)
∗ (*1* + *3* ^ *card* (*atms-of-ms A*)) ∗ *2*
+ *conflicting-bj-clss-yet* (*card* (*atms-of-ms A*)) *T* ∗ *2*
< ((*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*)) − $\mu_C{}'$ *A S*)
∗ (*1* + *3* ^ *card* (*atms-of-ms A*)) ∗ *2*
+ *conflicting-bj-clss-yet* (*card* (*atms-of-ms A*)) *S* ∗ *2*
by *linarith*
then show *?case* **unfolding** $\mu_{CDCL}{}'$*-def cl-T-S* **by** *presburger*
next
case (*learn C F' K F C' L T*) **note** *clss-S-C* = *this*(*1*) **and** *atms-C* = *this*(*2*) **and** *dist* = *this*(*3*)
and *tauto* = *this*(*4*) **and** *learn-restr* = *this*(*5*) **and** *tr-S* = *this*(*6*) **and** *C'* = *this*(*7*) **and**
F-C = *this*(*8*) **and** *C-new* = *this*(*9*) **and** *T* =*this*(*10*)
have *insert C* (*conflicting-bj-clss S*) ⊆ *build-all-simple-clss* (*atms-of-ms A*)
proof −
have *C* ∈ *build-all-simple-clss* (*atms-of-ms A*)
by (*metis* (*no-types*, *hide-lams*) *Un-subset-iff atms-of-ms-finite build-all-simple-clss-mono contra-subsetD dist distinct-mset-not-tautology-implies-in-build-all-simple-clss dual-order.trans fin-A atms-C atms-clss atms-trail tauto*)
moreover have *conflicting-bj-clss S* ⊆ *build-all-simple-clss* (*atms-of-ms A*)
unfolding *conflicting-bj-clss-def*

proof
fix x :: $'v$ *literal multiset*
assume $x \in \{C + \{\#L\#\} \mid C\ L.\ C + \{\#L\#\} \in\#$ *clauses* S
$\wedge$ *distinct-mset* $(C + \{\#L\#\}) \wedge \neg$ *tautology* $(C + \{\#L\#\})$
$\wedge$ $(\exists F'\ K\ F.$ *trail* $S = F'$ @ *Marked* K () # $F \wedge F \models as$ *CNot* $C)\}$
then have $\exists\, m\ l.\ x = m + \{\#l\#\} \wedge m + \{\#l\#\} \in\#$ *clauses* S
$\wedge$ *distinct-mset* $(m + \{\#l\#\}) \wedge \neg$ *tautology* $(m + \{\#l\#\})$
$\wedge$ $(\exists\, ms\ l\ msa.$ *trail* $S = ms$ @ *Marked* l () # $msa \wedge msa \models as$ *CNot* $m)$
by *blast*
then show $x \in$ *build-all-simple-clss* (*atms-of-ms* A)
by (*meson atms-clss atms-of-atms-of-ms-mono atms-of-ms-finite build-all-simple-clss-mono distinct-mset-not-tautology-implies-in-build-all-simple-clss fin-A finite-subset mem-set-mset-iff set-rev-mp*)
qed
ultimately show *?thesis*
by *auto*
qed
then have *card* (*insert* C (*conflicting-bj-clss* S)) ≤ 3 ^ (*card* (*atms-of-ms* A))
by (*meson Nat.le-trans atms-of-ms-finite build-all-simple-clss-card build-all-simple-clss-finite card-mono fin-A*)
moreover have [*simp*]: *card* (*insert* C (*conflicting-bj-clss* S))
$=$ *Suc* (*card* ((*conflicting-bj-clss* S)))
by (*metis* (*no-types*) C' *C-new card-insert-if conflicting-bj-clss-incl-clauses contra-subsetD finite-conflicting-bj-clss mem-set-mset-iff*)
moreover have [*simp*]: *conflicting-bj-clss* ($add\text{-}cls_{NOT}$ C S) $=$ *conflicting-bj-clss* $S \cup \{C\}$
using *dist tauto F-C n-d* **by** (*subst* $conflicting\text{-}bj\text{-}clss\text{-}add\text{-}cls_{NOT}$)
(*force simp add*: *ac-simps* C' *tr-S*)+
ultimately have [*simp*]: *conflicting-bj-clss-yet* (*card* (*atms-of-ms* A)) S
$=$ *Suc* (*conflicting-bj-clss-yet* (*card* (*atms-of-ms* A)) ($add\text{-}cls_{NOT}$ C S))
by *simp*
have *1*: *clauses* $T =$ *clauses* ($add\text{-}cls_{NOT}$ C S) **using** T **by** *auto*
have *2*: *conflicting-bj-clss-yet* (*card* (*atms-of-ms* A)) T
$=$ *conflicting-bj-clss-yet* (*card* (*atms-of-ms* A)) ($add\text{-}cls_{NOT}$ C S)
using T **unfolding** *conflicting-bj-clss-def* **by** *auto*
have *3*: $\mu_C{}'$ A $T = \mu_C{}'$ A ($add\text{-}cls_{NOT}$ C S)
using T **unfolding** $\mu_C{}'$-*def* **by** *auto*
have ((*2* + *card* (*atms-of-ms* A)) ^ (*1* + *card* (*atms-of-ms* A)) $-$ $\mu_C{}'$ A ($add\text{-}cls_{NOT}$ C S))
$*$ (*1* + *3* ^ *card* (*atms-of-ms* A)) $*$ *2*
$=$ ((*2* + *card* (*atms-of-ms* A)) ^ (*1* + *card* (*atms-of-ms* A)) $-$ $\mu_C{}'$ A S)
$*$ (*1* + *3* ^ *card* (*atms-of-ms* A)) $*$ *2*
using *n-d* **unfolding** $\mu_C{}'$-*def* **by** *auto*
moreover
have *conflicting-bj-clss-yet* (*card* (*atms-of-ms* A)) ($add\text{-}cls_{NOT}$ C S)
$*$ *2*
$+$ *card* (*set-mset* (*clauses* ($add\text{-}cls_{NOT}$ C S)))
$<$ *conflicting-bj-clss-yet* (*card* (*atms-of-ms* A)) S $*$ *2*
$+$ *card* (*set-mset* (*clauses* S))
by (*simp add*: C' *C-new n-d*)
ultimately show *?case* **unfolding** $\mu_{CDCL}{}'$-*def 1 2 3* **by** *presburger*
next
case ($forget_{NOT}$ C T) **note** $T =$ *this*(*4*)
have [*simp*]: $\mu_C{}'$ A ($remove\text{-}cls_{NOT}$ C S) $=$ $\mu_C{}'$ A S
unfolding $\mu_C{}'$-*def* **by** *auto*
have $forget_{NOT}$ S T
apply (*rule* $forget_{NOT}$.*intros*) **using** $forget_{NOT}$ **by** *auto*

then have *conflicting-bj-clss T* = *conflicting-bj-clss S*
using *do-not-forget-before-backtrack-rule-clause-learned-clause-untouched* **by** *blast*
moreover have *card* (*set-mset* (*clauses T*)) < *card* (*set-mset* (*clauses S*))
by (*metis T card-Diff1-less* $clauses\text{-}remove\text{-}cls_{NOT}$ *finite-set-mset* $forget_{NOT}$*.hyps*(*2*)
mem-set-mset-iff order-refl set-mset-minus-replicate-mset(*1*) $state\text{-}eq_{NOT}$*-clauses*)
ultimately show *?case* **unfolding** $\mu_{CDCL}{}'$*-def*
by (*metis* (*no-types*) *T* ‹$\mu_C{}'$ *A* ($remove\text{-}cls_{NOT}$ *C S*) = $\mu_C{}'$ *A S*› *add-le-cancel-left*
$\mu_C{}'$*-def not-le* $state\text{-}eq_{NOT}$*-trail*)
qed

lemma $cdcl_{NOT}$*-clauses-bound*:
assumes
$cdcl_{NOT}$ *S T* **and**
inv S **and**
atms-of-msu (*clauses S*) ⊆ *A* **and**
atm-of ‘(*lits-of* (*trail S*)) ⊆ *A* **and**
n-d: *no-dup* (*trail S*) **and**
fin-A[*simp*]: *finite A*
shows *set-mset* (*clauses T*) ⊆ *set-mset* (*clauses S*) ∪ *build-all-simple-clss A*
using *assms*
proof (*induction rule*: $cdcl_{NOT}$*-learn-all-induct*)
case *dpll-bj*
then show *?case* **using** *dpll-bj-clauses* **by** *simp*
next
case $forget_{NOT}$
then show *?case* **using** $clauses\text{-}remove\text{-}cls_{NOT}$ **unfolding** $state\text{-}eq_{NOT}$*-def* **by** *auto*
next
case (*learn C F K d F′ C′ L*) **note** *atms-C* = *this*(*2*) **and** *dist* = *this*(*3*) **and** *tauto* = *this*(*4*) **and**
T = *this*(*10*) **and** *atms-clss-S* = *this*(*12*) **and** *atms-trail-S* = *this*(*13*)
have *atms-of C* ⊆ *A*
using *atms-C atms-clss-S atms-trail-S* **by** *auto*
then have *build-all-simple-clss* (*atms-of C*) ⊆ *build-all-simple-clss A*
by (*simp add*: *build-all-simple-clss-mono*)
then have *C* ∈ *build-all-simple-clss A*
using *finite dist tauto*
by (*auto dest*: *distinct-mset-not-tautology-implies-in-build-all-simple-clss*)
then show *?case* **using** *T n-d* **by** *auto*
qed

lemma *rtranclp-*$cdcl_{NOT}$*-clauses-bound*:
assumes
$cdcl_{NOT}{}^{**}$ *S T* **and**
inv S **and**
atms-of-msu (*clauses S*) ⊆ *A* **and**
atm-of ‘(*lits-of* (*trail S*)) ⊆ *A* **and**
n-d: *no-dup* (*trail S*) **and**
finite: *finite A*
shows *set-mset* (*clauses T*) ⊆ *set-mset* (*clauses S*) ∪ *build-all-simple-clss A*
using *assms*(*1*−*5*)
proof *induction*
case *base*
then show *?case* **by** *simp*
next
case (*step T U*) **note** *st* = *this*(*1*) **and** $cdcl_{NOT}$ = *this*(*2*) **and** *IH* = *this*(*3*)[*OF this*(*4*−*7*)] **and**
inv = *this*(*4*) **and** *atms-clss-S* = *this*(*5*) **and** *atms-trail-S* = *this*(*6*) **and** *finite-cls-S* = *this*(*7*)

have *inv T*
using *rtranclp-cdcl*$_{NOT}$*-inv st inv* **by** *blast*
moreover have *atms-of-msu* (*clauses T*) $\subseteq$ *A* **and** *atm-of* ' *lits-of* (*trail T*) $\subseteq$ *A*
using *rtranclp-cdcl*$_{NOT}$*-trail-clauses-bound*[*OF st*] *inv atms-clss-S atms-trail-S n-d* **by** *blast+*
moreover have *no-dup* (*trail T*)
using *rtranclp-cdcl*$_{NOT}$*-no-dup*[*OF st* ‹*inv S*› *n-d*] **by** *simp*
ultimately have *set-mset* (*clauses U*) $\subseteq$ *set-mset* (*clauses T*) $\cup$ *build-all-simple-clss A*
using *cdcl*$_{NOT}$ *finite n-d* **by** (*auto simp*: *cdcl*$_{NOT}$*-clauses-bound*)
then show *?case* **using** *IH* **by** *auto*
qed

lemma *rtranclp-cdcl*$_{NOT}$*-card-clauses-bound*:
assumes
cdcl$_{NOT}$** *S T* **and**
inv S **and**
atms-of-msu (*clauses S*) $\subseteq$ *A* **and**
atm-of '(*lits-of* (*trail S*)) $\subseteq$ *A* **and**
n-d: *no-dup* (*trail S*) **and**
finite: *finite A*
shows *card* (*set-mset* (*clauses T*)) $\leq$ *card* (*set-mset* (*clauses S*)) + *3* ^ (*card A*)
using *rtranclp-cdcl*$_{NOT}$*-clauses-bound*[*OF assms*] *finite* **by** (*meson Nat.le-trans*
build-all-simple-clss-card build-all-simple-clss-finite card-Un-le card-mono finite-UnI
finite-set-mset nat-add-left-cancel-le)

lemma *rtranclp-cdcl*$_{NOT}$*-card-clauses-bound'*:
assumes
cdcl$_{NOT}$** *S T* **and**
inv S **and**
atms-of-msu (*clauses S*) $\subseteq$ *A* **and**
atm-of '(*lits-of* (*trail S*)) $\subseteq$ *A* **and**
n-d: *no-dup* (*trail S*) **and**
finite: *finite A*
shows *card* {*C*|*C*. *C* ∈# *clauses T* ∧ (*tautology C* ∨ ¬*distinct-mset C*)}
≤ *card* {*C*|*C*. *C*∈# *clauses S* ∧ (*tautology C* ∨ ¬*distinct-mset C*)} + *3* ^ (*card A*)
(**is** *card ?T* ≤ *card ?S* + -)
using *rtranclp-cdcl*$_{NOT}$*-clauses-bound*[*OF assms*] *finite*
proof −
have *?T* ⊆ *?S* ∪ *build-all-simple-clss A*
using *rtranclp-cdcl*$_{NOT}$*-clauses-bound*[*OF assms*] **by** *force*
then have *card ?T* ≤ *card* (*?S* ∪ *build-all-simple-clss A*)
using *finite* **by** (*simp add*: *assms*(*5*) *build-all-simple-clss-finite card-mono*)
then show *?thesis*
by (*meson le-trans build-all-simple-clss-card card-Un-le local.finite nat-add-left-cancel-le*)
qed

lemma *rtranclp-cdcl*$_{NOT}$*-card-simple-clauses-bound*:
assumes
cdcl$_{NOT}$** *S T* **and**
inv S **and**
atms-of-msu (*clauses S*) $\subseteq$ *A* **and**
atm-of '(*lits-of* (*trail S*)) $\subseteq$ *A* **and**
n-d: *no-dup* (*trail S*) **and**
finite: *finite A*
shows *card* (*set-mset* (*clauses T*))

$\leq$ *card* $\{C.\ C \in\#$ *clauses* $S \wedge$ (*tautology* $C \vee \neg$*distinct-mset* C)$\}$ + *3* ^ (*card* A)
(**is** *card* *?T* $\leq$ *card* *?S* + -)
using *rtranclp-cdcl*$_{NOT}$*-clauses-bound*[*OF assms*] *finite*
proof −
have $\bigwedge x.\ x \in\#$ *clauses* $T \Longrightarrow\neg$ *tautology* $x \Longrightarrow$ *distinct-mset* $x \Longrightarrow x \in$ *build-all-simple-clss* A
using *rtranclp-cdcl*$_{NOT}$*-clauses-bound*[*OF assms*] **by** (*metis* (*no-types*, *hide-lams*) *Un-iff assms*(*3*)
atms-of-atms-of-ms-mono build-all-simple-clss-mono contra-subsetD
distinct-mset-not-tautology-implies-in-build-all-simple-clss local.finite mem-set-mset-iff
subset-trans)
then have *set-mset* (*clauses* T) $\subseteq$ *?S* $\cup$ *build-all-simple-clss* A
using *rtranclp-cdcl*$_{NOT}$*-clauses-bound*[*OF assms*] **by** *auto*
then have *card*(*set-mset* (*clauses* T)) $\leq$ *card* (*?S* $\cup$ *build-all-simple-clss* A)
using *finite* **by** (*simp add*: *assms*(*5*) *build-all-simple-clss-finite card-mono*)
then show *?thesis*
by (*meson le-trans build-all-simple-clss-card card-Un-le local.finite nat-add-left-cancel-le*)
qed

definition $\mu_{CDCL}{}'$*-bound* :: *'v literal multiset set* $\Rightarrow$ *'st* $\Rightarrow$ *nat* **where**
$\mu_{CDCL}{}'$*-bound* A S =
((*2* + *card* (*atms-of-ms* A)) ^ (*1* + *card* (*atms-of-ms* A))) $*$ (*1* + *3* ^ *card* (*atms-of-ms* A)) $*$ *2*
+ *2*$*$*3* ^ (*card* (*atms-of-ms* A))
+ *card* $\{C.\ C \in\#$ *clauses* $S \wedge$ (*tautology* $C \vee \neg$*distinct-mset* C)$\}$ + *3* ^ (*card* (*atms-of-ms* A))

lemma $\mu_{CDCL}{}'$*-bound-reduce-trail-to*$_{NOT}$[*simp*]:
$\mu_{CDCL}{}'$*-bound* A (*reduce-trail-to*$_{NOT}$ M S) = $\mu_{CDCL}{}'$*-bound* A S
unfolding $\mu_{CDCL}{}'$*-bound-def* **by** *auto*

lemma *rtranclp-cdcl*$_{NOT}$-$\mu_{CDCL}{}'$*-bound-reduce-trail-to*$_{NOT}$:
assumes
cdcl$_{NOT}{}^{**}$ S T **and**
inv S **and**
atms-of-msu (*clauses* S) $\subseteq$ *atms-of-ms* A **and**
atm-of ‘(*lits-of* (*trail* S)) $\subseteq$ *atms-of-ms* A **and**
n-d: *no-dup* (*trail* S) **and**
finite: *finite* (*atms-of-ms* A) **and**
U: $U \sim$ *reduce-trail-to*$_{NOT}$ M T
shows $\mu_{CDCL}{}'$ A $U \leq \mu_{CDCL}{}'$*-bound* A S
proof −
have ((*2* + *card* (*atms-of-ms* A)) ^ (*1* + *card* (*atms-of-ms* A)) − $\mu_C{}'$ A U)
$\leq$ (*2* + *card* (*atms-of-ms* A)) ^ (*1* + *card* (*atms-of-ms* A))
by *auto*
then have ((*2* + *card* (*atms-of-ms* A)) ^ (*1* + *card* (*atms-of-ms* A)) − $\mu_C{}'$ A U)
$*$ (*1* + *3* ^ *card* (*atms-of-ms* A)) $*$ *2*
$\leq$ (*2* + *card* (*atms-of-ms* A)) ^ (*1* + *card* (*atms-of-ms* A)) $*$ (*1* + *3* ^ *card* (*atms-of-ms* A)) $*$ *2*
using *mult-le-mono1* **by** *blast*
moreover
have *conflicting-bj-clss-yet* (*card* (*atms-of-ms* A)) T $*$ *2* $\leq$ *2* $*$ *3* ^ *card* (*atms-of-ms* A)
by *linarith*
moreover have *card* (*set-mset* (*clauses* U))
$\leq$ *card* $\{C.\ C \in\#$ *clauses* $S \wedge$ (*tautology* $C \vee \neg$*distinct-mset* C)$\}$ + *3* ^ *card* (*atms-of-ms* A)
using *rtranclp-cdcl*$_{NOT}$*-card-simple-clauses-bound*[*OF assms*(*1*−*6*)] U **by** *auto*
ultimately show *?thesis*
unfolding $\mu_{CDCL}{}'$*-def* $\mu_{CDCL}{}'$*-bound-def* **by** *linarith*
qed

lemma *rtranclp-cdcl*$_{NOT}$*-*μ_{CDCL}*'-bound*:
assumes
cdcl$_{NOT}$** S T **and**
inv S **and**
atms-of-msu (*clauses* S) $\subseteq$ *atms-of-ms* A **and**
atm-of ‘(*lits-of* (*trail* S)) $\subseteq$ *atms-of-ms* A **and**
n-d: *no-dup* (*trail* S) **and**
finite: *finite* (*atms-of-ms* A)
shows μ_{CDCL}*'* A T $\leq$ μ_{CDCL}*'-bound* A S
proof –
have μ_{CDCL}*'* A (*reduce-trail-to*$_{NOT}$ (*trail* T) T) = μ_{CDCL}*'* A T
unfolding μ_{CDCL}*'-def* μ_C*'-def conflicting-bj-clss-def* **by** *auto*
then show *?thesis* **using** *rtranclp-cdcl*$_{NOT}$*-*μ_{CDCL}*'-bound-reduce-trail-to*$_{NOT}$[*OF assms*, *of - trail* T]
state-eq$_{NOT}$*-ref* **by** *fastforce*
qed

lemma *rtranclp-*μ_{CDCL}*'-bound-decreasing*:
assumes
cdcl$_{NOT}$** S T **and**
inv S **and**
atms-of-msu (*clauses* S) $\subseteq$ *atms-of-ms* A **and**
atm-of ‘(*lits-of* (*trail* S)) $\subseteq$ *atms-of-ms* A **and**
n-d: *no-dup* (*trail* S) **and**
finite[*simp*]: *finite* (*atms-of-ms* A)
shows μ_{CDCL}*'-bound* A T $\leq$ μ_{CDCL}*'-bound* A S
proof –
have {C. C $\in\#$ *clauses* T $\wedge$ (*tautology* C $\vee$ $\neg$ *distinct-mset* C)}
$\subseteq$ {C. C $\in\#$ *clauses* S $\wedge$ (*tautology* C $\vee$ $\neg$ *distinct-mset* C)} (**is** *?T* $\subseteq$ *?S*)
proof (*rule Set.subsetI*)
fix C **assume** C $\in$ *?T*
then have *C-T*: C $\in\#$ *clauses* T **and** *t-d*: *tautology* C $\vee$ $\neg$ *distinct-mset* C
by *auto*
then have C $\notin$ *build-all-simple-clss* (*atms-of-ms* A)
by (*auto dest*: *build-all-simple-clssE*)
then show C $\in$ *?S*
using *C-T rtranclp-cdcl*$_{NOT}$*-clauses-bound*[*OF assms*] *t-d* **by** *force*
qed
then have *card* {C. C $\in\#$ *clauses* T $\wedge$ (*tautology* C $\vee$ $\neg$ *distinct-mset* C)} $\leq$
card {C. C $\in\#$ *clauses* S $\wedge$ (*tautology* C $\vee$ $\neg$ *distinct-mset* C)}
by (*simp add*: *card-mono*)
then show *?thesis*
unfolding μ_{CDCL}*'-bound-def* **by** *auto*
qed

end — end of *conflict-driven-clause-learning-learning-before-backjump-only-distinct-learnt*

14.7 CDCL with restarts

14.7.1 Definition

locale *restart-ops* =
fixes
cdcl$_{NOT}$ :: *'st* $\Rightarrow$ *'st* $\Rightarrow$ *bool* **and**
restart :: *'st* $\Rightarrow$ *'st* $\Rightarrow$ *bool*
begin
inductive *cdcl*$_{NOT}$*-raw-restart* :: *'st* $\Rightarrow$ *'st* $\Rightarrow$ *bool* **where**

$cdcl_{NOT}$ *S T* $\Longrightarrow$ $cdcl_{NOT}$*-raw-restart S T* |
restart S T $\Longrightarrow$ $cdcl_{NOT}$*-raw-restart S T*

end

locale *conflict-driven-clause-learning-with-restarts* =
conflict-driven-clause-learning trail clauses prepend-trail tl-trail add-cls$_{NOT}$ *remove-cls*$_{NOT}$
propagate-conds inv backjump-conds learn-cond forget-cond
for
trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
clauses :: *'st* ⇒ *'v clauses* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-cls$_{NOT}$ *remove-cls*$_{NOT}$:: *'v clause* ⇒ *'st* ⇒ *'st* **and**
propagate-conds :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *bool* **and**
inv :: *'st* ⇒ *bool* **and**
backjump-conds :: *'v clause* ⇒ *'v clause* ⇒ *'v literal* ⇒ *'st* ⇒ *'st* ⇒ *bool* **and**
learn-cond forget-cond :: *'v clause* ⇒ *'st* ⇒ *bool*
begin

lemma $cdcl_{NOT}$*-iff-*$cdcl_{NOT}$*-raw-restart-no-restarts*:
$cdcl_{NOT}$ *S T* ⟷ *restart-ops.*$cdcl_{NOT}$*-raw-restart* $cdcl_{NOT}$ (λ- -. *False*) *S T*
(**is** *?C S T* ⟷ *?R S T*)
proof
fix *S T*
assume *?C S T*
then show *?R S T* **by** (*simp add*: *restart-ops.*$cdcl_{NOT}$*-raw-restart.intros*(*1*))
next
fix *S T*
assume *?R S T*
then show *?C S T*
apply (*cases rule*: *restart-ops.*$cdcl_{NOT}$*-raw-restart.cases*)
using ‹*?R S T*› **by** *fast+*
qed

lemma $cdcl_{NOT}$*-*$cdcl_{NOT}$*-raw-restart*:
$cdcl_{NOT}$ *S T* $\Longrightarrow$ *restart-ops.*$cdcl_{NOT}$*-raw-restart* $cdcl_{NOT}$ *restart S T*
by (*simp add*: *restart-ops.*$cdcl_{NOT}$*-raw-restart.intros*(*1*))
end

14.7.2 Increasing restarts

To add restarts we needs some assumptions on the predicate (called $cdcl_{NOT}$ here):

- a function *f* that is strictly monotonic. The first step is actually only used as a restart to clean the state (e.g. to ensure that the trail is empty). Then we assume that $(1::'a) \leq f\ n$ for $(1::'a) \leq n$: it means that between two consecutive restarts, at least one step will be done. This is necessary to avoid sequence. like: full – restart – full – ...
- a measure μ: it should decrease under the assumptions *bound-inv*, whenever a $cdcl_{NOT}$ or a *restart* is done. A parameter is given to μ: for conflict- driven clause learning, it is an upper-bound of the clauses. We are assuming that such a bound can be found after a restart whenever the invariant holds.
- we also assume that the measure decrease after any $cdcl_{NOT}$ step.

- an invariant on the states $cdcl_{NOT}$*-inv* that also holds after restarts.
- it is *not required* that the measure decrease with respect to restarts, but the measure has to be bound by some function μ*-bound* taking the same parameter as μ and the initial state of the considered $cdcl_{NOT}$ chain.

locale $cdcl_{NOT}$*-increasing-restarts-ops* =
  restart-ops $cdcl_{NOT}$ *restart* **for**
    restart :: *'st* ⇒ *'st* ⇒ *bool* **and**
    $cdcl_{NOT}$ :: *'st* ⇒ *'st* ⇒ *bool* +
  fixes
    f :: *nat* ⇒ *nat* **and**
    bound-inv :: *'bound* ⇒ *'st* ⇒ *bool* **and**
    μ :: *'bound* ⇒ *'st* ⇒ *nat* **and**
    $cdcl_{NOT}$*-inv* :: *'st* ⇒ *bool* **and**
    μ*-bound* :: *'bound* ⇒ *'st* ⇒ *nat*
  assumes
    f: *unbounded f* **and**
    f-ge-1:$\bigwedge n.\ n \geq 1 \Longrightarrow f\ n \neq 0$ **and**
    bound-inv: $\bigwedge A\ S\ T.\ cdcl_{NOT}\text{-}inv\ S \Longrightarrow bound\text{-}inv\ A\ S \Longrightarrow cdcl_{NOT}\ S\ T \Longrightarrow bound\text{-}inv\ A\ T$ **and**
    $cdcl_{NOT}$*-measure*: $\bigwedge A\ S\ T.\ cdcl_{NOT}\text{-}inv\ S \Longrightarrow bound\text{-}inv\ A\ S \Longrightarrow cdcl_{NOT}\ S\ T \Longrightarrow \mu\ A\ T < \mu\ A\ S$ **and**
    measure-bound2: $\bigwedge A\ T\ U.\ cdcl_{NOT}\text{-}inv\ T \Longrightarrow bound\text{-}inv\ A\ T \Longrightarrow cdcl_{NOT}^{**}\ T\ U$
      $\Longrightarrow \mu\ A\ U \leq \mu\text{-}bound\ A\ T$ **and**
    measure-bound4: $\bigwedge A\ T\ U.\ cdcl_{NOT}\text{-}inv\ T \Longrightarrow bound\text{-}inv\ A\ T \Longrightarrow cdcl_{NOT}^{**}\ T\ U$
      $\Longrightarrow \mu\text{-}bound\ A\ U \leq \mu\text{-}bound\ A\ T$ **and**
    $cdcl_{NOT}$*-restart-inv*: $\bigwedge A\ U\ V.\ cdcl_{NOT}\text{-}inv\ U \Longrightarrow restart\ U\ V \Longrightarrow bound\text{-}inv\ A\ U \Longrightarrow bound\text{-}inv\ A\ V$
      and
    exists-bound: $\bigwedge R\ S.\ cdcl_{NOT}\text{-}inv\ R \Longrightarrow restart\ R\ S \Longrightarrow \exists A.\ bound\text{-}inv\ A\ S$ **and**
    $cdcl_{NOT}$*-inv*: $\bigwedge S\ T.\ cdcl_{NOT}\text{-}inv\ S \Longrightarrow cdcl_{NOT}\ S\ T \Longrightarrow cdcl_{NOT}\text{-}inv\ T$ **and**
    $cdcl_{NOT}$*-inv-restart*: $\bigwedge S\ T.\ cdcl_{NOT}\text{-}inv\ S \Longrightarrow restart\ S\ T \Longrightarrow cdcl_{NOT}\text{-}inv\ T$
begin

lemma $cdcl_{NOT}$-$cdcl_{NOT}$*-inv*:
  assumes
    $(cdcl_{NOT}\widehat{\ }\widehat{\ }n)\ S\ T$ **and**
    $cdcl_{NOT}$*-inv S*
  shows $cdcl_{NOT}$*-inv T*
  using *assms* **by** (*induction n arbitrary*: *T*) (*auto intro*:*bound-inv* $cdcl_{NOT}$*-inv*)

lemma $cdcl_{NOT}$*-bound-inv*:
  assumes
    $(cdcl_{NOT}\widehat{\ }\widehat{\ }n)\ S\ T$ **and**
    $cdcl_{NOT}$*-inv S*
    bound-inv A S
  shows *bound-inv A T*
  using *assms* **by** (*induction n arbitrary*: *T*) (*auto intro*:*bound-inv* $cdcl_{NOT}$-$cdcl_{NOT}$*-inv*)

lemma *rtranclp*-$cdcl_{NOT}$-$cdcl_{NOT}$*-inv*:
  assumes
    $cdcl_{NOT}^{**}\ S\ T$ **and**
    $cdcl_{NOT}$*-inv S*
  shows $cdcl_{NOT}$*-inv T*
  using *assms* **by** *induction* (*auto intro*: $cdcl_{NOT}$*-inv*)

lemma *rtranclp-cdcl*$_{NOT}$*-bound-inv*:
  assumes
    $cdcl_{NOT}{}^{**}$ S T **and**
    bound-inv A S **and**
    $cdcl_{NOT}$*-inv* S
  shows *bound-inv* A T
  using *assms* **by** *induction* (*auto intro*:*bound-inv rtranclp-cdcl*$_{NOT}$*-cdcl*$_{NOT}$*-inv*)

lemma *cdcl*$_{NOT}$*-comp-n-le*:
  assumes
    $(cdcl_{NOT}\hat{}\hat{}(Suc\ n))$ S T **and**
    bound-inv A S
    $cdcl_{NOT}$*-inv* S
  shows $\mu\ A\ T < \mu\ A\ S - n$
  using *assms*
proof (*induction n arbitrary*: T)
  case *0*
  then show *?case* **using** *cdcl*$_{NOT}$*-measure* **by** *auto*
next
  case (*Suc n*) **note** *IH* =*this*(*1*)[*OF* - *this*(*3*) *this*(*4*)] **and** *S-T* =*this*(*2*) **and** *b-inv* = *this*(*3*) **and**
  c-inv = *this*(*4*)
  obtain U :: *'st* **where** *S-U*: $(cdcl_{NOT}\hat{}\hat{}(Suc\ n))$ S U **and** *U-T*: $cdcl_{NOT}$ U T **using** *S-T* **by** *auto*
  then have $\mu\ A\ U < \mu\ A\ S - n$ **using** *IH*[*of U*] **by** *simp*
  moreover
    have *bound-inv* A U
      using *S-U b-inv cdcl*$_{NOT}$*-bound-inv c-inv* **by** *blast*
    then have $\mu\ A\ T < \mu\ A\ U$ **using** *cdcl*$_{NOT}$*-measure*[*OF* - - *U-T*] *S-U c-inv cdcl*$_{NOT}$*-cdcl*$_{NOT}$*-inv*
by *auto*
  ultimately show *?case* **by** *linarith*
qed

lemma *wf-cdcl*$_{NOT}$:
  wf $\{(T, S).\ cdcl_{NOT}\ S\ T \wedge cdcl_{NOT}\text{-}inv\ S \wedge bound\text{-}inv\ A\ S\}$ (**is** *wf ?A*)
  apply (*rule wfP-if-measure2*[*of* - - μ *A*])
  using *cdcl*$_{NOT}$*-comp-n-le*[*of 0* - - *A*] **by** *auto*

lemma *rtranclp-cdcl*$_{NOT}$*-measure*:
  assumes
    $cdcl_{NOT}{}^{**}$ S T **and**
    bound-inv A S **and**
    $cdcl_{NOT}$*-inv* S
  shows $\mu\ A\ T \leq \mu\ A\ S$
  using *assms*
proof (*induction rule*: *rtranclp-induct*)
  case *base*
  then show *?case* **by** *auto*
next
  case (*step T U*) **note** *IH* =*this*(*3*)[*OF this*(*4*) *this*(*5*)] **and** *st* =*this*(*1*) **and** $cdcl_{NOT}$= *this*(*2*) **and**
    b-inv = *this*(*4*) **and** *c-inv* = *this*(*5*)
  have *bound-inv* A T
    by (*meson cdcl*$_{NOT}$*-bound-inv rtranclp-imp-relpowp st step.prems*)
  moreover have $cdcl_{NOT}$*-inv* T
    using *c-inv rtranclp-cdcl*$_{NOT}$*-cdcl*$_{NOT}$*-inv st* **by** *blast*
  ultimately have $\mu\ A\ U < \mu\ A\ T$ **using** *cdcl*$_{NOT}$*-measure*[*OF* - - $cdcl_{NOT}$] **by** *auto*

then show *?case* **using** *IH* **by** *linarith*
qed

lemma $cdcl_{NOT}$*-comp-bounded*:
assumes
bound-inv A S **and** $cdcl_{NOT}$*-inv S* **and** $m \geq 1+\mu$ *A S*
shows $\neg(cdcl_{NOT} \,\widehat{}\widehat{}\, m)$ *S T*
using *assms* $cdcl_{NOT}$*-comp-n-le*[*of* $m-1$ *S T A*] **by** *fastforce*

- *f n* $<$ *m* ensures that at least one step has been done.

inductive $cdcl_{NOT}$*-restart* **where**
restart-step: $(cdcl_{NOT}\widehat{}\widehat{}m)$ *S T* $\Longrightarrow m \geq f\ n \Longrightarrow$ *restart T U*
$\Longrightarrow$ $cdcl_{NOT}$*-restart* (*S*, *n*) (*U*, *Suc n*) |
restart-full: *full1* $cdcl_{NOT}$ *S T* $\Longrightarrow$ $cdcl_{NOT}$*-restart* (*S*, *n*) (*T*, *Suc n*)

lemmas $cdcl_{NOT}$*-with-restart-induct* = $cdcl_{NOT}$*-restart.induct*[*split-format*(*complete*),
OF $cdcl_{NOT}$*-increasing-restarts-ops-axioms*]

lemma $cdcl_{NOT}$*-restart-*$cdcl_{NOT}$*-raw-restart*:
$cdcl_{NOT}$*-restart S T* $\Longrightarrow$ $cdcl_{NOT}$*-raw-restart*** (*fst S*) (*fst T*)
proof (*induction rule*: $cdcl_{NOT}$*-restart.induct*)
case (*restart-step m S T n U*)
then have $cdcl_{NOT}{}^{**}$ *S T* **by** (*meson relpowp-imp-rtranclp*)
then have $cdcl_{NOT}$*-raw-restart*** *S T* **using** $cdcl_{NOT}$*-raw-restart.intros*(*1*)
rtranclp-mono[*of* $cdcl_{NOT}$ $cdcl_{NOT}$*-raw-restart*] **by** *blast*
moreover have $cdcl_{NOT}$*-raw-restart T U*
using ‹*restart T U*› $cdcl_{NOT}$*-raw-restart.intros*(*2*) **by** *blast*
ultimately show *?case* **by** *auto*
next
case (*restart-full S T*)
then have $cdcl_{NOT}{}^{**}$ *S T* **unfolding** *full1-def* **by** *auto*
then show *?case* **using** $cdcl_{NOT}$*-raw-restart.intros*(*1*)
rtranclp-mono[*of* $cdcl_{NOT}$ $cdcl_{NOT}$*-raw-restart*] **by** *auto*
qed

lemma $cdcl_{NOT}$*-with-restart-bound-inv*:
assumes
$cdcl_{NOT}$*-restart S T* **and**
bound-inv A (*fst S*) **and**
$cdcl_{NOT}$*-inv* (*fst S*)
shows *bound-inv A* (*fst T*)
using *assms* **apply** (*induction rule*: $cdcl_{NOT}$*-restart.induct*)
prefer *2* **apply** (*metis rtranclp-unfold fstI full1-def rtranclp-*$cdcl_{NOT}$*-bound-inv*)
by (*metis* $cdcl_{NOT}$*-bound-inv* $cdcl_{NOT}$*-*$cdcl_{NOT}$*-inv* $cdcl_{NOT}$*-restart-inv fst-conv*)

lemma $cdcl_{NOT}$*-with-restart-*$cdcl_{NOT}$*-inv*:
assumes
$cdcl_{NOT}$*-restart S T* **and**
$cdcl_{NOT}$*-inv* (*fst S*)
shows $cdcl_{NOT}$*-inv* (*fst T*)
using *assms* **apply** *induction*
apply (*metis* $cdcl_{NOT}$*-*$cdcl_{NOT}$*-inv* $cdcl_{NOT}$*-inv-restart fst-conv*)
apply (*metis fstI full-def full-unfold rtranclp-*$cdcl_{NOT}$*-*$cdcl_{NOT}$*-inv*)
done

lemma *rtranclp-cdcl$_{NOT}$-with-restart-cdcl$_{NOT}$-inv*:
assumes
cdcl$_{NOT}$-restart** S T **and**
cdcl$_{NOT}$-inv (fst S)
shows *cdcl$_{NOT}$-inv (fst T)*
using *assms* **by** *induction (auto intro: cdcl$_{NOT}$-with-restart-cdcl$_{NOT}$-inv)*

lemma *rtranclp-cdcl$_{NOT}$-with-restart-bound-inv*:
assumes
cdcl$_{NOT}$-restart** S T **and**
cdcl$_{NOT}$-inv (fst S) **and**
bound-inv A (fst S)
shows *bound-inv A (fst T)*
using *assms* **apply** *induction*
apply *(simp add: cdcl$_{NOT}$-cdcl$_{NOT}$-inv cdcl$_{NOT}$-with-restart-bound-inv)*
using *cdcl$_{NOT}$-with-restart-bound-inv rtranclp-cdcl$_{NOT}$-with-restart-cdcl$_{NOT}$-inv* **by** *blast*

lemma *cdcl$_{NOT}$-with-restart-increasing-number*:
cdcl$_{NOT}$-restart S T $\Longrightarrow$ snd T = 1 + snd S
by *(induction rule: cdcl$_{NOT}$-restart.induct) auto*
end

locale *cdcl$_{NOT}$-increasing-restarts =*
cdcl$_{NOT}$-increasing-restarts-ops restart cdcl$_{NOT}$ f bound-inv μ cdcl$_{NOT}$-inv μ-bound
for
trail :: 'st $\Rightarrow$ ('v, unit, unit) marked-lits **and**
clauses :: 'st $\Rightarrow$ 'v clauses **and**
prepend-trail :: ('v, unit, unit) marked-lit $\Rightarrow$ 'st $\Rightarrow$ 'st **and**
tl-trail :: 'st $\Rightarrow$ 'st **and**
add-cls$_{NOT}$ remove-cls$_{NOT}$:: 'v clause $\Rightarrow$ 'st $\Rightarrow$ 'st **and**
f :: nat $\Rightarrow$ nat **and**
restart :: 'st $\Rightarrow$ 'st $\Rightarrow$ bool **and**
bound-inv :: 'bound $\Rightarrow$ 'st $\Rightarrow$ bool **and**
μ :: 'bound $\Rightarrow$ 'st $\Rightarrow$ nat **and**
cdcl$_{NOT}$ :: 'st $\Rightarrow$ 'st $\Rightarrow$ bool **and**
cdcl$_{NOT}$-inv :: 'st $\Rightarrow$ bool **and**
μ-bound :: 'bound $\Rightarrow$ 'st $\Rightarrow$ nat +
assumes
measure-bound: $\bigwedge$A T V n. cdcl$_{NOT}$-inv T $\Longrightarrow$ bound-inv A T
$\Longrightarrow$ cdcl$_{NOT}$-restart (T, n) (V, Suc n) $\Longrightarrow$ μ A V $\leq$ μ-bound A T **and**
cdcl$_{NOT}$-raw-restart-μ-bound:
cdcl$_{NOT}$-restart (T, a) (V, b) $\Longrightarrow$ cdcl$_{NOT}$-inv T $\Longrightarrow$ bound-inv A T
$\Longrightarrow$ μ-bound A V $\leq$ μ-bound A T
begin

lemma *rtranclp-cdcl$_{NOT}$-raw-restart-μ-bound*:
cdcl$_{NOT}$-restart** (T, a) (V, b) $\Longrightarrow$ cdcl$_{NOT}$-inv T $\Longrightarrow$ bound-inv A T
$\Longrightarrow$ μ-bound A V $\leq$ μ-bound A T
apply *(induction rule: rtranclp-induct2)*
apply *simp*
by *(metis cdcl$_{NOT}$-raw-restart-μ-bound dual-order.trans fst-conv*
rtranclp-cdcl$_{NOT}$-with-restart-bound-inv rtranclp-cdcl$_{NOT}$-with-restart-cdcl$_{NOT}$-inv)

lemma *cdcl$_{NOT}$-raw-restart-measure-bound*:

$cdcl_{NOT}$*-restart* (T, a) (V, b) $\Longrightarrow$ $cdcl_{NOT}$*-inv* T $\Longrightarrow$ *bound-inv* A T
$\Longrightarrow$ μ A V $\leq$ μ*-bound* A T
apply (*cases rule*: $cdcl_{NOT}$*-restart.cases*)
apply *simp*
using *measure-bound relpowp-imp-rtranclp* **apply** *fastforce*
by (*metis full-def full-unfold measure-bound2 prod.inject*)

lemma *rtranclp-*$cdcl_{NOT}$*-raw-restart-measure-bound*:
$cdcl_{NOT}$*-restart*** (T, a) (V, b) $\Longrightarrow$ $cdcl_{NOT}$*-inv* T $\Longrightarrow$ *bound-inv* A T
$\Longrightarrow$ μ A V $\leq$ μ*-bound* A T
apply (*induction rule*: *rtranclp-induct2*)
apply (*simp add*: *measure-bound2*)
by (*metis dual-order.trans fst-conv measure-bound2 r-into-rtranclp rtranclp.rtrancl-refl*
rtranclp-$cdcl_{NOT}$*-with-restart-bound-inv rtranclp-*$cdcl_{NOT}$*-with-restart-*$cdcl_{NOT}$*-inv*
rtranclp-$cdcl_{NOT}$*-raw-restart-*μ*-bound*)

lemma *wf-*$cdcl_{NOT}$*-restart*:
wf $\{(T, S).$ $cdcl_{NOT}$*-restart* S T $\wedge$ $cdcl_{NOT}$*-inv* (*fst* S)$\}$ (**is** *wf ?A*)
proof (*rule ccontr*)
assume $\neg$ *?thesis*
then obtain g **where**
g: $\bigwedge i.$ $cdcl_{NOT}$*-restart* $(g\ i)$ $(g\ (Suc\ i))$ **and**
$cdcl_{NOT}$*-inv-g*: $\bigwedge i.$ $cdcl_{NOT}$*-inv* (*fst* $(g\ i)$)
unfolding *wf-iff-no-infinite-down-chain* **by** *fast*

have *snd-g*: $\bigwedge i.$ *snd* $(g\ i) = i +$ *snd* $(g\ 0)$
apply (*induct-tac* i)
apply *simp*
by (*metis Suc-eq-plus1-left add.commute add.left-commute*
$cdcl_{NOT}$*-with-restart-increasing-number* g)
then have *snd-g-0*: $\bigwedge i.$ $i > 0 \Longrightarrow$ *snd* $(g\ i) = i +$ *snd* $(g\ 0)$
by *blast*
have *unbounded-f-g*: *unbounded* $(\lambda i.\ f$ (*snd* $(g\ i)$))
using f **unfolding** *bounded-def* **by** (*metis add.commute f less-or-eq-imp-le snd-g*
not-bounded-nat-exists-larger not-le ordered-cancel-comm-monoid-diff-class.le-iff-add)

{ fix i
have H: $\bigwedge T\ Ta\ m.$ $(cdcl_{NOT}$ ^^ $m)$ T Ta $\Longrightarrow$ *no-step* $cdcl_{NOT}$ T $\Longrightarrow$ $m = 0$
apply (*case-tac* m) **apply** *simp* **by** (*meson relpowp-E2*)
have $\exists$ T $m.$ $(cdcl_{NOT}$ ^^ $m)$ (*fst* $(g\ i)$) T $\wedge$ $m \geq f$ (*snd* $(g\ i)$)
using g[*of i*] **apply** (*cases rule*: $cdcl_{NOT}$*-restart.cases*)
apply *auto*[]
using g[*of Suc i*] *f-ge-1* **apply** (*cases rule*: $cdcl_{NOT}$*-restart.cases*)
apply (*auto simp add*: *full1-def full-def dest*: *H dest*: *tranclpD*)
using *H Suc-leI leD* **by** *blast*
} note $H = this$
obtain A **where** *bound-inv* A (*fst* $(g\ 1)$)
using g[*of 0*] $cdcl_{NOT}$*-inv-g*[*of 0*] **apply** (*cases rule*: $cdcl_{NOT}$*-restart.cases*)
apply (*metis One-nat-def* $cdcl_{NOT}$*-inv exists-bound fst-conv relpowp-imp-rtranclp*
rtranclp-induct)
using H[*of 1*] **unfolding** *full1-def* **by** (*metis One-nat-def Suc-eq-plus1 diff-is-0-eq' diff-zero*
f-ge-1 fst-conv le-add2 relpowp-E2 snd-conv)
let $?j = \mu$*-bound* A (*fst* $(g\ 1)$) $+ 1$
obtain j **where**
j: f (*snd* $(g\ j)$) $>$ $?j$ **and** $j > 1$

using *unbounded-f-g not-bounded-nat-exists-larger* **by** *blast*
{
fix *i j*
have $cdcl_{NOT}$*-with-restart*: $j \geq i \Longrightarrow cdcl_{NOT}\text{-}restart^{**}$ (*g i*) (*g j*)
apply (*induction j*)
apply *simp*
by (*metis g le-Suc-eq rtranclp.rtrancl-into-rtrancl rtranclp.rtrancl-refl*)
} **note** $cdcl_{NOT}$*-restart* = *this*
have $cdcl_{NOT}$*-inv* (*fst* (*g* (*Suc 0*)))
by (*simp add*: $cdcl_{NOT}$*-inv-g*)
have $cdcl_{NOT}\text{-}restart^{**}$ (*fst* (*g 1*), *snd* (*g 1*)) (*fst* (*g j*), *snd* (*g j*))
using ‹*j*> *1*› **by** (*simp add*: $cdcl_{NOT}$*-restart*)
have μ *A* (*fst* (*g j*)) $\leq$ μ*-bound A* (*fst* (*g 1*))
apply (*rule rtranclp-*$cdcl_{NOT}$*-raw-restart-measure-bound*)
using ‹$cdcl_{NOT}\text{-}restart^{**}$ (*fst* (*g 1*), *snd* (*g 1*)) (*fst* (*g j*), *snd* (*g j*))› **apply** *blast*
apply (*simp add*: $cdcl_{NOT}$*-inv-g*)
using ‹*bound-inv A* (*fst* (*g 1*))› **apply** *simp*
done
then have μ *A* (*fst* (*g j*)) $\leq$ *?j*
by *auto*
have *inv*: *bound-inv A* (*fst* (*g j*))
using ‹*bound-inv A* (*fst* (*g 1*))› ‹$cdcl_{NOT}$*-inv* (*fst* (*g* (*Suc 0*)))›
‹$cdcl_{NOT}\text{-}restart^{**}$ (*fst* (*g 1*), *snd* (*g 1*)) (*fst* (*g j*), *snd* (*g j*))›
rtranclp-$cdcl_{NOT}$*-with-restart-bound-inv* **by** *auto*
obtain *T m* **where**
$cdcl_{NOT}$*-m*: ($cdcl_{NOT}$ ^^ *m*) (*fst* (*g j*)) *T* **and**
f-m: *f* (*snd* (*g j*)) $\leq$ *m*
using *H*[*of j*] **by** *blast*
have *?j* < *m*
using *f-m j Nat.le-trans* **by** *linarith*

then show *False*
using ‹μ *A* (*fst* (*g j*)) $\leq$ μ*-bound A* (*fst* (*g 1*))›
$cdcl_{NOT}$*-comp-bounded*[*OF inv* $cdcl_{NOT}$*-inv-g*, *of*] $cdcl_{NOT}$*-inv-g* $cdcl_{NOT}$*-m*
‹*?j* < *m*› **by** *auto*
qed

lemma $cdcl_{NOT}$*-restart-steps-bigger-than-bound*:
assumes
$cdcl_{NOT}$*-restart S T* **and**
bound-inv A (*fst S*) **and**
$cdcl_{NOT}$*-inv* (*fst S*) **and**
f (*snd S*) > μ*-bound A* (*fst S*)
shows *full1* $cdcl_{NOT}$ (*fst S*) (*fst T*)
using *assms*
proof (*induction rule*: $cdcl_{NOT}$*-restart.induct*)
case *restart-full*
then show *?case* **by** *auto*
next
case (*restart-step m S T n U*) **note** *st* = *this*(*1*) **and** *f* = *this*(*2*) **and** *bound-inv* = *this*(*4*) **and**
$cdcl_{NOT}$*-inv* =*this*(*5*) **and** μ = *this*(*6*)
then obtain *m'* **where** *m*: *m* = *Suc m'* **by** (*cases m*) *auto*
have μ *A S* − *m'* = *0*
using *f bound-inv* $cdcl_{NOT}$*-inv* μ *m rtranclp-*$cdcl_{NOT}$*-raw-restart-measure-bound* **by** *fastforce*
then have *False* **using** $cdcl_{NOT}$*-comp-n-le*[*of m' S T A*] *restart-step* **unfolding** *m* **by** *simp*

then show *?case* **by** *fast*
qed

lemma *rtranclp-cdcl$_{NOT}$-with-inv-inv-rtranclp-cdcl$_{NOT}$*:
assumes
inv: *cdcl$_{NOT}$-inv S* **and**
binv: *bound-inv A S*
shows $(\lambda S\ T.\ cdcl_{NOT}\ S\ T \wedge cdcl_{NOT}\text{-}inv\ S \wedge bound\text{-}inv\ A\ S)^{**}\ S\ T \longleftrightarrow cdcl_{NOT}{}^{**}\ S\ T$
(**is** $?A^{**}\ S\ T \longleftrightarrow ?B^{**}\ S\ T$)
apply (*rule iffI*)
using *rtranclp-mono*[*of ?A ?B*] **apply** *blast*
apply (*induction rule*: *rtranclp-induct*)
using *inv binv* **apply** *simp*
by (*metis* (*mono-tags*, *lifting*) *binv inv rtranclp.simps rtranclp-cdcl$_{NOT}$-bound-inv*
rtranclp-cdcl$_{NOT}$-cdcl$_{NOT}$-inv)

lemma *no-step-cdcl$_{NOT}$-restart-no-step-cdcl$_{NOT}$*:
assumes
n-s: *no-step cdcl$_{NOT}$-restart S* **and**
inv: *cdcl$_{NOT}$-inv* (*fst S*) **and**
binv: *bound-inv A* (*fst S*)
shows *no-step cdcl$_{NOT}$* (*fst S*)
proof (*rule ccontr*)
assume ¬ *?thesis*
then obtain *T* **where** *T*: *cdcl$_{NOT}$* (*fst S*) *T*
by *blast*
then obtain *U* **where** *U*: *full* $(\lambda S\ T.\ cdcl_{NOT}\ S\ T \wedge cdcl_{NOT}\text{-}inv\ S \wedge bound\text{-}inv\ A\ S)$ *T U*
using *wf-exists-normal-form-full*[*OF wf-cdcl$_{NOT}$*, *of A T*] **by** *auto*
moreover have *inv-T*: *cdcl$_{NOT}$-inv T*
using ‹*cdcl$_{NOT}$* (*fst S*) *T*› *cdcl$_{NOT}$-inv inv* **by** *blast*
moreover have *b-inv-T*: *bound-inv A T*
using ‹*cdcl$_{NOT}$* (*fst S*) *T*› *binv bound-inv inv* **by** *blast*
ultimately have *full cdcl$_{NOT}$ T U*
using *rtranclp-cdcl$_{NOT}$-with-inv-inv-rtranclp-cdcl$_{NOT}$ rtranclp-cdcl$_{NOT}$-bound-inv*
rtranclp-cdcl$_{NOT}$-cdcl$_{NOT}$-inv **unfolding** *full-def* **by** *blast*
then have *full1 cdcl$_{NOT}$* (*fst S*) *U*
using *T full-fullI* **by** *metis*
then show *False* **by** (*metis n-s prod.collapse restart-full*)
qed

end

14.8 Merging backjump and learning

locale *cdcl$_{NOT}$-merge-bj-learn-ops* =
dpll-state trail clauses prepend-trail tl-trail add-cls$_{NOT}$ remove-cls$_{NOT}$ +
decide-ops trail clauses prepend-trail tl-trail add-cls$_{NOT}$ remove-cls$_{NOT}$ +
forget-ops trail clauses prepend-trail tl-trail add-cls$_{NOT}$ remove-cls$_{NOT}$ forget-cond +
propagate-ops trail clauses prepend-trail tl-trail add-cls$_{NOT}$ remove-cls$_{NOT}$ propagate-conds
for
trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
clauses :: *'st* ⇒ *'v clauses* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-cls$_{NOT}$ remove-cls$_{NOT}$:: *'v clause* ⇒ *'st* ⇒ *'st* **and**
propagate-conds :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *bool* **and**

forget-cond :: *'v clause* ⇒ *'st* ⇒ *bool* +
fixes *backjump-l-cond* :: *'v clause* ⇒ *'v clause* ⇒ *'v literal* ⇒ *'st* ⇒ *bool*
begin
inductive *backjump-l* **where**
backjump-l: *trail S* = *F'* @ *Marked K* () # *F*
⟹ *no-dup* (*trail S*)
⟹ *T* ∼ *prepend-trail* (*Propagated L* ()) (*reduce-trail-to*$_{NOT}$ *F* (*add-cls*$_{NOT}$ (*C'* + {#*L*#}) *S*))
⟹ *C* ∈# *clauses S*
⟹ *trail S* ⊨*as CNot C*
⟹ *undefined-lit F L*
⟹ *atm-of L* ∈ *atms-of-msu* (*clauses S*) ∪ *atm-of* ‘ (*lits-of* (*trail S*))
⟹ *clauses S* ⊨*pm C'* + {#*L*#}
⟹ *F* ⊨*as CNot C'*
⟹ *backjump-l-cond C C' L T*
⟹ *backjump-l S T*
inductive-cases *backjump-lE*: *backjump-l S T*

inductive *cdcl*$_{NOT}$*-merged-bj-learn* :: *'st* ⇒ *'st* ⇒ *bool* **for** *S* :: *'st* **where**
cdcl$_{NOT}$*-merged-bj-learn-decide*$_{NOT}$: *decide*$_{NOT}$ *S S'* ⟹ *cdcl*$_{NOT}$*-merged-bj-learn S S'* |
cdcl$_{NOT}$*-merged-bj-learn-propagate*$_{NOT}$: *propagate*$_{NOT}$ *S S'* ⟹ *cdcl*$_{NOT}$*-merged-bj-learn S S'* |
cdcl$_{NOT}$*-merged-bj-learn-backjump-l*: *backjump-l S S'* ⟹ *cdcl*$_{NOT}$*-merged-bj-learn S S'* |
cdcl$_{NOT}$*-merged-bj-learn-forget*$_{NOT}$: *forget*$_{NOT}$ *S S'* ⟹ *cdcl*$_{NOT}$*-merged-bj-learn S S'*

lemma *cdcl*$_{NOT}$*-merged-bj-learn-no-dup-inv*:
cdcl$_{NOT}$*-merged-bj-learn S T* ⟹ *no-dup* (*trail S*) ⟹ *no-dup* (*trail T*)
apply (*induction rule*: *cdcl*$_{NOT}$*-merged-bj-learn.induct*)
using *defined-lit-map* **apply** *fastforce*
using *defined-lit-map* **apply** *fastforce*
apply (*force simp*: *defined-lit-map elim*!: *backjump-lE*)[]
using *forget*$_{NOT}$*.simps* **apply** *auto*[*1*]
done
end

locale *cdcl*$_{NOT}$*-merge-bj-learn-proxy* =
cdcl$_{NOT}$*-merge-bj-learn-ops trail clauses prepend-trail tl-trail add-cls*$_{NOT}$ *remove-cls*$_{NOT}$
propagate-conds forget-conds λ*C C' L' S*. *backjump-l-cond C C' L' S*
∧ *distinct-mset* (*C'* + {#*L'*#}) ∧ ¬*tautology* (*C'* + {#*L'*#})
for
trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
clauses :: *'st* ⇒ *'v clauses* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-cls$_{NOT}$ *remove-cls*$_{NOT}$:: *'v clause* ⇒ *'st* ⇒ *'st* **and**
propagate-conds :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *bool* **and**
forget-conds :: *'v clause* ⇒ *'st* ⇒ *bool* **and**
backjump-l-cond :: *'v clause* ⇒ *'v clause* ⇒ *'v literal* ⇒ *'st* ⇒ *bool* +
fixes
inv :: *'st* ⇒ *bool*
assumes
bj-can-jump:
⋀*S C F' K F L*.
inv S
⟹ *trail S* = *F'* @ *Marked K* () # *F*
⟹ *C* ∈# *clauses S*
⟹ *trail S* ⊨*as CNot C*

$\Longrightarrow$ *undefined-lit F L*
$\Longrightarrow$ *atm-of L* $\in$ *atms-of-msu* (*clauses S*) $\cup$ *atm-of* ‘ (*lits-of* (*F'* @ *Marked K* () # *F*))
$\Longrightarrow$ *clauses S* $\models$*pm C'* + {#*L*#}
$\Longrightarrow$ *F* $\models$*as CNot C'*
$\Longrightarrow$ $\neg$*no-step backjump-l S* **and**
cdcl-merged-inv: $\bigwedge S\ T.$ *cdcl*$_{NOT}$*-merged-bj-learn S T* $\Longrightarrow$ *inv S* $\Longrightarrow$ *inv T*
begin
abbreviation *backjump-conds* **where**
backjump-conds $\equiv$ λ- *C L* - -. *distinct-mset* (*C* + {#*L*#}) $\wedge$ $\neg$*tautology* (*C* + {#*L*#})

sublocale *dpll-with-backjumping-ops trail clauses prepend-trail tl-trail add-cls*$_{NOT}$ *remove-cls*$_{NOT}$
propagate-conds inv backjump-conds
proof (*unfold-locales*, *goal-cases*)
case *1*
{ fix *S S'*
assume *bj*: *backjump-l S S'* **and** *no-dup* (*trail S*)
then obtain *F' K F L C' C* **where**
S': *S'* $\sim$ *prepend-trail* (*Propagated L* ()) (*reduce-trail-to*$_{NOT}$ *F*
(*tl-trail*(*add-cls*$_{NOT}$ (*C'* + {#*L*#}) *S*)))
and
tr-S: *trail S* = *F'* @ *Marked K* () # *F* **and**
C: *C* $\in$# *clauses S* **and**
tr-S-C: *trail S* $\models$*as CNot C* **and**
undef-L: *undefined-lit F L* **and**
atm-L: *atm-of L* $\in$ *atms-of-msu* (*clauses S*) $\cup$ *atm-of* ‘ *lits-of* (*trail S*) **and**
cls-S-C': *clauses S* $\models$*pm C'* + {#*L*#} **and**
F-C': *F* $\models$*as CNot C'* **and**
dist: *distinct-mset* (*C'* + {#*L*#}) **and**
not-tauto: $\neg$ *tautology* (*C'* + {#*L*#})
by (*elim backjump-lE*) *simp*

have $\exists$ *S'*. *backjumping-ops.backjump trail clauses prepend-trail tl-trail backjump-conds S S'*
apply *rule*
apply (*rule backjumping-ops.backjump.intros*)
apply *unfold-locales*
using *tr-S* **apply** *simp*
apply (*rule state-eq*$_{NOT}$*-ref*)
using *C* **apply** *simp*
using *tr-S-C* **apply** *simp*
using *undef-L* **apply** *simp*
using *atm-L* **apply** *simp*
using *cls-S-C'* **apply** *simp*
using *F-C'* **apply** *simp*
using *dist not-tauto* **apply** *simp*
done
} note *H* = *this*(*1*)
then show *?case* **using** *1 bj-can-jump* **by** *meson*
qed

end

locale *cdcl*$_{NOT}$*-merge-bj-learn-proxy2* =
cdcl$_{NOT}$*-merge-bj-learn-proxy trail clauses prepend-trail tl-trail add-cls*$_{NOT}$ *remove-cls*$_{NOT}$
propagate-conds forget-conds backjump-l-cond inv
for

trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
clauses :: *'st* ⇒ *'v clauses* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-cls$_{NOT}$ *remove-cls*$_{NOT}$:: *'v clause* ⇒ *'st* ⇒ *'st* **and**
propagate-conds :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *bool* **and**
inv :: *'st* ⇒ *bool* **and**
forget-conds :: *'v clause* ⇒ *'st* ⇒ *bool* **and**
backjump-l-cond :: *'v clause* ⇒ *'v clause* ⇒ *'v literal* ⇒ *'st* ⇒ *bool*
begin

sublocale *conflict-driven-clause-learning-ops trail clauses prepend-trail tl-trail add-cls*$_{NOT}$ *remove-cls*$_{NOT}$ *propagate-conds inv backjump-conds* λC *-. distinct-mset* $C \wedge \neg$*tautology* C *forget-conds*
by *unfold-locales*
end

locale *cdcl*$_{NOT}$*-merge-bj-learn* =
cdcl$_{NOT}$*-merge-bj-learn-proxy2 trail clauses prepend-trail tl-trail add-cls*$_{NOT}$ *remove-cls*$_{NOT}$ *propagate-conds inv forget-conds backjump-l-cond*
for
trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
clauses :: *'st* ⇒ *'v clauses* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-cls$_{NOT}$ *remove-cls*$_{NOT}$:: *'v clause* ⇒ *'st* ⇒ *'st* **and**
propagate-conds :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *bool* **and**
inv :: *'st* ⇒ *bool* **and**
forget-conds :: *'v clause* ⇒ *'st* ⇒ *bool* **and**
backjump-l-cond :: *'v clause* ⇒ *'v clause* ⇒ *'v literal* ⇒ *'st* ⇒ *bool* +
assumes
dpll-bj-inv: $\bigwedge S\ T.$ *dpll-bj* $S\ T \Longrightarrow$ *inv* $S \Longrightarrow$ *inv* T **and**
learn-inv: $\bigwedge S\ T.$ *learn* $S\ T \Longrightarrow$ *inv* $S \Longrightarrow$ *inv* T
begin

interpretation *cdcl*$_{NOT}$:
conflict-driven-clause-learning trail clauses prepend-trail tl-trail add-cls$_{NOT}$ *remove-cls*$_{NOT}$ *propagate-conds inv backjump-conds* λC *-. distinct-mset* $C \wedge \neg$*tautology* C *forget-conds*
apply *unfold-locales*
apply (*simp only*: *cdcl*$_{NOT}$*.simps*)
using *cdcl*$_{NOT}$*-merged-bj-learn-forget*$_{NOT}$ *cdcl-merged-inv learn-inv*
by (*auto simp add*: *cdcl*$_{NOT}$*.simps dpll-bj-inv*)

lemma *backjump-l-learn-backjump*:
assumes *bt*: *backjump-l* $S\ T$ **and** *inv*: *inv* S **and** *n-d*: *no-dup* (*trail* S)
shows $\exists\, C'\ L.$ *learn* S (*add-cls*$_{NOT}$ (C' + {#L#}) S)
$\wedge$ *backjump* (*add-cls*$_{NOT}$ (C' + {#L#}) S) T
$\wedge$ *atms-of* (C' + {#L#}) $\subseteq$ *atms-of-msu* (*clauses* S) $\cup$ *atm-of* ‘ (*lits-of* (*trail* S))
proof −
obtain $C\ F'\ K\ F\ L\ l\ C'$ **where**
tr-S: *trail* S = F' @ *Marked* K () # F **and**
T: $T \sim$ *prepend-trail* (*Propagated* L l) (*reduce-trail-to*$_{NOT}$ F (*add-cls*$_{NOT}$ (C' + {#L#}) S)) **and**
C-cls-S: $C \in\#$ *clauses* S **and**
tr-S-CNot-C: *trail* $S \models$*as CNot* C **and**
undef: *undefined-lit* F L **and**

atm-L: *atm-of* $L \in$ *atms-of-msu* (*clauses* S) $\cup$ *atm-of* ‘ (*lits-of* (*trail* S)) **and**
clss-C: *clauses* S $\models$*pm* $C' + \{\#L\#\}$ **and**
F $\models$*as* *CNot* C' **and**
distinct: *distinct-mset* ($C' + \{\#L\#\}$) **and**
not-tauto: $\neg$ *tautology* ($C' + \{\#L\#\}$)
using *bt inv* **by** (*elim backjump-lE*) *simp*
have *atms-C'*: *atms-of* $C' \subseteq$ *atm-of* ‘ (*lits-of* F)
proof −
obtain *ll* :: $'v \Rightarrow ('v$ *literal* $\Rightarrow 'v) \Rightarrow 'v$ *literal set* $\Rightarrow 'v$ *literal* **where**
$\forall v\, f\, L.\ v \notin f$ ‘ $L \lor v = f\ (ll\ v\ f\ L) \land ll\ v\ f\ L \in L$
by *moura*
then show *?thesis* **unfolding** *tr-S*
by (*metis* (*no-types*) ‹F $\models$*as CNot* C'› *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set*
atms-of-def in-CNot-implies-uminus(*2*) *mem-set-mset-iff subsetI*)
qed
then have *atms-of* ($C' + \{\#L\#\}$) $\subseteq$ *atms-of-msu* (*clauses* S) $\cup$ *atm-of* ‘ (*lits-of* (*trail* S))
using *atm-L tr-S* **by** *auto*
moreover have *learn*: *learn* S (*add-cls*$_{NOT}$ ($C' + \{\#L\#\}$) S)
apply (*rule learn.intros*)
apply (*rule clss-C*)
using *atms-C' atm-L* **apply** (*fastforce simp add*: *tr-S in-plus-implies-atm-of-on-atms-of-ms*)[]
apply *standard*
apply (*rule distinct*)
apply (*rule not-tauto*)
apply *simp*
done
moreover have *bj*: *backjump* (*add-cls*$_{NOT}$ ($C' + \{\#L\#\}$) S) T
apply (*rule backjump.intros*)
using ‹F $\models$*as CNot* C'› *C-cls-S tr-S-CNot-C undef T distinct not-tauto n-d*
by (*auto simp*: *tr-S state-eq*$_{NOT}$*-def simp del*: *state-simp*$_{NOT}$)
ultimately show *?thesis* **by** *auto*
qed

lemma *cdcl*$_{NOT}$*-merged-bj-learn-is-tranclp-cdcl*$_{NOT}$:
cdcl$_{NOT}$*-merged-bj-learn* S T $\Longrightarrow$ *inv* S $\Longrightarrow$ *no-dup* (*trail* S) $\Longrightarrow$ *cdcl*$_{NOT}{}^{++}$ S T
proof (*induction rule*: *cdcl*$_{NOT}$*-merged-bj-learn.induct*)
case (*cdcl*$_{NOT}$*-merged-bj-learn-decide*$_{NOT}$ T)
then have *cdcl*$_{NOT}$ S T
using *bj-decide*$_{NOT}$ *cdcl*$_{NOT}$*.simps* **by** *fastforce*
then show *?case* **by** *auto*
next
case (*cdcl*$_{NOT}$*-merged-bj-learn-propagate*$_{NOT}$ T)
then have *cdcl*$_{NOT}$ S T
using *bj-propagate*$_{NOT}$ *cdcl*$_{NOT}$*.simps* **by** *fastforce*
then show *?case* **by** *auto*
next
case (*cdcl*$_{NOT}$*-merged-bj-learn-forget*$_{NOT}$ T)
then have *cdcl*$_{NOT}$ S T
using *c-forget*$_{NOT}$ **by** *blast*
then show *?case* **by** *auto*
next
case (*cdcl*$_{NOT}$*-merged-bj-learn-backjump-l* T) **note** *bt* = *this*(*1*) **and** *inv* = *this*(*2*) **and**
n-d = *this*(*3*)
obtain C' :: $'v$ *literal multiset* **and** L :: $'v$ *literal* **where**
f3: *learn* S (*add-cls*$_{NOT}$ ($C' + \{\#L\#\}$) S) $\land$

backjump (*add-cls*$_{NOT}$ (C' + {#*L*#}) *S*) *T* ∧
atms-of (C' + {#*L*#}) ⊆ *atms-of-msu* (*clauses S*) ∪ *atm-of* ' *lits-of* (*trail S*)
using *n-d backjump-l-learn-backjump*[*OF bt inv*] **by** *blast*
then have *f4*: *cdcl*$_{NOT}$ *S* (*add-cls*$_{NOT}$ (C' + {#*L*#}) *S*)
using *n-d c-learn* **by** *blast*
have *cdcl*$_{NOT}$ (*add-cls*$_{NOT}$ (C' + {#*L*#}) *S*) *T*
using *f3 n-d bj-backjump c-dpll-bj* **by** *blast*
then show *?case*
using *f4* **by** (*meson tranclp.r-into-trancl tranclp.trancl-into-trancl*)
qed

lemma *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-is-rtranclp-cdcl*$_{NOT}$*-and-inv*:
cdcl$_{NOT}$*-merged-bj-learn*** *S T* ⟹ *inv S* ⟹ *no-dup* (*trail S*) ⟹ *cdcl*$_{NOT}{}^{**}$ *S T* ∧ *inv T*
proof (*induction rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *auto*
next
case (*step T U*) **note** *st* =*this*(*1*) **and** *cdcl*$_{NOT}$ = *this*(*2*) **and** *IH* = *this*(*3*)[*OF this*(*4*−)] **and** *inv* = *this*(*4*) **and** *n-d* = *this*(*5*)
have *cdcl*$_{NOT}{}^{**}$ *T U*
using *cdcl*$_{NOT}$*-merged-bj-learn-is-tranclp-cdcl*$_{NOT}$[*OF cdcl*$_{NOT}$] *IH*
cdcl$_{NOT}$*.rtranclp-cdcl*$_{NOT}$*-no-dup inv n-d* **by** *auto*
then have *cdcl*$_{NOT}{}^{**}$ *S U* **using** *IH* **by** *fastforce*
moreover have *inv U* **using** *n-d IH* ‹*cdcl*$_{NOT}{}^{**}$ *T U*› *cdcl*$_{NOT}$*.rtranclp-cdcl*$_{NOT}$*-inv* **by** *blast*
ultimately show *?case* **using** *st* **by** *fast*
qed

lemma *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-is-rtranclp-cdcl*$_{NOT}$:
cdcl$_{NOT}$*-merged-bj-learn*** *S T* ⟹ *inv S* ⟹*no-dup* (*trail S*) ⟹ *cdcl*$_{NOT}{}^{**}$ *S T*
using *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-is-rtranclp-cdcl*$_{NOT}$*-and-inv* **by** *blast*

lemma *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-inv*:
cdcl$_{NOT}$*-merged-bj-learn*** *S T* ⟹ *inv S* ⟹ *no-dup* (*trail S*) ⟹ *inv T*
using *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-is-rtranclp-cdcl*$_{NOT}$*-and-inv* **by** *blast*

definition $\mu_C{}'$ :: *'v literal multiset set* ⇒ *'st* ⇒ *nat* **where**
$\mu_C{}'$ *A T* ≡ μ_C (*1*+*card* (*atms-of-ms A*)) (*2*+*card* (*atms-of-ms A*)) (*trail-weight T*)

definition $\mu_{CDCL}{}'$*-merged* :: *'v literal multiset set* ⇒ *'st* ⇒ *nat* **where**
$\mu_{CDCL}{}'$*-merged A T* ≡
((*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*)) − $\mu_C{}'$ *A T*) ∗ *2* + *card* (*set-mset* (*clauses T*))

lemma *cdcl*$_{NOT}$*-decreasing-measure'*:
assumes
cdcl$_{NOT}$*-merged-bj-learn S T* **and**
inv: *inv S* **and**
atm-clss: *atms-of-msu* (*clauses S*) ⊆ *atms-of-ms A* **and**
atm-trail: *atm-of* ' *lits-of* (*trail S*) ⊆ *atms-of-ms A* **and**
n-d: *no-dup* (*trail S*) **and**
fin-A: *finite A*
shows $\mu_{CDCL}{}'$*-merged A T* < $\mu_{CDCL}{}'$*-merged A S*
using *assms*(*1*)
proof *induction*
case (*cdcl*$_{NOT}$*-merged-bj-learn-decide*$_{NOT}$ *T*)
have *clauses S* = *clauses T*

using *$cdcl_{NOT}$-merged-bj-learn-decide$_{NOT}$.hyps* **by** *auto*
moreover have
$(2 + card\ (atms\text{-}of\text{-}ms\ A))\ \hat{}\ (1 + card\ (atms\text{-}of\text{-}ms\ A))$
$- \mu_C\ (1 + card\ (atms\text{-}of\text{-}ms\ A))\ (2 + card\ (atms\text{-}of\text{-}ms\ A))\ (trail\text{-}weight\ T)$
$< (2 + card\ (atms\text{-}of\text{-}ms\ A))\ \hat{}\ (1 + card\ (atms\text{-}of\text{-}ms\ A))$
$- \mu_C\ (1 + card\ (atms\text{-}of\text{-}ms\ A))\ (2 + card\ (atms\text{-}of\text{-}ms\ A))\ (trail\text{-}weight\ S)$
apply (*rule dpll-bj-trail-mes-decreasing-prop*)
using *$cdcl_{NOT}$-merged-bj-learn-decide$_{NOT}$ fin-A atm-clss atm-trail n-d inv*
by (*simp-all add*: *bj-decide$_{NOT}$ $cdcl_{NOT}$-merged-bj-learn-decide$_{NOT}$.hyps*)
ultimately show *?case*
unfolding *μ_{CDCL}'-merged-def μ_C'-def* **by** *simp*
next
case (*$cdcl_{NOT}$-merged-bj-learn-propagate$_{NOT}$ T*)
have *clauses S* = *clauses T*
using *$cdcl_{NOT}$-merged-bj-learn-propagate$_{NOT}$.hyps*
by (*simp add*: *bj-propagate$_{NOT}$ inv dpll-bj-clauses*)
moreover have
$(2 + card\ (atms\text{-}of\text{-}ms\ A))\ \hat{}\ (1 + card\ (atms\text{-}of\text{-}ms\ A))$
$- \mu_C\ (1 + card\ (atms\text{-}of\text{-}ms\ A))\ (2 + card\ (atms\text{-}of\text{-}ms\ A))\ (trail\text{-}weight\ T)$
$< (2 + card\ (atms\text{-}of\text{-}ms\ A))\ \hat{}\ (1 + card\ (atms\text{-}of\text{-}ms\ A))$
$- \mu_C\ (1 + card\ (atms\text{-}of\text{-}ms\ A))\ (2 + card\ (atms\text{-}of\text{-}ms\ A))\ (trail\text{-}weight\ S)$
apply (*rule dpll-bj-trail-mes-decreasing-prop*)
using *inv n-d atm-clss atm-trail fin-A* **by** (*simp-all add*: *bj-propagate$_{NOT}$*
$cdcl_{NOT}$-merged-bj-learn-propagate$_{NOT}$.hyps)
ultimately show *?case*
unfolding *μ_{CDCL}'-merged-def μ_C'-def* **by** *simp*
next
case (*$cdcl_{NOT}$-merged-bj-learn-forget$_{NOT}$ T*)
have *card* (*set-mset* (*clauses T*)) < *card* (*set-mset* (*clauses S*))
using ‹*forget$_{NOT}$ S T*› **by** (*metis card-Diff1-less*
$cdcl_{NOT}$-merged-bj-learn-forget$_{NOT}$.hyps clauses-remove-cls$_{NOT}$ finite-set-mset forgetE
mem-set-mset-iff order-refl set-mset-minus-replicate-mset(*1*) *state-eq$_{NOT}$-clauses*)
moreover
have *trail S* = *trail T*
using ‹*forget$_{NOT}$ S T*› **by** (*auto elim*: *forgetE*)
then have
$(2 + card\ (atms\text{-}of\text{-}ms\ A))\ \hat{}\ (1 + card\ (atms\text{-}of\text{-}ms\ A))$
$- \mu_C\ (1 + card\ (atms\text{-}of\text{-}ms\ A))\ (2 + card\ (atms\text{-}of\text{-}ms\ A))\ (trail\text{-}weight\ T)$
$= (2 + card\ (atms\text{-}of\text{-}ms\ A))\ \hat{}\ (1 + card\ (atms\text{-}of\text{-}ms\ A))$
$- \mu_C\ (1 + card\ (atms\text{-}of\text{-}ms\ A))\ (2 + card\ (atms\text{-}of\text{-}ms\ A))\ (trail\text{-}weight\ S)$
by *auto*
ultimately show *?case*
unfolding *μ_{CDCL}'-merged-def μ_C'-def* **by** *simp*
next
case (*$cdcl_{NOT}$-merged-bj-learn-backjump-l T*) **note** *bj-l* = *this*(*1*)
obtain *C' L* **where**
learn: *learn S* (*add-cls$_{NOT}$* (*C'* + {#*L*#}) *S*) **and**
bj: *backjump* (*add-cls$_{NOT}$* (*C'* + {#*L*#}) *S*) *T* **and**
atms-C: *atms-of* (*C'* + {#*L*#}) ⊆ *atms-of-msu* (*clauses S*) ∪ *atm-of* ‘ (*lits-of* (*trail S*))
using *bj-l inv backjump-l-learn-backjump n-d atm-clss atm-trail* **by** *blast*
have *card-T-S*: *card* (*set-mset* (*clauses T*)) ≤ *1*+ *card* (*set-mset* (*clauses S*))
using *bj-l inv* **by** (*force elim*!: *backjump-lE simp*: *card-insert-if*)
have
$((2 + card\ (atms\text{-}of\text{-}ms\ A))\ \hat{}\ (1 + card\ (atms\text{-}of\text{-}ms\ A))$
$- \mu_C\ (1 + card\ (atms\text{-}of\text{-}ms\ A))\ (2 + card\ (atms\text{-}of\text{-}ms\ A))\ (trail\text{-}weight\ T))$

< ((*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*))
− μ_C (*1* + *card* (*atms-of-ms A*)) (*2* + *card* (*atms-of-ms A*))
(*trail-weight* (*add-cls*$_{NOT}$ (C' + {#*L*#}) *S*)))
apply (*rule dpll-bj-trail-mes-decreasing-prop*)
using *bj bj-backjump* **apply** *blast*
using *cdcl*$_{NOT}$.*c-learn cdcl*$_{NOT}$.*cdcl*$_{NOT}$*-inv inv learn* **apply** *blast*
using *atms-C atm-clss atm-trail n-d clauses-add-cls*$_{NOT}$ **apply** *simp* **apply** *fast*
using *atm-trail n-d* **apply** *simp*
apply (*simp add*: *n-d*)
using *fin-A* **apply** *simp*
done
then have ((*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*))
− μ_C (*1* + *card* (*atms-of-ms A*)) (*2* + *card* (*atms-of-ms A*)) (*trail-weight T*))
< ((*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*))
− μ_C (*1* + *card* (*atms-of-ms A*)) (*2* + *card* (*atms-of-ms A*)) (*trail-weight S*))
using *n-d* **by** *auto*
then show *?case*
using *card-T-S* **unfolding** $\mu_{CDCL}{}'$*-merged-def* $\mu_C{}'$*-def* **by** *linarith*
qed

lemma *wf-cdcl*$_{NOT}$*-merged-bj-learn*:
assumes
fin-A: *finite A*
shows *wf* {(*T*, *S*).
(*inv S* ∧ *atms-of-msu* (*clauses S*) ⊆ *atms-of-ms A* ∧ *atm-of* ‘ *lits-of* (*trail S*) ⊆ *atms-of-ms A*
∧ *no-dup* (*trail S*))
∧ *cdcl*$_{NOT}$*-merged-bj-learn S T*}
apply (*rule wfP-if-measure*[*of - -* $\mu_{CDCL}{}'$*-merged A*])
using *cdcl*$_{NOT}$*-decreasing-measure*$'$ *fin-A* **by** *simp*

lemma *tranclp-cdcl*$_{NOT}$*-cdcl*$_{NOT}$*-tranclp*:
assumes
cdcl$_{NOT}$*-merged-bj-learn*$^{++}$ *S T* **and**
inv: *inv S* **and**
atm-clss: *atms-of-msu* (*clauses S*) ⊆ *atms-of-ms A* **and**
atm-trail: *atm-of* ‘ *lits-of* (*trail S*) ⊆ *atms-of-ms A* **and**
n-d: *no-dup* (*trail S*) **and**
fin-A[*simp*]: *finite A*
shows (*T*, *S*) ∈ {(*T*, *S*).
(*inv S* ∧ *atms-of-msu* (*clauses S*) ⊆ *atms-of-ms A* ∧ *atm-of* ‘ *lits-of* (*trail S*) ⊆ *atms-of-ms A*
∧ *no-dup* (*trail S*))
∧ *cdcl*$_{NOT}$*-merged-bj-learn S T*}$^+$ (**is** - ∈ *?P*$^+$)
using *assms*(*1*)
proof (*induction rule*: *tranclp-induct*)
case *base*
then show *?case* **using** *n-d atm-clss atm-trail inv* **by** *auto*
next
case (*step T U*) **note** *st* = *this*(*1*) **and** *cdcl*$_{NOT}$ = *this*(*2*) **and** *IH* = *this*(*3*)
have *cdcl*$_{NOT}{}^{**}$ *S T*
apply (*rule rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-is-rtranclp-cdcl*$_{NOT}$)
using *st cdcl*$_{NOT}$ *inv n-d atm-clss atm-trail inv* **by** *auto*
have *inv T*
apply (*rule rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-inv*)
using *inv st cdcl*$_{NOT}$ *n-d atm-clss atm-trail inv* **by** *auto*
moreover have *atms-of-msu* (*clauses T*) ⊆ *atms-of-ms A*

using $cdcl_{NOT}$*.rtranclp-*$cdcl_{NOT}$*-trail-clauses-bound*[*OF* ‹$cdcl_{NOT}{}^{**}$ *S T*› *inv n-d atm-clss atm-trail*]
by *fast*
moreover have *atm-of* ‘ (*lits-of* (*trail T*))⊆ *atms-of-ms A*
using $cdcl_{NOT}$*.rtranclp-*$cdcl_{NOT}$*-trail-clauses-bound*[*OF* ‹$cdcl_{NOT}{}^{**}$ *S T*› *inv n-d atm-clss atm-trail*]
by *fast*
moreover have *no-dup* (*trail T*)
using $cdcl_{NOT}$*.rtranclp-*$cdcl_{NOT}$*-no-dup*[*OF* ‹$cdcl_{NOT}{}^{**}$ *S T*› *inv n-d*] **by** *fast*
ultimately have (*U*, *T*) ∈ *?P*
using $cdcl_{NOT}$ **by** *auto*
then show *?case* **using** *IH* **by** (*simp add*: *trancl-into-trancl2*)
qed

lemma *wf-tranclp-*$cdcl_{NOT}$*-merged-bj-learn*:
assumes *finite A*
shows *wf* {(*T*, *S*).
(*inv S* ∧ *atms-of-msu* (*clauses S*) ⊆ *atms-of-ms A* ∧ *atm-of* ‘ *lits-of* (*trail S*) ⊆ *atms-of-ms A*
∧ *no-dup* (*trail S*))
∧ $cdcl_{NOT}$*-merged-bj-learn*$^{++}$ *S T*}
apply (*rule wf-subset*)
apply (*rule wf-trancl*[*OF wf-*$cdcl_{NOT}$*-merged-bj-learn*])
using *assms* **apply** *simp*
using *tranclp-*$cdcl_{NOT}$*-*$cdcl_{NOT}$*-tranclp*[*OF - - - - -* ‹*finite A*›] **by** *auto*

lemma *backjump-no-step-backjump-l*:
backjump S T ⟹ *inv S* ⟹ ¬*no-step backjump-l S*
apply (*elim backjumpE*)
apply (*rule bj-can-jump*)
apply *auto*[*7*]
by *blast*

lemma $cdcl_{NOT}$*-merged-bj-learn-final-state*:
fixes *A* :: ′*v literal multiset set* **and** *S T* :: ′*st*
assumes
n-s: *no-step* $cdcl_{NOT}$*-merged-bj-learn S* **and**
atms-S: *atms-of-msu* (*clauses S*) ⊆ *atms-of-ms A* **and**
atms-trail: *atm-of* ‘ *lits-of* (*trail S*) ⊆ *atms-of-ms A* **and**
n-d: *no-dup* (*trail S*) **and**
finite A **and**
inv: *inv S* **and**
decomp: *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
shows *unsatisfiable* (*set-mset* (*clauses S*))
∨ (*trail S* ⊨*asm clauses S* ∧ *satisfiable* (*set-mset* (*clauses S*)))
proof −
let *?N* = *set-mset* (*clauses S*)
let *?M* = *trail S*
consider
(*sat*) *satisfiable ?N* **and** *?M* ⊨*as ?N*
| (*sat*′) *satisfiable ?N* **and** ¬ *?M* ⊨*as ?N*
| (*unsat*) *unsatisfiable ?N*
by *auto*
then show *?thesis*
proof *cases*
case *sat*′ **note** *sat* = *this*(*1*) **and** *M* = *this*(*2*)
obtain *C* **where** *C* ∈ *?N* **and** ¬*?M* ⊨*a C* **using** *M* **unfolding** *true-annots-def* **by** *auto*
obtain *I* :: ′*v literal set* **where**

$I \models s$ *?N* **and**
cons: *consistent-interp I* **and**
tot: *total-over-m I ?N* **and**
atm-I-N: *atm-of* ' $I \subseteq$ *atms-of-ms ?N*
using *sat* **unfolding** *satisfiable-def-min* **by** *auto*
let $?I = I \cup \{P \mid P.\ P \in$ *lits-of ?M* $\wedge$ *atm-of P* $\notin$ *atm-of* ' $I\}$
let $?O = \{\{\#$*lit-of L*$\#\} \mid L.$ *is-marked L* $\wedge$ $L \in$ *set ?M* $\wedge$ *atm-of* (*lit-of L*) $\notin$ *atms-of-ms ?N*$\}$
have *cons-I'*: *consistent-interp ?I*
using *cons* **using** ⟨*no-dup ?M*⟩ **unfolding** *consistent-interp-def*
by (*auto simp add*: *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set lits-of-def*
dest!: *no-dup-cannot-not-lit-and-uminus*)
have *tot-I'*: *total-over-m ?I* (*?N* $\cup$ ($\lambda a.\ \{\#$*lit-of a*$\#\}$) ' *set ?M*)
using *tot atms-of-s-def* **unfolding** *total-over-m-def total-over-set-def*
by *fastforce*
have $\{P \mid P.\ P \in$ *lits-of ?M* $\wedge$ *atm-of P* $\notin$ *atm-of* ' $I\} \models s$ *?O*
using ⟨$I \models s$ *?N*⟩ *atm-I-N* **by** (*auto simp add*: *atm-of-eq-atm-of true-clss-def lits-of-def*)
then have *I'-N*: *?I* $\models s$ *?N* $\cup$ *?O*
using ⟨$I \models s$ *?N*⟩ *true-clss-union-increase* **by** *force*
have *tot'*: *total-over-m ?I* (*?N*$\cup$*?O*)
using *atm-I-N tot* **unfolding** *total-over-m-def total-over-set-def*
by (*force simp*: *image-iff lits-of-def dest*!: *is-marked-ex-Marked*)

have *atms-N-M*: *atms-of-ms ?N* $\subseteq$ *atm-of* ' *lits-of ?M*
proof (*rule ccontr*)
assume $\neg$ *?thesis*
then obtain $l :: 'v$ **where**
l-N: $l \in$ *atms-of-ms ?N* **and**
l-M: $l \notin$ *atm-of* ' *lits-of ?M*
by *auto*
have *undefined-lit ?M* (*Pos l*)
using *l-M* **by** (*metis Marked-Propagated-in-iff-in-lits-of*
atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set literal.sel(*1*))
have $decide_{NOT}$ *S* (*prepend-trail* (*Marked* (*Pos l*) ()) *S*)
by (*metis* ⟨*undefined-lit ?M* (*Pos l*)⟩ $decide_{NOT}$*.intros l-N literal.sel*(*1*)
state-eq$_{NOT}$*-ref*)
then show *False*
using $cdcl_{NOT}$*-merged-bj-learn-decide*$_{NOT}$ *n-s* **by** *blast*
qed

have *?M* $\models as$ *CNot C*
by (*metis atms-N-M* ⟨$C \in$ *?N*⟩ ⟨$\neg$ *?M* $\models a$ *C*⟩ *all-variables-defined-not-imply-cnot*
atms-of-atms-of-ms-mono atms-of-ms-CNot-atms-of atms-of-ms-CNot-atms-of-ms subsetCE)
have $\exists\, l \in$ *set ?M*. *is-marked l*
proof (*rule ccontr*)
let $?O = \{\{\#$*lit-of L*$\#\} \mid L.$ *is-marked L* $\wedge$ $L \in$ *set ?M* $\wedge$ *atm-of* (*lit-of L*) $\notin$ *atms-of-ms ?N*$\}$
have ϑ[*iff*]: $\bigwedge I.$ *total-over-m I* (*?N* $\cup$ *?O* $\cup$ ($\lambda a.\ \{\#$*lit-of a*$\#\}$) ' *set ?M*)
$\longleftrightarrow$ *total-over-m I* (*?N* $\cup$($\lambda a.\ \{\#$*lit-of a*$\#\}$) ' *set ?M*)
unfolding *total-over-set-def total-over-m-def atms-of-ms-def* **by** *auto*
assume $\neg$ *?thesis*
then have [*simp*]:$\{\{\#$*lit-of L*$\#\} \mid L.$ *is-marked L* $\wedge$ $L \in$ *set ?M*$\}$
$= \{\{\#$*lit-of L*$\#\} \mid L.$ *is-marked L* $\wedge$ $L \in$ *set ?M* $\wedge$ *atm-of* (*lit-of L*) $\notin$ *atms-of-ms ?N*$\}$
by *auto*
then have *?N* $\cup$ *?O* $\models ps$ ($\lambda a.\ \{\#$*lit-of a*$\#\}$) ' *set ?M*
using *all-decomposition-implies-propagated-lits-are-implied*[*OF decomp*] **by** *auto*

then have *?I* ⊨*s* (λ*a*. {#*lit-of a*#}) ‘ *set ?M*
using *cons-I′ I′-N tot-I′* ‹*?I* ⊨*s ?N* ∪ *?O*› **unfolding** ϑ *true-clss-clss-def* **by** *blast*
then have *lits-of ?M* ⊆ *?I*
unfolding *true-clss-def lits-of-def* **by** *auto*
then have *?M* ⊨*as ?N*
using *I′-N* ‹*C* ∈ *?N*› ‹¬ *?M* ⊨*a C*› *cons-I′ atms-N-M*
by (*meson* ‹*trail S* ⊨*as CNot C*› *consistent-CNot-not rev-subsetD sup-ge1 true-annot-def true-annots-def true-cls-mono-set-mset-l true-clss-def*)
then show *False* **using** *M* **by** *fast*
qed
from *List.split-list-first-propE*[*OF this*] **obtain** *K* :: *′v literal* **and** *d* :: *unit* **and**
F F′ :: (*′v*, *unit*, *unit*) *marked-lit list* **where**
M-K: *?M* = *F′* @ *Marked K* () # *F* **and**
nm: ∀*f*∈*set F′*. ¬*is-marked f*
unfolding *is-marked-def* **by** (*metis* (*full-types*) *old.unit.exhaust*)
let *?K* = *Marked K* ()::(*′v*, *unit*, *unit*) *marked-lit*
have *?K* ∈ *set ?M*
unfolding *M-K* **by** *auto*
let *?C* = *image-mset lit-of* {#*L*∈#*mset ?M*. *is-marked L* ∧ *L*≠*?K*#} :: *′v literal multiset*
let *?C′* = *set-mset* (*image-mset* (λ*L*::*′v literal*. {#*L*#}) (*?C*+{#*lit-of ?K*#}))
have *?N* ∪ {{#*lit-of L*#} |*L*. *is-marked L* ∧ *L* ∈ *set ?M*} ⊨*ps* (λ*a*. {#*lit-of a*#}) ‘ *set ?M*
using *all-decomposition-implies-propagated-lits-are-implied*[*OF decomp*] **.**
moreover have *C′*: *?C′* = {{#*lit-of L*#} |*L*. *is-marked L* ∧ *L* ∈ *set ?M*}
unfolding *M-K* **apply** *standard*
apply *force*
using *IntI* **by** *auto*
ultimately have *N-C-M*: *?N* ∪ *?C′* ⊨*ps* (λ*a*. {#*lit-of a*#}) ‘ *set ?M*
by *auto*
have *N-M-False*: *?N* ∪ (λ*L*. {#*lit-of L*#}) ‘ (*set ?M*) ⊨*ps* {{#}}
using *M* ‹*?M* ⊨*as CNot C*› ‹*C*∈*?N*› **unfolding** *true-clss-clss-def true-annots-def Ball-def true-annot-def* **by** (*metis consistent-CNot-not sup.orderE sup-commute true-clss-def true-clss-singleton-lit-of-implies-incl true-clss-union true-clss-union-increase*)

have *undefined-lit F K* **using** ‹*no-dup ?M*› **unfolding** *M-K* **by** (*simp add*: *defined-lit-map*)
moreover
have *?N* ∪ *?C′* ⊨*ps* {{#}}
proof −
have *A*: *?N* ∪ *?C′* ∪ (λ*a*. {#*lit-of a*#}) ‘ *set ?M* =
?N ∪ (λ*a*. {#*lit-of a*#}) ‘ *set ?M*
unfolding *M-K* **by** *auto*
show *?thesis*
using *true-clss-clss-left-right*[*OF N-C-M*, *of* {{#}}] *N-M-False* **unfolding** *A* **by** *auto*
qed
have *?N* ⊨*p image-mset uminus ?C* + {#−*K*#}
unfolding *true-clss-cls-def true-clss-clss-def total-over-m-def*
proof (*intro allI impI*)
fix *I*
assume
tot: *total-over-set I* (*atms-of-ms* (*?N* ∪ {*image-mset uminus ?C*+ {#− *K*#}})) **and**
cons: *consistent-interp I* **and**
I ⊨*s ?N*
have (*K* ∈ *I* ∧ −*K* ∉ *I*) ∨ (−*K* ∈ *I* ∧ *K* ∉ *I*)
using *cons tot* **unfolding** *consistent-interp-def* **by** (*cases K*) *auto*
have *tot′*: *total-over-set I*
(*atm-of* ‘ *lit-of* ‘ (*set ?M* ∩ {*L*. *is-marked L* ∧ *L* ≠ *Marked K* ()}))

using *tot* **by** (*auto simp add*: *atms-of-uminus-lit-atm-of-lit-of*)
{ fix *x* :: ('*v*, *unit*, *unit*) *marked-lit*
assume
a3: *lit-of* $x \notin I$ **and**
a1: $x \in$ *set ?M* **and**
a4: *is-marked x* **and**
a5: $x \neq$ *Marked K* ()
then have *Pos* (*atm-of* (*lit-of x*)) $\in I \vee$ *Neg* (*atm-of* (*lit-of x*)) $\in I$
using *a5 a4 tot' a1* **unfolding** *total-over-set-def atms-of-s-def* **by** *blast*
moreover have *f6*: *Neg* (*atm-of* (*lit-of x*)) $= -$ *Pos* (*atm-of* (*lit-of x*))
by *simp*
ultimately have $-$ *lit-of* $x \in I$
using *f6 a3* **by** (*metis* (*no-types*) *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set literal.sel*(*1*))
} **note** *H* = *this*

have $\neg I \models s$ *?C'*
using ‹*?N* ∪ *?C'* ⊨*ps* {{#}}› *tot cons* ‹*I* ⊨*s ?N*›
unfolding *true-clss-clss-def total-over-m-def*
by (*simp add*: *atms-of-uminus-lit-atm-of-lit-of atms-of-ms-single-image-atm-of-lit-of*)
then show *I* ⊨ *image-mset uminus ?C* + {#− *K*#}
unfolding *true-clss-def true-cls-def Bex-mset-def*
using ‹($K \in I \wedge -K \notin I$) ∨ ($-K \in I \wedge K \notin I$)›
by (*auto dest*!: *H*)
qed
moreover have *F* ⊨*as CNot* (*image-mset uminus ?C*)
using *nm* **unfolding** *true-annots-def CNot-def M-K* **by** (*auto simp add*: *lits-of-def*)
ultimately have *False*
using *bj-can-jump*[*of S F' K F C* −*K*
image-mset uminus (*image-mset lit-of* {# *L* :# *mset ?M*. *is-marked L* ∧ *L* ≠ *Marked K* ()#})]
‹*C*∈*?N*› *n-s* ‹*?M* ⊨*as CNot C*› *bj-backjump inv* **unfolding** *M-K*
by (*auto simp*: $cdcl_{NOT}$*-merged-bj-learn.simps*)
then show *?thesis* **by** *fast*
qed *auto*
qed

lemma *full-*$cdcl_{NOT}$*-merged-bj-learn-final-state*:
fixes *A* :: '*v literal multiset set* **and** *S T* :: '*st*
assumes
full: *full* $cdcl_{NOT}$*-merged-bj-learn S T* **and**
atms-S: *atms-of-msu* (*clauses S*) ⊆ *atms-of-ms A* **and**
atms-trail: *atm-of* ' *lits-of* (*trail S*) ⊆ *atms-of-ms A* **and**
n-d: *no-dup* (*trail S*) **and**
finite A **and**
inv: *inv S* **and**
decomp: *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
shows *unsatisfiable* (*set-mset* (*clauses T*))
∨ (*trail T* ⊨*asm clauses T* ∧ *satisfiable* (*set-mset* (*clauses T*)))
proof −
have *st*: $cdcl_{NOT}$*-merged-bj-learn*** *S T* **and** *n-s*: *no-step* $cdcl_{NOT}$*-merged-bj-learn T*
using *full* **unfolding** *full-def* **by** *blast+*
then have *st*: $cdcl_{NOT}{}^{**}$ *S T*
using *inv rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-is-rtranclp-*$cdcl_{NOT}$*-and-inv n-d* **by** *auto*
have *atms-of-msu* (*clauses T*) ⊆ *atms-of-ms A* **and** *atm-of* ' *lits-of* (*trail T*) ⊆ *atms-of-ms A*
using $cdcl_{NOT}$*.rtranclp-*$cdcl_{NOT}$*-trail-clauses-bound*[*OF st inv n-d atms-S atms-trail*] **by** *blast+*

moreover have *no-dup* (*trail* T)
  using $cdcl_{NOT}$*.rtranclp-*$cdcl_{NOT}$*-no-dup inv n-d st* **by** *blast*
moreover have *inv* T
  using $cdcl_{NOT}$*.rtranclp-*$cdcl_{NOT}$*-inv inv st* **by** *blast*
moreover have *all-decomposition-implies-m* (*clauses* T) (*get-all-marked-decomposition* (*trail* T))
  using $cdcl_{NOT}$*.rtranclp-*$cdcl_{NOT}$*-all-decomposition-implies inv st decomp n-d* **by** *blast*
ultimately show *?thesis*
  using $cdcl_{NOT}$*-merged-bj-learn-final-state*[*of T A*] ‹*finite A*› *n-s* **by** *fast*
qed

end

14.8.1 Instantiations

locale $cdcl_{NOT}$*-with-backtrack-and-restarts* =
  conflict-driven-clause-learning-learning-before-backjump-only-distinct-learnt trail clauses
    prepend-trail tl-trail add-cls$_{NOT}$ *remove-cls*$_{NOT}$ *propagate-conds inv backjump-conds*
    learn-restrictions forget-restrictions
  for
    trail :: $'st \Rightarrow$ ($'v$::*linorder*, *unit*, *unit*) *marked-lits* **and**
    clauses :: $'st \Rightarrow$ $'v$::*linorder clauses* **and**
    prepend-trail :: ($'v$, *unit*, *unit*) *marked-lit* $\Rightarrow$ $'st \Rightarrow$ $'st$ **and**
    tl-trail :: $'st \Rightarrow$ $'st$ **and**
    add-cls$_{NOT}$ *remove-cls*$_{NOT}$:: $'v$ *clause* $\Rightarrow$ $'st \Rightarrow$ $'st$ **and**
    propagate-conds :: ($'v$, *unit*, *unit*) *marked-lit* $\Rightarrow$ $'st \Rightarrow$ *bool* **and**
    inv :: $'st \Rightarrow$ *bool* **and**
    backjump-conds :: $'v$ *clause* $\Rightarrow$ $'v$ *clause* $\Rightarrow$ $'v$ *literal* $\Rightarrow$ $'st \Rightarrow$ $'st \Rightarrow$ *bool* **and**
    learn-restrictions forget-restrictions :: $'v$::*linorder clause* $\Rightarrow$ $'st \Rightarrow$ *bool*
    +
  fixes *f* :: *nat* $\Rightarrow$ *nat*
  assumes
    unbounded: *unbounded f* **and** *f-ge-1*: $\bigwedge n.\ n \geq 1 \Longrightarrow f\ n \geq 1$ **and**
    inv-restart: $\bigwedge S\ T.$ *inv* $S \Longrightarrow T \sim$ *reduce-trail-to*$_{NOT}$ ([]::$'a$ *list*) $S \Longrightarrow$ *inv* T
begin

lemma *bound-inv-inv*:
  assumes
    inv S **and**
    n-d: *no-dup* (*trail S*) **and**
    atms-clss-S-A: *atms-of-msu* (*clauses S*) $\subseteq$ *atms-of-ms A* **and**
    atms-trail-S-A:*atm-of* ‘ *lits-of* (*trail S*) $\subseteq$ *atms-of-ms A* **and**
    finite A **and**
    $cdcl_{NOT}$: $cdcl_{NOT}$ S T
  shows
    atms-of-msu (*clauses T*) $\subseteq$ *atms-of-ms A* **and**
    atm-of ‘ *lits-of* (*trail T*) $\subseteq$ *atms-of-ms A* **and**
    finite A
proof –
  have $cdcl_{NOT}$ S T
    using ‹*inv S*› $cdcl_{NOT}$ **by** *linarith*
  then have *atms-of-msu* (*clauses T*) $\subseteq$ *atms-of-msu* (*clauses S*) $\cup$ *atm-of* ‘ *lits-of* (*trail S*)
    using ‹*inv S*›
    by (*meson conflict-driven-clause-learning-ops.*$cdcl_{NOT}$*-atms-of-ms-clauses-decreasing*
      conflict-driven-clause-learning-ops-axioms n-d)
  then show *atms-of-msu* (*clauses T*) $\subseteq$ *atms-of-ms A*
    using *atms-clss-S-A atms-trail-S-A* **by** *blast*

next
  show *atm-of ‘ lits-of* (*trail T*) ⊆ *atms-of-ms A*
    by (*meson* ⟨*inv S*⟩ *atms-clss-S-A atms-trail-S-A* $cdcl_{NOT}$ $cdcl_{NOT}$*-atms-in-trail-in-set n-d*)
next
  show *finite A*
    using ⟨*finite A*⟩ **by** *simp*
qed

sublocale $cdcl_{NOT}$*-increasing-restarts-ops* $\lambda S\ T.\ T \sim$ *reduce-trail-to*$_{NOT}$ ([]::′*a list*) *S* $cdcl_{NOT}$ *f*
  $\lambda A\ S.$ *atms-of-msu* (*clauses S*) ⊆ *atms-of-ms A* ∧ *atm-of ‘ lits-of* (*trail S*) ⊆ *atms-of-ms A* ∧
  finite A
  $\mu_{CDCL}{}'$ $\lambda S.$ *inv S* ∧ *no-dup* (*trail S*)
  $\mu_{CDCL}{}'$*-bound*
  apply *unfold-locales*
          apply (*simp add*: *unbounded*)
         using *f-ge-1* **apply** *force*
        using *bound-inv-inv* **apply** *meson*
       apply (*rule* $cdcl_{NOT}$*-decreasing-measure*′; *simp*)
       apply (*rule rtranclp-*$cdcl_{NOT}$*-*$\mu_{CDCL}{}'$*-bound*; *simp*)
      apply (*rule rtranclp-*$\mu_{CDCL}{}'$*-bound-decreasing*; *simp*)
     apply *auto*[]
    apply *auto*[]
   using $cdcl_{NOT}$*-inv* $cdcl_{NOT}$*-no-dup* **apply** *blast*
  using *inv-restart* **apply** *auto*[]
  done

abbreviation $cdcl_{NOT}$*-l* **where**
$cdcl_{NOT}$*-l* ≡
  conflict-driven-clause-learning-ops.$cdcl_{NOT}$ *trail clauses prepend-trail tl-trail add-cls*$_{NOT}$
  remove-cls$_{NOT}$ *propagate-conds* (λ- - - *S T. backjump S T*)
  (λ*C S. distinct-mset C* ∧ ¬ *tautology C* ∧ *learn-restrictions C S*
    ∧ (∃ *F K F′ C′ L. trail S* = *F′* @ *Marked K* () # *F* ∧ *C* = *C′* + {#*L*#}
       ∧ *F* ⊨*as CNot C′* ∧ *C′* + {#*L*#} ∉# *clauses S*))
  (λ*C S.* ¬ (∃ *F′ F K L. trail S* = *F′* @ *Marked K* () # *F* ∧ *F* ⊨*as CNot* (*C* − {#*L*#}))
  ∧ *forget-restrictions C S*)

lemma $cdcl_{NOT}$*-with-restart-*$\mu_{CDCL}{}'$*-le-*$\mu_{CDCL}{}'$*-bound*:
  assumes
    $cdcl_{NOT}$: $cdcl_{NOT}$*-restart* (*T*, *a*) (*V*, *b*) **and**
    $cdcl_{NOT}$*-inv*:
      inv T
      no-dup (*trail T*) **and**
    bound-inv:
      atms-of-msu (*clauses T*) ⊆ *atms-of-ms A*
      atm-of ‘ lits-of (*trail T*) ⊆ *atms-of-ms A*
      finite A
  shows $\mu_{CDCL}{}'$ *A V* ≤ $\mu_{CDCL}{}'$*-bound A T*
  using $cdcl_{NOT}$*-inv bound-inv*
proof (*induction rule*: $cdcl_{NOT}$*-with-restart-induct*[*OF* $cdcl_{NOT}$])
  case (*1 m S T n U*) **note** *U* = *this*(*3*)
  show *?case*
    apply (*rule rtranclp-*$cdcl_{NOT}$*-*$\mu_{CDCL}{}'$*-bound-reduce-trail-to*$_{NOT}$[*of S T*])
        using ⟨($cdcl_{NOT}$ ^^ *m*) *S T*⟩ **apply** (*fastforce dest*!: *relpowp-imp-rtranclp*)
       using *1* **by** *auto*
next

case (*2 S T n*) **note** *full* = *this*(*2*)
show *?case*
apply (*rule rtranclp-cdcl*$_{NOT}$-μ_{CDCL}′*-bound*)
using *full 2* **unfolding** *full1-def* **by** *force+*
qed

lemma *cdcl*$_{NOT}$*-with-restart-*μ_{CDCL}′*-bound-le-*μ_{CDCL}′*-bound*:
assumes
cdcl$_{NOT}$: *cdcl*$_{NOT}$*-restart* (*T*, *a*) (*V*, *b*) **and**
cdcl$_{NOT}$*-inv*:
inv T
no-dup (*trail T*) **and**
bound-inv:
atms-of-msu (*clauses T*) ⊆ *atms-of-ms A*
atm-of ‘ *lits-of* (*trail T*) ⊆ *atms-of-ms A*
finite A
shows μ_{CDCL}′*-bound A V* ≤ μ_{CDCL}′*-bound A T*
using *cdcl*$_{NOT}$*-inv bound-inv*
proof (*induction rule*: *cdcl*$_{NOT}$*-with-restart-induct*[*OF cdcl*$_{NOT}$])
case (*1 m S T n U*) **note** *U* = *this*(*3*)
have μ_{CDCL}′*-bound A T* ≤ μ_{CDCL}′*-bound A S*
apply (*rule rtranclp-*μ_{CDCL}′*-bound-decreasing*)
using ‹(*cdcl*$_{NOT}$ ^^ *m*) *S T*› **apply** (*fastforce dest*: *relpowp-imp-rtranclp*)
using *1* **by** *auto*
then show *?case* **using** *U* **unfolding** μ_{CDCL}′*-bound-def* **by** *auto*
next
case (*2 S T n*) **note** *full* = *this*(*2*)
show *?case*
apply (*rule rtranclp-*μ_{CDCL}′*-bound-decreasing*)
using *full 2* **unfolding** *full1-def* **by** *force+*
qed

sublocale *cdcl*$_{NOT}$*-increasing-restarts - - - - - - f*
λ*S T*. *T* ∼ *reduce-trail-to*$_{NOT}$ ([]::′*a list*) *S*
λ*A S*. *atms-of-msu* (*clauses S*) ⊆ *atms-of-ms A*
∧ *atm-of* ‘ *lits-of* (*trail S*) ⊆ *atms-of-ms A* ∧ *finite A*
μ_{CDCL}′ *cdcl*$_{NOT}$
λ*S*. *inv S* ∧ *no-dup* (*trail S*)
μ_{CDCL}′*-bound*
apply *unfold-locales*
using *cdcl*$_{NOT}$*-with-restart-*μ_{CDCL}′*-le-*μ_{CDCL}′*-bound* **apply** *simp*
using *cdcl*$_{NOT}$*-with-restart-*μ_{CDCL}′*-bound-le-*μ_{CDCL}′*-bound* **apply** *simp*
done

lemma *cdcl*$_{NOT}$*-restart-all-decomposition-implies*:
assumes *cdcl*$_{NOT}$*-restart S T* **and**
inv (*fst S*) **and**
no-dup (*trail* (*fst S*))
all-decomposition-implies-m (*clauses* (*fst S*)) (*get-all-marked-decomposition* (*trail* (*fst S*)))
shows
all-decomposition-implies-m (*clauses* (*fst T*)) (*get-all-marked-decomposition* (*trail* (*fst T*)))
using *assms* **apply** (*induction*)
using *rtranclp-cdcl*$_{NOT}$*-all-decomposition-implies* **by** (*auto dest*!: *tranclp-into-rtranclp*
simp: *full1-def*)

lemma *rtranclp-cdcl$_{NOT}$-restart-all-decomposition-implies*:
  assumes *cdcl$_{NOT}$-restart** S T* **and**
    inv: *inv* (*fst S*) **and**
    n-d: *no-dup* (*trail* (*fst S*)) **and**
    decomp:
      all-decomposition-implies-m (*clauses* (*fst S*)) (*get-all-marked-decomposition* (*trail* (*fst S*)))
  shows
    all-decomposition-implies-m (*clauses* (*fst T*)) (*get-all-marked-decomposition* (*trail* (*fst T*)))
  using *assms*(*1*)
proof (*induction rule*: *rtranclp-induct*)
  case *base*
  then show *?case* **using** *decomp* **by** *simp*
next
  case (*step T u*) **note** *st* = *this*(*1*) **and** *r* = *this*(*2*) **and** *IH* = *this*(*3*)
  have *inv* (*fst T*)
    using *rtranclp-cdcl$_{NOT}$-with-restart-cdcl$_{NOT}$-inv*[*OF st*] *inv n-d* **by** *blast*
  moreover have *no-dup* (*trail* (*fst T*))
    using *rtranclp-cdcl$_{NOT}$-with-restart-cdcl$_{NOT}$-inv*[*OF st*] *inv n-d* **by** *blast*
  ultimately show *?case*
    using *cdcl$_{NOT}$-restart-all-decomposition-implies r IH n-d* **by** *fast*
qed

lemma *cdcl$_{NOT}$-restart-sat-ext-iff*:
  assumes
    st: *cdcl$_{NOT}$-restart S T* **and**
    n-d: *no-dup* (*trail* (*fst S*)) **and**
    inv: *inv* (*fst S*)
  shows *I* $\models$*sextm clauses* (*fst S*) $\longleftrightarrow$ *I* $\models$*sextm clauses*(*fst T*)
  using *assms*
proof (*induction*)
  case (*restart-step m S T n U*)
  then show *?case*
    using *rtranclp-cdcl$_{NOT}$-bj-sat-ext-iff n-d* **by** (*fastforce dest*!: *relpowp-imp-rtranclp*)
next
  case *restart-full*
  then show *?case* **using** *rtranclp-cdcl$_{NOT}$-bj-sat-ext-iff* **unfolding** *full1-def*
  by (*fastforce dest*!: *tranclp-into-rtranclp*)
qed

lemma *rtranclp-cdcl$_{NOT}$-restart-sat-ext-iff*:
  assumes
    st: *cdcl$_{NOT}$-restart** S T* **and**
    n-d: *no-dup* (*trail* (*fst S*)) **and**
    inv: *inv* (*fst S*)
  shows *I* $\models$*sextm clauses* (*fst S*) $\longleftrightarrow$ *I* $\models$*sextm clauses*(*fst T*)
  using *st*
proof (*induction*)
  case *base*
  then show *?case* **by** *simp*
next
  case (*step T U*) **note** *st* = *this*(*1*) **and** *r* = *this*(*2*) **and** *IH* = *this*(*3*)
  have *inv* (*fst T*)
    using *rtranclp-cdcl$_{NOT}$-with-restart-cdcl$_{NOT}$-inv*[*OF st*] *inv n-d* **by** *blast+*
  moreover have *no-dup* (*trail* (*fst T*))
    using *rtranclp-cdcl$_{NOT}$-with-restart-cdcl$_{NOT}$-inv rtranclp-cdcl$_{NOT}$-no-dup st inv n-d* **by** *blast*

ultimately show *?case*
  using $cdcl_{NOT}$*-restart-sat-ext-iff*[*OF r*] *IH* **by** *blast*
qed

theorem *full-*$cdcl_{NOT}$*-restart-backjump-final-state*:
  fixes *A* :: *'v literal multiset set* **and** *S T* :: *'st*
  assumes
    full: *full* $cdcl_{NOT}$*-restart* (*S*, *n*) (*T*, *m*) **and**
    atms-S: *atms-of-msu* (*clauses S*) ⊆ *atms-of-ms A* **and**
    atms-trail: *atm-of* ' *lits-of* (*trail S*) ⊆ *atms-of-ms A* **and**
    n-d: *no-dup* (*trail S*) **and**
    fin-A[*simp*]: *finite A* **and**
    inv: *inv S* **and**
    decomp: *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
  shows *unsatisfiable* (*set-mset* (*clauses S*))
    ∨ (*lits-of* (*trail T*) ⊨*sextm clauses S* ∧ *satisfiable* (*set-mset* (*clauses S*)))
proof −
  have *st*: $cdcl_{NOT}$*-restart*** (*S*, *n*) (*T*, *m*) **and**
    n-s: *no-step* $cdcl_{NOT}$*-restart* (*T*, *m*)
    using *full* **unfolding** *full-def* **by** *fast+*
  have *binv-T*: *atms-of-msu* (*clauses T*) ⊆ *atms-of-ms A atm-of* ' *lits-of* (*trail T*) ⊆ *atms-of-ms A*
    using *rtranclp-*$cdcl_{NOT}$*-with-restart-bound-inv*[*OF st*, *of A*] *inv n-d atms-S atms-trail*
    by *auto*
  moreover have *inv-T*: *no-dup* (*trail T*) *inv T*
    using *rtranclp-*$cdcl_{NOT}$*-with-restart-*$cdcl_{NOT}$*-inv*[*OF st*] *inv n-d* **by** *auto*
  moreover have *all-decomposition-implies-m* (*clauses T*) (*get-all-marked-decomposition* (*trail T*))
    using *rtranclp-*$cdcl_{NOT}$*-restart-all-decomposition-implies*[*OF st*] *inv n-d*
    decomp **by** *auto*
  ultimately have *T*: *unsatisfiable* (*set-mset* (*clauses T*))
    ∨ (*trail T* ⊨*asm clauses T* ∧ *satisfiable* (*set-mset* (*clauses T*)))
    using *no-step-*$cdcl_{NOT}$*-restart-no-step-*$cdcl_{NOT}$[*of* (*T*, *m*) *A*] *n-s*
    $cdcl_{NOT}$*-final-state*[*of T A*] **unfolding** $cdcl_{NOT}$*-NOT-all-inv-def* **by** *auto*
  have *eq-sat-S-T*:⋀*I*. *I* ⊨*sextm clauses S* ⟷ *I* ⊨*sextm clauses T*
    using *rtranclp-*$cdcl_{NOT}$*-restart-sat-ext-iff*[*OF st*] *inv n-d atms-S*
      atms-trail **by** *auto*
  have *cons-T*: *consistent-interp* (*lits-of* (*trail T*))
    using *inv-T*(*1*) *distinctconsistent-interp* **by** *blast*
  consider
      (*unsat*) *unsatisfiable* (*set-mset* (*clauses T*))
    | (*sat*) *trail T* ⊨*asm clauses T* **and** *satisfiable* (*set-mset* (*clauses T*))
    using *T* **by** *blast*
  then show *?thesis*
    proof *cases*
      case *unsat*
      then have *unsatisfiable* (*set-mset* (*clauses S*))
        using *eq-sat-S-T consistent-true-clss-ext-satisfiable true-clss-imp-true-cls-ext*
        unfolding *satisfiable-def* **by** *blast*
      then show *?thesis* **by** *fast*
    next
      case *sat*
      then have *lits-of* (*trail T*) ⊨*sextm clauses S*
        using *rtranclp-*$cdcl_{NOT}$*-restart-sat-ext-iff*[*OF st*] *inv n-d atms-S*
        atms-trail **by** (*auto simp*: *true-clss-imp-true-cls-ext true-annots-true-cls*)
      moreover then have *satisfiable* (*set-mset* (*clauses S*))
          using *cons-T consistent-true-clss-ext-satisfiable* **by** *blast*

ultimately show *?thesis* **by** *blast*
qed
qed
end — end of $cdcl_{NOT}$*-with-backtrack-and-restarts* locale

locale *most-general-*$cdcl_{NOT}$ =
dpll-state trail clauses prepend-trail tl-trail add-cls_{NOT} *remove-*cls_{NOT} +
propagate-ops trail clauses prepend-trail tl-trail add-cls_{NOT} *remove-*cls_{NOT} *propagate-conds* +
backjumping-ops trail clauses prepend-trail tl-trail add-cls_{NOT} *remove-*cls_{NOT} λ- - - - -. *True*
for
trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
clauses :: *'st* ⇒ *'v clauses* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-cls_{NOT} *remove-*cls_{NOT}:: *'v clause* ⇒ *'st* ⇒ *'st* **and**
propagate-conds :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *bool* **and**
inv :: *'st* ⇒ *bool*
begin
lemma *backjump-bj-can-jump*:
assumes
tr-S: *trail S* = *F'* @ *Marked K* () # *F* **and**
C: *C* ∈# *clauses S* **and**
tr-S-C: *trail S* ⊨*as CNot C* **and**
undef: *undefined-lit F L* **and**
atm-L: *atm-of L* ∈ *atms-of-msu* (*clauses S*) ∪ *atm-of* ‘ (*lits-of* (*F'* @ *Marked K* () # *F*)) **and**
cls-S-C': *clauses S* ⊨*pm C'* + {#*L*#} **and**
F-C': *F* ⊨*as CNot C'*
shows ¬*no-step backjump S*
using *backjump.intros*[*OF tr-S - C tr-S-C undef - cls-S-C' F-C'*,
of prepend-trail (*Propagated L* -) (*reduce-trail-to*$_{NOT}$ *F S*)] *atm-L* **unfolding** *tr-S*
by (*auto simp*: *state-eq*$_{NOT}$*-def simp del*: *state-simp*$_{NOT}$)

sublocale *dpll-with-backjumping-ops* - - - - - - - *inv* λ- - - - -. *True*
using *backjump-bj-can-jump* **by** *unfold-locales auto*
end

The restart does only reset the trail, contrary to Weidenbach's version. But there is a forget rule.

locale $cdcl_{NOT}$*-merge-bj-learn-with-backtrack-restarts* =
$cdcl_{NOT}$*-merge-bj-learn trail clauses prepend-trail tl-trail add-*cls_{NOT} *remove-*cls_{NOT}
propagate-conds inv forget-conds
λ*C C' L' S*. *distinct-mset* (*C'* + {#*L'*#}) ∧ *backjump-l-cond C C' L' S*
for
trail :: *'st* ⇒ (*'v*::*linorder*, *unit*, *unit*) *marked-lits* **and**
clauses :: *'st* ⇒ *'v*::*linorder clauses* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-cls_{NOT} *remove-*cls_{NOT}:: *'v clause* ⇒ *'st* ⇒ *'st* **and**
propagate-conds :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *bool* **and**
inv :: *'st* ⇒ *bool* **and**
forget-conds :: *'v clause* ⇒ *'st* ⇒ *bool* **and**
backjump-l-cond :: *'v clause* ⇒ *'v clause* ⇒ *'v literal* ⇒ *'st* ⇒ *bool*
+
fixes *f* :: *nat* ⇒ *nat*
assumes

unbounded: *unbounded f* **and** *f-ge-1*: $\bigwedge n.\ n \geq 1 \Longrightarrow f\ n \geq 1$ **and**
inv-restart: $\bigwedge S\ T.\ inv\ S \Longrightarrow T \sim$ *reduce-trail-to*$_{NOT}$ [] $S \Longrightarrow inv\ T$
begin

interpretation *cdcl*$_{NOT}$:
conflict-driven-clause-learning-ops trail clauses prepend-trail tl-trail add-cls$_{NOT}$ *remove-cls*$_{NOT}$
propagate-conds inv backjump-conds (λC -. *distinct-mset* $C \wedge \neg$ *tautology* C) *forget-conds*
by *unfold-locales*

interpretation *cdcl*$_{NOT}$:
conflict-driven-clause-learning trail clauses prepend-trail tl-trail add-cls$_{NOT}$ *remove-cls*$_{NOT}$
propagate-conds inv backjump-conds (λC -. *distinct-mset* $C \wedge \neg$ *tautology* C) *forget-conds*
apply *unfold-locales*
using *cdcl*$_{NOT}$*-merged-bj-learn-forget*$_{NOT}$ *cdcl-merged-inv learn-inv*
by (*auto simp add*: *cdcl*$_{NOT}$*.simps dpll-bj-inv*)

definition *not-simplified-cls* $A = \{\#C \in\# A.$ *tautology* $C \vee \neg$*distinct-mset* $C\#\}$

lemma *build-all-simple-clss-or-not-simplified-cls*:
assumes *atms-of-msu* (*clauses* S) $\subseteq$ *atms-of-ms* A **and**
$x \in\#$ *clauses* S **and** *finite* A
shows $x \in$ *build-all-simple-clss* (*atms-of-ms* A) $\vee$ $x \in\#$ *not-simplified-cls* (*clauses* S)
proof –
consider
(*simpl*) $\neg$*tautology* x **and** *distinct-mset* x
| (*n-simp*) *tautology* $x \vee \neg$*distinct-mset* x
by *auto*
then show *?thesis*
proof *cases*
case *simpl*
then have $x \in$ *build-all-simple-clss* (*atms-of-ms* A)
by (*meson assms atms-of-atms-of-ms-mono atms-of-ms-finite build-all-simple-clss-mono*
distinct-mset-not-tautology-implies-in-build-all-simple-clss finite-subset
mem-set-mset-iff subsetCE)
then show *?thesis* **by** *blast*
next
case *n-simp*
then have $x \in\#$ *not-simplified-cls* (*clauses* S)
using ‹$x \in\#$ *clauses* S› **unfolding** *not-simplified-cls-def* **by** *auto*
then show *?thesis* **by** *blast*
qed
qed

lemma *cdcl*$_{NOT}$*-merged-bj-learn-clauses-bound*:
assumes
cdcl$_{NOT}$*-merged-bj-learn* S T **and**
inv: *inv* S **and**
atms-clss: *atms-of-msu* (*clauses* S) $\subseteq$ *atms-of-ms* A **and**
atms-trail: *atm-of* ‘(*lits-of* (*trail* S)) $\subseteq$ *atms-of-ms* A **and**
n-d: *no-dup* (*trail* S) **and**
fin-A[*simp*]: *finite* A
shows *set-mset* (*clauses* T) $\subseteq$ *set-mset* (*not-simplified-cls* (*clauses* S))
$\cup$ *build-all-simple-clss* (*atms-of-ms* A)

using *assms*
proof (*induction rule*: $cdcl_{NOT}$*-merged-bj-learn.induct*)
case $cdcl_{NOT}$*-merged-bj-learn-decide*$_{NOT}$
then show *?case* **using** *dpll-bj-clauses* **by** (*force dest*!: *build-all-simple-clss-or-not-simplified-cls*)
next
case $cdcl_{NOT}$*-merged-bj-learn-propagate*$_{NOT}$
then show *?case* **using** *dpll-bj-clauses* **by** (*force dest*!: *build-all-simple-clss-or-not-simplified-cls*)
next
case $cdcl_{NOT}$*-merged-bj-learn-forget*$_{NOT}$
then show *?case* **using** *clauses-remove-cls*$_{NOT}$ **unfolding** *state-eq*$_{NOT}$*-def*
by (*force elim*!: *forgetE dest*: *build-all-simple-clss-or-not-simplified-cls*)
next
case ($cdcl_{NOT}$*-merged-bj-learn-backjump-l T*) **note** *bj* = *this*(*1*) **and** *inv* = *this*(*2*) **and** *atms-clss* = *this*(*3*) **and** *atms-trail* = *this*(*4*) **and** *n-d* = *this*(*5*)

have $cdcl_{NOT}{}^{**}$ *S T*
apply (*rule rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-is-rtranclp-cdcl*$_{NOT}$)
using ‹*backjump-l S T*› *inv* $cdcl_{NOT}$*-merged-bj-learn.simps n-d* **by** *blast+*
have *atm-of* ‘(*lits-of* (*trail T*)) ⊆ *atms-of-ms A*
using $cdcl_{NOT}$*.rtranclp-cdcl*$_{NOT}$*-trail-clauses-bound*[*OF* ‹$cdcl_{NOT}{}^{**}$ *S T*›] *inv atms-trail atms-clss n-d* **by** *auto*
have *atms-of-msu* (*clauses T*) ⊆ *atms-of-ms A*
using $cdcl_{NOT}$*.rtranclp-cdcl*$_{NOT}$*-trail-clauses-bound*[*OF* ‹$cdcl_{NOT}{}^{**}$ *S T*› *inv n-d atms-clss atms-trail*]
by *fast*
moreover have *no-dup* (*trail T*)
using $cdcl_{NOT}$*.rtranclp-cdcl*$_{NOT}$*-no-dup*[*OF* ‹$cdcl_{NOT}{}^{**}$ *S T*› *inv n-d*] **by** *fast*

obtain *F′ K F L l C′ C* **where**
tr-S: *trail S* = *F′* @ *Marked K* () # *F* **and**
T: *T* ∼ *prepend-trail* (*Propagated L l*) (*reduce-trail-to*$_{NOT}$ *F* (*add-cls*$_{NOT}$ (*C′* + {#*L*#}) *S*)) **and**
C ∈# *clauses S* **and**
trail S ⊨*as CNot C* **and**
undef: *undefined-lit F L* **and**
atm-of L = *atm-of K* ∨ *atm-of L* ∈ *atms-of-msu* (*clauses S*)
∨ *atm-of L* ∈ *atm-of* ‘ (*lits-of F′* ∪ *lits-of F*) **and**
clauses S ⊨*pm C′* + {#*L*#} **and**
F ⊨*as CNot C′* **and**
dist: *distinct-mset* (*C′* + {#*L*#}) **and**
tauto: ¬ *tautology* (*C′* + {#*L*#}) **and**
backjump-l-cond C C′ L T
using ‹*backjump-l S T*› **apply** (*induction rule*: *backjump-l.induct*) **by** *auto*

have *atms-of C′* ⊆ *atm-of* ‘ (*lits-of F*)
using ‹*F* ⊨*as CNot C′*› **by** (*simp add*: *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set atms-of-def image-subset-iff in-CNot-implies-uminus*(*2*))
then have *atms-of* (*C′*+{#*L*#}) ⊆ *atms-of-ms A*
using *T* ‹*atm-of* ‘ *lits-of* (*trail T*) ⊆ *atms-of-ms A*› *tr-S undef n-d* **by** *auto*
then have *build-all-simple-clss* (*atms-of* (*C′* + {#*L*#})) ⊆ *build-all-simple-clss* (*atms-of-ms A*)
apply − **by** (*rule build-all-simple-clss-mono*) (*simp-all*)
then have *C′* + {#*L*#} ∈ *build-all-simple-clss* (*atms-of-ms A*)
using *distinct-mset-not-tautology-implies-in-build-all-simple-clss*[*OF dist tauto*]
by *auto*
then show *?case*
using *T inv atms-clss undef tr-S n-d*
by (*force dest*!: *build-all-simple-clss-or-not-simplified-cls*)

qed

lemma $cdcl_{NOT}$*-merged-bj-learn-not-simplified-decreasing*:
assumes $cdcl_{NOT}$*-merged-bj-learn* S T
shows (*not-simplified-cls* (*clauses* T)) ⊆# (*not-simplified-cls* (*clauses* S))
using *assms* **apply** *induction*
prefer *4*
unfolding *not-simplified-cls-def* **apply** (*auto elim*!: *backjump-lE forgetE*)[*3*]
by (*elim backjump-lE*) *auto*

lemma *rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-not-simplified-decreasing*:
assumes $cdcl_{NOT}$*-merged-bj-learn*** S T
shows (*not-simplified-cls* (*clauses* T)) ⊆# (*not-simplified-cls* (*clauses* S))
using *assms* **apply** *induction*
apply *simp*
by (*drule* $cdcl_{NOT}$*-merged-bj-learn-not-simplified-decreasing*) *auto*

lemma *rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-clauses-bound*:
assumes
$cdcl_{NOT}$*-merged-bj-learn*** S T **and**
inv S **and**
atms-of-msu (*clauses* S) ⊆ *atms-of-ms* A **and**
atm-of ‘(*lits-of* (*trail* S)) ⊆ *atms-of-ms* A **and**
n-d: *no-dup* (*trail* S) **and**
finite[*simp*]: *finite* A
shows *set-mset* (*clauses* T) ⊆ *set-mset* (*not-simplified-cls* (*clauses* S))
∪ *build-all-simple-clss* (*atms-of-ms* A)
using *assms*(*1*−*5*)
proof *induction*
case *base*
then show *?case* **by** (*auto dest*!: *build-all-simple-clss-or-not-simplified-cls*)
next
case (*step* T U) **note** *st* = *this*(*1*) **and** $cdcl_{NOT}$ = *this*(*2*) **and** *IH* = *this*(*3*)[*OF this*(*4*−*7*)] **and**
inv = *this*(*4*) **and** *atms-clss-S* = *this*(*5*) **and** *atms-trail-S* = *this*(*6*) **and** *finite-cls-S* = *this*(*7*)
have *st'*: $cdcl_{NOT}{}^{**}$ S T
using *inv rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-is-rtranclp-*$cdcl_{NOT}$*-and-inv st n-d* **by** *blast*
have *inv* T
using *inv rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-inv st n-d* **by** *blast*
moreover
have *atms-of-msu* (*clauses* T) ⊆ *atms-of-ms* A **and**
atm-of ‘ *lits-of* (*trail* T) ⊆ *atms-of-ms* A
using $cdcl_{NOT}$*.rtranclp-*$cdcl_{NOT}$*-trail-clauses-bound*[*OF st'*] *inv atms-clss-S atms-trail-S n-d*
by *blast+*
moreover moreover have *no-dup* (*trail* T)
using $cdcl_{NOT}$*.rtranclp-*$cdcl_{NOT}$*-no-dup*[*OF* ‹$cdcl_{NOT}{}^{**}$ S T› *inv n-d*] **by** *fast*
ultimately have *set-mset* (*clauses* U)
⊆ *set-mset* (*not-simplified-cls* (*clauses* T)) ∪ *build-all-simple-clss* (*atms-of-ms* A)
using $cdcl_{NOT}$ *finite* $cdcl_{NOT}$*-merged-bj-learn-clauses-bound*
by (*auto intro*!: $cdcl_{NOT}$*-merged-bj-learn-clauses-bound*)
moreover have *set-mset* (*not-simplified-cls* (*clauses* T))
⊆ *set-mset* (*not-simplified-cls* (*clauses* S))
using *rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-not-simplified-decreasing*[*OF st*] **by** *auto*
ultimately show *?case* **using** *IH inv atms-clss-S*
by (*auto dest*!: *build-all-simple-clss-or-not-simplified-cls*)
qed

abbreviation $\mu_{CDCL}{}'$*-bound* **where**
$\mu_{CDCL}{}'$*-bound A T* == ((*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*))) ∗ *2*
  + *card* (*set-mset* (*not-simplified-cls*(*clauses T*)))
  + *3* ^ *card* (*atms-of-ms A*)

lemma *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-clauses-bound-card*:
  assumes
    cdcl$_{NOT}$*-merged-bj-learn*** *S T* **and**
    inv S **and**
    atms-of-msu (*clauses S*) ⊆ *atms-of-ms A* **and**
    atm-of ‘(*lits-of* (*trail S*)) ⊆ *atms-of-ms A* **and**
    n-d: *no-dup* (*trail S*) **and**
    finite: *finite A*
  shows $\mu_{CDCL}{}'$*-merged A T* ≤ $\mu_{CDCL}{}'$*-bound A S*
proof −
  have *set-mset* (*clauses T*) ⊆ *set-mset* (*not-simplified-cls*(*clauses S*))
    ∪ *build-all-simple-clss* (*atms-of-ms A*)
    using *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-clauses-bound*[*OF assms*] **.**
  moreover have *card* (*set-mset* (*not-simplified-cls*(*clauses S*))
      ∪ *build-all-simple-clss* (*atms-of-ms A*))
    ≤ *card* (*set-mset* (*not-simplified-cls*(*clauses S*))) + *3* ^ *card* (*atms-of-ms A*)
    by (*meson Nat.le-trans atms-of-ms-finite build-all-simple-clss-card card-Un-le finite*
      nat-add-left-cancel-le)
  ultimately have *card* (*set-mset* (*clauses T*))
    ≤ *card* (*set-mset* (*not-simplified-cls*(*clauses S*))) + *3* ^ *card* (*atms-of-ms A*)
    by (*meson build-all-simple-clss-finite card-mono dual-order.trans finite-UnI finite-set-mset*)
  moreover have ((*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*)) − $\mu_C{}'$ *A T*) ∗ *2*
    ≤ (*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*)) ∗ *2*
    by *auto*
  ultimately show *?thesis* **unfolding** $\mu_{CDCL}{}'$*-merged-def* **by** *auto*
qed

sublocale *cdcl*$_{NOT}$*-increasing-restarts-ops* λ*S T*. *T* ∼ *reduce-trail-to*$_{NOT}$ ([]::′*a list*) *S*
   cdcl$_{NOT}$*-merged-bj-learn f*
   λ*A S*. *atms-of-msu* (*clauses S*) ⊆ *atms-of-ms A*
     ∧ *atm-of* ‘ *lits-of* (*trail S*) ⊆ *atms-of-ms A* ∧ *finite A*
   $\mu_{CDCL}{}'$*-merged*
   λ*S*. *inv S* ∧ *no-dup* (*trail S*)
   $\mu_{CDCL}{}'$*-bound*
   apply *unfold-locales*
             using *unbounded* **apply** *simp*
            using *f-ge-1* **apply** *force*
           apply (*blast dest*!: *cdcl*$_{NOT}$*-merged-bj-learn-is-tranclp-cdcl*$_{NOT}$ *tranclp-into-rtranclp*
             cdcl$_{NOT}$*.rtranclp-cdcl*$_{NOT}$*-trail-clauses-bound*)
          apply (*simp add*: *cdcl*$_{NOT}$*-decreasing-measure*′)
         using *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-clauses-bound-card* **apply** *blast*
         apply (*drule rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-not-simplified-decreasing*)
         apply (*auto dest*!: *simp*: *card-mono set-mset-mono*)[]
      apply *simp*
     apply *auto*[]
    using *cdcl*$_{NOT}$*-merged-bj-learn-no-dup-inv cdcl-merged-inv* **apply** *blast*
   apply (*auto simp*: *inv-restart*)[]
   done

lemma $cdcl_{NOT}$*-restart-*$\mu_{CDCL}{}'$*-merged-le-*$\mu_{CDCL}{}'$*-bound*:
assumes
$cdcl_{NOT}$*-restart T V*
inv (*fst T*) **and**
no-dup (*trail* (*fst T*)) **and**
atms-of-msu (*clauses* (*fst T*)) $\subseteq$ *atms-of-ms A* **and**
atm-of ‘ *lits-of* (*trail* (*fst T*)) $\subseteq$ *atms-of-ms A* **and**
finite A
shows $\mu_{CDCL}{}'$*-merged A* (*fst V*) $\leq$ $\mu_{CDCL}{}'$*-bound A* (*fst T*)
using *assms*
proof *induction*
case (*restart-full S T n*)
show *?case*
unfolding *fst-conv*
apply (*rule rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-clauses-bound-card*)
using *restart-full* **unfolding** *full1-def* **by** (*force dest*!: *tranclp-into-rtranclp*)+
next
case (*restart-step m S T n U*) **note** *st* = *this*(*1*) **and** *U* = *this*(*3*) **and** *inv* = *this*(*4*) **and** *n-d* = *this*(*5*) **and** *atms-clss* = *this*(*6*) **and** *atms-trail* = *this*(*7*) **and** *finite* = *this*(*8*)
then have *st'*: $cdcl_{NOT}$*-merged-bj-learn*** *S T*
by (*blast dest*: *relpowp-imp-rtranclp*)
then have *st''*: $cdcl_{NOT}{}^{**}$ *S T*
using *inv n-d* **apply** − **by** (*rule rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-is-rtranclp-*$cdcl_{NOT}$) *auto*
have *inv T*
apply (*rule rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-inv*)
using *inv st' n-d* **by** *auto*
then have *inv U*
using *U* **by** (*auto simp*: *inv-restart*)
have *atms-of-msu* (*clauses T*) $\subseteq$ *atms-of-ms A*
using $cdcl_{NOT}$*.rtranclp-*$cdcl_{NOT}$*-trail-clauses-bound*[*OF st''*] *inv atms-clss atms-trail n-d*
by *simp*
then have *atms-of-msu* (*clauses U*) $\subseteq$ *atms-of-ms A*
using *U* **by** *simp*
have *not-simplified-cls* (*clauses U*) $\subseteq$# *not-simplified-cls* (*clauses T*)
using ‹*U* $\sim$ *reduce-trail-to*$_{NOT}$ [] *T*› **by** *auto*
moreover have *not-simplified-cls* (*clauses T*) $\subseteq$# *not-simplified-cls* (*clauses S*)
apply (*rule rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-not-simplified-decreasing*)
using ‹($cdcl_{NOT}$*-merged-bj-learn* ^^ *m*) *S T*› **by** (*auto dest*!: *relpowp-imp-rtranclp*)
ultimately have *U-S*: *not-simplified-cls* (*clauses U*) $\subseteq$# *not-simplified-cls* (*clauses S*)
by *auto*

have (*set-mset* (*clauses U*))
$\subseteq$ *set-mset* (*not-simplified-cls* (*clauses U*)) $\cup$ *build-all-simple-clss* (*atms-of-ms A*)
apply (*rule rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-clauses-bound*)
apply *simp*
using ‹*inv U*› **apply** *simp*
using ‹*atms-of-msu* (*clauses U*) $\subseteq$ *atms-of-ms A*› **apply** *simp*
using *U* **apply** *simp*
using *U* **apply** *simp*
using *finite* **apply** *simp*
done
then have *f1*: *card* (*set-mset* (*clauses U*)) $\leq$ *card* (*set-mset* (*not-simplified-cls* (*clauses U*)) $\cup$ *build-all-simple-clss* (*atms-of-ms A*))
by (*meson build-all-simple-clss-finite card-mono finite-UnI finite-set-mset*)

moreover have *set-mset* (*not-simplified-cls* (*clauses* U)) ∪ *build-all-simple-clss* (*atms-of-ms* A)
⊆ *set-mset* (*not-simplified-cls* (*clauses* S)) ∪ *build-all-simple-clss* (*atms-of-ms* A)
using *U-S* **by** *auto*
then have *f2*:
card (*set-mset* (*not-simplified-cls* (*clauses* U)) ∪ *build-all-simple-clss* (*atms-of-ms* A))
≤ *card* (*set-mset* (*not-simplified-cls* (*clauses* S)) ∪ *build-all-simple-clss* (*atms-of-ms* A))
by (*meson build-all-simple-clss-finite card-mono finite-UnI finite-set-mset*)

moreover have *card* (*set-mset* (*not-simplified-cls* (*clauses* S))
∪ *build-all-simple-clss* (*atms-of-ms* A))
≤ *card* (*set-mset* (*not-simplified-cls* (*clauses* S))) + *card* (*build-all-simple-clss* (*atms-of-ms* A))
using *card-Un-le* **by** *blast*
moreover have *card* (*build-all-simple-clss* (*atms-of-ms* A)) ≤ *3* ^ *card* (*atms-of-ms* A)
using *atms-of-ms-finite build-all-simple-clss-card local.finite* **by** *blast*
ultimately have *card* (*set-mset* (*clauses* U))
≤ *card* (*set-mset* (*not-simplified-cls* (*clauses* S))) + *3* ^ *card* (*atms-of-ms* A)
by *linarith*
then show *?case* **unfolding** μ_{CDCL}*'-merged-def* **by** *auto*
qed

lemma *cdcl*$_{NOT}$*-restart-*μ_{CDCL}*'-bound-le-*μ_{CDCL}*'-bound*:
assumes
cdcl$_{NOT}$*-restart* T V **and**
no-dup (*trail* (*fst* T)) **and**
inv (*fst* T) **and**
fin: *finite* A
shows μ_{CDCL}*'-bound* A (*fst* V) ≤ μ_{CDCL}*'-bound* A (*fst* T)
using *assms*(*1*−*3*)
proof *induction*
case (*restart-full* S T n)
have *not-simplified-cls* (*clauses* T) ⊆# *not-simplified-cls* (*clauses* S)
apply (*rule rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-not-simplified-decreasing*)
using ‹*full1 cdcl*$_{NOT}$*-merged-bj-learn* S T› **unfolding** *full1-def*
by (*auto dest*: *tranclp-into-rtranclp*)
then show *?case* **by** (*auto simp*: *card-mono set-mset-mono*)
next
case (*restart-step* m S T n U) **note** *st* = *this*(*1*) **and** U = *this*(*3*) **and** *n-d* =*this*(*4*) **and** *inv* = *this*(*5*)
then have *st'*: *cdcl*$_{NOT}$*-merged-bj-learn*** S T
by (*blast dest*: *relpowp-imp-rtranclp*)
then have *st''*: *cdcl*$_{NOT}$** S T
using *inv n-d* **apply** − **by** (*rule rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-is-rtranclp-cdcl*$_{NOT}$) *auto*
have *inv* T
apply (*rule rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-inv*)
using *inv st' n-d* **by** *auto*
then have *inv* U
using U **by** (*auto simp*: *inv-restart*)
have *not-simplified-cls* (*clauses* U) ⊆# *not-simplified-cls* (*clauses* T)
using ‹U ∼ *reduce-trail-to*$_{NOT}$ [] T› **by** *auto*
moreover have *not-simplified-cls* (*clauses* T) ⊆# *not-simplified-cls* (*clauses* S)
apply (*rule rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-not-simplified-decreasing*)
using ‹(*cdcl*$_{NOT}$*-merged-bj-learn* ^^ m) S T› **by** (*auto dest*!: *relpowp-imp-rtranclp*)
ultimately have *U-S*: *not-simplified-cls* (*clauses* U) ⊆# *not-simplified-cls* (*clauses* S)
by *auto*
then show *?case* **by** (*auto simp*: *card-mono set-mset-mono*)

qed

sublocale *cdcl*$_{NOT}$*-increasing-restarts - - - - - - f* $\lambda S\ T.\ T \sim$ *reduce-trail-to*$_{NOT}$ ([]::'*a list*) *S*
  $\lambda A\ S.$ *atms-of-msu* (*clauses S*) $\subseteq$ *atms-of-ms A*
   $\wedge$ *atm-of* ' *lits-of* (*trail S*) $\subseteq$ *atms-of-ms A* $\wedge$ *finite A*
  μ_{CDCL}'*-merged cdcl*$_{NOT}$*-merged-bj-learn*
   $\lambda S.$ *inv S* $\wedge$ *no-dup* (*trail S*)
  $\lambda A\ T.$ ((*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*))) $*$ *2*
   + *card* (*set-mset* (*not-simplified-cls*(*clauses T*)))
   + *3* ^ *card* (*atms-of-ms A*)
  apply *unfold-locales*
   using *cdcl*$_{NOT}$*-restart-*μ_{CDCL}'*-merged-le-*μ_{CDCL}'*-bound* **apply** *force*
  using *cdcl*$_{NOT}$*-restart-*μ_{CDCL}'*-bound-le-*μ_{CDCL}'*-bound* **by** *fastforce*

lemma *cdcl*$_{NOT}$*-restart-eq-sat-iff*:
 assumes
  cdcl$_{NOT}$*-restart S T* **and**
  no-dup (*trail* (*fst S*))
  inv (*fst S*)
 shows *I*$\models$*sextm clauses* (*fst S*) $\longleftrightarrow$ *I* $\models$*sextm clauses* (*fst T*)
 using *assms*
proof (*induction rule*: *cdcl*$_{NOT}$*-restart.induct*)
 case (*restart-full S T n*)
 then have *cdcl*$_{NOT}$*-merged-bj-learn*** *S T*
  by (*simp add*: *tranclp-into-rtranclp full1-def*)
 then show *?case*
  using *cdcl*$_{NOT}$*.rtranclp-cdcl*$_{NOT}$*-bj-sat-ext-iff restart-full.prems*(*1*,*2*)
  rtranclp-cdcl$_{NOT}$*-merged-bj-learn-is-rtranclp-cdcl*$_{NOT}$ **by** *auto*
next
 case (*restart-step m S T n U*)
 then have *cdcl*$_{NOT}$*-merged-bj-learn*** *S T*
  by (*auto simp*: *tranclp-into-rtranclp full1-def dest*!: *relpowp-imp-rtranclp*)
 then have *I* $\models$*sextm clauses S* $\longleftrightarrow$ *I* $\models$*sextm clauses T*
  using *cdcl*$_{NOT}$*.rtranclp-cdcl*$_{NOT}$*-bj-sat-ext-iff restart-step.prems*(*1*,*2*)
  rtranclp-cdcl$_{NOT}$*-merged-bj-learn-is-rtranclp-cdcl*$_{NOT}$ **by** *auto*
 moreover have *I* $\models$*sextm clauses T* $\longleftrightarrow$ *I* $\models$*sextm clauses U*
  using *restart-step.hyps*(*3*) **by** *auto*
 ultimately show *?case* **by** *auto*
qed

lemma *rtranclp-cdcl*$_{NOT}$*-restart-eq-sat-iff*:
 assumes
  cdcl$_{NOT}$*-restart*** *S T* **and**
  inv: *inv* (*fst S*) **and** *n-d*: *no-dup*(*trail* (*fst S*))
 shows *I*$\models$*sextm clauses* (*fst S*) $\longleftrightarrow$ *I* $\models$*sextm clauses* (*fst T*)
 using *assms*(*1*)
proof (*induction rule*: *rtranclp-induct*)
 case *base*
 then show *?case* **by** *simp*
next
 case (*step T U*) **note** *st* = *this*(*1*) **and** *cdcl* = *this*(*2*) **and** *IH* = *this*(*3*)
 have *inv* (*fst T*) **and** *no-dup* (*trail* (*fst T*))
  using *rtranclp-cdcl*$_{NOT}$*-with-restart-cdcl*$_{NOT}$*-inv* **using** *st inv n-d* **by** *blast*+
 then have *I*$\models$*sextm clauses* (*fst T*) $\longleftrightarrow$ *I* $\models$*sextm clauses* (*fst U*)

using $cdcl_{NOT}$*-restart-eq-sat-iff cdcl* **by** *blast*
then show *?case* **using** *IH* **by** *blast*
qed

lemma $cdcl_{NOT}$*-restart-all-decomposition-implies-m*:
assumes
$cdcl_{NOT}$*-restart S T* **and**
inv: *inv* (*fst S*) **and** *n-d*: *no-dup*(*trail* (*fst S*)) **and**
all-decomposition-implies-m (*clauses* (*fst S*))
(*get-all-marked-decomposition* (*trail* (*fst S*)))
shows *all-decomposition-implies-m* (*clauses* (*fst T*))
(*get-all-marked-decomposition* (*trail* (*fst T*)))
using *assms*
proof (*induction*)
case (*restart-full S T n*) **note** *full* = *this*(*1*) **and** *inv* = *this*(*2*) **and** *n-d* = *this*(*3*) **and**
decomp = *this*(*4*)
have *st*: $cdcl_{NOT}$*-merged-bj-learn*** *S T* **and**
n-s: *no-step* $cdcl_{NOT}$*-merged-bj-learn T*
using *full* **unfolding** *full1-def* **by** (*fast dest*: *tranclp-into-rtranclp*)+
have *st'*: $cdcl_{NOT}{}^{**}$ *S T*
using *inv rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-is-rtranclp-*$cdcl_{NOT}$*-and-inv st n-d* **by** *auto*
have *inv T*
using *rtranclp-*$cdcl_{NOT}$*-*$cdcl_{NOT}$*-inv*[*OF st*] *inv n-d* **by** *auto*
then show *?case*
using $cdcl_{NOT}$*.rtranclp-*$cdcl_{NOT}$*-all-decomposition-implies*[*OF - - n-d decomp*] *st' inv* **by** *auto*
next
case (*restart-step m S T n U*) **note** *st* = *this*(*1*) **and** *U* = *this*(*3*) **and** *inv* = *this*(*4*) **and**
n-d = *this*(*5*) **and** *decomp* = *this*(*6*)
show *?case* **using** *U* **by** *auto*
qed

lemma *rtranclp-*$cdcl_{NOT}$*-restart-all-decomposition-implies-m*:
assumes
$cdcl_{NOT}$*-restart*** *S T* **and**
inv: *inv* (*fst S*) **and** *n-d*: *no-dup*(*trail* (*fst S*)) **and**
decomp: *all-decomposition-implies-m* (*clauses* (*fst S*))
(*get-all-marked-decomposition* (*trail* (*fst S*)))
shows *all-decomposition-implies-m* (*clauses* (*fst T*))
(*get-all-marked-decomposition* (*trail* (*fst T*)))
using *assms*
proof (*induction*)
case *base*
then show *?case* **using** *decomp* **by** *simp*
next
case (*step T U*) **note** *st* = *this*(*1*) **and** *cdcl* = *this*(*2*) **and** *IH* = *this*(*3*)[*OF this*(*4*−)] **and**
inv = *this*(*4*) **and** *n-d* = *this*(*5*) **and** *decomp* = *this*(*6*)
have *inv* (*fst T*) **and** *no-dup* (*trail* (*fst T*))
using *rtranclp-*$cdcl_{NOT}$*-with-restart-*$cdcl_{NOT}$*-inv* **using** *st inv n-d* **by** *blast*+
then show *?case*
using $cdcl_{NOT}$*-restart-all-decomposition-implies-m*[*OF cdcl*] *IH* **by** *auto*
qed

lemma *full-*$cdcl_{NOT}$*-restart-normal-form*:
assumes
full: *full* $cdcl_{NOT}$*-restart S T* **and**

inv: *inv* (*fst S*) **and** *n-d*: *no-dup*(*trail* (*fst S*)) **and**
decomp: *all-decomposition-implies-m* (*clauses* (*fst S*))
(*get-all-marked-decomposition* (*trail* (*fst S*))) **and**
atms-cls: *atms-of-msu* (*clauses* (*fst S*)) ⊆ *atms-of-ms A* **and**
atms-trail: *atm-of* ' *lits-of* (*trail* (*fst S*)) ⊆ *atms-of-ms A* **and**
fin: *finite A*
shows *unsatisfiable* (*set-mset* (*clauses* (*fst S*)))
∨ *lits-of* (*trail* (*fst T*)) ⊨*sextm clauses* (*fst S*) ∧ *satisfiable* (*set-mset* (*clauses* (*fst S*)))
proof −
have *inv-T*: *inv* (*fst T*) **and** *n-d-T*: *no-dup* (*trail* (*fst T*))
using *rtranclp-cdcl*$_{NOT}$*-with-restart-cdcl*$_{NOT}$*-inv* **using** *full inv n-d* **unfolding** *full-def* **by** *blast+*
moreover have
atms-cls-T: *atms-of-msu* (*clauses* (*fst T*)) ⊆ *atms-of-ms A* **and**
atms-trail-T: *atm-of* ' *lits-of* (*trail* (*fst T*)) ⊆ *atms-of-ms A*
using *rtranclp-cdcl*$_{NOT}$*-with-restart-bound-inv*[*of S T A*] *full atms-cls atms-trail fin inv n-d*
unfolding *full-def* **by** *blast+*
ultimately have *no-step cdcl*$_{NOT}$*-merged-bj-learn* (*fst T*)
apply −
apply (*rule no-step-cdcl*$_{NOT}$*-restart-no-step-cdcl*$_{NOT}$[*of - A*])
using *full* **unfolding** *full-def* **apply** *simp*
apply *simp*
using *fin* **apply** *simp*
done
moreover have *all-decomposition-implies-m* (*clauses* (*fst T*))
(*get-all-marked-decomposition* (*trail* (*fst T*)))
using *rtranclp-cdcl*$_{NOT}$*-restart-all-decomposition-implies-m*[*of S T*] *inv n-d decomp*
full **unfolding** *full-def* **by** *auto*
ultimately have *unsatisfiable* (*set-mset* (*clauses* (*fst T*)))
∨ *trail* (*fst T*) ⊨*asm clauses* (*fst T*) ∧ *satisfiable* (*set-mset* (*clauses* (*fst T*)))
apply −
apply (*rule cdcl*$_{NOT}$*-merged-bj-learn-final-state*)
using *atms-cls-T atms-trail-T fin n-d-T fin inv-T* **by** *blast+*
then consider
(*unsat*) *unsatisfiable* (*set-mset* (*clauses* (*fst T*)))
| (*sat*) *trail* (*fst T*) ⊨*asm clauses* (*fst T*) **and** *satisfiable* (*set-mset* (*clauses* (*fst T*)))
by *auto*
then show *unsatisfiable* (*set-mset* (*clauses* (*fst S*)))
∨ *lits-of* (*trail* (*fst T*)) ⊨*sextm clauses* (*fst S*) ∧ *satisfiable* (*set-mset* (*clauses* (*fst S*)))
proof *cases*
case *unsat*
then have *unsatisfiable* (*set-mset* (*clauses* (*fst S*)))
unfolding *satisfiable-def* **apply** *auto*
using *rtranclp-cdcl*$_{NOT}$*-restart-eq-sat-iff*[*of S T*] *full inv n-d*
consistent-true-clss-ext-satisfiable true-clss-imp-true-cls-ext
unfolding *satisfiable-def full-def* **by** *blast*
then show *?thesis* **by** *blast*
next
case *sat*
then have *lits-of* (*trail* (*fst T*)) ⊨*sextm clauses* (*fst T*)
using *true-clss-imp-true-cls-ext* **by** (*auto simp*: *true-annots-true-cls*)
then have *lits-of* (*trail* (*fst T*)) ⊨*sextm clauses* (*fst S*)
using *rtranclp-cdcl*$_{NOT}$*-restart-eq-sat-iff*[*of S T*] *full inv n-d* **unfolding** *full-def* **by** *blast*
moreover then have *satisfiable* (*set-mset* (*clauses* (*fst S*)))
using *consistent-true-clss-ext-satisfiable distinctconsistent-interp n-d-T* **by** *fast*
ultimately show *?thesis* **by** *fast*

qed
qed

corollary *full-cdcl$_{NOT}$-restart-normal-form-init-state*:
assumes
init-state: *trail S* = [] *clauses S* = *N* **and**
full: *full cdcl$_{NOT}$-restart* (*S*, *0*) *T* **and**
inv: *inv S*
shows *unsatisfiable* (*set-mset N*)
∨ *lits-of* (*trail* (*fst T*)) $\models$*sextm N* ∧ *satisfiable* (*set-mset N*)
using *full-cdcl$_{NOT}$-restart-normal-form*[*of* (*S*, *0*) *T*] *assms* **by** *auto*

end

end
theory *DPLL-NOT*
imports *CDCL-NOT*
begin

15 DPLL as an instance of NOT

15.1 DPLL with simple backtrack

locale *dpll-with-backtrack*
begin
inductive *backtrack* :: (*'v*, *unit*, *unit*) *marked-lit list* × *'v clauses*
⇒ (*'v*, *unit*, *unit*) *marked-lit list* × *'v clauses* ⇒ *bool* **where**
backtrack-split (*fst S*) = (*M'*, *L* # *M*) ⟹ *is-marked L* ⟹ *D* ∈# *snd S*
⟹ *fst S* $\models$*as CNot D* ⟹ *backtrack S* (*Propagated* (− (*lit-of L*)) () # *M*, *snd S*)

inductive-cases *backtrackE*[*elim*]: *backtrack* (*M*, *N*) (*M'*, *N'*)
lemma *backtrack-is-backjump*:
fixes *M M'* :: (*'v*, *unit*, *unit*) *marked-lit list*
assumes
backtrack: *backtrack* (*M*, *N*) (*M'*, *N'*) **and**
no-dup: (*no-dup* ∘ *fst*) (*M*, *N*) **and**
decomp: *all-decomposition-implies-m N* (*get-all-marked-decomposition M*)
shows
∃ *C F' K F L l C'*.
M = *F'* @ *Marked K* () # *F* ∧
M' = *Propagated L l* # *F* ∧ *N* = *N'* ∧ *C* ∈# *N* ∧ *F'* @ *Marked K d* # *F* $\models$*as CNot C* ∧
undefined-lit F L ∧ *atm-of L* ∈ *atms-of-msu N* ∪ *atm-of* ' *lits-of* (*F'* @ *Marked K d* # *F*) ∧
N $\models$*pm C'* + {#*L*#} ∧ *F* $\models$*as CNot C'*
proof −
let *?S* = (*M*, *N*)
let *?T* = (*M'*, *N'*)
obtain *F F' P L D* **where**
b-sp: *backtrack-split M* = (*F'*, *L* # *F*) **and**
is-marked L **and**
D ∈# *snd ?S* **and**
M $\models$*as CNot D* **and**
bt: *backtrack ?S* (*Propagated* (− (*lit-of L*)) *P* # *F*, *N*) **and**
M': *M'* = *Propagated* (− (*lit-of L*)) *P* # *F* **and**
[*simp*]: *N'* = *N*
using *backtrackE*[*OF backtrack*] **by** (*metis backtrack fstI sndI*)

let *?K = lit-of L*
let *?C = image-mset lit-of* $\{\#K{\in}\#mset\ M.\ is\text{-}marked\ K \wedge K{\neq}L\#\}$ :: *'v literal multiset*
let *?C' = set-mset* (*image-mset single* (*?C*+{#*?K*#}))
obtain *K* **where** *L*: *L = Marked K* () **using** ‹*is-marked L*› **by** (*cases L*) *auto*

have *M*: *M = F' @ Marked K* () # *F*
  using *b-sp* **by** (*metis L backtrack-split-list-eq fst-conv snd-conv*)
moreover have *F' @ Marked K* () # *F* $\models$*as CNot D*
  using ‹*M*$\models$*as CNot D*› **unfolding** *M* **.**
moreover have *undefined-lit F* (−*?K*)
  using *no-dup* **unfolding** *M L* **by** (*simp add*: *defined-lit-map*)
moreover have *atm-of* (−*K*) ∈ *atms-of-msu N* ∪ *atm-of* ‘ *lits-of* (*F' @ Marked K d* # *F*)
  by *auto*
moreover
  have *set-mset N* ∪ *?C'* $\models$*ps* {{#}}
    proof −
      have *A*: *set-mset N* ∪ *?C'* ∪ (λ*a*. {#*lit-of a*#}) ‘ *set M* =
        set-mset N ∪ (λ*a*. {#*lit-of a*#}) ‘ *set M*
        unfolding *M L* **by** *auto*
      have *set-mset N* ∪ {{#*lit-of L*#} |*L*. *is-marked L* ∧ *L* ∈ *set M*}
          $\models$*ps* (λ*a*. {#*lit-of a*#}) ‘ *set M*
        using *all-decomposition-implies-propagated-lits-are-implied*[*OF decomp*] **.**
      moreover have *C'*: *?C'* = {{#*lit-of L*#} |*L*. *is-marked L* ∧ *L* ∈ *set M*}
        unfolding *M L* **apply** *standard*
          apply *force*
        using *IntI* **by** *auto*
      ultimately have *N-C-M*: *set-mset N* ∪ *?C'* $\models$*ps* (λ*a*. {#*lit-of a*#}) ‘ *set M*
        by *auto*
      have *set-mset N* ∪ (λ*L*. {#*lit-of L*#}) ‘ (*set M*) $\models$*ps* {{#}}
        unfolding *true-clss-clss-def*
        proof (*intro allI impI*, *goal-cases*)
          case (*1 I*) **note** *tot* = *this*(*1*) **and** *cons* = *this*(*2*) **and** *I-N-M* = *this*(*3*)
          have *I* $\models$ *D*
            using *I-N-M* ‹*D* ∈# *snd ?S*› **unfolding** *true-clss-def* **by** *auto*
          moreover have *I* $\models$*s CNot D*
            using ‹*M* $\models$*as CNot D*› **unfolding** *M* **by** (*metis 1*(*3*) ‹*M* $\models$*as CNot D*›
              true-annots-true-cls true-cls-mono-set-mset-l true-clss-def
              true-clss-singleton-lit-of-implies-incl true-clss-union)
          ultimately show *?case* **using** *cons consistent-CNot-not* **by** *blast*
        qed
      then show *?thesis*
        using *true-clss-clss-left-right*[*OF N-C-M*, *of* {{#}}] **unfolding** *A* **by** *auto*
    qed
  have *N* $\models$*pm image-mset uminus ?C* + {#−*?K*#}
    unfolding *true-clss-cls-def true-clss-clss-def total-over-m-def*
    proof (*intro allI impI*)
      fix *I*
      assume
        tot: *total-over-set I* (*atms-of-ms* (*set-mset N* ∪ {*image-mset uminus ?C* + {#− *?K*#}})) **and**
        cons: *consistent-interp I* **and**
        I $\models$*sm N*
      have (*K* ∈ *I* ∧ −*K* ∉ *I*) ∨ (−*K* ∈ *I* ∧ *K* ∉ *I*)
        using *cons tot* **unfolding** *consistent-interp-def L* **by** (*cases K*) *auto*
      have *total-over-set I* (*atm-of* ‘ *lit-of* ‘ (*set M* ∩ {*L*. *is-marked L* ∧ *L* ≠ *Marked K d*}))
        using *tot* **by** (*auto simp add*: *L atms-of-uminus-lit-atm-of-lit-of*)

then have *H*: $\bigwedge x.$
lit-of $x \notin I \Longrightarrow x \in$ *set* $M \Longrightarrow$*is-marked* x
$\Longrightarrow x \neq$ *Marked* K $d \Longrightarrow -$*lit-of* $x \in I$

unfolding *total-over-set-def atms-of-s-def*
proof −
fix x :: ($'v$, *unit*, *unit*) *marked-lit*
assume *a1*: $x \in$ *set* M
assume *a2*: $\forall l\in$*atm-of* ‘ *lit-of* ‘ (*set* $M \cap \{L.$ *is-marked* $L \wedge L \neq$ *Marked* K $d\}$).
Pos $l \in I \vee$ *Neg* $l \in I$
assume *a3*: *lit-of* $x \notin I$
assume *a4*: *is-marked* x
assume *a5*: $x \neq$ *Marked* K d
have *f6*: *Neg* (*atm-of* (*lit-of* x)) = − *Pos* (*atm-of* (*lit-of* x))
by *simp*
have *Pos* (*atm-of* (*lit-of* x)) $\in I \vee$ *Neg* (*atm-of* (*lit-of* x)) $\in I$
using *a5 a4 a2 a1* **by** *blast*
then show − *lit-of* $x \in I$
using *f6 a3* **by** (*metis* (*no-types*) *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set*
literal.sel(*1*))
qed
have $\neg I \models s$ *?C'*
using ‹*set-mset* $N \cup$ *?C'* $\models ps$ {{#}}› *tot cons* ‹$I \models sm$ N›
unfolding *true-clss-clss-def total-over-m-def*
by (*simp add*: *atms-of-uminus-lit-atm-of-lit-of atms-of-ms-single-image-atm-of-lit-of*)
then show $I \models$ *image-mset uminus ?C* + {#− *lit-of* L#}
unfolding *true-clss-def true-cls-def Bex-mset-def*
using ‹($K \in I \wedge -K \notin I$) $\vee$ ($-K \in I \wedge K \notin I$)›
unfolding L **by** (*auto dest*!: H)
qed
moreover
have *set* $F' \cap \{K.$ *is-marked* $K \wedge K \neq L\} = \{\}$
using *backtrack-split-fst-not-marked*[*of - M*] *b-sp* **by** *auto*
then have $F \models as$ *CNot* (*image-mset uminus ?C*)
unfolding *M CNot-def true-annots-def* **by** (*auto simp add*: *L lits-of-def*)
ultimately show *?thesis*
using M' ‹$D \in\#$ *snd ?S*› L **by** *force*
qed

lemma *backtrack-is-backjump'*:
fixes M M' :: ($'v$, *unit*, *unit*) *marked-lit list*
assumes
backtrack: *backtrack* S T **and**
no-dup: (*no-dup* ∘ *fst*) S **and**
decomp: *all-decomposition-implies-m* (*snd* S) (*get-all-marked-decomposition* (*fst* S))
shows
$\exists C\ F'\ K\ F\ L\ l\ C'.$
fst $S = F'$ @ *Marked* K () # F ∧
T = (*Propagated* L l # F, *snd* S) ∧ $C \in\#$ *snd* S ∧ *fst* $S \models as$ *CNot* C
∧ *undefined-lit* F L ∧ *atm-of* $L \in$ *atms-of-msu* (*snd* S) ∪ *atm-of* ‘ *lits-of* (*fst* S) ∧
snd $S \models pm$ C' + {#L#} ∧ $F \models as$ *CNot* C'
apply (*cases* S, *cases* T)
using *backtrack-is-backjump*[*of fst S snd S fst T snd T*] *assms* **by** *fastforce*

sublocale *dpll-state fst snd* $\lambda L\ (M,\ N).\ (L\ \#\ M,\ N)\ \lambda(M,\ N).\ (tl\ M,\ N)$
$\lambda C\ (M,\ N).\ (M,\ \{\#C\#\} + N)\ \lambda C\ (M,\ N).\ (M,\ remove\text{-}mset\ C\ N)$
by *unfold-locales auto*

sublocale *backjumping-ops fst snd* $\lambda L\ (M,\ N).\ (L\ \#\ M,\ N)\ \lambda(M,\ N).\ (tl\ M,\ N)$
$\lambda C\ (M,\ N).\ (M,\ \{\#C\#\} + N)\ \lambda C\ (M,\ N).\ (M,\ remove\text{-}mset\ C\ N)\ \lambda\text{- - -}\ S\ T.\ backtrack\ S\ T$
by *unfold-locales*

lemma *backtrack-is-backjump''*:
fixes $M\ M' :: ('v,\ unit,\ unit)\ marked\text{-}lit\ list$
assumes
backtrack: *backtrack S T* **and**
no-dup: $(no\text{-}dup \circ fst)\ S$ **and**
decomp: *all-decomposition-implies-m* (*snd S*) (*get-all-marked-decomposition* (*fst S*))
shows *backjump S T*
proof −
obtain $C\ F'\ K\ F\ L\ l\ C'$ **where**
1: $fst\ S = F'\ @\ Marked\ K\ ()\ \#\ F$ **and**
2: $T = (Propagated\ L\ l\ \#\ F,\ snd\ S)$ **and**
3: $C \in\#\ snd\ S$ **and**
4: $fst\ S \models as\ CNot\ C$ **and**
5: *undefined-lit F L* **and**
6: $atm\text{-}of\ L \in atms\text{-}of\text{-}msu\ (snd\ S) \cup atm\text{-}of\ `\ lits\text{-}of\ (fst\ S)$ **and**
7: $snd\ S \models pm\ C' + \{\#L\#\}$ **and**
8: $F \models as\ CNot\ C'$
using *backtrack-is-backjump'*[*OF assms*] **by** *blast*
show *?thesis*
using *backjump.intros*[*OF 1 - 3 4 5 6 7 8*] *2 backtrack 1 5*
by (*auto simp*: $state\text{-}eq_{NOT}\text{-}def$ *simp del*: $state\text{-}simp_{NOT}$)
qed

lemma *can-do-bt-step*:
assumes
M: $fst\ S = F'\ @\ Marked\ K\ d\ \#\ F$ **and**
$C \in\#\ snd\ S$ **and**
C: $fst\ S \models as\ CNot\ C$
shows $\neg$ *no-step backtrack S*
proof −
obtain $L\ G'\ G$ **where**
$backtrack\text{-}split\ (fst\ S) = (G',\ L\ \#\ G)$
unfolding *M* **by** (*induction F' rule*: *marked-lit-list-induct*) *auto*
moreover then have *is-marked L*
by (*metis backtrack-split-snd-hd-marked list.distinct*(*1*) *list.sel*(*1*) *snd-conv*)
ultimately show *?thesis*
using *backtrack.intros*[*of S G' L G C*] $\langle C \in\#\ snd\ S\rangle$ *C* **unfolding** *M* **by** *auto*
qed

end

sublocale $dpll\text{-}with\text{-}backtrack \subseteq dpll\text{-}with\text{-}backjumping\text{-}ops\ fst\ snd\ \lambda L\ (M,\ N).\ (L\ \#\ M,\ N)$
$\lambda(M,\ N).\ (tl\ M,\ N)\ \lambda C\ (M,\ N).\ (M,\ \{\#C\#\} + N)\ \lambda C\ (M,\ N).\ (M,\ remove\text{-}mset\ C\ N)\ \lambda\text{- -}.\ True$
$\lambda(M,\ N).\ no\text{-}dup\ M \wedge all\text{-}decomposition\text{-}implies\text{-}m\ N\ (get\text{-}all\text{-}marked\text{-}decomposition\ M)$
$\lambda\text{- - -}\ S\ T.\ backtrack\ S\ T$
by *unfold-locales* (*metis* (*mono-tags*, *lifting*) *dpll-with-backtrack.backtrack-is-backjump''*
dpll-with-backtrack.can-do-bt-step prod.case-eq-if comp-apply)

sublocale *dpll-with-backtrack* ⊆ *dpll-with-backjumping fst snd* λ*L* (*M*, *N*). (*L* # *M*, *N*)
λ(*M*, *N*). (*tl M*, *N*) λ*C* (*M*, *N*). (*M*, {#*C*#} + *N*) λ*C* (*M*, *N*). (*M*, *remove-mset C N*) λ- -. *True*
λ(*M*, *N*). *no-dup M* ∧ *all-decomposition-implies-m N* (*get-all-marked-decomposition M*)
λ- - - *S T*. *backtrack S T*
apply *unfold-locales*
using *dpll-bj-no-dup dpll-bj-all-decomposition-implies-inv* **apply** *fastforce*
done

sublocale *dpll-with-backtrack* ⊆ *conflict-driven-clause-learning-ops*
fst snd λ*L* (*M*, *N*). (*L* # *M*, *N*)
λ(*M*, *N*). (*tl M*, *N*) λ*C* (*M*, *N*). (*M*, {#*C*#} + *N*) λ*C* (*M*, *N*). (*M*, *remove-mset C N*) λ- -. *True*
λ(*M*, *N*). *no-dup M* ∧ *all-decomposition-implies-m N* (*get-all-marked-decomposition M*)
λ- - - *S T*. *backtrack S T* λ- -. *False* λ- -. *False*
by *unfold-locales*

sublocale *dpll-with-backtrack* ⊆ *conflict-driven-clause-learning*
fst snd λ*L* (*M*, *N*). (*L* # *M*, *N*)
λ(*M*, *N*). (*tl M*, *N*) λ*C* (*M*, *N*). (*M*, {#*C*#} + *N*) λ*C* (*M*, *N*). (*M*, *remove-mset C N*) λ- -. *True*
λ(*M*, *N*). *no-dup M* ∧ *all-decomposition-implies-m N* (*get-all-marked-decomposition M*)
λ- - - *S T*. *backtrack S T* λ- -. *False* λ- -. *False*
apply *unfold-locales*
using $cdcl_{NOT}$*.simps dpll-bj-inv forgetE learnE* **by** *blast*

context *dpll-with-backtrack*
begin
lemma *wf-tranclp-dpll-inital-state*:
assumes *fin*: *finite A*
shows *wf* {((*M*′::(′*v*, *unit*, *unit*) *marked-lits*, *N*′::′*v clauses*), ([], *N*))|*M*′ *N*′ *N*.
dpll-bj^{++} ([], *N*) (*M*′, *N*′) ∧ *atms-of-msu N* ⊆ *atms-of-ms A*}
using *wf-tranclp-dpll-bj*[*OF assms*(*1*)] **by** (*rule wf-subset*) *auto*

corollary *full-dpll-final-state-conclusive*:
fixes *M M*′ :: (′*v*, *unit*, *unit*) *marked-lit list*
assumes
full: *full dpll-bj* ([], *N*) (*M*′, *N*′)
shows *unsatisfiable* (*set-mset N*) ∨ (*M*′ ⊨*asm N* ∧ *satisfiable* (*set-mset N*))
using *assms full-dpll-backjump-final-state*[*of* ([],*N*) (*M*′, *N*′) *set-mset N*] **by** *auto*

corollary *full-dpll-normal-form-from-init-state*:
fixes *M M*′ :: (′*v*, *unit*, *unit*) *marked-lit list*
assumes
full: *full dpll-bj* ([], *N*) (*M*′, *N*′)
shows *M*′ ⊨*asm N* ⟷ *satisfiable* (*set-mset N*)
proof −
have *no-dup M*′
using *rtranclp-dpll-bj-no-dup*[*of* ([], *N*) (*M*′, *N*′)]
full **unfolding** *full-def* **by** *auto*
then have *M*′ ⊨*asm N* ⟹ *satisfiable* (*set-mset N*)
using *distinctconsistent-interp satisfiable-carac*′ *true-annots-true-cls* **by** *blast*
then show *?thesis*
using *full-dpll-final-state-conclusive*[*OF full*] **by** *auto*
qed

lemma $cdcl_{NOT}$*-is-dpll*:

$cdcl_{NOT}$ *S T* ⟷ *dpll-bj S T*
by (*auto simp*: $cdcl_{NOT}$*.simps learn.simps forget*$_{NOT}$*.simps*)

Another proof of termination:

lemma *wf* {(*T*, *S*). *dpll-bj S T* ∧ $cdcl_{NOT}$*-NOT-all-inv A S*}
unfolding $cdcl_{NOT}$*-is-dpll*[*symmetric*]
by (*rule wf-cdcl*$_{NOT}$*-no-learn-and-forget-infinite-chain*)
(*auto simp*: *learn.simps forget*$_{NOT}$*.simps*)
end

15.2 Adding restarts

locale *dpll-withbacktrack-and-restarts* =
dpll-with-backtrack +
fixes *f* :: *nat* ⇒ *nat*
assumes *unbounded*: *unbounded f* **and** *f-ge-1*:⋀*n*. *n*≥ *1* ⟹ *f n* ≥ *1*
begin
sublocale $cdcl_{NOT}$*-increasing-restarts fst snd* λ*L* (*M*, *N*). (*L* # *M*, *N*) λ(*M*, *N*). (*tl M*, *N*)
λ*C* (*M*, *N*). (*M*, {#*C*#} + *N*) λ*C* (*M*, *N*). (*M*, *remove-mset C N*) *f* λ(-, *N*) *S*. *S* = ([], *N*)
λ*A* (*M*, *N*). *atms-of-msu N* ⊆ *atms-of-ms A* ∧ *atm-of* ‘ *lits-of M* ⊆ *atms-of-ms A* ∧ *finite A*
∧ *all-decomposition-implies-m N* (*get-all-marked-decomposition M*)
λ*A T*. (*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*))
− μ_C (*1*+*card* (*atms-of-ms A*)) (*2*+*card* (*atms-of-ms A*)) (*trail-weight T*) *dpll-bj*
λ(*M*, *N*). *no-dup M* ∧ *all-decomposition-implies-m N* (*get-all-marked-decomposition M*)
λ*A* -. (*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*))
apply *unfold-locales*
apply (*rule unbounded*)
using *f-ge-1* **apply** *fastforce*
apply (*smt dpll-bj-all-decomposition-implies-inv dpll-bj-atms-in-trail-in-set*
dpll-bj-clauses dpll-bj-no-dup prod.case-eq-if)
apply (*rule dpll-bj-trail-mes-decreasing-prop*; *auto*)
apply (*case-tac T*, *simp*)
apply (*case-tac U*, *simp*)
using *dpll-bj-clauses dpll-bj-all-decomposition-implies-inv dpll-bj-no-dup* **by** *fastforce+*
end

end
theory *DPLL-W*
imports *Main Partial-Clausal-Logic Partial-Annotated-Clausal-Logic List-More Wellfounded-More*
DPLL-NOT
begin

16 DPLL

16.1 Rules

type-synonym *'a dpll*$_W$*-marked-lit* = (*'a*, *unit*, *unit*) *marked-lit*
type-synonym *'a dpll*$_W$*-marked-lits* = (*'a*, *unit*, *unit*) *marked-lits*
type-synonym *'v dpll*$_W$*-state* = *'v dpll*$_W$*-marked-lits* × *'v clauses*

abbreviation *trail* :: *'v dpll*$_W$*-state* ⇒ *'v dpll*$_W$*-marked-lits* **where**
trail ≡ *fst*
abbreviation *clauses* :: *'v dpll*$_W$*-state* ⇒ *'v clauses* **where**
clauses ≡ *snd*

The definition of DPLL is given in figure 2.13 page 70 of CW.

inductive $dpll_W$ :: $'v$ $dpll_W$*-state* $\Rightarrow$ $'v$ $dpll_W$*-state* $\Rightarrow$ *bool* **where**
propagate: $C + \{\#L\#\} \in\#$ *clauses* $S \Longrightarrow$ *trail* $S \models as$ *CNot* $C \Longrightarrow$ *undefined-lit* (*trail* S) L
$\Longrightarrow dpll_W$ S (*Propagated* L () # *trail* S, *clauses* S) |
decided: *undefined-lit* (*trail* S) $L \Longrightarrow$ *atm-of* $L \in$ *atms-of-msu* (*clauses* S)
$\Longrightarrow dpll_W$ S (*Marked* L () # *trail* S, *clauses* S) |
backtrack: *backtrack-split* (*trail* S) $= (M', L \# M) \Longrightarrow$ *is-marked* $L \Longrightarrow D \in\#$ *clauses* S
$\Longrightarrow$ *trail* $S \models as$ *CNot* $D \Longrightarrow dpll_W$ S (*Propagated* ($-$ (*lit-of* L)) () # M, *clauses* S)

16.2 Invariants

lemma $dpll_W$*-distinct-inv*:
assumes $dpll_W$ S S'
and *no-dup* (*trail* S)
shows *no-dup* (*trail* S')
using *assms*
proof (*induct rule*: $dpll_W$*.induct*)
case (*decided* L S)
then show *?case* **using** *defined-lit-map* **by** *force*
next
case (*propagate* C L S)
then show *?case* **using** *defined-lit-map* **by** *force*
next
case (*backtrack* S M' L M D) **note** *extracted* = *this*(*1*) **and** *no-dup* = *this*(*5*)
show *?case*
using *no-dup backtrack-split-list-eq*[*of trail* S, *symmetric*] **unfolding** *extracted* **by** *auto*
qed

lemma $dpll_W$*-consistent-interp-inv*:
assumes $dpll_W$ S S'
and *consistent-interp* (*lits-of* (*trail* S))
and *no-dup* (*trail* S)
shows *consistent-interp* (*lits-of* (*trail* S'))
using *assms*
proof (*induct rule*: $dpll_W$*.induct*)
case (*backtrack* S M' L M D) **note** *extracted* = *this*(*1*) **and** *marked* = *this*(*2*) **and** D = *this*(*4*) **and**
cons = *this*(*5*) **and** *no-dup* = *this*(*6*)
have *no-dup'*: *no-dup* M
by (*metis* (*no-types*) *backtrack-split-list-eq distinct.simps*(*2*) *distinct-append extracted*
list.simps(*9*) *map-append no-dup snd-conv*)
then have *insert* (*lit-of* L) (*lits-of* M) $\subseteq$ *lits-of* (*trail* S)
using *backtrack-split-list-eq*[*of trail* S, *symmetric*] **unfolding** *extracted* **by** *auto*
then have *cons*: *consistent-interp* (*insert* (*lit-of* L) (*lits-of* M))
using *consistent-interp-subset cons* **by** *blast*
moreover
have *lit-of* $L \notin$ *lits-of* M
using *no-dup backtrack-split-list-eq*[*of trail* S, *symmetric*] *extracted*
unfolding *lits-of-def* **by** *force*
moreover
have *atm-of* ($-$*lit-of* L) $\notin$ ($\lambda m.$ *atm-of* (*lit-of* m)) ‘ *set* M
using *no-dup backtrack-split-list-eq*[*of trail* S, *symmetric*] **unfolding** *extracted* **by** *force*
then have $-$*lit-of* $L \notin$ *lits-of* M
unfolding *lits-of-def* **by** *force*
ultimately show *?case* **by** *simp*
qed (*auto intro*: *consistent-add-undefined-lit-consistent*)

lemma $dpll_W$*-vars-in-snd-inv*:

assumes *dpll*$_W$ *S S'*
and *atm-of* ' (*lits-of* (*trail S*)) ⊆ *atms-of-msu* (*clauses S*)
shows *atm-of* ' (*lits-of* (*trail S'*)) ⊆ *atms-of-msu* (*clauses S'*)
using *assms*
proof (*induct rule*: *dpll*$_W$*.induct*)
case (*backtrack S M' L M D*)
then have *atm-of* (*lit-of L*) ∈ *atms-of-msu* (*clauses S*)
using *backtrack-split-list-eq*[*of trail S*, *symmetric*] **by** *auto*
moreover
have *atm-of* ' *lits-of* (*trail S*) ⊆ *atms-of-msu* (*clauses S*)
using *backtrack*(*5*) **by** *simp*
then have ⋀*xb. xb* ∈ *set M* ⟹ *atm-of* (*lit-of xb*) ∈ *atms-of-msu* (*clauses S*)
using *backtrack-split-list-eq*[*symmetric*, *of trail S*] *backtrack.hyps*(*1*)
unfolding *lits-of-def* **by** *auto*
ultimately show *?case* **by** (*auto simp* : *lits-of-def*)
qed (*auto simp*: *in-plus-implies-atm-of-on-atms-of-ms*)

lemma *atms-of-ms-lit-of-atms-of*: *atms-of-ms* ((λ*a*. {#*lit-of a*#}) ' *c*) = *atm-of* ' *lit-of* ' *c*
unfolding *atms-of-ms-def* **using** *image-iff* **by** *force*

Lemma theorem 2.8.2 page 71 of CW

lemma *dpll*$_W$*-propagate-is-conclusion*:
assumes *dpll*$_W$ *S S'*
and *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
and *atm-of* ' *lits-of* (*trail S*) ⊆ *atms-of-msu* (*clauses S*)
shows *all-decomposition-implies-m* (*clauses S'*) (*get-all-marked-decomposition* (*trail S'*))
using *assms*
proof (*induct rule*: *dpll*$_W$*.induct*)
case (*decided L S*)
then show *?case* **unfolding** *all-decomposition-implies-def* **by** *simp*
next
case (*propagate C L S*) **note** *inS* = *this*(*1*) **and** *cnot* = *this*(*2*) **and** *IH* = *this*(*4*) **and** *undef* = *this*(*3*) **and** *atms-incl* = *this*(*5*)
let *?I* = *set* (*map* (λ*a*. {#*lit-of a*#}) (*trail S*)) ∪ *set-mset* (*clauses S*)
have *?I* ⊨*p C* + {#*L*#} **by** (*auto simp add*: *inS*)
moreover have *?I* ⊨*ps CNot C* **using** *true-annots-true-clss-cls cnot* **by** *fastforce*
ultimately have *?I* ⊨*p* {#*L*#} **using** *true-clss-cls-plus-CNot*[*of ?I C L*] *inS* **by** *blast*
{
assume *get-all-marked-decomposition* (*trail S*) = []
then have *?case* **by** *blast*
}
moreover {
assume *n*: *get-all-marked-decomposition* (*trail S*) ≠ []
have *1*: ⋀*a b*. (*a*, *b*) ∈ *set* (*tl* (*get-all-marked-decomposition* (*trail S*)))
⟹ ((λ*a*. {#*lit-of a*#}) ' *set a* ∪ *set-mset* (*clauses S*)) ⊨*ps* (λ*a*. {#*lit-of a*#}) ' *set b*
using *IH* **unfolding** *all-decomposition-implies-def* **by** (*fastforce simp add*: *list.set-sel*(*2*) *n*)
moreover have *2*: ⋀*a c*. *hd* (*get-all-marked-decomposition* (*trail S*)) = (*a*, *c*)
⟹ ((λ*a*. {#*lit-of a*#}) ' *set a* ∪ *set-mset* (*clauses S*)) ⊨*ps* ((λ*a*. {#*lit-of a*#}) ' *set c*)
by (*metis IH all-decomposition-implies-cons-pair all-decomposition-implies-single list.collapse n*)
moreover have *3*: ⋀*a c*. *hd* (*get-all-marked-decomposition* (*trail S*)) = (*a*, *c*)
⟹ ((λ*a*. {#*lit-of a*#}) ' *set a* ∪ *set-mset* (*clauses S*)) ⊨*p* {#*L*#}
proof −
fix *a c*
assume *h*: *hd* (*get-all-marked-decomposition* (*trail S*)) = (*a*, *c*)

have h': *trail* S = c @ a **using** *get-all-marked-decomposition-decomp* h **by** *blast*
have I: *set* (*map* (λa. {#*lit-of* a#}) a) ∪ *set-mset* (*clauses* S)
∪ (λa. {#*lit-of* a#}) ' *set* c ⊨*ps* *CNot* C
using ‹?*I* ⊨*ps* *CNot* C› **unfolding** h' **by** (*simp add*: *Un-commute Un-left-commute*)
have
atms-of-ms (*CNot* C) ⊆ *atms-of-ms* (*set* (*map* (λa. {#*lit-of* a#}) a) ∪ *set-mset* (*clauses* S))
and
atms-of-ms ((λa. {#*lit-of* a#}) ' *set* c) ⊆ *atms-of-ms* (*set* (*map* (λa. {#*lit-of* a#}) a)
∪ *set-mset* (*clauses* S))
apply (*metis CNot-plus Un-subset-iff atms-of-atms-of-ms-mono atms-of-ms-CNot-atms-of atms-of-ms-union inS mem-set-mset-iff sup.coboundedI2*)
using *inS atms-of-atms-of-ms-mono atms-incl* **by** (*fastforce simp*: h')

then have (λa. {#*lit-of* a#}) ' *set* a ∪ *set-mset* (*clauses* S) ⊨*ps* *CNot* C
using *true-clss-clss-left-right*[*OF* - I] h *2* **by** *auto*
then show (λa. {#*lit-of* a#}) ' *set* a ∪ *set-mset* (*clauses* S) ⊨*p* {#L#}
by (*metis* (*no-types*) *Un-insert-right inS insertI1 mk-disjoint-insert inS mem-set-mset-iff true-clss-cls-in true-clss-cls-plus-CNot*)
qed
ultimately have ?*case*
by (*case-tac hd* (*get-all-marked-decomposition* (*trail* S)))
(*auto simp add*: *all-decomposition-implies-def*)
}
ultimately show ?*case* **by** *auto*
next
case (*backtrack* S M' L M D) **note** *extracted* = *this*(*1*) **and** *marked* = *this*(*2*) **and** D = *this*(*3*) **and** *cnot* = *this*(*4*) **and** *cons* = *this*(*4*) **and** *IH* = *this*(*5*) **and** *atms-incl* = *this*(*6*)
have S: *trail* S = M' @ L # M
using *backtrack-split-list-eq*[*of trail* S] **unfolding** *extracted* **by** *auto*
have M': ∀ l ∈ *set* M'. ¬*is-marked* l
using *extracted backtrack-split-fst-not-marked*[*of* - *trail* S] **by** *simp*
have n: *get-all-marked-decomposition* (*trail* S) ≠ [] **by** *auto*
then have *all-decomposition-implies-m* (*clauses* S) ((L # M, M')
tl (*get-all-marked-decomposition* (*trail* S)))
by (*metis* (*no-types*) *IH extracted get-all-marked-decomposition-backtrack-split list.exhaust-sel*)
then have *1*: (λa. {#*lit-of* a#}) ' *set* (L # M) ∪ *set-mset* (*clauses* S) ⊨*ps*(λa.{#*lit-of* a#}) ' *set* M'
by *simp*
moreover
have (λa. {#*lit-of* a#}) ' *set* (L # M) ∪ (λa. {#*lit-of* a#}) ' *set* M' ⊨*ps* *CNot* D
by (*metis* (*mono-tags*, *lifting*) S *Un-commute cons image-Un set-append true-annots-true-clss-clss*)
then have *2*: (λa. {#*lit-of* a#}) ' *set* (L # M) ∪ *set-mset* (*clauses* S) ∪ (λa. {#*lit-of* a#}) ' *set* M'
⊨*ps* *CNot* D
by (*metis* (*no-types*, *lifting*) *Un-assoc Un-left-commute true-clss-clss-union-l-r*)
ultimately
have *set* (*map* (λa. {#*lit-of* a#}) (L # M)) ∪ *set-mset* (*clauses* S) ⊨*ps* *CNot* D
using *true-clss-clss-left-right* **by** *fastforce*
then have *set* (*map* (λa. {#*lit-of* a#}) (L # M)) ∪ *set-mset* (*clauses* S) ⊨*p* {#}
by (*metis* (*mono-tags*, *lifting*) D *Un-def mem-Collect-eq set-mset-def true-clss-clss-contradiction-true-clss-cls-false*)
then have *IL*: (λa. {#*lit-of* a#}) ' *set* M ∪ *set-mset* (*clauses* S) ⊨*p* {#−*lit-of* L#}
using *true-clss-clss-false-left-right* **by** *auto*
show ?*case* **unfolding** S *all-decomposition-implies-def*

proof
  fix *x P level*
  assume *x*: $x \in$ *set* (*get-all-marked-decomposition*
    (*fst* (*Propagated* (− *lit-of* *L*) *P* # *M*, *clauses* *S*)))
  let *?M′* = *Propagated* (− *lit-of* *L*) *P* # *M*
  let *?hd* = *hd* (*get-all-marked-decomposition* *?M′*)
  let *?tl* = *tl* (*get-all-marked-decomposition* *?M′*)
  have *x* = *?hd* ∨ *x* ∈ *set* *?tl*
    using *x*
    by (*cases get-all-marked-decomposition ?M′*)
      auto
  moreover {
    assume *x′*: *x* ∈ *set* *?tl*
    have *L′*: *Marked* (*lit-of* *L*) () = *L* **using** *marked* **by** (*case-tac* *L*, *auto*)
    have *x* ∈ *set* (*get-all-marked-decomposition* (*M′* @ *L* # *M*))
      using *x′* *get-all-marked-decomposition-except-last-choice-equal*[*of* *M′* *lit-of* *L* *P* *M*]
      L′ **by** (*metis* (*no-types*) *M′* *list.set-sel*(*2*) *tl-Nil*)
    then have *case* *x* *of* (*Ls*, *seen*) ⇒ (λ*a*. {#*lit-of* *a*#}) ‘ *set* *Ls* ∪ *set-mset* (*clauses* *S*)
      ⊨*ps* (λ*a*. {#*lit-of* *a*#}) ‘ *set* *seen*
      using *marked* *IH* **by** (*case-tac* *L*) (*auto simp add*: *S* *all-decomposition-implies-def*)
  }
  moreover {
    assume *x′*: *x* = *?hd*
    have *tl*: *tl* (*get-all-marked-decomposition* (*M′* @ *L* # *M*)) ≠ []
      proof −
        have *f1*: ⋀*ms*. *length* (*get-all-marked-decomposition* (*M′* @ *ms*))
          = *length* (*get-all-marked-decomposition* *ms*)
          by (*simp add*: *M′* *get-all-marked-decomposition-remove-unmarked-length*)
        have *Suc* (*length* (*get-all-marked-decomposition* *M*)) ≠ *Suc* *0*
          by *blast*
        then show *?thesis*
          using *f1* *marked* **by** (*metis* (*no-types*) *get-all-marked-decomposition.simps*(*1*) *length-tl*
            list.sel(*3*) *list.size*(*3*) *marked-lit.collapse*(*1*))
      qed
    obtain *M0′* *M0* **where**
      L0: *hd* (*tl* (*get-all-marked-decomposition* (*M′* @ *L* # *M*))) = (*M0*, *M0′*)
      by (*cases* *hd* (*tl* (*get-all-marked-decomposition* (*M′* @ *L* # *M*))))
    have *x″*: *x* = (*M0*, *Propagated* (−*lit-of* *L*) *P* # *M0′*)
      unfolding *x′* **using** *get-all-marked-decomposition-last-choice* *tl* *M′* *L0*
      by (*metis* *marked* *marked-lit.collapse*(*1*))
    obtain *l-get-all-marked-decomposition* **where**
      get-all-marked-decomposition (*trail* *S*) = (*L* # *M*, *M′*) # (*M0*, *M0′*) #
        l-get-all-marked-decomposition
      using *get-all-marked-decomposition-backtrack-split* *extracted* **by** (*metis* (*no-types*) *L0* *S*
        hd-Cons-tl *n* *tl*)
    then have *M* = *M0′* @ *M0* **using** *get-all-marked-decomposition-hd-hd* **by** *fastforce*
    then have *IL′*: (λ*a*. {#*lit-of* *a*#}) ‘ *set* *M0* ∪ *set-mset* (*clauses* *S*)
      ∪ (λ*a*. {#*lit-of* *a*#}) ‘ *set* *M0′* ⊨*ps* {{#− *lit-of* *L*#}}
      using *IL* **by** (*simp add*: *Un-commute* *Un-left-commute* *image-Un*)
    moreover have *H*: (λ*a*. {#*lit-of* *a*#}) ‘ *set* *M0* ∪ *set-mset* (*clauses* *S*)
      ⊨*ps* (λ*a*. {#*lit-of* *a*#}) ‘ *set* *M0′*
      using *IH* *x″* **unfolding** *all-decomposition-implies-def* **by** (*metis* (*no-types*, *lifting*) *L0* *S*
        list.set-sel(*1*) *list.set-sel*(*2*) *old.prod.case* *tl* *tl-Nil*)
    ultimately have *case* *x* *of* (*Ls*, *seen*) ⇒ (λ*a*. {#*lit-of* *a*#}) ‘ *set* *Ls* ∪ *set-mset* (*clauses* *S*)
      ⊨*ps* (λ*a*. {#*lit-of* *a*#}) ‘ *set* *seen*

using *true-clss-clss-left-right* **unfolding** x'' **by** *auto*
}
ultimately show *case x of* (*Ls*, *seen*) ⇒
(λ*a*. {#*lit-of a*#}) ' *set Ls* ∪ *set-mset* (*snd* (*?M'*, *clauses S*))
⊨*ps* (λ*a*. {#*lit-of a*#}) ' *set seen*
unfolding *snd-conv* **by** *blast*
qed
qed

Lemma theorem 2.8.3 page 72 of CW

theorem *dpll$_W$-propagate-is-conclusion-of-decided*:
assumes *dpll$_W$ S S'*
and *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
and *atm-of* ' *lits-of* (*trail S*) ⊆ *atms-of-msu* (*clauses S*)
shows *set-mset* (*clauses S'*) ∪ {{#*lit-of L*#} |*L*. *is-marked L* ∧ *L* ∈ *set* (*trail S'*)}
⊨*ps* (λ*a*. {#*lit-of a*#}) ' ⋃(*set* ' *snd* ' *set* (*get-all-marked-decomposition* (*trail S'*)))
using *all-decomposition-implies-trail-is-implied*[*OF dpll$_W$-propagate-is-conclusion*[*OF assms*]] **.**

Lemma theorem 2.8.4 page 72 of CW

lemma *only-propagated-vars-unsat*:
assumes *marked*: ∀ *x* ∈ *set M*. ¬ *is-marked x*
and *DN*: *D* ∈ *N* **and** *D*: *M* ⊨*as CNot D*
and *inv*: *all-decomposition-implies N* (*get-all-marked-decomposition M*)
and *atm-incl*: *atm-of* ' *lits-of M* ⊆ *atms-of-ms N*
shows *unsatisfiable N*
proof (*rule ccontr*)
assume ¬ *unsatisfiable N*
then obtain *I* **where**
I: *I* ⊨*s N* **and**
cons: *consistent-interp I* **and**
tot: *total-over-m I N*
unfolding *satisfiable-def* **by** *auto*
then have *I-D*: *I* ⊨ *D*
using *DN* **unfolding** *true-clss-def* **by** *auto*

have *l0*: {{#*lit-of L*#} |*L*. *is-marked L* ∧ *L* ∈ *set M*} = {} **using** *marked* **by** *auto*
have *atms-of-ms* (*N* ∪ (λ*a*. {#*lit-of a*#}) ' *set M*) = *atms-of-ms N*
using *atm-incl* **unfolding** *atms-of-ms-def lits-of-def* **by** *auto*

then have *total-over-m I* (*N* ∪ (λ*a*. {#*lit-of a*#}) ' (*set M*))
using *tot* **unfolding** *total-over-m-def* **by** *auto*
then have *I* ⊨*s* (λ*a*. {#*lit-of a*#}) ' (*set M*)
using *all-decomposition-implies-propagated-lits-are-implied*[*OF inv*] *cons I*
unfolding *true-clss-clss-def l0* **by** *auto*
then have *IM*: *I* ⊨*s* (λ*a*. {#*lit-of a*#}) ' *set M* **by** *auto*
{
fix *K*
assume *K* ∈# *D*
then have −*K* ∈ *lits-of M*
by (*auto split*: *split-if-asm*
intro: *allE*[*OF D*[*unfolded true-annots-def Ball-def*], *of* {#−*K*#}])
then have −*K* ∈ *I* **using** *IM true-clss-singleton-lit-of-implies-incl* **by** *fastforce*
}
then have ¬ *I* ⊨ *D* **using** *cons* **unfolding** *true-cls-def consistent-interp-def* **by** *auto*
then show *False* **using** *I-D* **by** *blast*

qed

lemma *dpll*$_W$*-same-clauses*:
  assumes *dpll*$_W$ *S S'*
  shows *clauses S* = *clauses S'*
  using *assms* **by** (*induct rule*: *dpll*$_W$*.induct*, *auto*)

lemma *rtranclp-dpll*$_W$*-inv*:
  assumes *rtranclp dpll*$_W$ *S S'*
  and *inv*: *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
  and *atm-incl*: *atm-of ' lits-of* (*trail S*) ⊆ *atms-of-msu* (*clauses S*)
  and *consistent-interp* (*lits-of* (*trail S*))
  and *no-dup* (*trail S*)
  shows *all-decomposition-implies-m* (*clauses S'*) (*get-all-marked-decomposition* (*trail S'*))
  and *atm-of ' lits-of* (*trail S'*) ⊆ *atms-of-msu* (*clauses S'*)
  and *clauses S* = *clauses S'*
  and *consistent-interp* (*lits-of* (*trail S'*))
  and *no-dup* (*trail S'*)
  using *assms*
proof (*induct rule*: *rtranclp-induct*)
  case *base*
  show
    all-decomposition-implies-m (*clauses S*) (*get-all-marked-decomposition* (*trail S*)) **and**
    atm-of ' lits-of (*trail S*) ⊆ *atms-of-msu* (*clauses S*) **and**
    clauses S = *clauses S* **and**
    consistent-interp (*lits-of* (*trail S*)) **and**
    no-dup (*trail S*) **using** *assms* **by** *auto*
next
  case (*step S' S''*) **note** *dpll*$_W$*Star* = *this*(*1*) **and** *IH* = *this*(*3*,*4*,*5*,*6*,*7*) **and**
    dpll$_W$ = *this*(*2*)
  moreover
    assume
      inv: *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*)) **and**
      atm-incl: *atm-of ' lits-of* (*trail S*) ⊆ *atms-of-msu* (*clauses S*) **and**
      cons: *consistent-interp* (*lits-of* (*trail S*)) **and**
      no-dup (*trail S*)
  ultimately have *decomp*: *all-decomposition-implies-m* (*clauses S'*)
    (*get-all-marked-decomposition* (*trail S'*)) **and**
    atm-incl': *atm-of ' lits-of* (*trail S'*) ⊆ *atms-of-msu* (*clauses S'*) **and**
    snd: *clauses S* = *clauses S'* **and**
    cons': *consistent-interp* (*lits-of* (*trail S'*)) **and**
    no-dup': *no-dup* (*trail S'*) **by** *blast+*
  show *clauses S* = *clauses S''* **using** *dpll*$_W$*-same-clauses*[*OF dpll*$_W$] *snd* **by** *metis*

  show *all-decomposition-implies-m* (*clauses S''*) (*get-all-marked-decomposition* (*trail S''*))
    using *dpll*$_W$*-propagate-is-conclusion*[*OF dpll*$_W$] *decomp atm-incl'* **by** *auto*
  show *atm-of ' lits-of* (*trail S''*) ⊆ *atms-of-msu* (*clauses S''*)
    using *dpll*$_W$*-vars-in-snd-inv*[*OF dpll*$_W$] *atm-incl atm-incl'* **by** *auto*
  show *no-dup* (*trail S''*) **using** *dpll*$_W$*-distinct-inv*[*OF dpll*$_W$] *no-dup' dpll*$_W$ **by** *auto*
  show *consistent-interp* (*lits-of* (*trail S''*))
    using *cons' no-dup' dpll*$_W$*-consistent-interp-inv*[*OF dpll*$_W$] **by** *auto*
qed

definition *dpll*$_W$*-all-inv S* ≡
  (*all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*))

$\wedge$ *atm-of* ` *lits-of* (*trail S*) $\subseteq$ *atms-of-msu* (*clauses S*)
$\wedge$ *consistent-interp* (*lits-of* (*trail S*))
$\wedge$ *no-dup* (*trail S*))

lemma *dpll*$_W$*-all-inv-dest*[*dest*]:
assumes *dpll*$_W$*-all-inv S*
shows *all-decomposition-implies-m* (*clauses S*) (*get-all-marked-decomposition* (*trail S*))
and *atm-of* ` *lits-of* (*trail S*) $\subseteq$ *atms-of-msu* (*clauses S*)
and *consistent-interp* (*lits-of* (*trail S*)) $\wedge$ *no-dup* (*trail S*)
using *assms* **unfolding** *dpll*$_W$*-all-inv-def lits-of-def* **by** *auto*

lemma *rtranclp-dpll*$_W$*-all-inv*:
assumes *rtranclp dpll*$_W$ *S S'*
and *dpll*$_W$*-all-inv S*
shows *dpll*$_W$*-all-inv S'*
using *assms rtranclp-dpll*$_W$*-inv*[*OF assms*(*1*)] **unfolding** *dpll*$_W$*-all-inv-def lits-of-def* **by** *blast*

lemma *dpll*$_W$*-all-inv*:
assumes *dpll*$_W$ *S S'*
and *dpll*$_W$*-all-inv S*
shows *dpll*$_W$*-all-inv S'*
using *assms rtranclp-dpll*$_W$*-all-inv* **by** *blast*

lemma *rtranclp-dpll*$_W$*-inv-starting-from-0*:
assumes *rtranclp dpll*$_W$ *S S'*
and *inv*: *trail S* = []
shows *dpll*$_W$*-all-inv S'*
proof −
have *dpll*$_W$*-all-inv S*
using *assms* **unfolding** *all-decomposition-implies-def dpll*$_W$*-all-inv-def* **by** *auto*
then show *?thesis* **using** *rtranclp-dpll*$_W$*-all-inv*[*OF assms*(*1*)] **by** *blast*
qed

lemma *dpll*$_W$*-can-do-step*:
assumes *consistent-interp* (*set M*)
and *distinct M*
and *atm-of* ` (*set M*) $\subseteq$ *atms-of-msu N*
shows *rtranclp dpll*$_W$ ([], *N*) (*map* (λ*M*. *Marked M* ()) *M*, *N*)
using *assms*
proof (*induct M*)
case *Nil*
then show *?case* **by** *auto*
next
case (*Cons L M*)
then have *undefined-lit* (*map* (λ*M*. *Marked M* ()) *M*) *L*
unfolding *defined-lit-def consistent-interp-def* **by** *auto*
moreover have *atm-of L* $\in$ *atms-of-msu N* **using** *Cons.prems*(*3*) **by** *auto*
ultimately have *dpll*$_W$ (*map* (λ*M*. *Marked M* ()) *M*, *N*) (*map* (λ*M*. *Marked M* ()) (*L* # *M*), *N*)
using *dpll*$_W$*.decided* **by** *auto*
moreover have *consistent-interp* (*set M*) **and** *distinct M* **and** *atm-of* ` *set M* $\subseteq$ *atms-of-msu N*
using *Cons.prems* **unfolding** *consistent-interp-def* **by** *auto*
ultimately show *?case* **using** *Cons.hyps* **by** *auto*
qed

definition *conclusive-dpll*$_W$*-state* (*S*:: *'v dpll*$_W$*-state*) $\longleftrightarrow$

(*trail* S $\models$*asm clauses* S $\vee$ (($\forall L \in$ *set* (*trail* S). $\neg$*is-marked* L)
$\wedge$ ($\exists C \in\#$ *clauses* S. *trail* S $\models$*as CNot* C)))

lemma *dpll*$_W$*-strong-completeness*:
assumes *set* M $\models$*sm* N
and *consistent-interp* (*set* M)
and *distinct* M
and *atm-of* ‘ (*set* M) $\subseteq$ *atms-of-msu* N
shows *dpll*$_W$** ([], N) (*map* (λM. *Marked* M ()) M, N)
and *conclusive-dpll*$_W$*-state* (*map* (λM. *Marked* M ()) M, N)
proof −
show *rtranclp dpll*$_W$ ([], N) (*map* (λM. *Marked* M ()) M, N) **using** *dpll*$_W$*-can-do-step assms* **by** *auto*
have *map* (λM. *Marked* M ()) M $\models$*asm* N **using** *assms*(*1*) *true-annots-marked-true-cls* **by** *auto*
then show *conclusive-dpll*$_W$*-state* (*map* (λM. *Marked* M ()) M, N)
unfolding *conclusive-dpll*$_W$*-state-def* **by** *auto*
qed

lemma *dpll*$_W$*-sound*:
assumes
rtranclp dpll$_W$ ([], N) (M, N) **and**
$\forall S$. $\neg$*dpll*$_W$ (M, N) S
shows M $\models$*asm* N $\longleftrightarrow$ *satisfiable* (*set-mset* N) (**is** *?A* $\longleftrightarrow$ *?B*)
proof
let *?M'*= *lits-of* M
assume *?A*
then have *?M'* $\models$*sm* N **by** (*simp add*: *true-annots-true-cls*)
moreover have *consistent-interp ?M'*
using *rtranclp-dpll*$_W$*-inv-starting-from-0*[*OF assms*(*1*)] **by** *auto*
ultimately show *?B* **by** *auto*
next
assume *?B*
show *?A*
proof (*rule ccontr*)
assume *n*: $\neg$ *?A*
have ($\exists L$. *undefined-lit* M L $\wedge$ *atm-of* L $\in$ *atms-of-msu* N) $\vee$ ($\exists D \in\# N$. M $\models$*as CNot* D)
proof −
obtain D :: *'a clause* **where** D: $D \in\#$ N **and** $\neg$ M $\models$*a* D
using *n* **unfolding** *true-annots-def Ball-def* **by** *auto*
then have ($\exists L$. *undefined-lit* M L $\wedge$ *atm-of* L $\in$ *atms-of* D) $\vee$ M $\models$*as CNot* D
unfolding *true-annots-def Ball-def CNot-def true-annot-def*
using *atm-of-lit-in-atms-of true-annot-iff-marked-or-true-lit true-cls-def* **by** *blast*
then show *?thesis*

using D **apply** *auto* **by** (*meson atms-of-atms-of-ms-mono mem-set-mset-iff subset-eq*)
qed
moreover {
assume $\exists L$. *undefined-lit* M L $\wedge$ *atm-of* L $\in$ *atms-of-msu* N
then have *False* **using** *assms*(*2*) *decided* **by** *fastforce*
}
moreover {
assume $\exists D \in\# N$. M $\models$*as CNot* D
then obtain D **where** *DN*: $D \in\#$ N **and** *MD*: M $\models$*as CNot* D **by** *auto*
{

assume ∀ *l* ∈ *set M*. ¬ *is-marked l*
moreover have *dpll*$_W$*-all-inv* ([], *N*)
using *assms* **unfolding** *all-decomposition-implies-def dpll*$_W$*-all-inv-def* **by** *auto*
ultimately have *unsatisfiable* (*set-mset N*)
using *only-propagated-vars-unsat*[*of M D set-mset N*] *DN MD*
rtranclp-dpll$_W$*-all-inv*[*OF assms*(*1*)] **by** *force*
then have *False* **using** ‹*?B*› **by** *blast*
}
moreover {
assume *l*: ∃ *l* ∈ *set M*. *is-marked l*
then have *False*
using *backtrack*[*of* (*M*, *N*) - - - *D*] *DN MD assms*(*2*)
backtrack-split-some-is-marked-then-snd-has-hd[*OF l*]
by (*metis backtrack-split-snd-hd-marked fst-conv list.distinct*(*1*) *list.sel*(*1*) *snd-conv*)
}
ultimately have *False* **by** *blast*
}
ultimately show *False* **by** *blast*
qed
qed

16.3 Termination

definition *dpll*$_W$*-mes M n* =
map (λ*l*. *if is-marked l then 2 else* (*1*::*nat*)) (*rev M*) @ *replicate* (*n* − *length M*) *3*

lemma *length-dpll*$_W$*-mes*:
assumes *length M* ≤ *n*
shows *length* (*dpll*$_W$*-mes M n*) = *n*
using *assms* **unfolding** *dpll*$_W$*-mes-def* **by** *auto*

lemma *distinctcard-atm-of-lit-of-eq-length*:
assumes *no-dup S*
shows *card* (*atm-of* ‘ *lits-of S*) = *length S*
using *assms* **by** (*induct S*) (*auto simp add*: *image-image lits-of-def*)

lemma *dpll*$_W$*-card-decrease*:
assumes *dpll*: *dpll*$_W$ *S S'* **and** *length* (*trail S'*) ≤ *card vars*
and *length* (*trail S*) ≤ *card vars*
shows (*dpll*$_W$*-mes* (*trail S'*) (*card vars*), *dpll*$_W$*-mes* (*trail S*) (*card vars*))
∈ *lexn* {(*a*, *b*). *a* < *b*} (*card vars*)
using *assms*
proof (*induct rule*: *dpll*$_W$*.induct*)
case (*propagate C L S*)
have *m*: *map* (λ*l*. *if is-marked l then 2 else 1*) (*rev* (*trail S*))
@ *replicate* (*card vars* − *length* (*trail S*)) *3*
= *map* (λ*l*. *if is-marked l then 2 else 1*) (*rev* (*trail S*)) @ *3*
replicate (*card vars* − *Suc* (*length* (*trail S*))) *3*
using *propagate.prems*[*simplified*] **using** *Suc-diff-le* **by** *fastforce*
then show *?case*
using *propagate.prems*(*1*) **unfolding** *dpll*$_W$*-mes-def* **by** (*fastforce simp add*: *lexn-conv assms*(*2*))
next
case (*decided S L*)
have *m*: *map* (λ*l*. *if is-marked l then 2 else 1*) (*rev* (*trail S*))
@ *replicate* (*card vars* − *length* (*trail S*)) *3*
= *map* (λ*l*. *if is-marked l then 2 else 1*) (*rev* (*trail S*)) @ *3*

replicate (*card vars* − *Suc* (*length* (*trail S*))) *3*
using *decided.prems*[*simplified*] **using** *Suc-diff-le* **by** *fastforce*
then show *?case*
using *decided.prems* **unfolding** $dpll_W$*-mes-def* **by** (*force simp add*: *lexn-conv assms*(*2*))
next
case (*backtrack S M′ L M D*)
have *L*: *is-marked L* **using** *backtrack.hyps*(*2*) **by** *auto*
have *S*: *trail S* = *M′* @ *L* # *M*
using *backtrack.hyps*(*1*) *backtrack-split-list-eq*[*of trail S*] **by** *auto*
show *?case*
using *backtrack.prems L* **unfolding** $dpll_W$*-mes-def S* **by** (*fastforce simp add*: *lexn-conv assms*(*2*))
qed

Proposition theorem 2.8.7 page 73 of CW

lemma $dpll_W$*-card-decrease′*:
assumes *dpll*: $dpll_W$ *S S′*
and *atm-incl*: *atm-of* ‘ *lits-of* (*trail S*) ⊆ *atms-of-msu* (*clauses S*)
and *no-dup*: *no-dup* (*trail S*)
shows ($dpll_W$*-mes* (*trail S′*) (*card* (*atms-of-msu* (*clauses S′*))),
$dpll_W$*-mes* (*trail S*) (*card* (*atms-of-msu* (*clauses S*)))) ∈ *lex* {(*a*, *b*). *a* < *b*}
proof −
have *finite* (*atms-of-msu* (*clauses S*)) **unfolding** *atms-of-ms-def* **by** *auto*
then have *1*: *length* (*trail S*) ≤ *card* (*atms-of-msu* (*clauses S*))
using *distinctcard-atm-of-lit-of-eq-length*[*OF no-dup*] *atm-incl card-mono* **by** *metis*

moreover
have *no-dup′*: *no-dup* (*trail S′*) **using** *dpll* $dpll_W$*-distinct-inv no-dup* **by** *blast*
have *SS′*: *clauses S′* = *clauses S* **using** *dpll* **by** (*auto dest*!: $dpll_W$*-same-clauses*)
have *atm-incl′*: *atm-of* ‘ *lits-of* (*trail S′*) ⊆ *atms-of-msu* (*clauses S′*)
using *atm-incl dpll* $dpll_W$*-vars-in-snd-inv*[*OF dpll*] **by** *force*
have *finite* (*atms-of-msu* (*clauses S′*))
unfolding *atms-of-ms-def* **by** *auto*
then have *2*: *length* (*trail S′*) ≤ *card* (*atms-of-msu* (*clauses S*))
using *distinctcard-atm-of-lit-of-eq-length*[*OF no-dup′*] *atm-incl′ card-mono SS′* **by** *metis*

ultimately have ($dpll_W$*-mes* (*trail S′*) (*card* (*atms-of-msu* (*clauses S*))),
$dpll_W$*-mes* (*trail S*) (*card* (*atms-of-msu* (*clauses S*))))
∈ *lexn* {(*a*, *b*). *a* < *b*} (*card* (*atms-of-msu* (*clauses S*)))
using $dpll_W$*-card-decrease*[*OF assms*(*1*), *of atms-of-msu* (*clauses S*)] **by** *blast*
then have ($dpll_W$*-mes* (*trail S′*) (*card* (*atms-of-msu* (*clauses S*))),
$dpll_W$*-mes* (*trail S*) (*card* (*atms-of-msu* (*clauses S*)))) ∈ *lex* {(*a*, *b*). *a* < *b*}
unfolding *lex-def* **by** *auto*
then show ($dpll_W$*-mes* (*trail S′*) (*card* (*atms-of-msu* (*clauses S′*))),
$dpll_W$*-mes* (*trail S*) (*card* (*atms-of-msu* (*clauses S*)))) ∈ *lex* {(*a*, *b*). *a* < *b*}
using $dpll_W$*-same-clauses*[*OF assms*(*1*)] **by** *auto*
qed

lemma *wf-lexn*: *wf* (*lexn* {(*a*, *b*). (*a*::*nat*) < *b*} (*card* (*atms-of-msu* (*clauses S*))))
proof −
have *m*: {(*a*, *b*). *a* < *b*} = *measure id* **by** *auto*
show *?thesis* **apply** (*rule wf-lexn*) **unfolding** *m* **by** *auto*
qed

lemma $dpll_W$*-wf*:
wf {(*S′*, *S*). $dpll_W$*-all-inv S* ∧ $dpll_W$ *S S′*}

apply (*rule wf-wf-if-measure'*[*OF wf-lex-less*, *of - -*
λS. *dpll*$_W$*-mes* (*trail S*) (*card* (*atms-of-msu* (*clauses S*)))])
using *dpll*$_W$*-card-decrease'* **by** *fast*

lemma *dpll*$_W$*-tranclp-star-commute*:
$\{(S', S).$ *dpll*$_W$*-all-inv* $S \wedge$ *dpll*$_W$ $S\ S'\}^+ = \{(S', S).$ *dpll*$_W$*-all-inv* $S \wedge$ *tranclp dpll*$_W$ $S\ S'\}$
(**is** *?A* = *?B*)
proof
{ **fix** $S\ S'$
assume $(S, S') \in$ *?A*
then have $(S, S') \in$ *?B*
by (*induct rule*: *trancl.induct*, *auto*)
}
then show *?A* $\subseteq$ *?B* **by** *blast*
{ **fix** $S\ S'$
assume $(S, S') \in$ *?B*
then have *dpll*$_W$$^{++}$ $S'\ S$ **and** *dpll*$_W$*-all-inv* S' **by** *auto*
then have $(S, S') \in$ *?A*
proof (*induct rule*: *tranclp.induct*)
case *r-into-trancl*
then show *?case* **by** (*simp-all add*: *r-into-trancl'*)
next
case (*trancl-into-trancl* $S\ S'\ S''$)
then have $(S', S) \in \{a.$ *case a of* $(S', S) \Rightarrow$ *dpll*$_W$*-all-inv* $S \wedge$ *dpll*$_W$ $S\ S'\}^+$ **by** *blast*
moreover have *dpll*$_W$*-all-inv* S'
using *rtranclp-dpll*$_W$*-all-inv*[*OF tranclp-into-rtranclp*[*OF trancl-into-trancl.hyps*(*1*)]]
trancl-into-trancl.prems **by** *auto*
ultimately have $(S'', S') \in \{(pa, p).$ *dpll*$_W$*-all-inv* $p \wedge$ *dpll*$_W$ $p\ pa\}^+$
using ‹*dpll*$_W$*-all-inv* S'› *trancl-into-trancl.hyps*(*3*) **by** *blast*
then show *?case*
using ‹$(S', S) \in \{a.$ *case a of* $(S', S) \Rightarrow$ *dpll*$_W$*-all-inv* $S \wedge$ *dpll*$_W$ $S\ S'\}^+$› **by** *auto*
qed
}
then show *?B* $\subseteq$ *?A* **by** *blast*
qed

lemma *dpll*$_W$*-wf-tranclp*: *wf* $\{(S', S).$ *dpll*$_W$*-all-inv* $S \wedge$ *dpll*$_W$$^{++}$ $S\ S'\}$
unfolding *dpll*$_W$*-tranclp-star-commute*[*symmetric*] **by** (*simp add*: *dpll*$_W$*-wf wf-trancl*)

lemma *dpll*$_W$*-wf-plus*:
shows *wf* $\{(S', ([], N))|\ S'.$ *dpll*$_W$$^{++}$ $([], N)\ S'\}$ (**is** *wf ?P*)
apply (*rule wf-subset*[*OF dpll*$_W$*-wf-tranclp*, *of ?P*])
using *assms* **unfolding** *dpll*$_W$*-all-inv-def* **by** *auto*

16.4 Final States

lemma *dpll*$_W$*-no-more-step-is-a-conclusive-state*:
assumes $\forall S'.\ \neg$*dpll*$_W$ $S\ S'$
shows *conclusive-dpll*$_W$*-state S*
proof –
have *vars*: $\forall s \in$ *atms-of-msu* (*clauses S*). $s \in$ *atm-of* ‘ *lits-of* (*trail S*)
proof (*rule ccontr*)
assume ¬ ($\forall s \in$*atms-of-msu* (*clauses S*). $s \in$ *atm-of* ‘ *lits-of* (*trail S*))
then obtain L **where**
L-in-atms: $L \in$ *atms-of-msu* (*clauses S*) **and**

L-notin-trail: $L \notin$ *atm-of* ‘ *lits-of* (*trail S*) **by** *metis*
obtain L' **where** L': *atm-of* $L' = L$ **by** (*meson literal.sel*(*2*))
then have *undefined-lit* (*trail S*) L'
unfolding *Marked-Propagated-in-iff-in-lits-of* **by** (*metis L-notin-trail atm-of-uminus imageI*)
then show *False* **using** $dpll_W$*.decided assms*(*1*) *L-in-atms* L' **by** *blast*
qed
show *?thesis*
proof (*rule ccontr*)
assume *not-final*: $\neg$ *?thesis*
then have
$\neg$ *trail S* $\models$*asm clauses S* **and**
($\exists$ $L\in$*set* (*trail S*). *is-marked L*) $\vee$ ($\forall$ $C\in\#$*clauses S*. $\neg$*trail S* $\models$*as CNot C*)
unfolding *conclusive-*$dpll_W$*-state-def* **by** *auto*
moreover {
assume $\exists$ $L\in$*set* (*trail S*). *is-marked L*
then obtain *L* M' *M* **where** *L*: *backtrack-split* (*trail S*) = (M', *L* # *M*)
using *backtrack-split-some-is-marked-then-snd-has-hd* **by** *blast*
obtain *D* **where** *D* $\in\#$ *clauses S* **and** $\neg$ *trail S* $\models$*a D*
using ‹$\neg$ *trail S* $\models$*asm clauses S*› **unfolding** *true-annots-def* **by** *auto*
then have $\forall$ $s\in$*atms-of-ms* {*D*}. *s* $\in$ *atm-of* ‘ *lits-of* (*trail S*)
using *vars* **unfolding** *atms-of-ms-def* **by** *auto*
then have *trail S* $\models$*as CNot D*
using *all-variables-defined-not-imply-cnot*[*of D*] ‹$\neg$ *trail S* $\models$*a D*› **by** *auto*
moreover have *is-marked L*
using *L* **by** (*metis backtrack-split-snd-hd-marked list.distinct*(*1*) *list.sel*(*1*) *snd-conv*)
ultimately have *False*
using *assms*(*1*) $dpll_W$*.backtrack L* ‹*D* $\in\#$ *clauses S*› ‹*trail S* $\models$*as CNot D*› **by** *blast*
}
moreover {
assume *tr*: $\forall$ $C\in\#$*clauses S*. $\neg$*trail S* $\models$*as CNot C*
obtain *C* **where** *C-in-cls*: *C* $\in\#$ *clauses S* **and** *trC*: $\neg$ *trail S* $\models$*a C*
using ‹$\neg$ *trail S* $\models$*asm clauses S*› **unfolding** *true-annots-def* **by** *auto*
have $\forall$ $s\in$*atms-of-ms* {*C*}. *s* $\in$ *atm-of* ‘ *lits-of* (*trail S*)
using *vars* ‹*C* $\in\#$ *clauses S*› **unfolding** *atms-of-ms-def* **by** *auto*
then have *trail S* $\models$*as CNot C*
by (*meson C-in-cls tr trC all-variables-defined-not-imply-cnot*)
then have *False* **using** *tr C-in-cls* **by** *auto*
}
ultimately show *False* **by** *blast*
qed
qed

lemma $dpll_W$*-conclusive-state-correct*:
assumes $dpll_W^{**}$ ([], *N*) (*M*, *N*) **and** *conclusive-*$dpll_W$*-state* (*M*, *N*)
shows *M* $\models$*asm N* $\longleftrightarrow$ *satisfiable* (*set-mset N*) (**is** *?A* $\longleftrightarrow$ *?B*)
proof
let *?M'*= *lits-of M*
assume *?A*
then have *?M'* $\models$*sm N* **by** (*simp add*: *true-annots-true-cls*)
moreover have *consistent-interp ?M'*
using *rtranclp-*$dpll_W$*-inv-starting-from-0*[*OF assms*(*1*)] **by** *auto*
ultimately show *?B* **by** *auto*
next
assume *?B*
show *?A*

proof (*rule ccontr*)
assume *n*: ¬ *?A*
have *no-mark*: ∀ *L*∈*set M*. ¬ *is-marked L* ∃ *C* ∈# *N*. *M* ⊨*as CNot C*
using *n assms*(*2*) **unfolding** *conclusive-dpll$_W$-state-def* **by** *auto*
moreover obtain *D* **where** *DN*: *D* ∈# *N* **and** *MD*: *M* ⊨*as CNot D* **using** *no-mark* **by** *auto*
ultimately have *unsatisfiable* (*set-mset N*)
using *only-propagated-vars-unsat rtranclp-dpll$_W$-all-inv*[*OF assms*(*1*)]
unfolding *dpll$_W$-all-inv-def* **by** *force*
then show *False* **using** ‹*?B*› **by** *blast*
qed
qed

16.5 Link with NOT's DPLL

interpretation *dpll$_W$-$_{NOT}$*: *dpll-with-backtrack* **.**

lemma *state-eq$_{NOT}$-iff-eq*[*iff*, *simp*]: *dpll$_W$-$_{NOT}$.state-eq$_{NOT}$ S T* ⟷ *S* = *T*
unfolding *dpll$_W$-$_{NOT}$.state-eq$_{NOT}$-def* **by** (*cases S*, *cases T*) *auto*

declare *dpll$_W$-$_{NOT}$.state-simp$_{NOT}$*[*simp del*]

lemma *dpll$_W$-dpll$_W$-bj*:
assumes *inv*: *dpll$_W$-all-inv S* **and** *dpll*: *dpll$_W$ S T*
shows *dpll$_W$-$_{NOT}$.dpll-bj S T*
using *dpll inv*
apply (*induction rule*: *dpll$_W$.induct*)
using *dpll$_W$-$_{NOT}$.dpll-bj.simps* **apply** *fastforce*
using *dpll$_W$-$_{NOT}$.bj-decide$_{NOT}$* **apply** *fastforce*
apply (*frule dpll$_W$-$_{NOT}$.backtrack.intros*[*of - - - - -*], *simp-all*)
apply (*rule dpll$_W$-$_{NOT}$.dpll-bj.bj-backjump*)
apply (*rule dpll$_W$-$_{NOT}$.backtrack-is-backjump''*,
simp-all add: *dpll$_W$-all-inv-def*)
done

lemma *dpll$_W$-bj-dpll*:
assumes *inv*: *dpll$_W$-all-inv S* **and** *dpll*: *dpll$_W$-$_{NOT}$.dpll-bj S T*
shows *dpll$_W$ S T*
using *dpll*
apply (*induction rule*: *dpll$_W$-$_{NOT}$.dpll-bj.induct*)
apply (*elim dpll$_W$-$_{NOT}$.decideE*, *cases S*)
using *decided* **apply** *fastforce*
apply (*elim dpll$_W$-$_{NOT}$.propagateE*, *cases S*)
using *dpll$_W$.simps* **apply** *fastforce*
apply (*elim dpll$_W$-$_{NOT}$.backjumpE*, *cases S*)
by (*simp add*: *dpll$_W$.simps dpll-with-backtrack.backtrack.simps*)

lemma *rtranclp-dpll$_W$-rtranclp-dpll$_W$-$_{NOT}$*:
assumes *dpll$_W$** S T* **and** *dpll$_W$-all-inv S*
shows *dpll$_W$-$_{NOT}$.dpll-bj** S T*
using *assms* **apply** (*induction*)
apply *simp*
by (*auto intro*: *rtranclp-dpll$_W$-all-inv dpll$_W$-dpll$_W$-bj rtranclp.rtrancl-into-rtrancl*)

lemma *rtranclp-dpll-rtranclp-dpll$_W$*:
assumes *dpll$_W$-$_{NOT}$.dpll-bj** S T* **and** *dpll$_W$-all-inv S*
shows *dpll$_W$** S T*

using *assms* **apply** (*induction*)
apply *simp*
by (*auto intro*: *dpll$_W$-bj-dpll rtranclp.rtrancl-into-rtrancl rtranclp-dpll$_W$-all-inv*)

lemma *dpll-conclusive-state-correctness*:
assumes *dpll$_W$-$_{NOT}$.dpll-bj*** ([], *N*) (*M*, *N*) **and** *conclusive-dpll$_W$-state* (*M*, *N*)
shows $M \models asm\ N \longleftrightarrow$ *satisfiable* (*set-mset N*)
proof −
have *dpll$_W$-all-inv* ([], *N*)
unfolding *dpll$_W$-all-inv-def* **by** *auto*
show *?thesis*
apply (*rule dpll$_W$-conclusive-state-correct*)
apply (*simp add*: ‹*dpll$_W$-all-inv* ([], *N*)› *assms*(*1*) *rtranclp-dpll-rtranclp-dpll$_W$*)
using *assms*(*2*) **by** *simp*
qed

end
theory *CDCL-W-Level*
imports *Partial-Annotated-Clausal-Logic*
begin

16.5.1 Level of literals and clauses

Getting the level of a variable, implies that the list has to be reversed. Here is the funtion after reversing.

fun *get-rev-level* :: (*'v*, *nat*, *'a*) *marked-lits* ⇒ *nat* ⇒ *'v literal* ⇒ *nat* **where**
get-rev-level [] - - = *0* |
get-rev-level (*Marked l level* # *Ls*) *n L* =
(*if atm-of l* = *atm-of L then level else get-rev-level Ls level L*) |
get-rev-level (*Propagated l* - # *Ls*) *n L* =
(*if atm-of l* = *atm-of L then n else get-rev-level Ls n L*)

abbreviation *get-level M L* ≡ *get-rev-level* (*rev M*) *0 L*

lemma *get-rev-level-uminus*[*simp*]: *get-rev-level M n*(−*L*) = *get-rev-level M n L*
by (*induct arbitrary*: *n rule*: *get-rev-level.induct*) *auto*

lemma *atm-of-notin-get-rev-level-eq-0*[*simp*]:
assumes *atm-of L* ∉ *atm-of* ‘ *lits-of M*
shows *get-rev-level M n L* = *0*
using *assms* **by** (*induct M arbitrary*: *n rule*: *marked-lit-list-induct*) *auto*

lemma *get-rev-level-ge-0-atm-of-in*:
assumes *get-rev-level M n L* > *n*
shows *atm-of L* ∈ *atm-of* ‘ *lits-of M*
using *assms* **by** (*induct M arbitrary*: *n rule*: *marked-lit-list-induct*) *fastforce+*

In *get-rev-level* (resp. *get-level*), the beginning (resp. the end) can be skipped if the literal is not in the beginning (resp. the end).

lemma *get-rev-level-skip*[*simp*]:
assumes *atm-of L* ∉ *atm-of* ‘ *lits-of M*
shows *get-rev-level* (*M* @ *Marked K i* # *M'*) *n L* = *get-rev-level* (*Marked K i* # *M'*) *i L*
using *assms* **by** (*induct M arbitrary*: *n i rule*: *marked-lit-list-induct*) *auto*

lemma *get-rev-level-notin-end*[*simp*]:

assumes *atm-of* $L \notin$ *atm-of* ' *lits-of* M'
shows *get-rev-level* $(M\ @\ M')$ n L = *get-rev-level* M n L
using *assms* **by** (*induct M arbitrary*: *n rule*: *marked-lit-list-induct*) *auto*

If the literal is at the beginning, then the end can be skipped

lemma *get-rev-level-skip-end*[*simp*]:
assumes *atm-of* $L \in$ *atm-of* ' *lits-of* M
shows *get-rev-level* $(M\ @\ M')$ n L = *get-rev-level* M n L
using *assms* **by** (*induct arbitrary*: *n rule*: *marked-lit-list-induct*) *auto*

lemma *get-level-skip-beginning*:
assumes *atm-of* $L' \neq$ *atm-of* (*lit-of* K)
shows *get-level* $(K \# M)$ L' = *get-level* M L'
using *assms* **by** *auto*

lemma *get-level-skip-beginning-not-marked-rev*:
assumes *atm-of* $L \notin$ *atm-of* ' *lit-of* '(*set* S)
and $\forall s \in$ *set* S. $\neg$*is-marked* s
shows *get-level* $(M\ @$ *rev* $S)$ L = *get-level* M L
using *assms* **by** (*induction S rule*: *marked-lit-list-induct*) *auto*

lemma *get-level-skip-beginning-not-marked*[*simp*]:
assumes *atm-of* $L \notin$ *atm-of* ' *lit-of* '(*set* S)
and $\forall s \in$ *set* S. $\neg$*is-marked* s
shows *get-level* $(M\ @\ S)$ L = *get-level* M L
using *get-level-skip-beginning-not-marked-rev*[*of L rev S M*] *assms* **by** *auto*

lemma *get-rev-level-skip-beginning-not-marked*[*simp*]:
assumes *atm-of* $L \notin$ *atm-of* ' *lit-of* '(*set* S)
and $\forall s \in$ *set* S. $\neg$*is-marked* s
shows *get-rev-level* (*rev* S @ *rev* M) *0* L = *get-level* M L
using *get-level-skip-beginning-not-marked-rev*[*of L rev S M*] *assms* **by** *auto*

lemma *get-level-skip-in-all-not-marked*:
fixes M :: ('*a*, *nat*, '*b*) *marked-lit list* **and** L :: '*a literal*
assumes $\forall m \in$ *set* M. $\neg$ *is-marked* m
and *atm-of* $L \in$ *atm-of* ' *lit-of* ' (*set* M)
shows *get-rev-level* M n $L = n$
using *assms* **by** (*induction M rule*: *marked-lit-list-induct*) *auto*

lemma *get-level-skip-all-not-marked*[*simp*]:
fixes M
defines $M' \equiv$ *rev* M
assumes $\forall m \in$ *set* M. $\neg$ *is-marked* m
shows *get-level* M L = *0*
proof −
have M: M = *rev* M'
unfolding M'-*def* **by** *auto*
show *?thesis*
using *assms* **unfolding** M **by** (*induction M' rule*: *marked-lit-list-induct*) *auto*
qed

abbreviation *MMax* $M \equiv$ *Max* (*set-mset* M)

the {#*0*::'*a*#} is there to ensures that the set is not empty.

definition *get-maximum-level* :: ($'a$, *nat*, $'b$) *marked-lit list* ⇒ $'a$ *literal multiset* ⇒ *nat*
where
get-maximum-level M D = *MMax* ({#*0*#} + *image-mset* (*get-level M*) *D*)

lemma *get-maximum-level-ge-get-level*:
L ∈# *D* ⟹ *get-maximum-level M D* ≥ *get-level M L*
unfolding *get-maximum-level-def* **by** *auto*

lemma *get-maximum-level-empty*[*simp*]:
get-maximum-level M {#} = *0*
unfolding *get-maximum-level-def* **by** *auto*

lemma *get-maximum-level-exists-lit-of-max-level*:
D ≠ {#} ⟹ ∃ *L*∈# *D*. *get-level M L* = *get-maximum-level M D*
unfolding *get-maximum-level-def*
apply (*induct D*)
apply *simp*
by (*rename-tac D x*, *case-tac D* = {#}) (*auto simp add*: *max-def*)

lemma *get-maximum-level-empty-list*[*simp*]:
get-maximum-level [] *D* = *0*
unfolding *get-maximum-level-def* **by** (*simp add*: *image-constant-conv*)

lemma *get-maximum-level-single*[*simp*]:
get-maximum-level M {#*L*#} = *get-level M L*
unfolding *get-maximum-level-def* **by** *simp*

lemma *get-maximum-level-plus*:
get-maximum-level M (*D* + *D'*) = *max* (*get-maximum-level M D*) (*get-maximum-level M D'*)
by (*induct D*) (*auto simp add*: *get-maximum-level-def*)

lemma *get-maximum-level-exists-lit*:
assumes *n*: *n* > *0*
and *max*: *get-maximum-level M D* = *n*
shows ∃ *L* ∈#*D*. *get-level M L* = *n*
proof −
have *f*: *finite* (*insert 0* ((λ*L*. *get-level M L*) ‘ *set-mset D*)) **by** *auto*
then have *n* ∈ ((λ*L*. *get-level M L*) ‘ *set-mset D*)
using *n max Max-in*[*OF f*] **unfolding** *get-maximum-level-def* **by** *simp*
then show ∃ *L* ∈# *D*. *get-level M L* = *n* **by** *auto*
qed

lemma *get-maximum-level-skip-first*[*simp*]:
assumes *atm-of L* ∉ *atms-of D*
shows *get-maximum-level* (*Propagated L C* # *M*) *D* = *get-maximum-level M D*
using *assms* **unfolding** *get-maximum-level-def atms-of-def*
atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
by (*smt atm-of-in-atm-of-set-in-uminus get-level-skip-beginning image-iff marked-lit.sel*(*2*)
multiset.map-cong0)

lemma *get-maximum-level-skip-beginning*:
assumes *DH*: *atms-of D* ⊆ *atm-of* ‘*lits-of H*
shows *get-maximum-level* (*c* @ *Marked Kh i* # *H*) *D* = *get-maximum-level H D*
proof −

have (*get-rev-level* (*rev H* @ *Marked Kh i* # *rev c*) *0*) ' *set-mset D*
= (*get-rev-level* (*rev H*) *0*) ' *set-mset D*
using *DH* **unfolding** *atms-of-def*
by (*metis* (*no-types*, *lifting*) *get-rev-level-skip-end image-cong image-subset-iff lits-of-rev*)+
then show *?thesis* **using** *DH* **unfolding** *get-maximum-level-def* **by** *auto*
qed

lemma *get-maximum-level-D-single-propagated*:
get-maximum-level [*Propagated x21 x22*] *D* = *0*
proof –
have *A*: *insert 0* ((λ*L*. *0*) ' (*set-mset D* ∩ {*L*. *atm-of x21* = *atm-of L*})
∪ (λ*L*. *0*) ' (*set-mset D* ∩ {*L*. *atm-of x21* ≠ *atm-of L*})) = {*0*}
by *auto*
show *?thesis* **unfolding** *get-maximum-level-def* **by** (*simp add*: *A*)
qed

lemma *get-maximum-level-skip-notin*:
assumes *D*: ∀ *L*∈#*D*. *atm-of L* ∈ *atm-of* '*lits-of M*
shows *get-maximum-level M D* = *get-maximum-level* (*Propagated x21 x22* # *M*) *D*
proof –
have *A*: (*get-rev-level* (*rev M* @ [*Propagated x21 x22*]) *0*) ' *set-mset D*
= (*get-rev-level* (*rev M*) *0*) ' *set-mset D*
using *D* **by** (*auto intro*!: *image-cong simp add*: *lits-of-def*)
show *?thesis* **unfolding** *get-maximum-level-def* **by** (*auto simp*: *A*)
qed

lemma *get-maximum-level-skip-un-marked-not-present*:
assumes ∀ *L*∈#*D*. *atm-of L* ∈ *atm-of* ' *lits-of aa* **and**
∀ *m*∈*set M*. ¬ *is-marked m*
shows *get-maximum-level aa D* = *get-maximum-level* (*M* @ *aa*) *D*
using *assms* **by** (*induction M rule*: *marked-lit-list-induct*)
(*auto intro*!: *get-maximum-level-skip-notin*[*of D* - @ *aa*] *simp add*: *image-Un*)

fun *get-maximum-possible-level*:: ('*b*, *nat*, '*c*) *marked-lit list* ⇒ *nat* **where**
get-maximum-possible-level [] = *0* |
get-maximum-possible-level (*Marked K i* # *l*) = *max i* (*get-maximum-possible-level l*) |
get-maximum-possible-level (*Propagated* - - # *l*) = *get-maximum-possible-level l*

lemma *get-maximum-possible-level-append*[*simp*]:
get-maximum-possible-level (*M*@*M'*)
= *max* (*get-maximum-possible-level M*) (*get-maximum-possible-level M'*)
by (*induct M rule*: *marked-lit-list-induct*) *auto*

lemma *get-maximum-possible-level-rev*[*simp*]:
get-maximum-possible-level (*rev M*) = *get-maximum-possible-level M*
by (*induct M rule*: *marked-lit-list-induct*) *auto*

lemma *get-maximum-possible-level-ge-get-rev-level*:
max (*get-maximum-possible-level M*) *i* ≥ *get-rev-level M i L*
by (*induct M arbitrary*: *i rule*: *marked-lit-list-induct*) (*auto simp add*: *le-max-iff-disj*)

lemma *get-maximum-possible-level-ge-get-level*[*simp*]:
get-maximum-possible-level M ≥ *get-level M L*
using *get-maximum-possible-level-ge-get-rev-level*[*of rev* - *0*] **by** *auto*

lemma *get-maximum-possible-level-ge-get-maximum-level*[*simp*]:
get-maximum-possible-level M $\geq$ *get-maximum-level M D*
using *get-maximum-level-exists-lit-of-max-level* **unfolding** *Bex-mset-def*
by (*metis get-maximum-level-empty get-maximum-possible-level-ge-get-level le0*)

fun *get-all-mark-of-propagated* **where**
get-all-mark-of-propagated [] = [] |
get-all-mark-of-propagated (*Marked - - # L*) = *get-all-mark-of-propagated L* |
get-all-mark-of-propagated (*Propagated - mark # L*) = *mark # get-all-mark-of-propagated L*

lemma *get-all-mark-of-propagated-append*[*simp*]:
get-all-mark-of-propagated (*A @ B*) = *get-all-mark-of-propagated A @ get-all-mark-of-propagated B*
by (*induct A rule*: *marked-lit-list-induct*) *auto*

16.5.2 Properties about the levels

fun *get-all-levels-of-marked* :: ('*b*, '*a*, '*c*) *marked-lit list* $\Rightarrow$ '*a list* **where**
get-all-levels-of-marked [] = [] |
get-all-levels-of-marked (*Marked l level # Ls*) = *level # get-all-levels-of-marked Ls* |
get-all-levels-of-marked (*Propagated - - # Ls*) = *get-all-levels-of-marked Ls*

lemma *get-all-levels-of-marked-nil-iff-not-is-marked*:
get-all-levels-of-marked xs = [] $\longleftrightarrow$ ($\forall$ *x* $\in$ *set xs*. $\neg$*is-marked x*)
using *assms* **by** (*induction xs rule*: *marked-lit-list-induct*) *auto*

lemma *get-all-levels-of-marked-cons*:
get-all-levels-of-marked (*a # b*) =
(*if is-marked a then* [*level-of a*] *else* []) @ *get-all-levels-of-marked b*
by (*cases a*) *simp-all*

lemma *get-all-levels-of-marked-append*[*simp*]:
get-all-levels-of-marked (*a @ b*) = *get-all-levels-of-marked a @ get-all-levels-of-marked b*
by (*induct a*) (*simp-all add*: *get-all-levels-of-marked-cons*)

lemma *in-get-all-levels-of-marked-iff-decomp*:
i $\in$ *set* (*get-all-levels-of-marked M*) $\longleftrightarrow$ ($\exists$ *c K c'*. *M* = *c @ Marked K i # c'*) (**is** *?A* $\longleftrightarrow$ *?B*)
proof
assume *?B*
then show *?A* **by** *auto*
next
assume *?A*
then show *?B*
apply (*induction M rule*: *marked-lit-list-induct*)
apply *auto*[]
apply (*metis append-Cons append-Nil get-all-levels-of-marked.simps*(*2*) *set-ConsD*)
by (*metis append-Cons get-all-levels-of-marked.simps*(*3*))
qed

lemma *get-rev-level-less-max-get-all-levels-of-marked*:
get-rev-level M n L $\leq$ *Max* (*set* (*n # get-all-levels-of-marked M*))
by (*induct M arbitrary*: *n rule*: *get-all-levels-of-marked.induct*)
(*simp-all add*: *max.coboundedI2*)

lemma *get-rev-level-ge-min-get-all-levels-of-marked*:
assumes *atm-of L* $\in$ *atm-of* ‘ *lits-of M*
shows *get-rev-level M n L* $\geq$ *Min* (*set* (*n # get-all-levels-of-marked M*))

using *assms* **by** (*induct M arbitrary*: *n rule*: *get-all-levels-of-marked.induct*)
(*auto simp add*: *min-le-iff-disj*)

lemma *get-all-levels-of-marked-rev-eq-rev-get-all-levels-of-marked*[*simp*]:
get-all-levels-of-marked (*rev M*) = *rev* (*get-all-levels-of-marked M*)
by (*induct M rule*: *get-all-levels-of-marked.induct*)
(*simp-all add*: *max.coboundedI2*)

lemma *get-maximum-possible-level-max-get-all-levels-of-marked*:
get-maximum-possible-level M = *Max* (*insert 0* (*set* (*get-all-levels-of-marked M*)))
by (*induct M rule*: *marked-lit-list-induct*) (*auto simp*: *insert-commute*)

lemma *get-rev-level-in-levels-of-marked*:
get-rev-level M n L ∈ {*0*, *n*} ∪ *set* (*get-all-levels-of-marked M*)
by (*induction M arbitrary*: *n rule*: *marked-lit-list-induct*) (*force simp add*: *atm-of-eq-atm-of*)+

lemma *get-rev-level-in-atms-in-levels-of-marked*:
atm-of L ∈ *atm-of* ‘ (*lits-of M*) ⟹ *get-rev-level M n L* ∈ {*n*} ∪ *set* (*get-all-levels-of-marked M*)
by (*induction M arbitrary*: *n rule*: *marked-lit-list-induct*) (*auto simp add*: *atm-of-eq-atm-of*)

lemma *get-all-levels-of-marked-no-marked*:
(∀ *l*∈*set Ls*. ¬ *is-marked l*) ⟷ *get-all-levels-of-marked Ls* = []
by (*induction Ls*) (*auto simp add*: *get-all-levels-of-marked-cons*)

lemma *get-level-in-levels-of-marked*:
get-level M L ∈ {*0*} ∪ *set* (*get-all-levels-of-marked M*)
using *get-rev-level-in-levels-of-marked*[*of rev M 0 L*] **by** *auto*

The zero is here to avoid empty-list issues with *last*:

lemma *get-level-get-rev-level-get-all-levels-of-marked*:
assumes *atm-of L* ∉ *atm-of* ‘ (*lits-of M*)
shows *get-level* (*K* @ *M*) *L* = *get-rev-level* (*rev K*) (*last* (*0* # *get-all-levels-of-marked* (*rev M*)))
L
using *assms*
proof (*induct M arbitrary*: *K*)
case *Nil*
then show *?case* **by** *auto*
next
case (*Cons a M*)
then have *H*: ⋀*K*. *get-level* (*K* @ *M*) *L*
= *get-rev-level* (*rev K*) (*last* (*0* # *get-all-levels-of-marked* (*rev M*))) *L*
by *auto*
have *get-level* ((*K* @ [*a*])@ *M*) *L*
= *get-rev-level* (*a* # *rev K*) (*last* (*0* # *get-all-levels-of-marked* (*rev M*))) *L*
using *H*[*of K* @ [*a*]] **by** *simp*
then show *?case* **using** *Cons*(*2*) **by** (*cases a*) *auto*
qed

lemma *get-rev-level-can-skip-correctly-ordered*:
assumes
no-dup M **and**
atm-of L ∉ *atm-of* ‘ (*lits-of M*) **and**
get-all-levels-of-marked M = *rev* [*Suc 0*..<*Suc* (*length* (*get-all-levels-of-marked M*))]
shows *get-rev-level* (*rev M* @ *K*) *0 L* = *get-rev-level K* (*length* (*get-all-levels-of-marked M*)) *L*

using *assms*
proof (*induct M arbitrary*: *K rule*: *marked-lit-list-induct*)
  case *nil*
  then show *?case* **by** *simp*
next
  case (*marked* L' *i M K*)
  then have
    i: i = *Suc* (*length* (*get-all-levels-of-marked M*)) **and**
    get-all-levels-of-marked M = *rev* [*Suc 0*..<*Suc* (*length* (*get-all-levels-of-marked M*))]
    by *auto*
  then have *get-rev-level* (*rev M* @ (*Marked* L' *i* # *K*)) *0 L*
    = *get-rev-level* (*Marked* L' *i* # *K*) (*length* (*get-all-levels-of-marked M*)) *L*
    using *marked* **by** *auto*
  then show *?case* **using** *marked* **unfolding** *i* **by** *auto*
next
  case (*proped* L' *D M K*)
  then have *get-all-levels-of-marked M* = *rev* [*Suc 0*..<*Suc* (*length* (*get-all-levels-of-marked M*))]
    by *auto*
  then have *get-rev-level* (*rev M* @ (*Propagated* L' *D* # *K*)) *0 L*
    = *get-rev-level* (*Propagated* L' *D* # *K*) (*length* (*get-all-levels-of-marked M*)) *L*
    using *proped* **by** *auto*
  then show *?case* **using** *proped* **by** *auto*
qed

lemma *get-level-skip-beginning-hd-get-all-levels-of-marked*:
  assumes *atm-of L* $\notin$ *atm-of* ' *lits-of S*
  and *get-all-levels-of-marked S* $\neq$ []
  shows *get-level* (*M*@ *S*) *L* = *get-rev-level* (*rev M*) (*hd* (*get-all-levels-of-marked S*)) *L*
  using *assms*
proof (*induction S arbitrary*: *M rule*: *marked-lit-list-induct*)
  case *nil*
  then show *?case* **by** (*auto simp add*: *lits-of-def*)
next
  case (*marked K m*) **note** *notin* = *this*(*2*)
  then show *?case* **by** (*auto simp add*: *lits-of-def*)
next
  case (*proped L l*) **note** *IH* = *this*(*1*) **and** *L* = *this*(*2*) **and** *neq* = *this*(*3*)
  show *?case* **using** *IH*[*of M*@[*Propagated L l*]] *L neq* **by** (*auto simp add*: *atm-of-eq-atm-of*)
qed

end
theory *CDCL-W*
imports *Partial-Annotated-Clausal-Logic List-More CDCL-W-Level Wellfounded-More*

begin
declare *set-mset-minus-replicate-mset*[*simp*]

lemma *Bex-set-set-Bex-set*[*iff*]: ($\exists x \in$*set-mset C*. *P*) $\longleftrightarrow$ ($\exists x \in \# C$. *P*)
  by *auto*

17 Weidenbach's CDCL

sledgehammer-params[*verbose*, *e spass cvc4 z3 verit*]
declare *upt.simps*(*2*)[*simp del*]

17.1 The State

locale *state*$_W$ =
fixes
trail :: $'st \Rightarrow ('v, nat, 'v\ clause)$ *marked-lits* **and**
init-clss :: $'st \Rightarrow 'v$ *clauses* **and**
learned-clss :: $'st \Rightarrow 'v$ *clauses* **and**
backtrack-lvl :: $'st \Rightarrow nat$ **and**
conflicting :: $'st \Rightarrow 'v$ *clause option* **and**

cons-trail :: $('v, nat, 'v\ clause)$ *marked-lit* $\Rightarrow 'st \Rightarrow 'st$ **and**
tl-trail :: $'st \Rightarrow 'st$ **and**
add-init-cls :: $'v\ clause \Rightarrow 'st \Rightarrow 'st$ **and**
add-learned-cls :: $'v\ clause \Rightarrow 'st \Rightarrow 'st$ **and**
remove-cls :: $'v\ clause \Rightarrow 'st \Rightarrow 'st$ **and**
update-backtrack-lvl :: $nat \Rightarrow 'st \Rightarrow 'st$ **and**
update-conflicting :: $'v$ *clause option* $\Rightarrow 'st \Rightarrow 'st$ **and**

init-state :: $'v\ clauses \Rightarrow 'st$ **and**
restart-state :: $'st \Rightarrow 'st$
assumes
trail-cons-trail[*simp*]:
$\bigwedge L\ st.$ *undefined-lit* (*trail st*) (*lit-of L*) $\Longrightarrow$ *trail* (*cons-trail L st*) = *L* # *trail st* **and**
trail-tl-trail[*simp*]: $\bigwedge st.$ *trail* (*tl-trail st*) = *tl* (*trail st*) **and**
trail-add-init-cls[*simp*]:
$\bigwedge st\ C.$ *no-dup* (*trail st*) $\Longrightarrow$ *trail* (*add-init-cls C st*) = *trail st* **and**
trail-add-learned-cls[*simp*]:
$\bigwedge C\ st.$ *no-dup* (*trail st*) $\Longrightarrow$ *trail* (*add-learned-cls C st*) = *trail st* **and**
trail-remove-cls[*simp*]:
$\bigwedge C\ st.$ *trail* (*remove-cls C st*) = *trail st* **and**
trail-update-backtrack-lvl[*simp*]: $\bigwedge st\ C.$ *trail* (*update-backtrack-lvl C st*) = *trail st* **and**
trail-update-conflicting[*simp*]: $\bigwedge C\ st.$ *trail* (*update-conflicting C st*) = *trail st* **and**

init-clss-cons-trail[*simp*]:
$\bigwedge M\ st.$ *undefined-lit* (*trail st*) (*lit-of M*)$\Longrightarrow$ *init-clss* (*cons-trail M st*) = *init-clss st*
and
init-clss-tl-trail[*simp*]:
$\bigwedge st.$ *init-clss* (*tl-trail st*) = *init-clss st* **and**
init-clss-add-init-cls[*simp*]:
$\bigwedge st\ C.$ *no-dup* (*trail st*) $\Longrightarrow$ *init-clss* (*add-init-cls C st*) = {#*C*#} + *init-clss st* **and**
init-clss-add-learned-cls[*simp*]:
$\bigwedge C\ st.$ *no-dup* (*trail st*) $\Longrightarrow$ *init-clss* (*add-learned-cls C st*) = *init-clss st* **and**
init-clss-remove-cls[*simp*]:
$\bigwedge C\ st.$ *init-clss* (*remove-cls C st*) = *remove-mset C* (*init-clss st*) **and**
init-clss-update-backtrack-lvl[*simp*]:
$\bigwedge st\ C.$ *init-clss* (*update-backtrack-lvl C st*) = *init-clss st* **and**
init-clss-update-conflicting[*simp*]:
$\bigwedge C\ st.$ *init-clss* (*update-conflicting C st*) = *init-clss st* **and**

learned-clss-cons-trail[*simp*]:
$\bigwedge M\ st.$ *undefined-lit* (*trail st*) (*lit-of M*) $\Longrightarrow$
learned-clss (*cons-trail M st*) = *learned-clss st* **and**
learned-clss-tl-trail[*simp*]:
$\bigwedge st.$ *learned-clss* (*tl-trail st*) = *learned-clss st* **and**
learned-clss-add-init-cls[*simp*]:
$\bigwedge st\ C.$ *no-dup* (*trail st*) $\Longrightarrow$ *learned-clss* (*add-init-cls C st*) = *learned-clss st* **and**

learned-clss-add-learned-cls[*simp*]:
$\bigwedge C\ st.$ *no-dup* (*trail st*) $\Longrightarrow$ *learned-clss* (*add-learned-cls C st*) = {#*C*#} + *learned-clss st*
and
learned-clss-remove-cls[*simp*]:
$\bigwedge C\ st.$ *learned-clss* (*remove-cls C st*) = *remove-mset C* (*learned-clss st*) **and**
learned-clss-update-backtrack-lvl[*simp*]:
$\bigwedge st\ C.$ *learned-clss* (*update-backtrack-lvl C st*) = *learned-clss st* **and**
learned-clss-update-conflicting[*simp*]:
$\bigwedge C\ st.$ *learned-clss* (*update-conflicting C st*) = *learned-clss st* **and**

backtrack-lvl-cons-trail[*simp*]:
$\bigwedge M\ st.$ *undefined-lit* (*trail st*) (*lit-of M*) $\Longrightarrow$
backtrack-lvl (*cons-trail M st*) = *backtrack-lvl st* **and**
backtrack-lvl-tl-trail[*simp*]:
$\bigwedge st.$ *backtrack-lvl* (*tl-trail st*) = *backtrack-lvl st* **and**
backtrack-lvl-add-init-cls[*simp*]:
$\bigwedge st\ C.$ *no-dup* (*trail st*) $\Longrightarrow$ *backtrack-lvl* (*add-init-cls C st*) = *backtrack-lvl st* **and**
backtrack-lvl-add-learned-cls[*simp*]:
$\bigwedge C\ st.$ *no-dup* (*trail st*) $\Longrightarrow$ *backtrack-lvl* (*add-learned-cls C st*) = *backtrack-lvl st* **and**
backtrack-lvl-remove-cls[*simp*]:
$\bigwedge C\ st.$ *backtrack-lvl* (*remove-cls C st*) = *backtrack-lvl st* **and**
backtrack-lvl-update-backtrack-lvl[*simp*]:
$\bigwedge st\ k.$ *backtrack-lvl* (*update-backtrack-lvl k st*) = *k* **and**
backtrack-lvl-update-conflicting[*simp*]:
$\bigwedge C\ st.$ *backtrack-lvl* (*update-conflicting C st*) = *backtrack-lvl st* **and**

conflicting-cons-trail[*simp*]:
$\bigwedge M\ st.$ *undefined-lit* (*trail st*) (*lit-of M*) $\Longrightarrow$
conflicting (*cons-trail M st*) = *conflicting st* **and**
conflicting-tl-trail[*simp*]:
$\bigwedge st.$ *conflicting* (*tl-trail st*) = *conflicting st* **and**
conflicting-add-init-cls[*simp*]:
$\bigwedge st\ C.$ *no-dup* (*trail st*) $\Longrightarrow$ *conflicting* (*add-init-cls C st*) = *conflicting st* **and**
conflicting-add-learned-cls[*simp*]:
$\bigwedge C\ st.$ *no-dup* (*trail st*) $\Longrightarrow$ *conflicting* (*add-learned-cls C st*) = *conflicting st* **and**
conflicting-remove-cls[*simp*]:
$\bigwedge C\ st.$ *conflicting* (*remove-cls C st*) = *conflicting st* **and**
conflicting-update-backtrack-lvl[*simp*]:
$\bigwedge st\ C.$ *conflicting* (*update-backtrack-lvl C st*) = *conflicting st* **and**
conflicting-update-conflicting[*simp*]:
$\bigwedge C\ st.$ *conflicting* (*update-conflicting C st*) = *C* **and**

init-state-trail[*simp*]: $\bigwedge N.$ *trail* (*init-state N*) = [] **and**
init-state-clss[*simp*]: $\bigwedge N.$ *init-clss* (*init-state N*) = *N* **and**
init-state-learned-clss[*simp*]: $\bigwedge N.$ *learned-clss* (*init-state N*) = {#} **and**
init-state-backtrack-lvl[*simp*]: $\bigwedge N.$ *backtrack-lvl* (*init-state N*) = *0* **and**
init-state-conflicting[*simp*]: $\bigwedge N.$ *conflicting* (*init-state N*) = *None* **and**

trail-restart-state[*simp*]: *trail* (*restart-state S*) = [] **and**
init-clss-restart-state[*simp*]: *init-clss* (*restart-state S*) = *init-clss S* **and**
learned-clss-restart-state[*intro*]: *learned-clss* (*restart-state S*) $\subseteq$# *learned-clss S* **and**
backtrack-lvl-restart-state[*simp*]: *backtrack-lvl* (*restart-state S*) = *0* **and**
conflicting-restart-state[*simp*]: *conflicting* (*restart-state S*) = *None*
begin

definition *clauses* :: *'st* ⇒ *'v clauses* **where**
clauses S = *init-clss S* + *learned-clss S*

lemma
shows
clauses-cons-trail[*simp*]:
undefined-lit (*trail S*) (*lit-of M*) ⟹ *clauses* (*cons-trail M S*) = *clauses S* **and**

clss-tl-trail[*simp*]: *clauses* (*tl-trail S*) = *clauses S* **and**
clauses-add-learned-cls-unfolded:
no-dup (*trail S*) ⟹ *clauses* (*add-learned-cls U S*) = {#*U*#} + *learned-clss S* + *init-clss S*
and
clauses-add-init-cls[*simp*]:
no-dup (*trail S*) ⟹ *clauses* (*add-init-cls N S*) = {#*N*#} + *init-clss S* + *learned-clss S* **and**
clauses-update-backtrack-lvl[*simp*]: *clauses* (*update-backtrack-lvl k S*) = *clauses S* **and**
clauses-update-conflicting[*simp*]: *clauses* (*update-conflicting D S*) = *clauses S* **and**
clauses-remove-cls[*simp*]:
clauses (*remove-cls C S*) = *clauses S* − *replicate-mset* (*count* (*clauses S*) *C*) *C* **and**
clauses-add-learned-cls[*simp*]:
no-dup (*trail S*) ⟹ *clauses* (*add-learned-cls C S*) = {#*C*#} + *clauses S* **and**
clauses-restart[*simp*]: *clauses* (*restart-state S*) ⊆# *clauses S* **and**
clauses-init-state[*simp*]: ⋀*N*. *clauses* (*init-state N*) = *N*
prefer *9* **using** *clauses-def learned-clss-restart-state* **apply** *fastforce*
by (*auto simp*: *ac-simps replicate-mset-plus clauses-def intro*: *multiset-eqI*)

abbreviation *state* :: *'st* ⇒ (*'v*, *nat*, *'v clause*) *marked-lit list* × *'v clauses* × *'v clauses*
× *nat* × *'v clause option* **where**
state S ≡ (*trail S*, *init-clss S*, *learned-clss S*, *backtrack-lvl S*, *conflicting S*)

abbreviation *incr-lvl* :: *'st* ⇒ *'st* **where**
incr-lvl S ≡ *update-backtrack-lvl* (*backtrack-lvl S* + *1*) *S*

definition *state-eq* :: *'st* ⇒ *'st* ⇒ *bool* (**infix** ∼ *50*) **where**
S ∼ *T* ⟷ *state S* = *state T*

lemma *state-eq-ref*[*simp*, *intro*]:
S ∼ *S*
unfolding *state-eq-def* **by** *auto*

lemma *state-eq-sym*:
S ∼ *T* ⟷ *T* ∼ *S*
unfolding *state-eq-def* **by** *auto*

lemma *state-eq-trans*:
S ∼ *T* ⟹ *T* ∼ *U* ⟹ *S* ∼ *U*
unfolding *state-eq-def* **by** *auto*

lemma
shows
state-eq-trail: *S* ∼ *T* ⟹ *trail S* = *trail T* **and**
state-eq-init-clss: *S* ∼ *T* ⟹ *init-clss S* = *init-clss T* **and**
state-eq-learned-clss: *S* ∼ *T* ⟹ *learned-clss S* = *learned-clss T* **and**
state-eq-backtrack-lvl: *S* ∼ *T* ⟹ *backtrack-lvl S* = *backtrack-lvl T* **and**
state-eq-conflicting: *S* ∼ *T* ⟹ *conflicting S* = *conflicting T* **and**
state-eq-clauses: *S* ∼ *T* ⟹ *clauses S* = *clauses T* **and**

state-eq-undefined-lit: $S \sim T \Longrightarrow$ *undefined-lit* (*trail* S) L = *undefined-lit* (*trail* T) L
unfolding *state-eq-def clauses-def* **by** *auto*

lemmas *state-simp*[*simp*] = *state-eq-trail state-eq-init-clss state-eq-learned-clss state-eq-backtrack-lvl state-eq-conflicting state-eq-clauses state-eq-undefined-lit*

lemma *atms-of-ms-learned-clss-restart-state-in-atms-of-ms-learned-clssI*[*intro*]:
$x \in$ *atms-of-msu* (*learned-clss* (*restart-state* S)) $\Longrightarrow x \in$ *atms-of-msu* (*learned-clss* S)
by (*meson atms-of-ms-mono learned-clss-restart-state set-mset-mono subsetCE*)

function *reduce-trail-to* :: *'a list* $\Rightarrow$ *'st* $\Rightarrow$ *'st* **where**
reduce-trail-to F S =
(*if length* (*trail* S) = *length* F $\vee$ *trail* S = [] *then* S *else reduce-trail-to* F (*tl-trail* S))
by *fast+*
termination
by (*relation measure* (λ(-, S). *length* (*trail* S))) *simp-all*

declare *reduce-trail-to.simps*[*simp del*]

lemma
shows
reduce-trail-to-nil[*simp*]: *trail* S = [] $\Longrightarrow$ *reduce-trail-to* F S = S **and**
reduce-trail-to-eq-length[*simp*]: *length* (*trail* S) = *length* F $\Longrightarrow$ *reduce-trail-to* F S = S
by (*auto simp*: *reduce-trail-to.simps*)

lemma *reduce-trail-to-length-ne*:
length (*trail* S) $\neq$ *length* F $\Longrightarrow$ *trail* S $\neq$ [] $\Longrightarrow$
reduce-trail-to F S = *reduce-trail-to* F (*tl-trail* S)
by (*auto simp*: *reduce-trail-to.simps*)

lemma *trail-reduce-trail-to-length-le*:
assumes *length* F > *length* (*trail* S)
shows *trail* (*reduce-trail-to* F S) = []
using *assms* **apply** (*induction* F S *rule*: *reduce-trail-to.induct*)
by (*metis* (*no-types*, *hide-lams*) *length-tl less-imp-diff-less less-irrefl trail-tl-trail reduce-trail-to.simps*)

lemma *trail-reduce-trail-to-nil*[*simp*]:
trail (*reduce-trail-to* [] S) = []
apply (*induction* []:: (*'v*, *nat*, *'v clause*) *marked-lits* S *rule*: *reduce-trail-to.induct*)
by (*metis length-0-conv reduce-trail-to-length-ne reduce-trail-to-nil*)

lemma *clauses-reduce-trail-to-nil*:
clauses (*reduce-trail-to* [] S) = *clauses* S
proof (*induction* [] S *rule*: *reduce-trail-to.induct*)
case (*1 Sa*)
then have *clauses* (*reduce-trail-to* ([]::*'a list*) (*tl-trail Sa*)) = *clauses* (*tl-trail Sa*)
$\vee$ *trail Sa* = []
by *fastforce*
then show *clauses* (*reduce-trail-to* ([]::*'a list*) *Sa*) = *clauses Sa*
by (*metis* (*no-types*) *length-0-conv reduce-trail-to-eq-length clss-tl-trail reduce-trail-to-length-ne*)
qed

lemma *reduce-trail-to-skip-beginning*:

assumes *trail S* = *F'* @ *F*
shows *trail* (*reduce-trail-to F S*) = *F*
using *assms* **by** (*induction F' arbitrary*: *S*) (*auto simp*: *reduce-trail-to-length-ne*)

lemma *clauses-reduce-trail-to*[*simp*]:
clauses (*reduce-trail-to F S*) = *clauses S*
apply (*induction F S rule*: *reduce-trail-to.induct*)
by (*metis clss-tl-trail reduce-trail-to.simps*)

lemma *conflicting-update-trial*[*simp*]:
conflicting (*reduce-trail-to F S*) = *conflicting S*
apply (*induction F S rule*: *reduce-trail-to.induct*)
by (*metis conflicting-tl-trail reduce-trail-to.simps*)

lemma *backtrack-lvl-update-trial*[*simp*]:
backtrack-lvl (*reduce-trail-to F S*) = *backtrack-lvl S*
apply (*induction F S rule*: *reduce-trail-to.induct*)
by (*metis backtrack-lvl-tl-trail reduce-trail-to.simps*)

lemma *init-clss-update-trial*[*simp*]:
init-clss (*reduce-trail-to F S*) = *init-clss S*
apply (*induction F S rule*: *reduce-trail-to.induct*)
by (*metis init-clss-tl-trail reduce-trail-to.simps*)

lemma *learned-clss-update-trial*[*simp*]:
learned-clss (*reduce-trail-to F S*) = *learned-clss S*
apply (*induction F S rule*: *reduce-trail-to.induct*)
by (*metis learned-clss-tl-trail reduce-trail-to.simps*)

lemma *trail-eq-reduce-trail-to-eq*:
trail S = *trail T* $\Longrightarrow$ *trail* (*reduce-trail-to F S*) = *trail* (*reduce-trail-to F T*)
apply (*induction F S arbitrary*: *T rule*: *reduce-trail-to.induct*)
by (*metis trail-tl-trail reduce-trail-to.simps*)

lemma *reduce-trail-to-state-eq*$_{NOT}$*-compatible*:
assumes *ST*: *S* $\sim$ *T*
shows *reduce-trail-to F S* $\sim$ *reduce-trail-to F T*
proof −
have *trail* (*reduce-trail-to F S*) = *trail* (*reduce-trail-to F T*)
using *trail-eq-reduce-trail-to-eq*[*of S T F*] *ST* **by** *auto*
then show *?thesis* **using** *ST* **by** (*auto simp del*: *state-simp simp*: *state-eq-def*)
qed

lemma *reduce-trail-to-trail-tl-trail-decomp*[*simp*]:
trail S = *F'* @ *Marked K d* # *F* $\Longrightarrow$ (*trail* (*reduce-trail-to F S*)) = *F*
apply (*rule reduce-trail-to-skip-beginning*[*of - F'* @ *Marked K d* # []])
by (*cases F'*) (*auto simp add*:*tl-append reduce-trail-to-skip-beginning*)

lemma *reduce-trail-to-add-learned-cls*[*simp*]:
no-dup (*trail S*) $\Longrightarrow$
trail (*reduce-trail-to F* (*add-learned-cls C S*)) = *trail* (*reduce-trail-to F S*)
by (*rule trail-eq-reduce-trail-to-eq*) *auto*

lemma *reduce-trail-to-add-init-cls*[*simp*]:
no-dup (*trail S*) $\Longrightarrow$

trail (*reduce-trail-to F* (*add-init-cls C S*)) = *trail* (*reduce-trail-to F S*)
by (*rule trail-eq-reduce-trail-to-eq*) *auto*

lemma *reduce-trail-to-remove-learned-cls*[*simp*]:
trail (*reduce-trail-to F* (*remove-cls C S*)) = *trail* (*reduce-trail-to F S*)
by (*rule trail-eq-reduce-trail-to-eq*) *auto*

lemma *reduce-trail-to-update-conflicting*[*simp*]:
trail (*reduce-trail-to F* (*update-conflicting C S*)) = *trail* (*reduce-trail-to F S*)
by (*rule trail-eq-reduce-trail-to-eq*) *auto*

lemma *reduce-trail-to-update-backtrack-lvl*[*simp*]:
trail (*reduce-trail-to F* (*update-backtrack-lvl C S*)) = *trail* (*reduce-trail-to F S*)
by (*rule trail-eq-reduce-trail-to-eq*) *auto*

lemma *in-get-all-marked-decomposition-marked-or-empty*:
assumes (*a*, *b*) ∈ *set* (*get-all-marked-decomposition M*)
shows *a* = [] ∨ (*is-marked* (*hd a*))
using *assms*
proof (*induct M arbitrary*: *a b*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons m M*)
show *?case*
proof (*cases m*)
case (*Marked l mark*)
then show *?thesis* **using** *Cons* **by** *auto*
next
case (*Propagated l mark*)
then show *?thesis* **using** *Cons* **by** (*cases get-all-marked-decomposition M*) *force+*
qed
qed

lemma *in-get-all-marked-decomposition-trail-update-trail*[*simp*]:
assumes *H*: (*L* # *M1*, *M2*) ∈ *set* (*get-all-marked-decomposition* (*trail S*))
shows *trail* (*reduce-trail-to M1 S*) = *M1*
proof −
obtain *K mark* **where**
L: *L* = *Marked K mark*
using *H* **by** (*cases L*) (*auto dest*!: *in-get-all-marked-decomposition-marked-or-empty*)
obtain *c* **where**
tr-S: *trail S* = *c* @ *M2* @ *L* # *M1*
using *H* **by** *auto*
show *?thesis*
by (*rule reduce-trail-to-trail-tl-trail-decomp*[*of - c* @ *M2 K mark*])
(*auto simp*: *tr-S L*)
qed

fun *append-trail* **where**
append-trail [] *S* = *S* |
append-trail (*L* # *M*) *S* = *append-trail M* (*cons-trail L S*)

lemma *trail-append-trail*:
no-dup (*M* @ *trail S*) ⟹ *trail* (*append-trail M S*) = *rev M* @ *trail S*
by (*induction M arbitrary*: *S*) (*auto simp*: *defined-lit-map*)

lemma *init-clss-append-trail*:
no-dup $(M\ @\ trail\ S) \Longrightarrow init\text{-}clss\ (append\text{-}trail\ M\ S) = init\text{-}clss\ S$
by (*induction M arbitrary*: *S*) (*auto simp*: *defined-lit-map*)

lemma *learned-clss-append-trail*:
no-dup $(M\ @\ trail\ S) \Longrightarrow learned\text{-}clss\ (append\text{-}trail\ M\ S) = learned\text{-}clss\ S$
by (*induction M arbitrary*: *S*) (*auto simp*: *defined-lit-map*)

lemma *conflicting-append-trail*:
no-dup $(M\ @\ trail\ S) \Longrightarrow conflicting\ (append\text{-}trail\ M\ S) = conflicting\ S$
by (*induction M arbitrary*: *S*) (*auto simp*: *defined-lit-map*)

lemma *backtrack-lvl-append-trail*:
no-dup $(M\ @\ trail\ S) \Longrightarrow backtrack\text{-}lvl\ (append\text{-}trail\ M\ S) = backtrack\text{-}lvl\ S$
by (*induction M arbitrary*: *S*) (*auto simp*: *defined-lit-map*)

lemma *clauses-append-trail*:
no-dup $(M\ @\ trail\ S) \Longrightarrow clauses\ (append\text{-}trail\ M\ S) = clauses\ S$
by (*induction M arbitrary*: *S*) (*auto simp*: *defined-lit-map*)

lemmas *state-access-simp* =
trail-append-trail init-clss-append-trail learned-clss-append-trail backtrack-lvl-append-trail
clauses-append-trail conflicting-append-trail

This function is useful for proofs to speak of a global trail change, but is a bad for programs and code in general.

fun *delete-trail-and-rebuild* **where**
delete-trail-and-rebuild M S = *append-trail* (*rev M*) (*reduce-trail-to* ([]:: *'v list*) *S*)

end

17.2 Special Instantiation: using Triples as State

17.3 CDCL Rules

Because of the strategy we will later use, we distinguish propagate, conflict from the other rules

locale
$cdcl_W$*-ops* =
$state_W$ *trail init-clss learned-clss backtrack-lvl conflicting cons-trail tl-trail add-init-cls*
add-learned-cls remove-cls update-backtrack-lvl update-conflicting init-state
restart-state
for
trail :: *'st* ⇒ (*'v*, *nat*, *'v clause*) *marked-lits* **and**
init-clss :: *'st* ⇒ *'v clauses* **and**
learned-clss :: *'st* ⇒ *'v clauses* **and**
backtrack-lvl :: *'st* ⇒ *nat* **and**
conflicting :: *'st* ⇒*'v clause option* **and**

cons-trail :: (*'v*, *nat*, *'v clause*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-init-cls :: *'v clause* ⇒ *'st* ⇒ *'st* **and**
add-learned-cls :: *'v clause* ⇒ *'st* ⇒ *'st* **and**
remove-cls :: *'v clause* ⇒ *'st* ⇒ *'st* **and**
update-backtrack-lvl :: *nat* ⇒ *'st* ⇒ *'st* **and**

update-conflicting :: *'v clause option* ⇒ *'st* ⇒ *'st* **and**

init-state :: *'v clauses* ⇒ *'st* **and**
restart-state :: *'st* ⇒ *'st*
begin

inductive *propagate* :: *'st* ⇒ *'st* ⇒ *bool* **where**
propagate-rule[*intro*]:
$state\ S = (M,\ N,\ U,\ k,\ None) \Longrightarrow\ C + \{\#L\#\} \in\#\ clauses\ S \Longrightarrow M \models as\ CNot\ C$
$\Longrightarrow undefined\text{-}lit\ (trail\ S)\ L$
$\Longrightarrow T \sim cons\text{-}trail\ (Propagated\ L\ (C + \{\#L\#\}))\ S$
$\Longrightarrow propagate\ S\ T$
inductive-cases *propagateE*[*elim*]: *propagate S T*
thm *propagateE*

inductive *conflict* :: *'st* ⇒ *'st* ⇒ *bool* **where**
conflict-rule[*intro*]: $state\ S = (M,\ N,\ U,\ k,\ None) \Longrightarrow D \in\#\ clauses\ S \Longrightarrow M \models as\ CNot\ D$
$\Longrightarrow T \sim update\text{-}conflicting\ (Some\ D)\ S$
$\Longrightarrow conflict\ S\ T$

inductive-cases *conflictE*[*elim*]: *conflict S S'*

inductive *backtrack* :: *'st* ⇒ *'st* ⇒ *bool* **where**
backtrack-rule[*intro*]: $state\ S = (M,\ N,\ U,\ k,\ Some\ (D + \{\#L\#\}))$
$\Longrightarrow (Marked\ K\ (i+1)\ \#\ M1,\ M2) \in set\ (get\text{-}all\text{-}marked\text{-}decomposition\ M)$
$\Longrightarrow get\text{-}level\ M\ L = k$
$\Longrightarrow get\text{-}level\ M\ L = get\text{-}maximum\text{-}level\ M\ (D+\{\#L\#\})$
$\Longrightarrow get\text{-}maximum\text{-}level\ M\ D = i$
$\Longrightarrow T \sim cons\text{-}trail\ (Propagated\ L\ (D+\{\#L\#\}))$
$(reduce\text{-}trail\text{-}to\ M1$
$(add\text{-}learned\text{-}cls\ (D + \{\#L\#\})$
$(update\text{-}backtrack\text{-}lvl\ i$
$(update\text{-}conflicting\ None\ S))))$
$\Longrightarrow backtrack\ S\ T$
inductive-cases *backtrackE*[*elim*]: *backtrack S S'*
thm *backtrackE*

inductive *decide* :: *'st* ⇒ *'st* ⇒ *bool* **where**
decide-rule[*intro*]: $state\ S = (M,\ N,\ U,\ k,\ None)$
$\Longrightarrow undefined\text{-}lit\ M\ L \Longrightarrow atm\text{-}of\ L \in atms\text{-}of\text{-}msu\ (init\text{-}clss\ S)$
$\Longrightarrow T \sim cons\text{-}trail\ (Marked\ L\ (k+1))\ (incr\text{-}lvl\ S)$
$\Longrightarrow decide\ S\ T$
inductive-cases *decideE*[*elim*]: *decide S S'*
thm *decideE*

inductive *skip* :: *'st* ⇒ *'st* ⇒ *bool* **where**
skip-rule[*intro*]: $state\ S = (Propagated\ L\ C'\ \#\ M,\ N,\ U,\ k,\ Some\ D) \Longrightarrow\ -L \notin\#\ D \Longrightarrow D \neq \{\#\}$
$\Longrightarrow T \sim tl\text{-}trail\ S$
$\Longrightarrow skip\ S\ T$
inductive-cases *skipE*[*elim*]: *skip S S'*
thm *skipE*

$get\text{-}maximum\text{-}level\ (Propagated\ L\ (C + \{\#L\#\})\ \#\ M)\ D = k \vee k = 0$ is equivalent to $get\text{-}maximum\text{-}level\ (Propagated\ L\ (C + \{\#L\#\})\ \#\ M)\ D = k$

inductive *resolve* :: *'st* ⇒ *'st* ⇒ *bool* **where**

resolve-rule[*intro*]:
  state S = (*Propagated* L (C + {#L#}) # M, N, U, k, *Some* (D + {#−L#}))
  $\Longrightarrow$ *get-maximum-level* (*Propagated* L (C + {#L#}) # M) D = k
  $\Longrightarrow$ T ∼ *update-conflicting* (*Some* (D #∪ C)) (*tl-trail* S)
  $\Longrightarrow$ *resolve* S T
inductive-cases *resolveE*[*elim*]: *resolve* S S'
thm *resolveE*

inductive *restart* :: $'st \Rightarrow 'st \Rightarrow$ *bool* **where**
restart: *state* S = (M, N, U, k, *None*) $\Longrightarrow$ ¬M ⊨*asm clauses* S
$\Longrightarrow$ T ∼ *restart-state* S
$\Longrightarrow$ *restart* S T
inductive-cases *restartE*[*elim*]: *restart* S T
thm *restartE*

We add the condition C ∉# *init-clss* S, to maintain consistency even without the strategy.

inductive *forget* :: $'st \Rightarrow 'st \Rightarrow$ *bool* **where**
forget-rule: *state* S = (M, N, {#C#} + U, k, *None*)
  $\Longrightarrow$ ¬M ⊨*asm clauses* S
  $\Longrightarrow$ C ∉ *set* (*get-all-mark-of-propagated* (*trail* S))
  $\Longrightarrow$ C ∉# *init-clss* S
  $\Longrightarrow$ C ∈# *learned-clss* S
  $\Longrightarrow$ T ∼ *remove-cls* C S
  $\Longrightarrow$ *forget* S T
inductive-cases *forgetE*[*elim*]: *forget* S T

inductive $cdcl_W$*-rf* :: $'st \Rightarrow 'st \Rightarrow$ *bool* **for** S :: $'st$ **where**
restart: *restart* S T $\Longrightarrow$ $cdcl_W$*-rf* S T |
forget: *forget* S T $\Longrightarrow$ $cdcl_W$*-rf* S T

inductive $cdcl_W$*-bj* :: $'st \Rightarrow 'st \Rightarrow$ *bool* **where**
skip[*intro*]: *skip* S S' $\Longrightarrow$ $cdcl_W$*-bj* S S' |
resolve[*intro*]: *resolve* S S' $\Longrightarrow$ $cdcl_W$*-bj* S S' |
backtrack[*intro*]: *backtrack* S S' $\Longrightarrow$ $cdcl_W$*-bj* S S'

inductive-cases $cdcl_W$*-bjE*: $cdcl_W$*-bj* S T

inductive $cdcl_W$*-o*:: $'st \Rightarrow 'st \Rightarrow$ *bool* **for** S :: $'st$ **where**
decide[*intro*]: *decide* S S' $\Longrightarrow$ $cdcl_W$*-o* S S' |
bj[*intro*]: $cdcl_W$*-bj* S S' $\Longrightarrow$ $cdcl_W$*-o* S S'

inductive $cdcl_W$ :: $'st \Rightarrow 'st \Rightarrow$ *bool* **for** S :: $'st$ **where**
propagate: *propagate* S S' $\Longrightarrow$ $cdcl_W$ S S' |
conflict: *conflict* S S' $\Longrightarrow$ $cdcl_W$ S S' |
other: $cdcl_W$*-o* S S' $\Longrightarrow$ $cdcl_W$ S S'|
rf: $cdcl_W$*-rf* S S' $\Longrightarrow$ $cdcl_W$ S S'

lemma *rtranclp-propagate-is-rtranclp-*$cdcl_W$:
  propagate** S S' $\Longrightarrow$ $cdcl_W{}^{**}$ S S'
  by (*induction rule*: *rtranclp-induct*) (*fastforce dest*!: *propagate*)+

lemma $cdcl_W$*-all-rules-induct*[*consumes 1*, *case-names propagate conflict forget restart decide skip resolve backtrack*]:
  fixes S :: $'st$
  assumes

$cdcl_W$: $cdcl_W$ S S' **and**
$propagate$: $\bigwedge T.$ $propagate$ S $T \Longrightarrow P$ S T **and**
$conflict$: $\bigwedge T.$ $conflict$ S $T \Longrightarrow P$ S T **and**
$forget$: $\bigwedge T.$ $forget$ S $T \Longrightarrow P$ S T **and**
$restart$: $\bigwedge T.$ $restart$ S $T \Longrightarrow P$ S T **and**
$decide$: $\bigwedge T.$ $decide$ S $T \Longrightarrow P$ S T **and**
$skip$: $\bigwedge T.$ $skip$ S $T \Longrightarrow P$ S T **and**
$resolve$: $\bigwedge T.$ $resolve$ S $T \Longrightarrow P$ S T **and**
$backtrack$: $\bigwedge T.$ $backtrack$ S $T \Longrightarrow P$ S T
shows P S S'
using *assms(1)*
proof (*induct* S' *rule*: $cdcl_W$*.induct*)
case (*propagate* S') **note** *propagate* = *this(1)*
then show *?case* **using** *assms(2)* **by** *auto*
next
case (*conflict* S')
then show *?case* **using** *assms(3)* **by** *auto*
next
case (*other* S')
then show *?case*
proof (*induct rule*: $cdcl_W$*-o.induct*)
case (*decide* U)
then show *?case* **using** *assms(6)* **by** *auto*
next
case (*bj* S')
then show *?case* **using** *assms(7−9)* **by** (*induction rule*: $cdcl_W$*-bj.induct*) *auto*
qed
next
case (*rf* S')
then show *?case*
by (*induct rule*: $cdcl_W$*-rf.induct*) (*fast dest*: *forget restart*)+
qed

lemma $cdcl_W$*-all-induct*[*consumes 1*, *case-names propagate conflict forget restart decide skip resolve backtrack*]:
fixes S :: *'st*
assumes
$cdcl_W$: $cdcl_W$ S S' **and**
propagateH: $\bigwedge C\ L\ T.\ C + \{\#L\#\} \in\#$ *clauses* $S \Longrightarrow$ *trail* $S \models as$ *CNot* C
$\Longrightarrow$ *undefined-lit* (*trail* S) $L \Longrightarrow$ *conflicting* S = *None*
$\Longrightarrow T \sim$ *cons-trail* (*Propagated* L ($C + \{\#L\#\}$)) S
$\Longrightarrow P$ S T **and**
conflictH: $\bigwedge D\ T.\ D \in\#$ *clauses* $S \Longrightarrow$ *conflicting* S = *None* $\Longrightarrow$ *trail* $S \models as$ *CNot* D
$\Longrightarrow T \sim$ *update-conflicting* (*Some* D) S
$\Longrightarrow P$ S T **and**
forgetH: $\bigwedge C\ T.\ \neg$*trail* $S \models asm$ *clauses* S
$\Longrightarrow C \notin$ *set* (*get-all-mark-of-propagated* (*trail* S))
$\Longrightarrow C \notin\#$ *init-clss* S
$\Longrightarrow C \in\#$ *learned-clss* S
$\Longrightarrow$ *conflicting* S = *None*
$\Longrightarrow T \sim$ *remove-cls* C S
$\Longrightarrow P$ S T **and**
restartH: $\bigwedge T.\ \neg$*trail* $S \models asm$ *clauses* S
$\Longrightarrow$ *conflicting* S = *None*
$\Longrightarrow T \sim$ *restart-state* S

$\Longrightarrow P\ S\ T$ **and**
decideH: $\bigwedge L\ T.$ *conflicting* $S = None \Longrightarrow$ *undefined-lit* (*trail* S) L
$\Longrightarrow$ *atm-of* $L \in$ *atms-of-msu* (*init-clss* S)
$\Longrightarrow T \sim$ *cons-trail* (*Marked* L (*backtrack-lvl* S +*1*)) (*incr-lvl* S)
$\Longrightarrow P\ S\ T$ **and**
skipH: $\bigwedge L\ C'\ M\ D\ T.$ *trail* S = *Propagated* $L\ C'$ # M
$\Longrightarrow$ *conflicting* S = *Some* $D \Longrightarrow -L \notin\#\ D \Longrightarrow D \neq \{\#\}$
$\Longrightarrow T \sim$ *tl-trail* S
$\Longrightarrow P\ S\ T$ **and**
resolveH: $\bigwedge L\ C\ M\ D\ T.$
trail S = *Propagated* L ((C + {#L#})) # M
$\Longrightarrow$ *conflicting* S = *Some* (D + {#$-L$#})
$\Longrightarrow$ *get-maximum-level* (*Propagated* L (C + {#L#}) # M) D = *backtrack-lvl* S
$\Longrightarrow T \sim$ (*update-conflicting* (*Some* (D #∪ C)) (*tl-trail* S))
$\Longrightarrow P\ S\ T$ **and**
backtrackH: $\bigwedge K\ i\ M1\ M2\ L\ D\ T.$
(*Marked* K (*Suc* i) # *M1*, *M2*) $\in$ *set* (*get-all-marked-decomposition* (*trail* S))
$\Longrightarrow$ *get-level* (*trail* S) L = *backtrack-lvl* S
$\Longrightarrow$ *conflicting* S = *Some* (D + {#L#})
$\Longrightarrow$ *get-maximum-level* (*trail* S) (D+{#L#}) = *get-level* (*trail* S) L
$\Longrightarrow$ *get-maximum-level* (*trail* S) $D \equiv i$
$\Longrightarrow T \sim$ *cons-trail* (*Propagated* L (D+{#L#}))
(*reduce-trail-to M1*
(*add-learned-cls* (D + {#L#})
(*update-backtrack-lvl* i
(*update-conflicting None* S))))
$\Longrightarrow P\ S\ T$
shows $P\ S\ S'$
using $cdcl_W$
proof (*induct* $S\ S'$ *rule*: $cdcl_W$*-all-rules-induct*)
case (*propagate* S')
then show *?case* **by** (*elim propagateE*) (*frule propagateH*; *simp*)
next
case (*conflict* S')
then show *?case* **by** (*elim conflictE*) (*frule conflictH*; *simp*)
next
case (*restart* S')
then show *?case* **by** (*elim restartE*) (*frule restartH*; *simp*)
next
case (*decide* T)
then show *?case* **by** (*elim decideE*) (*frule decideH*; *simp*)
next
case (*backtrack* S')
then show *?case* **by** (*elim backtrackE*) (*frule backtrackH*; *simp del*: *state-simp add*: *state-eq-def*)
next
case (*forget* S')
then show *?case* **using** *forgetH* **by** *auto*
next
case (*skip* S')
then show *?case* **using** *skipH* **by** *auto*
next
case (*resolve* S')
then show *?case* **by** (*elim resolveE*) (*frule resolveH*; *simp*)
qed

lemma $cdcl_W$*-o-induct*[*consumes 1*, *case-names decide skip resolve backtrack*]:
fixes S :: $'st$
assumes $cdcl_W$: $cdcl_W$*-o* S T **and**
decideH: $\bigwedge L\ T.$ *conflicting* S = *None* $\Longrightarrow$ *undefined-lit* (*trail* S) L
$\Longrightarrow$ *atm-of* $L \in$ *atms-of-msu* (*init-clss* S)
$\Longrightarrow$ $T \sim$ *cons-trail* (*Marked* L (*backtrack-lvl* S *+1*)) (*incr-lvl* S)
$\Longrightarrow$ $P\ S\ T$ **and**
skipH: $\bigwedge L\ C'\ M\ D\ T.$ *trail* S = *Propagated* L C' # M
$\Longrightarrow$ *conflicting* S = *Some* D $\Longrightarrow$ $-L \notin\#$ D $\Longrightarrow$ $D \neq \{\#\}$
$\Longrightarrow$ $T \sim$ *tl-trail* S
$\Longrightarrow$ $P\ S\ T$ **and**
resolveH: $\bigwedge L\ C\ M\ D\ T.$
trail S = *Propagated* L ((C + {#L#})) # M
$\Longrightarrow$ *conflicting* S = *Some* (D + {#$-L$#})
$\Longrightarrow$ *get-maximum-level* (*Propagated* L (C + {#L#}) # M) D = *backtrack-lvl* S
$\Longrightarrow$ $T \sim$ *update-conflicting* (*Some* (D #∪ C)) (*tl-trail* S)
$\Longrightarrow$ $P\ S\ T$ **and**
backtrackH: $\bigwedge K\ i\ M1\ M2\ L\ D\ T.$
(*Marked* K (*Suc* i) # *M1*, *M2*) $\in$ *set* (*get-all-marked-decomposition* (*trail* S))
$\Longrightarrow$ *get-level* (*trail* S) L = *backtrack-lvl* S
$\Longrightarrow$ *conflicting* S = *Some* (D + {#L#})
$\Longrightarrow$ *get-level* (*trail* S) L = *get-maximum-level* (*trail* S) (D+{#L#})
$\Longrightarrow$ *get-maximum-level* (*trail* S) $D \equiv i$
$\Longrightarrow$ $T \sim$ *cons-trail* (*Propagated* L (D+{#L#}))
(*reduce-trail-to M1*
(*add-learned-cls* (D + {#L#})
(*update-backtrack-lvl* i
(*update-conflicting None* S))))
$\Longrightarrow$ $P\ S\ T$
shows $P\ S\ T$
using $cdcl_W$ **apply** (*induct* T *rule*: $cdcl_W$*-o.induct*)
using *assms*(*2*) **apply** *auto*[*1*]
apply (*elim* $cdcl_W$*-bjE skipE resolveE backtrackE*)
apply (*frule skipH*; *simp*)
apply (*frule resolveH*; *simp*)
apply (*frule backtrackH*; *simp-all del*: *state-simp add*: *state-eq-def*)
done

thm $cdcl_W$*-o.induct*
lemma $cdcl_W$*-o-all-rules-induct*[*consumes 1*, *case-names decide backtrack skip resolve*]:
fixes S T :: $'st$
assumes
$cdcl_W$*-o* S T **and**
$\bigwedge T.$ *decide* S T $\Longrightarrow$ $P\ S\ T$ **and**
$\bigwedge T.$ *backtrack* S T $\Longrightarrow$ $P\ S\ T$ **and**
$\bigwedge T.$ *skip* S T $\Longrightarrow$ $P\ S\ T$ **and**
$\bigwedge T.$ *resolve* S T $\Longrightarrow$ $P\ S\ T$
shows $P\ S\ T$
using *assms* **by** (*induct* T *rule*: $cdcl_W$*-o.induct*) (*auto simp*: $cdcl_W$*-bj.simps*)

lemma $cdcl_W$*-o-rule-cases*[*consumes 1*, *case-names decide backtrack skip resolve*]:
fixes S T :: $'st$
assumes
$cdcl_W$*-o* S T **and**
decide S T $\Longrightarrow$ P **and**

backtrack S $T \Longrightarrow P$ **and**
skip S $T \Longrightarrow P$ **and**
resolve S $T \Longrightarrow P$
shows P
using *assms* **by** (*auto simp*: $cdcl_W$*-o.simps* $cdcl_W$*-bj.simps*)

17.4 Invariants

17.4.1 Properties of the trail

We here establish that: * the marks are exactly 1..k where k is the level * the consistency of the trail * the fact that there is no duplicate in the trail.

lemma *backtrack-lit-skiped*:
assumes *L*: *get-level* (*trail S*) L = *backtrack-lvl S*
and *M1*: (*Marked K* (i + *1*) # *M1*, *M2*) ∈ *set* (*get-all-marked-decomposition* (*trail S*))
and *no-dup*: *no-dup* (*trail S*)
and *bt-l*: *backtrack-lvl S* = *length* (*get-all-levels-of-marked* (*trail S*))
and *order*: *get-all-levels-of-marked* (*trail S*)
= *rev* ([*1*..<(*1*+*length* (*get-all-levels-of-marked* (*trail S*)))])
shows *atm-of L* ∉ *atm-of* ‘ *lits-of M1*
proof
let *?M* = *trail S*
assume *L-in-M1*: *atm-of L* ∈ *atm-of* ‘ *lits-of M1*
obtain *c* **where** *Mc*: *trail S* = *c* @ *M2* @ *Marked K* (i + *1*) # *M1* **using** *M1* **by** *blast*
have *atm-of L* ∉ *atm-of* ‘ *lits-of c*
using *L-in-M1 no-dup mk-disjoint-insert* **unfolding** *Mc lits-of-def* **by** *force*
have *g-M-eq-g-M1*: *get-level ?M L* = *get-level M1 L*
using *L-in-M1* **unfolding** *Mc* **by** *auto*
have *g*: *get-all-levels-of-marked M1* = *rev* [*1*..<*Suc i*]
using *order* **unfolding** *Mc*
by (*auto simp del*: *upt-simps dest*!: *append-cons-eq-upt-length-i*
simp add: *rev-swap*[*symmetric*])
then have *Max* (*set* (*0* # *get-all-levels-of-marked* (*rev M1*))) < *Suc i* **by** *auto*
then have *get-level M1 L* < *Suc i*
using *get-rev-level-less-max-get-all-levels-of-marked*[*of rev M1 0 L*] **by** *linarith*
moreover have *Suc i* ≤ *backtrack-lvl S* **using** *bt-l* **by** (*simp add*: *Mc g*)
ultimately show *False* **using** *L g-M-eq-g-M1* **by** *auto*
qed

lemma $cdcl_W$*-distinctinv-1*:
assumes
$cdcl_W$ S S' **and**
no-dup (*trail S*) **and**
backtrack-lvl S = *length* (*get-all-levels-of-marked* (*trail S*)) **and**
get-all-levels-of-marked (*trail S*) = *rev* [*1*..<*1*+*length* (*get-all-levels-of-marked* (*trail S*))]
shows *no-dup* (*trail S'*)
using *assms*
proof (*induct rule*: $cdcl_W$*-all-induct*)
case (*backtrack K i M1 M2 L D T*) **note** *decomp* = *this*(*1*) **and** *L* = *this*(*2*) **and** *T* = *this*(*6*) **and** *n-d* = *this*(*7*)
obtain *c* **where** *Mc*: *trail S* = *c* @ *M2* @ *Marked K* (i + *1*) # *M1*
using *decomp* **by** *auto*
have *no-dup* (*M2* @ *Marked K* (i + *1*) # *M1*)
using *Mc n-d* **by** *fastforce*
moreover have *atm-of L* ∉ (*λl. atm-of* (*lit-of l*)) ‘ *set M1*

using *backtrack-lit-skiped*[*of S L K i M1 M2*] *L decomp backtrack.prems*
by (*fastforce simp*: *lits-of-def*)
moreover then have *undefined-lit M1 L*
by (*simp add*: *defined-lit-map*)
ultimately show *?case* **using** *decomp T n-d* **by** *simp*
qed (*auto simp*: *defined-lit-map*)

lemma $cdcl_W$*-consistent-inv-2*:
assumes
$cdcl_W$ S S' **and**
no-dup (*trail S*) **and**
backtrack-lvl S = *length* (*get-all-levels-of-marked* (*trail S*)) **and**
get-all-levels-of-marked (*trail S*) = *rev* [*1..<1+length* (*get-all-levels-of-marked* (*trail S*))]
shows *consistent-interp* (*lits-of* (*trail S'*))
using $cdcl_W$*-distinctinv-1*[*OF assms*] *distinctconsistent-interp* **by** *fast*

lemma $cdcl_W$*-o-bt*:
assumes
$cdcl_W$*-o S S'* **and**
backtrack-lvl S = *length* (*get-all-levels-of-marked* (*trail S*)) **and**
get-all-levels-of-marked (*trail S*) =
rev ([*1..<*(*1+length* (*get-all-levels-of-marked* (*trail S*)))]) **and**
n-d[*simp*]: *no-dup* (*trail S*)
shows *backtrack-lvl S'* = *length* (*get-all-levels-of-marked* (*trail S'*))
using *assms*
proof (*induct rule*: $cdcl_W$*-o-induct*)
case (*backtrack K i M1 M2 L D T*) **note** *decomp* = *this*(*1*) **and** *T* = *this*(*6*) **and** *level* = *this*(*8*)
have [*simp*]: *trail* (*reduce-trail-to M1 S*) = *M1*
using *decomp* **by** *auto*
obtain *c* **where** *M*: *trail S* = *c* @ *M2* @ *Marked K* (*i* + *1*) # *M1* **using** *decomp* **by** *auto*
have *rev* (*get-all-levels-of-marked* (*trail S*))
= [*1..<1+* (*length* (*get-all-levels-of-marked* (*trail S*)))]
using *level* **by** (*auto simp*: *rev-swap*[*symmetric*])
moreover have *atm-of L* ∉ (λ*l. atm-of* (*lit-of l*)) ‘ *set M1*
using *backtrack-lit-skiped*[*of S L K i M1 M2*] *backtrack*(*2*,*7*,*8*,*9*) *decomp*
by (*fastforce simp add*: *lits-of-def*)
moreover then have *undefined-lit M1 L*
by (*simp add*: *defined-lit-map*)
moreover then have *no-dup* (*trail T*)
using *T decomp n-d* **by** (*auto simp*: *defined-lit-map M*)
ultimately show *?case*
using *T n-d* **unfolding** *M* **by** (*auto dest*!: *append-cons-eq-upt-length simp del*: *upt-simps*)
qed *auto*

lemma $cdcl_W$*-rf-bt*:
assumes
$cdcl_W$*-rf S S'* **and**
backtrack-lvl S = *length* (*get-all-levels-of-marked* (*trail S*)) **and**
get-all-levels-of-marked (*trail S*) = *rev* [*1..<*(*1+length* (*get-all-levels-of-marked* (*trail S*)))]
shows *backtrack-lvl S'* = *length* (*get-all-levels-of-marked* (*trail S'*))
using *assms* **by** (*induct rule*: $cdcl_W$*-rf.induct*) *auto*

lemma $cdcl_W$*-bt*:
assumes
$cdcl_W$ S S' **and**

backtrack-lvl S = *length* (*get-all-levels-of-marked* (*trail S*)) **and**
get-all-levels-of-marked (*trail S*)
= *rev* ([*1*..<(*1*+*length* (*get-all-levels-of-marked* (*trail S*)))]) **and**
no-dup (*trail S*)
shows *backtrack-lvl S'* = *length* (*get-all-levels-of-marked* (*trail S'*))
using *assms* **by** (*induct rule*: $cdcl_W$.*induct*) (*auto simp add*: $cdcl_W$-*o-bt* $cdcl_W$-*rf-bt*)

lemma $cdcl_W$-*bt-level'*:
assumes
$cdcl_W$ *S S'* **and**
backtrack-lvl S = *length* (*get-all-levels-of-marked* (*trail S*)) **and**
get-all-levels-of-marked (*trail S*)
= *rev* ([*1*..<(*1*+*length* (*get-all-levels-of-marked* (*trail S*)))]) **and**
n-d: *no-dup* (*trail S*)
shows *get-all-levels-of-marked* (*trail S'*)
= *rev* ([*1*..<(*1*+*length* (*get-all-levels-of-marked* (*trail S'*)))])
using *assms*
proof (*induct rule*: $cdcl_W$-*all-induct*)
case (*decide L T*) **note** *undef* = *this*(*2*) **and** *T* = *this*(*4*)
let *?k* = *backtrack-lvl S*
let *?M* = *trail S*
let *?M'* = *Marked L* (*?k* + *1*) # *trail S*
have *H*: *get-all-levels-of-marked ?M* = *rev* [*Suc 0*..<*1*+*length* (*get-all-levels-of-marked ?M*)]
using *decide.prems* **by** *simp*
have *k*: *?k* = *length* (*get-all-levels-of-marked ?M*)
using *decide.prems* **by** *auto*
have *get-all-levels-of-marked ?M'* = *Suc ?k* # *get-all-levels-of-marked ?M* **by** *simp*
then have *get-all-levels-of-marked ?M'* = *Suc ?k* #
rev [*Suc 0*..<*1*+*length* (*get-all-levels-of-marked ?M*)]
using *H* **by** *auto*
moreover have ... = *rev* [*Suc 0*..< *Suc* (*1*+*length* (*get-all-levels-of-marked ?M*))]
unfolding *k* **by** *simp*
finally show *?case* **using** *T undef* **by** (*auto simp add*: *defined-lit-map*)
next
case (*backtrack K i M1 M2 L D T*) **note** *decomp* = *this*(*1*) **and** *confli* = *this*(*2*) **and** *T* =*this*(*6*) **and**
all-marked = *this*(*8*) **and** *bt-lvl* = *this*(*7*)
have *atm-of L* ∉ (λ*l*. *atm-of* (*lit-of l*)) ` *set M1*
using *backtrack-lit-skiped*[*of S L K i M1 M2*] *backtrack*(*2*,*7*,*8*,*9*) *decomp*
by (*fastforce simp add*: *lits-of-def*)
moreover then have *undefined-lit M1 L*
by (*simp add*: *defined-lit-map*)
then have [*simp*]: *trail T* = *Propagated L* (*D* + {#*L*#}) # *M1*
using *T decomp n-d* **by** *auto*
obtain *c* **where** *M*: *trail S* = *c* @ *M2* @ *Marked K* (*i* + *1*) # *M1* **using** *decomp* **by** *auto*
have *get-all-levels-of-marked* (*rev* (*trail S*))
= [*Suc 0*..<*2*+*length* (*get-all-levels-of-marked c*) + (*length* (*get-all-levels-of-marked M2*)
+ *length* (*get-all-levels-of-marked M1*))]
using *all-marked bt-lvl* **unfolding** *M* **by** (*auto simp add*: *rev-swap*[*symmetric*] *simp del*: *upt-simps*)
then show *?case*
using *T* **by** (*auto simp add*: *rev-swap M dest*!: *append-cons-eq-upt*(*1*) *simp del*: *upt-simps*)
qed *auto*

We write *1* + *length* (*get-all-levels-of-marked* (*trail S*)) instead of *backtrack-lvl S* to avoid non termination of rewriting.

definition $cdcl_W$*-M-level-inv* (S:: $'st$) $\longleftrightarrow$
consistent-interp (*lits-of* (*trail* S))
$\wedge$ *no-dup* (*trail* S)
$\wedge$ *backtrack-lvl* S = *length* (*get-all-levels-of-marked* (*trail* S))
$\wedge$ *get-all-levels-of-marked* (*trail* S)
= *rev* ([*1..<1+length* (*get-all-levels-of-marked* (*trail* S))])

lemma $cdcl_W$*-M-level-inv-decomp*:
assumes $cdcl_W$*-M-level-inv* S
shows *consistent-interp* (*lits-of* (*trail* S))
and *no-dup* (*trail* S)
using *assms* **unfolding** $cdcl_W$*-M-level-inv-def* **by** *fastforce+*

lemma $cdcl_W$*-consistent-inv*:
fixes S S' :: $'st$
assumes
$cdcl_W$ S S' **and**
$cdcl_W$*-M-level-inv* S
shows $cdcl_W$*-M-level-inv* S'
using *assms* $cdcl_W$*-consistent-inv-2* $cdcl_W$*-distinctinv-1* $cdcl_W$*-bt* $cdcl_W$*-bt-level'*
unfolding $cdcl_W$*-M-level-inv-def* **by** *meson+*

lemma *rtranclp-*$cdcl_W$*-consistent-inv*:
assumes ${cdcl_W}^{**}$ S S'
and $cdcl_W$*-M-level-inv* S
shows $cdcl_W$*-M-level-inv* S'
using *assms* **by** (*induct rule*: *rtranclp-induct*)
(*auto intro*: $cdcl_W$*-consistent-inv*)

lemma *tranclp-*$cdcl_W$*-consistent-inv*:
assumes ${cdcl_W}^{++}$ S S'
and $cdcl_W$*-M-level-inv* S
shows $cdcl_W$*-M-level-inv* S'
using *assms* **by** (*induct rule*: *tranclp-induct*)
(*auto intro*: $cdcl_W$*-consistent-inv*)

lemma $cdcl_W$*-M-level-inv-S0-*$cdcl_W$[*simp*]:
$cdcl_W$*-M-level-inv* (*init-state* N)
unfolding $cdcl_W$*-M-level-inv-def* **by** *auto*

lemma $cdcl_W$*-M-level-inv-get-level-le-backtrack-lvl*:
assumes *inv*: $cdcl_W$*-M-level-inv* S
shows *get-level* (*trail* S) L $\leq$ *backtrack-lvl* S
proof −
have *get-all-levels-of-marked* (*trail* S) = *rev* [*1..<1* + *backtrack-lvl* S]
using *inv* **unfolding** $cdcl_W$*-M-level-inv-def* **by** *auto*
then show *?thesis*
using *get-rev-level-less-max-get-all-levels-of-marked*[*of rev* (*trail* S) *0* L]
by (*auto simp*: *Max-n-upt*)
qed

lemma *backtrack-ex-decomp*:
assumes *M-l*: $cdcl_W$*-M-level-inv* S
and *i-S*: i < *backtrack-lvl* S
shows $\exists$ K *M1* *M2*. (*Marked* K (i + *1*) # *M1*, *M2*) $\in$ *set* (*get-all-marked-decomposition* (*trail* S))

proof −
  let *?M* = *trail S*
  have
    g: *get-all-levels-of-marked* (*trail S*) = *rev* [*Suc 0*..<*Suc* (*backtrack-lvl S*)]
    using *M-l* **unfolding** $cdcl_W$*-M-level-inv-def* **by** *simp-all*
  then have *i+1* ∈ *set* (*get-all-levels-of-marked* (*trail S*))
    using *i-S* **by** *auto*

  then obtain *c K c′* **where** *tr-S*: *trail S* = *c* @ *Marked K* (*i* + *1*) # *c′*
    using *in-get-all-levels-of-marked-iff-decomp*[*of i+1 trail S*] **by** *auto*

  obtain *M1 M2* **where** (*Marked K* (*i* + *1*) # *M1*, *M2*) ∈ *set* (*get-all-marked-decomposition* (*trail S*))
    unfolding *tr-S* **apply** (*induct c rule*: *marked-lit-list-induct*)
      apply *auto*[*2*]
    apply (*case-tac hd* (*get-all-marked-decomposition* (*xs* @ *Marked K* (*Suc i*) # *c′*)))
    apply (*case-tac get-all-marked-decomposition* (*xs* @ *Marked K* (*Suc i*) # *c′*))
    by *auto*
  then show *?thesis* **by** *blast*
qed

17.4.2 Better-Suited Induction Principle

We generalise the induction principle defined previously: the induction case for *backtrack* now includes the assumption that *undefined-lit M1 L*. This helps the simplifier and thus the automation.

lemma *backtrack-induction-lev*[*consumes 1*, *case-names M-devel-inv backtrack*]:
  assumes
    bt: *backtrack S T* **and**
    inv: $cdcl_W$*-M-level-inv S* **and**
    backtrackH: ⋀*K i M1 M2 L D T*.
      (*Marked K* (*Suc i*) # *M1*, *M2*) ∈ *set* (*get-all-marked-decomposition* (*trail S*))
      ⟹ *get-level* (*trail S*) *L* = *backtrack-lvl S*
      ⟹ *conflicting S* = *Some* (*D* + {#*L*#})
      ⟹ *get-level* (*trail S*) *L* = *get-maximum-level* (*trail S*) (*D*+{#*L*#})
      ⟹ *get-maximum-level* (*trail S*) *D* ≡ *i*
      ⟹ *undefined-lit M1 L*
      ⟹ *T* ∼ *cons-trail* (*Propagated L* (*D*+{#*L*#}))
            (*reduce-trail-to M1*
              (*add-learned-cls* (*D* + {#*L*#})
                (*update-backtrack-lvl i*
                  (*update-conflicting None S*))))
      ⟹ *P S T*
  shows *P S T*
proof −
  obtain *K i M1 M2 L D* **where**
    decomp: (*Marked K* (*Suc i*) # *M1*, *M2*) ∈ *set* (*get-all-marked-decomposition* (*trail S*)) **and**
    L: *get-level* (*trail S*) *L* = *backtrack-lvl S* **and**
    confl: *conflicting S* = *Some* (*D* + {#*L*#}) **and**
    lev-L: *get-level* (*trail S*) *L* = *get-maximum-level* (*trail S*) (*D*+{#*L*#}) **and**
    lev-D: *get-maximum-level* (*trail S*) *D* ≡ *i* **and**
    T: *T* ∼ *cons-trail* (*Propagated L* (*D*+{#*L*#}))
            (*reduce-trail-to M1*
              (*add-learned-cls* (*D* + {#*L*#})
                (*update-backtrack-lvl i*
                  (*update-conflicting None S*))))

using *bt* **by** (*elim backtrackE*) *metis*

have $atm\text{-}of\ L \notin (\lambda l.\ atm\text{-}of\ (lit\text{-}of\ l))$ ` *set M1*
using *backtrack-lit-skiped*[*of S L K i M1 M2*] *L decomp bt confl lev-L lev-D inv*
unfolding $cdcl_W$*-M-level-inv-def*
by (*fastforce simp add*: *lits-of-def*)
then have *undefined-lit M1 L*
by (*auto simp*: *defined-lit-map*)
then show *?thesis*
using *backtrackH*[*OF decomp L confl lev-L lev-D - T*] **by** *simp*
qed

lemmas *backtrack-induction-lev2* = *backtrack-induction-lev*[*consumes 2*, *case-names backtrack*]

lemma $cdcl_W$*-all-induct-lev-full*:
fixes S :: $'st$
assumes
$cdcl_W$: $cdcl_W\ S\ S'$ **and**
inv[*simp*]: $cdcl_W$*-M-level-inv S* **and**
propagateH: $\bigwedge C\ L\ T.\ C + \{\#L\#\} \in\#\ clauses\ S \Longrightarrow trail\ S \models as\ CNot\ C$
$\Longrightarrow$ *undefined-lit* (*trail S*) $L \Longrightarrow$ *conflicting S* = *None*
$\Longrightarrow T \sim cons\text{-}trail\ (Propagated\ L\ (C + \{\#L\#\}))\ S$
$\Longrightarrow cdcl_W\text{-}M\text{-}level\text{-}inv\ S$
$\Longrightarrow P\ S\ T$ **and**
conflictH: $\bigwedge D\ T.\ D \in\#\ clauses\ S \Longrightarrow conflicting\ S = None \Longrightarrow trail\ S \models as\ CNot\ D$
$\Longrightarrow T \sim update\text{-}conflicting\ (Some\ D)\ S$
$\Longrightarrow P\ S\ T$ **and**
forgetH: $\bigwedge C\ T.\ \neg trail\ S \models asm\ clauses\ S$
$\Longrightarrow C \notin set\ (get\text{-}all\text{-}mark\text{-}of\text{-}propagated\ (trail\ S))$
$\Longrightarrow C \notin\#\ init\text{-}clss\ S$
$\Longrightarrow C \in\#\ learned\text{-}clss\ S$
$\Longrightarrow conflicting\ S = None$
$\Longrightarrow T \sim remove\text{-}cls\ C\ S$
$\Longrightarrow cdcl_W\text{-}M\text{-}level\text{-}inv\ S$
$\Longrightarrow P\ S\ T$ **and**
restartH: $\bigwedge T.\ \neg trail\ S \models asm\ clauses\ S$
$\Longrightarrow conflicting\ S = None$
$\Longrightarrow T \sim restart\text{-}state\ S$
$\Longrightarrow cdcl_W\text{-}M\text{-}level\text{-}inv\ S$
$\Longrightarrow P\ S\ T$ **and**
decideH: $\bigwedge L\ T.\ conflicting\ S = None \Longrightarrow undefined\text{-}lit\ (trail\ S)\ L$
$\Longrightarrow atm\text{-}of\ L \in atms\text{-}of\text{-}msu\ (init\text{-}clss\ S)$
$\Longrightarrow T \sim cons\text{-}trail\ (Marked\ L\ (backtrack\text{-}lvl\ S + 1))\ (incr\text{-}lvl\ S)$
$\Longrightarrow cdcl_W\text{-}M\text{-}level\text{-}inv\ S$
$\Longrightarrow P\ S\ T$ **and**
skipH: $\bigwedge L\ C'\ M\ D\ T.\ trail\ S = Propagated\ L\ C'\ \#\ M$
$\Longrightarrow conflicting\ S = Some\ D \Longrightarrow -L \notin\#\ D \Longrightarrow D \neq \{\#\}$
$\Longrightarrow T \sim tl\text{-}trail\ S$
$\Longrightarrow cdcl_W\text{-}M\text{-}level\text{-}inv\ S$
$\Longrightarrow P\ S\ T$ **and**
resolveH: $\bigwedge L\ C\ M\ D\ T.$
$trail\ S = Propagated\ L\ (\ (C + \{\#L\#\}))\ \#\ M$
$\Longrightarrow conflicting\ S = Some\ (D + \{\#-L\#\})$
$\Longrightarrow get\text{-}maximum\text{-}level\ (Propagated\ L\ (C + \{\#L\#\})\ \#\ M)\ D = backtrack\text{-}lvl\ S$
$\Longrightarrow T \sim (update\text{-}conflicting\ (Some\ (D\ \#\cup\ C))\ (tl\text{-}trail\ S))$

$\Longrightarrow$ *cdcl*$_W$*-M-level-inv S*
$\Longrightarrow$ *P S T* **and**
backtrackH: $\bigwedge$*K i M1 M2 L D T*.
(*Marked K* (*Suc i*) # *M1*, *M2*) $\in$ *set* (*get-all-marked-decomposition* (*trail S*))
$\Longrightarrow$ *get-level* (*trail S*) *L* = *backtrack-lvl S*
$\Longrightarrow$ *conflicting S* = *Some* (*D* + {#*L*#})
$\Longrightarrow$ *get-maximum-level* (*trail S*) (*D*+{#*L*#}) = *get-level* (*trail S*) *L*
$\Longrightarrow$ *get-maximum-level* (*trail S*) *D* $\equiv$ *i*
$\Longrightarrow$ *undefined-lit M1 L*
$\Longrightarrow$ *T* $\sim$ *cons-trail* (*Propagated L* (*D*+{#*L*#}))
(*reduce-trail-to M1*
(*add-learned-cls* (*D* + {#*L*#})
(*update-backtrack-lvl i*
(*update-conflicting None S*))))
$\Longrightarrow$ *cdcl*$_W$*-M-level-inv S*
$\Longrightarrow$ *P S T*
shows *P S S'*
using *cdcl*$_W$
proof (*induct S' rule*: *cdcl*$_W$*-all-rules-induct*)
case (*propagate S'*)
then show *?case* **by** (*elim propagateE*) (*frule propagateH*; *simp*)
next
case (*conflict S'*)
then show *?case* **by** (*elim conflictE*) (*frule conflictH*; *simp*)
next
case (*restart S'*)
then show *?case* **by** (*elim restartE*) (*frule restartH*; *simp*)
next
case (*decide T*)
then show *?case* **by** (*elim decideE*) (*frule decideH*; *simp*)
next
case (*backtrack S'*)
then show *?case*
apply (*induction rule*: *backtrack-induction-lev*)
apply (*rule inv*)
by (*rule backtrackH*;
fastforce simp del: *state-simp simp add*: *state-eq-def dest*!: *HOL.meta-eq-to-obj-eq*)
next
case (*forget S'*)
then show *?case* **using** *forgetH* **by** *auto*
next
case (*skip S'*)
then show *?case* **using** *skipH* **by** *auto*
next
case (*resolve S'*)
then show *?case* **by** (*elim resolveE*) (*frule resolveH*; *simp*)
qed

lemmas *cdcl*$_W$*-all-induct-lev2* = *cdcl*$_W$*-all-induct-lev-full*[*consumes 2*, *case-names propagate conflict forget restart decide skip resolve backtrack*]

lemmas *cdcl*$_W$*-all-induct-lev* = *cdcl*$_W$*-all-induct-lev-full*[*consumes 1*, *case-names lev-inv propagate conflict forget restart decide skip resolve backtrack*]

thm *cdcl*$_W$*-o-induct*

lemma $cdcl_W$*-o-induct-lev*[*consumes 1*, *case-names M-lev decide skip resolve backtrack*]:
fixes S :: *'st*
assumes
$cdcl_W$: $cdcl_W$*-o S T* **and**
inv[*simp*]: $cdcl_W$*-M-level-inv S* **and**
decideH: $\bigwedge L\ T.$ *conflicting S = None* $\Longrightarrow$ *undefined-lit* (*trail S*) *L*
$\Longrightarrow$ *atm-of L* $\in$ *atms-of-msu* (*init-clss S*)
$\Longrightarrow$ $T \sim$ *cons-trail* (*Marked L* (*backtrack-lvl S +1*)) (*incr-lvl S*)
$\Longrightarrow$ $cdcl_W$*-M-level-inv S*
$\Longrightarrow$ *P S T* **and**
skipH: $\bigwedge L\ C'\ M\ D\ T.$ *trail S = Propagated L C' # M*
$\Longrightarrow$ *conflicting S = Some D* $\Longrightarrow$ $-L \notin\#$ *D* $\Longrightarrow$ $D \neq \{\#\}$
$\Longrightarrow$ $T \sim$ *tl-trail S*
$\Longrightarrow$ $cdcl_W$*-M-level-inv S*
$\Longrightarrow$ *P S T* **and**
resolveH: $\bigwedge L\ C\ M\ D\ T.$
trail S = Propagated L ((*C* + {#*L*#})) # *M*
$\Longrightarrow$ *conflicting S = Some* (*D* + {#$-L$#})
$\Longrightarrow$ *get-maximum-level* (*Propagated L* (*C* + {#*L*#}) # *M*) *D = backtrack-lvl S*
$\Longrightarrow$ $T \sim$ *update-conflicting* (*Some* (*D* #$\cup$ *C*)) (*tl-trail S*)
$\Longrightarrow$ $cdcl_W$*-M-level-inv S*
$\Longrightarrow$ *P S T* **and**
backtrackH: $\bigwedge K\ i\ M1\ M2\ L\ D\ T.$
(*Marked K* (*Suc i*) # *M1*, *M2*) $\in$ *set* (*get-all-marked-decomposition* (*trail S*))
$\Longrightarrow$ *get-level* (*trail S*) *L = backtrack-lvl S*
$\Longrightarrow$ *conflicting S = Some* (*D* + {#*L*#})
$\Longrightarrow$ *get-level* (*trail S*) *L = get-maximum-level* (*trail S*) (*D*+{#*L*#})
$\Longrightarrow$ *get-maximum-level* (*trail S*) *D* $\equiv$ *i*
$\Longrightarrow$ *undefined-lit M1 L*
$\Longrightarrow$ $T \sim$ *cons-trail* (*Propagated L* (*D*+{#*L*#}))
(*reduce-trail-to M1*
(*add-learned-cls* (*D* + {#*L*#})
(*update-backtrack-lvl i*
(*update-conflicting None S*))))
$\Longrightarrow$ $cdcl_W$*-M-level-inv S*
$\Longrightarrow$ *P S T*
shows *P S T*
using $cdcl_W$
proof (*induct S T rule*: $cdcl_W$*-o-all-rules-induct*)
case (*decide T*)
then show *?case* **by** (*elim decideE*) (*frule decideH*; *simp*)
next
case (*backtrack S'*)
then show *?case*
using *inv* **apply** (*induction rule*: *backtrack-induction-lev2*)
by (*rule backtrackH*)
(*fastforce simp del*: *state-simp simp add*: *state-eq-def dest*!: *HOL.meta-eq-to-obj-eq*)+
next
case (*skip S'*)
then show *?case* **using** *skipH* **by** *auto*
next
case (*resolve S'*)
then show *?case* **by** (*elim resolveE*) (*frule resolveH*; *simp*)
qed

lemmas *cdcl*$_W$*-o-induct-lev2* = *cdcl*$_W$*-o-induct-lev*[*consumes 2*, *case-names decide skip resolve backtrack*]

17.4.3 Compatibility with *op* ∼

lemma *propagate-state-eq-compatible*:
assumes
propagate S T **and**
$S \sim S'$ **and**
$T \sim T'$
shows *propagate S' T'*
using *assms* **apply** (*elim propagateE*)
apply (*rule propagate-rule*)
by (*auto simp*: *state-eq-def clauses-def simp del*: *state-simp*)

lemma *conflict-state-eq-compatible*:
assumes
conflict S T **and**
$S \sim S'$ **and**
$T \sim T'$
shows *conflict S' T'*
using *assms* **apply** (*elim conflictE*)
apply (*rule conflict-rule*)
by (*auto simp*: *state-eq-def clauses-def simp del*: *state-simp*)

lemma *backtrack-state-eq-compatible*:
assumes
backtrack S T **and**
$S \sim S'$ **and**
$T \sim T'$ **and**
inv: *cdcl*$_W$*-M-level-inv S*
shows *backtrack S' T'*
using *assms* **apply** (*induction rule*: *backtrack-induction-lev*)
using *inv* **apply** *simp*
apply (*rule backtrack-rule*)
apply *auto*[*5*]
by (*auto simp*: *state-eq-def clauses-def cdcl*$_W$*-M-level-inv-def simp del*: *state-simp*)

lemma *decide-state-eq-compatible*:
assumes
decide S T **and**
$S \sim S'$ **and**
$T \sim T'$
shows *decide S' T'*
using *assms* **apply** (*elim decideE*)
apply (*rule decide-rule*)
by (*auto simp*: *state-eq-def clauses-def simp del*: *state-simp*)

lemma *skip-state-eq-compatible*:
assumes
skip S T **and**
$S \sim S'$ **and**
$T \sim T'$
shows *skip S' T'*
using *assms* **apply** (*elim skipE*)
apply (*rule skip-rule*)

by (*auto simp*: *state-eq-def clauses-def HOL.eq-sym-conv*[*of* - # - *trail* -]
simp del: *state-simp dest*: *arg-cong*[*of* - # *trail* - *trail* - *tl*])

lemma *resolve-state-eq-compatible*:
assumes
resolve S T **and**
$S \sim S'$ **and**
$T \sim T'$
shows *resolve* S' T'
using *assms* **apply** (*elim resolveE*)
apply (*rule resolve-rule*)
by (*auto simp*: *state-eq-def clauses-def HOL.eq-sym-conv*[*of* - # - *trail* -]
simp del: *state-simp dest*: *arg-cong*[*of* - # *trail* - *trail* - *tl*])

lemma *forget-state-eq-compatible*:
assumes
forget S T **and**
$S \sim S'$ **and**
$T \sim T'$
shows *forget* S' T'
using *assms* **apply** (*elim forgetE*)
apply (*rule forget-rule*)
by (*auto simp*: *state-eq-def clauses-def HOL.eq-sym-conv*[*of* {#-#} + - -]
simp del: *state-simp dest*: *arg-cong*[*of* - # *trail* - *trail* - *tl*])

lemma $cdcl_W$*-state-eq-compatible*:
assumes
$cdcl_W$ *S T* **and** ¬*restart S T* **and**
$S \sim S'$ **and**
$T \sim T'$ **and**
inv: $cdcl_W$*-M-level-inv S*
shows $cdcl_W$ S' T'
using *assms* **by** (*meson assms backtrack-state-eq-compatible bj* $cdcl_W$*.simps* $cdcl_W$*-bj.simps*
$cdcl_W$*-o-rule-cases* $cdcl_W$*-rf.cases* $cdcl_W$*-rf.restart conflict-state-eq-compatible decide*
decide-state-eq-compatible forget forget-state-eq-compatible
propagate-state-eq-compatible resolve-state-eq-compatible
skip-state-eq-compatible)

lemma $cdcl_W$*-bj-state-eq-compatible*:
assumes
$cdcl_W$*-bj S T* **and** $cdcl_W$*-M-level-inv S*
$S \sim S'$ **and**
$T \sim T'$
shows $cdcl_W$*-bj* S' T'
using *assms*
by *induction* (*auto*
intro: *skip-state-eq-compatible backtrack-state-eq-compatible resolve-state-eq-compatible*)

lemma *tranclp-*$cdcl_W$*-bj-state-eq-compatible*:
assumes
$cdcl_W$*-bj*$^{++}$ *S T* **and** *inv*: $cdcl_W$*-M-level-inv S* **and**
$S \sim S'$ **and**
$T \sim T'$
shows $cdcl_W$*-bj*$^{++}$ S' T'
using *assms*

proof (*induction arbitrary*: S' T')
  case *base*
  then show *?case*
    using $cdcl_W$*-bj-state-eq-compatible* **by** *blast*
next
  case (*step T U*) **note** *IH* = *this*(*3*)[*OF this*(*4−5*)]
  have $cdcl_W^{++}$ *S T*
    using *tranclp-mono*[*of* $cdcl_W$*-bj* $cdcl_W$] *other step.hyps*(*1*) **by** *blast*
  then have $cdcl_W$*-M-level-inv T*
    using *inv tranclp-*$cdcl_W$*-consistent-inv* **by** *blast*
  then have $cdcl_W$*-bj*$^{++}$ T T'
    using ‹$U \sim T'$› $cdcl_W$*-bj-state-eq-compatible*[*of T U*] ‹$cdcl_W$*-bj T U*› **by** *auto*
  then show *?case*
    using *IH*[*of T*] **by** *auto*
qed

17.4.4 Conservation of some Properties

lemma *level-of-marked-ge-1*:
  assumes
    $cdcl_W$ S S' **and**
    inv: $cdcl_W$*-M-level-inv S* **and**
    $\forall L\ l.\ Marked\ L\ l \in set\ (trail\ S) \longrightarrow l > 0$
  shows $\forall L\ l.\ Marked\ L\ l \in set\ (trail\ S') \longrightarrow l > 0$
  using *assms* **apply** (*induct rule*: $cdcl_W$*-all-induct-lev2*)
  by (*auto dest*: *union-in-get-all-marked-decomposition-is-subset simp*: $cdcl_W$*-M-level-inv-decomp*)

lemma $cdcl_W$*-o-no-more-init-clss*:
  assumes
    $cdcl_W$*-o* S S' **and**
    inv: $cdcl_W$*-M-level-inv S*
  shows *init-clss S* = *init-clss* S'
  using *assms* **by** (*induct rule*: $cdcl_W$*-o-induct-lev2*) (*auto simp*: $cdcl_W$*-M-level-inv-decomp*)

lemma *tranclp-*$cdcl_W$*-o-no-more-init-clss*:
  assumes
    $cdcl_W$*-o*$^{++}$ S S' **and**
    inv: $cdcl_W$*-M-level-inv S*
  shows *init-clss S* = *init-clss* S'
  using *assms* **apply** (*induct rule*: *tranclp.induct*)
  by (*auto dest*: $cdcl_W$*-o-no-more-init-clss*
    dest!: *tranclp-*$cdcl_W$*-consistent-inv dest*: *tranclp-mono-explicit*[*of* $cdcl_W$*-o - -* $cdcl_W$]
    simp: *other*)

lemma *rtranclp-*$cdcl_W$*-o-no-more-init-clss*:
  assumes
    $cdcl_W$*-o*** S S' **and**
    inv: $cdcl_W$*-M-level-inv S*
  shows *init-clss S* = *init-clss* S'
  using *assms* **unfolding** *rtranclp-unfold* **by** (*auto intro*: *tranclp-*$cdcl_W$*-o-no-more-init-clss*)

lemma $cdcl_W$*-init-clss*:
  $cdcl_W$ S $T \Longrightarrow$ $cdcl_W$*-M-level-inv S* $\Longrightarrow$ *init-clss S* = *init-clss T*
  by (*induct rule*: $cdcl_W$*-all-induct-lev2*) (*auto simp*: $cdcl_W$*-M-level-inv-def*)

lemma *rtranclp-*$cdcl_W$*-init-clss*:

$cdcl_W^{**}$ S T $\Longrightarrow$ $cdcl_W$*-M-level-inv* S $\Longrightarrow$ *init-clss* S = *init-clss* T
by (*induct rule*: *rtranclp-induct*) (*auto dest*: $cdcl_W$*-init-clss rtranclp-*$cdcl_W$*-consistent-inv*)

lemma *tranclp-*$cdcl_W$*-init-clss*:
$cdcl_W^{++}$ S T $\Longrightarrow$ $cdcl_W$*-M-level-inv* S $\Longrightarrow$ *init-clss* S = *init-clss* T
using *rtranclp-*$cdcl_W$*-init-clss*[*of* S T] **unfolding** *rtranclp-unfold* **by** *auto*

17.4.5 Learned Clause

This invariant shows that:

- the learned clauses are entailed by the initial set of clauses.
- the conflicting clause is entailed by the initial set of clauses.
- the marks are entailed by the clauses. A more precise version would be to show that either these marked are learned or are in the set of clauses

definition $cdcl_W$*-learned-clause* (S:: $'st$) $\longleftrightarrow$
(*init-clss* S $\models$*psm learned-clss* S
$\wedge$ ($\forall$ T. *conflicting* S = *Some* T $\longrightarrow$ *init-clss* S $\models$*pm* T)
$\wedge$ *set* (*get-all-mark-of-propagated* (*trail* S)) $\subseteq$ *set-mset* (*clauses* S))

lemma $cdcl_W$*-learned-clause-S0-*$cdcl_W$[*simp*]:
$cdcl_W$*-learned-clause* (*init-state* N)
unfolding $cdcl_W$*-learned-clause-def* **by** *auto*

lemma $cdcl_W$*-learned-clss*:
assumes
$cdcl_W$ S S' **and**
learned: $cdcl_W$*-learned-clause* S **and**
lev-inv: $cdcl_W$*-M-level-inv* S
shows $cdcl_W$*-learned-clause* S'
using *assms*(*1*) *lev-inv learned*
proof (*induct rule*: $cdcl_W$*-all-induct-lev2*)
case (*backtrack K i M1 M2 L D T*) **note** *decomp* = *this*(*1*) **and** *confl* = *this*(*3*) **and** *undef* = *this*(*6*)
and T =*this*(*7*)
show *?case*
using *decomp confl learned undef T lev-inv* **unfolding** $cdcl_W$*-learned-clause-def*
by (*auto dest*!: *get-all-marked-decomposition-exists-prepend*
simp: *clauses-def* $cdcl_W$*-M-level-inv-decomp dest*: *true-clss-clss-left-right*)
next
case (*resolve L C M D*) **note** *trail* = *this*(*1*) **and** *confl* = *this*(*2*) **and** *lvl* = *this*(*3*) **and**
T =*this*(*4*)
moreover
have *init-clss* S $\models$*psm learned-clss* S
using *learned trail* **unfolding** $cdcl_W$*-learned-clause-def clauses-def* **by** *auto*
then have *init-clss* S $\models$*pm* C + {#L#}
using *trail learned* **unfolding** $cdcl_W$*-learned-clause-def clauses-def*
by (*auto dest*: *true-clss-clss-in-imp-true-clss-cls*)
ultimately show *?case*
using *learned*
by (*auto dest*: *mk-disjoint-insert true-clss-clss-left-right*
simp add: $cdcl_W$*-learned-clause-def clauses-def*

intro: *true-clss-cls-union-mset-true-clss-cls-or-not-true-clss-cls-or*)
next
case (*restart T*)
then show *?case*
using *learned-clss-restart-state*[*of T*]
by (*auto dest*!: *get-all-marked-decomposition-exists-prepend*
simp: *clauses-def state-eq-def* $cdcl_W$*-learned-clause-def*
simp del: *state-simp*
dest: *true-clss-clssm-subsetE*)
next
case *propagate*
then show *?case* **using** *learned* **by** (*auto simp*: $cdcl_W$*-learned-clause-def clauses-def*)
next
case *conflict*
then show *?case* **using** *learned*
by (*auto simp*: $cdcl_W$*-learned-clause-def clauses-def true-clss-clss-in-imp-true-clss-cls*)
next
case *forget*
then show *?case*
using *learned* **by** (*auto simp*: $cdcl_W$*-learned-clause-def clauses-def split*: *split-if-asm*)
qed (*auto simp*: $cdcl_W$*-learned-clause-def clauses-def*)

lemma *rtranclp-*$cdcl_W$*-learned-clss*:
assumes
$cdcl_W^{**}$ S S' **and**
$cdcl_W$*-M-level-inv* S
$cdcl_W$*-learned-clause* S
shows $cdcl_W$*-learned-clause* S'
using *assms* **by** *induction* (*auto dest*: $cdcl_W$*-learned-clss intro*: *rtranclp-*$cdcl_W$*-consistent-inv*)

17.4.6 No alien atom in the state

This invariant means that all the literals are in the set of clauses.

definition *no-strange-atm* $S' \longleftrightarrow$ (
($\forall$ T. *conflicting* S' = *Some* T $\longrightarrow$ *atms-of* $T \subseteq$ *atms-of-msu* (*init-clss* S'))
$\wedge$ ($\forall$ L *mark*. *Propagated* L *mark* $\in$ *set* (*trail* S')
$\longrightarrow$ *atms-of* (*mark*) $\subseteq$ *atms-of-msu* (*init-clss* S'))
$\wedge$ *atms-of-msu* (*learned-clss* S') $\subseteq$ *atms-of-msu* (*init-clss* S')
$\wedge$ *atm-of* ‘ (*lits-of* (*trail* S')) $\subseteq$ *atms-of-msu* (*init-clss* S'))

lemma *no-strange-atm-decomp*:
assumes *no-strange-atm* S
shows *conflicting* S = *Some* T $\Longrightarrow$ *atms-of* $T \subseteq$ *atms-of-msu* (*init-clss* S)
and ($\forall$ L *mark*. *Propagated* L *mark* $\in$ *set* (*trail* S)
$\longrightarrow$ *atms-of* (*mark*) $\subseteq$ *atms-of-msu* (*init-clss* S))
and *atms-of-msu* (*learned-clss* S) $\subseteq$ *atms-of-msu* (*init-clss* S)
and *atm-of* ‘ (*lits-of* (*trail* S)) $\subseteq$ *atms-of-msu* (*init-clss* S)
using *assms* **unfolding** *no-strange-atm-def* **by** *blast+*

lemma *no-strange-atm-S0* [*simp*]: *no-strange-atm* (*init-state* N)
unfolding *no-strange-atm-def* **by** *auto*

lemma $cdcl_W$*-no-strange-atm-explicit*:
assumes
$cdcl_W$ S S' **and**

lev: *cdcl*$_W$*-M-level-inv S* **and**
conf: ∀ *T*. *conflicting S* = *Some T* ⟶ *atms-of T* ⊆ *atms-of-msu* (*init-clss S*) **and**
marked: ∀ *L mark*. *Propagated L mark* ∈ *set* (*trail S*)
⟶ *atms-of mark* ⊆ *atms-of-msu* (*init-clss S*) **and**
learned: *atms-of-msu* (*learned-clss S*) ⊆ *atms-of-msu* (*init-clss S*) **and**
trail: *atm-of* ‘ (*lits-of* (*trail S*)) ⊆ *atms-of-msu* (*init-clss S*)
shows (∀ *T*. *conflicting S'* = *Some T* ⟶ *atms-of T* ⊆ *atms-of-msu* (*init-clss S'*)) ∧
(∀ *L mark*. *Propagated L mark* ∈ *set* (*trail S'*)
⟶ *atms-of* (*mark*) ⊆ *atms-of-msu* (*init-clss S'*)) ∧
atms-of-msu (*learned-clss S'*) ⊆ *atms-of-msu* (*init-clss S'*) ∧
atm-of ‘ (*lits-of* (*trail S'*)) ⊆ *atms-of-msu* (*init-clss S'*) (**is** *?C S'* ∧ *?M S'* ∧ *?U S'* ∧ *?V S'*)
using *assms*(*1*,*2*)
proof (*induct rule*: *cdcl*$_W$*-all-induct-lev2*)
case (*propagate C L T*) **note** *C-L* = *this*(*1*) **and** *undef* = *this*(*3*) **and** *confl* = *this*(*4*) **and** *T* =*this*(*5*)
have *?C* (*cons-trail* (*Propagated L* (*C* + {#*L*#})) *S*) **using** *confl undef* **by** *auto*
moreover
have *atms-of* (*C* + {#*L*#}) ⊆ *atms-of-msu* (*init-clss S*)
by (*metis* (*no-types*) *atms-of-atms-of-ms-mono atms-of-ms-union clauses-def mem-set-mset-iff*
C-L learned set-mset-union sup.orderE)
then have *?M* (*cons-trail* (*Propagated L* (*C* + {#*L*#})) *S*) **using** *undef*
by (*simp add*: *marked*)
moreover have *?U* (*cons-trail* (*Propagated L* (*C* + {#*L*#})) *S*)
using *learned undef* **by** *auto*
moreover have *?V* (*cons-trail* (*Propagated L* (*C* + {#*L*#})) *S*)
using *C-L learned trail undef* **unfolding** *clauses-def*
by (*auto simp*: *in-plus-implies-atm-of-on-atms-of-ms*)
ultimately show *?case* **using** *T* **by** *auto*
next
case (*decide L*)
then show *?case* **using** *learned marked conf trail* **unfolding** *clauses-def* **by** *auto*
next
case (*skip L C M D*)
then show *?case* **using** *learned marked conf trail* **by** *auto*
next
case (*conflict D T*) **note** *T* =*this*(*4*)
have *D*: *atm-of* ‘ *set-mset D* ⊆ ⋃(*atms-of* ‘ (*set-mset* (*clauses S*)))
using ‹*D* ∈# *clauses S*› **by** (*auto simp add*: *atms-of-def atms-of-ms-def*)
moreover {
fix *xa* :: *'v literal*
assume *a1*: *atm-of* ‘ *set-mset D* ⊆ (⋃*x*∈*set-mset* (*init-clss S*). *atms-of x*)
∪ (⋃*x*∈*set-mset* (*learned-clss S*). *atms-of x*)
assume *a2*: (⋃*x*∈*set-mset* (*learned-clss S*). *atms-of x*) ⊆ (⋃*x*∈*set-mset* (*init-clss S*). *atms-of x*)
assume *xa* ∈# *D*
then have *atm-of xa* ∈ *UNION* (*set-mset* (*init-clss S*)) *atms-of*
using *a2 a1* **by** (*metis* (*no-types*) *Un-iff atm-of-lit-in-atms-of atms-of-def subset-Un-eq*)
then have ∃ *m*∈*set-mset* (*init-clss S*). *atm-of xa* ∈ *atms-of m*
by *blast*
} **note** *H* = *this*
ultimately show *?case* **using** *conflict.prems T learned marked conf trail*
unfolding *atms-of-def atms-of-ms-def clauses-def*
by (*auto simp add*: *H*)
next
case (*restart T*)
then show *?case* **using** *learned marked conf trail* **by** *auto*
next

case (*forget C T*) **note** *C* = *this*(*3*) **and** *C-le* = *this*(*4*) **and** *confl* = *this*(*5*) **and**
T = *this*(*6*)
have *H*: $\bigwedge$*L mark. Propagated L mark* ∈ *set* (*trail S*) ⟹ *atms-of mark* ⊆ *atms-of-msu* (*init-clss S*)
using *marked* **by** *simp*
show *?case* **unfolding** *clauses-def* **apply** *standard*
using *conf T trail C* **unfolding** *clauses-def* **apply** (*auto dest*!: *H*)[]
apply *standard*
using *T trail C* **apply** (*auto dest*!: *H*)[]
apply *standard*
using *T learned C C-le atms-of-ms-remove-subset*[*of set-mset* (*learned-clss S*)] **apply** (*auto*)[]
using *T trail C* **apply** (*auto simp*: *clauses-def lits-of-def*)[]
done
next
case (*backtrack K i M1 M2 L D T*) **note** *decomp* = *this*(*1*) **and** *confl* = *this*(*3*) **and** *undef*= *this*(*6*)
and *T* =*this*(*7*)
have *?C T*
using *conf T decomp undef lev* **by** (*auto simp*: $cdcl_W$*-M-level-inv-decomp*)
moreover have *set M1* ⊆ *set* (*trail S*)
using *backtrack.hyps*(*1*) **by** *auto*
then have *M*: *?M T*
using *marked conf undef confl T decomp lev*
by (*auto simp*: *image-subset-iff clauses-def* $cdcl_W$*-M-level-inv-decomp*)
moreover have *?U T*
using *learned decomp conf confl T undef lev* **unfolding** *clauses-def*
by (*auto simp*: $cdcl_W$*-M-level-inv-decomp*)
moreover have *?V T*
using *M conf confl trail T undef decomp lev* **by** (*force simp*: $cdcl_W$*-M-level-inv-decomp*)
ultimately show *?case* **by** *blast*
next
case (*resolve L C M D T*) **note** *trail-S* = *this*(*1*) **and** *confl* = *this*(*2*) **and** *T* = *this*(*4*)
let *?T* = *update-conflicting* (*Some* (*remdups-mset* (*D* + *C*))) (*tl-trail S*)
have *?C ?T*
using *confl trail-S conf marked* **by** *simp*
moreover have *?M ?T*
using *confl trail-S conf marked* **by** *auto*
moreover have *?U ?T*
using *trail learned* **by** *auto*
moreover have *?V ?T*
using *confl trail-S trail* **by** *auto*
ultimately show *?case* **using** *T* **by** *auto*
qed

lemma $cdcl_W$*-no-strange-atm-inv*:
assumes $cdcl_W$ *S S'* **and** *no-strange-atm S* **and** $cdcl_W$*-M-level-inv S*
shows *no-strange-atm S'*
using $cdcl_W$*-no-strange-atm-explicit*[*OF assms*(*1*)] *assms*(*2*,*3*) **unfolding** *no-strange-atm-def* **by** *fast*

lemma *rtranclp-*$cdcl_W$*-no-strange-atm-inv*:
assumes $cdcl_W^{**}$ *S S'* **and** *no-strange-atm S* **and** $cdcl_W$*-M-level-inv S*
shows *no-strange-atm S'*
using *assms* **by** *induction* (*auto intro*: $cdcl_W$*-no-strange-atm-inv rtranclp-*$cdcl_W$*-consistent-inv*)

17.4.7 No duplicates all around

This invariant shows that there is no duplicate (no literal appearing twice in the formula). The last part could be proven using the previous invariant moreover.

definition *distinct-cdcl$_W$-state* (*S*::*'st*)
 $\longleftrightarrow$ ((∀ *T*. *conflicting S* = *Some T* $\longrightarrow$ *distinct-mset T*)
  ∧ *distinct-mset-mset* (*learned-clss S*)
  ∧ *distinct-mset-mset* (*init-clss S*)
  ∧ (∀ *L mark*. (*Propagated L mark* ∈ *set* (*trail S*) $\longrightarrow$ *distinct-mset* (*mark*))))

lemma *distinct-cdcl$_W$-state-decomp*:
 assumes *distinct-cdcl$_W$-state* (*S*::*'st*)
 shows ∀ *T*. *conflicting S* = *Some T* $\longrightarrow$ *distinct-mset T*
 and *distinct-mset-mset* (*learned-clss S*)
 and *distinct-mset-mset* (*init-clss S*)
 and ∀ *L mark*. (*Propagated L mark* ∈ *set* (*trail S*) $\longrightarrow$ *distinct-mset* (*mark*))
 using *assms* **unfolding** *distinct-cdcl$_W$-state-def* **by** *blast+*

lemma *distinct-cdcl$_W$-state-decomp-2*:
 assumes *distinct-cdcl$_W$-state* (*S*::*'st*)
 shows *conflicting S* = *Some T* $\Longrightarrow$ *distinct-mset T*
 using *assms* **unfolding** *distinct-cdcl$_W$-state-def* **by** *auto*

lemma *distinct-cdcl$_W$-state-S0-cdcl$_W$*[*simp*]:
 distinct-mset-mset N $\Longrightarrow$ *distinct-cdcl$_W$-state* (*init-state N*)
 unfolding *distinct-cdcl$_W$-state-def* **by** *auto*

lemma *distinct-cdcl$_W$-state-inv*:
 assumes
  cdcl$_W$ S S' **and**
  cdcl$_W$-M-level-inv S **and**
  distinct-cdcl$_W$-state S
 shows *distinct-cdcl$_W$-state S'*
 using *assms*
proof (*induct rule*: *cdcl$_W$-all-induct-lev2*)
 case (*backtrack K i M1 M2 L D*)
 then show *?case*
  unfolding *distinct-cdcl$_W$-state-def*
  by (*fastforce dest*: *get-all-marked-decomposition-incl simp*: *cdcl$_W$-M-level-inv-decomp*)
next
 case *restart*
 then show *?case* **unfolding** *distinct-cdcl$_W$-state-def distinct-mset-set-def clauses-def*
 using *learned-clss-restart-state*[*of S*] **by** *auto*
next
 case *resolve*
 then show *?case*
  by (*auto simp add*: *distinct-cdcl$_W$-state-def distinct-mset-set-def clauses-def*
   distinct-mset-single-add
   intro!: *distinct-mset-union-mset*)
qed (*auto simp add*: *distinct-cdcl$_W$-state-def distinct-mset-set-def clauses-def*)

lemma *rtanclp-distinct-cdcl$_W$-state-inv*:
 assumes
  cdcl$_W$** S S' **and**
  cdcl$_W$-M-level-inv S **and**

distinct-cdcl$_W$-state S
shows *distinct-cdcl$_W$-state S'*
using *assms* **apply** (*induct rule*: *rtranclp-induct*)
using *distinct-cdcl$_W$-state-inv rtranclp-cdcl$_W$-consistent-inv* **by** *blast+*

17.4.8 Conflicts and co

This invariant shows that each mark contains a contradiction only related to the previously defined variable.

abbreviation *every-mark-is-a-conflict* :: *'st* ⇒ *bool* **where**
every-mark-is-a-conflict S ≡
∀ *L mark a b*. *a* @ *Propagated L mark* # *b* = (*trail S*)
⟶ (*b* ⊨*as CNot* (*mark* − {#*L*#}) ∧ *L* ∈# *mark*)

definition *cdcl$_W$-conflicting S* ≡
(∀ *T*. *conflicting S* = *Some T* ⟶ *trail S* ⊨*as CNot T*)
∧ *every-mark-is-a-conflict S*

lemma *backtrack-atms-of-D-in-M1*:
fixes *M1* :: (*'v*, *nat*, *'v clause*) *marked-lits*
assumes
inv: *cdcl$_W$-M-level-inv S* **and**
undef: *undefined-lit M1 L* **and**
i: *get-maximum-level* (*trail S*) *D* = *i* **and**
decomp: (*Marked K* (*Suc i*) # *M1*, *M2*)
∈ *set* (*get-all-marked-decomposition* (*trail S*)) **and**
S-lvl: *backtrack-lvl S* = *get-maximum-level* (*trail S*) (*D* + {#*L*#}) **and**
S-confl: *conflicting S* = *Some* (*D* + {#*L*#}) **and**
undef: *undefined-lit M1 L* **and**
T: *T* ∼ (*cons-trail* (*Propagated L* (*D*+{#*L*#}))
(*reduce-trail-to M1*
(*add-learned-cls* (*D* + {#*L*#})
(*update-backtrack-lvl i*
(*update-conflicting None S*))))) **and**
confl: ∀ *T*. *conflicting S* = *Some T* ⟶ *trail S* ⊨*as CNot T*
shows *atms-of D* ⊆ *atm-of* ‘ *lits-of* (*tl* (*trail T*))
proof (*rule ccontr*)
let *?k* = *get-maximum-level* (*trail S*) (*D* + {#*L*#})
have *trail S* ⊨*as CNot D* **using** *confl S-confl* **by** *auto*
then have *vars-of-D*: *atms-of D* ⊆ *atm-of* ‘ *lits-of* (*trail S*) **unfolding** *atms-of-def*
by (*meson image-subsetI mem-set-mset-iff true-annots-CNot-all-atms-defined*)

obtain *M0* **where** *M*: *trail S* = *M0* @ *M2* @ *Marked K* (*Suc i*) # *M1*
using *decomp* **by** *auto*

have *max*: *get-maximum-level* (*trail S*) (*D* + {#*L*#})
= *length* (*get-all-levels-of-marked* (*M0* @ *M2* @ *Marked K* (*Suc i*) # *M1*))
using *inv* **unfolding** *cdcl$_W$-M-level-inv-def S-lvl M* **by** *simp*
assume *a*: ¬ *?thesis*
then obtain *L'* **where**
L': *L'* ∈ *atms-of D* **and**
L'-notin-M1: *L'* ∉ *atm-of* ‘ *lits-of M1*
using *T undef decomp inv* **by** (*auto simp*: *cdcl$_W$-M-level-inv-decomp*)
then have *L'-in*: *L'* ∈ *atm-of* ‘ *lits-of* (*M0* @ *M2* @ *Marked K* (*i* + *1*) # [])
using *vars-of-D* **unfolding** *M* **by** *force*

then obtain L'' **where**
L'' ∈# D **and**
L'': L' = atm-of L''
using L' L'-notin-M1 **unfolding** atms-of-def **by** auto
have lev-L'':
get-level (trail S) L'' = get-rev-level (Marked K (Suc i) # rev M2 @ rev M0) (Suc i) L''
using L'-notin-M1 L'' M **by** (auto simp del: get-rev-level.simps)
have get-all-levels-of-marked (trail S) = rev [1..<1+?k]
using inv S-lvl **unfolding** $cdcl_W$-M-level-inv-def **by** auto
then have get-all-levels-of-marked (M0 @ M2)
= rev [Suc (Suc i)..<Suc (get-maximum-level (trail S) (D + {#L#}))]
unfolding M **by** (auto simp:rev-swap[symmetric] dest!: append-cons-eq-upt-length-i-end)

then have M: get-all-levels-of-marked M0 @ get-all-levels-of-marked M2
= rev [Suc (Suc i)..<Suc (length (get-all-levels-of-marked (M0 @ M2 @ Marked K (Suc i) # M1)))]
unfolding max **unfolding** M **by** simp

have get-rev-level (Marked K (Suc i) # rev (M0 @ M2)) (Suc i) L''
≥ Min (set ((Suc i) # get-all-levels-of-marked (Marked K (Suc i) # rev (M0 @ M2))))
using get-rev-level-ge-min-get-all-levels-of-marked[of L''
rev (M0 @ M2 @ [Marked K (Suc i)]) Suc i] L'-in
unfolding L'' **by** (fastforce simp add: lits-of-def)
also have Min (set ((Suc i) # get-all-levels-of-marked (Marked K (Suc i) # rev (M0 @ M2))))
= Min (set ((Suc i) # get-all-levels-of-marked (rev (M0 @ M2)))) **by** auto
also have ... = Min (set ((Suc i) # get-all-levels-of-marked M0 @ get-all-levels-of-marked M2))
by (simp add: Un-commute)
also have ... = Min (set ((Suc i) # [Suc (Suc i)..<2 + length (get-all-levels-of-marked M0)
+ (length (get-all-levels-of-marked M2) + length (get-all-levels-of-marked M1))]))
unfolding M **by** (auto simp add: Un-commute)
also have ... = Suc i **by** (auto intro: Min-eqI)
finally have get-rev-level (Marked K (Suc i) # rev (M0 @ M2)) (Suc i) L'' ≥ Suc i **.**
then have get-level (trail S) L'' ≥ i + 1
using lev-L'' **by** simp
then have get-maximum-level (trail S) D ≥ i + 1
using get-maximum-level-ge-get-level[OF ‹L'' ∈# D›, of trail S] **by** auto
then show False **using** i **by** auto
qed

lemma distinct-atms-of-incl-not-in-other:
assumes
a1: no-dup (M @ M') **and** a2:
atms-of D ⊆ atm-of ` lits-of M'
shows ∀ x∈atms-of D. x ∉ atm-of ` lits-of M
proof −
{ fix aa :: 'a
have ff1: ⋀l ms. undefined-lit ms l ∨ atm-of l
∈ set (map (λm. atm-of (lit-of (m::('a, 'b, 'c) marked-lit))) ms)
by (simp add: defined-lit-map)
have ff2: ⋀a. a ∉ atms-of D ∨ a ∈ atm-of ` lits-of M'
using a2 **by** (meson subsetCE)
have ff3: ⋀a. a ∉ set (map (λm. atm-of (lit-of m)) M')
∨ a ∉ set (map (λm. atm-of (lit-of m)) M)
using a1 **by** (metis (lifting) IntI distinct-append empty-iff map-append)
have ∀ L a f. ∃ l. ((a::'a) ∉ f ` L ∨ (l::'a literal) ∈ L) ∧ (a ∉ f ` L ∨ f l = a)
by blast

then have *aa* $\notin$ *atms-of D* $\vee$ *aa* $\notin$ *atm-of* ' *lits-of M*
using *ff3 ff2 ff1* **by** (*metis* (*no-types*) *Marked-Propagated-in-iff-in-lits-of*) **}**
then show *?thesis*
by *blast*
qed

lemma *cdcl$_W$-propagate-is-conclusion*:
assumes
cdcl$_W$ S S' **and**
inv: *cdcl$_W$-M-level-inv S* **and**
decomp: *all-decomposition-implies-m* (*init-clss S*) (*get-all-marked-decomposition* (*trail S*)) **and**
learned: *cdcl$_W$-learned-clause S* **and**
confl: $\forall$ *T*. *conflicting S* = *Some T* $\longrightarrow$ *trail S* $\models$*as CNot T* **and**
alien: *no-strange-atm S*
shows *all-decomposition-implies-m* (*init-clss S'*) (*get-all-marked-decomposition* (*trail S'*))
using *assms*(*1*,*2*)
proof (*induct rule*: *cdcl$_W$-all-induct-lev2*)
case *restart*
then show *?case* **by** *auto*
next
case *forget*
then show *?case* **using** *decomp* **by** *auto*
next
case *conflict*
then show *?case* **using** *decomp* **by** *auto*
next
case (*resolve L C M D*) **note** *tr* = *this*(*1*) **and** *T* = *this*(*4*)
let *?decomp* = *get-all-marked-decomposition M*
have *M*: *set ?decomp* = *insert* (*hd ?decomp*) (*set* (*tl ?decomp*))
by (*cases ?decomp*) *auto*
show *?case*
using *decomp tr T* **unfolding** *all-decomposition-implies-def*
by (*cases hd* (*get-all-marked-decomposition M*))
(*auto simp*: *M*)
next
case (*skip L C' M D*) **note** *tr* = *this*(*1*) **and** *T* = *this*(*5*)
have *M*: *set* (*get-all-marked-decomposition M*)
= *insert* (*hd* (*get-all-marked-decomposition M*)) (*set* (*tl* (*get-all-marked-decomposition M*)))
by (*cases get-all-marked-decomposition M*) *auto*
show *?case*
using *decomp tr T* **unfolding** *all-decomposition-implies-def*
by (*cases hd* (*get-all-marked-decomposition M*))
(*auto simp add*: *M*)
next
case *decide* **note** *S* = *this*(*1*) **and** *undef* = *this*(*2*) **and** *T* = *this*(*4*)
show *?case* **using** *decomp T undef* **unfolding** *S all-decomposition-implies-def* **by** *auto*
next
case (*propagate C L T*) **note** *propa* = *this*(*2*) **and** *undef* = *this*(*3*) **and** *T* =*this*(*5*)
obtain *a y* **where** *ay*: *hd* (*get-all-marked-decomposition* (*trail S*)) = (*a*, *y*)
by (*cases hd* (*get-all-marked-decomposition* (*trail S*)))
then have *M*: *trail S* = *y* @ *a* **using** *get-all-marked-decomposition-decomp* **by** *blast*
have *M'*: *set* (*get-all-marked-decomposition* (*trail S*))
= *insert* (*a*, *y*) (*set* (*tl* (*get-all-marked-decomposition* (*trail S*))))
using *ay* **by** (*cases get-all-marked-decomposition* (*trail S*)) *auto*
have (λ*a*. {#*lit-of a*#}) ' *set a* $\cup$ *set-mset* (*init-clss S*) $\models$*ps* (λ*a*. {#*lit-of a*#}) ' *set y*

using *decomp ay* **unfolding** *all-decomposition-implies-def*
by (*cases get-all-marked-decomposition* (*trail S*)) *fastforce+*
then have *a-Un-N-M*: ($\lambda a.$ {#*lit-of a*#}) ' *set a* ∪ *set-mset* (*init-clss S*)
⊨*ps* ($\lambda a.$ {#*lit-of a*#}) ' *set* (*trail S*)
unfolding *M* **by** (*auto simp add*: *all-in-true-clss-clss image-Un*)

have ($\lambda a.$ {#*lit-of a*#}) ' *set a* ∪ *set-mset* (*init-clss S*) ⊨*p* {#*L*#} (**is** *?I* ⊨*p* -)
proof (*rule true-clss-cls-plus-CNot*)
show *?I* ⊨*p C* + {#*L*#}
using *propa propagate.prems learned confl* **unfolding** *M*
by (*metis Un-iff* $cdcl_W$*-learned-clause-def clauses-def mem-set-mset-iff propagate.hyps*(*1*)
set-mset-union true-clss-clss-in-imp-true-clss-cls true-clss-cs-mono-l2
union-trus-clss-clss)
next
have ($\lambda m.$ {#*lit-of m*#}) ' *set* (*trail S*) ⊨*ps CNot C*
using ‹(*trail S*) ⊨*as CNot C*› *true-annots-true-clss-clss* **by** *blast*
then show *?I* ⊨*ps CNot C*
using *a-Un-N-M true-clss-clss-left-right true-clss-clss-union-l-r* **by** *blast*
qed
moreover have $\bigwedge$*aa b*.
∀ (*Ls*, *seen*)∈*set* (*get-all-marked-decomposition* (*y* @ *a*)).
($\lambda a.$ {#*lit-of a*#}) ' *set Ls* ∪ *set-mset* (*init-clss S*) ⊨*ps* ($\lambda a.$ {#*lit-of a*#}) ' *set seen*
⟹ (*aa*, *b*) ∈ *set* (*tl* (*get-all-marked-decomposition* (*y* @ *a*)))
⟹ ($\lambda a.$ {#*lit-of a*#}) ' *set aa* ∪ *set-mset* (*init-clss S*) ⊨*ps* ($\lambda a.$ {#*lit-of a*#}) ' *set b*
by (*metis* (*no-types*, *lifting*) *case-prod-conv get-all-marked-decomposition-never-empty-sym*
list.collapse list.set-intros(*2*))

ultimately show *?case*
using *decomp T undef* **unfolding** *ay all-decomposition-implies-def*
using *M* ‹($\lambda a.$ {#*lit-of a*#}) ' *set a* ∪ *set-mset* (*init-clss S*) ⊨*ps* ($\lambda a.$ {#*lit-of a*#}) ' *set y*›
ay **by** *auto*
next
case (*backtrack K i M1 M2 L D T*) **note** *decomp'* = *this*(*1*) **and** *lev-L* = *this*(*2*) **and** *conf* = *this*(*3*)
and
undef = *this*(*6*) **and** *T* = *this*(*7*)
have ∀ *l* ∈ *set M2*. ¬*is-marked l*
using *get-all-marked-decomposition-snd-not-marked backtrack.hyps*(*1*) **by** *blast*
obtain *M0* **where** *M*: *trail S* = *M0* @ *M2* @ *Marked K* (*i* + *1*) # *M1*
using *decomp'* **by** *auto*
show *?case* **unfolding** *all-decomposition-implies-def*
proof
fix *x*
assume *x* ∈*set* (*get-all-marked-decomposition* (*trail T*))
then have *x*: *x* ∈ *set* (*get-all-marked-decomposition* (*Propagated L* ((*D* + {#*L*#})) # *M1*))
using *T decomp' undef inv* **by** (*simp add*: $cdcl_W$*-M-level-inv-decomp*)
let *?m* = *get-all-marked-decomposition* (*Propagated L* ((*D* + {#*L*#})) # *M1*)
let *?hd* = *hd ?m*
let *?tl* = *tl ?m*
have *x* = *?hd* ∨ *x* ∈ *set ?tl*
using *x* **by** (*case-tac ?m*) *auto*
moreover {
assume *x* ∈ *set ?tl*
then have *x* ∈ *set* (*get-all-marked-decomposition* (*trail S*))
using *tl-get-all-marked-decomposition-skip-some*[*of x*] **by** (*simp add*: *list.set-sel*(*2*) *M*)
then have *case x of* (*Ls*, *seen*) ⇒ ($\lambda a.$ {#*lit-of a*#}) ' *set Ls*

$\cup$ *set-mset* (*init-clss* (*T*))
$\models$*ps* ($\lambda a.$ {#*lit-of a*#}) ‘ *set seen*
using *decomp learned decomp confl alien inv T undef M*
unfolding *all-decomposition-implies-def* $cdcl_W$*-M-level-inv-def*
by *auto*
}
moreover {
assume *x = ?hd*
obtain *M1′ M1′′* **where** *M1*: *hd* (*get-all-marked-decomposition M1*) = (*M1′*, *M1′′*)
by (*cases hd* (*get-all-marked-decomposition M1*))
then have *x′*: *x* = (*M1′*, *Propagated L* ((*D* + {#*L*#})) # *M1′′*)
using ‹*x= ?hd*› **by** *auto*
have (*M1′*, *M1′′*) ∈ *set* (*get-all-marked-decomposition* (*trail S*))
using *M1*[*symmetric*] *hd-get-all-marked-decomposition-skip-some*[*OF M1*[*symmetric*],
of M0 @ *M2* - *i+1*] **unfolding** *M* **by** *fastforce*
then have *1*: ($\lambda a.$ {#*lit-of a*#}) ‘ *set M1′* ∪ *set-mset* (*init-clss S*)
$\models$*ps* ($\lambda a.$ {#*lit-of a*#}) ‘ *set M1′′*
using *decomp* **unfolding** *all-decomposition-implies-def* **by** *auto*
moreover
have *trail S* $\models$*as CNot D* **using** *conf confl* **by** *auto*
then have *vars-of-D*: *atms-of D* ⊆ *atm-of* ‘ *lits-of* (*trail S*)
unfolding *atms-of-def*
by (*meson image-subsetI mem-set-mset-iff true-annots-CNot-all-atms-defined*)
have *vars-of-D*: *atms-of D* ⊆ *atm-of* ‘ *lits-of M1*
using *backtrack-atms-of-D-in-M1*[*of S M1 L D i K M2 T*] *backtrack inv conf confl*
by (*auto simp*: $cdcl_W$*-M-level-inv-decomp*)
have *no-dup* (*trail S*) **using** *inv* **by** (*auto simp*: $cdcl_W$*-M-level-inv-decomp*)
then have *vars-in-M1*:
∀ *x* ∈ *atms-of D*. *x* ∉ *atm-of* ‘ *lits-of* (*M0* @ *M2* @ *Marked K* (*i* + *1*) # [])
using *vars-of-D distinct-atms-of-incl-not-in-other*[*of M0* @*M2* @ *Marked K* (*i* + *1*) # []
M1]
unfolding *M* **by** *auto*
have *M1* $\models$*as CNot D*
using *vars-in-M1 true-annots-remove-if-notin-vars*[*of M0* @ *M2* @ *Marked K* (*i* + *1*) # []
M1 CNot D] ‹*trail S* $\models$*as CNot D*› **unfolding** *M lits-of-def* **by** *simp*
have *M1* = *M1′′* @ *M1′* **by** (*simp add*: *M1 get-all-marked-decomposition-decomp*)
have *TT*: ($\lambda a.$ {#*lit-of a*#}) ‘ *set M1′* ∪ *set-mset* (*init-clss S*) $\models$*ps CNot D*
using *true-annots-true-clss-cls*[*OF* ‹*M1* $\models$*as CNot D*›] *true-clss-clss-left-right*[*OF 1*,
of CNot D] **unfolding** ‹*M1* = *M1′′* @ *M1′*› **by** (*auto simp add*: *inf-sup-aci*(*5*,*7*))
have *init-clss S* $\models$*pm D* + {#*L*#}
using *conf learned* $cdcl_W$*-learned-clause-def confl* **by** *blast*
then have *T′*: ($\lambda a.$ {#*lit-of a*#}) ‘ *set M1′* ∪ *set-mset* (*init-clss S*) $\models$*p D* + {#*L*#} **by** *auto*
have *atms-of* (*D* + {#*L*#}) ⊆ *atms-of-msu* (*clauses S*)
using *alien conf* **unfolding** *no-strange-atm-def clauses-def* **by** *auto*
then have ($\lambda a.$ {#*lit-of a*#}) ‘ *set M1′* ∪ *set-mset* (*init-clss S*) $\models$*p* {#*L*#}
using *true-clss-cls-plus-CNot*[*OF T′ TT*] **by** *auto*
ultimately
have *case x of* (*Ls*, *seen*) ⇒ ($\lambda a.$ {#*lit-of a*#}) ‘ *set Ls*
∪ *set-mset* (*init-clss T*)
$\models$*ps* ($\lambda a.$ {#*lit-of a*#}) ‘ *set seen* **using** *T′ T decomp′ undef inv* **unfolding** *x′*
by (*simp add*: $cdcl_W$*-M-level-inv-decomp*)
}
ultimately show *case x of* (*Ls*, *seen*) ⇒ ($\lambda a.$ {#*lit-of a*#}) ‘ *set Ls* ∪ *set-mset* (*init-clss T*)
$\models$*ps* ($\lambda a.$ {#*lit-of a*#}) ‘ *set seen* **using** *T* **by** *auto*
qed

qed

lemma *cdcl*$_W$*-propagate-is-false*:
  assumes
    cdcl$_W$ *S S′* **and**
    lev: *cdcl*$_W$*-M-level-inv S* **and**
    learned: *cdcl*$_W$*-learned-clause S* **and**
    decomp: *all-decomposition-implies-m* (*init-clss S*) (*get-all-marked-decomposition* (*trail S*)) **and**
    confl: ∀ *T*. *conflicting S* = *Some T* ⟶ *trail S* ⊨*as CNot T* **and**
    alien: *no-strange-atm S* **and**
    mark-confl: *every-mark-is-a-conflict S*
  shows *every-mark-is-a-conflict S′*
  using *assms*(*1*,*2*)
proof (*induct rule*: *cdcl*$_W$*-all-induct-lev2*)
  case (*propagate C L T*) **note** *undef* = *this*(*3*) **and** *T* =*this*(*5*)
  show *?case*
    proof (*intro allI impI*)
      fix *L′ mark a b*
      assume *a* @ *Propagated L′ mark* # *b* = *trail T*
      then have (*a*=[] ∧ *L* = *L′* ∧ *mark* = *C* + {#*L*#} ∧ *b* = *trail S*)
        ∨ *tl a* @ *Propagated L′ mark* # *b* = *trail S*
        using *T undef* **by** (*cases a*) *fastforce+*
      moreover {
        assume *tl a* @ *Propagated L′ mark* # *b* = *trail S*
        then have *b* ⊨*as CNot* (*mark* − {#*L′*#}) ∧ *L′* ∈# *mark*
          using *mark-confl* **by** *auto*
      }
      moreover {
        assume *a*=[] **and** *L* = *L′* **and** *mark* = *C* + {#*L*#} **and** *b* = *trail S*
        then have *b* ⊨*as CNot* (*mark* − {#*L*#}) ∧ *L* ∈# *mark*
          using ‹*trail S* ⊨*as CNot C*› **by** *auto*
      }
      ultimately show *b* ⊨*as CNot* (*mark* − {#*L′*#}) ∧ *L′* ∈# *mark* **by** *blast*
    qed
next
  case (*decide L*) **note** *undef*[*simp*] = *this*(*2*) **and** *T* = *this*(*4*)
  have ⋀*a La mark b*. *a* @ *Propagated La mark* # *b* = *Marked L* (*backtrack-lvl S*+*1*) # *trail S*
    ⟹ *tl a* @ *Propagated La mark* # *b* = *trail S* **by** (*case-tac a*, *auto*)
  then show *?case* **using** *mark-confl T* **unfolding** *decide.hyps*(*1*) **by** *fastforce*
next
  case (*skip L C′ M D T*) **note** *tr* = *this*(*1*) **and** *T* = *this*(*5*)
  show *?case*
    proof (*intro allI impI*)
      fix *L′ mark a b*
      assume *a* @ *Propagated L′ mark* # *b* = *trail T*
      then have *a* @ *Propagated L′ mark* # *b* = *M* **using** *tr T* **by** *simp*
      then have (*Propagated L C′* # *a*) @ *Propagated L′ mark* # *b* = *Propagated L C′* # *M* **by** *auto*
      moreover have ∀ *La mark a b*. *a* @ *Propagated La mark* # *b* = *Propagated L C′* # *M*
        ⟶ *b* ⊨*as CNot* (*mark* − {#*La*#}) ∧ *La* ∈# *mark*
        using *mark-confl* **unfolding** *skip.hyps*(*1*) **by** *simp*
      ultimately show *b* ⊨*as CNot* (*mark* − {#*L′*#}) ∧ *L′* ∈# *mark* **by** *blast*
    qed
next
  case (*conflict D*)
  then show *?case* **using** *mark-confl* **by** *simp*

next
  case (*resolve L C M D T*) **note** *tr-S* = *this*(*1*) **and** *T* = *this*(*4*)
  show *?case* **unfolding** *resolve.hyps*(*1*)
    proof (*intro allI impI*)
      fix *L′ mark a b*
      assume *a* @ *Propagated L′ mark* # *b* = *trail T*
      then have *Propagated L* ((*C* + {#*L*#})) # *M*
        = (*Propagated L* ((*C* + {#*L*#})) # *a*) @ *Propagated L′ mark* # *b*
        using *T tr-S* **by** *auto*
      then show *b* ⊨*as CNot* (*mark* − {#*L′*#}) ∧ *L′* ∈# *mark*
        using *mark-confl* **unfolding** *resolve.hyps*(*1*) **by** *presburger*
    qed
next
  case *restart*
  then show *?case* **by** *auto*
next
  case *forget*
  then show *?case* **using** *mark-confl* **by** *auto*
next
  case (*backtrack K i M1 M2 L D T*) **note** *decomp* = *this*(*1*) **and** *conf* = *this*(*3*) **and** *undef* = *this*(*6*)
and
    T =*this*(*7*)
  have ∀ *l* ∈ *set M2*. ¬*is-marked l*
    using *get-all-marked-decomposition-snd-not-marked backtrack.hyps*(*1*) **by** *blast*
  obtain *M0* **where** *M*: *trail S* = *M0* @ *M2* @ *Marked K* (*i* + *1*) # *M1*
    using *backtrack.hyps*(*1*) **by** *auto*
  have [*simp*]: *trail* (*reduce-trail-to M1* (*add-learned-cls* (*D* + {#*L*#})
  (*update-backtrack-lvl i* (*update-conflicting None S*)))) = *M1*
    using *decomp lev* **by** (*auto simp*: $cdcl_W$*-M-level-inv-decomp*)
  show *?case*
    proof (*intro allI impI*)
      fix *La mark a b*
      assume *a* @ *Propagated La mark* # *b* = *trail T*
      then have (*a* = [] ∧ *Propagated La mark* = *Propagated L* (*D* + {#*L*#}) ∧ *b* = *M1*)
        ∨ *tl a* @ *Propagated La mark* # *b* = *M1*
        using *M T decomp undef* **by** (*cases a*) (*auto*)
      moreover {
        assume *A*: *a* = [] **and**
          P: *Propagated La mark* = *Propagated L* ((*D* + {#*L*#})) **and**
          b: *b* = *M1*
        have *trail S* ⊨*as CNot D* **using** *conf confl* **by** *auto*
        then have *vars-of-D*: *atms-of D* ⊆ *atm-of* ‘ *lits-of* (*trail S*)
          unfolding *atms-of-def*
          by (*meson image-subsetI mem-set-mset-iff true-annots-CNot-all-atms-defined*)
        have *vars-of-D*: *atms-of D* ⊆ *atm-of* ‘ *lits-of M1*
          using *backtrack-atms-of-D-in-M1*[*of S M1 L D i K M2 T*] *T backtrack lev confl* **by** *auto*
        have *no-dup* (*trail S*) **using** *lev* **by** (*auto simp*: $cdcl_W$*-M-level-inv-decomp*)
        then have *vars-in-M1*: ∀ *x* ∈ *atms-of D*. *x* ∉
          atm-of ‘ *lits-of* (*M0* @ *M2* @ *Marked K* (*i* + *1*) # [])
          using *vars-of-D distinct-atms-of-incl-not-in-other*[*of M0* @ *M2* @ *Marked K* (*i* + *1*) # []
            M1] **unfolding** *M* **by** *auto*
        have *M1* ⊨*as CNot D*
          using *vars-in-M1 true-annots-remove-if-notin-vars*[*of M0* @ *M2* @ *Marked K* (*i* + *1*) # [] *M1*
            CNot D] ⟨*trail S* ⊨*as CNot D*⟩ **unfolding** *M lits-of-def* **by** *simp*
        then have *b* ⊨*as CNot* (*mark* − {#*La*#}) ∧ *La* ∈# *mark*

using *P b* **by** *auto*
}
moreover {
assume *tl a @ Propagated La mark # b = M1*
then obtain c' **where** c' *@ Propagated La mark # b = trail S* **unfolding** *M* **by** *auto*
then have *b* $\models$*as CNot* (*mark* − {#*La*#}) ∧ *La* ∈# *mark*
using *mark-confl* **by** *blast*
}
ultimately show *b* $\models$*as CNot* (*mark* − {#*La*#}) ∧ *La* ∈# *mark* **by** *fast*
qed
qed

lemma *cdcl*$_W$*-conflicting-is-false*:
assumes
cdcl$_W$ *S S'* **and**
M-lev: *cdcl*$_W$*-M-level-inv S* **and**
confl-inv: ∀ *T*. *conflicting S = Some T* ⟶ *trail S* $\models$*as CNot T* **and**
marked-confl: ∀ *L mark a b*. *a @ Propagated L mark # b* = (*trail S*)
⟶ (*b* $\models$*as CNot* (*mark* − {#*L*#}) ∧ *L* ∈# *mark*) **and**
dist: *distinct-cdcl*$_W$*-state S*
shows ∀ *T*. *conflicting S' = Some T* ⟶ *trail S'* $\models$*as CNot T*
using *assms*(*1*,*2*)
proof (*induct rule*: *cdcl*$_W$*-all-induct-lev2*)
case (*skip L C' M D*) **note** *tr-S = this*(*1*) **and** *T =this*(*5*)
then have *Propagated L C' # M* $\models$*as CNot D* **using** *assms skip* **by** *auto*
moreover
have *L* ∉# *D*
proof (*rule ccontr*)
assume ¬ *?thesis*
then have − *L* ∈ *lits-of M*
using *in-CNot-implies-uminus*(*2*)[*of D L Propagated L C' # M*]
⟨*Propagated L C' # M* $\models$*as CNot D*⟩ **by** *simp*
then show *False*
by (*metis M-lev cdcl*$_W$*-M-level-inv-decomp*(*1*) *consistent-interp-def insert-iff*
lits-of-cons marked-lit.sel(*2*) *skip.hyps*(*1*))
qed
ultimately show *?case*
using *skip.hyps*(*1−3*) *true-annots-CNot-lit-of-notin-skip T* **unfolding** *cdcl*$_W$*-M-level-inv-def*
by *fastforce*
next
case (*resolve L C M D T*) **note** *tr = this*(*1*) **and** *confl = this*(*2*) **and** *T = this*(*4*)
show *?case*
proof (*intro allI impI*)
fix *T'*
have *tl* (*trail S*) $\models$*as CNot C* **using** *tr assms*(*4*) **by** *fastforce*
moreover
have *distinct-mset* (*D* + {#− *L*#}) **using** *confl dist*
unfolding *distinct-cdcl*$_W$*-state-def* **by** *auto*
then have −*L* ∉# *D* **unfolding** *distinct-mset-def* **by** *auto*
have *M* $\models$*as CNot D*
proof −
have *Propagated L* ((*C* + {#*L*#})) # *M* $\models$*as CNot D* ∪ *CNot* {#− *L*#}
using *confl tr confl-inv* **by** *force*
then show *?thesis*
using *M-lev* ⟨− *L* ∉# *D*⟩ *tr true-annots-lit-of-notin-skip*

unfolding *cdcl*$_W$*-M-level-inv-def* **by** *force*
qed
moreover assume *conflicting* $T = $ *Some* T'
ultimately
show *trail* $T \models$*as CNot* T'
using *tr* T **by** *auto*
qed
qed (*auto simp*: *assms*(*2*) *cdcl*$_W$*-M-level-inv-decomp*)

lemma *cdcl*$_W$*-conflicting-decomp*:
assumes *cdcl*$_W$*-conflicting* S
shows $\forall$ T. *conflicting* S = *Some* $T \longrightarrow$ *trail* $S \models$*as CNot* T
and $\forall$ L *mark a b*. *a* @ *Propagated L mark* # *b* = (*trail* S)
$\longrightarrow$ ($b \models$*as CNot* (*mark* $-$ {#L#}) $\wedge$ $L \in$# *mark*)
using *assms* **unfolding** *cdcl*$_W$*-conflicting-def* **by** *blast+*

lemma *cdcl*$_W$*-conflicting-decomp2*:
assumes *cdcl*$_W$*-conflicting* S **and** *conflicting* S = *Some* T
shows *trail* $S \models$*as CNot* T
using *assms* **unfolding** *cdcl*$_W$*-conflicting-def* **by** *blast+*

lemma *cdcl*$_W$*-conflicting-decomp2'*:
assumes
cdcl$_W$*-conflicting* S **and**
conflicting S = *Some* D
shows *trail* $S \models$*as CNot* D
using *assms* **unfolding** *cdcl*$_W$*-conflicting-def* **by** *auto*

lemma *cdcl*$_W$*-conflicting-S0-cdcl*$_W$[*simp*]:
cdcl$_W$*-conflicting* (*init-state* N)
unfolding *cdcl*$_W$*-conflicting-def* **by** *auto*

17.4.9 Putting all the invariants together

lemma *cdcl*$_W$*-all-inv*:
assumes *cdcl*$_W$: *cdcl*$_W$ S S' **and**
1: *all-decomposition-implies-m* (*init-clss* S) (*get-all-marked-decomposition* (*trail* S)) **and**
2: *cdcl*$_W$*-learned-clause* S **and**
4: *cdcl*$_W$*-M-level-inv* S **and**
5: *no-strange-atm* S **and**
7: *distinct-cdcl*$_W$*-state* S **and**
8: *cdcl*$_W$*-conflicting* S
shows *all-decomposition-implies-m* (*init-clss* S') (*get-all-marked-decomposition* (*trail* S'))
and *cdcl*$_W$*-learned-clause* S'
and *cdcl*$_W$*-M-level-inv* S'
and *no-strange-atm* S'
and *distinct-cdcl*$_W$*-state* S'
and *cdcl*$_W$*-conflicting* S'
proof −
show *S1*: *all-decomposition-implies-m* (*init-clss* S') (*get-all-marked-decomposition* (*trail* S'))
using *cdcl*$_W$*-propagate-is-conclusion*[*OF cdcl*$_W$ *4 1 2 - 5*] *8* **unfolding** *cdcl*$_W$*-conflicting-def*
by *blast*
show *S2*: *cdcl*$_W$*-learned-clause* S' **using** *cdcl*$_W$*-learned-clss*[*OF cdcl*$_W$ *2 4*] **.**
show *S4*: *cdcl*$_W$*-M-level-inv* S' **using** *cdcl*$_W$*-consistent-inv*[*OF cdcl*$_W$ *4*] **.**
show *S5*: *no-strange-atm* S' **using** *cdcl*$_W$*-no-strange-atm-inv*[*OF cdcl*$_W$ *5 4*] **.**
show *S7*: *distinct-cdcl*$_W$*-state* S' **using** *distinct-cdcl*$_W$*-state-inv*[*OF cdcl*$_W$ *4 7*] **.**

show *S8*: $cdcl_W$*-conflicting* S'
using $cdcl_W$*-conflicting-is-false*[*OF* $cdcl_W$ *4* - - *7*] *8* $cdcl_W$*-propagate-is-false*[*OF* $cdcl_W$ *4* *2* *1* - *5*]
unfolding $cdcl_W$*-conflicting-def* **by** *fast*
qed

lemma *rtranclp-*$cdcl_W$*-all-inv*:
assumes
$cdcl_W$: *rtranclp* $cdcl_W$ S S' **and**
1: *all-decomposition-implies-m* (*init-clss* S) (*get-all-marked-decomposition* (*trail* S)) **and**
2: $cdcl_W$*-learned-clause* S **and**
4: $cdcl_W$*-M-level-inv* S **and**
5: *no-strange-atm* S **and**
7: *distinct-*$cdcl_W$*-state* S **and**
8: $cdcl_W$*-conflicting* S
shows
all-decomposition-implies-m (*init-clss* S') (*get-all-marked-decomposition* (*trail* S')) **and**
$cdcl_W$*-learned-clause* S' **and**
$cdcl_W$*-M-level-inv* S' **and**
no-strange-atm S' **and**
distinct-$cdcl_W$*-state* S' **and**
$cdcl_W$*-conflicting* S'
using *assms*
proof (*induct rule*: *rtranclp-induct*)
case *base*
case *1* **then show** *?case* **by** *blast*
case *2* **then show** *?case* **by** *blast*
case *3* **then show** *?case* **by** *blast*
case *4* **then show** *?case* **by** *blast*
case *5* **then show** *?case* **by** *blast*
case *6* **then show** *?case* **by** *blast*
next
case (*step* S' S'') **note** $H = this$
case *1* **with** $H(3-7)$[*OF* $this(1-6)$] **show** *?case* **using** $cdcl_W$*-all-inv*[*OF* $H(2)$] H **by** *presburger*
case *2* **with** $H(3-7)$[*OF* $this(1-6)$] **show** *?case* **using** $cdcl_W$*-all-inv*[*OF* $H(2)$] H **by** *presburger*
case *3* **with** $H(3-7)$[*OF* $this(1-6)$] **show** *?case* **using** $cdcl_W$*-all-inv*[*OF* $H(2)$] H **by** *presburger*
case *4* **with** $H(3-7)$[*OF* $this(1-6)$] **show** *?case* **using** $cdcl_W$*-all-inv*[*OF* $H(2)$] H **by** *presburger*
case *5* **with** $H(3-7)$[*OF* $this(1-6)$] **show** *?case* **using** $cdcl_W$*-all-inv*[*OF* $H(2)$] H **by** *presburger*
case *6* **with** $H(3-7)$[*OF* $this(1-6)$] **show** *?case* **using** $cdcl_W$*-all-inv*[*OF* $H(2)$] H **by** *presburger*
qed

lemma *all-invariant-S0-*$cdcl_W$:
assumes *distinct-mset-mset* N
shows *all-decomposition-implies-m* (*init-clss* (*init-state* N))
(*get-all-marked-decomposition* (*trail* (*init-state* N)))
and $cdcl_W$*-learned-clause* (*init-state* N)
and $\forall T$. *conflicting* (*init-state* N) = *Some* $T \longrightarrow$ (*trail* (*init-state* N))$\models$*as CNot* T
and *no-strange-atm* (*init-state* N)
and *consistent-interp* (*lits-of* (*trail* (*init-state* N)))

and $\forall$ *L mark a b. a @ Propagated L mark # b = trail (init-state N)* $\longrightarrow$
(*b* $\models$*as CNot (mark* $-$ $\{$#*L*#$\}$) $\wedge$ *L* $\in$# *mark*)
and *distinct-cdcl*$_W$*-state (init-state N)*
using *assms* **by** *auto*

lemma *cdcl*$_W$*-only-propagated-vars-unsat*:
assumes
marked: $\forall$ *x* $\in$ *set M*. $\neg$ *is-marked x* **and**
DN: *D* $\in$# *clauses S* **and**
D: *M* $\models$*as CNot D* **and**
inv: *all-decomposition-implies-m N (get-all-marked-decomposition M)* **and**
state: *state S = (M, N, U, k, C)* **and**
learned-cl: *cdcl*$_W$*-learned-clause S* **and**
atm-incl: *no-strange-atm S*
shows *unsatisfiable (set-mset N)*
proof (*rule ccontr*)
assume $\neg$ *unsatisfiable (set-mset N)*
then obtain *I* **where**
I: *I* $\models$*s set-mset N* **and**
cons: *consistent-interp I* **and**
tot: *total-over-m I (set-mset N)*
unfolding *satisfiable-def* **by** *auto*
have *atms-of-msu N* $\cup$ *atms-of-msu U = atms-of-msu N*
using *atm-incl state* **unfolding** *total-over-m-def no-strange-atm-def*
by (*auto simp add*: *clauses-def*)
then have *total-over-m I (set-mset N)* **using** *tot* **unfolding** *total-over-m-def* **by** *auto*
moreover have *N* $\models$*psm U* **using** *learned-cl state* **unfolding** *cdcl*$_W$*-learned-clause-def* **by** *auto*
ultimately have *I-D*: *I* $\models$ *D*
using *I DN cons state* **unfolding** *true-clss-clss-def true-clss-def Ball-def*
by (*metis Un-iff* ‹*atms-of-msu N* $\cup$ *atms-of-msu U = atms-of-msu N*› *atms-of-ms-union clauses-def mem-set-mset-iff prod.inject set-mset-union total-over-m-def*)

have *l0*: $\{\{$#*lit-of L*#$\}$ |*L. is-marked L* $\wedge$ *L* $\in$ *set M*$\}$ = $\{\}$ **using** *marked* **by** *auto*
have *atms-of-ms (set-mset N* $\cup$ (λ*a*. $\{$#*lit-of a*#$\}$) ‘ *set M) = atms-of-msu N*
using *atm-incl state* **unfolding** *no-strange-atm-def* **by** *auto*
then have *total-over-m I (set-mset N* $\cup$ (λ*a*. $\{$#*lit-of a*#$\}$) ‘ (*set M*))
using *tot* **unfolding** *total-over-m-def* **by** *auto*
then have *I* $\models$*s* (λ*a*. $\{$#*lit-of a*#$\}$) ‘ (*set M*)
using *all-decomposition-implies-propagated-lits-are-implied*[*OF inv*] *cons I*
unfolding *true-clss-clss-def l0* **by** *auto*
then have *IM*: *I* $\models$*s* (λ*a*. $\{$#*lit-of a*#$\}$) ‘ *set M* **by** *auto*
{
fix *K*
assume *K* $\in$# *D*
then have $-K$ $\in$ *lits-of M*
using *D* **unfolding** *true-annots-def Ball-def CNot-def true-annot-def true-cls-def true-lit-def Bex-mset-def* **by** (*metis (mono-tags, lifting) count-single less-not-refl mem-Collect-eq*)
then have $-K$ $\in$ *I* **using** *IM true-clss-singleton-lit-of-implies-incl lits-of-def* **by** *fastforce*
}
then have $\neg$ *I* $\models$ *D* **using** *cons* **unfolding** *true-cls-def true-lit-def consistent-interp-def* **by** *auto*
then show *False* **using** *I-D* **by** *blast*
qed

We have actually a much stronger theorem, namely *all-decomposition-implies ?N (get-all-marked-decomposition ?M)* $\Longrightarrow$ *?N* $\cup$ $\{\{$#*lit-of L*#$\}$ |*L. is-marked L* $\wedge$ *L* $\in$ *set ?M*$\}$ $\models$*ps* (λ*a*. $\{$#*lit-of a*#$\}$) ‘ *set*

?M, that show that the only choices we made are marked in the formula

lemma
  assumes *all-decomposition-implies-m N* (*get-all-marked-decomposition M*)
  and $\forall$ *m* $\in$ *set M*. $\neg$*is-marked m*
  shows *set-mset N* $\models$*ps* (λ*a*. {#*lit-of a*#}) ' *set M*
proof −
  have *T*: {{#*lit-of L*#} |*L*. *is-marked L* $\wedge$ *L* $\in$ *set M*} = {} **using** *assms*(*2*) **by** *auto*
  then show *?thesis*
    using *all-decomposition-implies-propagated-lits-are-implied*[*OF assms*(*1*)] **unfolding** *T* **by** *simp*
qed

lemma *conflict-with-false-implies-unsat*:
  assumes
    $cdcl_W$: $cdcl_W$ *S S'* **and**
    lev: $cdcl_W$*-M-level-inv S* **and**
    [*simp*]: *conflicting S'* = *Some* {#} **and**
    learned: $cdcl_W$*-learned-clause S*
  shows *unsatisfiable* (*set-mset* (*init-clss S*))
  using *assms*
proof −
  have $cdcl_W$*-learned-clause S'* **using** $cdcl_W$*-learned-clss* $cdcl_W$ *learned lev* **by** *auto*
  then have *init-clss S'* $\models$*pm* {#} **using** *assms*(*3*) **unfolding** $cdcl_W$*-learned-clause-def* **by** *auto*
  then have *init-clss S* $\models$*pm* {#}
    using $cdcl_W$*-init-clss*[*OF assms*(*1*) *lev*] **by** *auto*
  then show *?thesis* **unfolding** *satisfiable-def true-clss-cls-def* **by** *auto*
qed

lemma *conflict-with-false-implies-terminated*:
  assumes $cdcl_W$ *S S'*
  and *conflicting S* = *Some* {#}
  shows *False*
  using *assms* **by** (*induct rule*: $cdcl_W$*-all-induct*) *auto*

17.4.10 No tautology is learned

This is a simple consequence of all we have shown previously. It is not strictly necessary, but helps finding a better bound on the number of learned clauses.

lemma *learned-clss-are-not-tautologies*:
  assumes
    $cdcl_W$ *S S'* **and**
    lev: $cdcl_W$*-M-level-inv S* **and**
    conflicting: $cdcl_W$*-conflicting S* **and**
    no-tauto: $\forall$ *s* $\in$# *learned-clss S*. $\neg$*tautology s*
  shows $\forall$ *s* $\in$# *learned-clss S'*. $\neg$*tautology s*
  using *assms*
proof (*induct rule*: $cdcl_W$*-all-induct-lev2*)
  case (*backtrack K i M1 M2 L D*) **note** *confl* = *this*(*3*)
  have *consistent-interp* (*lits-of* (*trail S*)) **using** *lev* **by** (*auto simp*: $cdcl_W$*-M-level-inv-decomp*)
  moreover
    have *trail S* $\models$*as CNot* (*D* + {#*L*#})
      using *conflicting confl* **unfolding** $cdcl_W$*-conflicting-def* **by** *auto*
    then have *lits-of* (*trail S*) $\models$*s CNot* (*D* + {#*L*#}) **using** *true-annots-true-cls* **by** *blast*
  ultimately have $\neg$*tautology* (*D* + {#*L*#}) **using** *consistent-CNot-not-tautology* **by** *blast*
  then show *?case* **using** *backtrack no-tauto*

by (*auto simp*: *cdcl*$_W$*-M-level-inv-decomp split*: *split-if-asm*)
next
case *restart*
then show *?case* **using** *learned-clss-restart-state state-eq-learned-clss no-tauto*
by (*metis* (*no-types*, *lifting*) *ball-msetE ball-msetI mem-set-mset-iff set-mset-mono subsetCE*)
qed *auto*

definition *final-cdcl*$_W$*-state* (S:: $'st$)
$\longleftrightarrow$ (*trail* S $\models$*asm init-clss* S
$\vee$ (($\forall L \in$ *set* (*trail* S). $\neg$*is-marked* L) $\wedge$
($\exists C \in\#$ *init-clss* S. *trail* S $\models$*as CNot* C)))

definition *termination-cdcl*$_W$*-state* (S:: $'st$)
$\longleftrightarrow$ (*trail* S $\models$*asm init-clss* S
$\vee$ (($\forall L \in$ *atms-of-msu* (*init-clss* S). $L \in$ *atm-of* ‘ *lits-of* (*trail* S))
$\wedge$ ($\exists C \in\#$ *init-clss* S. *trail* S $\models$*as CNot* C)))

17.5 CDCL Strong Completeness

fun *mapi* :: ($'a \Rightarrow nat \Rightarrow 'b$) $\Rightarrow$ *nat* $\Rightarrow$ $'a$ *list* $\Rightarrow$ $'b$ *list* **where**
mapi - - [] = [] |
mapi f n (x # xs) = f x n # *mapi* f ($n - 1$) xs

lemma *mark-not-in-set-mapi*[*simp*]: $L \notin$ *set* $M \Longrightarrow$ *Marked* L $k \notin$ *set* (*mapi Marked* i M)
by (*induct* M *arbitrary*: i) *auto*

lemma *propagated-not-in-set-mapi*[*simp*]: $L \notin$ *set* $M \Longrightarrow$ *Propagated* L $k \notin$ *set* (*mapi Marked* i M)
by (*induct* M *arbitrary*: i) *auto*

lemma *image-set-mapi*:
f ‘ *set* (*mapi* g i M) = *set* (*mapi* (λx i. f (g x i)) i M)
by (*induction* M *arbitrary*: i) *auto*

lemma *mapi-map-convert*:
$\forall x$ i j. f x i = f x j $\Longrightarrow$ *mapi* f i M = *map* (λx. f x 0) M
by (*induction* M *arbitrary*: i) *auto*

lemma *defined-lit-mapi*: *defined-lit* (*mapi Marked* i M) $L \longleftrightarrow$ *atm-of* $L \in$ *atm-of* ‘ *set* M
by (*induction* M) (*auto simp*: *defined-lit-map image-set-mapi mapi-map-convert*)

lemma *cdcl*$_W$*-can-do-step*:
assumes
consistent-interp (*set* M) **and**
distinct M **and**
atm-of ‘ (*set* M) $\subseteq$ *atms-of-msu* N
shows $\exists S$. *rtranclp cdcl*$_W$ (*init-state* N) S
$\wedge$ *state* S = (*mapi Marked* (*length* M) M, N, {#}, *length* M, *None*)
using *assms*
proof (*induct* M)
case *Nil*
then show *?case* **by** *auto*
next
case (*Cons* L M) **note** IH = *this*(1)
have *consistent-interp* (*set* M) **and** *distinct* M **and** *atm-of* ‘ *set* $M \subseteq$ *atms-of-msu* N
using *Cons.prems*($1-3$) **unfolding** *consistent-interp-def* **by** *auto*
then obtain S **where**

    st: $cdcl_W^{**}$ (*init-state N*) *S* **and**
    S: *state S* = (*mapi Marked* (*length M*) *M*, *N*, {#}, *length M*, *None*)
    using *IH* **by** *auto*
  let *?S*$_0$ = *incr-lvl* (*cons-trail* (*Marked L* (*length M* +*1*)) *S*)
  have *undefined-lit* (*mapi Marked* (*length M*) *M*) *L*
    using *Cons.prems*(*1*,*2*) **unfolding** *defined-lit-def consistent-interp-def* **by** *fastforce*
  moreover have *init-clss S* = *N*
    using *S* **by** *blast*
  moreover have *atm-of L* ∈ *atms-of-msu N* **using** *Cons.prems*(*3*) **by** *auto*
  moreover have *undef*: *undefined-lit* (*trail S*) *L*
    using *S* ‹*distinct* (*L#M*)› *calculation*(*1*) **by** (*auto simp*: *defined-lit-mapi defined-lit-map*)
  ultimately have $cdcl_W$ *S ?S*$_0$
    using $cdcl_W$*.other*[*OF* $cdcl_W$*-o.decide*[*OF decide-rule*[*OF S*,
      of L ?S$_0$]]] *S* **by** (*auto simp*: *state-eq-def simp del*: *state-simp*)
  then show *?case*
    using *st S undef* **by** (*auto intro*!: *exI*[*of - ?S*$_0$])
qed

lemma $cdcl_W$*-strong-completeness*:
  assumes
    set M ⊨*s set-mset N* **and**
    consistent-interp (*set M*) **and**
    distinct M **and**
    atm-of ‘ (*set M*) ⊆ *atms-of-msu N*
  obtains *S* **where**
    state S = (*mapi Marked* (*length M*) *M*, *N*, {#}, *length M*, *None*) **and**
    rtranclp $cdcl_W$ (*init-state N*) *S* **and**
    final-$cdcl_W$*-state S*
proof –
  obtain *S* **where**
    st: *rtranclp* $cdcl_W$ (*init-state N*) *S* **and**
    S: *state S* = (*mapi Marked* (*length M*) *M*, *N*, {#}, *length M*, *None*)
    using $cdcl_W$*-can-do-step*[*OF assms*(*2*–*4*)] **by** *auto*
  have *lits-of* (*mapi Marked* (*length M*) *M*) = *set M*
    by (*induct M*, *auto*)
  then have *mapi Marked* (*length M*) *M* ⊨*asm N* **using** *assms*(*1*) *true-annots-true-cls* **by** *metis*
  then have *final*-$cdcl_W$*-state S*
    using *S* **unfolding** *final*-$cdcl_W$*-state-def* **by** *auto*
  then show *?thesis* **using** *that st S* **by** *blast*
qed

17.6 Higher level strategy

The rules described previously do not lead to a conclusive state. We have to add a strategy.

17.6.1 Definition

lemma *tranclp-conflict-iff*[*iff*]:
  full1 conflict S S′ ⟷ *conflict S S*′
proof –
  have *tranclp conflict S S*′ ⟹ *conflict S S*′
    unfolding *full1-def* **by** (*induct rule*: *tranclp.induct*) *force*+
  then have *tranclp conflict S S*′ ⟹ *conflict S S*′ **by** (*meson rtranclpD*)
  then show *?thesis* **unfolding** *full1-def* **by** (*metis conflictE option.simps*(*3*)
    conflicting-update-conflicting state-eq-conflicting tranclp.intros(*1*))

qed

inductive *cdcl*$_W$*-cp* :: *'st* ⇒ *'st* ⇒ *bool* **where**
conflict'[*intro*]: *conflict S S'* ⟹ *cdcl*$_W$*-cp S S'* |
propagate': *propagate S S'* ⟹ *cdcl*$_W$*-cp S S'*

lemma *rtranclp-cdcl*$_W$*-cp-rtranclp-cdcl*$_W$:
  cdcl$_W$*-cp*** *S T* ⟹ *cdcl*$_W$** *S T*
  by (*induction rule*: *rtranclp-induct*) (*auto simp*: *cdcl*$_W$*-cp.simps dest*: *cdcl*$_W$*.intros*)

lemma *cdcl*$_W$*-cp-state-eq-compatible*:
  assumes
    cdcl$_W$*-cp S T* **and**
    S ∼ *S'* **and**
    T ∼ *T'*
  shows *cdcl*$_W$*-cp S' T'*
  using *assms*
  apply (*induction*)
    using *conflict-state-eq-compatible* **apply** *auto*[*1*]
  using *propagate' propagate-state-eq-compatible* **by** *auto*

lemma *tranclp-cdcl*$_W$*-cp-state-eq-compatible*:
  assumes
    cdcl$_W$*-cp*$^{++}$ *S T* **and**
    S ∼ *S'* **and**
    T ∼ *T'*
  shows *cdcl*$_W$*-cp*$^{++}$ *S' T'*
  using *assms*
proof *induction*
  case *base*
  then show *?case*
    using *cdcl*$_W$*-cp-state-eq-compatible* **by** *blast*
next
  case (*step U V*)
  obtain *ss* :: *'st* **where**
    cdcl$_W$*-cp S ss* ∧ *cdcl*$_W$*-cp*** *ss U*
    by (*metis* (*no-types*) *step*(*1*) *tranclpD*)
  then show *?case*
    by (*meson cdcl*$_W$*-cp-state-eq-compatible rtranclp.rtrancl-into-rtrancl rtranclp-into-tranclp2*
      state-eq-ref step(*2*) *step*(*4*) *step*(*5*))
qed

lemma *option-full-cdcl*$_W$*-cp*:
  conflicting S ≠ *None* ⟹ *full cdcl*$_W$*-cp S S*
unfolding *full-def rtranclp-unfold tranclp-unfold* **by** (*auto simp add*: *cdcl*$_W$*-cp.simps*)

lemma *skip-unique*:
  skip S T ⟹ *skip S T'* ⟹ *T* ∼ *T'*
  by (*fastforce simp*: *state-eq-def simp del*: *state-simp*)

lemma *resolve-unique*:
  resolve S T ⟹ *resolve S T'* ⟹ *T* ∼ *T'*
  by (*fastforce simp*: *state-eq-def simp del*: *state-simp*)

lemma *cdcl*$_W$*-cp-no-more-clauses*:

assumes *cdcl*$_W$*-cp S S'*
shows *clauses S* = *clauses S'*
using *assms* **by** (*induct rule*: *cdcl*$_W$*-cp.induct*) (*auto elim*!: *conflictE propagateE*)

lemma *tranclp-cdcl*$_W$*-cp-no-more-clauses*:
assumes *cdcl*$_W$*-cp*$^{++}$ *S S'*
shows *clauses S* = *clauses S'*
using *assms* **by** (*induct rule*: *tranclp.induct*) (*auto dest*: *cdcl*$_W$*-cp-no-more-clauses*)

lemma *rtranclp-cdcl*$_W$*-cp-no-more-clauses*:
assumes *cdcl*$_W$*-cp*** *S S'*
shows *clauses S* = *clauses S'*
using *assms* **by** (*induct rule*: *rtranclp-induct*) (*fastforce dest*: *cdcl*$_W$*-cp-no-more-clauses*)+

lemma *no-conflict-after-conflict*:
conflict S T $\Longrightarrow$ ¬*conflict T U*
by *fastforce*

lemma *no-propagate-after-conflict*:
conflict S T $\Longrightarrow$ ¬*propagate T U*
by *fastforce*

lemma *tranclp-cdcl*$_W$*-cp-propagate-with-conflict-or-not*:
assumes *cdcl*$_W$*-cp*$^{++}$ *S U*
shows (*propagate*$^{++}$ *S U* ∧ *conflicting U* = *None*)
∨ (∃ *T D*. *propagate*** *S T* ∧ *conflict T U* ∧ *conflicting U* = *Some D*)
proof −
have *propagate*$^{++}$ *S U* ∨ (∃ *T*. *propagate*** *S T* ∧ *conflict T U*)
using *assms* **by** *induction*
(*force simp*: *cdcl*$_W$*-cp.simps tranclp-into-rtranclp dest*: *no-conflict-after-conflict*
no-propagate-after-conflict)+
moreover
have *propagate*$^{++}$ *S U* $\Longrightarrow$ *conflicting U* = *None*
unfolding *tranclp-unfold-end* **by** *auto*
moreover
have $\bigwedge$*T*. *conflict T U* $\Longrightarrow$ ∃ *D*. *conflicting U* = *Some D*
by *auto*
ultimately show *?thesis* **by** *meson*
qed

lemma *cdcl*$_W$*-cp-conflicting-not-empty*[*simp*]: *conflicting S* = *Some D* $\Longrightarrow$ ¬*cdcl*$_W$*-cp S S'*
proof
assume *cdcl*$_W$*-cp S S'* **and** *conflicting S* = *Some D*
then show *False* **by** (*induct rule*: *cdcl*$_W$*-cp.induct*) *auto*
qed

lemma *no-step-cdcl*$_W$*-cp-no-conflict-no-propagate*:
assumes *no-step cdcl*$_W$*-cp S*
shows *no-step conflict S* **and** *no-step propagate S*
using *assms conflict'* **apply** *blast*
by (*meson assms conflict' propagate'*)

CDCL with the reasonable strategy: we fully propagate the conflict and propagate, then we apply any other possible rule *cdcl*$_W$*-o S S'* and re-apply conflict and propagate *full cdcl*$_W$*-cp S' S''*

inductive *cdcl*$_W$*-stgy* :: *'st* ⇒ *'st* ⇒ *bool* **for** *S* :: *'st* **where**
conflict': *full1 cdcl*$_W$*-cp S S'* ⟹ *cdcl*$_W$*-stgy S S'* |
other': *cdcl*$_W$*-o S S'* ⟹ *no-step cdcl*$_W$*-cp S* ⟹ *full cdcl*$_W$*-cp S' S''* ⟹ *cdcl*$_W$*-stgy S S''*

17.6.2 Invariants

These are the same invariants as before, but lifted

lemma *cdcl*$_W$*-cp-learned-clause-inv*:
assumes *cdcl*$_W$*-cp S S'*
shows *learned-clss S* = *learned-clss S'*
using *assms* **by** (*induct rule*: *cdcl*$_W$*-cp.induct*) *fastforce+*

lemma *rtranclp-cdcl*$_W$*-cp-learned-clause-inv*:
assumes *cdcl*$_W$*-cp*** *S S'*
shows *learned-clss S* = *learned-clss S'*
using *assms* **by** (*induct rule*: *rtranclp-induct*) (*fastforce dest*: *cdcl*$_W$*-cp-learned-clause-inv*)+

lemma *tranclp-cdcl*$_W$*-cp-learned-clause-inv*:
assumes *cdcl*$_W$*-cp*$^{++}$ *S S'*
shows *learned-clss S* = *learned-clss S'*
using *assms* **by** (*simp add*: *rtranclp-cdcl*$_W$*-cp-learned-clause-inv tranclp-into-rtranclp*)

lemma *cdcl*$_W$*-cp-backtrack-lvl*:
assumes *cdcl*$_W$*-cp S S'*
shows *backtrack-lvl S* = *backtrack-lvl S'*
using *assms* **by** (*induct rule*: *cdcl*$_W$*-cp.induct*) *fastforce+*

lemma *rtranclp-cdcl*$_W$*-cp-backtrack-lvl*:
assumes *cdcl*$_W$*-cp*** *S S'*
shows *backtrack-lvl S* = *backtrack-lvl S'*
using *assms* **by** (*induct rule*: *rtranclp-induct*) (*fastforce dest*: *cdcl*$_W$*-cp-backtrack-lvl*)+

lemma *cdcl*$_W$*-cp-consistent-inv*:
assumes *cdcl*$_W$*-cp S S'*
and *cdcl*$_W$*-M-level-inv S*
shows *cdcl*$_W$*-M-level-inv S'*
using *assms*
proof (*induct rule*: *cdcl*$_W$*-cp.induct*)
case (*conflict'*)
then show *?case* **using** *cdcl*$_W$*-consistent-inv cdcl*$_W$*.conflict* **by** *blast*
next
case (*propagate' S S'*)
have *cdcl*$_W$ *S S'*
using *propagate'.hyps*(*1*) *propagate* **by** *blast*
then show *cdcl*$_W$*-M-level-inv S'*
using *propagate'.prems*(*1*) *cdcl*$_W$*-consistent-inv propagate* **by** *blast*
qed

lemma *full1-cdcl*$_W$*-cp-consistent-inv*:
assumes *full1 cdcl*$_W$*-cp S S'*
and *cdcl*$_W$*-M-level-inv S*
shows *cdcl*$_W$*-M-level-inv S'*
using *assms* **unfolding** *full1-def*
proof −
have *cdcl*$_W$*-cp*$^{++}$ *S S'* **and** *cdcl*$_W$*-M-level-inv S* **using** *assms* **unfolding** *full1-def* **by** *auto*

then show *?thesis* **by** (*induct rule*: *tranclp.induct*) (*blast intro*: $cdcl_W$*-cp-consistent-inv*)+
qed

lemma *rtranclp-*$cdcl_W$*-cp-consistent-inv*:
assumes *rtranclp* $cdcl_W$*-cp* S S'
and $cdcl_W$*-M-level-inv* S
shows $cdcl_W$*-M-level-inv* S'
using *assms* **unfolding** *full1-def*
by (*induction rule*: *rtranclp-induct*) (*blast intro*: $cdcl_W$*-cp-consistent-inv*)+

lemma $cdcl_W$*-stgy-consistent-inv*:
assumes $cdcl_W$*-stgy* S S'
and $cdcl_W$*-M-level-inv* S
shows $cdcl_W$*-M-level-inv* S'
using *assms* **apply** (*induct rule*: $cdcl_W$*-stgy.induct*)
unfolding *full-unfold* **by** (*blast intro*: $cdcl_W$*-consistent-inv full1-*$cdcl_W$*-cp-consistent-inv*
$cdcl_W$*.other*)+

lemma *rtranclp-*$cdcl_W$*-stgy-consistent-inv*:
assumes $cdcl_W$*-stgy*** S S'
and $cdcl_W$*-M-level-inv* S
shows $cdcl_W$*-M-level-inv* S'
using *assms* **by** *induction* (*auto dest*!: $cdcl_W$*-stgy-consistent-inv*)

lemma $cdcl_W$*-cp-no-more-init-clss*:
assumes $cdcl_W$*-cp* S S'
shows *init-clss* S = *init-clss* S'
using *assms* **by** (*induct rule*: $cdcl_W$*-cp.induct*) *auto*

lemma *tranclp-*$cdcl_W$*-cp-no-more-init-clss*:
assumes $cdcl_W$*-cp*$^{++}$ S S'
shows *init-clss* S = *init-clss* S'
using *assms* **by** (*induct rule*: *tranclp.induct*) (*auto dest*: $cdcl_W$*-cp-no-more-init-clss*)

lemma $cdcl_W$*-stgy-no-more-init-clss*:
assumes $cdcl_W$*-stgy* S S' **and** $cdcl_W$*-M-level-inv* S
shows *init-clss* S = *init-clss* S'
using *assms*
apply (*induct rule*: $cdcl_W$*-stgy.induct*)
unfolding *full1-def full-def* **apply** (*blast dest*: *tranclp-*$cdcl_W$*-cp-no-more-init-clss*
tranclp-$cdcl_W$*-o-no-more-init-clss*)
by (*metis* $cdcl_W$*-o-no-more-init-clss rtranclp-unfold tranclp-*$cdcl_W$*-cp-no-more-init-clss*)

lemma *rtranclp-*$cdcl_W$*-stgy-no-more-init-clss*:
assumes $cdcl_W$*-stgy*** S S' **and** $cdcl_W$*-M-level-inv* S
shows *init-clss* S = *init-clss* S'
using *assms*
apply (*induct rule*: *rtranclp-induct*, *simp*)
using $cdcl_W$*-stgy-no-more-init-clss* **by** (*simp add*: *rtranclp-*$cdcl_W$*-stgy-consistent-inv*)

lemma $cdcl_W$*-cp-dropWhile-trail'*:
assumes $cdcl_W$*-cp* S S'
obtains M **where** *trail* S' = M @ *trail* S **and** ($\forall l \in$ *set* M. $\neg$*is-marked* l)
using *assms* **by** *induction fastforce*+

lemma *rtranclp-cdcl*$_W$*-cp-dropWhile-trail'*:
  assumes *cdcl*$_W$*-cp*** *S S'*
  obtains *M* :: (*'v*, *nat*, *'v clause*) *marked-lit list* **where**
    trail S' = *M* @ *trail S* **and** ∀ *l* ∈ *set M*. ¬*is-marked l*
  using *assms* **by** *induction* (*fastforce dest*!: *cdcl*$_W$*-cp-dropWhile-trail'*)+

lemma *cdcl*$_W$*-cp-dropWhile-trail*:
  assumes *cdcl*$_W$*-cp S S'*
  shows ∃ *M*. *trail S'* = *M* @ *trail S* ∧ (∀ *l* ∈ *set M*. ¬*is-marked l*)
  using *assms* **by** *induction fastforce*+

lemma *rtranclp-cdcl*$_W$*-cp-dropWhile-trail*:
  assumes *cdcl*$_W$*-cp*** *S S'*
  shows ∃ *M*. *trail S'* = *M* @ *trail S* ∧ (∀ *l* ∈ *set M*. ¬*is-marked l*)
  using *assms* **by** *induction* (*fastforce dest*: *cdcl*$_W$*-cp-dropWhile-trail*)+

This theorem can be seen a a termination theorem for *cdcl*$_W$*-cp*.

lemma *length-model-le-vars*:
  assumes
    no-strange-atm S **and**
    no-d: *no-dup* (*trail S*) **and**
    finite (*atms-of-msu* (*init-clss S*))
  shows *length* (*trail S*) ≤ *card* (*atms-of-msu* (*init-clss S*))
proof −
  obtain *M N U k D* **where** *S*: *state S* = (*M*, *N*, *U*, *k*, *D*) **by** (*cases state S*, *auto*)
  have *finite* (*atm-of* ‘ *lits-of* (*trail S*))
    using *assms*(*1*,*3*) **unfolding** *S* **by** (*auto simp add*: *finite-subset*)
  have *length* (*trail S*) = *card* (*atm-of* ‘ *lits-of* (*trail S*))
    using *no-dup-length-eq-card-atm-of-lits-of no-d* **by** *blast*
  then show *?thesis* **using** *assms*(*1*) **unfolding** *no-strange-atm-def*
  by (*auto simp add*: *assms*(*3*) *card-mono*)
qed

lemma *cdcl*$_W$*-cp-decreasing-measure*:
  assumes
    cdcl$_W$: *cdcl*$_W$*-cp S T* **and**
    M-lev: *cdcl*$_W$*-M-level-inv S* **and**
    alien: *no-strange-atm S*
  shows (λ*S*. *card* (*atms-of-msu* (*init-clss S*)) − *length* (*trail S*)
      + (*if conflicting S* = *None then 1 else 0*)) *S*
    > (λ*S*. *card* (*atms-of-msu* (*init-clss S*)) − *length* (*trail S*)
      + (*if conflicting S* = *None then 1 else 0*)) *T*
  using *assms*
proof −
  have *length* (*trail T*) ≤ *card* (*atms-of-msu* (*init-clss T*))
    apply (*rule length-model-le-vars*)
       using *cdcl*$_W$*-no-strange-atm-inv alien M-lev* **apply** (*meson cdcl*$_W$ *cdcl*$_W$*.simps cdcl*$_W$*-cp.cases*)
      using *M-lev cdcl*$_W$ *cdcl*$_W$*-cp-consistent-inv cdcl*$_W$*-M-level-inv-def* **apply** *blast*
      using *cdcl*$_W$ **by** (*auto simp*: *cdcl*$_W$*-cp.simps*)
  with *assms*
  show *?thesis* **by** *induction* (*auto split*: *split-if-asm*)+
qed

lemma *cdcl*$_W$*-cp-wf*: *wf* {(*b*,*a*). (*cdcl*$_W$*-M-level-inv a* ∧ *no-strange-atm a*)
  ∧ *cdcl*$_W$*-cp a b*}

apply (*rule wf-wf-if-measure′*[*of less-than - -*
(*λS. card* (*atms-of-msu* (*init-clss S*)) − *length* (*trail S*)
+ (*if conflicting S* = *None then 1 else 0*))])
apply *simp*
using $cdcl_W$*-cp-decreasing-measure* **unfolding** *less-than-iff* **by** *blast*

lemma *rtranclp-*$cdcl_W$*-all-struct-inv-*$cdcl_W$*-cp-iff-rtranclp-*$cdcl_W$*-cp*:
assumes
lev: $cdcl_W$*-M-level-inv S* **and**
alien: *no-strange-atm S*
shows (*λa b.* ($cdcl_W$*-M-level-inv a* ∧ *no-strange-atm a*) ∧ $cdcl_W$*-cp a b*)** *S T*
⟷ $cdcl_W$*-cp*** *S T*
(**is** *?I S T* ⟷ *?C S T*)
proof
assume
?I S T
then show *?C S T* **by** *induction auto*
next
assume
?C S T
then show *?I S T*
proof *induction*
case *base*
then show *?case* **by** *simp*
next
case (*step T U*) **note** *st* = *this*(*1*) **and** *cp* = *this*(*2*) **and** *IH* = *this*(*3*)
have $cdcl_W$** *S T*
by (*metis rtranclp-unfold* $cdcl_W$*-cp-conflicting-not-empty cp st*
rtranclp-propagate-is-rtranclp-$cdcl_W$ *tranclp-*$cdcl_W$*-cp-propagate-with-conflict-or-not*)
then have
$cdcl_W$*-M-level-inv T* **and**
no-strange-atm T
using ‹$cdcl_W$** *S T*› **apply** (*simp add*: *assms*(*1*) *rtranclp-*$cdcl_W$*-consistent-inv*)
using ‹$cdcl_W$** *S T*› *alien rtranclp-*$cdcl_W$*-no-strange-atm-inv lev* **by** *blast*
then have (*λa b.* ($cdcl_W$*-M-level-inv a* ∧ *no-strange-atm a*)
∧ $cdcl_W$*-cp a b*)** *T U*
using *cp* **by** *auto*
then show *?case* **using** *IH* **by** *auto*
qed
qed

lemma $cdcl_W$*-cp-normalized-element*:
assumes
lev: $cdcl_W$*-M-level-inv S* **and**
no-strange-atm S
obtains *T* **where** *full* $cdcl_W$*-cp S T*
proof −
let *?inv* = *λa.* ($cdcl_W$*-M-level-inv a* ∧ *no-strange-atm a*)
obtain *T* **where** *T*: *full* (*λa b. ?inv a* ∧ $cdcl_W$*-cp a b*) *S T*
using $cdcl_W$*-cp-wf wf-exists-normal-form*[*of λa b. ?inv a* ∧ $cdcl_W$*-cp a b*]
unfolding *full-def* **by** *blast*
then have $cdcl_W$*-cp*** *S T*
using *rtranclp-*$cdcl_W$*-all-struct-inv-*$cdcl_W$*-cp-iff-rtranclp-*$cdcl_W$*-cp assms* **unfolding** *full-def*
by *blast*
moreover

then have $cdcl_W^{**}$ *S T*
using *rtranclp-cdcl*$_W$*-cp-rtranclp-cdcl*$_W$ **by** *blast*
then have
cdcl$_W$*-M-level-inv T* **and**
no-strange-atm T
using ⟨$cdcl_W^{**}$ *S T*⟩ **apply** (*simp add*: *assms*(*1*) *rtranclp-cdcl*$_W$*-consistent-inv*)
using ⟨$cdcl_W^{**}$ *S T*⟩ *assms*(*2*) *rtranclp-cdcl*$_W$*-no-strange-atm-inv lev* **by** *blast*
then have *no-step cdcl*$_W$*-cp T*
using *T* **unfolding** *full-def* **by** *auto*
ultimately show *thesis* **using** *that* **unfolding** *full-def* **by** *blast*
qed

lemma *in-atms-of-implies-atm-of-on-atms-of-ms*:
$C + \{\#L\#\} \in\# A \Longrightarrow x \in atms\text{-}of\ C \Longrightarrow x \in atms\text{-}of\text{-}msu\ A$
by (*metis add.commute atm-iff-pos-or-neg-lit atms-of-atms-of-ms-mono contra-subsetD mem-set-mset-iff multi-member-skip*)

lemma *propagate-no-stange-atm*:
assumes
propagate S S′ **and**
no-strange-atm S
shows *no-strange-atm S′*
using *assms* **by** *induction*
(*auto simp add*: *no-strange-atm-def clauses-def in-plus-implies-atm-of-on-atms-of-ms in-atms-of-implies-atm-of-on-atms-of-ms*)

lemma *always-exists-full-cdcl*$_W$*-cp-step*:
assumes *no-strange-atm S*
shows ∃ *S″*. *full cdcl*$_W$*-cp S S″*
using *assms*
proof (*induct card* (*atms-of-msu* (*init-clss S*) − *atm-of* ‘*lits-of* (*trail S*)) *arbitrary*: *S*)
case *0* **note** *card* = *this*(*1*) **and** *alien* = *this*(*2*)
then have *atm*: *atms-of-msu* (*init-clss S*) = *atm-of* ‘ *lits-of* (*trail S*)
unfolding *no-strange-atm-def* **by** *auto*
{ assume *a*: ∃ *S′*. *conflict S S′*
then obtain *S′* **where** *S′*: *conflict S S′* **by** *metis*
then have ∀ *S″*. ¬*cdcl*$_W$*-cp S′ S″* **by** *auto*
then have *?case* **using** *a S′ cdcl*$_W$*-cp.conflict′* **unfolding** *full-def* **by** *blast*
}
moreover {
assume *a*: ∃ *S′*. *propagate S S′*
then obtain *S′* **where** *propagate S S′* **by** *blast*
then obtain *M N U k C L* **where** *S*: *state S* = (*M*, *N*, *U*, *k*, *None*)
and *S′*: *state S′* = (*Propagated L* ((*C* + {#*L*#})) # *M*, *N*, *U*, *k*, *None*)
and *C* + {#*L*#} ∈# *clauses S*
and *M* ⊨*as CNot C*
and *undefined-lit M L*
using *propagate* **by** *auto*
have *atms-of-msu U* ⊆ *atms-of-msu N* **using** *alien S* **unfolding** *no-strange-atm-def* **by** *auto*
then have *atm-of L* ∈ *atms-of-msu* (*init-clss S*)
using ⟨*C* + {#*L*#} ∈# *clauses S*⟩ *S* **unfolding** *atms-of-ms-def clauses-def* **by** *force+*
then have *False* **using** ⟨*undefined-lit M L*⟩ *S* **unfolding** *atm* **unfolding** *lits-of-def*
by (*auto simp add*: *defined-lit-map*)
}
ultimately show *?case* **by** (*metis cdcl*$_W$*-cp.cases full-def rtranclp.rtrancl-refl*)

```
next
  case (Suc n) note IH = this(1) and card = this(2) and alien = this(3)
  { assume a: ∃S'. conflict S S'
    then obtain S' where S': conflict S S' by metis
    then have ∀S''. ¬cdcl_W-cp S' S'' by auto
    then have ?case  unfolding full-def Ex-def using S' cdcl_W-cp.conflict' by blast
  }
  moreover {
    assume a: ∃S'. propagate S S'
    then obtain S' where propagate: propagate S S' by blast
    then obtain M N U k C L where
      S: state S = (M, N, U, k, None) and
      S': state S' = (Propagated L ( (C + {#L#})) # M, N, U, k, None) and
      C + {#L#} ∈# clauses S and
      M ⊨as CNot C and
      undefined-lit M L
      by fastforce
    then have atm-of L ∉ atm-of ` lits-of M
      unfolding lits-of-def by (auto simp add: defined-lit-map)
    moreover
      have no-strange-atm S' using alien propagate propagate-no-stange-atm by blast
      then have atm-of L ∈ atms-of-msu N using S' unfolding no-strange-atm-def by auto
      then have ⋀A. {atm-of L} ⊆ atms-of-msu N − A ∨ atm-of L ∈ A by force
    moreover have Suc n − card {atm-of L} = n by simp
    moreover have card (atms-of-msu N − atm-of ` lits-of M) = Suc n
     using card S S' by simp
    ultimately
      have card (atms-of-msu N − atm-of ` insert L (lits-of M)) = n
        by (metis (no-types) Diff-insert card-Diff-subset finite.emptyI finite.insertI image-insert)
      then have n = card (atms-of-msu (init-clss S') − atm-of ` lits-of (trail S'))
        using card S S' by simp
    then have a1: Ex (full cdcl_W-cp S') using IH ‹no-strange-atm S'›  by blast
    have ?case
      proof −
        obtain S'' :: 'st where
          ff1: cdcl_W-cp** S' S'' ∧ no-step cdcl_W-cp S''
          using a1 unfolding full-def by blast
        have cdcl_W-cp** S S''
          using ff1 cdcl_W-cp.intros(2)[OF propagate]
          by (metis (no-types) converse-rtranclp-into-rtranclp)
        then have ∃S''. cdcl_W-cp** S S'' ∧ (∀S'''. ¬ cdcl_W-cp S'' S''')
          using ff1 by blast
        then show ?thesis unfolding full-def
          by meson
      qed
    }
  ultimately show ?case unfolding full-def by (metis cdcl_W-cp.cases rtranclp.rtrancl-refl)
qed
```

17.6.3 Literal of highest level in conflicting clauses

One important property of the $cdcl_W$ with strategy is that, whenever a conflict takes place, there is at least a literal of level k involved (except if we have derived the false clause). The reason is that we apply conflicts before a decision is taken.

abbreviation *no-clause-is-false* :: *'st* ⇒ *bool* **where**

no-clause-is-false $\equiv$
  $\lambda S.$ (*conflicting* S = *None* $\longrightarrow$ ($\forall D \in\#$ *clauses* S. $\neg$*trail* S $\models$*as* *CNot* D))

abbreviation *conflict-is-false-with-level* :: *'st* $\Rightarrow$ *bool* **where**
conflict-is-false-with-level $S \equiv \forall D$. *conflicting* S = *Some* $D \longrightarrow D \neq \{\#\}$
  $\longrightarrow$ ($\exists L \in\# D$. *get-level* (*trail* S) L = *backtrack-lvl* S)

lemma *not-conflict-not-any-negated-init-clss*:
  assumes $\forall S'$. $\neg$*conflict* S S'
  shows *no-clause-is-false* S
  using *assms* *state-eq-ref* **by** *blast*

lemma *full-cdcl*$_W$*-cp-not-any-negated-init-clss*:
  assumes *full* *cdcl*$_W$*-cp* S S'
  shows *no-clause-is-false* S'
  using *assms* *not-conflict-not-any-negated-init-clss* **unfolding** *full-def* **by** *blast*

lemma *full1-cdcl*$_W$*-cp-not-any-negated-init-clss*:
  assumes *full1* *cdcl*$_W$*-cp* S S'
  shows *no-clause-is-false* S'
  using *assms* *not-conflict-not-any-negated-init-clss* **unfolding** *full1-def* **by** *blast*

lemma *cdcl*$_W$*-stgy-not-non-negated-init-clss*:
  assumes *cdcl*$_W$*-stgy* S S'
  shows *no-clause-is-false* S'
  using *assms* **apply** (*induct rule*: *cdcl*$_W$*-stgy.induct*)
  using *full1-cdcl*$_W$*-cp-not-any-negated-init-clss* *full-cdcl*$_W$*-cp-not-any-negated-init-clss* **by** *metis+*

lemma *rtranclp-cdcl*$_W$*-stgy-not-non-negated-init-clss*:
  assumes *cdcl*$_W$*-stgy*** S S' **and** *no-clause-is-false* S
  shows *no-clause-is-false* S'
  using *assms* **by** (*induct rule*: *rtranclp-induct*) (*auto simp*: *cdcl*$_W$*-stgy-not-non-negated-init-clss*)

lemma *cdcl*$_W$*-stgy-conflict-ex-lit-of-max-level*:
  assumes *cdcl*$_W$*-cp* S S'
  and *no-clause-is-false* S
  and *cdcl*$_W$*-M-level-inv* S
  shows *conflict-is-false-with-level* S'
  using *assms*
proof (*induct rule*: *cdcl*$_W$*-cp.induct*)
  case *conflict'*
  then show *?case* **by** *auto*
next
  case *propagate'*
  then show *?case* **by** *auto*
qed

lemma *no-chained-conflict*:
  assumes *conflict* S S'
  and *conflict* S' S''
  shows *False*
  using *assms* **by** *fastforce*

lemma *rtranclp-cdcl*$_W$*-cp-propa-or-propa-confl*:
  assumes *cdcl*$_W$*-cp*** S U

shows *propagate*** *S U* ∨ (∃ *T*. *propagate*** *S T* ∧ *conflict T U*)
using *assms*
proof *induction*
case *base*
then show *?case* **by** *auto*
next
case (*step U V*) **note** *SU* = *this*(*1*) **and** *UV* = *this*(*2*) **and** *IH* = *this*(*3*)
consider (*confl*) *T* **where** *propagate*** *S T* **and** *conflict T U*
| (*propa*) *propagate*** *S U* **using** *IH* **by** *auto*
then show *?case*
proof *cases*
case *confl*
then have *False* **using** *UV* **by** *auto*
then show *?thesis* **by** *fast*
next
case *propa*
also have *conflict U V* ∨ *propagate U V* **using** *UV* **by** (*auto simp add*: *cdcl*$_W$*-cp.simps*)
ultimately show *?thesis* **by** *force*
qed
qed

lemma *rtranclp-cdcl*$_W$*-co-conflict-ex-lit-of-max-level*:
assumes *full*: *full cdcl*$_W$*-cp S U*
and *cls-f*: *no-clause-is-false S*
and *conflict-is-false-with-level S*
and *lev*: *cdcl*$_W$*-M-level-inv S*
shows *conflict-is-false-with-level U*
proof (*intro allI impI*)
fix *D*
assume *confl*: *conflicting U* = *Some D* **and**
D: *D* ≠ {#}
consider (*CT*) *conflicting S* = *None* | (*SD*) *D*′ **where** *conflicting S* = *Some D*′
by (*cases conflicting S*) *auto*
then show ∃ *L*∈#*D*. *get-level* (*trail U*) *L* = *backtrack-lvl U*
proof *cases*
case *SD*
then have *S* = *U*
by (*metis* (*no-types*) *assms*(*1*) *cdcl*$_W$*-cp-conflicting-not-empty full-def rtranclpD tranclpD*)
then show *?thesis* **using** *assms*(*3*) *confl D* **by** *blast*−
next
case *CT*
have *init-clss U* = *init-clss S* **and** *learned-clss U* = *learned-clss S*
using *assms*(*1*) **unfolding** *full-def*
apply (*metis* (*no-types*) *rtranclpD tranclp-cdcl*$_W$*-cp-no-more-init-clss*)
by (*metis* (*mono-tags*, *lifting*) *assms*(*1*) *full-def rtranclp-cdcl*$_W$*-cp-learned-clause-inv*)
obtain *T* **where** *propagate*** *S T* **and** *TU*: *conflict T U*
proof −
have *f5*: *U* ≠ *S*
using *confl CT* **by** *force*
then have *cdcl*$_W$*-cp*$^{++}$ *S U*
by (*metis full full-def rtranclpD*)
have ⋀*p pa*. ¬ *propagate p pa* ∨ *conflicting pa* =
(*None*::′*v literal multiset option*)
by *auto*
then show *?thesis*

using *f5 that tranclp-cdcl*$_W$*-cp-propagate-with-conflict-or-not*[*OF* ‹*cdcl*$_W$*-cp*$^{++}$ *S U*›]
full confl CT **unfolding** *full-def* **by** *auto*
qed
have *init-clss T* = *init-clss S* **and** *learned-clss T* = *learned-clss S*
using *TU* ‹*init-clss U* = *init-clss S*› ‹*learned-clss U* = *learned-clss S*› **by** *auto*
then have *D* ∈# *clauses S*
using *TU confl* **by** (*fastforce simp*: *clauses-def*)
then have ¬ *trail S* ⊨*as CNot D*
using *cls-f CT* **by** *simp*
moreover
obtain *M* **where** *tr-U*: *trail U* = *M* @ *trail S* **and** *nm*: ∀ *m*∈*set M*. ¬*is-marked m*
by (*metis* (*mono-tags*, *lifting*) *assms*(*1*) *full-def rtranclp-cdcl*$_W$*-cp-dropWhile-trail*)
have *trail U* ⊨*as CNot D*
using *TU confl* **by** *auto*
ultimately obtain *L* **where** *L* ∈# *D* **and** −*L* ∈ *lits-of M*
unfolding *tr-U CNot-def true-annots-def Ball-def true-annot-def true-cls-def* **by** *auto*

moreover have *inv-U*: *cdcl*$_W$*-M-level-inv U*
by (*metis cdcl*$_W$*-stgy.conflict′ cdcl*$_W$*-stgy-consistent-inv full full-unfold lev*)
moreover
have *backtrack-lvl U* = *backtrack-lvl S*
using *full* **unfolding** *full-def* **by** (*auto dest*: *rtranclp-cdcl*$_W$*-cp-backtrack-lvl*)

moreover
have *no-dup* (*trail U*)
using *inv-U* **unfolding** *cdcl*$_W$*-M-level-inv-def* **by** *auto*
{ fix *x* :: (′*v*, *nat*, ′*v literal multiset*) *marked-lit* **and**
xb :: (′*v*, *nat*, ′*v literal multiset*) *marked-lit*
assume *a1*: *atm-of L* = *atm-of* (*lit-of xb*)
moreover assume *a2*: − *L* = *lit-of x*
moreover assume *a3*: (λ*l*. *atm-of* (*lit-of l*)) ‘ *set M*
∩ (λ*l*. *atm-of* (*lit-of l*)) ‘ *set* (*trail S*) = {}
moreover assume *a4*: *x* ∈ *set M*
moreover assume *a5*: *xb* ∈ *set* (*trail S*)
moreover have *atm-of* (− *L*) = *atm-of L*
by *auto*
ultimately have *False*
by *auto*
}
then have *LS*: *atm-of L* ∉ *atm-of* ‘ *lits-of* (*trail S*)
using ‹−*L* ∈ *lits-of M*› ‹*no-dup* (*trail U*)› **unfolding** *tr-U lits-of-def* **by** *auto*
ultimately have *get-level* (*trail U*) *L* = *backtrack-lvl U*
proof (*cases get-all-levels-of-marked* (*trail S*) ≠ [], *goal-cases*)
case *2* **note** *LD* = *this*(*1*) **and** *LM* = *this*(*2*) **and** *inv-U* = *this*(*3*) **and** *US* = *this*(*4*) **and**
LS = *this*(*5*) **and** *ne* = *this*(*6*)
have *backtrack-lvl S* = *0*
using *lev ne* **unfolding** *cdcl*$_W$*-M-level-inv-def* **by** *auto*
moreover have *get-rev-level* (*rev M*) *0 L* = *0*
using *nm* **by** *auto*
ultimately show *?thesis* **using** *LS ne US* **unfolding** *tr-U*
by (*simp add*: *get-all-levels-of-marked-nil-iff-not-is-marked lits-of-def*)
next
case *1* **note** *LD* = *this*(*1*) **and** *LM* = *this*(*2*) **and** *inv-U* = *this*(*3*) **and** *US* = *this*(*4*) **and**
LS = *this*(*5*) **and** *ne* = *this*(*6*)

have *hd* (*get-all-levels-of-marked* (*trail S*)) = *backtrack-lvl S*
using *ne lev* **unfolding** $cdcl_W$*-M-level-inv-def*
by (*cases get-all-levels-of-marked* (*trail S*)) *auto*
moreover have *atm-of L* ∈ *atm-of* ‘ *lits-of M*
using ⟨−*L* ∈ *lits-of M*⟩ **by** (*simp add*: *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set lits-of-def*)
ultimately show *?thesis*
using *nm ne* **unfolding** *tr-U*
using *get-level-skip-beginning-hd-get-all-levels-of-marked*[*OF LS*, *of M*] *get-level-skip-in-all-not-marked*[*of rev M L backtrack-lvl S*]
unfolding *lits-of-def US*
by *auto*
qed
then show ∃ *L*∈#*D*. *get-level* (*trail U*) *L* = *backtrack-lvl U*
using ⟨*L* ∈# *D*⟩ **by** *blast*
qed
qed

17.6.4 Literal of highest level in marked literals

definition *mark-is-false-with-level* :: ′*st* ⇒ *bool* **where**
mark-is-false-with-level S′ ≡
∀ *D M1 M2 L*. *M1* @ *Propagated L D* # *M2* = *trail S′* ⟶ *D* − {#*L*#} ≠ {#}
⟶ (∃ *L*. *L* ∈# *D* ∧ *get-level* (*trail S′*) *L* = *get-maximum-possible-level M1*)

definition *no-more-propagation-to-do*:: ′*st* ⇒ *bool* **where**
no-more-propagation-to-do S ≡
∀ *D M M′ L*. *D* + {#*L*#} ∈# *clauses S* ⟶ *trail S* = *M′* @ *M* ⟶ *M* ⊨*as CNot D*
⟶ *undefined-lit M L* ⟶ *get-maximum-possible-level M* < *backtrack-lvl S*
⟶ (∃ *L*. *L* ∈# *D* ∧ *get-level* (*trail S*) *L* = *get-maximum-possible-level M*)

lemma *propagate-no-more-propagation-to-do*:
assumes *propagate*: *propagate S S′*
and *H*: *no-more-propagation-to-do S*
and *M*: $cdcl_W$*-M-level-inv S*
shows *no-more-propagation-to-do S′*
using *assms*
proof −
obtain *M N U k C L* **where**
S: *state S* = (*M*, *N*, *U*, *k*, *None*) **and**
S′: *state S′* = (*Propagated L* ((*C* + {#*L*#})) # *M*, *N*, *U*, *k*, *None*) **and**
C + {#*L*#} ∈# *clauses S* **and**
M ⊨*as CNot C* **and**
undefined-lit M L
using *propagate* **by** *auto*
let *?M′* = *Propagated L* ((*C* + {#*L*#})) # *M*
show *?thesis* **unfolding** *no-more-propagation-to-do-def*
proof (*intro allI impI*)
fix *D M1 M2 L′*
assume *D-L*: *D* + {#*L′*#} ∈# *clauses S′*
and *trail S′* = *M2* @ *M1*
and *get-max*: *get-maximum-possible-level M1* < *backtrack-lvl S′*
and *M1* ⊨*as CNot D*
and *undef*: *undefined-lit M1 L′*
have *tl M2* @ *M1* = *trail S* ∨ (*M2* = [] ∧ *M1* = *Propagated L* ((*C* + {#*L*#})) # *M*)
using ⟨*trail S′* = *M2* @ *M1*⟩ *S′ S* **by** (*cases M2*) *auto*

moreover {
assume *tl M2* @ *M1* = *trail S*
moreover have $D + \{\#L'\#\} \in\#$ *clauses S* **using** *D-L S S'* **unfolding** *clauses-def* **by** *auto*
moreover have *get-maximum-possible-level M1* < *backtrack-lvl S*
using *get-max S S'* **by** *auto*
ultimately obtain L' **where** $L' \in\# D$ **and**
get-level (*trail S*) L' = *get-maximum-possible-level M1*
using *H* ⟨*M1* ⊨*as CNot D*⟩ *undef* **unfolding** *no-more-propagation-to-do-def* **by** *metis*
moreover
{ **have** $cdcl_W$*-M-level-inv S'*
using $cdcl_W$*-consistent-inv*[*OF - M*] $cdcl_W$*.propagate*[*OF propagate*] **by** *blast*
then have *no-dup ?M'* **using** *S'* **unfolding** $cdcl_W$*-M-level-inv-def* **by** *auto*
moreover
have *atm-of* $L' \in$ *atm-of* ' (*lits-of M1*)
using ⟨$L' \in\# D$⟩ ⟨*M1* ⊨*as CNot D*⟩ **by** (*metis atm-of-uminus image-eqI*
in-CNot-implies-uminus(*2*))
then have *atm-of* $L' \in$ *atm-of* ' (*lits-of M*)
using ⟨*tl M2* @ *M1* = *trail S*⟩ *S* **by** *auto*
ultimately have *atm-of L* ≠ *atm-of* L' **unfolding** *lits-of-def* **by** *auto*
}
ultimately have $\exists L' \in\# D$. *get-level* (*trail S'*) L' = *get-maximum-possible-level M1*
using *S S'* **by** *auto*
}
moreover {
assume *M2* = [] **and** *M1*: *M1* = *Propagated L* ((*C* + {#*L*#})) # *M*
have $cdcl_W$*-M-level-inv S'*
using $cdcl_W$*-consistent-inv*[*OF - M*] $cdcl_W$*.propagate*[*OF propagate*] **by** *blast*
then have *get-all-levels-of-marked* (*trail S'*) = *rev* ([*Suc 0*..<(*Suc 0*+*k*)])
using *S'* **unfolding** $cdcl_W$*-M-level-inv-def* **by** *auto*
then have *get-maximum-possible-level M1* = *backtrack-lvl S'*
using *get-maximum-possible-level-max-get-all-levels-of-marked*[*of M1*] *S' M1*
by (*auto intro*: *Max-eqI*)
then have *False* **using** *get-max* **by** *auto*
}
ultimately show $\exists L.\ L \in\# D \wedge$ *get-level* (*trail S'*) *L* = *get-maximum-possible-level M1* **by** *fast*
qed
qed

lemma *conflict-no-more-propagation-to-do*:
assumes *conflict*: *conflict S S'*
and *H*: *no-more-propagation-to-do S*
and *M*: $cdcl_W$*-M-level-inv S*
shows *no-more-propagation-to-do S'*
using *assms* **unfolding** *no-more-propagation-to-do-def conflict.simps* **by** *force*

lemma $cdcl_W$*-cp-no-more-propagation-to-do*:
assumes *conflict*: $cdcl_W$*-cp S S'*
and *H*: *no-more-propagation-to-do S*
and *M*: $cdcl_W$*-M-level-inv S*
shows *no-more-propagation-to-do S'*
using *assms*
proof (*induct rule*: $cdcl_W$*-cp.induct*)
case (*conflict' S S'*)
then show *?case* **using** *conflict-no-more-propagation-to-do*[*of S S'*] **by** *blast*
next

case (*propagate' S S'*) **note** $S = this$
show *1*: *no-more-propagation-to-do S'*
using *propagate-no-more-propagation-to-do*[*of S S'*] *S* **by** *blast*
qed

lemma $cdcl_W$*-then-exists-*$cdcl_W$*-stgy-step*:
assumes
o: $cdcl_W$*-o S S'* **and**
alien: *no-strange-atm S* **and**
lev: $cdcl_W$*-M-level-inv S*
shows $\exists S'.$ $cdcl_W$*-stgy S S'*
proof −
obtain *S''* **where** *full* $cdcl_W$*-cp S' S''*
using *always-exists-full-*$cdcl_W$*-cp-step alien* $cdcl_W$*-no-strange-atm-inv* $cdcl_W$*-o-no-more-init-clss o other lev* **by** (*meson* $cdcl_W$*-consistent-inv*)
then show *?thesis*
using *assms* **by** (*metis always-exists-full-*$cdcl_W$*-cp-step* $cdcl_W$*-stgy.conflict' full-unfold other'*)
qed

lemma *backtrack-no-decomp*:
assumes *S*: *state* $S = (M, N, U, k, Some\ (D + \{\#L\#\}))$
and *L*: *get-level M L* $= k$
and *D*: *get-maximum-level M D* $< k$
and *M-L*: $cdcl_W$*-M-level-inv S*
shows $\exists S'.$ $cdcl_W$*-o S S'*
proof −
have *L-D*: *get-level M L* = *get-maximum-level M* $(D + \{\#L\#\})$
using *L D* **by** (*simp add*: *get-maximum-level-plus*)
let *?i* = *get-maximum-level M D*
obtain *K M1 M2* **where** *K*: (*Marked K* (*?i* + *1*) # *M1*, *M2*) $\in$ *set* (*get-all-marked-decomposition M*)
using *backtrack-ex-decomp*[*OF M-L*, *of ?i*] *D S* **by** *auto*
show *?thesis* **using** *backtrack-rule*[*OF S K L L-D*] **by** (*meson bj* $cdcl_W$*-bj.simps state-eq-ref*)
qed

lemma $cdcl_W$*-stgy-final-state-conclusive*:
assumes *termi*: $\forall S'. \neg$ $cdcl_W$*-stgy S S'*
and *decomp*: *all-decomposition-implies-m* (*init-clss S*) (*get-all-marked-decomposition* (*trail S*))
and *learned*: $cdcl_W$*-learned-clause S*
and *level-inv*: $cdcl_W$*-M-level-inv S*
and *alien*: *no-strange-atm S*
and *no-dup*: *distinct-*$cdcl_W$*-state S*
and *confl*: $cdcl_W$*-conflicting S*
and *confl-k*: *conflict-is-false-with-level S*
shows (*conflicting S* = *Some* {#} $\wedge$ *unsatisfiable* (*set-mset* (*init-clss S*)))
$\vee$ (*conflicting S* = *None* $\wedge$ *trail S* $\models$*as set-mset* (*init-clss S*))
proof −
let *?M* = *trail S*
let *?N* = *init-clss S*
let *?k* = *backtrack-lvl S*
let *?U* = *learned-clss S*
have *conflicting S* = *Some* {#}
$\vee$ *conflicting S* = *None*
$\vee$ ($\exists$ *D L*. *conflicting S* = *Some* $(D + \{\#L\#\})$)
apply (*cases conflicting S*, *auto*)

```
  apply (rename-tac C)
  by (case-tac C, auto)
moreover {
  assume conflicting S = Some {#}
  then have unsatisfiable (set-mset (init-clss S))
    using assms(3) unfolding cdcl_W-learned-clause-def true-clss-cls-def
    by (metis (no-types, lifting) Un-insert-right atms-of-empty satisfiable-def
      sup-bot.right-neutral total-over-m-insert total-over-set-empty true-cls-empty)
}
moreover {
  assume conflicting S = None
  { assume ¬?M ⊨asm ?N
    have atm-of ‘ (lits-of ?M) = atms-of-msu ?N (is ?A = ?B)
      proof
        show ?A ⊆ ?B using alien unfolding no-strange-atm-def by auto
        show ?B ⊆ ?A
          proof (rule ccontr)
            assume ¬?B ⊆ ?A
            then obtain l where l ∈ ?B and l ∉ ?A by auto
            then have undefined-lit ?M (Pos l)
              using ‹l ∉ ?A› unfolding lits-of-def by (auto simp add: defined-lit-map)
            then have ∃S'. cdcl_W-o S S'
              using cdcl_W-o.decide decide.intros ‹l ∈ ?B› no-strange-atm-def
              by (metis ‹conflicting S = None› literal.sel(1) state-eq-def)
            then show False
              using termi cdcl_W-then-exists-cdcl_W-stgy-step[OF - alien] level-inv by blast
          qed
      qed
    obtain D where ¬ ?M ⊨a D and D ∈# ?N
       using ‹¬?M ⊨asm ?N› unfolding lits-of-def true-annots-def Ball-def by auto
    have atms-of D ⊆ atm-of ‘ (lits-of ?M)
      using ‹D ∈# ?N› unfolding ‹atm-of ‘ (lits-of ?M) = atms-of-msu ?N› atms-of-ms-def
      by (auto simp add: atms-of-def)
    then have a1: atm-of ‘ set-mset D ⊆ atm-of ‘ lits-of (trail S)
      by (auto simp add: atms-of-def lits-of-def)
    have total-over-m (lits-of ?M) {D}
      using ‹atms-of D ⊆ atm-of ‘ (lits-of ?M)› atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
      by (fastforce simp: total-over-set-def)
    then have ?M ⊨as CNot D
      using total-not-true-cls-true-clss-CNot ‹¬ trail S ⊨a D› true-annot-def
      true-annots-true-cls by fastforce
    then have False
      proof -
        obtain S' where
          f2: full cdcl_W-cp S S'
          by (meson alien always-exists-full-cdcl_W-cp-step level-inv)
        then have S' = S
          using cdcl_W-stgy.conflict'[of S] by (metis (no-types) full-unfold termi)
        then show ?thesis
          using f2 ‹D ∈# init-clss S› ‹conflicting S = None› ‹trail S ⊨as CNot D›
          clauses-def full-cdcl_W-cp-not-any-negated-init-clss by auto
      qed
  }
  then have ?M ⊨asm ?N by blast
}
```

moreover {
  assume $\exists$ *D L. conflicting S* = *Some* (*D* + {#*L*#})
  then obtain *D L* **where** *LD*: *conflicting S* = *Some* (*D* + {#*L*#}) **and** *lev-L*: *get-level ?M L* = *?k*
    by (*metis* (*mono-tags*) *bex-msetE confl-k insert-DiffM2 multi-self-add-other-not-self*
      union-eq-empty)
  let *?D* = *D* + {#*L*#}
  have *?D* ≠ {#} **by** *auto*
  have *?M* ⊨*as CNot ?D* **using** *confl LD* **unfolding** $cdcl_W$*-conflicting-def* **by** *auto*
  then have *?M* ≠ [] **unfolding** *true-annots-def Ball-def true-annot-def true-cls-def* **by** *force*
  { **have** *M*: *?M* = *hd ?M* # *tl ?M* **using** ‹*?M* ≠ []› *list.collapse* **by** *fastforce*
    assume *marked*: *is-marked* (*hd ?M*)
    then obtain *k'* **where** *k'*: *k'* + *1* = *?k*
      using *level-inv M* **unfolding** $cdcl_W$*-M-level-inv-def*
      by (*cases hd* (*trail S*); *cases trail S*) *auto*
    obtain *L' l'* **where** *L'*: *hd ?M* = *Marked L' l'* **using** *marked* **by** (*case-tac hd ?M*) *auto*
    have *marked-hd-tl*: *get-all-levels-of-marked* (*hd* (*trail S*) # *tl* (*trail S*))
      = *rev* [*1*..<*1* + *length* (*get-all-levels-of-marked ?M*)]
      using *level-inv lev-L M* **unfolding** $cdcl_W$*-M-level-inv-def M*[*symmetric*]
      by *blast*
    then have *l'-tl*: *l'* # *get-all-levels-of-marked* (*tl ?M*)
      = *rev* [*1*..<*1* + *length* (*get-all-levels-of-marked ?M*)] **unfolding** *L'* **by** *simp*
    moreover have ... = *length* (*get-all-levels-of-marked ?M*)
      # *rev* [*1*..<*length* (*get-all-levels-of-marked ?M*)]
      using *M Suc-le-mono calculation* **by** (*fastforce simp add*: *upt.simps*(*2*))
    finally have
      l' = *?k* **and**
      g-r: *get-all-levels-of-marked* (*tl* (*trail S*))
        = *rev* [*1*..<*length* (*get-all-levels-of-marked* (*trail S*))]
      using *level-inv lev-L M* **unfolding** $cdcl_W$*-M-level-inv-def* **by** *auto*
    have ∗: ⋀*list. no-dup list* ⟹
        − *L* ∈ *lits-of list* ⟹ *atm-of L* ∈ *atm-of* ‘ *lits-of list*
      by (*metis atm-of-uminus imageI*)
    have *L'* = −*L*
      proof (*rule ccontr*)
        assume ¬ *?thesis*
        moreover have −*L* ∈ *lits-of ?M* **using** *confl LD* **unfolding** $cdcl_W$*-conflicting-def* **by** *auto*
        ultimately have *get-level* (*hd* (*trail S*) # *tl* (*trail S*)) *L* = *get-level* (*tl ?M*) *L*
          using $cdcl_W$*-M-level-inv-decomp*(*1*)[*OF level-inv*] **unfolding** *L' consistent-interp-def*
          by (*metis* (*no-types, lifting*) *L' M atm-of-eq-atm-of get-level-skip-beginning insert-iff*
            lits-of-cons marked-lit.sel(*1*))

        moreover
          have *length* (*get-all-levels-of-marked* (*trail S*)) = *?k*
            using *level-inv* **unfolding** $cdcl_W$*-M-level-inv-def* **by** *auto*
          then have *Max* (*set* (*0*#*get-all-levels-of-marked* (*tl* (*trail S*)))) = *?k* − *1*
            unfolding *g-r* **by** (*auto simp add*: *Max-n-upt*)
          then have *get-level* (*tl ?M*) *L* < *?k*
            using *get-maximum-possible-level-ge-get-level*[*of tl ?M L*]
            by (*metis One-nat-def add.right-neutral add-Suc-right diff-add-inverse2*
              get-maximum-possible-level-max-get-all-levels-of-marked k' le-imp-less-Suc
              list.simps(*15*))
        finally show *False* **using** *lev-L M* **by** *auto*
      qed
    have *L*: *hd ?M* = *Marked* (−*L*) *?k* **using** ‹*l'* = *?k*› ‹*L'* = −*L*› *L'* **by** *auto*

have *g-a-l*: *get-all-levels-of-marked ?M* = *rev* [*1..<1* + *?k*]
  using *level-inv lev-L M* **unfolding** *cdcl*$_W$*-M-level-inv-def* **by** *auto*
have *g-k*: *get-maximum-level* (*trail S*) *D* ≤ *?k*
  using *get-maximum-possible-level-ge-get-maximum-level*[*of ?M*]
    get-maximum-possible-level-max-get-all-levels-of-marked[*of ?M*]
  by (*auto simp add*: *Max-n-upt g-a-l*)
have *get-maximum-level* (*trail S*) *D* < *?k*
  proof (*rule ccontr*)
    assume ¬ *?thesis*
    then have *get-maximum-level* (*trail S*) *D* = *?k* **using** *M g-k* **unfolding** *L* **by** *auto*
    then obtain *L'* **where** *L'* ∈# *D* **and** *L-k*: *get-level ?M L'* = *?k*
      using *get-maximum-level-exists-lit*[*of ?k ?M D*] **unfolding** *k'*[*symmetric*] **by** *auto*
    have *L* ≠ *L'* **using** *no-dup* ‹*L'* ∈# *D*›
      unfolding *distinct-cdcl*$_W$*-state-def LD* **by** (*metis add.commute add-eq-self-zero*
        count-single count-union less-not-refl3 distinct-mset-def union-single-eq-member)
    have *L'* = −*L*
      proof (*rule ccontr*)
        assume ¬ *?thesis*
        then have *get-level ?M L'* = *get-level* (*tl ?M*) *L'*
          using *M* ‹*L* ≠ *L'*› *get-level-skip-beginning*[*of L' hd ?M tl ?M*] **unfolding** *L*
          by (*auto simp*: *atm-of-eq-atm-of*)
        moreover have ... < *?k*
          proof −
            { **assume** *a1*: *get-level* (*tl* (*trail S*)) *L'* = *backtrack-lvl S*
              assume *a2*: *rev* (*get-all-levels-of-marked* (*tl* (*trail S*))) =
                [*Suc 0..<backtrack-lvl S*]
              have *k'* + *Suc 0* = *backtrack-lvl S*
                using *k'* **by** *presburger*
              then have *False*
                using *a2 a1* **by** (*metis* (*no-types*) *Max-n-upt Zero-neq-Suc add-diff-cancel-left'*
                  add-diff-cancel-right' diff-is-0-eq
                  get-all-levels-of-marked-rev-eq-rev-get-all-levels-of-marked
                  get-rev-level-less-max-get-all-levels-of-marked list.set(*2*) *set-upt*)
            }
            then show *?thesis*
              using *g-r get-rev-level-less-max-get-all-levels-of-marked*[*of rev* (*tl ?M*) *0 L*]
              l'-tl calculation[*symmetric*] *g-a-l L-k*
              by (*auto simp*: *Max-n-upt cdcl*$_W$*-M-level-inv-def rev-swap*[*symmetric*])
          qed
        finally show *False* **using** *L-k* **by** *simp*
      qed
    then have *taut*: *tautology* (*D* + {#*L*#})
      using ‹*L'* ∈# *D*› **by** (*metis add.commute mset-leD mset-le-add-left multi-member-this*
        tautology-minus)
    have *consistent-interp* (*lits-of ?M*)
      using *level-inv* **unfolding** *cdcl*$_W$*-M-level-inv-def* **by** *auto*
    then have ¬*?M* ⊨*as CNot ?D*
      using *taut* **by** (*metis* (*no-types*) ‹*L'* = − *L*› ‹*L'* ∈# *D*› *add.commute consistent-interp-def*
        in-CNot-implies-uminus(*2*) *mset-leD mset-le-add-left multi-member-this*)
    moreover have *?M* ⊨*as CNot ?D*
      using *confl no-dup LD* **unfolding** *cdcl*$_W$*-conflicting-def* **by** *auto*
    ultimately show *False* **by** *blast*
  qed
then have *False*
  using *backtrack-no-decomp*[*OF* - ‹*get-level* (*trail S*) *L* = *backtrack-lvl S*› - *level-inv*]

```
    LD  alien termi by (metis cdcl_W-then-exists-cdcl_W-stgy-step level-inv)
  }
  moreover {
    assume ¬is-marked (hd ?M)
    then obtain L' C where L'C: hd ?M = Propagated L' C by (case-tac hd ?M, auto)
    then have M: ?M = Propagated L' C # tl ?M using ‹?M ≠ []›  list.collapse by fastforce
    then obtain C' where C':  C = C' + {#L'#}
      using confl unfolding cdcl_W-conflicting-def by (metis append-Nil diff-single-eq-union)
    { assume −L' ∉# ?D
      then have False
        using bj[OF cdcl_W-bj.skip[OF skip-rule[OF - ‹−L' ∉# ?D› ‹?D ≠ {#}›, of S C tl (trail S) -
          ]]]
        termi M by (metis LD alien cdcl_W-then-exists-cdcl_W-stgy-step state-eq-def level-inv)
    }
    moreover {
      assume −L' ∈# ?D
      then obtain D' where D': ?D = D' + {#−L'#} by (metis insert-DiffM2)
      have g-r: get-all-levels-of-marked (Propagated L' C # tl (trail S))
        = rev [Suc 0..<Suc (length (get-all-levels-of-marked (trail S)))]
        using level-inv M unfolding cdcl_W-M-level-inv-def by auto
      have Max (insert 0 (set (get-all-levels-of-marked (Propagated L' C # tl (trail S))))) = ?k
        using level-inv M unfolding g-r cdcl_W-M-level-inv-def set-rev
        by (auto simp add:Max-n-upt)
      then have get-maximum-level (Propagated L' C # tl ?M) D' ≤ ?k
        using get-maximum-possible-level-ge-get-maximum-level[of Propagated L' C # tl ?M]
        unfolding get-maximum-possible-level-max-get-all-levels-of-marked by auto
      then have get-maximum-level (Propagated L' C # tl ?M) D' = ?k
        ∨ get-maximum-level (Propagated L' C # tl ?M) D' < ?k
        using le-neq-implies-less by blast
      moreover {
        assume g-D'-k: get-maximum-level (Propagated L' C # tl ?M) D' = ?k
        have False
          proof −
            have f1: get-maximum-level (trail S) D' = backtrack-lvl S
              using M g-D'-k by auto
            have (trail S, init-clss S, learned-clss S, backtrack-lvl S, Some (D + {#L#}))
              = state S
              by (metis (no-types) LD)
            then have cdcl_W-o S (update-conflicting (Some (D' #∪ C')) (tl-trail S))
              using f1 bj[OF cdcl_W-bj.resolve[OF resolve-rule[of S L' C' tl ?M ?N ?U ?k D']]]
              C' D' M by (metis state-eq-def)
            then show ?thesis
              by (meson alien cdcl_W-then-exists-cdcl_W-stgy-step termi level-inv)
          qed
      }
      moreover {
        assume get-maximum-level (Propagated L' C # tl ?M) D' < ?k
        then have False
          proof −
            assume a1: get-maximum-level (Propagated L' C # tl (trail S)) D' < backtrack-lvl S
            obtain mm :: 'v literal multiset and ll :: 'v literal where
              f2: conflicting S = Some (mm + {#ll#})
                  get-level (trail S) ll = backtrack-lvl S
              using LD ‹get-level (trail S) L = backtrack-lvl S› by blast
            then have f3: get-maximum-level (trail S) D' ≤ get-level (trail S) ll
```

using *M a1* **by** *force*
have *lev-neq*: *get-level* (*trail S*) *ll* $\neq$ *get-maximum-level* (*trail S*) D'
using *f2 M calculation*(*2*) **by** *presburger*
have *f1*: *trail S* = *Propagated* L' *C* # *tl* (*trail S*)
conflicting S = *Some* (D' + {#− L'#})
using D' *LD M* **by** *force+*
have *f2*: *conflicting S* = *Some* (*mm* + {#*ll*#})
get-level (*trail S*) *ll* = *backtrack-lvl S*
using *f2* **by** *force+*
have *ll* = − L'
by (*metis* (*no-types*) D' *LD lev-neq option.inject f2 f3 le-antisym*
get-maximum-level-ge-get-level insert-noteq-member)
then show *?thesis*
using *f2 f1 M backtrack-no-decomp*[*of S*]
by (*metis* (*no-types*) *a1 alien* $cdcl_W$*-then-exists-*$cdcl_W$*-stgy-step level-inv termi*)
qed
}
ultimately have *False* **by** *blast*
}
ultimately have *False* **by** *blast*
}
ultimately have *False* **by** *blast*
}
ultimately show *?thesis* **by** *blast*
qed

lemma $cdcl_W$*-cp-tranclp-*$cdcl_W$:
$cdcl_W$*-cp S S'* $\Longrightarrow$ ${cdcl_W}^{++}$ *S S'*
apply (*induct rule*: $cdcl_W$*-cp.induct*)
by (*meson* $cdcl_W$*.conflict* $cdcl_W$*.propagate tranclp.r-into-trancl tranclp.trancl-into-trancl*)+

lemma *tranclp-*$cdcl_W$*-cp-tranclp-*$cdcl_W$:
$cdcl_W$*-cp*$^{++}$ *S S'* $\Longrightarrow$ ${cdcl_W}^{++}$ *S S'*
apply (*induct rule*: *tranclp.induct*)
apply (*simp add*: $cdcl_W$*-cp-tranclp-*$cdcl_W$)
by (*meson* $cdcl_W$*-cp-tranclp-*$cdcl_W$ *tranclp-trans*)

lemma $cdcl_W$*-stgy-tranclp-*$cdcl_W$:
$cdcl_W$*-stgy S S'* $\Longrightarrow$ ${cdcl_W}^{++}$ *S S'*
proof (*induct rule*: $cdcl_W$*-stgy.induct*)
case *conflict'*
then show *?case*
unfolding *full1-def* **by** (*simp add*: *tranclp-*$cdcl_W$*-cp-tranclp-*$cdcl_W$)
next
case (*other' S' S''*)
then have $S' = S'' \lor$ $cdcl_W$*-cp*$^{++}$ *S' S''*
by (*simp add*: *rtranclp-unfold full-def*)
then show *?case*
using *other'* **by** (*meson* $cdcl_W$*-ops.other* $cdcl_W$*-ops-axioms tranclp.r-into-trancl*
tranclp-$cdcl_W$*-cp-tranclp-*$cdcl_W$ *tranclp-trans*)
qed

lemma *tranclp-*$cdcl_W$*-stgy-tranclp-*$cdcl_W$:
$cdcl_W$*-stgy*$^{++}$ *S S'* $\Longrightarrow$ ${cdcl_W}^{++}$ *S S'*
apply (*induct rule*: *tranclp.induct*)

using $cdcl_W$*-stgy-tranclp-*$cdcl_W$ **apply** *blast*
by (*meson* $cdcl_W$*-stgy-tranclp-*$cdcl_W$ *tranclp-trans*)

lemma *rtranclp-*$cdcl_W$*-stgy-rtranclp-*$cdcl_W$:
$cdcl_W$*-stgy*** S $S' \Longrightarrow cdcl_W{}^{**}$ S S'
using *rtranclp-unfold*[*of* $cdcl_W$*-stgy* S S'] *tranclp-*$cdcl_W$*-stgy-tranclp-*$cdcl_W$[*of* S S'] **by** *auto*

lemma $cdcl_W$*-o-conflict-is-false-with-level-inv*:
assumes
$cdcl_W$*-o* S S' **and**
lev: $cdcl_W$*-M-level-inv* S **and**
confl-inv: *conflict-is-false-with-level* S **and**
n-d: *distinct-*$cdcl_W$*-state* S **and**
conflicting: $cdcl_W$*-conflicting* S
shows *conflict-is-false-with-level* S'
using *assms*(*1*,*2*)
proof (*induct rule*: $cdcl_W$*-o-induct-lev2*)
case (*resolve L C M D T*) **note** *tr-S* = *this*(*1*) **and** *confl* = *this*(*2*) **and** T = *this*(*4*)
have $-L \notin\# D$ **using** *n-d confl* **unfolding** *distinct-*$cdcl_W$*-state-def distinct-mset-def* **by** *auto*
moreover have $L \notin\# D$
proof (*rule ccontr*)
assume ¬ *?thesis*
moreover have *Propagated L* (C + {#L#}) # M ⊨*as CNot D*
using *conflicting confl tr-S* **unfolding** $cdcl_W$*-conflicting-def* **by** *auto*
ultimately have $-L \in$ *lits-of* (*Propagated L* ((C + {#L#})) # M)
using *in-CNot-implies-uminus*(*2*) **by** *blast*
moreover have *no-dup* (*Propagated L* ((C + {#L#})) # M)
using *lev tr-S* **unfolding** $cdcl_W$*-M-level-inv-def* **by** *auto*
ultimately show *False* **unfolding** *lits-of-def* **by** (*metis consistent-interp-def image-eqI list.set-intros*(*1*) *lits-of-def marked-lit.sel*(*2*) *distinctconsistent-interp*)
qed

ultimately
have *g-D*: *get-maximum-level* (*Propagated L* (C + {#L#}) # M) D
= *get-maximum-level M D*
proof −
have ∀ *a f L*. ((*a*::'*v*) ∈ *f* ‘ *L*) = (∃ *l*. (*l*::'*v literal*) ∈ *L* ∧ *a* = *f l*)
by *blast*
then show *?thesis*
using *get-maximum-level-skip-first*[*of L D* (C + {#L#}) M] **unfolding** *atms-of-def*
by (*metis* (*no-types*) ‹− $L \notin\# D$› ‹$L \notin\# D$› *atm-of-eq-atm-of mem-set-mset-iff*)
qed
{ assume
get-maximum-level (*Propagated L* (C + {#L#}) # M) D = *backtrack-lvl S* **and**
backtrack-lvl S > *0*
then have *D*: *get-maximum-level M D* = *backtrack-lvl S* **unfolding** *g-D* **by** *blast*
then have *?case*
using *tr-S* ‹*backtrack-lvl S* > *0*› *get-maximum-level-exists-lit*[*of backtrack-lvl S M D*] T
by *auto*
}
moreover {
assume [*simp*]: *backtrack-lvl S* = *0*
have ⋀*L*. *get-level M L* = *0*
proof −
fix *L*

have *atm-of L* $\notin$ *atm-of* ‘ (*lits-of M*) $\Longrightarrow$ *get-level M L* = *0* **by** *auto*
moreover {
assume *atm-of L* $\in$ *atm-of* ‘ (*lits-of M*)
have *g-r*: *get-all-levels-of-marked M* = *rev* [*Suc 0*..<*Suc* (*backtrack-lvl S*)]
using *lev tr-S* **unfolding** *cdcl*$_W$*-M-level-inv-def* **by** *auto*
have *Max* (*insert 0* (*set* (*get-all-levels-of-marked M*))) = (*backtrack-lvl S*)
unfolding *g-r* **by** (*simp add*: *Max-n-upt*)
then have *get-level M L* = *0*
using *get-maximum-possible-level-ge-get-level*[*of M L*]
unfolding *get-maximum-possible-level-max-get-all-levels-of-marked* **by** *auto*
}
ultimately show *get-level M L* = *0* **by** *blast*
qed
then have *?case* **using** *get-maximum-level-exists-lit-of-max-level*[*of D*#∪*C M*] *tr-S T*
by (*auto simp*: *Bex-mset-def*)
}
ultimately show *?case* **using** *resolve.hyps*(*3*) **by** *blast*
next
case (*skip L C' M D T*) **note** *tr-S* = *this*(*1*) **and** *D* = *this*(*2*) **and** *T* =*this*(*5*)
then obtain *La* **where** *La* ∈# *D* **and** *get-level* (*Propagated L C'* # *M*) *La* = *backtrack-lvl S*
using *skip confl-inv* **by** *auto*
moreover
have *atm-of La* $\neq$ *atm-of L*
proof (*rule ccontr*)
assume ¬ *?thesis*
then have *La*: *La* = *L* **using** ‹*La* ∈# *D*› ‹− *L* ∉# *D*› **by** (*auto simp add*: *atm-of-eq-atm-of*)
have *Propagated L C'* # *M* $\models$*as CNot D*
using *conflicting tr-S D* **unfolding** *cdcl*$_W$*-conflicting-def* **by** *auto*
then have −*L* $\in$ *lits-of M*
using ‹*La* ∈# *D*› *in-CNot-implies-uminus*(*2*)[*of D L Propagated L C'* # *M*] **unfolding** *La*
by *auto*
then show *False* **using** *lev tr-S* **unfolding** *cdcl*$_W$*-M-level-inv-def consistent-interp-def* **by** *auto*
qed
then have *get-level* (*Propagated L C'* # *M*) *La* = *get-level M La* **by** *auto*
ultimately show *?case* **using** *D tr-S T* **by** *auto*
qed (*auto split*: *split-if-asm simp*: *cdcl*$_W$*-M-level-inv-decomp*)

17.6.5 Strong completeness

lemma *cdcl*$_W$*-cp-propagate-confl*:
assumes *cdcl*$_W$*-cp S T*
shows *propagate*** *S T* ∨ (∃*S'*. *propagate*** *S S'* ∧ *conflict S' T*)
using *assms* **by** *induction blast+*

lemma *rtranclp-cdcl*$_W$*-cp-propagate-confl*:
assumes *cdcl*$_W$*-cp*** *S T*
shows *propagate*** *S T* ∨ (∃*S'*. *propagate*** *S S'* ∧ *conflict S' T*)
by (*simp add*: *assms rtranclp-cdcl*$_W$*-cp-propa-or-propa-confl*)

lemma *cdcl*$_W$*-cp-propagate-completeness*:
assumes *MN*: *set M* $\models$*s set-mset N* **and**
cons: *consistent-interp* (*set M*) **and**
tot: *total-over-m* (*set M*) (*set-mset N*) **and**
lits-of (*trail S*) ⊆ *set M* **and**
init-clss S = *N* **and**
propagate** *S S'* **and**

learned-clss $S = \{\#\}$
shows *length* (*trail* S) $\leq$ *length* (*trail* S') $\wedge$ *lits-of* (*trail* S') $\subseteq$ *set* M
using *assms*(*6*,*4*,*5*,*7*)
proof (*induction rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *auto*
next
case (*step* Y Z)
note *st* = *this*(*1*) **and** *propa* = *this*(*2*) **and** *IH* = *this*(*3*) **and** *lits'* = *this*(*4*) **and** *NS* = *this*(*5*) **and** *learned* = *this*(*6*)
then have *len*: *length* (*trail* S) $\leq$ *length* (*trail* Y) **and** *LM*: *lits-of* (*trail* Y) $\subseteq$ *set* M
by *blast+*

obtain M' N' U k C L **where**
Y: *state* $Y = (M', N', U, k, None)$ **and**
Z: *state* $Z = (Propagated\ L\ (C + \{\#L\#\})\ \#\ M', N', U, k, None)$ **and**
C: $C + \{\#L\#\} \in\#$ *clauses* Y **and**
M'-C: $M' \models as$ *CNot* C **and**
undefined-lit (*trail* Y) L
using *propa* **by** *auto*
have *init-clss* S = *init-clss* Y
using *st* **by** *induction auto*
then have [*simp*]: $N' = N$ **using** *NS* Y Z **by** *simp*
have *learned-clss* $Y = \{\#\}$
using *st learned* **by** *induction auto*
then have [*simp*]: $U = \{\#\}$ **using** Y **by** *auto*
have *set* $M \models s$ *CNot* C
using M'-C *LM* Y **unfolding** *true-annots-def Ball-def true-annot-def true-clss-def true-cls-def*
by *force*
moreover
have *set* $M \models C + \{\#L\#\}$
using *MN* C *learned* Y **unfolding** *true-clss-def clauses-def*
by (*metis NS* ‹*init-clss* S = *init-clss* Y› ‹*learned-clss* $Y = \{\#\}$› *add.right-neutral mem-set-mset-iff*)
ultimately have $L \in$ *set* M **by** (*simp add*: *cons consistent-CNot-not*)
then show *?case* **using** *LM len* Y Z **by** *auto*
qed

lemma *completeness-is-a-full1-propagation*:
fixes S :: *'st* **and** M :: *'v literal list*
assumes *MN*: *set* $M \models s$ *set-mset* N
and *cons*: *consistent-interp* (*set* M)
and *tot*: *total-over-m* (*set* M) (*set-mset* N)
and *alien*: *no-strange-atm* S
and *learned*: *learned-clss* $S = \{\#\}$
and *clsS*[*simp*]: *init-clss* $S = N$
and *lits*: *lits-of* (*trail* S) $\subseteq$ *set* M
shows $\exists S'.$ *propagate*** S S' $\wedge$ *full* $cdcl_W$*-cp* S S'
proof –
obtain S' **where** *full*: *full* $cdcl_W$*-cp* S S'
using *always-exists-full-*$cdcl_W$*-cp-step alien* **by** *blast*
then consider (*propa*) *propagate*** S S'
| (*confl*) $\exists X.$ *propagate*** S X $\wedge$ *conflict* X S'
using *rtranclp-*$cdcl_W$*-cp-propagate-confl* **unfolding** *full-def* **by** *blast*
then show *?thesis*

proof *cases*
  case *propa* **then show** *?thesis* **using** *full* **by** *blast*
next
  case *confl*
  then obtain *X* **where**
    X: *propagate*** *S X* **and**
    Xconf: *conflict X S'*
  by *blast*
  have *clsX*: *init-clss X = init-clss S*
    using *X* **by** *induction auto*
  have *learnedX*: *learned-clss X = {#}* **using** *X learned* **by** *induction auto*
  obtain *E* **where**
    E: *E ∈# init-clss X + learned-clss X* **and**
    Not-E: *trail X ⊨as CNot E*
    using *Xconf* **by** (*auto simp add*: *conflict.simps clauses-def*)
  have *lits-of* (*trail X*) ⊆ *set M*
    using $cdcl_W$*-cp-propagate-completeness*[*OF assms*(*1−3*) *lits - X learned*] *learned* **by** *auto*
  then have *MNE*: *set M ⊨s CNot E*
    using *Not-E*
    by (*fastforce simp add*: *true-annots-def true-annot-def true-clss-def true-cls-def*)
  have ¬ *set M ⊨s set-mset N*
    using *E consistent-CNot-not*[*OF cons MNE*]
    unfolding *learnedX true-clss-def* **unfolding** *clsX clsS* **by** *auto*
  then show *?thesis* **using** *MN* **by** *blast*
qed
qed

See also $cdcl_W$*-cp*** *?S ?S'* ⟹ ∃ *M*. *trail ?S'* = *M* @ *trail ?S* ∧ (∀ *l*∈*set M*. ¬ *is-marked l*)

lemma *rtranclp-propagate-is-trail-append*:
  *propagate*** *S T* ⟹ ∃ *c*. *trail T* = *c* @ *trail S*
  by (*induction rule*: *rtranclp-induct*) *auto*

lemma *rtranclp-propagate-is-update-trail*:
  *propagate*** *S T* ⟹ $cdcl_W$*-M-level-inv S* ⟹ *T* ∼ *delete-trail-and-rebuild* (*trail T*) *S*
proof (*induction rule*: *rtranclp-induct*)
  case *base*
  then show *?case* **unfolding** *state-eq-def* **by** (*auto simp*: $cdcl_W$*-M-level-inv-decomp state-access-simp*)
next
  case (*step T U*) **note** *IH=this*(*3*)[*OF this*(*4*)]
  moreover have $cdcl_W$*-M-level-inv U*
    using *rtranclp-*$cdcl_W$*-consistent-inv* ⟨*propagate*** *S T*⟩ ⟨*propagate T U*⟩
    rtranclp-mono[*of propagate* $cdcl_W$] $cdcl_W$*-cp-consistent-inv propagate'*
    rtranclp-propagate-is-rtranclp-$cdcl_W$ *step.prems* **by** *blast*
    then have *no-dup* (*trail U*) **unfolding** $cdcl_W$*-M-level-inv-def* **by** *auto*
  ultimately show *?case* **using** ⟨*propagate T U*⟩ **unfolding** *state-eq-def*
    by (*fastforce simp*: *state-access-simp*)
qed

lemma $cdcl_W$*-stgy-strong-completeness-n*:
  assumes
    MN: *set M ⊨s set-mset N* **and**
    cons: *consistent-interp* (*set M*) **and**
    tot: *total-over-m* (*set M*) (*set-mset N*) **and**
    atm-incl: *atm-of* ‘ (*set M*) ⊆ *atms-of-msu N* **and**
    distM: *distinct M* **and**

length: $n \leq$ *length* M
shows
$\exists M'\ k\ S.$ *length* $M' \geq n \wedge$
lits-of $M' \subseteq$ *set* $M \wedge$
no-dup $M' \wedge$
$S \sim$ *update-backtrack-lvl* k (*append-trail* (*rev* M') (*init-state* N)) $\wedge$
cdcl$_W$*-stgy*** (*init-state* N) S
using *length*
proof (*induction* n)
case *0*
have *update-backtrack-lvl 0* (*append-trail* (*rev* []) (*init-state* N)) $\sim$ *init-state* N
by (*auto simp*: *state-eq-def simp del*: *state-simp*)
moreover have
$0 \leq$ *length* [] **and**
lits-of [] $\subseteq$ *set* M **and**
cdcl$_W$*-stgy*** (*init-state* N) (*init-state* N)
and *no-dup* []
by (*auto simp*: *state-eq-def simp del*: *state-simp*)
ultimately show *?case* **using** *state-eq-sym* **by** *blast*
next
case (*Suc* n) **note** *IH* = *this*(*1*) **and** n = *this*(*2*)
then obtain $M'\ k\ S$ **where**
l-M': *length* $M' \geq n$ **and**
M': *lits-of* $M' \subseteq$ *set* M **and**
n-d[*simp*]: *no-dup* M' **and**
S: $S \sim$ *update-backtrack-lvl* k (*append-trail* (*rev* M') (*init-state* N)) **and**
st: *cdcl*$_W$*-stgy*** (*init-state* N) S
by *auto*
have
M: *cdcl*$_W$*-M-level-inv* S **and**
alien: *no-strange-atm* S
using *rtranclp-cdcl*$_W$*-consistent-inv*[*OF rtranclp-cdcl*$_W$*-stgy-rtranclp-cdcl*$_W$[*OF st*]]
rtranclp-cdcl$_W$*-no-strange-atm-inv*[*OF rtranclp-cdcl*$_W$*-stgy-rtranclp-cdcl*$_W$[*OF st*]]
S **unfolding** *state-eq-def cdcl*$_W$*-M-level-inv-def no-strange-atm-def* **by** *auto*
{ assume *no-step*: $\neg$*no-step propagate* S

obtain S' **where** S': *propagate*** $S\ S'$ **and** *full*: *full cdcl*$_W$*-cp* $S\ S'$
using *completeness-is-a-full1-propagation*[*OF assms*(*1*$-$*3*), *of* S] *alien* M' S
by (*auto simp*: *state-access-simp*)
have *lev*: *cdcl*$_W$*-M-level-inv* S'
using M S' *rtranclp-cdcl*$_W$*-consistent-inv rtranclp-propagate-is-rtranclp-cdcl*$_W$ **by** *blast*
then have *n-d'*[*simp*]: *no-dup* (*trail* S')
unfolding *cdcl*$_W$*-M-level-inv-def* **by** *auto*
have *length* (*trail* S) $\leq$ *length* (*trail* S') $\wedge$ *lits-of* (*trail* S') $\subseteq$ *set* M
using S' *full cdcl*$_W$*-cp-propagate-completeness*[*OF assms*(*1*$-$*3*), *of* S] M' S
by (*auto simp*: *state-access-simp*)
moreover
have *full*: *full1 cdcl*$_W$*-cp* $S\ S'$
using *full no-step no-step-cdcl*$_W$*-cp-no-conflict-no-propagate*(*2*) **unfolding** *full1-def full-def*
rtranclp-unfold **by** *blast*
then have *cdcl*$_W$*-stgy* $S\ S'$ **by** (*simp add*: *cdcl*$_W$*-stgy.conflict'*)
moreover
have *propa*: *propagate*$^{++}$ $S\ S'$ **using** S' *full* **unfolding** *full1-def* **by** (*metis rtranclpD tranclpD*)
have *trail* $S = M'$ **using** S **by** (*auto simp*: *state-access-simp*)
with *propa* **have** *length* (*trail* S') $> n$

using *l-M′ propa* **by** (*induction rule*: *tranclp.induct*) *auto*
moreover
have *stS′*: $cdcl_W$*-stgy*** (*init-state N*) *S′*
using *st* $cdcl_W$*-stgy.conflict′*[*OF full*] **by** *auto*
then have *init-clss S′ = N* **using** *stS′ rtranclp-*$cdcl_W$*-stgy-no-more-init-clss* **by** *fastforce*
moreover
have
[*simp*]:*learned-clss S′* = {#} **and**
[*simp*]: *init-clss S′ = init-clss S* **and**
[*simp*]: *conflicting S′ = None*
using *tranclp-into-rtranclp*[*OF* ‹*propagate*$^{++}$ *S S′*›] *S*
rtranclp-propagate-is-update-trail[*of S S′*] *S M* **unfolding** *state-eq-def*
by (*auto simp*: *state-access-simp*)
have *S-S′*: *S′* ∼ *update-backtrack-lvl* (*backtrack-lvl S′*)
(*append-trail* (*rev* (*trail S′*)) (*init-state N*)) **using** *S*
by (*auto simp*: *state-eq-def state-access-simp simp del*: *state-simp*)
have $cdcl_W$*-stgy*** (*init-state* (*init-clss S′*)) *S′*
apply (*rule rtranclp.rtrancl-into-rtrancl*)
using *st* **unfolding** ‹*init-clss S′ = N*› **apply** *simp*
using ‹$cdcl_W$*-stgy S S′*› **by** *simp*
ultimately have *?case*
apply −
apply (*rule exI*[*of - trail S′*], *rule exI*[*of - backtrack-lvl S′*], *rule exI*[*of - S′*])
using *S-S′* **by** (*auto simp*: *state-eq-def simp del*: *state-simp*)
}
moreover {
assume *no-step*: *no-step propagate S*
have *?case*
proof (*cases length M′* ≥ *Suc n*)
case *True*
then show *?thesis* **using** *l-M′ M′ st M alien S* **by** *fastforce*
next
case *False*
then have *n′*: *length M′ = n* **using** *l-M′* **by** *auto*
have *no-confl*: *no-step conflict S*
proof −
{ fix *D*
assume *D* ∈# *N* **and** *M′* ⊨*as CNot D*
then have *set M* ⊨ *D* **using** *MN* **unfolding** *true-clss-def* **by** *auto*
moreover have *set M* ⊨*s CNot D*
using ‹*M′* ⊨*as CNot D*› *M′*
by (*metis le-iff-sup true-annots-true-cls true-clss-union-increase*)
ultimately have *False* **using** *cons consistent-CNot-not* **by** *blast*
}
then show *?thesis* **using** *S* **by** (*auto simp*: *conflict.simps true-clss-def state-access-simp*)
qed
have *lenM*: *length M = card* (*set M*) **using** *distM* **by** (*induction M*) *auto*
have *no-dup M′* **using** *S M* **unfolding** $cdcl_W$*-M-level-inv-def* **by** *auto*
then have *card* (*lits-of M′*) = *length M′*
by (*induction M′*) (*auto simp add*: *lits-of-def card-insert-if*)
then have *lits-of M′* ⊂ *set M*
using *n M′ n′ lenM* **by** *auto*
then obtain *m* **where** *m*: *m* ∈ *set M* **and** *undef-m*: *m* ∉ *lits-of M′* **by** *auto*
moreover have *undef*: *undefined-lit M′ m*
using *M′ Marked-Propagated-in-iff-in-lits-of calculation*(*1*,*2*) *cons*

consistent-interp-def **by** *blast*
moreover have *atm-of m* $\in$ *atms-of-msu* (*init-clss S*)
using *atm-incl calculation S* **by** (*auto simp*: *state-access-simp*)
ultimately
have *dec*: *decide S* (*cons-trail* (*Marked m* (*k+1*)) (*incr-lvl S*))
using *decide.intros*[*of S rev M′ N - k m*
cons-trail (*Marked m* (*k + 1*)) (*incr-lvl S*)] *S*
by (*auto simp*: *state-access-simp*)
let *?S′* = *cons-trail* (*Marked m* (*k+1*)) (*incr-lvl S*)
have *lits-of* (*trail ?S′*) $\subseteq$ *set M* **using** *m M′ S undef* **by** (*auto simp*: *state-access-simp*)
moreover have *no-strange-atm ?S′*
using *alien dec M* **by** (*meson* $cdcl_W$*-no-strange-atm-inv decide other*)
ultimately obtain *S″* **where** *S″*: *propagate*** *?S′ S″* **and** *full*: *full* $cdcl_W$*-cp ?S′ S″*
using *completeness-is-a-full1-propagation*[*OF assms*(*1−3*), *of ?S′*] *S undef*
by (*auto simp*: *state-access-simp*)
have $cdcl_W$*-M-level-inv ?S′*
using *M dec rtranclp-mono*[*of decide* $cdcl_W$] **by** (*meson* $cdcl_W$*-consistent-inv decide other*)
then have *lev″*: $cdcl_W$*-M-level-inv S″*
using *S″ rtranclp-*$cdcl_W$*-consistent-inv rtranclp-propagate-is-rtranclp-*$cdcl_W$ **by** *blast*
then have *n-d″*: *no-dup* (*trail S″*)
unfolding $cdcl_W$*-M-level-inv-def* **by** *auto*
have *length* (*trail ?S′*) $\leq$ *length* (*trail S″*) $\wedge$ *lits-of* (*trail S″*) $\subseteq$ *set M*
using *S″ full* $cdcl_W$*-cp-propagate-completeness*[*OF assms*(*1−3*), *of ?S′ S″*] *m M′ S undef*
by (*simp add*: *state-access-simp*)
then have *Suc n* $\leq$ *length* (*trail S″*) $\wedge$ *lits-of* (*trail S″*) $\subseteq$ *set M*
using *l-M′ S undef* **by** (*auto simp*: *state-access-simp*)
moreover
have $cdcl_W$*-M-level-inv* (*cons-trail* (*Marked m* (*Suc* (*backtrack-lvl S*)))
(*update-backtrack-lvl* (*Suc* (*backtrack-lvl S*)) *S*))
using *S* ‹$cdcl_W$*-M-level-inv* (*cons-trail* (*Marked m* (*k + 1*)) (*incr-lvl S*))› **by** *auto*
then have *S″*: *S″* $\sim$ *update-backtrack-lvl* (*backtrack-lvl S″*)
(*append-trail* (*rev* (*trail S″*)) (*init-state N*))
using *rtranclp-propagate-is-update-trail*[*OF S″*] *S undef n-d″ lev″*
by (*auto simp del*: *state-simp simp*: *state-eq-def state-access-simp*)
then have $cdcl_W$*-stgy*** (*init-state N*) *S″*
using $cdcl_W$*-stgy.intros*(*2*)[*OF decide*[*OF dec*] *- full*] *no-step no-confl st*
by (*auto simp*: $cdcl_W$*-cp.simps*)
ultimately show *?thesis* **using** *S″ n-d″* **by** *blast*
qed
}
ultimately show *?case* **by** *blast*
qed

lemma $cdcl_W$*-stgy-strong-completeness*:
assumes *MN*: *set M* $\models$*s set-mset N*
and *cons*: *consistent-interp* (*set M*)
and *tot*: *total-over-m* (*set M*) (*set-mset N*)
and *atm-incl*: *atm-of* ‘ (*set M*) $\subseteq$ *atms-of-msu N*
and *distM*: *distinct M*
shows
$\exists$ *M′ k S*.
lits-of M′ = *set M* $\wedge$
S $\sim$ *update-backtrack-lvl k* (*append-trail* (*rev M′*) (*init-state N*)) $\wedge$
$cdcl_W$*-stgy*** (*init-state N*) *S* $\wedge$
final-$cdcl_W$*-state S*

proof −
  from *cdcl*$_W$*-stgy-strong-completeness-n*[*OF assms*, *of length M*]
  obtain *M′ k T* **where**
    l: *length M* ≤ *length M′* **and**
    M′-M: *lits-of M′* ⊆ *set M* **and**
    no-dup: *no-dup M′* **and**
    T: *T* ∼ *update-backtrack-lvl k* (*append-trail* (*rev M′*) (*init-state N*)) **and**
    st: *cdcl*$_W$*-stgy*** (*init-state N*) *T*
    by *auto*
  have *card* (*set M*) = *length M* **using** *distM* **by** (*simp add*: *distinct-card*)
  moreover
    have *cdcl*$_W$*-M-level-inv T*
      using *rtranclp-cdcl*$_W$*-stgy-consistent-inv*[*OF st*] *T* **by** *auto*
    then have *card* (*set* ((*map* (λ*l*. *atm-of* (*lit-of l*)) *M′*))) = *length M′*
      using *distinct-card no-dup* **by** *fastforce*
  moreover have *card* (*lits-of M′*) = *card* (*set* ((*map* (λ*l*. *atm-of* (*lit-of l*)) *M′*)))
    using *no-dup* **unfolding** *lits-of-def* **apply** (*induction M′*) **by** (*auto simp add*: *card-insert-if*)
  ultimately have *card* (*set M*) ≤ *card* (*lits-of M′*) **using** *l* **unfolding** *lits-of-def* **by** *auto*
  then have *set M* = *lits-of M′*
    using *M′-M card-seteq* **by** *blast*
  moreover
    then have *M′* ⊨*asm N*
      using *MN* **unfolding** *true-annots-def Ball-def true-annot-def true-clss-def* **by** *auto*
    then have *final-cdcl*$_W$*-state T*
      using *T no-dup* **unfolding** *final-cdcl*$_W$*-state-def* **by** (*auto simp*: *state-access-simp*)
  ultimately show *?thesis* **using** *st T* **by** *blast*
qed

17.6.6 No conflict with only variables of level less than backtrack level

This invariant is stronger than the previous argument in the sense that it is a property about all possible conflicts.

definition *no-smaller-confl* (*S*::*′st*) ≡
  (∀ *M K i M′ D*. *M′* @ *Marked K i* # *M* = *trail S* ⟶ *D* ∈# *clauses S*
    ⟶ ¬*M* ⊨*as CNot D*)

lemma *no-smaller-confl-init-sate*[*simp*]:
  no-smaller-confl (*init-state N*) **unfolding** *no-smaller-confl-def* **by** *auto*

lemma *cdcl*$_W$*-o-no-smaller-confl-inv*:
  fixes *S S′* :: *′st*
  assumes
    cdcl$_W$*-o S S′* **and**
    lev: *cdcl*$_W$*-M-level-inv S* **and**
    max-lev: *conflict-is-false-with-level S* **and**
    smaller: *no-smaller-confl S* **and**
    no-f: *no-clause-is-false S*
  shows *no-smaller-confl S′*
  using *assms*(*1*,*2*) **unfolding** *no-smaller-confl-def*
proof (*induct rule*: *cdcl*$_W$*-o-induct-lev2*)
  case (*decide L T*) **note** *confl* = *this*(*1*) **and** *undef* = *this*(*2*) **and** *T* = *this*(*4*)
  have [*simp*]: *clauses T* = *clauses S*
    using *T undef* **by** *auto*
  show *?case*
    proof (*intro allI impI*)

```
      fix M'' K i M' Da
      assume M'' @ Marked K i # M' = trail T
      and D: Da ∈# local.clauses T
      then have tl M'' @ Marked K i # M' = trail S
        ∨ (M'' = [] ∧ Marked K i # M' = Marked L (backtrack-lvl S + 1) # trail S)
        using T undef by (cases M'') auto
      moreover {
        assume tl M'' @ Marked K i # M' = trail S
        then have ¬M' ⊨as CNot Da
          using D T undef no-f confl smaller unfolding no-smaller-confl-def smaller by fastforce
      }
      moreover {
        assume Marked K i # M' = Marked L (backtrack-lvl S + 1) # trail S
        then have ¬M' ⊨as CNot Da using no-f D confl T by auto
      }
      ultimately show ¬M' ⊨as CNot Da by fast
    qed
next
  case resolve
  then show ?case using smaller no-f max-lev unfolding no-smaller-confl-def by auto
next
  case skip
  then show ?case using smaller no-f max-lev unfolding no-smaller-confl-def by auto
next
  case (backtrack K i M1 M2 L D T) note decomp = this(1) and confl = this(3) and undef = this(6)
    and T =this(7)
  obtain c where M: trail S = c @ M2 @ Marked K (i+1) # M1
    using decomp by auto

  show ?case
    proof (intro allI impI)
      fix M ia K' M' Da
      assume M' @ Marked K' ia # M = trail T
      then have tl M' @ Marked K' ia # M = M1
        using T decomp undef lev by (cases M') (auto simp: cdcl_W-M-level-inv-decomp)
      assume D: Da ∈# clauses T
      moreover{
        assume Da ∈# clauses S
        then have ¬M ⊨as CNot Da using ‹tl M' @ Marked K' ia # M = M1› M confl undef smaller
          unfolding no-smaller-confl-def by auto
      }
      moreover {
        assume Da: Da = D + {#L#}
        have ¬M ⊨as CNot Da
          proof (rule ccontr)
            assume ¬ ?thesis
            then have −L ∈ lits-of M unfolding Da by auto
            then have −L ∈ lits-of (Propagated L ((D + {#L#})) # M1)
              using UnI2 ‹tl M' @ Marked K' ia # M = M1›
              by auto
            moreover
              have backtrack S
                (cons-trail (Propagated L (D + {#L#}))
                  (reduce-trail-to M1 (add-learned-cls (D + {#L#})
                  (update-backtrack-lvl i (update-conflicting None S)))))
```

using *backtrack.intros*[*of S*] *backtrack.hyps*
by (*force simp*: *state-eq-def simp del*: *state-simp*)
then have $cdcl_W$*-M-level-inv*
(*cons-trail* (*Propagated L* (*D* + {#*L*#}))
(*reduce-trail-to M1* (*add-learned-cls* (*D* + {#*L*#})
(*update-backtrack-lvl i* (*update-conflicting None S*)))))
using $cdcl_W$*-consistent-inv*[*OF - lev*] *other*[*OF bj*] **by** *auto*
then have *no-dup* (*Propagated L* (*D* + {#*L*#}) # *M1*)
using *decomp undef lev* **unfolding** $cdcl_W$*-M-level-inv-def* **by** *auto*
ultimately show *False* **by** (*metis consistent-interp-def distinctconsistent-interp*
insertCI lits-of-cons marked-lit.sel(*2*))
qed
}
ultimately show ¬*M* ⊨*as CNot Da*
using *T undef* ‹*Da* = *D* + {#*L*#} ⟹ ¬ *M* ⊨*as CNot Da*› *decomp lev*
unfolding $cdcl_W$*-M-level-inv-def* **by** *fastforce*
qed
qed

lemma *conflict-no-smaller-confl-inv*:
assumes *conflict S S'*
and *no-smaller-confl S*
shows *no-smaller-confl S'*
using *assms* **unfolding** *no-smaller-confl-def* **by** *fastforce*

lemma *propagate-no-smaller-confl-inv*:
assumes *propagate*: *propagate S S'*
and *n-l*: *no-smaller-confl S*
shows *no-smaller-confl S'*
unfolding *no-smaller-confl-def*
proof (*intro allI impI*)
fix *M' K i M'' D*
assume *M'*: *M''* @ *Marked K i* # *M'* = *trail S'*
and *D* ∈# *clauses S'*
obtain *M N U k C L* **where**
S: *state S* = (*M*, *N*, *U*, *k*, *None*) **and**
S': *state S'* = (*Propagated L* ((*C* + {#*L*#})) # *M*, *N*, *U*, *k*, *None*) **and**
C + {#*L*#} ∈# *clauses S* **and**
M ⊨*as CNot C* **and**
undefined-lit M L
using *propagate* **by** *auto*
have *tl M''* @ *Marked K i* # *M'* = *trail S* **using** *M' S S'*
by (*metis Pair-inject list.inject list.sel*(*3*) *marked-lit.distinct*(*1*) *self-append-conv2*
tl-append2)
then have ¬*M'* ⊨*as CNot D*
using ‹*D* ∈# *clauses S'*› *n-l S S' clauses-def* **unfolding** *no-smaller-confl-def* **by** *auto*
then show ¬*M'* ⊨*as CNot D* **by** *auto*
qed

lemma $cdcl_W$*-cp-no-smaller-confl-inv*:
assumes *propagate*: $cdcl_W$*-cp S S'*
and *n-l*: *no-smaller-confl S*
shows *no-smaller-confl S'*
using *assms*
proof (*induct rule*: $cdcl_W$*-cp.induct*)

case (*conflict'* S S')
then show *?case* **using** *conflict-no-smaller-confl-inv*[*of* S S'] **by** *blast*
next
case (*propagate'* S S')
then show *?case* **using** *propagate-no-smaller-confl-inv*[*of* S S'] **by** *fastforce*
qed

lemma *rtrancp-cdcl*$_W$*-cp-no-smaller-confl-inv*:
assumes *propagate*: *cdcl*$_W$*-cp*** S S'
and *n-l*: *no-smaller-confl* S
shows *no-smaller-confl* S'
using *assms*
proof (*induct rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *simp*
next
case (*step* S' S'')
then show *?case* **using** *cdcl*$_W$*-cp-no-smaller-confl-inv*[*of* S' S''] **by** *fast*
qed

lemma *trancp-cdcl*$_W$*-cp-no-smaller-confl-inv*:
assumes *propagate*: *cdcl*$_W$*-cp*$^{++}$ S S'
and *n-l*: *no-smaller-confl* S
shows *no-smaller-confl* S'
using *assms*
proof (*induct rule*: *tranclp.induct*)
case (*r-into-trancl* S S')
then show *?case* **using** *cdcl*$_W$*-cp-no-smaller-confl-inv*[*of* S S'] **by** *blast*
next
case (*trancl-into-trancl* S S' S'')
then show *?case* **using** *cdcl*$_W$*-cp-no-smaller-confl-inv*[*of* S' S''] **by** *fast*
qed

lemma *full-cdcl*$_W$*-cp-no-smaller-confl-inv*:
assumes *full cdcl*$_W$*-cp* S S'
and *n-l*: *no-smaller-confl* S
shows *no-smaller-confl* S'
using *assms* **unfolding** *full-def*
using *rtrancp-cdcl*$_W$*-cp-no-smaller-confl-inv*[*of* S S'] **by** *blast*

lemma *full1-cdcl*$_W$*-cp-no-smaller-confl-inv*:
assumes *full1 cdcl*$_W$*-cp* S S'
and *n-l*: *no-smaller-confl* S
shows *no-smaller-confl* S'
using *assms* **unfolding** *full1-def*
using *trancp-cdcl*$_W$*-cp-no-smaller-confl-inv*[*of* S S'] **by** *blast*

lemma *cdcl*$_W$*-stgy-no-smaller-confl-inv*:
assumes *cdcl*$_W$*-stgy* S S'
and *n-l*: *no-smaller-confl* S
and *conflict-is-false-with-level* S
and *cdcl*$_W$*-M-level-inv* S
shows *no-smaller-confl* S'
using *assms*
proof (*induct rule*: *cdcl*$_W$*-stgy.induct*)

case (*conflict′ S′*)
then show *?case* using *full1-cdcl*$_W$*-cp-no-smaller-confl-inv*[*of S S′*] by *blast*
next
case (*other′ S′ S′′*)
have *no-smaller-confl S′*
using *cdcl*$_W$*-o-no-smaller-confl-inv*[*OF other′.hyps*(*1*) *other′.prems*(*3*,*2*,*1*)]
not-conflict-not-any-negated-init-clss other′.hyps(*2*) by *blast*
then show *?case* using *full-cdcl*$_W$*-cp-no-smaller-confl-inv*[*of S′ S′′*] *other′.hyps* by *blast*
qed

lemma *conflict-conflict-is-no-clause-is-false-test*:
assumes *conflict S S′*
and (∀ *D* ∈# *init-clss S* + *learned-clss S*. *trail S* ⊨*as CNot D*
⟶ (∃ *L*. *L* ∈# *D* ∧ *get-level* (*trail S*) *L* = *backtrack-lvl S*))
shows ∀ *D* ∈# *init-clss S′* + *learned-clss S′*. *trail S′* ⊨*as CNot D*
⟶ (∃ *L*. *L* ∈# *D* ∧ *get-level* (*trail S′*) *L* = *backtrack-lvl S′*)
using *assms* by *auto*

lemma *is-conflicting-exists-conflict*:
assumes ¬(∀ *D*∈#*init-clss S′* + *learned-clss S′*. ¬ *trail S′* ⊨*as CNot D*)
and *conflicting S′* = *None*
shows ∃ *S′′*. *conflict S′ S′′*
using *assms clauses-def not-conflict-not-any-negated-init-clss* by *fastforce*

lemma *cdcl*$_W$*-o-conflict-is-no-clause-is-false*:
fixes *S S′* :: *′st*
assumes
cdcl$_W$*-o S S′* and
lev: *cdcl*$_W$*-M-level-inv S* and
max-lev: *conflict-is-false-with-level S* and
no-f: *no-clause-is-false S* and
no-l: *no-smaller-confl S*
shows *no-clause-is-false S′*
∨ (*conflicting S′* = *None*
⟶ (∀ *D* ∈# *clauses S′*. *trail S′* ⊨*as CNot D*
⟶ (∃ *L*. *L* ∈# *D* ∧ *get-level* (*trail S′*) *L* = *backtrack-lvl S′*)))
using *assms*(*1*,*2*)
proof (*induct rule*: *cdcl*$_W$*-o-induct-lev2*)
case (*decide L T*) note *S* = *this*(*1*) and *undef* = *this*(*2*) and *T* =*this*(*4*)
show *?case*
proof (*rule HOL.disjI2*, *clarify*)
fix *D*
assume *D*: *D* ∈# *clauses T* and *M-D*: *trail T* ⊨*as CNot D*
let *?M* = *trail S*
let *?M′* = *trail T*
let *?k* = *backtrack-lvl S*
have ¬*?M* ⊨*as CNot D*
using *no-f D S T undef* by *auto*
have −*L* ∈# *D*
proof (*rule ccontr*)
assume ¬ *?thesis*
have *?M* ⊨*as CNot D*
unfolding *true-annots-def Ball-def true-annot-def CNot-def true-cls-def*
proof (*intro allI impI*)

fix x
assume x: $x \in \{\{\#- L\#\} \mid L.\ L \in\# D\}$

then obtain L' **where** L': $x = \{\#-L'\#\}$ $L' \in\# D$ **by** *auto*
obtain L'' **where** $L'' \in\# x$ **and** *lits-of* (*Marked* L (*?k* + *1*) # *?M*) $\models l$ L''
using *M-D x T undef* **unfolding** *true-annots-def Ball-def true-annot-def CNot-def true-cls-def Bex-mset-def* **by** *auto*
show $\exists L \in\# x$. *lits-of ?M* $\models l$ L **unfolding** *Bex-mset-def*
by (*metis* ‹$- L \notin\# D$› ‹$L'' \in\# x$› L' ‹*lits-of* (*Marked* L (*?k* + *1*) # *?M*) $\models l$ L''›
count-single insertE less-numeral-extra(*3*) *lits-of-cons marked-lit.sel*(*1*)
true-lit-def uminus-of-uminus-id)
qed
then show *False* **using** ‹$\neg$ *?M* $\models as$ *CNot D*› **by** *auto*
qed
have *atm-of L* $\notin$ *atm-of* ‘ (*lits-of ?M*)
using *undef defined-lit-map* **unfolding** *lits-of-def* **by** *fastforce*
then have *get-level* (*Marked* L (*?k* + *1*) # *?M*) ($-L$) = *?k* + *1* **by** *simp*
then show $\exists La$. $La \in\# D \wedge$ *get-level ?M' La* = *backtrack-lvl T*
using ‹$-L \in\# D$› *T undef* **by** *auto*
qed
next
case *resolve*
then show *?case* **by** *auto*
next
case *skip*
then show *?case* **by** *auto*
next
case (*backtrack K i M1 M2 L D T*) **note** *decomp* = *this*(*1*) **and** *undef* = *this*(*6*) **and** *T* =*this*(*7*)
show *?case*
proof (*rule HOL.disjI2*, *clarify*)
fix *Da*
assume *Da*: *Da* $\in\#$ *clauses T*
and *M-D*: *trail T* $\models as$ *CNot Da*
obtain *c* **where** *M*: *trail S* = *c* @ *M2* @ *Marked K* (*i* + *1*) # *M1*
using *decomp* **by** *auto*
have *tr-T*: *trail T* = *Propagated L* (*D* + {#*L*#}) # *M1*
using *T decomp undef lev* **by** (*auto simp*: $cdcl_W$*-M-level-inv-decomp*)
have *backtrack S T*
using *backtrack.intros backtrack.hyps T* **by** (*force simp del*: *state-simp simp*: *state-eq-def*)
then have *lev'*: $cdcl_W$*-M-level-inv T*
using $cdcl_W$*-consistent-inv lev other* **by** *blast*
then have $- L \notin$ *lits-of M1*
unfolding $cdcl_W$*-M-level-inv-def lits-of-def*
proof –
have *consistent-interp* (*lits-of* (*trail S*)) $\wedge$ *no-dup* (*trail S*)
$\wedge$ *backtrack-lvl S* = *length* (*get-all-levels-of-marked* (*trail S*))
$\wedge$ *get-all-levels-of-marked* (*trail S*)
= *rev* [*1*..<*1* + *length* (*get-all-levels-of-marked* (*trail S*))]
using *lev* $cdcl_W$*-M-level-inv-def* **by** *blast*
then show $- L \notin$ *lit-of* ‘ *set M1*
by (*metis* (*no-types*) *One-nat-def add.right-neutral add-Suc-right*
atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set backtrack.hyps(*2*)
$cdcl_W$*-ops.backtrack-lit-skiped* $cdcl_W$*-ops-axioms decomp lits-of-def*)
qed
{ **assume** *Da* $\in\#$ *clauses S*

```
      then have ¬M1 ⊨as CNot Da using no-l M unfolding no-smaller-confl-def by auto
    }
    moreover {
      assume Da: Da = D + {#L#}
      have ¬M1 ⊨as CNot Da using ‹− L ∉ lits-of M1› unfolding Da by simp
    }
    ultimately have ¬M1 ⊨as CNot Da
      using Da T undef decomp lev by (fastforce simp: cdcl_W-M-level-inv-decomp)
    then have −L ∈# Da
      using M-D ‹− L ∉ lits-of M1› in-CNot-implies-uminus(2)
        true-annots-CNot-lit-of-notin-skip T unfolding tr-T
      by (smt insert-iff lits-of-cons marked-lit.sel(2))
    have g-M1: get-all-levels-of-marked M1 = rev [1..<i+1]
      using lev lev' T decomp undef unfolding cdcl_W-M-level-inv-def by auto
    have no-dup (Propagated L (D + {#L#}) # M1)
      using lev lev' T decomp undef unfolding cdcl_W-M-level-inv-def by auto
    then have L: atm-of L ∉ atm-of ‘ lits-of M1 unfolding lits-of-def by auto
    have get-level (Propagated L ((D + {#L#})) # M1) (−L) = i
      using get-level-get-rev-level-get-all-levels-of-marked[OF L,
        of [Propagated L ((D + {#L#}))]]
      by (simp add: g-M1 split: if-splits)
    then show ∃La. La ∈# Da ∧ get-level (trail T) La = backtrack-lvl T
      using ‹−L ∈# Da› T decomp undef lev by (auto simp: cdcl_W-M-level-inv-def)
  qed
qed

lemma full1-cdcl_W-cp-exists-conflict-decompose:
  assumes confl: ∃D∈#clauses S. trail S ⊨as CNot D
  and full: full cdcl_W-cp S U
  and no-confl: conflicting S = None
  shows ∃T. propagate** S T ∧ conflict T U
proof −
  consider (propa) propagate** S U
      | (confl) T where propagate** S T and conflict T U
   using full unfolding full-def by (blast dest:rtranclp-cdcl_W-cp-propa-or-propa-confl)
  then show ?thesis
    proof cases
      case confl
      then show ?thesis by blast
    next
      case propa
      then have conflicting U = None
        using no-confl by induction auto
      moreover have [simp]: learned-clss U = learned-clss S and
        [simp]: init-clss U = init-clss S
        using propa by induction auto
      moreover
        obtain D where D: D∈#clauses U and
          trS: trail S ⊨as CNot D
          using confl clauses-def by auto
        obtain M where M: trail U = M @ trail S
          using full rtranclp-cdcl_W-cp-dropWhile-trail unfolding full-def by meson
        have tr-U: trail U ⊨as CNot D
          apply (rule true-annots-mono)
          using trS unfolding M by simp-all
```

have $\exists$ *V. conflict U V*
using ⟨*conflicting U = None*⟩ *D clauses-def not-conflict-not-any-negated-init-clss tr-U*
by *blast*
then have *False* using *full* $cdcl_W$*-cp.conflict'* unfolding *full-def* by *blast*
then show *?thesis* by *fast*
qed
qed

lemma *full1-*$cdcl_W$*-cp-exists-conflict-full1-decompose*:
assumes *confl*: $\exists$ *D*$\in$#*clauses S. trail S* $\models$*as CNot D*
and *full*: *full* $cdcl_W$*-cp S U*
and *no-confl*: *conflicting S = None*
shows $\exists$ *T D. propagate*** *S T* $\wedge$ *conflict T U*
$\wedge$ *trail T* $\models$*as CNot D* $\wedge$ *conflicting U = Some D* $\wedge$ *D* $\in$# *clauses S*
proof −
obtain *T* **where** *propa*: *propagate*** *S T* **and** *conf*: *conflict T U*
using *full1-*$cdcl_W$*-cp-exists-conflict-decompose*[*OF assms*] **by** *blast*
have *p*: *learned-clss T = learned-clss S init-clss T = init-clss S*
using *propa* **by** *induction auto*
have *c*: *learned-clss U = learned-clss T init-clss U = init-clss T*
using *conf* **by** *induction auto*
obtain *D* **where** *trail T* $\models$*as CNot D* $\wedge$ *conflicting U = Some D* $\wedge$ *D* $\in$# *clauses S*
using *conf p c* **by** (*fastforce simp*: *clauses-def*)
then show *?thesis*
using *propa conf* **by** *blast*
qed

lemma $cdcl_W$*-stgy-no-smaller-confl*:
assumes $cdcl_W$*-stgy S S'*
and *n-l*: *no-smaller-confl S*
and *conflict-is-false-with-level S*
and $cdcl_W$*-M-level-inv S*
and *no-clause-is-false S*
and *distinct-*$cdcl_W$*-state S*
and $cdcl_W$*-conflicting S*
shows *no-smaller-confl S'*
using *assms*
proof (*induct rule*: $cdcl_W$*-stgy.induct*)
case (*conflict' S'*)
show *no-smaller-confl S'*
using *conflict'.hyps conflict'.prems*(*1*) *full1-*$cdcl_W$*-cp-no-smaller-confl-inv* **by** *blast*
next
case (*other' S' S''*)
have *lev'*: $cdcl_W$*-M-level-inv S'*
using $cdcl_W$*-consistent-inv other other'.hyps*(*1*) *other'.prems*(*3*) **by** *blast*
show *no-smaller-confl S''*
using $cdcl_W$*-stgy-no-smaller-confl-inv*[*OF* $cdcl_W$*-stgy.other'*[*OF other'.hyps*(*1−3*)]]
other'.prems(*1−3*) **by** *blast*
qed

lemma $cdcl_W$*-stgy-ex-lit-of-max-level*:
assumes $cdcl_W$*-stgy S S'*
and *n-l*: *no-smaller-confl S*
and *conflict-is-false-with-level S*
and $cdcl_W$*-M-level-inv S*

```
  and no-clause-is-false S
  and distinct-cdcl_W-state S
  and cdcl_W-conflicting S
  shows conflict-is-false-with-level S'
  using assms
proof (induct rule: cdcl_W-stgy.induct)
  case (conflict' S')
  have no-smaller-confl S'
    using conflict'.hyps conflict'.prems(1) full1-cdcl_W-cp-no-smaller-confl-inv by blast
  moreover have conflict-is-false-with-level S'
    using conflict'.hyps conflict'.prems(2-4)
    rtranclp-cdcl_W-co-conflict-ex-lit-of-max-level[of S S']
    unfolding full-def full1-def rtranclp-unfold by presburger
  then show ?case by blast
next
  case (other' S' S'')
  have lev': cdcl_W-M-level-inv S'
    using cdcl_W-consistent-inv other other'.hyps(1) other'.prems(3) by blast
  moreover
    have no-clause-is-false S'
      ∨ (conflicting S' = None ⟶ (∀ D∈#clauses S'. trail S' ⊨as CNot D
          ⟶ (∃ L. L ∈# D ∧ get-level (trail S') L = backtrack-lvl S')))
      using cdcl_W-o-conflict-is-no-clause-is-false[of S S'] other'.hyps(1) other'.prems(1-4) by fast
  moreover {
    assume no-clause-is-false S'
    {
      assume conflicting S' = None
      then have conflict-is-false-with-level S' by auto
      moreover have full cdcl_W-cp S' S''
        by (metis (no-types) other'.hyps(3))
      ultimately have conflict-is-false-with-level S''
        using rtranclp-cdcl_W-co-conflict-ex-lit-of-max-level[of S' S''] lev' ‹no-clause-is-false S'›
        by blast
    }
    moreover
    {
      assume c: conflicting S' ≠ None
      have conflicting S ≠ None using other'.hyps(1) c
        by (induct rule: cdcl_W-o-induct) auto
      then have conflict-is-false-with-level S'
        using cdcl_W-o-conflict-is-false-with-level-inv[OF other'.hyps(1)]
        other'.prems(3,5,6,2) by blast
      moreover have cdcl_W-cp** S' S'' using other'.hyps(3) unfolding full-def by auto
      then have S' = S'' using c
        by (induct rule: rtranclp-induct)
          (fastforce intro: option.exhaust)+
      ultimately have conflict-is-false-with-level S'' by auto
    }
    ultimately have conflict-is-false-with-level S'' by blast
  }
  moreover {
     assume
       confl: conflicting S' = None and
       D-L: ∀ D ∈# clauses S'. trail S' ⊨as CNot D
         ⟶ (∃ L. L ∈# D ∧ get-level (trail S') L = backtrack-lvl S')
```

{ **assume** $\forall D\in\#$ *clauses* S'. $\neg$ *trail* $S' \models as$ *CNot D*
  then have *no-clause-is-false* S' **using** *confl* **by** *simp*
  then have *conflict-is-false-with-level* S'' **using** *calculation*(*3*) **by** *presburger*
}
moreover {
  assume $\neg(\forall D\in\#$ *clauses* S'. $\neg$ *trail* $S' \models as$ *CNot D*)
  then obtain *T D* **where**
    *propagate*** S' *T* **and**
    conflict T S'' **and**
    D: $D \in\#$ *clauses* S' **and**
    trail $S'' \models as$ *CNot D* **and**
    conflicting S'' = *Some D*
    using *full1-cdcl*$_W$*-cp-exists-conflict-full1-decompose*[*OF - - confl*]
    other'(*3*) **by** (*metis* (*mono-tags*, *lifting*) *ball-msetI bex-msetI conflictE state-eq-trail*
      trail-update-conflicting)
  obtain *M* **where** *M*: *trail* S'' = *M* @ *trail* S' **and** *nm*: $\forall m\in$*set M*. $\neg$*is-marked m*
    using *rtranclp-cdcl*$_W$*-cp-dropWhile-trail other'*(*3*) **unfolding** *full-def* **by** *meson*
  have *btS*: *backtrack-lvl* S'' = *backtrack-lvl* S'
    using *other'.hyps*(*3*) **unfolding** *full-def* **by** (*metis rtranclp-cdcl*$_W$*-cp-backtrack-lvl*)
  have *inv*: *cdcl*$_W$*-M-level-inv* S''
    by (*metis* (*no-types*) *cdcl*$_W$*-stgy.conflict' cdcl*$_W$*-stgy-consistent-inv full-unfold lev'*
      other'.hyps(*3*))
  then have *nd*: *no-dup* (*trail* S'')
    by (*metis* (*no-types*) *cdcl*$_W$*-M-level-inv-decomp*(*2*))
  have *conflict-is-false-with-level* S''
    proof *cases*
      assume *trail* $S' \models as$ *CNot D*
      moreover then obtain *L* **where**
        $L \in\# D$ **and**
        lev-L: *get-level* (*trail* S') *L* = *backtrack-lvl* S'
        using *D-L D* **by** *blast*
      moreover
        have *LS'*: $-L \in$ *lits-of* (*trail* S')
          using ‹*trail* $S' \models as$ *CNot D*› ‹$L \in\# D$› *in-CNot-implies-uminus*(*2*) **by** *blast*
        { **fix** *x* :: (*'v*, *nat*, *'v literal multiset*) *marked-lit* **and**
            xb :: (*'v*, *nat*, *'v literal multiset*) *marked-lit*
          assume *a1*: $x \in$ *set* (*trail* S') **and**
            a2: $xb \in$ *set M* **and**
            a3: (λl. *atm-of* (*lit-of l*)) ‘ *set M* $\cap$ (λl. *atm-of* (*lit-of l*)) ‘ *set* (*trail* S')
              = {} **and**
             a4: $-$ *L* = *lit-of x* **and**
             a5: *atm-of L* = *atm-of* (*lit-of xb*)
          moreover have *atm-of* (*lit-of x*) = *atm-of L*
            using *a4* **by** (*metis* (*no-types*) *atm-of-uminus*)
          ultimately have *False*
            using *a5 a3 a2 a1* **by** *auto*
        }
        then have *atm-of L* $\notin$ *atm-of* ‘ *lits-of M*
          using *nd LS'* **unfolding** *M* **by** (*auto simp add*: *lits-of-def*)
        then have *get-level* (*trail* S'') *L* = *get-level* (*trail* S') *L*
          unfolding *M* **by** (*simp add*: *lits-of-def*)
      ultimately show *?thesis* **using** *btS* ‹*conflicting* S'' = *Some D*› **by** *auto*
    next
      assume $\neg$*trail* $S' \models as$ *CNot D*
      then obtain *L* **where** $L \in\# D$ **and** *LM*: $-L \in$ *lits-of M*

using ‹*trail S''* ⊨*as CNot D*›
by (*auto simp add*: *CNot-def true-cls-def M true-annots-def true-annot-def*
split: *split-if-asm*)
{ fix *x* :: (*'v*, *nat*, *'v literal multiset*) *marked-lit* **and**
xb :: (*'v*, *nat*, *'v literal multiset*) *marked-lit*
assume *a1*: *xb* ∈ *set* (*trail S'*) **and**
a2: *x* ∈ *set M* **and**
a3: *atm-of L* = *atm-of* (*lit-of xb*) **and**
a4: − *L* = *lit-of x* **and**
a5: (*λl. atm-of* (*lit-of l*)) ' *set M* ∩ (*λl. atm-of* (*lit-of l*)) ' *set* (*trail S'*)
= {}
moreover have *atm-of* (*lit-of xb*) = *atm-of* (− *L*)
using *a3* **by** *simp*
ultimately have *False*
by *auto* **}**
then have *LS'*: *atm-of L* ∉ *atm-of* ' *lits-of* (*trail S'*)
using *nd* ‹*L* ∈# *D*› *LM* **unfolding** *M* **by** (*auto simp add*: *lits-of-def*)
show *?thesis*
proof *cases*
assume *ne*: *get-all-levels-of-marked* (*trail S'*) = []
have *backtrack-lvl S''* = *0*
using *inv ne nm* **unfolding** *cdcl$_W$-M-level-inv-def M*
by (*simp add*: *get-all-levels-of-marked-nil-iff-not-is-marked*)
moreover
have *a1*: *get-level M L* = *0*
using *nm* **by** *auto*
then have *get-level* (*M* @ *trail S'*) *L* = *0*
by (*metis LS' get-all-levels-of-marked-nil-iff-not-is-marked*
get-level-skip-beginning-not-marked lits-of-def ne)
ultimately show *?thesis* **using** ‹*conflicting S''* = *Some D*› ‹*L* ∈# *D*› **unfolding** *M*
by *auto*
next
assume *ne*: *get-all-levels-of-marked* (*trail S'*) ≠ []
have *hd* (*get-all-levels-of-marked* (*trail S'*)) = *backtrack-lvl S'*
using *ne lev' M nm* **unfolding** *cdcl$_W$-M-level-inv-def*
by (*cases get-all-levels-of-marked* (*trail S'*))
(*simp-all add*: *get-all-levels-of-marked-nil-iff-not-is-marked*[*symmetric*])
moreover have *atm-of L* ∈ *atm-of* ' *lits-of M*
using ‹−*L* ∈ *lits-of M*›
by (*simp add*: *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set lits-of-def*)
ultimately show *?thesis*
using *nm ne* ‹*L*∈#*D*› ‹*conflicting S''* = *Some D*›
get-level-skip-beginning-hd-get-all-levels-of-marked[*OF LS'*, *of M*]
get-level-skip-in-all-not-marked[*of rev M L backtrack-lvl S'*]
unfolding *lits-of-def btS M*
by *auto*
qed
qed
}
ultimately have *conflict-is-false-with-level S''* **by** *blast*
}
moreover
{
assume *conflicting S'* ≠ *None*
have *no-clause-is-false S'* **using** ‹*conflicting S'* ≠ *None*› **by** *auto*

then have *conflict-is-false-with-level* S'' **using** *calculation*(*3*) **by** *presburger*
}
ultimately show *?case* **by** *fast*
qed

lemma *rtranclp-cdcl*$_W$*-stgy-no-smaller-confl-inv*:
assumes
cdcl$_W$*-stgy*** S S' **and**
n-l: *no-smaller-confl* S **and**
cls-false: *conflict-is-false-with-level* S **and**
lev: *cdcl*$_W$*-M-level-inv* S **and**
no-f: *no-clause-is-false* S **and**
dist: *distinct-cdcl*$_W$*-state* S **and**
conflicting: *cdcl*$_W$*-conflicting* S **and**
decomp: *all-decomposition-implies-m* (*init-clss* S) (*get-all-marked-decomposition* (*trail* S)) **and**
learned: *cdcl*$_W$*-learned-clause* S **and**
alien: *no-strange-atm* S
shows *no-smaller-confl* S' ∧ *conflict-is-false-with-level* S'
using *assms*(*1*)
proof (*induct rule*: *rtranclp-induct*)
case *base*
then show *?case* **using** *n-l cls-false* **by** *auto*
next
case (*step* S' S'') **note** *st* = *this*(*1*) **and** *cdcl* = *this*(*2*) **and** *IH* = *this*(*3*)
have *no-smaller-confl* S' **and** *conflict-is-false-with-level* S'
using *IH* **by** *blast+*
moreover have *cdcl*$_W$*-M-level-inv* S'
using *st lev rtranclp-cdcl*$_W$*-stgy-rtranclp-cdcl*$_W$
by (*blast intro*: *rtranclp-cdcl*$_W$*-consistent-inv*)+
moreover have *no-clause-is-false* S'
using *st no-f rtranclp-cdcl*$_W$*-stgy-not-non-negated-init-clss* **by** *presburger*
moreover have *distinct-cdcl*$_W$*-state* S'
using *rtanclp-distinct-cdcl*$_W$*-state-inv*[*of* S S'] *lev rtranclp-cdcl*$_W$*-stgy-rtranclp-cdcl*$_W$[*OF st*]
dist **by** *auto*
moreover have *cdcl*$_W$*-conflicting* S'
using *rtranclp-cdcl*$_W$*-all-inv*(*6*)[*of* S S'] *st alien conflicting decomp dist learned lev*
rtranclp-cdcl$_W$*-stgy-rtranclp-cdcl*$_W$ **by** *blast*
ultimately show *?case*
using *cdcl*$_W$*-stgy-no-smaller-confl*[*OF cdcl*] *cdcl*$_W$*-stgy-ex-lit-of-max-level*[*OF cdcl*] **by** *fast*
qed

17.6.7 Final States are Conclusive

lemma *full-cdcl*$_W$*-stgy-final-state-conclusive-non-false*:
fixes S' :: *'st*
assumes *full*: *full cdcl*$_W$*-stgy* (*init-state* N) S'
and *no-d*: *distinct-mset-mset* N
and *no-empty*: $\forall D \in \# N.\ D \neq \{\#\}$
shows (*conflicting* S' = *Some* {#} ∧ *unsatisfiable* (*set-mset* (*init-clss* S')))
∨ (*conflicting* S' = *None* ∧ *trail* S' ⊨*asm init-clss* S')
proof −
let *?S* = *init-state* N
have
termi: $\forall S''.\ \neg$*cdcl*$_W$*-stgy* S' S'' **and**
step: *cdcl*$_W$*-stgy*** (*init-state* N) S' **using** *full* **unfolding** *full-def* **by** *auto*
moreover have

learned: $cdcl_W$*-learned-clause* S' **and**
level-inv: $cdcl_W$*-M-level-inv* S' **and**
alien: *no-strange-atm* S' **and**
no-dup: *distinct-*$cdcl_W$*-state* S' **and**
confl: $cdcl_W$*-conflicting* S' **and**
decomp: *all-decomposition-implies-m* (*init-clss* S') (*get-all-marked-decomposition* (*trail* S'))
using *no-d tranclp-*$cdcl_W$*-stgy-tranclp-*$cdcl_W$[*of ?S* S'] *step rtranclp-*$cdcl_W$*-all-inv*(*1*−*6*)[*of ?S* S']
unfolding *rtranclp-unfold* **by** *auto*
moreover
have $\forall$ *D*∈#*N*. ¬ [] ⊨*as CNot D* **using** *no-empty* **by** *auto*
then have *confl-k*: *conflict-is-false-with-level* S'
using *rtranclp-*$cdcl_W$*-stgy-no-smaller-confl-inv*[*OF step*] *no-d* **by** *auto*
show *?thesis*
using $cdcl_W$*-stgy-final-state-conclusive*[*OF termi decomp learned level-inv alien no-dup confl confl-k*] **.**
qed

lemma *conflict-is-full1-*$cdcl_W$*-cp*:
assumes *cp*: *conflict S* S'
shows *full1* $cdcl_W$*-cp S* S'
proof −
have $cdcl_W$*-cp S* S' **and** *conflicting* $S' \neq$ *None* **using** *cp* $cdcl_W$*-cp.intros* **by** *auto*
then have $cdcl_W$*-cp*$^{++}$ *S* S' **by** *blast*
moreover have *no-step* $cdcl_W$*-cp* S'
using ‹*conflicting* $S' \neq$ *None*› **by** (*metis* $cdcl_W$*-cp-conflicting-not-empty option.exhaust*)
ultimately show *full1* $cdcl_W$*-cp S* S' **unfolding** *full1-def* **by** *blast+*
qed

lemma $cdcl_W$*-cp-fst-empty-conflicting-false*:
assumes $cdcl_W$*-cp S* S'
and *trail S* = []
and *conflicting S* ≠ *None*
shows *False*
using *assms* **by** (*induct rule*: $cdcl_W$*-cp.induct*) *auto*

lemma $cdcl_W$*-o-fst-empty-conflicting-false*:
assumes $cdcl_W$*-o S* S'
and *trail S* = []
and *conflicting S* ≠ *None*
shows *False*
using *assms* **by** (*induct rule*: $cdcl_W$*-o-induct*) *auto*

lemma $cdcl_W$*-stgy-fst-empty-conflicting-false*:
assumes $cdcl_W$*-stgy S* S'
and *trail S* = []
and *conflicting S* ≠ *None*
shows *False*
using *assms* **apply** (*induct rule*: $cdcl_W$*-stgy.induct*)
using *tranclpD* $cdcl_W$*-cp-fst-empty-conflicting-false* **unfolding** *full1-def* **apply** *metis*
using $cdcl_W$*-o-fst-empty-conflicting-false* **by** *blast*
thm $cdcl_W$*-cp.induct*[*split-format*(*complete*)]

lemma $cdcl_W$*-cp-conflicting-is-false*:

$cdcl_W$*-cp S S'* $\Longrightarrow$ *conflicting S* = *Some* {#} $\Longrightarrow$ *False*
by (*induction rule*: $cdcl_W$*-cp.induct*) *auto*

lemma *rtranclp-*$cdcl_W$*-cp-conflicting-is-false*:
$cdcl_W$*-cp*$^{++}$ *S S'* $\Longrightarrow$ *conflicting S* = *Some* {#} $\Longrightarrow$ *False*
apply (*induction rule*: *tranclp.induct*)
by (*auto dest*: $cdcl_W$*-cp-conflicting-is-false*)

lemma $cdcl_W$*-o-conflicting-is-false*:
$cdcl_W$*-o S S'* $\Longrightarrow$ *conflicting S* = *Some* {#} $\Longrightarrow$ *False*
by (*induction rule*: $cdcl_W$*-o-induct*) *auto*

lemma $cdcl_W$*-stgy-conflicting-is-false*:
$cdcl_W$*-stgy S S'* $\Longrightarrow$ *conflicting S* = *Some* {#} $\Longrightarrow$ *False*
apply (*induction rule*: $cdcl_W$*-stgy.induct*)
unfolding *full1-def* **apply** (*metis* (*no-types*) $cdcl_W$*-cp-conflicting-not-empty tranclpD*)
unfolding *full-def* **by** (*metis conflict-with-false-implies-terminated other*)

lemma *rtranclp-*$cdcl_W$*-stgy-conflicting-is-false*:
$cdcl_W$*-stgy*** *S S'* $\Longrightarrow$ *conflicting S* = *Some* {#} $\Longrightarrow$ *S'* = *S*
apply (*induction rule*: *rtranclp-induct*)
apply *simp*
using $cdcl_W$*-stgy-conflicting-is-false* **by** *blast*

lemma *full-*$cdcl_W$*-init-clss-with-false-normal-form*:
assumes
∀ *m*∈ *set M*. ¬*is-marked m* **and**
E = *Some D* **and**
state S = (*M*, *N*, *U*, *0*, *E*)
full $cdcl_W$*-stgy S S'* **and**
all-decomposition-implies-m (*init-clss S*) (*get-all-marked-decomposition* (*trail S*))
$cdcl_W$*-learned-clause S*
$cdcl_W$*-M-level-inv S*
no-strange-atm S
distinct-$cdcl_W$*-state S*
$cdcl_W$*-conflicting S*
shows ∃ *M''*. *state S'* = (*M''*, *N*, *U*, *0*, *Some* {#})
using *assms*(*10*,*9*,*8*,*7*,*6*,*5*,*4*,*3*,*2*,*1*)
proof (*induction M arbitrary*: *E D S*)
case *Nil*
then show *?case*
using *rtranclp-*$cdcl_W$*-stgy-conflicting-is-false* **unfolding** *full-def* $cdcl_W$*-conflicting-def* **by** *auto*
next
case (*Cons L M*) **note** *IH* = *this*(*1*) **and** *full* = *this*(*8*) **and** *E* = *this*(*10*) **and** *inv* = *this*(*2*−*7*) **and** *S* = *this*(*9*) **and** *nm* = *this*(*11*)
obtain *K p* **where** *K*: *L* = *Propagated K p*
using *nm* **by** (*cases L*) *auto*
have *every-mark-is-a-conflict S* **using** *inv* **unfolding** $cdcl_W$*-conflicting-def* **by** *auto*
then have *MpK*: *M* ⊨*as CNot* (*p* − {#*K*#}) **and** *Kp*: *K* ∈# *p*
using *S* **unfolding** *K* **by** *fastforce+*
then have *p*: *p* = (*p* − {#*K*#}) + {#*K*#}
by (*auto simp add*: *multiset-eq-iff*)
then have *K'*: *L* = *Propagated K* (((*p* − {#*K*#}) + {#*K*#}))
using *K* **by** *auto*

consider (D) $D = \{\#\}$ | (D') $D \neq \{\#\}$ **by** *blast*
then show *?case*
  proof *cases*
    case D
    then show *?thesis*
      using *full rtranclp-cdcl$_W$-stgy-conflicting-is-false S* **unfolding** *full-def E D* **by** *auto*
  next
    case D'
    then have *no-p*: *no-step propagate S* **and** *no-c*: *no-step conflict S*
      using *S E* **by** *auto*
    then have *no-step cdcl$_W$-cp S* **by** (*auto simp*: *cdcl$_W$-cp.simps*)
    have *res-skip*: $\exists$ *T*. (*resolve S T* $\wedge$ *no-step skip S* $\wedge$ *full cdcl$_W$-cp T T*)
      $\vee$ (*skip S T* $\wedge$ *no-step resolve S* $\wedge$ *full cdcl$_W$-cp T T*)
      proof *cases*
        assume $-$*lit-of L* $\notin\#$ *D*
        then obtain *T* **where** *sk*: *skip S T* **and** *res*: *no-step resolve S*
        using *S that D' K* **unfolding** *skip.simps E* **by** *fastforce*
        have *full cdcl$_W$-cp T T*
          using *sk* **by** (*auto simp add*: *option-full-cdcl$_W$-cp*)
        then show *?thesis*
          using *sk res* **by** *blast*
      next
        assume *LD*: $\neg-$*lit-of L* $\notin\#$ *D*
        then have *D*: *Some D = Some* ((*D* $-$ $\{\#-$*lit-of L*$\#\}$) + $\{\#-$*lit-of L*$\#\}$)
          by (*auto simp add*: *multiset-eq-iff*)

        have $\bigwedge$*L*. *get-level M L = 0*
          by (*simp add*: *nm*)
          then have *get-maximum-level* (*Propagated K* (*p* $-$ $\{\#K\#\}$ + $\{\#K\#\}$) # *M*) (*D* $-$ $\{\#-$
K$\#\}$) = *0*
          using *LD get-maximum-level-exists-lit-of-max-level*
          proof $-$
            obtain *L'* **where** *get-level* (*L#M*) *L'* = *get-maximum-level* (*L#M*) *D*
              using *LD get-maximum-level-exists-lit-of-max-level*[*of D L#M*] **by** *fastforce*
            then show *?thesis* **by** (*metis* (*mono-tags*) *K' bex-msetE get-level-skip-all-not-marked*
              get-maximum-level-exists-lit nm not-gr0)
          qed
        then obtain *T* **where** *sk*: *resolve S T* **and** *res*: *no-step skip S*
          using *resolve-rule*[*of S K p* $-$ $\{\#K\#\}$ *M N U 0* (*D* $-$ $\{\#-K\#\}$)
          update-conflicting (*Some* (*remdups-mset* (*D* $-$ $\{\#-$ $K\#\}$ + (*p* $-$ $\{\#K\#\}$)))) (*tl-trail S*)]
          S **unfolding** *K' D E* **by** *fastforce*
        have *full cdcl$_W$-cp T T*
          using *sk* **by** (*auto simp add*: *option-full-cdcl$_W$-cp*)
        then show *?thesis*
         using *sk res* **by** *blast*
      qed
    then have *step-s*: $\exists$ *T*. *cdcl$_W$-stgy S T*
      using ‹*no-step cdcl$_W$-cp S*› *other'* **by** (*meson bj resolve skip*)
    have *get-all-marked-decomposition* (*L* # *M*) = [([], *L#M*)]
      using *nm* **unfolding** *K* **apply** (*induction M rule*: *marked-lit-list-induct*, *simp*)
        by (*case-tac hd* (*get-all-marked-decomposition xs*), *auto*)+
    then have *no-b*: *no-step backtrack S*
      using *nm S* **by** *auto*
    have *no-d*: *no-step decide S*

using *S E* **by** *auto*

have *full-S-S*: *full* $cdcl_W$*-cp S S*
using *S E* **by** (*auto simp add*: *option-full-*$cdcl_W$*-cp*)
then have *no-f*: *no-step* (*full1* $cdcl_W$*-cp*) *S*
unfolding *full-def full1-def rtranclp-unfold* **by** (*meson tranclpD*)
obtain *T* **where**
s: $cdcl_W$*-stgy S T* **and** *st*: $cdcl_W$*-stgy*** *T S'*
using *full step-s full* **unfolding** *full-def* **by** (*metis rtranclp-unfold tranclpD*)
have *resolve S T* ∨ *skip S T*
using *s no-b no-d res-skip full-S-S* **unfolding** $cdcl_W$*-stgy.simps* $cdcl_W$*-o.simps full-unfold full1-def*
by (*auto dest*!: *tranclpD simp*: $cdcl_W$*-bj.simps*)
then obtain *D'* **where** *T*: *state T* = (*M*, *N*, *U*, *0*, *Some D'*)
using *S E* **by** *auto*

have *st-c*: $cdcl_W$** *S T*
using *E T rtranclp-*$cdcl_W$*-stgy-rtranclp-*$cdcl_W$ *s* **by** *blast*
have $cdcl_W$*-conflicting T*
using *rtranclp-*$cdcl_W$*-all-inv*(*6*)[*OF st-c inv*(*6*,*5*,*4*,*3*,*2*,*1*)] **.**
show *?thesis*
apply (*rule IH*[*of T*])
using *rtranclp-*$cdcl_W$*-all-inv*(*6*)[*OF st-c inv*(*6*,*5*,*4*,*3*,*2*,*1*)] **apply** *blast*
using *rtranclp-*$cdcl_W$*-all-inv*(*5*)[*OF st-c inv*(*6*,*5*,*4*,*3*,*2*,*1*)] **apply** *blast*
using *rtranclp-*$cdcl_W$*-all-inv*(*4*)[*OF st-c inv*(*6*,*5*,*4*,*3*,*2*,*1*)] **apply** *blast*
using *rtranclp-*$cdcl_W$*-all-inv*(*3*)[*OF st-c inv*(*6*,*5*,*4*,*3*,*2*,*1*)] **apply** *blast*
using *rtranclp-*$cdcl_W$*-all-inv*(*2*)[*OF st-c inv*(*6*,*5*,*4*,*3*,*2*,*1*)] **apply** *blast*
using *rtranclp-*$cdcl_W$*-all-inv*(*1*)[*OF st-c inv*(*6*,*5*,*4*,*3*,*2*,*1*)] **apply** *blast*
apply (*metis full-def st full*)
using *T E* **apply** *blast*
apply *auto*[]
using *nm* **by** *simp*
qed
qed

lemma *full-*$cdcl_W$*-stgy-final-state-conclusive-is-one-false*:
fixes *S'* :: *'st*
assumes *full*: *full* $cdcl_W$*-stgy* (*init-state N*) *S'*
and *no-d*: *distinct-mset-mset N*
and *empty*: {#} ∈# *N*
shows *conflicting S'* = *Some* {#} ∧ *unsatisfiable* (*set-mset* (*init-clss S'*))
proof −
let *?S* = *init-state N*
have $cdcl_W$*-stgy*** *?S S'* **and** *no-step* $cdcl_W$*-stgy S'* **using** *full* **unfolding** *full-def* **by** *auto*
then have *plus-or-eq*: $cdcl_W$*-stgy*$^{++}$ *?S S'* ∨ *S'* = *?S* **unfolding** *rtranclp-unfold* **by** *auto*
have ∃ *S''*. *conflict ?S S''* **using** *empty not-conflict-not-any-negated-init-clss* **by** *force*

then have $cdcl_W$*-stgy*: ∃ *S'*. $cdcl_W$*-stgy ?S S'*
using $cdcl_W$*-cp.conflict'*[*of ?S*] *conflict-is-full1-*$cdcl_W$*-cp* $cdcl_W$*-stgy.intros*(*1*) **by** *metis*
have *S'* ≠ *?S* **using** ‹*no-step* $cdcl_W$*-stgy S'*› $cdcl_W$*-stgy* **by** *blast*

then obtain *St*:: *'st* **where** *St*: $cdcl_W$*-stgy ?S St* **and** $cdcl_W$*-stgy*** *St S'*
using *plus-or-eq* **by** (*metis* (*no-types*) ‹$cdcl_W$*-stgy*** *?S S'*› *converse-rtranclpE*)
have *st*: $cdcl_W$** *?S St*
by (*simp add*: *rtranclp-unfold* ‹$cdcl_W$*-stgy ?S St*› $cdcl_W$*-stgy-tranclp-*$cdcl_W$)

have ∃ *T*. *conflict ?S T*
using *empty not-conflict-not-any-negated-init-clss* **by** *force*
then have *fullSt*: *full1* $cdcl_W$*-cp ?S St*
using *St* **unfolding** $cdcl_W$*-stgy.simps* **by** *blast*
then have *bt*: *backtrack-lvl St* = (*0*::*nat*)
using *rtranclp-*$cdcl_W$*-cp-backtrack-lvl* **unfolding** *full1-def*
by (*fastforce dest*!: *tranclp-into-rtranclp*)
have *cls-St*: *init-clss St* = *N*
using *fullSt* $cdcl_W$*-stgy-no-more-init-clss*[*OF St*] **by** *auto*
have *conflicting St* ≠ *None*
proof (*rule ccontr*)
assume ¬ *?thesis*
then have ∃ *T*. *conflict St T*
using *empty cls-St*[] *conflict-rule*[*of St trail St N learned-clss St backtrack-lvl St* {#}]
by (*auto simp*: *clauses-def*)
then show *False* **using** *fullSt* **unfolding** *full1-def* **by** *blast*
qed

have *1*: ∀ *m*∈*set* (*trail St*). ¬ *is-marked m*
using *fullSt* **unfolding** *full1-def* **by** (*auto dest*!: *tranclp-into-rtranclp*
rtranclp-$cdcl_W$*-cp-dropWhile-trail*)
have *2*: *full* $cdcl_W$*-stgy St S'*
using ‹$cdcl_W$*-stgy*** *St S'*› ‹*no-step* $cdcl_W$*-stgy S'*› *bt* **unfolding** *full-def* **by** *auto*
have *3*: *all-decomposition-implies-m*
(*init-clss St*)
(*get-all-marked-decomposition*
(*trail St*))
using *rtranclp-*$cdcl_W$*-all-inv*(*1*)[*OF st*] *no-d bt* **by** *simp*
have *4*: $cdcl_W$*-learned-clause St*
using *rtranclp-*$cdcl_W$*-all-inv*(*2*)[*OF st*] *no-d bt bt* **by** *simp*
have *5*: $cdcl_W$*-M-level-inv St*
using *rtranclp-*$cdcl_W$*-all-inv*(*3*)[*OF st*] *no-d bt* **by** *simp*
have *6*: *no-strange-atm St*
using *rtranclp-*$cdcl_W$*-all-inv*(*4*)[*OF st*] *no-d bt* **by** *simp*
have *7*: *distinct-*$cdcl_W$*-state St*
using *rtranclp-*$cdcl_W$*-all-inv*(*5*)[*OF st*] *no-d bt* **by** *simp*
have *8*: $cdcl_W$*-conflicting St*
using *rtranclp-*$cdcl_W$*-all-inv*(*6*)[*OF st*] *no-d bt* **by** *simp*
have *init-clss S'* = *init-clss St* **and** *conflicting S'* = *Some* {#}
using ‹*conflicting St* ≠ *None*› *full-*$cdcl_W$*-init-clss-with-false-normal-form*[*OF 1*, *of - - St*]
2 3 4 5 6 7 8 St **apply** (*metis* ‹$cdcl_W$*-stgy*** *St S'*› *rtranclp-*$cdcl_W$*-stgy-no-more-init-clss*)
using ‹*conflicting St* ≠ *None*› *full-*$cdcl_W$*-init-clss-with-false-normal-form*[*OF 1*, *of - - St - - S'*] *2 3 4 5 6 7 8* **by** (*metis bt option.exhaust prod.inject*)

moreover have *init-clss S'* = *N*
using ‹$cdcl_W$*-stgy*** (*init-state N*) *S'*› *rtranclp-*$cdcl_W$*-stgy-no-more-init-clss* **by** *fastforce*
moreover have *unsatisfiable* (*set-mset N*)
by (*meson empty mem-set-mset-iff satisfiable-def true-cls-empty true-clss-def*)
ultimately show *?thesis* **by** *auto*
qed

lemma *full-*$cdcl_W$*-stgy-final-state-conclusive*:

fixes S' :: $'st$
assumes *full*: *full cdcl*$_W$*-stgy* (*init-state N*) S' **and** *no-d*: *distinct-mset-mset N*
shows (*conflicting* S' = *Some* {#} ∧ *unsatisfiable* (*set-mset* (*init-clss* S')))
∨ (*conflicting* S' = *None* ∧ *trail* S' ⊨*asm init-clss* S')
using *assms full-cdcl*$_W$*-stgy-final-state-conclusive-is-one-false*
full-cdcl$_W$*-stgy-final-state-conclusive-non-false* **by** *blast*

lemma *full-cdcl*$_W$*-stgy-final-state-conclusive-from-init-state*:
fixes S' :: $'st$
assumes *full*: *full cdcl*$_W$*-stgy* (*init-state N*) S'
and *no-d*: *distinct-mset-mset N*
shows (*conflicting* S' = *Some* {#} ∧ *unsatisfiable* (*set-mset N*))
∨ (*conflicting* S' = *None* ∧ *trail* S' ⊨*asm N* ∧ *satisfiable* (*set-mset N*))
proof −
have *N*: *init-clss* S' = *N*
using *full* **unfolding** *full-def* **by** (*auto dest*: *rtranclp-cdcl*$_W$*-stgy-no-more-init-clss*)
consider
(*confl*) *conflicting* S' = *Some* {#} **and** *unsatisfiable* (*set-mset* (*init-clss* S'))
| (*sat*) *conflicting* S' = *None* **and** *trail* S' ⊨*asm init-clss* S'
using *full-cdcl*$_W$*-stgy-final-state-conclusive*[*OF assms*] **by** *auto*
then show *?thesis*
proof *cases*
case *confl*
then show *?thesis* **by** (*auto simp*: *N*)
next
case *sat*
have *cdcl*$_W$*-M-level-inv* (*init-state N*) **by** *auto*
then have *cdcl*$_W$*-M-level-inv* S'
using *full rtranclp-cdcl*$_W$*-stgy-consistent-inv* **unfolding** *full-def* **by** *blast*
then have *consistent-interp* (*lits-of* (*trail* S')) **unfolding** *cdcl*$_W$*-M-level-inv-def* **by** *blast*
moreover have *lits-of* (*trail* S') ⊨*s set-mset* (*init-clss* S')
using *sat*(*2*) **by** (*auto simp add*: *true-annots-def true-annot-def true-clss-def*)
ultimately have *satisfiable* (*set-mset* (*init-clss* S')) **by** *simp*
then show *?thesis* **using** *sat* **unfolding** *N* **by** *blast*
qed
qed
end
end
theory *CDCL-W-Termination*
imports *CDCL-W*
begin

context *cdcl*$_W$*-ops*
begin

17.7 Termination

The condition that no learned clause is a tautology is overkill (in the sense that the no-duplicate condition is enough), but we can reuse *build-all-simple-clss*.

The invariant contains all the structural invariants that holds,

definition *cdcl*$_W$*-all-struct-inv* **where**
cdcl$_W$*-all-struct-inv S* =
(*no-strange-atm S* ∧ *cdcl*$_W$*-M-level-inv S*
∧ (∀ *s* ∈# *learned-clss S*. ¬*tautology s*)
∧ *distinct-cdcl*$_W$*-state S* ∧ *cdcl*$_W$*-conflicting S*

$\wedge$ *all-decomposition-implies-m* (*init-clss S*) (*get-all-marked-decomposition* (*trail S*))
$\wedge$ $cdcl_W$*-learned-clause S*)

lemma $cdcl_W$*-all-struct-inv-inv*:
assumes $cdcl_W$ *S S'* **and** $cdcl_W$*-all-struct-inv S*
shows $cdcl_W$*-all-struct-inv S'*
unfolding $cdcl_W$*-all-struct-inv-def*
proof (*intro HOL.conjI*)
show *no-strange-atm S'*
using $cdcl_W$*-all-inv*[*OF assms*(*1*)] *assms*(*2*) **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *auto*
show $cdcl_W$*-M-level-inv S'*
using $cdcl_W$*-all-inv*[*OF assms*(*1*)] *assms*(*2*) **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *fast*
show *distinct-*$cdcl_W$*-state S'*
using $cdcl_W$*-all-inv*[*OF assms*(*1*)] *assms*(*2*) **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *fast*
show $cdcl_W$*-conflicting S'*
using $cdcl_W$*-all-inv*[*OF assms*(*1*)] *assms*(*2*) **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *fast*
show *all-decomposition-implies-m* (*init-clss S'*) (*get-all-marked-decomposition* (*trail S'*))
using $cdcl_W$*-all-inv*[*OF assms*(*1*)] *assms*(*2*) **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *fast*
show $cdcl_W$*-learned-clause S'*
using $cdcl_W$*-all-inv*[*OF assms*(*1*)] *assms*(*2*) **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *fast*

show $\forall$ *s*$\in$#*learned-clss S'*. $\neg$ *tautology s*
using *assms*(*1*)[*THEN learned-clss-are-not-tautologies*] *assms*(*2*)
unfolding $cdcl_W$*-all-struct-inv-def* **by** *fast*
qed

lemma *rtranclp-*$cdcl_W$*-all-struct-inv-inv*:
assumes $cdcl_W^{**}$ *S S'* **and** $cdcl_W$*-all-struct-inv S*
shows $cdcl_W$*-all-struct-inv S'*
using *assms* **by** *induction* (*auto intro*: $cdcl_W$*-all-struct-inv-inv*)

lemma $cdcl_W$*-stgy-*$cdcl_W$*-all-struct-inv*:
$cdcl_W$*-stgy S T* $\Longrightarrow$ $cdcl_W$*-all-struct-inv S* $\Longrightarrow$ $cdcl_W$*-all-struct-inv T*
by (*meson* $cdcl_W$*-stgy-tranclp-*$cdcl_W$ *rtranclp-*$cdcl_W$*-all-struct-inv-inv rtranclp-unfold*)

lemma *rtranclp-*$cdcl_W$*-stgy-*$cdcl_W$*-all-struct-inv*:
$cdcl_W$*-stgy*** *S T* $\Longrightarrow$ $cdcl_W$*-all-struct-inv S* $\Longrightarrow$ $cdcl_W$*-all-struct-inv T*
by (*induction rule*: *rtranclp-induct*) (*auto intro*: $cdcl_W$*-stgy-*$cdcl_W$*-all-struct-inv*)

17.8 No Relearning of a clause

lemma $cdcl_W$*-o-new-clause-learned-is-backtrack-step*:
assumes *learned*: *D* $\in$# *learned-clss T* **and**
new: *D* $\notin$# *learned-clss S* **and**
$cdcl_W$: $cdcl_W$*-o S T* **and**
lev: $cdcl_W$*-M-level-inv S*
shows *backtrack S T* $\wedge$ *conflicting S* = *Some D*
using $cdcl_W$ *lev learned new*
proof (*induction rule*: $cdcl_W$*-o-induct-lev2*)
case (*backtrack K i M1 M2 L C T*) **note** *decomp* =*this*(*1*) **and** *undef* = *this*(*6*) **and** *T* = *this*(*7*)
and
D-T = *this*(*9*) **and** *D-S* = *this*(*10*)
then have *D* = *C* + {#*L*#}
using *not-gr0 lev* **by** (*auto simp*: $cdcl_W$*-M-level-inv-decomp*)
then show *?case*
using *T backtrack.hyps*(*1*−*5*) *backtrack.intros* **by** *auto*

qed *auto*

lemma $cdcl_W$*-cp-new-clause-learned-has-backtrack-step*:
assumes *learned*: $D \in\# $ *learned-clss* T **and**
new: $D \notin\# $ *learned-clss* S **and**
$cdcl_W$: $cdcl_W$*-stgy* S T **and**
lev: $cdcl_W$*-M-level-inv* S
shows $\exists S'$. *backtrack* S S' $\wedge$ $cdcl_W$*-stgy*** S' T $\wedge$ *conflicting* S = *Some* D
using $cdcl_W$ *learned new*
proof (*induction rule*: $cdcl_W$*-stgy.induct*)
case (*conflict'* S')
then show *?case*
unfolding *full1-def* **by** (*metis* (*mono-tags*, *lifting*) *rtranclp-*$cdcl_W$*-cp-learned-clause-inv* *tranclp-into-rtranclp*)
next
case (*other'* S' S'')
then have $D \in\# $ *learned-clss* S'
unfolding *full-def* **by** (*auto dest*: *rtranclp-*$cdcl_W$*-cp-learned-clause-inv*)
then show *?case*
using $cdcl_W$*-o-new-clause-learned-is-backtrack-step*[*OF* - ‹$D \notin\# $ *learned-clss* S› ‹$cdcl_W$*-o* S S'›]
‹*full* $cdcl_W$*-cp* S' S''› *lev* **by** (*metis* $cdcl_W$*-stgy.conflict'* *full-unfold r-into-rtranclp* *rtranclp.rtrancl-refl*)
qed

lemma *rtranclp-*$cdcl_W$*-cp-new-clause-learned-has-backtrack-step*:
assumes *learned*: $D \in\# $ *learned-clss* T **and**
new: $D \notin\# $ *learned-clss* S **and**
$cdcl_W$: $cdcl_W$*-stgy*** S T **and**
lev: $cdcl_W$*-M-level-inv* S
shows $\exists S'$ S''. $cdcl_W$*-stgy*** S S' $\wedge$ *backtrack* S' S'' $\wedge$ *conflicting* S' = *Some* D $\wedge$
$cdcl_W$*-stgy*** S'' T
using $cdcl_W$ *learned new*
proof (*induction rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *blast*
next
case (*step* T U) **note** *st* =*this*(*1*) **and** *o* = *this*(*2*) **and** *IH* = *this*(*3*) **and**
D-U = *this*(*4*) **and** *D-S* = *this*(*5*)
show *?case*
proof (*cases* $D \in\# $ *learned-clss* T)
case *True*
then obtain S' S'' **where**
st': $cdcl_W$*-stgy*** S S' **and**
bt: *backtrack* S' S'' **and**
confl: *conflicting* S' = *Some* D **and**
st'': $cdcl_W$*-stgy*** S'' T
using *IH D-S* **by** *metis*
then show *?thesis* **using** *o* **by** (*meson rtranclp.simps*)
next
case *False*
have $cdcl_W$*-M-level-inv* T
using *lev rtranclp-*$cdcl_W$*-stgy-consistent-inv st* **by** *blast*
then obtain S' **where**
bt: *backtrack* T S' **and**
st': $cdcl_W$*-stgy*** S' U **and**

confl: *conflicting* T = *Some* D
using $cdcl_W$*-cp-new-clause-learned-has-backtrack-step*[*OF D-U False o*]
by *metis*
then have $cdcl_W$*-stgy*** S T **and**
backtrack T S' **and**
conflicting T = *Some* D **and**
$cdcl_W$*-stgy*** S' U
using *o st* **by** *auto*
then show *?thesis* **by** *blast*
qed
qed

lemma *propagate-no-more-Marked-lit*:
assumes *propagate* S S'
shows *Marked* K i ∈ *set* (*trail* S) ⟷ *Marked* K i ∈ *set* (*trail* S')
using *assms* **by** *auto*

lemma *conflict-no-more-Marked-lit*:
assumes *conflict* S S'
shows *Marked* K i ∈ *set* (*trail* S) ⟷ *Marked* K i ∈ *set* (*trail* S')
using *assms* **by** *auto*

lemma $cdcl_W$*-cp-no-more-Marked-lit*:
assumes $cdcl_W$*-cp* S S'
shows *Marked* K i ∈ *set* (*trail* S) ⟷ *Marked* K i ∈ *set* (*trail* S')
using *assms* **apply** (*induct rule*: $cdcl_W$*-cp.induct*)
using *conflict-no-more-Marked-lit propagate-no-more-Marked-lit* **by** *auto*

lemma *rtranclp-*$cdcl_W$*-cp-no-more-Marked-lit*:
assumes $cdcl_W$*-cp*** S S'
shows *Marked* K i ∈ *set* (*trail* S) ⟷ *Marked* K i ∈ *set* (*trail* S')
using *assms* **apply** (*induct rule*: *rtranclp-induct*)
using $cdcl_W$*-cp-no-more-Marked-lit* **by** *blast+*

lemma $cdcl_W$*-o-no-more-Marked-lit*:
assumes $cdcl_W$*-o* S S' **and** $cdcl_W$*-M-level-inv* S **and** ¬*decide* S S'
shows *Marked* K i ∈ *set* (*trail* S') ⟶ *Marked* K i ∈ *set* (*trail* S)
using *assms*
proof (*induct rule*: $cdcl_W$*-o-induct-lev2*)
case *backtrack* **note** *decomp* = *this*(*1*) **and** *undef* = *this*(*6*) **and** T =*this*(*7*) **and** *lev* = *this*(*8*)
then show *?case*
by (*auto simp*: $cdcl_W$*-M-level-inv-decomp*)
next
case (*decide* L T)
then show *?case* **by** *blast*
qed *auto*

lemma $cdcl_W$*-new-marked-at-beginning-is-decide*:
assumes $cdcl_W$*-stgy* S S' **and**
lev: $cdcl_W$*-M-level-inv* S **and**
trail S' = M' @ *Marked* L i # M **and**
trail S = M
shows ∃ T. *decide* S T ∧ *no-step* $cdcl_W$*-cp* S
using *assms*
proof (*induct rule*: $cdcl_W$*-stgy.induct*)

case (*conflict′ S′*) **note** *st* =*this*(*1*) **and** *no-dup* = *this*(*2*) **and** *S′* = *this*(*3*) **and** *S* = *this*(*4*)
have $cdcl_W$*-M-level-inv S′*
using *full1-*$cdcl_W$*-cp-consistent-inv no-dup st* **by** *blast*
then have *Marked L i* ∈ *set* (*trail S′*) **and** *Marked L i* ∉ *set* (*trail S*)
using *no-dup* **unfolding** *S S′* $cdcl_W$*-M-level-inv-def* **by** (*auto simp add*: *rev-image-eqI*)
then have *False*
using *st rtranclp-*$cdcl_W$*-cp-no-more-Marked-lit*[*of S S′*]
unfolding *full1-def rtranclp-unfold* **by** *blast*
then show *?case* **by** *fast*
next
case (*other′ T U*) **note** *o* = *this*(*1*) **and** *ns* = *this*(*2*) **and** *st* = *this*(*3*) **and** *no-dup* = *this*(*4*) **and** *S′* = *this*(*5*) **and** *S* = *this*(*6*)
have $cdcl_W$*-M-level-inv U*
by (*metis* (*full-types*) *lev* $cdcl_W$*.simps* $cdcl_W$*-consistent-inv full-def o other′.hyps*(*3*) *rtranclp-*$cdcl_W$*-cp-consistent-inv*)
then have *Marked L i* ∈ *set* (*trail U*) **and** *Marked L i* ∉ *set* (*trail S*)
using *no-dup* **unfolding** *S S′* $cdcl_W$*-M-level-inv-def* **by** (*auto simp add*: *rev-image-eqI*)
then have *Marked L i* ∈ *set* (*trail T*)
using *st rtranclp-*$cdcl_W$*-cp-no-more-Marked-lit* **unfolding** *full-def* **by** *blast*
then show *?case*
using $cdcl_W$*-o-no-more-Marked-lit*[*OF o*] ‹*Marked L i* ∉ *set* (*trail S*)› *ns lev* **by** *meson*
qed

lemma $cdcl_W$*-o-is-decide*:
assumes $cdcl_W$*-o S′ T* **and** $cdcl_W$*-M-level-inv S′*
trail T = *drop* (*length* M_0) *M′* @ *Marked L i* # *H* @ *M***and**
¬ (∃ *M′*. *trail S′* = *M′* @ *Marked L i* # *H* @ *M*)
shows *decide S′ T*
using *assms*
proof (*induction rule*:$cdcl_W$*-o-induct-lev2*)
case (*backtrack K i M1 M2 L D*)
then obtain *c* **where** *trail S′* = *c* @ *M2* @ *Marked K* (*Suc i*) # *M1*
by *auto*
then show *?case*
using *backtrack* **by** (*cases drop* (*length* M_0) *M′*) (*auto simp*: $cdcl_W$*-M-level-inv-def*)
next
case *decide*
show *?case* **using** *decide-rule*[*of S′*] *decide*(*1*−*4*) **by** *auto*
qed *auto*

lemma *rtranclp-*$cdcl_W$*-new-marked-at-beginning-is-decide*:
assumes $cdcl_W$*-stgy*** *R U* **and**
trail U = *M′* @ *Marked L i* # *H* @ *M* **and**
trail R = *M* **and**
$cdcl_W$*-M-level-inv R*
shows
∃ *S T T′*. $cdcl_W$*-stgy*** *R S* ∧ *decide S T* ∧ $cdcl_W$*-stgy*** *T U* ∧ $cdcl_W$*-stgy*** *S U* ∧ *no-step* $cdcl_W$*-cp S* ∧ *trail T* = *Marked L i* # *H* @ *M* ∧ *trail S* = *H* @ *M* ∧ $cdcl_W$*-stgy S T′* ∧ $cdcl_W$*-stgy*** *T′ U*
using *assms*
proof (*induct arbitrary*: *M H M′ i rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *auto*
next
case (*step T U*) **note** *st* = *this*(*1*) **and** *IH* = *this*(*3*) **and** *s* = *this*(*2*) **and**

$U = this(4)$ **and** $S = this(5)$ **and** $lev = this(6)$
show *?case*
proof (*cases* $\exists M'$. *trail* $T = M'$ @ *Marked* L i # H @ M)
case *False*
with *s* **show** *?thesis* **using** U *s st* S
proof *induction*
case (*conflict'* W) **note** $cp = this(1)$ **and** $nd = this(2)$ **and** $W = this(3)$
then obtain M_0 **where** *trail* $W = M_0$ @ *trail* T **and** *nmarked*: $\forall l \in set\ M_0$. ¬ *is-marked* l
using *rtranclp-cdcl$_W$-cp-dropWhile-trail* **unfolding** *full1-def rtranclp-unfold* **by** *meson*
then have MV: M' @ *Marked* L i # H @ $M = M_0$ @ *trail* T **unfolding** W **by** *simp*
then have V: *trail* T = *drop* (*length* M_0) (M' @ *Marked* L i # H @ M)
by *auto*
have *takeWhile* (*Not o is-marked*) $M' = M_0$ @ *takeWhile* (*Not* ∘ *is-marked*) (*trail* T)
using *arg-cong*[*OF MV*, *of takeWhile* (*Not o is-marked*)] *nmarked*
by (*simp add*: *takeWhile-tail*)
from *arg-cong*[*OF this*, *of length*] **have** *length* $M_0 \leq$ *length* M'
unfolding *length-append* **by** (*metis* (*no-types*, *lifting*) *Nat.le-trans le-add1*
length-takeWhile-le)
then have *False* **using** *nd* V **by** *auto*
then show *?case* **by** *fast*
next
case (*other'* T' U) **note** $o = this(1)$ **and** $ns = this(2)$ **and** $cp = this(3)$ **and** $nd = this(4)$
and $U = this(5)$ **and** $st = this(6)$
obtain M_0 **where** *trail* $U = M_0$ @ *trail* T' **and** *nmarked*: $\forall l \in set\ M_0$. ¬ *is-marked* l
using *rtranclp-cdcl$_W$-cp-dropWhile-trail cp* **unfolding** *full-def* **by** *meson*
then have MV: M' @ *Marked* L i # H @ $M = M_0$ @ *trail* T' **unfolding** U **by** *simp*
then have V: *trail* T' = *drop* (*length* M_0) (M' @ *Marked* L i # H @ M)
by *auto*
have *takeWhile* (*Not o is-marked*) $M' = M_0$ @ *takeWhile* (*Not* ∘ *is-marked*) (*trail* T')
using *arg-cong*[*OF MV*, *of takeWhile* (*Not o is-marked*)] *nmarked*
by (*simp add*: *takeWhile-tail*)
from *arg-cong*[*OF this*, *of length*] **have** *length* $M_0 \leq$ *length* M'
unfolding *length-append* **by** (*metis* (*no-types*, *lifting*) *Nat.le-trans le-add1*
length-takeWhile-le)
then have *tr-T'*: *trail* T' = *drop* (*length* M_0) M' @ *Marked* L i # H @ M **using** V **by** *auto*
then have *LT'*: *Marked* L $i \in$ *set* (*trail* T') **by** *auto*
moreover
have *cdcl$_W$-M-level-inv* T
using *lev rtranclp-cdcl$_W$-stgy-consistent-inv step.hyps*(*1*) **by** *blast*
then have *decide* T T' **using** *o nd tr-T' cdcl$_W$-o-is-decide* **by** *metis*
ultimately have *decide* T T' **using** *cdcl$_W$-o-no-more-Marked-lit*[*OF o*] **by** *blast*
then have *1*: *cdcl$_W$-stgy*** R T **and** *2*: *decide* T T' **and** *3*: *cdcl$_W$-stgy*** T' U
using *st other'.prems*(*4*)
by (*metis cdcl$_W$-stgy.conflict' cp full-unfold r-into-rtranclp rtranclp.rtrancl-refl*)+
have [*simp*]: *drop* (*length* M_0) M' = []
using ‹*decide* T T'› ‹*Marked* L $i \in$ *set* (*trail* T')› *nd tr-T'*
by (*auto simp add*: *Cons-eq-append-conv*)
have T': *drop* (*length* M_0) M' @ *Marked* L i # H @ M = *Marked* L i # *trail* T
using ‹*decide* T T'› ‹*Marked* L $i \in$ *set* (*trail* T')› *nd tr-T'*
by *auto*
have *trail* T' = *Marked* L i # *trail* T
using ‹*decide* T T'› ‹*Marked* L $i \in$ *set* (*trail* T')› *tr-T'*
by *auto*
then have *5*: *trail* T' = *Marked* L i # H @ M
using *append.simps*(*1*) *list.sel*(*3*) *local.other'*(*5*) *tl-append2* **by** (*simp add*: *tr-T'*)

have *6*: *trail* $T = H$ @ M
by (*metis* (*no-types*) ⟨*trail* $T' =$ *Marked* L i # *trail* T⟩
⟨*trail* $T' =$ *drop* (*length* M_0) M' @ *Marked* L i # H @ M⟩ *append-Nil list.sel*(*3*) *nd*
tl-append2)
have *7*: $cdcl_W$*-stgy*** T U **using** *other'.prems*(*4*) *st* **by** *auto*
have *8*: $cdcl_W$*-stgy* T U $cdcl_W$*-stgy*** U U
using $cdcl_W$*-stgy.other'*[*OF other'*(*1−3*)] **by** *simp-all*
show *?case* **apply** (*rule exI*[*of* - T], *rule exI*[*of* - T'], *rule exI*[*of* - U])
using *ns 1 2 3 5 6 7 8* **by** *fast*
qed
next
case *True*
then obtain M' **where** T: *trail* $T = M'$ @ *Marked* L i # H @ M **by** *metis*
from *IH*[*OF this S lev*] **obtain** S' S'' S''' **where**
1: $cdcl_W$*-stgy*** R S' **and**
2: *decide* S' S'' **and**
3: $cdcl_W$*-stgy*** S'' T **and**
4: *no-step* $cdcl_W$*-cp* S' **and**
6: *trail* $S'' =$ *Marked* L i # H @ M **and**
7: *trail* $S' = H$ @ M **and**
8: $cdcl_W$*-stgy*** S' T **and**
9: $cdcl_W$*-stgy* S' S''' **and**
10: $cdcl_W$*-stgy*** S''' T
by *blast*
have $cdcl_W$*-stgy*** S'' U **using** *s* ⟨$cdcl_W$*-stgy*** S'' T ⟩ **by** *auto*
moreover have $cdcl_W$*-stgy*** S' U **using** *8 s* **by** *auto*
moreover have $cdcl_W$*-stgy*** S''' U **using** *10 s* **by** *auto*
ultimately show *?thesis* **apply** − **apply** (*rule exI*[*of* - S'], *rule exI*[*of* - S''])
using *1 2 4 6 7 8 9* **by** *blast*
qed
qed

lemma *rtranclp-*$cdcl_W$*-new-marked-at-beginning-is-decide'*:
assumes $cdcl_W$*-stgy*** R U **and**
trail $U = M'$ @ *Marked* L i # H @ M **and**
trail $R = M$ **and**
$cdcl_W$*-M-level-inv* R
shows $\exists y\ y'.$ $cdcl_W$*-stgy*** R y ∧ $cdcl_W$*-stgy* y y' ∧ ¬ (∃ c. *trail* $y = c$ @ *Marked* L i # H @ M)
∧ (λa b. $cdcl_W$*-stgy* a b ∧ (∃ c. *trail* $a = c$ @ *Marked* L i # H @ M))** y' U
proof −
fix T'
obtain S' T T' **where**
st: $cdcl_W$*-stgy*** R S' **and**
decide S' T **and**
TU: $cdcl_W$*-stgy*** T U **and**
no-step $cdcl_W$*-cp* S' **and**
trT: *trail* $T =$ *Marked* L i # H @ M **and**
trS': *trail* $S' = H$ @ M **and**
S'U: $cdcl_W$*-stgy*** S' U **and**
S'T': $cdcl_W$*-stgy* S' T' **and**
T'U: $cdcl_W$*-stgy*** T' U
using *rtranclp-*$cdcl_W$*-new-marked-at-beginning-is-decide*[*OF assms*] **by** *blast*
have *n*: ¬ (∃ c. *trail* $S' = c$ @ *Marked* L i # H @ M) **using** *trS'* **by** *auto*
show *?thesis*
using *rtranclp-trans*[*OF st*] *rtranclp-exists-last-with-prop*[*of* $cdcl_W$*-stgy* S' T' -

λa -. $\neg(\exists c.$ *trail* $a = c$ @ *Marked* L i # H @ M), *OF* $S'T'$ $T'U$ n]
by *meson*
qed

lemma *beginning-not-marked-invert*:
assumes A: M @ $A = M'$ @ *Marked* K i # H **and**
nm: $\forall m\in set$ M. $\neg$*is-marked* m
shows $\exists M$. $A = M$ @ *Marked* K i # H
proof −
have $A =$ *drop* (*length* M) (M' @ *Marked* K i # H)
using *arg-cong*[*OF* A, *of drop* (*length* M)] **by** *auto*
moreover have *drop* (*length* M) (M' @ *Marked* K i # H) = *drop* (*length* M) M' @ *Marked* K i # H
using *nm* **by** (*metis* (*no-types*, *lifting*) A *drop-Cons'* *drop-append* *marked-lit.disc*(*1*) *not-gr0*
nth-append nth-append-length nth-mem zero-less-diff)
finally show *?thesis* **by** *fast*
qed

lemma $cdcl_W$*-stgy-trail-has-new-marked-is-decide-step*:
assumes $cdcl_W$*-stgy* S T
$\neg$ ($\exists c.$ *trail* $S = c$ @ *Marked* L i # H @ M) **and**
(λa b. $cdcl_W$*-stgy* a b $\wedge$ ($\exists c.$ *trail* $a = c$ @ *Marked* L i # H @ M))** T U **and**
$\exists M'$. *trail* $U = M'$ @ *Marked* L i # H @ M **and**
lev: $cdcl_W$*-M-level-inv* S
shows $\exists S'$. *decide* S S' $\wedge$ *full* $cdcl_W$*-cp* S' T $\wedge$ *no-step* $cdcl_W$*-cp* S
using *assms*(*3*,*1*,*2*,*4*,*5*)
proof *induction*
case (*step* T U)
then show *?case* **by** *fastforce*
next
case *base*
then show *?case*
proof (*induction rule*: $cdcl_W$*-stgy.induct*)
case (*conflict'* T) **note** *cp* = *this*(*1*) **and** *nd* = *this*(*2*) **and** M' =*this*(*3*) **and** *no-dup* = *this*(*3*)
then obtain M' **where** M': *trail* $T = M'$ @ *Marked* L i # H @ M **by** *metis*
obtain M'' **where** M'': *trail* $T = M''$ @ *trail* S **and** *nm*: $\forall m\in$ *set* M''. $\neg$*is-marked* m
using *cp* **unfolding** *full1-def*
by (*metis rtranclp-*$cdcl_W$*-cp-dropWhile-trail'* *tranclp-into-rtranclp*)
have *False*
using *beginning-not-marked-invert*[*of* M'' *trail* S M' L i H @ M] M' *nm nd* **unfolding** M''
by *fast*
then show *?case* **by** *fast*
next
case (*other'* T U') **note** *o* = *this*(*1*) **and** *ns* = *this*(*2*) **and** *cp* = *this*(*3*) **and** *nd* = *this*(*4*)
and trU' = *this*(*5*)
have $cdcl_W$*-cp*** T U' **using** *cp* **unfolding** *full-def* **by** *blast*
from *rtranclp-*$cdcl_W$*-cp-dropWhile-trail*[*OF this*]
have $\exists M'$. *trail* $T = M'$ @ *Marked* L i # H @ M
using trU' *beginning-not-marked-invert*[*of* - *trail* T - L i H @ M] **by** *metis*
then obtain M' **where** M': *trail* $T = M'$ @ *Marked* L i # H @ M
by *auto*
with *o lev nd cp ns*
show *?case*
proof (*induction rule*: $cdcl_W$*-o-induct-lev2*)
case (*decide* L) **note** *dec* = *this*(*1*) **and** *cp* = *this*(*5*) **and** *ns* = *this*(*4*)
then have *decide* S (*cons-trail* (*Marked* L (*backtrack-lvl* S +*1*)) (*incr-lvl* S))

using *decide.hyps decide.intros*[*of S*] **by** *force*
then show *?case* **using** *cp decide.prems* **by** (*meson decide-state-eq-compatible ns state-eq-ref state-eq-sym*)
next
case (*backtrack K j M1 M2 L' D T*) **note** *decomp* = *this*(*1*) **and** *cp* = *this*(*3*)
and *undef* = *this*(*6*) **and** *T* = *this*(*7*) **and** *trT* = *this*(*12*) **and** *ns* = *this*(*4*)
obtain *MS3* **where** *MS3*: *trail S* = *MS3* @ *M2* @ *Marked K* (*Suc j*) # *M1*
using *get-all-marked-decomposition-exists-prepend*[*OF decomp*] **by** *metis*
have *tl* (*M'* @ *Marked L i* # *H* @ *M*) = *tl M'* @ *Marked L i* # *H* @ *M*
using *lev trT T lev undef decomp* **by** (*cases M'*) (*auto simp*: $cdcl_W$*-M-level-inv-decomp*)
then have *M''*: *M1* = *tl M'* @ *Marked L i* # *H* @ *M*
using *arg-cong*[*OF trT*[*simplified*], *of tl*] *T decomp undef lev*
by (*simp add*: $cdcl_W$*-M-level-inv-decomp*)
have *False* **using** *nd MS3 T undef decomp* **unfolding** *M''* **by** *auto*
then show *?case* **by** *fast*
qed *auto*
qed
qed

lemma *rtranclp-*$cdcl_W$*-stgy-with-trail-end-has-trail-end*:
assumes ($\lambda a\ b.$ $cdcl_W$*-stgy a b* $\wedge$ ($\exists$ *c. trail a* = *c* @ *Marked L i* # *H* @ *M*))** *T U* **and**
$\exists$ *M'. trail U* = *M'* @ *Marked L i* # *H* @ *M*
shows $\exists$ *M'. trail T* = *M'* @ *Marked L i* # *H* @ *M*
using *assms* **by** (*induction rule*: *rtranclp-induct*) *auto*

lemma $cdcl_W$*-o-cannot-learn*:
assumes
$cdcl_W$*-o y z* **and**
lev: $cdcl_W$*-M-level-inv y* **and**
trM: *trail y* = *c* @ *Marked Kh i* # *H* **and**
DL: *D* + {#*L*#} $\notin$# *learned-clss y* **and**
DH: *atms-of D* $\subseteq$ *atm-of* ‘*lits-of H* **and**
LH: *atm-of L* $\notin$ *atm-of* ‘*lits-of H* **and**
learned: $\forall$ *T. conflicting y* = *Some T* $\longrightarrow$ *trail y* $\models$*as CNot T* **and**
z: *trail z* = *c'* @ *Marked Kh i* # *H*
shows *D* + {#*L*#} $\notin$# *learned-clss z*
using *assms*(*1*−*2*) *trM DL DH LH learned z*
proof (*induction rule*: $cdcl_W$*-o-induct-lev2*)
case (*backtrack K j M1 M2 L' D' T*) **note** *decomp* = *this*(*1*) **and** *confl* = *this*(*3*) **and** *levD* = *this*(*5*)
and *undef* = *this*(*6*) **and** *T* = *this*(*7*)
obtain *M3* **where** *M3*: *trail y* = *M3* @ *M2* @ *Marked K* (*Suc j*) # *M1*
using *decomp get-all-marked-decomposition-exists-prepend* **by** *metis*
have *M*: *trail y* = *c* @ *Marked Kh i* # *H* **using** *trM* **by** *simp*
have *H*: *get-all-levels-of-marked* (*trail y*) = *rev* [*1*..<*1* + *backtrack-lvl y*]
using *lev* **unfolding** $cdcl_W$*-M-level-inv-def* **by** *auto*
have *c'* @ *Marked Kh i* # *H* = *Propagated L'* (*D'* + {#*L'*#}) # *trail* (*reduce-trail-to M1 y*)
using *backtrack.prems*(*6*) *decomp undef T lev* **by** (*force simp*: $cdcl_W$*-M-level-inv-def*)
then obtain *d* **where** *d*: *M1* = *d* @ *Marked Kh i* # *H*
by (*metis* (*no-types*) *decomp in-get-all-marked-decomposition-trail-update-trail list.inject list.sel*(*3*) *marked-lit.distinct*(*1*) *self-append-conv2 tl-append2*)
have *i* $\in$ *set* (*get-all-levels-of-marked* (*M3* @ *M2* @ *Marked K* (*Suc j*) # *d* @ *Marked Kh i* # *H*))
by *auto*
then have *i* > *0* **unfolding** *H*[*unfolded M3 d*] **by** *auto*
show *?case*
proof

assume *D* + {#*L*#} ∈# *learned-clss T*
then have *DLD′*: *D* + {#*L*#} = *D′* + {#*L′*#}
  using *DL T neq0-conv undef decomp lev* **by** (*fastforce simp*: $cdcl_W$*-M-level-inv-def*)
have *L-cKh*: *atm-of L* ∈ *atm-of ‘lits-of* (*c* @ [*Marked Kh i*])
  using *LH learned M DLD′*[*symmetric*] *confl* **by** (*fastforce simp add*: *image-iff*)
have *get-all-levels-of-marked* (*M3* @ *M2* @ *Marked K* (*j* + *1*) # *M1*)
  = *rev* [*1*..<*1* + *backtrack-lvl y*]
  using *lev* **unfolding** $cdcl_W$*-M-level-inv-def M3* **by** *auto*
from *arg-cong*[*OF this*, *of* λ*a*. (*Suc j*) ∈ *set a*] **have** *backtrack-lvl y* ≥ *j* **by** *auto*

have *DD′*[*simp*]: *D* = *D′*
  proof (*rule ccontr*)
    assume *D* ≠ *D′*
    then have *L′* ∈# *D* **using** *DLD′* **by** (*metis add.left-neutral count-single count-union*
      diff-union-cancelR neq0-conv union-single-eq-member)
    then have *get-level* (*trail y*) *L′* ≤ *get-maximum-level* (*trail y*) *D*
      using *get-maximum-level-ge-get-level* **by** *blast*
    moreover {
      have *get-maximum-level* (*trail y*) *D* = *get-maximum-level H D*
        using *DH* **unfolding** *M* **by** (*simp add*: *get-maximum-level-skip-beginning*)
      moreover
        have *get-all-levels-of-marked* (*trail y*) = *rev* [*1*..<*1* + *backtrack-lvl y*]
          using *lev* **unfolding** $cdcl_W$*-M-level-inv-def* **by** *auto*
        then have *get-all-levels-of-marked H* = *rev* [*1*..< *i*]
          unfolding *M* **by** (*auto dest*: *append-cons-eq-upt-length-i*
            simp add: *rev-swap*[*symmetric*])
        then have *get-maximum-possible-level H* < *i*
          using *get-maximum-possible-level-max-get-all-levels-of-marked*[*of H*] ‹*i* > *0*› **by** *auto*
      ultimately have *get-maximum-level* (*trail y*) *D* < *i*
        by (*metis* (*full-types*) *dual-order.strict-trans nat-neq-iff not-le*
          get-maximum-possible-level-ge-get-maximum-level) **}**
    moreover
      have *L* ∈# *D′*
        by (*metis DLD′* ‹*D* ≠ *D′*› *add.left-neutral count-single count-union diff-union-cancelR*
          neq0-conv union-single-eq-member)
      then have *get-maximum-level* (*trail y*) *D′* ≥ *get-level* (*trail y*) *L*
        using *get-maximum-level-ge-get-level* **by** *blast*
    moreover {
      have *get-all-levels-of-marked* (*c* @ [*Marked Kh i*]) = *rev* [*i*..< *backtrack-lvl y*+*1*]
        using *append-cons-eq-upt-length-i-end*[*of rev* (*get-all-levels-of-marked H*) *i*
          rev (*get-all-levels-of-marked c*) *Suc 0 Suc* (*backtrack-lvl y*)] *H*
        unfolding *M* **apply** (*auto simp add*: *rev-swap*[*symmetric*])
          by (*metis* (*no-types*, *hide-lams*) *Nil-is-append-conv Suc-le-eq less-Suc-eq list.sel*(*1*)
            rev.simps(*2*) *rev-rev-ident upt-Suc upt-rec*)
      have *get-level* (*trail y*) *L* = *get-level* (*c* @ [*Marked Kh i*]) *L*
        using *L-cKh LH* **unfolding** *M* **by** *simp*
      have *get-level* (*c* @ [*Marked Kh i*]) *L* ≥ *i*
        using *L-cKh*
          ‹*get-all-levels-of-marked* (*c* @ [*Marked Kh i*]) = *rev* [*i*..<*backtrack-lvl y* + *1*]›
        backtrack.hyps(*2*) *calculation*(*1*,*2*) **by** *auto*
      then have *get-level* (*trail y*) *L* ≥ *i*
        using *M* ‹*get-level* (*trail y*) *L* = *get-level* (*c* @ [*Marked Kh i*]) *L*› **by** *auto* **}**
    moreover have *get-maximum-level* (*trail y*) *D′* < *get-level* (*trail y*) *L*
      using ‹*j* ≤ *backtrack-lvl y*› *backtrack.hyps*(*2*,*5*) *calculation*(*1*−*4*) **by** *linarith*
    ultimately show *False* **using** *backtrack.hyps*(*4*) **by** *linarith*

qed
then have LL': $L = L'$ using DLD' by *auto*
have *nd*: *no-dup* (*trail y*) using *lev* unfolding $cdcl_W$*-M-level-inv-def* by *auto*

{ assume *D*: $D' = \{\#\}$
then have *j*: $j = 0$ using *levD* by *auto*
have $\forall m \in set\ M1.\ \neg is\text{-}marked\ m$
using *H* unfolding *M3 j*
by (*auto simp add*: *rev-swap*[*symmetric*] *get-all-levels-of-marked-no-marked*
dest!: *append-cons-eq-upt-length-i*)
then have *False* using *d* by *auto*
}
moreover {
assume *D*[*simp*]: $D' \neq \{\#\}$
have $i \leq j$
using *H* unfolding *M3 d* by (*auto simp add*: *rev-swap*[*symmetric*]
dest: *upt-decomp-lt*)
have $j > 0$ apply (*rule ccontr*)
using *H* ‹ $i > 0$ › unfolding *M3 d*
by (*auto simp add*: *rev-swap*[*symmetric*] *dest*!: *upt-decomp-lt*)
obtain L'' where
$L'' \in\# D'$ and
$L''D'$: *get-level* (*trail y*) L'' = *get-maximum-level* (*trail y*) D'
using *get-maximum-level-exists-lit-of-max-level*[*OF D*, *of trail y*] by *auto*
have $L''M$: *atm-of* L'' ∈ *atm-of* ‘ *lits-of* (*trail y*)
using *get-rev-level-ge-0-atm-of-in*[*of 0 rev* (*trail y*) L''] ‹$j>0$› *levD* $L''D'$ by *auto*
then have L'' ∈ *lits-of* (*Marked Kh i # d*)
proof −
{
assume $L''H$: *atm-of* L'' ∈ *atm-of* ‘ *lits-of H*
have *get-all-levels-of-marked H* = *rev* [*1..<i*]
using *H* unfolding *M*
by (*auto simp add*: *rev-swap*[*symmetric*] *dest*!: *append-cons-eq-upt-length-i*)
moreover have *get-level* (*trail y*) L'' = *get-level H* L''
using $L''H$ unfolding *M* by *simp*
ultimately have *False*
using *levD* ‹$j>0$› *get-rev-level-in-levels-of-marked*[*of rev H 0* L''] ‹$i \leq j$›
unfolding $L''D'$[*symmetric*] *nd* by *auto*
}
then show *?thesis*
using DD' *DH* ‹$L'' \in\# D'$› *atm-of-lit-in-atms-of contra-subsetD* by *metis*
qed
then have *False*
using *DH* ‹$L'' \in\# D'$› *nd* unfolding *M3 d*
by (*auto simp add*: *atms-of-def image-iff image-subset-iff lits-of-def*)
}
ultimately show *False* by *blast*
qed
qed *auto*

lemma $cdcl_W$*-stgy-with-trail-end-has-not-been-learned*:
assumes $cdcl_W$*-stgy y z* and
$cdcl_W$*-M-level-inv y* and
trail y = *c* @ *Marked Kh i # H* and
$D + \{\#L\#\} \notin\#$ *learned-clss y* and

DH: *atms-of D* ⊆ *atm-of ‘lits-of H* **and**
LH: *atm-of L* ∉ *atm-of ‘lits-of H* **and**
∀ *T*. *conflicting y* = *Some T* ⟶ *trail y* ⊨*as CNot T* **and**
trail z = *c'* @ *Marked Kh i* # *H*
shows *D* + {#*L*#} ∉# *learned-clss z*
using *assms*
proof *induction*
case *conflict'*
then show *?case*
unfolding *full1-def* **using** *tranclp-cdcl*$_W$*-cp-learned-clause-inv* **by** *auto*
next
case (*other' T U*) **note** *o* = *this*(*1*) **and** *cp* = *this*(*3*) **and** *lev* = *this*(*4*) **and** *trY* = *this*(*5*) **and** *notin* = *this*(*6*) **and** *DH* = *this*(*7*) **and** *LH* = *this*(*8*) **and** *confl* = *this*(*9*) **and** *trU* = *this*(*10*)
obtain *c'* **where** *c'*: *trail T* = *c'* @ *Marked Kh i* # *H*
using *cp beginning-not-marked-invert*[*of - trail T c' Kh i H*]
rtranclp-cdcl$_W$*-cp-dropWhile-trail*[*of T U*] **unfolding** *trU full-def* **by** *fastforce*
show *?case*
using *cdcl*$_W$*-o-cannot-learn*[*OF o lev trY notin DH LH confl c'*]
rtranclp-cdcl$_W$*-cp-learned-clause-inv cp* **unfolding** *full-def* **by** *auto*
qed

lemma *rtranclp-cdcl*$_W$*-stgy-with-trail-end-has-not-been-learned*:
assumes (λ*a b*. *cdcl*$_W$*-stgy a b* ∧ (∃ *c*. *trail a* = *c* @ *Marked K i* # *H* @ []))** *S z* **and**
cdcl$_W$*-all-struct-inv S* **and**
trail S = *c* @ *Marked K i* # *H* **and**
D + {#*L*#} ∉# *learned-clss S* **and**
DH: *atms-of D* ⊆ *atm-of ‘lits-of H* **and**
LH: *atm-of L* ∉ *atm-of ‘lits-of H* **and**
∃ *c'*. *trail z* = *c'* @ *Marked K i* # *H*
shows *D* + {#*L*#} ∉# *learned-clss z*
using *assms*(*1*−*4*,*7*)
proof (*induction rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *auto*[*1*]
next
case (*step T U*) **note** *st* = *this*(*1*) **and** *s* = *this*(*2*) **and** *IH* = *this*(*3*)[*OF this*(*4*−*6*)]
and *lev* = *this*(*4*) **and** *trS* = *this*(*5*) **and** *DL-S* = *this*(*6*) **and** *trU* = *this*(*7*)
obtain *c* **where** *c*: *trail T* = *c* @ *Marked K i* # *H* **using** *s* **by** *auto*
obtain *c'* **where** *c'*: *trail U* = *c'* @ *Marked K i* # *H* **using** *trU* **by** *blast*
have *cdcl*$_W$** *S T*
proof −
have ∀ *p pa*. ∃ *s sa*. ∀ *sb sc sd se*. (¬ *p*** (*sb*::*'st*) *sc* ∨ *p s sa* ∨ *pa*** *sb sc*)
∧ (¬ *pa s sa* ∨ ¬ *p*** *sd se* ∨ *pa*** *sd se*)
by (*metis* (*no-types*) *mono-rtranclp*)
then have *cdcl*$_W$*-stgy*** *S T*
using *st* **by** *blast*
then show *?thesis*
using *rtranclp-cdcl*$_W$*-stgy-rtranclp-cdcl*$_W$ **by** *blast*
qed
then have *lev'*: *cdcl*$_W$*-all-struct-inv T*
using *rtranclp-cdcl*$_W$*-all-struct-inv-inv*[*of S T*] *lev* **by** *auto*
then have *confl'*: ∀ *Ta*. *conflicting T* = *Some Ta* ⟶ *trail T* ⊨*as CNot Ta*
unfolding *cdcl*$_W$*-all-struct-inv-def cdcl*$_W$*-conflicting-def* **by** *blast*
show *?case*
apply (*rule cdcl*$_W$*-stgy-with-trail-end-has-not-been-learned*[*OF - - c - DH LH confl' c'*])

using *s lev′ IH c* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *blast+*
qed

lemma $cdcl_W$*-stgy-new-learned-clause*:
assumes $cdcl_W$*-stgy S T* **and**
lev: $cdcl_W$*-M-level-inv S* **and**
$E \notin\#$ *learned-clss S* **and**
$E \in\#$ *learned-clss T*
shows $\exists S'$. *backtrack S S′* $\wedge$ *conflicting S = Some E* $\wedge$ *full* $cdcl_W$*-cp S′ T*
using *assms*
proof *induction*
case *conflict′*
then show *?case* **unfolding** *full1-def* **by** (*auto dest*: *tranclp-*$cdcl_W$*-cp-learned-clause-inv*)
next
case (*other′ T U*) **note** *o = this*(*1*) **and** *cp = this*(*3*) **and** *not-yet = this*(*5*) **and** *learned = this*(*6*)
have $E \in\#$ *learned-clss T*
using *learned cp rtranclp-*$cdcl_W$*-cp-learned-clause-inv* **unfolding** *full-def* **by** *auto*
then have *backtrack S T* **and** *conflicting S = Some E*
using $cdcl_W$*-o-new-clause-learned-is-backtrack-step*[*OF - not-yet o*] *lev* **by** *blast+*
then show *?case* **using** *cp* **by** *blast*
qed

lemma $cdcl_W$*-stgy-no-relearned-clause*:
assumes
invR: $cdcl_W$*-all-struct-inv R* **and**
st′: $cdcl_W$*-stgy*** *R S* **and**
bt: *backtrack S T* **and**
confl: *conflicting S = Some E* **and**
already-learned: $E \in\#$ *clauses S* **and**
R: *trail R* = []
shows *False*
proof −
have *M-lev*: $cdcl_W$*-M-level-inv R*
using *invR* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *auto*
have $cdcl_W$*-M-level-inv S*
using *M-lev assms*(*2*) *rtranclp-*$cdcl_W$*-stgy-consistent-inv* **by** *blast*
with *bt* **obtain** *D L M1 M2-loc K i* **where**
T: *T* $\sim$ *cons-trail* (*Propagated L* ((*D* + {#*L*#})))
(*reduce-trail-to M1* (*add-learned-cls* (*D* + {#*L*#})
(*update-backtrack-lvl* (*get-maximum-level* (*trail S*) *D*) (*update-conflicting None S*))))
and
decomp: (*Marked K* (*Suc* (*get-maximum-level* (*trail S*) *D*)) # *M1*, *M2-loc*) $\in$
set (*get-all-marked-decomposition* (*trail S*)) **and**
k: *get-level* (*trail S*) *L* = *backtrack-lvl S* **and**
level: *get-level* (*trail S*) *L* = *get-maximum-level* (*trail S*) (*D*+{#*L*#}) **and**
confl-S: *conflicting S = Some* (*D* + {#*L*#}) **and**
i: *i = get-maximum-level* (*trail S*) *D* **and**
undef: *undefined-lit M1 L*
by (*induction rule*: *backtrack-induction-lev2*) *metis*
obtain *M2* **where**
M: *trail S = M2* @ *Marked K* (*Suc i*) # *M1*
using *get-all-marked-decomposition-exists-prepend*[*OF decomp*] **unfolding** *i* **by** (*metis append-assoc*)

have *invS*: $cdcl_W$*-all-struct-inv S*
using *invR rtranclp-*$cdcl_W$*-all-struct-inv-inv rtranclp-*$cdcl_W$*-stgy-rtranclp-*$cdcl_W$ *st′* **by** *blast*

then have *conf*: *cdcl*$_W$*-conflicting S* **unfolding** *cdcl*$_W$*-all-struct-inv-def* **by** *blast*
then have *trail S* ⊨*as CNot* (*D* + {#*L*#}) **unfolding** *cdcl*$_W$*-conflicting-def confl-S* **by** *auto*
then have *MD*: *trail S* ⊨*as CNot D* **by** *auto*

have *lev'*: *cdcl*$_W$*-M-level-inv S* **using** *invS* **unfolding** *cdcl*$_W$*-all-struct-inv-def* **by** *blast*

have *get-lvls-M*: *get-all-levels-of-marked* (*trail S*) = *rev* [*1..<Suc* (*backtrack-lvl S*)]
  using *lev'* **unfolding** *cdcl*$_W$*-M-level-inv-def* **by** *auto*

have *lev*: *cdcl*$_W$*-M-level-inv R* **using** *invR* **unfolding** *cdcl*$_W$*-all-struct-inv-def* **by** *blast*
then have *vars-of-D*: *atms-of D* ⊆ *atm-of* ' *lits-of M1*
  using *backtrack-atms-of-D-in-M1*[*OF lev' undef - decomp - - - T*] *confl-S conf T decomp k level*
  lev' i undef **unfolding** *cdcl*$_W$*-conflicting-def* **by** (*auto simp*: *cdcl*$_W$*-M-level-inv-def*)
have *no-dup* (*trail S*) **using** *lev'* **by** (*auto simp*: *cdcl*$_W$*-M-level-inv-decomp*)
have *vars-in-M1*:
  ∀ *x* ∈ *atms-of D*. *x* ∉ *atm-of* ' *lits-of* (*M2* @ [*Marked K* (*get-maximum-level* (*trail S*) *D* + *1*)])
    apply (*rule vars-of-D distinct-atms-of-incl-not-in-other*[*of*
    M2 @ *Marked K* (*get-maximum-level* (*trail S*) *D* + *1*) # [] *M1 D*])
    using ‹*no-dup* (*trail S*)› *M vars-of-D* **by** *simp-all*
have *M1-D*: *M1* ⊨*as CNot D*
  using *vars-in-M1 true-annots-remove-if-notin-vars*[*of M2* @ *Marked K* (*i* + *1*) # [] *M1 CNot D*]
  ‹*trail S* ⊨*as CNot D*› *M* **by** *simp*

have *get-lvls-M*: *get-all-levels-of-marked* (*trail S*) = *rev* [*1..<Suc* (*backtrack-lvl S*)]
  using *lev'* **unfolding** *cdcl*$_W$*-M-level-inv-def* **by** *auto*
then have *backtrack-lvl S* > *0* **unfolding** *M* **by** (*auto split*: *split-if-asm simp add*: *upt.simps*(*2*))

obtain *M1' K' Ls* **where**
  M': *trail S* = *Ls* @ *Marked K'* (*backtrack-lvl S*) # *M1'* **and**
  Ls: ∀ *l* ∈ *set Ls*. ¬ *is-marked l* **and**
  set M1 ⊆ *set M1'*
  proof −
    let *?Ls* = *takeWhile* (*Not o is-marked*) (*trail S*)
    have *MLs*: *trail S* = *?Ls* @ *dropWhile* (*Not o is-marked*) (*trail S*)
      by *auto*
    have *dropWhile* (*Not o is-marked*) (*trail S*) ≠ [] **unfolding** *M* **by** *auto*
    moreover
      from *hd-dropWhile*[*OF this*] **have** *is-marked*(*hd* (*dropWhile* (*Not o is-marked*) (*trail S*)))
        by *simp*
    ultimately
      obtain *K' K'k* **where**
        K'k: *dropWhile* (*Not o is-marked*) (*trail S*)
          = *Marked K' K'k* # *tl* (*dropWhile* (*Not o is-marked*) (*trail S*))
        by (*cases dropWhile* (*Not* ∘ *is-marked*) (*trail S*);
            cases hd (*dropWhile* (*Not* ∘ *is-marked*) (*trail S*)))
          simp-all
    moreover have ∀ *l* ∈ *set ?Ls*. ¬*is-marked l* **using** *set-takeWhileD* **by** *force*
    moreover
      have *get-all-levels-of-marked* (*trail S*)
            = *K'k* # *get-all-levels-of-marked*(*tl* (*dropWhile* (*Not* ∘ *is-marked*) (*trail S*)))
        apply (*subst MLs*, *subst K'k*)
        using *calculation*(*2*) **by** (*auto simp add*: *get-all-levels-of-marked-no-marked*)
      then have *K'k* = *backtrack-lvl S*
      using *calculation*(*2*) **by** (*auto split*: *split-if-asm simp add*: *get-lvls-M upt.simps*(*2*))
    moreover have *set M1* ⊆ *set* (*tl* (*dropWhile* (*Not o is-marked*) (*trail S*)))

unfolding *M* **by** (*induction M2*) *auto*
ultimately show *?thesis* **using** *that MLs* **by** *metis*
qed

have *get-lvls-M*: *get-all-levels-of-marked* (*trail S*) = *rev* [*1..<Suc* (*backtrack-lvl S*)]
using *lev′* **unfolding** *cdcl*$_W$*-M-level-inv-def* **by** *auto*
then have *backtrack-lvl S* > *0* **unfolding** *M* **by** (*auto split*: *split-if-asm simp add*: *upt.simps*(*2*) *i*)

have *M1′-D*: *M1′* ⊨*as CNot D* **using** *M1-D* ‹*set M1* ⊆ *set M1′*› **by** (*auto intro*: *true-annots-mono*)
have −*L* ∈ *lits-of* (*trail S*) **using** *conf confl-S* **unfolding** *cdcl*$_W$*-conflicting-def* **by** *auto*
have *lvls-M1′*: *get-all-levels-of-marked M1′* = *rev* [*1..<backtrack-lvl S*]
using *get-lvls-M Ls* **by** (*auto simp add*: *get-all-levels-of-marked-no-marked M′*
split: *split-if-asm simp add*: *upt.simps*(*2*))
have *L-notin*: *atm-of L* ∈ *atm-of* ‘ *lits-of Ls* ∨ *atm-of L* = *atm-of K′*
proof (*rule ccontr*)
assume ¬ *?thesis*
then have *atm-of L* ∉ *atm-of* ‘ *lits-of* (*Marked K′* (*backtrack-lvl S*) # *rev Ls*) **by** *simp*
then have *get-level* (*trail S*) *L* = *get-level M1′ L*
unfolding *M′* **by** *auto*
then show *False* **using** *get-level-in-levels-of-marked*[*of M1′ L*] ‹*backtrack-lvl S* > *0*›
unfolding *k lvls-M1′* **by** *auto*
qed
obtain *Y Z* **where**
RY: *cdcl*$_W$*-stgy*** *R Y* **and**
YZ: *cdcl*$_W$*-stgy Y Z* **and**
nt: ¬ (∃ *c. trail Y* = *c* @ *Marked K′* (*backtrack-lvl S*) # *M1′* @ []) **and**
Z: (λ*a b. cdcl*$_W$*-stgy a b* ∧ (∃ *c. trail a* = *c* @ *Marked K′* (*backtrack-lvl S*) # *M1′* @ []))**
Z S
using *rtranclp-cdcl*$_W$*-new-marked-at-beginning-is-decide′*[*OF st′ - - lev*, *of Ls K′*
backtrack-lvl S M1′ []]
unfolding *R M′* **by** *auto*
have [*simp*]: *cdcl*$_W$*-M-level-inv Y*
using *RY lev rtranclp-cdcl*$_W$*-stgy-consistent-inv* **by** *blast*
obtain *M′* **where** *trZ*: *trail Z* = *M′* @ *Marked K′* (*backtrack-lvl S*) # *M1′*
using *rtranclp-cdcl*$_W$*-stgy-with-trail-end-has-trail-end*[*OF Z*] *M′* **by** *auto*
have *no-dup* (*trail Y*)
using *RY lev rtranclp-cdcl*$_W$*-stgy-consistent-inv* **unfolding** *cdcl*$_W$*-M-level-inv-def* **by** *blast*
then obtain *Y′* **where**
dec: *decide Y Y′* **and**
Y′Z: *full cdcl*$_W$*-cp Y′ Z* **and**
no-step cdcl$_W$*-cp Y*
using *cdcl*$_W$*-stgy-trail-has-new-marked-is-decide-step*[*OF YZ nt Z*] *M′* **by** *auto*
have *trY*: *trail Y* = *M1′*
proof −
obtain *M′* **where** *M*: *trail Z* = *M′* @ *Marked K′* (*backtrack-lvl S*) # *M1′*
using *rtranclp-cdcl*$_W$*-stgy-with-trail-end-has-trail-end*[*OF Z*] *M′* **by** *auto*
obtain *M″* **where** *M″*: *trail Z* = *M″* @ *trail Y′* **and** ∀ *m*∈*set M″*. ¬*is-marked m*
using *Y′Z rtranclp-cdcl*$_W$*-cp-dropWhile-trail′* **unfolding** *full-def* **by** *blast*
obtain *M‴* **where** *trail Y′* = *M‴* @ *Marked K′* (*backtrack-lvl S*) # *M1′*
using *M″* **unfolding** *M*
by (*metis* (*no-types*, *lifting*) ‹∀ *m*∈*set M″*. ¬ *is-marked m*› *beginning-not-marked-invert*)
then show *?thesis* **using** *dec nt* **by** (*induction M‴*) *auto*
qed
have *Y-CT*: *conflicting Y* = *None* **using** ‹*decide Y Y′*› **by** *auto*
have *cdcl*$_W$** *R Y* **by** (*simp add*: *RY rtranclp-cdcl*$_W$*-stgy-rtranclp-cdcl*$_W$)

then have *init-clss Y = init-clss R* **using** *rtranclp-cdcl$_W$-init-clss[of R Y] M-lev* **by** *auto*
{ assume *DL*: *D + {#L#} ∈# clauses Y*
have *atm-of L ∉ atm-of ‘ lits-of M1*
apply (*rule backtrack-lit-skiped[of S]*)
using *decomp i k lev'* **unfolding** *cdcl$_W$-M-level-inv-def* **by** *auto*
then have *LM1*: *undefined-lit M1 L*
by (*metis Marked-Propagated-in-iff-in-lits-of atm-of-uminus image-eqI*)
have *L-trY*: *undefined-lit* (*trail Y*) *L*
using *L-notin* ‹*no-dup* (*trail S*)› **unfolding** *defined-lit-map trY M'*
by (*auto simp add*: *image-iff lits-of-def*)
have ∃ *Y'. propagate Y Y'*
using *propagate-rule[of Y] DL M1'-D L-trY Y-CT trY DL* **by** (*metis state-eq-ref*)
then have *False* **using** ‹*no-step cdcl$_W$-cp Y*› *propagate'* **by** *blast*
}
moreover {
assume *DL*: *D + {#L#} ∉# clauses Y*
have *lY-lZ*: *learned-clss Y = learned-clss Z*
using *dec Y'Z rtranclp-cdcl$_W$-cp-learned-clause-inv[of Y' Z]* **unfolding** *full-def*
by *auto*
have *invZ*: *cdcl$_W$-all-struct-inv Z*
by (*meson RY YZ invR r-into-rtranclp rtranclp-cdcl$_W$-all-struct-inv-inv*
rtranclp-cdcl$_W$-stgy-rtranclp-cdcl$_W$)
have *D + {#L#} ∉#learned-clss S*
apply (*rule rtranclp-cdcl$_W$-stgy-with-trail-end-has-not-been-learned[OF Z invZ trZ]*)
using *DL lY-lZ* **unfolding** *clauses-def* **apply** *simp*
apply (*metis* (*no-types, lifting*) ‹*set M1 ⊆ set M1'*› *image-mono order-trans*
vars-of-D lits-of-def)
using *L-notin* ‹*no-dup* (*trail S*)› **unfolding** *M'* **by** (*auto simp add*: *image-iff lits-of-def*)
then have *False*
using *already-learned DL confl st' M-lev* **unfolding** *M'*
by (*simp add*: ‹*init-clss Y = init-clss R*› *clauses-def confl-S*
rtranclp-cdcl$_W$-stgy-no-more-init-clss)
}
ultimately show *False* **by** *blast*
qed

lemma *rtranclp-cdcl$_W$-stgy-distinct-mset-clauses*:
assumes
invR: *cdcl$_W$-all-struct-inv R* **and**
st: *cdcl$_W$-stgy** R S* **and**
dist: *distinct-mset* (*clauses R*) **and**
R: *trail R = []*
shows *distinct-mset* (*clauses S*)
using *st*
proof (*induction*)
case *base*
then show *?case* **using** *dist* **by** *simp*
next
case (*step S T*) **note** *st = this(1)* **and** *s = this(2)* **and** *IH = this(3)*
from *s* **show** *?case*
proof (*cases rule*: *cdcl$_W$-stgy.cases*)
case *conflict'*
then show *?thesis*
using *IH* **unfolding** *full1-def* **by** (*auto dest*: *tranclp-cdcl$_W$-cp-no-more-clauses*)
next

case (*other' S'*) **note** *o* = *this(1)* **and** *full* = *this(3)*
have [*simp*]: *clauses T* = *clauses S'*
using *full* **unfolding** *full-def* **by** (*auto dest*: *rtranclp-cdcl$_W$-cp-no-more-clauses*)
show *?thesis*
using *o IH*
proof (*cases rule*: *cdcl$_W$-o-rule-cases*)
case *backtrack*
moreover
have *cdcl$_W$-all-struct-inv S*
using *invR rtranclp-cdcl$_W$-stgy-cdcl$_W$-all-struct-inv st* **by** *blast*
then have *cdcl$_W$-M-level-inv S*
unfolding *cdcl$_W$-all-struct-inv-def* **by** *auto*
ultimately obtain *E* **where**
conflicting S = *Some E* **and**
cls-S': *clauses S'* = {#*E*#} + *clauses S*
using ‹*cdcl$_W$-M-level-inv S*›
by (*induction rule*: *backtrack-induction-lev2*) (*auto simp*: *cdcl$_W$-M-level-inv-decomp*)
then have *E* ∉# *clauses S*
using *cdcl$_W$-stgy-no-relearned-clause R invR local.backtrack st* **by** *blast*
then show *?thesis* **using** *IH* **by** (*simp add*: *distinct-mset-add-single cls-S'*)
qed *auto*
qed
qed

lemma *cdcl$_W$-stgy-distinct-mset-clauses*:
assumes
st: *cdcl$_W$-stgy*** (*init-state N*) *S* **and**
no-duplicate-clause: *distinct-mset N* **and**
no-duplicate-in-clause: *distinct-mset-mset N*
shows *distinct-mset* (*clauses S*)
using *rtranclp-cdcl$_W$-stgy-distinct-mset-clauses*[*OF - st*] *assms*
by (*auto simp*: *cdcl$_W$-all-struct-inv-def distinct-cdcl$_W$-state-def*)

17.9 Decrease of a measure

fun *cdcl$_W$-measure* **where**
cdcl$_W$-measure S =
[(*3*::*nat*) ^ (*card* (*atms-of-msu* (*init-clss S*))) − *card* (*set-mset* (*learned-clss S*)),
if conflicting S = *None then 1 else 0*,
if conflicting S = *None then card* (*atms-of-msu* (*init-clss S*)) − *length* (*trail S*)
else length (*trail S*)
]

lemma *length-model-le-vars-all-inv*:
assumes *cdcl$_W$-all-struct-inv S*
shows *length* (*trail S*) ≤ *card* (*atms-of-msu* (*init-clss S*))
using *assms length-model-le-vars*[*of S*] **unfolding** *cdcl$_W$-all-struct-inv-def*
by (*auto simp*: *cdcl$_W$-M-level-inv-decomp*)
end

locale *cdcl$_W$-termination* =
cdcl$_W$-ops trail init-clss learned-clss backtrack-lvl conflicting cons-trail tl-trail
add-init-cls
add-learned-cls remove-cls update-backtrack-lvl update-conflicting init-state
restart-state
for

trail :: *'st* ⇒ (*'v*::*linorder*, *nat*, *'v clause*) *marked-lits* **and**
init-clss :: *'st* ⇒ *'v clauses* **and**
learned-clss :: *'st* ⇒ *'v clauses* **and**
backtrack-lvl :: *'st* ⇒ *nat* **and**
conflicting :: *'st* ⇒*'v clause option* **and**

cons-trail :: (*'v*, *nat*, *'v clause*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-init-cls :: *'v clause* ⇒ *'st* ⇒ *'st* **and**
add-learned-cls :: *'v clause* ⇒ *'st* ⇒ *'st* **and**
remove-cls :: *'v clause* ⇒ *'st* ⇒ *'st* **and**
update-backtrack-lvl :: *nat* ⇒ *'st* ⇒ *'st* **and**
update-conflicting :: *'v clause option* ⇒ *'st* ⇒ *'st* **and**

init-state :: *'v clauses* ⇒ *'st* **and**
restart-state :: *'st* ⇒ *'st*
begin

lemma *learned-clss-less-upper-bound*:
fixes *S* :: *'st*
assumes
$distinct\text{-}cdcl_W\text{-}state$ *S* **and**
∀ *s* ∈# *learned-clss S*. ¬*tautology s*
shows *card*(*set-mset* (*learned-clss S*)) ≤ *3* ^ *card* (*atms-of-msu* (*learned-clss S*))
proof −
have *set-mset* (*learned-clss S*) ⊆ *build-all-simple-clss* (*atms-of-msu* (*learned-clss S*))
apply (*rule simplified-in-build-all*)
using *assms* **unfolding** $distinct\text{-}cdcl_W\text{-}state\text{-}def$ **by** *auto*
then have *card*(*set-mset* (*learned-clss S*))
≤ *card* (*build-all-simple-clss* (*atms-of-msu* (*learned-clss S*)))
by (*simp add*: *build-all-simple-clss-finite card-mono*)
then show *?thesis*
by (*meson atms-of-ms-finite build-all-simple-clss-card finite-set-mset order-trans*)
qed

lemma *lexn3*[*intro*!, *simp*]:
$a < a' \lor (a = a' \land b < b') \lor (a = a' \land b = b' \land c < c')$
$\Longrightarrow ([a::nat,\ b,\ c],\ [a',\ b',\ c']) \in lexn\ \{(x,\ y).\ x < y\}\ 3$
apply *auto*
unfolding *lexn-conv* **apply** *fastforce*
unfolding *lexn-conv* **apply** *auto*
apply (*metis append.simps*(*1*) *append.simps*(*2*))+
done

lemma $cdcl_W\text{-}measure\text{-}decreasing$:
fixes *S* :: *'st*
assumes
$cdcl_W$ *S S'* **and**
no-restart:
¬(*learned-clss S* ⊆# *learned-clss S'* ∧ [] = *trail S'* ∧ *conflicting S'* = *None*)
and
learned-clss S ⊆# *learned-clss S'* **and**
no-relearn: ⋀*S'*. *backtrack S S'* ⟹ ∀ *T*. *conflicting S* = *Some T* ⟶ *T* ∉# *learned-clss S*
and
alien: *no-strange-atm S* **and**

M-level: $cdcl_W$*-M-level-inv S* **and**
no-taut: $\forall$ *s* $\in\#$ *learned-clss S*. $\neg$*tautology s* **and**
no-dup: *distinct-*$cdcl_W$*-state S* **and**
confl: $cdcl_W$*-conflicting S*
shows ($cdcl_W$*-measure S'*, $cdcl_W$*-measure S*) $\in$ *lexn* $\{(a, b).\ a < b\}$ *3*
using *assms*(*1*) *M-level assms*(*2*,*3*)
proof (*induct rule*: $cdcl_W$*-all-induct-lev2*)
case (*propagate C L*) **note** *undef* = *this*(*3*) **and** *T* = *this*(*4*) **and** *conf* = *this*(*5*)
have *propa*: *propagate S* (*cons-trail* (*Propagated L* (*C* + {#*L*#})) *S*)
using *propagate-rule*[*OF - propagate.hyps*(*1*,*2*)] *propagate.hyps* **by** *auto*
then have *no-dup'*: *no-dup* (*Propagated L* ((*C* + {#*L*#})) # *trail S*)
by (*metis M-level* $cdcl_W$*-M-level-inv-decomp*(*2*) *marked-lit.sel*(*2*) *propagate'*
r-into-rtranclp rtranclp-$cdcl_W$*-cp-consistent-inv trail-cons-trail undef*)

let *?N* = *init-clss S*
have *no-strange-atm* (*cons-trail* (*Propagated L* (*C* + {#*L*#})) *S*)
using *alien* $cdcl_W$*.propagate* $cdcl_W$*-no-strange-atm-inv propa M-level* **by** *blast*
then have *atm-of ‘ lits-of* (*Propagated L* ((*C* + {#*L*#})) # *trail S*)
$\subseteq$ *atms-of-msu* (*init-clss S*)
using *undef* **unfolding** *no-strange-atm-def* **by** *auto*
then have *card* (*atm-of ‘ lits-of* (*Propagated L* ((*C* + {#*L*#})) # *trail S*))
$\leq$ *card* (*atms-of-msu* (*init-clss S*))
by (*meson atms-of-ms-finite card-mono finite-set-mset*)
then have *length* (*Propagated L* ((*C* + {#*L*#})) # *trail S*) $\leq$ *card* (*atms-of-msu ?N*)
using *no-dup-length-eq-card-atm-of-lits-of no-dup'* **by** *fastforce*
then have *H*: *card* (*atms-of-msu* (*init-clss S*)) − *length* (*trail S*)
= *Suc* (*card* (*atms-of-msu* (*init-clss S*)) − *Suc* (*length* (*trail S*)))
by *simp*
show *?case* **using** *conf T undef* **by** (*auto simp*: *H*)
next
case (*decide L*) **note** *conf* = *this*(*1*) **and** *undef* = *this*(*2*) **and** *T* = *this*(*4*)
moreover
have *dec*: *decide S* (*cons-trail* (*Marked L* (*backtrack-lvl S* + *1*)) (*incr-lvl S*))
using *decide.intros decide.hyps* **by** *force*
then have $cdcl_W$:$cdcl_W$ *S* (*cons-trail* (*Marked L* (*backtrack-lvl S* + *1*)) (*incr-lvl S*))
using $cdcl_W$*.simps* **by** *blast*
moreover
have *lev*: $cdcl_W$*-M-level-inv* (*cons-trail* (*Marked L* (*backtrack-lvl S* + *1*)) (*incr-lvl S*))
using $cdcl_W$ *M-level* $cdcl_W$*-consistent-inv*[*OF* $cdcl_W$] **by** *auto*
then have *no-dup*: *no-dup* (*Marked L* (*backtrack-lvl S* + *1*) # *trail S*)
using *undef* **unfolding** $cdcl_W$*-M-level-inv-def* **by** *auto*
have *no-strange-atm* (*cons-trail* (*Marked L* (*backtrack-lvl S* + *1*)) (*incr-lvl S*))
using *M-level alien calculation*(*4*) $cdcl_W$*-no-strange-atm-inv* **by** *blast*
then have *length* (*Marked L* ((*backtrack-lvl S*) + *1*) # (*trail S*))
$\leq$ *card* (*atms-of-msu* (*init-clss S*))
using *no-dup clauses-def undef*
length-model-le-vars[*of cons-trail* (*Marked L* (*backtrack-lvl S* + *1*)) (*incr-lvl S*)]
by *fastforce*
ultimately show *?case* **using** *conf* **by** *auto*
next
case (*skip L C' M D*) **note** *tr* = *this*(*1*) **and** *conf* = *this*(*2*) **and** *T* = *this*(*5*)
show *?case* **using** *conf T* **unfolding** *clauses-def* **by** (*simp add*: *tr*)
next
case *conflict*
then show *?case* **by** *simp*

next
  case *resolve*
  then show *?case* **using** *finite* **unfolding** *clauses-def* **by** *simp*
next
  case (*backtrack K i M1 M2 L D T*) **note** *decomp* = *this*(*1*) **and** *conf* = *this*(*3*) **and** *undef* = *this*(*6*)
and
    T =*this*(*7*) **and** *lev* = *this*(*8*)
  let *?S′* = *T*
  have *bt*: *backtrack S ?S′*
    using *backtrack.hyps backtrack.intros*[*of S - - - - D L K i*] **by** *auto*
  have $D + \{\#L\#\} \notin\#$ *learned-clss S*
    using *no-relearn conf bt* **by** *auto*
  then have *card-T*:
    card (*set-mset* ({#*D* + {#*L*#}#} + *learned-clss S*)) = *Suc* (*card* (*set-mset* (*learned-clss S*)))
    by (*simp add*:)
  have *distinct-cdcl*$_W$*-state ?S′*
    using *bt M-level distinct-cdcl*$_W$*-state-inv no-dup other* **by** *blast*
  moreover have ∀ *s*∈#*learned-clss ?S′*. ¬ *tautology s*
    using *learned-clss-are-not-tautologies*[*OF cdcl*$_W$*.other*[*OF cdcl*$_W$*-o.bj*[*OF*
      cdcl$_W$*-bj.backtrack*[*OF bt*]]]] *M-level no-taut confl* **by** *auto*
  ultimately have *card* (*set-mset* (*learned-clss T*)) ≤ *3* ^ *card* (*atms-of-msu* (*learned-clss T*))
      by (*auto simp*: *clauses-def learned-clss-less-upper-bound*)
    then have *H*: *card* (*set-mset* ({#*D* + {#*L*#}#} + *learned-clss S*))
      ≤ *3* ^ *card* (*atms-of-msu* ({#*D* + {#*L*#}#} + *learned-clss S*))
      using *T undef decomp lev* **by** (*auto simp*: *cdcl*$_W$*-M-level-inv-decomp*)
  moreover
    have *atms-of-msu* ({#*D* + {#*L*#}#} + *learned-clss S*) ⊆ *atms-of-msu* (*init-clss S*)
      using *alien conf* **unfolding** *no-strange-atm-def* **by** *auto*
    then have *card-f*: *card* (*atms-of-msu* ({#*D* + {#*L*#}#} + *learned-clss S*))
      ≤ *card* (*atms-of-msu* (*init-clss S*))
      by (*meson atms-of-ms-finite card-mono finite-set-mset*)
    then have (*3*::*nat*) ^ *card* (*atms-of-msu* ({#*D* + {#*L*#}#} + *learned-clss S*))
      ≤ *3* ^ *card* (*atms-of-msu* (*init-clss S*)) **by** *simp*
  ultimately have (*3*::*nat*) ^ *card* (*atms-of-msu* (*init-clss S*))
    ≥ *card* (*set-mset* ({#*D* + {#*L*#}#} + *learned-clss S*))
    using *le-trans* **by** *blast*
  then show *?case* **using** *decomp undef diff-less-mono2 card-T T lev*
    by (*auto simp*: *cdcl*$_W$*-M-level-inv-decomp*)
next
  case *restart*
  then show *?case* **using** *alien* **by** (*auto simp*: *state-eq-def simp del*: *state-simp*)
next
  case (*forget C T*)
  then have *C* ∈# *learned-clss S* **and** $C \notin\#$ *learned-clss T*
    by *auto*
  then show *?case* **using** *forget*(*9*) **by** (*simp add*: *mset-leD*)
qed

lemma *propagate-measure-decreasing*:
  fixes *S* :: *′st*
  assumes *propagate S S′* **and** *cdcl*$_W$*-all-struct-inv S*
  shows (*cdcl*$_W$*-measure S′*, *cdcl*$_W$*-measure S*) ∈ *lexn* {(*a*, *b*). *a* < *b*} *3*
  apply (*rule cdcl*$_W$*-measure-decreasing*)
  using *assms*(*1*) *propagate* **apply** *blast*
          using *assms*(*1*) **apply** (*auto simp add*: *propagate.simps*)[*3*]

using *assms(2)* **apply** (*auto simp add*: *cdcl*$_W$*-all-struct-inv-def*)
done

lemma *conflict-measure-decreasing*:
fixes *S* :: *'st*
assumes *conflict S S'* **and** *cdcl*$_W$*-all-struct-inv S*
shows (*cdcl*$_W$*-measure S'*, *cdcl*$_W$*-measure S*) $\in$ *lexn* {(*a*, *b*). *a* < *b*} *3*
apply (*rule cdcl*$_W$*-measure-decreasing*)
using *assms(1) conflict* **apply** *blast*
using *assms(1)* **apply** (*auto simp add*: *propagate.simps*)[*3*]
using *assms(2)* **apply** (*auto simp add*: *cdcl*$_W$*-all-struct-inv-def*)
done

lemma *decide-measure-decreasing*:
fixes *S* :: *'st*
assumes *decide S S'* **and** *cdcl*$_W$*-all-struct-inv S*
shows (*cdcl*$_W$*-measure S'*, *cdcl*$_W$*-measure S*) $\in$ *lexn* {(*a*, *b*). *a* < *b*} *3*
apply (*rule cdcl*$_W$*-measure-decreasing*)
using *assms(1) decide other* **apply** *blast*
using *assms(1)* **apply** (*auto simp add*: *propagate.simps*)[*3*]
using *assms(2)* **apply** (*auto simp add*: *cdcl*$_W$*-all-struct-inv-def*)
done

lemma *trans-le*:
trans {(*a*, (*b*::*nat*)). *a* < *b*}
unfolding *trans-def* **by** *auto*

lemma *cdcl*$_W$*-cp-measure-decreasing*:
fixes *S* :: *'st*
assumes *cdcl*$_W$*-cp S S'* **and** *cdcl*$_W$*-all-struct-inv S*
shows (*cdcl*$_W$*-measure S'*, *cdcl*$_W$*-measure S*) $\in$ *lexn* {(*a*, *b*). *a* < *b*} *3*
using *assms*
proof *induction*
case *conflict'*
then show *?case* **using** *conflict-measure-decreasing* **by** *blast*
next
case *propagate'*
then show *?case* **using** *propagate-measure-decreasing* **by** *blast*
qed

lemma *tranclp-cdcl*$_W$*-cp-measure-decreasing*:
fixes *S* :: *'st*
assumes *cdcl*$_W$*-cp*$^{++}$ *S S'* **and** *cdcl*$_W$*-all-struct-inv S*
shows (*cdcl*$_W$*-measure S'*, *cdcl*$_W$*-measure S*) $\in$ *lexn* {(*a*, *b*). *a* < *b*} *3*
using *assms*
proof *induction*
case *base*
then show *?case* **using** *cdcl*$_W$*-cp-measure-decreasing* **by** *blast*
next
case (*step T U*) **note** *st* = *this(1)* **and** *step* = *this(2)* **and** *IH* =*this(3)* **and** *inv* = *this(4)*
then have (*cdcl*$_W$*-measure T*, *cdcl*$_W$*-measure S*) $\in$ *lexn* {*a*. *case a of* (*a*, *b*) $\Rightarrow$ *a* < *b*} *3* **by** *blast*

moreover have (*cdcl*$_W$*-measure U*, *cdcl*$_W$*-measure T*) $\in$ *lexn* {*a*. *case a of* (*a*, *b*) $\Rightarrow$ *a* < *b*} *3*
using *cdcl*$_W$*-cp-measure-decreasing*[*OF step*] *rtranclp-cdcl*$_W$*-all-struct-inv-inv inv*
tranclp-cdcl$_W$*-cp-tranclp-cdcl*$_W$[*OF st*]

unfolding *trans-def rtranclp-unfold*
by *blast*
ultimately show *?case* **using** *lexn-transI*[*OF trans-le*] **unfolding** *trans-def* **by** *blast*
qed

lemma $cdcl_W$*-stgy-step-decreasing*:
fixes *R S T* :: *'st*
assumes $cdcl_W$*-stgy S T* **and**
$cdcl_W$*-stgy*** *R S*
trail R = [] **and**
$cdcl_W$*-all-struct-inv R*
shows ($cdcl_W$*-measure T*, $cdcl_W$*-measure S*) ∈ *lexn* {(*a*, *b*). *a* < *b*} *3*
proof −
have $cdcl_W$*-all-struct-inv S*
using *assms*
by (*metis rtranclp-unfold rtranclp-*$cdcl_W$*-all-struct-inv-inv tranclp-*$cdcl_W$*-stgy-tranclp-*$cdcl_W$)
with *assms* **show** *?thesis*
proof *induction*
case (*conflict' V*) **note** *cp* = *this*(*1*) **and** *inv* = *this*(*5*)
show *?case*
using *tranclp-*$cdcl_W$*-cp-measure-decreasing*[*OF HOL.conjunct1*[*OF cp*[*unfolded full1-def*]] *inv*]
.
next
case (*other' T U*) **note** *st* = *this*(*1*) **and** *H*= *this*(*4*,*5*,*6*,*7*) **and** *cp* = *this*(*3*)
have $cdcl_W$*-all-struct-inv T*
using $cdcl_W$*-all-struct-inv-inv other other'.hyps*(*1*) *other'.prems*(*4*) **by** *blast*
from *tranclp-*$cdcl_W$*-cp-measure-decreasing*[*OF - this*]
have *le-or-eq*: ($cdcl_W$*-measure U*, $cdcl_W$*-measure T*) ∈ *lexn* {*a*. *case a of* (*a*, *b*) ⇒ *a* < *b*} *3* ∨
$cdcl_W$*-measure U* = $cdcl_W$*-measure T*
using *cp* **unfolding** *full-def rtranclp-unfold* **by** *blast*
moreover
have $cdcl_W$*-M-level-inv S*
using $cdcl_W$*-all-struct-inv-def other'.prems*(*4*) **by** *blast*
with *st* **have** ($cdcl_W$*-measure T*, $cdcl_W$*-measure S*) ∈ *lexn* {*a*. *case a of* (*a*, *b*) ⇒ *a* < *b*} *3*
proof (*induction rule*:$cdcl_W$*-o-induct-lev2*)
case (*decide T*)
then show *?case* **using** *decide-measure-decreasing H* **by** *blast*
next
case (*backtrack K i M1 M2 L D T*) **note** *decomp* = *this*(*1*) **and** *undef* = *this*(*6*) **and** *T* = *this*(*7*)
have *bt*: *backtrack S T*
apply (*rule backtrack-rule*)
using *backtrack.hyps* **by** *auto*
then have *no-relearn*: ∀ *T*. *conflicting S* = *Some T* ⟶ *T* ∉# *learned-clss S*
using $cdcl_W$*-stgy-no-relearned-clause*[*of R S T*] *H*
unfolding $cdcl_W$*-all-struct-inv-def clauses-def* **by** *auto*
have *inv*: $cdcl_W$*-all-struct-inv S*
using ‹$cdcl_W$*-all-struct-inv S*› **by** *blast*
show *?case*
apply (*rule* $cdcl_W$*-measure-decreasing*)
using *bt* $cdcl_W$*-bj.backtrack* $cdcl_W$*-o.bj other* **apply** *simp*
using *bt T undef decomp inv* **unfolding** $cdcl_W$*-all-struct-inv-def*
$cdcl_W$*-M-level-inv-def* **apply** *auto*[]
using *bt T undef decomp inv* **unfolding** $cdcl_W$*-all-struct-inv-def*
$cdcl_W$*-M-level-inv-def* **apply** *auto*[]

using *bt no-relearn* **apply** *auto*[]
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* **apply** *simp*
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-M-level-inv-def* **apply** *simp*
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* **apply** *simp*
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* **apply** *simp*
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *simp*
next
case *skip*
then show *?case* **by** *force*
next
case *resolve*
then show *?case* **by** *force*
qed
ultimately show *?case*
by (*metis lexn-transI transD trans-le*)
qed
qed

lemma *tranclp-*$cdcl_W$*-stgy-decreasing*:
fixes $R\ S\ T$ :: *'st*
assumes $cdcl_W$*-stgy*$^{++}$ $R\ S$
trail R = [] **and**
$cdcl_W$*-all-struct-inv* R
shows ($cdcl_W$*-measure* S, $cdcl_W$*-measure* R) $\in$ *lexn* $\{(a,\ b).\ a < b\}$ *3*
using *assms*
apply *induction*
using $cdcl_W$*-stgy-step-decreasing*[*of R - R*] **apply** *blast*
using $cdcl_W$*-stgy-step-decreasing*[*of - - R*] *tranclp-into-rtranclp*[*of* $cdcl_W$*-stgy R*]
lexn-transI[*OF trans-le*, *of 3*] **unfolding** *trans-def* **by** *blast*

lemma *tranclp-*$cdcl_W$*-stgy-S0-decreasing*:
fixes $R\ S\ T$ :: *'st*
assumes *pl*: $cdcl_W$*-stgy*$^{++}$ (*init-state* N) S **and**
no-dup: *distinct-mset-mset* N
shows ($cdcl_W$*-measure* S, $cdcl_W$*-measure* (*init-state* N)) $\in$ *lexn* $\{(a,\ b).\ a < b\}$ *3*
proof –
have $cdcl_W$*-all-struct-inv* (*init-state* N)
using *no-dup* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *auto*
then show *?thesis* **using** *pl tranclp-*$cdcl_W$*-stgy-decreasing init-state-trail* **by** *blast*
qed

lemma *wf-tranclp-*$cdcl_W$*-stgy*:
wf $\{(S$::*'st*, *init-state* $N)|\ S\ N.$ *distinct-mset-mset* $N \land cdcl_W$*-stgy*$^{++}$ (*init-state* N) $S\}$
apply (*rule wf-wf-if-measure'-notation2*[*of lexn* $\{(a,\ b).\ a < b\}$ *3 - -* $cdcl_W$*-measure*])
apply (*simp add*: *wf wf-lexn*)
using *tranclp-*$cdcl_W$*-stgy-S0-decreasing* **by** *blast*
end

end
theory *DPLL-CDCL-W-Implementation*
imports *Partial-Annotated-Clausal-Logic*
begin

18 Simple Implementation of the DPLL and CDCL

18.1 Common Rules

18.1.1 Propagation

The following theorem holds:

lemma *lits-of-unfold*[*iff*]:
  $(\forall c \in set\ C.\ -c \in lits\text{-}of\ Ms) \longleftrightarrow Ms \models as\ CNot\ (mset\ C)$
  unfolding *true-annots-def Ball-def true-annot-def CNot-def mem-set-multiset-eq* **by** *auto*

The right-hand version is written at a high-level, but only the left-hand side is executable.

definition *is-unit-clause* :: *'a literal list* ⇒ (*'a*, *'b*, *'c*) *marked-lit list* ⇒ *'a literal option*
 where
 is-unit-clause l M =
  (*case List.filter* (λ*a. atm-of a* ∉ *atm-of* ‘ *lits-of M*) *l of*
   a # [] ⇒ *if M* ⊨*as CNot* (*mset l* − {#*a*#}) *then Some a else None*
  | - ⇒ *None*)

definition *is-unit-clause-code* :: *'a literal list* ⇒ (*'a*, *'b*, *'c*) *marked-lit list*
  ⇒ *'a literal option* **where**
 is-unit-clause-code l M =
  (*case List.filter* (λ*a. atm-of a* ∉ *atm-of* ‘ *lits-of M*) *l of*
   a # [] ⇒ *if* (∀ *c* ∈*set* (*remove1 a l*). −*c* ∈ *lits-of M*) *then Some a else None*
  | - ⇒ *None*)

lemma *is-unit-clause-is-unit-clause-code*[*code*]:
  is-unit-clause l M = *is-unit-clause-code l M*
proof −
  have *1*: ⋀*a*. (∀ *c*∈*set* (*remove1 a l*). − *c* ∈ *lits-of M*) ⟷ *M* ⊨*as CNot* (*mset l* − {#*a*#})
    using *lits-of-unfold*[*of remove1 - l*, *of - M*] **by** *simp*
  thus *?thesis*
    unfolding *is-unit-clause-code-def is-unit-clause-def 1* **by** *blast*
qed

lemma *is-unit-clause-some-undef*:
  assumes *is-unit-clause l M* = *Some a*
  shows *undefined-lit M a*
proof −
  have (*case* [*a*←*l* . *atm-of a* ∉ *atm-of* ‘ *lits-of M*] *of* [] ⇒ *None*
      | [*a*] ⇒ *if M* ⊨*as CNot* (*mset l* − {#*a*#}) *then Some a else None*
      | *a* # *ab* # *xa* ⇒ *Map.empty xa*) = *Some a*
    using *assms* **unfolding** *is-unit-clause-def* **.**
  hence *a* ∈ *set* [*a*←*l* . *atm-of a* ∉ *atm-of* ‘ *lits-of M*]
    apply (*case-tac* [*a*←*l* . *atm-of a* ∉ *atm-of* ‘ *lits-of M*])
     apply *simp*
    apply (*case-tac list*) **by** (*auto split*: *split-if-asm*)
  hence *atm-of a* ∉ *atm-of* ‘ *lits-of M* **by** *auto*
  thus *?thesis*
    by (*simp add*: *Marked-Propagated-in-iff-in-lits-of*
     atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)
qed

lemma *is-unit-clause-some-CNot*: *is-unit-clause l M* = *Some a* ⟹ *M* ⊨*as CNot* (*mset l* − {#*a*#})
  unfolding *is-unit-clause-def*
proof −

assume (*case* [*a*←*l* . *atm-of a* ∉ *atm-of* ‘ *lits-of M*] *of* [] ⇒ *None*
| [*a*] ⇒ *if M* ⊨*as CNot* (*mset l* − {#*a*#}) *then Some a else None*
| *a* # *ab* # *xa* ⇒ *Map.empty xa*) = *Some a*
thus *?thesis*
apply (*case-tac* [*a*←*l* . *atm-of a* ∉ *atm-of* ‘ *lits-of M*], *simp*)
apply *simp*
apply (*case-tac list*) **by** (*auto split*: *split-if-asm*)
qed

lemma *is-unit-clause-some-in*: *is-unit-clause l M* = *Some a* ⟹ *a* ∈ *set l*
unfolding *is-unit-clause-def*
proof −
assume (*case* [*a*←*l* . *atm-of a* ∉ *atm-of* ‘ *lits-of M*] *of* [] ⇒ *None*
| [*a*] ⇒ *if M* ⊨*as CNot* (*mset l* − {#*a*#}) *then Some a else None*
| *a* # *ab* # *xa* ⇒ *Map.empty xa*) = *Some a*
thus *a* ∈ *set l*
by (*case-tac* [*a*←*l* . *atm-of a* ∉ *atm-of* ‘ *lits-of M*])
(*fastforce dest*: *filter-eq-ConsD split*: *split-if-asm* *split*: *list.splits*)+
qed

lemma *is-unit-clause-nil*[*simp*]: *is-unit-clause* [] *M* = *None*
unfolding *is-unit-clause-def* **by** *auto*

18.1.2 Unit propagation for all clauses

Finding the first clause to propagate

fun *find-first-unit-clause* :: *'a literal list list* ⇒ (*'a*, *'b*, *'c*) *marked-lit list*
⇒ (*'a literal* × *'a literal list*) *option* **where**
find-first-unit-clause (*a* # *l*) *M* =
(*case is-unit-clause a M of*
None ⇒ *find-first-unit-clause l M*
| *Some L* ⇒ *Some* (*L*, *a*)) |
find-first-unit-clause [] - = *None*

lemma *find-first-unit-clause-some*:
find-first-unit-clause l M = *Some* (*a*, *c*)
⟹ *c* ∈ *set l* ∧ *M* ⊨*as CNot* (*mset c* − {#*a*#}) ∧ *undefined-lit M a* ∧ *a* ∈ *set c*
apply (*induction l*)
apply *simp*
by (*auto split*: *option.splits dest*: *is-unit-clause-some-in is-unit-clause-some-CNot*
is-unit-clause-some-undef)

lemma *propagate-is-unit-clause-not-None*:
assumes *dist*: *distinct c* **and**
M: *M* ⊨*as CNot* (*mset c* − {#*a*#}) **and**
undef: *undefined-lit M a* **and**
ac: *a* ∈ *set c*
shows *is-unit-clause c M* ≠ *None*
proof −
have [*a*←*c* . *atm-of a* ∉ *atm-of* ‘ *lits-of M*] = [*a*]
using *assms*
proof (*induction c*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*Cons ac c*)

show *?case*
proof (*cases* $a = ac$)
case *True*
thus *?thesis* **using** *Cons*
by (*auto simp del*: *lits-of-unfold*
simp add: *lits-of-unfold*[*symmetric*] *Marked-Propagated-in-iff-in-lits-of*
atm-of-eq-atm-of atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)
next
case *False*
hence *T*: $mset\ c + \{\#ac\#\} - \{\#a\#\} = mset\ c - \{\#a\#\} + \{\#ac\#\}$
by (*auto simp add*: *multiset-eq-iff*)
show *?thesis* **using** *False Cons*
by (*auto simp add*: *T atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set*)
qed
qed
thus *?thesis*
using *M* **unfolding** *is-unit-clause-def* **by** *auto*
qed

lemma *find-first-unit-clause-none*:
$distinct\ c \Longrightarrow c \in set\ l \Longrightarrow M \models as\ CNot\ (mset\ c - \{\#a\#\}) \Longrightarrow undefined\text{-}lit\ M\ a \Longrightarrow a \in set\ c$
$\Longrightarrow find\text{-}first\text{-}unit\text{-}clause\ l\ M \neq None$
by (*induction l*)
(*auto split*: *option.split simp add*: *propagate-is-unit-clause-not-None*)

18.1.3 Decide

fun *find-first-unused-var* :: $'a\ literal\ list\ list \Rightarrow 'a\ literal\ set \Rightarrow 'a\ literal\ option$ **where**
$find\text{-}first\text{-}unused\text{-}var\ (a\ \#\ l)\ M =$
$(case\ List.find\ (\lambda lit.\ lit \notin M \wedge -lit \notin M)\ a\ of$
$None \Rightarrow find\text{-}first\text{-}unused\text{-}var\ l\ M$
$|\ Some\ a \Rightarrow Some\ a)\ |$
$find\text{-}first\text{-}unused\text{-}var\ []\ \text{-} = None$

lemma *find-none*[*iff*]:
$List.find\ (\lambda lit.\ lit \notin M \wedge -lit \notin M)\ a = None \longleftrightarrow atm\text{-}of\ `\ set\ a \subseteq atm\text{-}of\ `\ M$
apply (*induct a*)
using *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set*
by (*force simp add*: *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set*)+

lemma *find-some*: $List.find\ (\lambda lit.\ lit \notin M \wedge -lit \notin M)\ a = Some\ b \Longrightarrow b \in set\ a \wedge b \notin M \wedge -b \notin M$
unfolding *find-Some-iff* **by** (*metis nth-mem*)

lemma *find-first-unused-var-None*[*iff*]:
$find\text{-}first\text{-}unused\text{-}var\ l\ M = None \longleftrightarrow (\forall a \in set\ l.\ atm\text{-}of\ `\ set\ a \subseteq atm\text{-}of\ `\ M)$
by (*induct l*)
(*auto split*: *option.splits dest*!: *find-some*
simp add: *image-subset-iff atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set*)

lemma *find-first-unused-var-Some-not-all-incl*:
assumes $find\text{-}first\text{-}unused\text{-}var\ l\ M = Some\ c$
shows $\neg(\forall a \in set\ l.\ atm\text{-}of\ `\ set\ a \subseteq atm\text{-}of\ `\ M)$
proof −
have $find\text{-}first\text{-}unused\text{-}var\ l\ M \neq None$
using *assms* **by** (*cases find-first-unused-var l M*) *auto*
thus $\neg(\forall a \in set\ l.\ atm\text{-}of\ `\ set\ a \subseteq atm\text{-}of\ `\ M)$ **by** *auto*

qed

lemma *find-first-unused-var-Some*:
  find-first-unused-var l M = Some a $\Longrightarrow$ ($\exists$ *m* $\in$ *set l. a* $\in$ *set m* $\wedge$ *a* $\notin$ *M* $\wedge$ $-$*a* $\notin$ *M*)
  by (*induct l*) (*auto split*: *option.splits dest*: *find-some*)

lemma *find-first-unused-var-undefined*:
  find-first-unused-var l (*lits-of Ms*) = *Some a* $\Longrightarrow$ *undefined-lit Ms a*
  using *find-first-unused-var-Some*[*of l lits-of Ms a*] *Marked-Propagated-in-iff-in-lits-of*
  by *blast*

end
theory *DPLL-W-Implementation*
imports *DPLL-CDCL-W-Implementation DPLL-W ~~/src/HOL/Library/Code-Target-Numeral*
begin

18.2 Simple Implementation of DPLL

18.2.1 Combining the propagate and decide: a DPLL step

definition *DPLL-step* :: *int* $dpll_W$*-marked-lits* $\times$ *int literal list list*
  $\Rightarrow$ *int* $dpll_W$*-marked-lits* $\times$ *int literal list list* **where**
DPLL-step = (λ(*Ms*, *N*).
  (*case find-first-unit-clause N Ms of*
    Some (*L*, -) $\Rightarrow$ (*Propagated L* () # *Ms*, *N*)
  | - $\Rightarrow$
    if $\exists$ *C* $\in$ *set N*. ($\forall$ *c* $\in$ *set C*. $-$*c* $\in$ *lits-of Ms*)
    then
      (*case backtrack-split Ms of*
        (-, *L* # *M*) $\Rightarrow$ (*Propagated* ($-$ (*lit-of L*)) () # *M*, *N*)
      | (-, -) $\Rightarrow$ (*Ms*, *N*)
)
    else
    (*case find-first-unused-var N* (*lits-of Ms*) *of*
        Some a $\Rightarrow$ (*Marked a* () # *Ms*, *N*)
      | *None* $\Rightarrow$ (*Ms*, *N*))))

Example of propagation:

value *DPLL-step* ([*Marked* (*Neg 1*) ()], [[*Pos* (*1*::*int*), *Neg 2*]])

We define the conversion function between the states as defined in *Prop-DPLL* (with multisets) and here (with lists).

abbreviation *toS* $\equiv$ λ(*Ms*::(*int*, *unit*, *unit*) *marked-lit list*)
                    (*N*:: *int literal list list*). (*Ms*, *mset* (*map mset N*))
abbreviation *toS'* $\equiv$ λ(*Ms*::(*int*, *unit*, *unit*) *marked-lit list*,
                    N:: *int literal list list*). (*Ms*, *mset* (*map mset N*))

Proof of correctness of *DPLL-step*

lemma *DPLL-step-is-a-*$dpll_W$*-step*:
  assumes *step*: (*Ms'*, *N'*) = *DPLL-step* (*Ms*, *N*)
  and *neq*: (*Ms*, *N*) $\neq$ (*Ms'*, *N'*)
  shows $dpll_W$ (*toS Ms N*) (*toS Ms' N'*)
proof $-$
  let *?S* = (*Ms*, *mset* (*map mset N*))
  { fix *L E*

assume *unit*: *find-first-unit-clause N Ms* = *Some* (*L*, *E*)
hence *Ms′N*: (*Ms′*, *N′*) = (*Propagated L* () # *Ms*, *N*)
using *step* **unfolding** *DPLL-step-def* **by** *auto*
obtain *C* **where**
C: *C* ∈ *set N* **and**
Ms: *Ms* ⊨*as CNot* (*mset C* − {#*L*#}) **and**
undef: *undefined-lit Ms L* **and**
L ∈ *set C* **using** *find-first-unit-clause-some*[*OF unit*] **by** *metis*
have $dpll_W$ (*Ms*, *mset* (*map mset N*))
(*Propagated L* () # *fst* (*Ms*, *mset* (*map mset N*)), *snd* (*Ms*, *mset* (*map mset N*)))
apply (*rule* $dpll_W$*.propagate*)
using *Ms undef C* ‹*L* ∈ *set C*› **unfolding** *mem-set-multiset-eq* **by** (*auto simp add*: *C*)
hence *?thesis* **using** *Ms′N* **by** *auto*
}
moreover
{ assume *unit*: *find-first-unit-clause N Ms* = *None*
assume *exC*: ∃ *C* ∈ *set N*. *Ms* ⊨*as CNot* (*mset C*)
then obtain *C* **where** *C*: *C* ∈ *set N* **and** *Ms*: *Ms* ⊨*as CNot* (*mset C*) **by** *auto*
then obtain *L M M′* **where** *bt*: *backtrack-split Ms* = (*M′*, *L* # *M*)
using *step exC neq* **unfolding** *DPLL-step-def prod.case unit*
by (*cases backtrack-split Ms*, *case-tac b*) *auto*
hence *is-marked L* **using** *backtrack-split-snd-hd-marked*[*of Ms*] **by** *auto*
have *1*: $dpll_W$ (*Ms*, *mset* (*map mset N*))
(*Propagated* (− *lit-of L*) () # *M*, *snd* (*Ms*, *mset* (*map mset N*)))
apply (*rule* $dpll_W$*.backtrack*[*OF* - ‹*is-marked L*›, *of*])
using *C Ms bt* **by** *auto*
moreover have (*Ms′*, *N′*) = (*Propagated* (− (*lit-of L*)) () # *M*, *N*)
using *step exC* **unfolding** *DPLL-step-def bt prod.case unit* **by** *auto*
ultimately have *?thesis* **by** *auto*
}
moreover
{ assume *unit*: *find-first-unit-clause N Ms* = *None*
assume *exC*: ¬ (∃ *C* ∈ *set N*. *Ms* ⊨*as CNot* (*mset C*))
obtain *L* **where** *unused*: *find-first-unused-var N* (*lits-of Ms*) = *Some L*
using *step exC neq* **unfolding** *DPLL-step-def prod.case unit*
by (*cases find-first-unused-var N* (*lits-of Ms*)) *auto*
have $dpll_W$ (*Ms*, *mset* (*map mset N*))
(*Marked L* () # *fst* (*Ms*, *mset* (*map mset N*)), *snd* (*Ms*, *mset* (*map mset N*)))
apply (*rule* $dpll_W$*.decided*[*of ?S L*])
using *find-first-unused-var-Some*[*OF unused*]
by (*auto simp add*: *Marked-Propagated-in-iff-in-lits-of atms-of-ms-def*)
moreover have (*Ms′*, *N′*) = (*Marked L* () # *Ms*, *N*)
using *step exC* **unfolding** *DPLL-step-def unused prod.case unit* **by** *auto*
ultimately have *?thesis* **by** *auto*
}
ultimately show *?thesis* **by** (*cases find-first-unit-clause N Ms*) *auto*
qed

lemma *DPLL-step-stuck-final-state*:
assumes *step*: (*Ms*, *N*) = *DPLL-step* (*Ms*, *N*)
shows *conclusive-*$dpll_W$*-state* (*toS Ms N*)
proof −
have *unit*: *find-first-unit-clause N Ms* = *None*
using *step* **unfolding** *DPLL-step-def* **by** (*auto split*:*option.splits*)

{ **assume** *n*: ∃ *C* ∈ *set N*. *Ms* ⊨*as CNot* (*mset C*)
hence *Ms*: (*Ms*, *N*) = (*case backtrack-split Ms of* (*x*, []) ⇒ (*Ms*, *N*)
| (*x*, *L* # *M*) ⇒ (*Propagated* (− *lit-of L*) () # *M*, *N*))
using *step* **unfolding** *DPLL-step-def* **by** (*simp add*:*unit*)

have *snd* (*backtrack-split Ms*) = []
proof (*cases backtrack-split Ms*, *cases snd* (*backtrack-split Ms*))
fix *a b*
assume *backtrack-split Ms* = (*a*, *b*) **and** *snd* (*backtrack-split Ms*) = []
thus *snd* (*backtrack-split Ms*) = [] **by** *blast*
next
fix *a b aa list*
assume
bt: *backtrack-split Ms* = (*a*, *b*) **and**
bt′: *snd* (*backtrack-split Ms*) = *aa* # *list*
hence *Ms*: *Ms* = *Propagated* (− *lit-of aa*) () # *list* **using** *Ms* **by** *auto*
have *is-marked aa* **using** *backtrack-split-snd-hd-marked*[*of Ms*] *bt bt′* **by** *auto*
moreover have *fst* (*backtrack-split Ms*) @ *aa* # *list* = *Ms*
using *backtrack-split-list-eq*[*of Ms*] *bt′* **by** *auto*
ultimately have *False* **unfolding** *Ms* **by** *auto*
thus *snd* (*backtrack-split Ms*) = [] **by** *blast*
qed

hence *?thesis*
using *n backtrack-snd-empty-not-marked*[*of Ms*] **unfolding** *conclusive-dpll*$_W$*-state-def*
by (*cases backtrack-split Ms*) *auto*
}
moreover {
assume *n*: ¬ (∃ *C* ∈ *set N*. *Ms* ⊨*as CNot* (*mset C*))
hence *find-first-unused-var N* (*lits-of Ms*) = *None*
using *step* **unfolding** *DPLL-step-def* **by** (*simp add*: *unit split*: *option.splits*)
hence *a*: ∀ *a* ∈ *set N*. *atm-of* ‘ *set a* ⊆ *atm-of* ‘ (*lits-of Ms*) **by** *auto*
have *fst* (*toS Ms N*) ⊨*asm snd* (*toS Ms N*) **unfolding** *true-annots-def CNot-def Ball-def*
proof *clarify*
fix *x*
assume *x*: *x* ∈ *set-mset* (*clauses* (*toS Ms N*))
hence ¬*Ms* ⊨*as CNot x* **using** *n* **unfolding** *true-annots-def CNot-def Ball-def* **by** *auto*
moreover have *total-over-m* (*lits-of Ms*) {*x*}
using *a x image-iff in-mono atms-of-s-def*
unfolding *total-over-m-def total-over-set-def lits-of-def* **by** *fastforce*
ultimately show *fst* (*toS Ms N*) ⊨*a x*
using *total-not-CNot*[*of lits-of Ms x*] **by** (*simp add*: *true-annot-def true-annots-true-cls*)
qed
hence *?thesis* **unfolding** *conclusive-dpll*$_W$*-state-def* **by** *blast*
}
ultimately show *?thesis* **by** *blast*
qed

18.2.2 Adding invariants

Invariant tested in the function **function** *DPLL-ci* :: *int dpll*$_W$*-marked-lits* ⇒ *int literal list list*
⇒ *int dpll*$_W$*-marked-lits* × *int literal list list* **where**
DPLL-ci Ms N =
(*if* ¬*dpll*$_W$*-all-inv* (*Ms*, *mset* (*map mset N*))
then (*Ms*, *N*)

else
let $(Ms', N') = $ *DPLL-step* (Ms, N) *in*
if $(Ms', N') = (Ms, N)$ *then* (Ms, N) *else* *DPLL-ci* Ms' N)
by *fast+*
termination
proof (*relation* $\{(S', S).\ (toS'\ S', toS'\ S) \in \{(S', S).\ dpll_W\text{-}all\text{-}inv\ S \wedge dpll_W\ S\ S'\}\}$)
show *wf* $\{(S', S).(toS'\ S', toS'\ S) \in \{(S', S).\ dpll_W\text{-}all\text{-}inv\ S \wedge dpll_W\ S\ S'\}\}$
using *wf-if-measure-f*[*OF* $dpll_W$*-wf*, *of toS'*] **by** *auto*
next
fix *Ms* :: *int* $dpll_W$*-marked-lits* **and** *N x xa y*
assume¬ ¬ $dpll_W$*-all-inv* (*toS Ms N*)
and *step*: $x = $ *DPLL-step* (Ms, N)
and *x*: $(xa, y) = x$
and $(xa, y) \neq (Ms, N)$
thus $((xa, N), Ms, N) \in \{(S', S).\ (toS'\ S', toS'\ S) \in \{(S', S).\ dpll_W\text{-}all\text{-}inv\ S \wedge dpll_W\ S\ S'\}\}$
using *DPLL-step-is-a-*$dpll_W$*-step* $dpll_W$*-same-clauses split-conv* **by** *fastforce*
qed

No invariant tested **function** (*domintros*) *DPLL-part*:: *int* $dpll_W$*-marked-lits* ⇒ *int literal list list* ⇒
int $dpll_W$*-marked-lits* × *int literal list list* **where**
DPLL-part Ms N =
(*let* $(Ms', N') = $ *DPLL-step* (Ms, N) *in*
if $(Ms', N') = (Ms, N)$ *then* (Ms, N) *else* *DPLL-part* Ms' N)
by *fast+*

lemma *snd-DPLL-step*[*simp*]:
snd (*DPLL-step* (Ms, N)) = *N*
unfolding *DPLL-step-def* **by** (*auto split*: *split-if option.splits prod.splits list.splits*)

lemma $dpll_W$*-all-inv-implieS-2-eq3-and-dom*:
assumes $dpll_W$*-all-inv* (*Ms*, *mset* (*map mset N*))
shows *DPLL-ci Ms N* = *DPLL-part Ms N* ∧ *DPLL-part-dom* (*Ms*, *N*)
using *assms*
proof (*induct rule*: *DPLL-ci.induct*)
case (*1 Ms N*)
have *snd* (*DPLL-step* (*Ms*, *N*)) = *N* **by** *auto*
then obtain Ms' **where** Ms': *DPLL-step* $(Ms, N) = (Ms', N)$ **by** (*case-tac DPLL-step* (*Ms*, *N*)) *auto*
have *inv'*: $dpll_W$*-all-inv* (*toS* Ms' *N*) **by** (*metis* (*mono-tags*) *1.prems DPLL-step-is-a-*$dpll_W$*-step* Ms'
$dpll_W$*-all-inv old.prod.inject*)
{ **assume** $(Ms', N) \neq (Ms, N)$
hence *DPLL-ci* Ms' *N* = *DPLL-part* Ms' *N* ∧ *DPLL-part-dom* (Ms', N) **using** *1*(*1*)[*of - Ms' N*] Ms'
1(*2*) *inv'* **by** *auto*
hence *DPLL-part-dom* (*Ms*, *N*) **using** *DPLL-part.domintros* Ms' **by** *fastforce*
moreover have *DPLL-ci Ms N* = *DPLL-part Ms N* **using** *1.prems DPLL-part.psimps* Ms'
‹*DPLL-ci* Ms' *N* = *DPLL-part* Ms' *N* ∧ *DPLL-part-dom* (Ms', N)› ‹*DPLL-part-dom* (*Ms*, *N*)› **by** *auto*
ultimately have *?case* **by** *blast*
}
moreover {
assume $(Ms', N) = (Ms, N)$
hence *?case* **using** *DPLL-part.domintros DPLL-part.psimps* Ms' **by** *fastforce*
}
ultimately show *?case* **by** *blast*

qed

lemma *DPLL-ci-dpll$_W$-rtranclp*:
assumes *DPLL-ci Ms N* = (*Ms'*, *N'*)
shows *dpll$_W$*** (*toS Ms N*) (*toS Ms' N*)
using *assms*
proof (*induct Ms N arbitrary*: *Ms' N' rule*: *DPLL-ci.induct*)
case (*1 Ms N Ms' N'*) **note** *IH* = *this*(*1*) **and** *step* = *this*(*2*)
obtain S_1 S_2 **where** *S*: (S_1, S_2) = *DPLL-step* (*Ms*, *N*) **by** (*case-tac DPLL-step* (*Ms*, *N*)) *auto*

{ **assume** ¬*dpll$_W$-all-inv* (*toS Ms N*)
hence (*Ms*, *N*) = (*Ms'*, *N*) **using** *step* **by** *auto*
hence *?case* **by** *auto*
}
moreover
{ **assume** *dpll$_W$-all-inv* (*toS Ms N*)
and (S_1, S_2) = (*Ms*, *N*)
hence *?case* **using** *S step* **by** *auto*
}
moreover
{ **assume** *dpll$_W$-all-inv* (*toS Ms N*)
and (S_1, S_2) ≠ (*Ms*, *N*)
moreover obtain S_1' S_2' **where** *DPLL-ci* S_1 *N* = (S_1', S_2') **by** (*case-tac DPLL-ci* S_1 *N*) *auto*
moreover have *DPLL-ci Ms N* = *DPLL-ci* S_1 *N* **using** *DPLL-ci.simps*[*of Ms N*] *calculation*
proof −
have (*case* (S_1, S_2) *of* (*ms*, *lss*) ⇒
if (*ms*, *lss*) = (*Ms*, *N*) *then* (*Ms*, *N*) *else DPLL-ci ms N*) = *DPLL-ci Ms N*
using *S DPLL-ci.simps*[*of Ms N*] *calculation* **by** *presburger*
hence (*if* (S_1, S_2) = (*Ms*, *N*) *then* (*Ms*, *N*) *else DPLL-ci* S_1 *N*) = *DPLL-ci Ms N*
by *fastforce*
thus *?thesis*
using *calculation*(*2*) **by** *presburger*
qed
ultimately have *dpll$_W$*** (*toS* S_1' *N*) (*toS Ms' N*) **using** *IH*[*of* (S_1, S_2) S_1 S_2] *S step* **by** *simp*

moreover have *dpll$_W$* (*toS Ms N*) (*toS* S_1 *N*)
by (*metis DPLL-step-is-a-dpll$_W$-step S* ‹(S_1, S_2) ≠ (*Ms*, *N*)› *prod.sel*(*2*) *snd-DPLL-step*)
ultimately have *?case* **by** (*metis* (*mono-tags*, *hide-lams*) *IH S* ‹(S_1, S_2) ≠ (*Ms*, *N*)›
‹*DPLL-ci Ms N* = *DPLL-ci* S_1 *N*› ‹*dpll$_W$-all-inv* (*toS Ms N*)› *converse-rtranclp-into-rtranclp*
local.step)
}
ultimately show *?case* **by** *blast*
qed

lemma *dpll$_W$-all-inv-dpll$_W$-tranclp-irrefl*:
assumes *dpll$_W$-all-inv* (*Ms*, *N*)
and *dpll$_W$*$^{++}$ (*Ms*, *N*) (*Ms*, *N*)
shows *False*
proof −
have *1*: *wf* {(*S'*, *S*). *dpll$_W$-all-inv S* ∧ *dpll$_W$*$^{++}$ *S S'*} **using** *dpll$_W$-wf-tranclp* **by** *auto*
have ((*Ms*, *N*), (*Ms*, *N*)) ∈ {(*S'*, *S*). *dpll$_W$-all-inv S* ∧ *dpll$_W$*$^{++}$ *S S'*} **using** *assms* **by** *auto*
thus *False* **using** *wf-not-refl*[*OF 1*] **by** *blast*
qed

lemma *DPLL-ci-final-state*:

assumes *step*: *DPLL-ci Ms N* = (*Ms*, *N*)
and *inv*: $dpll_W$*-all-inv* (*toS Ms N*)
shows *conclusive-*$dpll_W$*-state* (*toS Ms N*)
proof −
have *st*: $dpll_W^{**}$ (*toS Ms N*) (*toS Ms N*) **using** *DPLL-ci-*$dpll_W$*-rtranclp*[*OF step*] **.**
have *DPLL-step* (*Ms*, *N*) = (*Ms*, *N*)
proof (*rule ccontr*)
obtain *Ms′ N′* **where** *Ms′N*: (*Ms′*, *N′*) = *DPLL-step* (*Ms*, *N*)
by (*case-tac DPLL-step* (*Ms*, *N*)) *auto*
assume ¬ *?thesis*
hence *DPLL-ci Ms′ N* = (*Ms*, *N*) **using** *step inv st Ms′N*[*symmetric*] **by** *fastforce*
hence $dpll_W^{++}$ (*toS Ms N*) (*toS Ms N*)
by (*metis DPLL-ci-*$dpll_W$*-rtranclp DPLL-step-is-a-*$dpll_W$*-step Ms′N* ‹*DPLL-step* (*Ms*, *N*) ≠ (*Ms*, *N*)›
prod.sel(*2*) *rtranclp-into-tranclp2 snd-DPLL-step*)
thus *False* **using** $dpll_W$*-all-inv-*$dpll_W$*-tranclp-irrefl inv* **by** *auto*
qed
thus *?thesis* **using** *DPLL-step-stuck-final-state*[*of Ms N*] **by** *simp*
qed

lemma *DPLL-step-obtains*:
obtains *Ms′* **where** (*Ms′*, *N*) = *DPLL-step* (*Ms*, *N*)
unfolding *DPLL-step-def* **by** (*metis* (*no-types*, *lifting*) *DPLL-step-def prod.collapse snd-DPLL-step*)

lemma *DPLL-ci-obtains*:
obtains *Ms′* **where** (*Ms′*, *N*) = *DPLL-ci Ms N*
proof (*induct rule*: *DPLL-ci.induct*)
case (*1 Ms N*) **note** *IH* = *this*(*1*) **and** *that* = *this*(*2*)
obtain *S* **where** *SN*: (*S*, *N*) = *DPLL-step* (*Ms*, *N*) **using** *DPLL-step-obtains* **by** *metis*
{ assume ¬ $dpll_W$*-all-inv* (*toS Ms N*)
hence *?case* **using** *that* **by** *auto*
}
moreover {
assume *n*: (*S*, *N*) ≠ (*Ms*, *N*)
and *inv*: $dpll_W$*-all-inv* (*toS Ms N*)
have ∃ *ms*. *DPLL-step* (*Ms*, *N*) = (*ms*, *N*)
by (*metis* ‹⋀*thesisa*. (⋀*S*. (*S*, *N*) = *DPLL-step* (*Ms*, *N*) ⟹ *thesisa*) ⟹ *thesisa*›)
hence *?thesis*
using *IH that* **by** *fastforce*
}
moreover {
assume *n*: (*S*, *N*) = (*Ms*, *N*)
hence *?case* **using** *SN that* **by** *fastforce*
}
ultimately show *?case* **by** *blast*
qed

lemma *DPLL-ci-no-more-step*:
assumes *step*: *DPLL-ci Ms N* = (*Ms′*, *N′*)
shows *DPLL-ci Ms′ N′* = (*Ms′*, *N′*)
using *assms*
proof (*induct arbitrary*: *Ms′ N′ rule*: *DPLL-ci.induct*)
case (*1 Ms N Ms′ N′*) **note** *IH* = *this*(*1*) **and** *step* = *this*(*2*)
obtain S_1 **where** *S*: (S_1, *N*) = *DPLL-step* (*Ms*, *N*) **using** *DPLL-step-obtains* **by** *auto*

{ **assume** $\neg$*dpll*$_W$*-all-inv* (*toS Ms N*)
  hence *?case* **using** *step* **by** *auto*
}
moreover {
  assume *dpll*$_W$*-all-inv* (*toS Ms N*)
  and $(S_1, N) = (Ms, N)$
  hence *?case* **using** *S step* **by** *auto*
}
moreover
{ **assume** *inv*: *dpll*$_W$*-all-inv* (*toS Ms N*)
  assume *n*: $(S_1, N) \neq (Ms, N)$
  obtain S_1' **where** *SS*: (S_1', N) = *DPLL-ci* S_1 *N* **using** *DPLL-ci-obtains* **by** *blast*
  moreover have *DPLL-ci Ms N* = *DPLL-ci* S_1 *N*
    proof –
      have (*case* (S_1, N) *of* (*ms*, *lss*) $\Rightarrow$ *if* (*ms*, *lss*) = (*Ms*, *N*) *then* (*Ms*, *N*) *else DPLL-ci ms N*)
        = *DPLL-ci Ms N*
        using *S DPLL-ci.simps*[*of Ms N*] *calculation inv* **by** *presburger*
      hence (*if* $(S_1, N) = (Ms, N)$ *then* (*Ms*, *N*) *else DPLL-ci* S_1 *N*) = *DPLL-ci Ms N*
        by *fastforce*
      thus *?thesis*
        using *calculation n* **by** *presburger*
    qed
  moreover
    have *DPLL-ci* S_1' *N* = (S_1', N) **using** *step IH*[*OF - - S n SS*[*symmetric*]] *inv* **by** *blast*
  ultimately have *?case* **using** *step* **by** *fastforce*
}
ultimately show *?case* **by** *blast*
qed

lemma *DPLL-part-dpll*$_W$*-all-inv-final*:
  fixes *M Ms'*:: (*int*, *unit*, *unit*) *marked-lit list* **and**
    N :: *int literal list list*
  assumes *inv*: *dpll*$_W$*-all-inv* (*Ms*, *mset* (*map mset N*))
  and *MsN*: *DPLL-part Ms N* = (*Ms'*, *N*)
  shows *conclusive-dpll*$_W$*-state* (*toS Ms' N*) $\wedge$ *dpll*$_W$** (*toS Ms N*) (*toS Ms' N*)
proof –
  have *2*: *DPLL-ci Ms N* = *DPLL-part Ms N* **using** *inv dpll*$_W$*-all-inv-implieS-2-eq3-and-dom* **by** *blast*
  hence *star*: *dpll*$_W$** (*toS Ms N*) (*toS Ms' N*) **unfolding** *MsN* **using** *DPLL-ci-dpll*$_W$*-rtranclp* **by** *blast*
  hence *inv'*: *dpll*$_W$*-all-inv* (*toS Ms' N*) **using** *inv rtranclp-dpll*$_W$*-all-inv* **by** *blast*
  show *?thesis* **using** *star DPLL-ci-final-state*[*OF DPLL-ci-no-more-step inv'*] *2* **unfolding** *MsN* **by** *blast*
qed

Embedding the invariant into the type

Defining the type **typedef** *dpll*$_W$*-state* =
  {(*M*::(*int*, *unit*, *unit*) *marked-lit list*, *N*::*int literal list list*).
    dpll$_W$*-all-inv* (*toS M N*)}
  morphisms *rough-state-of state-of*
proof
  show ([],[]) $\in$ {(*M*, *N*). *dpll*$_W$*-all-inv* (*toS M N*)} **by** (*auto simp add*: *dpll*$_W$*-all-inv-def*)
qed

lemma
DPLL-part-dom ([], *N*)
using *assms* *dpll*$_W$*-all-inv-implieS-2-eq3-and-dom*[*of* [] *N*] **by** (*simp add*: *dpll*$_W$*-all-inv-def*)

Some type classes **instantiation** *dpll*$_W$*-state* :: *equal*
begin
definition *equal-dpll*$_W$*-state* :: *dpll*$_W$*-state* $\Rightarrow$ *dpll*$_W$*-state* $\Rightarrow$ *bool* **where**
equal-dpll$_W$*-state S S'* = (*rough-state-of S* = *rough-state-of S'*)
instance
by *standard* (*simp add*: *rough-state-of-inject equal-dpll*$_W$*-state-def*)
end

DPLL **definition** *DPLL-step'* :: *dpll*$_W$*-state* $\Rightarrow$ *dpll*$_W$*-state* **where**
DPLL-step' S = *state-of* (*DPLL-step* (*rough-state-of S*))

declare *rough-state-of-inverse*[*simp*]

lemma *DPLL-step-dpll*$_W$*-conc-inv*:
DPLL-step (*rough-state-of S*) $\in$ {(*M*, *N*). *dpll*$_W$*-all-inv* (*toS M N*)}
by (*smt DPLL-ci.simps DPLL-ci-dpll*$_W$*-rtranclp case-prodE case-prodI2 rough-state-of mem-Collect-eq old.prod.case prod.sel*(*2*) *rtranclp-dpll*$_W$*-all-inv snd-DPLL-step*)

lemma *rough-state-of-DPLL-step'-DPLL-step*[*simp*]:
rough-state-of (*DPLL-step' S*) = *DPLL-step* (*rough-state-of S*)
using *DPLL-step-dpll*$_W$*-conc-inv DPLL-step'-def state-of-inverse* **by** *auto*

function *DPLL-tot*:: *dpll*$_W$*-state* $\Rightarrow$ *dpll*$_W$*-state* **where**
DPLL-tot S =
(*let S'* = *DPLL-step' S in*
if S' = *S then S else DPLL-tot S'*)
by *fast+*
termination
proof (*relation* {(*T'*, *T*).
(*rough-state-of T'*, *rough-state-of T*)
$\in$ {(*S'*, *S*). (*toS' S'*, *toS' S*)
$\in$ {(*S'*, *S*). *dpll*$_W$*-all-inv S* $\wedge$ *dpll*$_W$ *S S'*}}})
show *wf* {(*b*, *a*).
(*rough-state-of b*, *rough-state-of a*)
$\in$ {(*b*, *a*). (*toS' b*, *toS' a*)
$\in$ {(*b*, *a*). *dpll*$_W$*-all-inv a* $\wedge$ *dpll*$_W$ *a b*}}}
using *wf-if-measure-f*[*OF wf-if-measure-f*[*OF dpll*$_W$*-wf*, *of toS'*], *of rough-state-of*] **.**
next
fix *S x*
assume *x*: *x* = *DPLL-step' S*
and *x* $\neq$ *S*
have *dpll*$_W$*-all-inv* (*case rough-state-of S of* (*Ms*, *N*) $\Rightarrow$ (*Ms*, *mset* (*map mset N*)))
by (*metis* (*no-types*, *lifting*) *case-prodE mem-Collect-eq old.prod.case rough-state-of*)
moreover have *dpll*$_W$ (*case rough-state-of S of* (*Ms*, *N*) $\Rightarrow$ (*Ms*, *mset* (*map mset N*)))
(*case rough-state-of* (*DPLL-step' S*) *of* (*Ms*, *N*) $\Rightarrow$ (*Ms*, *mset* (*map mset N*)))
proof −
obtain *Ms N* **where** *Ms*: (*Ms*, *N*) = *rough-state-of S* **by** (*cases rough-state-of S*) *auto*
have *dpll*$_W$*-all-inv* (*toS'* (*Ms*, *N*)) **using** *calculation* **unfolding** *Ms* **by** *blast*
moreover obtain *Ms' N'* **where** *Ms'*: (*Ms'*, *N'*) = *rough-state-of* (*DPLL-step' S*)
by (*cases rough-state-of* (*DPLL-step' S*)) *auto*
ultimately have *dpll*$_W$*-all-inv* (*toS'* (*Ms'*, *N'*)) **unfolding** *Ms'*

by (*metis* (*no-types*, *lifting*) *case-prod-unfold mem-Collect-eq rough-state-of*)

have $dpll_W$ (*toS Ms N*) (*toS Ms' N'*)
apply (*rule DPLL-step-is-a-dpll*$_W$*-step*[*of Ms' N' Ms N*])
unfolding *Ms Ms'* **using** ⟨$x \neq S$⟩ *rough-state-of-inject x* **by** *fastforce+*
thus *?thesis* **unfolding** *Ms*[*symmetric*] *Ms'*[*symmetric*] **by** *auto*
qed
ultimately show $(x, S) \in \{(T', T).$ (*rough-state-of* T', *rough-state-of* T)
$\in \{(S', S).$ (*toS'* S', *toS'* S) $\in \{(S', S).$ $dpll_W$*-all-inv* $S \wedge dpll_W$ S $S'\}\}\}$
by (*auto simp add*: *x*)
qed

lemma [*code*]:
DPLL-tot S =
(*let* S' = *DPLL-step'* S *in*
if $S' = S$ *then* S *else* *DPLL-tot* S') **by** *auto*

lemma *DPLL-tot-DPLL-step-DPLL-tot*[*simp*]: *DPLL-tot* (*DPLL-step'* S) = *DPLL-tot* S
apply (*cases DPLL-step'* $S = S$)
apply *simp*
unfolding *DPLL-tot.simps*[*of S*] **by** (*simp del*: *DPLL-tot.simps*)

lemma *DOPLL-step'-DPLL-tot*[*simp*]:
DPLL-step' (*DPLL-tot* S) = *DPLL-tot* S
by (*rule DPLL-tot.induct*[*of* λS. *DPLL-step'* (*DPLL-tot* S) = *DPLL-tot* S S])
(*metis* (*full-types*) *DPLL-tot.simps*)

lemma *DPLL-tot-final-state*:
assumes *DPLL-tot* $S = S$
shows *conclusive-dpll*$_W$*-state* (*toS'* (*rough-state-of* S))
proof −
have *DPLL-step'* $S = S$ **using** *assms*[*symmetric*] *DOPLL-step'-DPLL-tot* **by** *metis*
hence *DPLL-step* (*rough-state-of* S) = (*rough-state-of* S)
unfolding *DPLL-step'-def* **using** *DPLL-step-dpll*$_W$*-conc-inv rough-state-of-inverse*
by (*metis rough-state-of-DPLL-step'-DPLL-step*)
thus *?thesis*
by (*metis* (*mono-tags*, *lifting*) *DPLL-step-stuck-final-state old.prod.exhaust split-conv*)
qed

lemma *DPLL-tot-star*:
assumes *rough-state-of* (*DPLL-tot* S) = S'
shows $dpll_W{}^{**}$ (*toS'* (*rough-state-of* S)) (*toS'* S')
using *assms*
proof (*induction arbitrary*: S' *rule*: *DPLL-tot.induct*)
case (*1 S S'*)
let *?x* = *DPLL-step'* S
{ assume *?x* = S
then have *?case* **using** *1*(*2*) **by** *simp*
}
moreover {
assume S: *?x* $\neq S$
have *?case*
apply (*cases DPLL-step'* $S = S$)

    using *S* **apply** *blast*
  by (*smt 1.IH 1.prems DPLL-step-is-a-dpll$_W$-step DPLL-tot.simps case-prodE2 rough-state-of-DPLL-step'-DPLL-step rtranclp.rtrancl-into-rtrancl rtranclp.rtrancl-refl rtranclp-idemp split-conv*)
}
ultimately show *?case* **by** *auto*
qed

lemma *rough-state-of-rough-state-of-nil*[*simp*]:
rough-state-of (*state-of* ([], *N*)) = ([], *N*)
apply (*rule DPLL-W-Implementation.dpll$_W$-state.state-of-inverse*)
unfolding *dpll$_W$-all-inv-def* **by** *auto*

Theorem of correctness

lemma *DPLL-tot-correct*:
assumes *rough-state-of* (*DPLL-tot* (*state-of* (([], *N*)))) = (*M*, *N'*)
and (*M'*, *N''*) = *toS'* (*M*, *N'*)
shows *M'* $\models$*asm* *N''* $\longleftrightarrow$ *satisfiable* (*set-mset* *N''*)
proof –
have *dpll$_W$*** (*toS'* ([], *N*)) (*toS'* (*M*, *N'*)) **using** *DPLL-tot-star*[*OF assms*(*1*)] **by** *auto*
moreover have *conclusive-dpll$_W$-state* (*toS'* (*M*, *N'*))
  using *DPLL-tot-final-state* **by** (*metis* (*mono-tags*, *lifting*) *DOPLL-step'-DPLL-tot DPLL-tot.simps assms*(*1*))
ultimately show *?thesis* **using** *dpll$_W$-conclusive-state-correct* **by** (*smt DPLL-ci.simps DPLL-ci-dpll$_W$-rtranclp assms*(*2*) *dpll$_W$-all-inv-def prod.case prod.sel*(*1*) *prod.sel*(*2*) *rtranclp-dpll$_W$-inv*(*3*) *rtranclp-dpll$_W$-inv-starting-from-0*)
qed

18.2.3 Code export

A conversion to *DPLL-W-Implementation.dpll$_W$-state* **definition** *Con* :: (*int*, *unit*, *unit*) *marked-lit list* × *int literal list list*
  ⇒ *dpll$_W$-state* **where**
Con xs = *state-of* (*if dpll$_W$-all-inv* (*toS* (*fst xs*) (*snd xs*)) *then xs else* ([], []))
lemma [*code abstype*]:
Con (*rough-state-of S*) = *S*
using *rough-state-of*[*of S*] **unfolding** *Con-def* **by** *auto*

declare *rough-state-of-DPLL-step'-DPLL-step*[*code abstract*]

lemma *Con-DPLL-step-rough-state-of-state-of*[*simp*]:
Con (*DPLL-step* (*rough-state-of s*)) = *state-of* (*DPLL-step* (*rough-state-of s*))
unfolding *Con-def* **by** (*metis* (*mono-tags*, *lifting*) *DPLL-step-dpll$_W$-conc-inv mem-Collect-eq prod.case-eq-if*)

A slightly different version of *DPLL-tot* where the returned boolean indicates the result.

definition *DPLL-tot-rep* **where**
DPLL-tot-rep S =
(*let* (*M*, *N*) = (*rough-state-of* (*DPLL-tot S*)) *in* (∀ *A* ∈ *set N*. (∃ *a*∈*set A*. *a* ∈ *lits-of* (*M*)), *M*))

One version of the generated SML code is here, but not included in the generated document. The only differences are:

- export *'a literal* from the SML Module *Clausal-Logic*;
- export the constructor *Con* from *DPLL-W-Implementation*;

- export the *int* constructor from *Arith*.

  All these allows to test on the code on some examples.

end
theory *CDCL-W-Implementation*
imports *DPLL-CDCL-W-Implementation CDCL-W-Termination*
begin

notation *image-mset* (**infixr** *'# 90*)

type-synonym $'a$ $cdcl_W$*-mark* = $'a$ *clause*
type-synonym $cdcl_W$*-marked-level* = *nat*

type-synonym $'v$ $cdcl_W$*-marked-lit* = ($'v$, $cdcl_W$*-marked-level*, $'v$ $cdcl_W$*-mark*) *marked-lit*
type-synonym $'v$ $cdcl_W$*-marked-lits* = ($'v$, $cdcl_W$*-marked-level*, $'v$ $cdcl_W$*-mark*) *marked-lits*
type-synonym $'v$ $cdcl_W$*-state* =
  $'v$ $cdcl_W$*-marked-lits* × $'v$ *clauses* × $'v$ *clauses* × *nat* × $'v$ *clause option*

abbreviation *trail* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'a$ **where**
trail ≡ (λ(*M*, -). *M*)

abbreviation *cons-trail* :: $'a \Rightarrow 'a$ *list* $\times 'b \times 'c \times 'd \times 'e \Rightarrow 'a$ *list* $\times 'b \times 'c \times 'd \times 'e$
  where
cons-trail ≡ (λ*L* (*M*, *S*). (*L*#*M*, *S*))

abbreviation *tl-trail* :: $'a$ *list* $\times 'b \times 'c \times 'd \times 'e \Rightarrow 'a$ *list* $\times 'b \times 'c \times 'd \times 'e$ **where**
tl-trail ≡ (λ(*M*, *S*). (*tl M*, *S*))

abbreviation *clauses* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'b$ **where**
clauses ≡ λ(*M*, *N*, -). *N*

abbreviation *learned-clss* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'c$ **where**
learned-clss ≡ λ(*M*, *N*, *U*, -). *U*

abbreviation *backtrack-lvl* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'd$ **where**
backtrack-lvl ≡ λ(*M*, *N*, *U*, *k*, -). *k*

abbreviation *update-backtrack-lvl* :: $'d \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e$
  where
update-backtrack-lvl ≡ λ*k* (*M*, *N*, *U*, -, *S*). (*M*, *N*, *U*, *k*, *S*)

abbreviation *conflicting* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'e$ **where**
conflicting ≡ λ(*M*, *N*, *U*, *k*, *D*). *D*

abbreviation *update-conflicting* :: $'e \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e$
  where
update-conflicting ≡ λ*S* (*M*, *N*, *U*, *k*, -). (*M*, *N*, *U*, *k*, *S*)

abbreviation *S0-*$cdcl_W$ *N* ≡ (([], *N*, {#}, *0*, *None*):: $'v$ $cdcl_W$*-state*)

abbreviation *add-learned-cls* **where**
add-learned-cls ≡ λ*C* (*M*, *N*, *U*, *S*). (*M*, *N*, {#*C*#} + *U*, *S*)

abbreviation *remove-cls* **where**
remove-cls ≡ λ*C* (*M*, *N*, *U*, *S*). (*M*, *remove-mset C N*, *remove-mset C U*, *S*)

interpretation $cdcl_W$: $state_W$ *trail clauses learned-clss backtrack-lvl conflicting*
λL (M, S). $(L \# M, S)$
$\lambda(M, S)$. $(tl\ M, S)$
λC (M, N, S). $(M, \{\#C\#\} + N, S)$
λC (M, N, U, S). $(M, N, \{\#C\#\} + U, S)$
λC (M, N, U, S). $(M$, *remove-mset* C N, *remove-mset* C U, $S)$
$\lambda(k{::}nat)$ $(M, N, U, \text{-}, D)$. (M, N, U, k, D)
λD $(M, N, U, k, \text{-})$. (M, N, U, k, D)
λN. $([], N, \{\#\}, 0, None)$
$\lambda(\text{-}, N, U, \text{-})$. $([], N, U, 0, None)$
by *unfold-locales auto*

lemma *trail-conv*: *trail* $(M, N, U, k, D) = M$ **and**
clauses-conv: *clauses* $(M, N, U, k, D) = N$ **and**
learned-clss-conv: *learned-clss* $(M, N, U, k, D) = U$ **and**
conflicting-conv: *conflicting* $(M, N, U, k, D) = D$ **and**
backtrack-lvl-conv: *backtrack-lvl* $(M, N, U, k, D) = k$
by *auto*

lemma *state-conv*:
$S = ($*trail* S, *clauses* S, *learned-clss* S, *backtrack-lvl* S, *conflicting* $S)$
by (*cases* S) *auto*

interpretation $cdcl_W$*-termination trail clauses learned-clss backtrack-lvl conflicting*
λL (M, S). $(L \# M, S)$
$\lambda(M, S)$. $(tl\ M, S)$
λC (M, N, S). $(M, \{\#C\#\} + N, S)$
λC (M, N, U, S). $(M, N, \{\#C\#\} + U, S)$
λC (M, N, U, S). $(M$, *remove-mset* C N, *remove-mset* C U, $S)$
$\lambda(k{::}nat)$ $(M, N, U, \text{-}, D)$. (M, N, U, k, D)
λD $(M, N, U, k, \text{-})$. (M, N, U, k, D)
λN. $([], N, \{\#\}, 0, None)$
$\lambda(\text{-}, N, U, \text{-})$. $([], N, U, 0, None)$
by *intro-locales*

lemmas $cdcl_W$*.clauses-def*[*simp*]

lemma $cdcl_W$*-state-eq-equality*[*iff*]: $cdcl_W$*.state-eq* S $T \longleftrightarrow S = T$
unfolding $cdcl_W$*.state-eq-def* **by** (*cases* S, *cases* T) *auto*
declare $cdcl_W$*.state-simp*[*simp del*]

18.3 CDCL Implementation

18.3.1 Definition of the rules

Types **lemma** *true-clss-remdups*[*simp*]:
$I \models s$ (*mset* $\circ$ *remdups*) ‘ $N \longleftrightarrow I \models s$ *mset* ‘ N
by (*simp add*: *true-clss-def*)

lemma *satisfiable-mset-remdups*[*simp*]:
satisfiable ((*mset* $\circ$ *remdups*) ‘ N) $\longleftrightarrow$ *satisfiable* (*mset* ‘ N)
unfolding *satisfiable-carac*[*symmetric*] **by** *simp*

declare *mset-map*[*symmetric*, *simp*]

value *backtrack-split* [*Marked* (*Pos* (*Suc 0*)) ()]
value $\exists$ *C* $\in$ *set* [[*Pos* (*Suc 0*), *Neg* (*Suc 0*)]]. ($\forall$ *c* $\in$ *set C*. $-c \in$ *lits-of* [*Marked* (*Pos* (*Suc 0*)) ()])

type-synonym *cdcl*$_W$*-state-inv-st* = (*nat*, *nat*, *nat literal list*) *marked-lit list* $\times$
nat literal list list $\times$ *nat literal list list* $\times$ *nat* $\times$ *nat literal list option*

We need some functions to convert between our abstract state *nat cdcl*$_W$*-state* and the concrete state *cdcl*$_W$*-state-inv-st*.

fun *convert* :: ('*a*, '*b*, '*c list*) *marked-lit* $\Rightarrow$ ('*a*, '*b*, '*c multiset*) *marked-lit* **where**
convert (*Propagated L C*) = *Propagated L* (*mset C*) |
convert (*Marked K i*) = *Marked K i*

abbreviation *convertC* :: '*a list option* $\Rightarrow$ '*a multiset option* **where**
convertC $\equiv$ *map-option mset*

lemma *convert-Propagated*[*elim*!]:
convert z = *Propagated L C* $\Longrightarrow$ ($\exists$ *C'*. *z* = *Propagated L C'* $\wedge$ *C* = *mset C'*)
by (*cases z*) *auto*

lemma *get-rev-level-map-convert*:
get-rev-level (*map convert M*) *n x* = *get-rev-level M n x*
by (*induction M arbitrary*: *n rule*: *marked-lit-list-induct*) *auto*

lemma *get-level-map-convert*[*simp*]:
get-level (*map convert M*) = *get-level M*
using *get-rev-level-map-convert*[*of rev M*] **by** (*simp add*: *rev-map*)

lemma *get-maximum-level-map-convert*[*simp*]:
get-maximum-level (*map convert M*) *D* = *get-maximum-level M D*
by (*induction D*)
(*auto simp add*: *get-maximum-level-plus*)

lemma *get-all-levels-of-marked-map-convert*[*simp*]:
get-all-levels-of-marked (*map convert M*) = (*get-all-levels-of-marked M*)
by (*induction M rule*: *marked-lit-list-induct*) *auto*

Conversion function

fun *toS* :: *cdcl*$_W$*-state-inv-st* $\Rightarrow$ *nat cdcl*$_W$*-state* **where**
toS (*M*, *N*, *U*, *k*, *C*) = (*map convert M*, *mset* (*map mset N*), *mset* (*map mset U*), *k*, *convertC C*)

Definition an abstract type

typedef *cdcl*$_W$*-state-inv* = {*S*::*cdcl*$_W$*-state-inv-st*. *cdcl*$_W$*-all-struct-inv* (*toS S*)}
morphisms *rough-state-of state-of*
proof
show ([],[], [], *0*, *None*) $\in$ {*S*. *cdcl*$_W$*-all-struct-inv* (*toS S*)}
by (*auto simp add*: *cdcl*$_W$*-all-struct-inv-def*)
qed

instantiation *cdcl*$_W$*-state-inv* :: *equal*
begin
definition *equal-cdcl*$_W$*-state-inv* :: *cdcl*$_W$*-state-inv* $\Rightarrow$ *cdcl*$_W$*-state-inv* $\Rightarrow$ *bool* **where**
equal-cdcl$_W$*-state-inv S S'* = (*rough-state-of S* = *rough-state-of S'*)
instance
by *standard* (*simp add*: *rough-state-of-inject equal-cdcl*$_W$*-state-inv-def*)

end

lemma *lits-of-map-convert*[*simp*]: *lits-of* (*map convert M*) = *lits-of M*
  by (*induction M rule*: *marked-lit-list-induct*) *simp-all*

lemma *undefined-lit-map-convert*[*iff*]:
  undefined-lit (*map convert M*) *L* ⟷ *undefined-lit M L*
  by (*auto simp add*: *Marked-Propagated-in-iff-in-lits-of*)

lemma *true-annot-map-convert*[*simp*]: *map convert M* ⊨*a N* ⟷ *M* ⊨*a N*
  by (*induction M rule*: *marked-lit-list-induct*) (*simp-all add*: *true-annot-def*)

lemma *true-annots-map-convert*[*simp*]: *map convert M* ⊨*as N* ⟷ *M* ⊨*as N*
  unfolding *true-annots-def* **by** *auto*

lemmas *propagateE*
lemma *find-first-unit-clause-some-is-propagate*:
  assumes *H*: *find-first-unit-clause* (*N* @ *U*) *M* = *Some* (*L*, *C*)
  shows *propagate* (*toS* (*M*, *N*, *U*, *k*, *None*)) (*toS* (*Propagated L C* # *M*, *N*, *U*, *k*, *None*))
  using *assms*
  by (*auto dest*!: *find-first-unit-clause-some simp add*: *propagate.simps*
    intro!: *exI*[*of - mset C* − {#*L*#}])

18.3.2 The Transitions

Propagate **definition** *do-propagate-step* **where**
do-propagate-step S =
  (*case S of*
    (*M*, *N*, *U*, *k*, *None*) ⇒
      (*case find-first-unit-clause* (*N* @ *U*) *M of*
        Some (*L*, *C*) ⇒ (*Propagated L C* # *M*, *N*, *U*, *k*, *None*)
      | *None* ⇒ (*M*, *N*, *U*, *k*, *None*))
  | *S* ⇒ *S*)

lemma *do-propgate-step*:
  do-propagate-step S ≠ *S* ⟹ *propagate* (*toS S*) (*toS* (*do-propagate-step S*))
  apply (*cases S*, *cases conflicting S*)
  using *find-first-unit-clause-some-is-propagate*[*of clauses S learned-clss S trail S - -*
    backtrack-lvl S]
  by (*auto simp add*: *do-propagate-step-def split*: *option.splits*)

lemma *do-propagate-step-option*[*simp*]:
  conflicting S ≠ *None* ⟹ *do-propagate-step S* = *S*
  unfolding *do-propagate-step-def* **by** (*cases S*, *cases conflicting S*) *auto*

lemma *do-propagate-step-no-step*:
  assumes *dist*: ∀ *c*∈*set* (*clauses S* @ *learned-clss S*). *distinct c* **and**
  prop-step: *do-propagate-step S* = *S*
  shows *no-step propagate* (*toS S*)
proof (*standard*, *standard*)
  fix *T*
  assume *propagate* (*toS S*) *T*
  then obtain *M N U k C L* **where**
    toSS: *toS S* = (*M*, *N*, *U*, *k*, *None*) **and**
    T: *T* = (*Propagated L* (*C* + {#*L*#}) # *M*, *N*, *U*, *k*, *None*) **and**
    MC: *M* ⊨*as CNot C* **and**

undef: *undefined-lit M L* **and**
CL: $C + \{\#L\#\} \in\# N + U$
apply − **by** (*cases toS S*) *auto*
let *?M* = *trail S*
let *?N* = *clauses S*
let *?U* = *learned-clss S*
let *?k* = *backtrack-lvl S*
let *?D* = *None*
have *S*: *S* = (*?M*, *?N*, *?U*, *?k*, *?D*)
using *toSS* **by** (*cases S*, *cases conflicting S*) *simp-all*
have *S*: *toS S* = *toS* (*?M*, *?N*, *?U*, *?k*, *?D*)
unfolding *S*[*symmetric*] **by** *simp*

have
M: *M* = *map convert ?M* **and**
N: *N* = *mset* (*map mset ?N*) **and**
U: *U* = *mset* (*map mset ?U*)
using *toSS*[*unfolded S*] **by** *auto*

obtain *D* **where**
DCL: *mset D* = $C + \{\#L\#\}$ **and**
D: $D \in$ *set* (*?N* @ *?U*)
using *CL* **unfolding** *N U* **by** *auto*
obtain C' L' **where**
setD: *set D* = *set* (L' # C') **and**
C': *mset* $C' = C$ **and**
L: $L = L'$
using *DCL* **by** (*metis ex-mset mset.simps*(*2*) *mset-eq-setD*)
have *find-first-unit-clause* (*?N* @ *?U*) *?M* $\neq$ *None*
apply (*rule dist find-first-unit-clause-none*[*of D ?N* @ *?U ?M L*, *OF - D*])
using *D assms*(*1*) **apply** *auto*[*1*]
using *MC setD DCL M MC* **unfolding** C'[*symmetric*] **apply** *auto*[*1*]
using *M undef* **apply** *auto*[*1*]
unfolding *setD L* **by** *auto*
then show *False* **using** *prop-step S* **unfolding** *do-propagate-step-def* **by** (*cases S*) *auto*
qed

Conflict **fun** *find-conflict* **where**
find-conflict M [] = *None* |
find-conflict M (*N* # *Ns*) = (*if* ($\forall c \in$ *set N*. $-c \in$ *lits-of M*) *then Some N else find-conflict M Ns*)

lemma *find-conflict-Some*:
find-conflict M Ns = *Some N* $\Longrightarrow$ $N \in$ *set Ns* $\wedge$ *M* $\models$*as CNot* (*mset N*)
by (*induction Ns rule*: *find-conflict.induct*)
(*auto split*: *split-if-asm*)

lemma *find-conflict-None*:
find-conflict M Ns = *None* $\longleftrightarrow$ ($\forall N \in$ *set Ns*. $\neg$*M* $\models$*as CNot* (*mset N*))
by (*induction Ns*) *auto*

lemma *find-conflict-None-no-confl*:
find-conflict M (*N*@*U*) = *None* $\longleftrightarrow$ *no-step conflict* (*toS* (*M*, *N*, *U*, *k*, *None*))
by (*auto simp add*: *find-conflict-None conflict.simps*)

definition *do-conflict-step* **where**

do-conflict-step S =
  (*case S of*
    (*M*, *N*, *U*, *k*, *None*) ⇒
      (*case find-conflict M* (*N* @ *U*) *of*
        Some a ⇒ (*M*, *N*, *U*, *k*, *Some a*)
      | *None* ⇒ (*M*, *N*, *U*, *k*, *None*))
  | *S* ⇒ *S*)

lemma *do-conflict-step*:
  do-conflict-step S ≠ *S* ⟹ *conflict* (*toS S*) (*toS* (*do-conflict-step S*))
  apply (*cases S*, *cases conflicting S*)
  unfolding *conflict.simps do-conflict-step-def*
  by (*auto dest*!:*find-conflict-Some split*: *option.splits*)

lemma *do-conflict-step-no-step*:
  do-conflict-step S = *S* ⟹ *no-step conflict* (*toS S*)
  apply (*cases S*, *cases conflicting S*)
  unfolding *do-conflict-step-def*
  using *find-conflict-None-no-confl*[*of trail S clauses S learned-clss S*
    backtrack-lvl S]
  by (*auto split*: *option.splits*)

lemma *do-conflict-step-option*[*simp*]:
  conflicting S ≠ *None* ⟹ *do-conflict-step S* = *S*
  unfolding *do-conflict-step-def* **by** (*cases S*, *cases conflicting S*) *auto*

lemma *do-conflict-step-conflicting*[*dest*]:
  do-conflict-step S ≠ *S* ⟹ *conflicting* (*do-conflict-step S*) ≠ *None*
  unfolding *do-conflict-step-def* **by** (*cases S*, *cases conflicting S*) (*auto split*: *option.splits*)

definition *do-cp-step* **where**
do-cp-step S =
  (*do-propagate-step o do-conflict-step*) *S*

lemma *cp-step-is-cdcl*$_W$*-cp*:
  assumes *H*: *do-cp-step S* ≠ *S*
  shows *cdcl*$_W$*-cp* (*toS S*) (*toS* (*do-cp-step S*))
proof −
  show *?thesis*
  proof (*cases do-conflict-step S* ≠ *S*)
    case *True*
    then show *?thesis*
      by (*auto simp add*: *do-conflict-step do-conflict-step-conflicting do-cp-step-def*)
  next
    case *False*
    then have *confl*[*simp*]: *do-conflict-step S* = *S* **by** *simp*
    show *?thesis*
      proof (*cases do-propagate-step S* = *S*)
        case *True*
        then show *?thesis*
        using *H* **by** (*simp add*: *do-cp-step-def*)
      next
        case *False*
        let *?S* = *toS S*
        let *?T* = *toS* (*do-propagate-step S*)

let *?U* = *toS* (*do-conflict-step* (*do-propagate-step S*))
have *propa*: *propagate* (*toS S*) *?T* **using** *False do-propgate-step* **by** *blast*
moreover have *ns*: *no-step conflict* (*toS S*) **using** *confl do-conflict-step-no-step* **by** *blast*
ultimately show *?thesis*
using $cdcl_W$*-cp.intros*(*2*)[*of ?S ?T*] *confl* **unfolding** *do-cp-step-def* **by** *auto*
qed
qed
qed

lemma *do-cp-step-eq-no-prop-no-confl*:
do-cp-step S = *S* $\Longrightarrow$ *do-conflict-step S* = *S* $\wedge$ *do-propagate-step S* = *S*
by (*cases S*, *cases conflicting S*)
(*auto simp add*: *do-conflict-step-def do-propagate-step-def do-cp-step-def split*: *option.splits*)

lemma *no-*$cdcl_W$*-cp-iff-no-propagate-no-conflict*:
no-step $cdcl_W$*-cp S* $\longleftrightarrow$ *no-step propagate S* $\wedge$ *no-step conflict S*
by (*auto simp*: $cdcl_W$*-cp.simps*)

lemma *do-cp-step-eq-no-step*:
assumes *H*: *do-cp-step S* = *S* **and** $\forall$ *c* $\in$ *set* (*clauses S* @ *learned-clss S*). *distinct c*
shows *no-step* $cdcl_W$*-cp* (*toS S*)
unfolding *no-*$cdcl_W$*-cp-iff-no-propagate-no-conflict*
using *assms* **apply** (*cases S*, *cases conflicting S*)
using *do-propagate-step-no-step*[*of S*]
by (*auto dest*!: *do-cp-step-eq-no-prop-no-confl*[*simplified*] *do-conflict-step-no-step*
split: *option.splits*)

lemma $cdcl_W$*-cp-*$cdcl_W$*-st*: $cdcl_W$*-cp S S'* $\Longrightarrow$ $cdcl_W{}^{**}$ *S S'*
by (*simp add*: $cdcl_W$*-cp-tranclp-*$cdcl_W$ *tranclp-into-rtranclp*)

lemma $cdcl_W$*-cp-wf-all-inv*:
wf {(*S'*, *S*::'*v*::*linorder* $cdcl_W$*-state*). $cdcl_W$*-all-struct-inv S* $\wedge$ $cdcl_W$*-cp S S'*}
(**is** *wf ?R*)
proof (*rule wf-bounded-measure*[*of -* λS. *card* (*atms-of-msu* (*clauses S*))+*1*
λS. *length* (*trail S*) + (*if conflicting S* = *None then 0 else 1*)], *goal-cases*)
case (*1 S S'*)
then have $cdcl_W$*-all-struct-inv S* **and** $cdcl_W$*-cp S S'* **by** *auto*
moreover then have $cdcl_W$*-all-struct-inv S'*
using *rtranclp-*$cdcl_W$*-all-struct-inv-inv* $cdcl_W$*-cp-*$cdcl_W$*-st* **by** *blast*
ultimately show *?case*
by (*auto simp*:$cdcl_W$*-cp.simps elim*!: *conflictE propagateE*
dest: *length-model-le-vars-all-inv*)
qed

lemma $cdcl_W$*-all-struct-inv-rough-state*[*simp*]: $cdcl_W$*-all-struct-inv* (*toS* (*rough-state-of S*))
using *rough-state-of* **by** *auto*

lemma [*simp*]: $cdcl_W$*-all-struct-inv* (*toS S*) $\Longrightarrow$ *rough-state-of* (*state-of S*) = *S*
by (*simp add*: *state-of-inverse*)

lemma *rough-state-of-state-of-do-cp-step*[*simp*]:
rough-state-of (*state-of* (*do-cp-step* (*rough-state-of S*))) = *do-cp-step* (*rough-state-of S*)
proof –
have $cdcl_W$*-all-struct-inv* (*toS* (*do-cp-step* (*rough-state-of S*)))
apply (*cases do-cp-step* (*rough-state-of S*) = (*rough-state-of S*))

    apply *simp*
  using *cp-step-is-cdcl*$_W$*-cp*[*of rough-state-of S*] *cdcl*$_W$*-all-struct-inv-rough-state*[*of S*]
  cdcl$_W$*-cp-cdcl*$_W$*-st rtranclp-cdcl*$_W$*-all-struct-inv-inv* **by** *blast*
then show *?thesis* **by** *auto*
qed

Skip **fun** *do-skip-step* :: *cdcl*$_W$*-state-inv-st* $\Rightarrow$ *cdcl*$_W$*-state-inv-st* **where**
do-skip-step (*Propagated L C # Ls,N,U,k, Some D*) =
  (*if* $-L \notin$ *set D* $\wedge$ *D* $\neq$ []
  then (*Ls, N, U, k, Some D*)
  else (*Propagated L C #Ls, N, U, k, Some D*)) |
do-skip-step S = *S*

lemma *do-skip-step*:
  do-skip-step S $\neq$ *S* $\Longrightarrow$ *skip* (*toS S*) (*toS* (*do-skip-step S*))
  apply (*induction S rule*: *do-skip-step.induct*)
  by (*auto simp add*: *skip.simps*)

lemma *do-skip-step-no*:
  do-skip-step S = *S* $\Longrightarrow$ *no-step skip* (*toS S*)
  by (*induction S rule*: *do-skip-step.induct*)
    (*auto simp add*: *other split*: *split-if-asm*)

lemma *do-skip-step-trail-is-None*[*iff*]:
  do-skip-step S = (*a, b, c, d, None*) $\longleftrightarrow$ *S* = (*a, b, c, d, None*)
  by (*cases S rule*: *do-skip-step.cases*) *auto*

Resolve **fun** *maximum-level-code*:: *'a literal list* $\Rightarrow$ (*'a, nat, 'a literal list*) *marked-lit list* $\Rightarrow$ *nat*
  where
maximum-level-code [] - = *0* |
maximum-level-code (*L # Ls*) *M* = *max* (*get-level M L*) (*maximum-level-code Ls M*)

lemma *maximum-level-code-eq-get-maximum-level*[*code, simp*]:
  maximum-level-code D M = *get-maximum-level M* (*mset D*)
  by (*induction D*) (*auto simp add*: *get-maximum-level-plus*)

fun *do-resolve-step* :: *cdcl*$_W$*-state-inv-st* $\Rightarrow$ *cdcl*$_W$*-state-inv-st* **where**
do-resolve-step (*Propagated L C # Ls, N, U, k, Some D*) =
  (*if* $-L \in$ *set D* $\wedge$ *maximum-level-code* (*remove1* ($-L$) *D*) (*Propagated L C # Ls*) = *k*
  then (*Ls, N, U, k, Some* (*remdups* (*remove1 L C @ remove1* ($-L$) *D*)))
  else (*Propagated L C # Ls, N, U, k, Some D*)) |
do-resolve-step S = *S*

lemma *do-resolve-step*:
  cdcl$_W$*-all-struct-inv* (*toS S*) $\Longrightarrow$ *do-resolve-step S* $\neq$ *S*
  $\Longrightarrow$ *resolve* (*toS S*) (*toS* (*do-resolve-step S*))
proof (*induction S rule*: *do-resolve-step.induct*)
  case (*1 L C M N U k D*)
  then have
    $-$ *L* $\in$ *set D* **and**
    M: *maximum-level-code* (*remove1* ($-L$) *D*) (*Propagated L C # M*) = *k*
    by (*cases mset D* $-$ {#$-$ *L*#} = {#},
      auto dest!: *get-maximum-level-exists-lit-of-max-level*[*of - Propagated L C # M*]
      split: *split-if-asm*)+
  have *every-mark-is-a-conflict* (*toS* (*Propagated L C # M, N, U, k, Some D*))

using *1(1)* **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-conflicting-def* **by** *fast*
then have $L \in set\ C$ **by** *fastforce*
then obtain C' **where** C: $mset\ C = C' + \{\#L\#\}$
by (*metis add.commute in-multiset-in-set insert-DiffM*)
obtain D' **where** D: $mset\ D = D' + \{\#-L\#\}$
using ‹$- L \in set\ D$› **by** (*metis add.commute in-multiset-in-set insert-DiffM*)
have $D'L$: $D' + \{\#- L\#\} - \{\#-L\#\} = D'$ **by** (*auto simp add*: *multiset-eq-iff*)

have CL: $mset\ C - \{\#L\#\} + \{\#L\#\} = mset\ C$ **using** ‹$L \in set\ C$› **by** (*auto simp add*: *multiset-eq-iff*)
have *get-maximum-level* (*Propagated L* $(C' + \{\#L\#\})$ # *map convert M*) $D' = k$
using *M*[*simplified*] **unfolding** *maximum-level-code-eq-get-maximum-level C*[*symmetric*] *CL*
by (*metis D D'L convert.simps*(*1*) *get-maximum-level-map-convert list.simps*(*9*))
then have
resolve
(*map convert* (*Propagated L C # M*), *mset ‘# mset N*, *mset ‘# mset U*, *k*, *Some* (*mset D*))
(*map convert M*, *mset ‘# mset N*, *mset ‘# mset U*, *k*,
Some $(((mset\ D - \{\#-L\#\})\ \#\cup\ (mset\ C - \{\#L\#\}))))$
unfolding *resolve.simps*
by (*simp add*: *C D*)
moreover have
(*map convert* (*Propagated L C # M*), *mset ‘# mset N*, *mset ‘# mset U*, *k*, *Some* (*mset D*))
= *toS* (*Propagated L C # M*, *N*, *U*, *k*, *Some D*)
by *auto*
moreover
have *distinct-mset* (*mset C*) **and** *distinct-mset* (*mset D*)
using ‹$cdcl_W$*-all-struct-inv* (*toS* (*Propagated L C # M*, *N*, *U*, *k*, *Some D*))›
unfolding $cdcl_W$*-all-struct-inv-def distinct-*$cdcl_W$*-state-def*
by *auto*
then have $(mset\ C - \{\#L\#\})\ \#\cup\ (mset\ D - \{\#- L\#\}) =$
remdups-mset $(mset\ C - \{\#L\#\} + (mset\ D - \{\#- L\#\}))$
by (*auto simp*: *distinct-mset-rempdups-union-mset*)
then have (*map convert M*, *mset ‘# mset N*, *mset ‘# mset U*, *k*,
Some $((mset\ D - \{\#- L\#\})\ \#\cup\ (mset\ C - \{\#L\#\})))$
= *toS* (*do-resolve-step* (*Propagated L C # M*, *N*, *U*, *k*, *Some D*))
using ‹$- L \in set\ D$› *M* **by** (*auto simp*:*ac-simps*)
ultimately show *?case*
by *simp*
qed *auto*

lemma *do-resolve-step-no*:
do-resolve-step S = S $\Longrightarrow$ *no-step resolve* (*toS S*)
apply (*cases S*; *cases hd* (*trail S*); *cases conflicting S*)
by (*auto*
elim!: *resolveE split*: *split-if-asm*
dest!: *union-single-eq-member*
simp del: *in-multiset-in-set get-maximum-level-map-convert*
simp: *in-multiset-in-set*[*symmetric*] *get-maximum-level-map-convert*[*symmetric*])

lemma *rough-state-of-state-of-resolve*[*simp*]:
$cdcl_W$*-all-struct-inv* (*toS S*) $\Longrightarrow$ *rough-state-of* (*state-of* (*do-resolve-step S*)) = *do-resolve-step S*
apply (*rule state-of-inverse*)
apply (*cases do-resolve-step S = S*)
apply *simp*
by (*blast dest*: *other resolve bj do-resolve-step* $cdcl_W$*-all-struct-inv-inv*)

lemma *do-resolve-step-trail-is-None*[*iff*]:
do-resolve-step $S = (a, b, c, d, None) \longleftrightarrow S = (a, b, c, d, None)$
by (*cases S rule*: *do-resolve-step.cases*) *auto*

Backjumping **fun** *find-level-decomp* **where**
find-level-decomp M [] *D k* = *None* |
find-level-decomp M (L # *Ls*) *D k* =
(*case* (*get-level M L*, *maximum-level-code* (*D* @ *Ls*) *M*) *of*
(i, j) ⇒ *if* $i = k \wedge j < i$ *then Some* (L, j) *else find-level-decomp M Ls* (L#D) k
)

lemma *find-level-decomp-some*:
assumes *find-level-decomp M Ls D k* = *Some* (L, j)
shows $L \in$ *set Ls* ∧ *get-maximum-level M* (*mset* (*remove1 L* (*Ls* @ *D*))) = j ∧ *get-level M L* = k
using *assms*
proof (*induction Ls arbitrary*: *D*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons* L' *Ls*) **note** *IH* = *this*(*1*) **and** H = *this*(*2*)

def *find* ≡ (*if get-level M* $L' \neq k$ ∨ ¬ *get-maximum-level M* (*mset D* + *mset Ls*) < *get-level M* L'
then find-level-decomp M Ls (L' # D) k
else Some (L', *get-maximum-level M* (*mset D* + *mset Ls*)))
have *a1*: $\bigwedge D$. *find-level-decomp M Ls D k* = *Some* (L, j) ⟹
$L \in$ *set Ls* ∧ *get-maximum-level M* (*mset Ls* + *mset D* − {#L#}) = j ∧ *get-level M L* = k
using *IH* **by** *simp*
have *a2*: *find* = *Some* (L, j)
using H **unfolding** *find-def* **by** (*auto split*: *split-if-asm*)
{ **assume** *Some* (L', *get-maximum-level M* (*mset D* + *mset Ls*)) ≠ *find*
then have *f3*: $L \in$ *set Ls* **and** *get-maximum-level M* (*mset Ls* + *mset* (L' # D) − {#L#}) = j
using *a1 IH a2* **unfolding** *find-def* **by** *meson+*
moreover then have *mset Ls* + *mset D* − {#L#} + {#L'#} = {#L'#} + *mset D* + (*mset Ls* − {#L#})
by (*auto simp*: *ac-simps multiset-eq-iff Suc-leI*)
ultimately have *f4*: *get-maximum-level M* (*mset Ls* + *mset D* − {#L#} + {#L'#}) = j
by (*metis* (*no-types*) *diff-union-single-conv mem-set-multiset-eq mset.simps*(*2*) *union-commute*)
} **note** *f4* = *this*
have {#L'#} + (*mset Ls* + *mset D*) = *mset Ls* + (*mset D* + {#L'#})
by (*auto simp*: *ac-simps*)
then have
($L = L'$ ⟶ *get-maximum-level M* (*mset Ls* + *mset D*) = j ∧ *get-level M* L' = k) **and**
($L \neq L'$ ⟶ $L \in$ *set Ls* ∧ *get-maximum-level M* (*mset Ls* + *mset D* − {#L#} + {#L'#}) = j ∧
get-level M L = k)
using *f4 a2 a1*[*of* L' # D] **unfolding** *find-def* **by** (*metis* (*no-types*) *add-diff-cancel-left'*
mset.simps(*2*) *option.inject prod.inject union-commute*)+
then show *?case* **by** *simp*
qed

lemma *find-level-decomp-none*:
assumes *find-level-decomp M Ls E k* = *None* **and** *mset* (L#D) = *mset* (*Ls* @ *E*)
shows ¬($L \in$ *set Ls* ∧ *get-maximum-level M* (*mset D*) < k ∧ k = *get-level M L*)
using *assms*
proof (*induction Ls arbitrary*: *E L D*)

case *Nil*
then show *?case* **by** *simp*
next
case (*Cons* L' *Ls*) **note** *IH* = *this*(*1*) **and** *find-none* = *this*(*2*) **and** *LD* = *this*(*3*)
have *mset D* + {#L'#} = *mset E* + (*mset Ls* + {#L'#}) $\Longrightarrow$ *mset D* = *mset E* + *mset Ls*
by (*metis add-right-imp-eq union-assoc*)
then show *?case*
using *find-none IH*[*of* L' # *E L D*] *LD* **by** (*auto simp add*: *ac-simps split*: *split-if-asm*)
qed

fun *bt-cut* **where**
bt-cut i (*Propagated - - # Ls*) = *bt-cut i Ls* |
bt-cut i (*Marked K k # Ls*) = (*if k* = *Suc i then Some* (*Marked K k # Ls*) *else bt-cut i Ls*) |
bt-cut i [] = *None*

lemma *bt-cut-some-decomp*:
bt-cut i M = *Some* M' $\Longrightarrow$ $\exists$ *K M2 M1*. *M* = *M2* @ M' $\wedge$ M' = *Marked K* (*i+1*) # *M1*
by (*induction i M rule*: *bt-cut.induct*) (*auto split*: *split-if-asm*)

lemma *bt-cut-not-none*: *M* = *M2* @ *Marked K* (*Suc i*) # M' $\Longrightarrow$ *bt-cut i M* $\neq$ *None*
by (*induction M2 arbitrary*: *M rule*: *marked-lit-list-induct*) *auto*

lemma *get-all-marked-decomposition-ex*:
$\exists$ *N*. (*Marked K* (*Suc i*) # M', *N*) $\in$ *set* (*get-all-marked-decomposition* (*M2*@*Marked K* (*Suc i*) # M'))
apply (*induction M2 rule*: *marked-lit-list-induct*)
apply *auto*[*2*]
by (*case-tac get-all-marked-decomposition* (*xs* @ *Marked K* (*Suc i*) # M')) *auto*

lemma *bt-cut-in-get-all-marked-decomposition*:
bt-cut i M = *Some* M' $\Longrightarrow$ $\exists$ *M2*. (M', *M2*) $\in$ *set* (*get-all-marked-decomposition M*)
by (*auto dest*!: *bt-cut-some-decomp simp add*: *get-all-marked-decomposition-ex*)

fun *do-backtrack-step* **where**
do-backtrack-step (*M*, *N*, *U*, *k*, *Some D*) =
(*case find-level-decomp M D* [] *k of*
None ⇒ (*M*, *N*, *U*, *k*, *Some D*)
| *Some* (*L*, *j*) ⇒
(*case bt-cut j M of*
Some (*Marked - - # Ls*) ⇒ (*Propagated L D # Ls*, *N*, *D # U*, *j*, *None*)
| - ⇒ (*M*, *N*, *U*, *k*, *Some D*))
) |
do-backtrack-step S = *S*

lemma *get-all-marked-decomposition-map-convert*:
(*get-all-marked-decomposition* (*map convert M*)) =
map (λ(*a*, *b*). (*map convert a*, *map convert b*)) (*get-all-marked-decomposition M*)
apply (*induction M rule*: *marked-lit-list-induct*)
apply *simp*
by (*case-tac get-all-marked-decomposition xs*, *auto*)+

lemma *do-backtrack-step*:
assumes *db*: *do-backtrack-step S* $\neq$ *S*
and *inv*: $cdcl_W$*-all-struct-inv* (*toS S*)
shows *backtrack* (*toS S*) (*toS* (*do-backtrack-step S*))

proof (*cases S*, *cases conflicting S*, *goal-cases*)
  case (*1 M N U k E*)
  then show *?case* **using** *db* **by** *auto*
next
  case (*2 M N U k E C*) **note** *S* = *this*(*1*) **and** *confl* = *this*(*2*)
  have *E*: *E* = *Some C* **using** *S confl* **by** *auto*

  obtain *L j* **where** *fd*: *find-level-decomp M C* [] *k* = *Some* (*L*, *j*)
    using *db* **unfolding** *S E* **by** (*cases C*) (*auto split*: *split-if-asm option.splits*)
  have *L* ∈ *set C* **and** *get-maximum-level M* (*mset* (*remove1 L C*)) = *j* **and**
    levL: *get-level M L* = *k*
    using *find-level-decomp-some*[*OF fd*] **by** *auto*
  obtain *C′* **where** *C*: *mset C* = *mset C′* + {#*L*#}
    using ‹*L* ∈ *set C*› **by** (*metis add.commute ex-mset in-multiset-in-set insert-DiffM*)
  obtain M_2 **where** M_2: *bt-cut j M* = *Some* M_2
    using *db fd* **unfolding** *S E* **by** (*auto split*: *option.splits*)
  obtain *M1 K* **where** *M1*: M_2 = *Marked K* (*Suc j*) # *M1*
    using *bt-cut-some-decomp*[*OF* M_2] **by** (*cases* M_2) *auto*
  obtain *c* **where** *c*: *M* = *c* @ *Marked K* (*Suc j*) # *M1*
     using *bt-cut-in-get-all-marked-decomposition*[*OF* M_2]
     unfolding *M1* **by** *fastforce*
  have *get-all-levels-of-marked* (*map convert M*) = *rev* [*1*..<*Suc k*]
    using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-M-level-inv-def S* **by** *auto*
  from *arg-cong*[*OF this*, *of* λ*a*. *Suc j* ∈ *set a*] **have** *j* ≤ *k* **unfolding** *c* **by** *auto*
  have *max-l-j*: *maximum-level-code C′ M* = *j*
    using *db fd* M_2 *C* **unfolding** *S E* **by** (*auto*
        split: *option.splits list.splits marked-lit.splits*
        dest!: *find-level-decomp-some*)[*1*]
  have *get-maximum-level M* (*mset C*) ≥ *k*
    using ‹*L* ∈ *set C*› *get-maximum-level-ge-get-level levL* **by** *blast*
  moreover have *get-maximum-level M* (*mset C*) ≤ *k*
    using *get-maximum-level-exists-lit-of-max-level*[*of mset C M*] *inv*
      $cdcl_W$*-M-level-inv-get-level-le-backtrack-lvl*[*of toS S*]
    unfolding *C* $cdcl_W$*-all-struct-inv-def S* **by** (*auto dest*: *sym*[*of get-level - -*])
  ultimately have *get-maximum-level M* (*mset C*) = *k* **by** *auto*

  obtain *M2* **where** *M2*: (M_2, *M2*) ∈ *set* (*get-all-marked-decomposition M*)
    using *bt-cut-in-get-all-marked-decomposition*[*OF* M_2] **by** *metis*
  have *H*: ($cdcl_W$*.reduce-trail-to* (*map convert M1*)
    (*add-learned-cls* (*mset C′* + {#*L*#})
      (*map convert M*, *mset* (*map mset N*), *mset* (*map mset U*), *j*, *None*))) =
     (*map convert M1*, *mset* (*map mset N*), {#*mset C′* + {#*L*#}#} + *mset* (*map mset U*), *j*, *None*)
      apply (*subst state-conv*[*of* $cdcl_W$*.reduce-trail-to - -*])
    using *M2* **unfolding** *M1* **by** *auto*
  have
    backtrack
      (*map convert M*, *mset* ‘# *mset N*, *mset* ‘# *mset U*, *k*, *Some* (*mset C*))
      (*Propagated L* (*mset C*) # *map convert M1*, *mset* ‘# *mset N*, *mset* ‘# *mset U* + {#*mset C*#},
j,
        None)
    apply (*rule backtrack-rule*)
          unfolding *C* **apply** *simp*
        using *Set.imageI*[*of* (M_2, *M2*) *set* (*get-all-marked-decomposition M*)
                    (λ(*a*, *b*). (*map convert a*, *map convert b*))] *M2*
        apply (*auto simp*: *get-all-marked-decomposition-map-convert M1*)[*1*]

```
          using max-l-j levL ‹j ≤ k› apply (simp add: get-maximum-level-plus)
         using C ‹get-maximum-level M (mset C) = k› levL apply auto[1]
        using max-l-j apply simp
       apply (cases cdcl_W.reduce-trail-to (map convert M1)
          (add-learned-cls (mset C' + {#L#})
          (map convert M, mset (map mset N), mset (map mset U), j, None)))
      using M2 M1 H by (auto simp: ac-simps)
   then show ?case
     using M_2 fd unfolding S E M1 by auto
   obtain M2 where (M_2, M2) ∈ set (get-all-marked-decomposition M)
     using bt-cut-in-get-all-marked-decomposition[OF M_2] by metis
qed

lemma do-backtrack-step-no:
  assumes db: do-backtrack-step S = S
  and inv: cdcl_W-all-struct-inv (toS S)
  shows no-step backtrack (toS S)
proof (rule ccontr, cases S, cases conflicting S, goal-cases)
  case 1
  then show ?case using db by (auto split: option.splits)
next
  case (2 M N U k E C) note bt = this(1) and S = this(2) and confl = this(3)
  obtain D L K b z M1 j where
    levL: get-level M L = get-maximum-level M (D + {#L#}) and
    k: k = get-maximum-level M (D + {#L#}) and
    j: j = get-maximum-level M D and
    CE: convertC E = Some (D + {#L#}) and
    decomp: (z # M1, b) ∈ set (get-all-marked-decomposition M) and
    z: Marked K (Suc j) = convert z using bt unfolding S
      by (auto split: option.splits elim!: backtrackE
        simp: get-all-marked-decomposition-map-convert)
  have z: z = Marked K (Suc j) using z by (cases z) auto
  obtain c where c: M = c @ b @ Marked K (Suc j) # M1
    using decomp unfolding z by blast
  have get-all-levels-of-marked (map convert M) = rev [1..<Suc k]
    using inv unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def S by auto
  from arg-cong[OF this, of λa. Suc j ∈ set a] have k > j unfolding c by auto
  obtain C D' where
    E: E = Some C and
    C: mset C = mset (L # D')
    using CE apply (cases E)
      apply simp
    by (metis ex-mset mset.simps(2) option.inject option.simps(9))
  have D'D: mset D' = D
    using C CE E by auto
  have find-level-decomp M C [] k ≠ None
    apply rule
    apply (drule find-level-decomp-none[of - - - - L D'])
    using C ‹k > j› mset-eq-setD unfolding k[symmetric] D'D j[symmetric] levL by fastforce+
  then obtain L' j' where fd-some: find-level-decomp M C [] k = Some (L', j')
    by (cases find-level-decomp M C [] k) auto
  have L': L' = L
    proof (rule ccontr)
      assume ¬ ?thesis
      then have L' ∈# D
```

by (*metis C D'D fd-some find-level-decomp-some in-multiset-in-set insert-iff list.simps*(*15*))
then have *get-level M L'* $\leq$ *get-maximum-level M D*
using *get-maximum-level-ge-get-level* **by** *blast*
then show *False* **using** ⟨*k* > *j*⟩ *j find-level-decomp-some*[*OF fd-some*] **by** *auto*
qed
then have *j'*: *j'* = *j* **using** *find-level-decomp-some*[*OF fd-some*] *j C D'D* **by** *auto*

have *btc-none*: *bt-cut j M* $\neq$ *None*
apply (*rule bt-cut-not-none*[*of M - @ -*])
using *c* **by** *simp*
show *?case* **using** *db* **unfolding** *S E*
by (*auto split*: *option.splits list.splits marked-lit.splits*
simp add: *fd-some L' j' btc-none*
dest: *bt-cut-some-decomp*)
qed

lemma *rough-state-of-state-of-backtrack*[*simp*]:
assumes *inv*: $cdcl_W$*-all-struct-inv* (*toS S*)
shows *rough-state-of* (*state-of* (*do-backtrack-step S*))= *do-backtrack-step S*
proof (*rule state-of-inverse*)
have *f2*: *backtrack* (*toS S*) (*toS* (*do-backtrack-step S*)) $\vee$ *do-backtrack-step S* = *S*
using *do-backtrack-step inv* **by** *blast*
have $\bigwedge$*p*. $\neg$ $cdcl_W$*-o* (*toS S*) *p* $\vee$ $cdcl_W$*-all-struct-inv p*
using *inv* $cdcl_W$*-all-struct-inv-inv other* **by** *blast*
then have *do-backtrack-step S* = *S* $\vee$ $cdcl_W$*-all-struct-inv* (*toS* (*do-backtrack-step S*))
using *f2* **by** *blast*
then show *do-backtrack-step S* $\in$ {*S*. $cdcl_W$*-all-struct-inv* (*toS S*)}
using *inv* **by** *fastforce*
qed

Decide **fun** *do-decide-step* **where**
do-decide-step (*M*, *N*, *U*, *k*, *None*) =
(*case find-first-unused-var N* (*lits-of M*) *of*
None ⇒ (*M*, *N*, *U*, *k*, *None*)
| *Some L* ⇒ (*Marked L* (*Suc k*) # *M*, *N*, *U*, *k+1*, *None*)) |
do-decide-step S = *S*

lemma *do-decide-step*:
do-decide-step S $\neq$ *S* $\Longrightarrow$ *decide* (*toS S*) (*toS* (*do-decide-step S*))
apply (*cases S*, *cases conflicting S*)
defer
apply (*auto split*: *option.splits simp add*: *decide.simps Marked-Propagated-in-iff-in-lits-of*
dest: *find-first-unused-var-undefined find-first-unused-var-Some*
intro: *atms-of-atms-of-ms-mono*)[*1*]
proof –
fix *a* :: (*nat*, *nat*, *nat literal list*) *marked-lit list* **and**
b :: *nat literal list list* **and** *c* :: *nat literal list list* **and**
d :: *nat* **and** *e* :: *nat literal list option*
{
fix *a* :: (*nat*, *nat*, *nat literal list*) *marked-lit list* **and**
b :: *nat literal list list* **and** *c* :: *nat literal list list* **and**
d :: *nat* **and** *x2* :: *nat literal* **and** *m* :: *nat literal list*
assume *a1*: *m* $\in$ *set b*
assume *x2* $\in$ *set m*
then have *f2*: *atm-of x2* $\in$ *atms-of* (*mset m*)

by *simp*
have $\bigwedge f.\ (f\ m::nat\ literal\ multiset) \in f\ `\ set\ b$
using *a1* **by** *blast*
then have $\bigwedge f.\ (atms\text{-}of\ (f\ m)::nat\ set) \subseteq atms\text{-}of\text{-}ms\ (f\ `\ set\ b)$
using *atms-of-atms-of-ms-mono* **by** *blast*
then have $\bigwedge n\ f.\ (n::nat) \in atms\text{-}of\text{-}ms\ (f\ `\ set\ b) \vee n \notin atms\text{-}of\ (f\ m)$
by (*meson contra-subsetD*)
then have $atm\text{-}of\ x2 \in atms\text{-}of\text{-}ms\ (mset\ `\ set\ b)$
using *f2* **by** *blast*
} **note** $H = this$
{
fix $m :: nat\ literal\ list$ **and** *x2*
have $m \in set\ b \Longrightarrow x2 \in set\ m \Longrightarrow x2 \notin lits\text{-}of\ a \Longrightarrow -\ x2 \notin lits\text{-}of\ a \Longrightarrow$
$\exists\, aa \in set\ b.\ \neg\ atm\text{-}of\ `\ set\ aa \subseteq atm\text{-}of\ `\ lits\text{-}of\ a$
by (*meson atm-of-in-atm-of-set-in-uminus contra-subsetD rev-image-eqI*)
} **note** $H' = this$

assume $do\text{-}decide\text{-}step\ S \neq S$ **and**
$S = (a, b, c, d, e)$ **and**
$conflicting\ S = None$
then show $decide\ (toS\ S)\ (toS\ (do\text{-}decide\text{-}step\ S))$
using $H\ H'$ **by** (*auto split*: *option.splits simp add*: *decide.simps Marked-Propagated-in-iff-in-lits-of dest*!: *find-first-unused-var-Some*)
qed

lemma *do-decide-step-no*:
$do\text{-}decide\text{-}step\ S = S \Longrightarrow no\text{-}step\ decide\ (toS\ S)$
by (*cases S*, *cases conflicting S*)
(*fastforce simp*: *atms-of-ms-mset-unfold atm-of-eq-atm-of Marked-Propagated-in-iff-in-lits-of split*: *option.splits*)+

lemma *rough-state-of-state-of-do-decide-step*[*simp*]:
$cdcl_W\text{-}all\text{-}struct\text{-}inv\ (toS\ S) \Longrightarrow rough\text{-}state\text{-}of\ (state\text{-}of\ (do\text{-}decide\text{-}step\ S)) = do\text{-}decide\text{-}step\ S$
proof (*subst state-of-inverse*, *goal-cases*)
case *1*
then show *?case*
by (*cases* $do\text{-}decide\text{-}step\ S = S$)
(*auto dest*: *do-decide-step decide other intro*: $cdcl_W$*-all-struct-inv-inv*)
qed *simp*

lemma *rough-state-of-state-of-do-skip-step*[*simp*]:
$cdcl_W\text{-}all\text{-}struct\text{-}inv\ (toS\ S) \Longrightarrow rough\text{-}state\text{-}of\ (state\text{-}of\ (do\text{-}skip\text{-}step\ S)) = do\text{-}skip\text{-}step\ S$
apply (*subst state-of-inverse*, *cases* $do\text{-}skip\text{-}step\ S = S$)
apply *simp*
by (*blast dest*: *other skip bj do-skip-step* $cdcl_W$*-all-struct-inv-inv*)+

18.3.3 Code generation

Type definition There are two invariants: one while applying conflict and propagate and one for the other rules

declare *rough-state-of-inverse*[*simp add*]
definition *Con* **where**
$Con\ xs = state\text{-}of\ (if\ cdcl_W\text{-}all\text{-}struct\text{-}inv\ (toS\ (fst\ xs,\ snd\ xs))\ then\ xs$
$else\ ([], [], [], 0, None))$

lemma [*code abstype*]:
Con (*rough-state-of* S) = S
using *rough-state-of*[*of* S] **unfolding** *Con-def* **by** *simp*

definition *do-cp-step'* **where**
do-cp-step' S = *state-of* (*do-cp-step* (*rough-state-of* S))

typedef $cdcl_W$*-state-inv-from-init-state* = {S::$cdcl_W$*-state-inv-st*. $cdcl_W$*-all-struct-inv* (*toS* S)
∧ $cdcl_W$*-stgy*** (*S0-*$cdcl_W$ (*clauses* (*toS* S))) (*toS* S)}
morphisms *rough-state-from-init-state-of state-from-init-state-of*
proof
show ([],[], [], *0*, *None*) ∈ {S. $cdcl_W$*-all-struct-inv* (*toS* S)
∧ $cdcl_W$*-stgy*** (*S0-*$cdcl_W$ (*clauses* (*toS* S))) (*toS* S)}
by (*auto simp add*: $cdcl_W$*-all-struct-inv-def*)
qed

instantiation $cdcl_W$*-state-inv-from-init-state* :: *equal*
begin
definition *equal-*$cdcl_W$*-state-inv-from-init-state* :: $cdcl_W$*-state-inv-from-init-state* ⇒
$cdcl_W$*-state-inv-from-init-state* ⇒ *bool* **where**
equal-$cdcl_W$*-state-inv-from-init-state* S S' ⟷
(*rough-state-from-init-state-of* S = *rough-state-from-init-state-of* S')
instance
by *standard* (*simp add*: *rough-state-from-init-state-of-inject*
equal-$cdcl_W$*-state-inv-from-init-state-def*)
end

definition *ConI* **where**
ConI S = *state-from-init-state-of* (*if* $cdcl_W$*-all-struct-inv* (*toS* (*fst* S, *snd* S))
∧ $cdcl_W$*-stgy*** (*S0-*$cdcl_W$ (*clauses* (*toS* S))) (*toS* S) *then* S *else* ([], [], [], *0*, *None*))

lemma [*code abstype*]:
ConI (*rough-state-from-init-state-of* S) = S
using *rough-state-from-init-state-of*[*of* S] **unfolding** *ConI-def*
by (*simp add*: *rough-state-from-init-state-of-inverse*)

definition *id-of-I-to*:: $cdcl_W$*-state-inv-from-init-state* ⇒ $cdcl_W$*-state-inv* **where**
id-of-I-to S = *state-of* (*rough-state-from-init-state-of* S)

lemma [*code abstract*]:
rough-state-of (*id-of-I-to* S) = *rough-state-from-init-state-of* S
unfolding *id-of-I-to-def* **using** *rough-state-from-init-state-of* **by** *auto*

Conflict and Propagate **function** *do-full1-cp-step* :: $cdcl_W$*-state-inv* ⇒ $cdcl_W$*-state-inv* **where**
do-full1-cp-step S =
(*let* S' = *do-cp-step'* S *in*
if S = S' *then* S *else* *do-full1-cp-step* S')
by *auto*
termination
proof (*relation* {(T', T). (*rough-state-of* T', *rough-state-of* T) ∈ {(S', S).
(*toS* S', *toS* S) ∈ {(S', S). $cdcl_W$*-all-struct-inv* S ∧ $cdcl_W$*-cp* S S'}}}, *goal-cases*)
case *1*
show *?case*
using *wf-if-measure-f*[*OF wf-if-measure-f*[*OF* $cdcl_W$*-cp-wf-all-inv*, *of toS*], *of rough-state-of*] **.**
next

case (*2 S′ S*)
then show *?case*
unfolding *do-cp-step′-def*
apply *simp*
by (*metis cp-step-is-cdcl$_W$-cp rough-state-of-inverse*)
qed

lemma *do-full1-cp-step-fix-point-of-do-full1-cp-step*:
do-cp-step(*rough-state-of* (*do-full1-cp-step S*)) = (*rough-state-of* (*do-full1-cp-step S*))
by (*rule do-full1-cp-step.induct*[*of* λS. *do-cp-step*(*rough-state-of* (*do-full1-cp-step S*))
= (*rough-state-of* (*do-full1-cp-step S*))]])
(*metis* (*full-types*) *do-full1-cp-step.elims rough-state-of-state-of-do-cp-step do-cp-step′-def*)

lemma *in-clauses-rough-state-of-is-distinct*:
c∈*set* (*clauses* (*rough-state-of S*) @ *learned-clss* (*rough-state-of S*)) ⟹ *distinct c*
apply (*cases rough-state-of S*)
using *rough-state-of*[*of S*] **by** (*auto simp add*: *distinct-mset-set-distinct cdcl$_W$-all-struct-inv-def*
distinct-cdcl$_W$-state-def)

lemma *do-full1-cp-step-full*:
full cdcl$_W$-cp (*toS* (*rough-state-of S*))
(*toS* (*rough-state-of* (*do-full1-cp-step S*)))
unfolding *full-def* **apply** *standard*
apply (*induction S rule*: *do-full1-cp-step.induct*)
apply (*smt cp-step-is-cdcl$_W$-cp do-cp-step′-def do-full1-cp-step.simps*
rough-state-of-state-of-do-cp-step rtranclp.rtrancl-refl rtranclp-into-tranclp2
tranclp-into-rtranclp)

apply (*rule do-cp-step-eq-no-step*[*OF do-full1-cp-step-fix-point-of-do-full1-cp-step*[*of S*]])
using *in-clauses-rough-state-of-is-distinct* **unfolding** *do-cp-step′-def* **by** *blast*

lemma [*code abstract*]:
rough-state-of (*do-cp-step′ S*) = *do-cp-step* (*rough-state-of S*)
unfolding *do-cp-step′-def* **by** *auto*

The other rules **fun** *do-other-step* **where**

do-other-step S =
(*let T* = *do-skip-step S in*
if T ≠ *S*
then T
else
(*let U* = *do-resolve-step T in*
if U ≠ *T*
then U else
(*let V* = *do-backtrack-step U in*
if V ≠ *U then V else do-decide-step V*)))

lemma *do-other-step*:
assumes *inv*: *cdcl$_W$-all-struct-inv* (*toS S*) **and**
st: *do-other-step S* ≠ *S*
shows *cdcl$_W$-o* (*toS S*) (*toS* (*do-other-step S*))
using *st inv* **by** (*auto split*: *split-if-asm*
simp add: *Let-def*
intro: *do-skip-step do-resolve-step do-backtrack-step do-decide-step*)

lemma *do-other-step-no*:
assumes *inv*: $cdcl_W$*-all-struct-inv* (*toS S*) **and**
st: *do-other-step S* = *S*
shows *no-step* $cdcl_W$*-o* (*toS S*)
using *st inv* **by** (*auto split*: *split-if-asm elim*: $cdcl_W$*-bjE*
simp add: *Let-def* $cdcl_W$*-bj.simps elim!*: $cdcl_W$*-o.cases*
dest!: *do-skip-step-no do-resolve-step-no do-backtrack-step-no do-decide-step-no*)

lemma *rough-state-of-state-of-do-other-step*[*simp*]:
rough-state-of (*state-of* (*do-other-step* (*rough-state-of S*))) = *do-other-step* (*rough-state-of S*)
proof (*cases do-other-step* (*rough-state-of S*) = *rough-state-of S*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
have $cdcl_W$*-o* (*toS* (*rough-state-of S*)) (*toS* (*do-other-step* (*rough-state-of S*)))
by (*metis False* $cdcl_W$*-all-struct-inv-rough-state do-other-step*[*of rough-state-of S*])
then have $cdcl_W$*-all-struct-inv* (*toS* (*do-other-step* (*rough-state-of S*)))
using $cdcl_W$*-all-struct-inv-inv* $cdcl_W$*-all-struct-inv-rough-state other* **by** *blast*
then show *?thesis*
by (*simp add*: *CollectI state-of-inverse*)
qed

definition *do-other-step′* **where**
do-other-step′ S =
state-of (*do-other-step* (*rough-state-of S*))

lemma *rough-state-of-do-other-step′*[*code abstract*]:
rough-state-of (*do-other-step′ S*) = *do-other-step* (*rough-state-of S*)
apply (*cases do-other-step* (*rough-state-of S*) = *rough-state-of S*)
unfolding *do-other-step′-def* **apply** *simp*
using *do-other-step*[*of rough-state-of S*] **by** (*auto intro*: $cdcl_W$*-all-struct-inv-inv*
$cdcl_W$*-all-struct-inv-rough-state other state-of-inverse*)

definition *do-*$cdcl_W$*-stgy-step* **where**
do-$cdcl_W$*-stgy-step S* =
(*let T* = *do-full1-cp-step S in*
if T ≠ *S*
then T
else
(*let U* = (*do-other-step′ T*) *in*
(*do-full1-cp-step U*)))

definition *do-*$cdcl_W$*-stgy-step′* **where**
do-$cdcl_W$*-stgy-step′ S* = *state-from-init-state-of* (*rough-state-of* (*do-*$cdcl_W$*-stgy-step* (*id-of-I-to S*)))

lemma *toS-do-full1-cp-step-not-eq*: *do-full1-cp-step S* ≠ *S* ⟹
toS (*rough-state-of S*) ≠ *toS* (*rough-state-of* (*do-full1-cp-step S*))
proof −
assume *a1*: *do-full1-cp-step S* ≠ *S*
then have *S* ≠ *do-cp-step′ S*
by *fastforce*
then show *?thesis*
by (*metis* (*no-types*) *cp-step-is-*$cdcl_W$*-cp do-cp-step′-def do-cp-step-eq-no-step*
do-full1-cp-step-fix-point-of-do-full1-cp-step in-clauses-rough-state-of-is-distinct

rough-state-of-inverse)
qed

do-full1-cp-step should not be unfolded anymore:

declare *do-full1-cp-step.simps*[*simp del*]

Correction of the transformation **lemma** *do-cdcl$_W$-stgy-step*:
assumes *do-cdcl$_W$-stgy-step S* $\neq$ *S*
shows *cdcl$_W$-stgy* (*toS* (*rough-state-of S*)) (*toS* (*rough-state-of* (*do-cdcl$_W$-stgy-step S*)))
proof (*cases do-full1-cp-step S* = *S*)
case *False*
then show *?thesis*
using *assms do-full1-cp-step-full*[*of S*] **unfolding** *full-unfold do-cdcl$_W$-stgy-step-def*
by (*auto intro*!: *cdcl$_W$-stgy.intros dest*: *toS-do-full1-cp-step-not-eq*)
next
case *True*
have *cdcl$_W$-o* (*toS* (*rough-state-of S*)) (*toS* (*rough-state-of* (*do-other-step' S*)))
by (*smt True assms cdcl$_W$-all-struct-inv-rough-state do-cdcl$_W$-stgy-step-def do-other-step rough-state-of-do-other-step' rough-state-of-inverse*)
moreover
have
np: *no-step propagate* (*toS* (*rough-state-of S*)) **and**
nc: *no-step conflict* (*toS* (*rough-state-of S*))
apply (*metis True do-cp-step-eq-no-prop-no-confl do-full1-cp-step-fix-point-of-do-full1-cp-step do-propagate-step-no-step in-clauses-rough-state-of-is-distinct*)
by (*metis True do-conflict-step-no-step do-cp-step-eq-no-prop-no-confl do-full1-cp-step-fix-point-of-do-full1-cp-step*)
then have *no-step cdcl$_W$-cp* (*toS* (*rough-state-of S*))
by (*simp add*: *cdcl$_W$-cp.simps*)
moreover have *full cdcl$_W$-cp* (*toS* (*rough-state-of* (*do-other-step' S*)))
(*toS* (*rough-state-of* (*do-full1-cp-step* (*do-other-step' S*))))
using *do-full1-cp-step-full* **by** *auto*
ultimately show *?thesis*
using *assms True* **unfolding** *do-cdcl$_W$-stgy-step-def*
by (*auto intro*!: *cdcl$_W$-stgy.other' dest*: *toS-do-full1-cp-step-not-eq*)
qed

lemma *length-trail-toS*[*simp*]:
length (*trail* (*toS S*)) = *length* (*trail S*)
by (*cases S*) *auto*

lemma *conflicting-noTrue-iff-toS*[*simp*]:
conflicting (*toS S*) $\neq$ *None* $\longleftrightarrow$ *conflicting S* $\neq$ *None*
by (*cases S*) *auto*

lemma *trail-toS-neq-imp-trail-neq*:
trail (*toS S*) $\neq$ *trail* (*toS S'*) $\Longrightarrow$ *trail S* $\neq$ *trail S'*
by (*cases S*, *cases S'*) *auto*

lemma *do-skip-step-trail-changed-or-conflict*:
assumes *d*: *do-other-step S* $\neq$ *S*
and *inv*: *cdcl$_W$-all-struct-inv* (*toS S*)
shows *trail S* $\neq$ *trail* (*do-other-step S*)
proof −

```
  have M: ⋀M K M1 c. M = c @ K # M1 ⟹ Suc (length M1) ≤ length M
    by auto
  have cdcl_W-M-level-inv (toS S)
    using inv unfolding cdcl_W-all-struct-inv-def by auto
  have cdcl_W-o (toS S) (toS (do-other-step S)) using do-other-step[OF inv d] .
  then show ?thesis
    using ‹cdcl_W-M-level-inv (toS S)›
    proof (induction toS (do-other-step S) rule: cdcl_W-o-induct-lev2)
      case decide
      then show ?thesis
        by (auto simp add: trail-toS-neq-imp-trail-neq)[]
    next
    case (skip)
    then show ?case
      by (cases S; cases do-other-step S) force
    next
      case (resolve)
      then show ?case
         by (cases S, cases do-other-step S) force
    next
       case (backtrack K i M1 M2 L D) note decomp = this(1) and confl-S = this(3) and undef =
this(6) and
        U = this(7)
      have [simp]: cons-trail (Propagated L (D + {#L#}))
        (cdcl_W.reduce-trail-to M1
          (add-learned-cls (D + {#L#})
            (update-backtrack-lvl (get-maximum-level (trail (toS S)) D)
              (update-conflicting None (toS S)))))
        =
        (Propagated L (D + {#L#})# M1,mset (map mset (clauses S)),
          {#D + {#L#}#} + mset (map mset (learned-clss S)),
          get-maximum-level (trail (toS S)) D, None)
        apply (subst state-conv[of cons-trail - -])
        using decomp undef by (cases S) auto
      then show ?case
        apply (cases do-other-step S)
        apply (auto split: split-if-asm simp: Let-def)
            apply (cases S rule: do-skip-step.cases; auto split: split-if-asm)
           apply (cases S rule: do-skip-step.cases; auto split: split-if-asm)

          apply (cases S rule: do-backtrack-step.cases;
            auto split: split-if-asm option.splits list.splits marked-lit.splits
              dest!: bt-cut-some-decomp simp: Let-def)
        using d apply (cases S rule: do-decide-step.cases; auto split: option.splits)[]
        done
    qed
qed

lemma do-full1-cp-step-induct:
  (⋀S. (S ≠  do-cp-step' S ⟹ P (do-cp-step' S)) ⟹ P S) ⟹ P a0
  using do-full1-cp-step.induct by metis

lemma do-cp-step-neq-trail-increase:
  ∃c. trail (do-cp-step S) = c @ trail  S ∧(∀m ∈ set c. ¬ is-marked m)
  by (cases S, cases conflicting S)
```

(*auto simp add*: *do-cp-step-def do-conflict-step-def do-propagate-step-def split*: *option.splits*)

lemma *do-full1-cp-step-neq-trail-increase*:
$\exists$ *c. trail* (*rough-state-of* (*do-full1-cp-step S*)) = *c* @ *trail* (*rough-state-of S*)
$\wedge$ ($\forall$ *m* $\in$ *set c.* $\neg$ *is-marked m*)
apply (*induction rule*: *do-full1-cp-step-induct*)
apply (*case-tac do-cp-step' S = S*)
apply (*simp add*: *do-full1-cp-step.simps*)
by (*smt Un-iff append-assoc do-cp-step'-def do-cp-step-neq-trail-increase do-full1-cp-step.simps rough-state-of-state-of-do-cp-step set-append*)

lemma *do-cp-step-conflicting*:
conflicting (*rough-state-of S*) $\neq$ *None* $\Longrightarrow$ *do-cp-step' S = S*
unfolding *do-cp-step'-def do-cp-step-def* **by** *simp*

lemma *do-full1-cp-step-conflicting*:
conflicting (*rough-state-of S*) $\neq$ *None* $\Longrightarrow$ *do-full1-cp-step S = S*
unfolding *do-cp-step'-def do-cp-step-def*
apply (*induction rule*: *do-full1-cp-step-induct*)
by (*rename-tac S*, *case-tac S* $\neq$ *do-cp-step' S*)
(*auto simp add*: *do-full1-cp-step.simps do-cp-step-conflicting*)

lemma *do-decide-step-not-conflicting-one-more-decide*:
assumes
conflicting S = None **and**
do-decide-step S $\neq$ *S*
shows *Suc* (*length* (*filter is-marked* (*trail S*)))
= *length* (*filter is-marked* (*trail* (*do-decide-step S*)))
using *assms* **unfolding** *do-other-step'-def*
by (*cases S*) (*auto simp*: *Let-def split*: *split-if-asm option.splits*
dest!: *find-first-unused-var-Some-not-all-incl*)

lemma *do-decide-step-not-conflicting-one-more-decide-bt*:
assumes *conflicting S* $\neq$ *None* **and**
do-decide-step S $\neq$ *S*
shows *length* (*filter is-marked* (*trail S*)) < *length* (*filter is-marked* (*trail* (*do-decide-step S*)))
using *assms* **unfolding** *do-other-step'-def* **by** (*cases S*, *cases conflicting S*)
(*auto simp add*: *Let-def split*: *split-if-asm option.splits*)

lemma *do-other-step-not-conflicting-one-more-decide-bt*:
assumes *conflicting* (*rough-state-of S*) $\neq$ *None* **and**
conflicting (*rough-state-of* (*do-other-step' S*)) = *None* **and**
do-other-step' S $\neq$ *S*
shows *length* (*filter is-marked* (*trail* (*rough-state-of S*)))
> *length* (*filter is-marked* (*trail* (*rough-state-of* (*do-other-step' S*))))
proof (*cases S*, *goal-cases*)
case (*1 y*) **note** *S = this*(*1*) **and** *inv = this*(*2*)
obtain *M N U k E* **where** *y*: *y* = (*M*, *N*, *U*, *k*, *Some E*)
using *assms*(*1*) *S inv* **by** (*cases y*, *cases conflicting y*) *auto*
have *M*: *rough-state-of* (*state-of* (*M*, *N*, *U*, *k*, *Some E*)) = (*M*, *N*, *U*, *k*, *Some E*)
using *inv y* **by** (*auto simp add*: *state-of-inverse*)
have *bt*: *do-other-step' S = state-of* (*do-backtrack-step* (*rough-state-of S*))

using *assms*(*1*,*2*) **apply** (*cases rough-state-of* (*do-other-step' S*))
apply(*auto simp add*: *Let-def do-other-step'-def*)

apply (*cases rough-state-of S rule*: *do-decide-step.cases*)
apply *auto*
done
show *?case*
using *assms*(*2*) *S* **unfolding** *bt y inv*
apply *simp*
by (*auto simp add*: *M*
split: *option.splits*
dest: *bt-cut-some-decomp arg-cong*[*of - - λu. length* (*filter is-marked u*)])
qed

lemma *do-other-step-not-conflicting-one-more-decide*:
assumes *conflicting* (*rough-state-of S*) = *None* **and**
do-other-step′ S ≠ *S*
shows *1* + *length* (*filter is-marked* (*trail* (*rough-state-of S*)))
= *length* (*filter is-marked* (*trail* (*rough-state-of* (*do-other-step′ S*))))
proof (*cases S*, *goal-cases*)
case (*1 y*) **note** *S* = *this*(*1*) **and** *inv* = *this*(*2*)
obtain *M N U k* **where** *y*: *y* = (*M*, *N*, *U*, *k*, *None*) **using** *assms*(*1*) *S inv* **by** (*cases y*) *auto*
have *M*: *rough-state-of* (*state-of* (*M*, *N*, *U*, *k*, *None*)) = (*M*, *N*, *U*, *k*, *None*)
using *inv y* **by** (*auto simp add*: *state-of-inverse*)
have *state-of* (*do-decide-step* (*M*, *N*, *U*, *k*, *None*)) ≠ *state-of* (*M*, *N*, *U*, *k*, *None*)
using *assms*(*2*) **unfolding** *do-other-step′-def y inv S* **by** (*auto simp add*: *M*)
then have *f4*: *do-skip-step* (*rough-state-of S*) = *rough-state-of S*
unfolding *S M y* **by** (*metis* (*full-types*) *do-skip-step.simps*(*4*))
have *f5*: *do-resolve-step* (*rough-state-of S*) = *rough-state-of S*
unfolding *S M y* **by** (*metis* (*no-types*) *do-resolve-step.simps*(*4*))
have *f6*: *do-backtrack-step* (*rough-state-of S*) = *rough-state-of S*
unfolding *S M y* **by** (*metis* (*no-types*) *do-backtrack-step.simps*(*2*))
have *do-other-step* (*rough-state-of S*) ≠ *rough-state-of S*
using *assms*(*2*) **unfolding** *S M y do-other-step′-def* **by** (*metis* (*no-types*))
then show *?case*
using *f6 f5 f4* **by** (*simp add*: *assms*(*1*) *do-decide-step-not-conflicting-one-more-decide*
do-other-step′-def)
qed

lemma *rough-state-of-state-of-do-skip-step-rough-state-of*[*simp*]:
rough-state-of (*state-of* (*do-skip-step* (*rough-state-of S*))) = *do-skip-step* (*rough-state-of S*)
by (*smt do-other-step.simps rough-state-of-inverse rough-state-of-state-of-do-other-step*)

lemma *conflicting-do-resolve-step-iff*[*iff*]:
conflicting (*do-resolve-step S*) = *None* ⟷ *conflicting S* = *None*
by (*cases S rule*: *do-resolve-step.cases*)
(*auto simp add*: *Let-def split*: *option.splits*)

lemma *conflicting-do-skip-step-iff*[*iff*]:
conflicting (*do-skip-step S*) = *None* ⟷ *conflicting S* = *None*
by (*cases S rule*: *do-skip-step.cases*)
(*auto simp add*: *Let-def split*: *option.splits*)

lemma *conflicting-do-decide-step-iff*[*iff*]:
conflicting (*do-decide-step S*) = *None* ⟷ *conflicting S* = *None*
by (*cases S rule*: *do-decide-step.cases*)
(*auto simp add*: *Let-def split*: *option.splits*)

lemma *conflicting-do-backtrack-step-imp*[*simp*]:
  do-backtrack-step $S \neq S \Longrightarrow$ *conflicting* (*do-backtrack-step* S) = *None*
  by (*cases* S *rule*: *do-backtrack-step.cases*)
    (*auto simp add*: *Let-def split*: *list.splits option.splits marked-lit.splits*)

lemma *do-skip-step-eq-iff-trail-eq*:
  do-skip-step $S = S \longleftrightarrow$ *trail* (*do-skip-step* S) = *trail* S
  by (*cases* S *rule*: *do-skip-step.cases*) *auto*

lemma *do-decide-step-eq-iff-trail-eq*:
  do-decide-step $S = S \longleftrightarrow$ *trail* (*do-decide-step* S) = *trail* S
  by (*cases* S *rule*: *do-decide-step.cases*) (*auto split*: *option.split*)

lemma *do-backtrack-step-eq-iff-trail-eq*:
  do-backtrack-step $S = S \longleftrightarrow$ *trail* (*do-backtrack-step* S) = *trail* S
  by (*cases* S *rule*: *do-backtrack-step.cases*)
    (*auto split*: *option.split list.splits marked-lit.splits*
      dest!: *bt-cut-in-get-all-marked-decomposition*)

lemma *do-resolve-step-eq-iff-trail-eq*:
  do-resolve-step $S = S \longleftrightarrow$ *trail* (*do-resolve-step* S) = *trail* S
  by (*cases* S *rule*: *do-resolve-step.cases*) *auto*

lemma *do-other-step-eq-iff-trail-eq*:
  trail (*do-other-step* S) = *trail* $S \longleftrightarrow$ *do-other-step* $S = S$
  by (*auto simp add*: *Let-def do-skip-step-eq-iff-trail-eq*[*symmetric*]
    do-decide-step-eq-iff-trail-eq[*symmetric*] *do-backtrack-step-eq-iff-trail-eq*[*symmetric*]
    do-resolve-step-eq-iff-trail-eq[*symmetric*])

lemma *do-full1-cp-step-do-other-step′-normal-form*[*dest*!]:
  assumes H: *do-full1-cp-step* (*do-other-step′* S) = S
  shows *do-other-step′* $S = S \wedge$ *do-full1-cp-step* $S = S$
proof −
  let *?T* = *do-other-step′* S
  { assume *confl*: *conflicting* (*rough-state-of ?T*) $\neq$ *None*
    then have *tr*: *trail* (*rough-state-of* (*do-full1-cp-step ?T*)) = *trail* (*rough-state-of ?T*)
      using *do-full1-cp-step-conflicting* **by** *auto*
    have *trail* (*rough-state-of* (*do-full1-cp-step* (*do-other-step′* S))) = *trail* (*rough-state-of* S)
      using *arg-cong*[*OF* H, *of* λS. *trail* (*rough-state-of* S)] **.**
    then have *trail* (*rough-state-of* (*do-other-step′* S)) = *trail* (*rough-state-of* S)
      by (*auto simp add*: *do-full1-cp-step-conflicting confl*)
    then have *do-other-step′* $S = S$
      by (*simp add*: *do-other-step-eq-iff-trail-eq do-other-step′-def*
        del: *do-other-step.simps*)
  }
  moreover {
    assume *eq*[*simp*]: *do-other-step′* $S = S$
    obtain c **where** c: *trail* (*rough-state-of* (*do-full1-cp-step* S)) = c @ *trail* (*rough-state-of* S)
      using *do-full1-cp-step-neq-trail-increase* **by** *auto*

    moreover have *trail* (*rough-state-of* (*do-full1-cp-step* S)) = *trail* (*rough-state-of* S)
      using *arg-cong*[*OF* H, *of* λS. *trail* (*rough-state-of* S)] **by** *simp*
    finally have $c = []$ **by** *blast*
    then have *do-full1-cp-step* $S = S$ **using** *assms* **by** *auto*

}
moreover {
assume *confl*: *conflicting* (*rough-state-of ?T*) = *None* **and** *neq*: *do-other-step'* $S \neq S$
obtain *c* **where**
c: *trail* (*rough-state-of* (*do-full1-cp-step ?T*)) = *c* @ *trail* (*rough-state-of ?T*) **and**
nm: ∀ *m*∈*set c*. ¬ *is-marked m*
using *do-full1-cp-step-neq-trail-increase* **by** *auto*
have *length* (*filter is-marked* (*trail* (*rough-state-of* (*do-full1-cp-step ?T*))))
= *length* (*filter is-marked* (*trail* (*rough-state-of ?T*))) **using** *nm* **unfolding** *c* **by** *force*
moreover have *length* (*filter is-marked* (*trail* (*rough-state-of S*)))
≠ *length* (*filter is-marked* (*trail* (*rough-state-of ?T*)))
using *do-other-step-not-conflicting-one-more-decide*[*OF* - *neq*]
do-other-step-not-conflicting-one-more-decide-bt[*of S*, *OF* - *confl neq*]
by *linarith*
finally have *False* **unfolding** *H* **by** *blast*
}
ultimately show *?thesis* **by** *blast*
qed

lemma *do-cdcl*$_W$*-stgy-step-no*:
assumes *S*: *do-cdcl*$_W$*-stgy-step S* = *S*
shows *no-step cdcl*$_W$*-stgy* (*toS* (*rough-state-of S*))
proof −
{
fix *S'*
assume *full1 cdcl*$_W$*-cp* (*toS* (*rough-state-of S*)) *S'*
then have *False*
using *do-full1-cp-step-full*[*of S*] **unfolding** *full-def S rtranclp-unfold full1-def*
by (*smt assms do-cdcl*$_W$*-stgy-step-def tranclpD*)
}
moreover {
fix *S' S''*
assume *cdcl*$_W$*-o* (*toS* (*rough-state-of S*)) *S'* **and**
no-step propagate (*toS* (*rough-state-of S*)) **and**
no-step conflict (*toS* (*rough-state-of S*)) **and**
full cdcl$_W$*-cp S' S''*
then have *False*
using *assms* **unfolding** *do-cdcl*$_W$*-stgy-step-def*
by (*smt cdcl*$_W$*-all-struct-inv-rough-state do-full1-cp-step-do-other-step'-normal-form*
do-other-step-no rough-state-of-do-other-step')
}
ultimately show *?thesis* **using** *assms* **by** (*force simp*: *cdcl*$_W$*-cp.simps cdcl*$_W$*-stgy.simps*)
qed

lemma *toS-rough-state-of-state-of-rough-state-from-init-state-of*[*simp*]:
toS (*rough-state-of* (*state-of* (*rough-state-from-init-state-of S*)))
= *toS* (*rough-state-from-init-state-of S*)
using *rough-state-from-init-state-of*[*of S*] **by** (*auto simp add*: *state-of-inverse*)

lemma *cdcl*$_W$*-cp-is-rtranclp-cdcl*$_W$: *cdcl*$_W$*-cp S T* ⟹ *cdcl*$_W$** *S T*
apply (*induction rule*: *cdcl*$_W$*-cp.induct*)
using *conflict* **apply** *blast*
using *propagate* **by** *blast*

lemma *rtranclp-cdcl*$_W$*-cp-is-rtranclp-cdcl*$_W$: *cdcl*$_W$*-cp*** *S T* ⟹ *cdcl*$_W$** *S T*

apply (*induction rule*: *rtranclp-induct*)
  apply *simp*
by (*fastforce dest*!: $cdcl_W$*-cp-is-rtranclp-*$cdcl_W$)

lemma $cdcl_W$*-stgy-is-rtranclp-*$cdcl_W$:
$cdcl_W$*-stgy* S $T \Longrightarrow cdcl_W^{**}$ S T
apply (*induction rule*: $cdcl_W$*-stgy.induct*)
 using $cdcl_W$*-stgy.conflict′ rtranclp-*$cdcl_W$*-stgy-rtranclp-*$cdcl_W$ **apply** *blast*
unfolding *full-def* **by** (*fastforce dest*!:$cdcl_W$*.other rtranclp-*$cdcl_W$*-cp-is-rtranclp-*$cdcl_W$)

lemma $cdcl_W$*-stgy-init-clss*: $cdcl_W$*-stgy* S $T \Longrightarrow$ $cdcl_W$*-M-level-inv* $S \Longrightarrow$ *clauses* S = *clauses* T
using *rtranclp-*$cdcl_W$*-init-clss* $cdcl_W$*-stgy-is-rtranclp-*$cdcl_W$ **by** *fast*

lemma *clauses-toS-rough-state-of-do-*$cdcl_W$*-stgy-step*[*simp*]:
clauses (*toS* (*rough-state-of* (*do-*$cdcl_W$*-stgy-step* (*state-of* (*rough-state-from-init-state-of* S)))))
  = *clauses* (*toS* (*rough-state-from-init-state-of* S)) (**is** - = *clauses* (*toS* *?S*))
apply (*cases do-*$cdcl_W$*-stgy-step* (*state-of ?S*) = *state-of ?S*)
  apply *simp*
by (*smt* $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-all-struct-inv-rough-state* $cdcl_W$*-stgy-no-more-init-clss*
  do-$cdcl_W$*-stgy-step toS-rough-state-of-state-of-rough-state-from-init-state-of*)

lemma *rough-state-from-init-state-of-do-*$cdcl_W$*-stgy-step′*[*code abstract*]:
rough-state-from-init-state-of (*do-*$cdcl_W$*-stgy-step′* S) =
  rough-state-of (*do-*$cdcl_W$*-stgy-step* (*id-of-I-to* S))
proof −
let *?S* = (*rough-state-from-init-state-of* S)
have $cdcl_W$*-stgy*** (*S0-*$cdcl_W$ (*clauses* (*toS* (*rough-state-from-init-state-of* S))))
  (*toS* (*rough-state-from-init-state-of* S))
  using *rough-state-from-init-state-of*[*of* S] **by** *auto*
moreover have $cdcl_W$*-stgy***
    (*toS* (*rough-state-from-init-state-of* S))
    (*toS* (*rough-state-of* (*do-*$cdcl_W$*-stgy-step*
      (*state-of* (*rough-state-from-init-state-of* S)))))
  using *do-*$cdcl_W$*-stgy-step*[*of state-of ?S*]
  by (*cases do-*$cdcl_W$*-stgy-step* (*state-of ?S*) = *state-of ?S*) *auto*
ultimately show *?thesis*
  unfolding *do-*$cdcl_W$*-stgy-step′-def id-of-I-to-def*
  by (*auto intro*!: *state-from-init-state-of-inverse*)
qed

All rules together **function** *do-all-*$cdcl_W$*-stgy* **where**
do-all-$cdcl_W$*-stgy* S =
(*let* T = *do-*$cdcl_W$*-stgy-step′* S *in*
if T = S *then* S *else do-all-*$cdcl_W$*-stgy* T)
by *fast+*
termination
proof (*relation* {(T, S).
  ($cdcl_W$*-measure* (*toS* (*rough-state-from-init-state-of* T)),
  $cdcl_W$*-measure* (*toS* (*rough-state-from-init-state-of* S)))
    ∈ *lexn* {(a, b). $a < b$} *3*}, *goal-cases*)
case *1*
show *?case* **by** (*rule wf-if-measure-f*) (*auto intro*!: *wf-lexn wf-less*)
next
case (*2* S T) **note** T = *this*(*1*) **and** ST = *this*(*2*)
let *?S* = *rough-state-from-init-state-of* S

have *S*: $cdcl_W$*-stgy*** (*S0-*$cdcl_W$ (*clauses* (*toS ?S*))) (*toS ?S*)
using *rough-state-from-init-state-of*[*of S*] **by** *auto*
moreover have $cdcl_W$*-stgy* (*toS* (*rough-state-from-init-state-of S*))
(*toS* (*rough-state-from-init-state-of T*))
proof −
have $\bigwedge c$. *rough-state-of* (*state-of* (*rough-state-from-init-state-of c*)) =
rough-state-from-init-state-of c
using *rough-state-from-init-state-of* **by** *force*
then have *do-*$cdcl_W$*-stgy-step* (*state-of* (*rough-state-from-init-state-of S*))
$\neq$ *state-of* (*rough-state-from-init-state-of S*)
using *ST T* **by** (*metis* (*no-types*) *id-of-I-to-def rough-state-from-init-state-of-inject*
rough-state-from-init-state-of-do-$cdcl_W$*-stgy-step'*)
then show *?thesis*
using *do-*$cdcl_W$*-stgy-step id-of-I-to-def rough-state-from-init-state-of-do-*$cdcl_W$*-stgy-step' T*
by *fastforce*
qed
moreover
have $cdcl_W$*-all-struct-inv* (*toS* (*rough-state-from-init-state-of S*))
using *rough-state-from-init-state-of*[*of S*] **by** *auto*
then have $cdcl_W$*-all-struct-inv* (*S0-*$cdcl_W$ (*clauses* (*toS* (*rough-state-from-init-state-of S*))))
by (*cases rough-state-from-init-state-of S*)
(*auto simp add*: $cdcl_W$*-all-struct-inv-def distinct-*$cdcl_W$*-state-def*)
ultimately show *?case*
by (*auto intro*!: $cdcl_W$*-stgy-step-decreasing*[*of - - S0-*$cdcl_W$ (*clauses* (*toS ?S*))]
simp del: $cdcl_W$*-measure.simps*)
qed

thm *do-all-*$cdcl_W$*-stgy.induct*
lemma *do-all-*$cdcl_W$*-stgy-induct*:
($\bigwedge S$. (*do-*$cdcl_W$*-stgy-step' S* $\neq$ *S* $\Longrightarrow$ *P* (*do-*$cdcl_W$*-stgy-step' S*)) $\Longrightarrow$ *P S*) $\Longrightarrow$ *P a0*
using *do-all-*$cdcl_W$*-stgy.induct* **by** *metis*

lemma *no-step-*$cdcl_W$*-stgy-*$cdcl_W$*-all*:
no-step $cdcl_W$*-stgy* (*toS* (*rough-state-from-init-state-of* (*do-all-*$cdcl_W$*-stgy S*)))
apply (*induction S rule*:*do-all-*$cdcl_W$*-stgy-induct*)
apply (*case-tac do-*$cdcl_W$*-stgy-step' S* $\neq$ *S*)
proof −
fix *Sa* :: $cdcl_W$*-state-inv-from-init-state*
assume *a1*: ¬ *do-*$cdcl_W$*-stgy-step' Sa* $\neq$ *Sa*
{ **fix** *pp*
have (*if True then Sa else do-all-*$cdcl_W$*-stgy Sa*) = *do-all-*$cdcl_W$*-stgy Sa*
using *a1* **by** *auto*
then have ¬ $cdcl_W$*-stgy* (*toS* (*rough-state-from-init-state-of* (*do-all-*$cdcl_W$*-stgy Sa*))) *pp*
using *a1* **by** (*metis* (*no-types*) *do-*$cdcl_W$*-stgy-step-no id-of-I-to-def*
rough-state-from-init-state-of-do-$cdcl_W$*-stgy-step' rough-state-of-inverse*) }
then show *no-step* $cdcl_W$*-stgy* (*toS* (*rough-state-from-init-state-of* (*do-all-*$cdcl_W$*-stgy Sa*)))
by *fastforce*
next
fix *Sa* :: $cdcl_W$*-state-inv-from-init-state*
assume *a1*: *do-*$cdcl_W$*-stgy-step' Sa* $\neq$ *Sa*
$\Longrightarrow$ *no-step* $cdcl_W$*-stgy* (*toS* (*rough-state-from-init-state-of*
(*do-all-*$cdcl_W$*-stgy* (*do-*$cdcl_W$*-stgy-step' Sa*))))
assume *a2*: *do-*$cdcl_W$*-stgy-step' Sa* $\neq$ *Sa*
have *do-all-*$cdcl_W$*-stgy Sa* = *do-all-*$cdcl_W$*-stgy* (*do-*$cdcl_W$*-stgy-step' Sa*)
by (*metis* (*full-types*) *do-all-*$cdcl_W$*-stgy.simps*)

then show *no-step* $cdcl_W$*-stgy* (*toS* (*rough-state-from-init-state-of* (*do-all-cdcl*$_W$*-stgy Sa*)))
  using *a2 a1* **by** *presburger*
qed

lemma *do-all-cdcl*$_W$*-stgy-is-rtranclp-cdcl*$_W$*-stgy*:
cdcl$_W$*-stgy*** (*toS* (*rough-state-from-init-state-of S*))
  (*toS* (*rough-state-from-init-state-of* (*do-all-cdcl*$_W$*-stgy S*)))
proof (*induction S rule*: *do-all-cdcl*$_W$*-stgy-induct*)
 case (*1 S*) **note** *IH* = *this*(*1*)
 show *?case*
  proof (*cases do-cdcl*$_W$*-stgy-step′ S* = *S*)
   case *True*
   then show *?thesis* **by** *simp*
  next
   case *False*
   have *f2*: *do-cdcl*$_W$*-stgy-step* (*id-of-I-to S*) = *id-of-I-to S* $\longrightarrow$
    rough-state-from-init-state-of (*do-cdcl*$_W$*-stgy-step′ S*)
    = *rough-state-of* (*state-of* (*rough-state-from-init-state-of S*))
    using *id-of-I-to-def rough-state-from-init-state-of-do-cdcl*$_W$*-stgy-step′* **by** *presburger*
   have *f3*: *do-all-cdcl*$_W$*-stgy S* = *do-all-cdcl*$_W$*-stgy* (*do-cdcl*$_W$*-stgy-step′ S*)
    by (*metis* (*full-types*) *do-all-cdcl*$_W$*-stgy.simps*)
   have *cdcl*$_W$*-stgy* (*toS* (*rough-state-from-init-state-of S*))
     (*toS* (*rough-state-from-init-state-of* (*do-cdcl*$_W$*-stgy-step′ S*)))
    = *cdcl*$_W$*-stgy* (*toS* (*rough-state-of* (*id-of-I-to S*)))
     (*toS* (*rough-state-of* (*do-cdcl*$_W$*-stgy-step* (*id-of-I-to S*))))
    using *id-of-I-to-def rough-state-from-init-state-of-do-cdcl*$_W$*-stgy-step′*
    toS-rough-state-of-state-of-rough-state-from-init-state-of **by** *presburger*
   then show *?thesis*
    using *f3 f2 IH do-cdcl*$_W$*-stgy-step* **by** *fastforce*
  qed
qed

Final theorem:

lemma *DPLL-tot-correct*:
 assumes
  r: *rough-state-from-init-state-of* (*do-all-cdcl*$_W$*-stgy* (*state-from-init-state-of*
   (([], *map remdups N*, [], *0*, *None*)))) = *S* **and**
  S: (*M′*, *N′*, *U′*, *k*, *E*) = *toS S*
 shows (*E* ≠ *Some* {#} ∧ *satisfiable* (*set* (*map mset N*)))
  ∨ (*E* = *Some* {#} ∧ *unsatisfiable* (*set* (*map mset N*)))
proof −
 let *?N* = *map remdups N*
 have *inv*: *cdcl*$_W$*-all-struct-inv* (*toS* ([], *map remdups N*, [], *0*, *None*))
  unfolding *cdcl*$_W$*-all-struct-inv-def distinct-cdcl*$_W$*-state-def distinct-mset-set-def* **by** *auto*
 then have *S0*: *rough-state-of* (*state-of* ([], *map remdups N*, [], *0*, *None*))
  = ([], *map remdups N*, [], *0*, *None*) **by** *simp*
 have *1*: *full cdcl*$_W$*-stgy* (*toS* ([], *?N*, [], *0*, *None*)) (*toS S*)
  unfolding *full-def* **apply** *rule*
   using *do-all-cdcl*$_W$*-stgy-is-rtranclp-cdcl*$_W$*-stgy*[*of*
    state-from-init-state-of ([], *map remdups N*, [], *0*, *None*)] *inv*
    no-step-cdcl$_W$*-stgy-cdcl*$_W$*-all*
    by (*auto simp del*: *do-all-cdcl*$_W$*-stgy.simps simp*: *state-from-init-state-of-inverse*
     r[*symmetric*])+
 moreover have *2*: *finite* (*set* (*map mset ?N*)) **by** *auto*
 moreover have *3*: *distinct-mset-set* (*set* (*map mset ?N*))

unfolding *distinct-mset-set-def* **by** *auto*
moreover
have *cdcl*$_W$*-all-struct-inv* (*toS S*)
by (*metis* (*no-types*) *cdcl*$_W$*-all-struct-inv-rough-state r*
toS-rough-state-of-state-of-rough-state-from-init-state-of)
then have *cons*: *consistent-interp* (*lits-of M'*)
unfolding *cdcl*$_W$*-all-struct-inv-def cdcl*$_W$*-M-level-inv-def S*[*symmetric*] **by** *auto*
moreover
have *clauses* (*toS* ([], *?N*, [], *0*, *None*)) = *clauses* (*toS S*)
apply (*rule rtranclp-cdcl*$_W$*-init-clss*)
using *1* **unfolding** *full-def* **by** (*auto simp add*: *rtranclp-cdcl*$_W$*-stgy-rtranclp-cdcl*$_W$)
then have *N'*: *mset* (*map mset ?N*) = *N'*
using *S*[*symmetric*] **by** *auto*
have ($E \neq$ *Some* {#} ∧ *satisfiable* (*set* (*map mset ?N*)))
∨ (E = *Some* {#} ∧ *unsatisfiable* (*set* (*map mset ?N*)))
using *full-cdcl*$_W$*-stgy-final-state-conclusive* **unfolding** *N'* **apply** *rule*
using *1* **apply** *simp*
using *2* **apply** *simp*
using *3* **apply** *simp*
using *S*[*symmetric*] *N'* **apply** *auto*[*1*]
using *S*[*symmetric*] *N' cons* **by** (*fastforce simp*: *true-annots-true-cls*)
then show *?thesis* **by** *auto*
qed

The Code The SML code is skipped in the documentation, but stays to ensure that some version of the exported code is working. The only difference between the generated code and the one used here is the export of the constructor ConI.

end
theory *CDCL-WNOT*
imports *CDCL-W-Termination CDCL-NOT*
begin

19 Link between Weidenbach's and NOT's CDCL

19.1 Inclusion of the states

declare *upt.simps*(*2*)[*simp del*]
sledgehammer-params[*verbose*]

context *cdcl*$_W$*-ops*
begin

lemma *backtrack-levE*:
backtrack S S' ⟹ *cdcl*$_W$*-M-level-inv S* ⟹
(⋀*D L K M1 M2*.
(*Marked K* (*Suc* (*get-maximum-level* (*trail S*) *D*)) # *M1*, *M2*)
∈ *set* (*get-all-marked-decomposition* (*trail S*)) ⟹
get-level (*trail S*) *L* = *get-maximum-level* (*trail S*) (*D* + {#*L*#}) ⟹
undefined-lit M1 L ⟹
S' ∼ *cons-trail* (*Propagated L* (*D* + {#*L*#}))
(*reduce-trail-to M1* (*add-learned-cls* (*D* + {#*L*#})
(*update-backtrack-lvl* (*get-maximum-level* (*trail S*) *D*) (*update-conflicting None S*)))) ⟹
backtrack-lvl S = *get-maximum-level* (*trail S*) (*D* + {#*L*#}) ⟹
conflicting S = *Some* (*D* + {#*L*#}) ⟹ *P*) ⟹

P
using *assms* **by** (*induction rule*: *backtrack-induction-lev2*) *metis*

lemma *backtrack-no-cdcl$_W$-bj*:
assumes *cdcl*: *cdcl$_W$-bj T U* **and** *inv*: *cdcl$_W$-M-level-inv V*
shows ¬*backtrack V T*
using *cdcl inv*
apply (*induction rule*: *cdcl$_W$-bj.induct*)
apply (*elim skipE*, *force elim*!: *backtrack-levE*[*OF - inv*] *simp*: *cdcl$_W$-M-level-inv-def*)
apply (*elim resolveE*, *force elim*!: *backtrack-levE*[*OF - inv*] *simp*: *cdcl$_W$-M-level-inv-def*)
apply *standard*
apply (*elim backtrack-levE*[*OF - inv*], *elim backtrackE*)
apply (*force simp del*: *state-simp simp add*: *state-eq-conflicting cdcl$_W$-M-level-inv-decomp*)
done

abbreviation *skip-or-resolve* :: *'st* ⇒ *'st* ⇒ *bool* **where**
skip-or-resolve ≡ (*λS T*. *skip S T* ∨ *resolve S T*)

lemma *rtranclp-cdcl$_W$-bj-skip-or-resolve-backtrack*:
assumes *cdcl$_W$-bj** S U* **and** *inv*: *cdcl$_W$-M-level-inv S*
shows *skip-or-resolve** S U* ∨ (∃ *T*. *skip-or-resolve** S T* ∧ *backtrack T U*)
using *assms*
proof (*induction*)
case *base*
then show *?case* **by** *simp*
next
case (*step U V*) **note** *st* = *this*(*1*) **and** *bj* = *this*(*2*) **and** *IH* = *this*(*3*)[*OF this*(*4*)]
consider
(*SU*) *S* = *U*
| (*SUp*) *cdcl$_W$-bj^{++} S U*
using *st* **unfolding** *rtranclp-unfold* **by** *blast*
then show *?case*
proof *cases*
case *SUp*
have ⋀*T*. *skip-or-resolve** S T* ⟹ *cdcl$_W$** S T*
using *mono-rtranclp*[*of skip-or-resolve cdcl$_W$*] *other* **by** *blast*
then have *skip-or-resolve** S U*
using *bj IH inv backtrack-no-cdcl$_W$-bj rtranclp-cdcl$_W$-consistent-inv*[*OF - inv*] **by** *meson*
then show *?thesis*
using *bj* **by** (*metis* (*no-types*, *lifting*) *cdcl$_W$-bj.cases rtranclp.simps*)
next
case *SU*
then show *?thesis*
using *bj* **by** (*metis* (*no-types*, *lifting*) *cdcl$_W$-bj.cases rtranclp.simps*)
qed
qed

lemma *rtranclp-skip-or-resolve-rtranclp-cdcl$_W$*:
skip-or-resolve** S T ⟹ *cdcl$_W$** S T*
by (*induction rule*: *rtranclp-induct*) (*auto dest*!: *cdcl$_W$-bj.intros cdcl$_W$.intros cdcl$_W$-o.intros*)

definition *backjump-l-cond* :: *'v clause* ⇒ *'v clause* ⇒ *'v literal* ⇒ *'st* ⇒ *bool* **where**
backjump-l-cond ≡ *λC C' L' S*. *True*

definition inv_{NOT} :: *'st* ⇒ *bool* **where**
$inv_{NOT} \equiv \lambda S.$ *no-dup* (*trail S*)

declare inv_{NOT}*-def*[*simp*]
end

fun *convert-marked-lit-from-W* **where**
convert-marked-lit-from-W (*Propagated L -*) = *Propagated L* () |
convert-marked-lit-from-W (*Marked L -*) = *Marked L* ()

abbreviation *convert-trail-from-W* ::
('*v*, '*lvl*, '*a*) *marked-lit list*
⇒ ('*v*, *unit*, *unit*) *marked-lit list* **where**
convert-trail-from-W ≡ *map convert-marked-lit-from-W*

lemma *lits-of-convert-trail-from-W*[*simp*]:
lits-of (*convert-trail-from-W M*) = *lits-of M*
by (*induction rule*: *marked-lit-list-induct*) *simp-all*

lemma *lit-of-convert-trail-from-W*[*simp*]:
lit-of (*convert-marked-lit-from-W L*) = *lit-of L*
by (*cases L*) *auto*

lemma *no-dup-convert-from-W*[*simp*]:
no-dup (*convert-trail-from-W M*) ⟷ *no-dup M*
by (*auto simp*: *comp-def*)

lemma *convert-trail-from-W-true-annots*[*simp*]:
convert-trail-from-W M ⊨*as C* ⟷ *M* ⊨*as C*
by (*auto simp*: *true-annots-true-cls*)

lemma *defined-lit-convert-trail-from-W*[*simp*]:
defined-lit (*convert-trail-from-W S*) *L* ⟷ *defined-lit S L*
by (*auto simp*: *defined-lit-map image-comp*)

The values *0* and {#} are dummy values.

fun *convert-marked-lit-from-NOT*
:: ('*a*, '*e*, '*b*) *marked-lit* ⇒ ('*a*, *nat*, '*a literal multiset*) *marked-lit* **where**
convert-marked-lit-from-NOT (*Propagated L -*) = *Propagated L* {#} |
convert-marked-lit-from-NOT (*Marked L -*) = *Marked L 0*

abbreviation *convert-trail-from-NOT* **where**
convert-trail-from-NOT ≡ *map convert-marked-lit-from-NOT*

lemma *convert-trail-from-W-from-NOT*[*simp*]:
convert-trail-from-W (*convert-trail-from-NOT M*) = *M*
by (*induction rule*: *marked-lit-list-induct*) *auto*

lemma *convert-trail-from-W-convert-lit-from-NOT*[*simp*]:
convert-marked-lit-from-W (*convert-marked-lit-from-NOT L*) = *L*
by (*cases L*) *auto*

abbreviation $trail_{NOT}$ **where**
$trail_{NOT}$ *S* ≡ *convert-trail-from-W* (*fst S*)

lemma *undefined-lit-convert-trail-from-W*[*iff*]:
undefined-lit (*convert-trail-from-W M*) $L \longleftrightarrow$ *undefined-lit M L*
by (*auto simp*: *defined-lit-map image-comp*)

lemma *lit-of-convert-marked-lit-from-NOT*[*iff*]:
lit-of (*convert-marked-lit-from-NOT L*) = *lit-of L*
by (*cases L*) *auto*

sublocale $state_W \subseteq$ *dpll-state*
λS. *convert-trail-from-W* (*trail S*)
clauses
$\lambda L\ S$. *cons-trail* (*convert-marked-lit-from-NOT L*) *S*
λS. *tl-trail S*
$\lambda C\ S$. *add-learned-cls C S*
$\lambda C\ S$. *remove-cls C S*
by *unfold-locales* (*auto simp*: *map-tl o-def*)

context $state_W$
begin
declare $state\text{-}simp_{NOT}$[*simp del*]
end

sublocale $cdcl_W$*-ops* $\subseteq$ $cdcl_{NOT}$*-merge-bj-learn-ops*
λS. *convert-trail-from-W* (*trail S*)
clauses
$\lambda L\ S$. *cons-trail* (*convert-marked-lit-from-NOT L*) *S*
λS. *tl-trail S*
$\lambda C\ S$. *add-learned-cls C S*
$\lambda C\ S$. *remove-cls C S*
λ- -. *True*
λ- *S*. *conflicting S* = *None*
$\lambda C\ C'\ L'\ S$. *backjump-l-cond* $C\ C'\ L'\ S$ $\wedge$ *distinct-mset* $(C' + \{\#L'\#\})$ $\wedge$ $\neg$*tautology* $(C' + \{\#L'\#\})$
by *unfold-locales*

sublocale $cdcl_W$*-ops* $\subseteq$ $cdcl_{NOT}$*-merge-bj-learn-proxy*
λS. *convert-trail-from-W* (*trail S*)
clauses
$\lambda L\ S$. *cons-trail* (*convert-marked-lit-from-NOT L*) *S*
λS. *tl-trail S*
$\lambda C\ S$. *add-learned-cls C S*
$\lambda C\ S$. *remove-cls C S*
λ- -. *True*
λ- *S*. *conflicting S* = *None backjump-l-cond* inv_{NOT}
proof (*unfold-locales*, *goal-cases*)
case *2*
then show *?case* **using** $cdcl_{NOT}$*-merged-bj-learn-no-dup-inv* **by** (*auto simp*: *comp-def*)
next
case (*1* C' *S C* F' *K F L*)
moreover
let $?C'$ = *remdups-mset* C'
have $L \notin\# C'$
using ‹$F \models as$ *CNot* C'› ‹*undefined-lit F L*› *Marked-Propagated-in-iff-in-lits-of*
in-CNot-implies-uminus(*2*) **by** *blast*
then have *distinct-mset* $(?C' + \{\#L\#\})$
by (*metis count-mset-set*(*3*) *distinct-mset-remdups-mset distinct-mset-single-add*

less-irrefl-nat mem-set-mset-iff remdups-mset-def)
moreover
have *no-dup F*
using ‹inv_{NOT} *S*› ‹*convert-trail-from-W* (*trail S*) = *F*′ @ *Marked K* () # *F*›
unfolding inv_{NOT}*-def*
by (*smt comp-apply distinct.simps*(*2*) *distinct-append list.simps*(*9*) *map-append no-dup-convert-from-W*)
then have *consistent-interp* (*lits-of F*)
using *distinctconsistent-interp* **by** *blast*
then have ¬ *tautology* (*C*′)
using ‹*F* ⊨*as CNot C*′› *consistent-CNot-not-tautology true-annots-true-cls* **by** *blast*
then have ¬ *tautology* (*?C*′ + {#*L*#})
using ‹*F* ⊨*as CNot C*′› ‹*undefined-lit F L*› **by** (*metis CNot-remdups-mset Marked-Propagated-in-iff-in-lits-of add.commute in-CNot-uminus tautology-add-single tautology-remdups-mset true-annot-singleton true-annots-def*)
show *?case*
proof −
have *f2*: *no-dup* (*convert-trail-from-W* (*trail S*))
using ‹inv_{NOT} *S*› **unfolding** inv_{NOT}*-def* **by** (*simp add*: *o-def*)
have *f3*: *atm-of L* ∈ *atms-of-msu* (*clauses S*) ∪ *atm-of* ‘ *lits-of* (*convert-trail-from-W* (*trail S*))
using ‹*convert-trail-from-W* (*trail S*) = *F*′ @ *Marked K* () # *F*› ‹*atm-of L* ∈ *atms-of-msu* (*clauses S*) ∪ *atm-of* ‘ *lits-of* (*F*′ @ *Marked K* () # *F*)› **by** *auto*
have *f4*: *clauses S* ⊨*pm remdups-mset C*′ + {#*L*#}
by (*metis* (*no-types*) ‹*L* ∉# *C*′› ‹*clauses S* ⊨*pm C*′ + {#*L*#}› *remdups-mset-singleton-sum*(*2*) *true-clss-cls-remdups-mset union-commute*)
have *F* ⊨*as CNot* (*remdups-mset C*′)
by (*simp add*: ‹*F* ⊨*as CNot C*′›)
then show *?thesis*
using *f4 f3 f2* ‹¬ *tautology* (*remdups-mset C*′ + {#*L*#})› *backjump-l.intros*[*OF - f2*] *calculation*(*2*−*5*,*9*) *state-eq*$_{NOT}$*-ref* **unfolding** *backjump-l-cond-def* **by** *blast*
qed
qed

sublocale $cdcl_W$*-ops* ⊆ $cdcl_{NOT}$*-merge-bj-learn-proxy2*
λ*S*. *convert-trail-from-W* (*trail S*)
clauses
λ*L S*. *cons-trail* (*convert-marked-lit-from-NOT L*) *S*
λ*S*. *tl-trail S*
λ*C S*. *add-learned-cls C S*
λ*C S*. *remove-cls C S* λ- -. *True* inv_{NOT}
λ- *S*. *conflicting S* = *None backjump-l-cond*
by *unfold-locales*

sublocale $cdcl_W$*-ops* ⊆ $cdcl_{NOT}$*-merge-bj-learn*
λ*S*. *convert-trail-from-W* (*trail S*)
clauses
λ*L S*. *cons-trail* (*convert-marked-lit-from-NOT L*) *S*
λ*S*. *tl-trail S*
λ*C S*. *add-learned-cls C S*
λ*C S*. *remove-cls C S* λ- -. *True* inv_{NOT}
λ- *S*. *conflicting S* = *None backjump-l-cond*
apply *unfold-locales*
using *dpll-bj-no-dup* **apply** (*simp add*: *comp-def*)

using *cdcl*$_{NOT}$*-no-dup* **by** (*auto simp add*: *comp-def cdcl*$_{NOT}$*.simps*)

context *cdcl*$_W$*-ops*
begin

Notations are lost while proving locale inclusion:

notation *state-eq*$_{NOT}$ (**infix** $\sim_{NOT}$ *50*)

19.2 Additional Lemmas between NOT and W states

lemma *trail*$_W$*-eq-reduce-trail-to*$_{NOT}$*-eq*:
trail S = *trail T* $\Longrightarrow$ *trail* (*reduce-trail-to*$_{NOT}$ *F S*) = *trail* (*reduce-trail-to*$_{NOT}$ *F T*)
proof (*induction F S arbitrary*: *T rule*: *reduce-trail-to*$_{NOT}$*.induct*)
case (*1 F S T*) **note** *IH* = *this*(*1*) **and** *tr* = *this*(*2*)
then have [] = *convert-trail-from-W* (*trail S*)
∨ *length F* = *length* (*convert-trail-from-W* (*trail S*))
∨ *trail* (*reduce-trail-to*$_{NOT}$ *F* (*tl-trail S*)) = *trail* (*reduce-trail-to*$_{NOT}$ *F* (*tl-trail T*))
using *IH* **by** (*metis* (*no-types*) *trail-tl-trail*)
then show *trail* (*reduce-trail-to*$_{NOT}$ *F S*) = *trail* (*reduce-trail-to*$_{NOT}$ *F T*)
using *tr* **by** (*metis* (*no-types*) *reduce-trail-to*$_{NOT}$*.elims*)
qed

lemma *trail-reduce-trail-to*$_{NOT}$*-add-learned-cls*:
no-dup (*trail S*) $\Longrightarrow$
trail (*reduce-trail-to*$_{NOT}$ *M* (*add-learned-cls D S*)) = *trail* (*reduce-trail-to*$_{NOT}$ *M S*)
by (*rule trail*$_W$*-eq-reduce-trail-to*$_{NOT}$*-eq*) *simp*

lemma *reduce-trail-to*$_{NOT}$*-reduce-trail-convert*:
reduce-trail-to$_{NOT}$ *C S* = *reduce-trail-to* (*convert-trail-from-NOT C*) *S*
apply (*induction C S rule*: *reduce-trail-to*$_{NOT}$*.induct*)
apply (*subst reduce-trail-to*$_{NOT}$*.simps*, *subst reduce-trail-to.simps*)
by *auto*

lemma *reduce-trail-to-length*:
length M = *length M'* $\Longrightarrow$ *reduce-trail-to M S* = *reduce-trail-to M' S*
apply (*induction M S arbitrary*: *rule*: *reduce-trail-to.induct*)
apply (*case-tac trail S* ≠ [] ; *case-tac length* (*trail S*) ≠ *length M'*; *simp*)
by (*simp-all add*: *reduce-trail-to-length-ne*)

19.3 More lemmas conflict–propagate and backjumping

19.3.1 Termination

lemma *cdcl*$_W$*-cp-normalized-element-all-inv*:
assumes *inv*: *cdcl*$_W$*-all-struct-inv S*
obtains *T* **where** *full cdcl*$_W$*-cp S T*
using *assms cdcl*$_W$*-cp-normalized-element* **unfolding** *cdcl*$_W$*-all-struct-inv-def* **by** *blast*
thm *backtrackE*

lemma *cdcl*$_W$*-bj-measure*:
assumes *cdcl*$_W$*-bj S T* **and** *cdcl*$_W$*-M-level-inv S*
shows *length* (*trail S*) + (*if conflicting S* = *None then 0 else 1*)
> *length* (*trail T*) + (*if conflicting T* = *None then 0 else 1*)
using *assms* **by** (*induction rule*: *cdcl*$_W$*-bj.induct*)
(*force dest*:*arg-cong*[*of - - length*]
intro: *get-all-marked-decomposition-exists-prepend*

elim!: *backtrack-levE*
simp: $cdcl_W$*-M-level-inv-def*)+

lemma *wf-*$cdcl_W$*-bj*:
wf {(*b*,*a*). $cdcl_W$*-bj a b* ∧ $cdcl_W$*-M-level-inv a*}
apply (*rule wfP-if-measure*[*of* λ-. *True*
- λ*T*. *length* (*trail T*) + (*if conflicting T* = *None then 0 else 1*), *simplified*])
using $cdcl_W$*-bj-measure* **by** *blast*

lemma $cdcl_W$*-bj-exists-normal-form*:
assumes *lev*: $cdcl_W$*-M-level-inv S*
shows ∃ *T*. *full* $cdcl_W$*-bj S T*
proof −
obtain *T* **where** *T*: *full* (λ*a b*. $cdcl_W$*-bj a b* ∧ $cdcl_W$*-M-level-inv a*) *S T*
using *wf-exists-normal-form-full*[*OF wf-*$cdcl_W$*-bj*] **by** *auto*
then have $cdcl_W$*-bj*** *S T*
by (*auto dest*: *rtranclp-and-rtranclp-left simp*: *full-def*)
moreover
then have $cdcl_W$** *S T*
using *mono-rtranclp*[*of* $cdcl_W$*-bj* $cdcl_W$] $cdcl_W$*.simps* **by** *blast*
then have $cdcl_W$*-M-level-inv T*
using *rtranclp-*$cdcl_W$*-consistent-inv lev* **by** *auto*
ultimately show *?thesis* **using** *T* **unfolding** *full-def* **by** *auto*
qed

lemma *rtranclp-skip-state-decomp*:
assumes *skip*** *S T* **and** *no-dup* (*trail S*)
shows
∃ *M*. *trail S* = *M* @ *trail T* ∧ (∀ *m*∈*set M*. ¬*is-marked m*) **and**
T ∼ *delete-trail-and-rebuild* (*trail T*) *S*
using *assms* **by** (*induction rule*: *rtranclp-induct*)
(*auto simp del*: *state-simp simp*: *state-eq-def state-access-simp*)

19.3.2 More backjumping

Backjumping after skipping or jump directly **lemma** *rtranclp-skip-backtrack-backtrack*:
assumes
*skip*** *S T* **and**
backtrack T W **and**
$cdcl_W$*-all-struct-inv S*
shows *backtrack S W*
using *assms*
proof *induction*
case *base*
then show *?case* **by** *simp*
next
case (*step T V*) **note** *st* = *this*(*1*) **and** *skip* = *this*(*2*) **and** *IH* = *this*(*3*) **and** *bt* = *this*(*4*) **and**
inv = *this*(*5*)
have *skip*** *S V*
using *st skip* **by** *auto*
then have $cdcl_W$*-all-struct-inv V*
using *rtranclp-mono*[*of skip* $cdcl_W$] *assms*(*3*) *rtranclp-*$cdcl_W$*-all-struct-inv-inv mono-rtranclp*
by (*auto dest*!: *bj other* $cdcl_W$*-bj.skip*)
then have $cdcl_W$*-M-level-inv V*
unfolding $cdcl_W$*-all-struct-inv-def* **by** *auto*
then obtain *N k M1 M2 K D L U i* **where**

V: *state* V = (*trail* V, N, U, k, *Some* (D + {#L#})) **and**
W: *state* W = (*Propagated* L (D + {#L#}) # *M1*, N, {#D + {#L#}#} + U,
get-maximum-level (*trail* V) D, *None*) **and**
decomp: (*Marked* K (*Suc* i) # *M1*, *M2*)
$\in$ *set* (*get-all-marked-decomposition* (*trail* V)) **and**
k = *get-maximum-level* (*trail* V) (D + {#L#}) **and**
lev-L: *get-level* (*trail* V) L = k **and**
undef: *undefined-lit M1* L **and**
$W \sim$ *cons-trail* (*Propagated* L (D + {#L#}))
(*reduce-trail-to M1* (*add-learned-cls* (D + {#L#})
(*update-backtrack-lvl* (*get-maximum-level* (*trail* V) D) (*update-conflicting None* V))))**and**
lev-l-D: *backtrack-lvl* V = *get-maximum-level* (*trail* V) (D + {#L#}) **and**
conflicting V = *Some* (D + {#L#}) **and**
i: i = *get-maximum-level* (*trail* V) D
using *bt* **by** (*elim backtrack-levE*)
(*auto simp*: $cdcl_W$*-M-level-inv-decomp state-eq-def simp del*: *state-simp*)+
let *?D* = (D + {#L#})
obtain L' C' **where**
T: *state* T = (*Propagated* L' C' # *trail* V, N, U, k, *Some ?D*) **and**
$V \sim$ *tl-trail* T **and**
$-L' \notin\#$ *?D* **and**
?D $\neq$ {#}
using *skip* V **by** *force*

let *?M* = *Propagated* L' C' # *trail* V
have $cdcl_W{}^{**}$ S T **using** *bj* $cdcl_W$*-bj.skip mono-rtranclp*[*of skip* $cdcl_W$ S T] *other st* **by** *meson*
then have *inv'*: $cdcl_W$*-all-struct-inv* T
using *rtranclp-*$cdcl_W$*-all-struct-inv-inv inv* **by** *blast*
have *M-lev*: $cdcl_W$*-M-level-inv* T **using** *inv'* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *auto*
then have *n-d'*: *no-dup ?M*
using T **unfolding** $cdcl_W$*-M-level-inv-def* **by** *auto*

have $k > 0$
using *decomp M-lev* T V **unfolding** $cdcl_W$*-M-level-inv-def* **by** *auto*
then have *atm-of* $L \in$ *atm-of* ‘ *lits-of* (*trail* V)
using *lev-L get-rev-level-ge-0-atm-of-in* V **by** *fastforce*
then have *L-L'*: *atm-of* $L \neq$ *atm-of* L'
using *n-d'* **unfolding** *lits-of-def* **by** *auto*
have *L'-M*: *atm-of* $L' \notin$ *atm-of* ‘ *lits-of* (*trail* V)
using *n-d'* **unfolding** *lits-of-def* **by** *auto*
have *?M* $\models$*as CNot ?D*
using *inv'* T **unfolding** $cdcl_W$*-conflicting-def* $cdcl_W$*-all-struct-inv-def* **by** *auto*
then have $L' \notin\#$ *?D*
using *L-L' L'-M* **unfolding** *true-annots-def* **by** (*auto simp add*: *true-annot-def true-cls-def*
atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set Ball-mset-def
split: *split-if-asm*)
have [*simp*]: *trail* (*reduce-trail-to M1* T) = *M1*
by (*metis* (*mono-tags*, *lifting*) *One-nat-def Pair-inject* T ‹$V \sim$ *tl-trail* T› *decomp*
diff-less in-get-all-marked-decomposition-trail-update-trail length-greater-0-conv
length-tl lessI list.distinct(*1*) *reduce-trail-to-length-ne state-eq-trail*
trail-reduce-trail-to-length-le trail-tl-trail)
have $skip^{**}$ S V
using *st skip* **by** *auto*
have *no-dup* (*trail* S)
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-M-level-inv-def* **by** *auto*

then have [*simp*]: *init-clss* $S = N$ **and** [*simp*]: *learned-clss* $S = U$
using *rtranclp-skip-state-decomp*[*OF* ‹*skip*** *S V*›] *V*
by (*auto simp del*: *state-simp simp*: *state-eq-def state-access-simp*)
then have *W-S*: $W \sim$ *cons-trail* (*Propagated L* (D + {#*L*#})) (*reduce-trail-to M1*
(*add-learned-cls* (D + {#*L*#}) (*update-backtrack-lvl i* (*update-conflicting None T*))))
using *W i T undef M-lev* **by** (*auto simp del*: *state-simp simp*: *state-eq-def* $cdcl_W$*-M-level-inv-def*)

obtain *M2'* **where**
(*Marked K* (*i+1*) # *M1*, *M2'*) $\in$ *set* (*get-all-marked-decomposition ?M*)
using *decomp V* **by** (*cases hd* (*get-all-marked-decomposition* (*trail V*)),
cases get-all-marked-decomposition (*trail V*)) *auto*
moreover
from *L-L'* **have** *get-level ?M L* = *k*
using *lev-L* ‹$-L' \notin\#$ *?D*› *V* **by** (*auto split*: *split-if-asm*)
moreover
have *atm-of* $L' \notin$ *atms-of D*
using ‹$L' \notin\#$ *?D*› ‹$-L' \notin\#$ *?D*› **by** (*simp add*: *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set*
atms-of-def)
then have *get-level ?M L* = *get-maximum-level ?M* (D+{#*L*#})
using *lev-l-D*[*symmetric*] *L-L' V lev-L* **by** *simp*
moreover have i = *get-maximum-level ?M D*
using *i* ‹*atm-of* $L' \notin$ *atms-of D*› **by** *auto*
moreover

ultimately have *backtrack T W*
using *T*(*1*) *W-S* **by** *blast*
then show *?thesis* **using** *IH inv* **by** *blast*
qed

lemma *fst-get-all-marked-decomposition-prepend-not-marked*:
assumes $\forall$ *m*$\in$*set MS*. $\neg$ *is-marked m*
shows *set* (*map fst* (*get-all-marked-decomposition M*))
= *set* (*map fst* (*get-all-marked-decomposition* (*MS* @ *M*)))
using *assms* **apply** (*induction MS rule*: *marked-lit-list-induct*)
apply *auto*[*2*]
by (*case-tac get-all-marked-decomposition* (*xs* @ *M*)) *simp-all*

See also ⟦*skip*** *?S ?T*; *backtrack ?T ?W*; $cdcl_W$*-all-struct-inv ?S*⟧ $\Longrightarrow$ *backtrack ?S ?W*

lemma *rtranclp-skip-backtrack-backtrack-end*:
assumes
skip: *skip*** *S T* **and**
bt: *backtrack S W* **and**
inv: $cdcl_W$*-all-struct-inv S*
shows *backtrack T W*
using *assms*
proof −
have *M-lev*: $cdcl_W$*-M-level-inv S*
using *bt inv* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** (*auto elim*!: *backtrack-levE*)
then obtain *k M M1 M2 K i D L N U* **where**
S: *state S* = (*M*, *N*, *U*, *k*, *Some* (D + {#*L*#})) **and**
W: *state W* = (*Propagated L* (D + {#*L*#}) # *M1*, *N*, {#*D* + {#*L*#}#} + *U*, *get-maximum-level*
M D,
None) **and**
decomp: (*Marked K* (*i+1*) # *M1*, *M2*) $\in$ *set* (*get-all-marked-decomposition M*) **and**
lev-l: *get-level M L* = *k* **and**

lev-l-D: *get-level M L* = *get-maximum-level M* (*D*+{#*L*#}) **and**
i: *i* = *get-maximum-level M D* **and**
undef: *undefined-lit M1 L*
using *bt* **by** (*elim backtrack-levE*)
(*simp-all add*: *cdcl*$_W$*-M-level-inv-decomp state-eq-def del*: *state-simp*)
let *?D* = (*D* + {#*L*#})

have [*simp*]: *no-dup* (*trail S*)
using *M-lev* **by** (*auto simp*: *cdcl*$_W$*-M-level-inv-decomp*)
have *cdcl*$_W$*-all-struct-inv T*
using *mono-rtranclp*[*of skip cdcl*$_W$] **by** (*smt bj cdcl*$_W$*-bj.skip inv local.skip other rtranclp-cdcl*$_W$*-all-struct-inv-inv*)
then have [*simp*]: *no-dup* (*trail T*)
unfolding *cdcl*$_W$*-all-struct-inv-def cdcl*$_W$*-M-level-inv-def* **by** *auto*

obtain *MS* M_T **where** *M*: *M* = *MS* @ M_T **and** M_T: M_T = *trail T* **and** *nm*: ∀ *m*∈*set MS*. ¬*is-marked m*
using *rtranclp-skip-state-decomp*(*1*)[*OF skip*] *S M-lev* **by** *auto*
have *T*: *state T* = (M_T, *N*, *U*, *k*, *Some ?D*)
using M_T *rtranclp-skip-state-decomp*(*2*)[*of S T*] *skip S*
by (*auto simp del*: *state-simp simp*: *state-eq-def state-access-simp*)

have *cdcl*$_W$*-all-struct-inv T*
apply (*rule rtranclp-cdcl*$_W$*-all-struct-inv-inv*[*OF - inv*])
using *bj cdcl*$_W$*-bj.skip local.skip other rtranclp-mono*[*of skip cdcl*$_W$] **by** *blast*
then have M_T ⊨*as CNot ?D*
unfolding *cdcl*$_W$*-all-struct-inv-def cdcl*$_W$*-conflicting-def* **using** *T* **by** *blast*
have ∀ *L*∈#*?D*. *atm-of L* ∈ *atm-of* ‘ *lits-of* M_T
proof −
have *f1*: ⋀*l*. ¬ M_T ⊨*a* {#− *l*#} ∨ *atm-of l* ∈ *atm-of* ‘ *lits-of* M_T
by (*simp add*: *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set in-lit-of-true-annot lits-of-def*)
have ⋀*l*. *l* ∉# *D* ∨ − *l* ∈ *lits-of* M_T
using ‹M_T ⊨*as CNot* (*D* + {#*L*#})› *multi-member-split* **by** *fastforce*
then show *?thesis*
using *f1* **by** (*meson* ‹M_T ⊨*as CNot* (*D* + {#*L*#})› *ball-msetI true-annots-CNot-all-atms-defined*)
qed
moreover have *no-dup M*
using *inv S* **unfolding** *cdcl*$_W$*-all-struct-inv-def cdcl*$_W$*-M-level-inv-def* **by** *auto*
ultimately have ∀ *L*∈#*?D*. *atm-of L* ∉ *atm-of* ‘ *lits-of MS*
unfolding *M* **unfolding** *lits-of-def* **by** *auto*
then have *H*: ⋀*L*. *L*∈#*?D* ⟹ *get-level M L* = *get-level* M_T *L*
unfolding *M* **by** (*fastforce simp*: *lits-of-def*)
have [*simp*]: *get-maximum-level M ?D* = *get-maximum-level* M_T *?D*
by (*metis* ‹M_T ⊨*as CNot* (*D* + {#*L*#})› *M nm ball-msetI true-annots-CNot-all-atms-defined get-maximum-level-skip-un-marked-not-present*)

have *lev-l'*: *get-level* M_T *L* = *k*
using *lev-l* **by** (*auto simp*: *H*)
have [*simp*]: *trail* (*reduce-trail-to M1 T*) = *M1*
using *T decomp M nm* **by** (*smt* M_T *append-assoc beginning-not-marked-invert get-all-marked-decomposition-exists-prepend reduce-trail-to-trail-tl-trail-decomp*)
have *W*: *W* ∼ *cons-trail* (*Propagated L* (*D* + {#*L*#})) (*reduce-trail-to M1* (*add-learned-cls* (*D* + {#*L*#}) (*update-backtrack-lvl i* (*update-conflicting None T*))))
using *W T i decomp undef* **by** (*auto simp del*: *state-simp simp*: *state-eq-def*)

have *lev-l-D'*: *get-level* M_T *L* = *get-maximum-level* M_T (*D*+{#*L*#})
  using *lev-l-D* **by** (*auto simp*: *H*)
have [*simp*]: *get-maximum-level M D* = *get-maximum-level* M_T *D*
  proof −
    have ⋀*ms m*. ¬ (*ms*::('*v*, *nat*, '*v literal multiset*) *marked-lit list*) ⊨*as CNot m*
      ∨ (∀ *l*∈#*m*. *atm-of l* ∈ *atm-of* ‘ *lits-of ms*)
      by (*simp add*: *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set in-CNot-implies-uminus*(*2*))
    then have ∀ *l*∈#*D*. *atm-of l* ∈ *atm-of* ‘ *lits-of* M_T
      using ‹M_T ⊨*as CNot* (*D* + {#*L*#})› **by** *auto*
    then show *?thesis*
      by (*metis M get-maximum-level-skip-un-marked-not-present nm*)
  qed
then have *i'*: *i* = *get-maximum-level* M_T *D*
  using *i* **by** *auto*
have *Marked K* (*i* + *1*) # *M1* ∈ *set* (*map fst* (*get-all-marked-decomposition M*))
  using *Set.imageI*[*OF decomp*, *of fst*] **by** *auto*
then have *Marked K* (*i* + *1*) # *M1* ∈ *set* (*map fst* (*get-all-marked-decomposition* M_T))
  using *fst-get-all-marked-decomposition-prepend-not-marked*[*OF nm*] **unfolding** *M* **by** *auto*
then obtain *M2'* **where** *decomp'*:(*Marked K* (*i*+*1*) # *M1*, *M2'*) ∈ *set* (*get-all-marked-decomposition* M_T)
  by *auto*
then show *backtrack T W*
  using *backtrack.intros*[*OF T decomp' lev-l'*] *lev-l-D' i' W* **by** *force*
qed

lemma $cdcl_W$-*bj-decomp-resolve-skip-and-bj*:
  assumes $cdcl_W$-*bj*** *S T* **and** *inv*: $cdcl_W$-*M-level-inv S*
  shows (*skip-or-resolve*** *S T*
    ∨ (∃ *U*. *skip-or-resolve*** *S U* ∧ *backtrack U T*))
  using *assms*
proof *induction*
  case *base*
  then show *?case* **by** *simp*
next
  case (*step T U*) **note** *st* = *this*(*1*) **and** *bj* = *this*(*2*) **and** *IH* = *this*(*3*)
  have *IH*: *skip-or-resolve*** *S T*
    proof −
      { **assume** (∃ *U*. *skip-or-resolve*** *S U* ∧ *backtrack U T*)
        then obtain *V* **where**
          bt: *backtrack V T* **and**
          skip-or-resolve** *S V*
          by *blast*
        have $cdcl_W$** *S V*
          using ‹*skip-or-resolve*** *S V*› *rtranclp-skip-or-resolve-rtranclp-*$cdcl_W$ **by** *blast*
        then have $cdcl_W$-*M-level-inv V* **and** $cdcl_W$-*M-level-inv S*
          using *rtranclp-*$cdcl_W$-*consistent-inv inv* **by** *blast+*
        with *bj bt* **have** *False* **using** *backtrack-no-*$cdcl_W$-*bj* **by** *simp*
      }
      then show *?thesis* **using** *IH inv* **by** *blast*
    qed
  show *?case*
    using *bj*
    proof (*cases rule*: $cdcl_W$-*bj.cases*)
      case *backtrack*

then show *?thesis* **using** *IH* **by** *blast*
qed (*metis* (*no-types*, *lifting*) *IH rtranclp.simps*)+
qed

lemma *resolve-skip-deterministic*:
resolve S T $\Longrightarrow$ *skip S U* $\Longrightarrow$ *False*
by *fastforce*

lemma *backtrack-unique*:
assumes
bt-T: *backtrack S T* **and**
bt-U: *backtrack S U* **and**
inv: $cdcl_W$*-all-struct-inv S*
shows $T \sim U$
proof −
have *lev*: $cdcl_W$*-M-level-inv S*
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *auto*
then obtain *M N U′ k D L i K M1 M2* **where**
S: *state S* = (*M*, *N*, *U′*, *k*, *Some* (*D* + {#*L*#})) **and**
decomp: (*Marked K* (*i+1*) # *M1*, *M2*) ∈ *set* (*get-all-marked-decomposition M*) **and**
get-level M L = *k* **and**
get-level M L = *get-maximum-level M* (*D*+{#*L*#}) **and**
get-maximum-level M D = *i* **and**
T: *state T* = (*Propagated L* ((*D*+{#*L*#})) # *M1* , *N*, {#*D* + {#*L*#}#} + *U′*, *i*, *None*) **and**
undef: *undefined-lit M1 L*
using *bt-T* **by** (*elim backtrack-levE*)
(*force simp*: $cdcl_W$*-M-level-inv-def state-eq-def simp del*: *state-simp*)+

obtain *D′ L′ i′ K′ M1′ M2′* **where**
S′: *state S* = (*M*, *N*, *U′*, *k*, *Some* (*D′* + {#*L′*#})) **and**
decomp′: (*Marked K′* (*i′+1*) # *M1′*, *M2′*) ∈ *set* (*get-all-marked-decomposition M*) **and**
get-level M L′ = *k* **and**
get-level M L′ = *get-maximum-level M* (*D′*+{#*L′*#}) **and**
get-maximum-level M D′ = *i′* **and**
U: *state U* = (*Propagated L′* (*D′*+{#*L′*#}) # *M1′*, *N*, {#*D′* + {#*L′*#}#} +*U′*, *i′*, *None*) **and**
undef: *undefined-lit M1′ L′*
using *bt-U lev S* **by** (*elim backtrack-levE*)
(*force simp*: $cdcl_W$*-M-level-inv-def state-eq-def simp del*: *state-simp*)+
obtain *c* **where** *M*: *M* = *c* @ *M2* @ *Marked K* (*i* + *1*) # *M1*
using *decomp* **by** *auto*
obtain *c′* **where** *M′*: *M* = *c′* @ *M2′* @ *Marked K′* (*i′* + *1*) # *M1′*
using *decomp′* **by** *auto*
have *marked*: *get-all-levels-of-marked M* = *rev* [*1*..<*1*+*k*]
using *inv S* **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-M-level-inv-def* **by** *auto*
then have $i < k$
unfolding *M*
by (*force simp add*: *rev-swap*[*symmetric*] *dest*!: *arg-cong*[*of* - - *set*])

have [*simp*]: *L* = *L′*
proof (*rule ccontr*)
assume ¬ *?thesis*
then have *L′* ∈# *D*
using *S* **unfolding** *S′* **by** (*fastforce simp*: *multiset-eq-iff split*: *split-if-asm*)
then have *get-maximum-level M D* ≥ *k*
using ‹*get-level M L′* = *k*› *get-maximum-level-ge-get-level* **by** *blast*

then show *False* **using** ⟨*get-maximum-level M D* = *i*⟩ ⟨*i* < *k*⟩ **by** *auto*
qed
then have [*simp*]: $D = D'$
using *S S'* **by** *auto*
have [*simp*]: $i=i'$ **using** ⟨*get-maximum-level M D'* = *i'*⟩ ⟨*get-maximum-level M D* = *i*⟩ **by** *auto*

Automation in a step later...

have *H*: $\bigwedge a\ A\ B.$ *insert a A* = *B* $\Longrightarrow$ *a* : *B*
by *blast*
have *get-all-levels-of-marked* (*c*@*M2*) = *rev* [*i*+*2*..<*1*+*k*] **and**
get-all-levels-of-marked (*c'*@*M2'*) = *rev* [*i*+*2*..<*1*+*k*]
using *marked* **unfolding** *M*
using *marked* **unfolding** *M'*
unfolding *rev-swap*[*symmetric*] **by** (*auto dest*: *append-cons-eq-upt-length-i-end*)
from *arg-cong*[*OF this*(*1*), *of set*] *arg-cong*[*OF this*(*2*), *of set*]
have
dropWhile (λ*L*. ¬*is-marked L* ∨ *level-of L* ≠ *Suc i*) (*c* @ *M2*) = [] **and**
dropWhile (λ*L*. ¬*is-marked L* ∨ *level-of L* ≠ *Suc i*) (*c'* @ *M2'*) = []
unfolding *dropWhile-eq-Nil-conv Ball-def*
by (*intro allI*; *case-tac x*; *auto dest*!: *H simp add*: *in-set-conv-decomp*)+

then have *M1* = *M1'*
using *arg-cong*[*OF M*, *of dropWhile* (λ*L*. ¬*is-marked L* ∨ *level-of L* ≠ *Suc i*)]
unfolding *M'* **by** *auto*
then show *?thesis* **using** *T U* **by** (*auto simp del*: *state-simp simp*: *state-eq-def*)
qed

lemma *if-can-apply-backtrack-no-more-resolve*:
assumes
skip: *skip*** *S U* **and**
bt: *backtrack S T* **and**
inv: $cdcl_W$*-all-struct-inv S*
shows ¬*resolve U V*
proof (*rule ccontr*)
assume *resolve*: ¬¬*resolve U V*

obtain *L C M N U' k D* **where**
U: *state U* = (*Propagated L* ((*C* + {#*L*#})) # *M*, *N*, *U'*, *k*, *Some* (*D* + {#−*L*#}))**and**
get-maximum-level (*Propagated L* (*C* + {#*L*#}) # *M*) *D* = *k* **and**
state V = (*M*, *N*, *U'*, *k*, *Some* (*D* #∪ *C*))
using *resolve* **by** *auto*
have $cdcl_W$*-all-struct-inv U*
using *mono-rtranclp*[*of skip* $cdcl_W$] **by** (*meson bj* $cdcl_W$*-bj.skip inv local.skip other*
rtranclp-$cdcl_W$*-all-struct-inv-inv*)
then have [*iff*]: *no-dup* (*trail S*) $cdcl_W$*-M-level-inv S* **and** [*iff*]: *no-dup* (*trail U*)
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-M-level-inv-def* **by** *blast*+
then have
S: *init-clss S* = *N*
learned-clss S = *U'*
backtrack-lvl S = *k*
conflicting S = *Some* (*D* + {#−*L*#})
using *rtranclp-skip-state-decomp*(*2*)[*OF skip*] *U*
by (*auto simp del*: *state-simp simp*: *state-eq-def state-access-simp*)
obtain M_0 **where**
tr-S: *trail S* = M_0 @ *trail U* **and**

nm: ∀ *m*∈*set* M_0. ¬*is-marked m*
using *rtranclp-skip-state-decomp*[*OF skip*] **by** *blast*

obtain *M′ D′ L′ i K M1 M2* **where**
S′: *state S* = (*M′*, *N*, *U′*, *k*, *Some* (*D′* + {#*L′*#})) **and**
decomp: (*Marked K* (*i*+*1*) # *M1*, *M2*) ∈ *set* (*get-all-marked-decomposition M′*) **and**
get-level M′ L′ = *k* **and**
get-level M′ L′ = *get-maximum-level M′* (*D′*+{#*L′*#}) **and**
get-maximum-level M′ D′ = *i* **and**
undef: *undefined-lit M1 L′* **and**
T: *state T* = (*Propagated L′* (*D′*+{#*L′*#}) # *M1* , *N*, {#*D′* + {#*L′*#}#}+*U′*, *i*, *None*)
using *bt* **by** (*elim backtrack-levE*) (*fastforce simp*: *S state-eq-def simp del*:*state-simp*)+
obtain *c* **where** *M*: *M′* = *c* @ *M2* @ *Marked K* (*i* + *1*) # *M1*
using *get-all-marked-decomposition-exists-prepend*[*OF decomp*] **by** *auto*
have *marked*: *get-all-levels-of-marked M′* = *rev* [*1*..<*1*+*k*]
using *inv S′* **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-M-level-inv-def* **by** *auto*
then have *i* < *k*
unfolding *M* **by** (*force simp add*: *rev-swap*[*symmetric*] *dest*!: *arg-cong*[*of* - - *set*])

have *DD′*: *D′* + {#*L′*#} = *D* + {#−*L*#}
using *S S′* **by** *auto*
have [*simp*]: *L′* = −*L*
proof (*rule ccontr*)
assume ¬ *?thesis*
then have −*L* ∈# *D′*
using *DD′* **by** (*metis add-diff-cancel-right′ diff-single-trivial diff-union-swap*
multi-self-add-other-not-self)
moreover
have *M′*: *M′* = M_0 @ *Propagated L* ((*C* + {#*L*#})) # *M*
using *tr-S U S S′* **by** (*auto simp*: *lits-of-def*)
have *no-dup M′*
using *inv U S′* **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-M-level-inv-def* **by** *auto*
have *atm-L-notin-M*: *atm-of L* ∉ *atm-of* ‘ (*lits-of M*)
using ‹*no-dup M′*› *M′ U S S′* **by** (*auto simp*: *lits-of-def*)
have *get-all-levels-of-marked M′* = *rev* [*1*..<*1*+*k*]
using *inv U S′* **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-M-level-inv-def* **by** *auto*
then have *get-all-levels-of-marked M* = *rev* [*1*..<*1*+*k*]
using *nm M′ S′ U* **by** (*simp add*: *get-all-levels-of-marked-no-marked*)
then have *get-lev-L*:
get-level(*Propagated L* (*C* + {#*L*#}) # *M*) *L* = *k*
using *get-level-get-rev-level-get-all-levels-of-marked*[*OF atm-L-notin-M*,
of [*Propagated L* ((*C* + {#*L*#}))]] **by** *simp*
have *atm-of L* ∉ *atm-of* ‘ (*lits-of* (*rev* M_0))
using ‹*no-dup M′*› *M′ U S′* **by** (*auto simp*: *lits-of-def*)
then have *get-level M′ L* = *k*
using *get-rev-level-notin-end*[*of L rev* M_0
rev M @ *Propagated L* (*C* + {#*L*#}) # [] *0*]
using *tr-S get-lev-L M′ U S′* **by** (*simp add*:*nm lits-of-def*)
ultimately have *get-maximum-level M′ D′* ≥ *k*
by (*metis get-maximum-level-ge-get-level get-rev-level-uminus*)
then show *False*
using ‹*i* < *k*› **unfolding** ‹*get-maximum-level M′ D′* = *i*› **by** *auto*
qed
have [*simp*]: *D* = *D′* **using** *DD′* **by** *auto*
have $cdcl_W^{**}$ *S U*

using *bj* $cdcl_W$*-bj.skip local.skip mono-rtranclp*[*of skip* $cdcl_W$ *S U*] *other* **by** *meson*
then have $cdcl_W$*-all-struct-inv U*
using *inv rtranclp-*$cdcl_W$*-all-struct-inv-inv* **by** *blast*
then have *Propagated L* ((*C* + {#*L*#})) # *M* $\models$*as CNot* (*D'* + {#*L'*#})
using $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-conflicting-def U* **by** *auto*
then have $\forall$ *L'*$\in$#*D. atm-of L'* $\in$ *atm-of* ' *lits-of* (*Propagated L* ((*C* + {#*L*#})) # *M*)
by (*metis CNot-plus CNot-singleton Un-insert-right* ‹*D* = *D'*› *true-annots-insert ball-msetI atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set in-CNot-implies-uminus*(*2*) *sup-bot.comm-neutral*)
then have *get-maximum-level M' D* = *k*
using *tr-S nm U S'*
get-maximum-level-skip-un-marked-not-present[*of D*
Propagated L (*C* + {#*L*#}) # *M* M_0]
unfolding ‹*get-maximum-level* (*Propagated L* (*C* + {#*L*#}) # *M*) *D* = *k*›
unfolding ‹*D* = *D'*›
by *simp*
then show *False*
using ‹*get-maximum-level M' D'* = *i*› ‹*i* < *k*› **by** *auto*
qed

lemma *if-can-apply-resolve-no-more-backtrack*:
assumes
skip: *skip*** *S U* **and**
resolve: *resolve S T* **and**
inv: $cdcl_W$*-all-struct-inv S*
shows ¬*backtrack U V*
using *assms*
by (*meson if-can-apply-backtrack-no-more-resolve rtranclp.rtrancl-refl rtranclp-skip-backtrack-backtrack*)

lemma *if-can-apply-backtrack-skip-or-resolve-is-skip*:
assumes
bt: *backtrack S T* **and**
skip: *skip-or-resolve*** *S U* **and**
inv: $cdcl_W$*-all-struct-inv S*
shows *skip*** *S U*
using *assms*(*2*,*3*,*1*)
by *induction* (*simp-all add*: *if-can-apply-backtrack-no-more-resolve*)

lemma $cdcl_W$*-bj-bj-decomp*:
assumes $cdcl_W$*-bj*** *S W* **and** $cdcl_W$*-all-struct-inv S*
shows
($\exists$ *T U V*. (λ*S T. skip-or-resolve S T* $\wedge$ *no-step backtrack S*)** *S T*
$\wedge$ (λ*T U. resolve T U* $\wedge$ *no-step backtrack T*) *T U*
$\wedge$ *skip*** *U V* $\wedge$ *backtrack V W*)
$\vee$ ($\exists$ *T U*. (λ*S T. skip-or-resolve S T* $\wedge$ *no-step backtrack S*)** *S T*
$\wedge$ (λ*T U. resolve T U* $\wedge$ *no-step backtrack T*) *T U* $\wedge$ *skip*** *U W*)
$\vee$ ($\exists$ *T. skip*** *S T* $\wedge$ *backtrack T W*)
$\vee$ *skip*** *S W* (**is** *?RB S W* $\vee$ *?R S W* $\vee$ *?SB S W* $\vee$ *?S S W*)
using *assms*
proof *induction*
case *base*
then show *?case* **by** *simp*
next
case (*step W X*) **note** *st* = *this*(*1*) **and** *bj* = *this*(*2*) **and** *IH* = *this*(*3*)[*OF this*(*4*)] **and** *inv* = *this*(*4*)

have ¬*?RB S W* **and** ¬*?SB S W*
  proof (*clarify*, *goal-cases*)
    case (*1 T U V*)
    have *skip-or-resolve*** *S T*
      using *1*(*1*) **by** (*auto dest*!: *rtranclp-and-rtranclp-left*)
    then show *False*
      by (*metis* (*no-types*, *lifting*) *1*(*2*) *1*(*4*) *1*(*5*) *backtrack-no-cdcl*$_W$*-bj*
        cdcl$_W$*-all-struct-inv-def* *cdcl*$_W$*-all-struct-inv-inv* *cdcl*$_W$*-o.bj* *local.bj* *other*
        resolve *rtranclp-cdcl*$_W$*-all-struct-inv-inv* *rtranclp-skip-backtrack-backtrack*
        rtranclp-skip-or-resolve-rtranclp-cdcl$_W$ *step.prems*)
  next
    case *2*
    then show *?case* **by** (*meson assms*(*2*) *cdcl*$_W$*-all-struct-inv-def* *backtrack-no-cdcl*$_W$*-bj*
      local.bj *rtranclp-skip-backtrack-backtrack*)
  qed
then have *IH*: *?R S W* ∨ *?S S W* **using** *IH* **by** *blast*

have *cdcl*$_W$** *S W* **by** (*metis cdcl*$_W$*-o.bj* *mono-rtranclp* *other* *st*)
then have *inv-W*: *cdcl*$_W$*-all-struct-inv* *W* **by** (*simp add*: *rtranclp-cdcl*$_W$*-all-struct-inv-inv*
  step.prems)
consider
    (*BT*) *X′* **where** *backtrack W X′*
  | (*skip*) *no-step backtrack W* **and** *skip W X*
  | (*resolve*) *no-step backtrack W* **and** *resolve W X*
  using *bj* *cdcl*$_W$*-bj.cases* **by** *meson*
then show *?case*
  proof *cases*
    case (*BT X′*)
    then consider
        (*bt*) *backtrack W X*
      | (*sk*) *skip W X*
      using *bj* *if-can-apply-backtrack-no-more-resolve*[*of W W X′ X*] *inv-W* *cdcl*$_W$*-bj.cases* **by** *fast*
    then show *?thesis*
      proof *cases*
        case *bt*
        then show *?thesis* **using** *IH* **by** *auto*
      next
        case *sk*
        then show *?thesis* **using** *IH* **by** (*meson rtranclp-trans r-into-rtranclp*)
      qed
  next
    case *skip*
    then show *?thesis* **using** *IH* **by** (*meson rtranclp.rtrancl-into-rtrancl*)
  next
    case *resolve* **note** *no-bt* = *this*(*1*) **and** *res* = *this*(*2*)
    consider
        (*RS*) *T U* **where**
          (λ*S T*. *skip-or-resolve S T* ∧ *no-step backtrack S*)** *S T* **and**
          resolve T U **and**
          no-step backtrack T **and**
          *skip*** *U W*
      | (*S*) *skip*** *S W*
      using *IH* **by** *auto*
    then show *?thesis*

proof *cases*
  case $(RS\ T\ U)$
  have $cdcl_W^{**}\ S\ T$
    using $RS(1)$ $cdcl_W$*-bj.resolve* $cdcl_W$*-o.bj* *other skip*
    mono-rtranclp[*of* $(\lambda S\ T.$ *skip-or-resolve* $S\ T \wedge$ *no-step backtrack* $S)$ $cdcl_W\ S\ T$]
    by *meson*
  then have $cdcl_W$*-all-struct-inv* U
    by (*meson* $RS(2)$ $cdcl_W$*-all-struct-inv-inv* $cdcl_W$*-bj.resolve* $cdcl_W$*-o.bj other*
      rtranclp-$cdcl_W$*-all-struct-inv-inv step.prems*)
  { **fix** U'
    assume $skip^{**}\ U\ U'$ **and** $skip^{**}\ U'\ W$
    have $cdcl_W$*-all-struct-inv* U'
      using ‹$cdcl_W$*-all-struct-inv* U› ‹$skip^{**}\ U\ U'$› *rtranclp-*$cdcl_W$*-all-struct-inv-inv*
        $cdcl_W$*-o.bj rtranclp-mono*[*of skip* $cdcl_W$] *other skip* **by** *blast*
    then have *no-step backtrack* U'
      using *if-can-apply-backtrack-no-more-resolve*[*OF* ‹$skip^{**}\ U'\ W$›] *res* **by** *blast*
  }
  with ‹$skip^{**}\ U\ W$›
  have $(\lambda S\ T.$ *skip-or-resolve* $S\ T \wedge$ *no-step backtrack* $S)^{**}\ U\ W$
    proof *induction*
      case *base*
      then show *?case* **by** *simp*
    next
      case (*step* $V\ W$) **note** $st = this(1)$ **and** $skip = this(2)$ **and** $IH = this(3)$ **and** $H = this(4)$
      have $\bigwedge U'.\ skip^{**}\ U'\ V \Longrightarrow skip^{**}\ U'\ W$
        using *skip* **by** *auto*
      then have $(\lambda S\ T.$ *skip-or-resolve* $S\ T \wedge$ *no-step backtrack* $S)^{**}\ U\ V$
        using *IH H* **by** *blast*
      moreover have $(\lambda S\ T.$ *skip-or-resolve* $S\ T \wedge$ *no-step backtrack* $S)^{**}\ V\ W$

        by (*simp add*: *local.skip r-into-rtranclp st step.prems*)
      ultimately show *?case* **by** *simp*
    qed
  then show *?thesis*
    proof −
      have *f1*: $\forall p\ pa\ pb\ pc.\ \neg\ p\ (pa)\ pb \vee \neg\ p^{**}\ pb\ pc \vee p^{**}\ pa\ pc$
        by (*meson converse-rtranclp-into-rtranclp*)
      have *skip-or-resolve* $T\ U \wedge$ *no-step backtrack* T
        using $RS(2)$ $RS(3)$ **by** *force*
      then have $(\lambda p\ pa.$ *skip-or-resolve* $p\ pa \wedge$ *no-step backtrack* $p)^{**}\ T\ W$
        proof −
          have $(\exists\, vr19\ vr16\ vr17\ vr18.\ vr19\ (vr16{::}'st)\ vr17 \wedge vr19^{**}\ vr17\ vr18$
              $\wedge \neg\ vr19^{**}\ vr16\ vr18)$
            $\vee \neg$ (*skip-or-resolve* $T\ U \wedge$ *no-step backtrack* T)
            $\vee \neg$ $(\lambda uu\ uua.$ *skip-or-resolve* $uu\ uua \wedge$ *no-step backtrack* $uu)^{**}\ U\ W$
            $\vee$ $(\lambda uu\ uua.$ *skip-or-resolve* $uu\ uua \wedge$ *no-step backtrack* $uu)^{**}\ T\ W$
            by *force*
          then show *?thesis*
            by (*metis* (*no-types*) ‹$(\lambda S\ T.$ *skip-or-resolve* $S\ T \wedge$ *no-step backtrack* $S)^{**}\ U\ W$›
              ‹*skip-or-resolve* $T\ U \wedge$ *no-step backtrack* T› *f1*)
        qed
      then have $(\lambda p\ pa.$ *skip-or-resolve* $p\ pa \wedge$ *no-step backtrack* $p)^{**}\ S\ W$
        using $RS(1)$ **by** *force*
      then show *?thesis*
        using *no-bt res* **by** *blast*

qed
next
case *S*
{ fix U'
assume $skip^{**}$ S U' **and** $skip^{**}$ U' W
then have $cdcl_W{}^{**}$ S U'
using *mono-rtranclp*[*of skip* $cdcl_W$ S U'] **by** (*simp add*: $cdcl_W$*-o.bj other skip*)
then have $cdcl_W$*-all-struct-inv* U'
by (*metis* (*no-types*, *hide-lams*) ‹$cdcl_W$*-all-struct-inv* *S*› *rtranclp-*$cdcl_W$*-all-struct-inv-inv*)
then have *no-step backtrack* U'
using *if-can-apply-backtrack-no-more-resolve*[*OF* ‹$skip^{**}$ U' W›] *res* **by** *blast*
}
with *S*
have ($\lambda S\ T$. *skip-or-resolve S T* $\wedge$ *no-step backtrack S*)** S W
proof *induction*
case *base*
then show *?case* **by** *simp*
next
case (*step V W*) **note** *st* = *this*(*1*) **and** *skip* = *this*(*2*) **and** *IH* = *this*(*3*) **and** *H* = *this*(*4*)
have $\bigwedge U'$. $skip^{**}$ U' $V \Longrightarrow skip^{**}$ U' W
using *skip* **by** *auto*
then have ($\lambda S\ T$. *skip-or-resolve S T* $\wedge$ *no-step backtrack S*)** S V
using *IH H* **by** *blast*
moreover have ($\lambda S\ T$. *skip-or-resolve S T* $\wedge$ *no-step backtrack S*)** V W

by (*simp add*: *local.skip r-into-rtranclp st step.prems*)
ultimately show *?case* **by** *simp*
qed
then show *?thesis* **using** *res no-bt* **by** *blast*
qed
qed
qed

The case distinction is needed, since $T \sim V$ does not imply that R^{**} T V.

lemma $cdcl_W$*-bj-strongly-confluent*:
assumes
$cdcl_W$*-bj*** S V **and**
$cdcl_W$*-bj*** S T **and**
n-s: *no-step* $cdcl_W$*-bj* V **and**
inv: $cdcl_W$*-all-struct-inv* S
shows $T \sim V \vee cdcl_W$*-bj*** T V
using *assms*(*2*)
proof *induction*
case *base*
then show *?case* **by** (*simp add*: *assms*(*1*))
next
case (*step T U*) **note** *st* = *this*(*1*) **and** *s-o-r* = *this*(*2*) **and** *IH* = *this*(*3*)
have $cdcl_W{}^{**}$ S T
using *st mono-rtranclp*[*of* $cdcl_W$*-bj* $cdcl_W$] *other* **by** *blast*
then have *lev-T*: $cdcl_W$*-M-level-inv* T
using *inv rtranclp-*$cdcl_W$*-consistent-inv*[*of S T*]
unfolding $cdcl_W$*-all-struct-inv-def* **by** *auto*

consider
(*TV*) $T \sim V$

```
      | (bj-TV) cdcl_W-bj** T V
    using IH by blast
  then show ?case
    proof cases
      case TV
      have no-step cdcl_W-bj T
        using ‹cdcl_W-M-level-inv T› n-s cdcl_W-bj-state-eq-compatible[of T - V] TV by auto
      then show ?thesis
        using s-o-r by auto
    next
      case bj-TV
      then obtain U' where
        T-U': cdcl_W-bj T U' and
        cdcl_W-bj** U' V
        using IH n-s s-o-r by (metis rtranclp-unfold tranclpD)
      have cdcl_W** S T
        by (metis (no-types, hide-lams) bj mono-rtranclp[of cdcl_W-bj cdcl_W] other st)
      then have inv-T: cdcl_W-all-struct-inv T
        by (metis (no-types, hide-lams) inv rtranclp-cdcl_W-all-struct-inv-inv)

      have lev-U: cdcl_W-M-level-inv U
        using s-o-r cdcl_W-consistent-inv lev-T other by blast
      show ?thesis
        using s-o-r
        proof cases
          case backtrack
          then obtain V0 where skip** T V0 and backtrack V0 V
            using IH if-can-apply-backtrack-skip-or-resolve-is-skip[OF backtrack - inv-T]
             cdcl_W-bj-decomp-resolve-skip-and-bj
             by (meson bj-TV cdcl_W-bj.backtrack inv-T lev-T n-s
               rtranclp-skip-backtrack-backtrack-end)
          then have cdcl_W-bj** T V0 and cdcl_W-bj V0 V
            using rtranclp-mono[of skip cdcl_W-bj] by blast+
          then show ?thesis
            using ‹backtrack V0 V› ‹skip** T V0› backtrack-unique inv-T local.backtrack
            rtranclp-skip-backtrack-backtrack by auto
        next
          case resolve
          then have U ∼ U'
            by (meson T-U' cdcl_W-bj.simps if-can-apply-backtrack-no-more-resolve inv-T
              resolve-skip-deterministic resolve-unique rtranclp.rtrancl-refl)
          then show ?thesis
            using ‹cdcl_W-bj** U' V› unfolding rtranclp-unfold
            by (meson T-U' bj cdcl_W-consistent-inv lev-T other state-eq-ref state-eq-sym
              tranclp-cdcl_W-bj-state-eq-compatible)
        next
          case skip
          consider
              (sk) skip T U'
            | (bt) backtrack T U'
            using T-U' by (meson cdcl_W-bj.cases local.skip resolve-skip-deterministic)
          then show ?thesis
            proof cases
              case sk
              then show ?thesis
```

using ⟨$cdcl_W$-bj^{**} U' V⟩ **unfolding** *rtranclp-unfold*
by (*meson* T-U' *bj* $cdcl_W$-*all-inv*(*3*) $cdcl_W$-*all-struct-inv-def inv-T local.skip other tranclp-*$cdcl_W$*-bj-state-eq-compatible skip-unique state-eq-ref*)
next
case *bt*
have $skip^{++}$ T U
using *local.skip* **by** *blast*
then show *?thesis*
using *bt* **by** (*metis* ⟨$cdcl_W$-bj^{**} U' V⟩ *backtrack inv-T tranclp-unfold-begin rtranclp-skip-backtrack-backtrack-end tranclp-into-rtranclp*)
qed
qed
qed
qed

lemma $cdcl_W$-*bj-unique-normal-form*:
assumes
ST: $cdcl_W$-bj^{**} S T **and** *SU*: $cdcl_W$-bj^{**} S U **and**
n-s-U: *no-step* $cdcl_W$-*bj* U **and**
n-s-T: *no-step* $cdcl_W$-*bj* T **and**
inv: $cdcl_W$-*all-struct-inv* S
shows $T \sim U$
proof −
have $T \sim U \vee$ $cdcl_W$-bj^{**} T U
using *ST SU* $cdcl_W$-*bj-strongly-confluent inv n-s-U* **by** *blast*
then show *?thesis*
by (*metis* (*no-types*) *n-s-T rtranclp-unfold state-eq-ref tranclp-unfold-begin*)
qed

lemma *full-*$cdcl_W$*-bj-unique-normal-form*:
assumes *full* $cdcl_W$-*bj* S T **and** *full* $cdcl_W$-*bj* S U **and**
inv: $cdcl_W$-*all-struct-inv* S
shows $T \sim U$
using $cdcl_W$-*bj-unique-normal-form assms* **unfolding** *full-def* **by** *blast*

19.4 CDCL FW

inductive $cdcl_W$-*merge-restart* :: *'st* ⇒ *'st* ⇒ *bool* **where**
fw-r-propagate: *propagate* S S' ⟹ $cdcl_W$-*merge-restart* S S' |
fw-r-conflict: *conflict* S T ⟹ *full* $cdcl_W$-*bj* T U ⟹ $cdcl_W$-*merge-restart* S U |
fw-r-decide: *decide* S S' ⟹ $cdcl_W$-*merge-restart* S S'|
fw-r-rf: $cdcl_W$-*rf* S S' ⟹ $cdcl_W$-*merge-restart* S S'

lemma $cdcl_W$-*merge-restart*-$cdcl_W$:
assumes $cdcl_W$-*merge-restart* S T
shows $cdcl_W^{**}$ S T
using *assms*
proof *induction*
case (*fw-r-conflict* S T U) **note** *confl* = *this*(*1*) **and** *bj* = *this*(*2*)
have $cdcl_W$ S T **using** *confl* **by** (*simp add*: $cdcl_W$.*intros r-into-rtranclp*)
moreover
have $cdcl_W$-bj^{**} T U **using** *bj* **unfolding** *full-def* **by** *auto*
then have $cdcl_W^{**}$ T U **by** (*metis* $cdcl_W$-*o.bj mono-rtranclp other*)
ultimately show *?case* **by** *auto*
qed (*simp-all add*: $cdcl_W$-*o.intros* $cdcl_W$.*intros r-into-rtranclp*)

lemma $cdcl_W$*-merge-restart-conflicting-true-or-no-step*:
assumes $cdcl_W$*-merge-restart S T*
shows *conflicting T* = *None* ∨ *no-step* $cdcl_W$ *T*
using *assms*
proof *induction*
case (*fw-r-conflict S T U*) **note** *confl* = *this*(*1*) **and** *n-s* = *this*(*2*)
{ fix *D V*
assume $cdcl_W$ *U V* **and** *conflicting U* = *Some D*
then have *False*
using *n-s* **unfolding** *full-def*
by (*induction rule*: $cdcl_W$*-all-rules-induct*) (*auto dest*!: $cdcl_W$*-bj.intros*)
}
then show *?case* **by** (*cases conflicting U*) *fastforce+*
qed (*auto simp add*: $cdcl_W$*-rf.simps*)

inductive $cdcl_W$*-merge* :: *'st* ⇒ *'st* ⇒ *bool* **where**
fw-propagate: *propagate S S'* ⟹ $cdcl_W$*-merge S S'* |
fw-conflict: *conflict S T* ⟹ *full* $cdcl_W$*-bj T U* ⟹ $cdcl_W$*-merge S U* |
fw-decide: *decide S S'* ⟹ $cdcl_W$*-merge S S'*|
fw-forget: *forget S S'* ⟹ $cdcl_W$*-merge S S'*

lemma $cdcl_W$*-merge-*$cdcl_W$*-merge-restart*:
$cdcl_W$*-merge S T* ⟹ $cdcl_W$*-merge-restart S T*
by (*meson* $cdcl_W$*-merge.cases* $cdcl_W$*-merge-restart.simps forget*)

lemma *rtranclp-*$cdcl_W$*-merge-tranclp-*$cdcl_W$*-merge-restart*:
$cdcl_W$*-merge*** *S T* ⟹ $cdcl_W$*-merge-restart*** *S T*
using *rtranclp-mono*[*of* $cdcl_W$*-merge* $cdcl_W$*-merge-restart*] $cdcl_W$*-merge-*$cdcl_W$*-merge-restart* **by** *blast*

lemma $cdcl_W$*-merge-rtranclp-*$cdcl_W$:
$cdcl_W$*-merge S T* ⟹ $cdcl_W$** *S T*
using $cdcl_W$*-merge-*$cdcl_W$*-merge-restart* $cdcl_W$*-merge-restart-*$cdcl_W$ **by** *blast*

lemma *rtranclp-*$cdcl_W$*-merge-rtranclp-*$cdcl_W$:
$cdcl_W$*-merge*** *S T* ⟹ $cdcl_W$** *S T*
using *rtranclp-mono*[*of* $cdcl_W$*-merge* $cdcl_W$**] $cdcl_W$*-merge-rtranclp-*$cdcl_W$ **by** *auto*

lemma $cdcl_W$*-merge-is-*$cdcl_{NOT}$*-merged-bj-learn*:
assumes
inv: $cdcl_W$*-all-struct-inv S* **and**
$cdcl_W$:$cdcl_W$*-merge S T*
shows $cdcl_{NOT}$*-merged-bj-learn S T*
∨ (*no-step* $cdcl_W$*-merge T* ∧ *conflicting T* ≠ *None*)
using $cdcl_W$ *inv*
proof *induction*
case (*fw-propagate S T*) **note** *propa* = *this*(*1*)
then obtain *M N U k L C* **where**
H: *state S* = (*M*, *N*, *U*, *k*, *None*) **and**
CL: *C* + {#*L*#} ∈# *clauses S* **and**
M-C: *M* ⊨*as CNot C* **and**
undef: *undefined-lit* (*trail S*) *L* **and**
T: *T* ∼ *cons-trail* (*Propagated L* (*C* + {#*L*#})) *S*
using *propa* **by** *auto*
have $propagate_{NOT}$ *S T*

apply (*rule propagate*$_{NOT}$*.propagate*$_{NOT}$[*of - C L*])
using *H CL T undef M-C* **by** (*auto simp*: *state-eq*$_{NOT}$*-def state-eq-def clauses-def*
simp del: *state-simp*)
then show *?case*
using *cdcl*$_{NOT}$*-merged-bj-learn.intros*(*2*) **by** *blast*
next
case (*fw-decide S T*) **note** *dec* = *this*(*1*) **and** *inv* = *this*(*2*)
then obtain *L* **where**
undef-L: *undefined-lit* (*trail S*) *L* **and**
atm-L: *atm-of L* ∈ *atms-of-msu* (*init-clss S*) **and**
T: *T* ∼ *cons-trail* (*Marked L* (*Suc* (*backtrack-lvl S*)))
(*update-backtrack-lvl* (*Suc* (*backtrack-lvl S*)) *S*)
by *auto*
have *decide*$_{NOT}$ *S T*
apply (*rule decide*$_{NOT}$*.decide*$_{NOT}$)
using *undef-L* **apply** *simp*
using *atm-L inv* **unfolding** *cdcl*$_W$*-all-struct-inv-def no-strange-atm-def clauses-def* **apply** *auto*[]
using *T undef-L* **unfolding** *state-eq-def state-eq*$_{NOT}$*-def* **by** (*auto simp*: *clauses-def*)
then show *?case* **using** *cdcl*$_{NOT}$*-merged-bj-learn-decide*$_{NOT}$ **by** *blast*
next
case (*fw-forget S T*) **note** *rf* =*this*(*1*) **and** *inv* = *this*(*2*)
then obtain *M N C U k* **where**
S: *state S* = (*M*, *N*, {#*C*#} + *U*, *k*, *None*) **and**
¬ *M* ⊨*asm clauses S* **and**
C ∉ *set* (*get-all-mark-of-propagated* (*trail S*)) **and**
C-init: *C* ∉# *init-clss S* **and**
C-le: *C* ∈# *learned-clss S* **and**
T: *T* ∼ *remove-cls C S*
by *auto*
have *init-clss S* ⊨*pm C*
using *inv C-le* **unfolding** *cdcl*$_W$*-all-struct-inv-def cdcl*$_W$*-learned-clause-def*
by (*meson mem-set-mset-iff true-clss-clss-in-imp-true-clss-cls*)
then have *S-C*: *clauses S* − *replicate-mset* (*count* (*clauses S*) *C*) *C* ⊨*pm C*
using *C-init C-le* **unfolding** *clauses-def* **by** (*simp add*: *Un-Diff*)
moreover have *H*: *init-clss S* + (*learned-clss S* − *replicate-mset* (*count* (*learned-clss S*) *C*) *C*)
= *init-clss S* + *learned-clss S* − *replicate-mset* (*count* (*learned-clss S*) *C*) *C*
using *C-le C-init* **by** (*metis clauses-def clauses-remove-cls diff-zero gr0I*
init-clss-remove-cls learned-clss-remove-cls plus-multiset.rep-eq replicate-mset-0
semiring-normalization-rules(*5*))
have *forget*$_{NOT}$ *S T*
apply (*rule forget*$_{NOT}$*.forget*$_{NOT}$)
using *S-C* **apply** *blast*
using *S* **apply** *simp*
using ‹*C* ∈# *learned-clss S*› **apply** (*simp add*: *clauses-def*)
using *T C-le C-init* **by** (*auto*
simp: *state-eq-def Un-Diff state-eq*$_{NOT}$*-def clauses-def ac-simps H*
simp del: *state-simp*)
then show *?case* **using** *cdcl*$_{NOT}$*-merged-bj-learn-forget*$_{NOT}$ **by** *blast*
next
case (*fw-conflict S T U*) **note** *confl* = *this*(*1*) **and** *bj* = *this*(*2*) **and** *inv* = *this*(*3*)
obtain C_S **where**
confl-T: *conflicting T* = *Some* C_S **and**
C_S: C_S ∈# *clauses S* **and**
tr-S-C_S: *trail S* ⊨*as CNot* C_S
using *confl* **by** *auto*

```
have cdcl_W-all-struct-inv T
  using cdcl_W.simps cdcl_W-all-struct-inv-inv confl inv by blast
then have cdcl_W-M-level-inv T
  unfolding cdcl_W-all-struct-inv-def by auto
then consider
    (no-bt) skip-or-resolve** T U
  | (bt) T' where skip-or-resolve** T T' and backtrack T' U
  using bj rtranclp-cdcl_W-bj-skip-or-resolve-backtrack unfolding full-def by meson
then show ?case
  proof cases
    case no-bt
    then have conflicting U ≠ None
      using confl by (induction rule: rtranclp-induct) auto
    moreover then have no-step cdcl_W-merge U
      by (auto simp: cdcl_W-merge.simps)
    ultimately show ?thesis by blast
  next
    case bt note s-or-r = this(1) and bt = this(2)
    have cdcl_W** T T'
      using s-or-r mono-rtranclp[of skip-or-resolve cdcl_W] rtranclp-skip-or-resolve-rtranclp-cdcl_W
      by blast
    then have cdcl_W-M-level-inv T'
      using rtranclp-cdcl_W-consistent-inv ‹cdcl_W-M-level-inv T› by blast
    then obtain M1 M2 i D L K where
      confl-T': conflicting T' = Some (D + {#L#}) and
      M1-M2:(Marked K (i+1) # M1, M2) ∈ set (get-all-marked-decomposition (trail T')) and
      get-level (trail T') L = backtrack-lvl T' and
      get-level (trail T') L = get-maximum-level (trail T') (D+{#L#}) and
      get-maximum-level (trail T') D = i and
      undef-L: undefined-lit M1 L and
      U: U ∼ cons-trail (Propagated L (D+{#L#}))
              (reduce-trail-to M1
                   (add-learned-cls (D + {#L#})
                      (update-backtrack-lvl i
                         (update-conflicting None T'))))
      using bt by (auto elim: backtrack-levE)
    have [simp]: clauses S = clauses T
      using confl by auto
    have [simp]: clauses T = clauses T'
      using s-or-r
      proof (induction)
        case base
        then show ?case by simp
      next
        case (step U V) note st = this(1) and s-o-r = this(2) and IH = this(3)
        have clauses U = clauses V
          using s-o-r by auto
        then show ?case using IH by auto
      qed
    have inv-T: cdcl_W-all-struct-inv T
      by (meson cdcl_W-cp.simps confl inv r-into-rtranclp rtranclp-cdcl_W-all-struct-inv-inv
        rtranclp-cdcl_W-cp-rtranclp-cdcl_W)
    have cdcl_W** T T'
      using rtranclp-skip-or-resolve-rtranclp-cdcl_W s-or-r by blast
    have inv-T': cdcl_W-all-struct-inv T'
```

using ⟨$cdcl_W$** T T'⟩ *inv-T rtranclp-cdcl$_W$-all-struct-inv-inv* **by** *blast*
have *inv-U*: *cdcl$_W$-all-struct-inv U*
using *cdcl$_W$-merge-restart-cdcl$_W$ confl fw-r-conflict inv local.bj rtranclp-cdcl$_W$-all-struct-inv-inv* **by** *blast*

have [*simp*]: *init-clss S = init-clss T'*
using ⟨$cdcl_W$** T T'⟩ *cdcl$_W$-init-clss confl cdcl$_W$-all-struct-inv-def conflict inv*
by (*metis* ⟨*cdcl$_W$-M-level-inv T*⟩ *rtranclp-cdcl$_W$-init-clss*)
then have *atm-L*: *atm-of L ∈ atms-of-msu* (*clauses S*)
using *inv-T' confl-T'* **unfolding** *cdcl$_W$-all-struct-inv-def no-strange-atm-def clauses-def*
by *auto*
obtain *M* **where** *tr-T*: *trail T = M @ trail T'*
using *s-or-r* **by** (*induction rule*: *rtranclp-induct*) *auto*
obtain *M'* **where**
tr-T': *trail T' = M' @ Marked K* (*i+1*) *# tl* (*trail U*) **and**
tr-U: *trail U = Propagated L* (*D* + {#*L*#}) *# tl* (*trail U*)
using *U M1-M2 undef-L inv-T'* **unfolding** *cdcl$_W$-all-struct-inv-def cdcl$_W$-M-level-inv-def*
by *fastforce*
def *M'' ≡ M @ M'*
have *tr-T*: *trail S = M'' @ Marked K* (*i+1*) *# tl* (*trail U*)
using *tr-T tr-T' confl* **unfolding** *M''-def* **by** *auto*
have *init-clss T' + learned-clss S ⊨pm D +* {#*L*#}
using *inv-T' confl-T'* **unfolding** *cdcl$_W$-all-struct-inv-def cdcl$_W$-learned-clause-def clauses-def*
by *simp*
have *reduce-trail-to* (*convert-trail-from-NOT* (*convert-trail-from-W M1*)) *S =*
reduce-trail-to M1 S
by (*rule reduce-trail-to-length*) *simp*
moreover have *trail* (*reduce-trail-to M1 S*) *= M1*
apply (*rule reduce-trail-to-skip-beginning*[*of - M @ - @ M2 @* [*Marked K* (*Suc i*)]])
using *confl M1-M2* ⟨*trail T = M @ trail T'*⟩
apply (*auto dest*!: *get-all-marked-decomposition-exists-prepend*
elim!: *conflictE*)
by (*rule sym*) *auto*
ultimately have [*simp*]: *trail* (*reduce-trail-to$_{NOT}$* (*convert-trail-from-W M1*) *S*) *= M1*
using *M1-M2 confl* **by** (*auto simp add*: *reduce-trail-to$_{NOT}$-reduce-trail-convert*)
have *every-mark-is-a-conflict U*
using *inv-U* **unfolding** *cdcl$_W$-all-struct-inv-def cdcl$_W$-conflicting-def* **by** *simp*
then have *tl* (*trail U*) *⊨as CNot D*
by (*metis add-diff-cancel-left' append-self-conv2 tr-U union-commute*)
have *backjump-l S U*
apply (*rule backjump-l*[*of - - - - - L*])
using *tr-T* **apply** *simp*
using *inv* **unfolding** *cdcl$_W$-all-struct-inv-def cdcl$_W$-M-level-inv-def*
apply (*simp add*: *comp-def*)
using *U M1-M2 confl undef-L M1-M2 inv-T' inv* **unfolding** *cdcl$_W$-all-struct-inv-def*
cdcl$_W$-M-level-inv-def **apply** (*auto simp*: *state-eq$_{NOT}$-def*
trail-reduce-trail-to$_{NOT}$-add-learned-cls)[]
using C_S **apply** *simp*
using *tr-S-C_S* **apply** *simp*

using *U undef-L M1-M2 inv-T' inv* **unfolding** *cdcl$_W$-all-struct-inv-def*
cdcl$_W$-M-level-inv-def **apply** *auto*[]
using *undef-L atm-L* **apply** (*simp add*: *trail-reduce-trail-to$_{NOT}$-add-learned-cls*)
using ⟨*init-clss T' + learned-clss S ⊨pm D +* {#*L*#}⟩ **unfolding** *clauses-def* **apply** *simp*
apply (*metis* ⟨*tl* (*trail U*) *⊨as CNot D*⟩ *convert-trail-from-W-true-annots*)

using *inv-T' inv-U U confl-T' undef-L M1-M2* **unfolding** $cdcl_W$*-all-struct-inv-def distinct-*$cdcl_W$*-state-def* **by** (*simp add*: $cdcl_W$*-M-level-inv-decomp backjump-l-cond-def*)
then show *?thesis* **using** $cdcl_{NOT}$*-merged-bj-learn-backjump-l* **by** *fast*
qed
qed

abbreviation $cdcl_{NOT}$*-restart* **where**
$cdcl_{NOT}$*-restart* ≡ *restart-ops.*$cdcl_{NOT}$*-raw-restart* $cdcl_{NOT}$ *restart*

lemma $cdcl_W$*-merge-restart-is-*$cdcl_{NOT}$*-merged-bj-learn-restart-no-step*:
assumes
inv: $cdcl_W$*-all-struct-inv S* **and**
$cdcl_W$:$cdcl_W$*-merge-restart S T*
shows $cdcl_{NOT}$*-restart*** *S T* ∨ (*no-step* $cdcl_W$*-merge T* ∧ *conflicting T* ≠ *None*)
proof –
consider
(*fw*) $cdcl_W$*-merge S T*
| (*fw-r*) *restart S T*
using $cdcl_W$ **by** (*meson* $cdcl_W$*-merge-restart.simps* $cdcl_W$*-rf.cases fw-conflict fw-decide fw-forget fw-propagate*)
then show *?thesis*
proof *cases*
case *fw*
then have *IH*: $cdcl_{NOT}$*-merged-bj-learn S T* ∨ (*no-step* $cdcl_W$*-merge T* ∧ *conflicting T* ≠ *None*)
using *inv* $cdcl_W$*-merge-is-*$cdcl_{NOT}$*-merged-bj-learn* **by** *blast*
have *invS*: inv_{NOT} *S*
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-M-level-inv-def* **by** *auto*
have *ff2*: $cdcl_{NOT}{}^{++}$ *S T* ⟶ $cdcl_{NOT}{}^{**}$ *S T*
by (*meson tranclp-into-rtranclp*)
have *ff3*: *no-dup* (*convert-trail-from-W* (*trail S*))
using *invS* **by** (*simp add*: *comp-def*)
have $cdcl_{NOT}$ ≤ $cdcl_{NOT}$*-restart*
by (*auto simp*: *restart-ops.*$cdcl_{NOT}$*-raw-restart.simps*)
then show *?thesis*
using *ff3 ff2 IH* $cdcl_{NOT}$*-merged-bj-learn-is-tranclp-*$cdcl_{NOT}$ *rtranclp-mono*[*of* $cdcl_{NOT}$ $cdcl_{NOT}$*-restart*] *invS predicate2D* **by** *blast*
next
case *fw-r*
then show *?thesis* **by** (*blast intro*: *restart-ops.*$cdcl_{NOT}$*-raw-restart.intros*)
qed
qed

abbreviation μ_{FW} :: *'st* ⇒ *nat* **where**
μ_{FW} *S* ≡ (*if no-step* $cdcl_W$*-merge S then 0 else 1+*μ_{CDCL}*'-merged* (*set-mset* (*init-clss S*)) *S*)

lemma $cdcl_W$*-merge-*μ_{FW}*-decreasing*:
assumes
inv: $cdcl_W$*-all-struct-inv S* **and**
fw: $cdcl_W$*-merge S T*
shows μ_{FW} *T* < μ_{FW} *S*
proof –
let *?A* = *init-clss S*
have *atm-clauses*: *atms-of-msu* (*clauses S*) ⊆ *atms-of-msu ?A*
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def no-strange-atm-def clauses-def* **by** *auto*
have *atm-trail*: *atm-of* ‘ *lits-of* (*trail S*) ⊆ *atms-of-msu ?A*

using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def no-strange-atm-def clauses-def* **by** *auto*
have *n-d*: *no-dup* (*trail S*)
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** (*auto simp*: $cdcl_W$*-M-level-inv-decomp*)
have [*simp*]: ¬ *no-step* $cdcl_W$*-merge S*
using *fw* **by** *auto*
have [*simp*]: *init-clss S = init-clss T*
using $cdcl_W$*-merge-restart-*$cdcl_W$[*of S T*] *inv rtranclp-*$cdcl_W$*-init-clss*
unfolding $cdcl_W$*-all-struct-inv-def*
by (*meson* $cdcl_W$*-merge.simps* $cdcl_W$*-merge-restart.simps* $cdcl_W$*-rf.simps fw*)
consider
(*merged*) $cdcl_{NOT}$*-merged-bj-learn S T*
| (*n-s*) *no-step* $cdcl_W$*-merge T*
using $cdcl_W$*-merge-is-*$cdcl_{NOT}$*-merged-bj-learn inv fw* **by** *blast*
then show *?thesis*
proof *cases*
case *merged*
then show *?thesis*
using $cdcl_{NOT}$*-decreasing-measure'*[*OF - - atm-clauses*] *atm-trail n-d*
by (*auto split*: *split-if simp*: *comp-def*)
next
case *n-s*
then show *?thesis* **by** *simp*
qed
qed

lemma *wf-*$cdcl_W$*-merge*: *wf* {(*T*, *S*). $cdcl_W$*-all-struct-inv S* ∧ $cdcl_W$*-merge S T*}
apply (*rule wfP-if-measure*[*of - -* μ_{FW}])
using $cdcl_W$*-merge-*μ_{FW}*-decreasing* **by** *blast*

lemma $cdcl_W$*-all-struct-inv-tranclp-*$cdcl_W$*-merge-tranclp-*$cdcl_W$*-merge-*$cdcl_W$*-all-struct-inv*:
assumes
inv: $cdcl_W$*-all-struct-inv b*
$cdcl_W$*-merge*$^{++}$ *b a*
shows (λ*S T*. $cdcl_W$*-all-struct-inv S* ∧ $cdcl_W$*-merge S T*)$^{++}$ *b a*
using *assms*(*2*)
proof *induction*
case *base*
then show *?case* **using** *inv* **by** *auto*
next
case (*step c d*) **note** *st =this*(*1*) **and** *fw = this*(*2*) **and** *IH = this*(*3*)
have $cdcl_W$*-all-struct-inv c*
using *tranclp-into-rtranclp*[*OF st*] $cdcl_W$*-merge-rtranclp-*$cdcl_W$
assms(*1*) *rtranclp-*$cdcl_W$*-all-struct-inv-inv rtranclp-mono*[*of* $cdcl_W$*-merge* $cdcl_W$**] **by** *fastforce*
then have (λ*S T*. $cdcl_W$*-all-struct-inv S* ∧ $cdcl_W$*-merge S T*)$^{++}$ *c d*
using *fw* **by** *auto*
then show *?case* **using** *IH* **by** *auto*
qed

lemma *wf-tranclp-*$cdcl_W$*-merge*: *wf* {(*T*, *S*). $cdcl_W$*-all-struct-inv S* ∧ $cdcl_W$*-merge*$^{++}$ *S T*}
using *wf-trancl*[*OF wf-*$cdcl_W$*-merge*]
apply (*rule wf-subset*)
by (*auto simp*: *trancl-set-tranclp*
$cdcl_W$*-all-struct-inv-tranclp-*$cdcl_W$*-merge-tranclp-*$cdcl_W$*-merge-*$cdcl_W$*-all-struct-inv*)

lemma *backtrack-is-full1-*$cdcl_W$*-bj*:

assumes *bt*: *backtrack S T* **and** *inv*: $cdcl_W$*-M-level-inv S*
shows *full1* $cdcl_W$*-bj S T*
proof −
have *no-step* $cdcl_W$*-bj T*
using *bt inv backtrack-no-*$cdcl_W$*-bj* **by** *blast*
moreover have $cdcl_W$*-bj*$^{++}$ *S T*
using *bt* **by** *auto*
ultimately show *?thesis* **unfolding** *full1-def* **by** *blast*
qed

lemma *rtrancl-*$cdcl_W$*-conflicting-true-*$cdcl_W$*-merge-restart*:
assumes $cdcl_W{}^{**}$ *S V* **and** *inv*: $cdcl_W$*-M-level-inv S* **and** *conflicting S = None*
shows ($cdcl_W$*-merge-restart*** *S V* ∧ *conflicting V = None*)
∨ (∃ *T U*. $cdcl_W$*-merge-restart*** *S T* ∧ *conflicting V* ≠ *None* ∧ *conflict T U* ∧ $cdcl_W$*-bj*** *U V*)
using *assms*
proof *induction*
case *base*
then show *?case* **by** *simp*
next
case (*step U V*) **note** *st* = *this*(*1*) **and** $cdcl_W$ = *this*(*2*) **and** *IH* = *this*(*3*)[*OF this*(*4*−)] **and**
confl[*simp*] = *this*(*5*) **and** *inv* = *this*(*4*)
from $cdcl_W$
show *?case*
proof (*cases*)
case *propagate*
moreover then have *conflicting U = None*
by *auto*
moreover have *conflicting V = None*
using *propagate* **by** *auto*
ultimately show *?thesis* **using** *IH* $cdcl_W$*-merge-restart.fw-r-propagate*[*of U V*] **by** *auto*
next
case *conflict*
moreover then have *conflicting U = None*
by *auto*
moreover have *conflicting V* ≠ *None*
using *conflict* **by** *auto*
ultimately show *?thesis* **using** *IH* **by** *auto*
next
case *other*
then show *?thesis*
proof *cases*
case *decide*
moreover then have *conflicting U = None*
by *auto*
ultimately show *?thesis* **using** *IH* $cdcl_W$*-merge-restart.fw-r-decide*[*of U V*] **by** *auto*
next
case *bj*
moreover {
assume *skip-or-resolve U V*
have *f1*: $cdcl_W$*-bj*$^{++}$ *U V*
by (*simp add*: *local.bj tranclp.r-into-trancl*)
obtain *T T'* :: *'st* **where**
f2: $cdcl_W$*-merge-restart*** *S U*
∨ $cdcl_W$*-merge-restart*** *S T* ∧ *conflicting U* ≠ *None*
∧ *conflict T T'* ∧ $cdcl_W$*-bj*** *T' U*

using *IH confl* **by** *blast*
then have *?thesis*
proof –
have *conflicting* $V \neq$ *None* $\wedge$ *conflicting* $U \neq$ *None*
using ⟨*skip-or-resolve U V*⟩ **by** *auto*
then show *?thesis*
by (*metis* (*no-types*) *IH f1 rtranclp-trans tranclp-into-rtranclp*)
qed
}
moreover {
assume *backtrack U V*
then have *conflicting* $U \neq$ *None* **by** *auto*
then obtain T T' **where**
$cdcl_W$*-merge-restart*** S T **and**
conflicting $U \neq$ *None* **and**
conflict T T' **and**
$cdcl_W$*-bj*** T' U
using *IH confl* **by** *meson*
have *invU*: $cdcl_W$*-M-level-inv U*
using *inv rtranclp-*$cdcl_W$*-consistent-inv step.hyps*(*1*) **by** *blast*
then have *conflicting* $V =$ *None*
using ⟨*backtrack U V*⟩ *inv* **by** (*auto elim*: *backtrack-levE*
simp: $cdcl_W$*-M-level-inv-decomp*)
have *full* $cdcl_W$*-bj* T' V
apply (*rule rtranclp-fullI*[*of* $cdcl_W$*-bj* T' U V])
using ⟨$cdcl_W$*-bj*** T' U⟩ **apply** *fast*
using ⟨*backtrack U V*⟩ *backtrack-is-full1-*$cdcl_W$*-bj invU* **unfolding** *full1-def full-def*
by *blast*
then have *?thesis*
using $cdcl_W$*-merge-restart.fw-r-conflict*[*of* T T' V] ⟨*conflict* T T'⟩
⟨$cdcl_W$*-merge-restart*** S T⟩ ⟨*conflicting* $V =$ *None*⟩ **by** *auto*
}
ultimately show *?thesis* **by** (*auto simp*: $cdcl_W$*-bj.simps*)
qed
next
case *rf*
moreover then have *conflicting* $U =$ *None* **and** *conflicting* $V =$ *None*
by (*auto simp*: $cdcl_W$*-rf.simps*)
ultimately show *?thesis* **using** *IH* $cdcl_W$*-merge-restart.fw-r-rf*[*of* U V] **by** *auto*
qed
qed

lemma *no-step-*$cdcl_W$*-no-step-*$cdcl_W$*-merge-restart*: *no-step* $cdcl_W$ $S \Longrightarrow$ *no-step* $cdcl_W$*-merge-restart* S
by (*auto simp*: $cdcl_W$*.simps* $cdcl_W$*-merge-restart.simps* $cdcl_W$*-o.simps* $cdcl_W$*-bj.simps*)

lemma *no-step-*$cdcl_W$*-merge-restart-no-step-*$cdcl_W$:
assumes
conflicting $S =$ *None* **and**
$cdcl_W$*-M-level-inv* S **and**
no-step $cdcl_W$*-merge-restart* S
shows *no-step* $cdcl_W$ S
proof –
{ fix S'
assume *conflict* S S'

then have $cdcl_W$ *S S'* **using** $cdcl_W$*.conflict* **by** *auto*
then have $cdcl_W$*-M-level-inv S'*
using *assms(2)* $cdcl_W$*-consistent-inv* **by** *blast*
then obtain *S''* **where** *full* $cdcl_W$*-bj S' S''*
using $cdcl_W$*-bj-exists-normal-form[of S']* **by** *auto*
then have *False*
using ‹*conflict S S'*› *assms(3) fw-r-conflict* **by** *blast*
}
then show *?thesis*
using *assms* **unfolding** $cdcl_W$*.simps* $cdcl_W$*-merge-restart.simps* $cdcl_W$*-o.simps* $cdcl_W$*-bj.simps*
by *fastforce*
qed

lemma *rtranclp-*$cdcl_W$*-merge-restart-no-step-*$cdcl_W$*-bj*:
assumes
$cdcl_W$*-merge-restart*** *S T* **and**
conflicting S = None
shows *no-step* $cdcl_W$*-bj T*
using *assms*
apply (*induction rule*: *rtranclp-induct*)
apply (*fastforce simp*: $cdcl_W$*-bj.simps* $cdcl_W$*-rf.simps* $cdcl_W$*-merge-restart.simps full-def*)
apply (*fastforce simp*: $cdcl_W$*-bj.simps* $cdcl_W$*-rf.simps* $cdcl_W$*-merge-restart.simps full-def*)

done

If *conflicting S* $\neq$ *None*, we cannot say anything.

Remark that this theorem does not say anything about well-foundedness: even if you know that one relation is well-founded, it only states that the normal forms are shared.

lemma *conflicting-true-full-*$cdcl_W$*-iff-full-*$cdcl_W$*-merge*:
assumes *confl*: *conflicting S = None* **and** *lev*: $cdcl_W$*-M-level-inv S*
shows *full* $cdcl_W$ *S V* $\longleftrightarrow$ *full* $cdcl_W$*-merge-restart S V*
proof
assume *full*: *full* $cdcl_W$*-merge-restart S V*
then have *st*: $cdcl_W$** *S V*
using *rtranclp-mono[of* $cdcl_W$*-merge-restart* $cdcl_W$***]* $cdcl_W$*-merge-restart-*$cdcl_W$
unfolding *full-def* **by** *auto*

have *n-s*: *no-step* $cdcl_W$*-merge-restart V*
using *full* **unfolding** *full-def* **by** *auto*
have *n-s-bj*: *no-step* $cdcl_W$*-bj V*
using *rtranclp-*$cdcl_W$*-merge-restart-no-step-*$cdcl_W$*-bj confl full* **unfolding** *full-def* **by** *auto*
have $\bigwedge S'$. *conflict V S'* $\Longrightarrow$ $cdcl_W$*-M-level-inv S'*
using $cdcl_W$*.conflict* $cdcl_W$*-consistent-inv lev rtranclp-*$cdcl_W$*-consistent-inv st* **by** *blast*
then have $\bigwedge S'$. *conflict V S'* $\Longrightarrow$ *False*
using *n-s n-s-bj* $cdcl_W$*-bj-exists-normal-form* $cdcl_W$*-merge-restart.simps* **by** *meson*
then have *n-s-*$cdcl_W$: *no-step* $cdcl_W$ *V*
using *n-s n-s-bj* **by** (*auto simp*: $cdcl_W$*.simps* $cdcl_W$*-o.simps* $cdcl_W$*-merge-restart.simps*)
then show *full* $cdcl_W$ *S V* **using** *st* **unfolding** *full-def* **by** *auto*
next
assume *full*: *full* $cdcl_W$ *S V*
have *no-step* $cdcl_W$*-merge-restart V*
using *full no-step-*$cdcl_W$*-no-step-*$cdcl_W$*-merge-restart* **unfolding** *full-def* **by** *blast*
moreover
consider
(*fw*) $cdcl_W$*-merge-restart*** *S V* **and** *conflicting V = None*

| (*bj*) *T U* **where**
cdcl$_W$*-merge-restart*** *S T* **and**
conflicting V $\neq$ *None* **and**
conflict T U **and**
cdcl$_W$*-bj*** *U V*
using *full rtrancl-cdcl*$_W$*-conflicting-true-cdcl*$_W$*-merge-restart confl lev* **unfolding** *full-def*
by *meson*
then have *cdcl*$_W$*-merge-restart*** *S V*
proof *cases*
case *fw*
then show *?thesis* **by** *fast*
next
case (*bj T U*)
have *no-step cdcl*$_W$*-bj V*
using *full* **unfolding** *full-def* **by** (*meson cdcl*$_W$*-o.bj other*)
then have *full cdcl*$_W$*-bj U V*
using ‹ *cdcl*$_W$*-bj*** *U V*› **unfolding** *full-def* **by** *auto*
then have *cdcl*$_W$*-merge-restart T V*
using ‹*conflict T U*› *cdcl*$_W$*-merge-restart.fw-r-conflict* **by** *blast*
then show *?thesis* **using** ‹*cdcl*$_W$*-merge-restart*** *S T*› **by** *auto*
qed
ultimately show *full cdcl*$_W$*-merge-restart S V* **unfolding** *full-def* **by** *fast*
qed

lemma *init-state-true-full-cdcl*$_W$*-iff-full-cdcl*$_W$*-merge*:
shows *full cdcl*$_W$ (*init-state N*) *V* $\longleftrightarrow$ *full cdcl*$_W$*-merge-restart* (*init-state N*) *V*
by (*rule conflicting-true-full-cdcl*$_W$*-iff-full-cdcl*$_W$*-merge*) *auto*

19.5 FW with strategy

19.5.1 The intermediate step

inductive *cdcl*$_W$*-s'* :: *'st* $\Rightarrow$ *'st* $\Rightarrow$ *bool* **where**
conflict': *full1 cdcl*$_W$*-cp S S'* $\Longrightarrow$ *cdcl*$_W$*-s' S S'* |
decide': *decide S S'* $\Longrightarrow$ *no-step cdcl*$_W$*-cp S* $\Longrightarrow$ *full cdcl*$_W$*-cp S' S''* $\Longrightarrow$ *cdcl*$_W$*-s' S S''* |
bj': *full1 cdcl*$_W$*-bj S S'* $\Longrightarrow$ *no-step cdcl*$_W$*-cp S* $\Longrightarrow$ *full cdcl*$_W$*-cp S' S''* $\Longrightarrow$ *cdcl*$_W$*-s' S S''*

inductive-cases *cdcl*$_W$*-s'E*: *cdcl*$_W$*-s' S T*

lemma *rtranclp-cdcl*$_W$*-bj-full1-cdclp-cdcl*$_W$*-stgy*:
cdcl$_W$*-bj*** *S S'* $\Longrightarrow$ *full cdcl*$_W$*-cp S' S''* $\Longrightarrow$ *cdcl*$_W$*-stgy*** *S S''*
proof (*induction rule*: *converse-rtranclp-induct*)
case *base*
then show *?case* **by** (*metis cdcl*$_W$*-stgy.conflict' full-unfold rtranclp.simps*)
next
case (*step T U*) **note** *st* =*this*(*2*) **and** *bj* = *this*(*1*) **and** *IH* = *this*(*3*)[*OF this*(*4*)]
have *no-step cdcl*$_W$*-cp T*
using *bj* **by** (*auto simp add*: *cdcl*$_W$*-bj.simps*)
consider
(*U*) *U* = *S'*
| (*U'*) *U'* **where** *cdcl*$_W$*-bj U U'* **and** *cdcl*$_W$*-bj*** *U' S'*
using *st* **by** (*metis converse-rtranclpE*)
then show *?case*
proof *cases*
case *U*
then show *?thesis*

using ‹*no-step cdcl*$_W$*-cp T*› *cdcl*$_W$*-o.bj local.bj other*′ *step.prems* **by** (*meson r-into-rtranclp*)
next
case *U*′ **note** *U*′ = *this*(*1*)
have *no-step cdcl*$_W$*-cp U*
using *U*′ **by** (*fastforce simp*: *cdcl*$_W$*-cp.simps cdcl*$_W$*-bj.simps*)
then have *full cdcl*$_W$*-cp U U*
by (*simp add*: *full-unfold*)
then have *cdcl*$_W$*-stgy T U*
using ‹*no-step cdcl*$_W$*-cp T*› *cdcl*$_W$*-stgy.simps local.bj cdcl*$_W$*-o.bj* **by** *meson*
then show *?thesis* **using** *IH* **by** *auto*
qed
qed

lemma *cdcl*$_W$*-s*′*-is-rtranclp-cdcl*$_W$*-stgy*:
cdcl$_W$*-s*′ *S T* ⟹ *cdcl*$_W$*-stgy*** *S T*
apply (*induction rule*: *cdcl*$_W$*-s*′*.induct*)
apply (*auto intro*: *cdcl*$_W$*-stgy.intros*)[]
apply (*meson decide other*′ *r-into-rtranclp*)
by (*metis full1-def rtranclp-cdcl*$_W$*-bj-full1-cdclp-cdcl*$_W$*-stgy tranclp-into-rtranclp*)

lemma *cdcl*$_W$*-cp-cdcl*$_W$*-bj-bissimulation*:
assumes
full cdcl$_W$*-cp T U* **and**
cdcl$_W$*-bj*** *T T*′ **and**
cdcl$_W$*-all-struct-inv T* **and**
no-step cdcl$_W$*-bj T*′
shows *full cdcl*$_W$*-cp T*′ *U*
∨ (∃ *U*′ *U*″. *full cdcl*$_W$*-cp T*′ *U*″ ∧ *full1 cdcl*$_W$*-bj U U*′ ∧ *full cdcl*$_W$*-cp U*′ *U*″ ∧ *cdcl*$_W$*-s*′** *U U*″)
using *assms*(*2*,*1*,*3*,*4*)
proof (*induction rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *blast*
next
case (*step T*′ *T*″) **note** *st* = *this*(*1*) **and** *bj* = *this*(*2*) **and** *IH* = *this*(*3*)[*OF this*(*4*,*5*)] **and**
full = *this*(*4*) **and** *inv* = *this*(*5*)
have *cdcl*$_W$** *T T*″
by (*metis* (*no-types*, *lifting*) *cdcl*$_W$*-o.bj local.bj mono-rtranclp*[*of cdcl*$_W$*-bj cdcl*$_W$ *T T*″] *other*
st rtranclp.rtrancl-into-rtrancl)
then have *inv-T*″: *cdcl*$_W$*-all-struct-inv T*″
using *inv rtranclp-cdcl*$_W$*-all-struct-inv-inv* **by** *blast*
have *cdcl*$_W$*-bj*$^{++}$ *T T*″
using *local.bj st* **by** *auto*
have *full1 cdcl*$_W$*-bj T T*″
by (*metis* ‹*cdcl*$_W$*-bj*$^{++}$ *T T*″› *full1-def step.prems*(*3*))
then have *T* = *U*
proof −
obtain *Z* **where** *cdcl*$_W$*-bj T Z*
by (*meson tranclpD* ‹*cdcl*$_W$*-bj*$^{++}$ *T T*″›)
{ assume *cdcl*$_W$*-cp*$^{++}$ *T U*
then obtain *Z*′ **where** *cdcl*$_W$*-cp T Z*′
by (*meson tranclpD*)
then have *False*
using ‹*cdcl*$_W$*-bj T Z*› **by** (*fastforce simp*: *cdcl*$_W$*-bj.simps cdcl*$_W$*-cp.simps*)
}
then show *?thesis*

using *full* **unfolding** *full-def rtranclp-unfold* **by** *blast*
qed
obtain U'' **where** *full* $cdcl_W$*-cp* T'' U''
using $cdcl_W$*-cp-normalized-element-all-inv inv-*T'' **by** *blast*
moreover then have $cdcl_W$*-stgy*** U U''
by (*metis* ‹$T = U$› ‹$cdcl_W$*-bj*$^{++}$ T T''› *rtranclp-*$cdcl_W$*-bj-full1-cdclp-*$cdcl_W$*-stgy rtranclp-unfold*)
moreover have $cdcl_W$*-s*$'^{**}$ U U''
proof −
obtain *ss* :: $'st \Rightarrow 'st$ **where**
f1: $\forall$ *x2*. ($\exists$ *v3*. $cdcl_W$*-cp x2 v3*) = $cdcl_W$*-cp x2* (*ss x2*)
by *moura*
have ¬ $cdcl_W$*-cp* U (*ss* U)
by (*meson full full-def*)
then show *?thesis*
using *f1* **by** (*metis* (*no-types*) ‹$T = U$› ‹*full1* $cdcl_W$*-bj* T T''› *bj*$'$ *calculation*(*1*)
r-into-rtranclp)
qed
ultimately show *?case*
using ‹*full1* $cdcl_W$*-bj* T T''› ‹*full* $cdcl_W$*-cp* T'' U''› **unfolding** ‹$T = U$› **by** *blast*
qed

lemma $cdcl_W$*-cp-*$cdcl_W$*-bj-bissimulation*$'$:
assumes
full $cdcl_W$*-cp* T U **and**
$cdcl_W$*-bj*** T T' **and**
$cdcl_W$*-all-struct-inv* T **and**
no-step $cdcl_W$*-bj* T'
shows *full* $cdcl_W$*-cp* T' U
$\vee$ ($\exists$ U'. *full1* $cdcl_W$*-bj* U U' $\wedge$ ($\forall$ U''. *full* $cdcl_W$*-cp* U' U'' $\longrightarrow$ *full* $cdcl_W$*-cp* T' U''
$\wedge$ $cdcl_W$*-s*$'^{**}$ U U''))
using *assms*(*2*,*1*,*3*,*4*)
proof (*induction rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *blast*
next
case (*step* T' T'') **note** *st* = *this*(*1*) **and** *bj* = *this*(*2*) **and** *IH* = *this*(*3*)[*OF this*(*4*,*5*)] **and**
full = *this*(*4*) **and** *inv* = *this*(*5*)
have $cdcl_W{}^{**}$ T T''
by (*metis* (*no-types*, *lifting*) $cdcl_W$*-o.bj local.bj mono-rtranclp*[*of* $cdcl_W$*-bj* $cdcl_W$ T T''] *other st*
rtranclp.rtrancl-into-rtrancl)
then have *inv-*T'': $cdcl_W$*-all-struct-inv* T''
using *inv rtranclp-*$cdcl_W$*-all-struct-inv-inv* **by** *blast*
have $cdcl_W$*-bj*$^{++}$ T T''
using *local.bj st* **by** *auto*
have *full1* $cdcl_W$*-bj* T T''
by (*metis* ‹$cdcl_W$*-bj*$^{++}$ T T''› *full1-def step.prems*(*3*))
then have $T = U$
proof −
obtain Z **where** $cdcl_W$*-bj* T Z
by (*meson tranclpD* ‹$cdcl_W$*-bj*$^{++}$ T T''›)
{ **assume** $cdcl_W$*-cp*$^{++}$ T U
then obtain Z' **where** $cdcl_W$*-cp* T Z'
by (*meson tranclpD*)
then have *False*
using ‹$cdcl_W$*-bj* T Z› **by** (*fastforce simp*: $cdcl_W$*-bj.simps* $cdcl_W$*-cp.simps*)

}
then show *?thesis*
using *full* **unfolding** *full-def rtranclp-unfold* **by** *blast*
qed
{ fix U''
assume *full* $cdcl_W$*-cp* T'' U''
moreover then have $cdcl_W$*-stgy*** U U''
by (*metis* ‹$T = U$› ‹$cdcl_W$*-bj*$^{++}$ T T''› *rtranclp-*$cdcl_W$*-bj-full1-cdclp-*$cdcl_W$*-stgy rtranclp-unfold*)
moreover have $cdcl_W$*-s*$'^{**}$ U U''
proof −
obtain *ss* :: $'st \Rightarrow 'st$ **where**
f1: $\forall x2.\ (\exists v3.\ cdcl_W\text{-}cp\ x2\ v3) = cdcl_W\text{-}cp\ x2\ (ss\ x2)$
by *moura*
have $\neg$ $cdcl_W$*-cp* U (*ss* U)
by (*meson assms*(*1*) *full-def*)
then show *?thesis*
using *f1* **by** (*metis* (*no-types*) ‹$T = U$› ‹*full1* $cdcl_W$*-bj* T T''› *bj*$'$ *calculation*(*1*)
r-into-rtranclp)
qed
ultimately have *full1* $cdcl_W$*-bj* U T'' **and** $cdcl_W$*-s*$'^{**}$ T'' U''
using ‹*full1* $cdcl_W$*-bj* T T''› ‹*full* $cdcl_W$*-cp* T'' U''› **unfolding** ‹$T = U$›
apply *blast*
by (*metis* ‹*full* $cdcl_W$*-cp* T'' U''› $cdcl_W$*-s*$'$*.simps full-unfold rtranclp.simps*)
}
then show *?case*
using ‹*full1* $cdcl_W$*-bj* T T''› *full bj*$'$ **unfolding** ‹$T = U$› *full-def* **by** (*metis r-into-rtranclp*)
qed

lemma $cdcl_W$*-stgy-*$cdcl_W$*-s*$'$*-connected*:
assumes $cdcl_W$*-stgy* S U **and** $cdcl_W$*-all-struct-inv* S
shows $cdcl_W$*-s*$'$ S U
$\vee$ ($\exists U'$. *full1* $cdcl_W$*-bj* U U' $\wedge$ ($\forall U''$. *full* $cdcl_W$*-cp* U' U'' $\longrightarrow$ $cdcl_W$*-s*$'$ S U''))
using *assms*
proof (*induction rule*: $cdcl_W$*-stgy.induct*)
case (*conflict*$'$ T)
then have $cdcl_W$*-s*$'$ S T
using $cdcl_W$*-s*$'$*.conflict*$'$ **by** *blast*
then show *?case*
by *blast*
next
case (*other*$'$ T U) **note** o = *this*(*1*) **and** *n-s* = *this*(*2*) **and** *full* = *this*(*3*) **and** *inv* = *this*(*4*)
show *?case*
using o
proof *cases*
case *decide*
then show *?thesis* **using** $cdcl_W$*-s*$'$*.simps full n-s* **by** *blast*
next
case *bj*
have *inv-T*: $cdcl_W$*-all-struct-inv* T
using $cdcl_W$*-all-struct-inv-inv o other other*$'$*.prems* **by** *blast*
consider
(*cp*) *full* $cdcl_W$*-cp* T U **and** *no-step* $cdcl_W$*-bj* T
| (*fbj*) T' **where** *full1* $cdcl_W$*-bj* T T'
apply (*cases no-step* $cdcl_W$*-bj* T)
using *full* **apply** *blast*

using $cdcl_W$-*bj-exists-normal-form*[*of T*] *inv-T* **unfolding** $cdcl_W$-*all-struct-inv-def*
by (*metis full-unfold*)
then show *?thesis*
proof *cases*
case *cp*
then show *?thesis*
proof –
obtain *ss* :: *'st* ⇒ *'st* **where**
f1: ∀ *s sa sb*. (¬ *full1* $cdcl_W$-*bj s sa* ∨ $cdcl_W$-*cp s* (*ss s*) ∨ ¬ *full* $cdcl_W$-*cp sa sb*)
∨ $cdcl_W$-*s' s sb*
using *bj'* **by** *moura*
have *full1* $cdcl_W$-*bj S T*
by (*simp add*: *cp*(*2*) *full1-def local.bj tranclp.r-into-trancl*)
then show *?thesis*
using *f1 full n-s* **by** *blast*
qed
next
case (*fbj U'*)
then have *full1* $cdcl_W$-*bj S U'*
using *bj* **unfolding** *full1-def* **by** *auto*
moreover have *no-step* $cdcl_W$-*cp S*
using *n-s* **by** *blast*
moreover have *T* = *U*
using *full fbj* **unfolding** *full1-def full-def rtranclp-unfold*
by (*force dest*!: *tranclpD simp*:$cdcl_W$-*bj.simps*)
ultimately show *?thesis* **using** $cdcl_W$-*s'.bj'*[*of S U'*] **using** *fbj* **by** *blast*
qed
qed
qed

lemma $cdcl_W$-*stgy*-$cdcl_W$-*s'-connected'*:
assumes $cdcl_W$-*stgy S U* **and** $cdcl_W$-*all-struct-inv S*
shows $cdcl_W$-*s' S U*
∨ (∃ *U' U''*. $cdcl_W$-*s' S U''* ∧ *full1* $cdcl_W$-*bj U U'* ∧ *full* $cdcl_W$-*cp U' U''*)
using *assms*
proof (*induction rule*: $cdcl_W$-*stgy.induct*)
case (*conflict' T*)
then have $cdcl_W$-*s' S T*
using $cdcl_W$-*s'.conflict'* **by** *blast*
then show *?case*
by *blast*
next
case (*other' T U*) **note** *o* = *this*(*1*) **and** *n-s* = *this*(*2*) **and** *full* = *this*(*3*) **and** *inv* = *this*(*4*)
show *?case*
using *o*
proof *cases*
case *decide*
then show *?thesis* **using** $cdcl_W$-*s'.simps full n-s* **by** *blast*
next
case *bj*
have $cdcl_W$-*all-struct-inv T*
using $cdcl_W$-*all-struct-inv-inv o other other'.prems* **by** *blast*
then obtain *T'* **where** *T'*: *full* $cdcl_W$-*bj T T'*
using $cdcl_W$-*bj-exists-normal-form* **unfolding** *full-def* $cdcl_W$-*all-struct-inv-def* **by** *metis*
then have *full* $cdcl_W$-*bj S T'*

proof −
have *f1*: $cdcl_W$*-bj*** T T' ∧ *no-step* $cdcl_W$*-bj* T'
by (*metis* (*no-types*) T' *full-def*)
then have $cdcl_W$*-bj*** S T'
by (*meson converse-rtranclp-into-rtranclp local.bj*)
then show *?thesis*
using *f1* **by** (*simp add*: *full-def*)
qed
have $cdcl_W$*-bj*** T T'
using T' **unfolding** *full-def* **by** *simp*
have $cdcl_W$*-all-struct-inv* T
using $cdcl_W$*-all-struct-inv-inv o other other'.prems* **by** *blast*
then consider
($T'U$) *full* $cdcl_W$*-cp* T' U
| (U) U' U'' **where**
full $cdcl_W$*-cp* T' U'' **and**
full1 $cdcl_W$*-bj* U U' **and**
full $cdcl_W$*-cp* U' U'' **and**
$cdcl_W$*-s'*** U U''
using $cdcl_W$*-cp-*$cdcl_W$*-bj-bissimulation*[*OF full* ‹$cdcl_W$*-bj*** T T'›] T' **unfolding** *full-def*
by *blast*
then show *?thesis* **by** (*metis* T' $cdcl_W$*-s'.simps full-fullI local.bj n-s*)
qed
qed

lemma $cdcl_W$*-stgy-*$cdcl_W$*-s'-no-step*:
assumes $cdcl_W$*-stgy* S U **and** $cdcl_W$*-all-struct-inv* S **and** *no-step* $cdcl_W$*-bj* U
shows $cdcl_W$*-s'* S U
using $cdcl_W$*-stgy-*$cdcl_W$*-s'-connected*[*OF assms*(*1*,*2*)] *assms*(*3*)
by (*metis* (*no-types*, *lifting*) *full1-def tranclpD*)

lemma *rtranclp-*$cdcl_W$*-stgy-connected-to-rtranclp-*$cdcl_W$*-s'*:
assumes $cdcl_W$*-stgy*** S U **and** *inv*: $cdcl_W$*-M-level-inv* S
shows $cdcl_W$*-s'*** S U ∨ (∃ T. $cdcl_W$*-s'*** S T ∧ $cdcl_W$*-bj*$^{++}$ T U ∧ *conflicting* U ≠ *None*)
using *assms*(*1*)
proof *induction*
case *base*
then show *?case* **by** *simp*
next
case (*step* T V) **note** *st* = *this*(*1*) **and** *o* = *this*(*2*) **and** *IH* = *this*(*3*)
from *o* **show** *?case*
proof *cases*
case *conflict'*
then have *f2*: $cdcl_W$*-s'* T V
using $cdcl_W$*-s'.conflict'* **by** *blast*
obtain *ss* :: *'st* **where**
f3: S = T ∨ $cdcl_W$*-stgy*** S *ss* ∧ $cdcl_W$*-stgy ss* T
by (*metis* (*full-types*) *rtranclp.simps st*)
obtain *ssa* :: *'st* **where**
$cdcl_W$*-cp* T *ssa*
using *conflict'* **by** (*metis* (*no-types*) *full1-def tranclpD*)
then have S = T
using *f3* **by** (*metis* (*no-types*) $cdcl_W$*-stgy.simps full-def full1-def*)
then show *?thesis*
using *f2* **by** *blast*

next
  case (*other'* U) **note** o = *this*(*1*) **and** *n-s* = *this*(*2*) **and** *full* = *this*(*3*)
  then show *?thesis*
    using *o*
    proof (*cases rule*: $cdcl_W$*-o-rule-cases*)
      case *decide*
      then have $cdcl_W$*-s'*** S T
        using *IH* **by** *auto*
      then show *?thesis*
        by (*meson decide decide' full n-s rtranclp.rtrancl-into-rtrancl*)
    next
      case *backtrack*
      consider
          (*s'*) $cdcl_W$*-s'*** S T
        | (*bj*) S' **where** $cdcl_W$*-s'*** S S' **and** $cdcl_W$*-bj*$^{++}$ S' T **and** *conflicting* $T \neq$ *None*
        using *IH* **by** *blast*
      then show *?thesis*
        proof *cases*
          case *s'*
          moreover
            have $cdcl_W$*-M-level-inv* T
              using *inv local.step*(*1*) *rtranclp-*$cdcl_W$*-stgy-consistent-inv* **by** *auto*
            then have *full1* $cdcl_W$*-bj* T U
              using *backtrack-is-full1-*$cdcl_W$*-bj backtrack* **by** *blast*
            then have $cdcl_W$*-s'* T V
             using *full bj' n-s* **by** *blast*
          ultimately show *?thesis* **by** *auto*
        next
          case (*bj* S') **note** *S-S'* = *this*(*1*) **and** *bj-T* = *this*(*2*)
          have *no-step* $cdcl_W$*-cp* S'
            using *bj-T* **by** (*fastforce simp*: $cdcl_W$*-cp.simps* $cdcl_W$*-bj.simps dest*!: *tranclpD*)
          moreover
            have $cdcl_W$*-M-level-inv* T
              using *inv local.step*(*1*) *rtranclp-*$cdcl_W$*-stgy-consistent-inv* **by** *auto*
            then have *full1* $cdcl_W$*-bj* T U
              using *backtrack-is-full1-*$cdcl_W$*-bj backtrack* **by** *blast*
            then have *full1* $cdcl_W$*-bj* S' U
              using *bj-T* **unfolding** *full1-def* **by** *fastforce*
          ultimately have $cdcl_W$*-s'* S' V **using** *full* **by** (*simp add*: *bj'*)
          then show *?thesis* **using** *S-S'* **by** *auto*
        qed
    next
      case *skip*
      then have [*simp*]: $U = V$
        using *full converse-rtranclpE* **unfolding** *full-def* **by** *fastforce*

      consider
          (*s'*) $cdcl_W$*-s'*** S T
        | (*bj*) S' **where** $cdcl_W$*-s'*** S S' **and** $cdcl_W$*-bj*$^{++}$ S' T **and** *conflicting* $T \neq$ *None*
        using *IH* **by** *blast*
      then show *?thesis*
        proof *cases*
          case *s'*
          have $cdcl_W$*-bj*$^{++}$ T V
            using *skip* **by** *force*

moreover have *conflicting V* $\neq$ *None*
  using *skip* **by** *auto*
ultimately show *?thesis* **using** s' **by** *auto*
next
  case (*bj* S') **note** S-S' = *this*(*1*) **and** *bj-T* = *this*(*2*)
  have $cdcl_W$-bj^{++} S' V
    using *skip bj-T* **by** (*metis* ‹$U = V$› $cdcl_W$-*bj.skip tranclp.simps*)

  moreover have *conflicting V* $\neq$ *None*
    using *skip* **by** *auto*
  ultimately show *?thesis* **using** S-S' **by** *auto*
qed
next
  case *resolve*
  then have [*simp*]: $U = V$
    using *full converse-rtranclpE* **unfolding** *full-def* **by** *fastforce*
  consider
    (s') $cdcl_W$-s'^{**} S T
    | (*bj*) S' **where** $cdcl_W$-s'^{**} S S' **and** $cdcl_W$-bj^{++} S' T **and** *conflicting T* $\neq$ *None*
    using *IH* **by** *blast*
  then show *?thesis*
    proof *cases*
      case s'
      have $cdcl_W$-bj^{++} T V
        using *resolve* **by** *force*
      moreover have *conflicting V* $\neq$ *None*
        using *resolve* **by** *auto*
      ultimately show *?thesis* **using** s' **by** *auto*
    next
      case (*bj* S') **note** S-S' = *this*(*1*) **and** *bj-T* = *this*(*2*)
      have $cdcl_W$-bj^{++} S' V
        using *resolve bj-T* **by** (*metis* ‹$U = V$› $cdcl_W$-*bj.resolve tranclp.simps*)
      moreover have *conflicting V* $\neq$ *None*
        using *resolve* **by** *auto*
      ultimately show *?thesis* **using** S-S' **by** *auto*
    qed
  qed
qed
qed

lemma *n-step*-$cdcl_W$-*stgy-iff-no-step*-$cdcl_W$-*cl*-$cdcl_W$-*o*:
  assumes *inv*: $cdcl_W$-*all-struct-inv S*
  shows *no-step* $cdcl_W$-s' S $\longleftrightarrow$ *no-step* $cdcl_W$-*cp S* $\wedge$ *no-step* $cdcl_W$-*o S* (**is** $?S'$ S $\longleftrightarrow$ $?C$ S $\wedge$ $?O$ S)
proof
  assume $?C$ S $\wedge$ $?O$ S
  then show $?S'$ S
    by (*auto simp*: $cdcl_W$-s'.*simps full1-def tranclp-unfold-begin*)
next
  assume *n-s*: $?S'$ S
  have $?C$ S
    proof (*rule ccontr*)
      assume $\neg$ *?thesis*
      then obtain S' **where** $cdcl_W$-*cp S* S'
        by *auto*
      then obtain T **where** *full1* $cdcl_W$-*cp S T*

using *cdcl*$_W$*-cp-normalized-element-all-inv inv* **by** (*metis* (*no-types*, *lifting*) *full-unfold*)
then show *False* **using** *n-s cdcl*$_W$*-s'.conflict'* **by** *blast*
qed
moreover have *?O S*
proof (*rule ccontr*)
assume ¬ *?thesis*
then obtain *S'* **where** *cdcl*$_W$*-o S S'*
by *auto*
then obtain *T* **where** *full1 cdcl*$_W$*-cp S' T*
using *cdcl*$_W$*-cp-normalized-element-all-inv inv*
by (*meson cdcl*$_W$*-all-struct-inv-def n-s*
cdcl$_W$*-stgy-cdcl*$_W$*-s'-connected' cdcl*$_W$*-then-exists-cdcl*$_W$*-stgy-step*)
then show *False* **using** *n-s* **by** (*meson* ‹*cdcl*$_W$*-o S S'*› *cdcl*$_W$*-all-struct-inv-def*
cdcl$_W$*-stgy-cdcl*$_W$*-s'-connected' cdcl*$_W$*-then-exists-cdcl*$_W$*-stgy-step inv*)
qed
ultimately show *?C S* ∧ *?O S* **by** *auto*
qed

lemma *cdcl*$_W$*-s'-tranclp-cdcl*$_W$:
cdcl$_W$*-s' S S'* ⟹ *cdcl*$_W$$^{++}$ *S S'*
proof (*induct rule*: *cdcl*$_W$*-s'.induct*)
case *conflict'*
then show *?case*
by (*simp add*: *full1-def tranclp-cdcl*$_W$*-cp-tranclp-cdcl*$_W$)
next
case *decide'*
then show *?case*
using *cdcl*$_W$*-stgy.simps cdcl*$_W$*-stgy-tranclp-cdcl*$_W$ **by** (*meson cdcl*$_W$*-o.simps*)
next
case (*bj' Sa S'a S''*) **note** *a2* = *this*(*1*) **and** *a1* = *this*(*2*) **and** *n-s* = *this*(*3*)
obtain *ss* :: *'st* ⇒ *'st* ⇒ (*'st* ⇒ *'st* ⇒ *bool*) ⇒ *'st* **where**
∀ *x0 x1 x2*. (∃ *v3*. *x2 x1 v3* ∧ *x2*** *v3 x0*) = (*x2 x1* (*ss x0 x1 x2*) ∧ *x2*** (*ss x0 x1 x2*) *x0*)
by *moura*
then have *f3*: ∀ *p s sa*. ¬ *p*$^{++}$ *s sa* ∨ *p s* (*ss sa s p*) ∧ *p*** (*ss sa s p*) *sa*
by (*metis* (*full-types*) *tranclpD*)
have *cdcl*$_W$*-bj*$^{++}$ *Sa S'a* ∧ *no-step cdcl*$_W$*-bj S'a*
using *a2* **by** (*simp add*: *full1-def*)
then have *cdcl*$_W$*-bj Sa* (*ss S'a Sa cdcl*$_W$*-bj*) ∧ *cdcl*$_W$*-bj*** (*ss S'a Sa cdcl*$_W$*-bj*) *S'a*
using *f3* **by** *auto*
then show *cdcl*$_W$$^{++}$ *Sa S''*
using *a1 n-s* **by** (*meson bj other rtranclp-cdcl*$_W$*-bj-full1-cdclp-cdcl*$_W$*-stgy*
rtranclp-cdcl$_W$*-stgy-rtranclp-cdcl*$_W$ *rtranclp-into-tranclp2*)
qed

lemma *tranclp-cdcl*$_W$*-s'-tranclp-cdcl*$_W$:
cdcl$_W$*-s'*$^{++}$ *S S'* ⟹ *cdcl*$_W$$^{++}$ *S S'*
apply (*induct rule*: *tranclp.induct*)
using *cdcl*$_W$*-s'-tranclp-cdcl*$_W$ **apply** *blast*
by (*meson cdcl*$_W$*-s'-tranclp-cdcl*$_W$ *tranclp-trans*)

lemma *rtranclp-cdcl*$_W$*-s'-rtranclp-cdcl*$_W$:
cdcl$_W$*-s'*** *S S'* ⟹ *cdcl*$_W$** *S S'*
using *rtranclp-unfold*[*of cdcl*$_W$*-s' S S'*] *tranclp-cdcl*$_W$*-s'-tranclp-cdcl*$_W$[*of S S'*] **by** *auto*

lemma *full-cdcl*$_W$*-stgy-iff-full-cdcl*$_W$*-s'*:

assumes *inv*: $cdcl_W$*-all-struct-inv S*
shows *full* $cdcl_W$*-stgy S T* $\longleftrightarrow$ *full* $cdcl_W$*-s′ S T* (**is** *?S* $\longleftrightarrow$ *?S′*)
proof
assume *?S′*
then have $cdcl_W^{**}$ *S T*
using *rtranclp-*$cdcl_W$*-s′-rtranclp-*$cdcl_W$[*of S T*] **unfolding** *full-def* **by** *blast*
then have *inv′*: $cdcl_W$*-all-struct-inv T*
using *rtranclp-*$cdcl_W$*-all-struct-inv-inv inv* **by** *blast*
have $cdcl_W$*-stgy*** *S T*
using ‹*?S′*› **unfolding** *full-def*
using $cdcl_W$*-s′-is-rtranclp-*$cdcl_W$*-stgy rtranclp-mono*[*of* $cdcl_W$*-s′* $cdcl_W$*-stgy***] **by** *auto*
then show *?S*
using ‹*?S′*› *inv′* $cdcl_W$*-stgy-*$cdcl_W$*-s′-connected′* **unfolding** *full-def* **by** *blast*
next
assume *?S*
then have *inv-T*:$cdcl_W$*-all-struct-inv T*
by (*metis assms full-def rtranclp-*$cdcl_W$*-all-struct-inv-inv rtranclp-*$cdcl_W$*-stgy-rtranclp-*$cdcl_W$)

consider
(*s′*) $cdcl_W$*-s′*** *S T*
| (*st*) *S′* **where** $cdcl_W$*-s′*** *S S′* **and** $cdcl_W$*-bj*$^{++}$ *S′ T* **and** *conflicting T* $\neq$ *None*
using *rtranclp-*$cdcl_W$*-stgy-connected-to-rtranclp-*$cdcl_W$*-s′*[*of S T*] *inv* ‹*?S*›
unfolding *full-def* $cdcl_W$*-all-struct-inv-def*
by *blast*
then show *?S′*
proof *cases*
case *s′*
then show *?thesis*
by (*metis* ‹*full* $cdcl_W$*-stgy S T*› *inv-T* $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-s′.simps*
$cdcl_W$*-stgy.conflict′* $cdcl_W$*-then-exists-*$cdcl_W$*-stgy-step full-def*
n-step-$cdcl_W$*-stgy-iff-no-step-*$cdcl_W$*-cl-*$cdcl_W$*-o*)
next
case (*st S′*)
have *full* $cdcl_W$*-cp T T*
using *option-full-*$cdcl_W$*-cp st*(*3*) **by** *blast*
moreover
have *n-s*: *no-step* $cdcl_W$*-bj T*
by (*metis* ‹*full* $cdcl_W$*-stgy S T*› *bj inv-T* $cdcl_W$*-all-struct-inv-def*
$cdcl_W$*-then-exists-*$cdcl_W$*-stgy-step full-def*)
then have *full1* $cdcl_W$*-bj S′ T*
using *st*(*2*) **unfolding** *full1-def* **by** *blast*
moreover have *no-step* $cdcl_W$*-cp S′*
using *st*(*2*) **by** (*fastforce dest*!: *tranclpD simp*: $cdcl_W$*-cp.simps* $cdcl_W$*-bj.simps*)
ultimately have $cdcl_W$*-s′ S′ T*
using $cdcl_W$*-s′.bj′*[*of S′ T T*] **by** *blast*
then have $cdcl_W$*-s′*** *S T*
using *st*(*1*) **by** *auto*
moreover have *no-step* $cdcl_W$*-s′ T*
using *inv-T* **by** (*metis* ‹*full* $cdcl_W$*-cp T T*› ‹*full* $cdcl_W$*-stgy S T*› $cdcl_W$*-all-struct-inv-def*
$cdcl_W$*-then-exists-*$cdcl_W$*-stgy-step full-def n-step-*$cdcl_W$*-stgy-iff-no-step-*$cdcl_W$*-cl-*$cdcl_W$*-o*)
ultimately show *?thesis*
unfolding *full-def* **by** *blast*
qed
qed

lemma *conflict-step-cdcl*$_W$*-stgy-step*:
assumes
conflict S T
cdcl$_W$*-all-struct-inv S*
shows $\exists$ *T. cdcl*$_W$*-stgy S T*
proof −
obtain *U* **where** *full cdcl*$_W$*-cp S U*
using *cdcl*$_W$*-cp-normalized-element-all-inv assms* **by** *blast*
then have *full1 cdcl*$_W$*-cp S U*
by (*metis cdcl*$_W$*-cp.conflict′ assms*(*1*) *full-unfold*)
then show *?thesis* **using** *cdcl*$_W$*-stgy.conflict′* **by** *blast*
qed

lemma *decide-step-cdcl*$_W$*-stgy-step*:
assumes
decide S T
cdcl$_W$*-all-struct-inv S*
shows $\exists$ *T. cdcl*$_W$*-stgy S T*
proof −
obtain *U* **where** *full cdcl*$_W$*-cp T U*
using *cdcl*$_W$*-cp-normalized-element-all-inv* **by** (*meson assms*(*1*) *assms*(*2*) *cdcl*$_W$*-all-struct-inv-inv cdcl*$_W$*-cp-normalized-element-all-inv decide other*)
then show *?thesis*
by (*metis assms cdcl*$_W$*-cp-normalized-element-all-inv cdcl*$_W$*-stgy.conflict′ decide full-unfold other′*)
qed

lemma *rtranclp-cdcl*$_W$*-cp-conflicting-Some*:
cdcl$_W$*-cp*** *S T* $\Longrightarrow$ *conflicting S* = *Some D* $\Longrightarrow$ *S* = *T*
using *rtranclpD tranclpD* **by** *fastforce*

inductive *cdcl*$_W$*-merge-cp* :: *′st* $\Rightarrow$ *′st* $\Rightarrow$ *bool* **where**
conflict′[*intro*]: *conflict S T* $\Longrightarrow$ *full cdcl*$_W$*-bj T U* $\Longrightarrow$ *cdcl*$_W$*-merge-cp S U* |
propagate′[*intro*]: *propagate*$^{++}$ *S S′* $\Longrightarrow$ *cdcl*$_W$*-merge-cp S S′*

lemma *cdcl*$_W$*-merge-restart-cases*[*consumes 1*, *case-names conflict propagate*]:
assumes
cdcl$_W$*-merge-cp S U* **and**
$\bigwedge$*T. conflict S T* $\Longrightarrow$ *full cdcl*$_W$*-bj T U* $\Longrightarrow$ *P* **and**
propagate^{++} *S U* $\Longrightarrow$ *P*
shows *P*
using *assms* **unfolding** *cdcl*$_W$*-merge-cp.simps* **by** *auto*

lemma *cdcl*$_W$*-merge-cp-tranclp-cdcl*$_W$*-merge*:
cdcl$_W$*-merge-cp S T* $\Longrightarrow$ *cdcl*$_W$*-merge*$^{++}$ *S T*
apply (*induction rule*: *cdcl*$_W$*-merge-cp.induct*)
using *cdcl*$_W$*-merge.simps* **apply** *auto*[*1*]
using *tranclp-mono*[*of propagate cdcl*$_W$*-merge*] *fw-propagate* **by** *blast*

lemma *rtranclp-cdcl*$_W$*-merge-cp-rtranclp-cdcl*$_W$:
cdcl$_W$*-merge-cp*** *S T* $\Longrightarrow$ *cdcl*$_W$** *S T*
apply (*induction rule*: *rtranclp-induct*)
apply *simp*
unfolding *cdcl*$_W$*-merge-cp.simps* **by** (*meson cdcl*$_W$*-merge-restart-cdcl*$_W$ *fw-r-conflict rtranclp-propagate-is-rtranclp-cdcl*$_W$ *rtranclp-trans tranclp-into-rtranclp*)

lemma *full1-cdcl$_W$-bj-no-step-cdcl$_W$-bj*:
  full1 cdcl$_W$-bj S T $\Longrightarrow$ *no-step cdcl$_W$-cp S*
  by (*metis rtranclp-unfold cdcl$_W$-cp-conflicting-not-empty option.exhaust full1-def rtranclp-cdcl$_W$-merge-restart-no-step-cdcl$_W$-bj tranclpD*)

inductive *cdcl$_W$-s'-without-decide* **where**
conflict'-without-decide[*intro*]: *full1 cdcl$_W$-cp S S'* $\Longrightarrow$ *cdcl$_W$-s'-without-decide S S'* |
bj'-without-decide[*intro*]: *full1 cdcl$_W$-bj S S'* $\Longrightarrow$ *no-step cdcl$_W$-cp S* $\Longrightarrow$ *full cdcl$_W$-cp S' S''*
  $\Longrightarrow$ *cdcl$_W$-s'-without-decide S S''*

lemma *rtranclp-cdcl$_W$-s'-without-decide-rtranclp-cdcl$_W$*:
  cdcl$_W$-s'-without-decide** S T $\Longrightarrow$ *cdcl$_W$** S T*
  apply (*induction rule*: *rtranclp-induct*)
    apply *simp*
  by (*meson cdcl$_W$-s'.simps cdcl$_W$-s'-tranclp-cdcl$_W$ cdcl$_W$-s'-without-decide.simps rtranclp-tranclp-tranclp tranclp-into-rtranclp*)

lemma *rtranclp-cdcl$_W$-s'-without-decide-rtranclp-cdcl$_W$-s'*:
  cdcl$_W$-s'-without-decide** S T $\Longrightarrow$ *cdcl$_W$-s'** S T*
proof (*induction rule*: *rtranclp-induct*)
  case *base*
  then show *?case* **by** *simp*
next
  case (*step y z*) **note** *a2* = *this*(*2*) **and** *a1* = *this*(*3*)
  have *cdcl$_W$-s' y z*
    using *a2* **by** (*metis* (*no-types*) *bj' cdcl$_W$-s'.conflict' cdcl$_W$-s'-without-decide.cases*)
  then show *cdcl$_W$-s'** S z*
    using *a1* **by** (*meson r-into-rtranclp rtranclp-trans*)
qed

lemma *rtranclp-cdcl$_W$-merge-cp-is-rtranclp-cdcl$_W$-s'-without-decide*:
  assumes
    cdcl$_W$-merge-cp** S V
    conflicting S = None
  shows
    (*cdcl$_W$-s'-without-decide** S V*)
    $\vee$ ($\exists$ *T*. *cdcl$_W$-s'-without-decide** S T* $\wedge$ *propagate^{++} T V*)
    $\vee$ ($\exists$ *T U*. *cdcl$_W$-s'-without-decide** S T* $\wedge$ *full1 cdcl$_W$-bj T U* $\wedge$ *propagate** U V*)
  using *assms*
proof (*induction rule*: *rtranclp-induct*)
  case *base*
  then show *?case* **by** *simp*
next
  case (*step U V*) **note** *st* = *this*(*1*) **and** *cp* = *this*(*2*) **and** *IH* = *this*(*3*)[*OF this*(*4*)]
  from *cp* **show** *?case*
    proof (*cases rule*: *cdcl$_W$-merge-restart-cases*)
      case *propagate*
      then show *?thesis* **using** *IH* **by** (*meson rtranclp-tranclp-tranclp tranclp-into-rtranclp*)
    next
      case (*conflict U'*) **note** *confl* = *this*(*1*) **and** *bj* = *this*(*2*)
      have *full1-U-U'*: *full1 cdcl$_W$-cp U U'*
        by (*simp add*: *conflict-is-full1-cdcl$_W$-cp local.conflict*(*1*))
      consider
          (*s'*) *cdcl$_W$-s'-without-decide** S U*

| (*propa*) T' **where** $cdcl_W$*-s'-without-decide*** S T' **and** *propagate*$^{++}$ T' U
| (*bj-prop*) T' T'' **where**
$cdcl_W$*-s'-without-decide*** S T' **and**
full1 $cdcl_W$*-bj* T' T'' **and**
propagate** T'' U
using *IH* **by** *blast*
then show *?thesis*
proof *cases*
case *s'*
have $cdcl_W$*-s'-without-decide* U U'
using *full1-U-U' conflict'-without-decide* **by** *blast*
then have $cdcl_W$*-s'-without-decide*** S U'
using ‹$cdcl_W$*-s'-without-decide*** S U› **by** *auto*
moreover have $U' = V \lor$ *full1* $cdcl_W$*-bj* U' V
using *bj* **by** (*meson full-unfold*)
ultimately show *?thesis* **by** *blast*
next
case *propa* **note** $s' = this(1)$ **and** T'*-U* $= this(2)$
have *full1* $cdcl_W$*-cp* T' U'
using *rtranclp-mono*[*of propagate* $cdcl_W$*-cp*] T'*-U* $cdcl_W$*-cp.propagate' full1-U-U'*
rtranclp-full1I[*of* $cdcl_W$*-cp* T'] **by** (*metis* (*full-types*) *predicate2D predicate2I*
tranclp-into-rtranclp)
have $cdcl_W$*-s'-without-decide*** S U'
using ‹*full1* $cdcl_W$*-cp* T' U'› *conflict'-without-decide* s' **by** *force*
have *full1* $cdcl_W$*-bj* U' $V \lor V = U'$
by (*metis* (*lifting*) *full-unfold local.bj*)
then show *?thesis*
using ‹$cdcl_W$*-s'-without-decide*** S U'› **by** *blast*
next
case *bj-prop* **note** $s' = this(1)$ **and** *bj-*$T' = this(2)$ **and** T''*-U* $= this(3)$
have *no-step* $cdcl_W$*-cp* T'
using *bj-*T' *full1-*$cdcl_W$*-bj-no-step-*$cdcl_W$*-bj* **by** *blast*
moreover have *full1* $cdcl_W$*-cp* T'' U'
using *rtranclp-mono*[*of propagate* $cdcl_W$*-cp*] T''*-U* $cdcl_W$*-cp.propagate' full1-U-U'*
rtranclp-full1I[*of* $cdcl_W$*-cp* T''] **by** *blast*
ultimately have $cdcl_W$*-s'-without-decide* T' U'
using *bj'-without-decide*[*of* T' T'' U'] *bj-*T' **by** (*simp add*: *full-unfold*)
then have $cdcl_W$*-s'-without-decide*** S U'
using s' *rtranclp.intros*(*2*)[*of - S* T' U'] **by** *blast*
then show *?thesis*
by (*metis full-unfold local.bj rtranclp.rtrancl-refl*)
qed
qed
qed

lemma *rtranclp-*$cdcl_W$*-s'-without-decide-is-rtranclp-*$cdcl_W$*-merge-cp*:
assumes
$cdcl_W$*-s'-without-decide*** S V **and**
confl: *conflicting* S = *None*
shows
($cdcl_W$*-merge-cp*** S V $\land$ *conflicting* V = *None*)
$\lor$ ($cdcl_W$*-merge-cp*** S V $\land$ *conflicting* $V \neq$ *None* $\land$ *no-step* $cdcl_W$*-cp* V $\land$ *no-step* $cdcl_W$*-bj* V)
$\lor$ ($\exists$ T. $cdcl_W$*-merge-cp*** S T $\land$ *conflict* T V)
using *assms*(*1*)

proof (*induction*)
  case *base*
  then show *?case* **using** *confl* **by** *auto*
next
  case (*step U V*) **note** *st* = *this*(*1*) **and** *s* = *this*(*2*) **and** *IH* = *this*(*3*)
  from *s* **show** *?case*
    proof (*cases rule*: $cdcl_W$*-s'-without-decide.cases*)
      case *conflict'-without-decide*
      then have *rt*: $cdcl_W$*-cp*$^{++}$ *U V* **unfolding** *full1-def* **by** *fast*
      then have *conflicting U* = *None*
        using *tranclp-*$cdcl_W$*-cp-propagate-with-conflict-or-not*[*of U V*]
        conflict **by** (*auto dest*!: *tranclpD simp*: *rtranclp-unfold*)
      then have $cdcl_W$*-merge-cp*** *S U* **using** *IH* **by** *auto*
      consider
          (*propa*) *propagate*$^{++}$ *U V*
        | (*confl'*) *conflict U V*
        | (*propa-confl'*) *U'* **where** *propagate*$^{++}$ *U U'* *conflict U' V*
        using *tranclp-*$cdcl_W$*-cp-propagate-with-conflict-or-not*[*OF rt*] **unfolding** *rtranclp-unfold*
        by *fastforce*
      then show *?thesis*
        proof *cases*
          case *propa*
          then have $cdcl_W$*-merge-cp U V*
            by *auto*
          moreover have *conflicting V* = *None*
            using *propa* **unfolding** *tranclp-unfold-end* **by** *auto*
          ultimately show *?thesis* **using** ‹$cdcl_W$*-merge-cp*** *S U*› **by** *force*
        next
          case *confl'*
          then show *?thesis* **using** ‹$cdcl_W$*-merge-cp*** *S U*› **by** *auto*
        next
          case *propa-confl'* **note** *propa* = *this*(*1*) **and** *confl'* = *this*(*2*)
          then have $cdcl_W$*-merge-cp U U'* **by** *auto*
          then have $cdcl_W$*-merge-cp*** *S U'* **using** ‹$cdcl_W$*-merge-cp*** *S U*› **by** *auto*
          then show *?thesis* **using** ‹$cdcl_W$*-merge-cp*** *S U*› *confl'* **by** *auto*
        qed
    next
      case (*bj'-without-decide U'*) **note** *full-bj* = *this*(*1*) **and** *cp* = *this*(*3*)
      then have *conflicting U* ≠ *None*
        using *full-bj* **unfolding** *full1-def* **by** (*fastforce dest*!: *tranclpD simp*: $cdcl_W$*-bj.simps*)
      with *IH* **obtain** *T* **where**
        S-T: $cdcl_W$*-merge-cp*** *S T* **and** *T-U*: *conflict T U*
        using *full-bj* **unfolding** *full1-def* **by** (*blast dest*: *tranclpD*)
      then have $cdcl_W$*-merge-cp T U'*
        using $cdcl_W$*-merge-cp.conflict'*[*of T U U'*] *full-bj* **by** (*simp add*: *full-unfold*)
      then have *S-U'*: $cdcl_W$*-merge-cp*** *S U'* **using** *S-T* **by** *auto*
      consider
          (*n-s*) *U'* = *V*
        | (*propa*) *propagate*$^{++}$ *U' V*
        | (*confl'*) *conflict U' V*
        | (*propa-confl'*) *U''* **where** *propagate*$^{++}$ *U' U''* *conflict U'' V*
        using *tranclp-*$cdcl_W$*-cp-propagate-with-conflict-or-not cp*
        unfolding *rtranclp-unfold full-def* **by** *metis*
      then show *?thesis*
        proof *cases*

case *propa*
then have $cdcl_W$*-merge-cp* U' V **by** *auto*
moreover have *conflicting* V = *None*
using *propa* **unfolding** *tranclp-unfold-end* **by** *auto*
ultimately show *?thesis* **using** *S-U'* **by** *force*
next
case *confl'*
then show *?thesis* **using** *S-U'* **by** *auto*
next
case *propa-confl'* **note** *propa* = *this(1)* **and** *confl* = *this(2)*
have $cdcl_W$*-merge-cp* U' U'' **using** *propa* **by** *auto*
then show *?thesis* **using** *S-U' confl* **by** (*meson rtranclp.rtrancl-into-rtrancl*)
next
case *n-s*
then show *?thesis*
using *S-U'* **apply** (*cases conflicting* V = *None*)
using *full-bj* **apply** *simp*
by (*metis cp full-def full-unfold full-bj*)
qed
qed
qed

lemma *no-step-*$cdcl_W$*-s'-no-ste-*$cdcl_W$*-merge-cp*:
assumes
$cdcl_W$*-all-struct-inv S*
conflicting S = *None*
no-step $cdcl_W$*-s' S*
shows *no-step* $cdcl_W$*-merge-cp S*
using *assms* **apply** (*auto simp*: $cdcl_W$*-s'.simps* $cdcl_W$*-merge-cp.simps*)
using *conflict-is-full1-*$cdcl_W$*-cp* **apply** *blast*
using $cdcl_W$*-cp-normalized-element-all-inv* $cdcl_W$*-cp.propagate'* **by** (*metis* $cdcl_W$*-cp.propagate'*
full-unfold tranclpD)

The *no-step decide S* is needed, since $cdcl_W$*-merge-cp* is $cdcl_W$*-s'* without *decide*.

lemma *conflicting-true-no-step-*$cdcl_W$*-merge-cp-no-step-s'-without-decide*:
assumes
confl: *conflicting S* = *None* **and**
inv: $cdcl_W$*-M-level-inv S* **and**
n-s: *no-step* $cdcl_W$*-merge-cp S*
shows *no-step* $cdcl_W$*-s'-without-decide S*
proof (*rule ccontr*)
assume ¬ *no-step* $cdcl_W$*-s'-without-decide S*
then obtain *T* **where**
$cdcl_W$: $cdcl_W$*-s'-without-decide S T*
by *auto*
then have *inv-T*: $cdcl_W$*-M-level-inv T*
using *rtranclp-*$cdcl_W$*-s'-without-decide-rtranclp-*$cdcl_W$[*of S T*]
rtranclp-$cdcl_W$*-consistent-inv inv* **by** *blast*
from $cdcl_W$ **show** *False*
proof *cases*
case *conflict'-without-decide*
have *no-step propagate S*
using *n-s* **by** *blast*
then have *conflict S T*
using *local.conflict' tranclp-*$cdcl_W$*-cp-propagate-with-conflict-or-not*[*of S T*]

unfolding *full1-def* **by** (*metis full1-def local.conflict′-without-decide rtranclp-unfold tranclp-unfold-begin*)
moreover
then obtain T' **where** *full* $cdcl_W$*-bj* T T'
using $cdcl_W$*-bj-exists-normal-form inv-T* **by** *blast*
ultimately show *False* **using** $cdcl_W$*-merge-cp.conflict′ n-s* **by** *meson*
next
case (*bj′-without-decide* S')
then show *?thesis*
using *confl* **unfolding** *full1-def* **by** (*fastforce simp*: $cdcl_W$*-bj.simps dest*: *tranclpD*)
qed
qed

lemma *conflicting-true-no-step-s′-without-decide-no-step-*$cdcl_W$*-merge-cp*:
assumes
inv: $cdcl_W$*-all-struct-inv* S **and**
n-s: *no-step* $cdcl_W$*-s′-without-decide* S
shows *no-step* $cdcl_W$*-merge-cp* S
proof (*rule ccontr*)
assume ¬ *?thesis*
then obtain T **where** $cdcl_W$*-merge-cp* S T
by *auto*
then show *False*
proof *cases*
case (*conflict′* S')
then show *False* **using** *n-s conflict′-without-decide conflict-is-full1-*$cdcl_W$*-cp* **by** *blast*
next
case *propagate′*
moreover
have $cdcl_W$*-all-struct-inv* T
using *inv* **by** (*meson local.propagate′ rtranclp-*$cdcl_W$*-all-struct-inv-inv rtranclp-propagate-is-rtranclp-*$cdcl_W$ *tranclp-into-rtranclp*)
then obtain U **where** *full* $cdcl_W$*-cp* T U
using $cdcl_W$*-cp-normalized-element-all-inv* **by** *auto*
ultimately have *full1* $cdcl_W$*-cp* S U
using *tranclp-full-full1I*[*of* $cdcl_W$*-cp* S T U] $cdcl_W$*-cp.propagate′*
tranclp-mono[*of propagate* $cdcl_W$*-cp*] **by** *blast*
then show *False* **using** *conflict′-without-decide n-s* **by** *blast*
qed
qed

lemma *no-step-*$cdcl_W$*-merge-cp-no-step-*$cdcl_W$*-cp*:
no-step $cdcl_W$*-merge-cp* $S \Longrightarrow$ $cdcl_W$*-M-level-inv* $S \Longrightarrow$ *no-step* $cdcl_W$*-cp* S
using $cdcl_W$*-bj-exists-normal-form* $cdcl_W$*-consistent-inv*[*OF* $cdcl_W$*.conflict*, *of* S]
by (*metis* $cdcl_W$*-cp.cases* $cdcl_W$*-merge-cp.simps tranclp.intros*(*1*))

lemma *conflicting-not-true-rtranclp-*$cdcl_W$*-merge-cp-no-step-*$cdcl_W$*-bj*:
assumes
conflicting S = *None* **and**
$cdcl_W$*-merge-cp*** S T
shows *no-step* $cdcl_W$*-bj* T
using *assms*(*2*,*1*) **by** (*induction*)
(*fastforce simp*: $cdcl_W$*-merge-cp.simps full-def tranclp-unfold-end* $cdcl_W$*-bj.simps*)+

lemma *conflicting-true-full-*$cdcl_W$*-merge-cp-iff-full-*$cdcl_W$*-s′-without-decode*:

assumes
confl: *conflicting S* = *None* **and**
inv: $cdcl_W$*-all-struct-inv S*
shows
full $cdcl_W$*-merge-cp S V* ⟷ *full* $cdcl_W$*-s'-without-decide S V* (**is** *?fw* ⟷ *?s'*)
proof
assume *?fw*
then have *st*: $cdcl_W$*-merge-cp*** *S V* **and** *n-s*: *no-step* $cdcl_W$*-merge-cp V*
unfolding *full-def* **by** *blast+*
have *inv-V*: $cdcl_W$*-all-struct-inv V*
using *rtranclp-*$cdcl_W$*-merge-cp-rtranclp-*$cdcl_W$[*of S V*] ‹*?fw*› **unfolding** *full-def*
by (*simp add*: *inv rtranclp-*$cdcl_W$*-all-struct-inv-inv*)
consider
(*s'*) $cdcl_W$*-s'-without-decide*** *S V*
| (*propa*) *T* **where** $cdcl_W$*-s'-without-decide*** *S T* **and** *propagate*++ *T V*
| (*bj*) *T U* **where** $cdcl_W$*-s'-without-decide*** *S T* **and** *full1* $cdcl_W$*-bj T U* **and** *propagate*** *U V*
using *rtranclp-*$cdcl_W$*-merge-cp-is-rtranclp-*$cdcl_W$*-s'-without-decide confl st n-s* **by** *metis*
then have $cdcl_W$*-s'-without-decide*** *S V*
proof *cases*
case *s'*
then show *?thesis* **.**
next
case *propa* **note** *s'* = *this*(*1*) **and** *propa* = *this*(*2*)
have *no-step* $cdcl_W$*-cp V*
using *no-step-*$cdcl_W$*-merge-cp-no-step-*$cdcl_W$*-cp n-s inv-V*
unfolding $cdcl_W$*-all-struct-inv-def* **by** *blast*
then have *full1* $cdcl_W$*-cp T V*
using *propa tranclp-mono*[*of propagate* $cdcl_W$*-cp*] $cdcl_W$*-cp.propagate'* **unfolding** *full1-def*
by *blast*
then have $cdcl_W$*-s'-without-decide T V*
using *conflict'-without-decide* **by** *blast*
then show *?thesis* **using** *s'* **by** *auto*
next
case *bj* **note** *s'* = *this*(*1*) **and** *bj* = *this*(*2*) **and** *propa* = *this*(*3*)
have *no-step* $cdcl_W$*-cp V*
using *no-step-*$cdcl_W$*-merge-cp-no-step-*$cdcl_W$*-cp n-s inv-V*
unfolding $cdcl_W$*-all-struct-inv-def* **by** *blast*
then have *full* $cdcl_W$*-cp U V*
using *propa rtranclp-mono*[*of propagate* $cdcl_W$*-cp*] $cdcl_W$*-cp.propagate'* **unfolding** *full-def*
by *blast*
moreover have *no-step* $cdcl_W$*-cp T*
using *bj* **unfolding** *full1-def* **by** (*fastforce dest*!: *tranclpD simp*:$cdcl_W$*-bj.simps*)
ultimately have $cdcl_W$*-s'-without-decide T V*
using *bj'-without-decide*[*of T U V*] *bj* **by** *blast*
then show *?thesis* **using** *s'* **by** *auto*
qed
moreover have *no-step* $cdcl_W$*-s'-without-decide V*
proof (*cases conflicting V* = *None*)
case *False*
{ fix *ss* :: *'st*
have *ff1*: ∀ *s sa*. ¬ $cdcl_W$*-s' s sa* ∨ *full1* $cdcl_W$*-cp s sa*
∨ (∃ *sb*. *decide s sb* ∧ *no-step* $cdcl_W$*-cp s* ∧ *full* $cdcl_W$*-cp sb sa*)
∨ (∃ *sb*. *full1* $cdcl_W$*-bj s sb* ∧ *no-step* $cdcl_W$*-cp s* ∧ *full* $cdcl_W$*-cp sb sa*)
by (*metis* $cdcl_W$*-s'.cases*)
have *ff2*: (∀ *p s sa*. ¬ *full1 p* (*s*::*'st*) *sa* ∨ p^{++} *s sa* ∧ *no-step p sa*)

$\land$ ($\forall p\ s\ sa.$ ($\neg\ p^{++}$ (s::$'st$) sa $\lor$ ($\exists\ s.\ p\ sa\ s$)) $\lor$ *full1 p s sa*)
by (*meson full1-def*)
obtain *ssa* :: ($'st \Rightarrow 'st \Rightarrow bool$) $\Rightarrow$ $'st \Rightarrow 'st \Rightarrow 'st$ **where**
ff3: $\forall p\ s\ sa.$ $\neg\ p^{++}$ $s\ sa$ $\lor$ $p\ s$ ($ssa\ p\ s\ sa$) $\land$ p^{**} ($ssa\ p\ s\ sa$) sa
by (*metis* (*no-types*) *tranclpD*)
then have *a3*: $\neg$ $cdcl_W$-cp^{++} $V\ ss$
using *False* **by** (*metis option-full-cdcl*$_W$*-cp full-def*)
have $\bigwedge s.$ $\neg$ $cdcl_W$-bj^{++} $V\ s$
using *ff3 False* **by** (*metis confl st*
conflicting-not-true-rtranclp-cdcl$_W$*-merge-cp-no-step-cdcl*$_W$*-bj*)
then have $\neg$ $cdcl_W$-s'-*without-decide* $V\ ss$
using *ff1 a3 ff2* **by** (*metis* $cdcl_W$-s'-*without-decide.cases*)
}
then show *?thesis*
by *fastforce*
next
case *True*
then show *?thesis*
using *conflicting-true-no-step-cdcl*$_W$*-merge-cp-no-step-*s'*-without-decide n-s inv-V*
unfolding $cdcl_W$*-all-struct-inv-def* **by** *blast*
qed
ultimately show *?s'* **unfolding** *full-def* **by** *blast*
next
assume s': *?s'*
then have *st*: $cdcl_W$-s'-*without-decide*** $S\ V$ **and** *n-s*: *no-step* $cdcl_W$-s'-*without-decide* V
unfolding *full-def* **by** *auto*
then have $cdcl_W{}^{**}$ $S\ V$
using *rtranclp-cdcl*$_W$*-*s'*-without-decide-rtranclp-cdcl*$_W$ *st* **by** *blast*
then have *inv-V*: $cdcl_W$*-all-struct-inv V* **using** *inv rtranclp-cdcl*$_W$*-all-struct-inv-inv* **by** *blast*
then have *n-s-cp-V*: *no-step* $cdcl_W$*-cp V*
using $cdcl_W$*-cp-normalized-element-all-inv*[*of V*] *full-fullI*[*of* $cdcl_W$*-cp V*] *n-s*
conflict$'$*-without-decide conflicting-true-no-step-*s'*-without-decide-no-step-cdcl*$_W$*-merge-cp*
no-step-cdcl$_W$*-merge-cp-no-step-cdcl*$_W$*-cp*
unfolding $cdcl_W$*-all-struct-inv-def* **by** *presburger*
have *n-s-bj*: *no-step* $cdcl_W$*-bj V*
proof (*rule ccontr*)
assume $\neg$ *?thesis*
then obtain *W* **where** *W*: $cdcl_W$*-bj V W* **by** *blast*
have $cdcl_W$*-all-struct-inv W*
using *W* $cdcl_W$*.simps* $cdcl_W$*-all-struct-inv-inv inv-V* **by** *blast*
then obtain W' **where** *full1* $cdcl_W$*-bj V* W'
using $cdcl_W$*-bj-exists-normal-form*[*of W*] *full-fullI*[*of* $cdcl_W$*-bj V W*] *W*
unfolding $cdcl_W$*-all-struct-inv-def*
by *blast*
moreover
then have $cdcl_W{}^{++}$ $V\ W'$
using *tranclp-mono*[*of* $cdcl_W$*-bj* $cdcl_W$] $cdcl_W$*.other* $cdcl_W$*-o.bj* **unfolding** *full1-def* **by** *blast*
then have $cdcl_W$*-all-struct-inv* W'
by (*meson inv-V rtranclp-cdcl*$_W$*-all-struct-inv-inv tranclp-into-rtranclp*)
then obtain *X* **where** *full* $cdcl_W$*-cp* W' *X*
using $cdcl_W$*-cp-normalized-element-all-inv* **by** *blast*
ultimately show *False*
using bj'*-without-decide n-s-cp-V n-s* **by** *blast*
qed
from s' **consider**

(cp-$true$) $cdcl_W$-$merge$-cp^{**} S V **and** $conflicting$ $V = None$
| (cp-$false$) $cdcl_W$-$merge$-cp^{**} S V **and** $conflicting$ $V \neq None$ **and** no-$step$ $cdcl_W$-cp V **and** no-$step$ $cdcl_W$-bj V
| (cp-$confl$) T **where** $cdcl_W$-$merge$-cp^{**} S T $conflict$ T V
using $rtranclp$-$cdcl_W$-s'-$without$-$decide$-is-$rtranclp$-$cdcl_W$-$merge$-$cp[of\ S\ V]$ $confl$
unfolding $full$-def **by** $meson$
then have $cdcl_W$-$merge$-cp^{**} S V
proof $cases$
case cp-$confl$ **note** S-$T = this(1)$ **and** $conf$-$V = this(2)$
have $full$ $cdcl_W$-bj V V
using $conf$-V n-s-bj **unfolding** $full$-def **by** $fast$
then have $cdcl_W$-$merge$-cp T V
using $cdcl_W$-$merge$-$cp.conflict'$ $conf$-V **by** $auto$
then show $?thesis$ **using** S-T **by** $auto$
qed $fast+$
moreover
then have $cdcl_W^{**}$ S V **using** $rtranclp$-$cdcl_W$-$merge$-cp-$rtranclp$-$cdcl_W$ **by** $blast$
then have $cdcl_W$-all-$struct$-inv V
using inv $rtranclp$-$cdcl_W$-all-$struct$-inv-inv **by** $blast$
then have no-$step$ $cdcl_W$-$merge$-cp V
using $conflicting$-$true$-no-$step$-s'-$without$-$decide$-no-$step$-$cdcl_W$-$merge$-cp s'
unfolding $full$-def **by** $blast$
ultimately show $?fw$ **unfolding** $full$-def **by** $auto$
qed

lemma $conflicting$-$true$-$full1$-$cdcl_W$-$merge$-cp-iff-$full1$-$cdcl_W$-s'-$without$-$decode$:
assumes
$confl$: $conflicting$ $S = None$ **and**
inv: $cdcl_W$-all-$struct$-inv S
shows
$full1$ $cdcl_W$-$merge$-cp S $V \longleftrightarrow full1$ $cdcl_W$-s'-$without$-$decide$ S V
proof −
have $full$ $cdcl_W$-$merge$-cp S $V = full$ $cdcl_W$-s'-$without$-$decide$ S V
using $confl$ $conflicting$-$true$-$full$-$cdcl_W$-$merge$-cp-iff-$full$-$cdcl_W$-s'-$without$-$decode$ inv
by $blast$
then show $?thesis$ **unfolding** $full$-$unfold$ $full1$-def
by ($metis$ ($mono$-$tags$) $tranclp$-$unfold$-$begin$)
qed

lemma $conflicting$-$true$-$full1$-$cdcl_W$-$merge$-cp-imp-$full1$-$cdcl_W$-s'-$without$-$decode$:
assumes
fw: $full1$ $cdcl_W$-$merge$-cp S V **and**
inv: $cdcl_W$-all-$struct$-inv S
shows
$full1$ $cdcl_W$-s'-$without$-$decide$ S V
proof −
have $conflicting$ $S = None$
using fw **unfolding** $full1$-def **by** ($auto$ $dest!$: $tranclpD$ $simp$: $cdcl_W$-$merge$-$cp.simps$)
then show $?thesis$
using $conflicting$-$true$-$full1$-$cdcl_W$-$merge$-cp-iff-$full1$-$cdcl_W$-s'-$without$-$decode$ fw inv **by** $blast$
qed

inductive $cdcl_W$-$merge$-$stgy$ **where**
fw-s-$cp[intro]$: $full1$ $cdcl_W$-$merge$-cp S $T \Longrightarrow cdcl_W$-$merge$-$stgy$ S T |
fw-s-$decide[intro]$: $decide$ S $T \Longrightarrow no$-$step$ $cdcl_W$-$merge$-cp $S \Longrightarrow full$ $cdcl_W$-$merge$-cp T U

$\Longrightarrow$ *cdcl$_W$-merge-stgy S U*

lemma *cdcl$_W$-merge-stgy-tranclp-cdcl$_W$-merge*:
assumes *fw*: *cdcl$_W$-merge-stgy S T*
shows *cdcl$_W$-merge^{++} S T*
proof –
{ fix *S T*
assume *full1 cdcl$_W$-merge-cp S T*
then have *cdcl$_W$-merge^{++} S T*
using *tranclp-mono*[*of cdcl$_W$-merge-cp cdcl$_W$-merge^{++}*] *cdcl$_W$-merge-cp-tranclp-cdcl$_W$-merge*
unfolding *full1-def*
by *auto*
} note *full1-cdcl$_W$-merge-cp-cdcl$_W$-merge = this*
show *?thesis*
using *fw*
apply (*induction rule*: *cdcl$_W$-merge-stgy.induct*)
using *full1-cdcl$_W$-merge-cp-cdcl$_W$-merge* **apply** *simp*
unfolding *full-unfold* **by** (*auto dest*!: *full1-cdcl$_W$-merge-cp-cdcl$_W$-merge fw-decide*)
qed

lemma *rtranclp-cdcl$_W$-merge-stgy-rtranclp-cdcl$_W$-merge*:
assumes *fw*: *cdcl$_W$-merge-stgy** S T*
shows *cdcl$_W$-merge** S T*
using *fw cdcl$_W$-merge-stgy-tranclp-cdcl$_W$-merge rtranclp-mono*[*of cdcl$_W$-merge-stgy cdcl$_W$-merge^{++}*]
unfolding *tranclp-rtranclp-rtranclp* **by** *blast*

lemma *cdcl$_W$-merge-stgy-rtranclp-cdcl$_W$*:
cdcl$_W$-merge-stgy S T $\Longrightarrow$ *cdcl$_W$** S T*
apply (*induction rule*: *cdcl$_W$-merge-stgy.induct*)
using *rtranclp-cdcl$_W$-merge-cp-rtranclp-cdcl$_W$* **unfolding** *full1-def*
apply (*simp add*: *tranclp-into-rtranclp*)
using *rtranclp-cdcl$_W$-merge-cp-rtranclp-cdcl$_W$ cdcl$_W$-o.decide cdcl$_W$.other* **unfolding** *full-def*
by (*meson r-into-rtranclp rtranclp-trans*)

lemma *rtranclp-cdcl$_W$-merge-stgy-rtranclp-cdcl$_W$*:
cdcl$_W$-merge-stgy** S T $\Longrightarrow$ *cdcl$_W$** S T*
using *rtranclp-mono*[*of cdcl$_W$-merge-stgy cdcl$_W$***] *cdcl$_W$-merge-stgy-rtranclp-cdcl$_W$* **by** *auto*

lemma *cdcl$_W$-merge-stgy-cases*[*consumes 1*, *case-names fw-s-cp fw-s-decide*]:
assumes
cdcl$_W$-merge-stgy S U
full1 cdcl$_W$-merge-cp S U $\Longrightarrow$ *P*
$\bigwedge$*T. decide S T* $\Longrightarrow$ *no-step cdcl$_W$-merge-cp S* $\Longrightarrow$ *full cdcl$_W$-merge-cp T U* $\Longrightarrow$ *P*
shows *P*
using *assms* **by** (*auto simp*: *cdcl$_W$-merge-stgy.simps*)

inductive *cdcl$_W$-s'-w* :: *'st* $\Rightarrow$ *'st* $\Rightarrow$ *bool* **where**
conflict': *full1 cdcl$_W$-s'-without-decide S S'* $\Longrightarrow$ *cdcl$_W$-s'-w S S'* |
decide': *decide S S'* $\Longrightarrow$ *no-step cdcl$_W$-s'-without-decide S* $\Longrightarrow$ *full cdcl$_W$-s'-without-decide S' S''*
$\Longrightarrow$ *cdcl$_W$-s'-w S S''*

lemma *cdcl$_W$-s'-w-rtranclp-cdcl$_W$*:
cdcl$_W$-s'-w S T $\Longrightarrow$ *cdcl$_W$** S T*
apply (*induction rule*: *cdcl$_W$-s'-w.induct*)
using *rtranclp-cdcl$_W$-s'-without-decide-rtranclp-cdcl$_W$* **unfolding** *full1-def*

apply (*simp add*: *tranclp-into-rtranclp*)
using *rtranclp-cdcl$_W$-s'-without-decide-rtranclp-cdcl$_W$* **unfolding** *full-def*
by (*meson decide other rtranclp-into-tranclp2 tranclp-into-rtranclp*)

lemma *rtranclp-cdcl$_W$-s'-w-rtranclp-cdcl$_W$*:
cdcl$_W$-s'-w** S T $\Longrightarrow$ *cdcl$_W$** S T*
using *rtranclp-mono*[*of cdcl$_W$-s'-w cdcl$_W$***] *cdcl$_W$-s'-w-rtranclp-cdcl$_W$* **by** *auto*

lemma *no-step-cdcl$_W$-cp-no-step-cdcl$_W$-s'-without-decide*:
assumes *no-step cdcl$_W$-cp S* **and** *conflicting S = None* **and** *inv*: *cdcl$_W$-M-level-inv S*
shows *no-step cdcl$_W$-s'-without-decide S*
by (*metis assms cdcl$_W$-cp.conflict' cdcl$_W$-cp.propagate' cdcl$_W$-merge-restart-cases tranclpD conflicting-true-no-step-cdcl$_W$-merge-cp-no-step-s'-without-decide*)

lemma *no-step-cdcl$_W$-cp-no-step-cdcl$_W$-merge-restart*:
assumes *no-step cdcl$_W$-cp S* **and** *conflicting S = None*
shows *no-step cdcl$_W$-merge-cp S*
by (*metis assms*(*1*) *cdcl$_W$-cp.conflict' cdcl$_W$-cp.propagate' cdcl$_W$-merge-restart-cases tranclpD*)
lemma *after-cdcl$_W$-s'-without-decide-no-step-cdcl$_W$-cp*:
assumes *cdcl$_W$-s'-without-decide S T*
shows *no-step cdcl$_W$-cp T*
using *assms* **by** (*induction rule*: *cdcl$_W$-s'-without-decide.induct*) (*auto simp*: *full1-def full-def*)

lemma *no-step-cdcl$_W$-s'-without-decide-no-step-cdcl$_W$-cp*:
cdcl$_W$-all-struct-inv S $\Longrightarrow$ *no-step cdcl$_W$-s'-without-decide S* $\Longrightarrow$ *no-step cdcl$_W$-cp S*
by (*simp add*: *conflicting-true-no-step-s'-without-decide-no-step-cdcl$_W$-merge-cp no-step-cdcl$_W$-merge-cp-no-step-cdcl$_W$-cp cdcl$_W$-all-struct-inv-def*)

lemma *after-cdcl$_W$-s'-w-no-step-cdcl$_W$-cp*:
assumes *cdcl$_W$-s'-w S T* **and** *cdcl$_W$-all-struct-inv S*
shows *no-step cdcl$_W$-cp T*
using *assms*
proof (*induction rule*: *cdcl$_W$-s'-w.induct*)
case *conflict'*
then show *?case*
by (*auto simp*: *full1-def tranclp-unfold-end after-cdcl$_W$-s'-without-decide-no-step-cdcl$_W$-cp*)
next
case (*decide' S T U*)
moreover
then have *cdcl$_W$** S U*
using *rtranclp-cdcl$_W$-s'-without-decide-rtranclp-cdcl$_W$*[*of T U*] *cdcl$_W$.other*[*of S T*] *cdcl$_W$-o.decide* **unfolding** *full-def* **by** *auto*
then have *cdcl$_W$-all-struct-inv U*
using *decide'.prems rtranclp-cdcl$_W$-all-struct-inv-inv* **by** *blast*
ultimately show *?case*
using *no-step-cdcl$_W$-s'-without-decide-no-step-cdcl$_W$-cp* **unfolding** *full-def* **by** *blast*
qed

lemma *rtranclp-cdcl$_W$-s'-w-no-step-cdcl$_W$-cp-or-eq*:
assumes *cdcl$_W$-s'-w** S T* **and** *cdcl$_W$-all-struct-inv S*
shows *S = T* $\vee$ *no-step cdcl$_W$-cp T*
using *assms*
proof (*induction rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *simp*

next
  case (*step T U*)
  moreover have $cdcl_W$*-all-struct-inv T*
    using *rtranclp-*$cdcl_W$*-s'-w-rtranclp-*$cdcl_W$[*of S U*] *assms*(*2*) *rtranclp-*$cdcl_W$*-all-struct-inv-inv*
    rtranclp-$cdcl_W$*-s'-w-rtranclp-*$cdcl_W$ *step.hyps*(*1*) **by** *blast*
  ultimately show *?case* **using** *after-*$cdcl_W$*-s'-w-no-step-*$cdcl_W$*-cp* **by** *fast*
qed

lemma *rtranclp-*$cdcl_W$*-merge-stgy'-no-step-*$cdcl_W$*-cp-or-eq*:
  assumes $cdcl_W$*-merge-stgy*** *S T* **and** *inv*: $cdcl_W$*-all-struct-inv S*
  shows $S = T \vee$ *no-step* $cdcl_W$*-cp T*
  using *assms*
proof (*induction rule*: *rtranclp-induct*)
  case *base*
  then show *?case* **by** *simp*
next
  case (*step T U*)
  moreover have $cdcl_W$*-all-struct-inv T*
    using *rtranclp-*$cdcl_W$*-merge-stgy-rtranclp-*$cdcl_W$[*of S U*] *assms*(*2*) *rtranclp-*$cdcl_W$*-all-struct-inv-inv*
    rtranclp-$cdcl_W$*-s'-w-rtranclp-*$cdcl_W$ *step.hyps*(*1*)
    by (*meson rtranclp-*$cdcl_W$*-merge-stgy-rtranclp-*$cdcl_W$)
  ultimately show *?case*
    using *after-*$cdcl_W$*-s'-w-no-step-*$cdcl_W$*-cp inv* **unfolding** $cdcl_W$*-all-struct-inv-def*
    by (*metis* $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-merge-stgy.simps full1-def full-def*
      no-step-$cdcl_W$*-merge-cp-no-step-*$cdcl_W$*-cp rtranclp-*$cdcl_W$*-all-struct-inv-inv*
      rtranclp-$cdcl_W$*-merge-stgy-rtranclp-*$cdcl_W$ *tranclp.intros*(*1*) *tranclp-into-rtranclp*)
qed

lemma *no-step-*$cdcl_W$*-s'-without-decide-no-step-*$cdcl_W$*-bj*:
  assumes *no-step* $cdcl_W$*-s'-without-decide S* **and** *inv*: $cdcl_W$*-all-struct-inv S*
  shows *no-step* $cdcl_W$*-bj S*
proof (*rule ccontr*)
  assume ¬ *?thesis*
  then obtain *T* **where** *S-T*: $cdcl_W$*-bj S T*
    by *auto*
  have $cdcl_W$*-all-struct-inv T*
    using *S-T* $cdcl_W$*-all-struct-inv-inv inv other* **by** *blast*
  then obtain *T'* **where** *full1* $cdcl_W$*-bj S T'*
    using $cdcl_W$*-bj-exists-normal-form*[*of T*] *full-fullI S-T* **unfolding** $cdcl_W$*-all-struct-inv-def*
    by *metis*
  moreover
    then have $cdcl_W$** *S T'*
      using *rtranclp-mono*[*of* $cdcl_W$*-bj* $cdcl_W$] $cdcl_W$*.other* $cdcl_W$*-o.bj tranclp-into-rtranclp*[*of* $cdcl_W$*-bj*]
      unfolding *full1-def* **by** (*metis* (*full-types*) *predicate2D predicate2I*)
    then have $cdcl_W$*-all-struct-inv T'*
      using *inv rtranclp-*$cdcl_W$*-all-struct-inv-inv* **by** *blast*
    then obtain *U* **where** *full* $cdcl_W$*-cp T' U*
      using $cdcl_W$*-cp-normalized-element-all-inv* **by** *blast*
  moreover have *no-step* $cdcl_W$*-cp S*
    using *S-T* **by** (*auto simp*: $cdcl_W$*-bj.simps*)
  ultimately show *False*
  using *assms* $cdcl_W$*-s'-without-decide.intros*(*2*)[*of S T' U*] **by** *fast*
qed

lemma $cdcl_W$*-s'-w-no-step-*$cdcl_W$*-bj*:

assumes $cdcl_W$*-s′-w S T* **and** $cdcl_W$*-all-struct-inv S*
shows *no-step* $cdcl_W$*-bj T*
using *assms* **apply** *induction*
using *rtranclp-*$cdcl_W$*-s′-without-decide-rtranclp-*$cdcl_W$ *rtranclp-*$cdcl_W$*-all-struct-inv-inv*
no-step-$cdcl_W$*-s′-without-decide-no-step-*$cdcl_W$*-bj* **unfolding** *full1-def*
apply (*meson tranclp-into-rtranclp*)
using *rtranclp-*$cdcl_W$*-s′-without-decide-rtranclp-*$cdcl_W$ *rtranclp-*$cdcl_W$*-all-struct-inv-inv*
no-step-$cdcl_W$*-s′-without-decide-no-step-*$cdcl_W$*-bj* **unfolding** *full-def*
by (*meson* $cdcl_W$*-merge-restart-*$cdcl_W$ *fw-r-decide*)

lemma *rtranclp-*$cdcl_W$*-s′-w-no-step-*$cdcl_W$*-bj-or-eq*:
assumes $cdcl_W$*-s′-w*** *S T* **and** $cdcl_W$*-all-struct-inv S*
shows $S = T \lor$ *no-step* $cdcl_W$*-bj T*
using *assms* **apply** *induction*
apply *simp*
using *rtranclp-*$cdcl_W$*-s′-w-rtranclp-*$cdcl_W$ *rtranclp-*$cdcl_W$*-all-struct-inv-inv*
$cdcl_W$*-s′-w-no-step-*$cdcl_W$*-bj* **by** *meson*

lemma *rtranclp-*$cdcl_W$*-s′-no-step-*$cdcl_W$*-s′-without-decide-decomp-into-*$cdcl_W$*-merge*:
assumes
$cdcl_W$*-s′*** *R V* **and**
conflicting R = None **and**
inv: $cdcl_W$*-all-struct-inv R*
shows ($cdcl_W$*-merge-stgy*** *R V* $\land$ *conflicting V = None*)
$\lor$ ($cdcl_W$*-merge-stgy*** *R V* $\land$ *conflicting V* $\neq$ *None* $\land$ *no-step* $cdcl_W$*-bj V*)
$\lor$ ($\exists$ *S T U*. $cdcl_W$*-merge-stgy*** *R S* $\land$ *no-step* $cdcl_W$*-merge-cp S* $\land$ *decide S T*
$\land$ $cdcl_W$*-merge-cp*** *T U* $\land$ *conflict U V*)
$\lor$ ($\exists$ *S T*. $cdcl_W$*-merge-stgy*** *R S* $\land$ *no-step* $cdcl_W$*-merge-cp S* $\land$ *decide S T*
$\land$ $cdcl_W$*-merge-cp*** *T V*
$\land$ *conflicting V = None*)
$\lor$ ($cdcl_W$*-merge-cp*** *R V* $\land$ *conflicting V = None*)
$\lor$ ($\exists$ *U*. $cdcl_W$*-merge-cp*** *R U* $\land$ *conflict U V*)
using *assms*(*1*,*2*)
proof *induction*
case *base*
then show *?case* **by** *simp*
next
case (*step V W*) **note** *st = this*(*1*) **and** *s′ = this*(*2*) **and** *IH = this*(*3*)[*OF this*(*4*)] **and**
n-s-R = this(*4*)
from *s′*
show *?case*
proof *cases*
case *conflict′*
consider
(*s′*) $cdcl_W$*-merge-stgy*** *R V*
| (*dec-confl*) *S T U* **where** $cdcl_W$*-merge-stgy*** *R S* **and** *no-step* $cdcl_W$*-merge-cp S* **and**
decide S T **and** $cdcl_W$*-merge-cp*** *T U* **and** *conflict U V*
| (*dec*) *S T* **where** $cdcl_W$*-merge-stgy*** *R S* **and** *no-step* $cdcl_W$*-merge-cp S* **and** *decide S T*
and $cdcl_W$*-merge-cp*** *T V* **and** *conflicting V = None*
| (*cp*) $cdcl_W$*-merge-cp*** *R V*
| (*cp-confl*) *U* **where** $cdcl_W$*-merge-cp*** *R U* **and** *conflict U V*
using *IH* **by** *meson*
then show *?thesis*
proof *cases*
next

case *s'*
then have $R = V$
by (*metis full1-def inv local.conflict' tranclp-unfold-begin*
rtranclp-cdcl$_W$*-merge-stgy'-no-step-cdcl*$_W$*-cp-or-eq*)
consider
(*V-W*) $V = W$
| (*propa*) *propagate*$^{++}$ V W **and** *conflicting* W = *None*
| (*propa-confl*) V' **where** *propagate*** V V' **and** *conflict* V' W
using *tranclp-cdcl*$_W$*-cp-propagate-with-conflict-or-not*[*of* V W] *conflict'*
unfolding *full-unfold full1-def* **by** *meson*
then show *?thesis*
proof *cases*
case *V-W*
then show *?thesis* **using** ‹$R = V$› *n-s-R* **by** *simp*
next
case *propa*
then show *?thesis* **using** ‹$R = V$› **by** *auto*
next
case *propa-confl*
moreover
then have *cdcl*$_W$*-merge-cp*** V V'
by (*metis rtranclp-unfold cdcl*$_W$*-merge-cp.propagate' r-into-rtranclp*)
ultimately show *?thesis* **using** *s'* ‹$R = V$› **by** *blast*
qed
next
case *dec-confl* **note** - = *this*(*5*)
then have *False* **using** *conflict'* **unfolding** *full1-def* **by** (*auto dest*!: *tranclpD*)
then show *?thesis* **by** *fast*
next
case *dec* **note** *T-V* = *this*(*4*)
consider
(*propa*) *propagate*$^{++}$ V W **and** *conflicting* W = *None*
| (*propa-confl*) V' **where** *propagate*** V V' **and** *conflict* V' W
using *tranclp-cdcl*$_W$*-cp-propagate-with-conflict-or-not*[*of* V W] *conflict'*
unfolding *full1-def* **by** *meson*
then show *?thesis*
proof *cases*
case *propa*
then show *?thesis*
by (*meson T-V cdcl*$_W$*-merge-cp.propagate' dec rtranclp.rtrancl-into-rtrancl*)
next
case *propa-confl*
then have *cdcl*$_W$*-merge-cp*** T V'
using *T-V* **by** (*metis rtranclp-unfold cdcl*$_W$*-merge-cp.propagate' rtranclp.simps*)
then show *?thesis* **using** *dec propa-confl*(*2*) **by** *metis*
qed
next
case *cp*
consider
(*propa*) *propagate*$^{++}$ V W **and** *conflicting* W = *None*
| (*propa-confl*) V' **where** *propagate*** V V' **and** *conflict* V' W
using *tranclp-cdcl*$_W$*-cp-propagate-with-conflict-or-not*[*of* V W] *conflict'*
unfolding *full1-def* **by** *meson*
then show *?thesis*
proof *cases*

case *propa*
then show *?thesis* **by** (*meson* $cdcl_W$*-merge-cp.propagate'* *cp rtranclp.rtrancl-into-rtrancl*)
next
case *propa-confl*
then show *?thesis*
using *propa-confl*(*2*) **by** (*metis rtranclp-unfold* $cdcl_W$*-merge-cp.propagate'*
cp rtranclp.rtrancl-into-rtrancl)
qed
next
case *cp-confl*
then show *?thesis* **using** *conflict'* **unfolding** *full1-def* **by** (*fastforce dest*!: *tranclpD*)
qed
next
case (*decide' V'*)
then have *conf-V*: *conflicting V* = *None*
by *auto*
consider
(*s'*) $cdcl_W$*-merge-stgy*** *R V*
| (*dec-confl*) *S T U* **where** $cdcl_W$*-merge-stgy*** *R S* **and** *no-step* $cdcl_W$*-merge-cp S* **and**
decide S T **and** $cdcl_W$*-merge-cp*** *T U* **and** *conflict U V*
| (*dec*) *S T* **where** $cdcl_W$*-merge-stgy*** *R S* **and** *no-step* $cdcl_W$*-merge-cp S* **and** *decide S T*
and $cdcl_W$*-merge-cp*** *T V* **and** *conflicting V* = *None*
| (*cp*) $cdcl_W$*-merge-cp*** *R V*
| (*cp-confl*) *U* **where** $cdcl_W$*-merge-cp*** *R U* **and** *conflict U V*
using *IH* **by** *meson*
then show *?thesis*
proof *cases*
case *s'*
have *confl-V'*: *conflicting V'* = *None* **using** *decide'*(*1*) **by** *auto*
have *full*: *full1* $cdcl_W$*-cp V' W* ∨ (*V'* = *W* ∧ *no-step* $cdcl_W$*-cp W*)
using *decide'*(*3*) **unfolding** *full-unfold* **by** *blast*
consider
(*V'-W*) *V'* = *W*
| (*propa*) *propagate*$^{++}$ *V' W* **and** *conflicting W* = *None*
| (*propa-confl*) *V''* **where** *propagate*** *V' V''* **and** *conflict V'' W*
using *tranclp-*$cdcl_W$*-cp-propagate-with-conflict-or-not*[*of V W*] *decide'*
by (*metis* ‹*full1* $cdcl_W$*-cp V' W* ∨ *V'* = *W* ∧ *no-step* $cdcl_W$*-cp W*› *full1-def*
tranclp-$cdcl_W$*-cp-propagate-with-conflict-or-not*)
then show *?thesis*
proof *cases*
case *V'-W*
then show *?thesis*
using *confl-V' local.decide'*(*1*,*2*) *s' conf-V*
no-step-$cdcl_W$*-cp-no-step-*$cdcl_W$*-merge-restart*[*of V*] **by** *blast*
next
case *propa*
then show *?thesis* **using** *local.decide'*(*1*,*2*) *s'* **by** (*metis* $cdcl_W$*-merge-cp.simps conf-V*
no-step-$cdcl_W$*-cp-no-step-*$cdcl_W$*-merge-restart r-into-rtranclp*)
next
case *propa-confl*
then have $cdcl_W$*-merge-cp*** *V' V''*
by (*metis rtranclp-unfold* $cdcl_W$*-merge-cp.propagate' r-into-rtranclp*)
then show *?thesis*
using *local.decide'*(*1*,*2*) *propa-confl*(*2*) *s' conf-V*
no-step-$cdcl_W$*-cp-no-step-*$cdcl_W$*-merge-restart*

by *metis*
qed
next
case (*dec*) **note** $s' = this(1)$ **and** $dec = this(2)$ **and** $cp = this(3)$ **and** $ns\text{-}cp\text{-}T = this(4)$
have *full* $cdcl_W$*-merge-cp* T V
unfolding *full-def* **by** (*simp add*: *conf-V local.decide*$'(2)$
no-step-$cdcl_W$*-cp-no-step-*$cdcl_W$*-merge-restart ns-cp-T*)
moreover have *no-step* $cdcl_W$*-merge-cp* V
by (*simp add*: *conf-V local.decide*$'(2)$ *no-step-*$cdcl_W$*-cp-no-step-*$cdcl_W$*-merge-restart*)
moreover have *no-step* $cdcl_W$*-merge-cp* S
by (*metis dec*)
ultimately have $cdcl_W$*-merge-stgy* S V
using *cp* **by** *blast*
then have $cdcl_W$*-merge-stgy*** R V **using** s' **by** *auto*
consider
(V'-W) $V' = W$
| (*propa*) *propagate*$^{++}$ V' W **and** *conflicting* W = *None*
| (*propa-confl*) V'' **where** *propagate*** V' V'' **and** *conflict* V'' W
using *tranclp-*$cdcl_W$*-cp-propagate-with-conflict-or-not*[*of* V' W] *decide*$'$
unfolding *full-unfold full1-def* **by** *meson*
then show *?thesis*
proof *cases*
case V'-W
moreover have *conflicting* V' = *None*
using *decide*$'(1)$ **by** *auto*
ultimately show *?thesis*
using ⟨$cdcl_W$*-merge-stgy*** R V⟩ *decide*$'$ ⟨*no-step* $cdcl_W$*-merge-cp* V⟩ **by** *blast*
next
case *propa*
moreover then have $cdcl_W$*-merge-cp* V' W
by *auto*
ultimately show *?thesis*
using ⟨$cdcl_W$*-merge-stgy*** R V⟩ *decide*$'$ ⟨*no-step* $cdcl_W$*-merge-cp* V⟩
by (*meson r-into-rtranclp*)
next
case *propa-confl*
moreover then have $cdcl_W$*-merge-cp*** V' V''
by (*metis* $cdcl_W$*-merge-cp.propagate*$'$ *rtranclp-unfold tranclp-unfold-end*)
ultimately show *?thesis* **using** ⟨$cdcl_W$*-merge-stgy*** R V⟩ *decide*$'$
⟨*no-step* $cdcl_W$*-merge-cp* V⟩ **by** (*meson r-into-rtranclp*)
qed
next
case *cp*
have *no-step* $cdcl_W$*-merge-cp* V
using *conf-V local.decide*$'(2)$ *no-step-*$cdcl_W$*-cp-no-step-*$cdcl_W$*-merge-restart* **by** *blast*
then have *full* $cdcl_W$*-merge-cp* R V
unfolding *full-def* **using** *cp* **by** *fast*
then have $cdcl_W$*-merge-stgy*** R V
unfolding *full-unfold* **by** *auto*
have *full1* $cdcl_W$*-cp* V' W $\vee$ ($V' = W$ $\wedge$ *no-step* $cdcl_W$*-cp* W)
using *decide*$'(3)$ **unfolding** *full-unfold* **by** *blast*

consider
(V'-W) $V' = W$
| (*propa*) *propagate*$^{++}$ V' W **and** *conflicting* W = *None*

| (*propa-confl*) V'' **where** *propagate*** V' V'' **and** *conflict* V'' W
using *tranclp-cdcl*$_W$*-cp-propagate-with-conflict-or-not*[*of* V' W] *decide'*
unfolding *full-unfold full1-def* **by** *meson*
then show *?thesis*

proof *cases*
case V'-W
moreover have *conflicting* V' = *None*
using *decide'*(*1*) **by** *auto*
ultimately show *?thesis*
using ‹*cdcl*$_W$*-merge-stgy*** R V› *decide'* ‹*no-step cdcl*$_W$*-merge-cp* V› **by** *blast*
next
case *propa*
moreover then have *cdcl*$_W$*-merge-cp* V' W
by *auto*
ultimately show *?thesis* **using** ‹*cdcl*$_W$*-merge-stgy*** R V› *decide'*
‹*no-step cdcl*$_W$*-merge-cp* V› **by** (*meson r-into-rtranclp*)
next
case *propa-confl*
moreover then have *cdcl*$_W$*-merge-cp*** V' V''
by (*metis cdcl*$_W$*-merge-cp.propagate' rtranclp-unfold tranclp-unfold-end*)
ultimately show *?thesis* **using** ‹*cdcl*$_W$*-merge-stgy*** R V› *decide'*
‹*no-step cdcl*$_W$*-merge-cp* V› **by** (*meson r-into-rtranclp*)
qed
next
case (*dec-confl*)
show *?thesis* **using** *conf-V dec-confl*(*5*) **by** *auto*
next
case *cp-confl*
then show *?thesis* **using** *decide'* **apply** − **by** (*intro HOL.disjI2*) *fastforce*
qed
next
case (*bj'* V')
then have ¬*no-step cdcl*$_W$*-bj* V
by (*auto dest*: *tranclpD simp*: *full1-def*)
then consider
(*s'*) *cdcl*$_W$*-merge-stgy*** R V **and** *conflicting* V = *None*
| (*dec-confl*) S T U **where** *cdcl*$_W$*-merge-stgy*** R S **and** *no-step cdcl*$_W$*-merge-cp* S **and**
decide S T **and** *cdcl*$_W$*-merge-cp*** T U **and** *conflict* U V
| (*dec*) S T **where** *cdcl*$_W$*-merge-stgy*** R S **and** *no-step cdcl*$_W$*-merge-cp* S **and** *decide* S T
and *cdcl*$_W$*-merge-cp*** T V **and** *conflicting* V = *None*
| (*cp*) *cdcl*$_W$*-merge-cp*** R V **and** *conflicting* V = *None*
| (*cp-confl*) U **where** *cdcl*$_W$*-merge-cp*** R U **and** *conflict* U V
using *IH* **by** *meson*
then show *?thesis*
proof *cases*
case *s'* **note** - =*this*(*2*)
then have *False*
using *bj'*(*1*) **unfolding** *full1-def* **by** (*force dest*!: *tranclpD simp*: *cdcl*$_W$*-bj.simps*)
then show *?thesis* **by** *fast*
next
case *dec* **note** - = *this*(*5*)
then have *False*
using *bj'*(*1*) **unfolding** *full1-def* **by** (*force dest*!: *tranclpD simp*: *cdcl*$_W$*-bj.simps*)
then show *?thesis* **by** *fast*

next
  case *dec-confl*
  then have $cdcl_W$*-merge-cp* $U\ V'$
    using bj' $cdcl_W$*-merge-cp.intros*(*1*)[*of* $U\ V\ V'$] **by** (*simp add*: *full-unfold*)
  then have $cdcl_W$*-merge-cp*** $T\ V'$
    using *dec-confl*(*4*) **by** *simp*
  consider
      (V'-W) $V' = W$
    | (*propa*) *propagate*$^{++}$ $V'\ W$ **and** *conflicting* W = *None*
    | (*propa-confl*) V'' **where** *propagate*** $V'\ V''$ **and** *conflict* $V''\ W$
    using *tranclp-cdcl*$_W$*-cp-propagate-with-conflict-or-not*[*of* $V'\ W$] bj'(*3*)
    unfolding *full-unfold full1-def* **by** *meson*
  then show *?thesis*
    proof *cases*
      case V'-W
      then have *no-step* $cdcl_W$*-cp* V'
        using bj'(*3*) **unfolding** *full-def* **by** *auto*
      then have *no-step* $cdcl_W$*-merge-cp* V'
        by (*metis* $cdcl_W$*-cp.propagate'* $cdcl_W$*-merge-cp.cases tranclpD*
          no-step-cdcl$_W$*-cp-no-conflict-no-propagate*(*1*))
      then have *full1* $cdcl_W$*-merge-cp* $T\ V'$
        unfolding *full1-def* **using** ‹$cdcl_W$*-merge-cp* $U\ V'$› *dec-confl*(*4*) **by** *auto*
      then have *full* $cdcl_W$*-merge-cp* $T\ V'$
        by (*simp add*: *full-unfold*)
      then have $cdcl_W$*-merge-stgy* $S\ V'$
        using *dec-confl*(*3*) $cdcl_W$*-merge-stgy.fw-s-decide* ‹*no-step* $cdcl_W$*-merge-cp* S› **by** *blast*
      then have $cdcl_W$*-merge-stgy*** $R\ V'$
        using ‹$cdcl_W$*-merge-stgy*** $R\ S$› **by** *auto*
      show *?thesis*
        proof *cases*
          assume *conflicting* W = *None*
          then show *?thesis* **using** ‹$cdcl_W$*-merge-stgy*** $R\ V'$› ‹$V' = W$› **by** *auto*
        next
          assume *conflicting* $W \neq$ *None*
          then show *?thesis*
            using ‹$cdcl_W$*-merge-stgy*** $R\ V'$› ‹$V' = W$› **by** (*metis* ‹$cdcl_W$*-merge-cp* $U\ V'$›
              conflicting-not-true-rtranclp-cdcl$_W$*-merge-cp-no-step-cdcl*$_W$*-bj dec-confl*(*5*)
              r-into-rtranclp conflictE)
        qed
    next
      case *propa*
      moreover then have $cdcl_W$*-merge-cp* $V'\ W$
        by *auto*
      ultimately show *?thesis* **using** *decide'* **by** (*meson* ‹$cdcl_W$*-merge-cp*** $T\ V'$› *dec-confl*(*1*–*3*)
        rtranclp.rtrancl-into-rtrancl)
    next
      case *propa-confl*
      moreover then have $cdcl_W$*-merge-cp*** $V'\ V''$
        by (*metis* $cdcl_W$*-merge-cp.propagate'* *rtranclp-unfold tranclp-unfold-end*)
      ultimately show *?thesis* **by** (*meson* ‹$cdcl_W$*-merge-cp*** $T\ V'$› *dec-confl*(*1*–*3*) *rtranclp-trans*)
    qed
next
  case *cp* **note** - = *this*(*2*)
  then show *?thesis* **using** bj'(*1*) ‹¬ *no-step* $cdcl_W$*-bj* V›
    conflicting-not-true-rtranclp-cdcl$_W$*-merge-cp-no-step-cdcl*$_W$*-bj* **by** *auto*

next
  case *cp-confl*
  then have $cdcl_W$*-merge-cp* U V' **by** (*simp add*: $cdcl_W$*-merge-cp.conflict'* *full-unfold*
    local.bj'(*1*))
  consider
      (V'-W) $V' = W$
    | (*propa*) *propagate*$^{++}$ V' W **and** *conflicting* W = *None*
    | (*propa-confl*) V'' **where** *propagate*** V' V'' **and** *conflict* V'' W
    using *tranclp-*$cdcl_W$*-cp-propagate-with-conflict-or-not*[*of* V' W] *bj'*
    unfolding *full-unfold full1-def* **by** *meson*
  then show *?thesis*

    proof *cases*
      case V'-W
      show *?thesis*
        proof *cases*
          assume *conflicting* V' = *None*
          then show *?thesis*
            using V'-W ‹$cdcl_W$*-merge-cp* U V'› *cp-confl*(*1*) **by** *force*
        next
          assume *confl*: *conflicting* $V' \neq$ *None*
          then have *no-step* $cdcl_W$*-merge-stgy* V'
            by (*fastforce simp*: $cdcl_W$*-merge-stgy.simps full1-def full-def*
              $cdcl_W$*-merge-cp.simps dest*!: *tranclpD*)
          have *no-step* $cdcl_W$*-merge-cp* V'
            using *confl* **by** (*auto simp*: *full1-def full-def* $cdcl_W$*-merge-cp.simps*
            dest!: *tranclpD*)
          moreover have $cdcl_W$*-merge-cp* U W
            using V'-W ‹$cdcl_W$*-merge-cp* U V'› **by** *blast*
          ultimately have *full1* $cdcl_W$*-merge-cp* R V'
            using *cp-confl*(*1*) V'-W **unfolding** *full1-def* **by** *auto*
          then have $cdcl_W$*-merge-stgy* R V'
            by *auto*
          moreover have *no-step* $cdcl_W$*-merge-stgy* V'
            using *confl* ‹*no-step* $cdcl_W$*-merge-cp* V'› **by** (*auto simp*: $cdcl_W$*-merge-stgy.simps*
              full1-def dest!: *tranclpD*)
          ultimately have $cdcl_W$*-merge-stgy*** R V' **by** *auto*
          show *?thesis* **by** (*metis* V'-W ‹$cdcl_W$*-merge-cp* U V'› ‹$cdcl_W$*-merge-stgy*** R V'›
            conflicting-not-true-rtranclp-$cdcl_W$*-merge-cp-no-step-*$cdcl_W$*-bj cp-confl*(*1*)
            rtranclp.rtrancl-into-rtrancl step.prems)
        qed
    next
      case *propa*
      moreover then have $cdcl_W$*-merge-cp* V' W
        by *auto*
      ultimately show *?thesis* **using** ‹$cdcl_W$*-merge-cp* U V'› *cp-confl*(*1*) **by** *force*
    next
      case *propa-confl*
      moreover then have $cdcl_W$*-merge-cp*** V' V''
        by (*metis* $cdcl_W$*-merge-cp.propagate'* *rtranclp-unfold tranclp-unfold-end*)
      ultimately show *?thesis*
        using ‹$cdcl_W$*-merge-cp* U V'› *cp-confl*(*1*) **by** (*metis rtranclp.rtrancl-into-rtrancl*
          rtranclp-trans)
    qed
qed

```
    qed
qed

lemma decide-rtranclp-cdcl_W-s'-rtranclp-cdcl_W-s':
  assumes
    dec: decide S T and
    cdcl_W-s'** T U and
    n-s-S: no-step cdcl_W-cp S and
    no-step cdcl_W-cp U
  shows cdcl_W-s'** S U
  using assms(2,4)
proof induction
  case (step U V) note st =this(1) and s' = this(2) and IH = this(3) and n-s = this(4)
  consider
      (TU) T = U
    | (s'-st) T' where cdcl_W-s' T T' and cdcl_W-s'** T' U
    using st[unfolded rtranclp-unfold] by (auto dest!: tranclpD)
  then show ?case
    proof cases
      case TU
      then show ?thesis
        proof -
          assume a1: T = U
          then have f2: cdcl_W-s' T V
            using s' by force
          obtain ss :: 'st where
            cdcl_W-s'** S T ∨ cdcl_W-cp T ss
            using a1 step.IH by blast
          then show ?thesis
            using f2 by (metis (full-types) cdcl_W-s'.decide' cdcl_W-s'E dec full1-is-full n-s-S
              rtranclp-unfold tranclp-unfold-end)
        qed
    next
      case (s'-st T') note s'-T' = this(1) and st = this(2)
      have cdcl_W-s'** S T'
        using s'-T'
        proof cases
          case conflict'
          then have cdcl_W-s' S T'
             using dec cdcl_W-s'.decide' n-s-S by (simp add: full-unfold)
          then show ?thesis
             using st by auto
        next
          case (decide' T'')
          then have cdcl_W-s' S T
             using dec cdcl_W-s'.decide' n-s-S by (simp add: full-unfold)
          then show ?thesis using decide' s'-T' by auto
        next
          case bj'
          then have False
            using dec unfolding full1-def by (fastforce dest!: tranclpD simp: cdcl_W-bj.simps)
          then show ?thesis by fast
        qed
      then show ?thesis using s' st by auto
    qed
```

next
  case *base*
  then have *full* $cdcl_W$*-cp T T*
    by (*simp add*: *full-unfold*)
  then show *?case*
    using $cdcl_W$*-s'.simps dec n-s-S* **by** *auto*
qed

lemma *rtranclp-*$cdcl_W$*-merge-stgy-rtranclp-*$cdcl_W$*-s'*:
  assumes
    $cdcl_W$*-merge-stgy*** *R V* **and**
    inv: $cdcl_W$*-all-struct-inv R*
  shows $cdcl_W$*-s'*** *R V*
  using *assms*(*1*)
proof *induction*
  case *base*
  then show *?case* **by** *simp*
next
  case (*step S T*) **note** *st* = *this*(*1*) **and** *fw* = *this*(*2*) **and** *IH* = *this*(*3*)
  have $cdcl_W$*-all-struct-inv S*
    using *inv rtranclp-*$cdcl_W$*-all-struct-inv-inv rtranclp-*$cdcl_W$*-merge-stgy-rtranclp-*$cdcl_W$ *st* **by** *blast*
  from *fw* **show** *?case*
    proof (*cases rule*: $cdcl_W$*-merge-stgy-cases*)
      case *fw-s-cp*
      then show *?thesis*
        proof –
          assume *a1*: *full1* $cdcl_W$*-merge-cp S T*
          obtain *ss* :: $('st \Rightarrow 'st \Rightarrow bool) \Rightarrow 'st \Rightarrow 'st$ **where**
            f2: $\bigwedge p\ s\ sa\ pa\ sb\ sc\ sd\ pb\ se\ sf.\ (\neg\ full1\ p\ (s::'st)\ sa \vee p^{++}\ s\ sa)$
              $\wedge\ (\neg\ pa\ (sb::'st)\ sc \vee \neg\ full1\ pa\ sd\ sb) \wedge (\neg\ pb^{++}\ se\ sf \vee pb\ sf\ (ss\ pb\ sf)$
              $\vee\ full1\ pb\ se\ sf)$
            by (*metis* (*no-types*) *full1-def*)
          then have *f3*: $cdcl_W$*-merge-cp*$^{++}$ *S T*
            using *a1* **by** *auto*
          obtain *ssa* :: $('st \Rightarrow 'st \Rightarrow bool) \Rightarrow 'st \Rightarrow 'st \Rightarrow 'st$ **where**
            f4: $\bigwedge p\ s\ sa.\ \neg\ p^{++}\ s\ sa \vee p\ s\ (ssa\ p\ s\ sa)$
            by (*meson tranclp-unfold-begin*)
          then have *f5*: $\bigwedge s.\ \neg$ *full1* $cdcl_W$*-merge-cp s S*
            using *f3 f2* **by** (*metis* (*full-types*))
          have $\bigwedge s.\ \neg$ *full* $cdcl_W$*-merge-cp s S*
            using *f4 f3* **by** (*meson full-def*)
          then have *S* = *R*
            using *f5* **by** (*metis* (*no-types*) $cdcl_W$*-merge-stgy.simps rtranclp-unfold st*
              tranclp-unfold-end)
          then show *?thesis*
            using *f2 a1* **by** (*metis* (*no-types*) ‹$cdcl_W$*-all-struct-inv S*›
              conflicting-true-full1-$cdcl_W$*-merge-cp-imp-full1-*$cdcl_W$*-s'-without-decode*
              rtranclp-$cdcl_W$*-s'-without-decide-rtranclp-*$cdcl_W$*-s' rtranclp-unfold*)
        qed
    next
      case (*fw-s-decide S'*) **note** *dec* = *this*(*1*) **and** *n-S* = *this*(*2*) **and** *full* = *this*(*3*)
      moreover then have *conflicting S'* = *None*
        by *auto*
      ultimately have *full* $cdcl_W$*-s'-without-decide S' T*
        by (*meson* ‹$cdcl_W$*-all-struct-inv S*› $cdcl_W$*-merge-restart-*$cdcl_W$ *fw-r-decide*

rtranclp-cdcl$_W$-all-struct-inv-inv
conflicting-true-full-cdcl$_W$-merge-cp-iff-full-cdcl$_W$-s′-without-decode)
then have *a1*: *cdcl$_W$-s′** S′ T*
unfolding *full-def* **by** (*metis* (*full-types*)*rtranclp-cdcl$_W$-s′-without-decide-rtranclp-cdcl$_W$-s′*)
have *cdcl$_W$-merge-stgy** S T*
using *fw* **by** *blast*
then have *cdcl$_W$-s′** S T*
using *decide-rtranclp-cdcl$_W$-s′-rtranclp-cdcl$_W$-s′ a1* **by** (*metis* ‹*cdcl$_W$-all-struct-inv S*› *dec*
n-S no-step-cdcl$_W$-merge-cp-no-step-cdcl$_W$-cp cdcl$_W$-all-struct-inv-def
rtranclp-cdcl$_W$-merge-stgy′-no-step-cdcl$_W$-cp-or-eq)
then show *?thesis* **using** *IH* **by** *auto*
qed
qed

lemma *rtranclp-cdcl$_W$-merge-stgy-distinct-mset-clauses*:
assumes *invR*: *cdcl$_W$-all-struct-inv R* **and**
st: *cdcl$_W$-merge-stgy** R S* **and**
dist: *distinct-mset* (*clauses R*) **and**
R: *trail R* = []
shows *distinct-mset* (*clauses S*)
using *rtranclp-cdcl$_W$-stgy-distinct-mset-clauses*[*OF invR - dist R*]
invR st rtranclp-mono[*of cdcl$_W$-s′ cdcl$_W$-stgy***] *cdcl$_W$-s′-is-rtranclp-cdcl$_W$-stgy*
by (*auto dest*!: *cdcl$_W$-s′-is-rtranclp-cdcl$_W$-stgy rtranclp-cdcl$_W$-merge-stgy-rtranclp-cdcl$_W$-s′*)

lemma *no-step-cdcl$_W$-s′-no-step-cdcl$_W$-merge-stgy*:
assumes
inv: *cdcl$_W$-all-struct-inv R* **and** *s′*: *no-step cdcl$_W$-s′ R*
shows *no-step cdcl$_W$-merge-stgy R*
proof −
{ fix *ss* :: *′st*
obtain *ssa* :: *′st* ⇒ *′st* ⇒ *′st* **where**
ff1: $\bigwedge$*s sa*. ¬ *cdcl$_W$-merge-stgy s sa* ∨ *full1 cdcl$_W$-merge-cp s sa* ∨ *decide s* (*ssa s sa*)
using *cdcl$_W$-merge-stgy.cases* **by** *moura*
obtain *ssb* :: (*′st* ⇒ *′st* ⇒ *bool*) ⇒ *′st* ⇒ *′st* ⇒ *′st* **where**
ff2: $\bigwedge$*p s sa*. ¬ *p^{++} s sa* ∨ *p s* (*ssb p s sa*)
by (*meson tranclp-unfold-begin*)
obtain *ssc* :: *′st* ⇒ *′st* **where**
ff3: $\bigwedge$*s sa sb*. (¬ *cdcl$_W$-all-struct-inv s* ∨ ¬ *cdcl$_W$-cp s sa* ∨ *cdcl$_W$-s′ s* (*ssc s*))
∧ (¬ *cdcl$_W$-all-struct-inv s* ∨ ¬ *cdcl$_W$-o s sb* ∨ *cdcl$_W$-s′ s* (*ssc s*))
using *n-step-cdcl$_W$-stgy-iff-no-step-cdcl$_W$-cl-cdcl$_W$-o* **by** *moura*
then have *ff4*: $\bigwedge$*s*. ¬ *cdcl$_W$-o R s*
using *s′ inv* **by** *blast*
have *ff5*: $\bigwedge$*s*. ¬ *cdcl$_W$-cp^{++} R s*
using *ff3 ff2 s′* **by** (*metis inv*)
have $\bigwedge$*s*. ¬ *cdcl$_W$-bj^{++} R s*
using *ff4 ff2* **by** (*metis bj*)
then have $\bigwedge$*s*. ¬ *cdcl$_W$-s′-without-decide R s*
using *ff5* **by** (*simp add*: *cdcl$_W$-s′-without-decide.simps full1-def*)
then have ¬ *cdcl$_W$-s′-without-decide^{++} R ss*
using *ff2* **by** *blast*
then have ¬ *cdcl$_W$-merge-stgy R ss*
using *ff4 ff1* **by** (*metis* (*full-types*) *decide full1-def inv*
conflicting-true-full1-cdcl$_W$-merge-cp-imp-full1-cdcl$_W$-s′-without-decode) **}**
then show *?thesis*
by *fastforce*

qed

lemma *wf-cdcl$_W$-merge-cp*:
  wf{(*T*, *S*). *cdcl$_W$-all-struct-inv S* ∧ *cdcl$_W$-merge-cp S T*}
  using *wf-tranclp-cdcl$_W$-merge* **by** (*rule wf-subset*) (*auto simp*: *cdcl$_W$-merge-cp-tranclp-cdcl$_W$-merge*)

lemma *wf-cdcl$_W$-merge-stgy*:
  wf{(*T*, *S*). *cdcl$_W$-all-struct-inv S* ∧ *cdcl$_W$-merge-stgy S T*}
  using *wf-tranclp-cdcl$_W$-merge* **by** (*rule wf-subset*)
  (*auto simp add*: *cdcl$_W$-merge-stgy-tranclp-cdcl$_W$-merge*)

lemma *cdcl$_W$-merge-cp-obtain-normal-form*:
  assumes *inv*: *cdcl$_W$-all-struct-inv R*
  obtains *S* **where** *full cdcl$_W$-merge-cp R S*
proof −
  obtain *S* **where** *full* (λ*S T*. *cdcl$_W$-all-struct-inv S* ∧ *cdcl$_W$-merge-cp S T*) *R S*
    using *wf-exists-normal-form-full*[*OF wf-cdcl$_W$-merge-cp*] **by** *blast*
  then have
    st: (λ*S T*. *cdcl$_W$-all-struct-inv S* ∧ *cdcl$_W$-merge-cp S T*)** *R S* **and**
    n-s: *no-step* (λ*S T*. *cdcl$_W$-all-struct-inv S* ∧ *cdcl$_W$-merge-cp S T*) *S*
    unfolding *full-def* **by** *blast*+
  have *cdcl$_W$-merge-cp*** *R S*
    using *st* **by** *induction auto*
  moreover
    have *cdcl$_W$-all-struct-inv S*
      using *st inv*
      apply (*induction rule*: *rtranclp-induct*)
        apply *simp*
      by (*meson r-into-rtranclp rtranclp-cdcl$_W$-all-struct-inv-inv*
        rtranclp-cdcl$_W$-merge-cp-rtranclp-cdcl$_W$)
    then have *no-step cdcl$_W$-merge-cp S*
      using *n-s* **by** *auto*
  ultimately show *?thesis*
    using *that* **unfolding** *full-def* **by** *blast*
qed

lemma *no-step-cdcl$_W$-merge-stgy-no-step-cdcl$_W$-s′*:
  assumes
    inv: *cdcl$_W$-all-struct-inv R* **and**
    confl: *conflicting R* = *None* **and**
    n-s: *no-step cdcl$_W$-merge-stgy R*
  shows *no-step cdcl$_W$-s′ R*
proof (*rule ccontr*)
  assume ¬ *?thesis*
  then obtain *S* **where** *cdcl$_W$-s′ R S* **by** *auto*
  then show *False*
    proof *cases*
      case *conflict′*
      then obtain *S′* **where** *full1 cdcl$_W$-merge-cp R S′*
        by (*metis* (*full-types*) *cdcl$_W$-merge-cp-obtain-normal-form cdcl$_W$-s′-without-decide.simps confl*
          conflicting-true-no-step-cdcl$_W$-merge-cp-no-step-s′-without-decide full-def full-unfold inv
          cdcl$_W$-all-struct-inv-def)
      then show *False* **using** *n-s* **by** *blast*
    next
      case (*decide′ R′*)

then have *cdcl_W-all-struct-inv R'*
using *inv cdcl_W-all-struct-inv-inv cdcl_W.other cdcl_W-o.decide* **by** *meson*
then obtain *R''* **where** *full cdcl_W-merge-cp R' R''*
using *cdcl_W-merge-cp-obtain-normal-form* **by** *blast*
moreover have *no-step cdcl_W-merge-cp R*
by (*simp add*: *confl local.decide'*(*2*) *no-step-cdcl_W-cp-no-step-cdcl_W-merge-restart*)
ultimately show *False* **using** *n-s cdcl_W-merge-stgy.intros local.decide'*(*1*) **by** *blast*
next
case (*bj' R'*)
then show *False*
using *confl no-step-cdcl_W-cp-no-step-cdcl_W-s'-without-decide inv*
unfolding *cdcl_W-all-struct-inv-def* **by** *blast*
qed
qed

lemma *rtranclp-cdcl_W-merge-cp-no-step-cdcl_W-bj*:
assumes *conflicting R* = *None* **and** *cdcl_W-merge-cp*** *R S*
shows *no-step cdcl_W-bj S*
using *assms conflicting-not-true-rtranclp-cdcl_W-merge-cp-no-step-cdcl_W-bj* **by** *blast*

lemma *rtranclp-cdcl_W-merge-stgy-no-step-cdcl_W-bj*:
assumes *confl*: *conflicting R* = *None* **and** *cdcl_W-merge-stgy*** *R S*
shows *no-step cdcl_W-bj S*
using *assms*(*2*)
proof *induction*
case *base*
then show *?case*
using *confl* **by** (*auto simp*: *cdcl_W-bj.simps*)[]
next
case (*step S T*) **note** *st* = *this*(*1*) **and** *fw* = *this*(*2*) **and** *IH* = *this*(*3*)
have *confl-S*: *conflicting S* = *None*
using *fw* **apply** *cases*
by (*auto simp*: *full1-def cdcl_W-merge-cp.simps dest*!: *tranclpD*)
from *fw* **show** *?case*
proof *cases*
case *fw-s-cp*
then show *?thesis*
using *rtranclp-cdcl_W-merge-cp-no-step-cdcl_W-bj confl-S*
by (*simp add*: *full1-def tranclp-into-rtranclp*)
next
case (*fw-s-decide S'*)
moreover then have *conflicting S'* = *None* **by** *auto*
ultimately show *?thesis*
using *conflicting-not-true-rtranclp-cdcl_W-merge-cp-no-step-cdcl_W-bj*
unfolding *full-def* **by** *meson*
qed
qed

lemma *full-cdcl_W-s'-full-cdcl_W-merge-restart*:
assumes
conflicting R = *None* **and**
inv: *cdcl_W-all-struct-inv R*
shows *full cdcl_W-s' R V* ⟷ *full cdcl_W-merge-stgy R V* (**is** *?s'* ⟷ *?fw*)
proof
assume *?s'*

then have $cdcl_W$*-s*$'^{**}$ *R V* **unfolding** *full-def* **by** *blast*
have $cdcl_W$*-all-struct-inv V*
  using ‹$cdcl_W$*-s*$'^{**}$ *R V*› *inv rtranclp-*$cdcl_W$*-all-struct-inv-inv rtranclp-*$cdcl_W$*-s′-rtranclp-*$cdcl_W$
  by *blast*
then have *n-s*: *no-step* $cdcl_W$*-merge-stgy V*
  using *no-step-*$cdcl_W$*-s′-no-step-*$cdcl_W$*-merge-stgy* **by** (*meson* ‹*full* $cdcl_W$*-s′ R V*› *full-def*)
have *n-s-bj*: *no-step* $cdcl_W$*-bj V*
  by (*metis* ‹$cdcl_W$*-all-struct-inv V*› ‹*full* $cdcl_W$*-s′ R V*› *bj full-def*
    n-step-$cdcl_W$*-stgy-iff-no-step-*$cdcl_W$*-cl-*$cdcl_W$*-o*)
have *n-s-cp*: *no-step* $cdcl_W$*-merge-cp V*
  proof −
    { fix *ss* :: *′st*
      obtain *ssa* :: *′st* ⇒ *′st* **where**
        ff1: ∀ *s*. ¬ $cdcl_W$*-all-struct-inv s* ∨ $cdcl_W$*-s′-without-decide s* (*ssa s*)
          ∨ *no-step* $cdcl_W$*-merge-cp s*
        using *conflicting-true-no-step-s′-without-decide-no-step-*$cdcl_W$*-merge-cp* **by** *moura*
      have (∀ *p s sa*. ¬ *full p* (*s*::*′st*) *sa* ∨ p^{**} *s sa* ∧ *no-step p sa*) **and**
        (∀ *p s sa*. (¬ p^{**} (*s*::*′st*) *sa* ∨ (∃ *s*. *p sa s*)) ∨ *full p s sa*)
        by (*meson full-def*)+
      then have ¬ $cdcl_W$*-merge-cp V ss*
        using *ff1* **by** (*metis* (*no-types*) ‹$cdcl_W$*-all-struct-inv V*› ‹*full* $cdcl_W$*-s′ R V*› $cdcl_W$*-s′.simps*
          $cdcl_W$*-s′-without-decide.cases*) **}**
    then show *?thesis*
      by *blast*
  qed
consider
    (*fw-no-confl*) $cdcl_W$*-merge-stgy*** *R V* **and** *conflicting V* = *None*
  | (*fw-confl*) $cdcl_W$*-merge-stgy*** *R V* **and** *conflicting V* ≠ *None* **and** *no-step* $cdcl_W$*-bj V*
  | (*fw-dec-confl*) *S T U* **where** $cdcl_W$*-merge-stgy*** *R S* **and** *no-step* $cdcl_W$*-merge-cp S* **and**
     decide S T **and** $cdcl_W$*-merge-cp*** *T U* **and** *conflict U V*
  | (*fw-dec-no-confl*) *S T* **where** $cdcl_W$*-merge-stgy*** *R S* **and** *no-step* $cdcl_W$*-merge-cp S* **and**
     decide S T **and** $cdcl_W$*-merge-cp*** *T V* **and** *conflicting V* = *None*
  | (*cp-no-confl*) $cdcl_W$*-merge-cp*** *R V* **and** *conflicting V* = *None*
  | (*cp-confl*) *U* **where** $cdcl_W$*-merge-cp*** *R U* **and** *conflict U V*
  using *rtranclp-*$cdcl_W$*-s′-no-step-*$cdcl_W$*-s′-without-decide-decomp-into-*$cdcl_W$*-merge*[*OF*
   ‹$cdcl_W$*-s*$'^{**}$ *R V*› *assms*] **by** *auto*
then show *?fw*
  proof *cases*
    case *fw-no-confl*
    then show *?thesis* **using** *n-s* **unfolding** *full-def* **by** *blast*
  next
    case *fw-confl*
    then show *?thesis* **using** *n-s* **unfolding** *full-def* **by** *blast*
  next
    case *fw-dec-confl*
    have $cdcl_W$*-merge-cp U V*
      using *n-s-bj* **by** (*metis* $cdcl_W$*-merge-cp.simps full-unfold fw-dec-confl*(*5*))
    then have *full1* $cdcl_W$*-merge-cp T V*
      unfolding *full1-def* **by** (*metis fw-dec-confl*(*4*) *n-s-cp tranclp-unfold-end*)
    then have $cdcl_W$*-merge-stgy S V* **using** ‹*decide S T*› ‹*no-step* $cdcl_W$*-merge-cp S*› **by** *auto*
    then show *?thesis* **using** *n-s* ‹ $cdcl_W$*-merge-stgy*** *R S*› **unfolding** *full-def* **by** *auto*
  next
    case *fw-dec-no-confl*
    then have *full* $cdcl_W$*-merge-cp T V*
      using *n-s-cp* **unfolding** *full-def* **by** *blast*

    then have *cdcl*$_W$*-merge-stgy S V* **using** ‹*decide S T*› ‹*no-step cdcl*$_W$*-merge-cp S*› **by** *auto*
    then show *?thesis* **using** *n-s* ‹ *cdcl*$_W$*-merge-stgy*** *R S*› **unfolding** *full-def* **by** *auto*
  next
    case *cp-no-confl*
    then have *full cdcl*$_W$*-merge-cp R V*
      by (*simp add*: *full-def n-s-cp*)
    then have $R = V \lor$ *cdcl*$_W$*-merge-stgy*$^{++}$ *R V*
      by (*metis* (*no-types*) *full-unfold fw-s-cp rtranclp-unfold tranclp-unfold-end*)
    then show *?thesis*
      by (*simp add*: *full-def n-s rtranclp-unfold*)
  next
    case *cp-confl*
    have *full cdcl*$_W$*-bj V V*
      using *n-s-bj* **unfolding** *full-def* **by** *blast*
    then have *full1 cdcl*$_W$*-merge-cp R V*
      unfolding *full1-def* **by** (*meson cdcl*$_W$*-merge-cp.conflict*$'$ *cp-confl*(*1*,*2*) *n-s-cp*
        rtranclp-into-tranclp1)
    then show *?thesis* **using** *n-s* **unfolding** *full-def* **by** *auto*
  qed
next
  assume *?fw*
  then have *cdcl*$_W$** *R V* **using** *rtranclp-mono*[*of cdcl*$_W$*-merge-stgy cdcl*$_W$**]
    cdcl$_W$*-merge-stgy-rtranclp-cdcl*$_W$ **unfolding** *full-def* **by** *auto*
  then have *inv*$'$: *cdcl*$_W$*-all-struct-inv V* **using** *inv rtranclp-cdcl*$_W$*-all-struct-inv-inv* **by** *blast*
  have *cdcl*$_W$*-s*$'^{**}$ *R V*
    using ‹*?fw*› **by** (*simp add*: *full-def inv rtranclp-cdcl*$_W$*-merge-stgy-rtranclp-cdcl*$_W$*-s*$'$)
  moreover have *no-step cdcl*$_W$*-s*$'$ *V*
    proof *cases*
      assume *conflicting V* = *None*
      then show *?thesis*
        by (*metis inv*$'$ ‹*full cdcl*$_W$*-merge-stgy R V*› *full-def*
          no-step-cdcl$_W$*-merge-stgy-no-step-cdcl*$_W$*-s*$'$)
    next
      assume *confl-V*: *conflicting V* $\neq$ *None*
      then have *no-step cdcl*$_W$*-bj V*
      using *rtranclp-cdcl*$_W$*-merge-stgy-no-step-cdcl*$_W$*-bj* **by** (*meson* ‹*full cdcl*$_W$*-merge-stgy R V*›
        assms(*1*) *full-def*)
      then show *?thesis* **using** *confl-V* **by** (*fastforce simp*: *cdcl*$_W$*-s*$'$*.simps full1-def cdcl*$_W$*-cp.simps*
        dest!: *tranclpD*)
    qed
  ultimately show *?s*$'$ **unfolding** *full-def* **by** *blast*
qed

lemma *full-cdcl*$_W$*-stgy-full-cdcl*$_W$*-merge*:
  assumes
    conflicting R = *None* **and**
    inv: *cdcl*$_W$*-all-struct-inv R*
  shows *full cdcl*$_W$*-stgy R V* $\longleftrightarrow$ *full cdcl*$_W$*-merge-stgy R V*
  by (*simp add*: *assms*(*1*) *full-cdcl*$_W$*-s*$'$*-full-cdcl*$_W$*-merge-restart full-cdcl*$_W$*-stgy-iff-full-cdcl*$_W$*-s*$'$
    inv)

lemma *full-cdcl*$_W$*-merge-stgy-final-state-conclusive*$'$:
  fixes S' :: $'$*st*
  assumes *full*: *full cdcl*$_W$*-merge-stgy* (*init-state N*) S'
  and *no-d*: *distinct-mset-mset N*

shows (*conflicting* S' = *Some* {#} ∧ *unsatisfiable* (*set-mset* N))
∨ (*conflicting* S' = *None* ∧ *trail* S' ⊨*asm* N ∧ *satisfiable* (*set-mset* N))
proof −
have $cdcl_W$*-all-struct-inv* (*init-state* N)
using *no-d* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *auto*
moreover have *conflicting* (*init-state* N) = *None*
by *auto*
ultimately show *?thesis*
by (*simp add*: *full full-*$cdcl_W$*-stgy-final-state-conclusive-from-init-state*
full-$cdcl_W$*-stgy-full-*$cdcl_W$*-merge no-d*)
qed

end

19.6 Adding Restarts

locale $cdcl_W$*-ops-restart* =
$cdcl_W$*-ops trail init-clss learned-clss backtrack-lvl conflicting cons-trail tl-trail*
add-init-cls
add-learned-cls remove-cls update-backtrack-lvl update-conflicting init-state
restart-state
for
trail :: *'st* ⇒ (*'v*::*linorder*, *nat*, *'v clause*) *marked-lits* **and**
init-clss :: *'st* ⇒ *'v clauses* **and**
learned-clss :: *'st* ⇒ *'v clauses* **and**
backtrack-lvl :: *'st* ⇒ *nat* **and**
conflicting :: *'st* ⇒*'v clause option* **and**

cons-trail :: (*'v*, *nat*, *'v clause*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-init-cls :: *'v clause* ⇒ *'st* ⇒ *'st* **and**
add-learned-cls remove-cls :: *'v clause* ⇒ *'st* ⇒ *'st* **and**
update-backtrack-lvl :: *nat* ⇒ *'st* ⇒ *'st* **and**
update-conflicting :: *'v clause option* ⇒ *'st* ⇒ *'st* **and**

init-state :: *'v*::*linorder clauses* ⇒ *'st* **and**
restart-state :: *'st* ⇒ *'st* +
fixes f :: *nat* ⇒ *nat*
assumes f: *unbounded* f
begin

The condition of the differences of cardinality has to be strict. Otherwise, you could be in a strange state, where nothing remains to do, but a restart is done. See the proof of well-foundedness.

inductive $cdcl_W$*-merge-with-restart* **where**
restart-step:
($cdcl_W$*-merge-stgy*^^(*card* (*set-mset* (*learned-clss* T)) − *card* (*set-mset* (*learned-clss* S)))) S T
⟹ *card* (*set-mset* (*learned-clss* T)) − *card* (*set-mset* (*learned-clss* S)) > f n
⟹ *restart* T U ⟹ $cdcl_W$*-merge-with-restart* (S, n) (U, *Suc* n) |
restart-full: *full1* $cdcl_W$*-merge-stgy* S T ⟹ $cdcl_W$*-merge-with-restart* (S, n) (T, *Suc* n)

lemma $cdcl_W$*-merge-with-restart* S T ⟹ $cdcl_W$*-merge-restart*** (*fst* S) (*fst* T)
by (*induction rule*: $cdcl_W$*-merge-with-restart.induct*)
(*auto dest*!: *relpowp-imp-rtranclp* $cdcl_W$*-merge-stgy-tranclp-*$cdcl_W$*-merge tranclp-into-rtranclp*
rtranclp-$cdcl_W$*-merge-stgy-rtranclp-*$cdcl_W$*-merge rtranclp-*$cdcl_W$*-merge-tranclp-*$cdcl_W$*-merge-restart*

fw-r-rf $cdcl_W$*-rf.restart*
simp: *full1-def*)

lemma $cdcl_W$*-merge-with-restart-rtranclp-*$cdcl_W$:
$cdcl_W$*-merge-with-restart S T* $\Longrightarrow$ $cdcl_W{}^{**}$ (*fst S*) (*fst T*)
by (*induction rule*: $cdcl_W$*-merge-with-restart.induct*)
(*auto dest*!: *relpowp-imp-rtranclp rtranclp-*$cdcl_W$*-merge-stgy-rtranclp-*$cdcl_W$ $cdcl_W$*.rf*
$cdcl_W$*-rf.restart tranclp-into-rtranclp simp*: *full1-def*)

lemma $cdcl_W$*-merge-with-restart-increasing-number*:
$cdcl_W$*-merge-with-restart S T* $\Longrightarrow$ *snd T* = *1* + *snd S*
by (*induction rule*: $cdcl_W$*-merge-with-restart.induct*) *auto*

lemma *full1* $cdcl_W$*-merge-stgy S T* $\Longrightarrow$ $cdcl_W$*-merge-with-restart* (*S*, *n*) (*T*, *Suc n*)
using *restart-full* **by** *blast*

lemma $cdcl_W$*-all-struct-inv-learned-clss-bound*:
assumes *inv*: $cdcl_W$*-all-struct-inv S*
shows *set-mset* (*learned-clss S*) $\subseteq$ *build-all-simple-clss* (*atms-of-msu* (*init-clss S*))
proof
fix *C*
assume *C*: *C* $\in$ *set-mset* (*learned-clss S*)
have *distinct-mset C*
using *C inv* **unfolding** $cdcl_W$*-all-struct-inv-def distinct-*$cdcl_W$*-state-def distinct-mset-set-def*
by *auto*
moreover have $\neg$*tautology C*
using *C inv* **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-learned-clause-def* **by** *auto*
moreover
have *atms-of C* $\subseteq$ *atms-of-msu* (*learned-clss S*)
using *C* **by** *auto*
then have *atms-of C* $\subseteq$ *atms-of-msu* (*init-clss S*)
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def no-strange-atm-def* **by** *force*
moreover have *finite* (*atms-of-msu* (*init-clss S*))
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *auto*
ultimately show *C* $\in$ *build-all-simple-clss* (*atms-of-msu* (*init-clss S*))
using *distinct-mset-not-tautology-implies-in-build-all-simple-clss build-all-simple-clss-mono*
by *blast*
qed

lemma $cdcl_W$*-merge-with-restart-init-clss*:
$cdcl_W$*-merge-with-restart S T* $\Longrightarrow$ $cdcl_W$*-M-level-inv* (*fst S*) $\Longrightarrow$
init-clss (*fst S*) = *init-clss* (*fst T*)
using $cdcl_W$*-merge-with-restart-rtranclp-*$cdcl_W$ *rtranclp-*$cdcl_W$*-init-clss* **by** *blast*

lemma
wf {(*T*, *S*). $cdcl_W$*-all-struct-inv* (*fst S*) $\wedge$ $cdcl_W$*-merge-with-restart S T*}
proof (*rule ccontr*)
assume $\neg$ *?thesis*
then obtain *g* **where**
g: $\bigwedge$*i*. $cdcl_W$*-merge-with-restart* (*g i*) (*g* (*Suc i*)) **and**
inv: $\bigwedge$*i*. $cdcl_W$*-all-struct-inv* (*fst* (*g i*))
unfolding *wf-iff-no-infinite-down-chain* **by** *fast*
{ fix *i*
have *init-clss* (*fst* (*g i*)) = *init-clss* (*fst* (*g 0*))
apply (*induction i*)

apply *simp*
using *g inv* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** (*metis* $cdcl_W$*-merge-with-restart-init-clss*)
} **note** *init-g* = *this*
let *?S* = *g 0*
have *finite* (*atms-of-msu* (*init-clss* (*fst ?S*)))
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *auto*
have *snd-g*: $\bigwedge$*i. snd* (*g i*) = *i* + *snd* (*g 0*)
apply (*induct-tac i*)
apply *simp*
by (*metis Suc-eq-plus1-left add-Suc* $cdcl_W$*-merge-with-restart-increasing-number g*)
then have *snd-g-0*: $\bigwedge$*i. i* > *0* $\Longrightarrow$ *snd* (*g i*) = *i* + *snd* (*g 0*)
by *blast*
have *unbounded-f-g*: *unbounded* (λ*i. f* (*snd* (*g i*)))
using *f* **unfolding** *bounded-def* **by** (*metis add.commute f less-or-eq-imp-le snd-g*
not-bounded-nat-exists-larger not-le ordered-cancel-comm-monoid-diff-class.le-iff-add)

obtain *k* **where**
f-g-k: *f* (*snd* (*g k*)) > *card* (*build-all-simple-clss* (*atms-of-msu* (*init-clss* (*fst ?S*)))) **and**
k > *card* (*build-all-simple-clss* (*atms-of-msu* (*init-clss* (*fst ?S*))))
using *not-bounded-nat-exists-larger*[*OF unbounded-f-g*] **by** *blast*

The following does not hold anymore with the non-strict version of cardinality in the definition.

{ fix *i*
assume *no-step* $cdcl_W$*-merge-stgy* (*fst* (*g i*))
with *g*[*of i*]
have *False*
proof (*induction rule*: $cdcl_W$*-merge-with-restart.induct*)
case (*restart-step T S n*) **note** *H* = *this*(*1*) **and** *c* = *this*(*2*) **and** *n-s* = *this*(*4*)
obtain *S'* **where** $cdcl_W$*-merge-stgy S S'*
using *H c* **by** (*metis gr-implies-not0 relpowp-E2*)
then show *False* **using** *n-s* **by** *auto*
next
case (*restart-full S T*)
then show *False* **unfolding** *full1-def* **by** (*auto dest*: *tranclpD*)
qed
} **note** *H* = *this*
obtain *m T* **where**
m: *m* = *card* (*set-mset* (*learned-clss T*)) − *card* (*set-mset* (*learned-clss* (*fst* (*g k*)))) **and**
m > *f* (*snd* (*g k*)) **and**
restart T (*fst* (*g* (*k*+*1*))) **and**
$cdcl_W$*-merge-stgy*: ($cdcl_W$*-merge-stgy* ^^ *m*) (*fst* (*g k*)) *T*
using *g*[*of k*] *H*[*of Suc k*] **by** (*force simp*: $cdcl_W$*-merge-with-restart.simps full1-def*)
have $cdcl_W$*-merge-stgy*** (*fst* (*g k*)) *T*
using $cdcl_W$*-merge-stgy relpowp-imp-rtranclp* **by** *metis*
then have $cdcl_W$*-all-struct-inv T*
using *inv*[*of k*] *rtranclp-*$cdcl_W$*-all-struct-inv-inv rtranclp-*$cdcl_W$*-merge-stgy-rtranclp-*$cdcl_W$
by *blast*
moreover have *card* (*set-mset* (*learned-clss T*)) − *card* (*set-mset* (*learned-clss* (*fst* (*g k*))))
> *card* (*build-all-simple-clss* (*atms-of-msu* (*init-clss* (*fst ?S*))))
unfolding *m*[*symmetric*] **using** ‹*m* > *f* (*snd* (*g k*))› *f-g-k* **by** *linarith*
then have *card* (*set-mset* (*learned-clss T*))
> *card* (*build-all-simple-clss* (*atms-of-msu* (*init-clss* (*fst ?S*))))
by *linarith*
moreover
have *init-clss* (*fst* (*g k*)) = *init-clss T*

using ‹$cdcl_W$*-merge-stgy*** (*fst* (*g k*)) *T*› *rtranclp-*$cdcl_W$*-merge-stgy-rtranclp-*$cdcl_W$
rtranclp-$cdcl_W$*-init-clss inv* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *blast*
then have *init-clss* (*fst ?S*) = *init-clss T*
using *init-g*[*of k*] **by** *auto*
ultimately show *False*
using $cdcl_W$*-all-struct-inv-learned-clss-bound* **by** (*metis Suc-leI card-mono not-less-eq-eq build-all-simple-clss-finite*)
qed

lemma $cdcl_W$*-merge-with-restart-distinct-mset-clauses*:
assumes *invR*: $cdcl_W$*-all-struct-inv* (*fst R*) **and**
st: $cdcl_W$*-merge-with-restart R S* **and**
dist: *distinct-mset* (*clauses* (*fst R*)) **and**
R: *trail* (*fst R*) = []
shows *distinct-mset* (*clauses* (*fst S*))
using *assms*(*2*,*1*,*3*,*4*)
proof (*induction*)
case (*restart-full S T*)
then show *?case* **using** *rtranclp-*$cdcl_W$*-merge-stgy-distinct-mset-clauses*[*of S T*] **unfolding** *full1-def*
by (*auto dest*: *tranclp-into-rtranclp*)
next
case (*restart-step T S n U*)
then have *distinct-mset* (*clauses T*)
using *rtranclp-*$cdcl_W$*-merge-stgy-distinct-mset-clauses*[*of S T*] **unfolding** *full1-def*
by (*auto dest*: *relpowp-imp-rtranclp*)
then show *?case* **using** ‹*restart T U*› **by** (*metis clauses-restart distinct-mset-union fstI mset-le-exists-conv restart.cases state-eq-clauses*)
qed

inductive $cdcl_W$*-with-restart* **where**
restart-step:
($cdcl_W$*-stgy*^^(*card* (*set-mset* (*learned-clss T*)) − *card* (*set-mset* (*learned-clss S*)))) *S T* ⟹
card (*set-mset* (*learned-clss T*)) − *card* (*set-mset* (*learned-clss S*)) > *f n* ⟹
restart T U ⟹
$cdcl_W$*-with-restart* (*S*, *n*) (*U*, *Suc n*) |
restart-full: *full1* $cdcl_W$*-stgy S T* ⟹ $cdcl_W$*-with-restart* (*S*, *n*) (*T*, *Suc n*)

lemma $cdcl_W$*-with-restart-rtranclp-*$cdcl_W$:
$cdcl_W$*-with-restart S T* ⟹ $cdcl_W$** (*fst S*) (*fst T*)
apply (*induction rule*: $cdcl_W$*-with-restart.induct*)
by (*auto dest*!: *relpowp-imp-rtranclp tranclp-into-rtranclp fw-r-rf*
$cdcl_W$*-rf.restart rtranclp-*$cdcl_W$*-stgy-rtranclp-*$cdcl_W$ $cdcl_W$*-merge-restart-*$cdcl_W$
simp: *full1-def*)

lemma $cdcl_W$*-with-restart-increasing-number*:
$cdcl_W$*-with-restart S T* ⟹ *snd T* = *1* + *snd S*
by (*induction rule*: $cdcl_W$*-with-restart.induct*) *auto*

lemma *full1* $cdcl_W$*-stgy S T* ⟹ $cdcl_W$*-with-restart* (*S*, *n*) (*T*, *Suc n*)
using *restart-full* **by** *blast*

lemma $cdcl_W$*-with-restart-init-clss*:
$cdcl_W$*-with-restart S T* ⟹ $cdcl_W$*-M-level-inv* (*fst S*) ⟹ *init-clss* (*fst S*) = *init-clss* (*fst T*)
using $cdcl_W$*-with-restart-rtranclp-*$cdcl_W$ *rtranclp-*$cdcl_W$*-init-clss* **by** *blast*

lemma
wf {(T, S). $cdcl_W$*-all-struct-inv* (*fst* S) ∧ $cdcl_W$*-with-restart* S T}
proof (*rule ccontr*)
assume ¬ *?thesis*
then obtain g **where**
g: ⋀i. $cdcl_W$*-with-restart* (g i) (g (*Suc* i)) **and**
inv: ⋀i. $cdcl_W$*-all-struct-inv* (*fst* (g i))
unfolding *wf-iff-no-infinite-down-chain* **by** *fast*
{ fix i
have *init-clss* (*fst* (g i)) = *init-clss* (*fst* (g *0*))
apply (*induction* i)
apply *simp*
using g *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** (*metis* $cdcl_W$*-with-restart-init-clss*)
} note *init-g* = *this*
let *?S* = g *0*
have *finite* (*atms-of-msu* (*init-clss* (*fst ?S*)))
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *auto*
have *snd-g*: ⋀i. *snd* (g i) = i + *snd* (g *0*)
apply (*induct-tac* i)
apply *simp*
by (*metis Suc-eq-plus1-left add-Suc* $cdcl_W$*-with-restart-increasing-number* g)
then have *snd-g-0*: ⋀i. i > *0* ⟹ *snd* (g i) = i + *snd* (g *0*)
by *blast*
have *unbounded-f-g*: *unbounded* (λi. f (*snd* (g i)))
using f **unfolding** *bounded-def* **by** (*metis add.commute f less-or-eq-imp-le snd-g*
not-bounded-nat-exists-larger not-le ordered-cancel-comm-monoid-diff-class.le-iff-add)

obtain k **where**
f-g-k: f (*snd* (g k)) > *card* (*build-all-simple-clss* (*atms-of-msu* (*init-clss* (*fst ?S*)))) **and**
k > *card* (*build-all-simple-clss* (*atms-of-msu* (*init-clss* (*fst ?S*))))
using *not-bounded-nat-exists-larger*[*OF unbounded-f-g*] **by** *blast*

The following does not hold anymore with the non-strict version of cardinality in the definition.

{ fix i
assume *no-step* $cdcl_W$*-stgy* (*fst* (g i))
with g[*of* i]
have *False*
proof (*induction rule*: $cdcl_W$*-with-restart.induct*)
case (*restart-step* T S n) **note** H = *this*(*1*) **and** c = *this*(*2*) **and** *n-s* = *this*(*4*)
obtain S' **where** $cdcl_W$*-stgy* S S'
using H c **by** (*metis gr-implies-not0 relpowp-E2*)
then show *False* **using** *n-s* **by** *auto*
next
case (*restart-full* S T)
then show *False* **unfolding** *full1-def* **by** (*auto dest*: *tranclpD*)
qed
} note H = *this*
obtain m T **where**
m: m = *card* (*set-mset* (*learned-clss* T)) − *card* (*set-mset* (*learned-clss* (*fst* (g k)))) **and**
m > f (*snd* (g k)) **and**
restart T (*fst* (g (k+*1*))) **and**
$cdcl_W$*-merge-stgy*: ($cdcl_W$*-stgy* ^^ m) (*fst* (g k)) T
using g[*of* k] H[*of Suc* k] **by** (*force simp*: $cdcl_W$*-with-restart.simps full1-def*)
have $cdcl_W$*-stgy*** (*fst* (g k)) T
using $cdcl_W$*-merge-stgy relpowp-imp-rtranclp* **by** *metis*

then have *cdcl$_W$-all-struct-inv T*
using *inv*[*of k*] *rtranclp-cdcl$_W$-all-struct-inv-inv rtranclp-cdcl$_W$-stgy-rtranclp-cdcl$_W$* **by** *blast*
moreover have *card* (*set-mset* (*learned-clss T*)) − *card* (*set-mset* (*learned-clss* (*fst* (*g k*))))
> *card* (*build-all-simple-clss* (*atms-of-msu* (*init-clss* (*fst ?S*))))
unfolding *m*[*symmetric*] **using** ‹*m* > *f* (*snd* (*g k*))› *f-g-k* **by** *linarith*
then have *card* (*set-mset* (*learned-clss T*))
> *card* (*build-all-simple-clss* (*atms-of-msu* (*init-clss* (*fst ?S*))))
by *linarith*
moreover
have *init-clss* (*fst* (*g k*)) = *init-clss T*
using ‹*cdcl$_W$-stgy*** (*fst* (*g k*)) *T*› *rtranclp-cdcl$_W$-stgy-rtranclp-cdcl$_W$ rtranclp-cdcl$_W$-init-clss*
inv **unfolding** *cdcl$_W$-all-struct-inv-def*
by *blast*
then have *init-clss* (*fst ?S*) = *init-clss T*
using *init-g*[*of k*] **by** *auto*
ultimately show *False*
using *cdcl$_W$-all-struct-inv-learned-clss-bound* **by** (*metis Suc-leI card-mono not-less-eq-eq*
build-all-simple-clss-finite)
qed

lemma *cdcl$_W$-with-restart-distinct-mset-clauses*:
assumes *invR*: *cdcl$_W$-all-struct-inv* (*fst R*) **and**
st: *cdcl$_W$-with-restart R S* **and**
dist: *distinct-mset* (*clauses* (*fst R*)) **and**
R: *trail* (*fst R*) = []
shows *distinct-mset* (*clauses* (*fst S*))
using *assms*(*2*,*1*,*3*,*4*)
proof (*induction*)
case (*restart-full S T*)
then show *?case* **using** *rtranclp-cdcl$_W$-stgy-distinct-mset-clauses*[*of S T*] **unfolding** *full1-def*
by (*auto dest*: *tranclp-into-rtranclp*)
next
case (*restart-step T S n U*)
then have *distinct-mset* (*clauses T*) **using** *rtranclp-cdcl$_W$-stgy-distinct-mset-clauses*[*of S T*]
unfolding *full1-def* **by** (*auto dest*: *relpowp-imp-rtranclp*)
then show *?case* **using** ‹*restart T U*› **by** (*metis clauses-restart distinct-mset-union fstI*
mset-le-exists-conv restart.cases state-eq-clauses)
qed
end

locale *luby-sequence* =
fixes *ur* :: *nat*
assumes *ur* > *0*
begin

lemma *exists-luby-decomp*:
fixes *i* ::*nat*
shows ∃ *k*::*nat*. (*2* ^ (*k* − *1*) ≤ *i* ∧ *i* < *2* ^ *k* − *1*) ∨ *i* = *2* ^ *k* − *1*
proof (*induction i*)
case *0*
then show *?case*
by (*rule exI*[*of - 0*], *simp*)
next
case (*Suc n*)
then obtain *k* **where** *2* ^ (*k* − *1*) ≤ *n* ∧ *n* < *2* ^ *k* − *1* ∨ *n* = *2* ^ *k* − *1*

by *blast*
then consider
(*st-interv*) $2 \hat{} (k - 1) \leq n$ **and** $n \leq 2 \hat{} k - 2$
| (*end-interv*) $2 \hat{} (k - 1) \leq n$ **and** $n = 2 \hat{} k - 2$
| (*pow2*) $n = 2 \hat{} k - 1$
by *linarith*
then show *?case*
proof *cases*
case *st-interv*
then show *?thesis* **apply** − **apply** (*rule exI*[*of - k*])
by (*metis* (*no-types*, *lifting*) *One-nat-def Suc-diff-Suc Suc-lessI*
‹$2 \hat{} (k - 1) \leq n \wedge n < 2 \hat{} k - 1 \vee n = 2 \hat{} k - 1$› *diff-self-eq-0*
dual-order.trans le-SucI le-imp-less-Suc numeral-2-eq-2 one-le-numeral
one-le-power zero-less-numeral zero-less-power)
next
case *end-interv*
then show *?thesis* **apply** − **apply** (*rule exI*[*of - k*]) **by** *auto*
next
case *pow2*
then show *?thesis* **apply** − **apply** (*rule exI*[*of - k+1*]) **by** *auto*
qed
qed

Luby sequences are defined by:

- $2^k - 1$, if $i = (2::'a)^k - (1::'a)$
- *luby-sequence-core* $(i - 2^{k-1} + 1)$, if $(2::'a)^{k-1} \leq i$ and $i \leq (2::'a)^k - (1::'a)$

Then the sequence is then scaled by a constant unit run (called *ur* here), strictly positive.

function *luby-sequence-core* :: *nat* ⇒ *nat* **where**
luby-sequence-core $i =$
(*if* $\exists k.\ i = 2\hat{}k - 1$
then $2\hat{}((SOME\ k.\ i = 2\hat{}k - 1) - 1)$
else *luby-sequence-core* $(i - 2\hat{}((SOME\ k.\ 2\hat{}(k-1) \leq i \wedge i < 2\hat{}k - 1) - 1) + 1))$
by *auto*
termination
proof (*relation less-than*, *goal-cases*)
case *1*
then show *?case* **by** *auto*
next
case (*2 i*)
let $?k = (SOME\ k.\ 2 \hat{} (k - 1) \leq i \wedge i < 2 \hat{} k - 1)$
have $2 \hat{} (?k - 1) \leq i \wedge i < 2 \hat{} ?k - 1$
apply (*rule someI-ex*)
using *2 exists-luby-decomp* **by** *blast*
then show *?case*

proof −
have $\forall n\ na.\ \neg (1::nat) \leq n \vee 1 \leq n \hat{} na$
by (*meson one-le-power*)
then have *f1*: $(1::nat) \leq 2 \hat{} (?k - 1)$
using *one-le-numeral* **by** *blast*
have *f2*: $i - 2 \hat{} (?k - 1) + 2 \hat{} (?k - 1) = i$
using ‹$2 \hat{} (?k - 1) \leq i \wedge i < 2 \hat{} ?k - 1$› *le-add-diff-inverse2* **by** *blast*

have *f3*: $2 \verb|^| ?k - 1 \neq Suc\ 0$
  using *f1* ‹$2 \verb|^| (?k - 1) \leq i \wedge i < 2 \verb|^| ?k - 1$› **by** *linarith*
have $2 \verb|^| ?k - (1::nat) \neq 0$
  using ‹$2 \verb|^| (?k - 1) \leq i \wedge i < 2 \verb|^| ?k - 1$› *gr-implies-not0* **by** *blast*
then have *f4*: $2 \verb|^| ?k \neq (1::nat)$
  by *linarith*
have *f5*: $\forall n\ na.\ if\ na = 0\ then\ (n::nat) \verb|^| na = 1\ else\ n \verb|^| na = n * n \verb|^| (na - 1)$
  by (*simp add*: *power-eq-if*)
then have $?k \neq 0$
  using *f4* **by** *meson*
then have $2 \verb|^| (?k - 1) \neq Suc\ 0$
  using *f5 f3* **by** *presburger*
then have $Suc\ 0 < 2 \verb|^| (?k - 1)$
  using *f1* **by** *linarith*
then show *?thesis*
  using *f2 less-than-iff* **by** *presburger*
qed
qed

declare *luby-sequence-core.simps*[*simp del*]

lemma *two-pover-n-eq-two-power-n′-eq*:
  assumes *H*: $(2::nat) \verb|^| (k::nat) - 1 = 2 \verb|^| k' - 1$
  shows $k' = k$
proof −
  have $(2::nat) \verb|^| (k::nat) = 2 \verb|^| k'$
    using *H* **by** (*metis One-nat-def Suc-pred zero-less-numeral zero-less-power*)
  then show *?thesis* **by** *simp*
qed

lemma *luby-sequence-core-two-power-minus-one*:
  luby-sequence-core $(2\verb|^|k - 1) = 2\verb|^|(k-1)$ (**is** $?L = ?K$)
proof −
  have *decomp*: $\exists ka.\ 2 \verb|^| k - 1 = 2 \verb|^| ka - 1$
    by *auto*
  have $?L = 2\verb|^|((SOME\ k'.\ (2::nat)\verb|^|k - 1 = 2\verb|^|k' - 1) - 1)$
    apply (*subst luby-sequence-core.simps*, *subst decomp*)
    by *simp*
  moreover have $(SOME\ k'.\ (2::nat)\verb|^|k - 1 = 2\verb|^|k' - 1) = k$
    apply (*rule some-equality*)
      apply *simp*
      using *two-pover-n-eq-two-power-n′-eq* **by** *blast*
  ultimately show *?thesis* **by** *presburger*
qed

lemma *different-luby-decomposition-false*:
  assumes
    H: $2 \verb|^| (k - Suc\ 0) \leq i$ **and**
    k′: $i < 2 \verb|^| k' - Suc\ 0$ **and**
    k-k′: $k > k'$
  shows *False*
proof −
  have $2 \verb|^| k' - Suc\ 0 < 2 \verb|^| (k - Suc\ 0)$
    using *k-k′ less-eq-Suc-le* **by** *auto*
  then show *?thesis*

using *H k′* **by** *linarith*
qed

lemma *luby-sequence-core-not-two-power-minus-one*:
assumes
k-i: $2 \hat{} (k - 1) \leq i$ **and**
i-k: $i < 2\hat{}k - 1$
shows *luby-sequence-core* $i =$ *luby-sequence-core* $(i - 2 \hat{} (k - 1) + 1)$
proof −
have *H*: $\neg (\exists ka.\ i = 2 \hat{} ka - 1)$
proof (*rule ccontr*)
assume ¬ *?thesis*
then obtain *k′::nat* **where** *k′*: $i = 2 \hat{} k' - 1$ **by** *blast*
have $(2::nat) \hat{} k' - 1 < 2 \hat{} k - 1$
using *i-k* **unfolding** *k′* **.**
then have $(2::nat) \hat{} k' < 2 \hat{} k$
by *linarith*
then have $k' < k$
by *simp*
have $2 \hat{} (k - 1) \leq 2 \hat{} k' - (1::nat)$
using *k-i* **unfolding** *k′* **.**
then have $(2::nat) \hat{} (k-1) < 2 \hat{} k'$
by (*metis Suc-diff-1 not-le not-less-eq zero-less-numeral zero-less-power*)
then have $k-1 < k'$
by *simp*

show *False* **using** ‹$k' < k$› ‹$k-1 < k'$› **by** *linarith*
qed
have $\bigwedge k\ k'.\ 2 \hat{} (k - Suc\ 0) \leq i \Longrightarrow i < 2 \hat{} k - Suc\ 0 \Longrightarrow 2 \hat{} (k' - Suc\ 0) \leq i \Longrightarrow$
$i < 2 \hat{} k' - Suc\ 0 \Longrightarrow k = k'$
by (*meson different-luby-decomposition-false linorder-neqE-nat*)
then have *k*: $(SOME\ k.\ 2 \hat{} (k - Suc\ 0) \leq i \wedge i < 2 \hat{} k - Suc\ 0) = k$
using *k-i i-k* **by** *auto*
show *?thesis*
apply (*subst luby-sequence-core.simps*[*of i*], *subst H*)
by (*simp add*: *k*)
qed

lemma *unbounded-luby-sequence-core*: *unbounded luby-sequence-core*
unfolding *bounded-def*
proof
assume $\exists b.\ \forall n.$ *luby-sequence-core* $n \leq b$
then obtain *b* **where** *b*: $\bigwedge n.$ *luby-sequence-core* $n \leq b$
by *metis*
have *luby-sequence-core* $(2\hat{}(b+1) - 1) = 2\hat{}b$
using *luby-sequence-core-two-power-minus-one*[*of b+1*] **by** *simp*
moreover have $(2::nat)\hat{}b > b$
by (*induction b*) *auto*
ultimately show *False* **using** *b*[*of* $2\hat{}(b+1) - 1$] **by** *linarith*
qed

abbreviation *luby-sequence* :: *nat* ⇒ *nat* **where**
luby-sequence $n \equiv$ *ur* ∗ *luby-sequence-core* n

lemma *bounded-luby-sequence*: *unbounded luby-sequence*

using *bounded-const-product*[*of ur*] *luby-sequence-axioms*
luby-sequence-def unbounded-luby-sequence-core **by** *blast*

lemma *luby-sequence-core-0*: *luby-sequence-core 0* = *1*
proof −
have *0*: (*0*::*nat*) = *2*^*0*−*1*
by *auto*
show *?thesis*
by (*subst 0*, *subst luby-sequence-core-two-power-minus-one*) *simp*
qed

lemma *luby-sequence-core n* ≥ *1*
proof (*induction n rule*: *nat-less-induct-case*)
case *0*
then show *?case* **by** (*simp add*: *luby-sequence-core-0*)
next
case (*Suc n*) **note** *IH* = *this*

consider
(*interv*) *k* **where** *2* ^ (*k* − *1*) ≤ *Suc n* **and** *Suc n* < *2* ^ *k* − *1*
| (*pow2*) *k* **where** *Suc n* = *2* ^ *k* − *Suc 0*
using *exists-luby-decomp*[*of Suc n*] **by** *auto*

then show *?case*
proof *cases*
case *pow2*
show *?thesis*
using *luby-sequence-core-two-power-minus-one pow2* **by** *auto*
next
case *interv*
have *n*: *Suc n* − *2* ^ (*k* − *1*) + *1* < *Suc n*
by (*metis Suc-1 Suc-eq-plus1 add.commute add-diff-cancel-left′ add-less-mono1 gr0I interv*(*1*) *interv*(*2*) *le-add-diff-inverse2 less-Suc-eq not-le power-0 power-one-right power-strict-increasing-iff*)
show *?thesis*
apply (*subst luby-sequence-core-not-two-power-minus-one*[*OF interv*])
using *IH n* **by** *auto*
qed
qed
end

locale *luby-sequence-restart* =
luby-sequence ur +
$cdcl_W$*-ops trail init-clss learned-clss backtrack-lvl conflicting cons-trail tl-trail add-init-cls*
add-learned-cls remove-cls update-backtrack-lvl update-conflicting init-state restart-state
for
ur :: *nat* **and**
trail :: ′*st* ⇒ (′*v*::*linorder*, *nat*, ′*v clause*) *marked-lits* **and**
init-clss :: ′*st* ⇒ ′*v clauses* **and**
learned-clss :: ′*st* ⇒ ′*v clauses* **and**
backtrack-lvl :: ′*st* ⇒ *nat* **and**
conflicting :: ′*st* ⇒′*v clause option* **and**
cons-trail :: (′*v*, *nat*, ′*v clause*) *marked-lit* ⇒ ′*st* ⇒ ′*st* **and**

tl-trail :: *'st* ⇒ *'st* **and**
add-init-cls :: *'v clause* ⇒ *'st* ⇒ *'st* **and**
add-learned-cls remove-cls :: *'v clause* ⇒ *'st* ⇒ *'st* **and**
update-backtrack-lvl :: *nat* ⇒ *'st* ⇒ *'st* **and**
update-conflicting :: *'v clause option* ⇒ *'st* ⇒ *'st* **and**

init-state :: *'v::linorder clauses* ⇒ *'st* **and**
restart-state :: *'st* ⇒ *'st*
begin

sublocale $cdcl_W$*-ops-restart* - - - - - - - - - - - - - - *luby-sequence*
apply *unfold-locales*
using *bounded-luby-sequence* **by** *blast*

end

end
theory *CDCL-W-Incremental*
imports *CDCL-W-Termination*
begin

20 Incremental SAT solving

context $cdcl_W$*-ops*
begin

This invariant holds all the invariant related to the strategy. See the structural invariant in $cdcl_W$*-all-struct-inv*

definition $cdcl_W$*-stgy-invariant* **where**
$cdcl_W$*-stgy-invariant S* ⟷
conflict-is-false-with-level S
∧ *no-clause-is-false S*
∧ *no-smaller-confl S*
∧ *no-clause-is-false S*

lemma $cdcl_W$*-stgy-*$cdcl_W$*-stgy-invariant*:
assumes
$cdcl_W$: $cdcl_W$*-stgy S T* **and**
inv-s: $cdcl_W$*-stgy-invariant S* **and**
inv: $cdcl_W$*-all-struct-inv S*
shows
$cdcl_W$*-stgy-invariant T*
unfolding $cdcl_W$*-stgy-invariant-def* $cdcl_W$*-all-struct-inv-def* **apply** *standard*
apply (*rule* $cdcl_W$*-stgy-ex-lit-of-max-level*[*of S*])
using *assms* **unfolding** $cdcl_W$*-stgy-invariant-def* $cdcl_W$*-all-struct-inv-def* **apply** *auto*[*7*]
apply *standard*
using $cdcl_W$ $cdcl_W$*-stgy-not-non-negated-init-clss* **apply** *blast*
apply *standard*
apply (*rule* $cdcl_W$*-stgy-no-smaller-confl-inv*)
using *assms* **unfolding** $cdcl_W$*-stgy-invariant-def* $cdcl_W$*-all-struct-inv-def* **apply** *auto*[*4*]
using $cdcl_W$ $cdcl_W$*-stgy-not-non-negated-init-clss* **by** *auto*

lemma *rtranclp-*$cdcl_W$*-stgy-*$cdcl_W$*-stgy-invariant*:
assumes
$cdcl_W$: $cdcl_W$*-stgy*** *S T* **and**

inv-s: *cdcl$_W$-stgy-invariant S* **and**
inv: *cdcl$_W$-all-struct-inv S*
shows
cdcl$_W$-stgy-invariant T
using *assms* **apply** (*induction*)
apply *simp*
using *cdcl$_W$-stgy-cdcl$_W$-stgy-invariant rtranclp-cdcl$_W$-all-struct-inv-inv*
rtranclp-cdcl$_W$-stgy-rtranclp-cdcl$_W$ **by** *blast*

abbreviation *decr-bt-lvl* **where**
decr-bt-lvl S ≡ *update-backtrack-lvl* (*backtrack-lvl S* − *1*) *S*

When we add a new clause, we reduce the trail until we get to tho first literal included in C. Then we can mark the conflict.

fun *cut-trail-wrt-clause* **where**
cut-trail-wrt-clause C [] *S* = *S* |
cut-trail-wrt-clause C (*Marked L* - # *M*) *S* =
(*if* −*L* ∈# *C then S*
else cut-trail-wrt-clause C M (*decr-bt-lvl* (*tl-trail S*))) |
cut-trail-wrt-clause C (*Propagated L* - # *M*) *S* =
(*if* −*L* ∈# *C then S*
else cut-trail-wrt-clause C M (*tl-trail S*))

definition *add-new-clause-and-update* :: *'v literal multiset* ⇒ *'st* ⇒ *'st* **where**
add-new-clause-and-update C S =
(*if trail S* ⊨*as CNot C*
then update-conflicting (*Some C*) (*add-init-cls C* (*cut-trail-wrt-clause C* (*trail S*) *S*))
else add-init-cls C S)

thm *cut-trail-wrt-clause.induct*
lemma *init-clss-cut-trail-wrt-clause*[*simp*]:
init-clss (*cut-trail-wrt-clause C M S*) = *init-clss S*
by (*induction rule*: *cut-trail-wrt-clause.induct*) *auto*

lemma *learned-clss-cut-trail-wrt-clause*[*simp*]:
learned-clss (*cut-trail-wrt-clause C M S*) = *learned-clss S*
by (*induction rule*: *cut-trail-wrt-clause.induct*) *auto*

lemma *conflicting-clss-cut-trail-wrt-clause*[*simp*]:
conflicting (*cut-trail-wrt-clause C M S*) = *conflicting S*
by (*induction rule*: *cut-trail-wrt-clause.induct*) *auto*

lemma *trail-cut-trail-wrt-clause*:
∃ *M*. *trail S* = *M* @ *trail* (*cut-trail-wrt-clause C* (*trail S*) *S*)
proof (*induction trail S arbitrary*:*S rule*: *marked-lit-list-induct*)
case *nil*
then show *?case* **by** *simp*
next
case (*marked L l M*) **note** *IH* = *this*(*1*)[*of decr-bt-lvl* (*tl-trail S*)] **and** *M* = *this*(*2*)[*symmetric*]
then show *?case* **using** *Cons-eq-appendI* **by** *fastforce+*
next
case (*proped L l M*) **note** *IH* = *this*(*1*)[*of tl-trail S*] **and** *M* = *this*(*2*)[*symmetric*]
then show *?case* **using** *Cons-eq-appendI* **by** *fastforce+*
qed

lemma *n-dup-no-dup-trail-cut-trail-wrt-clause*[*simp*]:
  assumes *n-d*: *no-dup* (*trail T*)
  shows *no-dup* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*))
proof −
  obtain *M* **where**
    M: *trail T* = *M* @ *trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)
    using *trail-cut-trail-wrt-clause*[*of T C*] **by** *auto*
  show *?thesis*
    using *n-d* **unfolding** *arg-cong*[*OF M*, *of no-dup*] **by** *auto*
qed

lemma *cut-trail-wrt-clause-backtrack-lvl-length-marked*:
  assumes
     backtrack-lvl T = *length* (*get-all-levels-of-marked* (*trail T*))
  shows
  backtrack-lvl (*cut-trail-wrt-clause C* (*trail T*) *T*) =
     length (*get-all-levels-of-marked* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)))
  using *assms*
proof (*induction trail T arbitrary*:*T rule*: *marked-lit-list-induct*)
  case *nil*
  then show *?case* **by** *simp*
next
  case (*marked L l M*) **note** *IH* = *this*(*1*)[*of decr-bt-lvl* (*tl-trail T*)] **and** *M* = *this*(*2*)[*symmetric*]
    and *bt* = *this*(*3*)
  then show *?case* **by** *auto*
next
  case (*proped L l M*) **note** *IH* = *this*(*1*)[*of tl-trail T*] **and** *M* = *this*(*2*)[*symmetric*] **and** *bt* = *this*(*3*)
  then show *?case* **by** *auto*
qed

lemma *cut-trail-wrt-clause-get-all-levels-of-marked*:
  assumes *get-all-levels-of-marked* (*trail T*) = *rev* [*Suc 0*..<
    Suc (*length* (*get-all-levels-of-marked* (*trail T*)))]
  shows
    get-all-levels-of-marked (*trail* ((*cut-trail-wrt-clause C* (*trail T*) *T*))) = *rev* [*Suc 0*..<
    Suc (*length* (*get-all-levels-of-marked* (*trail* ((*cut-trail-wrt-clause C* (*trail T*) *T*)))))]
  using *assms*
proof (*induction trail T arbitrary*:*T rule*: *marked-lit-list-induct*)
  case *nil*
  then show *?case* **by** *simp*
next
  case (*marked L l M*) **note** *IH* = *this*(*1*)[*of decr-bt-lvl* (*tl-trail T*)] **and** *M* = *this*(*2*)[*symmetric*]
    and *bt* = *this*(*3*)
  then show *?case* **by** (*cases count C L* = *0*) *auto*
next
  case (*proped L l M*) **note** *IH* = *this*(*1*)[*of tl-trail T*] **and** *M* = *this*(*2*)[*symmetric*] **and** *bt* = *this*(*3*)
  then show *?case* **by** (*cases count C L* = *0*) *auto*
qed

lemma *cut-trail-wrt-clause-CNot-trail*:
  assumes *trail T* ⊨*as CNot C*
  shows
    (*trail* ((*cut-trail-wrt-clause C* (*trail T*) *T*))) ⊨*as CNot C*
  using *assms*
proof (*induction trail T arbitrary*:*T rule*: *marked-lit-list-induct*)

case *nil*
then show *?case* **by** *simp*
next
case (*marked L l M*) **note** *IH* = *this*(*1*)[*of decr-bt-lvl* (*tl-trail T*)] **and** *M* = *this*(*2*)[*symmetric*]
and *bt* = *this*(*3*)

then show *?case* **apply** (*cases count C* (−*L*) = *0*)
apply (*auto simp*: *true-annots-true-cls*)
by (*smt CNot-def One-nat-def count-single diff-Suc-1 in-CNot-uminus less-numeral-extra*(*4*)
marked.prems marked-lit.sel(*1*) *mem-Collect-eq true-annot-def true-annot-lit-of-notin-skip*
true-annots-def true-clss-def zero-less-diff)
next
case (*proped L l M*) **note** *IH* = *this*(*1*)[*of tl-trail T*] **and** *M* = *this*(*2*)[*symmetric*] **and** *bt* = *this*(*3*)
then show *?case*

apply (*cases count C* (−*L*) = *0*)
apply (*auto simp*: *true-annots-true-cls*)
by (*smt CNot-def One-nat-def count-single diff-Suc-1 in-CNot-uminus less-numeral-extra*(*4*)
proped.prems marked-lit.sel(*2*) *mem-Collect-eq true-annot-def true-annot-lit-of-notin-skip*
true-annots-def true-clss-def zero-less-diff)
qed

lemma *cut-trail-wrt-clause-hd-trail-in-or-empty-trail*:
((∀ *L* ∈#*C*. −*L* ∉ *lits-of* (*trail T*)) ∧ *trail* (*cut-trail-wrt-clause C* (*trail T*) *T*) = [])
∨ (−*lit-of* (*hd* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*))) ∈# *C*
∧ *length* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)) ≥ *1*)
using *assms*
proof (*induction trail T arbitrary*:*T rule*: *marked-lit-list-induct*)
case *nil*
then show *?case* **by** *simp*
next
case (*marked L l M*) **note** *IH* = *this*(*1*)[*of decr-bt-lvl* (*tl-trail T*)] **and** *M* = *this*(*2*)[*symmetric*]
then show *?case* **by** *simp force*
next
case (*proped L l M*) **note** *IH* = *this*(*1*)[*of tl-trail T*] **and** *M* = *this*(*2*)[*symmetric*]
then show *?case* **by** *simp force*
qed

We can fully run $cdcl_W$*-s* or add a clause. Remark that we use $cdcl_W$*-s* to avoid an explicit *skip*, *resolve*, and *backtrack* normalisation to get rid of the conflict *C* if possible.

inductive *incremental-*$cdcl_W$ :: *'st* ⇒ *'st* ⇒ *bool* **for** *S* **where**
add-confl:
trail S ⊨*asm init-clss S* ⟹ *distinct-mset C* ⟹ *conflicting S* = *None* ⟹
trail S ⊨*as CNot C* ⟹
full $cdcl_W$*-stgy*
(*update-conflicting* (*Some C*) (*add-init-cls C* (*cut-trail-wrt-clause C* (*trail S*) *S*))) *T* ⟹
incremental-$cdcl_W$ *S T* |
add-no-confl:
trail S ⊨*asm init-clss S* ⟹ *distinct-mset C* ⟹ *conflicting S* = *None* ⟹
¬*trail S* ⊨*as CNot C* ⟹
full $cdcl_W$*-stgy* (*add-init-cls C S*) *T* ⟹
incremental-$cdcl_W$ *S T*

inductive *add-learned-clss* :: *'st* ⇒ *'v clauses* ⇒ *'st* ⇒ *bool* **for** *S* :: *'st* **where**
add-learned-clss-nil: *add-learned-clss S* {#} *S* |

add-learned-clss-plus:
add-learned-clss S A T $\Longrightarrow$ *add-learned-clss S* ({#*x*#} + *A*) (*add-learned-cls x T*)
declare *add-learned-clss.intros*[*intro*]

lemma *Ex-add-learned-clss*:
$\exists$ *T*. *add-learned-clss S A T*
by (*induction A arbitrary*: *S rule*: *multiset-induct*) (*auto simp*: *union-commute*[*of* - {#-#}])

lemma *add-learned-clss-trail*:
assumes *add-learned-clss S U T* **and** *no-dup* (*trail S*)
shows *trail T* = *trail S*
using *assms* **by** (*induction rule*: *add-learned-clss.induct*) (*simp-all add*: *ac-simps*)

lemma *add-learned-clss-learned-clss*:
assumes *add-learned-clss S U T* **and** *no-dup* (*trail S*)
shows *learned-clss T* = *U* + *learned-clss S*
using *assms* **by** (*induction rule*: *add-learned-clss.induct*)
(*auto simp*: *ac-simps dest*: *add-learned-clss-trail*)

lemma *add-learned-clss-init-clss*:
assumes *add-learned-clss S U T* **and** *no-dup* (*trail S*)
shows *init-clss T* = *init-clss S*
using *assms* **by** (*induction rule*: *add-learned-clss.induct*)
(*auto simp*: *ac-simps dest*: *add-learned-clss-trail*)

lemma *add-learned-clss-conflicting*:
assumes *add-learned-clss S U T* **and** *no-dup* (*trail S*)
shows *conflicting T* = *conflicting S*
using *assms* **by** (*induction rule*: *add-learned-clss.induct*)
(*auto simp*: *ac-simps dest*: *add-learned-clss-trail*)

lemma *add-learned-clss-backtrack-lvl*:
assumes *add-learned-clss S U T* **and** *no-dup* (*trail S*)
shows *backtrack-lvl T* = *backtrack-lvl S*
using *assms* **by** (*induction rule*: *add-learned-clss.induct*)
(*auto simp*: *ac-simps dest*: *add-learned-clss-trail*)

lemma *add-learned-clss-init-state-mempty*[*dest*!]:
add-learned-clss (*init-state N*) {#} *T* $\Longrightarrow$ *T* = *init-state N*
by (*cases rule*: *add-learned-clss.cases*) (*auto simp*: *add-learned-clss.cases*)

For multiset larger that 1 element, there is no way to know in which order the clauses are added. But contrary to a definition *fold-mset*, there is an element.

lemma *add-learned-clss-init-state-single*[*dest*!]:
add-learned-clss (*init-state N*) {#*C*#} *T* $\Longrightarrow$ *T* = *add-learned-cls C* (*init-state N*)
by (*induction* {#*C*#} *T rule*: *add-learned-clss.induct*)
(*auto simp*: *add-learned-clss.cases ac-simps union-is-single split*: *split-if-asm*)

thm *rtranclp-cdcl*$_W$*-stgy-no-smaller-confl-inv cdcl*$_W$*-stgy-final-state-conclusive*
lemma *cdcl*$_W$*-all-struct-inv-add-new-clause-and-update-cdcl*$_W$*-all-struct-inv*:
assumes
inv-T: *cdcl*$_W$*-all-struct-inv T* **and**
tr-T-N[*simp*]: *trail T* $\models$*asm N* **and**
tr-C[*simp*]: *trail T* $\models$*as CNot C* **and**
[*simp*]: *distinct-mset C*

shows *$cdcl_W$-all-struct-inv* (*add-new-clause-and-update C T*) (**is** *$cdcl_W$-all-struct-inv ?T'*)
proof −
let *?T* = *update-conflicting* (*Some C*) (*add-init-cls C* (*cut-trail-wrt-clause C* (*trail T*) *T*))
obtain *M* **where**
M: *trail T* = *M* @ *trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)
using *trail-cut-trail-wrt-clause*[*of T C*] **by** *blast*
have *H*[*dest*]: $\bigwedge$*x*. *x* ∈ *lits-of* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)) ⟹
x ∈ *lits-of* (*trail T*)
using *inv-T arg-cong*[*OF M*, *of lits-of*] **by** *auto*
have *H'*[*dest*]: $\bigwedge$*x*. *x* ∈ *set* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)) ⟹ *x* ∈ *set* (*trail T*)
using *inv-T arg-cong*[*OF M*, *of set*] **by** *auto*

have *H-proped*:$\bigwedge$*x*. *x* ∈ *set* (*get-all-mark-of-propagated* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*))) ⟹ *x* ∈ *set* (*get-all-mark-of-propagated* (*trail T*))
using *inv-T arg-cong*[*OF M*, *of get-all-mark-of-propagated*] **by** *auto*

have [*simp*]: *no-strange-atm ?T*
using *inv-T* **unfolding** *$cdcl_W$-all-struct-inv-def no-strange-atm-def add-new-clause-and-update-def*
$cdcl_W$-M-level-inv-def
by (*auto dest*!: *H H'*)

have *M-lev*: *$cdcl_W$-M-level-inv T*
using *inv-T* **unfolding** *$cdcl_W$-all-struct-inv-def* **by** *blast*
then have *no-dup* (*M* @ *trail* (*cut-trail-wrt-clause C* (*trail T*) *T*))
unfolding *$cdcl_W$-M-level-inv-def* **unfolding** *M*[*symmetric*] **by** *auto*
then have [*simp*]: *no-dup* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*))
by *auto*

have *consistent-interp* (*lits-of* (*M* @ *trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)))
using *M-lev* **unfolding** *$cdcl_W$-M-level-inv-def* **unfolding** *M*[*symmetric*] **by** *auto*
then have [*simp*]: *consistent-interp* (*lits-of* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)))
unfolding *consistent-interp-def* **by** *auto*

have [*simp*]: *$cdcl_W$-M-level-inv ?T*

using *M-lev cut-trail-wrt-clause-get-all-levels-of-marked*[*of T C*]
unfolding *$cdcl_W$-M-level-inv-def* **by** (*auto dest*: *H H'*
simp: *M-lev $cdcl_W$-M-level-inv-def cut-trail-wrt-clause-backtrack-lvl-length-marked*)

have [*simp*]: $\bigwedge$*s*. *s* ∈# *learned-clss T* ⟹ ¬*tautology s*
using *inv-T* **unfolding** *$cdcl_W$-all-struct-inv-def* **by** *auto*

have *distinct-$cdcl_W$-state T*
using *inv-T* **unfolding** *$cdcl_W$-all-struct-inv-def* **by** *auto*
then have [*simp*]: *distinct-$cdcl_W$-state ?T*
unfolding *distinct-$cdcl_W$-state-def* **by** *auto*

have *$cdcl_W$-conflicting T*
using *inv-T* **unfolding** *$cdcl_W$-all-struct-inv-def* **by** *auto*
have *trail ?T* ⊨*as CNot C*
by (*simp add*: *cut-trail-wrt-clause-CNot-trail*)
then have [*simp*]: *$cdcl_W$-conflicting ?T*
unfolding *$cdcl_W$-conflicting-def* **apply** *simp*
by (*metis M* ‹*$cdcl_W$-conflicting T*› *append-assoc $cdcl_W$-conflicting-decomp*(*2*))

have
decomp-T: *all-decomposition-implies-m* (*init-clss* T) (*get-all-marked-decomposition* (*trail* T))
using *inv-T* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** *auto*
have *all-decomposition-implies-m* (*init-clss* *?T*)
(*get-all-marked-decomposition* (*trail* *?T*))
unfolding *all-decomposition-implies-def*
proof *clarify*
fix *a b*
assume (*a*, *b*) ∈ *set* (*get-all-marked-decomposition* (*trail* *?T*))
from *in-get-all-marked-decomposition-in-get-all-marked-decomposition-prepend*[*OF this*]
obtain b' **where**
(*a*, b' @ *b*) ∈ *set* (*get-all-marked-decomposition* (*trail* T))
using *M* **by** *simp metis*
then have (λ*a*. {#*lit-of a*#}) ' *set a* ∪ *set-mset* (*init-clss* *?T*)
⊨*ps* (λ*a*. {#*lit-of a*#}) ' *set* (*b* @ b')
using *decomp-T* **unfolding** *all-decomposition-implies-def*

apply *auto*
by (*metis* (*no-types*, *lifting*) *case-prodD set-append sup.commute true-clss-clss-insert-l*)

then show (λ*a*. {#*lit-of a*#}) ' *set a* ∪ *set-mset* (*init-clss* *?T*)
⊨*ps* (λ*a*. {#*lit-of a*#}) ' *set b*
by (*auto simp*: *image-Un*)
qed

have [*simp*]: $cdcl_W$*-learned-clause ?T*
using *inv-T* **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-learned-clause-def*
by (*auto dest*!: *H-proped simp*: *clauses-def*)
show *?thesis*
using ‹*all-decomposition-implies-m* (*init-clss* *?T*)
(*get-all-marked-decomposition* (*trail* *?T*))›
unfolding $cdcl_W$*-all-struct-inv-def* **by** (*auto simp*: *add-new-clause-and-update-def*)
qed

lemma $cdcl_W$*-all-struct-inv-add-new-clause-and-update-*$cdcl_W$*-stgy-inv*:
assumes
inv-s: $cdcl_W$*-stgy-invariant* T **and**
inv: $cdcl_W$*-all-struct-inv* T **and**
tr-T-N[*simp*]: *trail* T ⊨*asm* N **and**
tr-C[*simp*]: *trail* T ⊨*as CNot* C **and**
[*simp*]: *distinct-mset* C
shows $cdcl_W$*-stgy-invariant* (*add-new-clause-and-update* C T) (**is** $cdcl_W$*-stgy-invariant* *?T'*)
proof −
have $cdcl_W$*-all-struct-inv ?T'*
using $cdcl_W$*-all-struct-inv-add-new-clause-and-update-*$cdcl_W$*-all-struct-inv assms* **by** *blast*
then have
no-dup-cut-T[*simp*]: *no-dup* (*trail* (*cut-trail-wrt-clause* C (*trail* T) T)) **and**
n-d[*simp*]: *no-dup* (*trail* T)
using $cdcl_W$*-M-level-inv-decomp*(*2*) $cdcl_W$*-all-struct-inv-def inv*
n-dup-no-dup-trail-cut-trail-wrt-clause **by** *blast+*
then have *trail* (*add-new-clause-and-update* C T) ⊨*as CNot* C
by (*simp add*: *add-new-clause-and-update-def cut-trail-wrt-clause-CNot-trail*
$cdcl_W$*-M-level-inv-def* $cdcl_W$*-all-struct-inv-def*)
obtain *MT* **where**
MT: *trail* T = *MT* @ *trail* (*cut-trail-wrt-clause* C (*trail* T) T)

using *trail-cut-trail-wrt-clause* **by** *blast*
consider
(*false*) $\forall L \in \# C. - L \notin$ *lits-of* (*trail T*) **and***trail* (*cut-trail-wrt-clause C* (*trail T*) *T*) = []
| (*not-false*) − *lit-of* (*hd* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*))) $\in\#$ *C* **and**
1 ≤ *length* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*))
using *cut-trail-wrt-clause-hd-trail-in-or-empty-trail*[*of C T*] **by** *auto*
then show *?thesis*
proof *cases*
case *false* **note** *C* = *this*(*1*) **and** *empty-tr* = *this*(*2*)
then have [*simp*]: *C* = {#}
by (*simp add*: *in-CNot-implies-uminus*(*2*) *multiset-eqI*)
show *?thesis*
using *empty-tr* **unfolding** $cdcl_W$*-stgy-invariant-def no-smaller-confl-def*
$cdcl_W$*-all-struct-inv-def* **by** (*auto simp*: *add-new-clause-and-update-def*)
next
case *not-false* **note** *C* = *this*(*1*) **and** *l* = *this*(*2*)
let *?L* = − *lit-of* (*hd* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)))
have *get-all-levels-of-marked* (*trail* (*add-new-clause-and-update C T*)) =
rev [*1*..<*1* + *length* (*get-all-levels-of-marked* (*trail* (*add-new-clause-and-update C T*)))]
using ‹$cdcl_W$*-all-struct-inv ?T'*› **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-M-level-inv-def*
by *blast*
moreover
have *backtrack-lvl* (*cut-trail-wrt-clause C* (*trail T*) *T*) =
length (*get-all-levels-of-marked* (*trail* (*add-new-clause-and-update C T*)))
using ‹$cdcl_W$*-all-struct-inv ?T'*› **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-M-level-inv-def*
by (*auto simp*:*add-new-clause-and-update-def*)
moreover
have *no-dup* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*))
using ‹$cdcl_W$*-all-struct-inv ?T'*› **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-M-level-inv-def*
by (*auto simp*:*add-new-clause-and-update-def*)
then have *atm-of ?L* ∉ *atm-of* ‘ *lits-of* (*tl* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)))
apply (*cases trail* (*cut-trail-wrt-clause C* (*trail T*) *T*))
apply (*auto*)
using *Marked-Propagated-in-iff-in-lits-of defined-lit-map* **by** *blast*

ultimately have *L*: *get-level* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)) (−*?L*)
= *length* (*get-all-levels-of-marked* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)))
using *get-level-get-rev-level-get-all-levels-of-marked*[*OF*
‹*atm-of ?L* ∉ *atm-of* ‘ *lits-of* (*tl* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)))›,
of [*hd* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*))]]

apply (*cases trail* (*add-init-cls C* (*cut-trail-wrt-clause C* (*trail T*) *T*));
cases hd (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)))
using *l* **by** (*auto split*: *split-if-asm*
simp:*rev-swap*[*symmetric*] *add-new-clause-and-update-def*)

have *L'*: *length* (*get-all-levels-of-marked* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)))
= *backtrack-lvl* (*cut-trail-wrt-clause C* (*trail T*) *T*)
using ‹$cdcl_W$*-all-struct-inv ?T'*› **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-M-level-inv-def*
by (*auto simp*:*add-new-clause-and-update-def*)

have [*simp*]: *no-smaller-confl* (*update-conflicting* (*Some C*)
(*add-init-cls C* (*cut-trail-wrt-clause C* (*trail T*) *T*)))
unfolding *no-smaller-confl-def*
proof (*clarify*, *goal-cases*)

case (*1 M K i M' D*)
then consider
(*DC*) *D* = *C*
| (*D-T*) *D* ∈# *clauses T*
by (*auto simp*: *clauses-def split*: *split-if-asm*)
then show *False*
proof *cases*
case *D-T*
have *no-smaller-confl T*
using *inv-s* unfolding $cdcl_W$*-stgy-invariant-def* by *auto*
have (*MT* @ *M'*) @ *Marked K i* # *M* = *trail T*
using *MT 1*(*1*) by *auto*
thus *False* using *D-T* ‹*no-smaller-confl T*› *1*(*3*) unfolding *no-smaller-confl-def* by *blast*
next
case *DC* note -[*simp*] = *this*
then have *atm-of* (−*?L*) ∈ *atm-of* ‘ (*lits-of M*)
using *1*(*3*) *C in-CNot-implies-uminus*(*2*) by *blast*
moreover
have *lit-of* (*hd* (*M'* @ *Marked K i* # [])) = −*?L*
using *l 1*(*1*)[*symmetric*] *inv*
by (*cases trail* (*add-init-cls C* (*cut-trail-wrt-clause C* (*trail T*) *T*)))
(*auto dest*!: *arg-cong*[*of* - # - - *hd*] *simp*: *hd-append* $cdcl_W$*-all-struct-inv-def*
$cdcl_W$*-M-level-inv-def*)
from *arg-cong*[*OF this*, *of atm-of*]
have *atm-of* (−*?L*) ∈ *atm-of* ‘ (*lits-of* (*M'* @ *Marked K i* # []))
by (*cases* (*M'* @ *Marked K i* # [])) *auto*
moreover have *no-dup* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*))
using ‹$cdcl_W$*-all-struct-inv ?T'*› unfolding $cdcl_W$*-all-struct-inv-def*
$cdcl_W$*-M-level-inv-def* by (*auto simp*: *add-new-clause-and-update-def*)
ultimately show *False*
unfolding *1*(*1*)[*symmetric*, *simplified*]
apply *auto*
using *Marked-Propagated-in-iff-in-lits-of defined-lit-map* apply *blast*
by (*metis IntI Marked-Propagated-in-iff-in-lits-of defined-lit-map empty-iff*)
qed
qed
show *?thesis* using *L L' C*
unfolding $cdcl_W$*-stgy-invariant-def*
unfolding $cdcl_W$*-all-struct-inv-def* by (*auto simp*: *add-new-clause-and-update-def*)
qed
qed

lemma *full*-$cdcl_W$*-stgy-inv-normal-form*:
assumes
full: *full* $cdcl_W$*-stgy S T* and
inv-s: $cdcl_W$*-stgy-invariant S* and
inv: $cdcl_W$*-all-struct-inv S*
shows *conflicting T* = *Some* {#} ∧ *unsatisfiable* (*set-mset* (*init-clss S*))
∨ *conflicting T* = *None* ∧ *trail T* ⊨*asm init-clss S* ∧ *satisfiable* (*set-mset* (*init-clss S*))
proof −
have *no-step* $cdcl_W$*-stgy T*
using *full* unfolding *full-def* by *blast*
moreover have $cdcl_W$*-all-struct-inv T* and *inv-s*: $cdcl_W$*-stgy-invariant T*
apply (*metis* $cdcl_W$*-ops.rtranclp*-$cdcl_W$*-stgy-rtranclp*-$cdcl_W$ $cdcl_W$*-ops-axioms full full-def inv*
rtranclp-$cdcl_W$*-all-struct-inv-inv*)

by (*metis full full-def inv inv-s rtranclp-cdcl*$_W$*-stgy-cdcl*$_W$*-stgy-invariant*)
ultimately have *conflicting T* = *Some* {#} ∧ *unsatisfiable* (*set-mset* (*init-clss T*))
∨ *conflicting T* = *None* ∧ *trail T* ⊨*asm init-clss T*
using *cdcl*$_W$*-stgy-final-state-conclusive*[*of T*] *full*
unfolding *cdcl*$_W$*-all-struct-inv-def cdcl*$_W$*-stgy-invariant-def full-def* **by** *fast*
moreover have *consistent-interp* (*lits-of* (*trail T*))
using ‹*cdcl*$_W$*-all-struct-inv T*› **unfolding** *cdcl*$_W$*-all-struct-inv-def cdcl*$_W$*-M-level-inv-def*
by *auto*
moreover have *init-clss S* = *init-clss T*
using *inv* **unfolding** *cdcl*$_W$*-all-struct-inv-def*
by (*metis rtranclp-cdcl*$_W$*-stgy-no-more-init-clss full full-def*)
ultimately show *?thesis*
by (*metis satisfiable-carac′ true-annot-def true-annots-def true-clss-def*)
qed

lemma *incremental-cdcl*$_W$*-inv*:
assumes
inc: *incremental-cdcl*$_W$ *S T* **and**
inv: *cdcl*$_W$*-all-struct-inv S* **and**
s-inv: *cdcl*$_W$*-stgy-invariant S*
shows
cdcl$_W$*-all-struct-inv T* **and**
cdcl$_W$*-stgy-invariant T*
using *inc*
proof (*induction*)
case (*add-confl C T*)
let *?T* = (*update-conflicting* (*Some C*) (*add-init-cls C* (*cut-trail-wrt-clause C* (*trail S*) *S*)))
have *cdcl*$_W$*-all-struct-inv ?T* **and** *inv-s-T*: *cdcl*$_W$*-stgy-invariant ?T*
using *add-confl.hyps*(*1*,*2*,*4*) *add-new-clause-and-update-def*
cdcl$_W$*-all-struct-inv-add-new-clause-and-update-cdcl*$_W$*-all-struct-inv inv* **apply** *auto*[*1*]
using *add-confl.hyps*(*1*,*2*,*4*) *add-new-clause-and-update-def*
cdcl$_W$*-all-struct-inv-add-new-clause-and-update-cdcl*$_W$*-stgy-inv inv s-inv* **by** *auto*
case *1* **show** *?case*
by (*metis add-confl.hyps*(*1*,*2*,*4*,*5*) *add-new-clause-and-update-def*
cdcl$_W$*-all-struct-inv-add-new-clause-and-update-cdcl*$_W$*-all-struct-inv*
rtranclp-cdcl$_W$*-all-struct-inv-inv rtranclp-cdcl*$_W$*-stgy-rtranclp-cdcl*$_W$ *full-def inv*)

case *2* **show** *?case*
by (*metis inv-s-T add-confl.hyps*(*1*,*2*,*4*,*5*) *add-new-clause-and-update-def*
cdcl$_W$*-all-struct-inv-add-new-clause-and-update-cdcl*$_W$*-all-struct-inv full-def inv*
rtranclp-cdcl$_W$*-stgy-cdcl*$_W$*-stgy-invariant*)
next
case (*add-no-confl C T*)
case *1*
have *cdcl*$_W$*-all-struct-inv* (*add-init-cls C S*)
using *inv* ‹*distinct-mset C*› **unfolding** *cdcl*$_W$*-all-struct-inv-def no-strange-atm-def*
cdcl$_W$*-M-level-inv-def distinct-cdcl*$_W$*-state-def cdcl*$_W$*-conflicting-def cdcl*$_W$*-learned-clause-def*
by (*auto simp*: *all-decomposition-implies-insert-single clauses-def*)
then show *?case*
using *add-no-confl*(*5*) **unfolding** *full-def* **by** (*auto intro*: *rtranclp-cdcl*$_W$*-stgy-cdcl*$_W$*-all-struct-inv*)
case *2* **have** *cdcl*$_W$*-stgy-invariant* (*add-init-cls C S*)
using *s-inv* ‹¬ *trail S* ⊨*as CNot C*› *inv* **unfolding** *cdcl*$_W$*-stgy-invariant-def no-smaller-confl-def*
eq-commute[*of - trail -*] *cdcl*$_W$*-M-level-inv-def cdcl*$_W$*-all-struct-inv-def*
by (*auto simp*: *true-annots-true-cls-def-iff-negation-in-model clauses-def split*: *split-if-asm*)
then show *?case*

by (*metis* ‹$cdcl_W$*-all-struct-inv* (*add-init-cls C S*)› *add-no-confl.hyps*(*5*) *full-def*
rtranclp-$cdcl_W$*-stgy-*$cdcl_W$*-stgy-invariant*)
qed

lemma *rtranclp-incremental-*$cdcl_W$*-inv*:
assumes
inc: *incremental-*$cdcl_W{}^{**}$ *S T* **and**
inv: $cdcl_W$*-all-struct-inv S* **and**
s-inv: $cdcl_W$*-stgy-invariant S*
shows
$cdcl_W$*-all-struct-inv T* **and**
$cdcl_W$*-stgy-invariant T*
using *inc* **apply** *induction*
using *inv* **apply** *simp*
using *s-inv* **apply** *simp*
using *incremental-*$cdcl_W$*-inv* **by** *blast+*

lemma *incremental-conclusive-state*:
assumes
inc: *incremental-*$cdcl_W$ *S T* **and**
inv: $cdcl_W$*-all-struct-inv S* **and**
s-inv: $cdcl_W$*-stgy-invariant S*
shows *conflicting T* = *Some* {#} ∧ *unsatisfiable* (*set-mset* (*init-clss T*))
∨ *conflicting T* = *None* ∧ *trail T* ⊨*asm init-clss T* ∧ *satisfiable* (*set-mset* (*init-clss T*))
using *inc* **apply** *induction*

apply (*metis Nitpick.rtranclp-unfold add-confl full-*$cdcl_W$*-stgy-inv-normal-form full-def*
incremental-$cdcl_W$*-inv*(*1*) *incremental-*$cdcl_W$*-inv*(*2*) *inv s-inv*)
by (*metis* (*full-types*) *rtranclp-unfold add-no-confl full-*$cdcl_W$*-stgy-inv-normal-form*
full-def incremental-$cdcl_W$*-inv*(*1*) *incremental-*$cdcl_W$*-inv*(*2*) *inv s-inv*)

lemma *tranclp-incremental-correct*:
assumes
inc: *incremental-*$cdcl_W{}^{++}$ *S T* **and**
inv: $cdcl_W$*-all-struct-inv S* **and**
s-inv: $cdcl_W$*-stgy-invariant S*
shows *conflicting T* = *Some* {#} ∧ *unsatisfiable* (*set-mset* (*init-clss T*))
∨ *conflicting T* = *None* ∧ *trail T* ⊨*asm init-clss T* ∧ *satisfiable* (*set-mset* (*init-clss T*))
using *inc* **apply** *induction*
using *assms incremental-conclusive-state* **apply** *blast*
by (*meson incremental-conclusive-state inv rtranclp-incremental-*$cdcl_W$*-inv s-inv*
tranclp-into-rtranclp)

lemma *blocked-induction-with-marked*:
assumes
n-d: *no-dup* (*L* # *M*) **and**
nil: *P* [] **and**
append: $\bigwedge$*M L M'*. *P M* ⟹ *is-marked L* ⟹ ∀ *m* ∈ *set M'*. ¬*is-marked m* ⟹ *no-dup* (*L* # *M'* @
M) ⟹
P (*L* # *M'* @ *M*) **and**
L: *is-marked L*
shows
P (*L* # *M*)
using *n-d L*
proof (*induction card* {*L'* ∈ *set M*. *is-marked L'*} *arbitrary*: *L M*)

case *0* **note** $n = this(1)$ **and** $n\text{-}d = this(2)$ **and** $L = this(3)$
then have $\forall\, m \in set\ M.\ \neg is\text{-}marked\ m$ **by** *auto*
then show *?case* **using** $append[of\ []\ L\ M]\ L\ nil\ n\text{-}d$ **by** *auto*
next
case $(Suc\ n)$ **note** $IH = this(1)$ **and** $n = this(2)$ **and** $n\text{-}d = this(3)$ **and** $L = this(4)$
have $\exists\, L' \in set\ M.\ is\text{-}marked\ L'$
proof (*rule ccontr*)
assume $\neg ?thesis$
then have H: $\{L' \in set\ M.\ is\text{-}marked\ L'\} = \{\}$
by *auto*
show *False* **using** n **unfolding** H **by** *auto*
qed
then obtain $L'\ M'\ M''$ **where**
M: $M = M'\ @\ L'\ \#\ M''$ **and**
L': $is\text{-}marked\ L'$ **and**
nm: $\forall\, m{\in}set\ M'.\ \neg is\text{-}marked\ m$
by (*auto elim*!: *split-list-first-propE*)
have $Suc\ n = card\ \{L' \in set\ M.\ is\text{-}marked\ L'\}$
using n **.**
moreover have $\{L' \in set\ M.\ is\text{-}marked\ L'\} = \{L'\} \cup \{L' \in set\ M''.\ is\text{-}marked\ L'\}$
using $nm\ L'\ n\text{-}d$ **unfolding** M **by** *auto*
moreover have $L' \notin \{L' \in set\ M''.\ is\text{-}marked\ L'\}$
using $n\text{-}d$ **unfolding** M **by** *auto*
ultimately have $n = card\ \{L'' \in set\ M''.\ is\text{-}marked\ L''\}$
using $n\ L'$ **by** *auto*
then have $P\ (L'\ \#\ M'')$ **using** $IH\ L'\ n\text{-}d\ M$ **by** *auto*
then show *?case* **using** $append[of\ L'\ \#\ M''\ L\ M']\ nm\ L\ n\text{-}d$ **unfolding** M **by** *blast*
qed

lemma *trail-bloc-induction*:
assumes
$n\text{-}d$: $no\text{-}dup\ M$ **and**
nil: $P\ []$ **and**
$append$: $\bigwedge M\ L\ M'.\ P\ M \Longrightarrow is\text{-}marked\ L \Longrightarrow \forall\, m \in set\ M'.\ \neg is\text{-}marked\ m \Longrightarrow no\text{-}dup\ (L\ \#\ M'\ @\ M) \Longrightarrow$
$P\ (L\ \#\ M'\ @\ M)$ **and**
$append\text{-}nm$: $\bigwedge M'\ M''.\ P\ M' \Longrightarrow M = M''\ @\ M' \Longrightarrow \forall\, m{\in}set\ M''.\ \neg is\text{-}marked\ m \Longrightarrow P\ M$
shows
$P\ M$
proof (*cases* $\{L' \in set\ M.\ is\text{-}marked\ L'\} = \{\}$)
case *True*
then show *?thesis* **using** $append\text{-}nm[of\ []\ M]\ nil$ **by** *auto*
next
case *False*
then have $\exists\, L' \in set\ M.\ is\text{-}marked\ L'$
by *auto*
then obtain $L'\ M'\ M''$ **where**
M: $M = M'\ @\ L'\ \#\ M''$ **and**
L': $is\text{-}marked\ L'$ **and**
nm: $\forall\, m{\in}set\ M'.\ \neg is\text{-}marked\ m$
by (*auto elim*!: *split-list-first-propE*)
have $P\ (L'\ \#\ M'')$
apply (*rule blocked-induction-with-marked*)
using $n\text{-}d$ **unfolding** M **apply** *simp*
using nil **apply** *simp*

using *append* apply *simp*
using L' by *auto*
then show *?thesis*
using *append-nm*[*of* - M'] *nm* unfolding M by *simp*
qed

inductive *Tcons* :: ('v, nat, 'v clause) marked-lits ⇒('v, nat, 'v clause) marked-lits ⇒ bool
for M :: ('v, nat, 'v clause) marked-lits where
Tcons M [] |
Tcons $M\ M' \Longrightarrow M = M'' @ M' \Longrightarrow (\forall m \in set\ M''.\ \neg is\text{-}marked\ m) \Longrightarrow Tcons\ M\ (M'' @ M')$ |
Tcons $M\ M' \Longrightarrow is\text{-}marked\ L \Longrightarrow M = M''' @ L \# M'' @ M' \Longrightarrow (\forall m \in set\ M''.\ \neg is\text{-}marked\ m) \Longrightarrow$
$Tcons\ M\ (L \# M'' @ M')$

lemma *Tcons-same-end*: $Tcons\ M\ M' \Longrightarrow \exists M''.\ M = M'' @ M'$
by (*induction rule*: *Tcons.induct*) *auto*

end

end

21 2-Watched-Literal

theory *CDCL-Two-Watched-Literals*
imports *CDCL-WNOT*
begin

21.1 Datastructure and Access Functions

Only the 2-watched literals have to be verified here: the backtrack level and the trail that appear in the state are not related to the 2-watched algoritm.

datatype *'v twl-clause* =
TWL-Clause (*watched*: *'v*) (*unwatched*: *'v*)

abbreviation *raw-clause* :: *'v clause twl-clause* ⇒ *'v clause* where
raw-clause C ≡ *watched C* + *unwatched C*

datatype (*'a*, *'b*, *'c*, *'d*) *twl-state* =
TWL-State (*trail*: *'a list*) (*init-clss*: *'b*)
(*learned-clss*: *'b*) (*backtrack-lvl*: *'c*)
(*conflicting*: *'d option*)

type-synonym (*'v*, *'lvl*, *'mark*) *twl-state-abs* =
((*'v*, *'lvl*, *'mark*) *marked-lit*, *'v clause twl-clause multiset*, *'lvl*, *'v clause*) *twl-state*

abbreviation *raw-init-clss* where
raw-init-clss S ≡ *image-mset raw-clause* (*init-clss S*)

abbreviation *raw-learned-clss* where
raw-learned-clss S ≡ *image-mset raw-clause* (*learned-clss S*)

abbreviation *clauses* where
clauses S ≡ *init-clss S* + *learned-clss S*

abbreviation *raw-clauses* where

raw-clauses S $\equiv$ *image-mset raw-clause* (*clauses S*)

definition
candidates-propagate :: ('*v*, '*lvl*, '*mark*) *twl-state-abs* $\Rightarrow$ ('*v literal* $\times$ '*v clause*) *set*
where
candidates-propagate S =
{(*L*, *raw-clause C*) | *L C*.
C $\in$# *clauses S* $\wedge$ *watched C* $-$ *mset-set* (*uminus* ‘ *lits-of* (*trail S*)) = {#*L*#} $\wedge$
undefined-lit (*trail S*) *L*}

definition *candidates-conflict* :: ('*v*, '*lvl*, '*mark*) *twl-state-abs* $\Rightarrow$ '*v clause set* **where**
candidates-conflict S =
{*raw-clause C* | *C*. *C* $\in$# *clauses S* $\wedge$ *watched C* $\subseteq$# *mset-set* (*uminus* ‘ *lits-of* (*trail S*))}

primrec (*nonexhaustive*) *index* :: '*a list* $\Rightarrow$'*a* $\Rightarrow$ *nat* **where**
index (*a* # *l*) *c* = (*if a* = *c then 0 else 1*+*index l c*)

lemma *index-nth*:
a $\in$ *set l* $\Longrightarrow$ *l* ! (*index l a*) = *a*
by (*induction l*) *auto*

21.2 Invariants

We need the following property about updates: if there is a literal L with $-L$ in the trail, and L is not watched, then it stays unwatched; i.e., while updating with *rewatch* it does not get swap with a watched literal L' such that $-L'$ is in the trail.

primrec *watched-decided-most-recently* :: ('*v*, '*lvl*, '*mark*) *marked-lit list* $\Rightarrow$ '*v clause twl-clause*
$\Rightarrow$ *bool*
where
watched-decided-most-recently M (*TWL-Clause W UW*) $\longleftrightarrow$
($\forall$ *L'*$\in$#*W*. $\forall$ *L*$\in$#*UW*.
$-L'$ $\in$ *lits-of M* $\longrightarrow$ $-L$ $\in$ *lits-of M* $\longrightarrow$ *L* $\notin$# *W* $\longrightarrow$
index (*map lit-of M*) ($-L'$) $\leq$ *index* (*map lit-of M*) ($-L$))

Here are the invariant strictly related to the 2-WL data structure.

primrec *wf-twl-cls* :: ('*v*, '*lvl*, '*mark*) *marked-lit list* $\Rightarrow$ '*v clause twl-clause* $\Rightarrow$ *bool* **where**
wf-twl-cls M (*TWL-Clause W UW*) $\longleftrightarrow$
distinct-mset W $\wedge$ *size W* $\leq$ *2* $\wedge$ (*size W* < *2* $\longrightarrow$ *set-mset UW* $\subseteq$ *set-mset W*) $\wedge$
($\forall$ *L* $\in$# *W*. $-L$ $\in$ *lits-of M* $\longrightarrow$ ($\forall$ *L'* $\in$# *UW*. *L'* $\notin$# *W* $\longrightarrow$ $-L'$ $\in$ *lits-of M*)) $\wedge$
watched-decided-most-recently M (*TWL-Clause W UW*)

lemma $-L$ $\in$ *lits-of M* $\Longrightarrow$ {*i*. *map lit-of M*!*i* = $-L$} $\neq$ {}
unfolding *set-map-lit-of-lits-of*[*symmetric*] *set-conv-nth*
by (*smt Collect-empty-eq mem-Collect-eq*)

lemma *size-mset-2*: *size x1* = *2* $\longleftrightarrow$ ($\exists$ *a b*. *x1* = {#*a*, *b*#})
by (*metis* (*no-types*, *hide-lams*) *Suc-eq-plus1 one-add-one size-1-singleton-mset*
size-Diff-singleton size-Suc-Diff1 size-eq-Suc-imp-eq-union size-single union-single-eq-diff
union-single-eq-member)

lemma *distinct-mset-size-2*: *distinct-mset* {#*a*, *b*#} $\longleftrightarrow$ *a* $\neq$ *b*
unfolding *distinct-mset-def* **by** *auto*

lemma *wf-twl-cls-annotation-indepnedant*:
assumes *M*: *map lit-of M* = *map lit-of M'*

shows *wf-twl-cls M* (*TWL-Clause W UW*) $\longleftrightarrow$ *wf-twl-cls M′* (*TWL-Clause W UW*)
proof −
have *lits-of M* = *lits-of M′*
using *arg-cong*[*OF M*, *of set*] **by** (*simp add*: *lits-of-def*)
then show *?thesis*
by (*simp add*: *lits-of-def M*)
qed

lemma *wf-twl-cls-wf-twl-cls-tl*:
assumes *wf*: *wf-twl-cls M C* **and** *n-d*: *no-dup M*
shows *wf-twl-cls* (*tl M*) *C*
proof (*cases M*)
case *Nil*
then show *?thesis* **using** *wf*
by (*cases C*) (*simp add*: *wf-twl-cls.simps*[*of tl* -])
next
case (*Cons l M′*) **note** *M* = *this*(*1*)
obtain *W UW* **where** *C*: *C* = *TWL-Clause W UW*
by (*cases C*)
{ **fix** *L L′*
assume
LW: *L* ∈# *W* **and**
LM: − *L* ∈ *lits-of M′* **and**
L′UW: *L′* ∈# *UW* **and**
count W L′ = *0*
then have
L′M: − *L′* ∈ *lits-of M*
using *wf* **by** (*auto simp*: *C M*)
have *watched-decided-most-recently M C*
using *wf* **by** (*auto simp*: *C*)
then have
index (*map lit-of M*) (−*L*) ≤ *index* (*map lit-of M*) (−*L′*)
using *LM L′M L′UW LW* ‹*count W L′* = *0*›
by (*metis* (*no-types*, *lifting*) *C M bspec-mset insert-iff less-not-refl2 lits-of-cons watched-decided-most-recently.simps*)
then have − *L′* ∈ *lits-of M′*
using ‹*count W L′* = *0*› *LW L′M* **by** (*auto simp*: *C M split*: *split-if-asm*)
}
moreover
{
fix *L′ L*
assume
L′ ∈# *W* **and**
L ∈# *UW* **and**
L′M: − *L′* ∈ *lits-of M′* **and**
− *L* ∈ *lits-of M′* **and**
L ∉# *W*
moreover
have *lit-of l* ≠ − *L′*
using *n-d* **unfolding** *M*
by (*metis* (*no-types*) *L′M M Marked-Propagated-in-iff-in-lits-of defined-lit-map distinct.simps*(*2*) *list.simps*(*9*) *set-map*)
moreover have *watched-decided-most-recently M C*
using *wf* **by** (*auto simp*: *C*)
ultimately have *index* (*map lit-of M′*) (− *L′*) ≤ *index* (*map lit-of M′*) (− *L*)

by (*fastforce simp*: *M C split*: *split-if-asm*)
}
moreover have *distinct-mset W* **and** *size W* $\leq$ *2* **and** (*size W* $<$ *2* $\longrightarrow$ *set-mset UW* $\subseteq$ *set-mset W*)
using *wf* **by** (*auto simp*: *C M*)
ultimately show *?thesis* **by** (*auto simp add*: *M C*)
qed

definition *wf-twl-state* :: (*'v*, *'lvl*, *'mark*) *twl-state-abs* $\Rightarrow$ *bool* **where**
wf-twl-state S $\longleftrightarrow$ ($\forall$ *C* $\in\#$ *clauses S*. *wf-twl-cls* (*trail S*) *C*) $\wedge$ *no-dup* (*trail S*)

lemma *wf-candidates-propagate-sound*:
assumes *wf*: *wf-twl-state S* **and**
cand: (*L*, *C*) $\in$ *candidates-propagate S*
shows *trail S* $\models$*as CNot* (*mset-set* (*set-mset C* $-$ {*L*})) $\wedge$ *undefined-lit* (*trail S*) *L*
proof
def *M* $\equiv$ *trail S*
def *N* $\equiv$ *init-clss S*
def *U* $\equiv$ *learned-clss S*

note *MNU-defs* [*simp*] = *M-def N-def U-def*

obtain *Cw* **where** *cw*:
C = *raw-clause Cw*
Cw $\in\#$ *N* + *U*
watched Cw $-$ *mset-set* (*uminus* ‘ *lits-of M*) = {#*L*#}
undefined-lit M L
using *cand* **unfolding** *candidates-propagate-def MNU-defs* **by** *blast*

obtain *W UW* **where** *cw-eq*: *Cw* = *TWL-Clause W UW*
by (*case-tac Cw*, *blast*)

have *l-w*: *L* $\in\#$ *W*
by (*metis Multiset.diff-le-self cw*(*3*) *cw-eq mset-leD multi-member-last twl-clause.sel*(*1*))

have *wf-c*: *wf-twl-cls M Cw*
using *wf* ‹*Cw* $\in\#$ *N* + *U*› **unfolding** *wf-twl-state-def* **by** *simp*

have *w-nw*:
distinct-mset W
size W $<$ *2* $\Longrightarrow$ *set-mset UW* $\subseteq$ *set-mset W*
$\bigwedge$*L L'*. *L* $\in\#$ *W* $\Longrightarrow$ $-$*L* $\in$ *lits-of M* $\Longrightarrow$ *L'* $\in\#$ *UW* $\Longrightarrow$ *L'* $\notin\#$ *W* $\Longrightarrow$ $-$*L'* $\in$ *lits-of M*
using *wf-c* **unfolding** *cw-eq* **by** *auto*

have $\forall$ *L'* $\in$ *set-mset C* $-$ {*L*}. $-$*L'* $\in$ *lits-of M*
proof (*cases size W* $<$ *2*)
case *True*
moreover have *size W* $\neq$ *0*
using *cw*(*3*) *cw-eq* **by** *auto*
ultimately have *size W* = *1*
by *linarith*
then have *w*: *W* = {#*L*#}
by (*metis* (*no-types*, *lifting*) *Multiset.diff-le-self cw*(*3*) *cw-eq single-not-empty size-1-singleton-mset subset-mset.add-diff-inverse union-is-single twl-clause.sel*(*1*))
from *True* **have** *set-mset UW* $\subseteq$ *set-mset W*

```
    using w-nw(2) by blast
  then show ?thesis
    using w cw(1) cw-eq by auto
next
  case sz2: False
  show ?thesis
  proof
    fix L'
    assume l': L' ∈ set-mset C − {L}
    have ex-la: ∃ La. La ≠ L ∧ La ∈# W
    proof (cases W)
      case empty
      thus ?thesis
        using l-w by auto
    next
      case lb: (add W' Lb)
      show ?thesis
      proof (cases W')
        case empty
        thus ?thesis
          using lb sz2 by simp
      next
        case lc: (add W'' Lc)
        thus ?thesis
          by (metis add-gr-0 count-union distinct-mset-single-add lb union-single-eq-member
            w-nw(1))
      qed
    qed
    then obtain La where la: La ≠ L La ∈# W
      by blast
    then have La ∈# mset-set (uminus ‘ lits-of M)
      using cw(3)[unfolded cw-eq, simplified, folded M-def]
      by (metis count-diff count-single diff-zero not-gr0)
    then have nla: −La ∈ lits-of M
      by auto
    then show −L' ∈ lits-of M

    proof −
      have f1: L' ∈ set-mset C
        using l' by blast
      have f2: L' ∉ {L}
        using l' by fastforce
      have ⋀l L. − (l::'a literal) ∈ L ∨ l ∉ uminus ‘ L
        by force
      then have ⋀l. − l ∈ lits-of M ∨ count {#L#} l = count (C − UW) l
        by (metis (no-types) add-diff-cancel-right' count-diff count-mset-set(3) cw(1) cw(3)
              cw-eq diff-zero twl-clause.sel(2))
      then show ?thesis
        by (smt comm-monoid-add-class.add-0 cw(1) cw-eq diff-union-cancelR ex-la f1 f2 insertCI
          less-numeral-extra(3) mem-set-mset-iff plus-multiset.rep-eq single.rep-eq
          twl-clause.sel(1) twl-clause.sel(2) w-nw(3))
    qed
  qed
qed
then show trail S ⊨as CNot (mset-set (set-mset C − {L}))
```

unfolding *true-annots-def* **by** *auto*

show *undefined-lit* (*trail S*) *L*
using *cw*(*4*) *M-def* **by** *blast*
qed

lemma *wf-candidates-propagate-complete*:
assumes *wf*: *wf-twl-state S* **and**
c-mem: $C \in\#$ *raw-clauses S* **and**
l-mem: $L \in\# C$ **and**
unsat: *trail S* $\models as$ *CNot* (*mset-set* (*set-mset C* $-$ $\{L\}$)) **and**
undef: *undefined-lit* (*trail S*) *L*
shows $(L, C) \in$ *candidates-propagate S*
proof $-$
def $M \equiv$ *trail S*
def $N \equiv$ *init-clss S*
def $U \equiv$ *learned-clss S*

note *MNU-defs* [*simp*] = *M-def N-def U-def*

obtain *Cw* **where** *cw*: $C =$ *raw-clause Cw Cw* $\in\# N + U$
using *c-mem* **by** *force*

obtain *W UW* **where** *cw-eq*: $Cw =$ *TWL-Clause W UW*
by (*case-tac Cw*, *blast*)

have *wf-c*: *wf-twl-cls M Cw*
using *wf cw*(*2*) **unfolding** *wf-twl-state-def* **by** *simp*

have *w-nw*:
distinct-mset W
size $W < 2 \Longrightarrow$ *set-mset UW* $\subseteq$ *set-mset W*
$\bigwedge L\ L'.\ L \in\# W \Longrightarrow -L \in$ *lits-of M* $\Longrightarrow L' \in\# UW \Longrightarrow L' \notin\# W \Longrightarrow -L' \in$ *lits-of M*
using *wf-c* **unfolding** *cw-eq* **by** *auto*

have *unit-set*: *set-mset* (W $-$ *mset-set* (*uminus* ‘ *lits-of M*)) = $\{L\}$
proof
show *set-mset* (W $-$ *mset-set* (*uminus* ‘ *lits-of M*)) $\subseteq \{L\}$
proof
fix L'
assume l': $L' \in$ *set-mset* (W $-$ *mset-set* (*uminus* ‘ *lits-of M*))
hence *l′-mem-w*: $L' \in$ *set-mset W*
by *auto*
have $L' \notin$ *uminus* ‘ *lits-of M*
using *distinct-mem-diff-mset*[*OF w-nw*(*1*) *l′*] **by** *simp*
then have $\neg$ *M* $\models a$ $\{\#-L'\#\}$
using *image-iff* **by** *fastforce*
moreover have $L' \in\# C$
using *cw*(*1*) *cw-eq l′-mem-w* **by** *auto*
ultimately have $L' = L$
unfolding *M-def* **by** (*metis unsat*[*unfolded CNot-def true-annots-def*, *simplified*])
then show $L' \in \{L\}$
by *simp*
qed
next

show $\{L\} \subseteq$ *set-mset* $(W - $ *mset-set* $($*uminus* ‘ *lits-of* $M))$
proof *clarify*
have $L \in\# W$
proof $($*cases* $W)$
case *empty*
thus *?thesis*
using *w-nw*(2) *cw*(1) *cw-eq* *l-mem* **by** *auto*
next
case $($*add* W' *La*$)$
thus *?thesis*
proof $($*cases* $La = L)$
case *True*
thus *?thesis*
using *add* **by** *simp*
next
case *False*
have $-La \in$ *lits-of* M
using *False add cw*(1) *cw-eq unsat*[*unfolded CNot-def true-annots-def*, *simplified*]
by *fastforce*
then show *?thesis*
by $($*metis M-def Marked-Propagated-in-iff-in-lits-of add add.left-neutral count-union cw*(1) *cw-eq gr0I l-mem twl-clause.sel*(1) *twl-clause.sel*(2) *undef union-single-eq-member w-nw*$(3))$
qed
qed
moreover have $L \notin\#$ *mset-set* $($*uminus* ‘ *lits-of* $M)$
using *Marked-Propagated-in-iff-in-lits-of undef* **by** *auto*
ultimately show $L \in$ *set-mset* $(W - $ *mset-set* $($*uminus* ‘ *lits-of* $M))$
by *auto*
qed
qed
have *unit*: $W - $ *mset-set* $($*uminus* ‘ *lits-of* $M) = \{\#L\#\}$
by $($*metis distinct-mset-minus distinct-mset-set-mset-ident distinct-mset-singleton set-mset-single unit-set w-nw*$(1))$

show *?thesis*
unfolding *candidates-propagate-def* **using** *unit undef cw cw-eq* **by** *fastforce*
qed

lemma *wf-candidates-conflict-sound*:
assumes *wf*: *wf-twl-state* S **and**
cand: $C \in$ *candidates-conflict* S
shows *trail* $S \models$*as CNot* $C \wedge C \in\#$ *image-mset raw-clause* $($*clauses* $S)$
proof
def $M \equiv$ *trail* S
def $N \equiv$ *init-clss* S
def $U \equiv$ *learned-clss* S

note *MNU-defs* [*simp*] $=$ *M-def N-def U-def*

obtain *Cw* **where** *cw*:
$C =$ *raw-clause Cw*
$Cw \in\# N + U$
watched Cw $\subseteq\#$ *mset-set* $($*uminus* ‘ *lits-of* $($*trail* $S))$
using *cand*[*unfolded candidates-conflict-def*, *simplified*] **by** *auto*

obtain W UW **where** *cw-eq*: $Cw = TWL\text{-}Clause\ W\ UW$
  by (*case-tac Cw*, *blast*)

have *wf-c*: *wf-twl-cls M Cw*
  using *wf cw*(*2*) **unfolding** *wf-twl-state-def* **by** *simp*

have *w-nw*:
  distinct-mset W
  $size\ W < 2 \Longrightarrow set\text{-}mset\ UW \subseteq set\text{-}mset\ W$
  $\bigwedge L\ L'.\ L \in\#\ W \Longrightarrow -L \in lits\text{-}of\ M \Longrightarrow L' \in\#\ UW \Longrightarrow L' \notin\#\ W \Longrightarrow -L' \in lits\text{-}of\ M$
 using *wf-c* **unfolding** *cw-eq* **by** *auto*

have $\forall L \in\#\ C.\ -L \in lits\text{-}of\ M$
proof (*cases* $W = \{\#\}$)
  case *True*
  then have $C = \{\#\}$
    using *cw*(*1*) *cw-eq w-nw*(*2*) **by** *auto*
  then show *?thesis*
    by *simp*
next
  case *False*
  then obtain *La* **where** *la*: $La \in\#\ W$
    using *multiset-eq-iff* **by** *force*
  show *?thesis*
  proof
    fix *L*
    assume *l*: $L \in\#\ C$
    show $-L \in lits\text{-}of\ M$
    proof (*cases* $L \in\#\ W$)
      case *True*
      thus *?thesis*
        using *cw*(*3*) *cw-eq* **by** *fastforce*
    next
      case *False*
      thus *?thesis*
        by (*smt M-def l add-diff-cancel-left′ count-diff cw*(*1*) *cw*(*3*) *la cw-eq*
          diff-zero elem-mset-set finite-imageI finite-lits-of-def gr0I imageE mset-leD
          uminus-of-uminus-id twl-clause.sel(*1*) *twl-clause.sel*(*2*) *w-nw*(*3*))
    qed
  qed
qed
then show *trail S* $\models$*as CNot C*
  unfolding *CNot-def true-annots-def* **by** *auto*

show $C \in\#$ *image-mset raw-clause* (*clauses S*)
  using *cw* **by** *auto*
qed

lemma *wf-candidates-conflict-complete*:
  assumes *wf*: *wf-twl-state S* **and**
    c-mem: $C \in\#$ *raw-clauses S* **and**
    unsat: *trail S* $\models$*as CNot C*
  shows $C \in$ *candidates-conflict S*
proof −

def $M \equiv$ *trail S*
def $N \equiv$ *init-clss S*
def $U \equiv$ *learned-clss S*

note *MNU-defs* [*simp*] = *M-def N-def U-def*

obtain *Cw* **where** *cw*: C = *raw-clause Cw* $Cw \in\# N + U$
  using *c-mem* **by** *force*

obtain *W UW* **where** *cw-eq*: *Cw* = *TWL-Clause W UW*
  by (*case-tac Cw*, *blast*)

have *wf-c*: *wf-twl-cls M Cw*
  using *wf cw*(*2*) **unfolding** *wf-twl-state-def* **by** *simp*

have *w-nw*:
  distinct-mset W
  size $W < 2 \Longrightarrow$ *set-mset UW* $\subseteq$ *set-mset W*
  $\bigwedge L\ L'.\ L \in\# W \Longrightarrow -L \in$ *lits-of M* $\Longrightarrow L' \in\# UW \Longrightarrow L' \notin\# W \Longrightarrow -L' \in$ *lits-of M*
 using *wf-c* **unfolding** *cw-eq* **by** *auto*

have $\bigwedge L.\ L \in\# C \Longrightarrow -L \in$ *lits-of M*
  unfolding *M-def* **using** *unsat*[*unfolded CNot-def true-annots-def*, *simplified*] **by** *blast*
then have *set-mset C* $\subseteq$ *uminus ‘ lits-of M*
  by (*metis imageI mem-set-mset-iff subsetI uminus-of-uminus-id*)
then have *set-mset W* $\subseteq$ *uminus ‘ lits-of M*
  using *cw*(*1*) *cw-eq* **by** *auto*
then have *subset*: $W \subseteq\#$ *mset-set* (*uminus ‘ lits-of M*)
  by (*simp add*: *w-nw*(*1*))

have W = *watched Cw*
  using *cw-eq twl-clause.sel*(*1*) **by** *simp*
then show *?thesis*
  using *MNU-defs cw*(*1*) *cw*(*2*) *subset candidates-conflict-def* **by** *blast*
qed

typedef *'v wf-twl* = {*S*::(*'v*, *nat*, *'v clause*) *twl-state-abs*. *wf-twl-state S*}
morphisms *rough-state-of-twl twl-of-rough-state*
proof −
  have *TWL-State* ([]::(*'v*, *nat*, *'v clause*) *marked-lits*)
    {#} {#} *0 None* ∈ {*S*:: (*'v*, *nat*, *'v clause*) *twl-state-abs*. *wf-twl-state S*}
    by (*auto simp*: *wf-twl-state-def*)
  then show *?thesis* **by** *auto*
qed

lemma [*code abstype*]:
  twl-of-rough-state (*rough-state-of-twl S*) = *S*
  by (*fact CDCL-Two-Watched-Literals.wf-twl.rough-state-of-twl-inverse*)

lemma *wf-twl-state-rough-state-of-twl*[*simp*]: *wf-twl-state* (*rough-state-of-twl S*)
  using *rough-state-of-twl* **by** *auto*

abbreviation *candidates-conflict-twl* :: *'v wf-twl* ⇒ *'v literal multiset set* **where**
candidates-conflict-twl S ≡ *candidates-conflict* (*rough-state-of-twl S*)

abbreviation *candidates-propagate-twl* :: *'v wf-twl* ⇒ (*'v literal* × *'v clause*) *set* **where**
candidates-propagate-twl S ≡ *candidates-propagate* (*rough-state-of-twl S*)

abbreviation *trail-twl* :: *'a wf-twl* ⇒ (*'a*, *nat*, *'a literal multiset*) *marked-lit list* **where**
trail-twl S ≡ *trail* (*rough-state-of-twl S*)

abbreviation *clauses-twl* :: *'a wf-twl* ⇒ *'a literal multiset multiset* **where**
clauses-twl S ≡ *raw-clauses* (*rough-state-of-twl S*)

abbreviation *init-clss-twl* :: *'a wf-twl* ⇒ *'a literal multiset multiset* **where**
init-clss-twl S ≡ *raw-init-clss* (*rough-state-of-twl S*)

abbreviation *learned-clss-twl* :: *'a wf-twl* ⇒ *'a literal multiset multiset* **where**
learned-clss-twl S ≡ *raw-learned-clss* (*rough-state-of-twl S*)

abbreviation *backtrack-lvl-twl* **where**
backtrack-lvl-twl S ≡ *backtrack-lvl* (*rough-state-of-twl S*)

abbreviation *conflicting-twl* **where**
conflicting-twl S ≡ *conflicting* (*rough-state-of-twl S*)

lemma *wf-candidates-twl-conflict-complete*:
  assumes
    c-mem: *C* ∈# *clauses-twl S* **and**
    unsat: *trail-twl S* ⊨*as CNot C*
  shows *C* ∈ *candidates-conflict-twl S*
  using *c-mem unsat wf-candidates-conflict-complete wf-twl-state-rough-state-of-twl* **by** *blast*

21.3 Abstract 2-WL

definition *tl-trail* **where**
  tl-trail S =
   TWL-State (*tl* (*trail S*)) (*init-clss S*) (*learned-clss S*) (*backtrack-lvl S*) (*conflicting S*)

locale *abstract-twl* =
  fixes
    watch :: (*'v*, *nat*, *'v clause*) *twl-state-abs* ⇒ *'v clause* ⇒ *'v clause twl-clause* **and**
    rewatch :: (*'v*, *nat*, *'v literal multiset*) *marked-lit* ⇒ (*'v*, *nat*, *'v clause*) *twl-state-abs* ⇒
      'v clause twl-clause ⇒ *'v clause twl-clause* **and**
    linearize :: *'v clauses* ⇒ *'v clause list* **and**
    restart-learned :: (*'v*, *nat*, *'v clause*) *twl-state-abs* ⇒ *'v clause twl-clause multiset*
  assumes
    clause-watch: *no-dup*(*trail S*) ⟹ *raw-clause* (*watch S C*) = *C* **and**
    wf-watch: *no-dup* (*trail S*) ⟹ *wf-twl-cls* (*trail S*) (*watch S C*) **and**
    clause-rewatch: *raw-clause* (*rewatch L S C'*) = *raw-clause C'* **and**
    wf-rewatch:
      no-dup (*trail S*) ⟹ *undefined-lit* (*trail S*) (*lit-of L*) ⟹ *wf-twl-cls* (*trail S*) *C'* ⟹
        wf-twl-cls (*L* # *trail S*) (*rewatch L S C'*)
      and
    linearize: *mset* (*linearize N*) = *N* **and**
    restart-learned: *restart-learned S* ⊆# *learned-clss S*
begin

lemma *linearize-mempty*[*simp*]: *linearize* {#} = []
  using *linearize mset-zero-iff* **by** *blast*

definition
cons-trail :: ('*v*, *nat*, '*v clause*) *marked-lit* ⇒ ('*v*, *nat*, '*v clause*) *twl-state-abs* ⇒
('*v*, *nat*, '*v clause*) *twl-state-abs*
where
cons-trail L S =
TWL-State (*L* # *trail S*) (*image-mset* (*rewatch L S*) (*init-clss S*))
(*image-mset* (*rewatch L S*) (*learned-clss S*)) (*backtrack-lvl S*) (*conflicting S*)

definition
add-init-cls :: '*v clause* ⇒ ('*v*, *nat*, '*v clause*) *twl-state-abs* ⇒
('*v*, *nat*, '*v clause*) *twl-state-abs*
where
add-init-cls C S =
TWL-State (*trail S*) ({#*watch S C*#} + *init-clss S*) (*learned-clss S*) (*backtrack-lvl S*)
(*conflicting S*)

definition
add-learned-cls :: '*v clause* ⇒ ('*v*, *nat*, '*v clause*) *twl-state-abs* ⇒
('*v*, *nat*, '*v clause*) *twl-state-abs*
where
add-learned-cls C S =
TWL-State (*trail S*) (*init-clss S*) ({#*watch S C*#} + *learned-clss S*) (*backtrack-lvl S*)
(*conflicting S*)

definition
remove-cls :: '*v clause* ⇒ ('*v*, *nat*, '*v clause*) *twl-state-abs* ⇒
('*v*, *nat*, '*v clause*) *twl-state-abs*
where
remove-cls C S =
TWL-State (*trail S*) (*filter-mset* (λ*D*. *raw-clause D* ≠ *C*) (*init-clss S*))
(*filter-mset* (λ*D*. *raw-clause D* ≠ *C*) (*learned-clss S*)) (*backtrack-lvl S*)
(*conflicting S*)

definition *init-state* :: '*v clauses* ⇒ ('*v*, *nat*, '*v clause*) *twl-state-abs* **where**
init-state N = *fold add-init-cls* (*linearize N*) (*TWL-State* [] {#} {#} *0 None*)

lemma *unchanged-fold-add-init-cls*:
trail (*fold add-init-cls Cs* (*TWL-State M N U k C*)) = *M*
learned-clss (*fold add-init-cls Cs* (*TWL-State M N U k C*)) = *U*
backtrack-lvl (*fold add-init-cls Cs* (*TWL-State M N U k C*)) = *k*
conflicting (*fold add-init-cls Cs* (*TWL-State M N U k C*)) = *C*
by (*induct Cs arbitrary*: *N*) (*auto simp*: *add-init-cls-def*)

lemma *unchanged-init-state*[*simp*]:
trail (*init-state N*) = []
learned-clss (*init-state N*) = {#}
backtrack-lvl (*init-state N*) = *0*
conflicting (*init-state N*) = *None*
unfolding *init-state-def* **by** (*rule unchanged-fold-add-init-cls*)+

lemma *clauses-init-fold-add-init*:
no-dup M ⟹
image-mset raw-clause (*init-clss* (*fold add-init-cls Cs* (*TWL-State M N U k C*))) =
mset Cs + *image-mset raw-clause N*
by (*induct Cs arbitrary*: *N*) (*auto simp*: *add.assoc add-init-cls-def clause-watch*)

lemma *init-clss-init-state*[*simp*]: *image-mset raw-clause* (*init-clss* (*init-state N*)) = *N*
unfolding *init-state-def* **by** (*simp add*: *clauses-init-fold-add-init linearize*)

definition *update-backtrack-lvl* **where**
update-backtrack-lvl k S =
TWL-State (*trail S*) (*init-clss S*) (*learned-clss S*) *k* (*conflicting S*)

definition *update-conflicting* **where**
update-conflicting C S = *TWL-State* (*trail S*) (*init-clss S*) (*learned-clss S*) (*backtrack-lvl S*) *C*

definition *restart′* **where**
restart′ S = *TWL-State* [] (*init-clss S*) (*restart-learned S*) *0 None*
end

21.4 Instanciation of the previous locale

definition *pull* :: (*'a* ⇒ *bool*) ⇒ *'a list* ⇒ *'a list* **where**
pull p xs = *filter p xs* @ *filter* (*Not* ∘ *p*) *xs*

lemma *set-pull*[*simp*]: *set* (*pull p xs*) = *set xs*
unfolding *pull-def* **by** *auto*

lemma *mset-pull*[*simp*]: *mset* (*pull p xs*) = *mset xs*
by (*simp add*: *pull-def mset-filter-compl*)

lemma *mset-take-pull-sorted-list-of-set-subseteq*:
mset (*take n* (*pull p* (*sorted-list-of-set* (*set-mset A*)))) ⊆# *A*
by (*metis mset-pull mset-set-set-mset-subseteq mset-sorted-list-of-set mset-take-subseteq subset-mset.dual-order.trans*)

definition *watch-nat* :: (*nat*, *nat*, *nat clause*) *twl-state-abs* ⇒ *nat clause* ⇒
nat clause twl-clause **where**
watch-nat S C =
(*let*
C′ = *remdups* (*sorted-list-of-set* (*set-mset C*));
negation-not-assigned = *filter* (λ*L*. −*L* ∉ *lits-of* (*trail S*)) *C′*;
negation-assigned-sorted-by-trail = *filter* (λ*L*. *L* ∈# *C*) (*map* (λ*L*. −*lit-of L*) (*trail S*));
W = *take 2* (*negation-not-assigned* @ *negation-assigned-sorted-by-trail*);
UW = *sorted-list-of-multiset* (*C* − *mset W*)
in TWL-Clause (*mset W*) (*mset UW*))

lemma *list-cases2*:
fixes *l* :: *'a list*
assumes
l = [] ⟹ *P* **and**
⋀*x*. *l* = [*x*] ⟹ *P* **and**
⋀*x y xs*. *l* = *x* # *y* # *xs* ⟹ *P*
shows *P*
by (*metis assms list.collapse*)

lemma *filter-in-list-prop-verifiedD*:
assumes [*L*←*P* . *Q L*] = *l*
shows ∀ *x* ∈ *set l*. *x* ∈ *set P* ∧ *Q x*
using *assms* **by** *auto*

lemma *no-dup-filter-diff*:
  assumes *n-d*: *no-dup M* **and** *H*: $[L\leftarrow map\ (\lambda L.\ -\ lit\text{-}of\ L)\ M.\ L \in\#\ C] = l$
  shows *distinct l*
  unfolding *H*[*symmetric*]
  apply (*rule distinct-filter*)
  using *n-d* **by** (*induction M*) *auto*

lemma *watch-nat-lists-disjointD*:
  assumes
    l: $[L\leftarrow remdups\ (sorted\text{-}list\text{-}of\text{-}set\ (set\text{-}mset\ C))\ .\ -\ L \notin lits\text{-}of\ (trail\ S)] = l$ **and**
    l': $[L\leftarrow map\ (\lambda L.\ -\ lit\text{-}of\ L)\ (trail\ S)\ .\ L \in\#\ C] = l'$
  shows $\forall x \in set\ l.\ \forall y \in set\ l'.\ x \neq y$
  by (*auto simp*: *l*[*symmetric*] *l'*[*symmetric*] *lits-of-def*)

lemma *watch-nat-list-cases* [*consumes 1*, *case-names nil-nil nil-single nil-other single-nil single-other other*]:
  fixes
    $C :: 'v::linorder\ literal\ multiset$ **and**
    $S :: (('v, 'b, 'c)\ marked\text{-}lit, 'd, 'e, 'f)\ twl\text{-}state$
  defines
    $xs \equiv [L\leftarrow remdups\ (sorted\text{-}list\text{-}of\text{-}set\ (set\text{-}mset\ C))\ .\ -\ L \notin lits\text{-}of\ (trail\ S)]$ **and**
    $ys \equiv [L\leftarrow map\ (\lambda L.\ -\ lit\text{-}of\ L)\ (trail\ S)\ .\ L \in\#\ C]$
  assumes *n-d*: *no-dup* (*trail S*) **and**
    nil-nil: $xs = [] \Longrightarrow ys = [] \Longrightarrow P$ **and**
    nil-single:
      $\bigwedge a.\ xs = [] \Longrightarrow ys = [a] \Longrightarrow\ a \in\#\ C \Longrightarrow P$ **and**
    nil-other: $\bigwedge a\ b\ ys'.\ xs = [] \Longrightarrow ys = a\ \#\ b\ \#\ ys' \Longrightarrow a \neq b \Longrightarrow P$ **and**
    single-nil: $\bigwedge a.\ xs = [a] \Longrightarrow ys = [] \Longrightarrow P$ **and**
    single-other: $\bigwedge a\ b\ ys'.\ xs = [a] \Longrightarrow ys = b\ \#\ ys' \Longrightarrow a \neq b \Longrightarrow P$ **and**
    other: $\bigwedge a\ b\ xs'.\ xs = a\ \#\ b\ \#\ xs' \Longrightarrow a \neq b \Longrightarrow P$
  shows *P*
proof –
  note *xs-def*[*simp*] **and** *ys-def*[*simp*]
  have *dist*: *distinct* $[L\leftarrow remdups\ (sorted\text{-}list\text{-}of\text{-}set\ (set\text{-}mset\ C))\ .\ -\ L \notin lits\text{-}of\ (trail\ S)]$
    by *auto*
  then have *H*: $\bigwedge a\ xs.\ [L\leftarrow remdups\ (sorted\text{-}list\text{-}of\text{-}set\ (set\text{-}mset\ C))\ .\ -\ L \notin lits\text{-}of\ (trail\ S)]$
    $\neq a\ \#\ a\ \#\ xs$
    by *force*
  show *?thesis*
  apply (*cases* $[L\leftarrow remdups\ (sorted\text{-}list\text{-}of\text{-}set\ (set\text{-}mset\ C)).\ -\ L \notin lits\text{-}of\ (trail\ S)]$
        rule: *list-cases2*;
      cases $[L\leftarrow map\ (\lambda L.\ -\ lit\text{-}of\ L)\ (trail\ S)\ .\ L \in\#\ C]$ *rule*: *list-cases2*)
          using *nil-nil* **apply** *simp*
         using *nil-single* **apply** (*force dest*: *filter-in-list-prop-verifiedD*)
        using *nil-other*
        apply (*auto dest*: *filter-in-list-prop-verifiedD watch-nat-lists-disjointD*
          no-dup-filter-diff[*OF n-d*] *simp*: *H*)[]
       using *single-nil* **apply** *simp*
      using *single-other*
      apply (*auto dest*: *filter-in-list-prop-verifiedD watch-nat-lists-disjointD*
        no-dup-filter-diff[*OF n-d*] *simp*: *H*)[]
     using *single-other* **apply** (*auto dest*: *filter-in-list-prop-verifiedD watch-nat-lists-disjointD*
       no-dup-filter-diff[*OF n-d*] *simp*: *H*)[]
    using *other xs-def ys-def* **by** (*metis H*)+
qed

lemma *watch-nat-lists-set-union*:
fixes
C :: *'v*::*linorder literal multiset* **and**
S :: ((*'v*, *'b*, *'c*) *marked-lit*, *'d*, *'e*, *'f*) *twl-state*
defines
xs ≡ [*L*←*remdups* (*sorted-list-of-set* (*set-mset* *C*)) . − *L* ∉ *lits-of* (*trail* *S*)] **and**
ys ≡ [*L*←*map* (λ*L*. − *lit-of* *L*) (*trail* *S*) . *L* ∈# *C*]
assumes *n-d*: *no-dup* (*trail* *S*)
shows *set-mset* *C* = *set* *xs* ∪ *set* *ys*
using *n-d* **unfolding** *xs-def* *ys-def* **by** (*auto simp*: *lits-of-def* *uminus-lit-swap*)

definition
rewatch-nat ::
(*nat*, *nat*, *nat literal multiset*) *marked-lit* ⇒ (*nat*, *nat*, *nat clause*) *twl-state-abs* ⇒
nat clause twl-clause ⇒ *nat clause twl-clause*
where
rewatch-nat *L* *S* *C* =
(*if* − *lit-of* *L* ∈# *watched* *C* *then*
case filter (λ*L'*. *L'* ∉# *watched* *C* ∧ − *L'* ∉ *lits-of* (*L* # *trail* *S*))
(*sorted-list-of-multiset* (*unwatched* *C*)) *of*
[] ⇒ *C*
| *L'* # - ⇒
TWL-Clause (*watched* *C* − {#− *lit-of* *L*#} + {#*L'*#}) (*unwatched* *C* − {#*L'*#} + {#− *lit-of*
L#})
else
C)

lemma *mset-intersection-inclusion*: *A* + (*B* − *A*) = *B* ⟷ *A* ⊆# *B*
apply (*rule iffI*)
apply (*metis mset-le-add-left*)
by (*auto simp*: *ac-simps multiset-eq-iff subseteq-mset-def*)

lemma *clause-watch-nat*:
assumes *no-dup* (*trail* *S*)
shows *raw-clause* (*watch-nat* *S* *C*) = *C*
using *assms*
apply (*cases rule*: *watch-nat-list-cases*[*OF assms*(*1*), *of C*])
by (*auto dest*: *filter-in-list-prop-verifiedD simp*: *watch-nat-def Let-def*
mset-intersection-inclusion subseteq-mset-def)

lemma *distinct-pull*[*simp*]: *distinct* (*pull* *p* *xs*) = *distinct* *xs*
unfolding *pull-def* **by** (*induct xs*) *auto*

lemma *falsified-watiched-imp-unwatched-falsified*:
assumes
watched: *L* ∈ *set* (*take* *n* (*pull* (*Not* ∘ *fls*) (*sorted-list-of-set* (*set-mset* *C*)))) **and**
falsified: *fls* *L* **and**
not-watched: *L'* ∉ *set* (*take* *n* (*pull* (*Not* ∘ *fls*) (*sorted-list-of-set* (*set-mset* *C*)))) **and**
unwatched: *L'* ∈# *C* − *mset* (*take* *n* (*pull* (*Not* ∘ *fls*) (*sorted-list-of-set* (*set-mset* *C*))))
shows *fls* *L'*
proof −
let *?Ls* = *sorted-list-of-set* (*set-mset* *C*)
let *?W* = *take* *n* (*pull* (*Not* ∘ *fls*) *?Ls*)

have $n >$ *length* (*filter* (*Not* $\circ$ *fls*) *?Ls*)
using *watched falsified*
unfolding *pull-def comp-def*
apply *auto*
using *in-set-takeD* **apply** *fastforce*
by (*metis gr0I length-greater-0-conv length-pos-if-in-set take-0 zero-less-diff*)
then have $\bigwedge L.\ L \in set\ ?Ls \Longrightarrow \neg\ fls\ L \Longrightarrow L \in set\ ?W$
unfolding *pull-def* **by** *auto*
then show *?thesis*
by (*metis Multiset.diff-le-self finite-set-mset mem-set-mset-iff mset-leD not-watched sorted-list-of-set unwatched*)
qed

lemma *set-mset-is-single-in-mset-is-single*:
$set\text{-}mset\ C = \{a\} \Longrightarrow x \in\#\ C \Longrightarrow x = a$
by *fastforce*

lemma *index-uminus-index-map-uminus*:
$-a \in set\ L \Longrightarrow index\ L\ (-a) = index\ (map\ uminus\ L)\ (a::'a\ literal)$
by (*induction L*) *auto*

lemma *index-filter*:
$a \in set\ L \Longrightarrow b \in set\ L \Longrightarrow P\ a \Longrightarrow P\ b \Longrightarrow$
$index\ L\ a \leq index\ L\ b \longleftrightarrow index\ (filter\ P\ L)\ a \leq index\ (filter\ P\ L)\ b$
by (*induction L*) *auto*

lemma *wf-watch-nat*: $no\text{-}dup\ (trail\ S) \Longrightarrow wf\text{-}twl\text{-}cls\ (trail\ S)\ (watch\text{-}nat\ S\ C)$
apply (*simp only*: *watch-nat-def Let-def partition-filter-conv case-prod-beta fst-conv snd-conv*)
unfolding *wf-twl-cls.simps*
apply (*intro conjI*)
proof *goal-cases*
case *1*
then show *?case*
by (*cases rule*: *watch-nat-list-cases*[*of S C*]) (*auto dest*: *filter-in-list-prop-verifiedD simp*: *distinct-mset-add-single*)
next
case *2*
then show *?case* **by** *simp*
next
case *3*
then show *?case*
proof (*cases rule*: *watch-nat-list-cases*[*of S C*])
case *nil-nil*
then have *set-mset C* = *set* [] ∪ *set* []
using *3* **by** (*metis watch-nat-lists-set-union*)
then show *?thesis*
by *simp*
next
case *nil-single*
then show *?thesis*
using *watch-nat-lists-set-union*[*of S C*] *3* **by**(*auto dest*!: *arg-cong*[*of* - [] *set*])
next
case *nil-other*
then show *?thesis*
using *3* **by** (*auto dest*!: *arg-cong*[*of* - [] *set*])

next
  case *single-nil*
  show *?thesis*
  using *watch-nat-lists-set-union*[*of S C*] *3 mset-leD* **unfolding** *single-nil* **by** *auto*
next
  case *single-other*
  then show *?thesis*
    using *3* **by** (*auto dest*!: *arg-cong*[*of* - [] *set*])
next
  case *other*
  then show *?thesis*
    using *3* **by** (*auto dest*!: *arg-cong*[*of* - [] *set*])[]
qed
next
case *4* **note** -[*simp*] = *this*
{
fix *a* :: *nat literal* **and** *ys′* :: *nat literal list* **and** *L* :: *nat literal* **and**
  L′ :: *nat literal*
assume *a1*: [*L*←*remdups* (*insort L* (*sorted-list-of-set* (*insert a* (*set ys′*) − {*L*}))).
  − *L* ∉ *lits-of* (*trail S*)] = [*a*]
assume *a2*: *set-mset C* = *insert L* (*insert a* (*set ys′*))
assume *a3*: *L′* ∈# *C*
assume *a4*: *a* ≠ *L′*
have *set* (*L* # *a* # *ys′*) = *set-mset C*
  using *a2* **by** *auto*
then have *L′* ∉ *set* [*l*←*remdups* (*sorted-list-of-set* (*set-mset C*)) . − *l* ∉ *lits-of* (*trail S*)]
  using *a4 a1* **by** (*metis List.finite-set list.set*(*1*) *list.set*(*2*) *singleton-iff*
    sorted-list-of-set.insert-remove)
then have − *L′* ∈ *lits-of* (*trail S*)
  using *a3* **by** *simp*
  } **note** *H* =*this*
show *?case* **using** *4*
  apply (*cases rule*: *watch-nat-list-cases*[*of S C*])
    apply (*auto dest*: *filter-in-list-prop-verifiedD H simp*: *filter-empty-conv*)[*3*]
  using *watch-nat-lists-set-union*[*of S C*] **by** (*auto dest*: *filter-in-list-prop-verifiedD H*)
next
case *5*
then show *?case*
  proof (*cases rule*: *watch-nat-list-cases*[*of S C*])
    case *nil-nil*
    then show *?thesis* **by** *auto*
  next
    case *nil-single*
    then show *?thesis*
      using *watch-nat-lists-set-union*[*of S C*] *5* **by** *auto*
  next
    case *nil-other*
    then show *?thesis*
      unfolding *watched-decided-most-recently.simps Ball-mset-def*
      apply (*intro allI impI*)
      apply (*subst index-uminus-index-map-uminus*,
        simp add: *index-uminus-index-map-uminus lits-of-def o-def*)
      apply (*subst index-uminus-index-map-uminus*,
        simp add: *index-uminus-index-map-uminus lits-of-def o-def*)

apply (*subst index-filter*[*of* - - - $\lambda L.\ L \in\#\ C$])
by (*auto dest*: *filter-in-list-prop-verifiedD*
simp: *uminus-lit-swap lits-of-def o-def*)
next
case *single-nil*
then show *?thesis*
using *watch-nat-lists-set-union*[*of S C*] *5* **by** *auto*
next
case *single-other*
then show *?thesis*
unfolding *watched-decided-most-recently.simps Ball-mset-def*
apply (*clarify*)
apply (*subst index-uminus-index-map-uminus*,
simp add: *index-uminus-index-map-uminus lits-of-def o-def*)
apply (*subst index-uminus-index-map-uminus*,
simp add: *index-uminus-index-map-uminus lits-of-def o-def*)

apply (*subst index-filter*[*of* - - - $\lambda L.\ L \in\#\ C$])
by (*auto dest*: *filter-in-list-prop-verifiedD simp*: *uminus-lit-swap lits-of-def o-def*)
next
case *other*
then show *?thesis*
apply *clarsimp*
apply (*elim disjE*)
prefer *2* **apply** (*auto dest*: *filter-in-list-prop-verifiedD*)[]
apply (*subst index-uminus-index-map-uminus*,
simp add: *index-uminus-index-map-uminus lits-of-def o-def*)[*1*]
apply (*subst index-uminus-index-map-uminus*,
simp add: *index-uminus-index-map-uminus lits-of-def o-def*)[*1*]

apply (*subst index-filter*[*of* - - - $\lambda L.\ L \in\#\ C$])
by (*auto dest*: *filter-in-list-prop-verifiedD*
simp: *index-uminus-index-map-uminus lits-of-def o-def uminus-lit-swap*)
qed
qed

lemma *filter-sorted-list-of-multiset-eqD*:
assumes [$x \leftarrow$ *sorted-list-of-multiset A*. *p x*] = *x* # *xs* (**is** *?comp* = -)
shows $x \in\#\ A$
proof −
have $x \in$ *set ?comp*
using *assms* **by** *simp*
then have $x \in$ *set* (*sorted-list-of-multiset A*)
by *simp*
then show $x \in\#\ A$
by *simp*
qed

lemma *clause-rewatch-nat*: *raw-clause* (*rewatch-nat L S C*) = *raw-clause C*
apply (*auto simp*: *rewatch-nat-def Let-def split*: *list.split*)
apply (*subst subset-mset.add-diff-assoc2*, *simp*)
apply (*subst subset-mset.add-diff-assoc2*, *simp*)
apply (*subst subset-mset.add-diff-assoc2*)
apply (*auto dest*: *filter-sorted-list-of-multiset-eqD*)
by (*metis* (*no-types*, *lifting*) *add.assoc add-diff-cancel-right′ filter-sorted-list-of-multiset-eqD*

insert-DiffM mset-leD mset-le-add-left)

lemma *filter-sorted-list-of-multiset-Nil*:
$[x \leftarrow \textit{sorted-list-of-multiset } M.\ p\ x] = [] \longleftrightarrow (\forall x \in\# M.\ \neg\ p\ x)$
by *auto* (*metis empty-iff filter-set list.set*(*1*) *mem-set-mset-iff member-filter set-sorted-list-of-multiset*)

lemma *filter-sorted-list-of-multiset-ConsD*:
$[x \leftarrow \textit{sorted-list-of-multiset } M.\ p\ x] = x\ \#\ xs \Longrightarrow p\ x$
by (*metis filter-set insert-iff list.set*(*2*) *member-filter*)

lemma *mset-minus-single-eq-mempty*:
$a - \{\#b\#\} = \{\#\} \longleftrightarrow a = \{\#b\#\} \vee a = \{\#\}$
by (*metis Multiset.diff-cancel add.right-neutral diff-single-eq-union diff-single-trivial zero-diff*)

lemma *size-mset-le-2-cases*:
assumes *size* $W \leq 2$
shows $W = \{\#\} \vee (\exists a.\ W = \{\#a\#\}) \vee (\exists a\ b.\ W = \{\#a,b\#\})$
by (*metis One-nat-def Suc-1 Suc-eq-plus1-left assms linorder-not-less nat-less-le not-less-eq-eq ordered-cancel-comm-monoid-diff-class.le-iff-add size-1-singleton-mset size-eq-0-iff-empty size-mset-2*)

lemma *wf-rewatch-nat'*:
assumes
wf: *wf-twl-cls* (*trail S*) *C* **and**
n-d: *no-dup* (*trail S*) **and**
undef: *undefined-lit* (*trail S*) (*lit-of L*)
shows *wf-twl-cls* (*L* # *trail S*) (*rewatch-nat L S C*)
using *filter-sorted-list-of-multiset-Nil*[*simp*]
proof (*cases* − *lit-of L* ∈# *watched C*)
case *falsified*: *True*

let *?unwatched-nonfalsified* =
$[L' \leftarrow \textit{sorted-list-of-multiset } (\textit{unwatched } C)\ .\ L' \notin\# \textit{ watched } C \wedge - L' \notin \textit{lits-of } (L\ \#\ \textit{trail } S)]$
obtain *W UW* **where** *C*: *C* = *TWL-Clause W UW*
by (*cases C*)

show *?thesis*
proof (*cases ?unwatched-nonfalsified*)
case *Nil*
show *?thesis*
unfolding *rewatch-nat-def*
using *falsified Nil*
apply (*simp only*: *wf-twl-cls.simps if-True list.cases C*)
apply (*intro conjI*)
proof *goal-cases*
case *1*
then show *?case* **using** *wf C* **by** *simp*
next
case *2*
then show *?case* **using** *wf C* **by** *simp*
next
case *3*
then show *?case* **using** *wf C* **by** *simp*

next
case *4*
then show *?case* **using** *wf C* **by** *auto*
next
case *5*
then show *?case*
using *C* **apply** *simp*
using *wf* **by** (*smt ball-msetI bspec-mset not-gr0 uminus-of-uminus-id watched-decided-most-recently.simps wf-twl-cls.simps*)
qed
next
case (*Cons L′ Ls*)
show *?thesis*
unfolding *rewatch-nat-def C*
using *falsified Cons*
apply (*simp only*: *wf-twl-cls.simps if-True list.cases C*)
apply (*intro conjI*)
proof *goal-cases*
case *1*
then show *?case* **using** *wf C n-d*
by (*smt Multiset.diff-le-self distinct-mset-add-single distinct-mset-single-add filter-sorted-list-of-multiset-ConsD insert-DiffM mset-leD twl-clause.sel*(*1*) *wf-twl-cls.simps*)
next
case *2*
then show *?case* **using** *wf C* **by** (*metis insert-DiffM2 size-single size-union twl-clause.sel*(*1*) *wf-twl-cls.simps*)
next
case *3*
then show *?case*
using *wf C* **by** (*force simp*: *mset-minus-single-eq-mempty dest*: *subset-singletonD*)
next
case *4*
have *H*: $\forall L{\in}\#W.\ -L \in$ *lits-of* (*trail S*) $\longrightarrow$
($\forall L'{\in}\#UW.$ *count W L′* $= 0 \longrightarrow -L' \in$ *lits-of* (*trail S*))
using *wf* **by** (*auto simp*: *C*)
have *W*: *size W* ≤ 2 **and** *W-UW*: *size W* $< 2 \longrightarrow$ *set-mset UW* $\subseteq$ *set-mset W*
using *wf* **by** (*auto simp*: *C*)

have *distinct*: *distinct-mset W*
using *wf* **by** (*auto simp*: *C*)
show *?case*
using *4*
unfolding *C watched-decided-most-recently.simps Ball-mset-def twl-clause.sel*
apply (*intro allI impI*)
apply (*rename-tac xW xUW*)
apply (*case-tac* $-$ *lit-of L* $=$ *xW*; *case-tac xW* $=$ *xUW*; *case-tac L′* $=$ *xW*)
apply (*auto simp*: *uminus-lit-swap*)[*2*]
using *filter-sorted-list-of-multiset-ConsD* **apply** *blast*
using *H size-mset-le-2-cases*[*OF W*]
using *distinct* **apply** (*fastforce split*: *split-if-asm simp*: *distinct-mset-size-2*)
using *distinct* **apply** (*fastforce split*: *split-if-asm simp*: *distinct-mset-size-2*)
using *distinct* **apply** (*fastforce split*: *split-if-asm simp*: *distinct-mset-size-2*)
using *filter-sorted-list-of-multiset-ConsD* **apply** *blast*
using *size-mset-le-2-cases*[*OF W*] *H* **by** (*fastforce simp*: *uminus-lit-swap*

dest: *filter-sorted-list-of-multiset-ConsD filter-sorted-list-of-multiset-eqD*)

next
case *5*
have H: $\forall x.\ x \in\#\ W \longrightarrow -\ x \in$ *lits-of* (*trail S*) $\longrightarrow$ ($\forall x.\ x \in\#\ UW \longrightarrow$ *count W x* $= 0$ $\longrightarrow -\ x \in$ *lits-of* (*trail S*))
using *wf* **by** (*auto simp*: *C*)

show *?case*
using *5* **unfolding** *C watched-decided-most-recently.simps Ball-mset-def*
apply (*intro allI impI conjI*)
apply (*rename-tac xW x*)
apply (*case-tac* $-$ *lit-of L* $=$ *xW*; *case-tac xW* $=$ *x*)
apply (*auto simp*: *uminus-lit-swap*)[*3*]
apply (*case-tac* $-$ *lit-of L* $=$ *x*)
apply (*clarsimp*)
using *H* **apply** (*blast dest*: *filter-sorted-list-of-multiset-ConsD filter-sorted-list-of-multiset-eqD*)
apply (*clarsimp*)
using *H* **apply** (*blast dest*: *filter-sorted-list-of-multiset-ConsD filter-sorted-list-of-multiset-eqD*)
done
qed
qed
next
case *False*
then have *wf-twl-cls* (*L* # *trail S*) *C*
apply (*cases C*)
using *wf n-d undef* **apply** (*clarify*)
unfolding *wf-twl-cls.simps*
apply (*intro conjI*)
apply *blast*
apply *blast*
apply *blast*
apply (*smt ball-mset-cong bspec-mset insert-iff lits-of-cons nat-neq-iff twl-clause.sel*(*1*) *uminus-of-uminus-id*)
apply (*auto simp*: *Marked-Propagated-in-iff-in-lits-of*)
done
then show *?thesis*
unfolding *rewatch-nat-def* **using** *False* **by** *simp*
qed

interpretation *twl*: *abstract-twl watch-nat rewatch-nat sorted-list-of-multiset learned-clss*
apply *unfold-locales*
apply (*rule clause-watch-nat*; *simp*)
apply (*rule wf-watch-nat*; *simp*)
apply (*rule clause-rewatch-nat*)
apply (*rule wf-rewatch-nat'*; *simp*)
apply (*rule mset-sorted-list-of-multiset*)
apply (*rule subset-mset.order-refl*)
done

21.5 Interpretation for $cdcl_W$*-ops.*$cdcl_W$

context *abstract-twl*

begin

21.5.1 Direct Interpretation

interpretation *rough-cdcl*: $state_W$ *trail raw-init-clss raw-learned-clss backtrack-lvl conflicting cons-trail tl-trail add-init-cls add-learned-cls remove-cls update-backtrack-lvl update-conflicting init-state restart'*
apply *unfold-locales*
apply (*simp-all add*: *add-init-cls-def add-learned-cls-def clause-rewatch clause-watch cons-trail-def remove-cls-def restart'-def tl-trail-def update-backtrack-lvl-def update-conflicting-def*)
apply (*rule image-mset-subseteq-mono*[*OF restart-learned*])
done

interpretation *rough-cdcl*: $cdcl_W$*-ops trail raw-init-clss raw-learned-clss backtrack-lvl conflicting cons-trail tl-trail add-init-cls add-learned-cls remove-cls update-backtrack-lvl update-conflicting init-state restart'*
by *unfold-locales*

interpretation $cdcl_{NOT}$: $cdcl_{NOT}$*-merge-bj-learn-ops*
$\lambda S.$ *convert-trail-from-W* (*trail S*)
rough-cdcl.clauses
$\lambda L\ S.$ *cons-trail* (*convert-marked-lit-from-NOT L*) *S*
$\lambda S.$ *tl-trail S*
$\lambda C\ S.$ *add-learned-cls C S*
$\lambda C\ S.$ *remove-cls C S*
$\lambda L\ S.$ *lit-of* $L \in$ *fst* ` *candidates-propagate S*
λ- $S.$ *conflicting S = None*
$\lambda C\ C'\ L'\ S.\ C \in$ *candidates-conflict S* $\wedge$ *distinct-mset* $(C' + \{\#L'\#\})$ $\wedge$ $\neg$*tautology* $(C' + \{\#L'\#\})$
by *unfold-locales*

21.5.2 Opaque Type with Invariant

declare *rough-cdcl.state-simp*[*simp del*]

definition *cons-trail-twl* :: ('*v*, *nat*, '*v literal multiset*) *marked-lit* ⇒ '*v wf-twl* ⇒ '*v wf-twl*
where
cons-trail-twl L S ≡ *twl-of-rough-state* (*cons-trail L* (*rough-state-of-twl S*))

lemma *wf-twl-state-cons-trail*:
undefined-lit (*trail S*) (*lit-of L*) $\Longrightarrow$ *wf-twl-state S* $\Longrightarrow$ *wf-twl-state* (*cons-trail L S*)
unfolding *wf-twl-state-def* **by** (*auto simp*: *cons-trail-def wf-rewatch defined-lit-map*)

lemma *rough-state-of-twl-cons-trail*:
undefined-lit (*trail-twl S*) (*lit-of L*) $\Longrightarrow$
rough-state-of-twl (*cons-trail-twl L S*) = *cons-trail L* (*rough-state-of-twl S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-cons-trail*
unfolding *cons-trail-twl-def* **by** *blast*

abbreviation *add-init-cls-twl* **where**
add-init-cls-twl C S ≡ *twl-of-rough-state* (*add-init-cls C* (*rough-state-of-twl S*))

lemma *wf-twl-add-init-cls*: *wf-twl-state S* $\Longrightarrow$ *wf-twl-state* (*add-init-cls L S*)
unfolding *wf-twl-state-def* **by** (*auto simp*: *wf-watch add-init-cls-def split*: *split-if-asm*)

lemma *rough-state-of-twl-add-init-cls*:

rough-state-of-twl (*add-init-cls-twl* *L* *S*) = *add-init-cls* *L* (*rough-state-of-twl* *S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-add-init-cls* **by** *blast*

abbreviation *add-learned-cls-twl* **where**
add-learned-cls-twl *C* *S* ≡ *twl-of-rough-state* (*add-learned-cls* *C* (*rough-state-of-twl* *S*))

lemma *wf-twl-add-learned-cls*: *wf-twl-state* *S* ⟹ *wf-twl-state* (*add-learned-cls* *L* *S*)
unfolding *wf-twl-state-def* **by** (*auto simp*: *wf-watch add-learned-cls-def split*: *split-if-asm*)

lemma *rough-state-of-twl-add-learned-cls*:
rough-state-of-twl (*add-learned-cls-twl* *L* *S*) = *add-learned-cls* *L* (*rough-state-of-twl* *S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-add-learned-cls* **by** *blast*

abbreviation *remove-cls-twl* **where**
remove-cls-twl *C* *S* ≡ *twl-of-rough-state* (*remove-cls* *C* (*rough-state-of-twl* *S*))

lemma *wf-twl-remove-cls*: *wf-twl-state* *S* ⟹ *wf-twl-state* (*remove-cls* *L* *S*)
unfolding *wf-twl-state-def* **by** (*auto simp*: *wf-watch remove-cls-def split*: *split-if-asm*)

lemma *rough-state-of-twl-remove-cls*:
rough-state-of-twl (*remove-cls-twl* *L* *S*) = *remove-cls* *L* (*rough-state-of-twl* *S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-remove-cls* **by** *blast*

abbreviation *init-state-twl* **where**
init-state-twl *N* ≡ *twl-of-rough-state* (*init-state* *N*)

lemma *wf-twl-state-wf-twl-state-fold-add-init-cls*:
assumes *wf-twl-state* *S*
shows *wf-twl-state* (*fold add-init-cls* *N* *S*)
using *assms* **apply** (*induction* *N* *arbitrary*: *S*)
apply (*auto simp*: *wf-twl-state-def*)[]
by (*simp add*: *wf-twl-add-init-cls*)

lemma *wf-twl-state-epsilon-state*[*simp*]:
wf-twl-state (*TWL-State* [] {#} {#} *0* *None*)
by (*auto simp*: *wf-twl-state-def*)

lemma *wf-twl-init-state*: *wf-twl-state* (*init-state* *N*)
unfolding *init-state-def* **by** (*auto intro*!: *wf-twl-state-wf-twl-state-fold-add-init-cls*)

lemma *rough-state-of-twl-init-state*:
rough-state-of-twl (*init-state-twl* *N*) = *init-state* *N*
by (*simp add*: *twl-of-rough-state-inverse wf-twl-init-state*)

abbreviation *tl-trail-twl* **where**
tl-trail-twl *S* ≡ *twl-of-rough-state* (*tl-trail* (*rough-state-of-twl* *S*))

lemma *wf-twl-state-tl-trail*: *wf-twl-state* *S* ⟹ *wf-twl-state* (*tl-trail* *S*)
by (*simp add*: *twl-of-rough-state-inverse wf-twl-init-state wf-twl-cls-wf-twl-cls-tl*
tl-trail-def wf-twl-state-def distinct-tl map-tl)

lemma *rough-state-of-twl-tl-trail*:
rough-state-of-twl (*tl-trail-twl* *S*) = *tl-trail* (*rough-state-of-twl* *S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-tl-trail* **by** *blast*

abbreviation *update-backtrack-lvl-twl* **where**
update-backtrack-lvl-twl k S ≡ *twl-of-rough-state* (*update-backtrack-lvl k* (*rough-state-of-twl S*))

lemma *wf-twl-state-update-backtrack-lvl*:
wf-twl-state S ⟹ *wf-twl-state* (*update-backtrack-lvl k S*)
unfolding *wf-twl-state-def* **by** (*auto simp*: *update-backtrack-lvl-def*)

lemma *rough-state-of-twl-update-backtrack-lvl*:
rough-state-of-twl (*update-backtrack-lvl-twl k S*) = *update-backtrack-lvl k* (*rough-state-of-twl S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-update-backtrack-lvl* **by** *fast*

abbreviation *update-conflicting-twl* **where**
update-conflicting-twl k S ≡ *twl-of-rough-state* (*update-conflicting k* (*rough-state-of-twl S*))

lemma *wf-twl-state-update-conflicting*:
wf-twl-state S ⟹ *wf-twl-state* (*update-conflicting k S*)
unfolding *wf-twl-state-def* **by** (*auto simp*: *update-conflicting-def*)

lemma *rough-state-of-twl-update-conflicting*:
rough-state-of-twl (*update-conflicting-twl k S*) = *update-conflicting k* (*rough-state-of-twl S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-update-conflicting* **by** *fast*

abbreviation *raw-clauses-twl* **where**
raw-clauses-twl S ≡ *raw-clauses* (*rough-state-of-twl S*)

abbreviation *restart-twl* **where**
restart-twl S ≡ *twl-of-rough-state* (*restart′* (*rough-state-of-twl S*))

lemma *wf-wf-restart′*: *wf-twl-state S* ⟹ *wf-twl-state* (*restart′ S*)
unfolding *restart′-def wf-twl-state-def* **apply** *standard*
apply *clarify*
apply (*rename-tac x*)
apply (*subgoal-tac wf-twl-cls* (*trail S*) *x*)
apply (*case-tac x*)
using *restart-learned* **by** *fastforce+*

lemma *rough-state-of-twl-restart-twl*:
rough-state-of-twl (*restart-twl S*) = *restart′* (*rough-state-of-twl S*)
by (*simp add*: *twl-of-rough-state-inverse wf-wf-restart′*)

interpretation $cdcl_W$*-twl-NOT*: *dpll-state*
λS. convert-trail-from-W (*trail-twl S*)
raw-clauses-twl
λL S. cons-trail-twl (*convert-marked-lit-from-NOT L*) *S*
λS. tl-trail-twl S
λC S. add-learned-cls-twl C S
λC S. remove-cls-twl C S
apply *unfold-locales*
apply (*simp add*: *rough-state-of-twl-cons-trail*)
apply (*metis rough-state-of-twl-tl-trail rough-cdcl.tl-trail*)
apply (*metis rough-state-of-twl-add-learned-cls rough-cdcl.trail-add-cls*$_{NOT}$)
apply (*metis rough-state-of-twl-remove-cls rough-cdcl.trail-remove-cls*)

apply (*simp add*: *rough-state-of-twl-cons-trail*)
apply (*simp add*: *twl.rough-state-of-twl-tl-trail*)
using *rough-cdcl.clauses-add-cls*$_{NOT}$ *rough-cdcl.clauses-def rough-state-of-twl-add-learned-cls*
apply *auto*[*1*]
using *rough-cdcl.clauses-def rough-cdcl.clauses-remove-cls rough-state-of-twl-remove-cls* **by** *auto*

interpretation *cdcl*$_W$*-twl*: *state*$_W$
trail-twl
init-clss-twl
learned-clss-twl
backtrack-lvl-twl
conflicting-twl
cons-trail-twl
tl-trail-twl
add-init-cls-twl
add-learned-cls-twl
remove-cls-twl
update-backtrack-lvl-twl
update-conflicting-twl
init-state-twl
restart-twl
apply *unfold-locales*
by (*simp-all add*: *rough-state-of-twl-cons-trail rough-state-of-twl-tl-trail*
rough-state-of-twl-add-init-cls rough-state-of-twl-add-learned-cls rough-state-of-twl-remove-cls
rough-state-of-twl-update-backtrack-lvl rough-state-of-twl-update-conflicting
rough-state-of-twl-init-state rough-state-of-twl-restart-twl
rough-cdcl.learned-clss-restart-state)

interpretation *cdcl*$_W$*-twl*: *cdcl*$_W$*-ops*
trail-twl
init-clss-twl
learned-clss-twl
backtrack-lvl-twl
conflicting-twl
cons-trail-twl
tl-trail-twl
add-init-cls-twl
add-learned-cls-twl
remove-cls-twl
update-backtrack-lvl-twl
update-conflicting-twl
init-state-twl
restart-twl
by *unfold-locales*

abbreviation *state-eq-twl* (**infix** $\sim TWL$ *51*) **where**
state-eq-twl S S' $\equiv$ *rough-cdcl.state-eq* (*rough-state-of-twl* S) (*rough-state-of-twl* S')
notation *cdcl*$_W$*-twl.state-eq* (**infix** $\sim$ *51*)
declare *cdcl*$_W$*-twl.state-simp*[*simp del*]
cdcl$_W$*-twl-NOT.state-simp*$_{NOT}$[*simp del*]

To avoid ambiguities:

no-notation *CDCL-Two-Watched-Literals.twl.state-eq-twl* (**infix** $\sim TWL$ *51*)

definition *propagate-twl* **where**

propagate-twl S S' $\longleftrightarrow$
($\exists L$ C. (L, C) $\in$ *candidates-propagate-twl* S
$\wedge$ S' $\sim TWL$ *cons-trail-twl* (*Propagated* L C) S
$\wedge$ *conflicting-twl* S = *None*)

lemma *propagate-twl-iff-propagate*:
assumes *inv*: $cdcl_W$*-twl.cdcl*$_W$*-all-struct-inv* S
shows $cdcl_W$*-twl.propagate* S T $\longleftrightarrow$ *propagate-twl* S T (**is** *?P* $\longleftrightarrow$ *?T*)
proof
assume *?P*
then obtain C L **where**
conflicting (*rough-state-of-twl* S) = *None* **and**
CL-Clauses: C + {#L#} $\in$# $cdcl_W$*-twl.clauses* S **and**
tr-CNot: *trail-twl* S $\models$*as* *CNot* C **and**
undef-lot: *undefined-lit* (*trail-twl* S) L **and**
T $\sim$ *cons-trail-twl* (*Propagated* L (C + {#L#})) S
unfolding $cdcl_W$*-twl.propagate.simps* **by** *blast*
have *distinct-mset* (C + {#L#})
using *inv* *CL-Clauses* **unfolding** $cdcl_W$*-twl.cdcl*$_W$*-all-struct-inv-def*
$cdcl_W$*-twl.distinct-cdcl*$_W$*-state-def* $cdcl_W$*-twl.clauses-def* *distinct-mset-set-def*
by (*metis* (*no-types*, *lifting*) *add-gr-0* *mem-set-mset-iff* *plus-multiset.rep-eq*)
then have *C-L-L*: *mset-set* (*set-mset* (C + {#L#}) − {L}) = C
by (*metis* *Un-insert-right* *add-diff-cancel-left'* *add-diff-cancel-right'*
distinct-mset-set-mset-ident *finite-set-mset* *insert-absorb2* *mset-set.insert-remove*
set-mset-single *set-mset-union*)
have (L, C+{#L#}) $\in$ *candidates-propagate-twl* S
apply (*rule* *wf-candidates-propagate-complete*)
using *rough-state-of-twl* **apply** *auto*[]
using *CL-Clauses* **unfolding** $cdcl_W$*-twl.clauses-def* **apply** *auto*[]
apply *simp*
using *C-L-L* *tr-CNot* **apply** *simp*
using *undef-lot* **apply** *blast*
done
show *?T* **unfolding** *propagate-twl-def*
apply (*rule* *exI*[*of* - L], *rule* *exI*[*of* - C+{#L#}])
apply (*auto* *simp*: ‹(L, C+{#L#}) $\in$ *candidates-propagate-twl* S›
‹*conflicting* (*rough-state-of-twl* S) = *None*›)
using ‹T $\sim$ *cons-trail-twl* (*Propagated* L (C + {#L#})) S› $cdcl_W$*-twl.state-eq-backtrack-lvl*
$cdcl_W$*-twl.state-eq-conflicting* $cdcl_W$*-twl.state-eq-init-clss*
$cdcl_W$*-twl.state-eq-learned-clss* $cdcl_W$*-twl.state-eq-trail* *rough-cdcl.state-eq-def* **by** *blast*
next
assume *?T*
then obtain L C **where**
LC: (L, C) $\in$ *candidates-propagate-twl* S **and**
T: T $\sim TWL$ *cons-trail-twl* (*Propagated* L C) S **and**
confl: *conflicting* (*rough-state-of-twl* S) = *None*
unfolding *propagate-twl-def* **by** *auto*
have [*simp*]: C − {#L#} + {#L#} = C
using *LC* **unfolding** *candidates-propagate-def*
by *clarify* (*metis* *add.commute* *add-diff-cancel-right'* *count-diff* *insert-DiffM*
multi-member-last *not-gr0* *zero-diff*)
have C $\in$# *raw-clauses-twl* S
using *LC* **unfolding** *candidates-propagate-def* *rough-cdcl.clauses-def* **by** *auto*
then have *distinct-mset* C
using *inv* **unfolding** $cdcl_W$*-twl.cdcl*$_W$*-all-struct-inv-def* $cdcl_W$*-twl.distinct-cdcl*$_W$*-state-def*

cdcl$_W$*-twl.clauses-def distinct-mset-set-def rough-cdcl.clauses-def* **by** *auto*
then have *C-L-L*: *mset-set* (*set-mset* $C - \{L\}$) = $C - \{\#L\#\}$
by (*metis* ‹$C - \{\#L\#\} + \{\#L\#\} = C$› *add-left-imp-eq diff-single-trivial distinct-mset-set-mset-ident finite-set-mset mem-set-mset-iff mset-set.remove multi-self-add-other-not-self union-commute*)

show *?P*
apply (*rule cdcl*$_W$*-twl.propagate.intros*[*of - trail-twl S init-clss-twl S learned-clss-twl S backtrack-lvl-twl S* $C-\{\#L\#\}$ *L*])
using *confl* **apply** *auto*[]
using *LC* **unfolding** *candidates-propagate-def* **apply** (*auto simp*: *cdcl*$_W$*-twl.clauses-def*)[]
using *wf-candidates-propagate-sound*[*OF - LC*] *rough-state-of-twl* **apply** (*simp add*: *C-L-L*)
using *wf-candidates-propagate-sound*[*OF - LC*] *rough-state-of-twl* **apply** *simp*
using *T* **unfolding** *cdcl*$_W$*-twl.state-eq-def rough-cdcl.state-eq-def* **by** *auto*
qed

term *local.state-eq-twl*
term *CDCL-Two-Watched-Literals.twl.state-eq-twl*
definition *conflict-twl* **where**
conflict-twl $S\ S' \longleftrightarrow$
($\exists C.\ C \in$ *candidates-conflict-twl S*
$\wedge\ S' \sim TWL$ *update-conflicting-twl* (*Some C*) *S*
$\wedge$ *conflicting-twl* $S = None$)

lemma *conflict-twl-iff-conflict*:
shows *cdcl*$_W$*-twl.conflict S T* $\longleftrightarrow$ *conflict-twl S T* (**is** *?C* $\longleftrightarrow$ *?T*)
proof
assume *?C*
then obtain *M N U k C* **where**
S: *rough-cdcl.state* (*rough-state-of-twl S*) = (*M*, *N*, *U*, *k*, *None*) **and**
C: $C \in\#$ *cdcl*$_W$*-twl.clauses S* **and**
M-C: $M \models as$ *CNot C* **and**
T: $T \sim$ *update-conflicting-twl* (*Some C*) *S*
by *auto*
have $C \in$ *candidates-conflict-twl S*
apply (*rule wf-candidates-conflict-complete*)
apply *simp*
using *C* **apply** (*auto simp*: *cdcl*$_W$*-twl.clauses-def*)[]
using *M-C S* **by** *auto*
moreover have $T \sim TWL$ *twl-of-rough-state* (*update-conflicting* (*Some C*) (*rough-state-of-twl S*))
using *T* **unfolding** *rough-cdcl.state-eq-def cdcl*$_W$*-twl.state-eq-def* **by** *auto*
ultimately show *?T*
using *S* **unfolding** *conflict-twl-def* **by** *auto*
next
assume *?T*
then obtain *C* **where**
C: $C \in$ *candidates-conflict-twl S* **and**
T: $T \sim TWL$ *update-conflicting-twl* (*Some C*) *S* **and**
confl: *conflicting-twl* $S = None$
unfolding *conflict-twl-def* **by** *auto*
have $C \in\#$ *cdcl*$_W$*-twl.clauses S*
using *C* **unfolding** *candidates-conflict-def cdcl*$_W$*-twl.clauses-def* **by** *auto*
moreover have *trail-twl S* $\models as$ *CNot C*
using *wf-candidates-conflict-sound*[*OF - C*] **by** *auto*
ultimately show *?C* **apply** −

apply (*rule* $cdcl_W$*-twl.conflict.conflict-rule*[*of* - - - - - *C*])
using *confl T* **unfolding** *rough-cdcl.state-eq-def* $cdcl_W$*-twl.state-eq-def* **by** *auto*
qed

inductive $cdcl_W$*-twl* :: *'v wf-twl* ⇒ *'v wf-twl* ⇒ *bool* **for** *S* :: *'v wf-twl* **where**
propagate: *propagate-twl S S'* ⟹ $cdcl_W$*-twl S S'* |
conflict: *conflict-twl S S'* ⟹ $cdcl_W$*-twl S S'* |
other: $cdcl_W$*-twl*.$cdcl_W$*-o S S'* ⟹ $cdcl_W$*-twl S S'*|
rf: $cdcl_W$*-twl*.$cdcl_W$*-rf S S'* ⟹ $cdcl_W$*-twl S S'*

lemma $cdcl_W$*-twl-iff*-$cdcl_W$:
assumes $cdcl_W$*-twl*.$cdcl_W$*-all-struct-inv S*
shows $cdcl_W$*-twl S T* ⟷ $cdcl_W$*-twl*.$cdcl_W$ *S T*
by (*simp add*: *assms* $cdcl_W$*-twl*.$cdcl_W$*.simps* $cdcl_W$*-twl.simps conflict-twl-iff-conflict*
propagate-twl-iff-propagate)

lemma *rtranclp*-$cdcl_W$*-twl-all-struct-inv-inv*:
assumes $cdcl_W$*-twl*** *S T* **and** $cdcl_W$*-twl*.$cdcl_W$*-all-struct-inv S*
shows $cdcl_W$*-twl*.$cdcl_W$*-all-struct-inv T*
using *assms* **by** (*induction rule*: *rtranclp-induct*)
(*simp-all add*: $cdcl_W$*-twl-iff*-$cdcl_W$ $cdcl_W$*-twl*.$cdcl_W$*-all-struct-inv-inv*)

lemma *rtranclp*-$cdcl_W$*-twl-iff-rtranclp*-$cdcl_W$:
assumes $cdcl_W$*-twl*.$cdcl_W$*-all-struct-inv S*
shows $cdcl_W$*-twl*** *S T* ⟷ $cdcl_W$*-twl*.$cdcl_W$** *S T* (**is** *?T* ⟷ *?W*)
proof
assume *?W*
then show *?T*
proof (*induction rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *simp*
next
case (*step T U*) **note** *st* = *this*(*1*) **and** *cdcl* = *this*(*2*) **and** *IH* = *this*(*3*)
have $cdcl_W$*-twl T U*
using *assms st cdcl* $cdcl_W$*-twl.rtranclp*-$cdcl_W$*-all-struct-inv-inv* $cdcl_W$*-twl-iff*-$cdcl_W$
by *blast*
then show *?case* **using** *IH* **by** *auto*
qed
next
assume *?T*
then show *?W*
proof (*induction rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *simp*
next
case (*step T U*) **note** *st* = *this*(*1*) **and** *cdcl* = *this*(*2*) **and** *IH* = *this*(*3*)
have $cdcl_W$*-twl*.$cdcl_W$ *T U*
using *assms st cdcl rtranclp*-$cdcl_W$*-twl-all-struct-inv-inv* $cdcl_W$*-twl-iff*-$cdcl_W$
by *blast*
then show *?case* **using** *IH* **by** *auto*
qed
qed

interpretation $cdcl_{NOT}$*-twl*: *backjumping-ops*
λS. convert-trail-from-W (*trail-twl S*)

abstract-twl.raw-clauses-twl
λL (*S*:: *'v wf-twl*).
cons-trail-twl
(*convert-marked-lit-from-NOT L*) (*S*:: *'v wf-twl*)
tl-trail-twl
add-learned-cls-twl
remove-cls-twl
λC - - (*S*:: *'v wf-twl*) -. $C \in$ *candidates-conflict-twl S*
by *unfold-locales*

lemma *reduce-trail-to*$_{NOT}$*-skip-beginning-twl*:
assumes *trail-twl S* = *convert-trail-from-NOT* (F' @ F)
shows *trail-twl* (*cdcl*$_W$*-twl.reduce-trail-to*$_{NOT}$ *F S*) = *convert-trail-from-NOT F*
using *assms* **by** (*induction* F' *arbitrary*: *S*) *auto*

lemma *reduce-trail-to*$_{NOT}$*-trail-tl-trail-twl-decomp*[*simp*]:
trail-twl S = *convert-trail-from-NOT* (F' @ *Marked K* () # *F*) $\Longrightarrow$
trail-twl (*cdcl*$_W$*-twl.reduce-trail-to*$_{NOT}$ *F* (*tl-trail-twl S*)) = *convert-trail-from-NOT F*
apply (*rule reduce-trail-to*$_{NOT}$*-skip-beginning-twl*[*of - tl* (F' @ *Marked K* () # [])])
by (*cases* F') (*auto simp add*:*tl-append rough-cdcl.reduce-trail-to*$_{NOT}$*-skip-beginning*)

lemma *trail-twl-reduce-trail-to*$_{NOT}$*-drop*:
trail-twl (*cdcl*$_W$*-twl.reduce-trail-to*$_{NOT}$ *F S*) =
(*if length* (*trail-twl S*) $\geq$ *length F*
then drop (*length* (*trail-twl S*) $-$ *length F*) (*trail-twl S*)
else [])
apply (*induction F S rule*: *cdcl*$_W$*-twl.reduce-trail-to*$_{NOT}$*.induct*)
apply (*rename-tac F S*)
apply (*case-tac trail-twl S*)
apply *auto*[]
apply (*rename-tac list*)
apply (*case-tac Suc* (*length list*) > *length F*)
prefer *2* **apply** *simp*
apply (*subgoal-tac Suc* (*length list*) $-$ *length F* = *Suc* (*length list* $-$ *length F*))
apply *simp*
apply *simp*
done

lemma *undefined-lit-convert-trail-from-NOT*[*simp*]:
undefined-lit (*convert-trail-from-NOT F*) *L* $\longleftrightarrow$ *undefined-lit F L*
by (*induction F rule*: *marked-lit-list-induct*) (*auto simp*: *defined-lit-map*)

lemma *lits-of-convert-trail-from-NOT*:
lits-of (*convert-trail-from-NOT F*) = *lits-of F*
by (*induction F rule*: *marked-lit-list-induct*) *auto*

lemma *map-eq-cons-decomp*:
assumes *SF*: *map f l* = *xs* @ *ys*
shows $\exists$ xs' ys'. $l = xs'$ @ ys' $\wedge$ *map f* $xs' = xs$ $\wedge$ *map f* $ys' = ys$
proof $-$
let *?F'* = *take* (*length xs*) *l*
let *?G* = *drop* (*length xs*) *l*
have *tr1*: *l* = *?F'* @ *?G*

by *simp*
moreover
have [*simp*]: *length l = length xs + length ys*
using *arg-cong*[*OF SF*, *of length*] **by** *auto*
have *map f ?F′ = xs* **and** *map f ?G = ys*
using *arg-cong*[*OF SF*, *of take* (*length xs*)] **apply** (*subst* (*asm*) *tr1*)
unfolding *map-append* **apply** *simp*
using *arg-cong*[*OF SF*, *of drop* (*length xs*)] **apply** (*subst* (*asm*) *tr1*)
unfolding *map-append* **apply** *simp*
done
ultimately show *?thesis* **by** *blast*
qed

interpretation $cdcl_{NOT}$*-twl*: *dpll-with-backjumping-ops*
λS. convert-trail-from-W (*trail-twl S*)
abstract-twl.raw-clauses-twl
λL S.
cons-trail-twl
(*convert-marked-lit-from-NOT L*) *S*
tl-trail-twl
add-learned-cls-twl
remove-cls-twl
λL S. lit-of L ∈ fst ‘ candidates-propagate-twl S
λS. no-dup (*trail-twl S*)
λC - - S -. C ∈ candidates-conflict-twl S
proof (*unfold-locales*, *goal-cases*)
case (*1 C′ S C F′ K F L*) **note** *n-d = this*(*1*) **and** *n-d′ = this*(*2*) **and** *undef = this*(*6*)
let *?T′* = (*cons-trail* (*Propagated L* {#}) (*rough-state-of-twl* ($cdcl_W$*-twl.reduce-trail-to*$_{NOT}$ *F S*)))
let *?T* = (*cons-trail-twl* (*Propagated L* {#}) ($cdcl_W$*-twl.reduce-trail-to*$_{NOT}$ *F S*))
have *tr-F-S*: *map lit-of* (*trail-twl* ($cdcl_W$*-twl.reduce-trail-to*$_{NOT}$ *F S*)) =
map lit-of (*convert-trail-from-NOT F*)
apply (*subst trail-twl-reduce-trail-to*$_{NOT}$*-drop*[*of F S*])
using *1*(*1*) *arg-cong*[*OF 1*(*3*), *of length*] *arg-cong*[*OF 1*(*3*), *of map lit-of*]
by (*auto simp*: *o-def drop-map*[*symmetric*])

have *no-dup* (*trail-twl S*)
using *1*(*1*) **by** *blast*
have *wf-twl-state* (*rough-state-of-twl* ($cdcl_W$*-twl.reduce-trail-to*$_{NOT}$ *F S*))
using *wf-twl-state-rough-state-of-twl* **by** *blast*
moreover have *undef′*: *undefined-lit* (*trail-twl* ($cdcl_W$*-twl.reduce-trail-to*$_{NOT}$ *F S*)) *L*
using *undef arg-cong*[*OF tr-F-S*, *of map atm-of*] **unfolding** *defined-lit-map image-set*
by (*simp add*: *o-def*)
ultimately have *wf-twl-state ?T′*
by (*simp-all add*: *wf-twl-state-cons-trail*)
then have *init-clss-twl ?T = init-clss-twl* ($cdcl_W$*-twl.reduce-trail-to*$_{NOT}$ *F S*)
using *1*(*6*) **by** (*simp add*: *undef′*)
then have [*simp*]: *init-clss-twl ?T = init-clss-twl S*
by (*simp add*: $cdcl_W$*-twl.reduce-trail-to*$_{NOT}$*-reduce-trail-convert*)

have *learned-clss-twl ?T = learned-clss-twl* ($cdcl_W$*-twl.reduce-trail-to*$_{NOT}$ *F S*)
by (*smt 1*(*3*) *1*(*6*) *append-assoc* $cdcl_W$*-twl.learned-clss-cons-trail*
$cdcl_W$*-twl-NOT.reduce-trail-to*$_{NOT}$*-eq-length* $cdcl_W$*-twl-NOT.reduce-trail-to*$_{NOT}$*-nil*
$cdcl_W$*-twl-NOT.reduce-trail-to*$_{NOT}$*-skip-beginning comp-apply defined-lit-convert-trail-from-W*

list.sel(3) marked-lit.sel(2) rev.simps(2) rev-append rev-eq-Cons-iff
cons-trail-twl-def)
moreover have *learned-clss-twl* ($cdcl_W$*-twl.reduce-trail-to*$_{NOT}$ *F S*)
= *learned-clss-twl S*
by (*simp add*: $cdcl_W$*-twl.reduce-trail-to*$_{NOT}$*-reduce-trail-convert*)
ultimately have [*simp*]: *learned-clss-twl ?T* = *learned-clss-twl S*
by *simp*
have *tr-L-F-S*: *map lit-of* (*trail-twl ?T*)
= *map lit-of* (*Propagated L* {#} # *convert-trail-from-NOT F*)
using *undef' tr-F-S* **by** (*simp add*: *o-def*)
have *C-confl-cand*: *C* ∈ *candidates-conflict-twl S*
apply(*rule wf-candidates-twl-conflict-complete*)
using *1*(*1*,*4*) **apply** (*simp add*: *rough-cdcl.clauses-def*)
using *1*(*5*) **by** (*simp add*: *tr-L-F-S true-annots-true-cls lits-of-convert-trail-from-NOT*)

have $cdcl_{NOT}$*-twl.backjump S*
(*cons-trail-twl* (*convert-marked-lit-from-NOT* (*Propagated L* ()))
($cdcl_W$*-twl.reduce-trail-to*$_{NOT}$ *F S*))
apply (*rule* $cdcl_{NOT}$*-twl.backjump.intros*[*of S F' K F - L C*, *OF 1*(*3*) - *1*(*4*−*6*) - *1*(*8*−*9*)])
unfolding $cdcl_W$*-twl-NOT.state-eq*$_{NOT}$*-def* **apply** (*metis convert-marked-lit-from-NOT.simps*(*1*))
using *1*(*7*) *1*(*3*) **apply** *presburger*
using *C-confl-cand* **by** *simp*
then show *?case*
by *blast*
qed

interpretation $cdcl_{NOT}$*-twl*: *dpll-with-backjumping*
λ*S*. *convert-trail-from-W* (*trail-twl S*)
abstract-twl.raw-clauses-twl
λ*L* (*S*:: *'v wf-twl*).
cons-trail-twl
(*convert-marked-lit-from-NOT L*) (*S*:: *'v wf-twl*)
tl-trail-twl
add-learned-cls-twl
remove-cls-twl
λ*L S*. *lit-of L* ∈ *fst* ' *candidates-propagate-twl S*
λ*S*. *no-dup* (*trail-twl S*)
λ*C* - - (*S*:: *'v wf-twl*) -. *C* ∈ *candidates-conflict-twl S*
apply *unfold-locales*
using $cdcl_{NOT}$*-twl.dpll-bj-no-dup* **by** (*simp add*: *o-def*)
end

end
theory *Prop-Superposition*
imports *Partial-Clausal-Logic ../lib/Herbrand-Interpretation*
begin
sledgehammer-params[*verbose*]
no-notation *Herbrand-Interpretation.true-cls* (**infix** ⊨ *50*)
notation *Herbrand-Interpretation.true-cls* (**infix** ⊨*h 50*)

no-notation *Herbrand-Interpretation.true-clss* (**infix** ⊨*s 50*)
notation *Herbrand-Interpretation.true-clss* (**infix** ⊨*hs 50*)

lemma *herbrand-interp-iff-partial-interp-cls*:
S ⊨*h C* ⟷ {*Pos P*|*P*. *P*∈*S*} ∪ {*Neg P*|*P*. *P*∉*S*} ⊨ *C*

unfolding *Herbrand-Interpretation.true-cls-def Partial-Clausal-Logic.true-cls-def*
by *auto*

lemma *herbrand-consistent-interp*:
consistent-interp $(\{Pos\ P|P.\ P{\in}S\} \cup \{Neg\ P|P.\ P{\notin}S\})$
unfolding *consistent-interp-def* **by** *auto*

lemma *herbrand-total-over-set*:
total-over-set $(\{Pos\ P|P.\ P{\in}S\} \cup \{Neg\ P|P.\ P{\notin}S\})\ T$
unfolding *total-over-set-def* **by** *auto*

lemma *herbrand-total-over-m*:
total-over-m $(\{Pos\ P|P.\ P{\in}S\} \cup \{Neg\ P|P.\ P{\notin}S\})\ T$
unfolding *total-over-m-def* **by** (*auto simp add*: *herbrand-total-over-set*)

lemma *herbrand-interp-iff-partial-interp-clss*:
$S \models hs\ C \longleftrightarrow \{Pos\ P|P.\ P{\in}S\} \cup \{Neg\ P|P.\ P{\notin}S\} \models s\ C$
unfolding *true-clss-def Ball-def herbrand-interp-iff-partial-interp-cls*
Partial-Clausal-Logic.true-clss-def **by** *auto*

definition *clss-lt* :: *'a::wellorder clauses* $\Rightarrow$ *'a clause* $\Rightarrow$ *'a clauses* **where**
clss-lt $N\ C = \{D \in N.\ D\ \#{\subset}\#\ C\}$

notation (*latex* **output**)
clss-lt (-<^bsup>-<^esup>)

locale *selection* =
fixes S :: *'a clause* $\Rightarrow$ *'a clause*
assumes
S-selects-subseteq: $\bigwedge C.\ S\ C \leq\#\ C$ **and**
S-selects-neg-lits: $\bigwedge C\ L.\ L \in\#\ S\ C \Longrightarrow$ *is-neg* L

locale *ground-resolution-with-selection* =
selection S **for** S :: (*'a* :: *wellorder*) *clause* $\Rightarrow$ *'a clause*
begin

context
fixes N :: *'a clause set*
begin

We do not create an equivalent of δ, but we directly defined N_C by inlining the definition.

function
production :: *'a clause* $\Rightarrow$ *'a interp*
where
production C =
$\{A.\ C \in N \wedge C \neq \{\#\} \wedge Max\ (set\text{-}mset\ C) = Pos\ A \wedge count\ C\ (Pos\ A) \leq 1$
$\wedge \neg (\bigcup D \in \{D.\ D\ \#{\subset}\#\ C\}.\ production\ D) \models h\ C \wedge S\ C = \{\#\}\}$
by *auto*
termination by (*relation* $\{(D,\ C).\ D\ \#{\subset}\#\ C\}$) (*auto simp*: *wf-less-multiset*)

declare *production.simps*[*simp del*]

definition *interp* :: *'a clause* $\Rightarrow$ *'a interp* **where**
interp $C = (\bigcup D \in \{D.\ D\ \#{\subset}\#\ C\}.\ production\ D)$

lemma *production-unfold*:
  $production\ C = \{A.\ C \in N \wedge C \neq \{\#\} \wedge Max\ (set\text{-}mset\ C) = Pos\ A \wedge\ count\ C\ (Pos\ A) \leq 1 \wedge \neg\ interp\ C \models h\ C \wedge S\ C = \{\#\}\}$
  unfolding *interp-def* **by** (*rule production.simps*)

abbreviation $productive\ A \equiv (production\ A \neq \{\})$

abbreviation $produces :: 'a\ clause \Rightarrow 'a \Rightarrow bool$ **where**
  $produces\ C\ A \equiv production\ C = \{A\}$

lemma *producesD*:
  $produces\ C\ A \Longrightarrow C \in N \wedge C \neq \{\#\} \wedge Pos\ A = Max\ (set\text{-}mset\ C) \wedge count\ C\ (Pos\ A) \leq 1 \wedge \neg\ interp\ C \models h\ C \wedge S\ C = \{\#\}$
  unfolding *production-unfold* **by** *auto*

lemma $produces\ C\ A \Longrightarrow Pos\ A \in\#\ C$
  by (*simp add*: *Max-in-lits producesD*)

lemma *interp′-def-in-set*:
  $interp\ C = (\bigcup D \in \{D \in N.\ D \#\subset\#\ C\}.\ production\ D)$
  unfolding *interp-def* **apply** *auto*
  unfolding *production-unfold* **apply** *auto*
  done

lemma *production-iff-produces*:
  $produces\ D\ A \longleftrightarrow A \in production\ D$
  unfolding *production-unfold* **by** *auto*

definition $Interp :: 'a\ clause \Rightarrow 'a\ interp$ **where**
  $Interp\ C = interp\ C \cup production\ C$

lemma
  assumes *produces C P*
  shows $Interp\ C \models h\ C$
  unfolding *Interp-def assms* **using** *producesD*[*OF assms*]
  by (*metis Max-in-lits Un-insert-right insertI1 pos-literal-in-imp-true-cls*)

definition $INTERP :: 'a\ interp$ **where**
$INTERP = (\bigcup D \in N.\ production\ D)$

lemma *interp-subseteq-Interp*[*simp*]: $interp\ C \subseteq Interp\ C$
  unfolding *Interp-def* **by** *simp*

lemma *Interp-as-UNION*: $Interp\ C = (\bigcup D \in \{D.\ D \#\subseteq\#\ C\}.\ production\ D)$
  unfolding *Interp-def interp-def le-multiset-def* **by** *fast*

lemma *productive-not-empty*: $productive\ C \Longrightarrow C \neq \{\#\}$
  unfolding *production-unfold* **by** *auto*

lemma *productive-imp-produces-Max-literal*: $productive\ C \Longrightarrow produces\ C\ (atm\text{-}of\ (Max\ (set\text{-}mset\ C)))$
  unfolding *production-unfold* **by** (*auto simp del*: *atm-of-Max-lit*)

lemma *productive-imp-produces-Max-atom*: $productive\ C \Longrightarrow produces\ C\ (Max\ (atms\text{-}of\ C))$
  unfolding *atms-of-def Max-atm-of-set-mset-commute*[*OF productive-not-empty*]

by (*rule productive-imp-produces-Max-literal*)

lemma *produces-imp-Max-literal*: *produces* C $A \Longrightarrow A = $ *atm-of* (*Max* (*set-mset* C))
by (*metis Max-singleton insert-not-empty productive-imp-produces-Max-literal*)

lemma *produces-imp-Max-atom*: *produces* C $A \Longrightarrow A = $ *Max* (*atms-of* C)
by (*metis Max-singleton insert-not-empty productive-imp-produces-Max-atom*)

lemma *produces-imp-Pos-in-lits*: *produces* C $A \Longrightarrow$ *Pos* $A \in\# C$
by (*auto intro*: *Max-in-lits dest*!: *producesD*)

lemma *productive-in-N*: *productive* $C \Longrightarrow C \in N$
unfolding *production-unfold* **by** *auto*

lemma *produces-imp-atms-leq*: *produces* C $A \Longrightarrow B \in$ *atms-of* $C \Longrightarrow B \leq A$
by (*metis Max-ge finite-atms-of insert-not-empty productive-imp-produces-Max-atom singleton-inject*)

lemma *produces-imp-neg-notin-lits*: *produces* C $A \Longrightarrow \neg$ *Neg* $A \in\# C$
by (*auto intro*!: *pos-Max-imp-neg-notin dest*: *producesD simp del*: *not-gr0*)

lemma *less-eq-imp-interp-subseteq-interp*: $C \ \#\subseteq\# \ D \Longrightarrow$ *interp* $C \subseteq$ *interp* D
unfolding *interp-def* **by** *auto* (*metis multiset-order.order.strict-trans2*)

lemma *less-eq-imp-interp-subseteq-Interp*: $C \ \#\subseteq\# \ D \Longrightarrow$ *interp* $C \subseteq$ *Interp* D
unfolding *Interp-def* **using** *less-eq-imp-interp-subseteq-interp* **by** *blast*

lemma *less-imp-production-subseteq-interp*: $C \ \#\subset\# \ D \Longrightarrow$ *production* $C \subseteq$ *interp* D
unfolding *interp-def* **by** *fast*

lemma *less-eq-imp-production-subseteq-Interp*: $C \ \#\subseteq\# \ D \Longrightarrow$ *production* $C \subseteq$ *Interp* D
unfolding *Interp-def* **using** *less-imp-production-subseteq-interp*
by (*metis multiset-order.le-imp-less-or-eq le-supI1 sup-ge2*)

lemma *less-imp-Interp-subseteq-interp*: $C \ \#\subset\# \ D \Longrightarrow$ *Interp* $C \subseteq$ *interp* D
unfolding *Interp-def*
by (*auto simp*: *less-eq-imp-interp-subseteq-interp less-imp-production-subseteq-interp*)

lemma *less-eq-imp-Interp-subseteq-Interp*: $C \ \#\subseteq\# \ D \Longrightarrow$ *Interp* $C \subseteq$ *Interp* D
using *less-imp-Interp-subseteq-interp*
unfolding *Interp-def* **by** (*metis multiset-order.le-imp-less-or-eq le-supI2 subset-refl sup-commute*)

lemma *false-Interp-to-true-interp-imp-less-multiset*: $A \notin$ *Interp* $C \Longrightarrow A \in$ *interp* $D \Longrightarrow C \ \#\subset\# \ D$
using *less-eq-imp-interp-subseteq-Interp multiset-linorder.not-less* **by** *blast*

lemma *false-interp-to-true-interp-imp-less-multiset*: $A \notin$ *interp* $C \Longrightarrow A \in$ *interp* $D \Longrightarrow C \ \#\subset\# \ D$
using *less-eq-imp-interp-subseteq-interp multiset-linorder.not-less* **by** *blast*

lemma *false-Interp-to-true-Interp-imp-less-multiset*: $A \notin$ *Interp* $C \Longrightarrow A \in$ *Interp* $D \Longrightarrow C \ \#\subset\# \ D$
using *less-eq-imp-Interp-subseteq-Interp multiset-linorder.not-less* **by** *blast*

lemma *false-interp-to-true-Interp-imp-le-multiset*: $A \notin$ *interp* $C \Longrightarrow A \in$ *Interp* $D \Longrightarrow C \ \#\subseteq\# \ D$
using *less-imp-Interp-subseteq-interp multiset-linorder.not-less* **by** *blast*

lemma *interp-subseteq-INTERP*: *interp* $C \subseteq$ *INTERP*

unfolding *interp-def INTERP-def* **by** (*auto simp*: *production-unfold*)

lemma *production-subseteq-INTERP*: *production C* $\subseteq$ *INTERP*
unfolding *INTERP-def* **using** *production-unfold* **by** *blast*

lemma *Interp-subseteq-INTERP*: *Interp C* $\subseteq$ *INTERP*
unfolding *Interp-def* **by** (*auto intro*!: *interp-subseteq-INTERP production-subseteq-INTERP*)

This lemma corresponds to theorem 2.7.6 page 66 of CW.

lemma *produces-imp-in-interp*:
assumes *a-in-c*: *Neg A* $\in\#$ *C* **and** *d*: *produces D A*
shows *A* $\in$ *interp C*
proof −
from *d* **have** *Max* (*set-mset D*) = *Pos A*
using *production-unfold* **by** *blast*
hence *D* $\#\subset\#$ $\{\#$*Neg A*$\#\}$
by (*auto intro*: *Max-pos-neg-less-multiset*)
moreover have $\{\#$*Neg A*$\#\}$ $\#\subseteq\#$ *C*
by (*rule less-eq-imp-le-multiset*) (*rule mset-le-single*[*OF a-in-c*[*unfolded mem-set-mset-iff*]])
ultimately show *?thesis*
using *d* **by** (*blast dest*: *less-eq-imp-interp-subseteq-interp less-imp-production-subseteq-interp*)
qed

lemma *neg-notin-Interp-not-produce*: *Neg A* $\in\#$ *C* $\Longrightarrow$ *A* $\notin$ *Interp D* $\Longrightarrow$ *C* $\#\subseteq\#$ *D* $\Longrightarrow$ $\neg$ *produces* *D'' A*
by (*auto dest*: *produces-imp-in-interp less-eq-imp-interp-subseteq-Interp*)

lemma *in-production-imp-produces*: *A* $\in$ *production C* $\Longrightarrow$ *produces C A*
by (*metis insert-absorb productive-imp-produces-Max-atom singleton-insert-inj-eq'*)

lemma *not-produces-imp-notin-production*: $\neg$ *produces C A* $\Longrightarrow$ *A* $\notin$ *production C*
by (*metis in-production-imp-produces*)

lemma *not-produces-imp-notin-interp*: ($\bigwedge$*D*. $\neg$ *produces D A*) $\Longrightarrow$ *A* $\notin$ *interp C*
unfolding *interp-def* **by** (*fast intro*!: *in-production-imp-produces*)

The results below corresponds to Lemma 3.4.

Nitpicking: If $D = D'$ and D is productive, $I^D \subseteq I_{D'}$ does not hold.

lemma *true-Interp-imp-general*:
assumes
c-le-d: *C* $\#\subseteq\#$ *D* **and**
d-lt-d': *D* $\#\subset\#$ *D'* **and**
c-at-d: *Interp D* $\models h$ *C* **and**
subs: *interp D'* $\subseteq$ ($\bigcup$ *C* $\in$ *CC*. *production C*)
shows ($\bigcup$ *C* $\in$ *CC*. *production C*) $\models h$ *C*
proof (*cases* $\exists$ *A*. *Pos A* $\in\#$ *C* $\wedge$ *A* $\in$ *Interp D*)
case *True*
then obtain *A* **where** *a-in-c*: *Pos A* $\in\#$ *C* **and** *a-at-d*: *A* $\in$ *Interp D*
by *blast*
from *a-at-d* **have** *A* $\in$ *interp D'*
using *d-lt-d' less-imp-Interp-subseteq-interp* **by** *blast*
thus *?thesis*
using *subs a-in-c* **by** (*blast dest*: *contra-subsetD*)
next
case *False*

then obtain A **where** *a-in-c*: *Neg* $A \in\#$ C **and** $A \notin$ *Interp* D
  using *c-at-d* **unfolding** *true-cls-def* **by** *blast*
hence $\bigwedge D''$. $\neg$ *produces* D'' A
  using *c-le-d neg-notin-Interp-not-produce* **by** *simp*
thus *?thesis*
  using *a-in-c subs not-produces-imp-notin-production* **by** *auto*
qed

lemma *true-Interp-imp-interp*: C #⊆# $D \Longrightarrow D$ #⊂# $D' \Longrightarrow$ *Interp* $D \models h$ $C \Longrightarrow$ *interp* $D' \models h$ C
  using *interp-def true-Interp-imp-general* **by** *simp*

lemma *true-Interp-imp-Interp*: C #⊆# $D \Longrightarrow D$ #⊂# $D' \Longrightarrow$ *Interp* $D \models h$ $C \Longrightarrow$ *Interp* $D' \models h$ C
  using *Interp-as-UNION interp-subseteq-Interp true-Interp-imp-general* **by** *simp*

lemma *true-Interp-imp-INTERP*: C #⊆# $D \Longrightarrow$ *Interp* $D \models h$ $C \Longrightarrow$ *INTERP* $\models h$ C
  using *INTERP-def interp-subseteq-INTERP*
    true-Interp-imp-general[*OF - less-multiset-right-total*]
  by *simp*

lemma *true-interp-imp-general*:
  assumes
    c-le-d: C #⊆# D **and**
    d-lt-d': D #⊂# D' **and**
    c-at-d: *interp* $D \models h$ C **and**
    subs: *interp* $D' \subseteq (\bigcup C \in CC.$ *production* $C)$
  shows $(\bigcup C \in CC.$ *production* $C) \models h$ C
proof (*cases* $\exists A$. *Pos* $A \in\#$ $C \wedge A \in$ *interp* D)
  case *True*
  then obtain A **where** *a-in-c*: *Pos* $A \in\#$ C **and** *a-at-d*: $A \in$ *interp* D
    by *blast*
  from *a-at-d* **have** $A \in$ *interp* D'
    using *d-lt-d' less-eq-imp-interp-subseteq-interp*[*OF multiset-order.less-imp-le*] **by** *blast*
  thus *?thesis*
    using *subs a-in-c* **by** (*blast dest*: *contra-subsetD*)
next
  case *False*
  then obtain A **where** *a-in-c*: *Neg* $A \in\#$ C **and** $A \notin$ *interp* D
    using *c-at-d* **unfolding** *true-cls-def* **by** *blast*
  hence $\bigwedge D''$. $\neg$ *produces* D'' A
    using *c-le-d* **by** (*auto dest*: *produces-imp-in-interp less-eq-imp-interp-subseteq-interp*)
  thus *?thesis*
    using *a-in-c subs not-produces-imp-notin-production* **by** *auto*
qed

This lemma corresponds to theorem 2.7.6 page 66 of CW. Here the strict maximality is important

lemma *true-interp-imp-interp*: C #⊆# $D \Longrightarrow D$ #⊂# $D' \Longrightarrow$ *interp* $D \models h$ $C \Longrightarrow$ *interp* $D' \models h$ C
  using *interp-def true-interp-imp-general* **by** *simp*

lemma *true-interp-imp-Interp*: C #⊆# $D \Longrightarrow D$ #⊂# $D' \Longrightarrow$ *interp* $D \models h$ $C \Longrightarrow$ *Interp* $D' \models h$ C
  using *Interp-as-UNION interp-subseteq-Interp*[*of* D'] *true-interp-imp-general* **by** *simp*

lemma *true-interp-imp-INTERP*: C #⊆# $D \Longrightarrow$ *interp* $D \models h$ $C \Longrightarrow$ *INTERP* $\models h$ C
  using *INTERP-def interp-subseteq-INTERP*
    true-interp-imp-general[*OF - less-multiset-right-total*]
  by *simp*

lemma *productive-imp-false-interp*: *productive* $C \Longrightarrow \neg$ *interp* $C \models h$ C
  unfolding *production-unfold* **by** *auto*

This lemma corresponds to theorem 2.7.6 page 66 of CW. Here the strict maximality is important

lemma *cls-gt-double-pos-no-production*:
  assumes D: $\{\#Pos\ P,\ Pos\ P\#\}\ \#{\subset}\#\ C$
  shows $\neg$*produces* C P
proof –
  let $?D = \{\#Pos\ P,\ Pos\ P\#\}$
  note $D' = D$[*unfolded less-multiset*$_{HO}$]
  consider
    (P) *count* C (*Pos* P) ≥ 2
  | (Q) Q **where** $Q >$ *Pos* P **and** $Q \in\#\ C$
    using *HOL.spec*[*OF HOL.conjunct2*[*OF D'*], *of Pos P*] **by** *auto*
  thus *?thesis*
    proof *cases*
      case Q
      have $Q \in$ *set-mset* C
        using $Q(2)$ **by** (*auto split*: *split-if-asm*)
      then have *Max* (*set-mset* C) $>$ *Pos* P
        using $Q(1)$ *Max-gr-iff* **by** *blast*
      thus *?thesis*
        unfolding *production-unfold* **by** *auto*
    next
      case P
      thus *?thesis*
        unfolding *production-unfold* **by** *auto*
    qed
qed

This lemma corresponds to theorem 2.7.6 page 66 of CW.

lemma
  assumes D: $C+\{\#Neg\ P\#\}\ \#{\subset}\#\ D$
  shows *production* $D \neq \{P\}$
proof –
  note $D' = D$[*unfolded less-multiset*$_{HO}$]
  consider
    (P) *Neg* $P \in\#\ D$
  | (Q) Q **where** $Q >$ *Neg* P **and** *count* D $Q >$ *count* ($C + \{\#Neg\ P\#\}$) Q
    using *HOL.spec*[*OF HOL.conjunct2*[*OF D'*], *of Neg P*] **by** *fastforce*
  thus *?thesis*
    proof *cases*
      case Q
      have $Q \in$ *set-mset* D
        using $Q(2)$ **by** (*auto split*: *split-if-asm*)
      then have *Max* (*set-mset* D) $>$ *Neg* P
        using $Q(1)$ *Max-gr-iff* **by** *blast*
      hence *Max* (*set-mset* D) $>$ *Pos* P
        using *less-trans*[*of Pos P Neg P Max* (*set-mset D*)] **by** *auto*
      thus *?thesis*
        unfolding *production-unfold* **by** *auto*
    next
      case P
      hence *Max* (*set-mset* D) $>$ *Pos* P

by (*meson Max-ge finite-set-mset le-less-trans linorder-not-le mem-set-mset-iff pos-less-neg*)
thus *?thesis*
unfolding *production-unfold* **by** *auto*
qed
qed

lemma *in-interp-is-produced*:
assumes $P \in INTERP$
shows $\exists D.\ D +\{\#Pos\ P\#\} \in N \wedge produces\ (D +\{\#Pos\ P\#\})\ P$
using *assms* **unfolding** *INTERP-def UN-iff production-iff-produces Ball-def*
by (*metis ground-resolution-with-selection.produces-imp-Pos-in-lits insert-DiffM2 ground-resolution-with-selection-axioms not-produces-imp-notin-production*)

end
end

abbreviation $MMax\ M \equiv Max\ (set\text{-}mset\ M)$

21.6 We can now define the rules of the calculus

inductive *superposition-rules* :: $'a\ clause \Rightarrow {'a}\ clause \Rightarrow {'a}\ clause \Rightarrow bool$ **where**
factoring: $superposition\text{-}rules\ (C + \{\#Pos\ P\#\} + \{\#Pos\ P\#\})\ B\ (C + \{\#Pos\ P\#\})\ |$
superposition-l: $superposition\text{-}rules\ (C_1 + \{\#Pos\ P\#\})\ (C_2 + \{\#Neg\ P\#\})\ (C_1 + C_2)$

inductive *superposition* :: $'a\ clauses \Rightarrow {'a}\ clauses \Rightarrow bool$ **where**
superposition: $A \in N \Longrightarrow B \in N \Longrightarrow superposition\text{-}rules\ A\ B\ C$
$\Longrightarrow superposition\ N\ (N \cup \{C\})$

definition *abstract-red* :: $'a{::}wellorder\ clause \Rightarrow {'a}\ clauses \Rightarrow bool$ **where**
$abstract\text{-}red\ C\ N = (clss\text{-}lt\ N\ C \models p\ C)$

lemma *less-multiset*[*iff*]: $M < N \longleftrightarrow M\ \#{\subset}\#\ N$
unfolding *less-multiset-def* **by** *auto*

lemma *less-eq-multiset*[*iff*]: $M \leq N \longleftrightarrow M\ \#{\subseteq}\#\ N$
unfolding *less-eq-multiset-def* **by** *auto*

lemma *herbrand-true-clss-true-clss-cls-herbrand-true-clss*:
assumes
AB: $A \models hs\ B$ **and**
BC: $B \models p\ C$
shows $A \models h\ C$
proof −
let $?I = \{Pos\ P\ |P.\ P \in A\} \cup \{Neg\ P\ |P.\ P \notin A\}$
have *B*: $?I \models s\ B$ **using** *AB*
by (*auto simp add*: *herbrand-interp-iff-partial-interp-clss*)

have *IH*: $\bigwedge I.\ total\text{-}over\text{-}set\ I\ (atms\text{-}of\ C) \Longrightarrow total\text{-}over\text{-}m\ I\ B \Longrightarrow consistent\text{-}interp\ I \Longrightarrow I \models s\ B \Longrightarrow I \models C$ **using** *BC*
by (*auto simp add*: *true-clss-cls-def*)
show *?thesis*
unfolding *herbrand-interp-iff-partial-interp-cls*
by (*auto intro*: *IH*[*of ?I*] *simp add*: *herbrand-total-over-set herbrand-total-over-m herbrand-consistent-interp B*)

qed

lemma *abstract-red-subset-mset-abstract-red*:
  assumes
    abstr: *abstract-red C N* **and**
    c-lt-d: $C \subseteq\# D$
  shows *abstract-red D N*
proof −
  have $\{D \in N.\ D \#\subset\# C\} \subseteq \{D' \in N.\ D' \#\subset\# D\}$
    using *c-lt-d less-eq-imp-le-multiset* **by** *fastforce*
  thus *?thesis*
    using *abstr* **unfolding** *abstract-red-def clss-lt-def*
    by (*metis* (*no-types*, *lifting*) *c-lt-d subset-mset.diff-add true-clss-cls-mono-r′*
      true-clss-cls-subset)
qed

lemma *true-clss-cls-extended*:
  assumes
    $A \models p\ B$ **and**
    tot: *total-over-m I* (*A*) **and**
    cons: *consistent-interp I* **and**
    I-A: $I \models s\ A$
  shows $I \models B$
proof −
  let $?I = I \cup \{Pos\ P|P.\ P \in atms\text{-}of\ B \wedge P \notin atms\text{-}of\text{-}s\ I\}$
  have *consistent-interp ?I*
    using *cons* **unfolding** *consistent-interp-def atms-of-s-def atms-of-def*
      apply (*auto 1 5 simp add*: *image-iff*)
    by (*metis atm-of-uminus literal.sel*(*1*))
  moreover have *total-over-m ?I* $(A \cup \{B\})$
    proof −
      obtain *aa* :: ′*a set* ⇒ ′*a literal set* ⇒ ′*a* **where**
        f2: $\forall x0\ x1.\ (\exists v2.\ v2 \in x0 \wedge Pos\ v2 \notin x1 \wedge Neg\ v2 \notin x1)$
          $\longleftrightarrow (aa\ x0\ x1 \in x0 \wedge Pos\ (aa\ x0\ x1) \notin x1 \wedge Neg\ (aa\ x0\ x1) \notin x1)$
        by *moura*
      have $\forall a.\ a \notin atms\text{-}of\text{-}ms\ A \vee Pos\ a \in I \vee Neg\ a \in I$
        using *tot* **by** (*simp add*: *total-over-m-def total-over-set-def*)
      hence $aa\ (atms\text{-}of\text{-}ms\ A \cup atms\text{-}of\text{-}ms\ \{B\})\ (I \cup \{Pos\ a\ |a.\ a \in atms\text{-}of\ B \wedge a \notin atms\text{-}of\text{-}s\ I\})$
        $\notin atms\text{-}of\text{-}ms\ A \cup atms\text{-}of\text{-}ms\ \{B\} \vee Pos\ (aa\ (atms\text{-}of\text{-}ms\ A \cup atms\text{-}of\text{-}ms\ \{B\})$
          $(I \cup \{Pos\ a\ |a.\ a \in atms\text{-}of\ B \wedge a \notin atms\text{-}of\text{-}s\ I\})) \in I$
            $\cup \{Pos\ a\ |a.\ a \in atms\text{-}of\ B \wedge a \notin atms\text{-}of\text{-}s\ I\}$
          $\vee Neg\ (aa\ (atms\text{-}of\text{-}ms\ A \cup atms\text{-}of\text{-}ms\ \{B\})$
            $(I \cup \{Pos\ a\ |a.\ a \in atms\text{-}of\ B \wedge a \notin atms\text{-}of\text{-}s\ I\})) \in I$
            $\cup \{Pos\ a\ |a.\ a \in atms\text{-}of\ B \wedge a \notin atms\text{-}of\text{-}s\ I\}$
        by *auto*
     hence *total-over-set* $(I \cup \{Pos\ a\ |a.\ a \in atms\text{-}of\ B \wedge a \notin atms\text{-}of\text{-}s\ I\})\ (atms\text{-}of\text{-}ms\ A \cup atms\text{-}of\text{-}ms$
$\{B\})$
        using *f2* **by** (*meson total-over-set-def*)
      thus *?thesis*
        by (*simp add*: *total-over-m-def*)
    qed
  moreover have $?I \models s\ A$
    using *I-A* **by** *auto*
  ultimately have $?I \models B$

using ⟨$A \models pB$⟩ **unfolding** *true-clss-cls-def* **by** *auto*
thus *?thesis*
oops
lemma
assumes
CP: $\neg$ *clss-lt* N ($\{\#C\#\}$ + $\{\#E\#\}$) $\models p$ $\{\#C\#\}$ + $\{\#Neg\ P\#\}$ **and**
clss-lt N ($\{\#C\#\}$ + $\{\#E\#\}$) $\models p$ $\{\#E\#\}$ + $\{\#Pos\ P\#\}$ $\vee$ *clss-lt* N ($\{\#C\#\}$ + $\{\#E\#\}$) $\models p$ $\{\#C\#\}$ + $\{\#Neg\ P\#\}$
shows *clss-lt* N ($\{\#C\#\}$ + $\{\#E\#\}$) $\models p$ $\{\#E\#\}$ + $\{\#Pos\ P\#\}$

oops

locale *ground-ordered-resolution-with-redundancy* =
ground-resolution-with-selection +
fixes *redundant* :: $'a$::*wellorder clause* $\Rightarrow$ $'a$ *clauses* $\Rightarrow$ *bool*
assumes
redundant-iff-abstract: *redundant* A N $\longleftrightarrow$ *abstract-red* A N
begin
definition *saturated* :: $'a$ *clauses* $\Rightarrow$ *bool* **where**
saturated N $\longleftrightarrow$ ($\forall$ A B C. $A \in N \longrightarrow B \in N \longrightarrow \neg$*redundant* A N $\longrightarrow \neg$*redundant* B N
$\longrightarrow$ *superposition-rules* A B C $\longrightarrow$ *redundant* C N $\vee$ $C \in N$)

lemma
assumes
saturated: *saturated* N **and**
finite: *finite* N **and**
empty: $\{\#\} \notin N$
shows *INTERP* N $\models hs$ N
proof (*rule ccontr*)
let $?N_{\mathcal{I}}$ = *INTERP* N
assume $\neg$ *?thesis*
hence *not-empty*: $\{E{\in}N.\ \neg ?N_{\mathcal{I}} \models h\ E\} \neq \{\}$
unfolding *true-clss-def Ball-def* **by** *auto*
def $D \equiv$ *Min* $\{E{\in}N.\ \neg ?N_{\mathcal{I}} \models h\ E\}$
have [*simp*]: $D \in N$
unfolding *D-def*
by (*metis* (*mono-tags*, *lifting*) *Min-in not-empty finite mem-Collect-eq rev-finite-subset subsetI*)
have *not-d-interp*: $\neg ?N_{\mathcal{I}} \models h\ D$
unfolding *D-def*
by (*metis* (*mono-tags*, *lifting*) *Min-in finite mem-Collect-eq not-empty rev-finite-subset subsetI*)
have *cls-not-D*: $\bigwedge E.\ E \in N \Longrightarrow E \neq D \Longrightarrow \neg ?N_{\mathcal{I}} \models h\ E \Longrightarrow D \leq E$
using *finite D-def* **by** (*auto simp del*: *less-eq-multiset*)
obtain C L **where** D: $D = C + \{\#L\#\}$ **and** *LSD*: $L \in\#\ S\ D \vee (S\ D = \{\#\} \wedge$ *Max* (*set-mset* D) $= L$)
proof (*cases* $S\ D = \{\#\}$)
case *False*
then obtain L **where** $L \in\#\ S\ D$
using *Max-in-lits* **by** *blast*
moreover
hence $L \in\#\ D$
using *S-selects-subseteq*[*of* D] **by** *auto*
hence $D = (D - \{\#L\#\}) + \{\#L\#\}$
by *auto*
ultimately show *?thesis* **using** *that* **by** *blast*
next

let *?L* = *MMax D*
case *True*
moreover
have *?L* ∈# *D*
by (*metis* (*no-types*, *lifting*) *Max-in-lits* ‹*D* ∈ *N*› *empty*)
hence *D* = (*D* − {#*?L*#}) + {#*?L*#}
by *auto*
ultimately show *?thesis* **using** *that* **by** *blast*
qed
have *red*: ¬*redundant D N*
proof (*rule ccontr*)
assume *red*[*simplified*]: ~~*redundant D N*
have ∀ *E* < *D*. *E* ∈ *N* ⟶ *?N*$_{\mathcal{I}}$ ⊨*h E*
using *cls-not-D not-le* **by** *fastforce*
hence *?N*$_{\mathcal{I}}$ ⊨*hs clss-lt N D*
unfolding *clss-lt-def true-clss-def Ball-def* **by** *blast*
thus *False*
using *red not-d-interp* **unfolding** *abstract-red-def redundant-iff-abstract*
using *herbrand-true-clss-true-clss-cls-herbrand-true-clss* **by** *fast*
qed

consider
(*L*) *P* **where** *L* = *Pos P* **and** *S D* = {#} **and** *Max* (*set-mset D*) = *Pos P*
| (*Lneg*) *P* **where** *L* = *Neg P*
using *LSD S-selects-neg-lits*[*of D L*] **by** (*cases L*) *auto*
thus *False*
proof *cases*
case *L* **note** *P* = *this*(*1*) **and** *S* = *this*(*2*) **and** *max* = *this*(*3*)
have *count D L* > *1*
proof (*rule ccontr*)
assume ~ *?thesis*
hence *count*: *count D L* = *1*
unfolding *D* **by** *auto*
have ¬*?N*$_{\mathcal{I}}$⊨*h D*
using *not-d-interp true-interp-imp-INTERP ground-resolution-with-selection-axioms*
by *blast*
hence *produces N D P*
using *not-empty empty finite* ‹*D* ∈ *N*› *count L*
true-interp-imp-INTERP **unfolding** *production-iff-produces* **unfolding** *production-unfold*
by (*auto simp add*: *max not-empty*)
hence *INTERP N* ⊨*h D*
unfolding *D*
by (*metis pos-literal-in-imp-true-cls produces-imp-Pos-in-lits*
production-subseteq-INTERP singletonI subsetCE)
thus *False*
using *not-d-interp* **by** *blast*
qed
then obtain *C′* **where** *C′*:*D* = *C′* + {#*Pos P*#} + {#*Pos P*#}
unfolding *D* **by** (*metis P add.left-neutral add-less-cancel-right count-single count-union*
multi-member-split)
have *sup*: *superposition-rules D D* (*D* − {#*L*#})
unfolding *C′ L* **by** (*auto simp add*: *superposition-rules.simps*)
have *C′* + {#*Pos P*#} #⊂# *C′* + {#*Pos P*#} + {#*Pos P*#}
by *auto*
moreover have ¬*?N*$_{\mathcal{I}}$ ⊨*h* (*D* − {#*L*#})

using *not-d-interp* **unfolding** C' L **by** *auto*
ultimately have $C' + \{\#Pos\ P\#\} \notin N$
by (*metis* (*no-types*, *lifting*) C' P *add-diff-cancel-right'* *cls-not-D* *less-multiset* *multi-self-add-other-not-self* *not-le*)
have $D - \{\#L\#\}\ \#{\subset}\#\ D$
unfolding C' L **by** *auto*
have *c'-p-p*: $C' + \{\#Pos\ P\#\} + \{\#Pos\ P\#\} - \{\#Pos\ P\#\} = C' + \{\#Pos\ P\#\}$
by *auto*
have *redundant* $(C' + \{\#Pos\ P\#\})$ N
using *saturated red sup* ‹$D \in N$›‹$C' + \{\#Pos\ P\#\} \notin N$› **unfolding** *saturated-def* C' L *c'-p-p*
by *blast*
moreover have $C' + \{\#Pos\ P\#\}\ \subseteq\#\ C' + \{\#Pos\ P\#\} + \{\#Pos\ P\#\}$
by *auto*
ultimately show *False*
using *red* **unfolding** C' *redundant-iff-abstract* **by** (*blast dest*: *abstract-red-subset-mset-abstract-red*)
next
case *Lneg* **note** $L = this(1)$
have $P \in\ ?N_{\mathcal{I}}$
using *not-d-interp* **unfolding** D *true-cls-def* L **by** (*auto split*: *split-if-asm*)
then obtain E **where**
DPN: $E + \{\#Pos\ P\#\} \in N$ **and**
prod: *production* N $(E + \{\#Pos\ P\#\}) = \{P\}$
using *in-interp-is-produced* **by** *blast*
have *sup-EC*: *superposition-rules* $(E + \{\#Pos\ P\#\})$ $(C + \{\#Neg\ P\#\})$ $(E + C)$
using *superposition-l* **by** *fast*
hence *superposition* N $(N \cup \{E{+}C\})$
using *DPN* ‹$D \in N$› **unfolding** D L **by** (*auto simp add*: *superposition.simps*)
have
PMax: $Pos\ P = MMax\ (E + \{\#Pos\ P\#\})$ **and**
count $(E + \{\#Pos\ P\#\})$ $(Pos\ P) \leq 1$ **and**
$S\ (E + \{\#Pos\ P\#\}) = \{\#\}$ **and**
$\neg interp\ N\ (E + \{\#Pos\ P\#\}) \models h\ E + \{\#Pos\ P\#\}$
using *prod* **unfolding** *production-unfold* **by** *auto*
have $Neg\ P \notin\#\ E$
using *prod produces-imp-neg-notin-lits* **by** *force*
hence $\bigwedge y.\ y \in\#\ (E + \{\#Pos\ P\#\})$
$\Longrightarrow count\ (E + \{\#Pos\ P\#\})\ (Neg\ P) < count\ (C + \{\#Neg\ P\#\})\ (Neg\ P)$
by (*auto split*: *split-if-asm*)
moreover have $\bigwedge y.\ y \in\#\ (E + \{\#Pos\ P\#\}) \Longrightarrow y < Neg\ P$
using *PMax* **by** (*metis DPN Max-less-iff empty finite-set-mset mem-set-mset-iff pos-less-neg set-mset-eq-empty-iff*)
moreover have $E + \{\#Pos\ P\#\} \neq C + \{\#Neg\ P\#\}$
using *prod produces-imp-neg-notin-lits* **by** *force*
ultimately have $E + \{\#Pos\ P\#\}\ \#{\subset}\#\ C + \{\#Neg\ P\#\}$
unfolding $less\text{-}multiset_{HO}$ **by** (*metis add.left-neutral add-lessD1*)
have *ce-lt-d*: $C + E\ \#{\subset}\#\ D$
unfolding D L
by (*metis* (*mono-tags*, *lifting*) *Max-pos-neg-less-multiset One-nat-def PMax count-single less-multiset-plus-right-nonempty mult-less-trans single-not-empty union-less-mono2 zero-less-Suc*)
have $?N_{\mathcal{I}} \models h\ E + \{\#Pos\ P\#\}$
using ‹$P \in\ ?N_{\mathcal{I}}$› **by** *blast*
have $?N_{\mathcal{I}} \models h\ C{+}E \vee C{+}E \notin N$
using *ce-lt-d cls-not-D* **unfolding** *D-def* **by** *fastforce*

```
      have Pos P ∉# C+E
        using D ‹P ∈ ground-resolution-with-selection.INTERP S N›
          ‹count (E + {#Pos P#}) (Pos P) ≤ 1› multi-member-skip not-d-interp by auto
      hence ⋀y. y ∈# C+E
        ⟹ count (C+E) (Pos P) < count (E + {#Pos P#}) (Pos P)
        by (auto split: split-if-asm)

      have ¬redundant (C + E) N
        proof (rule ccontr)
          assume red'[simplified]: ¬ ?thesis
          have abs: clss-lt N (C + E) ⊨p C + E
            using redundant-iff-abstract red' unfolding abstract-red-def by auto
          have clss-lt N (C + E) ⊨p E + {#Pos P#} ∨ clss-lt N (C + E) ⊨p C + {#Neg P#}
            proof clarify
              assume CP: ¬ clss-lt N (C + E) ⊨p C + {#Neg P#}
              { fix I
                assume
                  total-over-m I (clss-lt N (C + E) ∪ {E + {#Pos P#}}) and
                  consistent-interp I and
                  I ⊨s clss-lt N (C + E)
                  hence I ⊨ C + E
                    using abs sorry
                  moreover have ¬ I ⊨ C + {#Neg P#}
                    using CP unfolding true-clss-cls-def
                    sorry
                  ultimately have I ⊨ E + {#Pos P#} by auto
              }
              then show clss-lt N (C + E) ⊨p E + {#Pos P#}
                unfolding true-clss-cls-def by auto
            qed
          moreover have clss-lt N (C + E) ⊆ clss-lt N (C + {#Neg P#})
            using ce-lt-d mult-less-trans unfolding clss-lt-def D L by force
          ultimately have redundant (C + {#Neg P#}) N ∨ clss-lt N (C + E) ⊨p E + {#Pos P#}
            unfolding redundant-iff-abstract abstract-red-def using true-clss-cls-subset by blast
          show False sorry
        qed
      moreover have ¬ redundant (E + {#Pos P#}) N
        sorry
      ultimately have CEN: C + E ∈ N
        using ‹D∈N› ‹E + {#Pos P#}∈N› saturated sup-EC red unfolding saturated-def D L
        by (metis union-commute)
      have CED: C + E ≠ D
        using D ce-lt-d by auto
      have interp: ¬ INTERP N ⊨h C + E
      sorry
      show False
         using cls-not-D[OF CEN CED interp] ce-lt-d unfolding INTERP-def less-eq-multiset-def by
auto
  qed
qed

end

lemma tautology-is-redundant:
  assumes tautology C
```

shows *abstract-red C N*
using *assms* **unfolding** *abstract-red-def true-clss-cls-def tautology-def* **by** *auto*

lemma *subsumed-is-redundant*:
assumes *AB*: $A \subset\# B$
and *AN*: $A \in N$
shows *abstract-red B N*
proof −
have $A \in$ *clss-lt N B* **using** *AN AB* **unfolding** *clss-lt-def*
by (*auto dest*: *less-eq-imp-le-multiset simp add*: *multiset-order.dual-order.order-iff-strict*)
thus *?thesis*
using *AB* **unfolding** *abstract-red-def true-clss-cls-def Partial-Clausal-Logic.true-clss-def*
by *blast*
qed

inductive *redundant* :: *'a clause* ⇒ *'a clauses* ⇒ *bool* **where**
subsumption: $A \in N \Longrightarrow A \subset\# B \Longrightarrow$ *redundant B N*

lemma *redundant-is-redundancy-criterion*:
fixes *A* :: *'a* :: *wellorder clause* **and** *N* :: *'a* :: *wellorder clauses*
assumes *redundant A N*
shows *abstract-red A N*
using *assms*
proof (*induction rule*: *redundant.induct*)
case (*subsumption A B N*)
thus *?case*
using *subsumed-is-redundant*[*of A N B*] **unfolding** *abstract-red-def clss-lt-def* **by** *auto*
qed

lemma *redundant-mono*:
redundant A N $\Longrightarrow A \subseteq\# B \Longrightarrow$ *redundant B N*
apply (*induction rule*: *redundant.induct*)
by (*meson subset-mset.less-le-trans subsumption*)

locale *truc*=
selection S **for** *S* :: *nat clause* ⇒ *nat clause*
begin

end

end
theory *Weidenbach-Book*
imports
Prop-Normalisation

Prop-Resolution

Prop-Superposition

CDCL-NOT DPLL-NOT DPLL-W-Implementation CDCL-W-Implementation CDCL-W-Incremental CDCL-WNOT CDCL-Two-Watched-Literals

begin

end

22 Implementation for 2 Watched-Literals

theory *CDCL-Two-Watched-Literals-Implementation*
imports *CDCL-Two-Watched-Literals DPLL-CDCL-W-Implementation*
begin

type-synonym *conc-twl-state* =
((*nat*, *nat*, *nat list*) *marked-lit*, *nat literal list twl-clause list*, *nat*, *nat literal list*)
twl-state

fun *convert* :: ('*a*, '*b*, '*c list*) *marked-lit* ⇒ ('*a*, '*b*, '*c multiset*) *marked-lit* **where**
convert (*Propagated L C*) = *Propagated L* (*mset C*) |
convert (*Marked K i*) = *Marked K i*

abbreviation *convert-tr* :: ('*a*, '*b*, '*c list*) *marked-lits* ⇒ ('*a*, '*b*, '*c multiset*) *marked-lits*
where
convert-tr ≡ *map convert*

abbreviation *convertC* :: '*a literal list option* ⇒ '*a clause option* **where**
convertC ≡ *map-option mset*

fun *raw-clause-l* :: '*v list twl-clause* ⇒ '*v multiset twl-clause* **where**
raw-clause-l (*TWL-Clause UW W*) = *TWL-Clause* (*mset W*) (*mset UW*)

abbreviation *convert-clss* :: '*v literal list twl-clause list* ⇒ '*v clause twl-clause multiset*
where
convert-clss S ≡ *mset* (*map raw-clause-l S*)

fun *raw-state-of-conc* :: *conc-twl-state* ⇒ (*nat*, *nat*, *nat multiset*) *twl-state-abs* **where**
raw-state-of-conc (*TWL-State M N U k C*) =
TWL-State (*convert-tr M*) (*convert-clss N*) (*convert-clss U*) *k* (*map-option mset C*)

22.1 Abstract Implementation

We define here a locale serving as proxy between the abstract transition defined using multiset and a more concrete version using a representation that can be converted to lists.

22.1.1 An Extend State

The more concrete state has some way to find candidates. This is abstracted, since it can be integrated to the data-structure (see 2-watched literals)

locale *conc-state*$_W$*-with-candidates* =
state$_W$ *trail init-clss learned-clss backtrack-lvl conflicting cons-trail tl-trail add-init-cls*
add-learned-cls remove-cls update-backtrack-lvl update-conflicting init-state
restart-state
for
trail :: '*st* ⇒ ('*v*, *nat*, '*v clause*) *marked-lits* **and**
init-clss :: '*st* ⇒ '*v clauses* **and**
learned-clss :: '*st* ⇒ '*v clauses* **and**
backtrack-lvl :: '*st* ⇒ *nat* **and**
conflicting :: '*st* ⇒'*v clause option* **and**

cons-trail :: ('*v*, *nat*, '*v clause*) *marked-lit* ⇒ '*st* ⇒ '*st* **and**
tl-trail :: '*st* ⇒'*st* **and**

add-init-cls :: *'v clause* ⇒ *'st* ⇒ *'st* **and**
add-learned-cls :: *'v clause* ⇒ *'st* ⇒ *'st* **and**
remove-cls :: *'v clause* ⇒ *'st* ⇒ *'st* **and**
update-backtrack-lvl :: *nat* ⇒ *'st* ⇒ *'st* **and**
update-conflicting :: *'v clause option* ⇒ *'st* ⇒ *'st* **and**

init-state :: *'v clauses* ⇒ *'st* **and**
restart-state :: *'st* ⇒ *'st* +
fixes
raw-trail :: *'conc-st* ⇒ *'trail* **and**
raw-init-clss :: *'conc-st* ⇒ *'clss* **and**
raw-learned-clss :: *'conc-st* ⇒ *'clss* **and**
raw-backtrack-lvl :: *'conc-st* ⇒ *nat* **and**
raw-conflicting :: *'conc-st* ⇒ *'cls option* **and**

raw-cons-trail :: (*'v*, *nat*, *'cls*) *marked-lit* ⇒ *'conc-st* ⇒ *'conc-st* **and**
raw-tl-trail :: *'conc-st* ⇒*'conc-st* **and**
raw-add-init-cls :: *'cls* ⇒ *'conc-st* ⇒ *'conc-st* **and**
raw-add-learned-cls :: *'cls* ⇒ *'conc-st* ⇒ *'conc-st* **and**
raw-remove-cls :: *'cls* ⇒ *'conc-st* ⇒ *'conc-st* **and**
raw-update-backtrack-lvl :: *nat* ⇒ *'conc-st* ⇒ *'conc-st* **and**
raw-update-conflicting :: *'cls option* ⇒ *'conc-st* ⇒ *'conc-st* **and**

raw-init-state :: *'clss* ⇒ *'conc-st* **and**
raw-restart-state :: *'conc-st* ⇒ *'conc-st* **and**
get-propagate-candidates :: *'conc-st* ⇒ (*'v literal* × *'cls*) *list* **and**
get-conflict-candidates :: *'conc-st* ⇒ *'cls list* **and**
get-not-decided :: *'conc-st* ⇒ *'v literal option* **and**

st-of-raw :: *'conc-st* ⇒ *'st* **and**
cls-of-raw-cls :: *'cls* ⇒ *'v clause* **and**
clss-of-raw-clss :: *'clss* ⇒ *'v clause list* **and**
raw-cls-union :: *'cls* ⇒ *'cls* ⇒ *'cls* **and**
remdups-raw-cls :: *'cls* ⇒ *'cls* **and**
marked-lit-of-raw :: (*'v*, *nat*, *'cls*) *marked-lit* ⇒ (*'v*, *nat*, *'v clause*) *marked-lit* **and**
maximum-level :: *'cls* ⇒ *'conc-st* ⇒ *nat* **and**
raw-hd-trail :: *'conc-st* ⇒ (*'v*, *nat*, *'cls*) *marked-lit* **and**
remove :: *'v literal* ⇒ *'cls* ⇒ *'cls*

assumes

raw-cons-trail[*simp*]:
⋀*L S*. *st-of-raw* (*raw-cons-trail L S*) = *cons-trail* (*marked-lit-of-raw L*) (*st-of-raw S*) **and**
raw-tl-trail[*simp*]:
⋀*S*. *st-of-raw* (*raw-tl-trail S*) = *tl-trail* (*st-of-raw S*) **and**
raw-add-init-cls[*simp*]:
⋀*C S*. *st-of-raw* (*raw-add-init-cls C S*) = *add-init-cls* (*cls-of-raw-cls C*) (*st-of-raw S*) **and**
raw-add-learned-cls[*simp*]:
⋀*C S*.
st-of-raw (*raw-add-learned-cls C S*) = *add-learned-cls* (*cls-of-raw-cls C*) (*st-of-raw S*) **and**
raw-backtrack-lvl:
raw-backtrack-lvl S = *backtrack-lvl* (*st-of-raw S*) **and**
raw-update-backtrack-lvl[*simp*]:
⋀*k S*. *st-of-raw* (*raw-update-backtrack-lvl k S*) = *update-backtrack-lvl k* (*st-of-raw S*) **and**
raw-update-conflicting[*simp*]:

$\bigwedge(C::'cls\ option)\ S.\ st\text{-}of\text{-}raw\ (raw\text{-}update\text{-}conflicting\ C\ S) =$
$update\text{-}conflicting\ (map\text{-}option\ cls\text{-}of\text{-}raw\text{-}cls\ C)\ (st\text{-}of\text{-}raw\ S)$ **and**
raw-init-state:
$\bigwedge N.\ st\text{-}of\text{-}raw\ (raw\text{-}init\text{-}state\ N) = init\text{-}state\ (mset\ (clss\text{-}of\text{-}raw\text{-}clss\ N))$ **and**
cls-of-raw-cls-raw-cls-union[*simp*]:
$cls\text{-}of\text{-}raw\text{-}cls\ (raw\text{-}cls\text{-}union\ a\ b) = cls\text{-}of\text{-}raw\text{-}cls\ a\ \#\cup\ cls\text{-}of\text{-}raw\text{-}cls\ b$ **and**
cls-of-raw-cls-remdups-raw-cls[*simp*]:
$cls\text{-}of\text{-}raw\text{-}cls\ (remdups\text{-}raw\text{-}cls\ a) = remdups\text{-}mset\ (cls\text{-}of\text{-}raw\text{-}cls\ a)$ **and**
conflicting-raw-conflicting:
$conflicting\ (st\text{-}of\text{-}raw\ S) = map\text{-}option\ cls\text{-}of\text{-}raw\text{-}cls\ (raw\text{-}conflicting\ S)$ **and**
marked-lit-of-raw[*simp*]:
$\bigwedge L\ C.\ marked\text{-}lit\text{-}of\text{-}raw\ (Propagated\ L\ C) = Propagated\ L\ (cls\text{-}of\text{-}raw\text{-}cls\ C)$
$\bigwedge L\ i.\ marked\text{-}lit\text{-}of\text{-}raw\ (Marked\ L\ i) = Marked\ L\ i$
and
maximum-level[*simp*]:
$maximum\text{-}level\ C\ S = get\text{-}maximum\text{-}level\ (trail\ (st\text{-}of\text{-}raw\ S))\ (cls\text{-}of\text{-}raw\text{-}cls\ C)$ **and**
raw-hd-trail:
$\bigwedge S.\ trail\ (st\text{-}of\text{-}raw\ S) \neq [] \Longrightarrow$
$marked\text{-}lit\text{-}of\text{-}raw\ (raw\text{-}hd\text{-}trail\ S) = hd\ (trail\ (st\text{-}of\text{-}raw\ S))$ **and**
remove[*simp*]:
$cls\text{-}of\text{-}raw\text{-}cls\ (remove\ L\ C) = cls\text{-}of\text{-}raw\text{-}cls\ C - \{\#L\#\}$ **and**

get-conflict-candidates-empty:
$\bigwedge S.\ get\text{-}conflict\text{-}candidates\ S = [] \longleftrightarrow$
$(\forall D \in\#\ clauses\ (st\text{-}of\text{-}raw\ S).\ \neg\ trail\ (st\text{-}of\text{-}raw\ S) \models as\ CNot\ D)$ **and**
get-conflict-candidates-in-clauses:
$\bigwedge S.\ \forall C \in set\ (get\text{-}conflict\text{-}candidates\ S).\ cls\text{-}of\text{-}raw\text{-}cls\ C \in\#\ clauses\ (st\text{-}of\text{-}raw\ S) \wedge$
$trail\ (st\text{-}of\text{-}raw\ S) \models as\ CNot\ (cls\text{-}of\text{-}raw\text{-}cls\ C)$ **and**
get-propagate-candidates-lit-in-cls:
$\bigwedge S.\ \forall (L,\ C) \in set\ (get\text{-}propagate\text{-}candidates\ S).\ undefined\text{-}lit\ (trail\ (st\text{-}of\text{-}raw\ S))\ L \wedge$
$cls\text{-}of\text{-}raw\text{-}cls\ C \in\#\ clauses\ (st\text{-}of\text{-}raw\ S)$
$\wedge\ trail\ (st\text{-}of\text{-}raw\ S) \models as\ CNot\ (cls\text{-}of\text{-}raw\text{-}cls\ C - \{\#L\#\}) \wedge L \in\#\ cls\text{-}of\text{-}raw\text{-}cls\ C$ **and**
get-propagate-candidates-empty:
$\bigwedge S.\ get\text{-}propagate\text{-}candidates\ S = [] \longleftrightarrow$
$\neg(\exists C\ L.\ undefined\text{-}lit\ (trail\ (st\text{-}of\text{-}raw\ S))\ L \wedge C + \{\#L\#\} \in\#\ clauses\ (st\text{-}of\text{-}raw\ S) \wedge$
$trail\ (st\text{-}of\text{-}raw\ S) \models as\ CNot\ C)$ **and**
get-not-decided-Some:
$\bigwedge S\ L.\ get\text{-}not\text{-}decided\ S = Some\ L \Longrightarrow$
$undefined\text{-}lit\ (trail\ (st\text{-}of\text{-}raw\ S))\ L \wedge atm\text{-}of\ L \in atms\text{-}of\text{-}msu\ (init\text{-}clss\ (st\text{-}of\text{-}raw\ S))$
and
get-not-decided-None:
$\bigwedge S.\ get\text{-}not\text{-}decided\ S = None \Longrightarrow$
$\neg(\exists L.\ undefined\text{-}lit\ (trail\ (st\text{-}of\text{-}raw\ S))\ L \wedge$
$atm\text{-}of\ L \in atms\text{-}of\text{-}msu\ (init\text{-}clss\ (st\text{-}of\text{-}raw\ S)))$

22.1.2 Lowering from Transitions to Functions

locale
$cdcl_W\text{-}cands =$
$conc\text{-}state_W\text{-}with\text{-}candidates$ *trail init-clss learned-clss backtrack-lvl conflicting cons-trail*
tl-trail
add-init-cls add-learned-cls remove-cls update-backtrack-lvl update-conflicting init-state
restart-state

raw-trail raw-init-clss raw-learned-clss raw-backtrack-lvl raw-conflicting raw-cons-trail

raw-tl-trail
raw-add-init-cls raw-add-learned-cls raw-remove-cls raw-update-backtrack-lvl
raw-update-conflicting raw-init-state
raw-restart-state

get-propagate-candidates get-conflict-candidates get-not-decided st-of-raw
cls-of-raw-cls clss-of-raw-clss
raw-cls-union remdups-raw-cls marked-lit-of-raw
maximum-level raw-hd-trail remove
for
trail :: $'st \Rightarrow ('v{::}linorder, nat, 'v{::}linorder\ clause)$ *marked-lits* **and**
init-clss :: $'st \Rightarrow 'v$ *clauses* **and**
learned-clss :: $'st \Rightarrow 'v$ *clauses* **and**
backtrack-lvl :: $'st \Rightarrow nat$ **and**
conflicting :: $'st \Rightarrow 'v$ *clause option* **and**

cons-trail :: $('v, nat, 'v\ clause)$ *marked-lit* $\Rightarrow 'st \Rightarrow 'st$ **and**
tl-trail :: $'st \Rightarrow 'st$ **and**
add-init-cls :: $'v\ clause \Rightarrow 'st \Rightarrow 'st$ **and**
add-learned-cls :: $'v\ clause \Rightarrow 'st \Rightarrow 'st$ **and**
remove-cls :: $'v\ clause \Rightarrow 'st \Rightarrow 'st$ **and**
update-backtrack-lvl :: $nat \Rightarrow 'st \Rightarrow 'st$ **and**
update-conflicting :: $'v$ *clause option* $\Rightarrow 'st \Rightarrow 'st$ **and**

init-state :: $'v\ clauses \Rightarrow 'st$ **and**
restart-state :: $'st \Rightarrow 'st$ **and**

raw-trail :: $'conc\text{-}st \Rightarrow 'trail$ **and**
raw-init-clss :: $'conc\text{-}st \Rightarrow 'clss$ **and**
raw-learned-clss :: $'conc\text{-}st \Rightarrow 'clss$ **and**
raw-backtrack-lvl :: $'conc\text{-}st \Rightarrow nat$ **and**
raw-conflicting :: $'conc\text{-}st \Rightarrow 'cls$ *option* **and**

raw-cons-trail :: $('v, nat, 'cls)$ *marked-lit* $\Rightarrow 'conc\text{-}st \Rightarrow 'conc\text{-}st$ **and**
raw-tl-trail :: $'conc\text{-}st \Rightarrow 'conc\text{-}st$ **and**
raw-add-init-cls :: $'cls \Rightarrow 'conc\text{-}st \Rightarrow 'conc\text{-}st$ **and**
raw-add-learned-cls :: $'cls \Rightarrow 'conc\text{-}st \Rightarrow 'conc\text{-}st$ **and**
raw-remove-cls :: $'cls \Rightarrow 'conc\text{-}st \Rightarrow 'conc\text{-}st$ **and**
raw-update-backtrack-lvl :: $nat \Rightarrow 'conc\text{-}st \Rightarrow 'conc\text{-}st$ **and**
raw-update-conflicting :: $'cls$ *option* $\Rightarrow 'conc\text{-}st \Rightarrow 'conc\text{-}st$ **and**

raw-init-state :: $'clss \Rightarrow 'conc\text{-}st$ **and**
raw-restart-state :: $'conc\text{-}st \Rightarrow 'conc\text{-}st$ **and**
get-propagate-candidates :: $'conc\text{-}st \Rightarrow ('v\ literal \times 'cls)$ *list* **and**
get-conflict-candidates :: $'conc\text{-}st \Rightarrow 'cls$ *list* **and**
get-not-decided :: $'conc\text{-}st \Rightarrow 'v$ *literal option* **and**

st-of-raw :: $'conc\text{-}st \Rightarrow 'st$ **and**
cls-of-raw-cls :: $'cls \Rightarrow 'v\ clause$ **and**
clss-of-raw-clss :: $'clss \Rightarrow 'v$ *clause list* **and**
raw-cls-union :: $'cls \Rightarrow 'cls \Rightarrow 'cls$ **and**
remdups-raw-cls :: $'cls \Rightarrow 'cls$ **and**
marked-lit-of-raw :: $('v, nat, 'cls)$ *marked-lit* $\Rightarrow ('v, nat, 'v\ clause)$ *marked-lit* **and**
maximum-level :: $'cls \Rightarrow 'conc\text{-}st \Rightarrow nat$ **and**
raw-hd-trail :: $'conc\text{-}st \Rightarrow ('v, nat, 'cls)$ *marked-lit* **and**

remove :: *'v literal* ⇒ *'cls* ⇒ *'cls*
begin

interpretation *cdcl*$_W$*-termination trail init-clss learned-clss backtrack-lvl conflicting cons-trail tl-trail add-init-cls add-learned-cls remove-cls update-backtrack-lvl update-conflicting init-state restart-state*
by *unfold-locales*

The transitions **definition** *do-conflict-step* :: *'conc-st* ⇒ *'conc-st option* **where**
do-conflict-step S =
(*case raw-conflicting S of*
Some - ⇒ *None*
| *None* ⇒
(*case get-conflict-candidates S of*
[] ⇒ *None*
| *a* # - ⇒ *Some* (*raw-update-conflicting* (*Some a*) *S*)))

lemma *do-conflict-step-Some*:
assumes *conf*: *do-conflict-step S* = *Some T*
shows *conflict* (*st-of-raw S*) (*st-of-raw T*)
proof (*cases raw-conflicting S*)
case *Some*
then show *?thesis* **using** *conf* **unfolding** *do-conflict-step-def* **by** *simp*
next
case *None*
then obtain *C* **where**
C: *C* ∈ *set* (*get-conflict-candidates S*) **and**
T: *T* = *raw-update-conflicting* (*Some C*) *S*
using *conf* **unfolding** *do-conflict-step-def* **by** (*auto split*: *list.splits*)
have
cls-of-raw-cls C ∈# *clauses* (*st-of-raw S*) **and**
trail (*st-of-raw S*) ⊨*as CNot* (*cls-of-raw-cls C*)
using *get-conflict-candidates-in-clauses* **by** (*simp-all add*: *C some-in-eq*)
then show *?thesis*
using *conflict-rule*[*of st-of-raw S trail* (*st-of-raw S*) *init-clss* (*st-of-raw S*)
learned-clss (*st-of-raw S*) *backtrack-lvl* (*st-of-raw S*) *cls-of-raw-cls C st-of-raw T*]
state-eq-ref T None
by (*auto simp*: *conflicting-raw-conflicting*)
qed

lemma *do-conflict-step-None*:
assumes *conf*: *do-conflict-step S* = *None*
shows *no-step conflict* (*st-of-raw S*)
proof (*cases conflicting* (*st-of-raw S*))
case *Some*
then show *?thesis* **by** *auto*
next
case *None*
then have *get-conflict-candidates S* = []
using *conf* **unfolding** *do-conflict-step-def*
by (*auto split*: *list.splits option.splits simp*: *conflicting-raw-conflicting*)
then show *?thesis*
using *get-conflict-candidates-empty* **by** *auto*
qed

We have a list of conflict candidates, but we take only the first element, in case a conflict appears. This is necessary for non-redundancy.

definition *do-propagate-step* :: *'conc-st* ⇒ *'conc-st option* **where**
do-propagate-step S =
  (*case raw-conflicting S of*
    Some - ⇒ *None*
  | *None* ⇒
      (*case get-propagate-candidates S of*
        [] ⇒ *None*
      | (*L*, *C*) # - ⇒ *Some* (*raw-cons-trail* (*Propagated L C*) *S*)))

lemma *do-propagate-step-Some*:
  assumes *conf*: *do-propagate-step S* = *Some T*
  shows *propagate* (*st-of-raw S*) (*st-of-raw T*)
proof (*cases conflicting* (*st-of-raw S*))
  case *Some*
  then show *?thesis*
    using *conf* **by** (*auto simp*: *do-propagate-step-def conflicting-raw-conflicting*
      split: *option.splits list.splits*)
next
  case *None*
  then obtain *L C* **where**
    C: (*L*, *C*) ∈ *set* (*get-propagate-candidates S*) **and**
    T: *T* = *raw-cons-trail* (*Propagated L C*) *S*
     using *conf* **unfolding** *do-propagate-step-def*
    by (*auto split*: *list.splits simp*: *conflicting-raw-conflicting*)
  have
    cls-of-raw-cls C ∈# *clauses* (*st-of-raw S*) **and**
    undef: *undefined-lit* (*trail* (*st-of-raw S*)) *L*
    trail (*st-of-raw S*) ⊨*as CNot* (*cls-of-raw-cls C* − {#*L*#}) **and**
    L ∈# *cls-of-raw-cls C*
    using *get-propagate-candidates-lit-in-cls C* **by** *auto*
  then show *?thesis*
    using *propagate-rule*[*of st-of-raw S trail* (*st-of-raw S*) *init-clss* (*st-of-raw S*)
      learned-clss (*st-of-raw S*) *backtrack-lvl* (*st-of-raw S*) *cls-of-raw-cls C* − {#*L*#} *L*
      st-of-raw T]
      state-eq-ref T None
    by (*auto simp*: *conflicting-raw-conflicting*)
qed

lemma *do-propagate-step-None*:
  assumes *conf*: *do-propagate-step S* = *None*
  shows *no-step propagate* (*st-of-raw S*)
proof (*cases conflicting* (*st-of-raw S*))
  case *Some*
  then show *?thesis* **by** *auto*
next
  case *None*
  then have *get-propagate-candidates S* = []
     using *conf* **unfolding** *do-propagate-step-def*
     by (*auto split*: *list.splits option.splits simp*: *conflicting-raw-conflicting*)
  then show *?thesis*
    unfolding *get-propagate-candidates-empty* **by** (*force elim*!: *propagateE*)
qed

definition *do-skip-step* :: *'conc-st* ⇒ *'conc-st option* **where**

```
do-skip-step S =
  (case conflicting (st-of-raw S) of
    None ⇒ None
  | Some D ⇒
    (case trail (st-of-raw S) of
      Propagated L C' # - ⇒
        if −L ∉# D ∧ D ≠ {#} then Some (raw-tl-trail S) else None
    | - ⇒ None))

lemma do-skip-step-Some:
  assumes conf: do-skip-step S = Some T
  shows skip (st-of-raw S) (st-of-raw T)
proof (cases conflicting (st-of-raw S))
  case None
  then show ?thesis
    using conf by (auto simp: do-skip-step-def)
next
  case (Some D)
  then obtain L C M  where
    C: trail (st-of-raw S) = Propagated L C # M and
    T:  −L ∉# D and
    D ≠ {#} and
    st-of-raw T = tl-trail (st-of-raw S)
    using conf unfolding do-skip-step-def
    by (auto split: list.splits marked-lit.splits split-if-asm simp: conflicting-raw-conflicting)
  then show ?thesis
    using skip-rule[of st-of-raw S L C M init-clss (st-of-raw S)
      learned-clss (st-of-raw S) backtrack-lvl (st-of-raw S)]
      state-eq-ref T Some
    by (auto simp: conflicting-raw-conflicting)
qed

lemma do-skip-step-None:
  assumes conf: do-skip-step S = None
  shows no-step skip (st-of-raw S)
proof (cases conflicting (st-of-raw S))
  case None
  then show ?thesis by auto
next
  case Some
  then show ?thesis
    using conf unfolding do-skip-step-def
    by (auto split: list.splits marked-lit.splits split-if-asm simp: conflicting-raw-conflicting)
qed

definition do-resolve-step :: 'conc-st ⇒ 'conc-st option where
do-resolve-step S =
  (case raw-conflicting S of
    None ⇒ None
  | Some D ⇒
    if trail (st-of-raw S) ≠ []
    then
      (case raw-hd-trail S of
        Propagated L C ⇒
          if −L ∈# cls-of-raw-cls D ∧ cls-of-raw-cls D ≠ {#} ∧
```

maximum-level (*remove* ($-L$) *D*) *S* = *raw-backtrack-lvl* *S*
then Some (*raw-update-conflicting*
(*Some* (*raw-cls-union* (*remove* ($-L$) *D*) (*remove L C*)))
(*raw-tl-trail S*))
else None
| - ⇒ *None*)
else None)

lemma *do-resolve-step-Some*:
assumes *conf*: *do-resolve-step S* = *Some T* **and** *inv*: $cdcl_W$*-all-struct-inv* (*st-of-raw S*)
shows *resolve* (*st-of-raw S*) (*st-of-raw T*)
proof (*cases raw-conflicting S*)
case *None*
then show *?thesis*
using *conf* **by** (*auto simp*: *do-resolve-step-def*)
next
case (*Some D*)
def *M* ≡ *tl* (*trail* (*st-of-raw S*))
obtain *L C* **where**
C: *raw-hd-trail S* = *Propagated L C* **and**
T: $-L$ ∈# *cls-of-raw-cls D* **and**
cls-of-raw-cls D ≠ {#} **and**
T =
raw-update-conflicting (*Some* (*raw-cls-union* (*remove* ($-L$) *D*) (*remove L C*))) (*raw-tl-trail S*) **and**
maximum-level (*remove* ($-L$) *D*) *S* = *raw-backtrack-lvl S* **and**
empty: *trail* (*st-of-raw S*) ≠ []
using *conf Some* **unfolding** *do-resolve-step-def*
by (*auto split*: *list.splits marked-lit.splits split-if-asm simp*: *conflicting-raw-conflicting*)
moreover have *trail* (*st-of-raw S*) = *Propagated L* (*cls-of-raw-cls C*) # *M*
using *empty raw-hd-trail*[*of S*] *C M-def* **by** (*cases trail* (*st-of-raw S*)) *simp-all*
moreover then have *L* ∈# *cls-of-raw-cls C*
using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-conflicting-def* **by** *force*
ultimately show *?thesis*
using *resolve-rule*[*of st-of-raw S L cls-of-raw-cls C* − {#*L*#} *tl* (*trail* (*st-of-raw S*))
init-clss (*st-of-raw S*)
learned-clss (*st-of-raw S*) *backtrack-lvl* (*st-of-raw S*) *cls-of-raw-cls D* − {#$-L$#}
st-of-raw T]
state-eq-ref T Some
by (*auto simp*: *conflicting-raw-conflicting raw-backtrack-lvl*)
qed

definition *do-backtrack-step* :: '*conc-st* ⇒ '*conc-st option* **where**
do-backtrack-step S = *None*

definition *do-bj-step* :: '*conc-st* ⇒ '*conc-st option* **where**
do-bj-step S =
(*case do-skip-step S of*
Some T ⇒ *Some T*
| *None* ⇒
(*case do-resolve-step S of*
Some T ⇒ *Some T*
| *None* ⇒ *do-backtrack-step S*))
end

22.2 Implementation as list

type-synonym $'a$ $cdcl_W$*-mark* = $'a$ *clause*
type-synonym $cdcl_W$*-marked-level* = *nat*

type-synonym $'v$ $cdcl_W$*-marked-lit* = ($'v$, $cdcl_W$*-marked-level*, $'v$ $cdcl_W$*-mark*) *marked-lit*
type-synonym $'v$ $cdcl_W$*-marked-lits* = ($'v$, $cdcl_W$*-marked-level*, $'v$ $cdcl_W$*-mark*) *marked-lits*
type-synonym $'v$ $cdcl_W$*-state* =
  $'v$ $cdcl_W$*-marked-lits* × $'v$ *clauses* × $'v$ *clauses* × *nat* × $'v$ *clause option*

abbreviation *trail* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'a$ **where**
trail ≡ (λ(*M*, -). *M*)

abbreviation *cons-trail* :: $'a \Rightarrow 'a$ *list* $\times 'b \times 'c \times 'd \times 'e \Rightarrow 'a$ *list* $\times 'b \times 'c \times 'd \times 'e$
  where
cons-trail ≡ (λ*L* (*M*, *S*). (*L*#*M*, *S*))

abbreviation *tl-trail* :: $'a$ *list* $\times 'b \times 'c \times 'd \times 'e \Rightarrow 'a$ *list* $\times 'b \times 'c \times 'd \times 'e$ **where**
tl-trail ≡ (λ(*M*, *S*). (*tl M*, *S*))

abbreviation *clauses* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'b$ **where**
clauses ≡ λ(*M*, *N*, -). *N*

abbreviation *learned-clss* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'c$ **where**
learned-clss ≡ λ(*M*, *N*, *U*, -). *U*

abbreviation *backtrack-lvl* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'd$ **where**
backtrack-lvl ≡ λ(*M*, *N*, *U*, *k*, -). *k*

abbreviation *update-backtrack-lvl* :: $'d \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e$
  where
update-backtrack-lvl ≡ λ*k* (*M*, *N*, *U*, -, *S*). (*M*, *N*, *U*, *k*, *S*)

abbreviation *conflicting* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'e$ **where**
conflicting ≡ λ(*M*, *N*, *U*, *k*, *D*). *D*

abbreviation *update-conflicting* :: $'e \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e$
  where
update-conflicting ≡ λ*C* (*M*, *N*, *U*, *k*, -). (*M*, *N*, *U*, *k*, *C*)

abbreviation *S0-*$cdcl_W$ *N* ≡ (([], *N*, {#}, *0*, *None*):: $'v$ $cdcl_W$*-state*)

abbreviation *add-learned-cls* **where**
add-learned-cls ≡ λ*C* (*M*, *N*, *U*, *S*). (*M*, *N*, {#*C*#} + *U*, *S*)

abbreviation *remove-cls* **where**
remove-cls ≡ λ*C* (*M*, *N*, *U*, *S*). (*M*, *remove-mset C N*, *remove-mset C U*, *S*)

lemma *convert-Propagated*[*elim*!]:
  convert z = *Propagated L C* ⟹ (∃ *C'*. *z* = *Propagated L C'* ∧ *C* = *mset C'*)
  by (*cases z*) *auto*

type-synonym $cdcl_W$*-state-inv-st* = (*nat*, *nat*, *nat clause*) *marked-lit list* ×
  nat literal list list × *nat literal list list* × *nat* × *nat literal list option*

fun *maximum-level-code*:: $'a$ *literal list* ⇒ ($'a$, *nat*, $'a$ *literal list*) *marked-lit list* ⇒ *nat*

where
maximum-level-code [] - = *0* |
maximum-level-code (*L* # *Ls*) *M* = *max* (*get-level M L*) (*maximum-level-code Ls M*)

lemma *maximum-level-code-eq-get-maximum-level*[*code*, *simp*]:
maximum-level-code D M = *get-maximum-level M* (*mset D*)
by (*induction D*) (*auto simp add*: *get-maximum-level-plus*)

lemma *get-rev-level-convert-tr*:
get-rev-level (*convert-tr M*) *n* = *get-rev-level M n*
by (*induction M arbitrary*: *n rule*: *marked-lit-list-induct*) *auto*

lemma *get-level-convert-tr*:
get-level (*convert-tr M*) = *get-level M*
by (*simp add*: *get-rev-level-convert-tr rev-map*)

lemma *get-maximum-level-convert-tr*[*simp*]:
get-maximum-level (*convert-tr M*) (*mset D*) = *get-maximum-level M* (*mset D*)
by (*induction D*) (*simp-all add*: *get-maximum-level-plus get-level-convert-tr*)

interpretation $cdcl_W$: $state_W$
trail
λS. *mset* (*clauses S*)
λS. *mset* (*learned-clss S*)
backtrack-lvl conflicting
λL (*M*, *S*). (*L* # *M*, *S*)
λ(*M*, *S*). (*tl M*, *S*)
λC (*M*, *N*, *S*). (*M*, *C* # *N*, *S*)
λC (*M*, *N*, *U*, *S*). (*M*, *N*, *C* # *U*, *S*)
λC (*M*, *N*, *U*, *S*). (*M*, *removeAll C N*, *removeAll C U*, *S*)
λ(*k*::*nat*) (*M*, *N*, *U*, -, *D*). (*M*, *N*, *U*, *k*, *D*)
λD (*M*, *N*, *U*, *k*, -). (*M*, *N*, *U*, *k*, *D*)
λN. ([], *sorted-list-of-multiset N*, [], *0*, *None*)
λ(-, *N*, *U*, -). ([], *N*, *U*, *0*, *None*)
by *unfold-locales* (*auto simp*: *add.commute*)

fun *find-conflict* **where**
find-conflict M [] = *None* |
find-conflict M (*N* # *Ns*) = (*if* ($\forall c \in$ *set N*. $-c \in$ *lits-of M*) *then Some N else find-conflict M Ns*)

lemma *find-conflict-Some*:
find-conflict M Ns = *Some N* $\Longrightarrow$ *N* $\in$ *set Ns* $\wedge$ *M* $\models$*as CNot* (*mset N*)
by (*induction Ns rule*: *find-conflict.induct*)
(*auto split*: *split-if-asm*)

lemma *find-conflict-None*:
find-conflict M Ns = *None* $\longleftrightarrow$ ($\forall N \in$ *set Ns*. $\neg$*M* $\models$*as CNot* (*mset N*))
by (*induction Ns*) *auto*

lemma *find-conflict-sorted-list-of-multiset-None*:
find-conflict M (*map sorted-list-of-multiset Ns*) = *None* $\longleftrightarrow$ ($\forall N \in$ *set Ns*. $\neg$*M* $\models$*as CNot N*)
by (*simp add*: *find-conflict-None*)

lemma *find-conflict-sorted-list-of-multiset-2-None*:
find-conflict M (*map sorted-list-of-multiset Ns* @ *map sorted-list-of-multiset Ns'*) = *None*

$\longleftrightarrow (\forall N \in set\ Ns \cup set\ Ns'.\ \neg M \models as\ CNot\ N)$
by (*metis find-conflict-sorted-list-of-multiset-None map-append set-append*)

declare $cdcl_W$*.state-simp*[*simp del*] $cdcl_W$*.clauses-def*[*simp add*]

lemma *mset-map-mset-removeAll-remove-mset*:
$C \in set\ N \Longrightarrow distinct\ (map\ mset\ N) \Longrightarrow$
$mset\ (map\ mset\ (removeAll\ C\ N)) = remove\text{-}mset\ (mset\ C)\ (mset\ (map\ mset\ N))$
proof (*induction N*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons a N*) **note** *IH* = *this*(*1*) **and** *C* = *this*(*2*) **and** *dist* = *this*(*3*)
have *dist'*: *distinct* (*map mset N*)
using *dist* **by** *auto*
have *H*: $mset\ (map\ mset\ (removeAll\ C\ N)) = remove\text{-}mset\ (mset\ C)\ (mset\ (map\ mset\ N))$
by (*metis C IH count-mset-0 diff-zero dist distinct.simps*(*2*) *list.simps*(*9*) *removeAll-id replicate-mset-0 set-ConsD*)
have *rall*: $mset\ (map\ mset\ (removeAll\ C\ (a\ \#\ N))) =$
$(if\ C = a\ then\ \{\#\}\ else\ \{\#mset\ a\#\}) + mset\ (map\ mset\ (removeAll\ C\ N))$
by (*auto simp*: *ac-simps*)
have *rmset*: $remove\text{-}mset\ (mset\ C)\ (mset\ (map\ mset\ (a\ \#\ N))) =$
$(if\ mset\ C = mset\ a\ then\ \{\#\}\ else\ \{\#mset\ a\#\}) + remove\text{-}mset\ (mset\ C)\ (mset\ (map\ mset\ N))$
proof −
{ assume *a1*: $mset\ C \neq mset\ a$
then have $remove\text{-}mset\ (mset\ C)\ (mset\ (map\ mset\ (a\ \#\ N))) - \{\#mset\ a\#\} + \{\#mset\ a\#\}$
$= remove\text{-}mset\ (mset\ C)\ (mset\ (map\ mset\ (a\ \#\ N))) - \{\#\}$
by *simp*
then have *?thesis*
using *a1* **by** (*simp-all add*: *Multiset.diff-right-commute add.commute*)**}**
then show *?thesis*
by (*cases* $mset\ C \neq mset\ a$) (*auto simp*: *ac-simps*)
qed
have $C \neq a \longrightarrow mset\ C \neq mset\ a$
by (*metis C dist distinct.simps*(*2*) *image-eqI list.simps*(*9*) *set-ConsD set-map*)
then show *?case*
unfolding *rall rmset H* **by** *simp*
qed

interpretation $cdcl_W{}'$: $state_W$
trail
clauses
learned-clss
backtrack-lvl conflicting
$\lambda L\ (M, S).\ (L\ \#\ M, S)$
$\lambda(M, S).\ (tl\ M, S)$
$\lambda C\ (M, N, S).\ (M, \{\#C\#\} + N, S)$
$\lambda C\ (M, N, U, S).\ (M, N, \{\#C\#\} + U, S)$
$\lambda C\ (M, N, U, S).\ (M, remove\text{-}mset\ C\ N, remove\text{-}mset\ C\ U, S)$
$\lambda k\ (M, N, (U{::}nat\ clauses), \text{-}, D).\ (M, N, U, k, D)$
$\lambda D\ (M, N, U, k, \text{-}).\ (M, N, U, k, D)$
$\lambda N.\ ([], N, \{\#\}, 0, None)$
$\lambda(\text{-}, N, U, \text{-}).\ ([], N, U, 0, None)$
by *unfold-locales auto*

fun *union-mset-list* :: *'a list* ⇒ *'a list* ⇒ *'a list* **where**
union-mset-list [] $l = l$ |
union-mset-list (a # l) $l' = a$ # *union-mset-list* l (*remove1* a l')

lemma *mset-union-mset-list*[*simp*]:
mset (*union-mset-list* l l') = *mset* l #∪ *mset* l'
by (*induction* l *arbitrary*: l') (*auto simp*: *multiset-eq-iff*)

lemma *union-mset-list* l [] = l
by (*induction* l) *auto*

interpretation $cdcl_W$: *conc-state*$_W$*-with-candidates*
trail
clauses
learned-clss
backtrack-lvl conflicting
λL (M, S). (L # M, S)
λ(M, S). (*tl* M, S)
λC (M, N, S). (M, {#C#} + N, S)
λC (M, N, U, S). (M, N, {#C#} + U, S)
λC (M, N, U, S). (M, *remove-mset* C N, *remove-mset* C U, S)
λk (M, N, (U::*nat clauses*), -, D). (M, N, U, k, D)
λD (M, N, U, k, -). (M, N, U, k, D)
λN. ([], N, {#}, *0*, *None*)
λ(-, N, U, -). ([], N, U, *0*, *None*)

trail
clauses
learned-clss
backtrack-lvl
conflicting
λL (M, S). (L # M, S)
λ(M, S). (*tl* M, S)
λC (M, N, S). (M, C # N, S)
λC (M, N, U, S). (M, N, C # U, S)
λC (M, N, U, S). (M, *removeAll* C N, *removeAll* C U, S)
λ(k::*nat*) (M, N, U, -, D). (M, N, U, k, D)
λD (M, N, U, k, -). (M, N, U, k, D)
λN. ([], N, [], *0*, *None*)
λ(-, N, U, -). ([], N, U, *0*, *None*)

λ(M, N, U, S).
case find-first-unit-clause (N @ U) M *of*
None ⇒ []
| *Some* (L, a) ⇒ [(L, a)]
λ(M, N, U, S).
case find-conflict M (N @ U) *of*
None ⇒ []
| *Some* a ⇒ [a]
λ(M, N, U, S). *find-first-unused-var* (N @ U) (*lits-of* M)
λ(M, N, U, k, C).
(*convert-tr* M, *mset* (*map mset* N), *mset* (*map mset* U), k, *map-option mset* C)

mset
λN. (*map mset* N)

$\lambda a\ b.$ (*union-mset-list a b*)
remdups
convert
λC (M, N, U, k, D). *maximum-level-code C M*
$\lambda S.$ (*hd* (*trail S*))
remove1
apply *unfold-locales*
apply (*auto simp*: *map-tl add.commute distinct-mset-rempdups-union-mset* $cdcl_W{}'$*.clauses-def*)[*12*]
apply (*auto split*: *option.splits simp*: *find-conflict-None* $cdcl_W{}'$*.clauses-def*)[*2*]
apply (*auto simp*:)[]
using *hd-map* **apply** *metis*
apply *auto*[]

sorry

definition *truc* :: (*nat*, *nat*, *nat literal list*) *marked-lit list* ×
nat literal list list ×
nat literal list list × *nat* × *nat literal list option*
⇒ ((*nat*, *nat*, *nat literal list*) *marked-lit list* ×
nat literal list list ×
nat literal list list ×
nat × *nat literal list option*) *option*
where
truc = $cdcl_W$*-cands.do-conflict-step* $(\lambda(M, N, U, k, D).\ D)$ $(\lambda C\ (M, N, U, k, \text{-}).\ (M, N, U, k, C))$
$(\lambda(M, N, U, S).$ *case find-conflict M* $(N\ @\ U)$ *of None* ⇒ [] | *Some a* ⇒ [*a*])

interpretation $gcdcl_W 2$: $cdcl_W$*-cands*
trail
clauses
learned-clss
backtrack-lvl conflicting
$\lambda L\ (M, S).\ (L\ \#\ M, S)$
$\lambda(M, S).\ (tl\ M, S)$
$\lambda C\ (M, N, S).\ (M, \{\#C\#\} + N, S)$
$\lambda C\ (M, N, U, S).\ (M, N, \{\#C\#\} + U, S)$
$\lambda C\ (M, N, U, S).$ (*M*, *remove-mset C N*, *remove-mset C U*, *S*)
λk (*M*, *N*, (*U*::*nat clauses*), -, *D*). (*M*, *N*, *U*, *k*, *D*)
$\lambda D\ (M, N, U, k, \text{-}).\ (M, N, U, k, D)$
$\lambda N.$ ([], *N*, {#}, *0*, *None*)
λ(-, *N*, *U*, -). ([], *N*, *U*, *0*, *None*)

trail
clauses
learned-clss
backtrack-lvl
conflicting
$\lambda L\ (M, S).\ (L\ \#\ M, S)$
$\lambda(M, S).\ (tl\ M, S)$
$\lambda C\ (M, N, S).\ (M, C\ \#\ N, S)$
$\lambda C\ (M, N, U, S).\ (M, N, C\ \#\ U, S)$
$\lambda C\ (M, N, U, S).$ (*M*, *removeAll C N*, *removeAll C U*, *S*)
λ(*k*::*nat*) (*M*, *N*, *U*, -, *D*). (*M*, *N*, *U*, *k*, *D*)
$\lambda D\ (M, N, U, k, \text{-}).\ (M, N, U, k, D)$
$\lambda N.$ ([], *N*, [], *0*, *None*)

λ(-, N, U, -). ([], N, U, *0*, *None*)

λ(M, N, U, S).
case find-first-unit-clause (N @ U) M *of*
None ⇒ []
| *Some* (L, a) ⇒ [(L, a)]
λ(M, N, U, S).
case find-conflict M (N @ U) *of*
None ⇒ []
| *Some* a ⇒ [a]
λ(M, N, U, S). *find-first-unused-var* (N @ U) (*lits-of* M)
λ(M, N, U, k, C).
(*convert-tr* M, *mset* (*map mset* N), *mset* (*map mset* U), k, *map-option mset* C)

mset
λN. (*map mset* N)
λa b. (*union-mset-list* a b)
remdups
convert
λC (M, N, U, k, D). *maximum-level-code* C M
λS. (*hd* (*trail* S))
remove1
rewrites
$cdcl_W$*-cands.do-conflict-step* (λ(M, N, U, k, D). D) (λC (M, N, U, k, -). (M, N, U, k, C))
(λ(M, N, U, S). *case find-conflict* M (N @ U) *of None* ⇒ [] | *Some* a ⇒ [a])
= *truc*
apply *unfold-locales*
using [[*show-abbrevs* = *false*]]
unfolding *truc-def* **apply** *simp*
sorry

term $cdcl_W$*-cands.do-conflict-step*
thm *truc-def*
declare [[*show-abbrevs* = *false*, *show-types* = *true*, *show-sorts*]]
thm $gcdcl_W 2$*.do-conflict-step-def*
declare $gcdcl_W 2$*.do-conflict-step-def*[*code*]
export-code $gcdcl_W 2$*.do-conflict-step* **in** *SML*

end