Formalisation of Ground Resolution and CDCL in Isabelle/HOL

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Contents

theory *Wellfounded-More*
imports *Main*

begin

1 Transitions

This theory contains more facts about closure, the definition of full transformations, and well-foundedness.

1.1 More theorems about Closures

This is the equivalent of $?r \leq ?s \Longrightarrow ?r^{**} \leq ?s^{**}$ for *tranclp*

lemma *tranclp-mono-explicit*:
$r^{++}\ a\ b \Longrightarrow r \leq s \Longrightarrow s^{++}\ a\ b$
using *rtranclp-mono* **by** (*auto dest*!: *tranclpD intro*: *rtranclp-into-tranclp2*)

lemma *tranclp-mono*:
assumes *mono*: $r \leq s$
shows $r^{++} \leq s^{++}$
using *rtranclp-mono*[*OF mono*] *mono* **by** (*auto dest*!: *tranclpD intro*: *rtranclp-into-tranclp2*)

lemma *tranclp-idemp-rel*:
$R^{++++}\ a\ b \longleftrightarrow R^{++}\ a\ b$
apply (*rule iffI*)
prefer *2* **apply** *blast*
by (*induction rule*: *tranclp-induct*) *auto*

Equivalent of $?r^{****} = ?r^{**}$

lemma *trancl-idemp*: $(r^+)^+ = r^+$
  by *simp*

lemmas *tranclp-idemp*[*simp*] = *trancl-idemp*[*to-pred*]

This theorem already exists as $?r^{**}\ ?a\ ?b \equiv\ ?a =\ ?b \lor\ ?r^{++}\ ?a\ ?b$ (and sledgehammer uses it), but it makes sense to duplicate it, because it is unclear how stable the lemmas in Nitpick are.

lemma *rtranclp-unfold*: $\mathit{rtranclp}\ r\ a\ b \longleftrightarrow (a = b \lor \mathit{tranclp}\ r\ a\ b)$
  by (*meson rtranclp.simps rtranclpD tranclp-into-rtranclp*)

lemma *tranclp-unfold-end*: $\mathit{tranclp}\ r\ a\ b \longleftrightarrow (\exists\, a'.\ \mathit{rtranclp}\ r\ a\ a' \land r\ a'\ b)$
  by (*metis rtranclp.rtrancl-refl rtranclp-into-tranclp1 tranclp.cases tranclp-into-rtranclp*)

lemma *tranclp-unfold-begin*: $\mathit{tranclp}\ r\ a\ b \longleftrightarrow (\exists\, a'.\ r\ a\ a' \land \mathit{rtranclp}\ r\ a'\ b)$
  by (*meson rtranclp-into-tranclp2 tranclpD*)

lemma *trancl-set-tranclp*: $(a,\ b) \in \{(b,a).\ P\ a\ b\}^+ \longleftrightarrow P^{++}\ b\ a$
  apply (*rule iffI*)
    apply (*induction rule*: *trancl-induct*; *simp*)
  apply (*induction rule*: *tranclp-induct*; *auto simp*: *trancl-into-trancl2*)
  done

lemma *tranclp-rtranclp-rtranclp-rel*: $R^{++**}\ a\ b \longleftrightarrow R^{**}\ a\ b$
  by (*simp add*: *rtranclp-unfold*)

lemma *tranclp-rtranclp-rtranclp*[*simp*]: $R^{++**} = R^{**}$
  by (*fastforce simp*: *rtranclp-unfold*)

lemma *rtranclp-exists-last-with-prop*:
  assumes $R\ x\ z$
  and $R^{**}\ z\ z'$ **and** $P\ x\ z$
  shows $\exists\, y\ y'.\ R^{**}\ x\ y \land R\ y\ y' \land P\ y\ y' \land (\lambda a\ b.\ R\ a\ b \land \neg P\ a\ b)^{**}\ y'\ z'$
  using *assms*(*2*,*1*,*3*)
proof (*induction arbitrary*:)
  case *base*
  then show *?case* **by** *auto*
next
  case (*step* $z'\ z''$) **note** $z = \mathit{this}(2)$ **and** $\mathit{IH} = \mathit{this}(3)[\mathit{OF}\ \mathit{this}(4{-}5)]$
  show *?case*
    apply (*cases* $P\ z'\ z''$)
      apply (*rule exI*[*of - z'*], *rule exI*[*of - z''*])
      using *z assms*(*1*) *step.hyps*(*1*) *step.prems*(*2*) **apply** *auto*[*1*]
    using *IH z rtranclp.rtrancl-into-rtrancl* **by** *fastforce*
qed

lemma *rtranclp-and-rtranclp-left*: $(\lambda\ a\ b.\ P\ a\ b \land Q\ a\ b)^{**}\ S\ T \Longrightarrow P^{**}\ S\ T$
  by (*induction rule*: *rtranclp-induct*) *auto*

1.2 Full Transitions

We define here properties to define properties after all possible transitions.

abbreviation *no-step step* $S \equiv (\forall\, S'.\ \neg \mathit{step}\ S\ S')$

definition *full1* :: $('a \Rightarrow 'a \Rightarrow \mathit{bool}) \Rightarrow 'a \Rightarrow 'a \Rightarrow \mathit{bool}$ **where**

$\mathit{full1}\ \mathit{transf} = (\lambda S\ S'.\ \mathit{tranclp}\ \mathit{transf}\ S\ S' \wedge (\forall S''.\ \neg\ \mathit{transf}\ S'\ S''))$

definition $\mathit{full}{::}\ ('a \Rightarrow 'a \Rightarrow \mathit{bool}) \Rightarrow 'a \Rightarrow 'a \Rightarrow \mathit{bool}$ **where**
$\mathit{full}\ \mathit{transf} = (\lambda S\ S'.\ \mathit{rtranclp}\ \mathit{transf}\ S\ S' \wedge (\forall S''.\ \neg\ \mathit{transf}\ S'\ S''))$

lemma *rtranclp-full1I*:
$R^{**}\ a\ b \Longrightarrow \mathit{full1}\ R\ b\ c \Longrightarrow \mathit{full1}\ R\ a\ c$
unfolding *full1-def* **by** *auto*

lemma *tranclp-full1I*:
$R^{++}\ a\ b \Longrightarrow \mathit{full1}\ R\ b\ c \Longrightarrow \mathit{full1}\ R\ a\ c$
unfolding *full1-def* **by** *auto*

lemma *rtranclp-fullI*:
$R^{**}\ a\ b \Longrightarrow \mathit{full}\ R\ b\ c \Longrightarrow \mathit{full}\ R\ a\ c$
unfolding *full-def* **by** *auto*

lemma *tranclp-full-full1I*:
$R^{++}\ a\ b \Longrightarrow \mathit{full}\ R\ b\ c \Longrightarrow \mathit{full1}\ R\ a\ c$
unfolding *full-def full1-def* **by** *auto*

lemma *full-fullI*:
$R\ a\ b \Longrightarrow \mathit{full}\ R\ b\ c \Longrightarrow \mathit{full1}\ R\ a\ c$
unfolding *full-def full1-def* **by** *auto*

lemma *full-unfold*:
$\mathit{full}\ r\ S\ S' \longleftrightarrow ((S = S' \wedge \mathit{no\text{-}step}\ r\ S') \vee \mathit{full1}\ r\ S\ S')$
unfolding *full-def full1-def* **by** (*auto simp add*: *rtranclp-unfold*)

lemma *full1-is-full*[*intro*]: $\mathit{full1}\ R\ S\ T \Longrightarrow \mathit{full}\ R\ S\ T$
by (*simp add*: *full-unfold*)

lemma *not-full1-rtranclp-relation*: $\neg\mathit{full1}\ R^{**}\ a\ b$
by (*meson full1-def rtranclp.rtrancl-refl*)

lemma *not-full-rtranclp-relation*: $\neg\mathit{full}\ R^{**}\ a\ b$
by (*meson full-fullI not-full1-rtranclp-relation rtranclp.rtrancl-refl*)

lemma *full1-tranclp-relation-full*:
$\mathit{full1}\ R^{++}\ a\ b \longleftrightarrow \mathit{full1}\ R\ a\ b$
by (*metis converse-tranclpE full1-def reflclp-tranclp rtranclpD rtranclp-idemp rtranclp-reflclp tranclp.r-into-trancl tranclp-into-rtranclp*)

lemma *full-tranclp-relation-full*:
$\mathit{full}\ R^{++}\ a\ b \longleftrightarrow \mathit{full}\ R\ a\ b$
by (*metis full-unfold full1-tranclp-relation-full tranclp.r-into-trancl tranclpD*)

lemma *rtranclp-full1-eq-or-full1*:
$(\mathit{full1}\ R)^{**}\ a\ b \longleftrightarrow (a = b \vee \mathit{full1}\ R\ a\ b)$
proof −
have $\forall p\ a\ aa.\ \neg\ p^{**}\ (a{::}'a)\ aa \vee a = aa \vee (\exists ab.\ p^{**}\ a\ ab \wedge p\ ab\ aa)$
by (*metis rtranclp.cases*)
then obtain $aa :: ('a \Rightarrow 'a \Rightarrow \mathit{bool}) \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a$ **where**
f1: $\forall p\ a\ ab.\ \neg\ p^{**}\ a\ ab \vee a = ab \vee p^{**}\ a\ (aa\ p\ a\ ab) \wedge p\ (aa\ p\ a\ ab)\ ab$
by *moura*

```
  { assume a ≠ b
    { assume ¬ full1 R a b ∧ a ≠ b
      then have a ≠ b ∧ a ≠ b ∧ ¬ full1 R (aa (full1 R) a b) b ∨ ¬ (full1 R)^** a b ∧ a ≠ b
        using f1 by (metis (no-types) full1-def full1-tranclp-relation-full)
      then have ?thesis
        using f1 by blast }
    then have ?thesis
      by auto }
  then show ?thesis
    by fastforce
qed

lemma tranclp-full1-full1:
  (full1 R)^++ a b ⟷ full1 R a b
  by (metis full1-def rtranclp-full1-eq-or-full1 tranclp-unfold-begin)
```

1.3 Well-Foundedness and Full Transitions

```
lemma wf-exists-normal-form:
  assumes wf:wf {(x, y). R y x}
  shows ∃ b. R^** a b ∧ no-step R b
proof (rule ccontr)
  assume ¬ ?thesis
  then have H: ⋀b. ¬ R^** a b ∨ ¬no-step R b
    by blast
  def F ≡ rec-nat a (λi b. SOME c. R b c)
  have [simp]: F 0 = a
    unfolding F-def by auto
  have [simp]: ⋀i. F (Suc i) = (SOME b. R (F i) b)
    using F-def by simp
  { fix i
    have ∀ j<i. R (F j) (F (Suc j))
      proof (induction i)
        case 0
        then show ?case by auto
      next
        case (Suc i)
        then have R^** a (F i)
          by (induction i) auto
        then have R (F i) (SOME b. R (F i) b)
          using H by (simp add: someI-ex)
        then have ∀ j < Suc i. R (F j) (F (Suc j))
          using H Suc by (simp add: less-Suc-eq)
        then show ?case by fast
      qed
  }
  then have ∀ j. R (F j) (F (Suc j)) by blast
  then show False
    using wf unfolding wfP-def wf-iff-no-infinite-down-chain by blast
qed

lemma wf-exists-normal-form-full:
  assumes wf:wf {(x, y). R y x}
  shows ∃ b. full R a b
  using wf-exists-normal-form[OF assms] unfolding full-def by blast
```

1.4 More Well-Foundedness

A little list of theorems that could be useful, but are hidden:

- link between *wf* and infinite chains: $wf\ ?r = (\nexists f.\ \forall i.\ (f\ (Suc\ i),\ f\ i) \in\ ?r)$, $[\![wf\ ?r;\ \bigwedge k.\ (?f\ (Suc\ k),\ ?f\ k) \notin\ ?r \Longrightarrow\ ?thesis]\!] \Longrightarrow\ ?thesis$

```
lemma wf-if-measure-in-wf:
  wf R ⟹ (⋀a b. (a, b) ∈ S ⟹ (ν a, ν b)∈R) ⟹ wf S
  by (metis in-inv-image wfE-min wfI-min wf-inv-image)

lemma wfP-if-measure: fixes f :: 'a ⇒ nat
shows (⋀x y. P x ⟹ g x y  ⟹ f y < f x) ⟹ wf {(y,x). P x ∧ g x y}
  apply(insert wf-measure[of f])
  apply(simp only: measure-def inv-image-def less-than-def less-eq)
  apply(erule wf-subset)
  apply auto
  done

lemma wf-if-measure-f:
assumes wf r
shows wf {(b, a). (f b, f a) ∈ r}
  using assms by (metis inv-image-def wf-inv-image)

lemma wf-wf-if-measure':
assumes wf r and H: (⋀x y. P x ⟹ g x y ⟹ (f y, f x) ∈ r)
shows wf {(y,x). P x ∧ g x y}
proof –
  have wf {(b, a). (f b, f a) ∈ r} using assms(1) wf-if-measure-f by auto
  then have wf {(b, a). P a ∧ g a b ∧ (f b, f a) ∈ r}
    using wf-subset[of - {(b, a). P a ∧ g a b ∧ (f b, f a) ∈ r}] by auto
  moreover have {(b, a). P a ∧ g a b ∧ (f b, f a) ∈ r} ⊆ {(b, a). (f b, f a) ∈ r} by auto
  moreover have {(b, a). P a ∧ g a b ∧ (f b, f a) ∈ r} = {(b, a). P a ∧ g a b} using H by auto
  ultimately show ?thesis using wf-subset by simp
qed

lemma wf-lex-less: wf (lex {(a, b). (a::nat) < b})
proof –
  have m: {(a, b). a < b} = measure id by auto
  show ?thesis apply (rule wf-lex) unfolding m by auto
qed

lemma wfP-if-measure2: fixes f :: 'a ⇒ nat
shows (⋀x y. P x y ⟹ g x y  ⟹ f x < f y) ⟹ wf {(x,y). P x y ∧ g x y}
  apply(insert wf-measure[of f])
  apply(simp only: measure-def inv-image-def less-than-def less-eq)
  apply(erule wf-subset)
  apply auto
  done

lemma lexord-on-finite-set-is-wf:
  assumes
    P-finite: ⋀U. P U ⟶ U ∈ A and
    finite: finite A and
    wf: wf R and
    trans: trans R
  shows wf {(T, S). (P S ∧ P T) ∧ (T, S) ∈ lexord R}
proof (rule wfP-if-measure2)
  fix T S
  assume P: P S ∧ P T and
  s-le-t: (T, S) ∈ lexord R
  let ?f = λS. {U. (U, S) ∈ lexord R ∧ P U ∧ P S}
  have ?f T ⊆ ?f S
     using s-le-t P lexord-trans trans by auto
  moreover have T ∈ ?f S
    using s-le-t P by auto
  moreover have T ∉ ?f T
    using s-le-t by (auto simp add: lexord-irreflexive local.wf)
  ultimately have {U. (U, T) ∈ lexord R ∧ P U ∧ P T} ⊂ {U. (U, S) ∈ lexord R ∧ P U ∧ P S}
    by auto
  moreover have finite {U. (U, S) ∈ lexord R ∧ P U ∧ P S}
    using finite by (metis (no-types, lifting) P-finite finite-subset mem-Collect-eq subsetI)
  ultimately show card (?f T) < card (?f S) by (simp add: psubset-card-mono)
qed


lemma wf-fst-wf-pair:
  assumes wf {(M', M). R M' M}
  shows wf {((M', N'), (M, N)). R M' M}
proof –
  have wf ({(M', M). R M' M} <*lex*> {})
    using assms by auto
  then show ?thesis
    by (rule wf-subset) auto
qed

lemma wf-snd-wf-pair:
  assumes wf {(M', M). R M' M}
  shows wf {((M', N'), (M, N)). R N' N}
proof –
  have wf: wf {((M', N'), (M, N)). R M' M}
    using assms wf-fst-wf-pair by auto
  then have wf: ⋀P. (∀x. (∀y. (y, x) ∈ {((M', N'), M, N). R M' M} ⟶ P y) ⟶ P x) ⟹ All P
    unfolding wf-def by auto
  show ?thesis
    unfolding wf-def
    proof (intro allI impI)
      fix P :: 'c × 'a ⇒ bool and x :: 'c × 'a
      assume H: ∀x. (∀y. (y, x) ∈ {((M', N'), M, y). R N' y} ⟶ P y) ⟶ P x
      obtain a b where x: x = (a, b) by (cases x)
      have P: P x = (P ∘ (λ(a, b). (b, a))) (b, a)
        unfolding x by auto
      show P x
        using wf[of P o (λ(a, b). (b, a))] apply rule
          using H apply simp
        unfolding P by blast
    qed
qed

lemma wf-if-measure-f-notation2:
  assumes wf r
  shows wf {(b, h a)|b a. (f b, f (h a)) ∈ r}
  apply (rule wf-subset)
  using wf-if-measure-f[OF assms, of f] by auto

lemma wf-wf-if-measure′-notation2:
assumes wf r and H: (⋀x y. P x ⟹ g x y ⟹ (f y, f (h x)) ∈ r)
shows wf {(y,h x)| y x. P x ∧ g x y}
proof −
  have wf {(b, h a)|b a. (f b, f (h a)) ∈ r} using assms(1) wf-if-measure-f-notation2 by auto
  then have wf {(b, h a)|b a. P a ∧ g a b ∧ (f b, f (h a)) ∈ r}
    using wf-subset[of - {(b, h a)| b a. P a ∧ g a b ∧ (f b, f (h a)) ∈ r}] by auto
  moreover have {(b, h a)|b a. P a ∧ g a b ∧ (f b, f (h a)) ∈ r}
    ⊆ {(b, h a)|b a. (f b, f (h a)) ∈ r} by auto
  moreover have {(b, h a)|b a. P a ∧ g a b ∧ (f b, f (h a)) ∈ r} = {(b, h a)|b a. P a ∧ g a b}
    using H by auto
  ultimately show ?thesis using wf-subset by simp
qed

end
theory List-More
imports Main ../lib/Multiset-More
begin
```

Sledgehammer parameters

```
sledgehammer-params[debug]
```

2 Various Lemmas

Close to $(\bigwedge n.\ \forall m{<}n.\ ?P\ m \Longrightarrow ?P\ n) \Longrightarrow ?P\ ?n$, but with a separation between zero and non-zero, and case names.

```
thm nat-less-induct
lemma nat-less-induct-case[case-names 0 Suc]:
  assumes
    P 0 and
    ⋀n. (∀m < Suc n. P m) ⟹ P (Suc n)
  shows P n
  apply (induction rule: nat-less-induct)
  by (rename-tac n, case-tac n) (auto intro: assms)
```

This is only proved in simple cases by auto. In assumptions, nothing happens, and $?P\ (\mathit{if}\ ?Q\ \mathit{then}\ ?x\ \mathit{else}\ ?y) = (\neg\ (?Q \wedge \neg\ ?P\ ?x \vee \neg\ ?Q \wedge \neg\ ?P\ ?y))$ can blow up goals (because of other if expression).

```
lemma if-0-1-ge-0[simp]:
  0 < (if P then a else (0::nat)) ⟷ P ∧ 0 < a
  by auto
```

Bounded function have not been defined in Isabelle.

```
definition bounded where
bounded f ⟷ (∃b. ∀n. f n ≤ b)

abbreviation unbounded :: ('a ⇒ 'b::ord) ⇒ bool where
unbounded f ≡ ¬ bounded f
```

```
lemma not-bounded-nat-exists-larger:
  fixes f :: nat ⇒ nat
  assumes unbound: unbounded f
  shows ∃ n. f n > m ∧ n > n_0
proof (rule ccontr)
  assume H: ¬ ?thesis
  have finite {f n|n. n ≤ n_0}
    by auto
  have ⋀n. f n ≤ Max ({f n|n. n ≤ n_0} ∪ {m})
    apply (case-tac n ≤ n_0)
    apply (metis (mono-tags, lifting) Max-ge Un-insert-right ‹finite {f n |n. n ≤ n_0}›
      finite-insert insertCI mem-Collect-eq sup-bot.right-neutral)
    by (metis (no-types, lifting) H Max-less-iff Un-insert-right ‹finite {f n |n. n ≤ n_0}›
      finite-insert insertI1 insert-not-empty leI sup-bot.right-neutral)
  then show False
    using unbound unfolding bounded-def by auto
qed
```

```
lemma bounded-const-product:
  fixes k :: nat and f :: nat ⇒ nat
  assumes k > 0
  shows bounded f ⟷ bounded (λi. k ∗ f i)
  unfolding bounded-def apply (rule iffI)
   using mult-le-mono2 apply blast
  by (meson assms le-less-trans less-or-eq-imp-le nat-mult-less-cancel-disj split-div-lemma)
```

This lemma is not used, but here to show that a property that can be expected from *bounded* holds.

```
lemma bounded-finite-linorder:
  fixes f :: 'a ⇒ 'a ::{finite, linorder}
  shows bounded f
proof −
  have ⋀x. f x ≤ Max {f x|x. True}
    by (metis (mono-tags) Max-ge finite mem-Collect-eq)
  then show ?thesis
    unfolding bounded-def by blast
qed
```

3 More List

3.1 *upt*

The simplification rules are not very handy, because [*?i*..<*Suc ?j*] = (*if ?i* ≤ *?j then* [*?i*..<*?j*] @ [*?j*] *else* []) leads to a case distinction, that we do not want if the condition is not in the context.

```
lemma upt-Suc-le-append: ¬i ≤ j ⟹ [i..<Suc j] = []
  by auto
```

```
lemmas upt-simps[simp] = upt-Suc-append upt-Suc-le-append
```

```
declare upt.simps(2)[simp del]
```

```
lemma
  assumes i ≤ n − m
  shows take i [m..<n] = [m..<m+i]
  by (metis Nat.le-diff-conv2 add.commute assms diff-is-0-eq′ linear take-upt upt-conv-Nil)
```

The counterpart for this lemma when $n - m < i$ is $\mathit{length}\ ?xs \le\ ?n \Longrightarrow \mathit{take}\ ?n\ ?xs =\ ?xs$. It is close to $?i +\ ?m \le\ ?n \Longrightarrow \mathit{take}\ ?m\ [?i..<?n] = [?i..<?i +\ ?m]$, but seems more general.

```
lemma take-upt-bound-minus[simp]:
  assumes i ≤ n − m
  shows take i [m..<n] = [m ..<m+i]
  using assms by (induction i) auto


lemma append-cons-eq-upt:
  assumes A @ B = [m..<n]
  shows A = [m ..<m+length A] and B = [m + length A..<n]
proof −
  have take (length A) (A @ B) = A by auto
  moreover
    have length A ≤ n − m using assms linear calculation by fastforce
    then have take (length A) [m..<n] = [m ..<m+length A] by auto
  ultimately show A = [m ..<m+length A] using assms by auto
  show B = [m + length A..<n] using assms by (metis append-eq-conv-conj drop-upt)
qed


lemma length-list-Suc-0:
  length W = Suc 0 ⟷ (∃ L. W = [L])
  apply (cases W)
    apply simp
  apply (rename-tac a W′, case-tac W′)
  apply auto
  done


lemma length-list-2: length S = 2 ⟷ (∃ a b. S = [a, b])
  apply (cases S)
   apply simp
  apply (rename-tac a S′)
  apply (case-tac S′)
  by simp-all
```

The converse of $?A\ @\ ?B = [?m..<?n] \Longrightarrow\ ?A = [?m..<?m + \mathit{length}\ ?A]$ $?A\ @\ ?B = [?m..<?n] \Longrightarrow\ ?B = [?m + \mathit{length}\ ?A..<?n]$ does not hold, for example if B is empty and A is $[0::'a]$:

```
lemma A @ B = [m..< n] ⟷ A = [m ..<m+length A] ∧ B = [m + length A..<n]

oops
```

A more restrictive version holds:

```
lemma B ≠ [] ⟹ A @ B = [m..< n] ⟷ A = [m ..<m+length A] ∧ B = [m + length A..<n]
  (is ?P ⟹ ?A = ?B)
proof
  assume ?A then show ?B by (auto simp add: append-cons-eq-upt)
next
  assume ?P and ?B
  then show ?A using append-eq-conv-conj by fastforce
```

```
qed

lemma append-cons-eq-upt-length-i:
  assumes A @ i # B = [m..<n]
  shows A = [m ..<i]
proof −
  have A = [m ..< m + length A] using assms append-cons-eq-upt by auto
  have (A @ i # B) ! (length A) = i by auto
  moreover have n − m = length (A @ i # B)
    using assms length-upt by presburger
  then have [m..<n] ! (length A) = m + length A by simp
  ultimately have i = m + length A using assms by auto
  then show ?thesis using ‹A = [m ..< m + length A]› by auto
qed

lemma append-cons-eq-upt-length:
  assumes A @ i # B = [m..<n]
  shows length A = i − m
  using assms
proof (induction A arbitrary: m)
  case Nil
  then show ?case by (metis append-Nil diff-is-0-eq list.size(3) order-refl upt-eq-Cons-conv)
next
  case (Cons a A)
  then have A: A @ i # B = [m + 1..<n] by (metis append-Cons upt-eq-Cons-conv)
  then have m < i by (metis Cons.prems append-cons-eq-upt-length-i upt-eq-Cons-conv)
  with Cons.IH[OF A] show ?case by auto
qed

lemma append-cons-eq-upt-length-i-end:
  assumes A @ i # B = [m..<n]
  shows B = [Suc i ..<n]
proof −
  have B = [Suc m + length A..<n] using assms append-cons-eq-upt[of A @ [i] B m n] by auto
  have (A @ i # B) ! (length A) = i by auto
  moreover have n − m = length (A @ i # B)
    using assms length-upt by auto
  then have [m..<n]! (length A) = m + length A by simp
  ultimately have i = m + length A using assms by auto
  then show ?thesis using ‹B = [Suc m + length A..<n]› by auto
qed

lemma Max-n-upt: Max (insert 0 {Suc 0..<n}) = n − Suc 0
proof (induct n)
  case 0
  then show ?case by simp
next
  case (Suc n) note IH = this
  have i: insert 0 {Suc 0..<Suc n} = insert 0 {Suc 0..< n} ∪ {n} by auto
  show ?case using IH unfolding i by auto
qed

lemma upt-decomp-lt:
  assumes H: xs @ i # ys @ j # zs = [m ..< n]
  shows i < j
```

proof −
have xs: xs = [m ..< i] **and** ys: ys = [Suc i ..< j] **and** zs: zs = [Suc j ..< n]
using H **by** (auto dest: append-cons-eq-upt-length-i append-cons-eq-upt-length-i-end)
show ?thesis
by (metis append-cons-eq-upt-length-i-end assms lessI less-trans self-append-conv2
upt-eq-Cons-conv upt-rec ys)
qed

3.2 Lexicographic Ordering

lemma lexn-Suc:
(x # xs, y # ys) ∈ lexn r (Suc n) ⟷
(length xs = n ∧ length ys = n) ∧ ((x, y) ∈ r ∨ (x = y ∧ (xs, ys) ∈ lexn r n))
by (auto simp: map-prod-def image-iff lex-prod-def)

lemma lexn-n:
n > 0 ⟹ (x # xs, y # ys) ∈ lexn r n ⟷
(length xs = n−1 ∧ length ys = n−1) ∧ ((x, y) ∈ r ∨ (x = y ∧ (xs, ys) ∈ lexn r (n − 1)))
apply (cases n)
apply simp
by (auto simp: map-prod-def image-iff lex-prod-def)

There is some subtle point in the proof here. 1 is converted to Suc 0, but 2 is not: meaning that 1 is automatically simplified by default using the default simplification rule lexn ?r 0 = {}
lexn ?r (Suc ?n) = map-prod (λ(x, xs). x # xs) (λ(x, xs). x # xs) ‘ (?r <*lex*> lexn ?r ?n) ∩ {(xs, ys). length xs = Suc ?n ∧ length ys = Suc ?n}. However, the latter needs additional simplification rule.

lemma lexn2-conv:
([a, b], [c, d]) ∈ lexn r 2 ⟷ (a, c) ∈ r ∨ (a = c ∧ (b, d) ∈r)
by (auto simp: lexn-n simp del: lexn.simps(2))

lemma lexn3-conv:
([a, b, c], [a′, b′, c′]) ∈ lexn r 3 ⟷
(a, a′) ∈ r ∨ (a = a′ ∧ (b, b′) ∈ r) ∨ (a = a′ ∧ b = b′ ∧ (c, c′) ∈ r)
by (auto simp: lexn-n simp del: lexn.simps(2))

3.3 Remove and Multiset equality

lemma remove1-mset-single-add:
a ≠ b ⟹ remove1-mset a ({#b#} + C) = {#b#} + remove1-mset a C
remove1-mset a ({#a#} + C) = C
by (auto simp: multiset-eq-iff)

This is the sams as remove1 under the assumptions of non-duplication inside a clause.

fun remove1-cond **where**
remove1-cond f [] = [] |
remove1-cond f (C′ # L) = (if f C′ then L else C′ # remove1-cond f L)

lemma mset-map-mset-remove1-cond:
mset (map mset (remove1-cond (λL. mset L = mset a) C)) =
remove1-mset (mset a) (mset (map mset C))
by (induction C) (auto simp: ac-simps remove1-mset-single-add)

fun removeAll-cond **where**

```
removeAll-cond f [] = [] |
removeAll-cond f (C' # L) =
  (if f C' then removeAll-cond f L else C' # removeAll-cond f L)

lemma mset-map-mset-removeAll-cond:
  mset (map mset (removeAll-cond (λb. mset b = mset a) C))
    = removeAll-mset (mset a) (mset (map mset C))
  by (induction C) (auto simp: ac-simps mset-less-eqI multiset-diff-union-assoc)
```

Take from `../lib/Multiset_More.thy`, but named:

```
abbreviation union-mset-list where
union-mset-list xs ys ≡ case-prod append (fold (λx (ys, zs). (remove1 x ys, x # zs)) xs (ys, []))

lemma union-mset-list:
  mset xs #∪ mset ys = mset (union-mset-list xs ys)
proof –
  have ⋀zs. mset (case-prod append (fold (λx (ys, zs). (remove1 x ys, x # zs)) xs (ys, zs))) =
      (mset xs #∪ mset ys) + mset zs
    by (induct xs arbitrary: ys) (simp-all add: multiset-eq-iff)
  then show ?thesis by simp
qed

end
theory Prop-Logic

imports Main

begin
```

4 Logics

In this section we define the syntax of the formula and an abstraction over it to have simpler proofs. After that we define some properties like subformula and rewriting.

4.1 Definition and abstraction

The propositional logic is defined inductively. The type parameter is the type of the variables.

```
datatype 'v propo =
  FT | FF | FVar 'v | FNot 'v propo | FAnd 'v propo 'v propo | FOr 'v propo 'v propo
  | FImp 'v propo 'v propo | FEq 'v propo 'v propo
```

We do not define any notation for the formula, to distinguish properly between the formulas and Isabelle's logic.

To ease the proofs, we will write the the formula on a homogeneous manner, namely a connecting argument and a list of arguments.

```
datatype 'v connective = CT | CF | CVar 'v | CNot | CAnd | COr | CImp | CEq

abbreviation nullary-connective ≡ {CF} ∪ {CT} ∪ {CVar x | x. True}
definition binary-connectives ≡ {CAnd, COr, CImp, CEq}
```

We define our own induction principal: instead of distinguishing every constructor, we group them by arity.

```
lemma propo-induct-arity[case-names nullary unary binary]:
  fixes φ ψ :: 'v propo
  assumes nullary: (⋀φ x. φ = FF ∨ φ = FT ∨ φ = FVar x ⟹ P φ)
  and unary: (⋀ψ. P ψ ⟹ P (FNot ψ))
  and binary: (⋀φ ψ1 ψ2. P ψ1 ⟹ P ψ2 ⟹ φ = FAnd ψ1 ψ2 ∨ φ = FOr ψ1 ψ2 ∨ φ = FImp
ψ1 ψ2
    ∨ φ = FEq ψ1 ψ2 ⟹ P φ)
  shows P ψ
  apply (induct rule: propo.induct)
  using assms by metis+
```

The function *conn* is the interpretation of our representation (connective and list of arguments). We define any thing that has no sense to be false

```
fun conn :: 'v connective ⇒ 'v propo list ⇒ 'v propo where
conn CT [] = FT |
conn CF [] = FF |
conn (CVar v) [] = FVar v |
conn CNot [φ] = FNot φ |
conn CAnd (φ # [ψ]) = FAnd φ ψ |
conn COr (φ # [ψ]) = FOr φ ψ |
conn CImp (φ # [ψ]) = FImp φ ψ |
conn CEq (φ # [ψ]) = FEq φ ψ |
conn - - = FF
```

We will often use case distinction, based on the arity of the *'v connective*, thus we define our own splitting principle.

```
lemma connective-cases-arity[case-names nullary binary unary]:
  assumes nullary: ⋀x. c = CT ∨ c = CF ∨ c = CVar x ⟹ P
  and binary: c ∈ binary-connectives ⟹ P
  and unary: c = CNot  ⟹ P
  shows P
  using assms by (cases c) (auto simp: binary-connectives-def)
```

```
lemma connective-cases-arity-2[case-names nullary unary binary]:
  assumes nullary: c ∈ nullary-connective ⟹ P
  and unary: c = CNot  ⟹ P
  and binary: c ∈ binary-connectives ⟹ P
  shows P
  using assms by (cases c, auto simp add: binary-connectives-def)
```

Our previous definition is not necessary correct (connective and list of arguments) , so we define an inductive predicate.

```
inductive wf-conn :: 'v connective ⇒ 'v propo list ⇒ bool for c :: 'v connective where
wf-conn-nullary[simp]: (c = CT ∨ c = CF ∨ c  = CVar v) ⟹ wf-conn c [] |
wf-conn-unary[simp]: c = CNot ⟹ wf-conn c [ψ] |
wf-conn-binary[simp]: c ∈ binary-connectives ⟹ wf-conn c (ψ # ψ' # [])
thm wf-conn.induct
lemma wf-conn-induct[consumes 1, case-names CT CF CVar CNot COr CAnd CImp CEq]:
  assumes wf-conn c x and
    (⋀v. c = CT ⟹ P []) and
    (⋀v. c = CF ⟹ P []) and
    (⋀v. c = CVar v ⟹ P []) and
    (⋀ψ. c = CNot ⟹ P [ψ]) and
    (⋀ψ ψ'. c = COr ⟹ P [ψ, ψ'])  and
    (⋀ψ ψ'. c = CAnd ⟹ P [ψ, ψ']) and
    (⋀ψ ψ'. c = CImp ⟹ P [ψ, ψ']) and
    (⋀ψ ψ'. c = CEq ⟹ P [ψ, ψ'])
  shows P x
  using assms by induction (auto simp: binary-connectives-def)
```

4.2 properties of the abstraction

First we can define simplification rules.

```
lemma wf-conn-conn[simp]:
  wf-conn CT l ⟹ conn CT l = FT
  wf-conn CF l ⟹ conn CF l = FF
  wf-conn (CVar x) l ⟹ conn (CVar x) l = FVar x
  apply (simp-all add: wf-conn.simps)
  unfolding binary-connectives-def by simp-all
```

```
lemma wf-conn-list-decomp[simp]:
  wf-conn CT l ⟷ l = []
  wf-conn CF l ⟷ l = []
  wf-conn (CVar x) l ⟷ l = []
  wf-conn CNot (ξ @ φ # ξ') ⟷ ξ = [] ∧ ξ' = []
  apply (simp-all add: wf-conn.simps)
      unfolding binary-connectives-def apply simp-all
  by (metis append-Nil append-is-Nil-conv list.distinct(1) list.sel(3) tl-append2)
```

```
lemma wf-conn-list:
  wf-conn c l ⟹ conn c l = FT ⟷ (c = CT ∧ l = [])
  wf-conn c l ⟹ conn c l = FF  ⟷ (c = CF ∧ l = [])
  wf-conn c l ⟹ conn c l = FVar x  ⟷ (c = CVar x ∧ l = [])
  wf-conn c l ⟹ conn c l = FAnd a b  ⟷ (c = CAnd ∧ l = a # b # [])
  wf-conn c l ⟹ conn c l = FOr a b ⟷ (c = COr ∧ l = a # b # [])
  wf-conn c l ⟹ conn c l = FEq a b ⟷ (c = CEq ∧ l = a # b # [])
  wf-conn c l ⟹ conn c l = FImp a b ⟷ (c = CImp ∧ l = a # b # [])
  wf-conn c l ⟹ conn c l = FNot a ⟷ (c = CNot ∧ l = a # [])
  apply (induct l rule: wf-conn.induct)
  unfolding binary-connectives-def by auto
```

In the binary connective cases, we will often decompose the list of arguments (of length 2) into two elements.

```
lemma list-length2-decomp: length l = 2 ⟹ (∃ a b. l = a # b # [])
  apply (induct l, auto)
  by (rename-tac l, case-tac l, auto)
```

wf-conn for binary operators means that there are two arguments.

```
lemma wf-conn-bin-list-length:
  fixes l :: 'v propo list
  assumes conn: c ∈ binary-connectives
  shows length l = 2 ⟷ wf-conn c l
proof
  assume length l = 2
  then show wf-conn c l using wf-conn-binary list-length2-decomp using conn by metis
next
```

```
  assume wf-conn c l
  then show length l = 2 (is ?P l)
    proof (cases rule: wf-conn.induct)
      case wf-conn-nullary
      then show ?P [] using conn binary-connectives-def
        using connective.distinct(11) connective.distinct(13) connective.distinct(9) by blast
    next
      fix ψ :: 'v propo
      case wf-conn-unary
      then show ?P [ψ] using conn binary-connectives-def
        using connective.distinct  by blast
    next
      fix ψ ψ':: 'v propo
      show ?P [ψ, ψ'] by auto
    qed
qed
```

```
lemma wf-conn-not-list-length[iff]:
  fixes l :: 'v propo list
  shows wf-conn CNot l ⟷ length l = 1
  apply auto
  apply (metis append-Nil connective.distinct(5,17,27) length-Cons list.size(3) wf-conn.simps
    wf-conn-list-decomp(4))
  by (simp add: length-Suc-conv wf-conn.simps)
```

Decomposing the Not into an element is moreover very useful.

```
lemma wf-conn-Not-decomp:
  fixes l :: 'v propo list and a :: 'v
  assumes corr: wf-conn CNot l
  shows ∃ a. l = [a]
  by (metis (no-types, lifting) One-nat-def Suc-length-conv corr length-0-conv
    wf-conn-not-list-length)
```

The *wf-conn* remains correct if the length of list does not change. This lemma is very useful when we do one rewriting step

```
lemma wf-conn-no-arity-change:
  length l = length l' ⟹ wf-conn c l ⟷ wf-conn c l'
proof –
  {
    fix l l'
    have length l = length l' ⟹ wf-conn c l ⟹ wf-conn c l'
      apply (cases c l rule: wf-conn.induct, auto)
      by (metis wf-conn-bin-list-length)
  }
  then show length l = length l' ⟹ wf-conn c l = wf-conn c l' by metis
qed
```

```
lemma wf-conn-no-arity-change-helper:
  length (ξ @ φ # ξ') = length (ξ @ φ' # ξ')
  by auto
```

The injectivity of *conn* is useful to prove equality of the connectives and the lists.

```
lemma conn-inj-not:
  assumes correct: wf-conn c l
  and conn: conn c l = FNot ψ
  shows c = CNot and l = [ψ]
  apply (cases c l rule: wf-conn.cases)
  using correct conn unfolding binary-connectives-def apply auto
  apply (cases c l rule: wf-conn.cases)
  using correct conn unfolding binary-connectives-def by auto
```

```
lemma conn-inj:
  fixes c ca :: 'v connective and l ψs :: 'v propo list
  assumes corr: wf-conn ca l
  and corr': wf-conn c ψs
  and eq: conn ca l = conn c ψs
  shows ca = c ∧ ψs = l
  using corr
proof (cases ca l rule: wf-conn.cases)
  case (wf-conn-nullary v)
  then show ca = c ∧ ψs = l using assms
      by (metis conn.simps(1) conn.simps(2) conn.simps(3) wf-conn-list(1−3))
next
  case (wf-conn-unary ψ')
  then have ∗: FNot ψ' = conn c ψs using conn-inj-not eq assms by auto
  then have c = ca by (metis conn-inj-not(1) corr' wf-conn-unary(2))
  moreover have ψs = l using ∗ conn-inj-not(2) corr' wf-conn-unary(1) by force
  ultimately show ca = c ∧ ψs = l by auto
next
  case (wf-conn-binary ψ' ψ'')
  then show ca = c ∧ ψs = l
    using eq corr' unfolding binary-connectives-def apply (cases ca, auto simp add: wf-conn-list)
    using wf-conn-list(4−7) corr' by metis+
qed
```

4.3 Subformulas and properties

A characterization using sub-formulas is interesting for rewriting: we will define our relation on the sub-term level, and then lift the rewriting on the term-level. So the rewriting takes place on a subformula.

```
inductive subformula :: 'v propo ⇒ 'v propo ⇒ bool (infix ≼ 45) for φ where
subformula-refl[simp]: φ ≼ φ |
subformula-into-subformula: ψ ∈ set l ⟹ wf-conn c l ⟹ φ ≼ ψ ⟹ φ ≼ conn c l
```

On the *subformula-into-subformula*, we can see why we use our *conn* representation: one case is enough to express the subformulas property instead of listing all the cases.

This is an example of a property related to subformulas.

```
lemma subformula-in-subformula-not:
shows b: FNot φ ≼ ψ ⟹ φ ≼ ψ
  apply (induct rule: subformula.induct)
  using subformula-into-subformula wf-conn-unary subformula-refl list.set-intros(1) subformula-refl
    by (fastforce intro: subformula-into-subformula)+
```

```
lemma subformula-in-binary-conn:
  assumes conn: c ∈ binary-connectives
  shows f ≼ conn c [f, g]
  and g ≼ conn c [f, g]
proof −
```

```
  have a: wf-conn c (f# [g]) using conn wf-conn-binary binary-connectives-def by auto
  moreover have b: f ≼ f using subformula-refl by auto
  ultimately show f ≼ conn c [f, g]
    by (metis append-Nil in-set-conv-decomp subformula-into-subformula)
next
  have a: wf-conn c ([f] @ [g]) using conn wf-conn-binary binary-connectives-def by auto
  moreover have b: g ≼ g using subformula-refl by auto
  ultimately  show g ≼ conn c [f, g] using subformula-into-subformula by force
qed

lemma subformula-trans:
 ψ ≼ ψ′ ⟹ φ ≼ ψ ⟹ φ ≼ ψ′
  apply (induct ψ′ rule: subformula.inducts)
  by (auto simp: subformula-into-subformula)

lemma subformula-leaf:
  fixes φ ψ :: ′v propo
  assumes incl: φ ≼  ψ
  and simple: ψ = FT ∨ ψ = FF ∨ ψ  = FVar x
  shows φ = ψ
  using incl simple
  by (induct rule: subformula.induct, auto simp: wf-conn-list)

lemma subfurmula-not-incl-eq:
  assumes φ ≼ conn c l
  and wf-conn c l
  and ∀ψ. ψ ∈ set l ⟶ ¬ φ ≼ ψ
  shows φ = conn c l
  using assms apply (induction conn c l rule: subformula.induct, auto)
  using conn-inj by blast

lemma wf-subformula-conn-cases:
  wf-conn c l  ⟹ φ ≼ conn c l ⟷ (φ = conn c l ∨ (∃ψ. ψ ∈ set l ∧ φ ≼ ψ))
  apply standard
    using subfurmula-not-incl-eq apply metis
  by (auto simp add: subformula-into-subformula)

lemma subformula-decomp-explicit[simp]:
  φ ≼ FAnd ψ ψ′ ⟷ (φ = FAnd ψ ψ′ ∨ φ ≼ ψ ∨ φ ≼ ψ′) (is ?P FAnd)
  φ ≼ FOr ψ ψ′ ⟷ (φ = FOr ψ ψ′ ∨ φ ≼ ψ ∨ φ ≼ ψ′)
  φ ≼ FEq ψ ψ′ ⟷ (φ = FEq ψ ψ′ ∨ φ ≼ ψ ∨ φ ≼ ψ′)
  φ ≼ FImp ψ ψ′ ⟷ (φ = FImp ψ ψ′ ∨ φ ≼ ψ ∨ φ ≼ ψ′)
proof –
  have wf-conn CAnd [ψ, ψ′] by (simp add: binary-connectives-def)
  then have φ ≼ conn CAnd [ψ, ψ′] ⟷
    (φ = conn CAnd [ψ, ψ′] ∨ (∃ψ′′. ψ′′ ∈ set [ψ, ψ′] ∧ φ ≼ ψ′′))
    using wf-subformula-conn-cases by metis
  then show ?P FAnd by auto
next
  have wf-conn COr [ψ, ψ′] by (simp add: binary-connectives-def)
  then have φ ≼ conn COr [ψ, ψ′] ⟷
    (φ = conn COr [ψ, ψ′] ∨ (∃ψ′′. ψ′′ ∈ set [ψ, ψ′] ∧ φ ≼ ψ′′))
    using wf-subformula-conn-cases by metis
  then show ?P FOr by auto
next
```

```
  have wf-conn CEq [ψ, ψ'] by (simp add: binary-connectives-def)
  then have φ ≼ conn CEq [ψ, ψ'] ⟷
    (φ = conn CEq [ψ, ψ'] ∨ (∃ψ''. ψ'' ∈ set [ψ, ψ'] ∧ φ ≼ ψ''))
    using wf-subformula-conn-cases by metis
  then show ?P FEq by auto
next
  have wf-conn CImp [ψ, ψ'] by (simp add: binary-connectives-def)
  then have φ ≼ conn CImp [ψ, ψ'] ⟷
    (φ = conn CImp [ψ, ψ'] ∨ (∃ψ''. ψ'' ∈ set [ψ, ψ'] ∧ φ ≼ ψ''))
    using wf-subformula-conn-cases by metis
  then show ?P FImp by auto
qed

lemma wf-conn-helper-facts[iff]:
  wf-conn CNot [φ]
  wf-conn CT []
  wf-conn CF []
  wf-conn (CVar x) []
  wf-conn CAnd [φ, ψ]
  wf-conn COr [φ, ψ]
  wf-conn CImp [φ, ψ]
  wf-conn CEq [φ, ψ]
  using wf-conn.intros unfolding binary-connectives-def by fastforce+

lemma exists-c-conn: ∃ c l. φ = conn c l ∧ wf-conn c l
  by (cases φ) force+

lemma subformula-conn-decomp[simp]:
  assumes wf: wf-conn c l
  shows φ ≼ conn c l ⟷ (φ = conn c l ∨ (∃ ψ∈ set l. φ ≼ ψ)) (is ?A ⟷ ?B)
proof (rule iffI)
  {
    fix ξ
    have φ ≼ ξ ⟹ ξ = conn c l ⟹ wf-conn c l ⟹ ∀x::'a propo∈set l. ¬ φ ≼ x ⟹ φ = conn c l
      apply (induct rule: subformula.induct)
        apply simp
      using conn-inj by blast
  }
  moreover assume ?A
  ultimately show ?B using wf by metis
next
  assume ?B
  then show φ ≼ conn c l using wf wf-subformula-conn-cases by blast
qed

lemma subformula-leaf-explicit[simp]:
  φ ≼ FT ⟷ φ = FT
  φ ≼ FF ⟷ φ = FF
  φ ≼ FVar x ⟷ φ = FVar x
  apply auto
  using subformula-leaf by metis +
```

The variables inside the formula gives precisely the variables that are needed for the formula.

```
primrec vars-of-prop:: 'v propo ⇒ 'v set where
vars-of-prop FT = {} |
vars-of-prop FF = {} |
vars-of-prop (FVar x) = {x} |
vars-of-prop (FNot φ) = vars-of-prop φ |
vars-of-prop (FAnd φ ψ) = vars-of-prop φ ∪ vars-of-prop ψ |
vars-of-prop (FOr φ ψ) = vars-of-prop φ ∪ vars-of-prop ψ |
vars-of-prop (FImp φ ψ) = vars-of-prop φ ∪ vars-of-prop ψ |
vars-of-prop (FEq φ ψ) = vars-of-prop φ ∪ vars-of-prop ψ

lemma vars-of-prop-incl-conn:
  fixes ξ ξ' :: 'v propo list and ψ :: 'v propo and c :: 'v connective
  assumes corr: wf-conn c l and incl: ψ ∈ set l
  shows vars-of-prop ψ ⊆ vars-of-prop  (conn c l)
proof (cases c rule: connective-cases-arity-2)
  case nullary
  then have False using corr incl by auto
  then show vars-of-prop ψ ⊆ vars-of-prop (conn c l) by blast
next
  case binary note c = this
  then obtain a b where ab: l = [a, b]
    using wf-conn-bin-list-length list-length2-decomp corr by metis
  then have ψ = a ∨ ψ = b using incl by auto
  then show vars-of-prop ψ ⊆ vars-of-prop (conn c l)
    using ab c unfolding binary-connectives-def by auto
next
  case unary note c = this
  fix φ :: 'v propo
  have l = [ψ] using corr c incl split-list by force
  then show vars-of-prop ψ ⊆ vars-of-prop (conn c l) using c by auto
qed
```

The set of variables is compatible with the subformula order.

```
lemma subformula-vars-of-prop:
  φ ≼ ψ ⟹ vars-of-prop φ ⊆ vars-of-prop ψ
  apply (induct rule: subformula.induct)
  apply simp
  using vars-of-prop-incl-conn by blast
```

4.4 Positions

Instead of 1 or 2 we use *L* or *R*

```
datatype sign = L | R
```

We use *nil* instead of *ε*.

```
fun pos :: 'v propo ⇒ sign list set where
pos FF = {[]} |
pos FT = {[]} |
pos (FVar x) = {[]} |
pos (FAnd φ ψ) = {[]} ∪ { L # p | p. p∈ pos φ} ∪ { R # p | p. p∈ pos ψ} |
pos (FOr φ ψ) = {[]} ∪ { L # p | p. p∈ pos φ} ∪ { R # p | p. p∈ pos ψ} |
pos (FEq φ ψ) = {[]} ∪ { L # p | p. p∈ pos φ} ∪ { R # p | p. p∈ pos ψ} |
pos (FImp φ ψ) = {[]} ∪ { L # p | p. p∈ pos φ} ∪ { R # p | p. p∈ pos ψ} |
pos (FNot φ) = {[]} ∪ { L # p | p. p∈ pos φ}

lemma finite-pos: finite (pos φ)
  by (induct φ, auto)
```

```
lemma finite-inj-comp-set:
  fixes s :: 'v set
  assumes finite: finite s
  and inj: inj f
  shows card ({f p |p. p ∈ s}) = card s
  using finite
proof (induct s rule: finite-induct)
  show card {f p |p. p ∈ {}} = card {} by auto
next
  fix x :: 'v and s:: 'v set
  assume f: finite s and notin: x ∉ s
  and IH: card {f p |p. p ∈ s} = card s
  have f': finite {f p |p. p ∈ insert x s} using f by auto
  have notin': f x ∉ {f p |p. p ∈ s} using notin inj injD by fastforce
  have {f p |p. p ∈ insert x s} = insert (f x) {f p |p. p∈ s} by auto
  then have card {f p |p. p ∈ insert x s} = 1 + card {f p |p. p ∈ s}
    using finite card-insert-disjoint f' notin' by auto
  moreover have ... = card (insert x s) using notin f IH by auto
  finally show card {f p |p. p ∈ insert x s} = card (insert x s) .
qed

lemma cons-inject:
  inj (op # s)
  by (meson injI list.inject)

lemma finite-insert-nil-cons:
  finite s ⟹ card (insert [] {L # p |p. p ∈ s}) = 1 + card {L # p |p. p ∈ s}
  using card-insert-disjoint by auto


lemma cord-not[simp]:
  card (pos (FNot φ)) = 1 + card (pos φ)
by (simp add: cons-inject finite-inj-comp-set finite-pos)

lemma card-seperate:
  assumes finite s1 and finite s2
  shows card ({L # p |p. p ∈ s1} ∪ {R # p |p. p ∈ s2}) = card ({L # p |p. p ∈ s1})
          + card({R # p |p. p ∈ s2}) (is card (?L∪?R) = card ?L + card ?R)
proof −
  have finite ?L using assms by auto
  moreover have finite ?R using assms by auto
  moreover have ?L ∩ ?R = {} by blast
  ultimately show ?thesis using assms card-Un-disjoint by blast
qed

definition prop-size where prop-size φ = card (pos φ)

lemma prop-size-vars-of-prop:
  fixes φ :: 'v propo
  shows card (vars-of-prop φ) ≤ prop-size φ

  unfolding prop-size-def apply (induct φ, auto simp add: cons-inject finite-inj-comp-set finite-pos)
proof −
  fix φ1 φ2 :: 'v propo
  assume IH1: card (vars-of-prop φ1) ≤ card (pos φ1)
  and IH2: card (vars-of-prop φ2) ≤ card (pos φ2)
  let ?L = {L # p |p. p ∈ pos φ1}
  let ?R = {R # p |p. p ∈ pos φ2}
  have card (?L ∪ ?R) =  card ?L +  card ?R
    using card-seperate finite-pos by blast
  moreover have ... = card (pos φ1) + card (pos φ2)
    by (simp add: cons-inject finite-inj-comp-set finite-pos)
  moreover have ... ≥ card (vars-of-prop φ1) + card (vars-of-prop φ2) using IH1 IH2 by arith
  then have ... ≥ card (vars-of-prop φ1 ∪ vars-of-prop φ2) using card-Un-le le-trans by blast
  ultimately
    show card (vars-of-prop φ1 ∪ vars-of-prop φ2) ≤ Suc (card (?L ∪ ?R))
         card (vars-of-prop φ1 ∪ vars-of-prop φ2) ≤ Suc (card (?L ∪ ?R))
         card (vars-of-prop φ1 ∪ vars-of-prop φ2) ≤ Suc (card (?L ∪ ?R))
         card (vars-of-prop φ1 ∪ vars-of-prop φ2) ≤ Suc (card (?L ∪ ?R))
    by auto
qed

value pos (FImp (FAnd (FVar P) (FVar Q)) (FOr (FVar P) (FVar Q)))

inductive path-to :: sign list ⇒ 'v propo ⇒ 'v propo ⇒ bool where
path-to-refl[intro]: path-to [] φ φ |
path-to-l: c∈binary-connectives ∨ c = CNot ⟹ wf-conn c (φ#l) ⟹ path-to p φ φ'⟹
  path-to (L#p) (conn c (φ#l)) φ' |
path-to-r: c∈binary-connectives ⟹ wf-conn c (ψ#φ#[]) ⟹ path-to p φ φ' ⟹
  path-to (R#p) (conn c (ψ#φ#[])) φ'
```

There is a deep link between subformulas and pathes: a (correct) path leads to a subformula and a subformula is associated to a given path.

```
lemma path-to-subformula:
  path-to p φ φ' ⟹ φ' ≼ φ
  apply (induct rule: path-to.induct)
    apply simp
   apply (metis list.set-intros(1) subformula-into-subformula)
  using subformula-trans subformula-in-binary-conn(2) by metis

lemma subformula-path-exists:
  fixes φ φ':: 'v propo
  shows φ' ≼ φ ⟹ ∃p. path-to p φ φ'
proof (induct rule: subformula.induct)
  case subformula-refl
  have path-to [] φ' φ' by auto
  then show ∃p. path-to p φ' φ' by metis
next
  case (subformula-into-subformula ψ l c)
  note wf = this(2) and IH = this(4) and ψ = this(1)
  then obtain p where p: path-to p ψ φ' by metis
  {
    fix x :: 'v
    assume c = CT ∨ c = CF ∨ c = CVar x
    then have False using subformula-into-subformula by auto
    then have ∃p. path-to p (conn c l) φ' by blast
  }
  moreover {
    assume c: c = CNot
    then have l = [ψ] using wf ψ wf-conn-Not-decomp by fastforce
    then have path-to (L # p) (conn c l) φ' by (metis c wf-conn-unary p path-to-l)
   then have ∃p. path-to p (conn c l) φ' by blast
  }
  moreover {
    assume c: c∈ binary-connectives
    obtain a b where ab: [a, b] = l using subformula-into-subformula c wf-conn-bin-list-length
      list-length2-decomp by metis
    then have a = ψ ∨ b = ψ using ψ by auto
    then have path-to (L # p) (conn c l) φ' ∨ path-to (R # p) (conn c l) φ' using c  path-to-l
      path-to-r p ab by (metis wf-conn-binary)
    then have ∃p. path-to p (conn c l) φ' by blast
  }
  ultimately show ∃p. path-to p (conn c l) φ' using connective-cases-arity by metis
qed

fun replace-at :: sign list ⇒ 'v propo ⇒ 'v propo ⇒ 'v propo where
replace-at [] - ψ = ψ |
replace-at (L # l) (FAnd φ φ') ψ = FAnd (replace-at l φ ψ) φ'|
replace-at (R # l) (FAnd φ φ') ψ = FAnd φ (replace-at l φ' ψ) |
replace-at (L # l) (FOr φ φ') ψ = FOr (replace-at l φ ψ) φ' |
replace-at (R # l) (FOr φ φ') ψ = FOr φ (replace-at l φ' ψ) |
replace-at (L # l) (FEq φ φ') ψ = FEq (replace-at l φ ψ) φ'|
replace-at (R # l) (FEq φ φ') ψ = FEq φ (replace-at l φ' ψ) |
replace-at (L # l) (FImp φ φ') ψ = FImp (replace-at l φ ψ) φ'|
replace-at (R # l) (FImp φ φ') ψ = FImp φ (replace-at l φ' ψ) |
replace-at (L # l) (FNot φ) ψ = FNot (replace-at l φ ψ)
```

5 Semantics over the syntax

Given the syntax defined above, we define a semantics, by defining an evaluation function *eval*. This function is the bridge between the logic as we define it here and the built-in logic of Isabelle.

```
fun eval :: ('v ⇒ bool) ⇒ 'v propo ⇒ bool (infix ⊨ 50) where
𝒜 ⊨ FT = True |
𝒜 ⊨ FF = False |
𝒜 ⊨ FVar v = (𝒜 v) |
𝒜 ⊨ FNot φ = (¬(𝒜⊨ φ)) |
𝒜 ⊨ FAnd φ₁ φ₂ = (𝒜⊨φ₁ ∧ 𝒜⊨φ₂) |
𝒜 ⊨ FOr φ₁ φ₂ = (𝒜⊨φ₁ ∨ 𝒜⊨φ₂) |
𝒜 ⊨ FImp φ₁ φ₂ = (𝒜⊨φ₁ ⟶ 𝒜⊨φ₂)  |
𝒜 ⊨ FEq φ₁ φ₂ = (𝒜⊨φ₁ ⟷ 𝒜 ⊨φ₂)

definition evalf (infix ⊨f 50) where
evalf φ ψ = (∀A. A ⊨ φ ⟶ A ⊨ ψ)
```

The deduction rule is in the book. And the proof looks like to the one of the book.

```
theorem deduction-theorem:
  (φ ⊨f ψ) ⟷ (∀A. (A ⊨ FImp φ ψ))
proof
  assume H: φ ⊨f ψ
  {
    fix A
    have A ⊨ FImp φ ψ
      proof (cases A ⊨ φ)
```

```
        case True
        then have A ⊨ ψ using H unfolding evalf-def by metis
        then show A ⊨ FImp φ ψ by auto
      next
        case False
        then show A ⊨ FImp φ ψ by auto
      qed
  }
  then show ∀ A. A ⊨ FImp φ ψ by blast
next
  assume A: ∀ A. A ⊨ FImp φ ψ
  show φ ⊨f ψ
    proof (rule ccontr)
      assume ¬ φ ⊨f ψ
      then obtain A where A ⊨ φ and ¬ A ⊨ ψ using evalf-def by metis
      then have ¬ A ⊨ FImp φ ψ by auto
      then show False using A by blast
    qed
qed
```

A shorter proof:

```
lemma φ ⊨f ψ ⟷ (∀ A. A⊨ FImp φ ψ)
  by (simp add: evalf-def)

definition same-over-set:: ('v ⇒ bool) ⇒('v ⇒ bool) ⇒ 'v set ⇒ bool where
same-over-set A B S = (∀ c∈S. A c = B c)
```

If two mapping *A* and *B* have the same value over the variables, then the same formula are satisfiable.

```
lemma same-over-set-eval:
  assumes same-over-set A B (vars-of-prop φ)
  shows A ⊨ φ ⟷ B ⊨ φ
  using assms unfolding same-over-set-def by (induct φ, auto)

end
theory Prop-Abstract-Transformation
imports Main Prop-Logic Wellfounded-More

begin
```

This file is devoted to abstract properties of the transformations, like consistency preservation and lifting from terms to proposition.

6 Rewrite systems and properties

6.1 Lifting of rewrite rules

We can lift a rewrite relation r over a full1 formula: the relation *r* works on terms, while *propo-rew-step* works on formulas.

```
inductive propo-rew-step :: ('v propo ⇒ 'v propo ⇒ bool) ⇒ 'v propo ⇒ 'v propo ⇒ bool
  for r :: 'v propo ⇒ 'v propo ⇒ bool where
global-rel: r φ ψ ⟹ propo-rew-step r φ ψ |
propo-rew-one-step-lift: propo-rew-step r φ φ' ⟹ wf-conn c (ψs @ φ # ψs')
```

```
  ⟹ propo-rew-step r (conn c (ψs @ φ # ψs')) (conn c (ψs @ φ'# ψs'))
```

Here is a more precise link between the lifting and the subformulas: if a rewriting takes place between φ and φ', then there are two subformulas ψ in φ and ψ' in φ', ψ' is the result of the rewriting of r on ψ.

This lemma is only a health condition:

```
lemma propo-rew-step-subformula-imp:
shows propo-rew-step r φ φ' ⟹ ∃ ψ ψ'. ψ ≼ φ ∧ ψ' ≼ φ' ∧ r ψ ψ'
  apply (induct rule: propo-rew-step.induct)
    using subformula.simps subformula-into-subformula apply blast
  using wf-conn-no-arity-change subformula-into-subformula wf-conn-no-arity-change-helper
  in-set-conv-decomp by metis
```

The converse is moreover true: if there is a ψ and ψ', then every formula φ containing ψ, can be rewritten into a formula φ', such that it contains φ'.

```
lemma propo-rew-step-subformula-rec:
  fixes ψ ψ' φ :: 'v propo
  shows ψ ≼ φ ⟹ r ψ ψ' ⟹ (∃φ'. ψ' ≼ φ' ∧ propo-rew-step r φ φ')
proof (induct φ rule: subformula.induct)
  case subformula-refl
  hence propo-rew-step r ψ ψ' using propo-rew-step.intros by auto
  moreover have ψ' ≼ ψ' using Prop-Logic.subformula-refl by auto
  ultimately show ∃φ'. ψ' ≼ φ' ∧ propo-rew-step r ψ φ' by fastforce
next
  case (subformula-into-subformula ψ'' l c)
  note IH = this(4) and r = this(5) and ψ'' = this(1) and wf = this(2) and incl = this(3)
  then obtain φ' where *: ψ' ≼ φ' ∧ propo-rew-step r ψ'' φ' by metis
  moreover obtain ξ ξ' :: 'v propo list where
    l: l = ξ @ ψ'' # ξ' using List.split-list ψ'' by metis
  ultimately have propo-rew-step r (conn c l) (conn c (ξ @ φ' # ξ'))
    using propo-rew-step.intros(2) wf by metis
  moreover have ψ' ≼ conn c (ξ @ φ' # ξ')
    using wf * wf-conn-no-arity-change Prop-Logic.subformula-into-subformula
    by (metis (no-types) in-set-conv-decomp l wf-conn-no-arity-change-helper)
  ultimately show ∃φ'. ψ' ≼ φ' ∧ propo-rew-step r (conn c l) φ' by metis
qed
```

```
lemma propo-rew-step-subformula:
  (∃ψ ψ'. ψ ≼ φ ∧ r ψ ψ') ⟷ (∃φ'. propo-rew-step r φ φ')
  using propo-rew-step-subformula-imp propo-rew-step-subformula-rec by metis+
```

```
lemma consistency-decompose-into-list:
  assumes wf: wf-conn c l and wf': wf-conn c l'
  and same: ∀n. (A ⊨ l ! n ⟷ (A ⊨ l' ! n))
  shows (A ⊨ conn c l) = (A ⊨ conn c l')
proof (cases c rule: connective-cases-arity-2)
  case nullary
  thus (A ⊨ conn c l) ⟷ (A ⊨ conn c l') using wf wf' by auto
next
  case unary note c = this
  then obtain a where l: l = [a] using wf-conn-Not-decomp wf by metis
  obtain a' where l': l' = [a'] using wf-conn-Not-decomp wf' c by metis
  have A ⊨ a ⟷ A ⊨ a' using l l' by (metis nth-Cons-0 same)
  thus A ⊨ conn c l ⟷ A ⊨ conn c l' using  l l' c by auto
```

```
next
  case binary note c = this
  then obtain a b where l: l = [a, b]
    using wf-conn-bin-list-length list-length2-decomp wf by metis
  obtain a' b' where l': l' = [a', b']
    using wf-conn-bin-list-length list-length2-decomp wf' c by metis

  have p: A ⊨ a ⟷ A ⊨ a' A ⊨ b ⟷ A ⊨ b'
    using l l' same by (metis diff-Suc-1 nth-Cons' nat.distinct(2))+
  show A ⊨ conn c l ⟷ A ⊨ conn c l'
    using wf c p unfolding binary-connectives-def l l' by auto
qed
```

Relation between *propo-rew-step* and the rewriting we have seen before: *propo-rew-step r φ φ'* means that we rewrite *ψ* inside *φ* (ie at a path *p*) into *ψ'*.

```
lemma propo-rew-step-rewrite:
  fixes φ φ' :: 'v propo and r :: 'v propo ⇒ 'v propo ⇒ bool
  assumes propo-rew-step r φ φ'
  shows ∃ψ ψ' p. r ψ ψ' ∧ path-to p φ ψ ∧ replace-at p φ ψ' = φ'
  using assms
proof (induct rule: propo-rew-step.induct)
  case(global-rel φ ψ)
  moreover have path-to [] φ φ by auto
  moreover have replace-at [] φ ψ = ψ by auto
  ultimately show ?case by metis
next
  case (propo-rew-one-step-lift φ φ' c ξ ξ') note rel = this(1) and IH0 = this(2) and corr = this(3)
  obtain ψ ψ' p where IH: r ψ ψ' ∧ path-to p φ ψ ∧ replace-at p φ ψ' = φ' using IH0 by metis

  {
    fix x :: 'v
    assume c = CT ∨ c = CF ∨ c = CVar x
    hence False using corr by auto
    hence ∃ψ ψ' p. r ψ ψ' ∧ path-to p (conn c (ξ@ (φ # ξ'))) ψ
                  ∧ replace-at p (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
      by fast
  }
  moreover {
    assume c: c = CNot
    hence empty: ξ =[] ξ'=[] using corr by auto
    have path-to (L#p) (conn c (ξ@ (φ # ξ'))) ψ
      using c empty IH wf-conn-unary path-to-l by fastforce
    moreover have replace-at (L#p) (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
      using c empty IH by auto
    ultimately have ∃ψ ψ' p. r ψ ψ' ∧ path-to p (conn c (ξ@ (φ # ξ'))) ψ
                        ∧ replace-at p (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
    using IH by metis
  }
  moreover {
    assume c: c ∈ binary-connectives
    have length (ξ@ φ # ξ') = 2 using wf-conn-bin-list-length corr c by metis
    hence length ξ + length ξ' = 1 by auto
    hence ld: (length ξ = 1 ∧ length ξ' = 0) ∨ (length ξ = 0 ∧ length ξ' = 1)  by arith
    obtain a b where ab: (ξ=[] ∧ ξ'=[b]) ∨ (ξ=[a] ∧ ξ'=[])
      using ld by (case-tac ξ, case-tac ξ', auto)
    {
      assume φ: ξ=[] ∧ ξ'=[b]
      have path-to (L#p) (conn c (ξ@ (φ # ξ'))) ψ
        using φ c IH ab  corr  by (simp add: path-to-l)
      moreover have replace-at (L#p) (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
        using c IH ab φ unfolding binary-connectives-def by auto
      ultimately have ∃ψ ψ' p. r ψ ψ' ∧ path-to p (conn c (ξ@ (φ # ξ'))) ψ
        ∧ replace-at p (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
        using IH by metis
    }
    moreover {
      assume φ: ξ=[a]  ξ'=[]
      hence path-to (R#p) (conn c (ξ@ (φ # ξ'))) ψ
        using c IH corr path-to-r corr φ  by (simp add: path-to-r)
      moreover have replace-at (R#p) (conn c (ξ@ (φ # ξ'))) ψ' = conn c (ξ@ (φ' # ξ'))
        using c IH ab φ unfolding binary-connectives-def by auto
      ultimately have ?case using IH by metis
    }
    ultimately have ?case using ab by blast
  }
  ultimately show ?case using connective-cases-arity by blast
qed
```

6.2 Consistency preservation

We define *preserves-un-sat*: it means that a relation preserves consistency.

```
definition preserves-un-sat where
preserves-un-sat r ⟷ (∀φ ψ. r φ ψ ⟶ (∀A. A ⊨ φ ⟷ A ⊨ ψ))
```

```
lemma propo-rew-step-preservers-val-explicit:
propo-rew-step r φ ψ ⟹ preserves-un-sat r ⟹ propo-rew-step r φ ψ ⟹ (∀A. A ⊨φ ⟷ A⊨ψ)
  unfolding preserves-un-sat-def
proof (induction rule: propo-rew-step.induct)
  case global-rel
  thus ?case by simp
next
  case (propo-rew-one-step-lift φ φ' c ξ ξ') note rel = this(1) and wf = this(2)
    and IH = this(3)[OF this(4) this(1)]  and consistent = this(4)
  {
    fix A
    from IH have ∀n. (A ⊨ (ξ @ φ # ξ') ! n) = (A ⊨ (ξ @ φ' # ξ') ! n)
      by (metis (mono-tags, hide-lams) list-update-length nth-Cons-0 nth-append-length-plus
        nth-list-update-neq)
    hence  (A ⊨ conn c (ξ @ φ # ξ')) = (A ⊨ conn c (ξ @ φ' # ξ'))
      by (meson consistency-decompose-into-list wf wf-conn-no-arity-change-helper
        wf-conn-no-arity-change)
  }
  thus ∀A. A ⊨ conn c (ξ @ φ # ξ') ⟷ A ⊨ conn c (ξ @ φ' # ξ') by auto
qed
```

```
lemma propo-rew-step-preservers-val':
  assumes preserves-un-sat r
  shows preserves-un-sat (propo-rew-step r)
```

using *assms* **by** (*simp add*: *preserves-un-sat-def propo-rew-step-preservers-val-explicit*)

lemma *preserves-un-sat-OO*[*intro*]:
preserves-un-sat f $\Longrightarrow$ *preserves-un-sat g* $\Longrightarrow$ *preserves-un-sat* (*f OO g*)
unfolding *preserves-un-sat-def* **by** *auto*

lemma *star-consistency-preservation-explicit*:
assumes (*propo-rew-step r*)^** φ ψ **and** *preserves-un-sat r*
shows $\forall A.\ A \models \varphi \longleftrightarrow A \models \psi$
using *assms* **by** (*induct rule*: *rtranclp-induct*)
(*auto simp add*: *propo-rew-step-preservers-val-explicit*)

lemma *star-consistency-preservation*:
preserves-un-sat r $\Longrightarrow$ *preserves-un-sat* (*propo-rew-step r*)^**
by (*simp add*: *star-consistency-preservation-explicit preserves-un-sat-def*)

6.3 Full Lifting

In the previous a relation was lifted to a formula, now we define the relation such it is applied as long as possible. The definition is thus simply: it can be derived and nothing more can be derived.

lemma *full-ropo-rew-step-preservers-val*[*simp*]:
preserves-un-sat r $\Longrightarrow$ *preserves-un-sat* (*full* (*propo-rew-step r*))
by (*metis full-def preserves-un-sat-def star-consistency-preservation*)

lemma *full-propo-rew-step-subformula*:
full (*propo-rew-step r*) φ' φ $\Longrightarrow$ $\neg(\exists\ \psi\ \psi'.\ \psi \preceq \varphi \land r\ \psi\ \psi')$
unfolding *full-def* **using** *propo-rew-step-subformula-rec* **by** *metis*

7 Transformation testing

7.1 Definition and first properties

To prove correctness of our transformation, we create a *all-subformula-st* predicate. It tests recursively all subformulas. At each step, the actual formula is tested. The aim of this *test-symb* function is to test locally some properties of the formulas (i.e. at the level of the connective or at first level). This allows a clause description between the rewrite relation and the *test-symb*

definition *all-subformula-st* :: (*'a propo* $\Rightarrow$ *bool*) $\Rightarrow$ *'a propo* $\Rightarrow$ *bool* **where**
all-subformula-st test-symb $\varphi \equiv \forall \psi.\ \psi \preceq \varphi \longrightarrow$ *test-symb* ψ

lemma *test-symb-imp-all-subformula-st*[*simp*]:
test-symb FT $\Longrightarrow$ *all-subformula-st test-symb FT*
test-symb FF $\Longrightarrow$ *all-subformula-st test-symb FF*
test-symb (*FVar x*) $\Longrightarrow$ *all-subformula-st test-symb* (*FVar x*)
unfolding *all-subformula-st-def* **using** *subformula-leaf* **by** *metis+*

lemma *all-subformula-st-test-symb-true-phi*:
all-subformula-st test-symb φ $\Longrightarrow$ *test-symb* φ
unfolding *all-subformula-st-def* **by** *auto*

```
lemma all-subformula-st-decomp-imp:
  wf-conn c l ⟹ (test-symb (conn c l) ∧ (∀φ∈ set l. all-subformula-st test-symb φ))
  ⟹ all-subformula-st test-symb (conn c l)
  unfolding all-subformula-st-def by auto
```

To ease the finding of proofs, we give some explicit theorem about the decomposition.

```
lemma all-subformula-st-decomp-rec:
  all-subformula-st test-symb  (conn c l) ⟹ wf-conn c l
    ⟹ (test-symb (conn c l) ∧ (∀φ∈ set l. all-subformula-st test-symb φ))
  unfolding all-subformula-st-def by auto

lemma all-subformula-st-decomp:
  fixes c  :: 'v connective and l :: 'v propo list
  assumes wf-conn c l
  shows all-subformula-st test-symb (conn c l)
    ⟷ (test-symb (conn c l) ∧ (∀φ∈ set l. all-subformula-st test-symb φ))
  using assms all-subformula-st-decomp-rec all-subformula-st-decomp-imp by metis

lemma helper-fact: c ∈ binary-connectives ⟷ (c = COr ∨ c = CAnd ∨ c = CEq ∨ c = CImp)
  unfolding binary-connectives-def by auto
lemma all-subformula-st-decomp-explicit[simp]:
  fixes φ ψ :: 'v propo
  shows all-subformula-st test-symb (FAnd φ ψ)
      ⟷ (test-symb (FAnd φ ψ) ∧ all-subformula-st test-symb φ ∧ all-subformula-st test-symb ψ)
  and all-subformula-st test-symb (FOr φ ψ)
     ⟷ (test-symb (FOr φ ψ) ∧  all-subformula-st test-symb φ ∧ all-subformula-st test-symb ψ)
  and all-subformula-st test-symb (FNot φ)
     ⟷ (test-symb (FNot φ) ∧  all-subformula-st test-symb φ)
  and all-subformula-st test-symb (FEq φ ψ)
     ⟷ (test-symb (FEq φ ψ) ∧  all-subformula-st test-symb φ ∧ all-subformula-st test-symb ψ)
  and all-subformula-st test-symb (FImp φ ψ)
     ⟷ (test-symb (FImp φ ψ) ∧ all-subformula-st test-symb φ ∧ all-subformula-st test-symb ψ)
proof –
  have all-subformula-st test-symb (FAnd φ ψ) ⟷ all-subformula-st test-symb (conn CAnd [φ, ψ])
    by auto
  moreover have ... ⟷test-symb (conn CAnd [φ, ψ])∧(∀ξ∈ set [φ, ψ]. all-subformula-st test-symb
ξ)
    using all-subformula-st-decomp wf-conn-helper-facts(5) by metis
  finally show all-subformula-st test-symb (FAnd φ ψ)
    ⟷ (test-symb (FAnd φ ψ) ∧ all-subformula-st test-symb φ ∧ all-subformula-st test-symb ψ)
    by simp

  have all-subformula-st test-symb (FOr φ ψ) ⟷ all-subformula-st test-symb (conn COr [φ, ψ])
    by auto
  moreover have ...⟷
    (test-symb (conn COr [φ, ψ]) ∧ (∀ξ∈ set [φ, ψ]. all-subformula-st test-symb ξ))
    using all-subformula-st-decomp wf-conn-helper-facts(6) by metis
  finally show all-subformula-st test-symb (FOr φ ψ)
    ⟷ (test-symb (FOr φ ψ) ∧ all-subformula-st test-symb φ ∧ all-subformula-st test-symb ψ)
    by simp

  have all-subformula-st test-symb (FEq φ ψ) ⟷ all-subformula-st test-symb (conn CEq [φ, ψ])
    by auto
  moreover have ...
```

```
    ⟷ (test-symb (conn CEq [φ, ψ]) ∧ (∀ξ∈ set [φ, ψ]. all-subformula-st test-symb ξ))
    using all-subformula-st-decomp wf-conn-helper-facts(8) by metis
  finally show all-subformula-st test-symb (FEq φ ψ)
    ⟷ (test-symb (FEq φ ψ) ∧ all-subformula-st test-symb φ ∧ all-subformula-st test-symb ψ)
    by simp

  have all-subformula-st test-symb (FImp φ ψ) ⟷ all-subformula-st test-symb (conn CImp [φ, ψ])
    by auto
  moreover have ...
    ⟷(test-symb (conn CImp [φ, ψ]) ∧ (∀ξ∈ set [φ, ψ]. all-subformula-st test-symb ξ))
    using all-subformula-st-decomp wf-conn-helper-facts(7) by metis
  finally show all-subformula-st test-symb (FImp φ ψ)
    ⟷ (test-symb (FImp φ ψ) ∧ all-subformula-st test-symb φ ∧ all-subformula-st test-symb ψ)
    by simp

  have all-subformula-st test-symb (FNot φ) ⟷ all-subformula-st test-symb (conn CNot [φ])
    by auto
  moreover have ... = (test-symb (conn CNot [φ]) ∧ (∀ξ∈ set [φ]. all-subformula-st test-symb ξ))
    using all-subformula-st-decomp wf-conn-helper-facts(1) by metis
  finally show all-subformula-st test-symb (FNot φ)
    ⟷ (test-symb (FNot φ) ∧ all-subformula-st test-symb φ) by simp
qed
```

As *all-subformula-st* tests recursively, the function is true on every subformula.

```
lemma subformula-all-subformula-st:
  ψ ≼ φ ⟹ all-subformula-st test-symb φ ⟹ all-subformula-st test-symb ψ
  by (induct rule: subformula.induct, auto simp add: all-subformula-st-decomp)
```

The following theorem *no-test-symb-step-exists* shows the link between the *test-symb* function and the corresponding rewrite relation *r*: if we assume that if every time *test-symb* is true, then a *r* can be applied, finally as long as ¬ *all-subformula-st test-symb φ*, then something can be rewritten in *φ*.

```
lemma no-test-symb-step-exists:
  fixes r:: 'v propo ⇒ 'v propo ⇒ bool and test-symb:: 'v propo ⇒ bool and x :: 'v
  and φ :: 'v propo
  assumes test-symb-false-nullary: ∀x. test-symb FF ∧ test-symb FT ∧ test-symb (FVar x)
  and ∀φ'. φ' ≼ φ ⟶ (¬test-symb φ') ⟶  (∃ ψ. r φ' ψ) and
  ¬ all-subformula-st test-symb φ
  shows (∃ψ ψ'. ψ ≼ φ ∧ r ψ ψ')
  using assms
proof (induct φ rule: propo-induct-arity)
  case (nullary φ x)
  thus ∃ψ ψ'. ψ ≼ φ ∧ r ψ ψ'
    using wf-conn-nullary test-symb-false-nullary by fastforce
next
   case (unary φ) note IH = this(1)[OF this(2)] and r = this(2)  and nst = this(3) and subf =
this(4)
  from r IH nst have H: ¬ all-subformula-st test-symb φ ⟹ ∃ψ. ψ ≼ φ ∧ (∃ψ'. r ψ ψ')
    by (metis subformula-in-subformula-not subformula-refl subformula-trans)
  {
    assume n: ¬test-symb (FNot φ)
    obtain ψ where r (FNot φ) ψ using subformula-refl r n nst by blast
    moreover have FNot φ ≼ FNot φ using subformula-refl by auto
    ultimately have ∃ψ ψ'. ψ ≼ FNot φ ∧ r ψ ψ' by metis
  }
  moreover {
    assume n: test-symb (FNot φ)
    hence ¬ all-subformula-st test-symb φ
      using all-subformula-st-decomp-explicit(3) nst subf by blast
    hence ∃ψ ψ'. ψ ≼ FNot φ ∧ r ψ ψ'
      using H subformula-in-subformula-not subformula-refl subformula-trans by blast
  }
  ultimately show ∃ψ ψ'. ψ ≼ FNot φ ∧ r ψ ψ' by blast
next
  case (binary φ φ1 φ2)
  note IHφ1-0 = this(1)[OF this(4)] and IHφ2-0 = this(2)[OF this(4)] and r = this(4)
    and φ = this(3) and le = this(5) and nst = this(6)

  obtain c :: 'v connective where
    c: (c = CAnd ∨ c = COr ∨ c = CImp ∨ c = CEq) ∧ conn c [φ1, φ2] = φ
    using φ by fastforce

  hence corr: wf-conn c [φ1, φ2] using wf-conn.simps unfolding binary-connectives-def by auto
  have inc: φ1 ≼ φ φ2 ≼ φ using binary-connectives-def c subformula-in-binary-conn by blast+
  from r IHφ1-0 have IHφ1: ¬ all-subformula-st test-symb φ1 ⟹ ∃ψ ψ'. ψ ≼ φ1 ∧ r ψ ψ'
    using inc(1) subformula-trans le by blast
  from r IHφ2-0 have IHφ2: ¬ all-subformula-st test-symb φ2 ⟹ ∃ψ. ψ ≼ φ2 ∧ (∃ψ'. r ψ ψ')
    using inc(2) subformula-trans le by blast
  have cases: ¬test-symb φ ∨ ¬all-subformula-st test-symb φ1 ∨ ¬all-subformula-st test-symb φ2
    using c nst by auto
  show ∃ψ ψ'. ψ ≼ φ ∧ r ψ ψ'
    using IHφ1 IHφ2 subformula-trans inc subformula-refl cases le by blast
qed
```

7.2 Invariant conservation

If two rewrite relation are independant (or at least independant enough), then the property characterizing the first relation *all-subformula-st test-symb* remains true. The next show the same property, with changes in the assumptions.

The assumption $\forall \varphi'\, \psi.\ \varphi' \preceq \Phi \longrightarrow r\ \varphi'\ \psi \longrightarrow \textit{all-subformula-st test-symb}\ \varphi' \longrightarrow \textit{all-subformula-st test-symb}\ \psi$ means that rewriting with r does not mess up the property we want to preserve locally.

The previous assumption is not enough to go from r to *propo-rew-step* r: we have to add the assumption that rewriting inside does not mess up the term: $\forall c\ \xi\ \varphi\ \xi'\ \varphi'.\ \varphi \preceq \Phi \longrightarrow \textit{propo-rew-step}\ r\ \varphi\ \varphi' \longrightarrow \textit{wf-conn}\ c\ (\xi\ @\ \varphi\ \#\ \xi') \longrightarrow \textit{test-symb}\ (\textit{conn}\ c\ (\xi\ @\ \varphi\ \#\ \xi')) \longrightarrow \textit{test-symb}\ \varphi' \longrightarrow \textit{test-symb}\ (\textit{conn}\ c\ (\xi\ @\ \varphi'\ \#\ \xi'))$

7.2.1 Invariant while lifting of the rewriting relation

The condition $\varphi \preceq \Phi$ (that will by used with $\Phi = \varphi$ most of the time) is here to ensure that the recursive conditions on Φ will moreover hold for the subterm we are rewriting. For example if there is no equivalence symbol in Φ, we do not have to care about equivalence symbols in the two previous assumptions.

```
lemma propo-rew-step-inv-stay':
  fixes r:: 'v propo ⇒ 'v propo ⇒ bool and test-symb:: 'v propo ⇒ bool and x :: 'v
  and φ ψ Φ:: 'v propo
  assumes H: ∀φ' ψ. φ' ≼ Φ ⟶ r φ' ψ ⟶ all-subformula-st test-symb φ'
    ⟶ all-subformula-st test-symb ψ
  and H': ∀(c:: 'v connective) ξ φ ξ' φ'. φ ≼ Φ ⟶ propo-rew-step r φ φ'
    ⟶ wf-conn c (ξ @ φ # ξ') ⟶ test-symb (conn c (ξ @ φ # ξ')) ⟶ test-symb φ'
    ⟶ test-symb (conn c (ξ @ φ' # ξ')) and
    propo-rew-step r φ ψ  and
    φ ≼ Φ  and
    all-subformula-st test-symb φ
  shows all-subformula-st test-symb ψ
  using assms(3−5)
proof (induct rule: propo-rew-step.induct)
  case global-rel
  thus ?case using H by simp
next
  case (propo-rew-one-step-lift φ φ' c ξ ξ')
  note rel = this(1) and φ = this(2) and corr = this(3) and Φ = this(4) and nst = this(5)
  have sq: φ ≼ Φ
    using Φ corr subformula-into-subformula subformula-refl subformula-trans
    by (metis in-set-conv-decomp)
  from corr have ∀ ψ. ψ ∈ set (ξ @ φ # ξ') ⟶ all-subformula-st test-symb ψ
    using all-subformula-st-decomp nst by blast
  hence *: ∀ψ. ψ ∈ set (ξ @ φ' # ξ') ⟶ all-subformula-st test-symb ψ using φ sq by fastforce
  hence test-symb φ' using all-subformula-st-test-symb-true-phi by auto
  moreover from corr nst have test-symb (conn c (ξ @ φ # ξ'))
    using all-subformula-st-decomp by blast
  ultimately have test-symb: test-symb (conn c (ξ @ φ' # ξ')) using H' sq corr rel by blast

  have wf-conn c (ξ @ φ' # ξ')
    by (metis wf-conn-no-arity-change-helper corr wf-conn-no-arity-change)
  thus all-subformula-st test-symb (conn c (ξ @ φ' # ξ'))
    using * test-symb by (metis all-subformula-st-decomp)
qed
```

The need for $\varphi \preceq \Phi$ is not always necessary, hence we moreover have a version without inclusion.

```
lemma propo-rew-step-inv-stay:
  fixes r:: 'v propo ⇒ 'v propo ⇒ bool and test-symb:: 'v propo ⇒ bool and x :: 'v
  and φ ψ :: 'v propo
  assumes
    H: ∀φ' ψ. r φ' ψ ⟶ all-subformula-st test-symb φ' ⟶ all-subformula-st test-symb ψ and
    H': ∀(c:: 'v connective) ξ φ ξ' φ'. wf-conn c (ξ @ φ # ξ') ⟶ test-symb (conn c (ξ @ φ # ξ'))
      ⟶ test-symb φ' ⟶ test-symb (conn c (ξ @ φ' # ξ')) and
    propo-rew-step r φ ψ and
    all-subformula-st test-symb φ
  shows all-subformula-st test-symb ψ
  using propo-rew-step-inv-stay'[of φ r test-symb φ ψ] assms subformula-refl by metis
```

The lemmas can be lifted to *full* (*propo-rew-step r*) instead of *propo-rew-step*

7.2.2 Invariant after all rewriting

```
lemma full-propo-rew-step-inv-stay-with-inc:
  fixes r:: 'v propo ⇒ 'v propo ⇒ bool and test-symb:: 'v propo ⇒ bool and x :: 'v
  and φ ψ :: 'v propo
  assumes
    H: ∀ φ ψ. propo-rew-step r φ ψ ⟶ all-subformula-st test-symb φ
```

```
      ⟶ all-subformula-st test-symb ψ and
    H′: ∀ (c:: ′v connective) ξ φ ξ′ φ′. φ ≼ Φ ⟶ propo-rew-step r φ φ′
      ⟶ wf-conn c (ξ @ φ # ξ′) ⟶ test-symb (conn c (ξ @ φ # ξ′)) ⟶ test-symb φ′
      ⟶ test-symb (conn c (ξ @ φ′ # ξ′)) and
      φ ≼ Φ and
    full: full (propo-rew-step r) φ ψ and
    init: all-subformula-st test-symb φ
  shows all-subformula-st test-symb ψ
  using assms unfolding full-def
proof –
  have rel: (propo-rew-step r)** φ ψ
    using full unfolding full-def by auto
  thus all-subformula-st test-symb ψ
    using init
    proof (induct rule: rtranclp-induct)
      case base
      then show all-subformula-st test-symb φ by blast
    next
      case (step b c) note star = this(1) and IH = this(3) and one = this(2) and all = this(4)
      then have all-subformula-st test-symb b by metis
      then show all-subformula-st test-symb c using propo-rew-step-inv-stay′ H H′ rel one by auto
    qed
qed

lemma full-propo-rew-step-inv-stay′:
  fixes r:: ′v propo ⇒ ′v propo ⇒ bool and test-symb:: ′v propo ⇒ bool and x :: ′v
  and φ ψ :: ′v propo
  assumes
    H: ∀ φ ψ. propo-rew-step r φ ψ ⟶ all-subformula-st test-symb φ
      ⟶ all-subformula-st test-symb ψ and
    H′: ∀ (c:: ′v connective) ξ φ ξ′ φ′. propo-rew-step r φ φ′ ⟶ wf-conn c (ξ @ φ # ξ′)
      ⟶ test-symb (conn c (ξ @ φ # ξ′)) ⟶ test-symb φ′ ⟶ test-symb (conn c (ξ @ φ′ # ξ′)) and
    full: full (propo-rew-step r) φ ψ and
    init: all-subformula-st test-symb φ
  shows all-subformula-st test-symb ψ
  using full-propo-rew-step-inv-stay-with-inc[of r test-symb φ] assms subformula-refl by metis

lemma full-propo-rew-step-inv-stay:
  fixes r:: ′v propo ⇒ ′v propo ⇒ bool and test-symb:: ′v propo ⇒ bool and x :: ′v
  and φ ψ :: ′v propo
  assumes
    H: ∀ φ ψ. r φ ψ ⟶ all-subformula-st test-symb φ ⟶ all-subformula-st test-symb ψ and
    H′: ∀ (c:: ′v connective) ξ φ ξ′ φ′. wf-conn c (ξ @ φ # ξ′) ⟶ test-symb (conn c (ξ @ φ # ξ′))
      ⟶ test-symb φ′ ⟶ test-symb (conn c (ξ @ φ′ # ξ′)) and
    full: full (propo-rew-step r) φ ψ and
    init: all-subformula-st test-symb φ
  shows all-subformula-st test-symb ψ
  unfolding full-def
proof –
  have rel: (propo-rew-step r)^** φ ψ
    using full unfolding full-def by auto
  thus all-subformula-st test-symb ψ
    using init
    proof (induct rule: rtranclp-induct)
      case base
      thus all-subformula-st test-symb φ by blast
    next
      case (step b c)
      note star = this(1) and IH = this(3) and one = this(2) and all = this(4)
      hence all-subformula-st test-symb b by metis
      thus all-subformula-st test-symb c
        using propo-rew-step-inv-stay subformula-refl H H' rel one by auto
    qed
qed
```

```
lemma full-propo-rew-step-inv-stay-conn:
  fixes r:: 'v propo ⇒ 'v propo ⇒ bool and test-symb:: 'v propo ⇒ bool and x :: 'v
  and φ ψ :: 'v propo
  assumes
    H: ∀ φ ψ. r φ ψ ⟶ all-subformula-st test-symb φ ⟶ all-subformula-st test-symb ψ and
    H': ∀ (c:: 'v connective) l l'. wf-conn c l ⟶ wf-conn c l'
      ⟶ (test-symb (conn c l) ⟷ test-symb (conn c l')) and
    full: full (propo-rew-step r) φ ψ and
    init: all-subformula-st test-symb φ
  shows all-subformula-st test-symb ψ
proof −
  have ⋀(c:: 'v connective) ξ φ ξ' φ'. wf-conn c (ξ @ φ # ξ')
    ⟹ test-symb (conn c (ξ @ φ # ξ')) ⟹ test-symb φ' ⟹ test-symb (conn c (ξ @ φ' # ξ'))
    using H'  by (metis wf-conn-no-arity-change-helper wf-conn-no-arity-change)
  thus all-subformula-st test-symb ψ
    using H  full init full-propo-rew-step-inv-stay by blast
qed
```

```
end
theory Prop-Normalisation
imports Main Prop-Logic Prop-Abstract-Transformation ../lib/Multiset-More
begin
```

Given the previous definition about abstract rewriting and theorem about them, we now have the detailed rule making the transformation into CNF/DNF.

8 Rewrite Rules

The idea of Christoph Weidenbach's book is to remove gradually the operators: first equivalencies, then implication, after that the unused true/false and finally the reorganizing the or/and. We will prove each transformation seperately.

8.1 Elimination of the equivalences

The first transformation consists in removing every equivalence symbol.

```
inductive elim-equiv :: 'v propo ⇒ 'v propo ⇒ bool where
elim-equiv[simp]: elim-equiv (FEq φ ψ) (FAnd (FImp φ ψ)  (FImp ψ φ))
```

```
lemma elim-equiv-transformation-consistent:
A ⊨ FEq φ ψ ⟷ A ⊨ FAnd (FImp φ ψ) (FImp ψ φ)
  by auto
```

```
lemma elim-equiv-explicit: elim-equiv φ ψ ⟹ ∀ A. A ⊨ φ ⟷ A ⊨ ψ
  by (induct rule: elim-equiv.induct, auto)

lemma elim-equiv-consistent: preserves-un-sat elim-equiv
  unfolding preserves-un-sat-def by (simp add: elim-equiv-explicit)

lemma elimEquv-lifted-consistant:
  preserves-un-sat (full (propo-rew-step elim-equiv))
  by (simp add: elim-equiv-consistent)
```

This function ensures that there is no equivalencies left in the formula tested by *no-equiv-symb*.

```
fun no-equiv-symb :: 'v propo ⇒ bool where
no-equiv-symb (FEq - -) = False |
no-equiv-symb - = True
```

Given the definition of *no-equiv-symb*, it does not depend on the formula, but only on the connective used.

```
lemma no-equiv-symb-conn-characterization[simp]:
  fixes c :: 'v connective and l :: 'v propo list
  assumes wf: wf-conn c l
  shows no-equiv-symb (conn c l) ⟷ c ≠ CEq
    by (metis connective.distinct(13,25,35,43) wf no-equiv-symb.elims(3) no-equiv-symb.simps(1)
      wf-conn.cases wf-conn-list(6))

definition no-equiv where no-equiv = all-subformula-st no-equiv-symb

lemma no-equiv-eq[simp]:
  fixes φ ψ :: 'v propo
  shows
    ¬no-equiv (FEq φ ψ)
    no-equiv FT
    no-equiv FF
  using no-equiv-symb.simps(1) all-subformula-st-test-symb-true-phi unfolding no-equiv-def by auto
```

The following lemma helps to reconstruct *no-equiv* expressions: this representation is easier to use than the set definition.

```
lemma all-subformula-st-decomp-explicit-no-equiv[iff]:
fixes φ ψ :: 'v propo
shows
  no-equiv (FNot φ) ⟷ no-equiv φ
  no-equiv (FAnd φ ψ) ⟷ (no-equiv φ ∧ no-equiv ψ)
  no-equiv (FOr φ ψ) ⟷ (no-equiv φ ∧ no-equiv ψ)
  no-equiv (FImp φ ψ) ⟷ (no-equiv φ ∧ no-equiv ψ)
  by (auto simp: no-equiv-def)
```

A theorem to show the link between the rewrite relation *elim-equiv* and the function *no-equiv-symb*. This theorem is one of the assumption we need to characterize the transformation.

```
lemma no-equiv-elim-equiv-step:
  fixes φ :: 'v propo
  assumes no-equiv: ¬ no-equiv φ
  shows ∃ψ ψ'. ψ ≼ φ ∧ elim-equiv ψ ψ'
proof -
  have test-symb-false-nullary:
    ∀x::'v. no-equiv-symb FF ∧ no-equiv-symb FT ∧ no-equiv-symb (FVar x)
```

unfolding *no-equiv-def* **by** *auto*
moreover {
fix *c*:: *'v connective* **and** *l* :: *'v propo list* **and** *ψ* :: *'v propo*
assume *a1*: *elim-equiv* (*conn c l*) *ψ*
have ⋀*p pa*. ¬ *elim-equiv* (*p*::*'v propo*) *pa* ∨ ¬ *no-equiv-symb p*
using *elim-equiv.cases no-equiv-symb.simps*(*1*) **by** *blast*
then have *elim-equiv* (*conn c l*) *ψ* ⟹ ¬*no-equiv-symb* (*conn c l*) **using** *a1* **by** *metis*
}
moreover have *H'*: ∀ *ψ*. ¬*elim-equiv FT ψ* ∀ *ψ*. ¬*elim-equiv FF ψ* ∀ *ψ x*. ¬*elim-equiv* (*FVar x*) *ψ*
using *elim-equiv.cases* **by** *auto*
moreover have ⋀*φ*. ¬ *no-equiv-symb φ* ⟹ ∃ *ψ*. *elim-equiv φ ψ*
by (*case-tac φ*, *auto simp*: *elim-equiv.simps*)
then have ⋀*φ'*. *φ'* ⪯ *φ* ⟹ ¬*no-equiv-symb φ'* ⟹ ∃ *ψ*. *elim-equiv φ' ψ* **by** *force*
ultimately show *?thesis*
using *no-test-symb-step-exists no-equiv test-symb-false-nullary* **unfolding** *no-equiv-def* **by** *blast*
qed

Given all the previous theorem and the characterization, once we have rewritten everything, there is no equivalence symbol any more.

lemma *no-equiv-full-propo-rew-step-elim-equiv*:
full (*propo-rew-step elim-equiv*) *φ ψ* ⟹ *no-equiv ψ*
using *full-propo-rew-step-subformula no-equiv-elim-equiv-step* **by** *blast*

8.2 Eliminate Implication

After that, we can eliminate the implication symbols.

inductive *elim-imp* :: *'v propo* ⇒ *'v propo* ⇒ *bool* **where**
[*simp*]: *elim-imp* (*FImp φ ψ*) (*FOr* (*FNot φ*) *ψ*)

lemma *elim-imp-transformation-consistent*:
A ⊨ *FImp φ ψ* ⟷ *A* ⊨ *FOr* (*FNot φ*) *ψ*
by *auto*

lemma *elim-imp-explicit*: *elim-imp φ ψ* ⟹ ∀ *A*. *A* ⊨ *φ* ⟷ *A* ⊨ *ψ*
by (*induct φ ψ rule*: *elim-imp.induct*, *auto*)

lemma *elim-imp-consistent*: *preserves-un-sat elim-imp*
unfolding *preserves-un-sat-def* **by** (*simp add*: *elim-imp-explicit*)

lemma *elim-imp-lifted-consistant*:
preserves-un-sat (*full* (*propo-rew-step elim-imp*))
by (*simp add*: *elim-imp-consistent*)

fun *no-imp-symb* **where**
no-imp-symb (*FImp - -*) = *False* |
no-imp-symb - = *True*

lemma *no-imp-symb-conn-characterization*:
wf-conn c l ⟹ *no-imp-symb* (*conn c l*) ⟷ *c* ≠ *CImp*
by (*induction rule*: *wf-conn-induct*) *auto*

definition *no-imp* **where** *no-imp* ≡ *all-subformula-st no-imp-symb*
declare *no-imp-def*[*simp*]

lemma *no-imp-Imp*[*simp*]:

```
  ¬no-imp (FImp φ ψ)
  no-imp FT
  no-imp FF
  unfolding no-imp-def by auto

lemma all-subformula-st-decomp-explicit-imp[simp]:
  fixes φ ψ :: 'v propo
  shows
    no-imp (FNot φ) ⟷ no-imp φ
    no-imp (FAnd φ ψ) ⟷ (no-imp φ ∧ no-imp ψ)
    no-imp (FOr φ ψ) ⟷ (no-imp φ ∧ no-imp ψ)
  by auto
```

Invariant of the *elim-imp* transformation

```
lemma elim-imp-no-equiv:
  elim-imp φ ψ ⟹ no-equiv φ ⟹  no-equiv ψ
  by (induct φ ψ rule: elim-imp.induct, auto)

lemma elim-imp-inv:
  fixes φ ψ :: 'v propo
  assumes full (propo-rew-step elim-imp) φ ψ and no-equiv φ
  shows no-equiv ψ
  using full-propo-rew-step-inv-stay-conn[of elim-imp no-equiv-symb φ ψ] assms elim-imp-no-equiv
    no-equiv-symb-conn-characterization unfolding no-equiv-def by metis

lemma no-no-imp-elim-imp-step-exists:
  fixes φ :: 'v propo
  assumes no-equiv: ¬ no-imp φ
  shows ∃ψ ψ'. ψ ≼ φ ∧ elim-imp ψ ψ'
proof –
  have test-symb-false-nullary: ∀x. no-imp-symb FF ∧ no-imp-symb FT ∧ no-imp-symb (FVar (x:: 'v))
    by auto
  moreover {
     fix c:: 'v connective and l :: 'v propo list and ψ :: 'v propo
     have H: elim-imp (conn c l) ψ ⟹ ¬no-imp-symb (conn c l)
       by (auto elim: elim-imp.cases)
   }
  moreover
    have  H': ∀ψ. ¬elim-imp FT ψ ∀ψ. ¬elim-imp FF ψ ∀ψ x. ¬elim-imp (FVar x) ψ
      by (auto elim: elim-imp.cases)+
  moreover
    have ⋀φ. ¬ no-imp-symb φ ⟹ ∃ψ. elim-imp φ ψ
      by (case-tac φ) (force simp: elim-imp.simps)+
    then have (⋀φ'. φ' ≼ φ ⟹ ¬no-imp-symb φ' ⟹  ∃ ψ. elim-imp φ' ψ) by force
  ultimately show ?thesis
    using no-test-symb-step-exists no-equiv test-symb-false-nullary unfolding no-imp-def by blast
qed

lemma no-imp-full-propo-rew-step-elim-imp: full (propo-rew-step elim-imp) φ ψ ⟹ no-imp ψ
  using full-propo-rew-step-subformula no-no-imp-elim-imp-step-exists by blast
```

8.3 Eliminate all the True and False in the formula

Contrary to the book, we have to give the transformation and the "commutative" transformation. The latter is implicit in the book.

```
inductive elimTB where
ElimTB1: elimTB (FAnd φ FT) φ |
ElimTB1': elimTB (FAnd FT φ) φ |

ElimTB2: elimTB (FAnd φ FF) FF |
ElimTB2': elimTB (FAnd FF φ) FF |

ElimTB3: elimTB (FOr φ FT) FT |
ElimTB3': elimTB (FOr FT φ) FT |

ElimTB4: elimTB (FOr φ FF) φ |
ElimTB4': elimTB (FOr FF φ) φ |

ElimTB5: elimTB (FNot FT) FF |
ElimTB6: elimTB (FNot FF) FT


lemma elimTB-consistent: preserves-un-sat elimTB
proof −
  {
    fix φ ψ:: 'b propo
    have elimTB φ ψ ⟹ ∀ A. A ⊨ φ ⟷ A ⊨ ψ by (induction rule: elimTB.inducts) auto
  }
  then show ?thesis using preserves-un-sat-def by auto
qed

inductive no-T-F-symb :: 'v propo ⇒ bool where
no-T-F-symb-comp: c ≠ CF ⟹ c ≠ CT ⟹ wf-conn c l ⟹ (∀ φ ∈ set l. φ ≠ FT ∧ φ ≠ FF)
  ⟹ no-T-F-symb (conn c l)


lemma wf-conn-no-T-F-symb-iff[simp]:
  wf-conn c ψs ⟹
    no-T-F-symb (conn c ψs) ⟷ (c ≠ CF ∧ c ≠ CT ∧ (∀ ψ∈set ψs. ψ ≠ FF ∧ ψ ≠ FT))
  unfolding no-T-F-symb.simps apply (cases c)
          using wf-conn-list(1) apply fastforce
         using wf-conn-list(2) apply fastforce
        using wf-conn-list(3) apply fastforce
       apply (metis (no-types, hide-lams) conn-inj connective.distinct(5,17))
      using conn-inj apply blast+
  done

lemma wf-conn-no-T-F-symb-iff-explicit[simp]:
  no-T-F-symb (FAnd φ ψ) ⟷ (∀ χ ∈ set [φ, ψ]. χ ≠ FF ∧ χ ≠ FT)
  no-T-F-symb (FOr φ ψ) ⟷ (∀ χ ∈ set [φ, ψ]. χ ≠ FF ∧ χ ≠ FT)
  no-T-F-symb (FEq φ ψ) ⟷ (∀ χ ∈ set [φ, ψ]. χ ≠ FF ∧ χ ≠ FT)
  no-T-F-symb (FImp φ ψ) ⟷ (∀ χ ∈ set [φ, ψ]. χ ≠ FF ∧ χ ≠ FT)
     apply (metis conn.simps(36) conn.simps(37) conn.simps(5) propo.distinct(19)
       wf-conn-helper-facts(5) wf-conn-no-T-F-symb-iff)
    apply (metis conn.simps(36) conn.simps(37) conn.simps(6) propo.distinct(22)
      wf-conn-helper-facts(6) wf-conn-no-T-F-symb-iff)
   using wf-conn-no-T-F-symb-iff apply fastforce
  by (metis conn.simps(36) conn.simps(37) conn.simps(7) propo.distinct(23) wf-conn-helper-facts(7)
    wf-conn-no-T-F-symb-iff)
```

```
lemma no-T-F-symb-false[simp]:
  fixes c :: 'v connective
  shows
    ¬no-T-F-symb (FT :: 'v propo)
    ¬no-T-F-symb (FF :: 'v propo)
    by (metis (no-types) conn.simps(1,2) wf-conn-no-T-F-symb-iff wf-conn-nullary)+

lemma no-T-F-symb-bool[simp]:
  fixes x :: 'v
  shows no-T-F-symb (FVar x)
  using no-T-F-symb-comp wf-conn-nullary by (metis connective.distinct(3, 15) conn.simps(3)
    empty-iff list.set(1))


lemma no-T-F-symb-fnot-imp:
  ¬no-T-F-symb (FNot φ) ⟹ φ = FT ∨ φ = FF
proof (rule ccontr)
  assume n: ¬ no-T-F-symb (FNot φ)
  assume ¬ (φ = FT ∨ φ = FF)
  then have ∀φ' ∈ set [φ]. φ'≠FT ∧ φ'≠FF by auto
  moreover have wf-conn CNot [φ] by simp
  ultimately have no-T-F-symb (FNot φ)
    using no-T-F-symb.intros by (metis conn.simps(4) connective.distinct(5,17))
  then show False using n by blast
qed

lemma no-T-F-symb-fnot[simp]:
  no-T-F-symb (FNot φ) ⟷ ¬(φ = FT ∨ φ = FF)
  using no-T-F-symb.simps no-T-F-symb-fnot-imp by (metis conn-inj-not(2) list.set-intros(1))
```

Actually it is not possible to remover every FT and FF: if the formula is equal to true or false, we can not remove it.

```
inductive no-T-F-symb-except-toplevel where
no-T-F-symb-except-toplevel-true[simp]: no-T-F-symb-except-toplevel FT |
no-T-F-symb-except-toplevel-false[simp]: no-T-F-symb-except-toplevel FF |
noTrue-no-T-F-symb-except-toplevel[simp]: no-T-F-symb φ ⟹ no-T-F-symb-except-toplevel φ

lemma no-T-F-symb-except-toplevel-bool:
  fixes x :: 'v
  shows no-T-F-symb-except-toplevel (FVar x)
  by simp

lemma no-T-F-symb-except-toplevel-not-decom:
  φ ≠ FT ⟹ φ ≠ FF ⟹ no-T-F-symb-except-toplevel (FNot φ)
  by simp

lemma no-T-F-symb-except-toplevel-bin-decom:
  fixes φ ψ :: 'v propo
  assumes φ ≠ FT and φ ≠ FF and ψ ≠ FT and ψ ≠ FF
  and c: c∈ binary-connectives
  shows no-T-F-symb-except-toplevel (conn c [φ, ψ])
  by (metis (no-types, lifting) assms c conn.simps(4) list.discI noTrue-no-T-F-symb-except-toplevel
    wf-conn-no-T-F-symb-iff no-T-F-symb-fnot set-ConsD wf-conn-binary wf-conn-helper-facts(1)
    wf-conn-list-decomp(1,2))
```

```
lemma no-T-F-symb-except-toplevel-if-is-a-true-false:
  fixes l :: 'v propo list and c :: 'v connective
  assumes corr: wf-conn c l
  and FT ∈ set l ∨ FF ∈ set l
  shows ¬no-T-F-symb-except-toplevel (conn c l)
  by (metis assms empty-iff no-T-F-symb-except-toplevel.simps wf-conn-no-T-F-symb-iff set-empty
    wf-conn-list(1,2))
```

```
lemma no-T-F-symb-except-top-level-false-example[simp]:
  fixes φ ψ :: 'v propo
  assumes φ = FT ∨ ψ = FT ∨ φ = FF ∨ ψ = FF
  shows
    ¬ no-T-F-symb-except-toplevel (FAnd φ ψ)
    ¬ no-T-F-symb-except-toplevel (FOr φ ψ)
    ¬ no-T-F-symb-except-toplevel (FImp φ ψ)
    ¬ no-T-F-symb-except-toplevel (FEq φ ψ)
  using assms no-T-F-symb-except-toplevel-if-is-a-true-false unfolding binary-connectives-def
    by (metis (no-types) conn.simps(5−8) insert-iff list.simps(14−15) wf-conn-helper-facts(5−8))+
```

```
lemma no-T-F-symb-except-top-level-false-not[simp]:
  fixes φ ψ :: 'v propo
  assumes φ = FT ∨ φ = FF
  shows
    ¬ no-T-F-symb-except-toplevel (FNot φ)
  by (simp add: assms no-T-F-symb-except-toplevel.simps)
```

This is the local extension of *no-T-F-symb-except-toplevel*.

```
definition no-T-F-except-top-level where
no-T-F-except-top-level ≡ all-subformula-st no-T-F-symb-except-toplevel
```

This is another property we will use. While this version might seem to be the one we want to prove, it is not since *FT* can not be reduced.

```
definition no-T-F where
no-T-F ≡ all-subformula-st no-T-F-symb
```

```
lemma no-T-F-except-top-level-false:
  fixes l :: 'v propo list and c :: 'v connective
  assumes wf-conn c l
  and FT ∈ set l ∨ FF ∈ set l
  shows ¬no-T-F-except-top-level (conn c l)
  by (simp add: all-subformula-st-decomp assms no-T-F-except-top-level-def
    no-T-F-symb-except-toplevel-if-is-a-true-false)
```

```
lemma no-T-F-except-top-level-false-example[simp]:
  fixes φ ψ :: 'v propo
  assumes φ = FT ∨ ψ = FT ∨ φ = FF ∨ ψ = FF
  shows
    ¬no-T-F-except-top-level (FAnd φ ψ)
    ¬no-T-F-except-top-level (FOr φ ψ)
    ¬no-T-F-except-top-level (FEq φ ψ)
    ¬no-T-F-except-top-level (FImp φ ψ)
  by (metis all-subformula-st-test-symb-true-phi assms no-T-F-except-top-level-def
    no-T-F-symb-except-top-level-false-example)+
```

lemma *no-T-F-symb-except-toplevel-no-T-F-symb*:
  no-T-F-symb-except-toplevel $\varphi \Longrightarrow \varphi \neq FF \Longrightarrow \varphi \neq FT \Longrightarrow$ *no-T-F-symb* φ
  by (*induct rule*: *no-T-F-symb-except-toplevel.induct*, *auto*)

The two following lemmas give the precise link between the two definitions.

lemma *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb*:
  no-T-F-except-top-level $\varphi \Longrightarrow \varphi \neq FF \Longrightarrow \varphi \neq FT \Longrightarrow$ *no-T-F* φ
  unfolding *no-T-F-except-top-level-def no-T-F-def* **apply** (*induct* φ)
  using *no-T-F-symb-fnot* **by** *fastforce+*

lemma *no-T-F-no-T-F-except-top-level*:
  no-T-F $\varphi \Longrightarrow$ *no-T-F-except-top-level* φ
  unfolding *no-T-F-except-top-level-def no-T-F-def*
  unfolding *all-subformula-st-def* **by** *auto*

lemma *no-T-F-except-top-level-simp*[*simp*]: *no-T-F-except-top-level FF no-T-F-except-top-level FT*
  unfolding *no-T-F-except-top-level-def* **by** *auto*

lemma *no-T-F-no-T-F-except-top-level'*[*simp*]:
  no-T-F-except-top-level $\varphi \longleftrightarrow (\varphi = FF \vee \varphi = FT \vee$ *no-T-F* $\varphi)$
  using *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb no-T-F-no-T-F-except-top-level*
  by *auto*

lemma *no-T-F-bin-decomp*[*simp*]:
  assumes *c*: $c \in$ *binary-connectives*
  shows *no-T-F* (*conn c* $[\varphi, \psi]) \longleftrightarrow ($*no-T-F* $\varphi \wedge$ *no-T-F* $\psi)$
proof –
  have *wf*: *wf-conn c* $[\varphi, \psi]$ **using** *c* **by** *auto*
  then have *no-T-F* (*conn c* $[\varphi, \psi]) \longleftrightarrow ($*no-T-F-symb* (*conn c* $[\varphi, \psi]) \wedge$ *no-T-F* $\varphi \wedge$ *no-T-F* $\psi)$
    by (*simp add*: *all-subformula-st-decomp no-T-F-def*)
  then show *no-T-F* (*conn c* $[\varphi, \psi]) \longleftrightarrow ($*no-T-F* $\varphi \wedge$ *no-T-F* $\psi)$
    using *c wf all-subformula-st-decomp list.discI no-T-F-def no-T-F-symb-except-toplevel-bin-decom*
      no-T-F-symb-except-toplevel-no-T-F-symb no-T-F-symb-false(*1*,*2*) *wf-conn-helper-facts*(*2*,*3*)
      wf-conn-list(*1*,*2*) **by** *metis*
qed

lemma *no-T-F-bin-decomp-expanded*[*simp*]:
  assumes *c*: $c = CAnd \vee c = COr \vee c = CEq \vee c = CImp$
  shows *no-T-F* (*conn c* $[\varphi, \psi]) \longleftrightarrow ($*no-T-F* $\varphi \wedge$ *no-T-F* $\psi)$
  using *no-T-F-bin-decomp assms* **unfolding** *binary-connectives-def* **by** *blast*

lemma *no-T-F-comp-expanded-explicit*[*simp*]:
  fixes $\varphi\ \psi$ :: *'v propo*
  shows
    no-T-F (*FAnd* $\varphi\ \psi) \longleftrightarrow ($*no-T-F* $\varphi \wedge$ *no-T-F* $\psi)$
    no-T-F (*FOr* $\varphi\ \psi) \longleftrightarrow ($*no-T-F* $\varphi \wedge$ *no-T-F* $\psi)$
    no-T-F (*FEq* $\varphi\ \psi) \longleftrightarrow ($*no-T-F* $\varphi \wedge$ *no-T-F* $\psi)$
    no-T-F (*FImp* $\varphi\ \psi) \longleftrightarrow ($*no-T-F* $\varphi \wedge$ *no-T-F* $\psi)$
  using *assms conn.simps*(*5*−*8*) *no-T-F-bin-decomp-expanded* **by** (*metis* (*no-types*))+

lemma *no-T-F-comp-not*[*simp*]:
  fixes $\varphi\ \psi$ :: *'v propo*
  shows *no-T-F* (*FNot* $\varphi) \longleftrightarrow$ *no-T-F* φ

```
  by (metis all-subformula-st-decomp-explicit(3) all-subformula-st-test-symb-true-phi no-T-F-def
    no-T-F-symb-false(1,2) no-T-F-symb-fnot-imp)

lemma no-T-F-decomp:
  fixes φ ψ :: 'v propo
  assumes φ: no-T-F (FAnd φ ψ) ∨ no-T-F (FOr φ ψ) ∨ no-T-F (FEq φ ψ) ∨ no-T-F (FImp φ ψ)
  shows no-T-F ψ and no-T-F φ
  using assms by auto

lemma no-T-F-decomp-not:
  fixes φ :: 'v propo
  assumes φ: no-T-F (FNot φ)
  shows  no-T-F φ
  using assms by auto

lemma no-T-F-symb-except-toplevel-step-exists:
  fixes φ ψ :: 'v propo
  assumes no-equiv φ and no-imp φ
  shows ψ ≼ φ ⟹ ¬ no-T-F-symb-except-toplevel ψ ⟹ ∃ψ'. elimTB ψ ψ'
proof (induct ψ rule: propo-induct-arity)
  case (nullary φ' x)
  then have False using no-T-F-symb-except-toplevel-true no-T-F-symb-except-toplevel-false by auto
  then show ?case by blast
next
  case (unary ψ)
  then have ψ = FF ∨ ψ = FT using  no-T-F-symb-except-toplevel-not-decom by blast
  then show ?case using ElimTB5 ElimTB6 by blast
next
  case (binary φ' ψ1 ψ2)
  note IH1 = this(1) and IH2 = this(2) and φ' = this(3) and Fφ = this(4) and n = this(5)
  {
    assume φ' = FImp ψ1 ψ2 ∨ φ' = FEq ψ1 ψ2
    then have False using n Fφ subformula-all-subformula-st assms
      by (metis (no-types) no-equiv-eq(1) no-equiv-def no-imp-Imp(1) no-imp-def)
    then have ?case by blast
  }
  moreover {
    assume φ': φ' = FAnd ψ1 ψ2 ∨ φ' = FOr ψ1 ψ2
    then have ψ1 = FT ∨ ψ2 = FT ∨ ψ1 = FF ∨ ψ2 = FF
      using no-T-F-symb-except-toplevel-bin-decom conn.simps(5,6) n unfolding binary-connectives-def
       by fastforce+
    then have ?case using elimTB.intros φ' by blast
  }
  ultimately show ?case using φ' by blast
qed

lemma no-T-F-except-top-level-rew:
  fixes φ :: 'v propo
  assumes noTB: ¬ no-T-F-except-top-level φ and no-equiv: no-equiv φ and no-imp: no-imp φ
  shows ∃ψ ψ'. ψ ≼ φ ∧ elimTB ψ ψ'
proof -
  have test-symb-false-nullary: ∀x. no-T-F-symb-except-toplevel (FF:: 'v propo)
    ∧ no-T-F-symb-except-toplevel FT ∧ no-T-F-symb-except-toplevel (FVar (x:: 'v)) by auto
  moreover {
     fix c:: 'v connective and  l :: 'v propo list and ψ :: 'v propo
    have H: elimTB (conn c l) ψ ⟹ ¬no-T-F-symb-except-toplevel (conn c l)
      by (cases (conn c l) rule: elimTB.cases, auto)
  }
  moreover {
    fix x :: 'v
    have  H': no-T-F-except-top-level FT  no-T-F-except-top-level FF
      no-T-F-except-top-level (FVar x)
      by (auto simp: no-T-F-except-top-level-def test-symb-false-nullary)
  }
  moreover {
    fix ψ
    have ψ ≼ φ ⟹ ¬ no-T-F-symb-except-toplevel ψ ⟹ ∃ψ'. elimTB ψ ψ'
      using no-T-F-symb-except-toplevel-step-exists no-equiv no-imp by auto
  }
  ultimately show ?thesis
    using no-test-symb-step-exists noTB unfolding no-T-F-except-top-level-def by blast
qed

lemma elimTB-inv:
  fixes φ ψ :: 'v propo
  assumes full (propo-rew-step elimTB) φ ψ
  and no-equiv φ and no-imp φ
  shows no-equiv ψ and no-imp ψ
proof –
  {
    fix φ ψ :: 'v propo
    have H: elimTB φ ψ ⟹ no-equiv φ ⟹  no-equiv ψ
      by (induct φ ψ rule: elimTB.induct, auto)
  }
  then show no-equiv ψ
    using full-propo-rew-step-inv-stay-conn[of elimTB no-equiv-symb φ ψ]
      no-equiv-symb-conn-characterization assms unfolding no-equiv-def by metis
next
  {
    fix φ ψ :: 'v propo
    have H: elimTB φ ψ ⟹ no-imp φ ⟹ no-imp ψ
      by (induct φ ψ rule: elimTB.induct, auto)
  }
  then show no-imp ψ
    using full-propo-rew-step-inv-stay-conn[of elimTB no-imp-symb φ ψ] assms
      no-imp-symb-conn-characterization unfolding no-imp-def by metis
qed

lemma elimTB-full-propo-rew-step:
  fixes φ ψ :: 'v propo
  assumes no-equiv φ and no-imp φ and full (propo-rew-step elimTB) φ ψ
  shows no-T-F-except-top-level ψ
  using full-propo-rew-step-subformula no-T-F-except-top-level-rew assms elimTB-inv by fastforce
```

8.4 PushNeg

Push the negation inside the formula, until the litteral.

```
inductive pushNeg where
PushNeg1[simp]: pushNeg (FNot (FAnd φ ψ)) (FOr (FNot φ) (FNot ψ)) |
PushNeg2[simp]: pushNeg (FNot (FOr φ ψ)) (FAnd (FNot φ) (FNot ψ)) |
PushNeg3[simp]: pushNeg (FNot (FNot φ)) φ
```

```
lemma pushNeg-transformation-consistent:
A ⊨ FNot (FAnd φ ψ) ⟷ A ⊨ (FOr (FNot φ) (FNot ψ))
A ⊨ FNot (FOr φ ψ)  ⟷ A ⊨ (FAnd (FNot φ) (FNot ψ))
A ⊨ FNot (FNot φ)   ⟷ A ⊨ φ
  by auto
```

```
lemma pushNeg-explicit: pushNeg φ ψ ⟹ ∀ A. A ⊨ φ ⟷ A ⊨ ψ
  by (induct φ ψ rule: pushNeg.induct, auto)

lemma pushNeg-consistent: preserves-un-sat pushNeg
  unfolding preserves-un-sat-def by (simp add: pushNeg-explicit)
```

```
lemma pushNeg-lifted-consistant:
preserves-un-sat (full (propo-rew-step pushNeg))
  by (simp add: pushNeg-consistent)

fun simple where
simple FT = True |
simple FF = True |
simple (FVar -) = True |
simple - = False

lemma simple-decomp:
  simple φ ⟷ (φ = FT ∨ φ = FF ∨ (∃ x. φ = FVar x))
  by (cases φ) auto

lemma subformula-conn-decomp-simple:
  fixes φ ψ :: 'v propo
  assumes s: simple ψ
  shows φ ≼ FNot ψ ⟷ (φ = FNot ψ ∨ φ = ψ)
proof −
  have φ ≼ conn CNot [ψ] ⟷ (φ = conn CNot [ψ] ∨ (∃ ψ∈ set [ψ]. φ ≼ ψ))
    using subformula-conn-decomp  wf-conn-helper-facts(1) by metis
  then show φ ≼ FNot ψ ⟷ (φ = FNot ψ ∨ φ = ψ) using s by (auto simp: simple-decomp)
qed

lemma subformula-conn-decomp-explicit[simp]:
  fixes φ :: 'v propo and x :: 'v
  shows
    φ ≼ FNot FT ⟷ (φ = FNot FT ∨ φ = FT)
    φ ≼ FNot FF ⟷ (φ = FNot FF ∨ φ = FF)
    φ ≼ FNot (FVar x) ⟷ (φ = FNot (FVar x) ∨ φ = FVar x)
  by (auto simp: subformula-conn-decomp-simple)
```

```
fun simple-not-symb where
simple-not-symb (FNot φ) = (simple φ) |
simple-not-symb - = True

definition simple-not where
```

```
simple-not = all-subformula-st simple-not-symb
declare simple-not-def[simp]

lemma simple-not-Not[simp]:
  ¬ simple-not (FNot (FAnd φ ψ))
  ¬ simple-not (FNot (FOr φ ψ))
  by auto

lemma simple-not-step-exists:
  fixes φ ψ :: 'v propo
  assumes no-equiv φ and no-imp φ
  shows ψ ≼ φ ⟹ ¬ simple-not-symb ψ ⟹ ∃ψ'. pushNeg ψ ψ'
  apply (induct ψ, auto)
  apply (rename-tac ψ, case-tac ψ, auto intro: pushNeg.intros)
  by (metis assms(1,2) no-imp-Imp(1) no-equiv-eq(1) no-imp-def no-equiv-def
    subformula-in-subformula-not subformula-all-subformula-st)+

lemma simple-not-rew:
  fixes φ :: 'v propo
  assumes noTB: ¬ simple-not φ and no-equiv: no-equiv φ and no-imp: no-imp φ
  shows ∃ψ ψ'. ψ ≼ φ ∧ pushNeg ψ ψ'
proof −
  have ∀x. simple-not-symb (FF:: 'v propo) ∧ simple-not-symb FT ∧ simple-not-symb (FVar (x:: 'v))
    by auto
  moreover {
     fix c:: 'v connective and  l :: 'v propo list and ψ :: 'v propo
     have H: pushNeg (conn c l) ψ ⟹ ¬simple-not-symb (conn c l)
       by (cases (conn c l) rule: pushNeg.cases) auto
  }
  moreover {
     fix x :: 'v
     have  H': simple-not FT simple-not FF simple-not (FVar x)
       by simp-all
  }
  moreover {
     fix ψ :: 'v propo
     have  ψ ≼ φ ⟹ ¬ simple-not-symb ψ ⟹ ∃ψ'. pushNeg ψ ψ'
       using simple-not-step-exists no-equiv no-imp by blast
  }
  ultimately show ?thesis using no-test-symb-step-exists noTB unfolding simple-not-def by blast
qed

lemma no-T-F-except-top-level-pushNeg1:
  no-T-F-except-top-level (FNot (FAnd φ ψ)) ⟹ no-T-F-except-top-level (FOr (FNot φ) (FNot ψ))
 using no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb  no-T-F-comp-not no-T-F-decomp(1)
   no-T-F-decomp(2) no-T-F-no-T-F-except-top-level by (metis no-T-F-comp-expanded-explicit(2)
     propo.distinct(5,17))

lemma no-T-F-except-top-level-pushNeg2:
  no-T-F-except-top-level (FNot (FOr φ ψ)) ⟹ no-T-F-except-top-level (FAnd (FNot φ) (FNot ψ))
  by auto

lemma no-T-F-symb-pushNeg:
  no-T-F-symb (FOr (FNot φ') (FNot ψ'))
  no-T-F-symb (FAnd (FNot φ') (FNot ψ'))
  no-T-F-symb (FNot (FNot φ'))
  by auto

lemma propo-rew-step-pushNeg-no-T-F-symb:
  propo-rew-step pushNeg φ ψ ⟹ no-T-F-except-top-level φ ⟹ no-T-F-symb φ ⟹ no-T-F-symb ψ
  apply (induct rule: propo-rew-step.induct)
  apply (cases rule: pushNeg.cases)
  apply simp-all
  apply (metis no-T-F-symb-pushNeg(1))
  apply (metis no-T-F-symb-pushNeg(2))
  apply (simp, metis all-subformula-st-test-symb-true-phi no-T-F-def)
proof –
  fix φ φ':: 'a propo and c:: 'a connective and ξ ξ':: 'a propo list
  assume rel: propo-rew-step pushNeg φ φ'
  and IH: no-T-F φ ⟹ no-T-F-symb φ ⟹ no-T-F-symb φ'
  and wf: wf-conn c (ξ @ φ # ξ')
  and n: conn c (ξ @ φ # ξ') = FF ∨ conn c (ξ @ φ # ξ') = FT ∨ no-T-F (conn c (ξ @ φ # ξ'))
  and x: c ≠ CF ∧ c ≠ CT ∧ φ ≠ FF ∧ φ ≠ FT ∧ (∀ψ ∈ set ξ ∪ set ξ'. ψ ≠ FF ∧ ψ ≠ FT)
  then have c ≠ CF ∧ c ≠ CF ∧ wf-conn c (ξ @ φ' # ξ')
    using wf-conn-no-arity-change-helper wf-conn-no-arity-change by metis
  moreover have n': no-T-F (conn c (ξ @ φ # ξ')) using n by (simp add: wf wf-conn-list(1,2))
  moreover
  {
    have no-T-F φ
      by (metis Un-iff all-subformula-st-decomp list.set-intros(1) n' wf no-T-F-def set-append)
    moreover then have no-T-F-symb φ
      by (simp add: all-subformula-st-test-symb-true-phi no-T-F-def)
    ultimately have φ' ≠ FF ∧ φ' ≠ FT
      using IH no-T-F-symb-false(1) no-T-F-symb-false(2) by blast
    then have ∀ψ∈ set (ξ @ φ' # ξ'). ψ ≠ FF ∧ ψ ≠ FT using x by auto
  }
  ultimately show no-T-F-symb (conn c (ξ @ φ' # ξ')) by (simp add: x)
qed

lemma propo-rew-step-pushNeg-no-T-F:
  propo-rew-step pushNeg φ ψ ⟹ no-T-F φ ⟹ no-T-F ψ
proof (induct rule: propo-rew-step.induct)
  case global-rel
  then show ?case
    by (metis (no-types, lifting) no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb
      no-T-F-def no-T-F-except-top-level-pushNeg1 no-T-F-except-top-level-pushNeg2
      no-T-F-no-T-F-except-top-level all-subformula-st-decomp-explicit(3) pushNeg.simps
      simple.simps(1,2,5,6))
next
  case (propo-rew-one-step-lift φ φ' c ξ ξ')
  note rel = this(1) and IH = this(2) and wf = this(3) and no-T-F = this(4)
  moreover have wf': wf-conn c (ξ @ φ' # ξ')
    using wf-conn-no-arity-change wf-conn-no-arity-change-helper wf by metis
  ultimately show no-T-F (conn c (ξ @ φ' # ξ'))
    using all-subformula-st-test-symb-true-phi
    by (fastforce simp: no-T-F-def all-subformula-st-decomp wf wf')
qed


lemma pushNeg-inv:
  fixes φ ψ :: 'v propo
  assumes full (propo-rew-step pushNeg) φ ψ
  and no-equiv φ and no-imp φ and no-T-F-except-top-level φ
  shows no-equiv ψ and no-imp ψ and no-T-F-except-top-level ψ
proof –
  {
    fix φ ψ :: 'v propo
    assume rel: propo-rew-step pushNeg φ ψ
    and  no: no-T-F-except-top-level φ
    then have  no-T-F-except-top-level ψ
      proof –
        {
          assume φ = FT ∨ φ = FF
          from rel this have False
            apply (induct rule: propo-rew-step.induct)
              using pushNeg.cases apply blast
            using wf-conn-list(1) wf-conn-list(2) by auto
          then have no-T-F-except-top-level ψ by blast
        }
        moreover {
          assume φ ≠ FT ∧ φ ≠ FF
          then have no-T-F φ
            by (metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
          then have no-T-F ψ
            using propo-rew-step-pushNeg-no-T-F rel by auto
          then have no-T-F-except-top-level ψ by (simp add: no-T-F-no-T-F-except-top-level)
        }
        ultimately show no-T-F-except-top-level ψ by metis
      qed
  }
  moreover {
    fix c :: 'v connective and ξ ξ' :: 'v propo list and ζ ζ' :: 'v propo
    assume rel: propo-rew-step pushNeg ζ ζ'
    and incl: ζ ≼ φ
    and corr: wf-conn c (ξ @ ζ # ξ')
    and no-T-F: no-T-F-symb-except-toplevel (conn c (ξ @ ζ # ξ'))
    and n: no-T-F-symb-except-toplevel ζ'
    have no-T-F-symb-except-toplevel (conn c (ξ @ ζ' # ξ'))
    proof
      have p: no-T-F-symb (conn c (ξ @ ζ # ξ'))
        using corr wf-conn-list(1) wf-conn-list(2) no-T-F-symb-except-toplevel-no-T-F-symb no-T-F
        by blast
      have l: ∀φ∈set (ξ @ ζ # ξ'). φ ≠ FT ∧ φ ≠ FF
        using corr wf-conn-no-T-F-symb-iff p by blast
      from rel incl have ζ'≠FT ∧ζ'≠FF
        apply (induction ζ ζ' rule: propo-rew-step.induct)
        apply (cases rule: pushNeg.cases, auto)
        by (metis assms(4) no-T-F-symb-except-top-level-false-not no-T-F-except-top-level-def
          all-subformula-st-test-symb-true-phi subformula-in-subformula-not
          subformula-all-subformula-st append-is-Nil-conv list.distinct(1)
          wf-conn-no-arity-change-helper wf-conn-list(1,2) wf-conn-no-arity-change)+
      then have ∀φ ∈ set (ξ @ ζ' # ξ'). φ ≠ FT ∧ φ ≠ FF using l by auto
      moreover have c ≠ CT ∧ c ≠ CF using corr by auto
      ultimately show no-T-F-symb (conn c (ξ @ ζ' # ξ'))
        by (metis corr no-T-F-symb-comp wf-conn-no-arity-change wf-conn-no-arity-change-helper)
```

qed
}
ultimately show *no-T-F-except-top-level ψ*
using *full-propo-rew-step-inv-stay-with-inc*[*of pushNeg no-T-F-symb-except-toplevel φ*] *assms*
subformula-refl **unfolding** *no-T-F-except-top-level-def full-unfold* **by** *metis*
next
{
fix *φ ψ* :: *'v propo*
have *H*: *pushNeg φ ψ* ⟹ *no-equiv φ* ⟹ *no-equiv ψ*
by (*induct φ ψ rule*: *pushNeg.induct*, *auto*)
}
then show *no-equiv ψ*
using *full-propo-rew-step-inv-stay-conn*[*of pushNeg no-equiv-symb φ ψ*]
no-equiv-symb-conn-characterization assms **unfolding** *no-equiv-def full-unfold* **by** *metis*
next
{
fix *φ ψ* :: *'v propo*
have *H*: *pushNeg φ ψ* ⟹ *no-imp φ* ⟹ *no-imp ψ*
by (*induct φ ψ rule*: *pushNeg.induct*, *auto*)
}
then show *no-imp ψ*
using *full-propo-rew-step-inv-stay-conn*[*of pushNeg no-imp-symb φ ψ*] *assms*
no-imp-symb-conn-characterization **unfolding** *no-imp-def full-unfold* **by** *metis*
qed

lemma *pushNeg-full-propo-rew-step*:
fixes *φ ψ* :: *'v propo*
assumes
no-equiv φ **and**
no-imp φ **and**
full (*propo-rew-step pushNeg*) *φ ψ* **and**
no-T-F-except-top-level φ
shows *simple-not ψ*
using *assms full-propo-rew-step-subformula pushNeg-inv*(*1*,*2*) *simple-not-rew* **by** *blast*

8.5 Push inside

inductive *push-conn-inside* :: *'v connective* ⇒ *'v connective* ⇒ *'v propo* ⇒ *'v propo* ⇒ *bool*
for *c c'*:: *'v connective* **where**
push-conn-inside-l[*simp*]: *c* = *CAnd* ∨ *c* = *COr* ⟹ *c'* = *CAnd* ∨ *c'* = *COr*
⟹ *push-conn-inside c c'* (*conn c* [*conn c'* [*φ1*, *φ2*], *ψ*])
(*conn c'* [*conn c* [*φ1*, *ψ*], *conn c* [*φ2*, *ψ*]]) |
push-conn-inside-r[*simp*]: *c* = *CAnd* ∨ *c* = *COr* ⟹ *c'* = *CAnd* ∨ *c'* = *COr*
⟹ *push-conn-inside c c'* (*conn c* [*ψ*, *conn c'* [*φ1*, *φ2*]])
(*conn c'* [*conn c* [*ψ*, *φ1*], *conn c* [*ψ*, *φ2*]])

lemma *push-conn-inside-explicit*: *push-conn-inside c c' φ ψ* ⟹ ∀ *A*. *A*⊨*φ* ⟷ *A*⊨*ψ*
by (*induct φ ψ rule*: *push-conn-inside.induct*, *auto*)

lemma *push-conn-inside-consistent*: *preserves-un-sat* (*push-conn-inside c c'*)
unfolding *preserves-un-sat-def* **by** (*simp add*: *push-conn-inside-explicit*)

lemma *propo-rew-step-push-conn-inside*[*simp*]:
¬*propo-rew-step* (*push-conn-inside c c'*) *FT ψ* ¬*propo-rew-step* (*push-conn-inside c c'*) *FF ψ*

```
 proof –
  {
   {
     fix φ ψ
     have push-conn-inside c c′ φ ψ ⟹ φ = FT ∨ φ = FF ⟹ False
       by (induct rule: push-conn-inside.induct, auto)
   } note H = this
   fix φ
   have propo-rew-step (push-conn-inside c c′) φ ψ ⟹ φ = FT ∨ φ = FF ⟹ False
     apply (induct rule: propo-rew-step.induct, auto simp: wf-conn-list(1) wf-conn-list(2))
     using H by blast+
  }
  then show
    ¬propo-rew-step (push-conn-inside c c′) FT ψ
    ¬propo-rew-step (push-conn-inside c c′) FF ψ by blast+
qed


inductive not-c-in-c′-symb:: ′v connective ⇒ ′v connective ⇒ ′v propo  ⇒ bool for c c′ where
not-c-in-c′-symb-l[simp]: wf-conn c [conn c′ [φ, φ′], ψ] ⟹ wf-conn c′ [φ, φ′]
  ⟹ not-c-in-c′-symb c c′ (conn c [conn c′ [φ, φ′], ψ]) |
not-c-in-c′-symb-r[simp]: wf-conn c [ψ, conn c′ [φ, φ′]] ⟹ wf-conn c′ [φ, φ′]
  ⟹ not-c-in-c′-symb c c′ (conn c [ψ, conn c′ [φ, φ′]])

abbreviation c-in-c′-symb c c′ φ ≡ ¬not-c-in-c′-symb c c′ φ


lemma c-in-c′-symb-simp:
  not-c-in-c′-symb c c′ ξ ⟹ ξ = FF ∨ ξ = FT ∨ ξ = FVar x ∨ ξ = FNot FF ∨ ξ = FNot FT
    ∨ ξ = FNot (FVar x)⟹ False
  apply (induct rule: not-c-in-c′-symb.induct, auto simp: wf-conn.simps wf-conn-list(1−3))
  using conn-inj-not(2) wf-conn-binary unfolding binary-connectives-def by fastforce+

lemma  c-in-c′-symb-simp′[simp]:
  ¬not-c-in-c′-symb c c′ FF
  ¬not-c-in-c′-symb c c′ FT
  ¬not-c-in-c′-symb c c′ (FVar x)
  ¬not-c-in-c′-symb c c′ (FNot FF)
  ¬not-c-in-c′-symb c c′ (FNot FT)
  ¬not-c-in-c′-symb c c′ (FNot (FVar x))
  using c-in-c′-symb-simp by metis+

definition c-in-c′-only where
c-in-c′-only c c′ ≡ all-subformula-st (c-in-c′-symb c c′)

lemma c-in-c′-only-simp[simp]:
  c-in-c′-only c c′ FF
  c-in-c′-only c c′ FT
  c-in-c′-only c c′ (FVar x)
  c-in-c′-only c c′ (FNot FF)
  c-in-c′-only c c′ (FNot FT)
  c-in-c′-only c c′ (FNot (FVar x))
  unfolding c-in-c′-only-def by auto
```

```
lemma not-c-in-c′-symb-commute:
  not-c-in-c′-symb c c′ ξ ⟹ wf-conn c [φ, ψ] ⟹ ξ = conn c [φ, ψ]
    ⟹ not-c-in-c′-symb c c′ (conn c [ψ, φ])
proof (induct rule: not-c-in-c′-symb.induct)
  case (not-c-in-c′-symb-r φ′ φ′′ ψ′) note H = this
  then have ψ: ψ = conn c′ [φ′′, ψ′] using conn-inj by auto
  have wf-conn c [conn c′ [φ′′, ψ′], φ]
    using H(1) wf-conn-no-arity-change length-Cons by metis
  then show not-c-in-c′-symb c c′ (conn c [ψ, φ])
    unfolding ψ using not-c-in-c′-symb.intros(1) H by auto
next
  case (not-c-in-c′-symb-l φ′ φ′′ ψ′) note H = this
  then have φ = conn c′ [φ′, φ′′] using conn-inj by auto
  moreover have wf-conn c [ψ′, conn c′ [φ′, φ′′]]
    using H(1) wf-conn-no-arity-change length-Cons by metis
  ultimately show  not-c-in-c′-symb c c′ (conn c [ψ, φ])
    using not-c-in-c′-symb.intros(2) conn-inj not-c-in-c′-symb-l.hyps
      not-c-in-c′-symb-l.prems(1,2) by blast
qed

lemma not-c-in-c′-symb-commute′:
  wf-conn c [φ, ψ] ⟹ c-in-c′-symb c c′ (conn c [φ, ψ])  ⟷ c-in-c′-symb c c′ (conn c [ψ, φ])
  using not-c-in-c′-symb-commute wf-conn-no-arity-change by (metis length-Cons)

lemma not-c-in-c′-comm:
  assumes wf: wf-conn c [φ, ψ]
  shows c-in-c′-only c c′ (conn c [φ, ψ]) ⟷ c-in-c′-only c c′ (conn c [ψ, φ]) (is ?A ⟷ ?B)
proof −
  have ?A ⟷ (c-in-c′-symb c c′ (conn c [φ, ψ])
              ∧ (∀ ξ ∈ set [φ, ψ]. all-subformula-st (c-in-c′-symb c c′) ξ))
    using all-subformula-st-decomp wf unfolding c-in-c′-only-def  by fastforce
  also have … ⟷ (c-in-c′-symb c c′ (conn c [ψ, φ])
                  ∧ (∀ ξ ∈ set [ψ, φ]. all-subformula-st (c-in-c′-symb c c′) ξ))
    using not-c-in-c′-symb-commute′ wf by auto
  also
    have wf-conn c [ψ, φ] using wf-conn-no-arity-change wf by (metis length-Cons)
    then have (c-in-c′-symb c c′ (conn c [ψ, φ])
             ∧ (∀ ξ ∈ set [ψ, φ]. all-subformula-st (c-in-c′-symb c c′) ξ))
           ⟷ ?B
      using all-subformula-st-decomp  unfolding c-in-c′-only-def by fastforce
  finally show ?thesis .
qed

lemma not-c-in-c′-simp[simp]:
  fixes φ1 φ2 ψ :: ′v propo and x :: ′v
  shows
  c-in-c′-symb c c′ FT
  c-in-c′-symb c c′ FF
  c-in-c′-symb c c′ (FVar x)
  wf-conn c [conn c′ [φ1, φ2], ψ] ⟹ wf-conn c′ [φ1, φ2]
    ⟹ ¬ c-in-c′-only c c′ (conn c [conn c′ [φ1, φ2], ψ])
  apply (simp-all add: c-in-c′-only-def)
  using all-subformula-st-test-symb-true-phi not-c-in-c′-symb-l by blast

lemma c-in-c′-symb-not[simp]:
  fixes c c' :: 'v connective and ψ :: 'v propo
  shows c-in-c'-symb c c' (FNot ψ)
proof –
  {
    fix ξ :: 'v propo
    have not-c-in-c'-symb c c' (FNot ψ) ⟹ False
      apply (induct FNot ψ rule: not-c-in-c'-symb.induct)
      using conn-inj-not(2) by blast+
  }
 then show ?thesis by auto
qed

lemma c-in-c'-symb-step-exists:
  fixes φ :: 'v propo
  assumes c: c = CAnd ∨ c = COr and c': c' = CAnd ∨ c' = COr
  shows ψ ≼ φ ⟹ ¬ c-in-c'-symb c c' ψ ⟹ ∃ψ'. push-conn-inside c c' ψ ψ'
  apply (induct ψ rule: propo-induct-arity)
  apply auto[2]
proof –
  fix ψ1 ψ2 φ':: 'v propo
  assume IHψ1: ψ1 ≼ φ ⟹ ¬ c-in-c'-symb c c' ψ1 ⟹ Ex (push-conn-inside c c' ψ1)
  and IHψ2: ψ1 ≼ φ ⟹ ¬ c-in-c'-symb c c' ψ1 ⟹ Ex (push-conn-inside c c' ψ1)
  and φ': φ' = FAnd ψ1 ψ2 ∨ φ' = FOr ψ1 ψ2 ∨ φ' = FImp ψ1 ψ2 ∨ φ' = FEq ψ1 ψ2
  and inφ: φ' ≼ φ and  n0: ¬c-in-c'-symb c c' φ'
  then have n: not-c-in-c'-symb c c' φ' by auto
  {
    assume φ': φ' = conn c [ψ1, ψ2]
    obtain a b where ψ1 = conn c' [a, b] ∨ ψ2 = conn c' [a, b]
      using n φ' apply (induct rule: not-c-in-c'-symb.induct)
      using c by force+
    then have Ex (push-conn-inside c c' φ')
      unfolding φ' apply auto
      using push-conn-inside.intros(1) c c' apply blast
      using push-conn-inside.intros(2) c c' by blast
  }
  moreover {
     assume φ': φ' ≠ conn c [ψ1, ψ2]
     have ∀φ c ca. ∃φ1 ψ1 ψ2 ψ1' ψ2' φ2'. conn (c::'v connective) [φ1, conn ca [ψ1, ψ2]] = φ
             ∨ conn c [conn ca [ψ1', ψ2'], φ2'] = φ ∨ c-in-c'-symb c ca φ
       by (metis not-c-in-c'-symb.cases)
     then have Ex (push-conn-inside c c' φ')
       by (metis (no-types) c c' n push-conn-inside-l push-conn-inside-r)
  }
  ultimately show Ex (push-conn-inside c c' φ') by blast
qed


lemma c-in-c'-symb-rew:
  fixes φ :: 'v propo
  assumes noTB: ¬c-in-c'-only c c' φ
  and c: c = CAnd ∨ c = COr and c': c' = CAnd ∨ c' = COr
  shows ∃ψ ψ'. ψ ≼ φ ∧ push-conn-inside c c' ψ ψ'
proof –
  have test-symb-false-nullary:
    ∀x. c-in-c'-symb c c' (FF:: 'v propo) ∧ c-in-c'-symb c c' FT
```

```
      ∧ c-in-c′-symb c c′ (FVar (x:: ′v))
    by auto
  moreover {
    fix x :: ′v
    have  H′: c-in-c′-symb c c′ FT c-in-c′-symb c c′ FF c-in-c′-symb c c′ (FVar x)
      by simp+
  }
  moreover {
    fix ψ :: ′v propo
    have ψ ⪯ φ ⟹ ¬ c-in-c′-symb c c′ ψ ⟹ ∃ψ′. push-conn-inside c c′ ψ ψ′
      by (auto simp: assms(2) c′ c-in-c′-symb-step-exists)
  }
  ultimately show ?thesis using noTB no-test-symb-step-exists[of c-in-c′-symb c c′]
    unfolding c-in-c′-only-def by metis
qed

lemma push-conn-insidec-in-c′-symb-no-T-F:
  fixes φ ψ :: ′v propo
  shows propo-rew-step (push-conn-inside c c′) φ ψ ⟹ no-T-F φ ⟹ no-T-F ψ
proof (induct rule: propo-rew-step.induct)
  case (global-rel φ ψ)
  then show no-T-F ψ
    by (cases rule: push-conn-inside.cases, auto)
next
  case (propo-rew-one-step-lift φ φ′ c ξ ξ′)
  note rel = this(1) and IH = this(2) and wf = this(3) and no-T-F = this(4)
  have no-T-F φ
    using wf no-T-F  no-T-F-def subformula-into-subformula subformula-all-subformula-st
    subformula-refl by (metis (no-types) in-set-conv-decomp)
  then have φ′: no-T-F φ′ using IH by blast

  have ∀ζ ∈ set (ξ @ φ # ξ′). no-T-F ζ by (metis wf no-T-F no-T-F-def all-subformula-st-decomp)
  then have n: ∀ζ ∈ set (ξ @ φ′ # ξ′). no-T-F ζ using φ′ by auto
  then have n′: ∀ζ ∈ set (ξ @ φ′ # ξ′). ζ ≠  FF ∧ ζ ≠ FT
    using φ′ by (metis no-T-F-symb-false(1) no-T-F-symb-false(2) no-T-F-def
      all-subformula-st-test-symb-true-phi)

  have wf′: wf-conn c (ξ @ φ′ # ξ′)
    using wf wf-conn-no-arity-change by (metis wf-conn-no-arity-change-helper)
  {
    fix x :: ′v
    assume c = CT ∨ c = CF ∨ c = CVar x
    then have False using wf by auto
    then have no-T-F (conn c (ξ @ φ′ # ξ′)) by blast
  }
  moreover {
    assume c: c = CNot
    then have ξ = [] ξ′ = [] using wf by auto
    then have no-T-F (conn c (ξ @ φ′ # ξ′))
      using c by (metis φ′ conn.simps(4) no-T-F-symb-false(1,2) no-T-F-symb-fnot no-T-F-def
        all-subformula-st-decomp-explicit(3) all-subformula-st-test-symb-true-phi self-append-conv2)
  }
  moreover {
    assume c: c ∈ binary-connectives
    then have no-T-F-symb (conn c (ξ @ φ′ # ξ′)) using wf′ n′ no-T-F-symb.simps by fastforce
    then have no-T-F (conn c (ξ @ φ' # ξ'))
      by (metis all-subformula-st-decomp-imp wf' n no-T-F-def)
  }
  ultimately show no-T-F (conn c (ξ @ φ' # ξ')) using connective-cases-arity by auto
qed


lemma simple-propo-rew-step-push-conn-inside-inv:
propo-rew-step (push-conn-inside c c') φ ψ ⟹ simple φ ⟹ simple ψ
  apply (induct rule: propo-rew-step.induct)
  apply (rename-tac φ, case-tac φ, auto simp: push-conn-inside.simps)[]
  by (metis append-is-Nil-conv list.distinct(1) simple.elims(2) wf-conn-list(1−3))


lemma simple-propo-rew-step-inv-push-conn-inside-simple-not:
  fixes c c' :: 'v connective and φ ψ :: 'v propo
  shows propo-rew-step (push-conn-inside c c') φ ψ ⟹ simple-not φ ⟹ simple-not ψ
proof (induct rule: propo-rew-step.induct)
  case (global-rel φ ψ)
  then show ?case by (cases φ, auto simp: push-conn-inside.simps)
next
  case (propo-rew-one-step-lift φ φ' ca ξ ξ') note rew = this(1) and IH = this(2) and wf = this(3)
   and simple = this(4)
  show ?case
    proof (cases ca rule: connective-cases-arity)
      case nullary
      then show ?thesis using propo-rew-one-step-lift by auto
    next
      case binary note ca = this
      obtain a b where ab: ξ @ φ' # ξ' = [a, b]
        using wf ca list-length2-decomp wf-conn-bin-list-length
        by (metis (no-types) wf-conn-no-arity-change-helper)
      have ∀ζ ∈ set (ξ @ φ # ξ'). simple-not ζ
        by (metis wf all-subformula-st-decomp simple simple-not-def)
      then have ∀ζ ∈ set (ξ @ φ' # ξ'). simple-not ζ using IH by simp
      moreover have simple-not-symb (conn ca (ξ @ φ' # ξ')) using ca
      by (metis ab conn.simps(5−8) helper-fact simple-not-symb.simps(5) simple-not-symb.simps(6)
          simple-not-symb.simps(7) simple-not-symb.simps(8))
      ultimately show ?thesis
        by (simp add: ab all-subformula-st-decomp ca)
    next
      case unary
      then show ?thesis
         using rew simple-propo-rew-step-push-conn-inside-inv[OF rew] IH local.wf simple by auto
    qed
qed

lemma propo-rew-step-push-conn-inside-simple-not:
  fixes φ φ' :: 'v propo and ξ ξ' :: 'v propo list and c :: 'v connective
  assumes
    propo-rew-step (push-conn-inside c c') φ φ' and
    wf-conn c (ξ @ φ # ξ') and
    simple-not-symb (conn c (ξ @ φ # ξ')) and
    simple-not-symb φ'
  shows simple-not-symb (conn c (ξ @ φ' # ξ'))
  using assms
proof (induction rule: propo-rew-step.induct)
print-cases
  case (global-rel)
  then show ?case
    by (metis conn.simps(12,17) list.discI push-conn-inside.cases simple-not-symb.elims(3)
      wf-conn-helper-facts(5) wf-conn-list(2) wf-conn-list(8) wf-conn-no-arity-change
      wf-conn-no-arity-change-helper)
next
  case (propo-rew-one-step-lift φ φ' c' χs χs') note tel = this(1) and wf = this(2) and
    IH = this(3) and wf' = this(4) and simple' = this(5) and simple = this(6)
  then show ?case
    proof (cases c' rule: connective-cases-arity)
      case nullary
      then show ?thesis using wf simple simple' by auto
    next
      case binary note c = this(1)
      have corr': wf-conn c (ξ @ conn c' (χs @ φ' # χs') # ξ')
        using wf wf-conn-no-arity-change
        by (metis wf' wf-conn-no-arity-change-helper)
      then show ?thesis
        using c propo-rew-one-step-lift wf
        by (metis conn.simps(17) connective.distinct(37) propo-rew-step-subformula-imp
          push-conn-inside.cases simple-not-symb.elims(3) wf-conn.simps wf-conn-list(2,8))
    next
      case unary
      then have empty: χs = [] χs' = [] using wf by auto
      then show ?thesis using simple unary simple' wf wf'
        by (metis connective.distinct(37) connective.distinct(39) propo-rew-step-subformula-imp
          push-conn-inside.cases simple-not-symb.elims(3) tel wf-conn-list(8)
          wf-conn-no-arity-change wf-conn-no-arity-change-helper)
    qed
qed

lemma push-conn-inside-not-true-false:
  push-conn-inside c c' φ ψ ⟹ ψ ≠ FT ∧ ψ ≠ FF
  by (induct rule: push-conn-inside.induct, auto)

lemma push-conn-inside-inv:
  fixes φ ψ :: 'v propo
  assumes full (propo-rew-step (push-conn-inside c c')) φ ψ
  and no-equiv φ and no-imp φ and no-T-F-except-top-level φ and simple-not φ
  shows no-equiv ψ and no-imp ψ and no-T-F-except-top-level ψ and simple-not ψ
proof -
  {
     {
        fix φ ψ :: 'v propo
        have H: push-conn-inside c c' φ ψ ⟹ all-subformula-st simple-not-symb φ
          ⟹ all-subformula-st simple-not-symb ψ
          by (induct φ ψ rule: push-conn-inside.induct, auto)
     } note H = this

    fix φ ψ :: 'v propo
    have H: propo-rew-step (push-conn-inside c c') φ ψ ⟹ all-subformula-st simple-not-symb φ
      ⟹ all-subformula-st simple-not-symb ψ
```

```
apply (induct φ ψ rule: propo-rew-step.induct)
using H apply simp
proof (rename-tac φ φ' ca ψs ψs', case-tac ca rule: connective-cases-arity)
  fix φ φ' :: 'v propo and c:: 'v connective and ξ ξ':: 'v propo list
  and x:: 'v
  assume wf-conn c (ξ @ φ # ξ')
  and  c = CT ∨ c = CF ∨ c = CVar x
  then have ξ @ φ # ξ' = [] by auto
  then have False by auto
  then show all-subformula-st simple-not-symb (conn c (ξ @ φ' # ξ')) by blast
next
  fix φ φ' :: 'v propo and ca:: 'v connective and ξ ξ':: 'v propo list
  and x :: 'v
  assume rel: propo-rew-step (push-conn-inside c c') φ φ'
  and φ-φ': all-subformula-st simple-not-symb φ ⟹ all-subformula-st simple-not-symb φ'
  and corr: wf-conn ca (ξ @ φ # ξ')
  and n: all-subformula-st simple-not-symb (conn ca (ξ @ φ # ξ'))
  and c: ca = CNot

  have empty: ξ = [] ξ' = [] using c corr by auto
  then have simple-not:all-subformula-st simple-not-symb (FNot φ) using corr c n by auto
  then have simple φ
    using all-subformula-st-test-symb-true-phi simple-not-symb.simps(1) by blast
  then have simple φ'
    using rel simple-propo-rew-step-push-conn-inside-inv by blast
  then show all-subformula-st simple-not-symb (conn ca (ξ @ φ' # ξ')) using c empty
    by (metis simple-not φ-φ' append-Nil conn.simps(4) all-subformula-st-decomp-explicit(3)
      simple-not-symb.simps(1))
next
  fix φ φ' :: 'v propo and ca :: 'v connective and ξ ξ' :: 'v propo list
  and x :: 'v
  assume rel: propo-rew-step (push-conn-inside c c') φ φ'
  and nφ: all-subformula-st simple-not-symb φ ⟹  all-subformula-st simple-not-symb φ'
  and corr: wf-conn ca (ξ @ φ # ξ')
  and n: all-subformula-st simple-not-symb (conn ca (ξ @ φ # ξ'))
  and c: ca ∈ binary-connectives

  have all-subformula-st simple-not-symb φ
    using n c corr all-subformula-st-decomp by fastforce
  then have φ':  all-subformula-st simple-not-symb φ' using nφ by blast
  obtain a b where ab: [a, b] = (ξ @ φ # ξ')
    using corr c list-length2-decomp wf-conn-bin-list-length by metis
  then have ξ @ φ' # ξ' = [a, φ'] ∨ (ξ @ φ' # ξ') = [φ', b]
    using ab by (metis (no-types, hide-lams) append-Cons append-Nil append-Nil2
      append-is-Nil-conv butlast.simps(2) butlast-append list.sel(3) tl-append2)
  moreover
  {
    fix χ :: 'v propo
    have wf': wf-conn ca [a, b]
      using ab corr by presburger
    have all-subformula-st simple-not-symb (conn ca [a, b])
      using ab n by presburger
    then have all-subformula-st simple-not-symb χ ∨ χ ∉ set (ξ @ φ' # ξ')
      using wf' by (metis (no-types) φ' all-subformula-st-decomp calculation insert-iff
        list.set(2))
        }
        then have ∀φ. φ ∈ set (ξ @ φ' # ξ') ⟶ all-subformula-st simple-not-symb φ
            by (metis (no-types))

        moreover have simple-not-symb (conn ca (ξ @ φ' # ξ'))
          using ab conn-inj-not(1) corr wf-conn-list-decomp(4) wf-conn-no-arity-change
            not-Cons-self2 self-append-conv2 simple-not-symb.elims(3) by (metis (no-types) c
            calculation(1) wf-conn-binary)
        moreover have wf-conn ca (ξ @ φ' # ξ') using c calculation(1) by auto
        ultimately show all-subformula-st simple-not-symb (conn ca (ξ @ φ' # ξ'))
          by (metis all-subformula-st-decomp-imp)
      qed
  }
  moreover {
     fix ca :: 'v connective and ξ ξ' :: 'v propo list and φ φ' :: 'v propo
     have propo-rew-step (push-conn-inside c c') φ φ' ⟹ wf-conn ca (ξ @ φ # ξ')
       ⟹ simple-not-symb (conn ca (ξ @ φ # ξ')) ⟹ simple-not-symb φ'
       ⟹ simple-not-symb (conn ca (ξ @ φ' # ξ'))
       by (metis append-self-conv2 conn.simps(4) conn-inj-not(1) simple-not-symb.elims(3)
         simple-not-symb.simps(1) simple-propo-rew-step-push-conn-inside-inv
         wf-conn-no-arity-change-helper wf-conn-list-decomp(4) wf-conn-no-arity-change)
  }
  ultimately show simple-not ψ
    using full-propo-rew-step-inv-stay'[of push-conn-inside c c' simple-not-symb] assms
    unfolding no-T-F-except-top-level-def simple-not-def full-unfold by metis
next
  {
    fix φ ψ :: 'v propo
    have H: propo-rew-step (push-conn-inside c c') φ ψ ⟹ no-T-F-except-top-level φ
      ⟹  no-T-F-except-top-level ψ
      proof −
        assume rel: propo-rew-step (push-conn-inside c c') φ ψ
        and no-T-F-except-top-level φ
        then have no-T-F φ ∨ φ = FF ∨ φ = FT
          by (metis no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
        moreover {
          assume φ = FF ∨ φ = FT
          then have False using rel propo-rew-step-push-conn-inside by blast
          then have no-T-F-except-top-level ψ by blast
        }
        moreover {
          assume no-T-F φ ∧ φ ≠ FF ∧ φ ≠ FT
          then have no-T-F ψ using rel push-conn-insidec-in-c'-symb-no-T-F by blast
          then have no-T-F-except-top-level ψ using no-T-F-no-T-F-except-top-level by blast
        }
        ultimately show no-T-F-except-top-level ψ by blast
      qed
  }
  moreover {
     fix ca :: 'v connective and ξ ξ' :: 'v propo list and φ φ' :: 'v propo
     assume rel: propo-rew-step (push-conn-inside c c') φ φ'
     assume corr: wf-conn ca (ξ @ φ # ξ')
     then have c: ca ≠ CT ∧ ca ≠ CF by auto
     assume no-T-F: no-T-F-symb-except-toplevel (conn ca (ξ @ φ # ξ'))
     have no-T-F-symb-except-toplevel (conn ca (ξ @ φ' # ξ'))
     proof
       have c: ca ≠ CT ∧ ca ≠ CF using corr by auto
       have ζ: ∀ζ∈ set (ξ @ φ # ξ'). ζ≠FT ∧ ζ ≠ FF
         using corr no-T-F no-T-F-symb-except-toplevel-if-is-a-true-false by blast
       then have φ ≠ FT ∧ φ ≠ FF by auto
       from rel this have  φ' ≠ FT ∧ φ' ≠ FF
         apply (induct rule: propo-rew-step.induct)
         by (metis append-is-Nil-conv conn.simps(2) conn-inj list.distinct(1)
           wf-conn-helper-facts(3) wf-conn-list(1) wf-conn-no-arity-change
           wf-conn-no-arity-change-helper push-conn-inside-not-true-false)+
       then have ∀ζ ∈ set (ξ @ φ' # ξ'). ζ ≠ FT ∧ ζ ≠ FF using ζ by auto
       moreover have wf-conn ca (ξ @ φ' # ξ')
         using corr wf-conn-no-arity-change by (metis wf-conn-no-arity-change-helper)
       ultimately show no-T-F-symb (conn ca (ξ @ φ' # ξ')) using no-T-F-symb.intros c by metis
     qed
   }
   ultimately show no-T-F-except-top-level ψ
     using full-propo-rew-step-inv-stay'[of push-conn-inside c c' no-T-F-symb-except-toplevel]
     assms unfolding no-T-F-except-top-level-def full-unfold by metis

next
  {
    fix φ ψ :: 'v propo
    have H: push-conn-inside c c' φ ψ ⟹ no-equiv φ ⟹ no-equiv ψ
      by (induct φ ψ rule: push-conn-inside.induct, auto)
  }
  then show no-equiv ψ
    using full-propo-rew-step-inv-stay-conn[of push-conn-inside c c' no-equiv-symb] assms
    no-equiv-symb-conn-characterization unfolding no-equiv-def by metis

next
  {
    fix φ ψ :: 'v propo
    have H: push-conn-inside c c' φ ψ ⟹ no-imp φ ⟹  no-imp ψ
      by (induct φ ψ rule: push-conn-inside.induct, auto)
  }
  then show no-imp ψ
    using full-propo-rew-step-inv-stay-conn[of push-conn-inside c c' no-imp-symb] assms
    no-imp-symb-conn-characterization unfolding no-imp-def by metis
qed


lemma push-conn-inside-full-propo-rew-step:
  fixes φ ψ :: 'v propo
  assumes
    no-equiv φ and
    no-imp φ and
    full (propo-rew-step (push-conn-inside c c')) φ ψ and
    no-T-F-except-top-level φ and
    simple-not φ and
    c = CAnd ∨ c = COr and
    c' = CAnd ∨ c' = COr
  shows c-in-c'-only c c' ψ
  using c-in-c'-symb-rew assms full-propo-rew-step-subformula by blast
```

8.5.1 Only one type of connective in the formula (+ not)

inductive *only-c-inside-symb* :: *'v connective* ⇒ *'v propo* ⇒ *bool* **for** *c*:: *'v connective* **where**
simple-only-c-inside[*simp*]: *simple* φ ⟹ *only-c-inside-symb c* φ |
simple-cnot-only-c-inside[*simp*]: *simple* φ ⟹ *only-c-inside-symb c* (*FNot* φ) |
only-c-inside-into-only-c-inside: *wf-conn c l* ⟹ *only-c-inside-symb c* (*conn c l*)

lemma *only-c-inside-symb-simp*[*simp*]:
only-c-inside-symb c FF only-c-inside-symb c FT only-c-inside-symb c (*FVar x*) **by** *auto*

definition *only-c-inside* **where** *only-c-inside c* = *all-subformula-st* (*only-c-inside-symb c*)

lemma *only-c-inside-symb-decomp*:
only-c-inside-symb c ψ ⟷ (*simple* ψ
∨ (∃ φ'. ψ = *FNot* φ' ∧ *simple* φ')
∨ (∃ *l*. ψ = *conn c l* ∧ *wf-conn c l*))
by (*auto simp*: *only-c-inside-symb.intros*(*3*)) (*induct rule*: *only-c-inside-symb.induct*, *auto*)

lemma *only-c-inside-symb-decomp-not*[*simp*]:
fixes *c* :: *'v connective*
assumes *c*: *c* ≠ *CNot*
shows *only-c-inside-symb c* (*FNot* ψ) ⟷ *simple* ψ
apply (*auto simp*: *only-c-inside-symb.intros*(*3*))
by (*induct FNot* ψ *rule*: *only-c-inside-symb.induct*, *auto simp*: *wf-conn-list*(*8*) *c*)

lemma *only-c-inside-decomp-not*[*simp*]:
assumes *c*: *c* ≠ *CNot*
shows *only-c-inside c* (*FNot* ψ) ⟷ *simple* ψ
by (*metis* (*no-types*, *hide-lams*) *all-subformula-st-def all-subformula-st-test-symb-true-phi c only-c-inside-def only-c-inside-symb-decomp-not simple-only-c-inside subformula-conn-decomp-simple*)

lemma *only-c-inside-decomp*:
only-c-inside c φ ⟷
(∀ ψ. ψ ≼ φ ⟶ (*simple* ψ ∨ (∃ φ'. ψ = *FNot* φ' ∧ *simple* φ')
∨ (∃ *l*. ψ = *conn c l* ∧ *wf-conn c l*)))
unfolding *only-c-inside-def* **by** (*auto simp*: *all-subformula-st-def only-c-inside-symb-decomp*)

lemma *only-c-inside-c-c'-false*:
fixes *c c'* :: *'v connective* **and** *l* :: *'v propo list* **and** φ :: *'v propo*
assumes *cc'*: *c* ≠ *c'* **and** *c*: *c* = *CAnd* ∨ *c* = *COr* **and** *c'*: *c'* = *CAnd* ∨ *c'* = *COr*
and *only*: *only-c-inside c* φ **and** *incl*: *conn c' l* ≼ φ **and** *wf*: *wf-conn c' l*
shows *False*
proof −
let *?*ψ = *conn c' l*
have *simple ?*ψ ∨ (∃ φ'. *?*ψ = *FNot* φ' ∧ *simple* φ') ∨ (∃ *l*. *?*ψ = *conn c l* ∧ *wf-conn c l*)
using *only-c-inside-decomp only incl* **by** *blast*
moreover have ¬ *simple ?*ψ
using *wf simple-decomp* **by** (*metis c' connective.distinct*(*19*) *connective.distinct*(*7*,*9*,*21*,*29*,*31*) *wf-conn-list*(*1*−*3*))
moreover
{
fix φ'

```
      have ?ψ ≠ FNot φ' using c' conn-inj-not(1) wf by blast
    }
  ultimately obtain l :: 'v propo list where ?ψ = conn c l ∧ wf-conn c l by metis
  then have c = c' using conn-inj wf by metis
  then show False using cc' by auto
qed

lemma only-c-inside-implies-c-in-c'-symb:
  assumes δ: c ≠ c' and c: c = CAnd ∨ c = COr and c': c' = CAnd ∨ c' = COr
  shows only-c-inside c φ ⟹ c-in-c'-symb c c' φ
  apply (rule ccontr)
  apply (cases rule: not-c-in-c'-symb.cases, auto)
  by (metis δ c c' connective.distinct(37,39) list.distinct(1) only-c-inside-c-c'-false
    subformula-in-binary-conn(1,2) wf-conn.simps)+

lemma c-in-c'-symb-decomp-level1:
  fixes l :: 'v propo list and c c' ca :: 'v connective
  shows wf-conn ca l ⟹ ca ≠ c ⟹ c-in-c'-symb c c' (conn ca l)
proof −
  have not-c-in-c'-symb c c' (conn ca l) ⟹ wf-conn ca l ⟹ ca = c
    by (induct conn ca l rule: not-c-in-c'-symb.induct, auto simp: conn-inj)
  then show wf-conn ca l ⟹ ca ≠ c ⟹ c-in-c'-symb c c' (conn ca l) by blast
qed


lemma only-c-inside-implies-c-in-c'-only:
  assumes δ: c ≠ c' and c: c = CAnd ∨ c = COr and c': c' = CAnd ∨ c' = COr
  shows only-c-inside c φ ⟹ c-in-c'-only c c' φ
  unfolding c-in-c'-only-def all-subformula-st-def
  using only-c-inside-implies-c-in-c'-symb
    by (metis all-subformula-st-def assms(1) c c' only-c-inside-def subformula-trans)


lemma c-in-c'-symb-c-implies-only-c-inside:
  assumes δ: c = CAnd ∨ c = COr c' = CAnd ∨ c' = COr c ≠ c' and wf: wf-conn c [φ, ψ]
  and inv: no-equiv (conn c l) no-imp (conn c l) simple-not (conn c l)
  shows wf-conn c l ⟹ c-in-c'-only c c' (conn c l) ⟹ (∀ψ∈ set l. only-c-inside c ψ)
using inv
proof (induct conn c l arbitrary: l rule: propo-induct-arity)
  case (nullary x)
  then show ?case by (auto simp: wf-conn-list assms)
next
  case (unary φ la)
  then have c = CNot ∧ la = [φ] by (metis (no-types) wf-conn-list(8))
  then show ?case using assms(2) assms(1) by blast
next
  case (binary φ1 φ2)
  note IHφ1 = this(1) and IHφ2 = this(2) and φ = this(3) and only = this(5) and wf = this(4)
    and no-equiv = this(6) and no-imp = this(7) and simple-not = this(8)
  then have l: l = [φ1, φ2] by (meson wf-conn-list(4−7))
  let ?φ = conn c l

  obtain c1 l1 c2 l2 where φ1: φ1 = conn c1 l1 and wfφ1: wf-conn c1 l1
    and φ2: φ2 = conn c2 l2 and wfφ2: wf-conn c2 l2 using exists-c-conn by metis
  then have c-in-onlyφ1: c-in-c'-only c c' (conn c1 l1) and c-in-c'-only c c' (conn c2 l2)
    using only l unfolding c-in-c′-only-def using assms(1) by auto
  have incφ1: φ1 ⪯ ?φ and incφ2: φ2 ⪯ ?φ
    using φ1 φ2 φ local.wf by (metis conn.simps(5−8) helper-fact subformula-in-binary-conn(1,2))+

  have c1-eq: c1 ≠ CEq and c2-eq: c2 ≠ CEq
    unfolding no-equiv-def using incφ1 incφ2 by (metis φ1 φ2 wfφ1 wfφ2 assms(1) no-equiv
      no-equiv-eq(1) no-equiv-symb.elims(3) no-equiv-symb-conn-characterization wf-conn-list(4,5)
      no-equiv-def subformula-all-subformula-st)+
  have c1-imp: c1 ≠ CImp and c2-imp: c2 ≠ CImp
    using no-imp by (metis φ1 φ2 all-subformula-st-decomp-explicit-imp(2,3) assms(1)
      conn.simps(5,6) l no-imp-Imp(1) no-imp-symb.elims(3) no-imp-symb-conn-characterization
      wfφ1 wfφ2 all-subformula-st-decomp no-imp-symb-conn-characterization)+
  have c1c: c1 ≠ c′
    proof
      assume c1c: c1 = c′
      then obtain ξ1 ξ2 where l1: l1 = [ξ1, ξ2]
        by (metis assms(2) connective.distinct(37,39) helper-fact wfφ1 wf-conn.simps
          wf-conn-list-decomp(1−3))
      have c-in-c′-only c c′ (conn c [conn c′ l1, φ2]) using c1c l only φ1 by auto
      moreover have not-c-in-c′-symb c c′ (conn c [conn c′ l1, φ2])
        using l1 φ1 c1c l local.wf not-c-in-c′-symb-l wfφ1 by blast
      ultimately show False using φ1 c1c l l1 local.wf not-c-in-c′-simp(4) wfφ1 by blast
   qed
  then have (φ1 = conn c l1 ∧ wf-conn c l1) ∨ (∃ψ1. φ1 = FNot ψ1) ∨ simple φ1
    by (metis φ1 assms(1−3) c1-eq c1-imp simple.elims(3) wfφ1 wf-conn-list(4) wf-conn-list(5−7))
  moreover {
    assume φ1 = conn c l1 ∧ wf-conn c l1
    then have only-c-inside c φ1
      by (metis IHφ1 φ1 all-subformula-st-decomp-imp incφ1 no-equiv no-equiv-def no-imp no-imp-def
        c-in-onlyφ1 only-c-inside-def only-c-inside-into-only-c-inside simple-not simple-not-def
        subformula-all-subformula-st)
  }
  moreover {
    assume ∃ψ1. φ1 = FNot ψ1
    then obtain ψ1 where φ1 = FNot ψ1 by metis
    then have only-c-inside c φ1
      by (metis all-subformula-st-def assms(1) connective.distinct(37,39) incφ1
        only-c-inside-decomp-not simple-not simple-not-def simple-not-symb.simps(1))
  }
  moreover {
    assume simple φ1
    then have only-c-inside c φ1
      by (metis all-subformula-st-decomp-explicit(3) assms(1) connective.distinct(37,39)
        only-c-inside-decomp-not only-c-inside-def)
  }
  ultimately have only-c-insideφ1: only-c-inside c φ1 by metis

  have c-in-onlyφ2: c-in-c′-only c c′ (conn c2 l2)
    using only l φ2 wfφ2 assms unfolding c-in-c′-only-def by auto
  have c2c: c2 ≠ c′
    proof
      assume c2c: c2 = c′
      then obtain ξ1 ξ2 where l2: l2 = [ξ1, ξ2]
       by (metis assms(2) wfφ2 wf-conn.simps connective.distinct(7,9,19,21,29,31,37,39))
      then have c-in-c′-symb c c′ (conn c [φ1, conn c′ l2])
```

```
      using c2c l only φ2 all-subformula-st-test-symb-true-phi unfolding c-in-c′-only-def by auto
    moreover have not-c-in-c′-symb c c′ (conn c [φ1, conn c′ l2])
      using assms(1) c2c l2 not-c-in-c′-symb-r wfφ2 wf-conn-helper-facts(5,6) by metis
    ultimately show False by auto
  qed
then have (φ2 = conn c l2 ∧ wf-conn c l2) ∨ (∃ψ2. φ2 = FNot ψ2) ∨ simple φ2
  using c2-eq by (metis φ2 assms(1−3) c2-eq c2-imp simple.elims(3) wfφ2 wf-conn-list(4−7))
moreover {
  assume φ2 = conn c l2 ∧ wf-conn c l2
  then have only-c-inside c φ2
    by (metis IHφ2 φ2 all-subformula-st-decomp incφ2 no-equiv no-equiv-def no-imp no-imp-def
      c-in-onlyφ2 only-c-inside-def only-c-inside-into-only-c-inside simple-not simple-not-def
      subformula-all-subformula-st)
}
moreover {
  assume ∃ψ2. φ2 = FNot ψ2
  then obtain ψ2 where φ2 = FNot ψ2 by metis
  then have only-c-inside c φ2
    by (metis all-subformula-st-def assms(1−3) connective.distinct(38,40) incφ2
      only-c-inside-decomp-not simple-not simple-not-def simple-not-symb.simps(1))
}
moreover {
  assume simple φ2
  then have only-c-inside c φ2
    by (metis all-subformula-st-decomp-explicit(3) assms(1) connective.distinct(37,39)
      only-c-inside-decomp-not only-c-inside-def)
}
ultimately have only-c-insideφ2: only-c-inside c φ2 by metis
show ?case using l only-c-insideφ1 only-c-insideφ2 by auto
qed
```

8.5.2 Push Conjunction

```
definition pushConj where pushConj = push-conn-inside CAnd COr
```

```
lemma pushConj-consistent: preserves-un-sat pushConj
  unfolding pushConj-def by (simp add: push-conn-inside-consistent)
```

```
definition and-in-or-symb where and-in-or-symb = c-in-c′-symb CAnd COr
```

```
definition and-in-or-only where
and-in-or-only = all-subformula-st (c-in-c′-symb CAnd COr)
```

```
lemma pushConj-inv:
  fixes φ ψ :: ′v propo
  assumes full (propo-rew-step pushConj) φ ψ
  and   no-equiv φ and no-imp φ and no-T-F-except-top-level φ and simple-not φ
  shows no-equiv ψ and no-imp ψ and no-T-F-except-top-level ψ and simple-not ψ
  using push-conn-inside-inv assms unfolding pushConj-def by metis+
```

```
lemma pushConj-full-propo-rew-step:
  fixes φ ψ :: ′v propo
  assumes
    no-equiv φ and
    no-imp φ and
    full (propo-rew-step pushConj) φ ψ and
    no-T-F-except-top-level φ and
    simple-not φ
  shows and-in-or-only ψ
  using assms push-conn-inside-full-propo-rew-step
  unfolding pushConj-def and-in-or-only-def c-in-c′-only-def by (metis (no-types))
```

8.5.3 Push Disjunction

```
definition pushDisj where pushDisj = push-conn-inside COr CAnd

lemma pushDisj-consistent: preserves-un-sat pushDisj
  unfolding pushDisj-def by (simp add: push-conn-inside-consistent)

definition or-in-and-symb where or-in-and-symb = c-in-c′-symb COr CAnd

definition or-in-and-only where
or-in-and-only = all-subformula-st (c-in-c′-symb COr CAnd)


lemma not-or-in-and-only-or-and[simp]:
  ~or-in-and-only (FOr (FAnd ψ1 ψ2) φ′)
  unfolding or-in-and-only-def
  by (metis all-subformula-st-test-symb-true-phi conn.simps(5−6) not-c-in-c′-symb-l
    wf-conn-helper-facts(5) wf-conn-helper-facts(6))

lemma pushDisj-inv:
  fixes φ ψ :: ′v propo
  assumes full (propo-rew-step pushDisj) φ ψ
  and no-equiv φ and no-imp φ and no-T-F-except-top-level φ and simple-not φ
  shows no-equiv ψ and no-imp ψ and no-T-F-except-top-level ψ and simple-not ψ
  using push-conn-inside-inv assms unfolding pushDisj-def by metis+

lemma pushDisj-full-propo-rew-step:
  fixes φ ψ :: ′v propo
  assumes
    no-equiv φ and
    no-imp φ and
    full (propo-rew-step pushDisj) φ ψ and
    no-T-F-except-top-level φ and
    simple-not φ
  shows or-in-and-only ψ
  using assms push-conn-inside-full-propo-rew-step
  unfolding pushDisj-def or-in-and-only-def c-in-c′-only-def by (metis (no-types))
```

9 The full transformations

9.1 Abstract Property characterizing that only some connective are inside the others

9.1.1 Definition

The normal is a super group of groups

```
inductive grouped-by :: ′a connective ⇒ ′a propo ⇒ bool for c where
simple-is-grouped[simp]: simple φ ⟹ grouped-by c φ |
simple-not-is-grouped[simp]: simple φ ⟹ grouped-by c (FNot φ) |
connected-is-group[simp]: grouped-by c φ ⟹  grouped-by c ψ ⟹ wf-conn c [φ, ψ]
  ⟹ grouped-by c (conn c [φ, ψ])

lemma simple-clause[simp]:
  grouped-by c FT
  grouped-by c FF
  grouped-by c (FVar x)
  grouped-by c (FNot FT)
  grouped-by c (FNot FF)
  grouped-by c (FNot (FVar x))
  by simp+

lemma only-c-inside-symb-c-eq-c':
  only-c-inside-symb c (conn c' [φ1, φ2]) ⟹ c' = CAnd ∨ c' = COr ⟹ wf-conn c' [φ1, φ2]
    ⟹ c' = c
  by (induct conn c' [φ1, φ2] rule: only-c-inside-symb.induct, auto simp: conn-inj)


lemma only-c-inside-c-eq-c':
  only-c-inside c (conn c' [φ1, φ2]) ⟹  c' = CAnd ∨ c' = COr ⟹ wf-conn c' [φ1, φ2] ⟹ c = c'
  unfolding only-c-inside-def all-subformula-st-def using only-c-inside-symb-c-eq-c' subformula-refl
  by blast

lemma only-c-inside-imp-grouped-by:
  assumes c: c ≠ CNot and c': c' = CAnd ∨ c' = COr
  shows only-c-inside c φ ⟹ grouped-by c φ (is ?O φ ⟹ ?G φ)
proof (induct φ rule: propo-induct-arity)
  case (nullary φ x)
  then show ?G φ by auto
next
  case (unary ψ)
  then show ?G (FNot ψ) by (auto simp: c)
next
  case (binary φ φ1 φ2)
  note IHφ1 = this(1) and IHφ2 = this(2) and φ = this(3) and only = this(4)
  have φ-conn: φ = conn c [φ1, φ2] and wf: wf-conn c [φ1, φ2]
    proof –
      obtain c'' l'' where φ-c'': φ = conn c'' l'' and wf: wf-conn c'' l''
        using exists-c-conn by metis
      then have l'': l'' = [φ1, φ2] using φ by (metis wf-conn-list(4–7))
      have only-c-inside-symb c (conn c'' [φ1, φ2])
        using only all-subformula-st-test-symb-true-phi
        unfolding only-c-inside-def φ-c'' l'' by metis
      then have c = c''
        by (metis φ φ-c'' conn-inj conn-inj-not(2) l'' list.distinct(1) list.inject wf
          only-c-inside-symb.cases simple.simps(5–8))
      then show φ = conn c [φ1, φ2] and wf-conn c [φ1, φ2] using φ-c'' wf l'' by auto
    qed
  have grouped-by c φ1  using wf IHφ1 IHφ2 φ-conn only φ unfolding only-c-inside-def by auto
  moreover have grouped-by c φ2
    using wf φ IHφ1 IHφ2 φ-conn only unfolding only-c-inside-def by auto
  ultimately show ?G φ  using φ-conn connected-is-group local.wf by blast
qed
```

```
lemma grouped-by-false:
  grouped-by c (conn c′ [φ, ψ]) ⟹ c ≠ c′ ⟹ wf-conn c′ [φ, ψ] ⟹ False
  apply (induct conn c′ [φ, ψ] rule: grouped-by.induct)
  apply (auto simp: simple-decomp wf-conn-list, auto simp: conn-inj)
  by (metis list.distinct(1) list.sel(3) wf-conn-list(8))+
```

Then the CNF form is a conjunction of clauses: every clause is in CNF form and two formulas in CNF form can be related by an and.

```
inductive super-grouped-by:: ′a connective ⇒ ′a connective ⇒ ′a propo ⇒ bool for c c′ where
grouped-is-super-grouped[simp]: grouped-by c φ ⟹ super-grouped-by c c′ φ  |
connected-is-super-group: super-grouped-by c c′ φ ⟹ super-grouped-by c c′ ψ ⟹ wf-conn c [φ, ψ]
  ⟹ super-grouped-by c c′ (conn c′ [φ, ψ])

lemma simple-cnf[simp]:
  super-grouped-by c c′ FT
  super-grouped-by c c′ FF
  super-grouped-by c c′ (FVar x)
  super-grouped-by c c′ (FNot FT)
  super-grouped-by c c′ (FNot FF)
  super-grouped-by c c′ (FNot (FVar x))
  by auto

lemma c-in-c′-only-super-grouped-by:
  assumes c: c = CAnd ∨ c = COr and c′: c′ = CAnd ∨ c′ = COr and  cc′: c ≠ c′
  shows no-equiv φ ⟹ no-imp φ ⟹ simple-not φ ⟹ c-in-c′-only c c′ φ
    ⟹ super-grouped-by c c′ φ
    (is ?NE φ ⟹ ?NI φ ⟹ ?SN φ ⟹ ?C φ ⟹ ?S φ)
proof (induct φ rule: propo-induct-arity)
  case (nullary φ x)
  then show ?S φ by auto
next
  case (unary φ)
  then have simple-not-symb (FNot φ)
    using all-subformula-st-test-symb-true-phi unfolding simple-not-def by blast
  then have φ = FT ∨ φ = FF ∨ (∃ x. φ = FVar x) by (cases φ, auto)
  then show ?S (FNot φ) by auto
next
  case (binary φ φ1 φ2)
  note IHφ1 = this(1) and IHφ2 = this(2) and no-equiv = this(4) and no-imp = this(5)
    and simpleN = this(6) and c-in-c′-only = this(7) and φ′ = this(3)
  {
    assume φ = FImp φ1 φ2 ∨ φ = FEq φ1 φ2
    then have False using no-equiv no-imp by auto
    then have ?S φ by auto
  }
  moreover {
    assume φ: φ = conn c′ [φ1, φ2] ∧ wf-conn c′ [φ1, φ2]
    have c-in-c′-only: c-in-c′-only c c′ φ1 ∧ c-in-c′-only c c′ φ2 ∧ c-in-c′-symb c c′ φ
      using c-in-c′-only φ′ unfolding c-in-c′-only-def by auto
    have super-grouped-by c c′ φ1 using φ c′ no-equiv no-imp simpleN IHφ1 c-in-c′-only by auto
    moreover have super-grouped-by c c′ φ2
      using φ c′ no-equiv no-imp simpleN IHφ2 c-in-c′-only by auto
    ultimately have ?S φ
      using super-grouped-by.intros(2) φ by (metis c wf-conn-helper-facts(5,6))
  }
  moreover {
    assume φ: φ = conn c [φ1, φ2] ∧ wf-conn c [φ1, φ2]
    then have only-c-inside c φ1 ∧ only-c-inside c φ2
      using c-in-c′-symb-c-implies-only-c-inside c c′ c-in-c′-only  list.set-intros(1)
        wf-conn-helper-facts(5,6)  no-equiv no-imp simpleN  last-ConsL last-ConsR last-in-set
        list.distinct(1) by (metis (no-types, hide-lams) cc′)
    then have only-c-inside c (conn c [φ1, φ2])
      unfolding only-c-inside-def using φ
      by (simp add: only-c-inside-into-only-c-inside all-subformula-st-decomp)
    then have grouped-by c φ using φ only-c-inside-imp-grouped-by c by blast
    then have ?S φ using super-grouped-by.intros(1) by metis
  }
  ultimately show ?S φ by (metis φ′ c c′ cc′ conn.simps(5,6) wf-conn-helper-facts(5,6))
qed
```

9.2 Conjunctive Normal Form

```
definition is-conj-with-TF where is-conj-with-TF == super-grouped-by COr CAnd

lemma or-in-and-only-conjunction-in-disj:
  shows no-equiv φ ⟹ no-imp φ ⟹ simple-not φ ⟹ or-in-and-only φ ⟹ is-conj-with-TF φ
  using c-in-c′-only-super-grouped-by
  unfolding is-conj-with-TF-def or-in-and-only-def c-in-c′-only-def
  by (simp add: c-in-c′-only-def c-in-c′-only-super-grouped-by)

definition is-cnf where
is-cnf φ ≡ is-conj-with-TF φ ∧ no-T-F-except-top-level φ
```

9.2.1 Full CNF transformation

The full1 CNF transformation consists simply in chaining all the transformation defined before.

```
definition cnf-rew where cnf-rew =
  (full (propo-rew-step elim-equiv)) OO
  (full (propo-rew-step elim-imp)) OO
  (full (propo-rew-step elimTB)) OO
  (full (propo-rew-step pushNeg)) OO
  (full (propo-rew-step pushDisj))

lemma cnf-rew-consistent: preserves-un-sat cnf-rew
  by (simp add: cnf-rew-def elimEquv-lifted-consistant elim-imp-lifted-consistant elimTB-consistent
    preserves-un-sat-OO pushDisj-consistent pushNeg-lifted-consistant)


lemma cnf-rew-is-cnf: cnf-rew φ φ′ ⟹ is-cnf φ′
  apply (unfold cnf-rew-def OO-def)
  apply auto
proof –
  fix φ φEq φImp φTB φNeg φDisj :: ′v propo
  assume Eq: full (propo-rew-step elim-equiv) φ φEq
  then have no-equiv: no-equiv φEq using no-equiv-full-propo-rew-step-elim-equiv by blast

  assume Imp: full (propo-rew-step elim-imp) φEq φImp
  then have no-imp: no-imp φImp using no-imp-full-propo-rew-step-elim-imp by blast
  have no-imp-inv: no-equiv φImp using no-equiv Imp elim-imp-inv by blast
  assume TB: full (propo-rew-step elimTB) φImp φTB
  then have noTB: no-T-F-except-top-level φTB
    using no-imp-inv no-imp elimTB-full-propo-rew-step by blast
  have noTB-inv: no-equiv φTB no-imp φTB using elimTB-inv TB no-imp no-imp-inv by blast+

  assume Neg: full (propo-rew-step pushNeg) φTB φNeg
  then have noNeg: simple-not φNeg
    using noTB-inv noTB pushNeg-full-propo-rew-step by blast
  have noNeg-inv: no-equiv φNeg no-imp φNeg no-T-F-except-top-level φNeg
    using pushNeg-inv Neg noTB noTB-inv by blast+

  assume Disj: full (propo-rew-step pushDisj) φNeg φDisj
  then have no-Disj: or-in-and-only φDisj
    using noNeg-inv noNeg pushDisj-full-propo-rew-step by blast
  have noDisj-inv: no-equiv φDisj no-imp φDisj no-T-F-except-top-level φDisj
    simple-not φDisj
  using pushDisj-inv Disj noNeg noNeg-inv by blast+

  moreover have is-conj-with-TF φDisj
    using or-in-and-only-conjunction-in-disj noDisj-inv no-Disj by blast
  ultimately show is-cnf φDisj unfolding is-cnf-def by blast
qed
```

9.3 Disjunctive Normal Form

```
definition is-disj-with-TF where is-disj-with-TF ≡ super-grouped-by CAnd COr

lemma and-in-or-only-conjunction-in-disj:
  shows no-equiv φ ⟹ no-imp φ ⟹ simple-not φ ⟹ and-in-or-only φ ⟹ is-disj-with-TF φ
  using c-in-c′-only-super-grouped-by
  unfolding is-disj-with-TF-def and-in-or-only-def c-in-c′-only-def
  by (simp add: c-in-c′-only-def c-in-c′-only-super-grouped-by)

definition is-dnf :: ′a propo ⇒ bool where
is-dnf φ ⟷ is-disj-with-TF φ ∧ no-T-F-except-top-level φ
```

9.3.1 Full DNF transform

The full1 DNF transformation consists simply in chaining all the transformation defined before.

```
definition dnf-rew where dnf-rew ≡
  (full (propo-rew-step elim-equiv)) OO
  (full (propo-rew-step elim-imp)) OO
  (full (propo-rew-step elimTB)) OO
  (full (propo-rew-step pushNeg)) OO
  (full (propo-rew-step pushConj))

lemma dnf-rew-consistent: preserves-un-sat dnf-rew
  by (simp add: dnf-rew-def elimEquv-lifted-consistant elim-imp-lifted-consistant elimTB-consistent
    preserves-un-sat-OO pushConj-consistent pushNeg-lifted-consistant)

theorem dnf-transformation-correction:
    dnf-rew φ φ′ ⟹ is-dnf φ′
  apply (unfold dnf-rew-def OO-def)
  by (meson and-in-or-only-conjunction-in-disj elimTB-full-propo-rew-step elimTB-inv(1,2)
```

elim-imp-inv is-dnf-def no-equiv-full-propo-rew-step-elim-equiv
no-imp-full-propo-rew-step-elim-imp pushConj-full-propo-rew-step pushConj-inv(*1*−*4*)
pushNeg-full-propo-rew-step pushNeg-inv(*1*−*3*))

10 More aggressive simplifications: Removing true and false at the beginning

10.1 Transformation

We should remove *FT* and *FF* at the beginning and not in the middle of the algorithm. To do this, we have to use more rules (one for each connective):

inductive *elimTBFull* **where**
ElimTBFull1[*simp*]: *elimTBFull* (*FAnd* φ *FT*) φ |
ElimTBFull1'[*simp*]: *elimTBFull* (*FAnd FT* φ) φ |

ElimTBFull2[*simp*]: *elimTBFull* (*FAnd* φ *FF*) *FF* |
ElimTBFull2'[*simp*]: *elimTBFull* (*FAnd FF* φ) *FF* |

ElimTBFull3[*simp*]: *elimTBFull* (*FOr* φ *FT*) *FT* |
ElimTBFull3'[*simp*]: *elimTBFull* (*FOr FT* φ) *FT* |

ElimTBFull4[*simp*]: *elimTBFull* (*FOr* φ *FF*) φ |
ElimTBFull4'[*simp*]: *elimTBFull* (*FOr FF* φ) φ |

ElimTBFull5[*simp*]: *elimTBFull* (*FNot FT*) *FF* |
ElimTBFull5'[*simp*]: *elimTBFull* (*FNot FF*) *FT* |

ElimTBFull6-l[*simp*]: *elimTBFull* (*FImp FT* φ) φ |
ElimTBFull6-l'[*simp*]: *elimTBFull* (*FImp FF* φ) *FT* |
ElimTBFull6-r[*simp*]: *elimTBFull* (*FImp* φ *FT*) *FT* |
ElimTBFull6-r'[*simp*]: *elimTBFull* (*FImp* φ *FF*) (*FNot* φ) |

ElimTBFull7-l[*simp*]: *elimTBFull* (*FEq FT* φ) φ |
ElimTBFull7-l'[*simp*]: *elimTBFull* (*FEq FF* φ) (*FNot* φ) |
ElimTBFull7-r[*simp*]: *elimTBFull* (*FEq* φ *FT*) φ |
ElimTBFull7-r'[*simp*]: *elimTBFull* (*FEq* φ *FF*) (*FNot* φ)

The transformation is still consistent.

lemma *elimTBFull-consistent*: *preserves-un-sat elimTBFull*
proof −
  {
    fix φ ψ:: *'b propo*
    have *elimTBFull* φ $\psi \Longrightarrow \forall A.\ A \models \varphi \longleftrightarrow A \models \psi$
      by (*induct-tac rule*: *elimTBFull.inducts*, *auto*)
  }
  then show *?thesis* **using** *preserves-un-sat-def* **by** *auto*
qed

Contrary to the theorem $[\![$*no-equiv ?*φ; *no-imp ?*φ; *?*$\psi \preceq$ *?*φ; $\neg$ *no-T-F-symb-except-toplevel ?*$\psi]\!] \Longrightarrow \exists\,\psi'$. *elimTB ?*$\psi$ ψ', we do not need the assumption *no-equiv* φ and *no-imp* φ, since our transformation is more general.

lemma *no-T-F-symb-except-toplevel-step-exists'*:
  fixes φ :: *'v propo*

```
  shows ψ ≼ φ ⟹ ¬ no-T-F-symb-except-toplevel ψ ⟹ ∃ψ'. elimTBFull ψ ψ'
proof (induct ψ rule: propo-induct-arity)
  case (nullary φ')
  then have False using no-T-F-symb-except-toplevel-true no-T-F-symb-except-toplevel-false by auto
  then show Ex (elimTBFull φ') by blast
next
  case (unary ψ)
  then have ψ = FF ∨ ψ = FT using no-T-F-symb-except-toplevel-not-decom by blast
  then show Ex (elimTBFull (FNot ψ)) using ElimTBFull5 ElimTBFull5' by blast
next
  case (binary φ' ψ1 ψ2)
  then have ψ1 = FT ∨ ψ2 = FT ∨ ψ1 = FF ∨ ψ2 = FF
    by (metis binary-connectives-def conn.simps(5−8) insertI1 insert-commute
      no-T-F-symb-except-toplevel-bin-decom binary.hyps(3))
  then show Ex (elimTBFull φ') using elimTBFull.intros binary.hyps(3) by blast
qed
```

The same applies here. We do not need the assumption, but the deep link between ¬ *no-T-F-except-top-level* *φ* and the existence of a rewriting step, still exists.

```
lemma no-T-F-except-top-level-rew':
  fixes φ :: 'v propo
  assumes noTB: ¬ no-T-F-except-top-level φ
  shows ∃ψ ψ'. ψ ≼ φ ∧ elimTBFull ψ ψ'
proof −
  have test-symb-false-nullary:
    ∀x. no-T-F-symb-except-toplevel (FF:: 'v propo) ∧ no-T-F-symb-except-toplevel FT
      ∧ no-T-F-symb-except-toplevel (FVar (x:: 'v))
    by auto
  moreover {
    fix c:: 'v connective and l :: 'v propo list and ψ :: 'v propo
    have H: elimTBFull (conn c l) ψ ⟹ ¬no-T-F-symb-except-toplevel (conn c l)
      by (cases (conn c l) rule: elimTBFull.cases) auto
  }
  ultimately show ?thesis
    using no-test-symb-step-exists[of no-T-F-symb-except-toplevel φ elimTBFull] noTB
    no-T-F-symb-except-toplevel-step-exists' unfolding no-T-F-except-top-level-def by metis
qed
```

```
lemma elimTBFull-full-propo-rew-step:
  fixes φ ψ :: 'v propo
  assumes full (propo-rew-step elimTBFull) φ ψ
  shows no-T-F-except-top-level ψ
  using full-propo-rew-step-subformula no-T-F-except-top-level-rew' assms by fastforce
```

10.2 More invariants

As the aim is to use the transformation as the first transformation, we have to show some more invariants for *elim-equiv* and *elim-imp*. For the other transformation, we have already proven it.

```
lemma propo-rew-step-ElimEquiv-no-T-F: propo-rew-step elim-equiv φ ψ ⟹ no-T-F φ ⟹ no-T-F ψ
proof (induct rule: propo-rew-step.induct)
  fix φ' :: 'v propo and ψ' :: 'v propo
  assume a1: no-T-F φ'
  assume a2: elim-equiv φ' ψ'
  have ∀ x0 x1. (¬ elim-equiv (x1 :: 'v propo) x0 ∨ (∃ v2 v3 v4 v5 v6 v7. x1 = FEq v2 v3
    ∧ x0 =  FAnd (FImp v4 v5) (FImp v6 v7) ∧ v2 = v4 ∧ v4 = v7 ∧ v3 = v5 ∧ v3 = v6))
      = (¬ elim-equiv x1 x0 ∨ (∃ v2 v3 v4 v5 v6 v7. x1 = FEq v2 v3
    ∧ x0 = FAnd (FImp v4 v5) (FImp v6 v7) ∧ v2 = v4 ∧ v4 = v7 ∧ v3 = v5 ∧ v3 = v6))
    by meson
  then have ∀ p pa. ¬ elim-equiv (p :: 'v propo) pa ∨ (∃ pb pc pd pe pf pg. p = FEq pb pc
    ∧ pa = FAnd (FImp pd pe) (FImp pf pg) ∧ pb = pd ∧ pd = pg ∧ pc = pe ∧ pc = pf)
    using elim-equiv.cases by force
  then show  no-T-F ψ' using a1 a2 by fastforce
next
  fix φ φ' :: 'v propo and ξ ξ' :: 'v propo list and c :: 'v connective
  assume rel: propo-rew-step elim-equiv φ φ'
  and IH: no-T-F φ ⟹ no-T-F φ'
  and corr: wf-conn c (ξ @ φ # ξ')
  and no-T-F: no-T-F (conn c (ξ @ φ # ξ'))
  {
    assume c: c = CNot
    then have empty: ξ = [] ξ' = [] using corr by auto
    then have no-T-F φ using no-T-F c no-T-F-decomp-not by auto
    then have no-T-F (conn c (ξ @ φ' # ξ')) using c empty no-T-F-comp-not IH by auto
  }
  moreover {
    assume c: c ∈ binary-connectives
    obtain a b where ab: ξ @ φ # ξ' = [a, b]
      using corr c list-length2-decomp wf-conn-bin-list-length by metis
    then have φ: φ = a ∨ φ = b
      by (metis append.simps(1) append-is-Nil-conv list.distinct(1) list.sel(3) nth-Cons-0
        tl-append2)
    have ζ: ∀ζ∈ set (ξ @ φ # ξ'). no-T-F ζ
      using no-T-F unfolding no-T-F-def using corr all-subformula-st-decomp by blast

    then have φ': no-T-F φ' using ab IH φ by auto
    have l': ξ @ φ' # ξ' = [φ', b] ∨ ξ @ φ' # ξ' = [a, φ']
      by (metis (no-types, hide-lams) ab append-Cons append-Nil append-Nil2 butlast.simps(2)
        butlast-append list.distinct(1) list.sel(3))
    then have ∀ ζ ∈ set (ξ @ φ' # ξ'). no-T-F ζ using ζ φ' ab by fastforce
    moreover
      have ∀ ζ ∈ set (ξ @ φ # ξ'). ζ ≠ FT ∧ ζ ≠ FF
        using ζ corr no-T-F no-T-F-except-top-level-false no-T-F-no-T-F-except-top-level by blast
      then have no-T-F-symb (conn c (ξ @ φ' # ξ'))
        by (metis φ' l' ab all-subformula-st-test-symb-true-phi c list.distinct(1)
          list.set-intros(1,2) no-T-F-symb-except-toplevel-bin-decom
          no-T-F-symb-except-toplevel-no-T-F-symb no-T-F-symb-false(1,2) no-T-F-def wf-conn-binary
          wf-conn-list(1,2))
    ultimately have no-T-F (conn c (ξ @ φ' # ξ'))
      by (metis l' all-subformula-st-decomp-imp c no-T-F-def wf-conn-binary)
  }
  moreover {
     fix x
     assume c = CVar x ∨ c = CF ∨ c = CT
     then have False using corr by auto
     then have  no-T-F (conn c (ξ @ φ' # ξ')) by auto
  }
  ultimately show no-T-F (conn c (ξ @ φ' # ξ')) using corr wf-conn.cases by metis
qed

lemma elim-equiv-inv':
  fixes φ ψ :: 'v propo
  assumes full (propo-rew-step elim-equiv) φ ψ and no-T-F-except-top-level φ
  shows no-T-F-except-top-level ψ
proof –
  {
    fix φ ψ :: 'v propo
    have propo-rew-step elim-equiv φ ψ ⟹ no-T-F-except-top-level φ
      ⟹ no-T-F-except-top-level ψ
      proof –
        assume rel: propo-rew-step elim-equiv φ ψ
        and no: no-T-F-except-top-level φ
        {
          assume φ = FT ∨ φ = FF
          from rel this have False
            apply (induct rule: propo-rew-step.induct, auto simp: wf-conn-list(1,2))
            using elim-equiv.simps by blast+
          then have no-T-F-except-top-level ψ by blast
        }
        moreover {
          assume φ ≠ FT ∧ φ ≠ FF
          then have no-T-F φ
            by (metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
          then have no-T-F ψ using propo-rew-step-ElimEquiv-no-T-F rel by blast
          then have no-T-F-except-top-level ψ by (simp add: no-T-F-no-T-F-except-top-level)
        }
        ultimately show no-T-F-except-top-level ψ by metis
      qed
  }
  moreover {
     fix c :: 'v connective and ξ ξ' :: 'v propo list and ζ ζ' :: 'v propo
     assume rel: propo-rew-step elim-equiv ζ ζ'
     and incl: ζ ⪯ φ
     and corr: wf-conn c (ξ @ ζ # ξ')
     and no-T-F: no-T-F-symb-except-toplevel (conn c (ξ @ ζ # ξ'))
     and n: no-T-F-symb-except-toplevel ζ'
     have no-T-F-symb-except-toplevel (conn c (ξ @ ζ' # ξ'))
     proof
       have p: no-T-F-symb (conn c (ξ @ ζ # ξ'))
         using corr wf-conn-list(1) wf-conn-list(2) no-T-F-symb-except-toplevel-no-T-F-symb no-T-F
         by blast
       have l: ∀φ∈set (ξ @ ζ # ξ'). φ ≠ FT ∧ φ ≠ FF
         using corr wf-conn-no-T-F-symb-iff p by blast
       from rel incl have ζ'≠FT ∧ζ'≠FF
         apply (induction ζ ζ' rule: propo-rew-step.induct)
         apply (cases rule: elim-equiv.cases, auto simp: elim-equiv.simps)
         by (metis append-is-Nil-conv list.distinct wf-conn-list(1,2) wf-conn-no-arity-change
           wf-conn-no-arity-change-helper)+
       then have ∀φ ∈ set (ξ @ ζ' # ξ'). φ ≠ FT ∧ φ ≠ FF using l by auto
       moreover have c ≠ CT ∧ c ≠ CF using corr by auto
       ultimately show no-T-F-symb (conn c (ξ @ ζ' # ξ'))
         by (metis corr wf-conn-no-arity-change wf-conn-no-arity-change-helper no-T-F-symb-comp)
    qed
  }
  ultimately show no-T-F-except-top-level ψ
    using full-propo-rew-step-inv-stay-with-inc[of elim-equiv no-T-F-symb-except-toplevel φ]
      assms subformula-refl unfolding no-T-F-except-top-level-def by metis
qed


lemma propo-rew-step-ElimImp-no-T-F: propo-rew-step elim-imp φ ψ ⟹ no-T-F φ ⟹  no-T-F ψ
proof (induct rule: propo-rew-step.induct)
  case (global-rel φ' ψ')
  then show no-T-F ψ'
    using elim-imp.cases no-T-F-comp-not no-T-F-decomp(1,2)
    by (metis no-T-F-comp-expanded-explicit(2))
next
  case (propo-rew-one-step-lift φ φ' c ξ ξ')
  note rel = this(1) and IH = this(2) and corr = this(3) and no-T-F = this(4)
  {
    assume c: c = CNot
    then have empty: ξ = [] ξ' = [] using corr by auto
    then have no-T-F φ using no-T-F c no-T-F-decomp-not by auto
    then have no-T-F (conn c (ξ @ φ' # ξ')) using c empty no-T-F-comp-not IH by auto
  }
  moreover {
    assume c: c ∈ binary-connectives
    then obtain a b where ab: ξ @ φ # ξ' = [a, b]
      using corr list-length2-decomp wf-conn-bin-list-length by metis
    then have φ: φ = a ∨ φ = b
      by (metis append-self-conv2 wf-conn-list-decomp(4) wf-conn-unary list.discI list.sel(3)
        nth-Cons-0 tl-append2)
    have ζ: ∀ζ ∈ set (ξ @ φ # ξ'). no-T-F ζ using ab c propo-rew-one-step-lift.prems by auto

    then have φ': no-T-F φ'
      using ab IH φ corr no-T-F no-T-F-def all-subformula-st-decomp-explicit by auto
    have χ: ξ @ φ' # ξ' = [φ', b] ∨ ξ @ φ' # ξ' = [a, φ']
      by (metis (no-types, hide-lams) ab append-Cons append-Nil append-Nil2 butlast.simps(2)
        butlast-append list.distinct(1) list.sel(3))
    then have ∀ζ∈ set (ξ @ φ' # ξ'). no-T-F ζ using ζ φ' ab by fastforce
    moreover
      have no-T-F (last (ξ @ φ' # ξ'))  by (simp add: calculation)
      then have no-T-F-symb (conn c (ξ @ φ' # ξ'))
        by (metis χ φ' ζ ab all-subformula-st-test-symb-true-phi c last.simps list.distinct(1)
          list.set-intros(1) no-T-F-bin-decomp no-T-F-def)
    ultimately have no-T-F (conn c (ξ @ φ' # ξ')) using c χ by fastforce
  }
  moreover {
    fix x
    assume c = CVar x ∨ c = CF ∨ c = CT
    then have False using corr by auto
    then have no-T-F (conn c (ξ @ φ' # ξ')) by auto
  }
  ultimately show no-T-F (conn c (ξ @ φ' # ξ')) using corr wf-conn.cases by blast
qed
```

```
lemma elim-imp-inv':
  fixes φ ψ :: 'v propo
  assumes full (propo-rew-step elim-imp) φ ψ and no-T-F-except-top-level φ
  showsno-T-F-except-top-level ψ
proof –
  {
    {
      fix φ ψ :: 'v propo
      have H: elim-imp φ ψ ⟹ no-T-F-except-top-level φ ⟹  no-T-F-except-top-level ψ
        by (induct φ ψ rule: elim-imp.induct, auto)
    } note H = this
    fix φ ψ :: 'v propo
    have propo-rew-step elim-imp φ ψ ⟹ no-T-F-except-top-level φ ⟹  no-T-F-except-top-level ψ
      proof –
        assume rel: propo-rew-step elim-imp φ ψ
        and no: no-T-F-except-top-level φ
        {
          assume φ = FT ∨ φ = FF
          from rel this have False
            apply (induct rule: propo-rew-step.induct)
            by (cases rule: elim-imp.cases, auto simp: wf-conn-list(1,2))
          then have no-T-F-except-top-level ψ by blast
        }
        moreover {
          assume φ ≠ FT ∧ φ ≠ FF
          then have no-T-F φ
            by (metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
          then have no-T-F ψ
            using rel propo-rew-step-ElimImp-no-T-F by blast
          then have no-T-F-except-top-level ψ by (simp add: no-T-F-no-T-F-except-top-level)
        }
        ultimately show no-T-F-except-top-level ψ by metis
      qed
  }
  moreover {
     fix c :: 'v connective and ξ ξ' :: 'v propo list and ζ ζ' :: 'v propo
     assume rel: propo-rew-step elim-imp ζ ζ'
     and incl: ζ ≼ φ
     and corr: wf-conn c (ξ @ ζ # ξ')
     and no-T-F: no-T-F-symb-except-toplevel (conn c (ξ @ ζ # ξ'))
     and n: no-T-F-symb-except-toplevel ζ'
     have no-T-F-symb-except-toplevel (conn c (ξ @ ζ' # ξ'))
     proof
       have p: no-T-F-symb (conn c (ξ @ ζ # ξ'))
         by (simp add: corr no-T-F no-T-F-symb-except-toplevel-no-T-F-symb wf-conn-list(1,2))

       have l: ∀φ∈set (ξ @ ζ # ξ'). φ ≠ FT ∧ φ ≠ FF
         using corr wf-conn-no-T-F-symb-iff p by blast
       from rel incl have ζ'≠FT ∧ζ'≠FF
         apply (induction ζ ζ' rule: propo-rew-step.induct)
         apply (cases rule: elim-imp.cases, auto)
         using wf-conn-list(1,2) wf-conn-no-arity-change wf-conn-no-arity-change-helper
         by (metis append-is-Nil-conv list.distinct(1))+
       then have ∀φ∈set (ξ @ ζ' # ξ'). φ ≠ FT ∧ φ ≠ FF using l by auto
       moreover have c ≠ CT ∧ c ≠ CF using corr by auto
      ultimately show no-T-F-symb (conn c (ξ @ ζ′ # ξ′))
        using corr wf-conn-no-arity-change no-T-F-symb-comp
        by (metis wf-conn-no-arity-change-helper)
    qed
  }
  ultimately show no-T-F-except-top-level ψ
    using full-propo-rew-step-inv-stay-with-inc[of elim-imp no-T-F-symb-except-toplevel φ]
    assms subformula-refl unfolding no-T-F-except-top-level-def by metis
qed
```

10.3 The new CNF and DNF transformation

The transformation is the same as before, but the order is not the same.

```
definition dnf-rew′ :: ′a propo ⇒ ′a propo ⇒ bool where
dnf-rew′ =
  (full (propo-rew-step elimTBFull)) OO
  (full (propo-rew-step elim-equiv)) OO
  (full (propo-rew-step elim-imp)) OO
  (full (propo-rew-step pushNeg)) OO
  (full (propo-rew-step pushConj))

lemma dnf-rew′-consistent: preserves-un-sat dnf-rew′
  by (simp add: dnf-rew′-def elimEquv-lifted-consistant elim-imp-lifted-consistant
    elimTBFull-consistent preserves-un-sat-OO pushConj-consistent pushNeg-lifted-consistant)

theorem cnf-transformation-correction:
    dnf-rew′ φ φ′ ⟹ is-dnf φ′
  unfolding dnf-rew′-def OO-def
  by (meson and-in-or-only-conjunction-in-disj elimTBFull-full-propo-rew-step elim-equiv-inv′
    elim-imp-inv elim-imp-inv′ is-dnf-def no-equiv-full-propo-rew-step-elim-equiv
    no-imp-full-propo-rew-step-elim-imp pushConj-full-propo-rew-step pushConj-inv(1−4)
    pushNeg-full-propo-rew-step pushNeg-inv(1−3))
```

Given all the lemmas before the CNF transformation is easy to prove:

```
definition cnf-rew′ :: ′a propo ⇒ ′a propo ⇒ bool where
cnf-rew′ =
  (full (propo-rew-step elimTBFull)) OO
  (full (propo-rew-step elim-equiv)) OO
  (full (propo-rew-step elim-imp)) OO
  (full (propo-rew-step pushNeg)) OO
  (full (propo-rew-step pushDisj))

lemma cnf-rew′-consistent: preserves-un-sat cnf-rew′
  by (simp add: cnf-rew′-def elimEquv-lifted-consistant elim-imp-lifted-consistant
    elimTBFull-consistent preserves-un-sat-OO pushDisj-consistent pushNeg-lifted-consistant)

theorem cnf′-transformation-correction:
  cnf-rew′ φ φ′ ⟹ is-cnf φ′
  unfolding cnf-rew′-def OO-def
  by (meson elimTBFull-full-propo-rew-step elim-equiv-inv′ elim-imp-inv elim-imp-inv′ is-cnf-def
    no-equiv-full-propo-rew-step-elim-equiv no-imp-full-propo-rew-step-elim-imp
    or-in-and-only-conjunction-in-disj pushDisj-full-propo-rew-step pushDisj-inv(1−4)
    pushNeg-full-propo-rew-step pushNeg-inv(1) pushNeg-inv(2) pushNeg-inv(3))

end
```

11 Partial Clausal Logic

theory *Partial-Clausal-Logic*
imports *../lib/Clausal-Logic List-More*
begin

11.1 Clauses

Clauses are (finite) multisets of literals.

type-synonym *'a clause = 'a literal multiset*
type-synonym *'v clauses = 'v clause set*

11.2 Partial Interpretations

type-synonym *'a interp = 'a literal set*

definition *true-lit* :: *'a interp* $\Rightarrow$ *'a literal* $\Rightarrow$ *bool* (**infix** $\models$*l 50*) **where**
  $I \models l\ L \longleftrightarrow L \in I$

declare *true-lit-def*[*simp*]

11.2.1 Consistency

definition *consistent-interp* :: *'a literal set* $\Rightarrow$ *bool* **where**
consistent-interp $I = (\forall L.\ \neg(L \in I \wedge -\ L \in I))$

lemma *consistent-interp-empty*[*simp*]:
  consistent-interp {} **unfolding** *consistent-interp-def* **by** *auto*

lemma *consistent-interp-single*[*simp*]:
  consistent-interp $\{L\}$ **unfolding** *consistent-interp-def* **by** *auto*

lemma *consistent-interp-subset*:
  assumes
    $A \subseteq B$ **and**
    consistent-interp B
  shows *consistent-interp* A
  using *assms* **unfolding** *consistent-interp-def* **by** *auto*

lemma *consistent-interp-change-insert*:
  $a \notin A \Longrightarrow -a \notin A \Longrightarrow$ *consistent-interp* (*insert* $(-a)$ A) $\longleftrightarrow$ *consistent-interp* (*insert* a A)
  unfolding *consistent-interp-def* **by** *fastforce*

lemma *consistent-interp-insert-pos*[*simp*]:
  $a \notin A \Longrightarrow$ *consistent-interp* (*insert* a A) $\longleftrightarrow$ *consistent-interp* $A \wedge -a \notin A$
  unfolding *consistent-interp-def* **by** *auto*

lemma *consistent-interp-insert-not-in*:
  consistent-interp $A \Longrightarrow a \notin A \Longrightarrow -a \notin A \Longrightarrow$ *consistent-interp* (*insert* a A)
  unfolding *consistent-interp-def* **by** *auto*

11.2.2 Atoms

definition *atms-of-ms* :: *'a literal multiset set* $\Rightarrow$ *'a set* **where**
atms-of-ms $\psi s = \bigcup$(*atms-of* ‘ ψs)

lemma *atms-of-mmltiset*[*simp*]:
  atms-of (*mset a*) = *atm-of* ‘ *set a*
  by (*induct a*) *auto*

lemma *atms-of-ms-mset-unfold*:
  atms-of-ms (*mset* ‘ *b*) = $(\bigcup x{\in}b.$ *atm-of* ‘ *set x*)
  unfolding *atms-of-ms-def* **by** *simp*

definition *atms-of-s* :: *'a literal set* ⇒ *'a set* **where**
  atms-of-s C = *atm-of* ‘ *C*

lemma *atms-of-ms-emtpy-set*[*simp*]:
  atms-of-ms {} = {}
  unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-memtpy*[*simp*]:
  atms-of-ms {{#}} = {}
  unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-mono*:
  A ⊆ *B* ⟹ *atms-of-ms A* ⊆ *atms-of-ms B*
  unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-finite*[*simp*]:
  finite ψs ⟹ *finite* (*atms-of-ms* ψs)
  unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-union*[*simp*]:
  atms-of-ms ($\psi s \cup \chi s$) = *atms-of-ms* ψs ∪ *atms-of-ms* χs
  unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-insert*[*simp*]:
  atms-of-ms (*insert* ψs χs) = *atms-of* ψs ∪ *atms-of-ms* χs
  unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-singleton*[*simp*]: *atms-of-ms* {*L*} = *atms-of L*
  unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-atms-of-ms-mono*[*simp*]:
  A ∈ ψ ⟹ *atms-of A* ⊆ *atms-of-ms* ψ
  unfolding *atms-of-ms-def* **by** *fastforce*

lemma *atms-of-ms-single-set-mset-atns-of*[*simp*]:
  atms-of-ms (*single* ‘ *set-mset B*) = *atms-of B*
  unfolding *atms-of-ms-def atms-of-def* **by** *auto*

lemma *atms-of-ms-remove-incl*:
  shows *atms-of-ms* (*Set.remove a* ψ) ⊆ *atms-of-ms* ψ
  unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-remove-subset*:
  atms-of-ms ($\varphi - \psi$) ⊆ *atms-of-ms* φ
  unfolding *atms-of-ms-def* **by** *auto*

lemma *finite-atms-of-ms-remove-subset*[*simp*]:

```
  finite (atms-of-ms A) ⟹ finite (atms-of-ms (A − C))
  using atms-of-ms-remove-subset[of A C] finite-subset by blast

lemma atms-of-ms-empty-iff:
  atms-of-ms A = {} ⟷ A = {{#}} ∨ A = {}
  apply (rule iffI)
   apply (metis (no-types, lifting) atms-empty-iff-empty atms-of-atms-of-ms-mono insert-absorb
    singleton-iff singleton-insert-inj-eq′ subsetI subset-empty)
  apply auto[]
  done

lemma in-implies-atm-of-on-atms-of-ms:
  assumes L ∈# C and C ∈ N
  shows atm-of L ∈ atms-of-ms N
  using atms-of-atms-of-ms-mono[of C N] assms by (simp add: atm-of-lit-in-atms-of subset-iff)

lemma in-plus-implies-atm-of-on-atms-of-ms:
  assumes C+{#L#} ∈ N
  shows atm-of L ∈ atms-of-ms N
  using in-implies-atm-of-on-atms-of-ms[of - C +{#L#}] assms by auto

lemma in-m-in-literals:
  assumes {#A#} + D ∈ ψs
  shows atm-of A ∈ atms-of-ms ψs
  using assms by (auto dest: atms-of-atms-of-ms-mono)

lemma atms-of-s-union[simp]:
  atms-of-s (Ia ∪ Ib) = atms-of-s Ia ∪ atms-of-s Ib
  unfolding atms-of-s-def by auto

lemma atms-of-s-single[simp]:
  atms-of-s {L} = {atm-of L}
  unfolding atms-of-s-def by auto

lemma atms-of-s-insert[simp]:
  atms-of-s (insert L Ib) = {atm-of L} ∪ atms-of-s Ib
  unfolding atms-of-s-def by auto

lemma in-atms-of-s-decomp[iff]:
  P ∈ atms-of-s I ⟷ (Pos P ∈ I ∨ Neg P ∈ I) (is ?P ⟷ ?Q)
proof
  assume ?P
  then show ?Q unfolding atms-of-s-def by (metis image-iff literal.exhaust-sel)
next
  assume ?Q
  then show ?P unfolding atms-of-s-def by force
qed

lemma atm-of-in-atm-of-set-in-uminus:
  atm-of L′ ∈ atm-of ‘ B ⟹ L′ ∈ B ∨ − L′ ∈ B
  using atms-of-s-def by (cases L′) fastforce+
```

11.2.3 Totality

```
definition total-over-set :: ′a interp ⇒ ′a set ⇒ bool where
total-over-set I S = (∀ l∈S. Pos l ∈ I ∨ Neg l ∈ I)
```

```
definition total-over-m  :: 'a literal set ⇒ 'a clause set ⇒ bool where
total-over-m I ψs = total-over-set I (atms-of-ms ψs)

lemma total-over-set-empty[simp]:
  total-over-set I {}
  unfolding total-over-set-def by auto

lemma total-over-m-empty[simp]:
  total-over-m I {}
  unfolding total-over-m-def by auto

lemma total-over-set-single[iff]:
  total-over-set I {L} ⟷ (Pos L ∈ I ∨ Neg L ∈ I)
  unfolding total-over-set-def by auto

lemma total-over-set-insert[iff]:
  total-over-set I (insert L Ls) ⟷ ((Pos L ∈ I ∨ Neg L ∈ I) ∧ total-over-set I Ls)
  unfolding total-over-set-def by auto

lemma total-over-set-union[iff]:
  total-over-set I (Ls ∪ Ls') ⟷ (total-over-set I Ls ∧ total-over-set I Ls')
  unfolding total-over-set-def by auto

lemma total-over-m-subset:
  A ⊆ B ⟹ total-over-m I B ⟹ total-over-m I A
  using atms-of-ms-mono[of A] unfolding total-over-m-def total-over-set-def by auto

lemma total-over-m-sum[iff]:
  shows total-over-m I {C + D} ⟷ (total-over-m I {C} ∧ total-over-m I {D})
  using assms unfolding total-over-m-def total-over-set-def by auto

lemma total-over-m-union[iff]:
  total-over-m I (A ∪ B) ⟷ (total-over-m I A ∧ total-over-m I B)
  unfolding total-over-m-def total-over-set-def by auto

lemma total-over-m-insert[iff]:
  total-over-m I (insert a A) ⟷ (total-over-set I (atms-of a) ∧ total-over-m I A)
  unfolding total-over-m-def total-over-set-def by fastforce

lemma total-over-m-extension:
  fixes I :: 'v literal set and A :: 'v clauses
  assumes total: total-over-m I A
  shows ∃ I'. total-over-m (I ∪ I') (A∪B)
    ∧ (∀ x∈I'. atm-of x ∈ atms-of-ms B ∧ atm-of x ∉ atms-of-ms A)
proof –
  let ?I' = {Pos v |v. v∈ atms-of-ms B ∧ v ∉ atms-of-ms A}
  have (∀ x∈?I'. atm-of x ∈ atms-of-ms B ∧ atm-of x ∉ atms-of-ms A) by auto
  moreover have total-over-m (I ∪ ?I') (A∪B)
    using total unfolding total-over-m-def total-over-set-def by auto
  ultimately show ?thesis by blast
qed

lemma total-over-m-consistent-extension:
  fixes I :: 'v literal set and A :: 'v clauses
  assumes total: total-over-m I A
  and cons: consistent-interp I
  shows ∃I′. total-over-m (I ∪ I′) (A ∪ B)
    ∧ (∀x∈I′. atm-of x ∈ atms-of-ms B ∧ atm-of x ∉ atms-of-ms A) ∧ consistent-interp (I ∪ I′)
proof −
  let ?I′ = {Pos v |v. v∈ atms-of-ms B ∧ v ∉ atms-of-ms A ∧ Pos v ∉ I ∧ Neg v ∉ I}
  have (∀x∈?I′. atm-of x ∈ atms-of-ms B ∧ atm-of x ∉ atms-of-ms A) by auto
  moreover have total-over-m (I ∪ ?I′) (A∪B)
    using total unfolding total-over-m-def total-over-set-def by auto
  moreover have consistent-interp (I ∪ ?I′)
    using cons unfolding consistent-interp-def by (intro allI) (rename-tac L, case-tac L, auto)
  ultimately show ?thesis by blast
qed

lemma total-over-set-atms-of-m[simp]:
  total-over-set Ia (atms-of-s Ia)
  unfolding total-over-set-def atms-of-s-def by (metis image-iff literal.exhaust-sel)

lemma total-over-set-literal-defined:
  assumes {#A#} + D ∈ ψs
  and total-over-set I (atms-of-ms ψs)
  shows A ∈ I ∨ −A ∈ I
  using assms unfolding total-over-set-def by (metis (no-types) Neg-atm-of-iff in-m-in-literals
    literal.collapse(1) uminus-Neg uminus-Pos)

lemma tot-over-m-remove:
  assumes total-over-m (I ∪ {L}) {ψ}
  and L: ¬ L ∈# ψ −L ∉# ψ
  shows total-over-m I {ψ}
  unfolding total-over-m-def total-over-set-def
proof
  fix l
  assume l: l ∈ atms-of-ms {ψ}
  then have Pos l ∈ I ∨ Neg l ∈ I ∨ l = atm-of L
    using assms unfolding total-over-m-def total-over-set-def by auto
  moreover have atm-of L ∉ atms-of-ms {ψ}
    proof (rule ccontr)
      assume ¬ ?thesis
      then have atm-of L ∈ atms-of ψ by auto
      then have Pos (atm-of L) ∈# ψ ∨ Neg (atm-of L) ∈# ψ
        using atm-imp-pos-or-neg-lit by metis
      then have L ∈# ψ ∨ − L ∈# ψ by (cases L) auto
      then show False using L by auto
    qed
  ultimately show  Pos l ∈ I ∨ Neg l ∈ I using l by metis
qed

lemma total-union:
  assumes total-over-m I ψ
  shows total-over-m (I ∪ I′) ψ
  using assms unfolding total-over-m-def total-over-set-def by auto

lemma total-union-2:
  assumes total-over-m I ψ
  and total-over-m I′ ψ′
```

shows *total-over-m* $(I \cup I')$ $(\psi \cup \psi')$
using *assms* **unfolding** *total-over-m-def total-over-set-def* **by** *auto*

11.2.4 Interpretations

definition *true-cls* :: *'a interp* $\Rightarrow$ *'a clause* $\Rightarrow$ *bool* (**infix** $\models$ *50*) **where**
$I \models C \longleftrightarrow (\exists L \in\# C.\ I \models l\ L)$

lemma *true-cls-empty*[*iff*]: $\neg\ I \models \{\#\}$
unfolding *true-cls-def* **by** *auto*

lemma *true-cls-singleton*[*iff*]: $I \models \{\#L\#\} \longleftrightarrow I \models l\ L$
unfolding *true-cls-def* **by** (*auto split*:*if-split-asm*)

lemma *true-cls-union*[*iff*]: $I \models C + D \longleftrightarrow I \models C \vee I \models D$
unfolding *true-cls-def* **by** *auto*

lemma *true-cls-mono-set-mset*: *set-mset* $C \subseteq$ *set-mset* $D \Longrightarrow I \models C \Longrightarrow I \models D$
unfolding *true-cls-def subset-eq Bex-def* **by** *metis*

lemma *true-cls-mono-leD*[*dest*]: $A \subseteq\# B \Longrightarrow I \models A \Longrightarrow I \models B$
unfolding *true-cls-def* **by** *auto*

lemma
assumes $I \models \psi$
shows *true-cls-union-increase*[*simp*]: $I \cup I' \models \psi$
and *true-cls-union-increase'*[*simp*]: $I' \cup I \models \psi$
using *assms* **unfolding** *true-cls-def* **by** *auto*

lemma *true-cls-mono-set-mset-l*:
assumes $A \models \psi$
and $A \subseteq B$
shows $B \models \psi$
using *assms* **unfolding** *true-cls-def* **by** *auto*

lemma *true-cls-replicate-mset*[*iff*]: $I \models$ *replicate-mset* $n\ L \longleftrightarrow n \neq 0 \wedge I \models l\ L$
by (*induct n*) *auto*

lemma *true-cls-empty-entails*[*iff*]: $\neg\ \{\} \models N$
by (*auto simp add*: *true-cls-def*)

lemma *true-cls-not-in-remove*:
assumes $L \notin\# \chi$
and $I \cup \{L\} \models \chi$
shows $I \models \chi$
using *assms* **unfolding** *true-cls-def* **by** *auto*

definition *true-clss* :: *'a interp* $\Rightarrow$ *'a clauses* $\Rightarrow$ *bool* (**infix** $\models s$ *50*) **where**
$I \models s\ CC \longleftrightarrow (\forall C \in CC.\ I \models C)$

lemma *true-clss-empty*[*simp*]: $I \models s\ \{\}$
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-singleton*[*iff*]: $I \models s\ \{C\} \longleftrightarrow I \models C$
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-empty-entails-empty*[*iff*]: {} ⊨*s* *N* ⟷ *N* = {}
unfolding *true-clss-def* **by** (*auto simp add*: *true-cls-def*)

lemma *true-cls-insert-l* [*simp*]:
M ⊨ *A* ⟹ *insert L M* ⊨ *A*
unfolding *true-cls-def* **by** *auto*

lemma *true-clss-union*[*iff*]: *I* ⊨*s* *CC* ∪ *DD* ⟷ *I* ⊨*s* *CC* ∧ *I* ⊨*s* *DD*
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-insert*[*iff*]: *I* ⊨*s* *insert C DD* ⟷ *I* ⊨ *C* ∧ *I* ⊨*s* *DD*
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-mono*: *DD* ⊆ *CC* ⟹ *I* ⊨*s* *CC* ⟹ *I* ⊨*s* *DD*
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-union-increase*[*simp*]:
assumes *I* ⊨*s* *ψ*
shows *I* ∪ *I′* ⊨*s* *ψ*
using *assms* **unfolding** *true-clss-def* **by** *auto*

lemma *true-clss-union-increase′*[*simp*]:
assumes *I′* ⊨*s* *ψ*
shows *I* ∪ *I′* ⊨*s* *ψ*
using *assms* **by** (*auto simp add*: *true-clss-def*)

lemma *true-clss-commute-l*:
(*I* ∪ *I′* ⊨*s* *ψ*) ⟷ (*I′* ∪ *I* ⊨*s* *ψ*)
by (*simp add*: *Un-commute*)

lemma *model-remove*[*simp*]: *I* ⊨*s* *N* ⟹ *I* ⊨*s* *Set.remove a N*
by (*simp add*: *true-clss-def*)

lemma *model-remove-minus*[*simp*]: *I* ⊨*s* *N* ⟹ *I* ⊨*s* *N* − *A*
by (*simp add*: *true-clss-def*)

lemma *notin-vars-union-true-cls-true-cls*:
assumes ∀ *x*∈*I′*. *atm-of x* ∉ *atms-of-ms A*
and *atms-of L* ⊆ *atms-of-ms A*
and *I* ∪ *I′* ⊨ *L*
shows *I* ⊨ *L*
using *assms* **unfolding** *true-cls-def true-lit-def Bex-def*
by (*metis Un-iff atm-of-lit-in-atms-of contra-subsetD*)

lemma *notin-vars-union-true-clss-true-clss*:
assumes ∀ *x*∈*I′*. *atm-of x* ∉ *atms-of-ms A*
and *atms-of-ms L* ⊆ *atms-of-ms A*
and *I* ∪ *I′* ⊨*s* *L*
shows *I* ⊨*s* *L*
using *assms* **unfolding** *true-clss-def true-lit-def Ball-def*
by (*meson atms-of-atms-of-ms-mono notin-vars-union-true-cls-true-cls subset-trans*)

11.2.5 Satisfiability

definition *satisfiable* :: *′a clause set* ⇒ *bool* **where**
satisfiable CC ≡ ∃ *I*. (*I* ⊨*s* *CC* ∧ *consistent-interp I* ∧ *total-over-m I CC*)

```
lemma satisfiable-single[simp]:
  satisfiable {{#L#}}
  unfolding satisfiable-def by fastforce

abbreviation unsatisfiable :: 'a clause set ⇒ bool where
  unsatisfiable CC ≡ ¬ satisfiable CC

lemma satisfiable-decreasing:
  assumes satisfiable (ψ ∪ ψ')
  shows satisfiable ψ
  using assms total-over-m-union unfolding satisfiable-def by blast

lemma satisfiable-def-min:
  satisfiable CC
    ⟷ (∃I. I ⊨s CC ∧ consistent-interp I ∧ total-over-m I CC ∧ atm-of`I = atms-of-ms CC)
    (is ?sat ⟷ ?B)
proof
  assume ?B then show ?sat by (auto simp add: satisfiable-def)
next
  assume ?sat
  then obtain I where
    I-CC: I ⊨s CC and
    cons: consistent-interp I and
    tot: total-over-m I CC
    unfolding satisfiable-def by auto
  let ?I = {P. P ∈ I ∧ atm-of P ∈ atms-of-ms CC}

  have I-CC: ?I ⊨s CC
    using I-CC in-implies-atm-of-on-atms-of-ms unfolding true-clss-def Ball-def true-cls-def
    Bex-def true-lit-def
    by blast

  moreover have cons: consistent-interp ?I
    using cons unfolding consistent-interp-def by auto
  moreover have total-over-m ?I CC
    using tot unfolding total-over-m-def total-over-set-def by auto
  moreover
    have atms-CC-incl: atms-of-ms CC ⊆ atm-of`I
      using tot unfolding total-over-m-def total-over-set-def atms-of-ms-def
      by (auto simp add: atms-of-def atms-of-s-def[symmetric])
    have  atm-of ` ?I = atms-of-ms CC
      using atms-CC-incl unfolding atms-of-ms-def by force
  ultimately show ?B by auto
qed
```

11.2.6 Entailment for Multisets of Clauses

```
definition true-cls-mset :: 'a interp ⇒ 'a clause multiset ⇒ bool (infix ⊨m 50) where
  I ⊨m CC ⟷ (∀C ∈# CC. I ⊨ C)

lemma true-cls-mset-empty[simp]: I ⊨m {#}
  unfolding true-cls-mset-def by auto

lemma true-cls-mset-singleton[iff]: I ⊨m {#C#} ⟷ I ⊨ C
  unfolding true-cls-mset-def by (auto split: if-split-asm)
```

```
lemma true-cls-mset-union[iff]: I ⊨m CC + DD ⟷ I ⊨m CC ∧ I ⊨m DD
  unfolding true-cls-mset-def by fastforce

lemma true-cls-mset-image-mset[iff]: I ⊨m image-mset f A ⟷ (∀ x ∈# A. I ⊨ f x)
  unfolding true-cls-mset-def by fastforce

lemma true-cls-mset-mono: set-mset DD ⊆ set-mset CC ⟹ I ⊨m CC ⟹ I ⊨m DD
  unfolding true-cls-mset-def subset-iff by auto

lemma true-clss-set-mset[iff]: I ⊨s set-mset CC ⟷ I ⊨m CC
  unfolding true-clss-def true-cls-mset-def by auto

lemma true-cls-mset-increasing-r[simp]:
  I ⊨m CC ⟹ I ∪ J ⊨m CC
  unfolding true-cls-mset-def by auto

theorem true-cls-remove-unused:
  assumes I ⊨ ψ
  shows {v ∈ I. atm-of v ∈ atms-of ψ} ⊨ ψ
  using assms unfolding true-cls-def atms-of-def by auto

theorem true-clss-remove-unused:
  assumes I ⊨s ψ
  shows {v ∈ I. atm-of v ∈ atms-of-ms ψ} ⊨s ψ
  unfolding true-clss-def atms-of-def Ball-def
proof (intro allI impI)
  fix x
  assume x ∈ ψ
  then have I ⊨ x
    using assms unfolding true-clss-def atms-of-def Ball-def by auto

  then have {v ∈ I. atm-of v ∈ atms-of x} ⊨ x
    by (simp only: true-cls-remove-unused[of I])
  moreover have {v ∈ I. atm-of v ∈ atms-of x} ⊆ {v ∈ I. atm-of v ∈ atms-of-ms ψ}
    using ‹x ∈ ψ› by (auto simp add: atms-of-ms-def)
  ultimately show {v ∈ I. atm-of v ∈ atms-of-ms ψ} ⊨ x
    using true-cls-mono-set-mset-l by blast
qed
```

A simple application of the previous theorem:

```
lemma true-clss-union-decrease:
  assumes II': I ∪ I' ⊨ ψ
  and H: ∀ v ∈ I'. atm-of v ∉ atms-of ψ
  shows I ⊨ ψ
proof –
  let ?I = {v ∈ I ∪ I'. atm-of v ∈ atms-of ψ}
  have ?I ⊨ ψ using true-cls-remove-unused II' by blast
  moreover have ?I ⊆ I using H by auto
  ultimately show ?thesis using true-cls-mono-set-mset-l by blast
qed

lemma multiset-not-empty:
  assumes M ≠ {#}
  and x ∈# M
  shows ∃ A. x = Pos A ∨ x = Neg A
  using assms literal.exhaust-sel by blast

lemma atms-of-ms-empty:
  fixes ψ :: 'v clauses
  assumes atms-of-ms ψ = {}
  shows ψ = {} ∨ ψ = {{#}}
  using assms by (auto simp add: atms-of-ms-def)

lemma consistent-interp-disjoint:
 assumes consI: consistent-interp I
 and disj: atms-of-s A ∩ atms-of-s I = {}
 and consA: consistent-interp A
 shows consistent-interp (A ∪ I)
proof (rule ccontr)
  assume ¬ ?thesis
  moreover have ⋀L. ¬ (L ∈ A ∧ −L ∈ I)
    using disj unfolding atms-of-s-def by (auto simp add: rev-image-eqI)
  ultimately show False
    using consA consI unfolding consistent-interp-def by (metis (full-types) Un-iff
      literal.exhaust-sel uminus-Neg uminus-Pos)
qed

lemma total-remove-unused:
  assumes total-over-m I ψ
  shows total-over-m {v ∈ I. atm-of v ∈ atms-of-ms ψ} ψ
  using assms unfolding total-over-m-def total-over-set-def
  by (metis (lifting) literal.sel(1,2) mem-Collect-eq)

lemma true-cls-remove-hd-if-notin-vars:
  assumes insert a M'⊨ D
  and atm-of a ∉ atms-of D
  shows M' ⊨ D
  using assms by (auto simp add: atm-of-lit-in-atms-of true-cls-def)

lemma total-over-set-atm-of:
  fixes I :: 'v interp and K :: 'v set
  shows total-over-set I K ⟷ (∀ l ∈ K. l ∈ (atm-of ‘ I))
  unfolding total-over-set-def by (metis atms-of-s-def in-atms-of-s-decomp)
```

11.2.7 Tautologies

```
definition tautology (ψ:: 'v clause) ≡ ∀ I. total-over-set I (atms-of ψ) ⟶ I ⊨ ψ

lemma tautology-Pos-Neg[intro]:
  assumes Pos p ∈# A and Neg p ∈# A
  shows tautology A
  using assms unfolding tautology-def total-over-set-def true-cls-def Bex-def
  by (meson atm-iff-pos-or-neg-lit true-lit-def)

lemma tautology-minus[simp]:
  assumes L ∈# A and −L ∈# A
  shows  tautology A
  by (metis assms literal.exhaust tautology-Pos-Neg uminus-Neg uminus-Pos)

lemma tautology-exists-Pos-Neg:
  assumes tautology ψ
  shows ∃p. Pos p ∈# ψ ∧ Neg p ∈# ψ
proof (rule ccontr)
  assume p: ¬ (∃p. Pos p ∈# ψ ∧ Neg p ∈# ψ)
  let ?I = {−L | L. L ∈# ψ}
  have total-over-set ?I (atms-of ψ)
    unfolding total-over-set-def using atm-imp-pos-or-neg-lit by force
  moreover have ¬ ?I ⊨ ψ
    unfolding true-cls-def true-lit-def Bex-def apply clarify
    using p by (rename-tac x L, case-tac L) fastforce+
  ultimately show False using assms unfolding tautology-def by auto
qed

lemma tautology-decomp:
  tautology ψ ⟷ (∃p. Pos p ∈# ψ ∧ Neg p ∈# ψ)
  using tautology-exists-Pos-Neg by auto

lemma tautology-false[simp]: ¬tautology {#}
  unfolding tautology-def by auto

lemma tautology-add-single:
  tautology ({#a#} + L) ⟷ tautology L ∨ −a ∈# L
  unfolding tautology-decomp by (cases a) auto

lemma minus-interp-tautology:
  assumes {−L | L. L∈# χ} ⊨ χ
  shows tautology χ
proof −
  obtain L where L ∈# χ ∧ −L ∈# χ
    using assms unfolding true-cls-def by auto
  then show ?thesis using tautology-decomp literal.exhaust uminus-Neg uminus-Pos by metis
qed

lemma remove-literal-in-model-tautology:
  assumes I ∪ {Pos P} ⊨ φ
  and I ∪ {Neg P} ⊨ φ
  shows I ⊨ φ ∨ tautology φ
  using assms unfolding true-cls-def by auto

lemma tautology-imp-tautology:
  fixes χ χ' :: 'v clause
  assumes ∀I. total-over-m I {χ} ⟶ I ⊨ χ ⟶ I ⊨ χ' and tautology χ
  shows tautology χ' unfolding tautology-def
proof (intro allI HOL.impI)
  fix I ::'v literal set
  assume totI: total-over-set I (atms-of χ')
  let ?I' = {Pos v |v. v∈ atms-of χ ∧ v ∉ atms-of-s I}
  have totI': total-over-m (I ∪ ?I') {χ} unfolding total-over-m-def total-over-set-def by auto
  then have χ: I ∪ ?I' ⊨ χ using assms(2) unfolding total-over-m-def tautology-def by simp
  then have I ∪ (?I'− I) ⊨ χ' using assms(1) totI' by auto
  moreover have ⋀L. L ∈# χ' ⟹ L ∉ ?I'
    using totI unfolding total-over-set-def by (auto dest: pos-lit-in-atms-of)
  ultimately show I ⊨ χ' unfolding true-cls-def by auto
qed
```

11.2.8 Entailment for clauses and propositions

definition *true-cls-cls* :: *'a clause* ⇒ *'a clause* ⇒ *bool* (**infix** ⊨*f* *49*) **where**
$\psi \models\!f\ \chi \longleftrightarrow (\forall I.\ \textit{total-over-m}\ I\ (\{\psi\} \cup \{\chi\}) \longrightarrow \textit{consistent-interp}\ I \longrightarrow I \models \psi \longrightarrow I \models \chi)$

definition *true-cls-clss* :: *'a clause* ⇒ *'a clauses* ⇒ *bool* (**infix** ⊨*fs* *49*) **where**
$\psi \models\!fs\ \chi \longleftrightarrow (\forall I.\ \textit{total-over-m}\ I\ (\{\psi\} \cup \chi) \longrightarrow \textit{consistent-interp}\ I \longrightarrow I \models \psi \longrightarrow I \models\!s\ \chi)$

definition *true-clss-cls* :: *'a clauses* ⇒ *'a clause* ⇒ *bool* (**infix** ⊨*p* *49*) **where**
$N \models\!p\ \chi \longleftrightarrow (\forall I.\ \textit{total-over-m}\ I\ (N \cup \{\chi\}) \longrightarrow \textit{consistent-interp}\ I \longrightarrow I \models\!s\ N \longrightarrow I \models \chi)$

definition *true-clss-clss* :: *'a clauses* ⇒ *'a clauses* ⇒ *bool* (**infix** ⊨*ps* *49*) **where**
$N \models\!ps\ N' \longleftrightarrow (\forall I.\ \textit{total-over-m}\ I\ (N \cup N') \longrightarrow \textit{consistent-interp}\ I \longrightarrow I \models\!s\ N \longrightarrow I \models\!s\ N')$

lemma *true-cls-cls-refl*[*simp*]:
$A \models\!f\ A$
unfolding *true-cls-cls-def* **by** *auto*

lemma *true-cls-cls-insert-l*[*simp*]:
$a \models\!f\ C \Longrightarrow \textit{insert}\ a\ A \models\!p\ C$
unfolding *true-cls-cls-def true-clss-cls-def true-clss-def* **by** *fastforce*

lemma *true-cls-clss-empty*[*iff*]:
$N \models\!fs\ \{\}$
unfolding *true-cls-clss-def* **by** *auto*

lemma *true-prop-true-clause*[*iff*]:
$\{\varphi\} \models\!p\ \psi \longleftrightarrow \varphi \models\!f\ \psi$
unfolding *true-cls-cls-def true-clss-cls-def* **by** *auto*

lemma *true-clss-clss-true-clss-cls*[*iff*]:
$N \models\!ps\ \{\psi\} \longleftrightarrow N \models\!p\ \psi$
unfolding *true-clss-clss-def true-clss-cls-def* **by** *auto*

lemma *true-clss-clss-true-cls-clss*[*iff*]:
$\{\chi\} \models\!ps\ \psi \longleftrightarrow \chi \models\!fs\ \psi$
unfolding *true-clss-clss-def true-cls-clss-def* **by** *auto*

lemma *true-clss-clss-empty*[*simp*]:
$N \models\!ps\ \{\}$
unfolding *true-clss-clss-def* **by** *auto*

lemma *true-clss-cls-subset*:
$A \subseteq B \Longrightarrow A \models\!p\ CC \Longrightarrow B \models\!p\ CC$
unfolding *true-clss-cls-def total-over-m-union* **by** (*simp add*: *total-over-m-subset true-clss-mono*)

lemma *true-clss-cs-mono-l*[*simp*]:
$A \models\!p\ CC \Longrightarrow A \cup B\ \models\!p\ CC$
by (*auto intro*: *true-clss-cls-subset*)

lemma *true-clss-cs-mono-l2*[*simp*]:
$B \models\!p\ CC \Longrightarrow A \cup B\ \models\!p\ CC$
by (*auto intro*: *true-clss-cls-subset*)

lemma *true-clss-cls-mono-r*[*simp*]:
$A \models\!p\ CC \Longrightarrow A \models\!p\ CC + CC'$

```
  unfolding true-clss-cls-def total-over-m-union total-over-m-sum by blast

lemma true-clss-cls-mono-r'[simp]:
  A ⊨p CC' ⟹ A ⊨p CC + CC'
  unfolding true-clss-cls-def total-over-m-union total-over-m-sum by blast

lemma true-clss-clss-union-l[simp]:
  A ⊨ps CC ⟹ A ∪ B  ⊨ps CC
  unfolding true-clss-clss-def total-over-m-union by fastforce

lemma true-clss-clss-union-l-r[simp]:
  B ⊨ps CC ⟹ A ∪ B ⊨ps CC
  unfolding true-clss-clss-def total-over-m-union by fastforce

lemma true-clss-cls-in[simp]:
  CC ∈ A ⟹ A ⊨p CC
  unfolding true-clss-cls-def true-clss-def total-over-m-union by fastforce

lemma true-clss-cls-insert-l[simp]:
  A ⊨p C ⟹ insert a A ⊨p C
  unfolding true-clss-cls-def true-clss-def using total-over-m-union
  by (metis Un-iff insert-is-Un sup.commute)

lemma true-clss-clss-insert-l[simp]:
  A ⊨ps C ⟹ insert a A ⊨ps C
  unfolding true-clss-cls-def true-clss-clss-def true-clss-def by blast

lemma true-clss-clss-union-and[iff]:
  A ⊨ps C ∪ D ⟷ (A ⊨ps C ∧ A ⊨ps D)
proof
  {
    fix A C D :: 'a clauses
    assume A: A ⊨ps C ∪ D
    have A ⊨ps C
        unfolding true-clss-clss-def true-clss-cls-def insert-def total-over-m-insert
      proof (intro allI impI)
        fix I
        assume
          totAC: total-over-m I (A ∪ C) and
          cons: consistent-interp I and
          I: I ⊨s A
        then have  tot: total-over-m I A and tot': total-over-m I  C by auto
        obtain I' where
          tot': total-over-m (I ∪ I') (A ∪ C ∪ D) and
          cons': consistent-interp (I ∪ I') and
          H: ∀ x∈I'. atm-of x ∈ atms-of-ms D ∧ atm-of x ∉ atms-of-ms (A ∪ C)
          using total-over-m-consistent-extension[OF - cons, of A ∪ C] tot tot' by blast
        moreover have I ∪ I' ⊨s A using I by simp
        ultimately have I ∪ I' ⊨s C ∪ D using A unfolding true-clss-clss-def  by auto
        then have I ∪ I' ⊨s C ∪ D by auto
        then show I ⊨s C using notin-vars-union-true-clss-true-clss[of I'] H by auto
      qed
  } note H = this
  assume A ⊨ps C ∪ D
  then show A ⊨ps C ∧ A ⊨ps D using H[of A] Un-commute[of C D] by metis
```

```
next
  assume A ⊨ps C ∧ A ⊨ps D
  then show A ⊨ps C ∪ D
    unfolding true-clss-clss-def by auto
qed

lemma true-clss-clss-insert[iff]:
  A ⊨ps insert L Ls ⟷ (A ⊨p L ∧ A ⊨ps Ls)
  using true-clss-clss-union-and[of A {L} Ls] by auto

lemma true-clss-clss-subset:
  A ⊆ B ⟹ A ⊨ps CC ⟹ B ⊨ps CC
  by (metis subset-Un-eq true-clss-clss-union-l)

lemma union-trus-clss-clss[simp]: A ∪ B ⊨ps B
  unfolding true-clss-clss-def by auto

lemma true-clss-clss-remove[simp]:
  A ⊨ps B ⟹ A⊨ps B − C
  by (metis Un-Diff-Int true-clss-clss-union-and)

lemma true-clss-clss-subsetE:
  N ⊨ps B ⟹ A ⊆ B ⟹ N ⊨ps A
  by (metis sup.orderE true-clss-clss-union-and)

lemma true-clss-clss-in-imp-true-clss-cls:
  assumes N ⊨ps U
  and A ∈ U
  shows N ⊨p A
  using assms mk-disjoint-insert by fastforce

lemma all-in-true-clss-clss: ∀ x ∈ B. x ∈ A ⟹ A ⊨ps B
  unfolding true-clss-clss-def true-clss-def by auto

lemma true-clss-clss-left-right:
  assumes A ⊨ps B
  and A ∪ B ⊨ps M
  shows A ⊨ps M ∪ B
  using assms unfolding true-clss-clss-def by auto

lemma true-clss-clss-generalise-true-clss-clss:
  A ∪ C ⊨ps D ⟹ B ⊨ps C ⟹ A ∪ B ⊨ps D
proof −
  assume a1: A ∪ C ⊨ps D
  assume B ⊨ps C
  then have f2: ⋀M. M ∪ B ⊨ps C
    by (meson true-clss-clss-union-l-r)
  have ⋀M. C ∪ (M ∪ A) ⊨ps D
    using a1 by (simp add: Un-commute sup-left-commute)
  then show ?thesis
    using f2 by (metis (no-types) Un-commute true-clss-clss-left-right true-clss-clss-union-and)
qed

lemma true-clss-cls-or-true-clss-cls-or-not-true-clss-cls-or:
  assumes D: N ⊨p D + {#− L#}
  and C: N ⊨p C + {#L#}
  shows N ⊨p D + C
  unfolding true-clss-cls-def
proof (intro allI impI)
  fix I
  assume
    tot: total-over-m I (N ∪ {D + C}) and
    consistent-interp I and
    I ⊨s N
  {
    assume L: L ∈ I ∨ −L ∈ I
    then have total-over-m I {D + {#− L#}}
      using tot by (cases L) auto
    then have I ⊨ D + {#− L#} using D ‹I ⊨s N› tot ‹consistent-interp I›
      unfolding true-clss-cls-def by auto
    moreover
      have total-over-m I {C + {#L#}}
        using L tot by (cases L) auto
      then have I ⊨ C + {#L#}
        using C ‹I ⊨s N› tot ‹consistent-interp I› unfolding true-clss-cls-def by auto
    ultimately have I ⊨ D + C using ‹consistent-interp I› consistent-interp-def by fastforce
  }
  moreover {
    assume L: L ∉ I ∧ −L ∉ I
    let ?I' = I ∪ {L}
    have consistent-interp ?I' using L ‹consistent-interp I› by auto
    moreover have total-over-m ?I' {D + {#− L#}}
      using tot unfolding total-over-m-def total-over-set-def by (auto simp add: atms-of-def)
    moreover have total-over-m ?I' N using tot using total-union by blast
    moreover have ?I' ⊨s N using ‹I ⊨s N› using true-clss-union-increase by blast
    ultimately have ?I' ⊨ D + {#− L#}
      using D unfolding true-clss-cls-def by blast
    then have ?I' ⊨ D using L by auto
    moreover
      have total-over-set I (atms-of (D + C)) using tot by auto
      then have L ∉# D ∧ −L ∉# D
        using L unfolding total-over-set-def atms-of-def by (cases L) force+
    ultimately have I ⊨ D + C unfolding true-cls-def by auto
  }
  ultimately show I ⊨ D + C by blast
qed

lemma true-cls-union-mset[iff]: I ⊨ C #∪ D ⟷ I ⊨ C ∨ I ⊨ D
  unfolding true-cls-def by force

lemma true-clss-cls-union-mset-true-clss-cls-or-not-true-clss-cls-or:
  assumes
    D: N ⊨p D + {#− L#} and
    C: N ⊨p C + {#L#}
  shows N ⊨p D #∪ C
  unfolding true-clss-cls-def
proof (intro allI impI)
  fix I
  assume
    tot: total-over-m I (N ∪ {D #∪ C}) and
    consistent-interp I and
    I ⊨s N
  {
    assume L: L ∈ I ∨ −L ∈ I
    then have total-over-m I {D + {#− L#}}
      using tot by (cases L) auto
    then have I ⊨ D + {#− L#}
      using D ‹I ⊨s N› tot ‹consistent-interp I› unfolding true-clss-cls-def by auto
    moreover
      have total-over-m I {C + {#L#}}
        using L tot by (cases L) auto
      then have I ⊨ C + {#L#}
        using C ‹I ⊨s N› tot ‹consistent-interp I› unfolding true-clss-cls-def by auto
    ultimately have I ⊨ D #∪ C using ‹consistent-interp I› unfolding consistent-interp-def
    by auto
  }
  moreover {
    assume L: L ∉ I ∧ −L ∉ I
    let ?I′ = I ∪ {L}
    have consistent-interp ?I′ using L ‹consistent-interp I› by auto
    moreover have total-over-m ?I′ {D + {#− L#}}
      using tot unfolding total-over-m-def total-over-set-def by (auto simp add: atms-of-def)
    moreover have total-over-m ?I′ N using tot using total-union by blast
    moreover have ?I′ ⊨s N using ‹I ⊨s N› using true-clss-union-increase by blast
    ultimately have ?I′ ⊨ D + {#− L#}
      using D unfolding true-clss-cls-def by blast
    then have ?I′ ⊨ D using L by auto
    moreover
      have total-over-set I (atms-of (D + C)) using tot by auto
      then have L ∉# D ∧ −L ∉# D
        using L unfolding total-over-set-def atms-of-def by (cases L) force+
    ultimately have I ⊨ D  #∪ C unfolding true-cls-def by auto
  }
  ultimately show I ⊨ D #∪ C by blast
qed

lemma satisfiable-carac[iff]:
  (∃ I. consistent-interp I ∧ I ⊨s φ) ⟷ satisfiable φ (is (∃ I. ?Q I) ⟷ ?S)
proof
  assume ?S
  then show ∃ I. ?Q I unfolding satisfiable-def by auto
next
  assume ∃ I. ?Q I
  then obtain I where cons: consistent-interp I and I: I ⊨s φ by metis
  let ?I′ = {Pos v |v. v ∉ atms-of-s I ∧ v ∈ atms-of-ms φ}
  have consistent-interp (I ∪ ?I′)
    using cons unfolding consistent-interp-def by (intro allI) (rename-tac L, case-tac L, auto)
  moreover have total-over-m (I ∪ ?I′) φ
    unfolding total-over-m-def total-over-set-def by auto
  moreover have I ∪ ?I′ ⊨s φ
    using I unfolding Ball-def true-clss-def true-cls-def by auto
  ultimately show ?S unfolding satisfiable-def by blast
qed

lemma satisfiable-carac′[simp]: consistent-interp I ⟹ I ⊨s φ ⟹ satisfiable φ
```

```
  using satisfiable-carac by metis
```

11.3 Subsumptions

```
lemma subsumption-total-over-m:
  assumes A ⊆# B
  shows total-over-m I {B} ⟹ total-over-m I {A}
  using assms unfolding subset-mset-def total-over-m-def total-over-set-def
  by (auto simp add: mset-le-exists-conv)

lemma atms-of-replicate-mset-replicate-mset-uminus[simp]:
  atms-of (D − replicate-mset (count D L) L  − replicate-mset (count D (−L)) (−L))
    = atms-of D − {atm-of L}
  by (fastforce simp: atm-of-eq-atm-of atms-of-def)

lemma subsumption-chained:
  assumes
    ∀ I. total-over-m I {D} ⟶ I ⊨ D ⟶ I ⊨ φ and
    C ⊆# D
  shows (∀ I. total-over-m I {C} ⟶ I ⊨ C ⟶ I ⊨ φ) ∨ tautology φ
  using assms
proof (induct card {Pos v | v. v ∈ atms-of D ∧ v ∉ atms-of C} arbitrary: D
    rule: nat-less-induct-case)
  case 0 note n = this(1) and  H = this(2) and incl = this(3)
  then have atms-of D ⊆ atms-of C by auto
  then have ∀ I. total-over-m I {C} ⟶ total-over-m I {D}
    unfolding total-over-m-def total-over-set-def by auto
  moreover have ∀ I. I ⊨ C ⟶ I ⊨ D using incl true-cls-mono-leD by blast
  ultimately show ?case using H by auto
next
  case (Suc n D) note IH = this(1) and card = this(2) and H = this(3) and incl = this(4)
  let ?atms = {Pos v |v. v ∈ atms-of D ∧ v ∉ atms-of C}
  have finite ?atms by auto
  then obtain L where  L: L ∈ ?atms
    using card by (metis (no-types, lifting) Collect-empty-eq card-0-eq mem-Collect-eq
      nat.simps(3))
  let ?D' = D − replicate-mset (count D L) L − replicate-mset (count D (−L)) (−L)
  have atms-of-D: atms-of-ms {D} ⊆ atms-of-ms {?D'} ∪ {atm-of L} by auto

  {
    fix I
    assume total-over-m I {?D'}
    then have tot: total-over-m (I ∪ {L}) {D}
      unfolding total-over-m-def total-over-set-def using atms-of-D by auto

    assume IDL: I ⊨ ?D'
    then have I ∪ {L} ⊨ D unfolding true-cls-def by force
    then have I ∪ {L} ⊨ φ using H tot by auto

    moreover
      have tot': total-over-m (I ∪ {−L}) {D}
        using tot unfolding total-over-m-def total-over-set-def by auto
      have I ∪ {−L} ⊨ D using IDL unfolding true-cls-def by force
      then have I ∪ {−L} ⊨ φ using H tot' by auto
    ultimately have I ⊨ φ ∨ tautology φ
      using L remove-literal-in-model-tautology by force
  } note H′ = this

  have L ∉# C  and −L ∉# C using L atm-iff-pos-or-neg-lit by force+
  then have C-in-D′: C ⊆# ?D′ using ‹C ⊆# D› by (auto simp: subseteq-mset-def not-in-iff)
  have card {Pos v |v. v ∈ atms-of ?D′ ∧ v ∉ atms-of C} <
    card {Pos v |v. v ∈ atms-of D ∧ v ∉ atms-of C}
    using L by (auto intro!: psubset-card-mono)
  then show ?case
    using IH C-in-D′ H′  unfolding card[symmetric] by blast
qed
```

11.4 Removing Duplicates

```
lemma tautology-remdups-mset[iff]:
  tautology (remdups-mset C) ⟷ tautology C
  unfolding tautology-decomp by auto

lemma atms-of-remdups-mset[simp]: atms-of (remdups-mset C) = atms-of C
  unfolding atms-of-def by auto

lemma true-cls-remdups-mset[iff]: I ⊨ remdups-mset C ⟷ I ⊨ C
  unfolding true-cls-def by auto

lemma true-clss-cls-remdups-mset[iff]: A ⊨p remdups-mset C ⟷ A ⊨p C
  unfolding true-clss-cls-def total-over-m-def by auto
```

11.5 Set of all Simple Clauses

```
definition simple-clss :: ′v set ⇒ ′v clause set where
simple-clss atms = {C. atms-of C ⊆ atms ∧ ¬tautology C ∧ distinct-mset C}

lemma simple-clss-empty[simp]:
  simple-clss {} = {{#}}
  unfolding simple-clss-def by auto

lemma simple-clss-insert:
  assumes l ∉ atms
  shows simple-clss (insert l atms) =
    (op + {#Pos l#}) ‘ (simple-clss atms)
    ∪ (op + {#Neg l#}) ‘ (simple-clss atms)
    ∪ simple-clss atms(is ?I = ?U)
proof (standard; standard)
  fix C
  assume C ∈ ?I
  then have
    atms: atms-of C ⊆ insert l atms and
    taut: ¬tautology C and
    dist: distinct-mset C
    unfolding simple-clss-def by auto
  have H: ⋀x. x ∈# C ⟹ atm-of x ∈ insert l atms
    using atm-of-lit-in-atms-of atms by blast
  consider
      (Add) L where L ∈# C and L = Neg l ∨ L = Pos l
    | (No)  Pos l ∉# C  Neg l ∉# C
    by auto
  then show C ∈ ?U
```

```
    proof cases
      case Add
      then have L ∉# C − {#L#}
        using dist unfolding distinct-mset-def by (auto simp: not-in-iff)
      moreover have −L ∉# C
        using taut Add by auto
      ultimately have atms-of (C − {#L#}) ⊆ atms
        using atms Add by (smt H atms-of-def imageE in-diffD insertE literal.exhaust-sel
          subset-iff uminus-Neg uminus-Pos)

      moreover have ¬ tautology (C − {#L#})
        using taut by (metis Add(1) insert-DiffM tautology-add-single)
      moreover have distinct-mset (C − {#L#})
        using dist by auto
      ultimately have (C − {#L#}) ∈ simple-clss atms
        using Add unfolding simple-clss-def by auto
      moreover have C = {#L#} + (C − {#L#})
        using Add by (auto simp: multiset-eq-iff)
      ultimately show ?thesis using Add by auto
    next
      case No
      then have C ∈ simple-clss atms
        using taut atms dist unfolding simple-clss-def
        by (auto simp: atm-iff-pos-or-neg-lit split: if-split-asm dest!: H)
      then show ?thesis by blast
    qed
next
  fix C
  assume C ∈ ?U
  then consider
      (Add) L C′ where C = {#L#} + C′ and C′ ∈ simple-clss atms and
        L = Pos l ∨ L = Neg l
    | (No) C ∈ simple-clss atms
    by auto
  then show C ∈ ?I
    proof cases
      case No
      then show ?thesis unfolding simple-clss-def by auto
    next
      case (Add L C′) note C′ = this(1) and C = this(2) and L = this(3)
      then have
        atms: atms-of C′ ⊆ atms and
        taut: ¬tautology C′ and
        dist: distinct-mset C′
        unfolding simple-clss-def by auto
      have atms-of C ⊆ insert l atms
        using atms C′ L by auto
      moreover have ¬ tautology C
        using taut C′ L by (metis assms atm-of-lit-in-atms-of atms literal.sel(1,2) subset-eq
          tautology-add-single uminus-Neg uminus-Pos)
      moreover have distinct-mset C
        using dist C′ L
        by (metis assms atm-of-lit-in-atms-of atms contra-subsetD distinct-mset-add-single
          literal.sel(1,2))
      ultimately show ?thesis unfolding simple-clss-def by blast
    qed
qed

lemma simple-clss-finite:
  fixes atms :: 'v set
  assumes finite atms
  shows finite (simple-clss atms)
  using assms by (induction rule: finite-induct) (auto simp: simple-clss-insert)

lemma simple-clssE:
  assumes
    x ∈ simple-clss atms
  shows atms-of x ⊆ atms ∧ ¬tautology x ∧ distinct-mset x
  using assms unfolding simple-clss-def by auto

lemma cls-in-simple-clss:
  shows {#} ∈ simple-clss s
  unfolding simple-clss-def by auto

lemma simple-clss-card:
  fixes atms :: 'v  set
  assumes finite atms
  shows card (simple-clss atms) ≤ (3::nat) ^ (card atms)
  using assms
proof (induct atms rule: finite-induct)
  case empty
  then show ?case by auto
next
  case (insert l C) note fin = this(1) and l = this(2) and IH = this(3)
  have notin:
    ⋀C'. {#Pos l#} + C' ∉ simple-clss C
    ⋀C'. {#Neg l#} + C' ∉ simple-clss C
    using l unfolding simple-clss-def by auto
  have H: ⋀C' D. {#Pos l#} + C' = {#Neg l#} + D ⟹ D ∈ simple-clss C ⟹ False
    proof −
      fix C' D
      assume C'D: {#Pos l#} + C' = {#Neg l#} + D and D: D ∈ simple-clss C
      then have Pos l ∈# D by (metis insert-noteq-member literal.distinct(1) union-commute)
      then have l ∈ atms-of D
        by (simp add: atm-iff-pos-or-neg-lit)
      then show False using D l unfolding simple-clss-def by auto
    qed
  let ?P = (op + {#Pos l#}) ‘ (simple-clss C)
  let ?N = (op + {#Neg l#}) ‘ (simple-clss C)
  let ?O = simple-clss C
  have card (?P ∪ ?N ∪ ?O) = card (?P ∪ ?N) + card ?O
    apply (subst card-Un-disjoint)
    using l fin by (auto simp: simple-clss-finite notin)
  moreover have card (?P ∪ ?N) = card ?P + card ?N
    apply (subst card-Un-disjoint)
    using l fin H by (auto simp: simple-clss-finite notin)
  moreover
    have card ?P = card ?O
      using inj-on-iff-eq-card[of ?O op + {#Pos l#}]
      by (auto simp: fin simple-clss-finite inj-on-def)
  moreover have card ?N = card ?O
    using inj-on-iff-eq-card[of ?O op + {#Neg l#}]
    by (auto simp: fin simple-clss-finite inj-on-def)
  moreover have (3::nat) ^ card (insert l C) = 3 ^ (card C) + 3 ^ (card C) + 3 ^ (card C)
   using l by (simp add: fin mult-2-right numeral-3-eq-3)
  ultimately show ?case using IH l by (auto simp: simple-clss-insert)
qed

lemma simple-clss-mono:
  assumes incl: atms ⊆ atms'
  shows simple-clss atms ⊆ simple-clss atms'
  using assms unfolding simple-clss-def by auto

lemma distinct-mset-not-tautology-implies-in-simple-clss:
  assumes distinct-mset χ and ¬tautology χ
  shows χ ∈ simple-clss (atms-of χ)
  using assms unfolding simple-clss-def by auto

lemma simplified-in-simple-clss:
  assumes distinct-mset-set ψ and ∀ χ ∈ ψ. ¬tautology χ
  shows ψ ⊆ simple-clss (atms-of-ms ψ)
  using assms unfolding simple-clss-def
  by (auto simp: distinct-mset-set-def atms-of-ms-def)
```

11.6 Experiment: Expressing the Entailments as Locales

```
locale entail =
  fixes entail :: 'a set ⇒ 'b ⇒ bool (infix ⊨e 50)
  assumes entail-insert[simp]: I ≠ {} ⟹ insert L I ⊨e x ⟷ {L} ⊨e x ∨ I ⊨e x
  assumes entail-union[simp]: I ⊨e A ⟹ I ∪ I' ⊨e A
begin

definition entails :: 'a set ⇒ 'b set ⇒ bool (infix ⊨es 50) where
  I ⊨es A ⟷ (∀ a ∈ A. I ⊨e a)

lemma entails-empty[simp]:
  I ⊨es {}
  unfolding entails-def by auto

lemma entails-single[iff]:
  I ⊨es {a} ⟷ I ⊨e a
  unfolding entails-def by auto

lemma entails-insert-l[simp]:
  M ⊨es A ⟹ insert L M ⊨es A
  unfolding entails-def by (metis Un-commute entail-union insert-is-Un)

lemma entails-union[iff]: I ⊨es CC ∪ DD ⟷ I ⊨es CC ∧ I ⊨es DD
  unfolding entails-def by blast

lemma entails-insert[iff]: I ⊨es insert C DD ⟷ I ⊨e C ∧ I ⊨es DD
  unfolding entails-def by blast

lemma entails-insert-mono: DD ⊆ CC ⟹ I ⊨es CC ⟹ I ⊨es DD
  unfolding entails-def by blast
```

```
lemma entails-union-increase[simp]:
 assumes I ⊨es ψ
 shows I ∪ I′ ⊨es ψ
 using assms unfolding entails-def by auto

lemma true-clss-commute-l:
  (I ∪ I′ ⊨es ψ) ⟷ (I′ ∪ I ⊨es ψ)
  by (simp add: Un-commute)

lemma entails-remove[simp]: I ⊨es N ⟹ I ⊨es Set.remove a N
  by (simp add: entails-def)

lemma entails-remove-minus[simp]: I ⊨es N ⟹ I ⊨es N − A
  by (simp add: entails-def)

end

interpretation true-cls: entail true-cls
  by standard (auto simp add: true-cls-def)
```

11.7 Entailment to be extended

```
definition true-clss-ext :: 'a literal set ⇒ 'a literal multiset set ⇒ bool (infix ⊨sext 49)
where
I ⊨sext N ⟷ (∀ J. I ⊆ J ⟶ consistent-interp J ⟶ total-over-m J N ⟶ J ⊨s N)

lemma true-clss-imp-true-cls-ext:
  I⊨s N ⟹ I ⊨sext N
  unfolding true-clss-ext-def by (metis sup.orderE true-clss-union-increase′)

lemma true-clss-ext-decrease-right-remove-r:
  assumes I ⊨sext N
  shows I ⊨sext N − {C}
  unfolding true-clss-ext-def
proof (intro allI impI)
  fix J
  assume
    I ⊆ J and
    cons: consistent-interp J and
    tot: total-over-m J (N − {C})
  let ?J = J ∪ {Pos (atm-of P)|P. P ∈# C ∧ atm-of P ∉ atm-of ‘ J}
  have I ⊆ ?J using ‹I ⊆ J› by auto
  moreover have consistent-interp ?J
    using cons unfolding consistent-interp-def apply (intro allI)
    by (rename-tac L, case-tac L) (fastforce simp add: image-iff)+
  moreover have total-over-m ?J N
    using tot unfolding total-over-m-def total-over-set-def atms-of-ms-def
    apply clarify
    apply (rename-tac l a, case-tac a ∈ N − {C})
      apply auto[]
    using atms-of-s-def atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
    by (fastforce simp: atms-of-def)
  ultimately have ?J ⊨s N
    using assms unfolding true-clss-ext-def by blast
  then have ?J ⊨s N − {C} by auto
  have {v ∈ ?J. atm-of v ∈ atms-of-ms (N − {C})} ⊆ J
```

```
    using tot unfolding total-over-m-def total-over-set-def
    by (auto intro!: rev-image-eqI)
  then show J ⊨s N − {C}
    using true-clss-remove-unused[OF ‹?J ⊨s N − {C}›] unfolding true-clss-def
    by (meson true-cls-mono-set-mset-l)
qed

lemma consistent-true-clss-ext-satisfiable:
  assumes consistent-interp I and I ⊨sext A
  shows satisfiable A
  by (metis Un-empty-left assms satisfiable-carac subset-Un-eq sup.left-idem
    total-over-m-consistent-extension total-over-m-empty true-clss-ext-def)

lemma not-consistent-true-clss-ext:
  assumes ¬consistent-interp I
  shows I⊨sext A
  by (meson assms consistent-interp-subset true-clss-ext-def)
end
theory Prop-Logic-Multiset
imports ../lib/Multiset-More Prop-Normalisation Partial-Clausal-Logic
begin
```

12 Link with Multiset Version

12.1 Transformation to Multiset

```
fun mset-of-conj :: 'a propo ⇒ 'a literal multiset where
mset-of-conj (FOr φ ψ) = mset-of-conj φ + mset-of-conj ψ |
mset-of-conj (FVar v) = {# Pos v #} |
mset-of-conj (FNot (FVar v)) = {# Neg v #} |
mset-of-conj FF = {#}

fun mset-of-formula :: 'a propo ⇒ 'a literal multiset set where
mset-of-formula (FAnd φ ψ) = mset-of-formula φ ∪ mset-of-formula ψ |
mset-of-formula (FOr φ ψ) = {mset-of-conj (FOr φ ψ)} |
mset-of-formula (FVar ψ) = {mset-of-conj (FVar ψ)} |
mset-of-formula (FNot ψ) = {mset-of-conj (FNot ψ)} |
mset-of-formula FF = {{#}} |
mset-of-formula FT = {}
```

12.2 Equisatisfiability of the two Version

```
lemma is-conj-with-TF-FNot:
  is-conj-with-TF (FNot φ) ⟷ (∃ v. φ = FVar v ∨ φ = FF ∨ φ = FT)
  unfolding is-conj-with-TF-def apply (rule iffI)
  apply (induction FNot φ rule: super-grouped-by.induct)
  apply (induction FNot φ rule: grouped-by.induct)
     apply simp
    apply (cases φ; simp)
  apply auto
  done

lemma grouped-by-COr-FNot:
  grouped-by COr (FNot φ) ⟷ (∃ v. φ = FVar v ∨ φ = FF ∨ φ = FT)
  unfolding is-conj-with-TF-def apply (rule iffI)
  apply (induction FNot φ rule: grouped-by.induct)
     apply simp
    apply (cases φ; simp)
  apply auto
  done

lemma
  shows no-T-F-FF[simp]: ¬no-T-F FF and
    no-T-F-FT[simp]: ¬no-T-F FT
  unfolding no-T-F-def all-subformula-st-def by auto

lemma grouped-by-CAnd-FAnd:
  grouped-by CAnd (FAnd φ1 φ2) ⟷ grouped-by CAnd φ1 ∧ grouped-by CAnd φ2
  apply (rule iffI)
  apply (induction FAnd φ1 φ2 rule: grouped-by.induct)
  using connected-is-group[of CAnd φ1 φ2] by auto

lemma grouped-by-COr-FOr:
  grouped-by COr (FOr φ1 φ2) ⟷ grouped-by COr φ1 ∧ grouped-by COr φ2
  apply (rule iffI)
  apply (induction FOr φ1 φ2 rule: grouped-by.induct)
  using connected-is-group[of COr φ1 φ2] by auto


lemma grouped-by-COr-FAnd[simp]: ¬ grouped-by COr (FAnd φ1 φ2)
  apply clarify
   apply (induction FAnd φ1 φ2 rule: grouped-by.induct)
   apply auto
  done

lemma grouped-by-COr-FEq[simp]: ¬ grouped-by COr (FEq φ1 φ2)
  apply clarify
   apply (induction FEq φ1 φ2 rule: grouped-by.induct)
   apply auto
  done

lemma [simp]: ¬grouped-by COr (FImp φ ψ)
  apply clarify
  by (induction FImp φ ψ rule: grouped-by.induct) simp-all

lemma [simp]: ¬ is-conj-with-TF (FImp φ ψ)
  unfolding is-conj-with-TF-def apply clarify
  by (induction FImp φ ψ rule: super-grouped-by.induct) simp-all

lemma [simp]: ¬grouped-by COr (FEq φ ψ)
  apply clarify
  by (induction FEq φ ψ rule: grouped-by.induct) simp-all

lemma [simp]: ¬ is-conj-with-TF (FEq φ ψ)
  unfolding is-conj-with-TF-def apply clarify
  by (induction FEq φ ψ rule: super-grouped-by.induct) simp-all

lemma is-conj-with-TF-Fand:
  is-conj-with-TF (FAnd φ1 φ2) ⟹ is-conj-with-TF φ1 ∧ is-conj-with-TF φ2
  unfolding is-conj-with-TF-def
  apply (induction FAnd φ1 φ2 rule: super-grouped-by.induct)
   apply (auto simp: grouped-by-CAnd-FAnd intro: grouped-is-super-grouped)[]
  apply auto[]
  done

lemma is-conj-with-TF-FOr:
  is-conj-with-TF (FOr φ1 φ2) ⟹ grouped-by COr φ1 ∧ grouped-by COr φ2
  unfolding is-conj-with-TF-def
  apply (induction FOr φ1 φ2 rule: super-grouped-by.induct)
   apply (auto simp: grouped-by-COr-FOr)[]
  apply auto[]
  done

lemma grouped-by-COr-mset-of-formula:
  grouped-by COr φ ⟹ mset-of-formula φ = (if φ = FT then {} else {mset-of-conj φ})
  by (induction φ) (auto simp add: grouped-by-COr-FNot)
```

When a formula is in CNF form, then there is equisatisfiability. Remark that the definition for the entailment are slightly different: *op* ⊨ uses a function assigning *True* or *False*, while *op* ⊨*s* uses a set where being in the list means entailment of a literal.

```
theorem
  fixes φ :: 'v propo
  assumes is-cnf φ
  shows eval A φ ⟷ Partial-Clausal-Logic.true-clss ({Pos v|v. A v} ∪ {Neg v|v. ¬A v})
    (mset-of-formula φ)
  using assms
proof (induction φ)
  case FF
  then show ?case by auto
next
  case FT
  then show ?case by auto
next
  case (FVar v)
  then show ?case by auto
next
  case (FAnd φ ψ)
  then show ?case
  unfolding is-cnf-def by (auto simp: is-conj-with-TF-FNot dest: is-conj-with-TF-Fand
    dest!:is-conj-with-TF-FOr)
next
  case (FOr φ ψ)
  then have [simp]: mset-of-formula φ = {mset-of-conj φ} mset-of-formula ψ = {mset-of-conj ψ}
    unfolding is-cnf-def by (auto dest!:is-conj-with-TF-FOr simp: grouped-by-COr-mset-of-formula
      split: if-splits)
  have is-conj-with-TF φ is-conj-with-TF ψ
    using FOr(3) unfolding is-cnf-def no-T-F-def
    by (metis grouped-is-super-grouped is-conj-with-TF-FOr is-conj-with-TF-def)+
  then show ?case using FOr
    unfolding is-cnf-def by simp
next
  case (FImp φ ψ)
  then show ?case
    unfolding is-cnf-def by auto
next
```

```
  case (FEq φ ψ)
  then show ?case
    unfolding is-cnf-def by auto
next
  case (FNot φ)
  then show ?case
    unfolding is-cnf-def by (auto simp: is-conj-with-TF-FNot)
qed

end
theory Prop-Resolution
imports Partial-Clausal-Logic List-More Wellfounded-More

begin
```

13 Resolution

13.1 Simplification Rules

```
inductive simplify :: 'v clauses ⇒ 'v clauses ⇒ bool for N :: 'v clause set where
tautology-deletion:
    (A + {#Pos P#} + {#Neg P#})∈ N ⟹ simplify  N (N − {A + {#Pos P#} + {#Neg P#}})|
condensation:
    (A + {#L#} + {#L#}) ∈ N ⟹ simplify N ( N− {A + {#L#} + {#L#}} ∪ {A + {#L#}}) |
subsumption:
    A ∈ N ⟹ A ⊂# B ⟹ B ∈ N ⟹ simplify N (N − {B})

lemma simplify-preserves-un-sat':
  fixes N N' :: 'v clauses
  assumes simplify N N'
  and total-over-m I N
  shows I ⊨s N' ⟶ I ⊨s N
  using assms
proof (induct rule: simplify.induct)
  case (tautology-deletion A P)
  then have I ⊨ A + {#Pos P#} + {#Neg P#}
    by (metis total-over-m-def total-over-set-literal-defined true-cls-singleton true-cls-union
      true-lit-def uminus-Neg union-commute)
  then show ?case by (metis Un-Diff-cancel2 true-clss-singleton true-clss-union)
next
  case (condensation A P)
  then show ?case by (metis Diff-insert-absorb Set.set-insert insertE true-cls-union true-clss-def
    true-clss-singleton true-clss-union)
next
  case (subsumption A B)
  have A ≠ B using subsumption.hyps(2) by auto
  then have I ⊨s N − {B} ⟹ I ⊨ A using ⟨A ∈ N⟩ by (simp add: true-clss-def)
  moreover have I ⊨ A ⟹ I ⊨ B using ⟨A <# B⟩ by auto
  ultimately show ?case by (metis insert-Diff-single true-clss-insert)
qed

lemma simplify-preserves-un-sat:
  fixes N N' :: 'v clauses
  assumes simplify N N'
  and total-over-m I N
  shows I ⊨s N ⟶ I ⊨s N′
  using assms apply (induct rule: simplify.induct)
  using true-clss-def by fastforce+

lemma simplify-preserves-un-sat″:
  fixes N N′ :: ′v clauses
  assumes simplify N N′
  and total-over-m I N′
  shows I ⊨s N ⟶ I ⊨s N′
  using assms apply (induct rule: simplify.induct)
  using true-clss-def by fastforce+

lemma simplify-preserves-un-sat-eq:
  fixes N N′ :: ′v clauses
  assumes simplify N N′
  and total-over-m I N
  shows I ⊨s N ⟷ I ⊨s N′
  using simplify-preserves-un-sat simplify-preserves-un-sat′ assms by blast

lemma simplify-preserves-finite:
 assumes simplify ψ ψ′
 shows finite ψ ⟷ finite ψ′
 using assms by (induct rule: simplify.induct, auto simp add: remove-def)

lemma rtranclp-simplify-preserves-finite:
 assumes rtranclp simplify ψ ψ′
 shows finite ψ ⟷ finite ψ′
 using assms by (induct rule: rtranclp-induct) (auto simp add: simplify-preserves-finite)

lemma simplify-atms-of-ms:
  assumes simplify ψ ψ′
  shows atms-of-ms ψ′ ⊆ atms-of-ms ψ
  using assms unfolding atms-of-ms-def
proof (induct rule: simplify.induct)
  case (tautology-deletion A P)
  then show ?case by auto
next
  case (condensation A P)
  moreover have A + {#P#} + {#P#} ∈ ψ ⟹ ∃x∈ψ. atm-of P ∈ atm-of ‘ set-mset x
    by (metis Un-iff atms-of-def atms-of-plus atms-of-singleton insert-iff)
  ultimately show ?case by (auto simp add: atms-of-def)
next
  case (subsumption A P)
  then show ?case by auto
qed

lemma rtranclp-simplify-atms-of-ms:
  assumes rtranclp simplify ψ ψ′
  shows atms-of-ms ψ′ ⊆ atms-of-ms ψ
  using assms apply (induct rule: rtranclp-induct)
   apply (fastforce intro: simplify-atms-of-ms)
  using simplify-atms-of-ms by blast

lemma factoring-imp-simplify:
  assumes {#L#} + {#L#} + C ∈ N
```

shows $\exists N'.$ *simplify* $N\ N'$
proof −
have $C + \{\#L\#\} + \{\#L\#\} \in N$ **using** *assms* **by** (*simp add*: *add.commute union-lcomm*)
from *condensation*[*OF this*] **show** *?thesis* **by** *blast*
qed

13.2 Unconstrained Resolution

type-synonym *'v uncon-state* = *'v clauses*
inductive *uncon-res* :: *'v uncon-state* ⇒ *'v uncon-state* ⇒ *bool* **where**
resolution:
$\{\#Pos\ p\#\} + C \in N \Longrightarrow \{\#Neg\ p\#\} + D \in N \Longrightarrow (\{\#Pos\ p\#\} + C, \{\#Neg\ p\#\} + D) \notin$ *already-used*
$\Longrightarrow$ *uncon-res* $(N)\ (N \cup \{C + D\})$ |
factoring: $\{\#L\#\} + \{\#L\#\} + C \in N \Longrightarrow$ *uncon-res* $N\ (N \cup\{C + \{\#L\#\}\})$

lemma *uncon-res-increasing*:
assumes *uncon-res* $S\ S'$ **and** $\psi \in S$
shows $\psi \in S'$
using *assms* **by** (*induct rule*: *uncon-res.induct*) *auto*

lemma *rtranclp-uncon-inference-increasing*:
assumes *rtranclp uncon-res* $S\ S'$ **and** $\psi \in S$
shows $\psi \in S'$
using *assms* **by** (*induct rule*: *rtranclp-induct*) (*auto simp add*: *uncon-res-increasing*)

13.2.1 Subsumption

definition *subsumes* :: *'a literal multiset* ⇒ *'a literal multiset* ⇒ *bool* **where**
subsumes $\chi\ \chi' \longleftrightarrow$
$(\forall I.$ *total-over-m* $I\ \{\chi'\} \longrightarrow$ *total-over-m* $I\ \{\chi\})$
$\wedge\ (\forall I.$ *total-over-m* $I\ \{\chi\} \longrightarrow I \models \chi \longrightarrow I \models \chi')$

lemma *subsumes-refl*[*simp*]:
subsumes $\chi\ \chi$
unfolding *subsumes-def* **by** *auto*

lemma *subsumes-subsumption*:
assumes *subsumes* $D\ \chi$
and $C \subset\# D$ **and** ¬*tautology* χ
shows *subsumes* $C\ \chi$ **unfolding** *subsumes-def*
using *assms subsumption-total-over-m subsumption-chained* **unfolding** *subsumes-def*
by (*blast intro*!: *subset-mset.less-imp-le*)

lemma *subsumes-tautology*:
assumes *subsumes* $(C + \{\#Pos\ P\#\} + \{\#Neg\ P\#\})\ \chi$
shows *tautology* χ
using *assms* **unfolding** *subsumes-def* **by** (*simp add*: *tautology-def*)

13.3 Inference Rule

type-synonym *'v state* = *'v clauses* × (*'v clause* × *'v clause*) *set*
inductive *inference-clause* :: *'v state* ⇒ *'v clause* × (*'v clause* × *'v clause*) *set* ⇒ *bool*
(**infix** $\Rightarrow_{\mathrm{Res}}$ *100*) **where**
resolution:

```
  {#Pos p#} + C ∈ N ⟹ {#Neg p#} + D ∈ N ⟹ ({#Pos p#} + C, {#Neg p#} + D) ∉
already-used
  ⟹ inference-clause (N, already-used) (C + D, already-used ∪ {({#Pos p#} + C, {#Neg p#} +
D)}) |
factoring: {#L#} + {#L#} + C ∈ N ⟹ inference-clause (N, already-used) (C + {#L#}, already-used)

inductive inference :: 'v state ⇒ 'v state ⇒ bool where
inference-step: inference-clause S (clause, already-used)
  ⟹ inference S (fst S ∪ {clause}, already-used)


abbreviation already-used-inv
  :: 'a literal multiset set × ('a literal multiset × 'a literal multiset) set ⇒ bool where
already-used-inv state ≡
  (∀ (A, B) ∈ snd state. ∃ p. Pos p ∈# A ∧ Neg p ∈# B ∧
          ((∃ χ ∈ fst state. subsumes χ ((A − {#Pos p#}) + (B − {#Neg p#})))
            ∨ tautology ((A − {#Pos p#}) + (B − {#Neg p#}))))

lemma inference-clause-preserves-already-used-inv:
  assumes inference-clause S S'
  and already-used-inv S
  shows already-used-inv (fst S ∪ {fst S'}, snd S')
  using assms apply (induct rule: inference-clause.induct)
  by fastforce+

lemma inference-preserves-already-used-inv:
  assumes inference S S'
  and already-used-inv S
  shows already-used-inv S'
  using assms
proof (induct rule: inference.induct)
  case (inference-step S clause already-used)
  then show ?case
    using inference-clause-preserves-already-used-inv[of S (clause, already-used)] by simp
qed

lemma rtranclp-inference-preserves-already-used-inv:
  assumes rtranclp inference S S'
  and already-used-inv S
  shows already-used-inv S'
  using assms apply (induct rule: rtranclp-induct, simp)
  using inference-preserves-already-used-inv unfolding tautology-def by fast

lemma subsumes-condensation:
  assumes subsumes (C + {#L#} + {#L#}) D
  shows subsumes (C + {#L#}) D
  using assms unfolding subsumes-def by simp

lemma simplify-preserves-already-used-inv:
  assumes simplify N N'
  and already-used-inv (N, already-used)
  shows already-used-inv (N', already-used)
  using assms
proof (induct rule: simplify.induct)
  case (condensation C L)
  then show ?case
    using subsumes-condensation by simp fast
next
  {
     fix a:: 'a and A :: 'a set and P
     have (∃x ∈ Set.remove a A. P x) ⟷ (∃x ∈ A. x ≠ a ∧ P x) by auto
  } note ex-member-remove = this
  {
    fix a a0 :: 'v clause and A :: 'v clauses and  y
    assume a ∈ A and a0 ⊂# a
    then have (∃x ∈ A. subsumes x y) ⟷ (subsumes a y  ∨ (∃x ∈ A. x ≠ a ∧ subsumes x y))
      by auto
  } note tt2 = this
  case (subsumption A B) note A = this(1) and AB = this(2) and B = this(3) and inv = this(4)
  show ?case
    proof (standard, standard)
      fix x a b
      assume x: x ∈ snd (N − {B}, already-used) and [simp]: x = (a, b)
      obtain p where p: Pos p ∈# a ∧ Neg p ∈# b and
        q: (∃χ∈N. subsumes χ (a − {#Pos p#} + (b − {#Neg p#})))
          ∨ tautology (a − {#Pos p#} + (b − {#Neg p#}))
        using inv x by fastforce
      consider (taut) tautology (a − {#Pos p#} + (b − {#Neg p#})) |
        (χ) χ where χ ∈ N subsumes χ (a − {#Pos p#} + (b − {#Neg p#}))
          ¬tautology (a − {#Pos p#} + (b − {#Neg p#}))
        using q by auto
      then show
        ∃p. Pos p ∈# a ∧ Neg p ∈# b
            ∧ ((∃χ∈fst (N − {B}, already-used). subsumes χ (a − {#Pos p#} + (b − {#Neg p#})))
                ∨ tautology (a − {#Pos p#} + (b − {#Neg p#})))
        proof cases
          case taut
          then show ?thesis using p by auto
        next
          case χ note H = this
          show ?thesis using p A AB B  subsumes-subsumption[OF - AB H(3)] H(1,2) by auto
        qed
    qed
next
  case (tautology-deletion C P)
  then show ?case apply clarify
  proof −
    fix a b
    assume C + {#Pos P#} + {#Neg P#} ∈ N
    assume already-used-inv (N, already-used)
    and (a, b) ∈ snd (N − {C + {#Pos P#} + {#Neg P#}}, already-used)
    then obtain p where
      Pos p ∈# a ∧ Neg p ∈# b ∧
        ((∃χ∈fst (N ∪ {C + {#Pos P#} + {#Neg P#}}, already-used).
              subsumes χ (a − {#Pos p#} + (b − {#Neg p#})))
          ∨ tautology (a − {#Pos p#} + (b − {#Neg p#})))
      by fastforce
    moreover have tautology (C + {#Pos P#} + {#Neg P#}) by auto
    ultimately show
      ∃p. Pos p ∈# a ∧ Neg p ∈# b
      ∧ ((∃χ∈fst (N − {C + {#Pos P#} + {#Neg P#}}, already-used).
            subsumes χ (a − {#Pos p#} + (b − {#Neg p#})))
          ∨ tautology (a − {#Pos p#} + (b − {#Neg p#})))
      by (metis (no-types) Diff-iff Un-insert-right empty-iff fst-conv insertE subsumes-tautology
        sup-bot.right-neutral)
  qed
qed


lemma
  factoring-satisfiable: I ⊨ {#L#} + {#L#} + C ⟷ I ⊨ {#L#} + C and
  resolution-satisfiable:
    consistent-interp I ⟹ I ⊨ {#Pos p#} + C ⟹ I ⊨ {#Neg p#} + D ⟹ I ⊨ C + D and
    factoring-same-vars: atms-of ({#L#} + {#L#} + C) = atms-of ({#L#} + C)
  unfolding true-cls-def consistent-interp-def by (fastforce split: if-split-asm)+

lemma inference-increasing:
  assumes inference S S′ and ψ ∈ fst S
  shows ψ ∈ fst S′
  using assms by (induct rule: inference.induct, auto)

lemma rtranclp-inference-increasing:
  assumes rtranclp inference S S′ and ψ ∈ fst S
  shows ψ ∈ fst S′
  using assms by (induct rule: rtranclp-induct, auto simp add: inference-increasing)

lemma inference-clause-already-used-increasing:
  assumes inference-clause S S′
  shows snd S ⊆ snd S′
  using assms by (induct rule:inference-clause.induct, auto)


lemma inference-already-used-increasing:
  assumes inference S S′
  shows snd S ⊆ snd S′
  using assms apply (induct rule:inference.induct)
  using inference-clause-already-used-increasing by fastforce

lemma inference-clause-preserves-un-sat:
  fixes N N′ :: ′v clauses
  assumes inference-clause T T′
  and  total-over-m I (fst T)
  and consistent: consistent-interp I
  shows I ⊨s fst T ⟷ I ⊨s fst T ∪ {fst T′}
  using assms apply (induct rule: inference-clause.induct)
  unfolding consistent-interp-def true-clss-def by auto force+


lemma inference-preserves-un-sat:
  fixes N N′ :: ′v clauses
  assumes inference T T′
  and  total-over-m I (fst T)
  and consistent: consistent-interp I
  shows I ⊨s fst T ⟷ I ⊨s fst T′
  using assms apply (induct rule: inference.induct)
  using inference-clause-preserves-un-sat by fastforce

lemma inference-clause-preserves-atms-of-ms:
  assumes inference-clause S S'
  shows atms-of-ms (fst (fst S ∪ {fst S'}, snd S')) ⊆ atms-of-ms (fst S)
  using assms apply (induct rule: inference-clause.induct)
   apply auto
     apply (metis Set.set-insert UnCI atms-of-ms-insert atms-of-plus)
    apply (metis Set.set-insert UnCI atms-of-ms-insert atms-of-plus)
   apply (simp add: in-m-in-literals union-assoc)
  unfolding atms-of-ms-def using assms by fastforce

lemma inference-preserves-atms-of-ms:
  fixes N N' :: 'v clauses
  assumes inference T T'
  shows atms-of-ms (fst T') ⊆ atms-of-ms (fst T)
  using assms apply (induct rule: inference.induct)
  using inference-clause-preserves-atms-of-ms by fastforce

lemma inference-preserves-total:
  fixes N N' :: 'v clauses
  assumes inference (N, already-used) (N', already-used')
  shows total-over-m I N ⟹ total-over-m I N'
    using assms inference-preserves-atms-of-ms unfolding total-over-m-def total-over-set-def
    by fastforce


lemma rtranclp-inference-preserves-total:
  assumes rtranclp inference T T'
  shows total-over-m I (fst T) ⟹ total-over-m I (fst T')
  using assms by (induct rule: rtranclp-induct, auto simp add: inference-preserves-total)

lemma rtranclp-inference-preserves-un-sat:
  assumes rtranclp inference N N'
  and  total-over-m I (fst N)
  and consistent: consistent-interp I
  shows I ⊨s fst N ⟷ I ⊨s fst N'
  using assms apply (induct rule: rtranclp-induct)
  apply (simp add: inference-preserves-un-sat)
  using inference-preserves-un-sat rtranclp-inference-preserves-total by blast

lemma inference-preserves-finite:
  assumes inference ψ ψ' and finite (fst ψ)
  shows finite (fst ψ')
  using assms by (induct rule: inference.induct, auto simp add: simplify-preserves-finite)


lemma inference-clause-preserves-finite-snd:
  assumes inference-clause ψ ψ' and finite (snd ψ)
  shows finite (snd ψ')
  using assms by (induct rule: inference-clause.induct, auto)


lemma inference-preserves-finite-snd:
  assumes inference ψ ψ' and finite (snd ψ)
  shows finite (snd ψ')
  using assms inference-clause-preserves-finite-snd by (induct rule: inference.induct, fastforce)


lemma rtranclp-inference-preserves-finite:
  assumes rtranclp inference ψ ψ' and finite (fst ψ)
  shows finite (fst ψ')
  using assms by (induct rule: rtranclp-induct)
    (auto simp add: simplify-preserves-finite inference-preserves-finite)

lemma consistent-interp-insert:
  assumes consistent-interp I
  and atm-of P ∉ atm-of ‘ I
  shows consistent-interp (insert P I)
proof –
  have P: insert P I = I ∪ {P} by auto
  show ?thesis unfolding P
  apply (rule consistent-interp-disjoint)
  using assms by (auto simp: image-iff)
qed

lemma simplify-clause-preserves-sat:
  assumes simp: simplify ψ ψ'
  and satisfiable ψ'
  shows satisfiable ψ
  using assms
proof induction
  case (tautology-deletion A P) note AP = this(1) and sat = this(2)
  let ?A' = A + {#Pos P#} + {#Neg P#}
  let ?ψ' = ψ – {?A'}
  obtain I where
    I: I ⊨s ?ψ' and
    cons: consistent-interp I  and
    tot: total-over-m I ?ψ'
    using sat unfolding satisfiable-def by auto
  { assume Pos P ∈ I ∨ Neg P ∈ I
    then have I ⊨ ?A' by auto
    then have I ⊨s ψ using I by (metis insert-Diff tautology-deletion.hyps true-clss-insert)
    then have ?case using cons tot by auto
  }
  moreover {
    assume Pos: Pos P ∉ I and Neg: Neg P ∉ I
    then have consistent-interp (I ∪ {Pos P}) using cons by simp
    moreover have I'A: I ∪ {Pos P} ⊨ ?A' by auto
    have {Pos P} ∪ I ⊨s ψ – {A + {#Pos P#} + {#Neg P#}}
      using ‹I ⊨s ψ – {A + {#Pos P#} + {#Neg P#}}› true-clss-union-increase' by blast
    then have I ∪ {Pos P} ⊨s ψ
      by (metis (no-types) Un-empty-right Un-insert-left Un-insert-right I'A insert-Diff
        sup-bot.left-neutral tautology-deletion.hyps true-clss-insert)
    ultimately have ?case using satisfiable-carac' by blast
  }
  ultimately show ?case by blast
next
  case (condensation A L) note AL = this(1) and sat = this(2)
  have f3: simplify ψ (ψ – {A + {#L#} + {#L#}} ∪ {A + {#L#}})
```

```
    using AL simplify.condensation by blast
  obtain LL :: 'a literal multiset set ⇒ 'a literal set where
    f4: LL (ψ − {A + {#L#} + {#L#}} ∪ {A + {#L#}}) ⊨s ψ − {A + {#L#} + {#L#}} ∪ {A
+ {#L#}}
      ∧ consistent-interp (LL (ψ − {A + {#L#} + {#L#}} ∪ {A + {#L#}}))
      ∧ total-over-m (LL (ψ − {A + {#L#} + {#L#}}
                  ∪ {A + {#L#}})) (ψ − {A + {#L#} + {#L#}} ∪ {A + {#L#}})
    using sat by (meson satisfiable-def)
  have f5: insert (A + {#L#} + {#L#}) (ψ − {A + {#L#} + {#L#}}) = ψ
    using AL by fastforce
  have atms-of (A + {#L#} + {#L#}) = atms-of ({#L#} + A)
    by simp
  then show ?case
    using f5 f4 f3 by (metis (no-types) add.commute satisfiable-def simplify-preserves-un-sat'
      total-over-m-insert total-over-m-union)
next
  case (subsumption A B) note A = this(1) and AB = this(2) and B = this(3) and sat = this(4)
  let ?ψ' = ψ − {B}
  obtain I where I: I ⊨s ?ψ' and cons: consistent-interp I  and tot: total-over-m I ?ψ'
    using sat unfolding satisfiable-def by auto
  have I ⊨ A using A I by (metis AB Diff-iff subset-mset.less-irrefl singletonD true-clss-def)
  then have I ⊨ B using AB subset-mset.less-imp-le true-cls-mono-leD by blast
  then have I ⊨s ψ using I by (metis insert-Diff-single true-clss-insert)
  then show ?case using cons satisfiable-carac' by blast
qed

lemma simplify-preserves-unsat:
  assumes inference ψ ψ'
  shows satisfiable (fst ψ') ⟶ satisfiable (fst ψ)
  using assms apply (induct rule: inference.induct)
  using satisfiable-decreasing by (metis fst-conv)+

lemma inference-preserves-unsat:
  assumes inference** S S'
  shows satisfiable (fst S') ⟶  satisfiable (fst S)
  using assms apply (induct rule: rtranclp-induct)
  apply simp-all
  using simplify-preserves-unsat by blast

datatype 'v sem-tree = Node 'v 'v sem-tree 'v sem-tree | Leaf

fun sem-tree-size :: 'v sem-tree ⇒ nat where
sem-tree-size Leaf = 0 |
sem-tree-size (Node - ag ad) = 1 + sem-tree-size ag + sem-tree-size ad

lemma sem-tree-size[case-names bigger]:
  (⋀xs:: 'v sem-tree. (⋀ys:: 'v sem-tree. sem-tree-size ys < sem-tree-size xs ⟹ P ys) ⟹ P xs)
  ⟹ P xs
  by (fact Nat.measure-induct-rule)


fun partial-interps :: 'v sem-tree ⇒ 'v interp ⇒ 'v clauses ⇒ bool where
partial-interps Leaf I ψ = (∃χ. ¬ I ⊨ χ ∧ χ ∈ ψ ∧ total-over-m I {χ}) |
partial-interps (Node v ag ad) I ψ ⟷
  (partial-interps ag (I ∪ {Pos v}) ψ ∧ partial-interps ad (I∪ {Neg v}) ψ)
```

```
lemma simplify-preserve-partial-leaf:
  simplify N N′ ⟹ partial-interps Leaf I N ⟹ partial-interps Leaf I N′
  apply (induct rule: simplify.induct)
    using union-lcomm apply auto[1]
   apply (simp, metis atms-of-plus total-over-set-union true-cls-union)
  apply simp
  by (metis atms-of-ms-singleton mset-le-exists-conv subset-mset-def true-cls-mono-leD
    total-over-m-def total-over-m-sum)
```

```
lemma simplify-preserve-partial-tree:
  assumes simplify N N′
  and partial-interps t I N
  shows partial-interps t I N′
  using assms apply (induct t arbitrary: I, simp)
  using simplify-preserve-partial-leaf by metis
```

```
lemma inference-preserve-partial-tree:
  assumes inference S S′
  and partial-interps t I (fst S)
  shows partial-interps t I (fst S′)
  using assms apply (induct t arbitrary: I, simp-all)
  by (meson inference-increasing)
```

```
lemma rtranclp-inference-preserve-partial-tree:
  assumes rtranclp inference N N′
  and partial-interps t I (fst N)
  shows partial-interps t I (fst N′)
  using assms apply (induct rule: rtranclp-induct, auto)
  using inference-preserve-partial-tree by force
```

```
function build-sem-tree :: ′v :: linorder set ⇒ ′v clauses ⇒ ′v sem-tree where
build-sem-tree atms ψ =
  (if atms = {} ∨ ¬ finite atms
  then Leaf
  else Node (Min atms) (build-sem-tree (Set.remove (Min atms) atms) ψ)
     (build-sem-tree (Set.remove (Min atms) atms) ψ))
by auto
termination
  apply (relation measure (λ(A, -). card A), simp-all)
  apply (metis Min-in card-Diff1-less remove-def)+
done
declare build-sem-tree.induct[case-names tree]
```

```
lemma unsatisfiable-empty[simp]:
  ¬unsatisfiable {}
   unfolding satisfiable-def apply auto
  using consistent-interp-def unfolding total-over-m-def total-over-set-def atms-of-ms-def by blast
```

```
lemma partial-interps-build-sem-tree-atms-general:
  fixes ψ :: 'v :: linorder clauses and p :: 'v literal list
  assumes unsat: unsatisfiable ψ and finite ψ and consistent-interp I
  and finite atms
  and atms-of-ms ψ = atms ∪ atms-of-s I and atms ∩ atms-of-s I = {}
  shows partial-interps (build-sem-tree atms ψ) I ψ
  using assms
proof (induct arbitrary: I rule: build-sem-tree.induct)
  case (1 atms ψ Ia) note IH1 = this(1) and IH2 = this(2) and unsat = this(3) and finite = this(4)
    and cons = this(5)  and f = this(6) and un = this(7) and disj = this(8)
  {
    assume atms: atms = {}
    then have atmsIa: atms-of-ms ψ = atms-of-s Ia using un by auto
    then have total-over-m Ia ψ unfolding total-over-m-def atmsIa by auto
    then have χ: ∃ χ ∈ ψ. ¬ Ia ⊨ χ
      using unsat cons unfolding true-clss-def satisfiable-def by auto
    then have build-sem-tree atms ψ = Leaf using atms by auto
    moreover
      have tot: ⋀χ. χ ∈ ψ ⟹ total-over-m Ia {χ}
      unfolding total-over-m-def total-over-set-def atms-of-ms-def atms-of-s-def
      using atmsIa atms-of-ms-def by fastforce
    have partial-interps Leaf Ia ψ
      using χ tot by (auto simp add: total-over-m-def total-over-set-def atms-of-ms-def)

      ultimately have ?case by metis
  }
  moreover {
    assume atms: atms ≠ {}
    have build-sem-tree atms ψ = Node (Min atms) (build-sem-tree (Set.remove (Min atms) atms) ψ)
       (build-sem-tree (Set.remove (Min atms) atms) ψ)
      using build-sem-tree.simps[of atms ψ] f atms by metis

    have consistent-interp (Ia ∪ {Pos (Min atms)}) unfolding consistent-interp-def
      by (metis Int-iff Min-in Un-iff atm-of-uminus atms cons consistent-interp-def disj empty-iff
        f in-atms-of-s-decomp insert-iff literal.distinct(1) literal.exhaust-sel literal.sel(2)
        uminus-Neg uminus-Pos)
   moreover have atms-of-ms ψ = Set.remove (Min atms) atms ∪ atms-of-s (Ia ∪ {Pos (Min atms)})
      using Min-in atms f un by fastforce
    moreover have disj': Set.remove (Min atms) atms ∩ atms-of-s (Ia ∪ {Pos (Min atms)}) = {}
      by simp (metis disj disjoint-iff-not-equal member-remove)
    moreover have finite (Set.remove (Min atms) atms) using f by (simp add: remove-def)
    ultimately have subtree1: partial-interps (build-sem-tree (Set.remove (Min atms) atms) ψ)
        (Ia ∪ {Pos (Min atms)}) ψ
      using IH1[of Ia ∪ {Pos (Min (atms))}] atms f unsat finite by metis

    have consistent-interp (Ia ∪ {Neg (Min atms)}) unfolding consistent-interp-def
      by (metis Int-iff Min-in Un-iff atm-of-uminus atms cons consistent-interp-def disj empty-iff
        f in-atms-of-s-decomp insert-iff literal.distinct(1) literal.exhaust-sel literal.sel(2)
        uminus-Neg)
   moreover have atms-of-ms ψ = Set.remove (Min atms) atms ∪ atms-of-s (Ia ∪ {Neg (Min atms)})
       using ‹atms-of-ms ψ = Set.remove (Min atms) atms ∪ atms-of-s (Ia ∪ {Pos (Min atms)})› by
blast

    moreover have disj': Set.remove (Min atms) atms ∩ atms-of-s (Ia ∪ {Neg (Min atms)}) = {}
      using disj by auto
    moreover have finite (Set.remove (Min atms) atms) using f by (simp add: remove-def)
    ultimately have subtree2: partial-interps (build-sem-tree (Set.remove (Min atms) atms) ψ)
        (Ia ∪ {Neg (Min atms)}) ψ
      using IH2[of Ia ∪ {Neg (Min (atms))}] atms f unsat finite by metis

    then have ?case
      using IH1 subtree1 subtree2 f local.finite unsat atms by simp
  }
  ultimately show ?case by metis
qed


lemma partial-interps-build-sem-tree-atms:
  fixes ψ :: 'v :: linorder clauses and p :: 'v literal list
  assumes unsat: unsatisfiable ψ and finite: finite ψ
  shows partial-interps (build-sem-tree (atms-of-ms ψ) ψ) {} ψ
proof –
  have consistent-interp {} unfolding consistent-interp-def by auto
  moreover have atms-of-ms ψ = atms-of-ms ψ ∪ atms-of-s {} unfolding atms-of-s-def by auto
  moreover have atms-of-ms ψ ∩ atms-of-s {} = {} unfolding atms-of-s-def by auto
  moreover have finite (atms-of-ms ψ) unfolding atms-of-ms-def using finite by simp
  ultimately show partial-interps (build-sem-tree (atms-of-ms ψ) ψ) {} ψ
    using partial-interps-build-sem-tree-atms-general[of ψ {} atms-of-ms ψ] assms by metis
qed

lemma can-decrease-count:
  fixes ψ'' :: 'v clauses × ('v clause × 'v clause × 'v) set
  assumes count χ L = n
  and L ∈# χ and χ ∈ fst ψ
  shows ∃ψ' χ'. inference** ψ ψ' ∧ χ' ∈ fst ψ' ∧ (∀L. L ∈# χ ⟷ L ∈# χ')
                ∧ count χ' L = 1
                ∧ (∀φ. φ ∈ fst ψ ⟶ φ ∈ fst ψ')
                ∧ (I ⊨ χ ⟷ I ⊨ χ')
                ∧ (∀I'. total-over-m I' {χ} ⟶ total-over-m I' {χ'})
  using assms
proof (induct n arbitrary: χ ψ)
  case 0
  then show ?case by (simp add: not-in-iff[symmetric])
next
  case (Suc n χ)
  note IH = this(1) and count = this(2) and L = this(3) and χ = this(4)
  {
    assume n = 0
    then have inference** ψ ψ
    and χ ∈ fst ψ
    and ∀L. (L ∈# χ) ⟷ (L ∈# χ)
    and count χ L = (1::nat)
    and ∀φ. φ ∈ fst ψ ⟶ φ ∈ fst ψ
      by (auto simp add: count L χ)
    then have ?case by metis
  }
  moreover {
    assume n > 0
    then have ∃C. χ = C + {#L, L#}
      by (smt L Suc-eq-plus1-left add.left-commute add-diff-cancel-left' add-diff-cancel-right'
        count-greater-zero-iff count-single local.count multi-member-split plus-multiset.rep-eq)
    then obtain C where C: χ = C + {#L, L#} by metis
    let ?χ' = C +{#L#}
    let ?ψ' = (fst ψ ∪ {?χ'}, snd ψ)
    have φ: ∀φ ∈ fst ψ. (φ ∈ fst ψ ∨ φ ≠ ?χ') ⟷ φ ∈ fst ?ψ' unfolding C by auto
    have inf: inference ψ ?ψ'
      using C factoring χ prod.collapse union-commute inference-step by metis
    moreover have count': count ?χ' L = n using C count by auto
    moreover have Lχ': L ∈# ?χ' by auto
    moreover have χ'ψ': ?χ' ∈ fst ?ψ' by auto
    ultimately obtain ψ'' and χ''
    where
      inference** ?ψ' ψ'' and
      α: χ'' ∈ fst ψ'' and
      ∀La. (La ∈# ?χ') ⟷ (La ∈# χ'') and
      β: count χ'' L = (1::nat) and
      φ': ∀φ. φ ∈ fst ?ψ' ⟶ φ ∈ fst ψ'' and
      Iχ: I ⊨ ?χ' ⟷ I ⊨ χ'' and
      tot: ∀I'. total-over-m I' {?χ'} ⟶ total-over-m I' {χ''}
      using IH[of ?χ' ?ψ'] count' Lχ' χ'ψ' by blast

    then have inference** ψ ψ''
    and ∀La. (La ∈# χ) ⟷ (La ∈# χ'')
    using inf unfolding C by auto
    moreover have ∀φ. φ ∈ fst ψ ⟶ φ ∈ fst ψ'' using φ φ' by metis
    moreover have I ⊨ χ ⟷ I ⊨ χ'' using Iχ unfolding true-cls-def C by auto
    moreover have ∀I'. total-over-m I' {χ} ⟶ total-over-m I' {χ''}
      using tot unfolding C total-over-m-def by auto
    ultimately have ?case using φ φ' α β  by metis
  }
  ultimately show ?case by auto
qed

lemma can-decrease-tree-size:
  fixes ψ :: 'v state and tree :: 'v sem-tree
  assumes finite (fst ψ)  and already-used-inv ψ
  and partial-interps tree I (fst ψ)
  shows ∃(tree':: 'v sem-tree) ψ'. inference** ψ ψ' ∧ partial-interps tree' I (fst ψ')
            ∧ (sem-tree-size tree' < sem-tree-size tree ∨ sem-tree-size tree = 0)
  using assms
proof (induct arbitrary: I rule: sem-tree-size)
  case (bigger xs I) note IH = this(1) and finite = this(2) and a-u-i = this(3) and part = this(4)

  {
    assume sem-tree-size xs = 0
    then have ?case using part by blast
  }

  moreover {
    assume sn0: sem-tree-size xs > 0
    obtain ag ad v where xs: xs = Node v ag ad using sn0 by (cases xs, auto)
    {
      assume sem-tree-size ag = 0 and sem-tree-size ad = 0
      then have ag: ag = Leaf and ad: ad = Leaf by (cases ag,  auto) (cases ad,  auto)

      then obtain χ χ' where
  χ: ¬ I ∪ {Pos v} ⊨ χ and
  totχ: total-over-m (I ∪ {Pos v}) {χ} and
  χψ: χ ∈ fst ψ and
  χ': ¬ I ∪ {Neg v} ⊨ χ'  and
  totχ': total-over-m (I ∪ {Neg v}) {χ'} and
  χ'ψ: χ' ∈ fst ψ
  using part unfolding xs by auto
have Posv: ¬Pos v ∈# χ using χ unfolding true-cls-def true-lit-def by auto
have Negv: ¬Neg v ∈# χ' using χ' unfolding true-cls-def true-lit-def by auto
{
  assume Negχ: ¬Neg v ∈# χ
  have ¬ I ⊨ χ using χ Posv unfolding true-cls-def true-lit-def by auto
  moreover have total-over-m I {χ}
    using Posv Negχ atm-imp-pos-or-neg-lit totχ unfolding total-over-m-def total-over-set-def
    by fastforce
  ultimately have partial-interps Leaf I (fst ψ)
  and sem-tree-size Leaf < sem-tree-size xs
  and inference** ψ ψ
    unfolding xs by (auto simp add: χψ)
}
moreover {
  assume Posχ: ¬Pos v ∈# χ'
  then have Iχ: ¬ I ⊨ χ' using χ' Posv unfolding true-cls-def true-lit-def by auto
  moreover have total-over-m I {χ'}
    using Negv Posχ atm-imp-pos-or-neg-lit totχ'
    unfolding total-over-m-def total-over-set-def by fastforce
  ultimately have partial-interps Leaf I (fst ψ) and
    sem-tree-size Leaf < sem-tree-size xs and
    inference** ψ ψ
    using χ'ψ Iχ unfolding xs by auto
}
moreover {
  assume neg: Neg v ∈# χ and pos: Pos v ∈# χ'
  then obtain ψ' χ2 where inf: rtranclp inference ψ ψ' and χ2incl: χ2 ∈ fst ψ'
    and χχ2-incl: ∀L. L ∈# χ ⟷ L ∈# χ2
    and countχ2: count χ2 (Neg v) = 1
    and φ: ∀φ::'v literal multiset. φ ∈ fst ψ ⟶ φ ∈ fst ψ'
    and Iχ: I ⊨ χ ⟷ I ⊨ χ2
    and tot-impχ: ∀I'. total-over-m I' {χ} ⟶ total-over-m I' {χ2}
    using can-decrease-count[of χ Neg v count χ (Neg v) ψ I] χψ χ'ψ by auto

  have χ' ∈ fst ψ' by (simp add: χ'ψ φ)
  with pos
  obtain ψ'' χ2' where
  inf': inference** ψ' ψ''
  and χ2'-incl: χ2' ∈ fst ψ''
  and χ'χ2-incl: ∀L::'v literal. (L ∈# χ') = (L ∈# χ2')
  and countχ2': count χ2' (Pos v) = (1::nat)
  and φ': ∀φ::'v literal multiset. φ ∈ fst ψ' ⟶ φ ∈ fst ψ''
  and Iχ': I ⊨ χ' ⟷ I ⊨ χ2'
  and tot-impχ': ∀I'. total-over-m I' {χ'} ⟶ total-over-m I' {χ2'}
  using can-decrease-count[of χ' Pos v  count χ' (Pos v) ψ' I] by auto

  obtain C where χ2: χ2 = C + {#Neg v#} and negC: Neg v ∉# C and posC: Pos v ∉# C
    proof -
    have ⋀m. Suc 0 − count m (Neg v) = count (χ2 − m) (Neg v)
      by (simp add: countχ2)
    then show ?thesis
      using that by (metis (no-types) One-nat-def Posv Suc-inject Suc-pred χχ2-incl
        count-diff count-single insert-DiffM2 mem-Collect-eq multi-member-skip neg
        not-gr0 set-mset-def union-commute)
  qed

obtain C′ where
  χ2′: χ2′ = C′ + {#Pos v#} and
  posC′: Pos v ∉# C′ and
  negC′: Neg v ∉# C′
  proof −
    assume a1: ⋀C′. ⟦χ2′ = C′ + {#Pos v#}; Pos v ∉# C′; Neg v ∉# C′⟧ ⟹ thesis
    have f2: ⋀n. (n::nat) − n = 0
      by simp
    have Neg v ∉# χ2′ − {#Pos v#}
      using Negv χ′χ2-incl by (auto simp: not-in-iff)
    have count {#Pos v#} (Pos v) = 1
      by simp
    then show ?thesis
      by (metis χ′χ2-incl ‹Neg v ∉# χ2′ − {#Pos v#}› a1 countχ2′ count-diff f2
        insert-DiffM2 less-numeral-extra(3) mem-Collect-eq pos set-mset-def)
  qed

have already-used-inv ψ′
  using rtranclp-inference-preserves-already-used-inv[of ψ ψ′] a-u-i inf by blast
then have a-u-i-ψ′′: already-used-inv ψ′′
  using rtranclp-inference-preserves-already-used-inv a-u-i inf′ unfolding tautology-def
  by simp

have totC: total-over-m I {C}
  using tot-impχ totχ tot-over-m-remove[of I Pos v C] negC posC unfolding χ2
  by (metis total-over-m-sum uminus-Neg uminus-of-uminus-id)
have totC′: total-over-m I {C′}
  using tot-impχ′ totχ′ total-over-m-sum tot-over-m-remove[of I Neg v C′] negC′ posC′
  unfolding χ2′ by (metis total-over-m-sum uminus-Neg)
have ¬ I ⊨ C + C′
  using χ Iχ χ′ Iχ′ unfolding χ2 χ2′ true-cls-def by auto
then have part-I-ψ′′′: partial-interps Leaf I (fst ψ′′ ∪ {C + C′})
  using totC totC′ by simp
    (metis ‹¬ I ⊨ C + C′› atms-of-ms-singleton total-over-m-def total-over-m-sum)
{
  assume ({#Pos v#} + C′, {#Neg v#} + C) ∉ snd ψ′′
  then have inf′′: inference ψ′′ (fst ψ′′ ∪ {C + C′}, snd ψ′′ ∪ {(χ2′, χ2)})
    using add.commute φ′ χ2incl ‹χ2′ ∈ fst ψ′′› unfolding χ2 χ2′
    by (metis prod.collapse inference-step resolution)
  have inference** ψ (fst ψ′′ ∪ {C + C′}, snd ψ′′ ∪ {(χ2′, χ2)})
    using inf inf′ inf′′ rtranclp-trans by auto
  moreover have sem-tree-size Leaf < sem-tree-size xs unfolding xs by auto
  ultimately have ?case using part-I-ψ′′′ by (metis fst-conv)
}
moreover {
  assume a: ({#Pos v#} + C′, {#Neg v#} + C) ∈ snd ψ′′
  then have (∃χ ∈ fst ψ′′. (∀I. total-over-m I {C+C′} ⟶ total-over-m I {χ})
              ∧ (∀I. total-over-m I {χ} ⟶ I ⊨ χ ⟶ I ⊨ C' + C))
          ∨ tautology (C' + C)
      proof –
        obtain p where p: Pos p ∈# ({#Pos v#} + C') and
        n: Neg p ∈# ({#Neg v#} + C) and
        decomp: ((∃χ∈fst ψ''.
                  (∀I. total-over-m I {({#Pos v#} + C') − {#Pos p#}
                        + (({#Neg v#} + C) − {#Neg p#})}
                    ⟶ total-over-m I {χ})
                  ∧ (∀I. total-over-m I {χ} ⟶ I ⊨ χ
                  ⟶ I ⊨ ({#Pos v#} + C') − {#Pos p#} + (({#Neg v#} + C) − {#Neg p#}))
                 )
             ∨ tautology (({#Pos v#} + C') − {#Pos p#} + (({#Neg v#} + C) − {#Neg p#})))
          using a by (blast intro: allE[OF a-u-i-ψ''[unfolded subsumes-def Ball-def],
             of ({#Pos v#} + C', {#Neg v#} + C)])
        { assume p ≠ v
          then have Pos p ∈# C' ∧ Neg p ∈# C using p n by force
          then have ?thesis unfolding Bex-def by auto
        }
        moreover {
          assume p = v
         then have ?thesis using decomp by (metis add.commute add-diff-cancel-left')
        }
        ultimately show ?thesis by auto
      qed
    moreover {
      assume ∃χ ∈ fst ψ''. (∀I. total-over-m I {C+C'} ⟶ total-over-m I {χ})
        ∧ (∀I. total-over-m I {χ} ⟶ I ⊨ χ ⟶ I ⊨ C' + C)
      then obtain ϑ where ϑ: ϑ ∈ fst ψ'' and
        tot-ϑ-CC': ∀I. total-over-m I {C+C'} ⟶ total-over-m I {ϑ} and
        ϑ-inv: ∀I. total-over-m I {ϑ} ⟶ I ⊨ ϑ ⟶ I ⊨ C' + C by blast
      have partial-interps Leaf I (fst ψ'')
        using tot-ϑ-CC' ϑ ϑ-inv totC totC' ‹¬ I ⊨ C + C'› total-over-m-sum by fastforce
      moreover have sem-tree-size Leaf < sem-tree-size xs unfolding xs by auto
      ultimately have ?case by (metis inf inf' rtranclp-trans)
    }
    moreover {
      assume tautCC': tautology (C' + C)
      have total-over-m I {C'+C} using totC totC' total-over-m-sum by auto
      then have ¬tautology (C' + C)
        using ‹¬ I ⊨ C + C'› unfolding add.commute[of C C'] total-over-m-def
        unfolding tautology-def by auto
      then have False using tautCC'  unfolding tautology-def by auto
    }
    ultimately have ?case by auto
  }
  ultimately have ?case by auto
}
ultimately have ?case using part by (metis (no-types) sem-tree-size.simps(1))
}
moreover {
  assume size-ag: sem-tree-size ag > 0
  have sem-tree-size ag < sem-tree-size xs unfolding xs by auto
  moreover have partial-interps ag (I ∪ {Pos v}) (fst ψ)
    and partad: partial-interps ad (I ∪ {Neg v}) (fst ψ)
```

```
        using part partial-interps.simps(2) unfolding xs by metis+
      moreover have sem-tree-size ag < sem-tree-size xs ⟶ finite (fst ψ) ⟶ already-used-inv ψ
        ⟶ ( partial-interps ag (I ∪ {Pos v}) (fst ψ) ⟶
        (∃ tree' ψ'. inference** ψ ψ' ∧ partial-interps tree' (I ∪ {Pos v}) (fst ψ')
          ∧ (sem-tree-size tree' < sem-tree-size ag ∨ sem-tree-size ag = 0)))
          using IH by auto
      ultimately obtain ψ' :: 'v state and  tree' :: 'v sem-tree  where
        inf: inference** ψ ψ'
        and part: partial-interps tree' (I ∪ {Pos v}) (fst ψ')
        and size: sem-tree-size tree' < sem-tree-size ag ∨ sem-tree-size ag = 0
        using finite part rtranclp.rtrancl-refl a-u-i by blast

      have partial-interps ad (I ∪ {Neg v}) (fst ψ')
        using rtranclp-inference-preserve-partial-tree inf partad by metis
      then have partial-interps (Node v tree' ad) I (fst ψ') using part by auto
      then have ?case using inf size size-ag part unfolding xs by fastforce
    }
    moreover {
      assume size-ad: sem-tree-size ad > 0
      have sem-tree-size ad < sem-tree-size xs unfolding xs by auto
      moreover have partag: partial-interps ag (I ∪ {Pos v}) (fst ψ) and
        partial-interps ad (I ∪ {Neg v}) (fst ψ)
        using part partial-interps.simps(2) unfolding xs by metis+
      moreover have sem-tree-size ad < sem-tree-size xs ⟶ finite (fst ψ) ⟶ already-used-inv ψ
        ⟶ ( partial-interps ad (I ∪ {Neg v}) (fst ψ)
        ⟶ (∃ tree' ψ'. inference** ψ ψ' ∧ partial-interps tree' (I ∪ {Neg v}) (fst ψ')
            ∧ (sem-tree-size tree' < sem-tree-size ad ∨ sem-tree-size ad = 0)))
        using IH by auto
      ultimately obtain ψ' :: 'v state and  tree' :: 'v sem-tree  where
        inf: inference** ψ ψ'
        and part: partial-interps tree' (I ∪ {Neg v}) (fst ψ')
        and size: sem-tree-size tree' < sem-tree-size ad ∨ sem-tree-size ad = 0
        using finite part  rtranclp.rtrancl-refl a-u-i by blast

      have partial-interps ag (I ∪ {Pos v}) (fst ψ')
        using rtranclp-inference-preserve-partial-tree inf partag by metis
      then have partial-interps (Node v ag tree') I (fst ψ') using part by auto
      then have ?case using inf size size-ad unfolding xs by fastforce
    }
    ultimately have ?case by auto
  }
  ultimately show ?case by auto
qed

lemma inference-completeness-inv:
  fixes ψ :: 'v ::linorder state
  assumes
    unsat: ¬satisfiable (fst ψ) and
    finite: finite (fst ψ) and
    a-u-v: already-used-inv ψ
  shows ∃ψ'. (inference** ψ ψ' ∧ {#} ∈ fst ψ')
proof -
  obtain tree where  partial-interps tree {} (fst ψ)
    using partial-interps-build-sem-tree-atms assms by metis
  then show ?thesis
```

```
    using unsat finite a-u-v
    proof (induct tree arbitrary: ψ rule: sem-tree-size)
      case (bigger tree ψ) note H = this
      {
        fix χ
        assume tree: tree = Leaf
        obtain χ where χ: ¬ {} ⊨ χ and totχ: total-over-m {} {χ} and χψ: χ ∈ fst ψ
          using H unfolding tree by auto
        moreover have {#} = χ
          using  totχ unfolding total-over-m-def total-over-set-def by fastforce
        moreover have inference** ψ ψ by auto
        ultimately have ?case by metis
      }
      moreover {
        fix v tree1 tree2
        assume tree: tree = Node v tree1 tree2
        obtain
          tree′ ψ′ where inf: inference** ψ ψ′ and
          part′: partial-interps tree′ {} (fst ψ′) and
          decrease: sem-tree-size tree′ < sem-tree-size tree ∨ sem-tree-size tree = 0
          using can-decrease-tree-size[of ψ] H(2,4,5)  unfolding tautology-def by meson
        have sem-tree-size tree′ < sem-tree-size tree using decrease unfolding tree by auto
        moreover have finite (fst ψ′) using rtranclp-inference-preserves-finite inf H(4) by metis
        moreover have unsatisfiable (fst ψ′)
          using inference-preserves-unsat inf bigger.prems(2) by blast
        moreover have already-used-inv ψ′
          using H(5) inf rtranclp-inference-preserves-already-used-inv[of ψ ψ′] by auto
        ultimately have ?case using inf rtranclp-trans part′ H(1) by fastforce
      }
      ultimately show ?case by (cases tree, auto)
   qed
qed

lemma inference-completeness:
  fixes ψ :: ′v ::linorder state
  assumes unsat: ¬satisfiable (fst ψ)
  and finite: finite (fst ψ)
  and snd ψ = {}
  shows ∃ψ′. (rtranclp inference ψ ψ′ ∧ {#} ∈ fst ψ′)
proof −
  have already-used-inv ψ unfolding assms by auto
  then show ?thesis using assms inference-completeness-inv by blast
qed

lemma inference-soundness:
  fixes ψ :: ′v ::linorder state
  assumes rtranclp inference ψ ψ′ and {#} ∈ fst ψ′
  shows unsatisfiable (fst ψ)
  using assms by (meson rtranclp-inference-preserves-un-sat satisfiable-def true-cls-empty
    true-clss-def)

lemma inference-soundness-and-completeness:
fixes ψ :: ′v ::linorder state
assumes finite: finite (fst ψ)
and snd ψ = {}
```

```
shows (∃ψ'. (inference** ψ ψ' ∧ {#} ∈ fst ψ')) ⟷ unsatisfiable (fst ψ)
  using assms inference-completeness inference-soundness by metis
```

13.4 Lemma about the simplified state

```
abbreviation simplified ψ ≡ (no-step simplify ψ)

lemma simplified-count:
  assumes simp: simplified ψ and χ: χ ∈ ψ
  shows count χ L ≤ 1
proof –
  {
    let ?χ' = χ – {#L, L#}
    assume count χ L ≥ 2
    then have f1: count (χ – {#L, L#} + {#L, L#}) L = count χ L
      by simp
    then have L ∈# χ – {#L#}
      by (metis (no-types) add.left-neutral add-diff-cancel-left' count-union diff-diff-add
        diff-single-trivial insert-DiffM mem-Collect-eq multi-member-this not-gr0 set-mset-def)
    then have χ': ?χ' + {#L#} + {#L#} = χ
      using f1 by (metis diff-diff-add diff-single-eq-union in-diffD)

    have ∃ψ'. simplify ψ ψ'
      by (metis (no-types, hide-lams) χ χ' add.commute factoring-imp-simplify union-assoc)
    then have False using simp by auto
  }
  then show ?thesis by arith
qed

lemma simplified-no-both:
  assumes  simp: simplified ψ and χ: χ ∈ ψ
  shows ¬ (L ∈# χ ∧ –L ∈# χ)
proof (rule ccontr)
  assume ¬ ¬ (L ∈# χ ∧ – L ∈# χ)
  then have L ∈# χ ∧ – L ∈# χ by metis
  then obtain  χ' where χ = χ' + {#Pos (atm-of L)#}+ {#Neg (atm-of L)#}
    by (metis Neg-atm-of-iff Pos-atm-of-iff diff-union-swap insert-DiffM2 uminus-Neg uminus-Pos)
  then show False using χ simp tautology-deletion by fastforce
qed

lemma simplified-not-tautology:
  assumes simplified {ψ}
  shows ~tautology ψ
proof (rule ccontr)
  assume ~ ?thesis
  then obtain p where Pos p ∈# ψ ∧ Neg p ∈# ψ using tautology-decomp by metis
  then obtain χ where ψ = χ + {#Pos p#} + {#Neg p#}
    by (metis insert-noteq-member literal.distinct(1) multi-member-split)
  then have ~ simplified {ψ} by (auto intro: tautology-deletion)
  then show False using assms by auto
qed

lemma simplified-remove:
  assumes simplified {ψ}
  shows simplified {ψ – {#l#}}
proof (rule ccontr)
```

```
  assume ns: ¬ simplified {ψ − {#l#}}
  {
    assume ¬ l∈# ψ
    then have ψ − {#l#} = ψ by simp
    then have False using ns assms by auto
  }
  moreover {
    assume lψ: l∈# ψ
    have A: ⋀A. A ∈ {ψ − {#l#}} ⟷ A + {#l#} ∈ {ψ}  by (auto simp add: lψ)
    obtain l′ where l′: simplify {ψ − {#l#}} l′ using ns by metis
    then have ∃ l′. simplify {ψ} l′
      proof (induction rule: simplify.induct)
        case (tautology-deletion A P)
        have {#Neg P#} + ({#Pos P#} + (A + {#l#})) ∈ {ψ}
          by (metis (no-types) A add.commute tautology-deletion.hyps union-lcomm)
        then show ?thesis
           by (metis simplify.tautology-deletion[of A+{#l#} P {ψ}] add.commute)
      next
        case (condensation A L)
        have A + {#L#} + {#L#} + {#l#} ∈ {ψ}
          using A condensation.hyps by blast
        then have {#L, L#} + (A + {#l#}) ∈ {ψ}
          by (metis (no-types) union-assoc union-commute)
        then show ?case
          using factoring-imp-simplify by blast
      next
        case (subsumption A B)
        then show ?case by blast
      qed
    then have False using assms(1) by blast
  }
  ultimately show False by auto
qed


lemma in-simplified-simplified:
  assumes simp: simplified ψ and incl: ψ′ ⊆ ψ
  shows simplified ψ′
proof (rule ccontr)
  assume ¬ ?thesis
  then obtain ψ′′ where simplify ψ′ ψ′′  by metis
    then have ∃ l′. simplify ψ l′
      proof (induction rule: simplify.induct)
        case (tautology-deletion A P)
        then show ?thesis using simplify.tautology-deletion[of A P ψ] incl by blast
      next
        case (condensation A L)
        then show ?case using simplify.condensation[of A L ψ] incl by blast
      next
        case (subsumption A B)
        then show ?case using simplify.subsumption[of A ψ B] incl by auto
      qed
  then show False using assms(1) by blast
qed
```

```
lemma simplified-in:
  assumes simplified ψ
  and N ∈ ψ
  shows simplified {N}
  using assms by (metis Set.set-insert empty-subsetI in-simplified-simplified insert-mono)

lemma subsumes-imp-formula:
  assumes ψ ≤# φ
  shows {ψ} ⊨p φ
  unfolding true-clss-cls-def apply auto
  using assms true-cls-mono-leD by blast

lemma simplified-imp-distinct-mset-tauto:
  assumes simp: simplified ψ'
  shows distinct-mset-set ψ' and  ∀χ ∈ ψ'. ¬tautology χ
proof –
  show ∀χ ∈ ψ'. ¬tautology χ
    using simp by (auto simp add: simplified-in simplified-not-tautology)

  show distinct-mset-set ψ'
    proof (rule ccontr)
      assume ¬?thesis
      then obtain χ where χ ∈ ψ' and ¬distinct-mset χ unfolding distinct-mset-set-def by auto
      then obtain L where count χ L ≥ 2
        unfolding distinct-mset-def
        by (meson count-greater-eq-one-iff le-antisym simp simplified-count)
      then show False by (metis Suc-1 ‹χ ∈ ψ'› not-less-eq-eq simp simplified-count)
    qed
qed

lemma simplified-no-more-full1-simplified:
  assumes simplified ψ
  shows ¬full1 simplify ψ ψ'
  using assms unfolding full1-def by (meson tranclpD)
```

13.5 Resolution and Invariants

```
inductive resolution :: 'v state ⇒ 'v state ⇒ bool where
full1-simp: full1 simplify N N' ⟹ resolution (N, already-used) (N', already-used) |
inferring: inference (N, already-used) (N', already-used') ⟹ simplified N
  ⟹ full simplify N' N'' ⟹ resolution (N, already-used) (N'', already-used')
```

13.5.1 Invariants

```
lemma resolution-finite:
  assumes resolution ψ ψ' and finite (fst ψ)
  shows finite (fst ψ')
  using assms by (induct rule: resolution.induct)
    (auto simp add: full1-def full-def  rtranclp-simplify-preserves-finite
      dest: tranclp-into-rtranclp inference-preserves-finite)

lemma rtranclp-resolution-finite:
  assumes resolution** ψ ψ' and finite (fst ψ)
  shows finite (fst ψ')
  using assms by (induct rule: rtranclp-induct, auto simp add: resolution-finite)
```

```
lemma resolution-finite-snd:
  assumes resolution ψ ψ' and finite (snd ψ)
  shows finite (snd ψ')
  using assms apply (induct rule: resolution.induct, auto simp add: inference-preserves-finite-snd)
  using inference-preserves-finite-snd snd-conv by metis

lemma rtranclp-resolution-finite-snd:
  assumes resolution** ψ ψ' and finite (snd ψ)
  shows finite (snd ψ')
  using assms by (induct rule: rtranclp-induct, auto simp add: resolution-finite-snd)

lemma resolution-always-simplified:
 assumes resolution ψ ψ'
 shows simplified (fst ψ')
 using assms by (induct rule: resolution.induct)
   (auto simp add: full1-def full-def)

lemma tranclp-resolution-always-simplified:
  assumes tranclp resolution ψ ψ'
  shows simplified (fst ψ')
  using assms by (induct rule: tranclp.induct, auto simp add: resolution-always-simplified)

lemma resolution-atms-of:
  assumes resolution ψ ψ' and finite (fst ψ)
  shows atms-of-ms (fst ψ') ⊆ atms-of-ms (fst ψ)
  using assms apply (induct rule: resolution.induct)
    apply(simp add: rtranclp-simplify-atms-of-ms tranclp-into-rtranclp full1-def )
  by (metis (no-types, lifting) contra-subsetD fst-conv full-def
    inference-preserves-atms-of-ms rtranclp-simplify-atms-of-ms subsetI)

lemma rtranclp-resolution-atms-of:
  assumes resolution** ψ ψ' and finite (fst ψ)
  shows atms-of-ms (fst ψ') ⊆ atms-of-ms (fst ψ)
  using assms apply (induct rule: rtranclp-induct)
  using resolution-atms-of rtranclp-resolution-finite by blast+

lemma resolution-include:
  assumes res: resolution ψ ψ' and finite: finite (fst ψ)
  shows fst ψ' ⊆ simple-clss (atms-of-ms (fst ψ))
proof −
  have finite': finite (fst ψ') using local.finite res resolution-finite by blast
  have simplified (fst ψ') using res finite' resolution-always-simplified by blast
  then have fst ψ' ⊆ simple-clss (atms-of-ms (fst ψ'))
    using simplified-in-simple-clss finite' simplified-imp-distinct-mset-tauto[of fst ψ'] by auto
  moreover have atms-of-ms (fst ψ') ⊆ atms-of-ms (fst ψ)
    using res finite resolution-atms-of[of ψ ψ'] by auto
  ultimately show ?thesis by (meson atms-of-ms-finite local.finite order.trans rev-finite-subset
    simple-clss-mono)
qed

lemma rtranclp-resolution-include:
  assumes res: tranclp resolution ψ ψ' and finite: finite (fst ψ)
  shows fst ψ' ⊆ simple-clss (atms-of-ms (fst ψ))
  using assms apply (induct rule: tranclp.induct)
    apply (simp add: resolution-include)
  by (meson simple-clss-mono order-class.le-trans resolution-include
    rtranclp-resolution-atms-of rtranclp-resolution-finite tranclp-into-rtranclp)

abbreviation already-used-all-simple
  :: ('a literal multiset × 'a literal multiset) set ⇒ 'a set ⇒ bool where
already-used-all-simple already-used vars ≡
(∀ (A, B) ∈ already-used. simplified {A} ∧ simplified {B} ∧ atms-of A ⊆ vars ∧ atms-of B ⊆ vars)

lemma already-used-all-simple-vars-incl:
  assumes vars ⊆ vars'
  shows already-used-all-simple a vars ⟹ already-used-all-simple a vars'
  using assms by fast

lemma inference-clause-preserves-already-used-all-simple:
  assumes inference-clause S S'
  and already-used-all-simple (snd S) vars
  and simplified (fst S)
  and atms-of-ms (fst S) ⊆ vars
  shows already-used-all-simple (snd (fst S ∪ {fst S'}, snd S')) vars
  using assms
proof (induct rule: inference-clause.induct)
  case (factoring L C N already-used)
  then show ?case by (simp add: simplified-in factoring-imp-simplify)
next
  case (resolution P C N D already-used) note H = this
  show ?case apply clarify
    proof −
      fix A B v
      assume (A, B) ∈ snd (fst (N, already-used)
        ∪ {fst (C + D, already-used ∪ {({#Pos P#} + C, {#Neg P#} + D)})},
          snd (C + D, already-used ∪ {({#Pos P#} + C, {#Neg P#} + D)}))
      then have (A, B) ∈ already-used ∨ (A, B) = ({#Pos P#} + C, {#Neg P#} + D) by auto
      moreover {
        assume (A, B) ∈ already-used
        then have simplified {A} ∧ simplified {B} ∧ atms-of A ⊆ vars ∧ atms-of B ⊆ vars
          using H(4) by auto
      }
      moreover {
        assume eq: (A, B) = ({#Pos P#} + C, {#Neg P#} + D)
        then have simplified {A} using simplified-in H(1,5) by auto
        moreover have simplified {B} using eq simplified-in H(2,5) by auto
        moreover have atms-of A ⊆ atms-of-ms N
          using eq H(1)
          using atms-of-atms-of-ms-mono[of A N] by auto
        moreover have atms-of B ⊆ atms-of-ms N
          using eq H(2) atms-of-atms-of-ms-mono[of B N] by auto
        ultimately have simplified {A} ∧ simplified {B} ∧ atms-of A ⊆ vars ∧ atms-of B ⊆ vars
          using H(6) by auto
      }
      ultimately show simplified {A} ∧ simplified {B} ∧ atms-of A ⊆ vars ∧ atms-of B ⊆ vars
        by fast
    qed
qed

lemma inference-preserves-already-used-all-simple:
  assumes inference S S'
  and already-used-all-simple (snd S) vars
  and simplified (fst S)
  and atms-of-ms (fst S) ⊆ vars
  shows already-used-all-simple (snd S') vars
  using assms
proof (induct rule: inference.induct)
  case (inference-step S clause already-used)
  then show ?case
    using inference-clause-preserves-already-used-all-simple[of S (clause, already-used) vars]
    by auto
qed

lemma already-used-all-simple-inv:
  assumes resolution S S'
  and already-used-all-simple (snd S) vars
  and atms-of-ms (fst S) ⊆ vars
  shows already-used-all-simple (snd S') vars
  using assms
proof (induct rule: resolution.induct)
  case (full1-simp N N')
  then show ?case by simp
next
  case (inferring N already-used N' already-used' N'')
  then show already-used-all-simple (snd (N'', already-used')) vars
    using inference-preserves-already-used-all-simple[of (N, already-used)] by simp
qed

lemma rtranclp-already-used-all-simple-inv:
  assumes resolution** S S'
  and already-used-all-simple (snd S) vars
  and atms-of-ms (fst S) ⊆ vars
  and finite (fst S)
  shows already-used-all-simple (snd S') vars
  using assms
proof (induct rule: rtranclp-induct)
  case base
  then show ?case by simp
next
  case (step S' S'') note infstar = this(1) and IH = this(3) and res = this(2) and
    already = this(4) and atms = this(5) and finite = this(6)
  have already-used-all-simple (snd S') vars using IH already atms finite by simp
  moreover have atms-of-ms (fst S') ⊆ atms-of-ms (fst S)
    by (simp add: infstar local.finite rtranclp-resolution-atms-of)
  then have atms-of-ms (fst S') ⊆ vars using atms by auto
  ultimately show ?case
    using already-used-all-simple-inv[OF res] by simp
qed

lemma inference-clause-simplified-already-used-subset:
  assumes inference-clause S S'
  and simplified (fst S)
  shows snd S ⊂ snd S'
  using assms apply (induct rule: inference-clause.induct, auto)
  using factoring-imp-simplify by blast
```

```
lemma inference-simplified-already-used-subset:
  assumes inference S S′
  and simplified (fst S)
  shows snd S ⊂ snd S′
  using assms apply (induct rule: inference.induct)
  by (metis inference-clause-simplified-already-used-subset snd-conv)

lemma resolution-simplified-already-used-subset:
  assumes resolution S S′
  and simplified (fst S)
  shows snd S ⊂ snd S′
  using assms apply (induct rule: resolution.induct, simp-all add: full1-def)
  apply (meson tranclpD)
  by (metis inference-simplified-already-used-subset fst-conv snd-conv)

lemma tranclp-resolution-simplified-already-used-subset:
  assumes tranclp resolution S S′
  and simplified (fst S)
  shows snd S ⊂ snd S′
  using assms apply (induct rule: tranclp.induct)
  using resolution-simplified-already-used-subset apply metis
  by (meson tranclp-resolution-always-simplified resolution-simplified-already-used-subset
    less-trans)

abbreviation already-used-top vars ≡ simple-clss vars × simple-clss vars

lemma already-used-all-simple-in-already-used-top:
  assumes already-used-all-simple s vars and finite vars
  shows s ⊆ already-used-top vars
proof
  fix x
  assume x-s: x ∈ s
  obtain A B where x: x = (A, B) by (cases x, auto)
  then have simplified {A} and atms-of A ⊆ vars using assms(1) x-s by fastforce+
  then have A: A ∈ simple-clss vars
    using simple-clss-mono[of atms-of A vars] x assms(2)
    simplified-imp-distinct-mset-tauto[of {A}]
    distinct-mset-not-tautology-implies-in-simple-clss by fast
  moreover have simplified {B} and atms-of B ⊆ vars using assms(1) x-s x by fast+
  then have B: B ∈ simple-clss vars
    using simplified-imp-distinct-mset-tauto[of {B}]
    distinct-mset-not-tautology-implies-in-simple-clss
    simple-clss-mono[of atms-of B vars] x assms(2) by fast
  ultimately show x ∈ simple-clss vars × simple-clss vars
    unfolding x by auto
qed

lemma already-used-top-finite:
  assumes finite vars
  shows finite (already-used-top vars)
  using simple-clss-finite assms by auto

lemma already-used-top-increasing:
  assumes var ⊆ var′ and finite var′
  shows already-used-top var ⊆ already-used-top var′
  using assms simple-clss-mono by auto

lemma already-used-all-simple-finite:
  fixes s :: (′a literal multiset × ′a literal multiset) set and vars :: ′a set
  assumes already-used-all-simple s vars and finite vars
  shows finite s
  using assms already-used-all-simple-in-already-used-top[OF assms(1)]
  rev-finite-subset[OF already-used-top-finite[of vars]] by auto

abbreviation card-simple vars ψ ≡ card (already-used-top vars − ψ)

lemma resolution-card-simple-decreasing:
  assumes res: resolution ψ ψ′
  and a-u-s: already-used-all-simple (snd ψ) vars
  and finite-v: finite vars
  and finite-fst: finite (fst ψ)
  and finite-snd: finite (snd ψ)
  and simp: simplified (fst ψ)
  and atms-of-ms (fst ψ) ⊆ vars
  shows card-simple vars (snd ψ′) < card-simple vars (snd ψ)
proof −
  let ?vars = vars
  let ?top = simple-clss ?vars × simple-clss ?vars
  have 1: card-simple vars (snd ψ) = card ?top − card (snd ψ)
    using card-Diff-subset finite-snd  already-used-all-simple-in-already-used-top[OF a-u-s]
    finite-v by metis
  have a-u-s′: already-used-all-simple (snd ψ′) vars
    using already-used-all-simple-inv res a-u-s assms(7) by blast
  have f: finite (snd ψ′) using already-used-all-simple-finite a-u-s′ finite-v by auto
  have 2: card-simple vars (snd ψ′) = card ?top − card (snd ψ′)
    using card-Diff-subset[OF f] already-used-all-simple-in-already-used-top[OF a-u-s′ finite-v]
    by auto
  have card (already-used-top vars) ≥ card (snd ψ′)
    using already-used-all-simple-in-already-used-top[OF a-u-s′ finite-v]
    card-mono[of already-used-top vars snd ψ′] already-used-top-finite[OF finite-v] by metis
  then show ?thesis
    using  psubset-card-mono[OF f resolution-simplified-already-used-subset[OF res simp]]
    unfolding 1 2 by linarith
qed


lemma tranclp-resolution-card-simple-decreasing:
  assumes tranclp resolution ψ ψ′ and finite-fst: finite (fst ψ)
  and already-used-all-simple (snd ψ) vars
  and atms-of-ms (fst ψ) ⊆ vars
  and finite-v: finite vars
  and finite-snd: finite (snd ψ)
  and simplified (fst ψ)
  shows card-simple vars (snd ψ′) < card-simple vars (snd ψ)
  using assms
proof (induct rule: tranclp-induct)
  case (base ψ′)
  then show ?case by (simp add: resolution-card-simple-decreasing)
next
```

```
  case (step ψ' ψ'') note res = this(1) and res' = this(2) and a-u-s = this(5) and
    atms = this(6) and f-v = this(7) and f-fst = this(4) and H = this
  then have card-simple vars (snd ψ') < card-simple vars (snd ψ) by auto
  moreover have a-u-s': already-used-all-simple (snd ψ') vars
    using rtranclp-already-used-all-simple-inv[OF tranclp-into-rtranclp[OF res] a-u-s atms f-fst] .
  have finite (fst ψ')
    by (meson finite-fst res rtranclp-resolution-finite tranclp-into-rtranclp)
  moreover have finite (snd ψ') using already-used-all-simple-finite[OF a-u-s' f-v] .
  moreover have simplified (fst ψ') using res tranclp-resolution-always-simplified by blast
  moreover have atms-of-ms (fst ψ') ⊆ vars
    by (meson atms f-fst order.trans res rtranclp-resolution-atms-of tranclp-into-rtranclp)
  ultimately show ?case
    using resolution-card-simple-decreasing[OF res' a-u-s' f-v] f-v
    less-trans[of card-simple vars (snd ψ'') card-simple vars (snd ψ')
      card-simple vars (snd ψ)]
    by blast
qed


lemma tranclp-resolution-card-simple-decreasing-2:
  assumes tranclp resolution ψ ψ'
  and finite-fst: finite (fst ψ)
  and empty-snd: snd ψ = {}
  and simplified (fst ψ)
  shows card-simple (atms-of-ms (fst ψ)) (snd ψ') < card-simple (atms-of-ms (fst ψ)) (snd ψ)
proof −
  let ?vars = (atms-of-ms (fst ψ))
  have already-used-all-simple (snd ψ) ?vars unfolding empty-snd by auto
  moreover have atms-of-ms (fst ψ) ⊆ ?vars by auto
  moreover have finite-v: finite ?vars using finite-fst by auto
  moreover have finite-snd: finite (snd ψ) unfolding empty-snd by auto
  ultimately show ?thesis
    using assms(1,2,4) tranclp-resolution-card-simple-decreasing[of ψ ψ'] by presburger
qed
```

13.5.2 well-foundness if the relation

```
lemma wf-simplified-resolution:
  assumes f-vars: finite vars
  shows wf {(y:: 'v:: linorder state, x). (atms-of-ms (fst x) ⊆ vars ∧ simplified (fst x)
    ∧ finite (snd x) ∧ finite (fst x) ∧ already-used-all-simple (snd x) vars) ∧ resolution x y}
proof −
  {
    fix a b :: 'v::linorder state
    assume (b, a) ∈ {(y, x). (atms-of-ms (fst x) ⊆ vars ∧ simplified (fst x) ∧ finite (snd x)
      ∧ finite (fst x) ∧ already-used-all-simple (snd x) vars) ∧ resolution x y}
    then have
      atms-of-ms (fst a) ⊆ vars and
      simp: simplified (fst a) and
      finite (snd a) and
      finite (fst a) and
      a-u-v: already-used-all-simple (snd a) vars and
      res: resolution a b by auto
    have finite (already-used-top vars) using f-vars already-used-top-finite by blast
    moreover have already-used-top vars ⊆ already-used-top vars by auto
    moreover have snd b ⊆ already-used-top vars
```

```
      using already-used-all-simple-in-already-used-top[of snd b vars]
      a-u-v already-used-all-simple-inv[OF res] ‹finite (fst a)› ‹atms-of-ms (fst a) ⊆ vars› f-vars
      by presburger
    moreover have snd a ⊂ snd b using resolution-simplified-already-used-subset[OF res simp] .
    ultimately have finite (already-used-top vars) ∧ already-used-top vars ⊆ already-used-top vars
      ∧ snd b ⊆ already-used-top vars ∧ snd a ⊂ snd b by metis
  }
  then show ?thesis using wf-bounded-set[of {(y:: 'v:: linorder state, x).
    (atms-of-ms (fst x) ⊆ vars
    ∧ simplified (fst x) ∧ finite (snd x) ∧ finite (fst x)∧ already-used-all-simple (snd x) vars)
    ∧ resolution x y} λ-. already-used-top vars snd] by auto
qed

lemma wf-simplified-resolution':
  assumes f-vars: finite vars
  shows wf {(y:: 'v:: linorder state, x). (atms-of-ms (fst x) ⊆ vars ∧ ¬simplified (fst x)
    ∧ finite (snd x) ∧ finite (fst x) ∧ already-used-all-simple (snd x) vars) ∧ resolution x y}
  unfolding wf-def
   apply (simp add: resolution-always-simplified)
  by (metis (mono-tags, hide-lams) fst-conv resolution-always-simplified)

lemma wf-resolution:
  assumes f-vars: finite vars
  shows wf ({(y:: 'v:: linorder state, x). (atms-of-ms (fst x) ⊆ vars ∧ simplified (fst x)
        ∧ finite (snd x) ∧ finite (fst x) ∧ already-used-all-simple (snd x) vars) ∧ resolution x y}
    ∪ {(y, x). (atms-of-ms (fst x) ⊆ vars ∧ ¬ simplified (fst x) ∧ finite (snd x) ∧ finite (fst x)
       ∧ already-used-all-simple (snd x) vars) ∧ resolution x y}) (is wf (?R ∪ ?S))
proof −
  have Domain ?R Int Range ?S = {} using resolution-always-simplified by auto blast
  then show wf (?R ∪ ?S)
    using wf-simplified-resolution[OF f-vars] wf-simplified-resolution'[OF f-vars] wf-Un[of ?R ?S]
    by fast
qed

lemma rtrancp-simplify-already-used-inv:
  assumes simplify** S S'
  and already-used-inv (S, N)
  shows already-used-inv (S', N)
  using assms apply induction
  using simplify-preserves-already-used-inv by fast+

lemma full1-simplify-already-used-inv:
  assumes full1 simplify S S'
  and already-used-inv (S, N)
  shows already-used-inv (S', N)
  using assms tranclp-into-rtranclp[of simplify S S'] rtrancp-simplify-already-used-inv
  unfolding full1-def by fast

lemma full-simplify-already-used-inv:
  assumes full simplify S S'
  and already-used-inv (S, N)
  shows already-used-inv (S', N)
  using assms  rtrancp-simplify-already-used-inv unfolding full-def by fast
lemma resolution-already-used-inv:
  assumes resolution S S'
```

and *already-used-inv S*
shows *already-used-inv S'*
using *assms*
proof *induction*
case (*full1-simp N N' already-used*)
then show *?case* **using** *full1-simplify-already-used-inv* **by** *fast*
next
case (*inferring N already-used N' already-used' N''*) **note** *inf* = *this*(*1*) **and** *full* = *this*(*3*) **and** *a-u-v* = *this*(*4*)
then show *?case*
using *inference-preserves-already-used-inv*[*OF inf a-u-v*] *full-simplify-already-used-inv full*
by *fast*
qed

lemma *rtranclp-resolution-already-used-inv*:
assumes *resolution*** *S S'*
and *already-used-inv S*
shows *already-used-inv S'*
using *assms* **apply** *induction*
using *resolution-already-used-inv* **by** *fast+*

lemma *rtanclp-simplify-preserves-unsat*:
assumes *simplify*** ψ ψ'
shows *satisfiable* ψ' ⟶ *satisfiable* ψ
using *assms* **apply** *induction*
using *simplify-clause-preserves-sat* **by** *blast+*

lemma *full1-simplify-preserves-unsat*:
assumes *full1 simplify* ψ ψ'
shows *satisfiable* ψ' ⟶ *satisfiable* ψ
using *assms rtanclp-simplify-preserves-unsat*[*of* ψ ψ'] *tranclp-into-rtranclp*
unfolding *full1-def* **by** *metis*

lemma *full-simplify-preserves-unsat*:
assumes *full simplify* ψ ψ'
shows *satisfiable* ψ' ⟶ *satisfiable* ψ
using *assms rtanclp-simplify-preserves-unsat*[*of* ψ ψ'] **unfolding** *full-def* **by** *metis*

lemma *resolution-preserves-unsat*:
assumes *resolution* ψ ψ'
shows *satisfiable* (*fst* ψ') ⟶ *satisfiable* (*fst* ψ)
using *assms* **apply** (*induct rule*: *resolution.induct*)
using *full1-simplify-preserves-unsat* **apply** (*metis fst-conv*)
using *full-simplify-preserves-unsat simplify-preserves-unsat* **by** *fastforce*

lemma *rtranclp-resolution-preserves-unsat*:
assumes *resolution*** ψ ψ'
shows *satisfiable* (*fst* ψ') ⟶ *satisfiable* (*fst* ψ)
using *assms* **apply** *induction*
using *resolution-preserves-unsat* **by** *fast+*

lemma *rtranclp-simplify-preserve-partial-tree*:
assumes *simplify*** *N N'*
and *partial-interps t I N*
shows *partial-interps t I N'*

```
  using assms apply (induction, simp)
  using simplify-preserve-partial-tree by metis

lemma full1-simplify-preserve-partial-tree:
  assumes full1 simplify N N′
  and partial-interps t I N
  shows partial-interps t I N′
  using assms rtranclp-simplify-preserve-partial-tree[of N N′ t I] tranclp-into-rtranclp
  unfolding full1-def by fast

lemma full-simplify-preserve-partial-tree:
  assumes full simplify N N′
  and partial-interps t I N
  shows partial-interps t I N′
  using assms rtranclp-simplify-preserve-partial-tree[of N N′ t I] tranclp-into-rtranclp
  unfolding full-def by fast

lemma resolution-preserve-partial-tree:
  assumes resolution S S′
  and partial-interps t I (fst S)
  shows partial-interps t I (fst S′)
  using assms apply induction
    using full1-simplify-preserve-partial-tree fst-conv apply metis
  using full-simplify-preserve-partial-tree inference-preserve-partial-tree by fastforce

lemma rtranclp-resolution-preserve-partial-tree:
  assumes resolution** S S′
  and partial-interps t I (fst S)
  shows partial-interps t I (fst S′)
  using assms apply induction
  using resolution-preserve-partial-tree by fast+
  thm nat-less-induct nat.induct

lemma nat-ge-induct[case-names 0 Suc]:
  assumes P 0
  and (⋀n. (⋀m. m<Suc n ⟹ P m) ⟹ P (Suc n))
  shows P n
  using assms apply (induct rule: nat-less-induct)
  by (rename-tac n, case-tac n) auto

lemma wf-always-more-step-False:
  assumes wf R
  shows (∀ x. ∃ z. (z, x)∈R) ⟹ False
 using assms unfolding wf-def by (meson Domain.DomainI assms wfE-min)

lemma finite-finite-mset-element-of-mset[simp]:
  assumes finite N
  shows finite {f φ L |φ L. φ ∈ N ∧ L ∈# φ ∧ P φ L}
  using assms
proof (induction N rule: finite-induct)
  case empty
  show ?case by auto
next
  case (insert x N) note finite = this(1) and IH = this(3)
  have {f φ L |φ L. (φ = x ∨ φ ∈ N) ∧ L ∈# φ ∧ P φ L} ⊆ {f x L | L. L ∈# x ∧ P x L}
```

```
    ∪ {f φ L |φ L. φ ∈ N ∧ L ∈# φ ∧ P φ L} by auto
  moreover have finite {f x L | L. L ∈# x} by auto
  ultimately show ?case using IH finite-subset by fastforce
qed


 value card
 value filter-mset
value {#count φ L |L ∈# φ. 2 ≤ count φ L#}
value (λφ. msetsum {#count φ L |L ∈# φ. 2 ≤ count φ L#})

syntax
  -comprehension1′-mset :: ′a ⇒ ′b ⇒ ′b multiset ⇒ ′a multiset
      (({#-/. - : setof -#}))
translations
  {#e. x: setof M#} == CONST set-mset (CONST image-mset (%x. e) M)
value {# a. a : setof {#1,1,2::int#}#} = {1,2}

definition sum-count-ge-2 :: ′a multiset set ⇒ nat (Ξ) where
sum-count-ge-2 ≡ folding.F (λφ. op +(msetsum {#count φ L |L ∈# φ. 2 ≤ count φ L#})) 0


interpretation sum-count-ge-2:
  folding (λφ. op +(msetsum {#count φ L |L ∈# φ. 2 ≤ count φ L#})) 0
rewrites
  folding.F (λφ. op +(msetsum {#count φ L |L ∈# φ. 2 ≤ count φ L#})) 0 = sum-count-ge-2
proof −
  show folding (λφ. op + (msetsum (image-mset (count φ) {# L ∈# φ. 2 ≤ count φ L#})))
    by standard auto
  then interpret sum-count-ge-2:
    folding (λφ. op +(msetsum {#count φ L |L ∈# φ. 2 ≤ count φ L#})) 0 .
  show folding.F (λφ. op + (msetsum (image-mset (count φ) {# L ∈# φ. 2 ≤ count φ L#}))) 0
    = sum-count-ge-2 by (auto simp add: sum-count-ge-2-def)
qed

lemma finite-incl-le-setsum:
 finite (B::′a multiset set) ⟹ A ⊆ B ⟹ Ξ A ≤ Ξ B
proof (induction arbitrary:A rule: finite-induct)
  case empty
  then show ?case by simp
next
  case (insert a F) note finite = this(1) and aF = this(2) and IH = this(3) and AF = this(4)
  show ?case
    proof (cases a ∈ A)
      assume a ∉ A
      then have A ⊆ F using AF by auto
      then show ?case using IH[of A] by (simp add: aF local.finite)
    next
      assume aA: a ∈ A
      then have A − {a} ⊆ F using AF by auto
      then have Ξ (A − {a}) ≤ Ξ F using IH by blast
      then show ?case
        proof −
          obtain nn :: nat ⇒ nat ⇒ nat where
            ∀ x0 x1. (∃ v2. x0 = x1 + v2) = (x0 = x1 + nn x0 x1)
```

```
             by moura
           then have Ξ F = Ξ (A − {a}) + nn (Ξ F) (Ξ (A − {a}))
             by (meson ‹Ξ (A − {a}) ≤ Ξ F› le-iff-add)
           then show ?thesis
             by (metis (no-types) le-iff-add aA aF add.assoc finite.insertI finite-subset
               insert.prems local.finite sum-count-ge-2.insert sum-count-ge-2.remove)
         qed
    qed
qed

lemma simplify-finite-measure-decrease:
  simplify N N′ ⟹ finite N ⟹ card N′ + Ξ N′ < card N + Ξ N
proof (induction rule: simplify.induct)
  case (tautology-deletion A P) note an = this(1) and fin = this(2)
  let ?N′ = N − {A + {#Pos P#} + {#Neg P#}}
  have card ?N′ < card N
    by (meson card-Diff1-less tautology-deletion.hyps tautology-deletion.prems)
  moreover have ?N′ ⊆ N by auto
  then have sum-count-ge-2 ?N′ ≤ sum-count-ge-2 N using finite-incl-le-setsum[OF fin] by blast
  ultimately show ?case by linarith
next
  case (condensation A L) note AN = this(1) and fin = this(2)
  let ?C′ = A + {#L#}
  let ?C = A + {#L#} + {#L#}
  let ?N′ = N − {?C} ∪ {?C′}
  have card ?N′ ≤ card N
    using AN by (metis (no-types, lifting) Diff-subset Un-empty-right Un-insert-right card.remove
      card-insert-if card-mono fin finite-Diff order-refl)
  moreover have Ξ {?C′} < Ξ {?C}
    proof −
      have mset-decomp:
        {# La ∈# A. (L = La ⟶ La ∈# A) ∧ (L ≠ La ⟶ 2 ≤ count A La)#}
        = {# La ∈# A. L ≠ La ∧ 2 ≤ count A La#} +
          {# La ∈# A. L = La ∧ Suc 0 ≤ count A L#}
           by (auto simp: multiset-eq-iff ac-simps)
      have mset-decomp2: {# La ∈# A. L ≠ La ⟶ 2 ≤ count A La#} =
        {# La ∈# A. L ≠ La ∧ 2 ≤ count A La#} + replicate-mset (count A L) L
        by (auto simp: multiset-eq-iff)
      show ?thesis
        by (auto simp: mset-decomp mset-decomp2 filter-mset-eq ac-simps)
   qed
  have Ξ ?N′ < Ξ N
    proof cases
      assume a1: ?C′ ∈ N
      then show ?thesis
        proof −
          have f2: ⋀m M. insert (m::′a literal multiset) (M − {m}) = M ∪ {} ∨ m ∉ M
            using Un-empty-right insert-Diff by blast
          have f3: ⋀m M Ma. insert (m::′a literal multiset) M − insert m Ma = M − insert m Ma
            by simp
          then have f4: ⋀M m. M − {m::′a literal multiset} = M ∪ {} ∨ m ∈ M
            using Diff-insert-absorb Un-empty-right by fastforce
          have f5: insert (A + {#L#} + {#L#}) N = N
            using f3 f2 Un-empty-right condensation.hyps insert-iff by fastforce
          have ⋀m M. insert (m::′a literal multiset) M = M ∪ {} ∨ m ∉ M
            using f3 f2 Un-empty-right add.right-neutral insert-iff by fastforce
          then have Ξ (N − {A + {#L#} + {#L#}}) < Ξ N
            using f5 f4 by (metis Un-empty-right ‹Ξ {A + {#L#}} < Ξ {A + {#L#} + {#L#}}›
              add.right-neutral add-diff-cancel-left′ add-gr-0 diff-less fin finite.emptyI not-le
              sum-count-ge-2.empty sum-count-ge-2.insert-remove trans-le-add2)
          then show ?thesis
            using f3 f2 a1 by (metis (no-types) Un-empty-right Un-insert-right condensation.hyps
              insert-iff multi-self-add-other-not-self)
        qed
    next
      assume ?C′ ∉ N
      have mset-decomp:
        {# La ∈# A. (L = La ⟶ Suc 0 ≤ count A La) ∧ (L ≠ La ⟶ 2 ≤ count A La)#}
        = {# La ∈# A. L ≠ La ∧ 2 ≤ count A La#} +
          {# La ∈# A. L = La ∧ Suc 0 ≤ count A L#}
           by (auto simp: multiset-eq-iff ac-simps)
      have mset-decomp2: {# La ∈# A. L ≠ La ⟶ 2 ≤ count A La#} =
        {# La ∈# A. L ≠ La ∧ 2 ≤ count A La#} + replicate-mset (count A L) L
        by (auto simp: multiset-eq-iff)

      show ?thesis
        using ‹Ξ {A + {#L#}} < Ξ {A + {#L#} + {#L#}}› condensation.hyps fin
        sum-count-ge-2.remove[of - A + {#L#} + {#L#}] ‹?C′ ∉ N›
        by (auto simp: mset-decomp mset-decomp2 filter-mset-eq)
    qed
  ultimately show ?case by linarith
next
  case (subsumption A B) note AN = this(1) and AB = this(2) and BN = this(3) and fin = this(4)
  have card (N − {B}) < card N using BN by (meson card-Diff1-less subsumption.prems)
  moreover have Ξ (N − {B}) ≤ Ξ N
    by (simp add: Diff-subset finite-incl-le-setsum subsumption.prems)
  ultimately show ?case by linarith
qed

lemma simplify-terminates:
  wf {(N′, N). finite N ∧ simplify N N′}
  using assms apply (rule wfP-if-measure[of finite simplify λN. card N + Ξ N])
  using simplify-finite-measure-decrease by blast



lemma wf-terminates:
  assumes wf r
  shows ∃N′.(N′, N)∈ r* ∧ (∀N″. (N″, N′)∉ r)
proof −
  let ?P = λN. (∃N′.(N′, N)∈ r* ∧ (∀N″. (N″, N′)∉ r))
  have (∀x. (∀y. (y, x) ∈ r ⟶ ?P y) ⟶ ?P x)
    proof clarify
      fix x
      assume H: ∀y. (y, x) ∈ r ⟶ ?P y
      { assume ∃y. (y, x) ∈ r
        then obtain y where y: (y, x) ∈ r by blast
        then have ?P y using H by blast
        then have ?P x using y by (meson rtrancl.rtrancl-into-rtrancl)
      }
      moreover {
        assume ¬(∃ y. (y, x) ∈ r)
        then have ?P x by auto
      }
      ultimately show ?P x by blast
    qed
  moreover have (∀ x. (∀ y. (y, x) ∈ r ⟶ ?P y) ⟶ ?P x) ⟶ All ?P
    using assms unfolding wf-def by (rule allE)
  ultimately have All ?P by blast
  then show ?P N by blast
qed

lemma rtranclp-simplify-terminates:
  assumes fin: finite N
  shows ∃ N'. simplify** N N' ∧ simplified N'
proof –
  have H: {(N', N). finite N ∧ simplify N N'} = {(N', N). simplify N N' ∧ finite N} by auto
  then have wf: wf {(N', N). simplify N N' ∧ finite N}
    using simplify-terminates by (simp add: H)
  obtain N' where N': (N', N)∈ {(b, a). simplify a b ∧ finite a}* and
    more: (∀ N''. (N'', N')∉ {(b, a). simplify a b ∧ finite a})
    using  Prop-Resolution.wf-terminates[OF wf, of N] by blast
  have 1: simplify** N N'
    using N' by (induction rule: rtrancl.induct) auto
  then have finite N' using fin rtranclp-simplify-preserves-finite by blast
  then have 2: ∀ N''. ¬simplify N' N'' using more by auto

  show ?thesis using 1 2 by blast
qed

lemma finite-simplified-full1-simp:
  assumes finite N
  shows simplified N ∨ (∃ N'. full1 simplify N N')
  using rtranclp-simplify-terminates[OF assms] unfolding full1-def
  by (metis Nitpick.rtranclp-unfold)

lemma finite-simplified-full-simp:
  assumes finite N
  shows ∃ N'. full simplify N N'
  using rtranclp-simplify-terminates[OF assms] unfolding full-def by metis

lemma can-decrease-tree-size-resolution:
  fixes ψ :: 'v state and tree :: 'v sem-tree
  assumes finite (fst ψ) and already-used-inv ψ
  and partial-interps tree I (fst ψ)
  and simplified (fst ψ)
  shows ∃ (tree':: 'v sem-tree) ψ'. resolution** ψ ψ' ∧ partial-interps tree' I (fst ψ')
    ∧ (sem-tree-size tree' < sem-tree-size tree ∨ sem-tree-size tree = 0)
  using assms
proof (induct arbitrary: I rule: sem-tree-size)
  case (bigger xs I) note IH = this(1) and finite = this(2) and a-u-i = this(3) and part = this(4)
    and simp = this(5)

  { assume sem-tree-size xs = 0
    then have ?case using part by blast
```

```
}

moreover {
  assume sn0: sem-tree-size xs > 0
  obtain ag ad v where xs: xs = Node v ag ad using sn0 by (cases xs, auto)
  {
     assume sem-tree-size ag = 0 ∧ sem-tree-size ad = 0
     then have ag: ag = Leaf and ad: ad = Leaf by (cases ag, auto, cases ad, auto)

     then obtain χ χ' where
       χ: ¬ I ∪ {Pos v} ⊨ χ and
       totχ: total-over-m (I ∪ {Pos v}) {χ} and
       χψ: χ ∈ fst ψ and
       χ': ¬ I ∪ {Neg v} ⊨ χ'  and
       totχ': total-over-m (I ∪ {Neg v}) {χ'} and χ'ψ: χ' ∈ fst ψ
       using part unfolding xs by auto
     have Posv: Pos v ∉# χ using χ unfolding true-cls-def true-lit-def by auto
     have Negv: Neg v ∉# χ' using χ' unfolding true-cls-def true-lit-def by auto
     {
       assume Negχ: ¬Neg v ∈# χ
       then have ¬ I ⊨ χ using χ Posv unfolding true-cls-def true-lit-def by auto
       moreover have total-over-m I {χ}
         using Posv Negχ atm-imp-pos-or-neg-lit totχ unfolding total-over-m-def total-over-set-def
         by fastforce
       ultimately have partial-interps Leaf I (fst ψ)
       and sem-tree-size Leaf < sem-tree-size xs
       and resolution** ψ ψ
         unfolding xs by (auto simp add: χψ)
     }
     moreover {
        assume Posχ: ¬Pos v ∈# χ'
        then have Iχ: ¬ I ⊨ χ' using χ' Posv unfolding true-cls-def true-lit-def by auto
        moreover have total-over-m I {χ'}
          using Negv Posχ atm-imp-pos-or-neg-lit totχ'
          unfolding total-over-m-def total-over-set-def by fastforce
        ultimately have  partial-interps Leaf I (fst ψ)
        and sem-tree-size Leaf < sem-tree-size xs
        and resolution** ψ ψ using χ'ψ Iχ unfolding xs by auto
     }
     moreover {
        assume neg: Neg v ∈# χ and pos: Pos v ∈# χ'
        have count χ (Neg v) = 1
          using simplified-count[OF simp χψ] neg
          by (simp add: dual-order.antisym)
        have count χ' (Pos v) = 1
          using simplified-count[OF simp χ'ψ] pos
          by (simp add: dual-order.antisym)

        obtain C where χC: χ = C + {#Neg v#} and negC: Neg v ∉# C and posC: Pos v ∉# C
          by (metis (no-types, lifting) One-nat-def Posv Suc-eq-plus1-left ‹count χ (Neg v) = 1›
            add-diff-cancel-left' count-diff count-greater-eq-one-iff count-single insert-DiffM
            insert-DiffM2 less-numeral-extra(3) multi-member-skip not-le not-less-eq-eq)

        obtain C' where
          χC': χ' = C' + {#Pos v#} and
```

```
      posC′: Pos v ∉# C′ and
      negC′: Neg v ∉# C′
      by (metis (no-types, lifting) One-nat-def Negv Suc-eq-plus1-left ‹count χ′ (Pos v) = 1›
        add-diff-cancel-left′ count-diff count-greater-eq-one-iff count-single insert-DiffM
        insert-DiffM2 less-numeral-extra(3) multi-member-skip not-le not-less-eq-eq)

    have totC: total-over-m I {C}
      using totχ tot-over-m-remove[of I Pos v C] negC posC unfolding χC
      by (metis total-over-m-sum uminus-Neg uminus-of-uminus-id)
    have totC′: total-over-m I {C′}
      using totχ′ total-over-m-sum tot-over-m-remove[of I Neg v C′] negC′ posC′
      unfolding χC′ by (metis total-over-m-sum uminus-Neg)
    have ¬ I ⊨ C + C′
      using χ χ′ χC χC′ by auto
    then have part-I-ψ′′′: partial-interps Leaf I (fst ψ ∪ {C + C′})
      using totC totC′ ‹¬ I ⊨ C + C′› by (metis Un-insert-right insertI1
        partial-interps.simps(1) total-over-m-sum)
    {
      assume ({#Pos v#} + C′, {#Neg v#} + C) ∉ snd ψ
      then have inf′′: inference ψ (fst ψ ∪ {C + C′}, snd ψ ∪ {(χ′, χ)})
        by (metis χ′ψ χC χC′ χψ add.commute inference-step prod.collapse resolution)
      obtain N′ where full: full simplify (fst ψ ∪ {C + C′}) N′
        by (metis finite-simplified-full-simp fst-conv inf′′ inference-preserves-finite
          local.finite)
      have resolution ψ (N′, snd ψ ∪ {(χ′, χ)})
        using resolution.intros(2)[OF - simp full, of snd ψ snd ψ ∪ {(χ′, χ)}] inf′′
        by (metis surjective-pairing)
      moreover have partial-interps Leaf I N′
        using full-simplify-preserve-partial-tree[OF full part-I-ψ′′′] .
      moreover have sem-tree-size Leaf < sem-tree-size xs unfolding xs by auto
      ultimately have ?case
        by (metis (no-types) prod.sel(1) rtranclp.rtrancl-into-rtrancl rtranclp.rtrancl-refl)
    }
    moreover {
      assume a: ({#Pos v#} + C′, {#Neg v#} + C) ∈ snd ψ
      then have (∃χ ∈ fst ψ. (∀I. total-over-m I {C+C′} ⟶ total-over-m I {χ})
          ∧ (∀I. total-over-m I {χ} ⟶ I ⊨ χ ⟶ I ⊨ C′ + C)) ∨ tautology (C′ + C)
        proof −
          obtain p where p: Pos p ∈# ({#Pos v#} + C′) ∧ Neg p ∈# ({#Neg v#} + C)
              ∧((∃χ∈fst ψ. (∀I. total-over-m I {({#Pos v#} + C′) − {#Pos p#} + (({#Neg v#}
+ C) − {#Neg p#})} ⟶ total-over-m I {χ}) ∧ (∀I. total-over-m I {χ} ⟶ I ⊨ χ ⟶ I ⊨ ({#Pos
v#} + C′) − {#Pos p#} + (({#Neg v#} + C) − {#Neg p#}))) ∨ tautology (({#Pos v#} + C′) −
{#Pos p#} + (({#Neg v#} + C) − {#Neg p#})))
            using a by (blast intro: allE[OF a-u-i[unfolded subsumes-def Ball-def],
                of ({#Pos v#} + C′, {#Neg v#} + C)])
          { assume p ≠ v
            then have Pos p ∈# C′ ∧ Neg p ∈# C using p by force
            then have ?thesis by auto
          }
          moreover {
            assume p = v
           then have ?thesis using p by (metis add.commute add-diff-cancel-left′)
          }
          ultimately show ?thesis by auto
        qed
        moreover {
          assume ∃χ ∈ fst ψ. (∀I. total-over-m I {C+C'} ⟶ total-over-m I {χ})
            ∧ (∀I. total-over-m I {χ} ⟶ I ⊨ χ ⟶ I ⊨ C' + C)
          then obtain ϑ where
            ϑ: ϑ ∈ fst ψ and
            tot-ϑ-CC': ∀I. total-over-m I {C+C'} ⟶ total-over-m I {ϑ} and
            ϑ-inv: ∀I. total-over-m I {ϑ} ⟶ I ⊨ ϑ ⟶ I ⊨ C' + C by blast
          have partial-interps Leaf I (fst ψ)
            using tot-ϑ-CC' ϑ ϑ-inv totC totC' ‹¬ I ⊨ C + C'› total-over-m-sum by fastforce
          moreover have sem-tree-size Leaf < sem-tree-size xs unfolding xs by auto
          ultimately have ?case by blast
        }
        moreover {
          assume tautCC': tautology (C' + C)
          have total-over-m I {C'+C} using totC totC' total-over-m-sum by auto
          then have ¬tautology (C' + C)
            using ‹¬ I ⊨ C + C'› unfolding add.commute[of C C'] total-over-m-def
            unfolding tautology-def by auto
          then have False using tautCC'  unfolding tautology-def by auto
        }
        ultimately have ?case by auto
      }
      ultimately have ?case by auto
   }
   ultimately have ?case using part by (metis (no-types) sem-tree-size.simps(1))
}
moreover {
  assume size-ag: sem-tree-size ag > 0
  have sem-tree-size ag < sem-tree-size xs unfolding xs by auto
  moreover have partial-interps ag (I ∪ {Pos v}) (fst ψ)
  and partad: partial-interps ad (I ∪ {Neg v}) (fst ψ)
    using part partial-interps.simps(2) unfolding xs by metis+
  moreover
    have sem-tree-size ag < sem-tree-size xs ⟹ finite (fst ψ) ⟹ already-used-inv ψ
      ⟹ partial-interps ag (I ∪ {Pos v}) (fst ψ) ⟹ simplified (fst ψ)
      ⟹ ∃tree' ψ'. resolution** ψ ψ' ∧ partial-interps tree' (I ∪ {Pos v}) (fst ψ')
          ∧ (sem-tree-size tree' < sem-tree-size ag ∨ sem-tree-size ag = 0)
      using IH[of ag I ∪ {Pos v}]  by auto
  ultimately obtain ψ' :: 'v state and  tree' :: 'v sem-tree  where
    inf: resolution** ψ ψ'
    and part: partial-interps tree' (I ∪ {Pos v}) (fst ψ')
    and size: sem-tree-size tree' < sem-tree-size ag ∨ sem-tree-size ag = 0
    using finite part rtranclp.rtrancl-refl a-u-i simp by blast

  have partial-interps ad (I ∪ {Neg v}) (fst ψ')
    using rtranclp-resolution-preserve-partial-tree inf partad by fast
  then have partial-interps (Node v tree' ad) I (fst ψ') using part by auto
  then have ?case using inf size size-ag part unfolding xs by fastforce
}
moreover {
  assume size-ad: sem-tree-size ad > 0
  have sem-tree-size ad < sem-tree-size xs unfolding xs by auto
  moreover
    have
      partag: partial-interps ag (I ∪ {Pos v}) (fst ψ) and
```

```
        partial-interps ad (I ∪ {Neg v}) (fst ψ)
        using part partial-interps.simps(2) unfolding xs by metis+
    moreover have sem-tree-size ad < sem-tree-size xs ⟶ finite (fst ψ) ⟶ already-used-inv ψ
      ⟶ ( partial-interps ad (I ∪ {Neg v}) (fst ψ) ⟶ simplified (fst ψ)
      ⟶ (∃ tree' ψ'. resolution** ψ ψ' ∧ partial-interps tree' (I ∪ {Neg v}) (fst ψ')
           ∧ (sem-tree-size tree' < sem-tree-size ad ∨ sem-tree-size ad = 0)))
      using IH by blast
    ultimately obtain ψ' :: 'v state and  tree' :: 'v sem-tree  where
      inf: resolution** ψ ψ'
      and part: partial-interps tree' (I ∪ {Neg v}) (fst ψ')
      and size: sem-tree-size tree' < sem-tree-size ad ∨ sem-tree-size ad = 0
      using finite part  rtranclp.rtrancl-refl a-u-i simp by blast

    have partial-interps ag (I ∪ {Pos v}) (fst ψ')
      using rtranclp-resolution-preserve-partial-tree inf partag by fast
    then have partial-interps (Node v ag tree') I (fst ψ') using part by auto
    then have ?case using inf size size-ad unfolding xs by fastforce
  }
   ultimately have ?case by auto
 }
 ultimately show ?case by auto
qed

lemma resolution-completeness-inv:
  fixes ψ :: 'v ::linorder state
  assumes
    unsat: ¬satisfiable (fst ψ) and
    finite: finite (fst ψ) and
    a-u-v: already-used-inv ψ
  shows ∃ ψ'. (resolution** ψ ψ' ∧ {#} ∈ fst ψ')
proof −
  obtain tree where  partial-interps tree {} (fst ψ)
    using partial-interps-build-sem-tree-atms assms by metis
  then show ?thesis
    using unsat finite a-u-v
    proof (induct tree arbitrary: ψ rule: sem-tree-size)
      case (bigger tree ψ) note H = this
      {
        fix χ
        assume tree: tree = Leaf
        obtain χ where χ: ¬ {} ⊨ χ and totχ: total-over-m {} {χ} and χψ: χ ∈ fst ψ
          using H unfolding tree by auto
        moreover have {#} = χ
          using H atms-empty-iff-empty totχ
          unfolding true-cls-def total-over-m-def total-over-set-def by fastforce
        moreover have resolution** ψ ψ by auto
        ultimately have ?case by metis
      }
      moreover {
        fix v tree1 tree2
        assume tree: tree = Node v tree1 tree2
        obtain ψ₀ where ψ₀: resolution** ψ ψ₀ and simp: simplified (fst ψ₀)
          proof −
            { assume simplified (fst ψ)
              moreover have resolution** ψ ψ by auto
              ultimately have thesis using that by blast
            }
            moreover {
              assume ¬simplified (fst ψ)
              then have ∃ψ'. full1 simplify (fst ψ) ψ'
                by (metis Nitpick.rtranclp-unfold bigger.prems(3) full1-def
                  rtranclp-simplify-terminates)
              then obtain N where full1 simplify (fst ψ) N by metis
              then have resolution ψ (N, snd ψ)
                using resolution.intros(1)[of fst ψ N snd ψ] by auto
              moreover have simplified N
                using ‹full1 simplify (fst ψ) N› unfolding full1-def by blast
              ultimately have ?thesis using that by force
            }
            ultimately show ?thesis by auto
          qed


        have p: partial-interps tree {} (fst ψ0)
        and uns: unsatisfiable (fst ψ0)
        and f: finite (fst ψ0)
        and a-u-v: already-used-inv ψ0
             using ψ0 bigger.prems(1) rtranclp-resolution-preserve-partial-tree apply blast
            using ψ0 bigger.prems(2) rtranclp-resolution-preserves-unsat apply blast
           using ψ0 bigger.prems(3) rtranclp-resolution-finite apply blast
          using rtranclp-resolution-already-used-inv[OF ψ0 bigger.prems(4)] by blast
        obtain tree' ψ' where
          inf: resolution** ψ0 ψ' and
          part': partial-interps tree' {} (fst ψ') and
          decrease: sem-tree-size tree' < sem-tree-size tree ∨ sem-tree-size tree = 0
          using can-decrease-tree-size-resolution[OF f a-u-v p simp] unfolding tautology-def
          by meson
        have s: sem-tree-size tree' < sem-tree-size tree using decrease unfolding tree by auto
        have fin: finite (fst ψ')
          using f inf rtranclp-resolution-finite by blast
        have unsat: unsatisfiable (fst ψ')
          using rtranclp-resolution-preserves-unsat inf uns by metis
        have a-u-i': already-used-inv ψ'
          using a-u-v inf rtranclp-resolution-already-used-inv[of ψ0 ψ'] by auto
        have ?case
          using inf rtranclp-trans[of resolution] H(1)[OF s part' unsat fin a-u-i'] ψ0 by blast
       }
       ultimately show ?case by (cases tree, auto)
       qed
qed

lemma resolution-preserves-already-used-inv:
        assumes resolution S S'
        and already-used-inv S
        shows already-used-inv S'
        using assms
        apply (induct rule: resolution.induct)
         apply (rule full1-simplify-already-used-inv; simp)
        apply (rule full-simplify-already-used-inv, simp)
        apply (rule inference-preserves-already-used-inv, simp)
        apply blast
        done

lemma rtranclp-resolution-preserves-already-used-inv:
        assumes resolution** S S'
        and already-used-inv S
        shows already-used-inv S'
        using assms
        apply (induct rule: rtranclp-induct)
         apply simp
        using resolution-preserves-already-used-inv by fast

lemma resolution-completeness:
        fixes ψ :: 'v ::linorder state
        assumes unsat: ¬satisfiable (fst ψ)
        and finite: finite (fst ψ)
        and snd ψ = {}
        shows ∃ψ'. (resolution** ψ ψ' ∧ {#} ∈ fst ψ')
proof -
        have already-used-inv ψ unfolding assms by auto
        then show ?thesis using assms resolution-completeness-inv by blast
qed

lemma rtranclp-preserves-sat:
        assumes simplify** S S'
        and satisfiable S
        shows satisfiable S'
        using assms apply induction
         apply simp
        by (meson satisfiable-carac satisfiable-def simplify-preserves-un-sat-eq)

lemma resolution-preserves-sat:
        assumes resolution S S'
        and satisfiable (fst S)
        shows satisfiable (fst S')
        using assms apply (induction rule: resolution.induct)
         using rtranclp-preserves-sat tranclp-into-rtranclp unfolding full1-def apply fastforce
        by (metis fst-conv full-def inference-preserves-un-sat rtranclp-preserves-sat
          satisfiable-carac' satisfiable-def)

lemma rtranclp-resolution-preserves-sat:
        assumes resolution** S S'
        and satisfiable (fst S)
        shows satisfiable (fst S')
        using assms apply (induction rule: rtranclp-induct)
         apply simp
        using resolution-preserves-sat by blast

lemma resolution-soundness:
        fixes ψ :: 'v ::linorder state
        assumes resolution** ψ ψ' and {#} ∈ fst ψ'
        shows unsatisfiable (fst ψ)
        using assms by (meson rtranclp-resolution-preserves-sat satisfiable-def true-cls-empty
          true-clss-def)
```

```
lemma resolution-soundness-and-completeness:
fixes ψ :: 'v ::linorder state
assumes finite: finite (fst ψ)
and snd: snd ψ = {}
shows (∃ψ'. (resolution** ψ ψ' ∧ {#} ∈ fst ψ')) ⟷ unsatisfiable (fst ψ)
  using assms resolution-completeness resolution-soundness by metis

lemma simplified-falsity:
  assumes simp: simplified ψ
  and {#} ∈ ψ
  shows ψ = {{#}}
proof (rule ccontr)
  assume H: ¬ ?thesis
  then obtain χ where χ ∈ ψ and χ ≠ {#} using assms(2) by blast
  then have {#} ⊂# χ by (simp add: mset-less-empty-nonempty)
  then have simplify ψ (ψ − {χ})
    using simplify.subsumption[OF assms(2) ‹{#} ⊂# χ› ‹χ ∈ ψ›] by blast
  then show False using simp by blast
qed


lemma simplify-falsity-in-preserved:
  assumes simplify χs χs'
  and {#} ∈ χs
  shows {#} ∈ χs'
  using assms
  by induction auto

lemma rtranclp-simplify-falsity-in-preserved:
  assumes simplify** χs χs'
  and {#} ∈ χs
  shows {#} ∈ χs'
  using assms
  by induction (auto intro: simplify-falsity-in-preserved)

lemma resolution-falsity-get-falsity-alone:
  assumes finite (fst ψ)
  shows (∃ψ'. (resolution** ψ ψ' ∧ {#} ∈ fst ψ')) ⟷ (∃a-u-v. resolution** ψ ({{#}}, a-u-v))
    (is ?A ⟷ ?B)
proof
  assume ?B
  then show ?A by auto
next
  assume ?A
  then obtain χs a-u-v where χs: resolution** ψ (χs, a-u-v) and F: {#} ∈ χs by auto
  { assume simplified χs
    then have ?B using simplified-falsity[OF - F] χs by blast
  }
  moreover {
    assume ¬ simplified χs
    then obtain χs' where full1 simplify χs χs'
       by (metis χs assms finite-simplified-full1-simp fst-conv rtranclp-resolution-finite)
    then have {#} ∈ χs'
      unfolding full1-def by (meson F rtranclp-simplify-falsity-in-preserved
        tranclp-into-rtranclp)
    then have ?B
      by (metis χs ‹full1 simplify χs χs'› fst-conv full1-simp resolution-always-simplified
        rtranclp.rtrancl-into-rtrancl simplified-falsity)
  }
  ultimately show ?B by blast
qed

lemma resolution-soundness-and-completeness':
  fixes ψ :: 'v ::linorder state
  assumes
    finite: finite (fst ψ)and
    snd: snd ψ = {}
  shows (∃ a-u-v. (resolution** ψ ({{#}}, a-u-v))) ⟷ unsatisfiable (fst ψ)
    using assms resolution-completeness resolution-soundness resolution-falsity-get-falsity-alone
    by metis

end

theory Partial-Annotated-Clausal-Logic
imports Partial-Clausal-Logic

begin
```

14 Partial Clausal Logic

We here define marked literals (that will be used in both DPLL and CDCL) and the entailment corresponding to it.

14.1 Marked Literals

14.1.1 Definition

```
datatype ('v, 'lvl, 'mark) marked-lit =
  is-marked: Marked (lit-of: 'v literal) (level-of: 'lvl) |
  is-proped: Propagated (lit-of: 'v literal) (mark-of: 'mark)

lemma marked-lit-list-induct[case-names nil marked proped]:
  assumes P [] and
  ⋀L l xs. P xs ⟹ P (Marked L l # xs) and
  ⋀L m xs. P xs ⟹ P (Propagated L m # xs)
  shows P xs
  using assms apply (induction xs, simp)
  by (rename-tac a xs, case-tac a) auto

lemma is-marked-ex-Marked:
  is-marked L ⟹ ∃ K lvl. L = Marked K lvl
  by (cases L) auto

type-synonym ('v, 'l, 'm) marked-lits = ('v, 'l, 'm) marked-lit list

definition lits-of :: ('a, 'b, 'c) marked-lit set ⇒ 'a literal set where
lits-of Ls = lit-of ` Ls

abbreviation lits-of-l :: ('a, 'b, 'c) marked-lit list ⇒ 'a literal set where
lits-of-l Ls ≡ lits-of (set Ls)
```

lemma *lits-of-l-empty*[*simp*]:
  lits-of {} = {}
  unfolding *lits-of-def* **by** *auto*

lemma *lits-of-insert*[*simp*]:
  lits-of (*insert L Ls*) = *insert* (*lit-of L*) (*lits-of Ls*)
  unfolding *lits-of-def* **by** *auto*

lemma *lits-of-l-append*[*simp*]:
  lits-of ($l \cup l'$) = *lits-of* $l \cup$ *lits-of* l'
  unfolding *lits-of-def* **by** *auto*

lemma *finite-lits-of-def*[*simp*]:
  finite (*lits-of-l L*)
  unfolding *lits-of-def* **by** *auto*

abbreviation *unmark* **where**
unmark ≡ (λ*a*. {#*lit-of a*#})

abbreviation *unmark-s* **where**
unmark-s M ≡ *unmark* ‘ *M*

abbreviation *unmark-l* **where**
unmark-l M ≡ *unmark-s* (*set M*)

lemma *atms-of-ms-lambda-lit-of-is-atm-of-lit-of*[*simp*]:
  atms-of-ms (*unmark-l M'*) = *atm-of* ‘ *lits-of-l M'*
  unfolding *atms-of-ms-def lits-of-def* **by** *auto*

lemma *lits-of-l-empty-is-empty*[*iff*]:
  lits-of-l M = {} ⟷ *M* = []
  by (*induct M*) (*auto simp*: *lits-of-def*)

14.1.2 Entailment

definition *true-annot* :: (*'a*, *'l*, *'m*) *marked-lits* ⇒ *'a clause* ⇒ *bool* (**infix** ⊨*a 49*) **where**
  I ⊨*a C* ⟷ (*lits-of-l I*) ⊨ *C*

definition *true-annots* :: (*'a*, *'l*, *'m*) *marked-lits* ⇒ *'a clauses* ⇒ *bool* (**infix** ⊨*as 49*) **where**
  I ⊨*as CC* ⟷ (∀ *C* ∈ *CC*. *I* ⊨*a C*)

lemma *true-annot-empty-model*[*simp*]:
  ¬[] ⊨*a ψ*
  unfolding *true-annot-def true-cls-def* **by** *simp*

lemma *true-annot-empty*[*simp*]:
  ¬*I* ⊨*a* {#}
  unfolding *true-annot-def true-cls-def* **by** *simp*

lemma *empty-true-annots-def*[*iff*]:
  [] ⊨*as ψ* ⟷ *ψ* = {}
  unfolding *true-annots-def* **by** *auto*

lemma *true-annots-empty*[*simp*]:
  I ⊨*as* {}

unfolding *true-annots-def* **by** *auto*

lemma *true-annots-single-true-annot*[*iff*]:
I ⊨*as* {*C*} ⟷ *I* ⊨*a C*
unfolding *true-annots-def* **by** *auto*

lemma *true-annot-insert-l*[*simp*]:
M ⊨*a A* ⟹ *L* # *M* ⊨*a A*
unfolding *true-annot-def* **by** *auto*

lemma *true-annots-insert-l* [*simp*]:
M ⊨*as A* ⟹ *L* # *M* ⊨*as A*
unfolding *true-annots-def* **by** *auto*

lemma *true-annots-union*[*iff*]:
M ⊨*as A* ∪ *B* ⟷ (*M* ⊨*as A* ∧ *M* ⊨*as B*)
unfolding *true-annots-def* **by** *auto*

lemma *true-annots-insert*[*iff*]:
M ⊨*as insert a A* ⟷ (*M* ⊨*a a* ∧ *M* ⊨*as A*)
unfolding *true-annots-def* **by** *auto*

Link between ⊨*as* and ⊨*s*:

lemma *true-annots-true-cls*:
I ⊨*as CC* ⟷ *lits-of-l I* ⊨*s CC*
unfolding *true-annots-def Ball-def true-annot-def true-clss-def* **by** *auto*

lemma *in-lit-of-true-annot*:
a ∈ *lits-of-l M* ⟷ *M* ⊨*a* {#*a*#}
unfolding *true-annot-def lits-of-def* **by** *auto*

lemma *true-annot-lit-of-notin-skip*:
L # *M* ⊨*a A* ⟹ *lit-of L* ∉# *A* ⟹ *M* ⊨*a A*
unfolding *true-annot-def true-cls-def* **by** *auto*

lemma *true-clss-singleton-lit-of-implies-incl*:
I ⊨*s unmark-l MLs* ⟹ *lits-of-l MLs* ⊆ *I*
unfolding *true-clss-def lits-of-def* **by** *auto*

lemma *true-annot-true-clss-cls*:
MLs ⊨*a ψ* ⟹ *set* (*map unmark MLs*) ⊨*p ψ*
unfolding *true-annot-def true-clss-cls-def true-cls-def*
by (*auto dest*: *true-clss-singleton-lit-of-implies-incl*)

lemma *true-annots-true-clss-cls*:
MLs ⊨*as ψ* ⟹ *set* (*map unmark MLs*) ⊨*ps ψ*
by (*auto*
dest: *true-clss-singleton-lit-of-implies-incl*
simp add: *true-clss-def true-annots-def true-annot-def lits-of-def true-cls-def*
true-clss-clss-def)

lemma *true-annots-marked-true-cls*[*iff*]:
map (*λM*. *Marked M a*) *M* ⊨*as N* ⟷ *set M* ⊨*s N*
proof −

```
  have ∗: lit-of ‘ (λM. Marked M a) ‘ set M = set M unfolding lits-of-def by force
  show ?thesis by (simp add: true-annots-true-cls ∗ lits-of-def)
qed

lemma true-annot-singleton[iff]: M ⊨a {#L#} ⟷ L ∈ lits-of-l M
  unfolding true-annot-def lits-of-def by auto

lemma true-annots-true-clss-clss:
  A ⊨as Ψ ⟹ unmark-l A ⊨ps Ψ
  unfolding true-clss-clss-def true-annots-def true-clss-def
  by (auto
    dest!: true-clss-singleton-lit-of-implies-incl
    simp add: lits-of-def true-annot-def true-cls-def)

lemma true-annot-commute:
  M @ M' ⊨a D ⟷ M' @ M ⊨a D
  unfolding true-annot-def by (simp add: Un-commute)

lemma true-annots-commute:
  M @ M' ⊨as D ⟷ M' @ M ⊨as D
  unfolding true-annots-def by (auto simp add: true-annot-commute)

lemma true-annot-mono[dest]:
  set I ⊆ set I' ⟹ I ⊨a N ⟹ I' ⊨a N
  using true-cls-mono-set-mset-l unfolding true-annot-def lits-of-def
  by (metis (no-types) Un-commute Un-upper1 image-Un sup.orderE)

lemma true-annots-mono:
  set I ⊆ set I' ⟹ I ⊨as N ⟹ I' ⊨as N
  unfolding true-annots-def by auto
```

14.1.3 Defined and undefined literals

```
definition defined-lit :: ('a, 'l, 'm) marked-lit list ⇒ 'a literal ⇒ bool
  where
defined-lit I L ⟷ (∃ l. Marked L l ∈ set I) ∨ (∃ P. Propagated L P ∈ set I)
  ∨ (∃ l. Marked (−L) l ∈ set I) ∨ (∃ P. Propagated (−L) P ∈ set I)

abbreviation undefined-lit :: ('a, 'l, 'm) marked-lit list ⇒ 'a literal ⇒ bool
where undefined-lit I L ≡ ¬defined-lit I L

lemma defined-lit-rev[simp]:
  defined-lit (rev M) L ⟷ defined-lit M L
  unfolding defined-lit-def by auto

lemma atm-imp-marked-or-proped:
  assumes x ∈ set I
  shows
    (∃ l. Marked (− lit-of x) l ∈ set I)
    ∨ (∃ l. Marked (lit-of x) l ∈ set I)
    ∨ (∃ l. Propagated (− lit-of x) l ∈ set I)
    ∨ (∃ l. Propagated (lit-of x) l ∈ set I)
  using assms marked-lit.exhaust-sel by metis

lemma literal-is-lit-of-marked:
  assumes L = lit-of x
  shows (∃ l. x = Marked L l) ∨ (∃ l'. x = Propagated L l')
  using assms by (cases x) auto

lemma true-annot-iff-marked-or-true-lit:
  defined-lit I L ⟷ ((lits-of-l I) ⊨l L ∨ (lits-of-l I) ⊨l −L)
  unfolding defined-lit-def by (auto simp add: lits-of-def rev-image-eqI
    dest!: literal-is-lit-of-marked)

lemma consistent-interp (lits-of-l I) ⟹ I ⊨as N ⟹ satisfiable N
  by (simp add: true-annots-true-cls)

lemma defined-lit-map:
  defined-lit Ls L ⟷ atm-of L ∈ (λl. atm-of (lit-of l)) ` set Ls
 unfolding defined-lit-def apply (rule iffI)
   using image-iff apply fastforce
 by (fastforce simp add: atm-of-eq-atm-of dest: atm-imp-marked-or-proped)

lemma defined-lit-uminus[iff]:
  defined-lit I (−L) ⟷ defined-lit I L
  unfolding defined-lit-def by auto

lemma Marked-Propagated-in-iff-in-lits-of-l:
  defined-lit I L ⟷ (L ∈ lits-of-l I ∨ −L ∈ lits-of-l I)
  unfolding lits-of-def defined-lit-def
  by (auto simp: rev-image-eqI) (rename-tac x, case-tac x, auto)+

lemma consistent-add-undefined-lit-consistent[simp]:
  assumes
    consistent-interp (lits-of-l Ls) and
    undefined-lit Ls L
  shows consistent-interp (insert L (lits-of-l Ls))
  using assms unfolding consistent-interp-def by (auto simp: Marked-Propagated-in-iff-in-lits-of-l)

lemma decided-empty[simp]:
  ¬defined-lit [] L
  unfolding defined-lit-def by simp
```

14.2 Backtracking

```
fun backtrack-split :: ('v, 'l, 'm) marked-lits
  ⇒ ('v, 'l, 'm) marked-lits × ('v, 'l, 'm) marked-lits where
backtrack-split [] = ([], []) |
backtrack-split (Propagated L P # mlits) = apfst ((op #) (Propagated L P)) (backtrack-split mlits) |
backtrack-split (Marked L l # mlits) = ([], Marked L l # mlits)

lemma backtrack-split-fst-not-marked: a ∈ set (fst (backtrack-split l)) ⟹ ¬is-marked a
  by (induct l rule: marked-lit-list-induct) auto

lemma backtrack-split-snd-hd-marked:
  snd (backtrack-split l) ≠ [] ⟹ is-marked (hd (snd (backtrack-split l)))
  by (induct l rule: marked-lit-list-induct) auto

lemma backtrack-split-list-eq[simp]:
  fst (backtrack-split l) @ (snd (backtrack-split l)) = l
  by (induct l rule: marked-lit-list-induct) auto
```

```
lemma backtrack-snd-empty-not-marked:
  backtrack-split M = (M'', []) ⟹ ∀ l∈set M. ¬ is-marked l
  by (metis append-Nil2 backtrack-split-fst-not-marked backtrack-split-list-eq snd-conv)

lemma backtrack-split-some-is-marked-then-snd-has-hd:
  ∃ l∈set M. is-marked l ⟹ ∃ M' L' M''. backtrack-split M = (M'', L' # M')
  by (metis backtrack-snd-empty-not-marked list.exhaust prod.collapse)
```

Another characterisation of the result of *backtrack-split*. This view allows some simpler proofs, since *takeWhile* and *dropWhile* are highly automated:

```
lemma backtrack-split-takeWhile-dropWhile:
  backtrack-split M = (takeWhile (Not o is-marked) M, dropWhile (Not o is-marked) M)
proof (induct M)
  case Nil show ?case by simp
next
  case (Cons L M) then show ?case by (cases L) auto
qed
```

14.3 Decomposition with respect to the marked literals

The pattern *get-all-marked-decomposition* [] = [([], [])] is necessary otherwise, we can call the *hd* function in the other pattern.

```
fun get-all-marked-decomposition :: ('a, 'l, 'm) marked-lits
  ⇒ (('a, 'l, 'm) marked-lits × ('a, 'l, 'm) marked-lits) list where
get-all-marked-decomposition (Marked L l # Ls) =
  (Marked L l # Ls, []) # get-all-marked-decomposition Ls |
get-all-marked-decomposition (Propagated L P# Ls) =
  (apsnd ((op #) (Propagated L P)) (hd (get-all-marked-decomposition Ls)))
    # tl (get-all-marked-decomposition Ls) |
get-all-marked-decomposition [] = [([], [])]

value get-all-marked-decomposition [Propagated A5 B5, Marked C4 D4, Propagated A3 B3,
  Propagated A2 B2, Marked C1 D1, Propagated A0 B0]


lemma get-all-marked-decomposition-never-empty[iff]:
  get-all-marked-decomposition M = [] ⟷ False
  by (induct M, simp) (rename-tac a xs, case-tac a, auto)

lemma get-all-marked-decomposition-never-empty-sym[iff]:
  [] = get-all-marked-decomposition M ⟷ False
  using get-all-marked-decomposition-never-empty[of M] by presburger

lemma get-all-marked-decomposition-decomp:
  hd (get-all-marked-decomposition S) = (a, c) ⟹ S = c @ a
proof (induct S arbitrary: a c)
  case Nil
  then show ?case by simp
next
  case (Cons x A)
  then show ?case by (cases x; cases hd (get-all-marked-decomposition A)) auto
qed

lemma get-all-marked-decomposition-backtrack-split:
  backtrack-split S = (M, M') ⟷ hd (get-all-marked-decomposition S) = (M', M)
```

```
proof (induction S arbitrary: M M′)
  case Nil
  then show ?case by auto
next
  case (Cons a S)
  then show ?case using backtrack-split-takeWhile-dropWhile by (cases a) force+
qed

lemma get-all-marked-decomposition-nil-backtrack-split-snd-nil:
  get-all-marked-decomposition S = [([], A)] ⟹ snd (backtrack-split S) = []
  by (simp add: get-all-marked-decomposition-backtrack-split sndI)

lemma get-all-marked-decomposition-length-1-fst-empty-or-length-1:
  assumes get-all-marked-decomposition M = (a, b) # []
  shows a = [] ∨ (length a = 1 ∧ is-marked (hd a) ∧ hd a ∈ set M)
  using assms
proof (induct M arbitrary: a b rule: marked-lit-list-induct)
  case nil then show ?case by simp
next
  case (marked L mark M)
  then show ?case by simp
next
  case (proped L mark M)
  then show ?case by (cases get-all-marked-decomposition M) force+
qed

lemma get-all-marked-decomposition-fst-empty-or-hd-in-M:
  assumes get-all-marked-decomposition M = (a, b) # l
  shows a = [] ∨ (is-marked (hd a) ∧ hd a ∈ set M)
  using assms apply (induct M arbitrary: a b rule: marked-lit-list-induct)
    apply auto[2]
  by (metis UnCI backtrack-split-snd-hd-marked get-all-marked-decomposition-backtrack-split
    get-all-marked-decomposition-decomp hd-in-set list.sel(1) set-append snd-conv)

lemma get-all-marked-decomposition-snd-not-marked:
  assumes (a, b) ∈ set (get-all-marked-decomposition M)
  and L ∈ set b
  shows ¬is-marked L
  using assms apply (induct M arbitrary: a b rule: marked-lit-list-induct, simp)
  by (rename-tac L′ l xs a b, case-tac get-all-marked-decomposition xs; fastforce)+

lemma tl-get-all-marked-decomposition-skip-some:
  assumes x ∈ set (tl (get-all-marked-decomposition M1))
  shows x ∈ set (tl (get-all-marked-decomposition (M0 @ M1)))
  using assms
  by (induct M0 rule: marked-lit-list-induct)
     (auto simp add: list.set-sel(2))

lemma hd-get-all-marked-decomposition-skip-some:
  assumes (x, y) = hd (get-all-marked-decomposition M1)
  shows (x, y) ∈ set (get-all-marked-decomposition (M0 @ Marked K i # M1))
  using assms
proof (induct M0)
  case Nil
  then show ?case by auto
```

```
next
  case (Cons L M0)
  then have xy: (x, y) ∈ set (get-all-marked-decomposition (M0 @ Marked K i # M1)) by blast
  show ?case
    proof (cases L)
      case (Marked l m)
      then show ?thesis using xy by auto
    next
      case (Propagated l m)
      then show ?thesis
        using xy Cons.prems
        by (cases get-all-marked-decomposition (M0 @ Marked K i # M1))
           (auto dest!: get-all-marked-decomposition-decomp
              arg-cong[of get-all-marked-decomposition -  - hd])
    qed
qed

lemma get-all-marked-decomposition-snd-union:
  set M = ⋃(set ‘ snd ‘ set (get-all-marked-decomposition M)) ∪ {L |L. is-marked L ∧ L ∈ set M}
  (is ?M M = ?U M ∪ ?Ls M)
proof (induct M arbitrary:)
  case Nil
  then show ?case by simp
next
  case (Cons L M)
  show ?case
    proof (cases L)
      case (Marked a l) note L = this
      then have L ∈ ?Ls (L#M) by auto
      moreover have ?U (L#M) = ?U M unfolding L by auto
      moreover have ?M M = ?U M ∪ ?Ls M using Cons.hyps by auto
      ultimately show ?thesis by auto
    next
      case (Propagated a P)
      then show ?thesis using Cons.hyps by (cases (get-all-marked-decomposition M)) auto
    qed
qed

lemma in-get-all-marked-decomposition-in-get-all-marked-decomposition-prepend:
  (a, b) ∈ set (get-all-marked-decomposition M′) ⟹
    ∃ b′. (a, b′ @ b) ∈ set (get-all-marked-decomposition (M @ M′))
  apply (induction M rule: marked-lit-list-induct)
    apply (metis append-Nil)
   apply auto[]
  by (rename-tac L′ m xs, case-tac get-all-marked-decomposition (xs @ M′)) auto

lemma get-all-marked-decomposition-remove-unmark-ssed-length:
  assumes ∀ l ∈ set M′. ¬is-marked l
  shows length (get-all-marked-decomposition (M′ @ M″))
    = length (get-all-marked-decomposition M″)
  using assms by (induct M′ arbitrary: M″ rule: marked-lit-list-induct) auto

lemma get-all-marked-decomposition-not-is-marked-length:
  assumes ∀ l ∈ set M′. ¬is-marked l
  shows 1 + length (get-all-marked-decomposition (Propagated (−L) P # M))
```

```
    = length (get-all-marked-decomposition (M' @ Marked L l # M))
 using assms get-all-marked-decomposition-remove-unmark-ssed-length by fastforce

lemma get-all-marked-decomposition-last-choice:
  assumes tl (get-all-marked-decomposition (M' @ Marked L l # M)) ≠ []
  and ∀ l ∈ set M'. ¬is-marked l
  and hd (tl (get-all-marked-decomposition (M' @ Marked L l # M))) = (M0', M0)
  shows hd (get-all-marked-decomposition (Propagated (−L) P # M)) = (M0', Propagated (−L) P #
M0)
  using assms by (induct M' rule: marked-lit-list-induct) auto

lemma get-all-marked-decomposition-except-last-choice-equal:
  assumes ∀ l ∈ set M'. ¬is-marked l
  shows tl (get-all-marked-decomposition (Propagated (−L) P # M))
    = tl (tl (get-all-marked-decomposition (M' @ Marked L l # M)))
  using assms by (induct M'  rule: marked-lit-list-induct) auto

lemma get-all-marked-decomposition-hd-hd:
  assumes get-all-marked-decomposition Ls = (M, C) # (M0, M0') # l
  shows tl M = M0' @ M0 ∧ is-marked (hd M)
  using assms
proof (induct Ls arbitrary: M C M0 M0' l)
  case Nil
  then show ?case by simp
next
  case (Cons a Ls M C M0 M0' l) note IH = this(1) and g = this(2)
  { fix L level
    assume a: a = Marked L level
    have Ls = M0' @ M0
      using g a by (force intro: get-all-marked-decomposition-decomp)
    then have tl M = M0' @ M0 ∧ is-marked (hd M) using g a by auto
  }
  moreover {
    fix L P
    assume a: a = Propagated L P
    have tl M = M0' @ M0 ∧ is-marked (hd M)
      using IH Cons.prems unfolding a by (cases get-all-marked-decomposition Ls) auto
  }
  ultimately show ?case by (cases a) auto
qed

lemma get-all-marked-decomposition-exists-prepend[dest]:
  assumes (a, b) ∈ set (get-all-marked-decomposition M)
  shows ∃ c. M = c @ b @ a
  using assms apply (induct M rule: marked-lit-list-induct)
    apply simp
  by (rename-tac L' m xs, case-tac get-all-marked-decomposition xs;
    auto dest!: arg-cong[of get-all-marked-decomposition - - hd]
      get-all-marked-decomposition-decomp)+

lemma get-all-marked-decomposition-incl:
  assumes (a, b) ∈ set (get-all-marked-decomposition M)
  shows set b ⊆ set M and set a ⊆ set M
  using assms get-all-marked-decomposition-exists-prepend by fastforce+
lemma get-all-marked-decomposition-exists-prepend':
  assumes (a, b) ∈ set (get-all-marked-decomposition M)
  obtains c where M = c @ b @ a
  using assms apply (induct M rule: marked-lit-list-induct)
    apply auto[1]
  by (rename-tac L' m xs, case-tac hd (get-all-marked-decomposition xs),
    auto dest!: get-all-marked-decomposition-decomp simp add: list.set-sel(2))+

lemma union-in-get-all-marked-decomposition-is-subset:
  assumes (a, b) ∈ set (get-all-marked-decomposition M)
  shows set a ∪ set b ⊆ set M
  using assms by force

lemma Marked-cons-in-get-all-marked-decomposition-append-Marked-cons:
  ∃ M1 M2. (Marked K i # M1, M2) ∈ set (get-all-marked-decomposition (c @ Marked K i # c'))
  apply (induction c rule: marked-lit-list-induct)
    apply auto[2]
  apply (rename-tac L m xs,
      case-tac hd (get-all-marked-decomposition (xs @ Marked K i # c')))
  apply (case-tac get-all-marked-decomposition (xs @ Marked K i # c'))
  by auto

definition all-decomposition-implies :: 'a literal multiset set
  ⇒ (('a, 'l, 'm) marked-lit list × ('a, 'l, 'm) marked-lit list) list ⇒ bool where
 all-decomposition-implies N S
  ⟷ (∀ (Ls, seen) ∈ set S. unmark-l Ls ∪ N ⊨ps unmark-l seen)

lemma all-decomposition-implies-empty[iff]:
  all-decomposition-implies N [] unfolding all-decomposition-implies-def by auto

lemma all-decomposition-implies-single[iff]:
  all-decomposition-implies N [(Ls, seen)]
    ⟷ unmark-l Ls ∪ N ⊨ps unmark-l seen
  unfolding all-decomposition-implies-def by auto

lemma all-decomposition-implies-append[iff]:
  all-decomposition-implies N (S @ S')
    ⟷ (all-decomposition-implies N S ∧ all-decomposition-implies N S')
  unfolding all-decomposition-implies-def by auto

lemma all-decomposition-implies-cons-pair[iff]:
  all-decomposition-implies N ((Ls, seen) # S')
    ⟷ (all-decomposition-implies N [(Ls, seen)] ∧ all-decomposition-implies N S')
  unfolding all-decomposition-implies-def by auto

lemma all-decomposition-implies-cons-single[iff]:
  all-decomposition-implies N (l # S') ⟷
    (unmark-l (fst l) ∪ N ⊨ps unmark-l (snd l) ∧
      all-decomposition-implies N S')
  unfolding all-decomposition-implies-def by auto

lemma all-decomposition-implies-trail-is-implied:
  assumes all-decomposition-implies N (get-all-marked-decomposition M)
  shows N ∪ {unmark L |L. is-marked L ∧ L ∈ set M}
    ⊨ps unmark ` ⋃(set ` snd ` set (get-all-marked-decomposition M))
```

```
using assms
proof (induct length (get-all-marked-decomposition M) arbitrary: M)
  case 0
  then show ?case by auto
next
  case (Suc n) note IH = this(1) and length = this(2) and decomp = this(3)
  consider
      (le1) length (get-all-marked-decomposition M) ≤ 1
    | (gt1) length (get-all-marked-decomposition M) > 1
    by arith
  then show ?case
    proof cases
      case le1
      then obtain a b where g: get-all-marked-decomposition M = (a, b) # []
        by (cases get-all-marked-decomposition M) auto
      moreover {
        assume a = []
        then have ?thesis using Suc.prems g by auto
      }
      moreover {
        assume l: length a = 1 and m: is-marked (hd a) and hd: hd a ∈ set M
        then have unmark (hd a) ∈ {unmark L |L. is-marked L ∧ L ∈ set M} by auto
        then have H: unmark-l a ∪ N ⊆ N ∪ {unmark L |L. is-marked L ∧ L ∈ set M}
          using l by (cases a) auto
        have f1: unmark-l a ∪ N ⊨ps unmark-l b
          using decomp unfolding all-decomposition-implies-def g by simp
        have ?thesis
          apply (rule true-clss-clss-subset) using f1 H g by auto
      }
      ultimately show ?thesis
        using get-all-marked-decomposition-length-1-fst-empty-or-length-1 by blast
    next
      case gt1
      then obtain Ls0 seen0 M' where
        Ls0: get-all-marked-decomposition M = (Ls0, seen0) # get-all-marked-decomposition M' and
        length': length (get-all-marked-decomposition M') = n and
        M'-in-M: set M' ⊆ set M
        using length by (induct M rule: marked-lit-list-induct) (auto simp: subset-insertI2)
      let ?d = ⋃(set ‘ snd ‘ set (get-all-marked-decomposition M'))
      let ?unM = {unmark L |L. is-marked L ∧ L ∈ set M}
      let ?unM' = {unmark L |L. is-marked L ∧ L ∈ set M'}
      {
        assume n = 0
        then have get-all-marked-decomposition M' = [] using length' by auto
        then have ?thesis using Suc.prems unfolding all-decomposition-implies-def Ls0 by auto
      }
      moreover {
        assume n: n > 0
        then obtain Ls1 seen1 l where
          Ls1: get-all-marked-decomposition M' = (Ls1, seen1) # l
          using length' by (induct M' rule: marked-lit-list-induct) auto

        have all-decomposition-implies N (get-all-marked-decomposition M')
          using decomp unfolding Ls0 by auto
        then have N: N ∪ ?unM' ⊨ps unmark-s ?d
```

```
         using IH length' by auto
        have l: N ∪ ?unM' ⊆ N ∪ ?unM
         using M'-in-M by auto
        from true-clss-clss-subset[OF this N]
        have ΨN: N ∪ ?unM ⊨ps unmark-s ?d by auto
        have is-marked (hd Ls0) and LS: tl Ls0 = seen1 @ Ls1
         using get-all-marked-decomposition-hd-hd[of M] unfolding Ls0 Ls1 by auto

        have LSM: seen1 @ Ls1 = M' using get-all-marked-decomposition-decomp[of M'] Ls1 by auto
        have M': set M' = ?d ∪ {L |L. is-marked L ∧ L ∈ set M'}
         using get-all-marked-decomposition-snd-union by auto

        {
         assume Ls0 ≠ []
         then have hd Ls0 ∈ set M
          using get-all-marked-decomposition-fst-empty-or-hd-in-M Ls0 by blast
         then have N ∪ ?unM ⊨p unmark (hd Ls0)
          using ‹is-marked (hd Ls0)› by (metis (mono-tags, lifting) UnCI mem-Collect-eq
           true-clss-cls-in)
        } note hd-Ls0 = this

        have l: unmark ` (?d ∪ {L |L. is-marked L ∧ L ∈ set M'}) = unmark-s ?d ∪ ?unM'
         by auto
        have N ∪ ?unM' ⊨ps unmark ` (?d ∪ {L |L. is-marked L ∧ L ∈ set M'})
         unfolding l using N by (auto simp: all-in-true-clss-clss)
        then have t: N ∪ ?unM' ⊨ps unmark-l (tl Ls0)
         using M' unfolding LS LSM by auto
        then have N ∪ ?unM ⊨ps unmark-l (tl Ls0)
         using M'-in-M true-clss-clss-subset[OF - t, of N ∪ ?unM] by auto
        then have N ∪ ?unM ⊨ps unmark-l Ls0
         using hd-Ls0 by (cases Ls0) auto

        moreover have unmark-l Ls0 ∪ N ⊨ps unmark-l seen0
         using decomp unfolding Ls0 by simp
        moreover have ⋀M Ma. (M::'a literal multiset set) ∪ Ma ⊨ps M
         by (simp add: all-in-true-clss-clss)
        ultimately have Ψ: N ∪ ?unM ⊨ps unmark-l seen0
         by (meson true-clss-clss-left-right true-clss-clss-union-and true-clss-clss-union-l-r)

        moreover have unmark ` (set seen0 ∪ ?d) = unmark-l seen0 ∪ unmark-s ?d
         by auto
        ultimately have ?thesis using ΨN unfolding Ls0 by simp
       }
       ultimately show ?thesis by auto
     qed
qed

lemma all-decomposition-implies-propagated-lits-are-implied:
 assumes all-decomposition-implies N (get-all-marked-decomposition M)
 shows N ∪ {unmark L |L. is-marked L ∧ L ∈ set M} ⊨ps unmark-l M
   (is ?I ⊨ps ?A)
proof –
 have ?I ⊨ps unmark-s {L |L. is-marked L ∧ L ∈ set M}
   by (auto intro: all-in-true-clss-clss)
 moreover have ?I ⊨ps unmark ` ⋃(set ` snd ` set (get-all-marked-decomposition M))
```

using *all-decomposition-implies-trail-is-implied assms* **by** *blast*
ultimately have *N* ∪ {*unmark m* |*m. is-marked m* ∧ *m* ∈ *set M*}
⊨*ps unmark* ‘ ⋃(*set* ‘ *snd* ‘ *set* (*get-all-marked-decomposition M*))
∪ *unmark* ‘ {*m* |*m. is-marked m* ∧ *m* ∈ *set M*}
by *blast*
then show *?thesis*
by (*metis* (*no-types*) *get-all-marked-decomposition-snd-union*[*of M*] *image-Un*)
qed

lemma *all-decomposition-implies-insert-single*:
all-decomposition-implies N M ⟹ *all-decomposition-implies* (*insert C N*) *M*
unfolding *all-decomposition-implies-def* **by** *auto*

14.4 Negation of Clauses

definition *CNot* :: *'v clause* ⇒ *'v clauses* **where**
CNot ψ = { {#−*L*#} | *L*. *L* ∈# *ψ* }

lemma *in-CNot-uminus*[*iff*]:
shows {#*L*#} ∈ *CNot ψ* ⟷ −*L* ∈# *ψ*
unfolding *CNot-def* **by** *force*

lemma
shows
CNot-singleton[*simp*]: *CNot* {#*L*#} = {{#−*L*#}} **and**
CNot-empty[*simp*]: *CNot* {#} = {} **and**
CNot-plus[*simp*]: *CNot* (*A* + *B*) = *CNot A* ∪ *CNot B*
unfolding *CNot-def* **by** *auto*

lemma *CNot-eq-empty*[*iff*]:
CNot D = {} ⟷ *D* = {#}
unfolding *CNot-def* **by** (*auto simp add*: *multiset-eqI*)

lemma *in-CNot-implies-uminus*:
assumes *L* ∈# *D* **and** *M* ⊨*as CNot D*
shows *M* ⊨*a* {#−*L*#} **and** −*L* ∈ *lits-of-l M*
using *assms* **by** (*auto simp*: *true-annots-def true-annot-def CNot-def*)

lemma *CNot-remdups-mset*[*simp*]:
CNot (*remdups-mset A*) = *CNot A*
unfolding *CNot-def* **by** *auto*

lemma *Ball-CNot-Ball-mset*[*simp*] :
(∀ *x*∈*CNot D*. *P x*) ⟷ (∀ *L*∈# *D*. *P* {#−*L*#})
unfolding *CNot-def* **by** *auto*

lemma *consistent-CNot-not*:
assumes *consistent-interp I*
shows *I* ⊨*s CNot φ* ⟹ ¬*I* ⊨ *φ*
using *assms* **unfolding** *consistent-interp-def true-clss-def true-cls-def* **by** *auto*

lemma *total-not-true-cls-true-clss-CNot*:
assumes *total-over-m I* {*φ*} **and** ¬*I* ⊨ *φ*
shows *I* ⊨*s CNot φ*
using *assms* **unfolding** *total-over-m-def total-over-set-def true-clss-def true-cls-def CNot-def*
apply *clarify*

```
  by (rename-tac x L, case-tac L) (force intro: pos-lit-in-atms-of neg-lit-in-atms-of)+

lemma total-not-CNot:
  assumes total-over-m I {φ} and ¬I ⊨s CNot φ
  shows I ⊨ φ
  using assms total-not-true-cls-true-clss-CNot by auto

lemma atms-of-ms-CNot-atms-of[simp]:
  atms-of-ms (CNot C) = atms-of C
  unfolding atms-of-ms-def atms-of-def CNot-def by fastforce

lemma true-clss-clss-contradiction-true-clss-cls-false:
  C ∈ D ⟹ D ⊨ps CNot C ⟹ D ⊨p {#}
  unfolding true-clss-clss-def true-clss-cls-def total-over-m-def
  by (metis Un-commute atms-of-empty atms-of-ms-CNot-atms-of atms-of-ms-insert atms-of-ms-union
    consistent-CNot-not insert-absorb sup-bot.left-neutral true-clss-def)

lemma true-annots-CNot-all-atms-defined:
  assumes M ⊨as CNot T and a1: L ∈# T
  shows atm-of L ∈ atm-of ‘ lits-of-l M
  by (metis assms atm-of-uminus image-eqI in-CNot-implies-uminus(1) true-annot-singleton)

lemma true-annots-CNot-all-uminus-atms-defined:
  assumes M ⊨as CNot T and a1: −L ∈# T
  shows atm-of L ∈ atm-of ‘ lits-of-l M
  by (metis assms atm-of-uminus image-eqI in-CNot-implies-uminus(1) true-annot-singleton)

lemma true-clss-clss-false-left-right:
  assumes {{#L#}} ∪ B ⊨p {#}
  shows B ⊨ps CNot {#L#}
  unfolding true-clss-clss-def true-clss-cls-def
proof (intro allI impI)
  fix I
  assume
    tot: total-over-m I (B ∪ CNot {#L#}) and
    cons: consistent-interp I and
    I: I ⊨s B
  have total-over-m I ({{#L#}} ∪ B) using tot by auto
  then have ¬I ⊨s insert {#L#} B
    using assms cons unfolding true-clss-cls-def by simp
  then show I ⊨s CNot {#L#}
    using tot I by (cases L) auto
qed

lemma true-annots-true-cls-def-iff-negation-in-model:
  M ⊨as CNot C ⟷ (∀ L ∈# C. −L ∈ lits-of-l M)
  unfolding CNot-def true-annots-true-cls true-clss-def by auto


lemma true-annot-CNot-diff:
  I ⊨as CNot C ⟹ I ⊨as CNot (C − C')
  by (auto simp: true-annots-true-cls-def-iff-negation-in-model dest: in-diffD)

lemma consistent-CNot-not-tautology:
  consistent-interp M ⟹ M ⊨s CNot D ⟹ ¬tautology D
  by (metis atms-of-ms-CNot-atms-of consistent-CNot-not satisfiable-carac' satisfiable-def
    tautology-def total-over-m-def)

lemma atms-of-ms-CNot-atms-of-ms: atms-of-ms (CNot CC) = atms-of-ms {CC}
  by simp

lemma total-over-m-CNot-toal-over-m[simp]:
  total-over-m I (CNot C) = total-over-set I (atms-of C)
  unfolding total-over-m-def total-over-set-def by auto
```

The following lemma is very useful when in the goal appears an axioms like $- L = K$: this lemma allows the simplifier to rewrite L.

```
lemma uminus-lit-swap: −(a::'a literal) = i ⟷ a = −i
  by auto

lemma true-clss-cls-plus-CNot:
  assumes CC-L: A ⊨p CC + {#L#}
  and CNot-CC: A ⊨ps CNot CC
  shows A ⊨p {#L#}
  unfolding true-clss-clss-def true-clss-cls-def CNot-def total-over-m-def
proof (intro allI impI)
  fix I
  assume
    tot: total-over-set I (atms-of-ms (A ∪ {{#L#}})) and
    cons: consistent-interp I and
    I: I ⊨s A
  let ?I = I ∪ {Pos P|P. P ∈ atms-of CC ∧ P ∉ atm-of ` I}
  have cons': consistent-interp ?I
    using cons unfolding consistent-interp-def
    by (auto simp: uminus-lit-swap atms-of-def rev-image-eqI)
  have I': ?I ⊨s A
    using I true-clss-union-increase by blast
  have tot-CNot: total-over-m ?I (A ∪ CNot CC)
    using tot atms-of-s-def by (fastforce simp: total-over-m-def total-over-set-def)

  then have tot-I-A-CC-L: total-over-m ?I (A ∪ {CC + {#L#}})
    using tot unfolding total-over-m-def total-over-set-atm-of by auto
  then have ?I ⊨ CC + {#L#} using CC-L cons' I' unfolding true-clss-cls-def by blast
  moreover
    have ?I ⊨s CNot CC using CNot-CC cons' I' tot-CNot unfolding true-clss-clss-def by auto
    then have ¬A ⊨p CC
      by (metis (no-types, lifting) I' atms-of-ms-CNot-atms-of-ms atms-of-ms-union cons'
        consistent-CNot-not tot-CNot total-over-m-def true-clss-cls-def)
    then have ¬?I ⊨ CC using ‹?I ⊨s CNot CC› cons' consistent-CNot-not by blast
  ultimately have ?I ⊨ {#L#} by blast
  then show I ⊨ {#L#}
    by (metis (no-types, lifting) atms-of-ms-union cons' consistent-CNot-not tot total-not-CNot
      total-over-m-def total-over-set-union true-clss-union-increase)
qed

lemma true-annots-CNot-lit-of-notin-skip:
  assumes LM: L # M ⊨as CNot A and LA: lit-of L ∉# A −lit-of L ∉# A
  shows M ⊨as CNot A
  using LM unfolding true-annots-def Ball-def
proof (intro allI impI)
```

```
  fix l
  assume H: ∀x. x ∈ CNot A ⟶ L # M ⊨a x  and l: l ∈ CNot A
  then have L # M ⊨a l by auto
  then show M ⊨a l using LA l by (cases L) (auto simp: CNot-def)
 qed

lemma true-clss-clss-union-false-true-clss-clss-cnot:
  A ∪ {B} ⊨ps {{#}} ⟷ A ⊨ps CNot B
  using total-not-CNot consistent-CNot-not unfolding total-over-m-def true-clss-clss-def
  by fastforce

lemma true-annot-remove-hd-if-notin-vars:
  assumes a # M'⊨a D and atm-of (lit-of a) ∉ atms-of D
  shows M' ⊨a D
  using assms true-cls-remove-hd-if-notin-vars unfolding true-annot-def by auto

lemma true-annot-remove-if-notin-vars:
  assumes M @ M'⊨a D and ∀x∈atms-of D. x ∉ atm-of ‘ lits-of-l M
  shows M' ⊨a D
  using assms apply (induct M, simp)
  using true-annot-remove-hd-if-notin-vars by force+

lemma true-annots-remove-if-notin-vars:
  assumes M @ M'⊨as D and ∀x∈atms-of-ms D. x ∉ atm-of ‘ lits-of-l M
  shows M' ⊨as D unfolding true-annots-def
  using assms true-annot-remove-if-notin-vars[of M M']
  unfolding true-annots-def atms-of-ms-def by force

lemma all-variables-defined-not-imply-cnot:
  assumes
    ∀s ∈ atms-of-ms {B}. s ∈ atm-of ‘ lits-of-l A and
    ¬ A ⊨a B
  shows A ⊨as CNot B
  unfolding true-annot-def true-annots-def Ball-def CNot-def true-lit-def
proof (clarify, rule ccontr)
  fix L
  assume LB: L ∈# B and  ¬ lits-of-l A ⊨l − L
  then have atm-of L ∈ atm-of ‘ lits-of-l A
    using assms(1) by (simp add: atm-of-lit-in-atms-of lits-of-def)
  then have L ∈ lits-of-l A ∨ −L ∈ lits-of-l A
    using atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set by metis
  then have L ∈ lits-of-l A using ‹ ¬ lits-of-l A ⊨l − L› by auto
  then show False
    using LB assms(2) unfolding true-annot-def true-lit-def true-cls-def Bex-def
    by blast
qed

lemma CNot-union-mset[simp]:
  CNot (A #∪ B) = CNot A ∪ CNot B
  unfolding CNot-def by auto
```

14.5 Other

```
abbreviation no-dup L ≡ distinct (map (λl. atm-of (lit-of l)) L)

lemma no-dup-rev[simp]:
  no-dup (rev M) ⟷ no-dup M
  by (auto simp: rev-map[symmetric])

lemma no-dup-length-eq-card-atm-of-lits-of-l:
  assumes no-dup M
  shows length M  = card (atm-of ‘ lits-of-l M)
  using assms unfolding lits-of-def by (induct M) (auto simp add: image-image)

lemma distinct-consistent-interp:
  no-dup M ⟹ consistent-interp (lits-of-l M)
proof (induct M)
  case Nil
  show ?case by auto
next
  case (Cons L M)
  then have a1: consistent-interp (lits-of-l M) by auto
  have a2: atm-of (lit-of L) ∉ (λl. atm-of (lit-of l)) ‘ set M using Cons.prems by auto
  have undefined-lit M (lit-of L)
    using a2 image-iff unfolding defined-lit-def by fastforce
  then show ?case
    using a1 by simp
qed

lemma distinct-get-all-marked-decomposition-no-dup:
  assumes (a, b) ∈ set (get-all-marked-decomposition M)
  and no-dup M
  shows no-dup (a @ b)
  using assms by force

lemma true-annots-lit-of-notin-skip:
  assumes L # M ⊨as CNot A
  and −lit-of L ∉# A
  and no-dup (L # M)
  shows M ⊨as CNot A
proof −
  have ∀ l ∈# A. −l ∈ lits-of-l (L # M)
    using assms(1) in-CNot-implies-uminus(2) by blast
  moreover
    have atm-of (lit-of L) ∉ atm-of ‘ lits-of-l M
      using assms(3) unfolding lits-of-def by force
    then have − lit-of L ∉ lits-of-l M unfolding lits-of-def
      by (metis (no-types) atm-of-uminus imageI)
  ultimately have ∀  l ∈# A. −l ∈ lits-of-l M
    using assms(2) by (metis insert-iff list.simps(15) lits-of-insert uminus-of-uminus-id)
  then show ?thesis by (auto simp add: true-annots-def)
qed

abbreviation true-annots-mset (infix ⊨asm 50) where
I ⊨asm C ≡ I ⊨as (set-mset C)

abbreviation true-clss-clss-m:: 'v clause multiset ⇒ 'v clause multiset ⇒ bool (infix ⊨psm 50)
where
I ⊨psm C ≡ set-mset I ⊨ps (set-mset C)
```

Analog of ⟦?N ⊨ps ?B; ?A ⊆ ?B⟧ ⟹ ?N ⊨ps ?A

lemma *true-clss-clssm-subsetE*: *N* ⊨*psm B* ⟹ *A* ⊆# *B* ⟹ *N* ⊨*psm A*
  using *set-mset-mono true-clss-clss-subsetE* **by** *blast*

abbreviation *true-clss-cls-m*:: *'a clause multiset* ⇒ *'a clause* ⇒ *bool* (**infix** ⊨*pm 50*) **where**
I ⊨*pm C* ≡ *set-mset I* ⊨*p C*

abbreviation *distinct-mset-mset* :: *'a multiset multiset* ⇒ *bool* **where**
distinct-mset-mset Σ ≡ *distinct-mset-set* (*set-mset* Σ)

abbreviation *all-decomposition-implies-m* **where**
all-decomposition-implies-m A B ≡ *all-decomposition-implies* (*set-mset A*) *B*

abbreviation *atms-of-mm* :: *'a literal multiset multiset* ⇒ *'a set* **where**
atms-of-mm U ≡ *atms-of-ms* (*set-mset U*)

Other definition using *Union-mset*

lemma *atms-of-mm U* ≡ *set-mset* (⋃# *image-mset* (*image-mset atm-of*) *U*)
  unfolding *atms-of-ms-def* **by** (*auto simp*: *atms-of-def*)

abbreviation *true-clss-m*:: *'a interp* ⇒ *'a clause multiset* ⇒ *bool* (**infix** ⊨*sm 50*) **where**
I ⊨*sm C* ≡ *I* ⊨*s set-mset C*

abbreviation *true-clss-ext-m* (**infix** ⊨*sextm 49*) **where**
I ⊨*sextm C* ≡ *I* ⊨*sext set-mset C*

end
theory *CDCL-Abstract-Clause-Representation*
imports *Main Partial-Clausal-Logic*
begin

type-synonym *'v clause* = *'v literal multiset*
type-synonym *'v clauses* = *'v clause multiset*

14.6 Abstract Clause Representation

We will abstract the representation of clause and clauses via two locales. We expect our representation to behave like multiset, but the internal representation can be done using list or whatever other representation.

We assume the following:

- there is an equivalent to adding and removing a literal and to taking the union of clauses.

locale *raw-cls* =
  fixes
    mset-cls:: *'cls* ⇒ *'v clause* **and**
    insert-cls :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**
    remove-lit :: *'v literal* ⇒ *'cls* ⇒ *'cls*
  assumes
    insert-cls[*simp*]: *mset-cls* (*insert-cls L C*) = *mset-cls C* + {#*L*#} **and**
    remove-lit[*simp*]: *mset-cls* (*remove-lit L C*) = *remove1-mset L* (*mset-cls C*)
begin
end

locale *raw-ccls-union* =

```
  fixes
    mset-cls:: 'cls ⇒ 'v clause and
    union-cls :: 'cls ⇒ 'cls ⇒ 'cls and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls
  assumes
    insert-ccls[simp]: mset-cls (insert-cls L C) = mset-cls C + {#L#} and
    mset-ccls-union-cls[simp]: mset-cls (union-cls C D) = mset-cls C #∪ mset-cls D and
    remove-clit[simp]: mset-cls (remove-lit L C) = remove1-mset L (mset-cls C)
begin
end
```

Instantiation of the previous locale, in an unnamed context to avoid polluating with simp rules

```
context
begin
  interpretation list-cls: raw-cls mset
    op # remove1
    by unfold-locales (auto simp: union-mset-list ex-mset)

  interpretation cls-cls: raw-cls id
    λL C. C + {#L#} remove1-mset
    by unfold-locales (auto simp: union-mset-list)

  interpretation list-cls: raw-ccls-union mset
    union-mset-list
    op # remove1
    by unfold-locales (auto simp: union-mset-list ex-mset)

  interpretation cls-cls: raw-ccls-union id
    op #∪ λL C. C + {#L#} remove1-mset
    by unfold-locales (auto simp: union-mset-list)
end
```

Over the abstract clauses, we have the following properties:

- We can insert a clause
- We can take the union (used only in proofs for the definition of *clauses*)
- there is an operator indicating whether the abstract clause is contained or not
- if a concrete clause is contained the abstract clauses, then there is an abstract clause

```
locale raw-clss =
  raw-cls mset-cls insert-cls remove-lit
  for
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls +
  fixes
    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss
  assumes
    insert-clss[simp]: mset-clss (insert-clss L C) = mset-clss C + {#mset-cls L#} and
    union-clss[simp]: mset-clss (union-clss C D) = mset-clss C + mset-clss D and
    mset-clss-union-clss[simp]: mset-clss (insert-clss C′ D) = {#mset-cls C′#} + mset-clss D and
    in-clss-mset-clss[dest]: in-clss a C ⟹ mset-cls a ∈# mset-clss C and
    in-mset-clss-exists-preimage: b ∈# mset-clss C ⟹ ∃ b′. in-clss b′ C ∧ mset-cls b′ = b and
    remove-from-clss-mset-clss[simp]:
      mset-clss (remove-from-clss a C) = mset-clss C − {#mset-cls a#} and
    in-clss-union-clss[simp]:
      in-clss a (union-clss C D) ⟷ in-clss a C ∨ in-clss a D
begin

end

experiment
begin
  fun remove-first where
  remove-first - [] = [] |
  remove-first C (C′ # L) = (if mset C = mset C′ then L else C′ # remove-first C L)

  lemma mset-map-mset-remove-first:
    mset (map mset (remove-first a C)) = remove1-mset (mset a) (mset (map mset C))
    by (induction C) (auto simp: ac-simps remove1-mset-single-add)

  interpretation clss-clss: raw-clss id λL C. C + {#L#} remove1-mset
    id op + op ∈# λL C. C + {#L#} remove1-mset
    by unfold-locales (auto simp: ac-simps)

  interpretation list-clss: raw-clss mset
    op # remove1 λL. mset (map mset L) op @ λL C. L ∈ set C op #
    remove-first
    by unfold-locales (auto simp: ac-simps union-mset-list mset-map-mset-remove-first ex-mset)
end

end
theory CDCL-WNOT-Measure
imports Main
begin
```

15 Measure

This measure show the termination of the core of CDCL: each step improves the number of literals we know for sure.

This measure can also be seen as the increasing lexicographic order: it is an order on bounded sequences, when each element is bounded. The proof involves a measure like the one defined here (the same?).

```
definition μC :: nat ⇒ nat ⇒ nat list ⇒ nat where
μC s b M ≡ (∑i=0..<length M. M!i ∗ b^ (s +i − length M))

lemma μC-nil[simp]:
  μC s b [] = 0
  unfolding μC-def by auto

lemma μC-single[simp]:
  μC s b [L] = L ∗ b ^ (s − Suc 0)
  unfolding μC-def by auto

lemma set-sum-atLeastLessThan-add:
  (∑ i=k..<k+(b::nat). f i) = (∑ i=0..<b. f (k+ i))
  by (induction b) auto

lemma set-sum-atLeastLessThan-Suc:
  (∑ i=1..<Suc j. f i) = (∑ i=0..<j. f (Suc i))
  using set-sum-atLeastLessThan-add[of - 1 j] by force

lemma μC-cons:
  μC s b (L # M) = L ∗ b ^ (s − 1 − length M) + μC s b M
proof −
  have μC s b (L # M) = (∑ i=0..<length (L#M). (L#M)!i ∗ b^ (s +i − length (L#M)))
    unfolding μC-def by blast
  also have ... = (∑ i=0..<1. (L#M)!i ∗ b^ (s +i − length (L#M)))
              + (∑ i=1..<length (L#M). (L#M)!i ∗ b^ (s +i − length (L#M)))
     by (rule setsum-add-nat-ivl[symmetric]) simp-all
  finally have μC s b (L # M)= L ∗ b ^ (s − 1 − length M)
               + (∑ i=1..<length (L#M). (L#M)!i ∗ b^ (s +i − length (L#M)))
     by auto
  moreover {
    have (∑ i=1..<length (L#M). (L#M)!i ∗ b^ (s +i − length (L#M))) =
          (∑ i=0..<length (M). (L#M)!(Suc i) ∗ b^ (s + (Suc i) − length (L#M)))
     unfolding length-Cons set-sum-atLeastLessThan-Suc by blast
    also have ... = (∑ i=0..<length (M). M!i ∗ b^ (s + i − length M))
      by auto
    finally have (∑ i=1..<length (L#M). (L#M)!i ∗ b^ (s +i − length (L#M))) = μC s b M
      unfolding μC-def .
    }
  ultimately show ?thesis by presburger
qed

lemma μC-append:
  assumes s ≥ length (M@M′)
  shows μC s b (M@M′) = μC (s − length M′) b M + μC s b M′
proof −
  have μC s b (M@M′) = (∑ i=0..<length (M@M′). (M@M′)!i ∗ b^ (s +i − length (M@M′)))
    unfolding μC-def by blast
  moreover then have ... = (∑ i=0..< length M. (M@M′)!i ∗ b^ (s +i − length (M@M′)))
              + (∑ i=length M..<length (M@M′). (M@M′)!i ∗ b^ (s +i − length (M@M′)))
    by (auto intro!: setsum-add-nat-ivl[symmetric])
  moreover
    have ∀ i∈{0..< length M}. (M@M′)!i ∗ b^ (s +i − length (M@M′)) = M ! i ∗ b ^ (s − length M′
      + i − length M)
      using ‹s ≥ length (M@M′)› by (auto simp add: nth-append ac-simps)
     then have μC (s − length M′) b M = (∑ i=0..< length M. (M@M′)!i ∗ b^ (s +i − length
(M@M′)))
      unfolding μC-def by auto
  ultimately have μC s b (M@M′)= μC (s − length M′) b M
               + (∑ i=length M..<length (M@M′). (M@M′)!i ∗ b^ (s +i − length (M@M′)))
     by auto
  moreover {
    have (∑ i=length M..<length (M@M′). (M@M′)!i ∗ b^ (s +i − length (M@M′))) =
        (∑ i=0..<length M'. M'!i ∗ b^ (s + i − length M'))
     unfolding length-append set-sum-atLeastLessThan-add by auto
    then have (∑ i=length M..<length (M@M'). (M@M')!i ∗ b^ (s +i − length (M@M'))) = μC s b
M'
      unfolding μC-def .
    }
  ultimately show ?thesis by presburger
qed

lemma μC-cons-non-empty-inf:
  assumes M-ge-1: ∀ i∈set M. i ≥ 1 and M: M ≠ []
  shows μC s b M ≥ b ^ (s − length M)
  using assms by (cases M) (auto simp: mult-eq-if μC-cons)
```

Copy of `~~/src/HOL/ex/NatSum.thy` (but generalized to $0 \le k$)

```
lemma sum-of-powers: 0 ≤ k ⟹ (k − 1) ∗ (∑ i=0..<n. k^i) = k^n − (1::nat)
  apply (cases k = 0)
    apply (cases n; simp)
  by (induct n) (auto simp: Nat.nat-distrib)
```

In the degenerated cases, we only have the large inequality holds. In the other cases, the following strict inequality holds:

```
lemma μC-bounded-non-degenerated:
  fixes b ::nat
  assumes
    b > 0 and
    M ≠ [] and
    M-le: ∀ i < length M. M!i < b and
    s ≥ length M
  shows μC s b M < b^s
proof −
  consider (b1) b= 1 | (b) b>1 using ‹b>0› by (cases b) auto
  then show ?thesis
    proof cases
      case b1
      then have ∀ i < length M. M!i = 0 using M-le by auto
      then have μC s b M = 0 unfolding μC-def by auto
      then show ?thesis using ‹b > 0› by auto
    next
      case b
      have ∀  i ∈ {0..<length M}. M!i ∗ b^ (s +i − length M) ≤ (b−1) ∗ b^ (s +i − length M)
        using M-le ‹b > 1› by auto
      then have μC s b M ≤  (∑ i=0..<length M. (b−1) ∗ b^ (s +i − length M))
         using ‹M≠[]› ‹b>0› unfolding μC-def by (auto intro: setsum-mono)
      also
       have ∀  i ∈ {0..<length M}. (b−1) ∗ b^ (s +i − length M) = (b−1) ∗ b^ i ∗ b^(s − length M)
         by (metis Nat.add-diff-assoc2 add.commute assms(4) mult.assoc power-add)
       then have (∑ i=0..<length M. (b−1) ∗ b^ (s +i − length M))
          = (∑ i=0..<length M. (b−1)∗ b^ i ∗ b^(s − length M))
         by (auto simp add: ac-simps)
      also have ... = (∑ i=0..<length M. b^ i) ∗ b^(s − length M) ∗ (b−1)
         by (simp add: setsum-left-distrib setsum-right-distrib ac-simps)
      finally have μC s b M ≤ (∑ i=0..<length M. b^ i) ∗ (b−1) ∗ b^(s − length M)
        by (simp add: ac-simps)
      also
        have (∑ i=0..<length M. b^ i)∗ (b−1) = b ^ (length M) − 1
          using sum-of-powers[of b length M] ‹b>1›
          by (auto simp add: ac-simps)
      finally have μ_C s b M ≤ (b ^ (length M) − 1) ∗ b ^ (s − length M)
        by auto
      also have ... < b ^ (length M) ∗ b ^ (s − length M)
        using ‹b>1› by auto
      also have ... = b ^ s
        by (metis assms(4) le-add-diff-inverse power-add)
      finally show ?thesis unfolding μ_C-def by (auto simp add: ac-simps)
    qed
qed
```

In the degenerate case $b = (0::'a)$, the list M is empty (since the list cannot contain any element).

```
lemma μ_C-bounded:
  fixes b ::nat
  assumes
    M-le: ∀ i < length M. M!i < b and
    s ≥ length M
    b > 0
  shows μ_C s b M < b ^ s
proof −
  consider (M0) M = [] | (M) b > 0 and M ≠ []
    using M-le by (cases b, cases M) auto
  then show ?thesis
    proof cases
      case M0
      then show ?thesis using M-le ‹b > 0› by auto
    next
      case M
      show ?thesis using μ_C-bounded-non-degenerated[OF M assms(1,2)] by arith
    qed
qed
```

When $b = 0$, we cannot show that the measure is empty, since $0^0 = 1$.

```
lemma μ_C-base-0:
  assumes length M ≤ s
  shows μ_C s 0 M ≤ M!0
proof −
  {
    assume s = length M
    moreover {
      fix n
      have (∑ i=0..<n. M ! i ∗ (0::nat) ^ i) ≤ M ! 0
        apply (induction n rule: nat-induct)
        by simp (rename-tac n, case-tac n, auto)
    }
    ultimately have ?thesis unfolding μ_C-def by auto
  }
  moreover
  {
    assume length M < s
    then have μ_C s 0 M = 0 unfolding μ_C-def by auto}
```

ultimately show *?thesis* **using** *assms* **unfolding** μ_C*-def* **by** *linarith*
qed

end
theory *CDCL-NOT*
imports *CDCL-Abstract-Clause-Representation List-More Wellfounded-More CDCL-WNOT-Measure Partial-Annotated-Clausal-Logic*
begin

16 NOT's CDCL

16.1 Auxiliary Lemmas and Measure

lemma *no-dup-cannot-not-lit-and-uminus*:
no-dup M ⟹ − *lit-of xa* = *lit-of x* ⟹ *x* ∈ *set M* ⟹ *xa* ∉ *set M*
by (*metis atm-of-uminus distinct-map inj-on-eq-iff uminus-not-id'*)

lemma *atms-of-ms-single-atm-of*[*simp*]:
atms-of-ms {*unmark L* |*L*. *P L*} = *atm-of* ' {*lit-of L* |*L*. *P L*}
unfolding *atms-of-ms-def* **by** *force*

lemma *atms-of-uminus-lit-atm-of-lit-of*:
atms-of {# −*lit-of x*. *x* ∈# *A*#} = *atm-of* ' (*lit-of* ' (*set-mset A*))
unfolding *atms-of-def* **by** (*auto simp add*: *Fun.image-comp*)

lemma *atms-of-ms-single-image-atm-of-lit-of*:
atms-of-ms (*unmark-s A*) = *atm-of* ' (*lit-of* ' *A*)
unfolding *atms-of-ms-def* **by** *auto*

16.2 Initial definitions

16.2.1 The state

We define here an abstraction over operation on the state we are manipulating.

locale *dpll-state-ops* =
raw-clss mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
for
mset-cls:: *'cls* ⇒ *'v clause* **and**
insert-cls :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**
remove-lit :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**
mset-clss:: *'clss* ⇒ *'v clauses* **and**
union-clss :: *'clss* ⇒ *'clss* ⇒ *'clss* **and**
in-clss :: *'cls* ⇒ *'clss* ⇒ *bool* **and**
insert-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**
remove-from-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* +
fixes
trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
raw-clauses :: *'st* ⇒ *'clss* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒*'st* **and**
add-cls$_{NOT}$ :: *'cls* ⇒ *'st* ⇒ *'st* **and**
remove-cls$_{NOT}$ :: *'cls* ⇒ *'st* ⇒ *'st*
begin

notation *insert-cls* (**infix** !++ *50*)

notation *in-clss* (**infix** !∈! *50*)
notation *union-clss* (**infix** ⊕ *50*)
notation *insert-clss* (**infix** !++! *50*)

abbreviation $clauses_{NOT}$ **where**
$clauses_{NOT}$ *S* ≡ *mset-clss* (*raw-clauses S*)

end

locale *dpll-state* =
  dpll-state-ops mset-cls insert-cls remove-lit — related to each clause
    mset-clss union-clss in-clss insert-clss remove-from-clss — related to the clauses

    trail raw-clauses prepend-trail tl-trail $add\text{-}cls_{NOT}$ $remove\text{-}cls_{NOT}$ — related to the state
  for
    mset-cls:: *'cls* ⇒ *'v clause* **and**
    insert-cls :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**
    remove-lit :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**
    mset-clss:: *'clss* ⇒ *'v clauses* **and**
    union-clss :: *'clss* ⇒ *'clss* ⇒ *'clss* **and**
    in-clss :: *'cls* ⇒ *'clss* ⇒ *bool* **and**
    insert-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**
    remove-from-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**
    trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
    raw-clauses :: *'st* ⇒ *'clss* **and**
    prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
    tl-trail :: *'st* ⇒*'st* **and**
    $add\text{-}cls_{NOT}$ :: *'cls* ⇒ *'st* ⇒ *'st* **and**
    $remove\text{-}cls_{NOT}$ :: *'cls* ⇒ *'st* ⇒ *'st* +
  assumes
    trail-prepend-trail[*simp*]:
      ⋀*st L*. *undefined-lit* (*trail st*) (*lit-of L*) ⟹ *trail* (*prepend-trail L st*) = *L* # *trail st*
      and
    tl-trail[*simp*]: *trail* (*tl-trail S*) = *tl* (*trail S*) **and**
    $trail\text{-}add\text{-}cls_{NOT}$[*simp*]: ⋀*st C*. *no-dup* (*trail st*) ⟹ *trail* ($add\text{-}cls_{NOT}$ *C st*) = *trail st* **and**
    $trail\text{-}remove\text{-}cls_{NOT}$[*simp*]: ⋀*st C*. *trail* ($remove\text{-}cls_{NOT}$ *C st*) = *trail st* **and**

    clauses-prepend-trail[*simp*]:
      ⋀*st L*. *undefined-lit* (*trail st*) (*lit-of L*) ⟹
        $clauses_{NOT}$ (*prepend-trail L st*) = $clauses_{NOT}$ *st*
      and
    clauses-tl-trail[*simp*]: ⋀*st*. $clauses_{NOT}$ (*tl-trail st*) = $clauses_{NOT}$ *st* **and**
    $clauses\text{-}add\text{-}cls_{NOT}$[*simp*]:
      ⋀*st C*. *no-dup* (*trail st*) ⟹ $clauses_{NOT}$ ($add\text{-}cls_{NOT}$ *C st*) = {#*mset-cls C*#} + $clauses_{NOT}$ *st*
and
    $clauses\text{-}remove\text{-}cls_{NOT}$[*simp*]:
      ⋀*st C*. $clauses_{NOT}$ ($remove\text{-}cls_{NOT}$ *C st*) = *removeAll-mset* (*mset-cls C*) ($clauses_{NOT}$ *st*)
begin

function $reduce\text{-}trail\text{-}to_{NOT}$ :: *'a list* ⇒ *'st* ⇒ *'st* **where**
$reduce\text{-}trail\text{-}to_{NOT}$ *F S* =
  (*if length* (*trail S*) = *length F* ∨ *trail S* = [] *then S else* $reduce\text{-}trail\text{-}to_{NOT}$ *F* (*tl-trail S*))
by *fast+*

termination by (*relation measure* (λ(-, *S*). *length* (*trail S*))) *auto*
declare *reduce-trail-to*$_{NOT}$*.simps*[*simp del*]

lemma
shows
reduce-trail-to$_{NOT}$*-nil*[*simp*]: *trail S* = [] ⟹ *reduce-trail-to*$_{NOT}$ *F S* = *S* **and**
reduce-trail-to$_{NOT}$*-eq-length*[*simp*]: *length* (*trail S*) = *length F* ⟹ *reduce-trail-to*$_{NOT}$ *F S* = *S*
by (*auto simp*: *reduce-trail-to*$_{NOT}$*.simps*)

lemma *reduce-trail-to*$_{NOT}$*-length-ne*[*simp*]:
length (*trail S*) ≠ *length F* ⟹ *trail S* ≠ [] ⟹
reduce-trail-to$_{NOT}$ *F S* = *reduce-trail-to*$_{NOT}$ *F* (*tl-trail S*)
by (*auto simp*: *reduce-trail-to*$_{NOT}$*.simps*)

lemma *trail-reduce-trail-to*$_{NOT}$*-length-le*:
assumes *length F* > *length* (*trail S*)
shows *trail* (*reduce-trail-to*$_{NOT}$ *F S*) = []
using *assms* **by** (*induction F S rule*: *reduce-trail-to*$_{NOT}$*.induct*)
(*simp add*: *less-imp-diff-less reduce-trail-to*$_{NOT}$*.simps*)

lemma *trail-reduce-trail-to*$_{NOT}$*-nil*[*simp*]:
trail (*reduce-trail-to*$_{NOT}$ [] *S*) = []
by (*induction* [] *S rule*: *reduce-trail-to*$_{NOT}$*.induct*)
(*simp add*: *less-imp-diff-less reduce-trail-to*$_{NOT}$*.simps*)

lemma *clauses-reduce-trail-to*$_{NOT}$*-nil*:
clauses$_{NOT}$ (*reduce-trail-to*$_{NOT}$ [] *S*) = *clauses*$_{NOT}$ *S*
by (*induction* [] *S rule*: *reduce-trail-to*$_{NOT}$*.induct*)
(*simp add*: *less-imp-diff-less reduce-trail-to*$_{NOT}$*.simps*)

lemma *trail-reduce-trail-to*$_{NOT}$*-drop*:
trail (*reduce-trail-to*$_{NOT}$ *F S*) =
(*if length* (*trail S*) ≥ *length F*
then drop (*length* (*trail S*) − *length F*) (*trail S*)
else [])
apply (*induction F S rule*: *reduce-trail-to*$_{NOT}$*.induct*)
apply (*rename-tac F S*, *case-tac trail S*)
 apply *auto*[]
apply (*rename-tac list*, *case-tac Suc* (*length list*) > *length F*)
 prefer *2* **apply** *simp*
apply (*subgoal-tac Suc* (*length list*) − *length F* = *Suc* (*length list* − *length F*))
 apply *simp*
apply *simp*
done

lemma *reduce-trail-to*$_{NOT}$*-skip-beginning*:
assumes *trail S* = *F'* @ *F*
shows *trail* (*reduce-trail-to*$_{NOT}$ *F S*) = *F*
using *assms* **by** (*auto simp*: *trail-reduce-trail-to*$_{NOT}$*-drop*)

lemma *reduce-trail-to*$_{NOT}$*-clauses*[*simp*]:
clauses$_{NOT}$ (*reduce-trail-to*$_{NOT}$ *F S*) = *clauses*$_{NOT}$ *S*
by (*induction F S rule*: *reduce-trail-to*$_{NOT}$*.induct*)
(*simp add*: *less-imp-diff-less reduce-trail-to*$_{NOT}$*.simps*)

abbreviation *trail-weight* **where**
trail-weight S ≡ *map* ((*λl. 1* + *length l*) *o snd*) (*get-all-marked-decomposition* (*trail S*))

definition *state-eq*$_{NOT}$ :: *'st* ⇒ *'st* ⇒ *bool* (**infix** ∼ *50*) **where**
S ∼ *T* ⟷ *trail S* = *trail T* ∧ *clauses*$_{NOT}$ *S* = *clauses*$_{NOT}$ *T*

lemma *state-eq*$_{NOT}$*-ref*[*simp*]:
  S ∼ *S*
  unfolding *state-eq*$_{NOT}$*-def* **by** *auto*

lemma *state-eq*$_{NOT}$*-sym*:
  S ∼ *T* ⟷ *T* ∼ *S*
  unfolding *state-eq*$_{NOT}$*-def* **by** *auto*

lemma *state-eq*$_{NOT}$*-trans*:
  S ∼ *T* ⟹ *T* ∼ *U* ⟹ *S* ∼ *U*
  unfolding *state-eq*$_{NOT}$*-def* **by** *auto*

lemma
  shows
    state-eq$_{NOT}$*-trail*: *S* ∼ *T* ⟹ *trail S* = *trail T* **and**
    state-eq$_{NOT}$*-clauses*: *S* ∼ *T* ⟹ *clauses*$_{NOT}$ *S* = *clauses*$_{NOT}$ *T*
  unfolding *state-eq*$_{NOT}$*-def* **by** *auto*

lemmas *state-simp*$_{NOT}$[*simp*] = *state-eq*$_{NOT}$*-trail state-eq*$_{NOT}$*-clauses*

lemma *trail-eq-reduce-trail-to*$_{NOT}$*-eq*:
  trail S = *trail T* ⟹ *trail* (*reduce-trail-to*$_{NOT}$ *F S*) = *trail* (*reduce-trail-to*$_{NOT}$ *F T*)
  apply (*induction F S arbitrary*: *T rule*: *reduce-trail-to*$_{NOT}$*.induct*)
  by (*metis tl-trail reduce-trail-to*$_{NOT}$*-eq-length reduce-trail-to*$_{NOT}$*-length-ne reduce-trail-to*$_{NOT}$*-nil*)

lemma *reduce-trail-to*$_{NOT}$*-state-eq*$_{NOT}$*-compatible*:
  assumes *ST*: *S* ∼ *T*
  shows *reduce-trail-to*$_{NOT}$ *F S* ∼ *reduce-trail-to*$_{NOT}$ *F T*
proof −
  have *clauses*$_{NOT}$ (*reduce-trail-to*$_{NOT}$ *F S*) = *clauses*$_{NOT}$ (*reduce-trail-to*$_{NOT}$ *F T*)
    using *ST* **by** *auto*
  moreover have *trail* (*reduce-trail-to*$_{NOT}$ *F S*) = *trail* (*reduce-trail-to*$_{NOT}$ *F T*)
    using *trail-eq-reduce-trail-to*$_{NOT}$*-eq*[*of S T F*] *ST* **by** *auto*
  ultimately show *?thesis* **by** (*auto simp del*: *state-simp*$_{NOT}$ *simp*: *state-eq*$_{NOT}$*-def*)
qed

lemma *trail-reduce-trail-to*$_{NOT}$*-add-cls*$_{NOT}$[*simp*]:
  no-dup (*trail S*) ⟹
    trail (*reduce-trail-to*$_{NOT}$ *F* (*add-cls*$_{NOT}$ *C S*)) = *trail* (*reduce-trail-to*$_{NOT}$ *F S*)
  by (*rule trail-eq-reduce-trail-to*$_{NOT}$*-eq*) *simp*

lemma *reduce-trail-to*$_{NOT}$*-trail-tl-trail-decomp*[*simp*]:
  trail S = *F'* @ *Marked K* () # *F* ⟹
     trail (*reduce-trail-to*$_{NOT}$ *F* (*tl-trail S*)) = *F*
  apply (*rule reduce-trail-to*$_{NOT}$*-skip-beginning*[*of - tl* (*F'* @ *Marked K* () # [])])
  by (*cases F'*) (*auto simp add*:*tl-append reduce-trail-to*$_{NOT}$*-skip-beginning*)

lemma *reduce-trail-to*$_{NOT}$*-length*:
  length M = *length M'* ⟹ *reduce-trail-to*$_{NOT}$ *M S* = *reduce-trail-to*$_{NOT}$ *M' S*

```
  apply (induction M S arbitrary: rule: reduce-trail-to_NOT.induct)
  by (simp add: reduce-trail-to_NOT.simps)

end
```

16.2.2 Definition of the operation

```
locale propagate-ops =
  dpll-state  mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT
  for
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    trail :: 'st ⇒ ('v, unit, unit) marked-lits and
    raw-clauses :: 'st ⇒ 'clss and
    prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒'st and
    add-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    remove-cls_NOT :: 'cls ⇒ 'st ⇒ 'st +
  fixes
    propagate-cond :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ bool
begin
inductive propagate_NOT :: 'st ⇒ 'st ⇒ bool where
propagate_NOT[intro]: C + {#L#} ∈# clauses_NOT S ⟹ trail S ⊨as CNot C
    ⟹ undefined-lit (trail S) L
    ⟹ propagate-cond (Propagated L ()) S
    ⟹ T ∼ prepend-trail (Propagated L ()) S
    ⟹ propagate_NOT S T
inductive-cases propagate_NOTE[elim]: propagate_NOT S T

end

locale decide-ops =
  dpll-state  mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT
  for
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    trail :: 'st ⇒ ('v, unit, unit) marked-lits and
    raw-clauses :: 'st ⇒ 'clss and
    prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒'st and
    add-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    remove-cls_NOT :: 'cls ⇒ 'st ⇒ 'st
begin
inductive decide_NOT ::  'st ⇒ 'st ⇒ bool where
decide_NOT[intro]: undefined-lit (trail S) L ⟹ atm-of L ∈ atms-of-mm (clauses_NOT S)
  ⟹ T ∼ prepend-trail (Marked L ()) S
  ⟹ decide_NOT S T

inductive-cases decide_NOTE[elim]: decide_NOT S S'
end

locale backjumping-ops =
  dpll-state  mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT
  for
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    trail :: 'st ⇒ ('v, unit, unit) marked-lits and
    raw-clauses :: 'st ⇒ 'clss and
    prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒'st and
    add-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    remove-cls_NOT :: 'cls ⇒ 'st ⇒ 'st +
  fixes
    backjump-conds :: 'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool
begin

inductive backjump where
trail S = F' @ Marked K ()# F
   ⟹ T ∼ prepend-trail (Propagated L ()) (reduce-trail-to_NOT F S)
   ⟹ C ∈# clauses_NOT S
   ⟹ trail S ⊨as CNot C
   ⟹ undefined-lit F L
   ⟹ atm-of L ∈ atms-of-mm (clauses_NOT S) ∪ atm-of ` (lits-of-l (trail S))
   ⟹ clauses_NOT S ⊨pm C' + {#L#}
   ⟹ F ⊨as CNot C'
   ⟹ backjump-conds C C' L S T
   ⟹ backjump S T
inductive-cases backjumpE: backjump S T
```

The condition *atm-of* $L \in$ *atms-of-mm* (*clauses*$_{NOT}$ S) $\cup$ *atm-of* ` *lits-of-l* (*trail* S) is not implied by the the condition *clauses*$_{NOT}$ S $\models$*pm* $C' + \{\#L\#\}$ (no negation).

end

16.3 DPLL with backjumping

```
locale dpll-with-backjumping-ops =
  propagate-ops mset-cls insert-cls remove-lit
```

mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-cls$_{NOT}$ *remove-cls*$_{NOT}$ *propagate-conds* +
decide-ops mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-cls$_{NOT}$ *remove-cls*$_{NOT}$ +
backjumping-ops mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-cls$_{NOT}$ *remove-cls*$_{NOT}$ *backjump-conds*
for
mset-cls:: *'cls* $\Rightarrow$ *'v clause* **and**
insert-cls :: *'v literal* $\Rightarrow$ *'cls* $\Rightarrow$ *'cls* **and**
remove-lit :: *'v literal* $\Rightarrow$ *'cls* $\Rightarrow$ *'cls* **and**
mset-clss:: *'clss* $\Rightarrow$ *'v clauses* **and**
union-clss :: *'clss* $\Rightarrow$ *'clss* $\Rightarrow$ *'clss* **and**
in-clss :: *'cls* $\Rightarrow$ *'clss* $\Rightarrow$ *bool* **and**
insert-clss :: *'cls* $\Rightarrow$ *'clss* $\Rightarrow$ *'clss* **and**
remove-from-clss :: *'cls* $\Rightarrow$ *'clss* $\Rightarrow$ *'clss* **and**
trail :: *'st* $\Rightarrow$ (*'v*, *unit*, *unit*) *marked-lits* **and**
raw-clauses :: *'st* $\Rightarrow$ *'clss* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* $\Rightarrow$ *'st* $\Rightarrow$ *'st* **and**
tl-trail :: *'st* $\Rightarrow$*'st* **and**
add-cls$_{NOT}$ :: *'cls* $\Rightarrow$ *'st* $\Rightarrow$ *'st* **and**
remove-cls$_{NOT}$ :: *'cls* $\Rightarrow$ *'st* $\Rightarrow$ *'st* **and**
inv :: *'st* $\Rightarrow$ *bool* **and**
backjump-conds :: *'v clause* $\Rightarrow$ *'v clause* $\Rightarrow$ *'v literal* $\Rightarrow$ *'st* $\Rightarrow$ *'st* $\Rightarrow$ *bool* **and**
propagate-conds :: (*'v*, *unit*, *unit*) *marked-lit* $\Rightarrow$ *'st* $\Rightarrow$ *bool* +
assumes
bj-can-jump:
$\bigwedge$*S C F' K F L.*
inv S $\Longrightarrow$
no-dup (*trail S*) $\Longrightarrow$
trail S = *F'* @ *Marked K* () # *F* $\Longrightarrow$
C $\in$# *clauses*$_{NOT}$ *S* $\Longrightarrow$
trail S $\models$*as CNot C* $\Longrightarrow$
undefined-lit F L $\Longrightarrow$
atm-of L $\in$ *atms-of-mm* (*clauses*$_{NOT}$ *S*) $\cup$ *atm-of* ‘ (*lits-of-l* (*F'* @ *Marked K* () # *F*)) $\Longrightarrow$
clauses$_{NOT}$ *S* $\models$*pm C'* + {#*L*#} $\Longrightarrow$
F $\models$*as CNot C'* $\Longrightarrow$
$\neg$*no-step backjump S*
begin

We cannot add a like condition *atms-of C'* $\subseteq$ *atms-of-ms N* because to ensure that we can backjump even if the last decision variable has disappeared.

The part of the condition *atm-of L* $\in$ *atm-of* ‘ *lits-of-l* (*F'* @ *Marked K* () # *F*) is important, otherwise you are not sure that you can backtrack.

16.3.1 Definition

We define dpll with backjumping:

inductive *dpll-bj* :: *'st* $\Rightarrow$ *'st* $\Rightarrow$ *bool* **for** *S* :: *'st* **where**
bj-decide$_{NOT}$: *decide*$_{NOT}$ *S S'* $\Longrightarrow$ *dpll-bj S S'* |
bj-propagate$_{NOT}$: *propagate*$_{NOT}$ *S S'* $\Longrightarrow$ *dpll-bj S S'* |
bj-backjump: *backjump S S'* $\Longrightarrow$ *dpll-bj S S'*

```
lemmas dpll-bj-induct = dpll-bj.induct[split-format(complete)]
thm dpll-bj-induct[OF dpll-with-backjumping-ops-axioms]
lemma dpll-bj-all-induct[consumes 2, case-names decide_NOT propagate_NOT backjump]:
  fixes S T :: 'st
  assumes
    dpll-bj S T and
    inv S
    ⋀L T. undefined-lit (trail S) L ⟹ atm-of L ∈ atms-of-mm (clauses_NOT S)
      ⟹ T ∼ prepend-trail (Marked L ()) S
      ⟹ P S T and
    ⋀C L T. C + {#L#} ∈# clauses_NOT S ⟹ trail S ⊨as CNot C ⟹ undefined-lit (trail S) L
      ⟹ T ∼ prepend-trail (Propagated L ()) S
      ⟹ P S T and
    ⋀C F' K F L C' T. C ∈# clauses_NOT S ⟹ F' @ Marked K () # F ⊨as CNot C
      ⟹ trail S = F' @ Marked K () # F
      ⟹ undefined-lit F L
      ⟹ atm-of L ∈ atms-of-mm (clauses_NOT S) ∪ atm-of ‘ (lits-of-l (F' @ Marked K () # F))
      ⟹ clauses_NOT S ⊨pm C' + {#L#}
      ⟹ F ⊨as CNot C'
      ⟹ T ∼ prepend-trail (Propagated L ()) (reduce-trail-to_NOT F S)
      ⟹ P S T
  shows P S T
  apply (induct T rule: dpll-bj-induct[OF local.dpll-with-backjumping-ops-axioms])
     apply (rule assms(1))
    using assms(3) apply blast
   apply (elim propagate_NOTE) using assms(4) apply blast
  apply (elim backjumpE) using assms(5) ‹inv S› by simp
```

16.3.2 Basic properties

First, some better suited induction principle

```
lemma dpll-bj-clauses:
  assumes dpll-bj S T and inv S
  shows clauses_NOT S = clauses_NOT T
  using assms by (induction rule: dpll-bj-all-induct) auto
```

No duplicates in the trail

```
lemma dpll-bj-no-dup:
  assumes dpll-bj S T and inv S
  and no-dup (trail S)
  shows no-dup (trail T)
  using assms by (induction rule: dpll-bj-all-induct)
  (auto simp add: defined-lit-map reduce-trail-to_NOT-skip-beginning)
```

Valuations

```
lemma dpll-bj-sat-iff:
  assumes dpll-bj S T and inv S
  shows I ⊨sm clauses_NOT S ⟷ I ⊨sm clauses_NOT T
  using assms by (induction rule: dpll-bj-all-induct) auto
```

Clauses

```
lemma dpll-bj-atms-of-ms-clauses-inv:
  assumes
    dpll-bj S T and
    inv S
  shows atms-of-mm (clauses_NOT S) = atms-of-mm (clauses_NOT T)
  using assms by (induction rule: dpll-bj-all-induct) auto
```

```
lemma dpll-bj-atms-in-trail:
  assumes
    dpll-bj S T and
    inv S and
    atm-of ‘ (lits-of-l (trail S)) ⊆ atms-of-mm (clauses_NOT S)
  shows atm-of ‘ (lits-of-l (trail T)) ⊆ atms-of-mm (clauses_NOT S)
  using assms by (induction rule: dpll-bj-all-induct)
  (auto simp: in-plus-implies-atm-of-on-atms-of-ms reduce-trail-to_NOT-skip-beginning)

lemma dpll-bj-atms-in-trail-in-set:
  assumes dpll-bj S Tand
    inv S and
  atms-of-mm (clauses_NOT S) ⊆ A and
  atm-of ‘ (lits-of-l (trail S)) ⊆ A
  shows atm-of ‘ (lits-of-l (trail T)) ⊆ A
  using assms by (induction rule: dpll-bj-all-induct)
  (auto simp: in-plus-implies-atm-of-on-atms-of-ms)

lemma dpll-bj-all-decomposition-implies-inv:
  assumes
    dpll-bj S T and
    inv: inv S and
    decomp: all-decomposition-implies-m (clauses_NOT S) (get-all-marked-decomposition (trail S))
  shows all-decomposition-implies-m (clauses_NOT T) (get-all-marked-decomposition (trail T))
  using assms(1,2)
proof (induction rule:dpll-bj-all-induct)
  case decide_NOT
  then show ?case using decomp by auto
next
  case (propagate_NOT C L T) note propa = this(1) and undef = this(3) and T = this(4)
  let ?M' = trail (prepend-trail (Propagated L ()) S)
  let ?N = clauses_NOT S
  obtain a y l where ay: get-all-marked-decomposition ?M' = (a, y) # l
    by (cases get-all-marked-decomposition ?M') fastforce+
  then have M': ?M' = y @ a using get-all-marked-decomposition-decomp[of ?M'] by auto
  have M: get-all-marked-decomposition (trail S) = (a, tl y) # l
    using ay undef by (cases  get-all-marked-decomposition (trail S)) auto
  have y_0: y = (Propagated L ()) # (tl y)
    using ay undef by (auto simp add: M)
  from arg-cong[OF this, of set] have y[simp]: set y = insert (Propagated L ()) (set (tl y))
    by simp
  have tr-S: trail S = tl y @ a
    using arg-cong[OF M', of tl] y_0 M get-all-marked-decomposition-decomp by force
  have a-Un-N-M: unmark-l a ∪ set-mset ?N ⊨ps unmark-l (tl y)
    using decomp ay unfolding all-decomposition-implies-def by (simp add: M)+

  moreover have unmark-l a ∪ set-mset ?N ⊨p {#L#} (is ?I ⊨p -)
    proof (rule true-clss-cls-plus-CNot)
      show ?I ⊨p C + {#L#}
        using propa propagate_NOT.prems by (auto dest!: true-clss-clss-in-imp-true-clss-cls)
    next
      have unmark-l ?M' ⊨ps CNot C
        using ‹trail S ⊨as CNot C› undef by (auto simp add: true-annots-true-clss-clss)
      have a1: unmark-l a ∪ unmark-l (tl y) ⊨ps CNot C
        using propagate_NOT.hyps(2) tr-S true-annots-true-clss-clss
        by (force simp add: image-Un sup-commute)
      then have unmark-l a ∪ set-mset (clauses_NOT S) ⊨ps unmark-l a ∪ unmark-l (tl y)
        using a-Un-N-M true-clss-clss-def by blast
      then show unmark-l a ∪ set-mset (clauses_NOT S) ⊨ps CNot C
        using a1 by (meson true-clss-clss-left-right true-clss-clss-union-and
          true-clss-clss-union-l-r)
    qed
  ultimately have unmark-l a ∪ set-mset ?N ⊨ps unmark-l ?M'
    unfolding M' by (auto simp add: all-in-true-clss-clss image-Un)
  then show ?case
    using decomp T M undef unfolding ay all-decomposition-implies-def by (auto simp add: ay)
next
  case (backjump C F' K F L D T) note confl = this(2) and tr = this(3) and undef = this(4)
    and L = this(5) and N-C = this(6) and vars-D = this(5) and T = this(8)
  have decomp: all-decomposition-implies-m (clauses_NOT S) (get-all-marked-decomposition F)
    using decomp unfolding tr all-decomposition-implies-def
    by (metis (no-types, lifting) get-all-marked-decomposition.simps(1)
      get-all-marked-decomposition-never-empty hd-Cons-tl insert-iff list.sel(3) list.set(2)
      tl-get-all-marked-decomposition-skip-some)

  obtain a b li where F: get-all-marked-decomposition F = (a, b) # li
    by (cases get-all-marked-decomposition F) auto
  have F = b @ a
    using get-all-marked-decomposition-decomp[of F a b] F by auto
  have a-N-b:unmark-l a ∪ set-mset (clauses_NOT S) ⊨ps unmark-l b
    using decomp unfolding all-decomposition-implies-def by (auto simp add: F)

  have F-D:unmark-l F ⊨ps CNot D
    using ⟨F ⊨as CNot D⟩ by (simp add: true-annots-true-clss-clss)
  then have unmark-l a ∪ unmark-l b ⊨ps CNot D
    unfolding ⟨F = b @ a⟩ by (simp add: image-Un sup.commute)
  have a-N-CNot-D: unmark-l a ∪ set-mset (clauses_NOT S) ⊨ps CNot D ∪ unmark-l b
    apply (rule true-clss-clss-left-right)
    using a-N-b F-D unfolding ⟨F = b @ a⟩ by (auto simp add: image-Un ac-simps)

  have a-N-D-L: unmark-l a ∪ set-mset (clauses_NOT S) ⊨p D+{#L#}
    by (simp add: N-C)
  have unmark-l a ∪ set-mset (clauses_NOT S) ⊨p {#L#}
    using a-N-D-L a-N-CNot-D by (blast intro: true-clss-cls-plus-CNot)
  then show ?case
    using decomp T tr undef unfolding all-decomposition-implies-def by (auto simp add: F)
qed
```

16.3.3 Termination

Using a proper measure

```
lemma length-get-all-marked-decomposition-append-Marked:
  length (get-all-marked-decomposition (F' @ Marked K () # F)) =
    length (get-all-marked-decomposition F')
    + length (get-all-marked-decomposition (Marked K () # F))
    − 1
  by (induction F' rule: marked-lit-list-induct) auto

lemma take-length-get-all-marked-decomposition-marked-sandwich:
  take (length (get-all-marked-decomposition F))
      (map (f o snd) (rev (get-all-marked-decomposition (F' @ Marked K () # F))))
      =
     map (f o snd) (rev (get-all-marked-decomposition F))
```

proof (*induction F′ rule*: *marked-lit-list-induct*)
  case *nil*
  then show *?case* **by** *auto*
next
  case (*marked K*)
  then show *?case* **by** (*simp add*: *length-get-all-marked-decomposition-append-Marked*)
next
  case (*proped L m F′*) **note** *IH* = *this*(*1*)
  obtain *a b l* **where** *F′*: *get-all-marked-decomposition* (*F′* @ *Marked K* () # *F*) = (*a*, *b*) # *l*
    by (*cases get-all-marked-decomposition* (*F′* @ *Marked K* () # *F*)) *auto*
  have *length* (*get-all-marked-decomposition F*) − *length l* = *0*
    using *length-get-all-marked-decomposition-append-Marked*[*of F′ K F*]
    unfolding *F′* **by** (*cases get-all-marked-decomposition F′*) *auto*
  then show *?case*
    using *IH* **by** (*simp add*: *F′*)
qed

lemma *length-get-all-marked-decomposition-length*:
  length (*get-all-marked-decomposition M*) ≤ *1* + *length M*
  by (*induction M rule*: *marked-lit-list-induct*) *auto*

lemma *length-in-get-all-marked-decomposition-bounded*:
  assumes *i*:*i* ∈ *set* (*trail-weight S*)
  shows *i* ≤ *Suc* (*length* (*trail S*))
proof −
  obtain *a b* **where**
    (*a*, *b*) ∈ *set* (*get-all-marked-decomposition* (*trail S*)) **and**
    ib: *i* = *Suc* (*length b*)
    using *i* **by** *auto*
  then obtain *c* **where** *trail S* = *c* @ *b* @ *a*
    using *get-all-marked-decomposition-exists-prepend′* **by** *metis*
  from *arg-cong*[*OF this*, *of length*] **show** *?thesis* **using** *i ib* **by** *auto*
qed

Well-foundedness The bounds are the following:

- *1* + *card* (*atms-of-ms A*): *card* (*atms-of-ms A*) is an upper bound on the length of the list. As *get-all-marked-decomposition* appends an possibly empty couple at the end, adding one is needed.
- *2* + *card* (*atms-of-ms A*): *card* (*atms-of-ms A*) is an upper bound on the number of elements, where adding one is necessary for the same reason as for the bound on the list, and one is needed to have a strict bound.

abbreviation *unassigned-lit* :: *′b literal multiset set* ⇒ *′a list* ⇒ *nat* **where**
  unassigned-lit N M ≡ *card* (*atms-of-ms N*) − *length M*
lemma *dpll-bj-trail-mes-increasing-prop*:
  fixes *M* :: (*′v*, *unit*, *unit*) *marked-lits* **and** *N* :: *′v clauses*
  assumes
    dpll-bj S T **and**
    inv S **and**
    NA: *atms-of-mm* (*clauses*$_{NOT}$ *S*) ⊆ *atms-of-ms A* **and**
    MA: *atm-of* ‘ *lits-of-l* (*trail S*) ⊆ *atms-of-ms A* **and**

```
    n-d: no-dup (trail S) and
    finite: finite A
  shows μ_C (1+card (atms-of-ms A)) (2+card (atms-of-ms A)) (trail-weight T)
    > μ_C (1+card (atms-of-ms A)) (2+card (atms-of-ms A)) (trail-weight S)
  using assms(1,2)
proof (induction rule: dpll-bj-all-induct)
  case (propagate_NOT C L) note CLN = this(1) and MC = this(2) and undef-L = this(3) and T = this(4)
  have incl: atm-of ` lits-of-l (Propagated L () # trail S) ⊆ atms-of-ms A
    using propagate_NOT dpll-bj-atms-in-trail-in-set bj-propagate_NOT NA MA CLN
    by (auto simp: in-plus-implies-atm-of-on-atms-of-ms)

  have no-dup: no-dup (Propagated L () # trail S)
    using defined-lit-map n-d undef-L by auto
  obtain a b l where M: get-all-marked-decomposition (trail S) = (a, b) # l
    by (cases get-all-marked-decomposition (trail S)) auto
  have b-le-M: length b ≤ length (trail S)
    using get-all-marked-decomposition-decomp[of trail S] by (simp add: M)
  have finite (atms-of-ms A) using finite by simp

  then have length (Propagated L () # trail S) ≤ card (atms-of-ms A)
    using incl finite unfolding no-dup-length-eq-card-atm-of-lits-of-l[OF no-dup]
    by (simp add: card-mono)
  then have latm: unassigned-lit A b = Suc (unassigned-lit A (Propagated L d # b))
    using b-le-M by auto
  then show ?case using T undef-L by (auto simp: latm M μ_C-cons)
next
  case (decide_NOT L) note undef-L = this(1) and MC = this(2) and T = this(3)
  have incl: atm-of ` lits-of-l (Marked L () # (trail S)) ⊆ atms-of-ms A
    using dpll-bj-atms-in-trail-in-set bj-decide_NOT decide_NOT.decide_NOT[OF decide_NOT.hyps] NA MA MC
    by auto

  have no-dup: no-dup (Marked L () # (trail S))
    using defined-lit-map n-d undef-L by auto
  obtain a b l where M: get-all-marked-decomposition (trail S) = (a, b) # l
    by (cases get-all-marked-decomposition (trail S)) auto

  then have length (Marked L () # (trail S)) ≤ card (atms-of-ms A)
    using incl finite unfolding no-dup-length-eq-card-atm-of-lits-of-l[OF no-dup]
    by (simp add: card-mono)
  show ?case using T undef-L by (simp add: μ_C-cons)
next
  case (backjump C F' K F L C' T) note undef-L = this(4) and MC = this(1) and tr-S = this(3) and
    L = this(5) and T = this(8)
  have incl: atm-of ` lits-of-l (Propagated L () # F) ⊆ atms-of-ms A
    using dpll-bj-atms-in-trail-in-set NA MA L by (auto simp: tr-S)

  have no-dup: no-dup (Propagated L () # F)
    using defined-lit-map n-d undef-L tr-S by auto
  obtain a b l where M: get-all-marked-decomposition (trail S) = (a, b) # l
    by (cases get-all-marked-decomposition (trail S)) auto
  have b-le-M: length b ≤ length (trail S)
    using get-all-marked-decomposition-decomp[of trail S] by (simp add: M)
  have fin-atms-A: finite (atms-of-ms A) using finite by simp

  then have F-le-A: length (Propagated L () # F) ≤  card (atms-of-ms A)
    using incl finite unfolding no-dup-length-eq-card-atm-of-lits-of-l[OF no-dup]
    by (simp add: card-mono)
  have tr-S-le-A: length (trail S) ≤  (card (atms-of-ms A))
    using n-d MA by (metis fin-atms-A card-mono no-dup-length-eq-card-atm-of-lits-of-l)
  obtain a b l where F: get-all-marked-decomposition F = (a, b) # l
    by (cases get-all-marked-decomposition F) auto
  then have F = b @ a
    using get-all-marked-decomposition-decomp[of Propagated L () # F a
      Propagated L () # b] by simp
  then have latm: unassigned-lit A b = Suc (unassigned-lit A (Propagated L () # b))
     using F-le-A by simp
  obtain rem where
    rem:map (λa. Suc (length (snd a))) (rev (get-all-marked-decomposition (F' @ Marked K () # F)))
    = map (λa. Suc (length (snd a))) (rev (get-all-marked-decomposition F)) @ rem
    using take-length-get-all-marked-decomposition-marked-sandwich[of F λa. Suc (length a) F' K]
    unfolding o-def by (metis append-take-drop-id)
  then have rem: map (λa. Suc (length (snd a)))
      (get-all-marked-decomposition (F' @ Marked K () # F))
    = rev rem @ map (λa. Suc (length (snd a))) ((get-all-marked-decomposition F))
    by (simp add: rev-map[symmetric] rev-swap)
  have length (rev rem @ map (λa. Suc (length (snd a))) (get-all-marked-decomposition F))
          ≤ Suc (card (atms-of-ms A))
    using arg-cong[OF rem, of length] tr-S-le-A
    length-get-all-marked-decomposition-length[of F' @ Marked K () # F] tr-S by auto
  moreover
    { fix i :: nat and xs :: 'a list
      have i < length xs ⟹ length xs − Suc i < length xs
        by auto
      then have H: i<length xs ⟹ rev xs ! i ∈ set xs
        using rev-nth[of i xs] unfolding in-set-conv-nth by (force simp add: in-set-conv-nth)
    } note H = this
    have ∀ i<length rem. rev rem ! i < card (atms-of-ms A) + 2
      using tr-S-le-A length-in-get-all-marked-decomposition-bounded[of - S] unfolding tr-S
      by (force simp add: o-def rem dest!: H intro: length-get-all-marked-decomposition-length)
  ultimately show ?case
    using μC-bounded[of rev rem card (atms-of-ms A)+2 unassigned-lit A l] T undef-L
    by (simp add: rem μC-append μC-cons F tr-S)
qed

lemma dpll-bj-trail-mes-decreasing-prop:
  assumes dpll: dpll-bj S T  and inv: inv S and
  N-A: atms-of-mm (clausesNOT S) ⊆ atms-of-ms A and
  M-A: atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A and
  nd: no-dup (trail S) and
  fin-A: finite A
  shows (2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A))
               − μC (1+card (atms-of-ms A)) (2+card (atms-of-ms A)) (trail-weight T)
            < (2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A))
               − μC (1+card (atms-of-ms A)) (2+card (atms-of-ms A)) (trail-weight S)
proof −
  let ?b = 2+card (atms-of-ms A)
  let ?s = 1+card (atms-of-ms A)
  let ?μ = μ_C ?s ?b
  have M'-A: atm-of ` lits-of-l (trail T) ⊆ atms-of-ms A
    by (meson M-A N-A dpll dpll-bj-atms-in-trail-in-set inv)
  have nd': no-dup (trail T)
    using ‹dpll-bj S T› dpll-bj-no-dup nd inv by blast
  { fix i :: nat and xs :: 'a list
    have i < length xs ⟹ length xs − Suc i < length xs
      by auto
    then have H: i<length xs ⟹  xs ! i ∈ set xs
      using rev-nth[of i xs] unfolding in-set-conv-nth by (force simp add: in-set-conv-nth)
  } note H = this

  have l-M-A: length (trail S) ≤ card (atms-of-ms A)
    by (simp add: fin-A M-A card-mono no-dup-length-eq-card-atm-of-lits-of-l nd)
  have l-M'-A: length (trail T) ≤ card (atms-of-ms A)
    by (simp add: fin-A M'-A card-mono no-dup-length-eq-card-atm-of-lits-of-l nd')
  have l-trail-weight-M: length (trail-weight T) ≤ 1+card (atms-of-ms A)
     using l-M'-A length-get-all-marked-decomposition-length[of trail T] by auto
  have bounded-M: ∀ i<length (trail-weight T). (trail-weight T)! i < card (atms-of-ms A) + 2
    using length-in-get-all-marked-decomposition-bounded[of - T] l-M'-A
    by (metis (no-types, lifting) H Nat.le-trans add-2-eq-Suc' not-le not-less-eq-eq)

  from dpll-bj-trail-mes-increasing-prop[OF dpll inv N-A M-A nd fin-A]
  have μ_C ?s ?b (trail-weight S) < μ_C ?s ?b (trail-weight T) by simp
  moreover from μ_C-bounded[OF bounded-M l-trail-weight-M]
    have μ_C ?s ?b (trail-weight T) ≤ ?b ^ ?s by auto
  ultimately show ?thesis by linarith
qed

lemma wf-dpll-bj:
  assumes fin: finite A
  shows wf {(T, S). dpll-bj S T
    ∧ atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A ∧ atm-of ` lits-of-l (trail S) ⊆ atms-of-ms A
    ∧ no-dup (trail S) ∧ inv S}
  (is wf ?A)
proof (rule wf-bounded-measure[of -
        λ-. (2 + card (atms-of-ms A))^(1 + card (atms-of-ms A))
        λS. μ_C (1+card (atms-of-ms A)) (2+card (atms-of-ms A)) (trail-weight S)])
  fix a b :: 'st
  let ?b = 2+card (atms-of-ms A)
  let ?s = 1+card (atms-of-ms A)
  let ?μ = μ_C ?s ?b
  assume ab: (b, a) ∈ ?A

  have fin-A: finite (atms-of-ms A)
    using fin by auto
  have
    dpll-bj: dpll-bj a b and
    N-A: atms-of-mm (clauses_NOT a) ⊆ atms-of-ms A and
    M-A: atm-of ` lits-of-l (trail a) ⊆ atms-of-ms A and
    nd: no-dup (trail a) and
    inv: inv a
    using ab by auto

  have M'-A: atm-of ` lits-of-l (trail b) ⊆ atms-of-ms A
```

```
    by (meson M-A N-A ‹dpll-bj a b› dpll-bj-atms-in-trail-in-set inv)
  have nd': no-dup (trail b)
    using ‹dpll-bj a b› dpll-bj-no-dup nd inv by blast
  { fix i :: nat and xs :: 'a list
    have i < length xs ⟹ length xs − Suc i < length xs
      by auto
    then have H: i<length xs ⟹  xs ! i ∈ set xs
      using rev-nth[of i xs] unfolding in-set-conv-nth by (force simp add: in-set-conv-nth)
  } note H = this

  have l-M-A: length (trail a) ≤ card (atms-of-ms A)
    by (simp add: fin-A M-A card-mono no-dup-length-eq-card-atm-of-lits-of-l nd)
  have l-M'-A: length (trail b) ≤ card (atms-of-ms A)
    by (simp add: fin-A M'-A card-mono no-dup-length-eq-card-atm-of-lits-of-l nd')
  have l-trail-weight-M: length (trail-weight b) ≤ 1+card (atms-of-ms A)
     using l-M'-A length-get-all-marked-decomposition-length[of trail b] by auto
  have bounded-M: ∀ i<length (trail-weight b). (trail-weight b)! i < card (atms-of-ms A) + 2
    using length-in-get-all-marked-decomposition-bounded[of - b] l-M'-A
    by (metis (no-types, lifting) Nat.le-trans One-nat-def Suc-1 add.right-neutral add-Suc-right
      le-imp-less-Suc less-eq-Suc-le nth-mem)

  from dpll-bj-trail-mes-increasing-prop[OF dpll-bj inv N-A M-A nd fin]
  have μ_C ?s ?b (trail-weight a) < μ_C ?s ?b (trail-weight b) by simp
  moreover from μ_C-bounded[OF bounded-M l-trail-weight-M]
    have μ_C ?s ?b (trail-weight b) ≤ ?b ^ ?s by auto
  ultimately show ?b ^ ?s ≤ ?b ^ ?s ∧
           μ_C ?s ?b (trail-weight b) ≤ ?b ^ ?s ∧
           μ_C ?s ?b (trail-weight a) < μ_C ?s ?b (trail-weight b)
    by blast
qed
```

16.3.4 Normal Forms

We prove that given a normal form of DPLL, with some invariants, the either N is satisfiable and the built valuation M is a model; or N is unsatisfiable.

Idea of the proof: We have to prove tat *satisfiable* N, $\neg\ M \models as\ N$ and there is no remaining step is incompatible.

1. The *decide* rules tells us that every variable in N has a value.
2. $\neg\ M \models as\ N$ tells us that there is conflict.
3. There is at least one decision in the trail (otherwise, M is a model of N).
4. Now if we build the clause with all the decision literals of the trail, we can apply the *backjump* rule.

   The assumption are saying that we have a finite upper bound A for the literals, that we cannot do any step *no-step dpll-bj S*

```
theorem dpll-backjump-final-state:
  fixes A :: 'v literal multiset set and S T :: 'st
  assumes
    atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A and
    atm-of ` lits-of-l (trail S) ⊆ atms-of-ms A and
    no-dup (trail S) and
    finite A and
    inv: inv S and
    n-s: no-step dpll-bj S and
    decomp: all-decomposition-implies-m (clauses_NOT S) (get-all-marked-decomposition (trail S))
  shows unsatisfiable (set-mset (clauses_NOT S))
    ∨ (trail S ⊨asm clauses_NOT S ∧ satisfiable (set-mset (clauses_NOT S)))
proof -
  let ?N = set-mset (clauses_NOT S)
  let ?M = trail S
  consider
      (sat) satisfiable ?N and ?M ⊨as ?N
    | (sat') satisfiable ?N and ¬ ?M ⊨as ?N
    | (unsat) unsatisfiable ?N
    by auto
  then show ?thesis
    proof cases
      case sat' note sat = this(1) and M = this(2)
      obtain C where C ∈ ?N and ¬?M ⊨a C using M unfolding true-annots-def by auto
      obtain I :: 'v literal set where
        I ⊨s ?N and
        cons: consistent-interp I and
        tot: total-over-m I ?N and
        atm-I-N: atm-of `I ⊆ atms-of-ms ?N
        using sat unfolding satisfiable-def-min by auto
      let ?I = I ∪ {P| P. P ∈ lits-of-l ?M ∧ atm-of P ∉ atm-of ` I}
      let ?O = {unmark L |L. is-marked L ∧ L ∈ set ?M ∧ atm-of (lit-of L) ∉ atms-of-ms ?N}
      have cons-I': consistent-interp ?I
        using cons using ‹no-dup ?M› unfolding consistent-interp-def
        by (auto simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set lits-of-def
          dest!: no-dup-cannot-not-lit-and-uminus)
      have tot-I': total-over-m ?I (?N ∪ unmark-l ?M)
        using tot atm-I-N unfolding total-over-m-def total-over-set-def
        by (fastforce simp: image-iff lits-of-def)
      have {P |P. P ∈ lits-of-l ?M ∧ atm-of P ∉ atm-of ` I} ⊨s ?O
        using ‹I⊨s ?N› atm-I-N by (auto simp add: atm-of-eq-atm-of true-clss-def lits-of-def)
      then have I'-N: ?I ⊨s ?N ∪ ?O
        using ‹I⊨s ?N› true-clss-union-increase by force
      have tot': total-over-m ?I (?N∪?O)
        using atm-I-N tot unfolding total-over-m-def total-over-set-def
        by (force simp: lits-of-def dest!: is-marked-ex-Marked)

      have atms-N-M: atms-of-ms ?N ⊆ atm-of ` lits-of-l ?M
        proof (rule ccontr)
          assume ¬ ?thesis
          then obtain l :: 'v where
            l-N: l ∈ atms-of-ms ?N and
            l-M: l ∉ atm-of ` lits-of-l ?M
            by auto
          have undefined-lit ?M (Pos l)
            using l-M by (metis Marked-Propagated-in-iff-in-lits-of-l
              atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set literal.sel(1))
          from bj-decide_NOT[OF decide_NOT[OF this]] show False
            using l-N n-s by (metis literal.sel(1) state-eq_NOT-ref)
        qed
```

```
have ?M ⊨as CNot C
  apply (rule all-variables-defined-not-imply-cnot)
  using ‹C ∈ set-mset (clauses_NOT S)› ‹¬ trail S ⊨a C›
     atms-N-M by (auto dest: atms-of-atms-of-ms-mono)
have ∃ l ∈ set ?M. is-marked l
  proof (rule ccontr)
    let ?O = {unmark L |L. is-marked L ∧ L ∈ set ?M ∧ atm-of (lit-of L) ∉ atms-of-ms ?N}
    have ϑ[iff]: ⋀I. total-over-m I (?N ∪ ?O ∪ unmark-l ?M)
      ⟷ total-over-m I (?N ∪unmark-l ?M)
      unfolding total-over-set-def total-over-m-def atms-of-ms-def by blast
    assume ¬ ?thesis
    then have [simp]:{unmark L |L. is-marked L ∧ L ∈ set ?M}
      = {unmark L |L. is-marked L ∧ L ∈ set ?M ∧ atm-of (lit-of L) ∉ atms-of-ms ?N}
      by auto
    then have ?N ∪ ?O ⊨ps unmark-l ?M
      using all-decomposition-implies-propagated-lits-are-implied[OF decomp] by auto

    then have ?I ⊨s unmark-l ?M
      using cons-I' I'-N tot-I' ‹?I ⊨s ?N ∪ ?O› unfolding ϑ true-clss-clss-def by blast
    then have lits-of-l ?M ⊆ ?I
      unfolding true-clss-def lits-of-def by auto
    then have ?M ⊨as ?N
      using I'-N ‹C ∈ ?N› ‹¬ ?M ⊨a C› cons-I' atms-N-M
      by (meson ‹trail S ⊨as CNot C› consistent-CNot-not rev-subsetD sup-ge1 true-annot-def
        true-annots-def true-cls-mono-set-mset-l true-clss-def)
    then show False using M by fast
  qed
from List.split-list-first-propE[OF this] obtain K :: 'v literal and
  F F' :: ('v, unit, unit) marked-lit list where
  M-K: ?M = F' @ Marked K () # F and
  nm: ∀f∈set F'. ¬is-marked f
  unfolding is-marked-def by (metis (full-types) old.unit.exhaust)
let ?K = Marked K () :: ('v, unit, unit) marked-lit
have ?K ∈ set ?M
  unfolding M-K by auto
let ?C = image-mset lit-of {#L∈#mset ?M. is-marked L ∧ L≠?K#} :: 'v literal multiset
let ?C' = set-mset (image-mset (λL::'v literal. {#L#}) (?C + unmark ?K))
have ?N ∪ {unmark L |L. is-marked L ∧ L ∈ set ?M} ⊨ps unmark-l ?M
  using all-decomposition-implies-propagated-lits-are-implied[OF decomp] .
moreover have C': ?C' = {unmark L |L. is-marked L ∧ L ∈ set ?M}
  unfolding M-K by standard force+
ultimately have N-C-M: ?N ∪ ?C' ⊨ps unmark-l ?M
  by auto
have N-M-False: ?N ∪ (λL. unmark L) ` (set ?M) ⊨ps {{#}}
  using M ‹?M ⊨as CNot C› ‹C∈?N› unfolding true-clss-clss-def true-annots-def Ball-def
  true-annot-def by (metis consistent-CNot-not sup.orderE sup-commute true-clss-def
    true-clss-singleton-lit-of-implies-incl true-clss-union true-clss-union-increase)

have undefined-lit F K using ‹no-dup ?M› unfolding M-K by (simp add: defined-lit-map)
moreover
  have ?N ∪ ?C' ⊨ps {{#}}
    proof −
      have A: ?N ∪ ?C' ∪ unmark-l ?M = ?N ∪ unmark-l ?M
        unfolding M-K by auto
      show ?thesis
```

```
            using true-clss-clss-left-right[OF N-C-M, of {{#}}] N-M-False unfolding A by auto
          qed
        have ?N ⊨p image-mset uminus ?C + {#−K#}
          unfolding true-clss-cls-def true-clss-clss-def total-over-m-def
          proof (intro allI impI)
            fix I
            assume
              tot: total-over-set I (atms-of-ms (?N ∪ {image-mset uminus ?C+ {#− K#}})) and
              cons: consistent-interp I and
              I ⊨s ?N
            have (K ∈ I ∧ −K ∉ I) ∨ (−K ∈ I ∧ K ∉ I)
              using cons tot unfolding consistent-interp-def by (cases K) auto
            have  {a ∈ set (trail S). is-marked a ∧ a ≠ Marked K ()} =
              set (trail S) ∩ {L. is-marked L ∧ L ≠ Marked K ()}
             by auto
            then have tot′: total-over-set I
               (atm-of ` lit-of ` (set ?M ∩ {L. is-marked L ∧ L ≠ Marked K ()}))
              using tot by (auto simp add: atms-of-uminus-lit-atm-of-lit-of)
            { fix x :: ('v, unit, unit) marked-lit
              assume
                a3: lit-of x ∉ I and
                a1: x ∈ set ?M and
                a4: is-marked x and
                a5: x ≠ Marked K ()
              then have Pos (atm-of (lit-of x)) ∈ I ∨ Neg (atm-of (lit-of x)) ∈ I
                using a5 a4 tot′ a1 unfolding total-over-set-def atms-of-s-def by blast
              moreover have f6: Neg (atm-of (lit-of x)) = − Pos (atm-of (lit-of x))
                by simp
              ultimately have − lit-of x ∈ I
                using f6 a3 by (metis (no-types) atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
                  literal.sel(1))
            } note H = this

            have ¬I ⊨s ?C′
              using ‹?N ∪ ?C′ ⊨ps {{#}}› tot cons ‹I ⊨s ?N›
              unfolding true-clss-clss-def total-over-m-def
              by (simp add: atms-of-uminus-lit-atm-of-lit-of atms-of-ms-single-image-atm-of-lit-of)
            then show I ⊨ image-mset uminus ?C + {#− K#}
              unfolding true-clss-def true-cls-def using ‹(K ∈ I ∧ −K ∉ I) ∨ (−K ∈ I ∧ K ∉ I)›
              by (auto dest!: H)
          qed
      moreover have F ⊨as CNot (image-mset uminus ?C)
        using nm unfolding true-annots-def CNot-def M-K by (auto simp add: lits-of-def)
      ultimately have False
        using bj-can-jump[of S F′ K F C −K
          image-mset uminus (image-mset lit-of {# L :# mset ?M. is-marked L ∧ L ≠ Marked K ()#})]
          ‹C∈?N› n-s ‹?M ⊨as CNot C› bj-backjump inv ‹no-dup (trail S)› unfolding M-K by auto
        then show ?thesis by fast
    qed auto
qed

end

locale dpll-with-backjumping =
  dpll-with-backjumping-ops mset-cls insert-cls remove-lit
```

mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-cls$_{NOT}$ *remove-cls*$_{NOT}$ *inv backjump-conds*
propagate-conds
for
mset-cls:: *'cls* ⇒ *'v clause* **and**
insert-cls :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**
remove-lit :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**
mset-clss:: *'clss* ⇒ *'v clauses* **and**
union-clss :: *'clss* ⇒ *'clss* ⇒ *'clss* **and**
in-clss :: *'cls* ⇒ *'clss* ⇒ *bool* **and**
insert-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**
remove-from-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**
trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
raw-clauses :: *'st* ⇒ *'clss* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒*'st* **and**
add-cls$_{NOT}$ :: *'cls* ⇒ *'st* ⇒ *'st* **and**
remove-cls$_{NOT}$ :: *'cls* ⇒ *'st* ⇒ *'st* **and**
inv :: *'st* ⇒ *bool* **and**
backjump-conds :: *'v clause* ⇒ *'v clause* ⇒ *'v literal* ⇒ *'st* ⇒ *'st* ⇒ *bool* **and**
propagate-conds :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *bool*
+
assumes *dpll-bj-inv*: ⋀*S T*. *dpll-bj S T* ⟹ *inv S* ⟹ *inv T*
begin

lemma *rtranclp-dpll-bj-inv*:
assumes *dpll-bj*** *S T* **and** *inv S*
shows *inv T*
using *assms* **by** (*induction rule*: *rtranclp-induct*)
(*auto simp add*: *dpll-bj-no-dup intro*: *dpll-bj-inv*)

lemma *rtranclp-dpll-bj-no-dup*:
assumes *dpll-bj*** *S T* **and** *inv S*
and *no-dup* (*trail S*)
shows *no-dup* (*trail T*)
using *assms* **by** (*induction rule*: *rtranclp-induct*)
(*auto simp add*: *dpll-bj-no-dup dest*: *rtranclp-dpll-bj-inv dpll-bj-inv*)

lemma *rtranclp-dpll-bj-atms-of-ms-clauses-inv*:
assumes
dpll-bj** *S T* **and** *inv S*
shows *atms-of-mm* (*clauses*$_{NOT}$ *S*) = *atms-of-mm* (*clauses*$_{NOT}$ *T*)
using *assms* **by** (*induction rule*: *rtranclp-induct*)
(*auto dest*: *rtranclp-dpll-bj-inv dpll-bj-atms-of-ms-clauses-inv*)

lemma *rtranclp-dpll-bj-atms-in-trail*:
assumes
dpll-bj** *S T* **and**
inv S **and**
atm-of ‘ (*lits-of-l* (*trail S*)) ⊆ *atms-of-mm* (*clauses*$_{NOT}$ *S*)
shows *atm-of* ‘ (*lits-of-l* (*trail T*)) ⊆ *atms-of-mm* (*clauses*$_{NOT}$ *T*)
using *assms* **apply** (*induction rule*: *rtranclp-induct*)
using *dpll-bj-atms-in-trail dpll-bj-atms-of-ms-clauses-inv rtranclp-dpll-bj-inv* **by** *auto*

lemma *rtranclp-dpll-bj-sat-iff*:

```
  assumes dpll-bj** S T and inv S
  shows I ⊨sm clauses_NOT S ⟷ I ⊨sm clauses_NOT T
  using assms by (induction rule: rtranclp-induct)
    (auto dest!: dpll-bj-sat-iff simp: rtranclp-dpll-bj-inv)

lemma rtranclp-dpll-bj-atms-in-trail-in-set:
  assumes
    dpll-bj** S T and
    inv S
    atms-of-mm (clauses_NOT S) ⊆ A and
    atm-of ‘ (lits-of-l (trail S)) ⊆ A
  shows atm-of ‘ (lits-of-l (trail T)) ⊆ A
  using assms by (induction rule: rtranclp-induct)
  (auto dest: rtranclp-dpll-bj-inv
    simp: dpll-bj-atms-in-trail-in-set rtranclp-dpll-bj-atms-of-ms-clauses-inv rtranclp-dpll-bj-inv)

lemma rtranclp-dpll-bj-all-decomposition-implies-inv:
  assumes
    dpll-bj** S T and
    inv S
    all-decomposition-implies-m (clauses_NOT S) (get-all-marked-decomposition (trail S))
  shows all-decomposition-implies-m (clauses_NOT T) (get-all-marked-decomposition (trail T))
  using assms by (induction rule: rtranclp-induct)
    (auto intro: dpll-bj-all-decomposition-implies-inv simp: rtranclp-dpll-bj-inv)

lemma rtranclp-dpll-bj-inv-incl-dpll-bj-inv-trancl:
  {(T, S). dpll-bj++ S T
    ∧ atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A ∧ atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A
    ∧ no-dup (trail S) ∧ inv S}
     ⊆ {(T, S). dpll-bj S T ∧ atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A
        ∧ atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A ∧ no-dup (trail S) ∧ inv S}+
    (is ?A ⊆ ?B+)
proof standard
  fix x
  assume x-A: x ∈ ?A
  obtain S T::'st where
    x[simp]: x = (T, S) by (cases x) auto
  have
    dpll-bj++ S T and
    atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A and
    atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A and
    no-dup (trail S) and
     inv S
    using x-A by auto
  then show x ∈ ?B+ unfolding x
    proof (induction rule: tranclp-induct)
      case base
      then show ?case by auto
    next
      case (step T U) note step = this(1) and ST = this(2) and IH = this(3)[OF this(4−7)]
        and N-A = this(4) and M-A = this(5) and nd = this(6) and inv = this(7)

      have [simp]: atms-of-mm (clauses_NOT S) = atms-of-mm (clauses_NOT T)
        using step rtranclp-dpll-bj-atms-of-ms-clauses-inv tranclp-into-rtranclp inv by fastforce
      have no-dup (trail T)
```

```
        using local.step nd rtranclp-dpll-bj-no-dup tranclp-into-rtranclp inv by fastforce
      moreover have atm-of ` (lits-of-l (trail T)) ⊆ atms-of-ms A
        by (metis inv M-A N-A local.step rtranclp-dpll-bj-atms-in-trail-in-set
          tranclp-into-rtranclp)
      moreover have inv T
         using inv local.step rtranclp-dpll-bj-inv tranclp-into-rtranclp by fastforce
      ultimately have (U, T) ∈ ?B using ST N-A M-A inv by auto
      then show ?case using IH by (rule trancl-into-trancl2)
    qed
qed

lemma wf-tranclp-dpll-bj:
  assumes fin: finite A
  shows wf {(T, S). dpll-bj^++ S T
    ∧ atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A ∧ atm-of ` lits-of-l (trail S) ⊆ atms-of-ms A
    ∧ no-dup (trail S) ∧ inv S}
  using wf-trancl[OF wf-dpll-bj[OF fin]] rtranclp-dpll-bj-inv-incl-dpll-bj-inv-trancl
  by (rule wf-subset)

lemma dpll-bj-sat-ext-iff:
  dpll-bj S T ⟹ inv S ⟹ I⊨sextm clauses_NOT S ⟷ I⊨sextm clauses_NOT T
  by (simp add: dpll-bj-clauses)

lemma rtranclp-dpll-bj-sat-ext-iff:
  dpll-bj^** S T ⟹ inv S ⟹ I⊨sextm clauses_NOT S ⟷ I⊨sextm clauses_NOT T
  by (induction rule: rtranclp-induct) (simp-all add: rtranclp-dpll-bj-inv dpll-bj-sat-ext-iff)

theorem full-dpll-backjump-final-state:
  fixes A :: 'v literal multiset set and S T :: 'st
  assumes
    full: full dpll-bj S T and
    atms-S: atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A and
    atms-trail: atm-of ` lits-of-l (trail S) ⊆ atms-of-ms A and
    n-d: no-dup (trail S) and
    finite A and
    inv: inv S and
    decomp: all-decomposition-implies-m (clauses_NOT S) (get-all-marked-decomposition (trail S))
  shows unsatisfiable (set-mset (clauses_NOT S))
  ∨ (trail T ⊨asm clauses_NOT S ∧ satisfiable (set-mset (clauses_NOT S)))
proof –
  have st: dpll-bj^** S T and no-step dpll-bj T
    using full unfolding full-def by fast+
  moreover have atms-of-mm (clauses_NOT T) ⊆ atms-of-ms A
    using atms-S inv rtranclp-dpll-bj-atms-of-ms-clauses-inv st by blast
  moreover have atm-of ` lits-of-l (trail T) ⊆ atms-of-ms A
     using atms-S atms-trail inv rtranclp-dpll-bj-atms-in-trail-in-set st by auto
  moreover have no-dup (trail T)
    using n-d inv rtranclp-dpll-bj-no-dup st by blast
  moreover have inv: inv T
    using inv rtranclp-dpll-bj-inv st by blast
  moreover
    have decomp: all-decomposition-implies-m (clauses_NOT T) (get-all-marked-decomposition (trail T))
      using ‹inv S› decomp rtranclp-dpll-bj-all-decomposition-implies-inv st by blast
  ultimately have unsatisfiable (set-mset (clauses_NOT T))
    ∨ (trail T ⊨asm clauses_NOT T ∧ satisfiable (set-mset (clauses_NOT T)))
    using ‹finite A› dpll-backjump-final-state by force
  then show ?thesis
    by (meson ‹inv S› rtranclp-dpll-bj-sat-iff satisfiable-carac st true-annots-true-cls)
qed

corollary full-dpll-backjump-final-state-from-init-state:
  fixes A :: 'v literal multiset set and S T :: 'st
  assumes
    full: full dpll-bj S T and
    trail S = [] and
    clauses_NOT S = N and
    inv S
  shows unsatisfiable (set-mset N) ∨ (trail T ⊨asm N ∧ satisfiable (set-mset N))
  using assms full-dpll-backjump-final-state[of S T set-mset N] by auto

lemma tranclp-dpll-bj-trail-mes-decreasing-prop:
  assumes dpll: dpll-bj^++ S T  and inv: inv S and
  N-A: atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A and
  M-A: atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A and
  n-d: no-dup (trail S) and
  fin-A: finite A
  shows (2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A))
              − μ_C (1+card (atms-of-ms A)) (2+card (atms-of-ms A)) (trail-weight T)
            < (2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A))
              − μ_C (1+card (atms-of-ms A)) (2+card (atms-of-ms A)) (trail-weight S)
  using dpll
proof (induction)
  case base
  then show ?case
    using N-A M-A n-d dpll-bj-trail-mes-decreasing-prop fin-A inv by blast
next
  case (step T U) note st = this(1) and dpll = this(2) and IH = this(3)
  have  atms-of-mm (clauses_NOT S) = atms-of-mm (clauses_NOT T)
    using rtranclp-dpll-bj-atms-of-ms-clauses-inv by (metis dpll-bj-clauses dpll-bj-inv inv st
      tranclpD)
  then have N-A': atms-of-mm (clauses_NOT T) ⊆ atms-of-ms A
     using N-A by auto
  moreover have M-A': atm-of ‘ lits-of-l (trail T) ⊆ atms-of-ms A
    by (meson M-A N-A inv rtranclp-dpll-bj-atms-in-trail-in-set st dpll
      tranclp.r-into-trancl tranclp-into-rtranclp tranclp-trans)
  moreover have nd: no-dup (trail T)
    by (metis inv n-d rtranclp-dpll-bj-no-dup st tranclp-into-rtranclp)
  moreover have inv T
    by (meson dpll dpll-bj-inv inv rtranclp-dpll-bj-inv st tranclp-into-rtranclp)
  ultimately show ?case
    using IH dpll-bj-trail-mes-decreasing-prop[of T U A] dpll fin-A by linarith
qed

end
```

16.4 CDCL

16.4.1 Learn and Forget

```
locale learn-ops =
  dpll-state mset-cls insert-cls remove-lit
```

```
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT
  for
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    trail :: 'st ⇒ ('v, unit, unit) marked-lits and
    raw-clauses :: 'st ⇒ 'clss and
    prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒'st and
    add-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    remove-cls_NOT :: 'cls ⇒ 'st ⇒ 'st +
  fixes
    learn-cond :: 'cls ⇒ 'st ⇒ bool
begin
inductive learn :: 'st ⇒ 'st ⇒ bool where
learn_NOT-rule: clauses_NOT S ⊨pm mset-cls C ⟹
  atms-of (mset-cls C) ⊆ atms-of-mm (clauses_NOT S) ∪ atm-of ` (lits-of-l (trail S)) ⟹
  learn-cond C S ⟹
  T ∼ add-cls_NOT C S ⟹
  learn S T
inductive-cases learn_NOT E: learn S T

lemma learn-μ_C-stable:
  assumes learn S T and no-dup (trail S)
  shows μ_C A B (trail-weight S) = μ_C A B (trail-weight T)
  using assms by (auto elim: learn_NOT E)
end

locale forget-ops =
  dpll-state mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT
  for
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    trail :: 'st ⇒ ('v, unit, unit) marked-lits and
    raw-clauses :: 'st ⇒ 'clss and
    prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒'st and
    add-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    remove-cls_NOT :: 'cls ⇒ 'st ⇒ 'st +
  fixes
    forget-cond :: 'cls ⇒ 'st ⇒ bool
```

```
begin
inductive forget_NOT :: 'st ⇒ 'st ⇒ bool where
forget_NOT:
  removeAll-mset (mset-cls C)(clauses_NOT S) ⊨pm mset-cls C ⟹
  forget-cond C S ⟹
  C !∈! raw-clauses S ⟹
  T ∼ remove-cls_NOT C S ⟹
  forget_NOT S T
inductive-cases forget_NOTE: forget_NOT S T

lemma forget-μ_C-stable:
  assumes forget_NOT S T
  shows μ_C A B (trail-weight S) = μ_C A B (trail-weight T)
  using assms by (auto elim!: forget_NOTE)
end

locale learn-and-forget_NOT =
  learn-ops mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT learn-cond +
  forget-ops mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT forget-cond
  for
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    trail :: 'st ⇒ ('v, unit, unit) marked-lits and
    raw-clauses :: 'st ⇒ 'clss and
    prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒'st and
    add-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    remove-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    learn-cond forget-cond :: 'cls ⇒ 'st ⇒ bool
begin
inductive learn-and-forget_NOT :: 'st ⇒ 'st ⇒ bool
where
lf-learn: learn S T ⟹ learn-and-forget_NOT S T |
lf-forget: forget_NOT S T ⟹ learn-and-forget_NOT S T
end
```

16.4.2 Definition of CDCL

```
locale conflict-driven-clause-learning-ops =
  dpll-with-backjumping-ops mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT
    inv backjump-conds propagate-conds +
  learn-and-forget_NOT  mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT learn-cond
```

forget-cond
for
mset-cls:: $'cls \Rightarrow 'v$ *clause* **and**
insert-cls :: $'v$ *literal* $\Rightarrow 'cls \Rightarrow 'cls$ **and**
remove-lit :: $'v$ *literal* $\Rightarrow 'cls \Rightarrow 'cls$ **and**
mset-clss:: $'clss \Rightarrow 'v$ *clauses* **and**
union-clss :: $'clss \Rightarrow 'clss \Rightarrow 'clss$ **and**
in-clss :: $'cls \Rightarrow 'clss \Rightarrow$ *bool* **and**
insert-clss :: $'cls \Rightarrow 'clss \Rightarrow 'clss$ **and**
remove-from-clss :: $'cls \Rightarrow 'clss \Rightarrow 'clss$ **and**
trail :: $'st \Rightarrow ('v,$ *unit*, *unit*) *marked-lits* **and**
raw-clauses :: $'st \Rightarrow 'clss$ **and**
prepend-trail :: $('v,$ *unit*, *unit*) *marked-lit* $\Rightarrow 'st \Rightarrow 'st$ **and**
tl-trail :: $'st \Rightarrow 'st$ **and**
$add\text{-}cls_{NOT}$ :: $'cls \Rightarrow 'st \Rightarrow 'st$ **and**
$remove\text{-}cls_{NOT}$ :: $'cls \Rightarrow 'st \Rightarrow 'st$ **and**
inv :: $'st \Rightarrow$ *bool* **and**
backjump-conds :: $'v$ *clause* $\Rightarrow 'v$ *clause* $\Rightarrow 'v$ *literal* $\Rightarrow 'st \Rightarrow 'st \Rightarrow$ *bool* **and**
propagate-conds :: $('v,$ *unit*, *unit*) *marked-lit* $\Rightarrow 'st \Rightarrow$ *bool* **and**
learn-cond forget-cond :: $'cls \Rightarrow 'st \Rightarrow$ *bool*
begin

inductive $cdcl_{NOT}$ :: $'st \Rightarrow 'st \Rightarrow$ *bool* **for** S :: $'st$ **where**
c-dpll-bj: *dpll-bj* $S\ S' \Longrightarrow cdcl_{NOT}\ S\ S'$ |
c-learn: *learn* $S\ S' \Longrightarrow cdcl_{NOT}\ S\ S'$ |
$c\text{-}forget_{NOT}$: $forget_{NOT}\ S\ S' \Longrightarrow cdcl_{NOT}\ S\ S'$

lemma $cdcl_{NOT}$*-all-induct*[*consumes 1*, *case-names dpll-bj learn* $forget_{NOT}$]:
fixes $S\ T$ :: $'st$
assumes $cdcl_{NOT}\ S\ T$ **and**
dpll: $\bigwedge T.$ *dpll-bj* $S\ T \Longrightarrow P\ S\ T$ **and**
learning:
$\bigwedge C\ T.\ clauses_{NOT}\ S \models pm$ *mset-cls* $C \Longrightarrow$
atms-of (*mset-cls* C) $\subseteq$ *atms-of-mm* ($clauses_{NOT}\ S$) $\cup$ *atm-of* ‘ (*lits-of-l* (*trail* S)) $\Longrightarrow$
$T \sim add\text{-}cls_{NOT}\ C\ S \Longrightarrow$
$P\ S\ T$ **and**
forgetting: $\bigwedge C\ T.$ *removeAll-mset* (*mset-cls* C) ($clauses_{NOT}\ S$) $\models pm$ *mset-cls* $C \Longrightarrow$
C !∈! *raw-clauses* $S \Longrightarrow$
$T \sim remove\text{-}cls_{NOT}\ C\ S \Longrightarrow$
$P\ S\ T$
shows $P\ S\ T$
using *assms*(*1*) **by** (*induction rule*: $cdcl_{NOT}$*.induct*)
(*auto intro*: *assms*(*2*, *3*, *4*) *elim*!: $learn_{NOT}E$ $forget_{NOT}E$)+

lemma $cdcl_{NOT}$*-no-dup*:
assumes
$cdcl_{NOT}\ S\ T$ **and**
inv S **and**
no-dup (*trail* S)
shows *no-dup* (*trail* T)
using *assms* **by** (*induction rule*: $cdcl_{NOT}$*-all-induct*) (*auto intro*: *dpll-bj-no-dup*)

Consistency of the trail **lemma** $cdcl_{NOT}$*-consistent*:
assumes
$cdcl_{NOT}\ S\ T$ **and**

```
    inv S and
    no-dup (trail S)
  shows consistent-interp (lits-of-l (trail T))
  using cdcl_NOT-no-dup[OF assms] distinct-consistent-interp by fast
```

The subtle problem here is that tautologies can be removed, meaning that some variable can disappear of the problem. It is also possible that some variable of the trail are not in the clauses anymore.

```
lemma cdcl_NOT-atms-of-ms-clauses-decreasing:
  assumes cdcl_NOT S Tand inv S and no-dup (trail S)
  shows atms-of-mm (clauses_NOT T) ⊆ atms-of-mm (clauses_NOT S) ∪ atm-of ` (lits-of-l (trail S))
  using assms by (induction rule: cdcl_NOT-all-induct)
    (auto dest!: dpll-bj-atms-of-ms-clauses-inv set-mp simp add: atms-of-ms-def Union-eq)

lemma cdcl_NOT-atms-in-trail:
  assumes cdcl_NOT S Tand inv S and no-dup (trail S)
  and atm-of ` (lits-of-l (trail S)) ⊆ atms-of-mm (clauses_NOT S)
  shows atm-of ` (lits-of-l (trail T)) ⊆ atms-of-mm (clauses_NOT S)
  using assms by (induction rule: cdcl_NOT-all-induct) (auto simp add: dpll-bj-atms-in-trail)

lemma cdcl_NOT-atms-in-trail-in-set:
  assumes
    cdcl_NOT S T and inv S and no-dup (trail S) and
    atms-of-mm (clauses_NOT S) ⊆ A and
    atm-of ` (lits-of-l (trail S)) ⊆ A
  shows atm-of ` (lits-of-l (trail T)) ⊆ A
  using assms
  by (induction rule: cdcl_NOT-all-induct)
     (simp-all add: dpll-bj-atms-in-trail-in-set dpll-bj-atms-of-ms-clauses-inv)

lemma cdcl_NOT-all-decomposition-implies:
  assumes cdcl_NOT S T and inv S and n-d[simp]: no-dup (trail S) and
    all-decomposition-implies-m (clauses_NOT S) (get-all-marked-decomposition (trail S))
  shows
    all-decomposition-implies-m (clauses_NOT T) (get-all-marked-decomposition (trail T))
  using assms(1,2,4)
proof (induction rule: cdcl_NOT-all-induct)
  case dpll-bj
  then show ?case
     using dpll-bj-all-decomposition-implies-inv n-d by blast
next
  case learn
  then show ?case by (auto simp add: all-decomposition-implies-def)
next
  case (forget_NOT C T) note cls-C = this(1) and C = this(2) and T = this(3) and iniv = this(4)
and
    decomp = this(5)
  show ?case
    unfolding all-decomposition-implies-def Ball-def
    proof (intro allI, clarify)
      fix a b
      assume (a, b) ∈ set (get-all-marked-decomposition (trail T))
      then have unmark-l a ∪ set-mset (clauses_NOT S) ⊨ps unmark-l b
        using decomp T by (auto simp add: all-decomposition-implies-def)
      moreover
```

```
        have a1:mset-cls C ∈ set-mset (clauses_NOT S)
          using C by blast
        have clauses_NOT T = clauses_NOT (remove-cls_NOT C S)
          using T state-eq_NOT-clauses by blast
        then have set-mset (clauses_NOT T) ⊨ps set-mset (clauses_NOT S)
          using a1 by (metis (no-types) clauses-remove-cls_NOT cls-C insert-Diff order-refl
          set-mset-minus-replicate-mset(1) true-clss-clss-def true-clss-clss-insert)
      ultimately show unmark-l a ∪ set-mset (clauses_NOT T)
        ⊨ps unmark-l b
        using true-clss-clss-generalise-true-clss-clss by blast
    qed
qed
```

Extension of models

```
lemma cdcl_NOT-bj-sat-ext-iff:
  assumes cdcl_NOT S Tand inv S and n-d: no-dup (trail S)
  shows I⊨sextm clauses_NOT S ⟷ I⊨sextm clauses_NOT T
  using assms
proof (induction rule:cdcl_NOT-all-induct)
  case dpll-bj
  then show ?case by (simp add: dpll-bj-clauses)
next
  case (learn C T) note T = this(3)
  { fix J
    assume
      I ⊨sextm clauses_NOT S and
      I ⊆ J and
      tot: total-over-m J (set-mset ({#mset-cls C#} + clauses_NOT S)) and
      cons: consistent-interp J
    then have J ⊨sm clauses_NOT S unfolding true-clss-ext-def by auto

    moreover
      with ‹clauses_NOT S⊨pm mset-cls C› have J ⊨ mset-cls C
        using tot cons unfolding true-clss-cls-def by auto
    ultimately have J ⊨sm {#mset-cls C#} + clauses_NOT S by auto
  }
  then have H: I ⊨sextm (clauses_NOT S) ⟹ I ⊨sext insert (mset-cls C) (set-mset (clauses_NOT S))
    unfolding true-clss-ext-def by auto
  show ?case
    apply standard
      using T n-d apply (auto simp add: H)[]
    using T n-d apply simp
    by (metis Diff-insert-absorb insert-subset subsetI subset-antisym
      true-clss-ext-decrease-right-remove-r)
next
  case (forget_NOT C T) note cls-C = this(1) and T = this(3)
  { fix J
    assume
      I ⊨sext set-mset (clauses_NOT S) − {mset-cls C} and
      I ⊆ J and
      tot: total-over-m J (set-mset (clauses_NOT S)) and
      cons: consistent-interp J
    then have J ⊨s set-mset (clauses_NOT S) − {mset-cls C}
      unfolding true-clss-ext-def by (meson Diff-subset total-over-m-subset)

    moreover
```

```
      with cls-C have J ⊨ mset-cls C
        using tot cons unfolding true-clss-cls-def
        by (metis Un-commute forget_NOT.hyps(2) in-clss-mset-clss insert-Diff insert-is-Un order-refl
          set-mset-minus-replicate-mset(1))
    ultimately have J ⊨sm (clauses_NOT S) by (metis insert-Diff-single true-clss-insert)
  }
  then have H: I ⊨sext set-mset (clauses_NOT S) − {mset-cls C} ⟹ I ⊨sextm (clauses_NOT S)
    unfolding true-clss-ext-def by blast
  show ?case using T by (auto simp: true-clss-ext-decrease-right-remove-r H)
qed

end — end of conflict-driven-clause-learning-ops
```

16.5 CDCL with invariant

```
locale conflict-driven-clause-learning =
  conflict-driven-clause-learning-ops +
  assumes cdcl_NOT-inv: ⋀S T. cdcl_NOT S T ⟹ inv S ⟹ inv T
begin
sublocale dpll-with-backjumping
  apply unfold-locales
  using cdcl_NOT.simps cdcl_NOT-inv by auto

lemma rtranclp-cdcl_NOT-inv:
  cdcl_NOT^** S T ⟹ inv S ⟹ inv T
  by (induction rule: rtranclp-induct) (auto simp add: cdcl_NOT-inv)

lemma rtranclp-cdcl_NOT-no-dup:
  assumes cdcl_NOT^** S T and inv S
  and no-dup (trail S)
  shows no-dup (trail T)
  using assms by (induction rule: rtranclp-induct) (auto intro: cdcl_NOT-no-dup rtranclp-cdcl_NOT-inv)

lemma rtranclp-cdcl_NOT-trail-clauses-bound:
  assumes
    cdcl: cdcl_NOT^** S T and
    inv: inv S and
    n-d: no-dup (trail S) and
    atms-clauses-S: atms-of-mm (clauses_NOT S) ⊆ A and
    atms-trail-S: atm-of ‘(lits-of-l (trail S)) ⊆ A
  shows atm-of ‘ (lits-of-l (trail T)) ⊆ A ∧ atms-of-mm (clauses_NOT T) ⊆  A
  using cdcl
proof (induction rule: rtranclp-induct)
  case base
  then show ?case using atms-clauses-S atms-trail-S by simp
next
  case (step T U) note st = this(1) and cdcl_NOT = this(2) and IH = this(3)
  have inv T using inv st rtranclp-cdcl_NOT-inv by blast
  have no-dup (trail T)
    using rtranclp-cdcl_NOT-no-dup[of S T] st cdcl_NOT inv n-d by blast
  then have atms-of-mm (clauses_NOT U) ⊆ A
    using cdcl_NOT-atms-of-ms-clauses-decreasing[OF cdcl_NOT] IH n-d ‹inv T› by fast
  moreover
    have atm-of ‘(lits-of-l (trail U)) ⊆ A
      using cdcl_NOT-atms-in-trail-in-set[OF cdcl_NOT, of A] ‹no-dup (trail T)›
      by (meson atms-trail-S atms-clauses-S IH ‹inv T› cdcl_NOT )
```

ultimately show *?case* **by** *fast*
qed

lemma *rtranclp-*$cdcl_{NOT}$*-all-decomposition-implies*:
assumes $cdcl_{NOT}{}^{**}$ *S T* **and** *inv S* **and** *no-dup* (*trail S*) **and**
all-decomposition-implies-m ($clauses_{NOT}$ *S*) (*get-all-marked-decomposition* (*trail S*))
shows
all-decomposition-implies-m ($clauses_{NOT}$ *T*) (*get-all-marked-decomposition* (*trail T*))
using *assms* **by** (*induction*)
(*auto intro*: *rtranclp-*$cdcl_{NOT}$*-inv* $cdcl_{NOT}$*-all-decomposition-implies* *rtranclp-*$cdcl_{NOT}$*-no-dup*)

lemma *rtranclp-*$cdcl_{NOT}$*-bj-sat-ext-iff*:
assumes $cdcl_{NOT}{}^{**}$ *S T***and** *inv S* **and** *no-dup* (*trail S*)
shows $I\models$*sextm* $clauses_{NOT}$ *S* $\longleftrightarrow$ $I\models$*sextm* $clauses_{NOT}$ *T*
using *assms* **apply** (*induction rule*: *rtranclp-induct*)
using $cdcl_{NOT}$*-bj-sat-ext-iff* **by** (*auto intro*: *rtranclp-*$cdcl_{NOT}$*-inv* *rtranclp-*$cdcl_{NOT}$*-no-dup*)

definition $cdcl_{NOT}$*-NOT-all-inv* **where**
$cdcl_{NOT}$*-NOT-all-inv A S* $\longleftrightarrow$ (*finite A* $\wedge$ *inv S* $\wedge$ *atms-of-mm* ($clauses_{NOT}$ *S*) $\subseteq$ *atms-of-ms A*
$\wedge$ *atm-of ‘ lits-of-l* (*trail S*) $\subseteq$ *atms-of-ms A* $\wedge$ *no-dup* (*trail S*))

lemma $cdcl_{NOT}$*-NOT-all-inv*:
assumes $cdcl_{NOT}{}^{**}$ *S T* **and** $cdcl_{NOT}$*-NOT-all-inv A S*
shows $cdcl_{NOT}$*-NOT-all-inv A T*
using *assms* **unfolding** $cdcl_{NOT}$*-NOT-all-inv-def*
by (*simp add*: *rtranclp-*$cdcl_{NOT}$*-inv* *rtranclp-*$cdcl_{NOT}$*-no-dup* *rtranclp-*$cdcl_{NOT}$*-trail-clauses-bound*)

abbreviation *learn-or-forget* **where**
learn-or-forget S T $\equiv$ *learn S T* $\vee$ $forget_{NOT}$ *S T*

lemma *rtranclp-learn-or-forget-*$cdcl_{NOT}$:
learn-or-forget** *S T* $\Longrightarrow$ $cdcl_{NOT}{}^{**}$ *S T*
using *rtranclp-mono*[*of learn-or-forget* $cdcl_{NOT}$] **by** (*blast intro*: $cdcl_{NOT}$*.c-learn* $cdcl_{NOT}$*.c-*$forget_{NOT}$)

lemma *learn-or-forget-dpll-*μ_C:
assumes
l-f: *learn-or-forget*** *S T* **and**
dpll: *dpll-bj T U* **and**
inv: $cdcl_{NOT}$*-NOT-all-inv A S*
shows (*2+card* (*atms-of-ms A*)) ^ (*1+card* (*atms-of-ms A*))
$-$ μ_C (*1+card* (*atms-of-ms A*)) (*2+card* (*atms-of-ms A*)) (*trail-weight U*)
< (*2+card* (*atms-of-ms A*)) ^ (*1+card* (*atms-of-ms A*))
$-$ μ_C (*1+card* (*atms-of-ms A*)) (*2+card* (*atms-of-ms A*)) (*trail-weight S*)
(**is** *?μ U* < *?μ S*)
proof $-$
have *?μ S* = *?μ T*
using *l-f*
proof (*induction*)
case *base*
then show *?case* **by** *simp*
next
case (*step T U*)
moreover then have *no-dup* (*trail T*)
using *rtranclp-*$cdcl_{NOT}$*-no-dup*[*of S T*] $cdcl_{NOT}$*-NOT-all-inv-def inv*
rtranclp-learn-or-forget-$cdcl_{NOT}$ **by** *auto*

```
      ultimately show ?case
        using forget-μ_C-stable learn-μ_C-stable inv unfolding cdcl_NOT-NOT-all-inv-def by presburger
    qed
  moreover have cdcl_NOT-NOT-all-inv A T
     using rtranclp-learn-or-forget-cdcl_NOT  cdcl_NOT-NOT-all-inv  l-f inv by blast
  ultimately show ?thesis
    using dpll-bj-trail-mes-decreasing-prop[of T U A, OF dpll] finite
    unfolding cdcl_NOT-NOT-all-inv-def by presburger
qed

lemma infinite-cdcl_NOT-exists-learn-and-forget-infinite-chain:
  assumes
    ⋀i. cdcl_NOT (f i) (f(Suc i)) and
    inv: cdcl_NOT-NOT-all-inv A (f 0)
  shows ∃j. ∀i≥j. learn-or-forget (f i) (f (Suc i))
  using assms
proof (induction (2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A))
    − μ_C (1+card (atms-of-ms A)) (2+card (atms-of-ms A)) (trail-weight (f 0))
    arbitrary: f
    rule: nat-less-induct-case)
  case (Suc n) note IH = this(1) and μ = this(2) and cdcl_NOT = this(3) and inv = this(4)
  consider
      (dpll-end) ∃j. ∀i≥j. learn-or-forget (f i) (f (Suc i))
    | (dpll-more) ¬(∃j. ∀i≥j. learn-or-forget (f i) (f (Suc i)))
    by blast
  then show ?case
    proof cases
      case dpll-end
      then show ?thesis by auto
    next
      case dpll-more
      then have j: ∃i. ¬ learn (f i) (f (Suc i)) ∧ ¬forget_NOT (f i) (f (Suc i))
        by blast
      obtain i where
        ¬learn (f i) (f (Suc i)) ∧ ¬forget_NOT (f i) (f (Suc i)) and
        ∀k<i. learn-or-forget (f k) (f (Suc k))
        proof −
          obtain i_0 where ¬ learn (f i_0) (f (Suc i_0)) ∧ ¬forget_NOT (f i_0) (f (Suc i_0))
            using j by auto
          then have {i. i≤i_0 ∧  ¬ learn (f i) (f (Suc i)) ∧ ¬forget_NOT (f i) (f (Suc i))} ≠ {}
            by auto
          let ?I = {i. i≤i_0 ∧  ¬ learn (f i) (f (Suc i)) ∧ ¬forget_NOT (f i) (f (Suc i))}
          let ?i = Min ?I
          have finite ?I
            by auto
          have ¬ learn (f ?i) (f (Suc ?i)) ∧ ¬forget_NOT (f ?i) (f (Suc ?i))
            using Min-in[OF ‹finite ?I› ‹?I ≠ {}›] by auto
          moreover have ∀k<?i. learn-or-forget (f k) (f (Suc k))
            using Min.coboundedI[of {i. i ≤ i_0 ∧ ¬ learn (f i) (f (Suc i)) ∧ ¬ forget_NOT (f i)
              (f (Suc i))}, simplified]
            by (meson ‹¬ learn (f i_0) (f (Suc i_0)) ∧ ¬ forget_NOT (f i_0) (f (Suc i_0))› less-imp-le
              dual-order.trans not-le)
          ultimately show ?thesis using that by blast
        qed
      def g ≡ λn. f (n + Suc i)
      have dpll-bj (f i) (g 0)
        using ‹¬ learn (f i) (f (Suc i)) ∧ ¬ forget_NOT (f i) (f (Suc i))› cdcl_NOT cdcl_NOT.cases
        g-def by auto
      {
        fix j
        assume j ≤ i
        then have learn-or-forget^** (f 0) (f j)
          apply (induction j)
           apply simp
          by (metis (no-types, lifting) Suc-leD Suc-le-lessD rtranclp.simps
            ‹∀ k<i. learn (f k) (f (Suc k)) ∨ forget_NOT (f k) (f (Suc k))›)
      }
      then have learn-or-forget^** (f 0) (f i) by blast
      then have (2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A))
          − μ_C (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A)) (trail-weight (g 0))
        < (2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A))
          − μ_C (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A)) (trail-weight (f 0))
        using learn-or-forget-dpll-μ_C[of f 0 f i g 0 A] inv ‹dpll-bj (f i) (g 0)›
        unfolding cdcl_NOT-NOT-all-inv-def by linarith

      moreover have cdcl_NOT-i: cdcl_NOT^** (f 0) (g 0)
        using rtranclp-learn-or-forget-cdcl_NOT[of f 0 f i] ‹learn-or-forget^** (f 0) (f i)›
        cdcl_NOT[of i] unfolding g-def by auto
      moreover have ⋀i. cdcl_NOT (g i) (g (Suc i))
        using cdcl_NOT g-def by auto
      moreover have cdcl_NOT-NOT-all-inv A (g 0)
        using inv cdcl_NOT-i rtranclp-cdcl_NOT-trail-clauses-bound g-def cdcl_NOT-NOT-all-inv by auto
      ultimately obtain j where j: ⋀i. i≥j ⟹ learn-or-forget (g i) (g (Suc i))
        using IH unfolding μ[symmetric] by presburger
      show ?thesis
        proof
          {
            fix k
            assume k ≥ j + Suc i
            then have learn-or-forget (f k) (f (Suc k))
              using j[of k−Suc i] unfolding g-def by auto
          }
          then show ∀ k≥j+Suc i. learn-or-forget (f k) (f (Suc k))
            by auto
        qed
    qed
next
  case 0 note H = this(1) and cdcl_NOT = this(2) and inv = this(3)
  show ?case
    proof (rule ccontr)
      assume ¬ ?case
      then have j: ∃ i. ¬ learn (f i) (f (Suc i)) ∧ ¬forget_NOT (f i) (f (Suc i))
        by blast
      obtain i where
        ¬learn (f i) (f (Suc i)) ∧ ¬forget_NOT (f i) (f (Suc i)) and
        ∀ k<i. learn-or-forget (f k) (f (Suc k))
        proof −
          obtain i_0 where ¬ learn (f i_0) (f (Suc i_0)) ∧ ¬forget_NOT (f i_0) (f (Suc i_0))
            using j by auto
          then have {i. i≤i_0 ∧ ¬ learn (f i) (f (Suc i)) ∧ ¬forget_NOT (f i) (f (Suc i))} ≠ {}
```

```
            by auto
          let ?I = {i. i≤i_0 ∧  ¬ learn (f i) (f (Suc i)) ∧ ¬forget_NOT (f i) (f (Suc i))}
          let ?i = Min ?I
          have finite ?I
            by auto
          have ¬ learn (f ?i) (f (Suc ?i)) ∧ ¬forget_NOT (f ?i) (f (Suc ?i))
            using Min-in[OF ‹finite ?I› ‹?I ≠ {}›] by auto
          moreover have ∀ k<?i. learn-or-forget (f k) (f (Suc k))
            using Min.coboundedI[of {i. i ≤ i_0 ∧ ¬ learn (f i) (f (Suc i)) ∧ ¬ forget_NOT (f i)
              (f (Suc i))}, simplified]
            by (meson ‹¬ learn (f i_0) (f (Suc i_0)) ∧ ¬ forget_NOT (f i_0) (f (Suc i_0))› less-imp-le
              dual-order.trans not-le)
          ultimately show ?thesis using that by blast
        qed
      have dpll-bj (f i) (f (Suc i))
        using ‹¬ learn (f i) (f (Suc i)) ∧ ¬ forget_NOT (f i) (f (Suc i))› cdcl_NOT cdcl_NOT.cases
        by blast
      {
        fix j
        assume j ≤ i
        then have learn-or-forget** (f 0) (f j)
          apply (induction j)
           apply simp
          by (metis (no-types, lifting) Suc-leD Suc-le-lessD rtranclp.simps
            ‹∀ k<i. learn (f k) (f (Suc k)) ∨ forget_NOT (f k) (f (Suc k))›)
      }
      then have learn-or-forget** (f 0) (f i) by blast

      then show False
        using learn-or-forget-dpll-μ_C[of f 0 f i f (Suc i) A] inv  0
        ‹dpll-bj (f i) (f (Suc i))› unfolding cdcl_NOT-NOT-all-inv-def by linarith
    qed
qed

lemma wf-cdcl_NOT-no-learn-and-forget-infinite-chain:
  assumes
    no-infinite-lf: ⋀f j. ¬ (∀ i≥j. learn-or-forget (f i) (f (Suc i)))
  shows wf {(T, S). cdcl_NOT S T ∧ cdcl_NOT-NOT-all-inv A S} (is wf {(T, S). cdcl_NOT S T
        ∧ ?inv S})
  unfolding wf-iff-no-infinite-down-chain
proof (rule ccontr)
  assume ¬ ¬ (∃f. ∀ i. (f (Suc i), f i) ∈ {(T, S). cdcl_NOT S T ∧ ?inv S})
  then obtain f where
    ∀ i. cdcl_NOT (f i) (f (Suc i)) ∧ ?inv (f i)
    by fast
  then have ∃ j. ∀ i≥j. learn-or-forget (f i) (f (Suc i))
    using infinite-cdcl_NOT-exists-learn-and-forget-infinite-chain[of f] by meson
  then show False using no-infinite-lf by blast
qed

lemma inv-and-tranclp-cdcl-_NOT-tranclp-cdcl_NOT-and-inv:
  cdcl_NOT^++ S T ∧ cdcl_NOT-NOT-all-inv A S ⟷ (λS T. cdcl_NOT S T ∧ cdcl_NOT-NOT-all-inv A
S)^++ S T
  (is ?A ∧ ?I ⟷ ?B)
proof
```

```
  assume ?A ∧ ?I
  then have ?A and ?I by blast+
  then show ?B
    apply induction
      apply (simp add: tranclp.r-into-trancl)
    by (subst tranclp.simps) (auto intro: cdcl_NOT-NOT-all-inv tranclp-into-rtranclp)
next
  assume ?B
  then have ?A by induction auto
  moreover have ?I using ‹?B› tranclpD by fastforce
  ultimately show ?A ∧ ?I by blast
qed

lemma wf-tranclp-cdcl_NOT-no-learn-and-forget-infinite-chain:
  assumes
    no-infinite-lf: ⋀f j. ¬ (∀ i≥j. learn-or-forget (f i) (f (Suc i)))
  shows wf {(T, S). cdcl_NOT^++ S T ∧ cdcl_NOT-NOT-all-inv A S}
  using wf-trancl[OF wf-cdcl_NOT-no-learn-and-forget-infinite-chain[OF no-infinite-lf]]
  apply (rule wf-subset)
  by (auto simp: trancl-set-tranclp inv-and-tranclp-cdcl-_NOT-tranclp-cdcl_NOT-and-inv)

lemma cdcl_NOT-final-state:
  assumes
    n-s: no-step cdcl_NOT S and
    inv: cdcl_NOT-NOT-all-inv A S and
    decomp: all-decomposition-implies-m (clauses_NOT S) (get-all-marked-decomposition (trail S))
  shows unsatisfiable (set-mset (clauses_NOT S))
    ∨ (trail S ⊨asm clauses_NOT S ∧ satisfiable (set-mset (clauses_NOT S)))
proof −
  have n-s': no-step dpll-bj S
    using n-s by (auto simp: cdcl_NOT.simps)
  show ?thesis
    apply (rule dpll-backjump-final-state[of S A])
    using inv decomp n-s' unfolding cdcl_NOT-NOT-all-inv-def by auto
qed

lemma full-cdcl_NOT-final-state:
  assumes
    full: full cdcl_NOT S T and
    inv: cdcl_NOT-NOT-all-inv A S and
    n-d: no-dup (trail S) and
    decomp: all-decomposition-implies-m (clauses_NOT S) (get-all-marked-decomposition (trail S))
  shows unsatisfiable (set-mset (clauses_NOT T))
    ∨ (trail T ⊨asm clauses_NOT T ∧ satisfiable (set-mset (clauses_NOT T)))
proof −
  have st: cdcl_NOT^** S T and n-s: no-step cdcl_NOT T
    using full unfolding full-def by blast+
  have n-s': cdcl_NOT-NOT-all-inv A T
    using cdcl_NOT-NOT-all-inv inv st by blast
 moreover have all-decomposition-implies-m (clauses_NOT T) (get-all-marked-decomposition (trail T))
    using cdcl_NOT-NOT-all-inv-def decomp inv rtranclp-cdcl_NOT-all-decomposition-implies st by auto
  ultimately show ?thesis
    using cdcl_NOT-final-state n-s by blast
qed
```

end — end of *conflict-driven-clause-learning*

16.6 Termination

16.6.1 Restricting learn and forget

locale *conflict-driven-clause-learning-learning-before-backjump-only-distinct-learnt* =
dpll-state mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-cls$_{NOT}$ remove-cls$_{NOT}$ +
conflict-driven-clause-learning mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-cls$_{NOT}$ remove-cls$_{NOT}$
inv backjump-conds propagate-conds
$\lambda C\ S.$ *distinct-mset* (*mset-cls* C) $\wedge\ \neg$*tautology* (*mset-cls* C) $\wedge$ *learn-restrictions* $C\ S\ \wedge$
($\exists F\ K\ d\ F'\ C'\ L.$ *trail* $S = F'$ @ *Marked* K () # $F\ \wedge$ *mset-cls* $C = C' + \{\#L\#\}\ \wedge\ F \models$*as CNot* C'
$\wedge\ C' + \{\#L\#\} \notin\#$ clauses$_{NOT}$ S)
$\lambda C\ S.\ \neg(\exists F'\ F\ K\ d\ L.$ *trail* $S = F'$ @ *Marked* K () # $F\ \wedge\ F \models$*as CNot* (*remove1-mset* L (*mset-cls* C)))
$\wedge$ *forget-restrictions* $C\ S$
for
mset-cls:: *'cls* $\Rightarrow$ *'v clause* **and**
insert-cls :: *'v literal* $\Rightarrow$ *'cls* $\Rightarrow$ *'cls* **and**
remove-lit :: *'v literal* $\Rightarrow$ *'cls* $\Rightarrow$ *'cls* **and**
mset-clss:: *'clss* $\Rightarrow$ *'v clauses* **and**
union-clss :: *'clss* $\Rightarrow$ *'clss* $\Rightarrow$ *'clss* **and**
in-clss :: *'cls* $\Rightarrow$ *'clss* $\Rightarrow$ *bool* **and**
insert-clss :: *'cls* $\Rightarrow$ *'clss* $\Rightarrow$ *'clss* **and**
remove-from-clss :: *'cls* $\Rightarrow$ *'clss* $\Rightarrow$ *'clss* **and**
trail :: *'st* $\Rightarrow$ (*'v*, *unit*, *unit*) *marked-lits* **and**
raw-clauses :: *'st* $\Rightarrow$ *'clss* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* $\Rightarrow$ *'st* $\Rightarrow$ *'st* **and**
tl-trail :: *'st* $\Rightarrow$*'st* **and**
add-cls$_{NOT}$ :: *'cls* $\Rightarrow$ *'st* $\Rightarrow$ *'st* **and**
remove-cls$_{NOT}$ :: *'cls* $\Rightarrow$ *'st* $\Rightarrow$ *'st* **and**
inv :: *'st* $\Rightarrow$ *bool* **and**
backjump-conds :: *'v clause* $\Rightarrow$ *'v clause* $\Rightarrow$ *'v literal* $\Rightarrow$ *'st* $\Rightarrow$ *'st* $\Rightarrow$ *bool* **and**
propagate-conds :: (*'v*, *unit*, *unit*) *marked-lit* $\Rightarrow$ *'st* $\Rightarrow$ *bool* **and**
learn-restrictions forget-restrictions :: *'cls* $\Rightarrow$ *'st* $\Rightarrow$ *bool*
begin

lemma cdcl$_{NOT}$-learn-all-induct[*consumes 1*, *case-names dpll-bj learn* forget$_{NOT}$]:
fixes $S\ T$ :: *'st*
assumes cdcl$_{NOT}$ $S\ T$ **and**
dpll: $\bigwedge T.$ *dpll-bj* $S\ T \Longrightarrow P\ S\ T$ **and**
learning:
$\bigwedge C\ F\ K\ F'\ C'\ L\ T.$ clauses$_{NOT}$ $S \models$*pm mset-cls* $C \Longrightarrow$
atms-of (*mset-cls* C) $\subseteq$ *atms-of-mm* (clauses$_{NOT}$ S) $\cup$ *atm-of* ‘ (*lits-of-l* (*trail* S)) $\Longrightarrow$
distinct-mset (*mset-cls* C) $\Longrightarrow$
$\neg$ *tautology* (*mset-cls* C) $\Longrightarrow$
learn-restrictions $C\ S \Longrightarrow$
trail $S = F'$ @ *Marked* K () # $F \Longrightarrow$
mset-cls $C = C' + \{\#L\#\} \Longrightarrow$
$F \models$*as CNot* $C' \Longrightarrow$
$C' + \{\#L\#\} \notin\#$ clauses$_{NOT}$ $S \Longrightarrow$

```
        T ∼ add-cls_NOT C S ⟹
        P S T and
    forgetting: ⋀C T. removeAll-mset (mset-cls C) (clauses_NOT S) ⊨pm mset-cls C ⟹
      C !∈! raw-clauses S ⟹
      ¬(∃F' F K L. trail S = F' @ Marked K () # F ∧ F ⊨as CNot (mset-cls C − {#L#})) ⟹
      T ∼ remove-cls_NOT C S ⟹
      forget-restrictions C S ⟹
      P S T
    shows P S T
  using assms(1)
  apply (induction rule: cdcl_NOT.induct)
    apply (auto dest: assms(2) simp add: learn-ops-axioms)[]
   apply (auto elim!: learn-ops.learn.cases[OF learn-ops-axioms] dest: assms(3))[]
  apply (auto elim!: forget-ops.forget_NOT.cases[OF forget-ops-axioms] dest!: assms(4))
  done

lemma rtranclp-cdcl_NOT-inv:
  cdcl_NOT** S T ⟹ inv S ⟹ inv T
  apply (induction rule: rtranclp-induct)
   apply simp
  using cdcl_NOT-inv unfolding conflict-driven-clause-learning-def
  conflict-driven-clause-learning-axioms-def by blast

lemma learn-always-simple-clauses:
  assumes
    learn: learn S T and
    n-d: no-dup (trail S)
  shows set-mset (clauses_NOT T − clauses_NOT S)
    ⊆ simple-clss (atms-of-mm (clauses_NOT S) ∪ atm-of ‘ lits-of-l (trail S))
proof
  fix C assume C: C ∈ set-mset (clauses_NOT T − clauses_NOT S)
  have distinct-mset C ¬tautology C using learn C n-d by (elim learn_NOTE; auto)+
  then have C ∈ simple-clss (atms-of C)
    using distinct-mset-not-tautology-implies-in-simple-clss by blast
  moreover have atms-of C ⊆ atms-of-mm (clauses_NOT S) ∪ atm-of ‘ lits-of-l (trail S)
    using learn C n-d by (elim learn_NOTE) (auto simp: atms-of-ms-def atms-of-def image-Un
      true-annots-CNot-all-atms-defined)
  moreover have finite (atms-of-mm (clauses_NOT S) ∪ atm-of ‘ lits-of-l (trail S))
     by auto
  ultimately show C ∈ simple-clss (atms-of-mm (clauses_NOT S) ∪ atm-of ‘ lits-of-l (trail S))
    using simple-clss-mono  by (metis (no-types) insert-subset mk-disjoint-insert)
qed

definition conflicting-bj-clss S ≡
   {C+{#L#} |C L. C+{#L#} ∈# clauses_NOT S ∧ distinct-mset (C+{#L#})
   ∧ ¬tautology (C+{#L#})
     ∧ (∃F' K F. trail S = F' @ Marked K () # F ∧ F ⊨as CNot C)}

lemma conflicting-bj-clss-remove-cls_NOT[simp]:
  conflicting-bj-clss (remove-cls_NOT C S) = conflicting-bj-clss S − {mset-cls C}
  unfolding conflicting-bj-clss-def by fastforce

lemma conflicting-bj-clss-remove-cls_NOT'[simp]:
  T ∼ remove-cls_NOT C S ⟹ conflicting-bj-clss T = conflicting-bj-clss S − {mset-cls C}
  unfolding conflicting-bj-clss-def by fastforce
```

lemma *conflicting-bj-clss-add-cls*$_{NOT}$*-state-eq*:
assumes
T: $T \sim \textit{add-cls}_{NOT}\ C'\ S$ **and**
n-d: $\textit{no-dup}\ (\textit{trail}\ S)$
shows $\textit{conflicting-bj-clss}\ T$
$= \textit{conflicting-bj-clss}\ S$
$\cup\ (\textit{if}\ \exists\, C\ L.\ \textit{mset-cls}\ C' = C + \{\#L\#\} \wedge \textit{distinct-mset}\ (C+\{\#L\#\}) \wedge \neg\textit{tautology}\ (C+\{\#L\#\})$
$\wedge\ (\exists\, F'\ K\ d\ F.\ \textit{trail}\ S = F'\ @\ \textit{Marked}\ K\ ()\ \#\ F \wedge F \models\!\textit{as}\ \textit{CNot}\ C)$
$\textit{then}\ \{\textit{mset-cls}\ C'\}\ \textit{else}\ \{\})$
proof −
def $P \equiv \lambda C\ L\ T.\ \ \textit{distinct-mset}\ (C + \{\#L\#\}) \wedge \neg\ \textit{tautology}\ (C + \{\#L\#\}) \wedge$
$(\exists\, F'\ K\ F.\ \textit{trail}\ T = F'\ @\ \textit{Marked}\ K\ ()\ \#\ F \wedge F \models\!\textit{as}\ \textit{CNot}\ C)$
have *conf*: $\bigwedge T.\ \textit{conflicting-bj-clss}\ T = \{C + \{\#L\#\}\ |C\ L.\ C + \{\#L\#\} \in\!\#\ \textit{clauses}_{NOT}\ T \wedge P\ C\ L\ T\}$
unfolding *conflicting-bj-clss-def P-def* **by** *auto*
have *P-S-T*: $\bigwedge C\ L.\ P\ C\ L\ T = P\ C\ L\ S$
using *T n-d* **unfolding** *P-def* **by** *auto*
have *P*: $\textit{conflicting-bj-clss}\ T\ = \{C + \{\#L\#\}\ |C\ L.\ C + \{\#L\#\} \in\!\#\ \textit{clauses}_{NOT}\ S \wedge P\ C\ L\ T\} \cup$
$\{C + \{\#L\#\}\ |C\ L.\ C + \{\#L\#\} \in\!\#\ \{\#\textit{mset-cls}\ C'\#\} \wedge P\ C\ L\ T\}$
using *T n-d* **unfolding** *conf* **by** *auto*
moreover have $\{C + \{\#L\#\}\ |C\ L.\ C + \{\#L\#\} \in\!\#\ \textit{clauses}_{NOT}\ S \wedge P\ C\ L\ T\} = \textit{conflicting-bj-clss}\ S$
using *T n-d* **unfolding** *P-def conflicting-bj-clss-def* **by** *auto*
moreover have $\{C + \{\#L\#\}\ |C\ L.\ C + \{\#L\#\} \in\!\#\ \{\#\textit{mset-cls}\ C'\#\} \wedge P\ C\ L\ T\} =$
$(\textit{if}\ \exists\, C\ L.\ \textit{mset-cls}\ C' = C + \{\#L\#\} \wedge P\ C\ L\ S\ \textit{then}\ \{\textit{mset-cls}\ C'\}\ \textit{else}\ \{\})$
using *n-d T* **by** (*force simp*: *P-S-T*)
ultimately show *?thesis* **unfolding** *P-def* **by** *presburger*
qed

lemma *conflicting-bj-clss-add-cls*$_{NOT}$:
$\textit{no-dup}\ (\textit{trail}\ S) \Longrightarrow$
$\textit{conflicting-bj-clss}\ (\textit{add-cls}_{NOT}\ C'\ S)$
$= \textit{conflicting-bj-clss}\ S$
$\cup\ (\textit{if}\ \exists\, C\ L.\ \textit{mset-cls}\ C' = C + \{\#L\#\}\wedge \textit{distinct-mset}\ (C+\{\#L\#\}) \wedge \neg\textit{tautology}\ (C+\{\#L\#\})$
$\wedge\ (\exists\, F'\ K\ d\ F.\ \textit{trail}\ S = F'\ @\ \textit{Marked}\ K\ ()\ \#\ F \wedge F \models\!\textit{as}\ \textit{CNot}\ C)$
$\textit{then}\ \{\textit{mset-cls}\ C'\}\ \textit{else}\ \{\})$
using *conflicting-bj-clss-add-cls*$_{NOT}$*-state-eq* **by** *auto*

lemma *conflicting-bj-clss-incl-clauses*:
$\textit{conflicting-bj-clss}\ S \subseteq \textit{set-mset}\ (\textit{clauses}_{NOT}\ S)$
unfolding *conflicting-bj-clss-def* **by** *auto*

lemma *finite-conflicting-bj-clss*[*simp*]:
$\textit{finite}\ (\textit{conflicting-bj-clss}\ S)$
using *conflicting-bj-clss-incl-clauses*[*of S*] *rev-finite-subset* **by** *blast*

lemma *learn-conflicting-increasing*:
$\textit{no-dup}\ (\textit{trail}\ S) \Longrightarrow \textit{learn}\ S\ T \Longrightarrow \textit{conflicting-bj-clss}\ S \subseteq \textit{conflicting-bj-clss}\ T$
apply (*elim learn*$_{NOT}$*E*)
by (*subst conflicting-bj-clss-add-cls*$_{NOT}$*-state-eq*[*of T*]) *auto*

abbreviation $\textit{conflicting-bj-clss-yet}\ b\ S \equiv$
$3\ \widehat{}\ b - \textit{card}\ (\textit{conflicting-bj-clss}\ S)$

```
abbreviation $\mu_L$ :: nat ⇒ 'st ⇒ nat × nat where
  $\mu_L$ b S ≡ (conflicting-bj-clss-yet b S, card (set-mset (clauses$_{NOT}$ S)))

lemma do-not-forget-before-backtrack-rule-clause-learned-clause-untouched:
  assumes forget$_{NOT}$ S T
  shows conflicting-bj-clss S = conflicting-bj-clss T
  using assms apply (elim forget$_{NOT}$E)
  apply auto
  unfolding conflicting-bj-clss-def
  apply clarify
  using diff-union-cancelR by (metis diff-union-cancelR)

lemma forget-$\mu_L$-decrease:
  assumes forget$_{NOT}$: forget$_{NOT}$ S T
  shows ($\mu_L$ b T, $\mu_L$ b S) ∈ less-than <*lex*> less-than
proof –
  have card (set-mset (clauses$_{NOT}$ S)) > 0
    using forget$_{NOT}$ by (elim forget$_{NOT}$E) (auto simp: size-mset-removeAll-mset-le-iff card-gt-0-iff)
  then have card (set-mset (clauses$_{NOT}$ T)) < card (set-mset (clauses$_{NOT}$ S))
    using forget$_{NOT}$ by (elim forget$_{NOT}$E) (auto simp: size-mset-removeAll-mset-le-iff)
  then show ?thesis
    unfolding do-not-forget-before-backtrack-rule-clause-learned-clause-untouched[OF forget$_{NOT}$]
    by auto
qed

lemma set-condition-or-split:
  {a. (a = b ∨ Q a) ∧ S a} = (if S b then {b} else {}) ∪ {a. Q a ∧ S a}
  by auto

lemma set-insert-neq:
  A ≠ insert a A ⟷ a ∉ A
  by auto

lemma learn-$\mu_L$-decrease:
  assumes learnST: learn S T and n-d: no-dup (trail S) and
   A: atms-of-mm (clauses$_{NOT}$ S) ∪ atm-of ‘ lits-of-l (trail S) ⊆ A and
   fin-A: finite A
  shows ($\mu_L$ (card A) T, $\mu_L$ (card A) S) ∈ less-than <*lex*> less-than
proof –
  have [simp]: (atms-of-mm (clauses$_{NOT}$ T) ∪ atm-of ‘ lits-of-l (trail T))
    = (atms-of-mm (clauses$_{NOT}$ S) ∪ atm-of ‘ lits-of-l (trail S))
    using learnST n-d by (elim learn$_{NOT}$E) auto

  then have card (atms-of-mm (clauses$_{NOT}$ T) ∪ atm-of ‘ lits-of-l (trail T))
    = card (atms-of-mm (clauses$_{NOT}$ S) ∪ atm-of ‘ lits-of-l (trail S))
    by (auto intro!: card-mono)
  then have 3: (3::nat) ^ card (atms-of-mm (clauses$_{NOT}$ T) ∪ atm-of ‘ lits-of-l (trail T))
    = 3 ^ card (atms-of-mm (clauses$_{NOT}$ S) ∪ atm-of ‘ lits-of-l (trail S))
    by (auto intro: power-mono)
  moreover have conflicting-bj-clss S ⊆ conflicting-bj-clss T
    using learnST n-d by (simp add: learn-conflicting-increasing)
  moreover have conflicting-bj-clss S ≠ conflicting-bj-clss T
    using learnST
    proof (elim learn$_{NOT}$E, goal-cases)
      case (1 C) note clss-S = this(1) and atms-C = this(2) and inv = this(3) and T = this(4)
      then obtain F K F' C' L where
        tr-S: trail S = F' @ Marked K () # F and
        C: mset-cls C = C' + {#L#} and
        F: F ⊨as CNot C' and
        C-S:C' + {#L#} ∉# clauses_NOT S
        by blast
      moreover have distinct-mset (mset-cls C) ¬ tautology (mset-cls C) using inv by blast+
      ultimately have C' + {#L#} ∈ conflicting-bj-clss T
        using T n-d unfolding conflicting-bj-clss-def by fastforce
      moreover have C' + {#L#} ∉ conflicting-bj-clss S
        using C-S unfolding conflicting-bj-clss-def by auto
      ultimately show ?case by blast
    qed
  moreover have fin-T: finite (conflicting-bj-clss T)
    using learnST by induction (auto simp add: conflicting-bj-clss-add-cls_NOT )
  ultimately have card (conflicting-bj-clss T) ≥ card (conflicting-bj-clss S)
    using card-mono by blast

  moreover
    have fin': finite (atms-of-mm (clauses_NOT T) ∪ atm-of ` lits-of-l (trail T))
      by auto
    have 1:atms-of-ms (conflicting-bj-clss T) ⊆ atms-of-mm (clauses_NOT T)
      unfolding conflicting-bj-clss-def atms-of-ms-def by auto
    have 2: ⋀x. x∈ conflicting-bj-clss T ⟹ ¬ tautology x ∧ distinct-mset x
      unfolding conflicting-bj-clss-def by auto
    have T: conflicting-bj-clss T
    ⊆ simple-clss (atms-of-mm (clauses_NOT T) ∪ atm-of ` lits-of-l (trail T))
      by standard (meson 1 2 fin' ‹finite (conflicting-bj-clss T)› simple-clss-mono
        distinct-mset-set-def simplified-in-simple-clss subsetCE sup.coboundedI1)
  moreover
    then have #: 3 ^ card (atms-of-mm (clauses_NOT T) ∪ atm-of ` lits-of-l (trail T))
        ≥ card (conflicting-bj-clss T)
      by (meson Nat.le-trans simple-clss-card simple-clss-finite card-mono fin')
    have atms-of-mm (clauses_NOT T) ∪ atm-of ` lits-of-l (trail T) ⊆ A
      using learn_NOTE[OF learnST] A by simp
    then have 3 ^ (card A) ≥ card (conflicting-bj-clss T)
      using # fin-A by (meson simple-clss-card simple-clss-finite
        simple-clss-mono calculation(2) card-mono dual-order.trans)
  ultimately show ?thesis
    using psubset-card-mono[OF fin-T ]
    unfolding less-than-iff lex-prod-def by clarify
     (meson ‹conflicting-bj-clss S ≠ conflicting-bj-clss T›
        ‹conflicting-bj-clss S ⊆ conflicting-bj-clss T›
        diff-less-mono2 le-less-trans not-le psubsetI)
qed
```

We have to assume the following:

- *inv S*: the invariant holds in the inital state.
- *A* is a (finite *finite A*) superset of the literals in the trail *atm-of ‘ lits-of-l* (*trail S*) ⊆ *atms-of-ms A* and in the clauses *atms-of-mm* ($clauses_{NOT}$ *S*) ⊆ *atms-of-ms A*. This can the the set of all the literals in the starting set of clauses.
- *no-dup* (*trail S*): no duplicate in the trail. This is invariant along the path.

```
definition $\mu_{CDCL}$ where
$\mu_{CDCL}$ A T ≡ ((2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A))
              − $\mu_C$ (1+card (atms-of-ms A)) (2+card (atms-of-ms A)) (trail-weight T),
           conflicting-bj-clss-yet (card (atms-of-ms A)) T, card (set-mset (clauses$_{NOT}$ T)))
lemma cdcl$_{NOT}$-decreasing-measure:
  assumes
    cdcl$_{NOT}$ S T and
    inv: inv S and
    atm-clss: atms-of-mm (clauses$_{NOT}$ S) ⊆ atms-of-ms A and
    atm-lits: atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A and
    n-d: no-dup (trail S) and
    fin-A: finite A
  shows ($\mu_{CDCL}$ A T, $\mu_{CDCL}$ A S)
          ∈ less-than <*lex*> (less-than <*lex*> less-than)
  using assms(1)
proof induction
  case (c-dpll-bj T)
  from dpll-bj-trail-mes-decreasing-prop[OF this(1) inv atm-clss atm-lits n-d fin-A]
  show ?case unfolding $\mu_{CDCL}$-def
    by (meson in-lex-prod less-than-iff)
next
  case (c-learn T) note learn = this(1)
  then have S: trail S =  trail T
    using inv atm-clss atm-lits n-d fin-A
    by (elim learn$_{NOT}$E) auto
  show ?case
    using learn-$\mu_L$-decrease[OF learn n-d, of atms-of-ms A] atm-clss atm-lits fin-A n-d
    unfolding S $\mu_{CDCL}$-def by auto
next
  case (c-forget$_{NOT}$ T) note forget$_{NOT}$ = this(1)
  have trail S = trail T using forget$_{NOT}$ by induction auto
  then show ?case
    using forget-$\mu_L$-decrease[OF forget$_{NOT}$] unfolding $\mu_{CDCL}$-def by auto
qed

lemma wf-cdcl$_{NOT}$-restricted-learning:
  assumes finite A
  shows wf {(T, S).
    (atms-of-mm (clauses$_{NOT}$ S) ⊆ atms-of-ms A ∧ atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A
    ∧ no-dup (trail S)
    ∧ inv S)
    ∧ cdcl$_{NOT}$ S T }
  by (rule wf-wf-if-measure'[of less-than <*lex*> (less-than <*lex*> less-than)])
     (auto intro: cdcl$_{NOT}$-decreasing-measure[OF - - - - - assms])

definition $\mu_C$' :: 'v literal multiset set ⇒ 'st ⇒ nat where
$\mu_C$' A T ≡ $\mu_C$ (1+card (atms-of-ms A)) (2+card (atms-of-ms A)) (trail-weight T)

definition $\mu_{CDCL}$' :: 'v literal multiset set ⇒ 'st ⇒ nat where
$\mu_{CDCL}$' A T ≡
  ((2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A)) − $\mu_C$' A T) * (1+ 3^card (atms-of-ms A)) *
2
  + conflicting-bj-clss-yet (card (atms-of-ms A)) T * 2
  + card (set-mset (clauses$_{NOT}$ T))
```

```
lemma cdcl_NOT-decreasing-measure':
  assumes
    cdcl_NOT S T and
    inv: inv S and
    atms-clss: atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A and
    atms-trail: atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A and
    n-d: no-dup (trail S) and
    fin-A: finite A
  shows μ_CDCL' A T < μ_CDCL' A S
  using assms(1)
proof (induction rule: cdcl_NOT-learn-all-induct)
  case (dpll-bj T)
  then have (2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A)) − μ_C' A T
    < (2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A)) − μ_C' A S
    using dpll-bj-trail-mes-decreasing-prop fin-A inv n-d atms-clss atms-trail
    unfolding μ_C'-def by blast
  then have XX: ((2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A)) − μ_C' A T) + 1
    ≤ (2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A)) − μ_C' A S
    by auto
  from mult-le-mono1[OF this, of (1 + 3 ^ card (atms-of-ms A))]
  have ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) − μ_C' A T) *
      (1 + 3 ^ card (atms-of-ms A)) + (1 + 3 ^ card (atms-of-ms A))
    ≤ ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) − μ_C' A S)
      * (1 + 3 ^ card (atms-of-ms A))
    unfolding Nat.add-mult-distrib
    by presburger
  moreover
    have cl-T-S: clauses_NOT T = clauses_NOT S
      using dpll-bj.hyps inv dpll-bj-clauses by auto
    have conflicting-bj-clss-yet (card (atms-of-ms A)) S < 1+ 3 ^ card (atms-of-ms A)
    by simp
  ultimately have ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) − μ_C' A T)
      * (1 + 3 ^ card (atms-of-ms A)) + conflicting-bj-clss-yet (card (atms-of-ms A)) T
    < ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) − μ_C' A S) *(1 + 3 ^ card (atms-of-ms
A))
    by linarith
  then have ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) − μ_C' A T)
        * (1 + 3 ^ card (atms-of-ms A))
      + conflicting-bj-clss-yet (card (atms-of-ms A)) T
    < ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) − μ_C' A S)
        * (1 + 3 ^ card (atms-of-ms A))
      + conflicting-bj-clss-yet (card (atms-of-ms A)) S
    by linarith
  then have ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) − μ_C' A T)
      * (1 + 3 ^ card (atms-of-ms A)) * 2
    + conflicting-bj-clss-yet (card (atms-of-ms A)) T * 2
    < ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) − μ_C' A S)
      * (1 + 3 ^ card (atms-of-ms A)) * 2
    + conflicting-bj-clss-yet (card (atms-of-ms A)) S * 2
    by linarith
  then show ?case unfolding μ_CDCL'-def cl-T-S by presburger
next
  case (learn C F' K F C' L T) note clss-S-C = this(1) and atms-C = this(2) and dist = this(3)
    and tauto = this(4) and learn-restr = this(5) and tr-S = this(6) and C' = this(7) and
    F-C = this(8) and C-new = this(9) and T = this(10)
  have insert (mset-cls C) (conflicting-bj-clss S) ⊆ simple-clss (atms-of-ms A)
    proof -
      have mset-cls C ∈ simple-clss (atms-of-ms A)
        using C'
        by (metis (no-types, hide-lams) Un-subset-iff simple-clss-mono
          contra-subsetD dist distinct-mset-not-tautology-implies-in-simple-clss
          dual-order.trans atms-C atms-clss atms-trail tauto)
      moreover have conflicting-bj-clss S ⊆ simple-clss (atms-of-ms A)
        proof
          fix x :: 'v literal multiset
          assume x ∈ conflicting-bj-clss S
          then have x ∈# clauses_NOT S ∧ distinct-mset x ∧ ¬ tautology x
            unfolding conflicting-bj-clss-def by blast
          then show x ∈ simple-clss (atms-of-ms A)
            by (meson atms-clss atms-of-atms-of-ms-mono atms-of-ms-finite simple-clss-mono
              distinct-mset-not-tautology-implies-in-simple-clss fin-A finite-subset
              set-rev-mp)
        qed
      ultimately show ?thesis
        by auto
    qed
  then have card (insert (mset-cls C) (conflicting-bj-clss S)) ≤ 3 ^ (card (atms-of-ms A))
    by (meson Nat.le-trans atms-of-ms-finite simple-clss-card simple-clss-finite
      card-mono fin-A)
  moreover have [simp]: card (insert (mset-cls C) (conflicting-bj-clss S))
    = Suc (card ((conflicting-bj-clss S)))
    by (metis (no-types) C' C-new card-insert-if conflicting-bj-clss-incl-clauses contra-subsetD
      finite-conflicting-bj-clss)
  moreover have [simp]: conflicting-bj-clss (add-cls_NOT C S) = conflicting-bj-clss S ∪ {mset-cls C}
    using dist tauto F-C by (subst conflicting-bj-clss-add-cls_NOT[OF n-d]) (force simp: C' tr-S n-d)
  ultimately have [simp]: conflicting-bj-clss-yet (card (atms-of-ms A)) S
    = Suc (conflicting-bj-clss-yet (card (atms-of-ms A)) (add-cls_NOT C S))
      by simp
  have 1: clauses_NOT T = clauses_NOT (add-cls_NOT C S) using T by auto
  have 2: conflicting-bj-clss-yet (card (atms-of-ms A)) T
    = conflicting-bj-clss-yet (card (atms-of-ms A)) (add-cls_NOT C S)
    using T unfolding conflicting-bj-clss-def by auto
  have 3: μ_C' A T = μ_C' A (add-cls_NOT C S)
    using T unfolding μ_C'-def by auto
  have ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) − μ_C' A (add-cls_NOT C S))
    ∗ (1 + 3 ^ card (atms-of-ms A)) ∗ 2
    = ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) − μ_C' A S)
    ∗ (1 + 3 ^ card (atms-of-ms A)) ∗ 2
      using n-d unfolding μ_C'-def by auto
  moreover
    have conflicting-bj-clss-yet (card (atms-of-ms A)) (add-cls_NOT C S)
        ∗ 2
      + card (set-mset (clauses_NOT (add-cls_NOT C S)))
      < conflicting-bj-clss-yet (card (atms-of-ms A)) S ∗ 2
      + card (set-mset (clauses_NOT S))
      by (simp add: C' C-new n-d)
  ultimately show ?case unfolding μ_CDCL'-def 1 2 3 by presburger
next
  case (forget_NOT C T) note T = this(4)
  have [simp]: μ_C' A (remove-cls_NOT C S) =  μ_C' A S
```

unfolding μ_C*'-def* **by** *auto*
have *forget*$_{NOT}$ *S T*
apply (*rule forget*$_{NOT}$*.intros*) **using** *forget*$_{NOT}$ **by** *auto*
then have *conflicting-bj-clss T* = *conflicting-bj-clss S*
using *do-not-forget-before-backtrack-rule-clause-learned-clause-untouched* **by** *blast*
moreover have *card* (*set-mset* (*clauses*$_{NOT}$ *T*)) < *card* (*set-mset* (*clauses*$_{NOT}$ *S*))
by (*metis T card-Diff1-less clauses-remove-cls*$_{NOT}$ *finite-set-mset forget*$_{NOT}$*.hyps*(*2*)
in-clss-mset-clss order-refl set-mset-minus-replicate-mset(*1*) *state-eq*$_{NOT}$*-clauses*)
ultimately show *?case* **unfolding** μ_{CDCL}*'-def*
using *T* ‹μ_C*' A* (*remove-cls*$_{NOT}$ *C S*) = μ_C*' A S*› **by** (*metis* (*no-types*) *add-le-cancel-left*
μ_C*'-def not-le state-eq*$_{NOT}$*-trail*)
qed

lemma *cdcl*$_{NOT}$*-clauses-bound*:
assumes
cdcl$_{NOT}$ *S T* **and**
inv S **and**
atms-of-mm (*clauses*$_{NOT}$ *S*) ⊆ *A* **and**
atm-of ‘(*lits-of-l* (*trail S*)) ⊆ *A* **and**
n-d: *no-dup* (*trail S*) **and**
fin-A[*simp*]: *finite A*
shows *set-mset* (*clauses*$_{NOT}$ *T*) ⊆ *set-mset* (*clauses*$_{NOT}$ *S*) ∪ *simple-clss A*
using *assms*
proof (*induction rule*: *cdcl*$_{NOT}$*-learn-all-induct*)
case *dpll-bj*
then show *?case* **using** *dpll-bj-clauses* **by** *simp*
next
case *forget*$_{NOT}$
then show *?case* **using** *clauses-remove-cls*$_{NOT}$ **unfolding** *state-eq*$_{NOT}$*-def* **by** *auto*
next
case (*learn C F K d F' C' L*) **note** *atms-C* = *this*(*2*) **and** *dist* = *this*(*3*) **and** *tauto* = *this*(*4*) **and**
T = *this*(*10*) **and** *atms-clss-S* = *this*(*12*) **and** *atms-trail-S* = *this*(*13*)
have *atms-of* (*mset-cls C*) ⊆ *A*
using *atms-C atms-clss-S atms-trail-S* **by** *fast*
then have *simple-clss* (*atms-of* (*mset-cls C*)) ⊆ *simple-clss A*
by (*simp add*: *simple-clss-mono*)
then have *mset-cls C* ∈ *simple-clss A*
using *finite dist tauto* **by** (*auto dest*: *distinct-mset-not-tautology-implies-in-simple-clss*)
then show *?case* **using** *T n-d* **by** *auto*
qed

lemma *rtranclp-cdcl*$_{NOT}$*-clauses-bound*:
assumes
cdcl$_{NOT}$** *S T* **and**
inv S **and**
atms-of-mm (*clauses*$_{NOT}$ *S*) ⊆ *A* **and**
atm-of ‘(*lits-of-l* (*trail S*)) ⊆ *A* **and**
n-d: *no-dup* (*trail S*) **and**
finite: *finite A*
shows *set-mset* (*clauses*$_{NOT}$ *T*) ⊆ *set-mset* (*clauses*$_{NOT}$ *S*) ∪ *simple-clss A*
using *assms*(*1*−*5*)
proof *induction*
case *base*
then show *?case* **by** *simp*
next

```
  case (step T U) note st = this(1) and cdcl_NOT = this(2) and IH = this(3)[OF this(4−7)] and
    inv = this(4) and atms-clss-S = this(5) and atms-trail-S = this(6) and finite-cls-S = this(7)
  have inv T
    using rtranclp-cdcl_NOT-inv st inv by blast
  moreover have atms-of-mm (clauses_NOT T) ⊆ A and atm-of ‘ lits-of-l (trail T) ⊆ A
    using rtranclp-cdcl_NOT-trail-clauses-bound[OF st] inv atms-clss-S atms-trail-S n-d by auto
  moreover have no-dup (trail T)
   using rtranclp-cdcl_NOT-no-dup[OF st ‹inv S› n-d] by simp
  ultimately have set-mset (clauses_NOT U) ⊆ set-mset (clauses_NOT T) ∪ simple-clss A
    using cdcl_NOT finite n-d by (auto simp: cdcl_NOT-clauses-bound)
  then show ?case using IH by auto
qed

lemma rtranclp-cdcl_NOT-card-clauses-bound:
  assumes
    cdcl_NOT** S T and
    inv S and
    atms-of-mm (clauses_NOT S) ⊆ A and
    atm-of ‘(lits-of-l (trail S)) ⊆ A and
    n-d: no-dup (trail S) and
    finite: finite A
  shows card (set-mset (clauses_NOT T)) ≤ card (set-mset (clauses_NOT S)) + 3 ^ (card A)
  using rtranclp-cdcl_NOT-clauses-bound[OF assms] finite by (meson Nat.le-trans
    simple-clss-card simple-clss-finite card-Un-le card-mono finite-UnI
    finite-set-mset nat-add-left-cancel-le)

lemma rtranclp-cdcl_NOT-card-clauses-bound':
  assumes
    cdcl_NOT** S T and
    inv S and
    atms-of-mm (clauses_NOT S) ⊆ A and
    atm-of ‘(lits-of-l (trail S)) ⊆ A and
    n-d: no-dup (trail S) and
    finite: finite A
  shows card {C|C. C ∈# clauses_NOT T ∧ (tautology C ∨ ¬distinct-mset C)}
    ≤ card {C|C. C∈# clauses_NOT S ∧ (tautology C ∨ ¬distinct-mset C)} + 3 ^ (card A)
    (is card ?T ≤ card ?S + -)
  using rtranclp-cdcl_NOT-clauses-bound[OF assms] finite
proof −
  have ?T ⊆ ?S ∪ simple-clss A
    using rtranclp-cdcl_NOT-clauses-bound[OF assms] by force
  then have card ?T ≤ card (?S ∪ simple-clss A)
    using finite by (simp add: assms(5) simple-clss-finite card-mono)
  then show ?thesis
    by (meson le-trans simple-clss-card card-Un-le local.finite nat-add-left-cancel-le)
qed

lemma rtranclp-cdcl_NOT-card-simple-clauses-bound:
  assumes
    cdcl_NOT** S T and
    inv S and
    NA: atms-of-mm (clauses_NOT S) ⊆ A and
    MA: atm-of ‘ (lits-of-l (trail S)) ⊆ A and
    n-d: no-dup (trail S) and
    finite: finite A
```

shows card (set-mset (clauses$_{NOT}$ T))
≤ card {C. C ∈# clauses$_{NOT}$ S ∧ (tautology C ∨ ¬distinct-mset C)} + 3 ^ (card A)
(**is** card ?T ≤ card ?S + -)
using rtranclp-cdcl$_{NOT}$-clauses-bound[OF assms] finite
proof –
have ⋀x. x ∈# clauses$_{NOT}$ T ⟹ ¬ tautology x ⟹ distinct-mset x ⟹ x ∈ simple-clss A
using rtranclp-cdcl$_{NOT}$-clauses-bound[OF assms] **by** (metis (no-types, hide-lams) Un-iff NA atms-of-atms-of-ms-mono simple-clss-mono contra-subsetD subset-trans distinct-mset-not-tautology-implies-in-simple-clss)
then have set-mset (clauses$_{NOT}$ T) ⊆ ?S ∪ simple-clss A
using rtranclp-cdcl$_{NOT}$-clauses-bound[OF assms] **by** auto
then have card(set-mset (clauses$_{NOT}$ T)) ≤ card (?S ∪ simple-clss A)
using finite **by** (simp add: assms(5) simple-clss-finite card-mono)
then show ?thesis
by (meson le-trans simple-clss-card card-Un-le local.finite nat-add-left-cancel-le)
qed

definition μ_{CDCL}'-bound :: 'v literal multiset set ⇒ 'st ⇒ nat **where**
μ_{CDCL}'-bound A S =
((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A))) ∗ (1 + 3 ^ card (atms-of-ms A)) ∗ 2
+ 2∗3 ^ (card (atms-of-ms A))
+ card {C. C ∈# clauses$_{NOT}$ S ∧ (tautology C ∨ ¬distinct-mset C)} + 3 ^ (card (atms-of-ms A))

lemma μ_{CDCL}'-bound-reduce-trail-to$_{NOT}$[simp]:
μ_{CDCL}'-bound A (reduce-trail-to$_{NOT}$ M S) = μ_{CDCL}'-bound A S
unfolding μ_{CDCL}'-bound-def **by** auto

lemma rtranclp-cdcl$_{NOT}$-μ_{CDCL}'-bound-reduce-trail-to$_{NOT}$:
assumes
$cdcl_{NOT}^{**}$ S T **and**
inv S **and**
atms-of-mm (clauses$_{NOT}$ S) ⊆ atms-of-ms A **and**
atm-of ‘(lits-of-l (trail S)) ⊆ atms-of-ms A **and**
n-d: no-dup (trail S) **and**
finite: finite (atms-of-ms A) **and**
U: U ∼ reduce-trail-to$_{NOT}$ M T
shows μ_{CDCL}' A U ≤ μ_{CDCL}'-bound A S
proof –
have ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) − μ_C' A U)
≤ (2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A))
by auto
then have ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) − μ_C' A U)
∗ (1 + 3 ^ card (atms-of-ms A)) ∗ 2
≤ (2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) ∗ (1 + 3 ^ card (atms-of-ms A)) ∗ 2
using mult-le-mono1 **by** blast
moreover
have conflicting-bj-clss-yet (card (atms-of-ms A)) T ∗ 2 ≤ 2 ∗ 3 ^ card (atms-of-ms A)
by linarith
moreover have card (set-mset (clauses$_{NOT}$ U))
≤ card {C. C ∈# clauses$_{NOT}$ S ∧ (tautology C ∨ ¬distinct-mset C)} + 3 ^ card (atms-of-ms A)
using rtranclp-cdcl$_{NOT}$-card-simple-clauses-bound[OF assms(1−6)] U **by** auto
ultimately show ?thesis
unfolding μ_{CDCL}'-def μ_{CDCL}'-bound-def **by** linarith
qed

lemma *rtranclp-cdcl*$_{NOT}$*-*μ_{CDCL}*'-bound*:
assumes
$\mathit{cdcl}_{NOT}{}^{**}$ *S T* **and**
inv S **and**
atms-of-mm (*clauses*$_{NOT}$ *S*) ⊆ *atms-of-ms A* **and**
atm-of '(*lits-of-l* (*trail S*)) ⊆ *atms-of-ms A* **and**
n-d: *no-dup* (*trail S*) **and**
finite: *finite* (*atms-of-ms A*)
shows μ_{CDCL}*' A T* ≤ μ_{CDCL}*'-bound A S*
proof −
have μ_{CDCL}*' A* (*reduce-trail-to*$_{NOT}$ (*trail T*) *T*) = μ_{CDCL}*' A T*
unfolding μ_{CDCL}*'-def* μ_{C}*'-def conflicting-bj-clss-def* **by** *auto*
then show *?thesis* **using** *rtranclp-cdcl*$_{NOT}$*-*μ_{CDCL}*'-bound-reduce-trail-to*$_{NOT}$[*OF assms*, *of - trail T*]
state-eq$_{NOT}$*-ref* **by** *fastforce*
qed

lemma *rtranclp-*μ_{CDCL}*'-bound-decreasing*:
assumes
$\mathit{cdcl}_{NOT}{}^{**}$ *S T* **and**
inv S **and**
atms-of-mm (*clauses*$_{NOT}$ *S*) ⊆ *atms-of-ms A* **and**
atm-of '(*lits-of-l* (*trail S*)) ⊆ *atms-of-ms A* **and**
n-d: *no-dup* (*trail S*) **and**
finite[*simp*]: *finite* (*atms-of-ms A*)
shows μ_{CDCL}*'-bound A T* ≤ μ_{CDCL}*'-bound A S*
proof −
have {*C*. *C* ∈# *clauses*$_{NOT}$ *T* ∧ (*tautology C* ∨ ¬ *distinct-mset C*)}
⊆ {*C*. *C* ∈# *clauses*$_{NOT}$ *S* ∧ (*tautology C* ∨ ¬ *distinct-mset C*)} (**is** *?T* ⊆ *?S*)
proof (*rule Set.subsetI*)
fix *C* **assume** *C* ∈ *?T*
then have *C-T*: *C* ∈# *clauses*$_{NOT}$ *T* **and** *t-d*: *tautology C* ∨ ¬ *distinct-mset C*
by *auto*
then have *C* ∉ *simple-clss* (*atms-of-ms A*)
by (*auto dest*: *simple-clssE*)
then show *C* ∈ *?S*
using *C-T rtranclp-cdcl*$_{NOT}$*-clauses-bound*[*OF assms*] *t-d* **by** *force*
qed
then have *card* {*C*. *C* ∈# *clauses*$_{NOT}$ *T* ∧ (*tautology C* ∨ ¬ *distinct-mset C*)} ≤
card {*C*. *C* ∈# *clauses*$_{NOT}$ *S* ∧ (*tautology C* ∨ ¬ *distinct-mset C*)}
by (*simp add*: *card-mono*)
then show *?thesis*
unfolding μ_{CDCL}*'-bound-def* **by** *auto*
qed

end — end of *conflict-driven-clause-learning-learning-before-backjump-only-distinct-learnt*

16.7 CDCL with restarts

16.7.1 Definition

locale *restart-ops* =
fixes
cdcl$_{NOT}$ :: *'st* ⇒ *'st* ⇒ *bool* **and**
restart :: *'st* ⇒ *'st* ⇒ *bool*
begin
inductive *cdcl*$_{NOT}$*-raw-restart* :: *'st* ⇒ *'st* ⇒ *bool* **where**

cdcl$_{NOT}$ *S T* $\Longrightarrow$ *cdcl*$_{NOT}$*-raw-restart S T* |
restart S T $\Longrightarrow$ *cdcl*$_{NOT}$*-raw-restart S T*

end

locale *conflict-driven-clause-learning-with-restarts* =
  conflict-driven-clause-learning mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls$_{NOT}$ *remove-cls*$_{NOT}$
    inv backjump-conds propagate-conds learn-cond forget-cond
    for
    mset-cls:: *'cls* ⇒ *'v clause* **and**
    insert-cls :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**
    remove-lit :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**
    mset-clss:: *'clss* ⇒ *'v clauses* **and**
    union-clss :: *'clss* ⇒ *'clss* ⇒ *'clss* **and**
    in-clss :: *'cls* ⇒ *'clss* ⇒ *bool* **and**
    insert-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**
    remove-from-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**
    trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
    raw-clauses :: *'st* ⇒ *'clss* **and**
    prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
    tl-trail :: *'st* ⇒*'st* **and**
    add-cls$_{NOT}$ :: *'cls* ⇒ *'st* ⇒ *'st* **and**
    remove-cls$_{NOT}$ :: *'cls* ⇒ *'st* ⇒ *'st* **and**
    inv :: *'st* ⇒ *bool* **and**
    backjump-conds :: *'v clause* ⇒ *'v clause* ⇒ *'v literal* ⇒ *'st* ⇒ *'st* ⇒ *bool* **and**
    propagate-conds :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *bool* **and**
    learn-cond forget-cond :: *'cls* ⇒ *'st* ⇒ *bool*
begin

lemma *cdcl*$_{NOT}$*-iff-cdcl*$_{NOT}$*-raw-restart-no-restarts*:
  cdcl$_{NOT}$ *S T* ⟷ *restart-ops.cdcl*$_{NOT}$*-raw-restart cdcl*$_{NOT}$ (λ- -. *False*) *S T*
  (**is** *?C S T* ⟷ *?R S T*)
proof
  fix *S T*
  assume *?C S T*
  then show *?R S T* **by** (*simp add*: *restart-ops.cdcl*$_{NOT}$*-raw-restart.intros*(*1*))
next
  fix *S T*
  assume *?R S T*
  then show *?C S T*
    apply (*cases rule*: *restart-ops.cdcl*$_{NOT}$*-raw-restart.cases*)
    using ‹*?R S T*› **by** *fast+*
qed

lemma *cdcl*$_{NOT}$*-cdcl*$_{NOT}$*-raw-restart*:
  cdcl$_{NOT}$ *S T* $\Longrightarrow$ *restart-ops.cdcl*$_{NOT}$*-raw-restart cdcl*$_{NOT}$ *restart S T*
  by (*simp add*: *restart-ops.cdcl*$_{NOT}$*-raw-restart.intros*(*1*))
end

16.7.2 Increasing restarts

To add restarts we needs some assumptions on the predicate (called *cdcl*$_{NOT}$ here):

- a function f that is strictly monotonic. The first step is actually only used as a restart to clean the state (e.g. to ensure that the trail is empty). Then we assume that $(1::'a) \leq f$ n for $(1::'a) \leq n$: it means that between two consecutive restarts, at least one step will be done. This is necessary to avoid sequence. like: full – restart – full – ...

- a measure μ: it should decrease under the assumptions *bound-inv*, whenever a $cdcl_{NOT}$ or a *restart* is done. A parameter is given to μ: for conflict- driven clause learning, it is an upper-bound of the clauses. We are assuming that such a bound can be found after a restart whenever the invariant holds.
- we also assume that the measure decrease after any $cdcl_{NOT}$ step.
- an invariant on the states $cdcl_{NOT}$*-inv* that also holds after restarts.
- it is *not required* that the measure decrease with respect to restarts, but the measure has to be bound by some function *μ-bound* taking the same parameter as μ and the initial state of the considered $cdcl_{NOT}$ chain.

locale $cdcl_{NOT}$*-increasing-restarts-ops* =
  restart-ops $cdcl_{NOT}$ *restart* **for**
    restart :: *'st ⇒ 'st ⇒ bool* **and**
    $cdcl_{NOT}$ :: *'st ⇒ 'st ⇒ bool* +
  fixes
    f :: *nat ⇒ nat* **and**
    bound-inv :: *'bound ⇒ 'st ⇒ bool* **and**
    μ :: *'bound ⇒ 'st ⇒ nat* **and**
    $cdcl_{NOT}$*-inv* :: *'st ⇒ bool* **and**
    μ-bound :: *'bound ⇒ 'st ⇒ nat*
  assumes
    f: *unbounded f* **and**
    f-ge-1:$\bigwedge n.\ n \geq 1 \Longrightarrow f\ n \neq 0$ **and**
    bound-inv: $\bigwedge A\ S\ T.\ cdcl_{NOT}\text{-}inv\ S \Longrightarrow bound\text{-}inv\ A\ S \Longrightarrow cdcl_{NOT}\ S\ T \Longrightarrow bound\text{-}inv\ A\ T$ **and**
    $cdcl_{NOT}$*-measure*: $\bigwedge A\ S\ T.\ cdcl_{NOT}\text{-}inv\ S \Longrightarrow bound\text{-}inv\ A\ S \Longrightarrow cdcl_{NOT}\ S\ T \Longrightarrow \mu\ A\ T < \mu$
$A\ S$ **and**
    measure-bound2: $\bigwedge A\ T\ U.\ cdcl_{NOT}\text{-}inv\ T \Longrightarrow bound\text{-}inv\ A\ T \Longrightarrow cdcl_{NOT}{}^{**}\ T\ U$
      $\Longrightarrow \mu\ A\ U \leq \mu\text{-}bound\ A\ T$ **and**
    measure-bound4: $\bigwedge A\ T\ U.\ cdcl_{NOT}\text{-}inv\ T \Longrightarrow bound\text{-}inv\ A\ T \Longrightarrow cdcl_{NOT}{}^{**}\ T\ U$
      $\Longrightarrow \mu\text{-}bound\ A\ U \leq \mu\text{-}bound\ A\ T$ **and**
    $cdcl_{NOT}$*-restart-inv*: $\bigwedge A\ U\ V.\ cdcl_{NOT}\text{-}inv\ U \Longrightarrow restart\ U\ V \Longrightarrow bound\text{-}inv\ A\ U \Longrightarrow bound\text{-}inv$
$A\ V$
      and
    exists-bound: $\bigwedge R\ S.\ cdcl_{NOT}\text{-}inv\ R \Longrightarrow restart\ R\ S \Longrightarrow \exists A.\ bound\text{-}inv\ A\ S$ **and**
    $cdcl_{NOT}$*-inv*: $\bigwedge S\ T.\ cdcl_{NOT}\text{-}inv\ S \Longrightarrow cdcl_{NOT}\ S\ T \Longrightarrow cdcl_{NOT}\text{-}inv\ T$ **and**
    $cdcl_{NOT}$*-inv-restart*: $\bigwedge S\ T.\ cdcl_{NOT}\text{-}inv\ S \Longrightarrow restart\ S\ T \Longrightarrow cdcl_{NOT}\text{-}inv\ T$
begin

lemma $cdcl_{NOT}$*-*$cdcl_{NOT}$*-inv*:
  assumes
    $(cdcl_{NOT}\widehat{\ }\widehat{\ }n)\ S\ T$ **and**
    $cdcl_{NOT}$*-inv S*
  shows $cdcl_{NOT}$*-inv T*
  using *assms* **by** (*induction n arbitrary*: *T*) (*auto intro*:*bound-inv* $cdcl_{NOT}$*-inv*)

lemma $cdcl_{NOT}$*-bound-inv*:
  assumes
    $(cdcl_{NOT}\widehat{\ }\widehat{\ }n)\ S\ T$ **and**

cdcl$_{NOT}$*-inv S*
bound-inv A S
shows *bound-inv A T*
using *assms* **by** (*induction n arbitrary*: *T*) (*auto intro*:*bound-inv cdcl*$_{NOT}$*-cdcl*$_{NOT}$*-inv*)

lemma *rtranclp-cdcl*$_{NOT}$*-cdcl*$_{NOT}$*-inv*:
assumes
cdcl$_{NOT}{}^{**}$ *S T* **and**
cdcl$_{NOT}$*-inv S*
shows *cdcl*$_{NOT}$*-inv T*
using *assms* **by** *induction* (*auto intro*: *cdcl*$_{NOT}$*-inv*)

lemma *rtranclp-cdcl*$_{NOT}$*-bound-inv*:
assumes
cdcl$_{NOT}{}^{**}$ *S T* **and**
bound-inv A S **and**
cdcl$_{NOT}$*-inv S*
shows *bound-inv A T*
using *assms* **by** *induction* (*auto intro*:*bound-inv rtranclp-cdcl*$_{NOT}$*-cdcl*$_{NOT}$*-inv*)

lemma *cdcl*$_{NOT}$*-comp-n-le*:
assumes
(*cdcl*$_{NOT}{}^{\frown\frown}$(*Suc n*)) *S T* **and**
bound-inv A S
cdcl$_{NOT}$*-inv S*
shows $\mu\ A\ T < \mu\ A\ S - n$
using *assms*
proof (*induction n arbitrary*: *T*)
case *0*
then show *?case* **using** *cdcl*$_{NOT}$*-measure* **by** *auto*
next
case (*Suc n*) **note** *IH* = *this*(*1*)[*OF - this*(*3*) *this*(*4*)] **and** *S-T* = *this*(*2*) **and** *b-inv* = *this*(*3*) **and** *c-inv* = *this*(*4*)
obtain *U* :: *'st* **where** *S-U*: (*cdcl*$_{NOT}{}^{\frown\frown}$(*Suc n*)) *S U* **and** *U-T*: *cdcl*$_{NOT}$ *U T* **using** *S-T* **by** *auto*
then have $\mu\ A\ U < \mu\ A\ S - n$ **using** *IH*[*of U*] **by** *simp*
moreover
have *bound-inv A U*
using *S-U b-inv cdcl*$_{NOT}$*-bound-inv c-inv* **by** *blast*
then have $\mu\ A\ T < \mu\ A\ U$ **using** *cdcl*$_{NOT}$*-measure*[*OF - - U-T*] *S-U c-inv cdcl*$_{NOT}$*-cdcl*$_{NOT}$*-inv* **by** *auto*
ultimately show *?case* **by** *linarith*
qed

lemma *wf-cdcl*$_{NOT}$:
wf $\{(T, S).$ *cdcl*$_{NOT}$ *S T* $\wedge$ *cdcl*$_{NOT}$*-inv S* $\wedge$ *bound-inv A S*$\}$ (**is** *wf ?A*)
apply (*rule wfP-if-measure2*[*of - -* μ *A*])
using *cdcl*$_{NOT}$*-comp-n-le*[*of 0 - - A*] **by** *auto*

lemma *rtranclp-cdcl*$_{NOT}$*-measure*:
assumes
cdcl$_{NOT}{}^{**}$ *S T* **and**
bound-inv A S **and**
cdcl$_{NOT}$*-inv S*
shows $\mu\ A\ T \leq \mu\ A\ S$
using *assms*

proof (*induction rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *auto*
next
case (*step T U*) **note** *IH* = *this*(*3*)[*OF this*(*4*) *this*(*5*)] **and** *st* = *this*(*1*) **and** *cdcl*$_{NOT}$ = *this*(*2*) **and**
b-inv = *this*(*4*) **and** *c-inv* = *this*(*5*)
have *bound-inv A T*
by (*meson cdcl*$_{NOT}$*-bound-inv rtranclp-imp-relpowp st step.prems*)
moreover have *cdcl*$_{NOT}$*-inv T*
using *c-inv rtranclp-cdcl*$_{NOT}$*-cdcl*$_{NOT}$*-inv st* **by** *blast*
ultimately have μ *A U* < μ *A T* **using** *cdcl*$_{NOT}$*-measure*[*OF - - cdcl*$_{NOT}$] **by** *auto*
then show *?case* **using** *IH* **by** *linarith*
qed

lemma *cdcl*$_{NOT}$*-comp-bounded*:
assumes
bound-inv A S **and** *cdcl*$_{NOT}$*-inv S* **and** $m \geq 1+\mu$ *A S*
shows ¬(*cdcl*$_{NOT}$ ^^ *m*) *S T*
using *assms cdcl*$_{NOT}$*-comp-n-le*[*of m*−*1 S T A*] **by** *fastforce*

- *f n* < *m* ensures that at least one step has been done.

inductive *cdcl*$_{NOT}$*-restart* **where**
restart-step: (*cdcl*$_{NOT}$^^*m*) *S T* ⟹ *m* ≥ *f n* ⟹ *restart T U*
⟹ *cdcl*$_{NOT}$*-restart* (*S*, *n*) (*U*, *Suc n*) |
restart-full: *full1 cdcl*$_{NOT}$ *S T* ⟹ *cdcl*$_{NOT}$*-restart* (*S*, *n*) (*T*, *Suc n*)

lemmas *cdcl*$_{NOT}$*-with-restart-induct* = *cdcl*$_{NOT}$*-restart.induct*[*split-format*(*complete*),
OF cdcl$_{NOT}$*-increasing-restarts-ops-axioms*]

lemma *cdcl*$_{NOT}$*-restart-cdcl*$_{NOT}$*-raw-restart*:
cdcl$_{NOT}$*-restart S T* ⟹ *cdcl*$_{NOT}$*-raw-restart*** (*fst S*) (*fst T*)
proof (*induction rule*: *cdcl*$_{NOT}$*-restart.induct*)
case (*restart-step m S T n U*)
then have *cdcl*$_{NOT}$** *S T* **by** (*meson relpowp-imp-rtranclp*)
then have *cdcl*$_{NOT}$*-raw-restart*** *S T* **using** *cdcl*$_{NOT}$*-raw-restart.intros*(*1*)
rtranclp-mono[*of cdcl*$_{NOT}$ *cdcl*$_{NOT}$*-raw-restart*] **by** *blast*
moreover have *cdcl*$_{NOT}$*-raw-restart T U*
using ‹*restart T U*› *cdcl*$_{NOT}$*-raw-restart.intros*(*2*) **by** *blast*
ultimately show *?case* **by** *auto*
next
case (*restart-full S T*)
then have *cdcl*$_{NOT}$** *S T* **unfolding** *full1-def* **by** *auto*
then show *?case* **using** *cdcl*$_{NOT}$*-raw-restart.intros*(*1*)
rtranclp-mono[*of cdcl*$_{NOT}$ *cdcl*$_{NOT}$*-raw-restart*] **by** *auto*
qed

lemma *cdcl*$_{NOT}$*-with-restart-bound-inv*:
assumes
cdcl$_{NOT}$*-restart S T* **and**
bound-inv A (*fst S*) **and**
cdcl$_{NOT}$*-inv* (*fst S*)
shows *bound-inv A* (*fst T*)
using *assms* **apply** (*induction rule*: *cdcl*$_{NOT}$*-restart.induct*)

prefer *2* **apply** (*metis rtranclp-unfold fstI full1-def rtranclp-cdcl$_{NOT}$-bound-inv*)
by (*metis cdcl$_{NOT}$-bound-inv cdcl$_{NOT}$-cdcl$_{NOT}$-inv cdcl$_{NOT}$-restart-inv fst-conv*)

lemma *cdcl$_{NOT}$-with-restart-cdcl$_{NOT}$-inv*:
assumes
cdcl$_{NOT}$-restart S T **and**
cdcl$_{NOT}$-inv (*fst S*)
shows *cdcl$_{NOT}$-inv* (*fst T*)
using *assms* **apply** *induction*
apply (*metis cdcl$_{NOT}$-cdcl$_{NOT}$-inv cdcl$_{NOT}$-inv-restart fst-conv*)
apply (*metis fstI full-def full-unfold rtranclp-cdcl$_{NOT}$-cdcl$_{NOT}$-inv*)
done

lemma *rtranclp-cdcl$_{NOT}$-with-restart-cdcl$_{NOT}$-inv*:
assumes
cdcl$_{NOT}$-restart** S T **and**
cdcl$_{NOT}$-inv (*fst S*)
shows *cdcl$_{NOT}$-inv* (*fst T*)
using *assms* **by** *induction* (*auto intro*: *cdcl$_{NOT}$-with-restart-cdcl$_{NOT}$-inv*)

lemma *rtranclp-cdcl$_{NOT}$-with-restart-bound-inv*:
assumes
cdcl$_{NOT}$-restart** S T **and**
cdcl$_{NOT}$-inv (*fst S*) **and**
bound-inv A (*fst S*)
shows *bound-inv A* (*fst T*)
using *assms* **apply** *induction*
apply (*simp add*: *cdcl$_{NOT}$-cdcl$_{NOT}$-inv cdcl$_{NOT}$-with-restart-bound-inv*)
using *cdcl$_{NOT}$-with-restart-bound-inv rtranclp-cdcl$_{NOT}$-with-restart-cdcl$_{NOT}$-inv* **by** *blast*

lemma *cdcl$_{NOT}$-with-restart-increasing-number*:
cdcl$_{NOT}$-restart S T $\Longrightarrow$ *snd T* = *1* + *snd S*
by (*induction rule*: *cdcl$_{NOT}$-restart.induct*) *auto*
end

locale *cdcl$_{NOT}$-increasing-restarts* =
cdcl$_{NOT}$-increasing-restarts-ops restart cdcl$_{NOT}$ f bound-inv μ cdcl$_{NOT}$-inv μ-bound +
dpll-state mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-cls$_{NOT}$ remove-cls$_{NOT}$
for
mset-cls:: *'cls* ⇒ *'v clause* **and**
insert-cls :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**
remove-lit :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**
mset-clss:: *'clss* ⇒ *'v clauses* **and**
union-clss :: *'clss* ⇒ *'clss* ⇒ *'clss* **and**
in-clss :: *'cls* ⇒ *'clss* ⇒ *bool* **and**
insert-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**
remove-from-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**
trail :: *'st* ⇒ (*'v*, *unit*, *unit*) *marked-lits* **and**
raw-clauses :: *'st* ⇒ *'clss* **and**
prepend-trail :: (*'v*, *unit*, *unit*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒*'st* **and**
add-cls$_{NOT}$ :: *'cls* ⇒ *'st* ⇒ *'st* **and**
remove-cls$_{NOT}$ :: *'cls* ⇒ *'st* ⇒ *'st* **and**

```
    f :: nat ⇒ nat and
    restart :: 'st ⇒ 'st ⇒ bool and
    bound-inv :: 'bound ⇒ 'st ⇒ bool and
    μ :: 'bound ⇒ 'st ⇒ nat and
    cdcl_NOT :: 'st ⇒ 'st ⇒ bool and
    cdcl_NOT-inv :: 'st ⇒ bool and
    μ-bound :: 'bound ⇒ 'st ⇒ nat +
  assumes
    measure-bound: ⋀A T V n. cdcl_NOT-inv T ⟹ bound-inv A T
      ⟹ cdcl_NOT-restart (T, n) (V, Suc n) ⟹ μ A V ≤ μ-bound A T and
    cdcl_NOT-raw-restart-μ-bound:
      cdcl_NOT-restart (T, a) (V, b) ⟹  cdcl_NOT-inv T ⟹ bound-inv A T
        ⟹ μ-bound A V ≤ μ-bound A T
begin

lemma rtranclp-cdcl_NOT-raw-restart-μ-bound:
  cdcl_NOT-restart** (T, a) (V, b) ⟹  cdcl_NOT-inv T ⟹ bound-inv A T
    ⟹ μ-bound A V ≤ μ-bound A T
  apply (induction rule: rtranclp-induct2)
   apply simp
  by (metis cdcl_NOT-raw-restart-μ-bound dual-order.trans fst-conv
    rtranclp-cdcl_NOT-with-restart-bound-inv rtranclp-cdcl_NOT-with-restart-cdcl_NOT-inv)

lemma cdcl_NOT-raw-restart-measure-bound:
  cdcl_NOT-restart (T, a) (V, b) ⟹  cdcl_NOT-inv T ⟹ bound-inv A T
    ⟹ μ A V ≤ μ-bound A T
  apply (cases rule: cdcl_NOT-restart.cases)
     apply simp
    using measure-bound relpowp-imp-rtranclp apply fastforce
   by (metis full-def full-unfold measure-bound2 prod.inject)

lemma rtranclp-cdcl_NOT-raw-restart-measure-bound:
  cdcl_NOT-restart** (T, a) (V, b) ⟹  cdcl_NOT-inv T ⟹ bound-inv A T
    ⟹ μ A V ≤ μ-bound A T
  apply (induction rule: rtranclp-induct2)
    apply (simp add: measure-bound2)
  by (metis dual-order.trans fst-conv measure-bound2 r-into-rtranclp rtranclp.rtrancl-refl
    rtranclp-cdcl_NOT-with-restart-bound-inv rtranclp-cdcl_NOT-with-restart-cdcl_NOT-inv
    rtranclp-cdcl_NOT-raw-restart-μ-bound)

lemma wf-cdcl_NOT-restart:
  wf {(T, S). cdcl_NOT-restart S T ∧ cdcl_NOT-inv (fst S)} (is wf ?A)
proof (rule ccontr)
  assume ¬ ?thesis
  then obtain g where
    g: ⋀i. cdcl_NOT-restart (g i) (g (Suc i)) and
    cdcl_NOT-inv-g: ⋀i. cdcl_NOT-inv (fst (g i))
    unfolding wf-iff-no-infinite-down-chain by fast

  have snd-g: ⋀i. snd (g i) = i + snd (g 0)
    apply (induct-tac i)
      apply simp
      by (metis Suc-eq-plus1-left add.commute add.left-commute
        cdcl_NOT-with-restart-increasing-number g)
  then have snd-g-0: ⋀i. i > 0 ⟹ snd (g i) = i + snd (g 0)
    by blast
  have unbounded-f-g: unbounded (λi. f (snd (g i)))
    using f unfolding bounded-def by (metis add.commute f less-or-eq-imp-le snd-g
      not-bounded-nat-exists-larger not-le le-iff-add)

  { fix i
    have H: ⋀T Ta m. (cdcl_NOT ^^ m) T Ta ⟹ no-step cdcl_NOT T ⟹ m = 0
      apply (case-tac m) by simp (meson relpowp-E2)
    have ∃ T m. (cdcl_NOT ^^ m) (fst (g i)) T ∧ m ≥ f (snd (g i))
      using g[of i] apply (cases rule: cdcl_NOT-restart.cases)
        apply auto[]
      using g[of Suc i] f-ge-1 apply (cases rule: cdcl_NOT-restart.cases)
      apply (auto simp add: full1-def full-def dest: H dest: tranclpD)
      using H Suc-leI leD by blast
  } note H = this
  obtain A where bound-inv A (fst (g 1))
    using g[of 0] cdcl_NOT-inv-g[of 0] apply (cases rule: cdcl_NOT-restart.cases)
      apply (metis One-nat-def cdcl_NOT-inv exists-bound fst-conv relpowp-imp-rtranclp
        rtranclp-induct)
      using H[of 1] unfolding full1-def by (metis One-nat-def Suc-eq-plus1 diff-is-0-eq' diff-zero
        f-ge-1 fst-conv le-add2 relpowp-E2 snd-conv)
  let ?j = μ-bound A (fst (g 1)) + 1
  obtain j where
    j: f (snd (g j)) > ?j and j > 1
    using unbounded-f-g not-bounded-nat-exists-larger by blast
  {
     fix i j
     have cdcl_NOT-with-restart: j ≥ i ⟹ cdcl_NOT-restart^** (g i) (g j)
       apply (induction j)
         apply simp
       by (metis g le-Suc-eq rtranclp.rtrancl-into-rtrancl rtranclp.rtrancl-refl)
  } note cdcl_NOT-restart = this
  have cdcl_NOT-inv (fst (g (Suc 0)))
    by (simp add: cdcl_NOT-inv-g)
  have cdcl_NOT-restart^** (fst (g 1), snd (g 1)) (fst (g j), snd (g j))
    using ‹j> 1› by (simp add: cdcl_NOT-restart)
  have μ A (fst (g j)) ≤ μ-bound A (fst (g 1))
    apply (rule rtranclp-cdcl_NOT-raw-restart-measure-bound)
    using ‹cdcl_NOT-restart^** (fst (g 1), snd (g 1)) (fst (g j), snd (g j))› apply blast
        apply (simp add: cdcl_NOT-inv-g)
       using ‹bound-inv A (fst (g 1))› apply simp
    done
  then have μ A (fst (g j)) ≤ ?j
    by auto
  have inv: bound-inv A (fst (g j))
    using ‹bound-inv A (fst (g 1))› ‹cdcl_NOT-inv (fst (g (Suc 0)))›
     ‹cdcl_NOT-restart^** (fst (g 1), snd (g 1)) (fst (g j), snd (g j))›
    rtranclp-cdcl_NOT-with-restart-bound-inv by auto
  obtain T m where
    cdcl_NOT-m: (cdcl_NOT ^^ m) (fst (g j)) T and
    f-m: f (snd (g j)) ≤ m
    using H[of j] by blast
  have ?j < m
    using f-m j Nat.le-trans by linarith
  then show False
    using ⟨μ A (fst (g j)) ≤ μ-bound A (fst (g 1))⟩
    cdcl_NOT-comp-bounded[OF inv cdcl_NOT-inv-g, of ] cdcl_NOT-inv-g cdcl_NOT-m
    ⟨?j < m⟩ by auto
qed

lemma cdcl_NOT-restart-steps-bigger-than-bound:
  assumes
    cdcl_NOT-restart S T and
    bound-inv A (fst S) and
    cdcl_NOT-inv (fst S) and
    f (snd S) > μ-bound A (fst S)
  shows full1 cdcl_NOT (fst S) (fst T)
  using assms
proof (induction rule: cdcl_NOT-restart.induct)
  case restart-full
  then show ?case by auto
next
  case (restart-step m S T n U) note st = this(1) and f = this(2) and bound-inv = this(4) and
    cdcl_NOT-inv = this(5) and μ = this(6)
  then obtain m' where m: m = Suc m' by (cases m) auto
  have μ A S − m' = 0
    using f bound-inv cdcl_NOT-inv μ m rtranclp-cdcl_NOT-raw-restart-measure-bound by fastforce
  then have False using cdcl_NOT-comp-n-le[of m' S T A] restart-step unfolding m by simp
  then show ?case by fast
qed

lemma rtranclp-cdcl_NOT-with-inv-inv-rtranclp-cdcl_NOT:
  assumes
    inv: cdcl_NOT-inv S and
    binv: bound-inv A S
  shows (λS T. cdcl_NOT S T ∧ cdcl_NOT-inv S ∧ bound-inv A S)** S T ⟷ cdcl_NOT** S T
  (is ?A** S T ⟷ ?B** S T)
  apply (rule iffI)
    using rtranclp-mono[of ?A ?B] apply blast
  apply (induction rule: rtranclp-induct)
    using inv binv apply simp
  by (metis (mono-tags, lifting) binv inv rtranclp.simps rtranclp-cdcl_NOT-bound-inv
    rtranclp-cdcl_NOT-cdcl_NOT-inv)

lemma no-step-cdcl_NOT-restart-no-step-cdcl_NOT:
  assumes
    n-s: no-step cdcl_NOT-restart S and
    inv: cdcl_NOT-inv (fst S) and
    binv: bound-inv A (fst S)
  shows no-step cdcl_NOT (fst S)
proof (rule ccontr)
  assume ¬ ?thesis
  then obtain T where T: cdcl_NOT (fst S) T
    by blast
  then obtain U where U: full (λS T. cdcl_NOT S T ∧ cdcl_NOT-inv S ∧ bound-inv A S) T U
     using wf-exists-normal-form-full[OF wf-cdcl_NOT, of A T] by auto
  moreover have inv-T: cdcl_NOT-inv T
    using ⟨cdcl_NOT (fst S) T⟩ cdcl_NOT-inv inv by blast
  moreover have b-inv-T: bound-inv A T
```

using ‹$cdcl_{NOT}$ (fst S) T› binv bound-inv inv **by** blast
ultimately have full $cdcl_{NOT}$ T U
using rtranclp-$cdcl_{NOT}$-with-inv-inv-rtranclp-$cdcl_{NOT}$ rtranclp-$cdcl_{NOT}$-bound-inv rtranclp-$cdcl_{NOT}$-$cdcl_{NOT}$-inv **unfolding** full-def **by** blast
then have full1 $cdcl_{NOT}$ (fst S) U
using T full-fullI **by** metis
then show False **by** (metis n-s prod.collapse restart-full)
qed

end

16.8 Merging backjump and learning

locale $cdcl_{NOT}$-merge-bj-learn-ops =
decide-ops mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-cls_{NOT} remove-cls_{NOT} +
forget-ops mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-cls_{NOT} remove-cls_{NOT} forget-cond +
propagate-ops mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-cls_{NOT} remove-cls_{NOT} propagate-conds
for
mset-cls:: 'cls ⇒ 'v clause **and**
insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls **and**
remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls **and**
mset-clss:: 'clss ⇒ 'v clauses **and**
union-clss :: 'clss ⇒ 'clss ⇒ 'clss **and**
in-clss :: 'cls ⇒ 'clss ⇒ bool **and**
insert-clss :: 'cls ⇒ 'clss ⇒ 'clss **and**
remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss **and**
trail :: 'st ⇒ ('v, unit, unit) marked-lits **and**
raw-clauses :: 'st ⇒ 'clss **and**
prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st **and**
tl-trail :: 'st ⇒'st **and**
add-cls_{NOT} :: 'cls ⇒ 'st ⇒ 'st **and**
remove-cls_{NOT} :: 'cls ⇒ 'st ⇒ 'st **and**
propagate-conds :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ bool **and**
forget-cond :: 'cls ⇒ 'st ⇒ bool +
fixes backjump-l-cond :: 'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool
begin
inductive backjump-l **where**
backjump-l: trail S = F' @ Marked K () # F
⟹ no-dup (trail S)
⟹ T ∼ prepend-trail (Propagated L ()) (reduce-trail-to_{NOT} F (add-cls_{NOT} C'' S))
⟹ C ∈# $clauses_{NOT}$ S
⟹ trail S ⊨as CNot C
⟹ undefined-lit F L
⟹ atm-of L ∈ atms-of-mm ($clauses_{NOT}$ S) ∪ atm-of ‘ (lits-of-l (trail S))
⟹ $clauses_{NOT}$ S ⊨pm C' + {#L#}
⟹ mset-cls C'' = C' + {#L#}
⟹ F ⊨as CNot C'
⟹ backjump-l-cond C C' L S T
⟹ backjump-l S T

Avoid (meaningless) simplification:

```
declare reduce-trail-to_NOT-length-ne[simp del] Set.Un-iff[simp del] Set.insert-iff[simp del]
inductive-cases backjump-lE: backjump-l S T
thm backjump-lE
declare reduce-trail-to_NOT-length-ne[simp]  Set.Un-iff[simp] Set.insert-iff[simp]

inductive cdcl_NOT-merged-bj-learn :: 'st ⇒ 'st ⇒ bool for S :: 'st where
cdcl_NOT-merged-bj-learn-decide_NOT:  decide_NOT S S' ⟹ cdcl_NOT-merged-bj-learn S S' |
cdcl_NOT-merged-bj-learn-propagate_NOT: propagate_NOT S S' ⟹ cdcl_NOT-merged-bj-learn S S' |
cdcl_NOT-merged-bj-learn-backjump-l:  backjump-l S S' ⟹ cdcl_NOT-merged-bj-learn S S' |
cdcl_NOT-merged-bj-learn-forget_NOT: forget_NOT S S' ⟹ cdcl_NOT-merged-bj-learn S S'

lemma cdcl_NOT-merged-bj-learn-no-dup-inv:
  cdcl_NOT-merged-bj-learn S T ⟹ no-dup (trail S) ⟹ no-dup (trail T)
  apply (induction rule: cdcl_NOT-merged-bj-learn.induct)
      using defined-lit-map apply fastforce
    using defined-lit-map apply fastforce
   apply (force simp: defined-lit-map elim!: backjump-lE)[]
  using forget_NOT.simps apply auto[1]
  done
end

locale cdcl_NOT-merge-bj-learn-proxy =
  cdcl_NOT-merge-bj-learn-ops mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT propagate-conds
    forget-cond
    λC C' L' S T. backjump-l-cond C C' L' S T
    ∧ distinct-mset (C' + {#L'#}) ∧ ¬tautology (C' + {#L'#})
  for
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    trail :: 'st ⇒ ('v, unit, unit) marked-lits and
    raw-clauses :: 'st ⇒ 'clss and
    prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒'st and
    add-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    remove-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    propagate-conds :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ bool and
    forget-cond :: 'cls ⇒ 'st ⇒ bool and
    backjump-l-cond :: 'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool +
  fixes
    inv :: 'st ⇒ bool
  assumes
     bj-merge-can-jump:
     ⋀S C F' K F L.
       inv S
       ⟹ trail S = F' @ Marked K () # F
       ⟹ C ∈# clauses_NOT S
       ⟹ trail S ⊨as CNot C
       ⟹ undefined-lit F L
       ⟹ atm-of L ∈ atms-of-mm (clauses_NOT S) ∪ atm-of ‘ (lits-of-l (F' @ Marked K () # F))
       ⟹ clauses_NOT S ⊨pm C' + {#L#}
       ⟹ F ⊨as CNot C'
       ⟹ ¬no-step backjump-l S and
     cdcl-merged-inv: ⋀S T. cdcl_NOT-merged-bj-learn S T ⟹ inv S ⟹ inv T
begin

abbreviation backjump-conds :: 'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool
  where
backjump-conds ≡ λC C' L' S T. distinct-mset (C' + {#L'#}) ∧ ¬tautology (C' + {#L'#})
```

Without additional knowledge on *backjump-l-cond*, it is impossible to have the same invariant.

```
sublocale dpll-with-backjumping-ops mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT inv
    backjump-conds propagate-conds
proof (unfold-locales, goal-cases)
  case 1
  { fix S S'
    assume bj: backjump-l S S' and no-dup (trail S)
    then obtain F' K F L C' C D where
      S': S' ∼ prepend-trail (Propagated L ()) (reduce-trail-to_NOT F (add-cls_NOT D S))
        and
      tr-S: trail S = F' @ Marked K () # F and
      C: C ∈# clauses_NOT S and
      tr-S-C: trail S ⊨as CNot C  and
      undef-L: undefined-lit F L and
      atm-L:
       atm-of L ∈ insert (atm-of K) (atms-of-mm (clauses_NOT S) ∪ atm-of ‘ (lits-of-l F' ∪ lits-of-l F))
       and
      cls-S-C': clauses_NOT S ⊨pm C' + {#L#} and
      F-C': F ⊨as CNot C' and
      dist: distinct-mset (C' + {#L#}) and
      not-tauto: ¬ tautology (C' + {#L#}) and
      cond: backjump-l-cond C C' L S S'
      mset-cls D = C' + {#L#}
      by (elim backjump-lE) metis
    interpret backjumping-ops mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT
    backjump-conds
      by unfold-locales
    have ∃ T. backjump S T
      apply rule
      apply (rule backjump.intros)
               using tr-S apply simp
              apply (rule state-eq_NOT-ref)
             using C apply simp
            using tr-S-C apply simp
          using undef-L apply simp
         using atm-L tr-S apply simp
        using cls-S-C' apply simp
      using F-C' apply simp
      using dist not-tauto cond apply simp
      done
    } note H = this(1)
  then show ?case using 1 bj-merge-can-jump by meson
qed

end

locale cdcl_NOT-merge-bj-learn-proxy2 =
  cdcl_NOT-merge-bj-learn-proxy mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT
    propagate-conds forget-cond backjump-l-cond inv
  for
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    trail :: 'st ⇒ ('v, unit, unit) marked-lits and
    raw-clauses :: 'st ⇒ 'clss and
    prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒'st and
    add-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    remove-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    propagate-conds :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ bool and
    forget-cond :: 'cls ⇒ 'st ⇒ bool and
    backjump-l-cond :: 'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool and
    inv :: 'st ⇒ bool
begin

sublocale conflict-driven-clause-learning-ops mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT
     inv backjump-conds propagate-conds
     λC -. distinct-mset (mset-cls C) ∧ ¬tautology (mset-cls C)
     forget-cond
  by unfold-locales
end

locale cdcl_NOT-merge-bj-learn =
  cdcl_NOT-merge-bj-learn-proxy2 mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT
    propagate-conds forget-cond backjump-l-cond inv
  for
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    trail :: 'st ⇒ ('v, unit, unit) marked-lits and
    raw-clauses :: 'st ⇒ 'clss and
    prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒'st and
    add-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    remove-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    backjump-l-cond :: 'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool and
    propagate-conds :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ bool and
    forget-cond :: 'cls ⇒ 'st ⇒ bool and
    inv :: 'st ⇒ bool +
  assumes
     dpll-merge-bj-inv: ⋀S T.  dpll-bj S T ⟹ inv S ⟹ inv T and
     learn-inv: ⋀S T. learn S T ⟹ inv S ⟹ inv T
begin

sublocale
   conflict-driven-clause-learning mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT
     inv backjump-conds propagate-conds
     λC -.  distinct-mset (mset-cls C) ∧ ¬tautology (mset-cls C)
     forget-cond
  apply unfold-locales
  using cdcl_NOT-merged-bj-learn-forget_NOT cdcl-merged-inv learn-inv
  by (auto simp add: cdcl_NOT.simps dpll-merge-bj-inv)

lemma backjump-l-learn-backjump:
  assumes bt: backjump-l S T and inv: inv S and n-d: no-dup (trail S)
  shows ∃ C' L D. learn S (add-cls_NOT D S)
    ∧ mset-cls D = (C' + {#L#})
    ∧ backjump (add-cls_NOT D S) T
    ∧ atms-of (C' + {#L#}) ⊆ atms-of-mm (clauses_NOT S) ∪ atm-of ‘ (lits-of-l (trail S))
proof –
   obtain C F' K F L l C' D where
     tr-S: trail S = F' @ Marked K () # F and
     T: T ∼ prepend-trail (Propagated L l) (reduce-trail-to_NOT F (add-cls_NOT D S)) and
     C-cls-S: C ∈# clauses_NOT S and
     tr-S-CNot-C: trail S ⊨as CNot C and
     undef: undefined-lit F L and
     atm-L: atm-of L ∈ atms-of-mm (clauses_NOT S) ∪ atm-of ‘ (lits-of-l (trail S)) and
     clss-C: clauses_NOT S ⊨pm mset-cls D and
     D: mset-cls D = C' + {#L#}
     F ⊨as CNot C' and
     distinct: distinct-mset (mset-cls D) and
     not-tauto: ¬ tautology (mset-cls D)
     using bt inv by (elim backjump-lE) simp
   have atms-C': atms-of C' ⊆ atm-of ‘ (lits-of-l F)
     by (metis D(2) atms-of-def image-subsetI true-annots-CNot-all-atms-defined)
   then have atms-of (C' + {#L#}) ⊆ atms-of-mm (clauses_NOT S) ∪ atm-of ‘ (lits-of-l (trail S))
     using atm-L tr-S by auto
   moreover have learn: learn S (add-cls_NOT D S)
     apply (rule learn.intros)
         apply (rule clss-C)
```

```
      using atms-C' atm-L D apply (fastforce simp add: tr-S in-plus-implies-atm-of-on-atms-of-ms)
    apply standard
     apply (rule distinct)
     apply (rule not-tauto)
     apply simp
    done
  moreover have bj: backjump (add-cls_NOT D S) T
    apply (rule backjump.intros)
    using ‹F ⊨as CNot C'› C-cls-S tr-S-CNot-C undef T distinct not-tauto n-d D
    by (auto simp: tr-S state-eq_NOT-def simp del: state-simp_NOT)
  ultimately show ?thesis using D by blast
qed

lemma cdcl_NOT-merged-bj-learn-is-tranclp-cdcl_NOT:
  cdcl_NOT-merged-bj-learn S T ⟹ inv S ⟹ no-dup (trail S) ⟹ cdcl_NOT^++ S T
proof (induction rule: cdcl_NOT-merged-bj-learn.induct)
  case (cdcl_NOT-merged-bj-learn-decide_NOT T)
  then have cdcl_NOT S T
    using bj-decide_NOT cdcl_NOT.simps by fastforce
  then show ?case by auto
next
  case (cdcl_NOT-merged-bj-learn-propagate_NOT T)
  then have cdcl_NOT S T
    using bj-propagate_NOT cdcl_NOT.simps by fastforce
  then show ?case by auto
next
   case (cdcl_NOT-merged-bj-learn-forget_NOT T)
   then have cdcl_NOT S T
     using c-forget_NOT by blast
   then show ?case by auto
next
   case (cdcl_NOT-merged-bj-learn-backjump-l T) note bt = this(1) and inv = this(2) and
     n-d = this(3)
   obtain C' :: 'v literal multiset and L :: 'v literal and D :: 'cls where
     f3: learn S (add-cls_NOT D S) ∧
       backjump (add-cls_NOT D S) T ∧
       atms-of (C' + {#L#}) ⊆ atms-of-mm (clauses_NOT S) ∪ atm-of ` lits-of-l (trail S) and
     D: mset-cls D = C' + {#L#}
     using n-d backjump-l-learn-backjump[OF bt inv] by blast
   then have f4: cdcl_NOT S (add-cls_NOT D S)
     using n-d c-learn by blast
   have cdcl_NOT (add-cls_NOT D S) T
     using f3 n-d bj-backjump c-dpll-bj by blast
   then show ?case
     using f4 by (meson tranclp.r-into-trancl tranclp.trancl-into-trancl)
qed

lemma rtranclp-cdcl_NOT-merged-bj-learn-is-rtranclp-cdcl_NOT-and-inv:
  cdcl_NOT-merged-bj-learn^** S T ⟹ inv S ⟹ no-dup (trail S) ⟹ cdcl_NOT^** S T ∧ inv T
proof (induction rule: rtranclp-induct)
  case base
  then show ?case by auto
next
  case (step T U) note st = this(1) and cdcl_NOT = this(2) and IH = this(3)[OF this(4−)] and
    inv = this(4) and n-d = this(5)
```

have *cdcl*$_{NOT}{}^{**}$ *T U*
using *cdcl*$_{NOT}$*-merged-bj-learn-is-tranclp-cdcl*$_{NOT}$[*OF cdcl*$_{NOT}$] *IH*
rtranclp-cdcl$_{NOT}$*-no-dup inv n-d* **by** *auto*
then have *cdcl*$_{NOT}{}^{**}$ *S U* **using** *IH* **by** *fastforce*
moreover have *inv U* **using** *n-d IH* ‹*cdcl*$_{NOT}{}^{**}$ *T U*› *rtranclp-cdcl*$_{NOT}$*-inv* **by** *blast*
ultimately show *?case* **using** *st* **by** *fast*
qed

lemma *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-is-rtranclp-cdcl*$_{NOT}$:
cdcl$_{NOT}$*-merged-bj-learn*** *S T* $\Longrightarrow$ *inv S* $\Longrightarrow$*no-dup* (*trail S*) $\Longrightarrow$ *cdcl*$_{NOT}{}^{**}$ *S T*
using *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-is-rtranclp-cdcl*$_{NOT}$*-and-inv* **by** *blast*

lemma *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-inv*:
cdcl$_{NOT}$*-merged-bj-learn*** *S T* $\Longrightarrow$ *inv S* $\Longrightarrow$ *no-dup* (*trail S*) $\Longrightarrow$ *inv T*
using *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-is-rtranclp-cdcl*$_{NOT}$*-and-inv* **by** *blast*

definition $\mu_C{}'$ :: *'v literal multiset set* ⇒ *'st* ⇒ *nat* **where**
$\mu_C{}'$ *A T* ≡ μ_C (*1+card* (*atms-of-ms A*)) (*2+card* (*atms-of-ms A*)) (*trail-weight T*)

definition $\mu_{CDCL}{}'$*-merged* :: *'v literal multiset set* ⇒ *'st* ⇒ *nat* **where**
$\mu_{CDCL}{}'$*-merged A T* ≡
((*2+card* (*atms-of-ms A*)) ^ (*1+card* (*atms-of-ms A*)) − $\mu_C{}'$ *A T*) ∗ *2* + *card* (*set-mset* (*clauses*$_{NOT}$ *T*))

lemma *cdcl*$_{NOT}$*-decreasing-measure'*:
assumes
cdcl$_{NOT}$*-merged-bj-learn S T* **and**
inv: *inv S* **and**
atm-clss: *atms-of-mm* (*clauses*$_{NOT}$ *S*) ⊆ *atms-of-ms A* **and**
atm-trail: *atm-of* ‘ *lits-of-l* (*trail S*) ⊆ *atms-of-ms A* **and**
n-d: *no-dup* (*trail S*) **and**
fin-A: *finite A*
shows $\mu_{CDCL}{}'$*-merged A T* < $\mu_{CDCL}{}'$*-merged A S*
using *assms*(*1*)
proof *induction*
case (*cdcl*$_{NOT}$*-merged-bj-learn-decide*$_{NOT}$ *T*)
have *clauses*$_{NOT}$ *S* = *clauses*$_{NOT}$ *T*
using *cdcl*$_{NOT}$*-merged-bj-learn-decide*$_{NOT}$*.hyps* **by** *auto*
moreover have
(*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*))
− μ_C (*1* + *card* (*atms-of-ms A*)) (*2* + *card* (*atms-of-ms A*)) (*trail-weight T*)
< (*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*))
− μ_C (*1* + *card* (*atms-of-ms A*)) (*2* + *card* (*atms-of-ms A*)) (*trail-weight S*)
apply (*rule dpll-bj-trail-mes-decreasing-prop*)
using *cdcl*$_{NOT}$*-merged-bj-learn-decide*$_{NOT}$ *fin-A atm-clss atm-trail n-d inv*
by (*simp-all add*: *bj-decide*$_{NOT}$ *cdcl*$_{NOT}$*-merged-bj-learn-decide*$_{NOT}$*.hyps*)
ultimately show *?case*
unfolding $\mu_{CDCL}{}'$*-merged-def* $\mu_C{}'$*-def* **by** *simp*
next
case (*cdcl*$_{NOT}$*-merged-bj-learn-propagate*$_{NOT}$ *T*)
have *clauses*$_{NOT}$ *S* = *clauses*$_{NOT}$ *T*
using *cdcl*$_{NOT}$*-merged-bj-learn-propagate*$_{NOT}$*.hyps*
by (*simp add*: *bj-propagate*$_{NOT}$ *inv dpll-bj-clauses*)
moreover have
(*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*))

```
      − μ_C (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A)) (trail-weight T)
     < (2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A))
      − μ_C (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A)) (trail-weight S)
    apply (rule dpll-bj-trail-mes-decreasing-prop)
    using inv n-d atm-clss atm-trail fin-A by (simp-all add: bj-propagate_NOT
      cdcl_NOT-merged-bj-learn-propagate_NOT.hyps)
  ultimately show ?case
    unfolding μ_CDCL'-merged-def μ_C'-def by simp
next
  case (cdcl_NOT-merged-bj-learn-forget_NOT T)
  have card (set-mset (clauses_NOT T)) < card (set-mset (clauses_NOT S))
    using ‹forget_NOT S T› by (metis card-Diff1-less clauses-remove-cls_NOT finite-set-mset
      forget_NOT.cases in-clss-mset-clss linear set-mset-minus-replicate-mset(1) state-eq_NOT-def)
  moreover
    have trail S = trail T
      using ‹forget_NOT S T› by (auto elim: forget_NOTE)
    then have
      (2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A))
       − μ_C (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A)) (trail-weight T)
      = (2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A))
       − μ_C (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A)) (trail-weight S)
      by auto
  ultimately show ?case
    unfolding μ_CDCL'-merged-def μ_C'-def by simp
next
  case (cdcl_NOT-merged-bj-learn-backjump-l T) note bj-l = this(1)
  obtain C' L D where
    learn: learn S (add-cls_NOT D S) and
    bj: backjump (add-cls_NOT D S) T and
    atms-C: atms-of (C' + {#L#}) ⊆ atms-of-mm (clauses_NOT S) ∪ atm-of ` (lits-of-l (trail S)) and
    D: mset-cls D =  C' + {#L#}
    using bj-l inv backjump-l-learn-backjump n-d atm-clss atm-trail by meson
  have card-T-S: card (set-mset (clauses_NOT T)) ≤ 1+ card (set-mset (clauses_NOT S))
    using bj-l inv by (force elim!: backjump-lE simp: card-insert-if)
  have
    ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A))
      − μ_C (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A)) (trail-weight T))
    < ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A))
      − μ_C (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A))
          (trail-weight (add-cls_NOT D S)))
    apply (rule dpll-bj-trail-mes-decreasing-prop)
         using bj bj-backjump apply blast
        using cdcl_NOT.c-learn cdcl_NOT-inv inv learn apply blast
       using atms-C atm-clss atm-trail D apply (simp add: n-d) apply fast
      using atm-trail n-d apply simp
     apply (simp add: n-d)
    using fin-A apply simp
    done
  then have ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A))
      − μ_C (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A)) (trail-weight T))
    < ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A))
      − μ_C (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A)) (trail-weight S))
    using n-d by auto
  then show ?case
    using card-T-S unfolding μ_CDCL'-merged-def μ_C'-def by linarith
```

```
qed

lemma wf-cdcl_NOT-merged-bj-learn:
  assumes
    fin-A: finite A
  shows wf {(T, S).
    (inv S ∧ atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A ∧ atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A
    ∧ no-dup (trail S))
    ∧ cdcl_NOT-merged-bj-learn S T}
  apply (rule wfP-if-measure[of - - μ_CDCL'-merged A])
  using cdcl_NOT-decreasing-measure' fin-A by simp

lemma tranclp-cdcl_NOT-cdcl_NOT-tranclp:
  assumes
    cdcl_NOT-merged-bj-learn⁺⁺ S T and
    inv: inv S and
    atm-clss: atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A and
    atm-trail: atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A and
    n-d: no-dup (trail S) and
    fin-A[simp]: finite A
  shows (T, S) ∈ {(T, S).
    (inv S ∧ atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A ∧ atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A
    ∧ no-dup (trail S))
    ∧ cdcl_NOT-merged-bj-learn S T}⁺ (is - ∈ ?P⁺)
  using assms(1)
proof (induction rule: tranclp-induct)
  case base
  then show ?case using n-d atm-clss atm-trail inv by auto
next
  case (step T U) note st = this(1) and cdcl_NOT = this(2) and IH = this(3)
  have cdcl_NOT** S T
    apply (rule rtranclp-cdcl_NOT-merged-bj-learn-is-rtranclp-cdcl_NOT)
    using st cdcl_NOT inv  n-d atm-clss atm-trail inv by auto
  have inv T
    apply (rule rtranclp-cdcl_NOT-merged-bj-learn-inv)
      using inv st cdcl_NOT n-d atm-clss atm-trail inv by auto
  moreover have atms-of-mm (clauses_NOT T) ⊆ atms-of-ms A
    using rtranclp-cdcl_NOT-trail-clauses-bound[OF ‹cdcl_NOT** S T› inv n-d atm-clss atm-trail]
    by fast
  moreover have atm-of ‘ (lits-of-l (trail T))⊆ atms-of-ms A
    using rtranclp-cdcl_NOT-trail-clauses-bound[OF ‹cdcl_NOT** S T› inv n-d atm-clss atm-trail]
    by fast
  moreover have no-dup (trail T)
    using rtranclp-cdcl_NOT-no-dup[OF  ‹cdcl_NOT** S T› inv n-d] by fast
  ultimately have (U, T) ∈ ?P
    using cdcl_NOT by auto
  then show ?case using IH by (simp add: trancl-into-trancl2)
qed

lemma wf-tranclp-cdcl_NOT-merged-bj-learn:
  assumes finite A
  shows wf {(T, S).
    (inv S ∧ atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A ∧ atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A
    ∧ no-dup (trail S))
    ∧ cdcl_NOT-merged-bj-learn⁺⁺ S T}
  apply (rule wf-subset)
   apply (rule wf-trancl[OF wf-cdcl_NOT-merged-bj-learn])
   using assms apply simp
  using tranclp-cdcl_NOT-cdcl_NOT-tranclp[OF - - - - - ‹finite A›] by auto

lemma backjump-no-step-backjump-l:
  backjump S T ⟹ inv S ⟹ ¬no-step backjump-l S
  apply (elim backjumpE)
  apply (rule bj-merge-can-jump)
    apply auto[7]
  by blast

lemma cdcl_NOT-merged-bj-learn-final-state:
  fixes A :: 'v literal multiset set and S T :: 'st
  assumes
    n-s: no-step cdcl_NOT-merged-bj-learn S and
    atms-S: atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A and
    atms-trail: atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A and
    n-d: no-dup (trail S) and
    finite A and
    inv: inv S and
    decomp: all-decomposition-implies-m (clauses_NOT S) (get-all-marked-decomposition (trail S))
  shows unsatisfiable (set-mset (clauses_NOT S))
    ∨ (trail S ⊨asm clauses_NOT S ∧ satisfiable (set-mset (clauses_NOT S)))
proof -
  let ?N = set-mset (clauses_NOT S)
  let ?M = trail S
  consider
      (sat) satisfiable ?N and ?M ⊨as ?N
    | (sat') satisfiable ?N and ¬ ?M ⊨as ?N
    | (unsat) unsatisfiable ?N
    by auto
  then show ?thesis
    proof cases
      case sat' note sat = this(1) and M = this(2)
      obtain C where C ∈ ?N and ¬?M ⊨a C using M unfolding true-annots-def by auto
      obtain I :: 'v literal set where
        I ⊨s ?N and
        cons: consistent-interp I and
        tot: total-over-m I ?N and
        atm-I-N: atm-of ‘I ⊆ atms-of-ms ?N
        using sat unfolding satisfiable-def-min by auto
      let ?I = I ∪ {P| P. P ∈ lits-of-l ?M ∧ atm-of P ∉ atm-of ‘ I}
      let ?O = {unmark L |L. is-marked L ∧ L ∈ set ?M ∧ atm-of (lit-of L) ∉ atms-of-ms ?N}
      have cons-I': consistent-interp ?I
        using cons using ‹no-dup ?M›  unfolding consistent-interp-def
        by (auto simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set lits-of-def
          dest!: no-dup-cannot-not-lit-and-uminus)
      have tot-I': total-over-m ?I (?N ∪ unmark-l ?M)
        using tot atms-of-s-def unfolding total-over-m-def total-over-set-def
        by (fastforce simp: image-iff)
      have {P |P. P ∈ lits-of-l ?M ∧ atm-of P ∉ atm-of ‘ I} ⊨s ?O
        using ‹I⊨s ?N› atm-I-N by (auto simp add: atm-of-eq-atm-of true-clss-def lits-of-def)
      then have I'-N: ?I ⊨s ?N ∪ ?O
        using ‹I⊨s ?N› true-clss-union-increase by force
```

```
have tot′: total-over-m ?I (?N∪?O)
  using atm-I-N tot unfolding total-over-m-def total-over-set-def
  by (force simp: image-iff lits-of-def dest!: is-marked-ex-Marked)

have atms-N-M: atms-of-ms ?N ⊆ atm-of ‘ lits-of-l ?M
  proof (rule ccontr)
    assume ¬ ?thesis
    then obtain l :: ′v where
      l-N: l ∈ atms-of-ms ?N and
      l-M: l ∉ atm-of ‘ lits-of-l ?M
      by auto
    have undefined-lit ?M (Pos l)
      using l-M by (metis Marked-Propagated-in-iff-in-lits-of-l
        atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set literal.sel(1))
    have decide_NOT S (prepend-trail (Marked (Pos l) ()) S)
      by (metis ‹undefined-lit ?M (Pos l)› decide_NOT.intros l-N literal.sel(1)
        state-eq_NOT-ref)
    then show False
      using cdcl_NOT-merged-bj-learn-decide_NOT n-s by blast
  qed

have ?M ⊨as CNot C
apply (rule all-variables-defined-not-imply-cnot)
  using atms-N-M ‹C ∈ ?N› ‹¬ ?M ⊨a C› atms-of-atms-of-ms-mono[OF ‹C ∈ ?N›]
  by (auto dest: atms-of-atms-of-ms-mono)
have ∃ l ∈ set ?M. is-marked l
  proof (rule ccontr)
    let ?O = {unmark L |L. is-marked L ∧ L ∈ set ?M ∧ atm-of (lit-of L) ∉ atms-of-ms ?N}
    have ϑ[iff]: ⋀I. total-over-m I (?N ∪ ?O ∪ unmark-l ?M)
      ⟷ total-over-m I (?N ∪unmark-l ?M)
      unfolding total-over-set-def total-over-m-def atms-of-ms-def by blast
    assume ¬ ?thesis
    then have [simp]:{unmark L |L. is-marked L ∧ L ∈ set ?M}
      = {unmark L |L. is-marked L ∧ L ∈ set ?M ∧ atm-of (lit-of L) ∉ atms-of-ms ?N}
      by auto
    then have ?N ∪ ?O ⊨ps unmark-l ?M
      using all-decomposition-implies-propagated-lits-are-implied[OF decomp] by auto

    then have ?I ⊨s unmark-l ?M
      using cons-I′ I′-N tot-I′ ‹?I ⊨s ?N ∪ ?O› unfolding ϑ true-clss-clss-def by blast
    then have lits-of-l ?M ⊆ ?I
      unfolding true-clss-def lits-of-def by auto
    then have ?M ⊨as ?N
      using I′-N ‹C ∈ ?N› ‹¬ ?M ⊨a C› cons-I′ atms-N-M
      by (meson ‹trail S ⊨as CNot C› consistent-CNot-not rev-subsetD sup-ge1 true-annot-def
        true-annots-def true-cls-mono-set-mset-l true-clss-def)
    then show False using M by fast
  qed
from List.split-list-first-propE[OF this] obtain K :: ′v literal and d :: unit and
  F F′ :: (′v, unit, unit) marked-lit list where
  M-K: ?M = F′ @ Marked K () # F and
  nm: ∀ f∈set F′. ¬is-marked f
  unfolding is-marked-def by (metis (full-types) old.unit.exhaust)
let ?K = Marked K ()::(′v, unit, unit) marked-lit
have ?K ∈ set ?M
```

```
  unfolding M-K by auto
let ?C = image-mset lit-of {#L∈#mset ?M. is-marked L ∧ L≠?K#} :: 'v literal multiset
let ?C' = set-mset (image-mset (λL::'v literal. {#L#}) (?C + unmark ?K))
have ?N ∪ {unmark L |L. is-marked L ∧ L ∈ set ?M} ⊨ps unmark-l ?M
  using all-decomposition-implies-propagated-lits-are-implied[OF decomp] .
moreover have C': ?C' = {unmark L |L. is-marked L ∧ L ∈ set ?M}
  unfolding M-K apply standard
    apply force
  using IntI by auto
ultimately have N-C-M: ?N ∪ ?C' ⊨ps unmark-l ?M
  by auto
have N-M-False: ?N ∪ (λL. unmark L) ` (set ?M) ⊨ps {{#}}
  using M ‹?M ⊨as CNot C› ‹C∈?N› unfolding true-clss-clss-def true-annots-def Ball-def
  true-annot-def by (metis consistent-CNot-not sup.orderE sup-commute true-clss-def
    true-clss-singleton-lit-of-implies-incl true-clss-union true-clss-union-increase)

have undefined-lit F K using ‹no-dup ?M› unfolding M-K by (simp add: defined-lit-map)
moreover
  have ?N ∪ ?C' ⊨ps {{#}}
    proof −
      have A: ?N ∪ ?C' ∪ unmark-l ?M = ?N ∪ unmark-l ?M
        unfolding M-K by auto
      show ?thesis
        using true-clss-clss-left-right[OF N-C-M, of {{#}}] N-M-False unfolding A by auto
    qed
  have ?N ⊨p image-mset uminus ?C + {#−K#}
    unfolding true-clss-cls-def true-clss-clss-def total-over-m-def
    proof (intro allI impI)
      fix I
      assume
        tot: total-over-set I (atms-of-ms (?N ∪ {image-mset uminus ?C+ {#− K#}})) and
        cons: consistent-interp I and
        I ⊨s ?N
      have (K ∈ I ∧ −K ∉ I) ∨ (−K ∈ I ∧ K ∉ I)
        using cons tot unfolding consistent-interp-def by (cases K) auto
      have  {a ∈ set (trail S). is-marked a ∧ a ≠ Marked K ()} =
       set (trail S) ∩ {L. is-marked L ∧ L ≠ Marked K ()}
       by auto
      then have tot': total-over-set I
         (atm-of ` lit-of ` (set ?M ∩ {L. is-marked L ∧ L ≠ Marked K ()}))
        using tot by (auto simp add: atms-of-uminus-lit-atm-of-lit-of)
      { fix x :: ('v, unit, unit) marked-lit
        assume
          a3: lit-of x ∉ I and
          a1: x ∈ set ?M and
          a4: is-marked x and
          a5: x ≠ Marked K ()
        then have Pos (atm-of (lit-of x)) ∈ I ∨ Neg (atm-of (lit-of x)) ∈ I
          using a5 a4 tot' a1 unfolding total-over-set-def atms-of-s-def by blast
        moreover have f6: Neg (atm-of (lit-of x)) = − Pos (atm-of (lit-of x))
          by simp
        ultimately have − lit-of x ∈ I
          using f6 a3 by (metis (no-types) atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
            literal.sel(1))
      } note H = this
```

```
          have ¬I ⊨s ?C'
            using ‹?N ∪ ?C' ⊨ps {{#}}› tot cons ‹I ⊨s ?N›
            unfolding true-clss-clss-def total-over-m-def
            by (simp add: atms-of-uminus-lit-atm-of-lit-of atms-of-ms-single-image-atm-of-lit-of)
          then show I ⊨ image-mset uminus ?C + {#− K#}
            unfolding true-clss-def true-cls-def Bex-def
            using ‹(K ∈ I ∧ −K ∉ I) ∨ (−K ∈ I ∧ K ∉ I)›
            by (auto dest!: H)
        qed
      moreover have F ⊨as CNot (image-mset uminus ?C)
        using nm unfolding true-annots-def CNot-def M-K by (auto simp add: lits-of-def)
      ultimately have False
        using bj-merge-can-jump[of S F' K F C −K
          image-mset uminus (image-mset lit-of {# L :# mset ?M. is-marked L ∧ L ≠ Marked K ()#})]
          ‹C∈?N› n-s ‹?M ⊨as CNot C› bj-backjump inv unfolding M-K
          by (auto simp: cdcl_NOT-merged-bj-learn.simps)
        then show ?thesis by fast
    qed auto
qed

lemma full-cdcl_NOT-merged-bj-learn-final-state:
  fixes A :: 'v literal multiset set and S T :: 'st
  assumes
    full: full cdcl_NOT-merged-bj-learn S T and
    atms-S: atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A and
    atms-trail: atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A and
    n-d: no-dup (trail S) and
    finite A and
    inv: inv S and
    decomp: all-decomposition-implies-m (clauses_NOT S) (get-all-marked-decomposition (trail S))
  shows unsatisfiable (set-mset (clauses_NOT T))
    ∨ (trail T ⊨asm clauses_NOT T ∧ satisfiable (set-mset (clauses_NOT T)))
proof −
  have st: cdcl_NOT-merged-bj-learn** S T and n-s: no-step cdcl_NOT-merged-bj-learn T
    using full unfolding full-def by blast+
  then have st: cdcl_NOT** S T
    using inv rtranclp-cdcl_NOT-merged-bj-learn-is-rtranclp-cdcl_NOT-and-inv n-d by auto
  have atms-of-mm (clauses_NOT T) ⊆ atms-of-ms A and atm-of ‘ lits-of-l (trail T) ⊆ atms-of-ms A
    using rtranclp-cdcl_NOT-trail-clauses-bound[OF st inv n-d atms-S atms-trail] by blast+
  moreover have no-dup (trail T)
    using rtranclp-cdcl_NOT-no-dup inv n-d st by blast
  moreover have inv T
    using rtranclp-cdcl_NOT-inv inv st by blast
  moreover have all-decomposition-implies-m (clauses_NOT T) (get-all-marked-decomposition (trail T))
    using rtranclp-cdcl_NOT-all-decomposition-implies inv st decomp n-d by blast
  ultimately show ?thesis
    using cdcl_NOT-merged-bj-learn-final-state[of T A] ‹finite A› n-s by fast
qed

end
```

16.8.1 Instantiations

```
locale cdcl_NOT-with-backtrack-and-restarts =
  conflict-driven-clause-learning-learning-before-backjump-only-distinct-learnt
```

```
    mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT
    inv backjump-conds propagate-conds learn-restrictions forget-restrictions
  for
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    trail :: 'st ⇒ ('v, unit, unit) marked-lits and
    raw-clauses :: 'st ⇒ 'clss and
    prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒'st and
    add-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    remove-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    inv :: 'st ⇒ bool and
    backjump-conds ::  'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool and
    propagate-conds :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ bool and
    learn-restrictions forget-restrictions :: 'cls ⇒ 'st ⇒ bool
    +
  fixes f :: nat ⇒ nat
  assumes
    unbounded: unbounded f and f-ge-1: ⋀n. n ≥ 1 ⟹ f n ≥ 1 and
    inv-restart:⋀S T. inv S ⟹ T ∼ reduce-trail-to_NOT ([]::'a list) S ⟹ inv T
begin

lemma bound-inv-inv:
  assumes
    inv S and
    n-d: no-dup (trail S) and
    atms-clss-S-A: atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A and
    atms-trail-S-A:atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A and
    finite A and
    cdcl_NOT: cdcl_NOT S T
  shows
    atms-of-mm (clauses_NOT T) ⊆ atms-of-ms A and
    atm-of ‘ lits-of-l (trail T) ⊆ atms-of-ms A and
    finite A
proof –
  have cdcl_NOT S T
    using ‹inv S› cdcl_NOT by linarith
  then have atms-of-mm (clauses_NOT T) ⊆ atms-of-mm (clauses_NOT S) ∪ atm-of ‘ lits-of-l (trail S)
    using ‹inv S›
    by (meson conflict-driven-clause-learning-ops.cdcl_NOT-atms-of-ms-clauses-decreasing
      conflict-driven-clause-learning-ops-axioms n-d)
  then show atms-of-mm (clauses_NOT T) ⊆ atms-of-ms A
    using atms-clss-S-A atms-trail-S-A by blast
next
  show atm-of ‘ lits-of-l (trail T) ⊆ atms-of-ms A
    by (meson ‹inv S› atms-clss-S-A atms-trail-S-A cdcl_NOT cdcl_NOT-atms-in-trail-in-set n-d)
next
```

```
  show finite A
    using ‹finite A› by simp
qed

sublocale cdcl_NOT-increasing-restarts-ops λS T. T ∼ reduce-trail-to_NOT ([]::'a list) S cdcl_NOT f
  λA S. atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A ∧ atm-of ` lits-of-l (trail S) ⊆ atms-of-ms A ∧
  finite A
  μ_CDCL' λS. inv S ∧ no-dup (trail S)
  μ_CDCL'-bound
  apply unfold-locales
           apply (simp add: unbounded)
          using f-ge-1 apply force
         using bound-inv-inv apply meson
        apply (rule cdcl_NOT-decreasing-measure'; simp)
        apply (rule rtranclp-cdcl_NOT-μ_CDCL'-bound; simp)
       apply (rule rtranclp-μ_CDCL'-bound-decreasing; simp)
      apply auto[]
    apply auto[]
   using cdcl_NOT-inv cdcl_NOT-no-dup apply blast
  using inv-restart apply auto[]
  done

lemma cdcl_NOT-with-restart-μ_CDCL'-le-μ_CDCL'-bound:
  assumes
    cdcl_NOT: cdcl_NOT-restart (T, a) (V, b) and
    cdcl_NOT-inv:
      inv T
      no-dup (trail T) and
    bound-inv:
      atms-of-mm (clauses_NOT T) ⊆ atms-of-ms A
      atm-of ` lits-of-l (trail T) ⊆ atms-of-ms A
      finite A
  shows μ_CDCL' A V ≤ μ_CDCL'-bound A T
  using cdcl_NOT-inv bound-inv
proof (induction rule: cdcl_NOT-with-restart-induct[OF cdcl_NOT])
  case (1 m S T n U) note U = this(3)
  show ?case
    apply (rule rtranclp-cdcl_NOT-μ_CDCL'-bound-reduce-trail-to_NOT[of S T])
         using ‹(cdcl_NOT ^^ m) S T› apply (fastforce dest!: relpowp-imp-rtranclp)
        using 1 by auto
next
  case (2 S T n) note full = this(2)
  show ?case
    apply (rule rtranclp-cdcl_NOT-μ_CDCL'-bound)
    using full 2 unfolding full1-def by force+
qed

lemma cdcl_NOT-with-restart-μ_CDCL'-bound-le-μ_CDCL'-bound:
  assumes
    cdcl_NOT: cdcl_NOT-restart (T, a) (V, b) and
    cdcl_NOT-inv:
      inv T
      no-dup (trail T) and
    bound-inv:
      atms-of-mm (clauses_NOT T) ⊆ atms-of-ms A
      atm-of ‘ lits-of-l (trail T) ⊆ atms-of-ms A
      finite A
  shows $\mu_{CDCL}$'-bound A V ≤ $\mu_{CDCL}$'-bound A T
  using cdcl$_{NOT}$-inv bound-inv
proof (induction rule: cdcl$_{NOT}$-with-restart-induct[OF cdcl$_{NOT}$])
  case (1 m S T n U) note U = this(3)
  have $\mu_{CDCL}$'-bound A T ≤ $\mu_{CDCL}$'-bound A S
     apply (rule rtranclp-$\mu_{CDCL}$'-bound-decreasing)
         using ‹(cdcl$_{NOT}$ ^^ m) S T› apply (fastforce dest: relpowp-imp-rtranclp)
        using 1 by auto
  then show ?case using U unfolding $\mu_{CDCL}$'-bound-def by auto
next
  case (2 S T n) note full = this(2)
  show ?case
    apply (rule rtranclp-$\mu_{CDCL}$'-bound-decreasing)
    using full 2 unfolding full1-def by force+
qed

sublocale cdcl$_{NOT}$-increasing-restarts - - - - - - - - - - - - - -
    f
    λS T. T ∼ reduce-trail-to$_{NOT}$ ([]::'a list) S
   λA S. atms-of-mm (clauses$_{NOT}$ S) ⊆ atms-of-ms A
     ∧ atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A ∧ finite A
   $\mu_{CDCL}$' cdcl$_{NOT}$
    λS. inv S ∧ no-dup (trail S)
   $\mu_{CDCL}$'-bound
  apply unfold-locales
   using cdcl$_{NOT}$-with-restart-$\mu_{CDCL}$'-le-$\mu_{CDCL}$'-bound apply simp
  using cdcl$_{NOT}$-with-restart-$\mu_{CDCL}$'-bound-le-$\mu_{CDCL}$'-bound apply simp
  done

lemma cdcl$_{NOT}$-restart-all-decomposition-implies:
  assumes cdcl$_{NOT}$-restart S T and
    inv (fst S) and
    no-dup (trail (fst S))
    all-decomposition-implies-m (clauses$_{NOT}$ (fst S)) (get-all-marked-decomposition (trail (fst S)))
  shows
    all-decomposition-implies-m (clauses$_{NOT}$ (fst T)) (get-all-marked-decomposition (trail (fst T)))
  using assms apply (induction)
  using rtranclp-cdcl$_{NOT}$-all-decomposition-implies by (auto dest!: tranclp-into-rtranclp
    simp: full1-def)

lemma rtranclp-cdcl$_{NOT}$-restart-all-decomposition-implies:
  assumes cdcl$_{NOT}$-restart** S T and
    inv: inv (fst S) and
    n-d: no-dup (trail (fst S)) and
    decomp:
      all-decomposition-implies-m (clauses$_{NOT}$ (fst S)) (get-all-marked-decomposition (trail (fst S)))
  shows
    all-decomposition-implies-m (clauses$_{NOT}$ (fst T)) (get-all-marked-decomposition (trail (fst T)))
  using assms(1)
proof (induction rule: rtranclp-induct)
  case base
  then show ?case using decomp by simp
next
```

```
  case (step T u) note st = this(1) and r = this(2) and IH = this(3)
  have inv (fst T)
    using rtranclp-cdcl_NOT-with-restart-cdcl_NOT-inv[OF st] inv n-d by blast
  moreover have no-dup (trail (fst T))
    using rtranclp-cdcl_NOT-with-restart-cdcl_NOT-inv[OF st] inv n-d by blast
  ultimately show ?case
    using cdcl_NOT-restart-all-decomposition-implies r IH n-d by fast
qed

lemma cdcl_NOT-restart-sat-ext-iff:
  assumes
    st: cdcl_NOT-restart S T and
    n-d: no-dup (trail (fst S)) and
    inv: inv (fst S)
  shows I ⊨sextm clauses_NOT (fst S) ⟷ I ⊨sextm clauses_NOT (fst T)
  using assms
proof (induction)
  case (restart-step m S T n U)
  then show ?case
    using rtranclp-cdcl_NOT-bj-sat-ext-iff n-d by (fastforce dest!: relpowp-imp-rtranclp)
next
  case restart-full
  then show ?case using rtranclp-cdcl_NOT-bj-sat-ext-iff unfolding full1-def
  by (fastforce dest!: tranclp-into-rtranclp)
qed

lemma rtranclp-cdcl_NOT-restart-sat-ext-iff:
  assumes
    st: cdcl_NOT-restart** S T and
    n-d: no-dup (trail (fst S)) and
    inv: inv (fst S)
  shows I ⊨sextm clauses_NOT (fst S) ⟷ I ⊨sextm clauses_NOT (fst T)
  using st
proof (induction)
  case base
  then show ?case by simp
next
  case (step T U) note st = this(1) and r = this(2) and IH = this(3)
  have inv (fst T)
    using rtranclp-cdcl_NOT-with-restart-cdcl_NOT-inv[OF st] inv n-d by blast+
  moreover have no-dup (trail (fst T))
    using rtranclp-cdcl_NOT-with-restart-cdcl_NOT-inv rtranclp-cdcl_NOT-no-dup st inv n-d by blast
  ultimately show ?case
    using cdcl_NOT-restart-sat-ext-iff[OF r] IH by blast
qed

theorem full-cdcl_NOT-restart-backjump-final-state:
  fixes A :: 'v literal multiset set and S T :: 'st
  assumes
    full: full cdcl_NOT-restart (S, n) (T, m) and
    atms-S: atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A and
    atms-trail: atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A and
    n-d: no-dup (trail S) and
    fin-A[simp]: finite A and
    inv: inv S and
    decomp: all-decomposition-implies-m (clauses_NOT S) (get-all-marked-decomposition (trail S))
  shows unsatisfiable (set-mset (clauses_NOT S))
    ∨ (lits-of-l (trail T) ⊨sextm clauses_NOT S ∧ satisfiable (set-mset (clauses_NOT S)))
proof −
  have st: cdcl_NOT-restart** (S, n) (T, m) and
    n-s: no-step cdcl_NOT-restart (T, m)
    using full unfolding full-def by fast+
  have binv-T: atms-of-mm (clauses_NOT T) ⊆ atms-of-ms A
    atm-of ‘ lits-of-l (trail T) ⊆ atms-of-ms A
    using rtranclp-cdcl_NOT-with-restart-bound-inv[OF st, of A] inv n-d atms-S atms-trail
    by auto
  moreover have inv-T: no-dup (trail T) inv T
    using rtranclp-cdcl_NOT-with-restart-cdcl_NOT-inv[OF st] inv n-d by auto
  moreover have all-decomposition-implies-m (clauses_NOT T) (get-all-marked-decomposition (trail T))
    using rtranclp-cdcl_NOT-restart-all-decomposition-implies[OF st] inv n-d
    decomp by auto
  ultimately have T: unsatisfiable (set-mset (clauses_NOT T))
    ∨ (trail T ⊨asm clauses_NOT T ∧ satisfiable (set-mset (clauses_NOT T)))
    using no-step-cdcl_NOT-restart-no-step-cdcl_NOT[of (T, m) A] n-s
    cdcl_NOT-final-state[of T A] unfolding cdcl_NOT-NOT-all-inv-def by auto
  have eq-sat-S-T:⋀I. I ⊨sextm clauses_NOT S ⟷ I ⊨sextm clauses_NOT T
    using rtranclp-cdcl_NOT-restart-sat-ext-iff[OF st] inv n-d atms-S
        atms-trail by auto
  have cons-T: consistent-interp (lits-of-l (trail T))
    using inv-T(1) distinct-consistent-interp by blast
  consider
      (unsat) unsatisfiable (set-mset (clauses_NOT T))
    | (sat) trail T ⊨asm clauses_NOT T and satisfiable (set-mset (clauses_NOT T))
    using T by blast
  then show ?thesis
    proof cases
      case unsat
      then have unsatisfiable (set-mset (clauses_NOT S))
        using eq-sat-S-T consistent-true-clss-ext-satisfiable true-clss-imp-true-cls-ext
        unfolding satisfiable-def by blast
      then show ?thesis by fast
    next
      case sat
      then have lits-of-l (trail T) ⊨sextm clauses_NOT S
        using rtranclp-cdcl_NOT-restart-sat-ext-iff[OF st] inv n-d atms-S
        atms-trail by (auto simp: true-clss-imp-true-cls-ext true-annots-true-cls)
      moreover then have satisfiable (set-mset (clauses_NOT S))
          using cons-T consistent-true-clss-ext-satisfiable by blast
      ultimately show ?thesis by blast
    qed
qed
end — end of cdcl_NOT-with-backtrack-and-restarts locale
```

The restart does only reset the trail, contrary to Weidenbach's version. But there is a forget rule.

```
locale cdcl_NOT-merge-bj-learn-with-backtrack-restarts =
  cdcl_NOT-merge-bj-learn mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-cls_NOT remove-cls_NOT
    λC C' L' S T. distinct-mset (C' + {#L'#}) ∧ backjump-l-cond C C' L' S T
    propagate-conds forget-conds inv
  for
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    trail :: 'st ⇒ ('v, unit, unit) marked-lits and
    raw-clauses :: 'st ⇒ 'clss and
    prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒'st and
    add-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    remove-cls_NOT :: 'cls ⇒ 'st ⇒ 'st and
    propagate-conds :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ bool and
    inv :: 'st ⇒ bool and
    forget-conds :: 'cls ⇒ 'st ⇒ bool and
    backjump-l-cond :: 'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool
    +
  fixes f :: nat ⇒ nat
  assumes
    unbounded: unbounded f and f-ge-1: ⋀n. n ≥ 1 ⟹ f n ≥ 1 and
    inv-restart:⋀S T. inv S ⟹ T ∼ reduce-trail-to_NOT [] S ⟹ inv T
begin

definition not-simplified-cls A = {#C ∈# A. tautology C ∨ ¬distinct-mset C#}

lemma simple-clss-or-not-simplified-cls:
  assumes atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A and
    x ∈# clauses_NOT S and finite A
  shows x ∈ simple-clss (atms-of-ms A) ∨ x ∈# not-simplified-cls (clauses_NOT S)
proof -
  consider
      (simpl) ¬tautology x and distinct-mset x
    | (n-simp) tautology x ∨ ¬distinct-mset x
    by auto
  then show ?thesis
    proof cases
      case simpl
      then have x ∈ simple-clss (atms-of-ms A)
        by (meson assms atms-of-atms-of-ms-mono atms-of-ms-finite simple-clss-mono
          distinct-mset-not-tautology-implies-in-simple-clss finite-subset
          subsetCE)
      then show ?thesis by blast
    next
      case n-simp
      then have x ∈# not-simplified-cls (clauses_NOT S)
        using ‹x ∈# clauses_NOT S› unfolding not-simplified-cls-def by auto
      then show ?thesis by blast
    qed
qed

lemma cdcl_NOT-merged-bj-learn-clauses-bound:
  assumes
    cdcl_NOT-merged-bj-learn S T and
    inv: inv S and
    atms-clss: atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A and
    atms-trail: atm-of ‘(lits-of-l (trail S)) ⊆ atms-of-ms A and
    n-d: no-dup (trail S) and
    fin-A[simp]: finite A
  shows set-mset (clauses_NOT T) ⊆ set-mset (not-simplified-cls (clauses_NOT S))
    ∪ simple-clss (atms-of-ms A)
  using assms
proof (induction rule: cdcl_NOT-merged-bj-learn.induct)
  case cdcl_NOT-merged-bj-learn-decide_NOT
  then show ?case using dpll-bj-clauses by (force dest!: simple-clss-or-not-simplified-cls)
next
  case cdcl_NOT-merged-bj-learn-propagate_NOT
  then show ?case using dpll-bj-clauses by (force dest!: simple-clss-or-not-simplified-cls)
next
  case cdcl_NOT-merged-bj-learn-forget_NOT
  then show ?case using clauses-remove-cls_NOT unfolding state-eq_NOT-def
    by (force elim!: forget_NOTE  dest: simple-clss-or-not-simplified-cls)
next
  case (cdcl_NOT-merged-bj-learn-backjump-l T) note bj = this(1) and inv = this(2) and
    atms-clss = this(3) and atms-trail = this(4) and n-d = this(5)

  have cdcl_NOT** S T
    apply (rule rtranclp-cdcl_NOT-merged-bj-learn-is-rtranclp-cdcl_NOT)
    using ‹backjump-l S T› inv cdcl_NOT-merged-bj-learn.simps n-d by blast+
  have atm-of ‘(lits-of-l (trail T)) ⊆ atms-of-ms A
    using rtranclp-cdcl_NOT-trail-clauses-bound[OF ‹cdcl_NOT** S T›] inv atms-trail atms-clss
    n-d by auto
  have atms-of-mm (clauses_NOT T) ⊆ atms-of-ms A
    using rtranclp-cdcl_NOT-trail-clauses-bound[OF ‹cdcl_NOT** S T› inv n-d atms-clss atms-trail]
    by fast
  moreover have no-dup (trail T)
    using rtranclp-cdcl_NOT-no-dup[OF ‹cdcl_NOT** S T› inv n-d] by fast

  obtain F' K F L l C' C D where
    tr-S: trail S = F' @ Marked K () # F and
    T: T ∼ prepend-trail (Propagated L l) (reduce-trail-to_NOT F (add-cls_NOT D S)) and
    C ∈# clauses_NOT S and
    trail S ⊨as CNot C and
    undef: undefined-lit F L and
    clauses_NOT S ⊨pm C' + {#L#} and
    F ⊨as CNot C' and
    D: mset-cls D = C' + {#L#} and
    dist: distinct-mset (C' + {#L#}) and
    tauto: ¬ tautology (C' + {#L#}) and
    backjump-l-cond C C' L S T
    using ‹backjump-l S T› apply (elim backjump-lE) by auto

  have atms-of C' ⊆ atm-of ‘ (lits-of-l F)
    using ‹F ⊨as CNot C'› by (simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
      atms-of-def image-subset-iff in-CNot-implies-uminus(2))
  then have atms-of (C'+{#L#}) ⊆ atms-of-ms A
    using T ‹atm-of ‘ lits-of-l (trail T) ⊆ atms-of-ms A› tr-S undef n-d by auto
  then have simple-clss (atms-of (C' + {#L#})) ⊆ simple-clss (atms-of-ms A)
    apply − by (rule simple-clss-mono) (simp-all)
  then have C' + {#L#} ∈ simple-clss (atms-of-ms A)
    using distinct-mset-not-tautology-implies-in-simple-clss[OF dist tauto]
    by auto
  then show ?case
    using T inv atms-clss undef tr-S n-d D by (force dest!: simple-clss-or-not-simplified-cls)
qed

lemma cdcl_NOT-merged-bj-learn-not-simplified-decreasing:
  assumes cdcl_NOT-merged-bj-learn S T
  shows (not-simplified-cls (clauses_NOT T)) ⊆# (not-simplified-cls (clauses_NOT S))
  using assms apply induction
  prefer 4
  unfolding not-simplified-cls-def apply (auto elim!: backjump-lE forget_NOT E)[3]
  by (elim backjump-lE) auto

lemma rtranclp-cdcl_NOT-merged-bj-learn-not-simplified-decreasing:
  assumes cdcl_NOT-merged-bj-learn** S T
  shows (not-simplified-cls (clauses_NOT T)) ⊆# (not-simplified-cls (clauses_NOT S))
  using assms apply induction
    apply simp
   by (drule cdcl_NOT-merged-bj-learn-not-simplified-decreasing) auto

lemma rtranclp-cdcl_NOT-merged-bj-learn-clauses-bound:
  assumes
    cdcl_NOT-merged-bj-learn** S T and
    inv S and
    atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A and
    atm-of `(lits-of-l (trail S)) ⊆ atms-of-ms A and
    n-d: no-dup (trail S) and
    finite[simp]: finite A
  shows set-mset (clauses_NOT T) ⊆ set-mset (not-simplified-cls (clauses_NOT S))
    ∪ simple-clss (atms-of-ms A)
  using assms(1−5)
proof induction
  case base
  then show ?case by (auto dest!: simple-clss-or-not-simplified-cls)
next
  case (step T U) note st = this(1) and cdcl_NOT = this(2) and IH = this(3)[OF this(4−7)] and
    inv = this(4) and atms-clss-S = this(5) and atms-trail-S = this(6) and finite-cls-S = this(7)
  have st': cdcl_NOT** S T
    using inv rtranclp-cdcl_NOT-merged-bj-learn-is-rtranclp-cdcl_NOT-and-inv st n-d by blast
  have inv T
    using inv rtranclp-cdcl_NOT-merged-bj-learn-inv st n-d by blast
  moreover
    have atms-of-mm (clauses_NOT T) ⊆ atms-of-ms A and
      atm-of ` lits-of-l (trail T) ⊆ atms-of-ms A
      using rtranclp-cdcl_NOT-trail-clauses-bound[OF st'] inv atms-clss-S atms-trail-S n-d
      by blast+
  moreover moreover have no-dup (trail T)
    using rtranclp-cdcl_NOT-no-dup[OF ‹cdcl_NOT** S T› inv n-d] by fast
  ultimately have set-mset (clauses_NOT U)
    ⊆ set-mset (not-simplified-cls (clauses_NOT T)) ∪ simple-clss (atms-of-ms A)
    using cdcl_NOT finite cdcl_NOT-merged-bj-learn-clauses-bound
```

by (*auto intro*!: *cdcl*$_{NOT}$*-merged-bj-learn-clauses-bound*)
moreover have *set-mset* (*not-simplified-cls* (*clauses*$_{NOT}$ *T*))
$\subseteq$ *set-mset* (*not-simplified-cls* (*clauses*$_{NOT}$ *S*))
using *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-not-simplified-decreasing*[*OF st*] **by** *auto*
ultimately show *?case* **using** *IH inv atms-clss-S*
by (*auto dest*!: *simple-clss-or-not-simplified-cls*)
qed

abbreviation μ_{CDCL}*'-bound* **where**
μ_{CDCL}*'-bound A T* $\equiv$ ((*2*+*card* (*atms-of-ms A*)) ^ (*1*+*card* (*atms-of-ms A*))) $*$ *2*
+ *card* (*set-mset* (*not-simplified-cls*(*clauses*$_{NOT}$ *T*)))
+ *3* ^ *card* (*atms-of-ms A*)

lemma *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-clauses-bound-card*:
assumes
cdcl$_{NOT}$*-merged-bj-learn*** *S T* **and**
inv S **and**
atms-of-mm (*clauses*$_{NOT}$ *S*) $\subseteq$ *atms-of-ms A* **and**
atm-of '(*lits-of-l* (*trail S*)) $\subseteq$ *atms-of-ms A* **and**
n-d: *no-dup* (*trail S*) **and**
finite: *finite A*
shows μ_{CDCL}*'-merged A T* $\leq$ μ_{CDCL}*'-bound A S*
proof –
have *set-mset* (*clauses*$_{NOT}$ *T*) $\subseteq$ *set-mset* (*not-simplified-cls*(*clauses*$_{NOT}$ *S*))
$\cup$ *simple-clss* (*atms-of-ms A*)
using *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-clauses-bound*[*OF assms*] **.**
moreover have *card* (*set-mset* (*not-simplified-cls*(*clauses*$_{NOT}$ *S*))
$\cup$ *simple-clss* (*atms-of-ms A*))
$\leq$ *card* (*set-mset* (*not-simplified-cls*(*clauses*$_{NOT}$ *S*))) + *3* ^ *card* (*atms-of-ms A*)
by (*meson Nat.le-trans atms-of-ms-finite simple-clss-card card-Un-le finite*
nat-add-left-cancel-le)
ultimately have *card* (*set-mset* (*clauses*$_{NOT}$ *T*))
$\leq$ *card* (*set-mset* (*not-simplified-cls*(*clauses*$_{NOT}$ *S*))) + *3* ^ *card* (*atms-of-ms A*)
by (*meson Nat.le-trans atms-of-ms-finite simple-clss-finite card-mono*
finite-UnI finite-set-mset local.finite)
moreover have ((*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*)) – μ_C*' A T*) $*$ *2*
$\leq$ (*2* + *card* (*atms-of-ms A*)) ^ (*1* + *card* (*atms-of-ms A*)) $*$ *2*
by *auto*
ultimately show *?thesis* **unfolding** μ_{CDCL}*'-merged-def* **by** *auto*
qed

sublocale *cdcl*$_{NOT}$*-increasing-restarts-ops* λ*S T. T* $\sim$ *reduce-trail-to*$_{NOT}$ ([]::*'a list*) *S*
cdcl$_{NOT}$*-merged-bj-learn f*
λ*A S. atms-of-mm* (*clauses*$_{NOT}$ *S*) $\subseteq$ *atms-of-ms A*
$\wedge$ *atm-of* ' *lits-of-l* (*trail S*) $\subseteq$ *atms-of-ms A* $\wedge$ *finite A*
μ_{CDCL}*'-merged*
λ*S. inv S* $\wedge$ *no-dup* (*trail S*)
μ_{CDCL}*'-bound*
apply *unfold-locales*
using *unbounded* **apply** *simp*
using *f-ge-1* **apply** *force*
apply (*blast dest*!: *cdcl*$_{NOT}$*-merged-bj-learn-is-tranclp-cdcl*$_{NOT}$ *tranclp-into-rtranclp*
rtranclp-cdcl$_{NOT}$*-trail-clauses-bound*)
apply (*simp add*: *cdcl*$_{NOT}$*-decreasing-measure'*)
using *rtranclp-cdcl*$_{NOT}$*-merged-bj-learn-clauses-bound-card* **apply** *blast*

```
        apply (drule rtranclp-cdcl$_{NOT}$-merged-bj-learn-not-simplified-decreasing)
        apply (auto dest!: simp: card-mono set-mset-mono )[]
      apply simp
     apply auto[]
    using cdcl$_{NOT}$-merged-bj-learn-no-dup-inv cdcl-merged-inv apply blast
   apply (auto simp: inv-restart)[]
   done

lemma cdcl$_{NOT}$-restart-$\mu_{CDCL}$'-merged-le-$\mu_{CDCL}$'-bound:
  assumes
   cdcl$_{NOT}$-restart T V
   inv (fst T) and
   no-dup (trail (fst T)) and
   atms-of-mm (clauses$_{NOT}$ (fst T)) ⊆ atms-of-ms A and
   atm-of ` lits-of-l (trail (fst T)) ⊆ atms-of-ms A and
   finite A
  shows $\mu_{CDCL}$'-merged A (fst V) ≤ $\mu_{CDCL}$'-bound A (fst T)
  using assms
proof induction
  case (restart-full S T n)
  show ?case
   unfolding fst-conv
   apply (rule rtranclp-cdcl$_{NOT}$-merged-bj-learn-clauses-bound-card)
   using  restart-full unfolding full1-def by (force dest!: tranclp-into-rtranclp)+
next
  case (restart-step m S T n U) note st = this(1) and U = this(3) and inv = this(4) and
   n-d = this(5) and atms-clss = this(6) and atms-trail = this(7) and finite = this(8)
  then have st': cdcl$_{NOT}$-merged-bj-learn$^{**}$ S T
   by (blast dest: relpowp-imp-rtranclp)
  then have st'': cdcl$_{NOT}$$^{**}$ S T
   using inv n-d apply - by (rule rtranclp-cdcl$_{NOT}$-merged-bj-learn-is-rtranclp-cdcl$_{NOT}$) auto
  have inv T
   apply (rule rtranclp-cdcl$_{NOT}$-merged-bj-learn-inv)
     using inv st' n-d by auto
  then have inv U
   using U by (auto simp: inv-restart)
  have atms-of-mm (clauses$_{NOT}$ T) ⊆ atms-of-ms A
   using rtranclp-cdcl$_{NOT}$-trail-clauses-bound[OF st''] inv atms-clss atms-trail n-d
   by simp
  then have atms-of-mm (clauses$_{NOT}$ U) ⊆ atms-of-ms A
   using U by simp
  have not-simplified-cls (clauses$_{NOT}$ U) ⊆# not-simplified-cls (clauses$_{NOT}$ T)
   using ‹U ∼ reduce-trail-to$_{NOT}$ [] T› by auto
  moreover have not-simplified-cls (clauses$_{NOT}$ T) ⊆# not-simplified-cls (clauses$_{NOT}$ S)
   apply (rule rtranclp-cdcl$_{NOT}$-merged-bj-learn-not-simplified-decreasing)
   using ‹(cdcl$_{NOT}$-merged-bj-learn ^^ m) S T› by (auto dest!: relpowp-imp-rtranclp)
  ultimately have U-S: not-simplified-cls (clauses$_{NOT}$ U) ⊆# not-simplified-cls (clauses$_{NOT}$ S)
   by auto

  have (set-mset (clauses$_{NOT}$ U))
   ⊆ set-mset (not-simplified-cls (clauses$_{NOT}$ U)) ∪ simple-clss (atms-of-ms A)
   apply (rule rtranclp-cdcl$_{NOT}$-merged-bj-learn-clauses-bound)
       apply simp
      using ‹inv U› apply simp
     using ‹atms-of-mm (clauses$_{NOT}$ U) ⊆ atms-of-ms A› apply simp
```

using *U* **apply** *simp*
using *U* **apply** *simp*
using *finite* **apply** *simp*
done
then have *f1*: *card* (*set-mset* ($clauses_{NOT}$ *U*)) ≤ *card* (*set-mset* (*not-simplified-cls* ($clauses_{NOT}$ *U*))
∪ *simple-clss* (*atms-of-ms* *A*))
by (*simp add*: *simple-clss-finite card-mono local.finite*)

moreover have *set-mset* (*not-simplified-cls* ($clauses_{NOT}$ *U*)) ∪ *simple-clss* (*atms-of-ms* *A*)
⊆ *set-mset* (*not-simplified-cls* ($clauses_{NOT}$ *S*)) ∪ *simple-clss* (*atms-of-ms* *A*)
using *U-S* **by** *auto*
then have *f2*:
card (*set-mset* (*not-simplified-cls* ($clauses_{NOT}$ *U*)) ∪ *simple-clss* (*atms-of-ms* *A*))
≤ *card* (*set-mset* (*not-simplified-cls* ($clauses_{NOT}$ *S*)) ∪ *simple-clss* (*atms-of-ms* *A*))
by (*simp add*: *simple-clss-finite card-mono local.finite*)

moreover have *card* (*set-mset* (*not-simplified-cls* ($clauses_{NOT}$ *S*))
∪ *simple-clss* (*atms-of-ms* *A*))
≤ *card* (*set-mset* (*not-simplified-cls* ($clauses_{NOT}$ *S*))) + *card* (*simple-clss* (*atms-of-ms* *A*))
using *card-Un-le* **by** *blast*
moreover have *card* (*simple-clss* (*atms-of-ms* *A*)) ≤ *3* ^ *card* (*atms-of-ms* *A*)
using *atms-of-ms-finite simple-clss-card local.finite* **by** *blast*
ultimately have *card* (*set-mset* ($clauses_{NOT}$ *U*))
≤ *card* (*set-mset* (*not-simplified-cls* ($clauses_{NOT}$ *S*))) + *3* ^ *card* (*atms-of-ms* *A*)
by *linarith*
then show *?case* **unfolding** μ_{CDCL}*'-merged-def* **by** *auto*
qed

lemma $cdcl_{NOT}$*-restart-*μ_{CDCL}*'-bound-le-*μ_{CDCL}*'-bound*:
assumes
$cdcl_{NOT}$*-restart* *T V* **and**
no-dup (*trail* (*fst* *T*)) **and**
inv (*fst* *T*) **and**
fin: *finite* *A*
shows μ_{CDCL}*'-bound* *A* (*fst* *V*) ≤ μ_{CDCL}*'-bound* *A* (*fst* *T*)
using *assms*(*1*−*3*)
proof *induction*
case (*restart-full* *S T n*)
have *not-simplified-cls* ($clauses_{NOT}$ *T*) ⊆# *not-simplified-cls* ($clauses_{NOT}$ *S*)
apply (*rule rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-not-simplified-decreasing*)
using ‹*full1* $cdcl_{NOT}$*-merged-bj-learn* *S T*› **unfolding** *full1-def*
by (*auto dest*: *tranclp-into-rtranclp*)
then show *?case* **by** (*auto simp*: *card-mono set-mset-mono*)
next
case (*restart-step* *m S T n U*) **note** *st* = *this*(*1*) **and** *U* = *this*(*3*) **and** *n-d* = *this*(*4*) **and**
inv = *this*(*5*)
then have *st'*: $cdcl_{NOT}$*-merged-bj-learn*** *S T*
by (*blast dest*: *relpowp-imp-rtranclp*)
then have *st''*: $cdcl_{NOT}{}^{**}$ *S T*
using *inv n-d* **apply** − **by** (*rule rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-is-rtranclp-*$cdcl_{NOT}$) *auto*
have *inv T*
apply (*rule rtranclp-*$cdcl_{NOT}$*-merged-bj-learn-inv*)
using *inv st' n-d* **by** *auto*
then have *inv U*
using *U* **by** (*auto simp*: *inv-restart*)

```
  have not-simplified-cls (clauses_NOT U) ⊆# not-simplified-cls (clauses_NOT T)
    using ‹U ∼ reduce-trail-to_NOT [] T› by auto
  moreover have not-simplified-cls (clauses_NOT T) ⊆# not-simplified-cls (clauses_NOT S)
    apply (rule rtranclp-cdcl_NOT-merged-bj-learn-not-simplified-decreasing)
    using ‹(cdcl_NOT-merged-bj-learn ^^ m) S T› by (auto dest!: relpowp-imp-rtranclp)
  ultimately have U-S: not-simplified-cls (clauses_NOT U) ⊆# not-simplified-cls (clauses_NOT S)
    by auto
  then show ?case by (auto simp: card-mono set-mset-mono)
qed


sublocale cdcl_NOT-increasing-restarts - - - - - - - - - - - - - - f
   λS T. T ∼ reduce-trail-to_NOT ([]::'a list) S
   λA S. atms-of-mm (clauses_NOT S) ⊆ atms-of-ms A
     ∧ atm-of ` lits-of-l (trail S) ⊆ atms-of-ms A ∧ finite A
   μ_CDCL'-merged cdcl_NOT-merged-bj-learn
    λS. inv S ∧ no-dup (trail S)
   λA T. ((2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A))) * 2
     + card (set-mset (not-simplified-cls(clauses_NOT T)))
     + 3 ^ card (atms-of-ms A)
   apply unfold-locales
     using cdcl_NOT-restart-μ_CDCL'-merged-le-μ_CDCL'-bound apply force
    using cdcl_NOT-restart-μ_CDCL'-bound-le-μ_CDCL'-bound by fastforce

lemma cdcl_NOT-restart-eq-sat-iff:
  assumes
    cdcl_NOT-restart S T and
    no-dup (trail (fst S))
    inv (fst S)
  shows I⊨sextm clauses_NOT (fst S) ⟷ I ⊨sextm clauses_NOT (fst T)
  using assms
proof (induction rule: cdcl_NOT-restart.induct)
  case (restart-full S T n)
  then have cdcl_NOT-merged-bj-learn** S T
    by (simp add: tranclp-into-rtranclp full1-def)
  then show ?case
    using rtranclp-cdcl_NOT-bj-sat-ext-iff restart-full.prems(1,2)
    rtranclp-cdcl_NOT-merged-bj-learn-is-rtranclp-cdcl_NOT by auto
next
  case (restart-step m S T n U)
  then have cdcl_NOT-merged-bj-learn** S T
    by (auto simp: tranclp-into-rtranclp full1-def dest!: relpowp-imp-rtranclp)
  then have I ⊨sextm clauses_NOT S ⟷ I ⊨sextm clauses_NOT T
    using rtranclp-cdcl_NOT-bj-sat-ext-iff restart-step.prems(1,2)
    rtranclp-cdcl_NOT-merged-bj-learn-is-rtranclp-cdcl_NOT by auto
  moreover have I ⊨sextm clauses_NOT T ⟷ I ⊨sextm clauses_NOT U
    using restart-step.hyps(3) by auto
  ultimately show ?case by auto
qed

lemma rtranclp-cdcl_NOT-restart-eq-sat-iff:
  assumes
    cdcl_NOT-restart** S T and
    inv: inv (fst S) and n-d: no-dup(trail (fst S))
  shows I⊨sextm clauses_NOT (fst S) ⟷ I ⊨sextm clauses_NOT (fst T)
  using assms(1)
proof (induction rule: rtranclp-induct)
  case base
  then show ?case by simp
next
  case (step T U) note st = this(1) and cdcl = this(2) and IH = this(3)
  have inv (fst T) and no-dup (trail (fst T))
    using rtranclp-cdcl_NOT-with-restart-cdcl_NOT-inv using st inv n-d by blast+
  then have I|=sextm clauses_NOT (fst T) ⟷ I |=sextm clauses_NOT (fst U)
    using  cdcl_NOT-restart-eq-sat-iff cdcl by blast
  then show ?case using IH by blast
qed

lemma cdcl_NOT-restart-all-decomposition-implies-m:
  assumes
    cdcl_NOT-restart S T and
    inv: inv (fst S) and n-d: no-dup(trail (fst S)) and
    all-decomposition-implies-m (clauses_NOT (fst S))
      (get-all-marked-decomposition (trail (fst S)))
  shows all-decomposition-implies-m (clauses_NOT (fst T))
      (get-all-marked-decomposition (trail (fst T)))
  using assms
proof (induction)
  case (restart-full S T n) note full = this(1) and inv = this(2) and n-d = this(3) and
    decomp = this(4)
  have st: cdcl_NOT-merged-bj-learn** S T and
    n-s: no-step cdcl_NOT-merged-bj-learn T
    using full unfolding full1-def by (fast dest: tranclp-into-rtranclp)+
  have st': cdcl_NOT** S T
    using inv rtranclp-cdcl_NOT-merged-bj-learn-is-rtranclp-cdcl_NOT-and-inv st n-d by auto
  have inv T
    using rtranclp-cdcl_NOT-cdcl_NOT-inv[OF st] inv n-d by auto
  then show ?case
    using rtranclp-cdcl_NOT-all-decomposition-implies[OF -  - n-d decomp] st' inv by auto
next
  case (restart-step m S T n U) note st = this(1) and U = this(3) and inv = this(4) and
    n-d = this(5) and decomp = this(6)
  show ?case using U by auto
qed

lemma rtranclp-cdcl_NOT-restart-all-decomposition-implies-m:
  assumes
    cdcl_NOT-restart** S T and
    inv: inv (fst S) and n-d: no-dup(trail (fst S)) and
    decomp: all-decomposition-implies-m (clauses_NOT (fst S))
      (get-all-marked-decomposition (trail (fst S)))
  shows all-decomposition-implies-m (clauses_NOT (fst T))
      (get-all-marked-decomposition (trail (fst T)))
  using assms
proof (induction)
  case base
  then show ?case using decomp by simp
next
  case (step T U) note st = this(1) and cdcl = this(2) and IH = this(3)[OF this(4−)] and
    inv = this(4) and n-d = this(5) and decomp = this(6)
  have inv (fst T) and no-dup (trail (fst T))
    using rtranclp-cdcl_NOT-with-restart-cdcl_NOT-inv using st inv n-d by blast+
  then show ?case
    using cdcl_NOT-restart-all-decomposition-implies-m[OF cdcl] IH by auto
qed

lemma full-cdcl_NOT-restart-normal-form:
  assumes
    full: full cdcl_NOT-restart S T and
    inv: inv (fst S) and n-d: no-dup(trail (fst S)) and
    decomp: all-decomposition-implies-m (clauses_NOT (fst S))
      (get-all-marked-decomposition (trail (fst S))) and
    atms-cls: atms-of-mm (clauses_NOT (fst S)) ⊆ atms-of-ms A and
    atms-trail: atm-of ‘ lits-of-l (trail (fst S)) ⊆ atms-of-ms A and
    fin: finite A
  shows unsatisfiable (set-mset (clauses_NOT (fst S)))
    ∨ lits-of-l (trail (fst T)) ⊨sextm clauses_NOT (fst S) ∧ satisfiable (set-mset (clauses_NOT (fst S)))
proof –
  have inv-T: inv (fst T) and n-d-T: no-dup (trail (fst T))
    using rtranclp-cdcl_NOT-with-restart-cdcl_NOT-inv using full inv n-d unfolding full-def by blast+
  moreover have
    atms-cls-T: atms-of-mm (clauses_NOT (fst T)) ⊆ atms-of-ms A and
    atms-trail-T: atm-of ‘ lits-of-l (trail (fst T)) ⊆ atms-of-ms A
    using rtranclp-cdcl_NOT-with-restart-bound-inv[of S T A] full atms-cls atms-trail fin inv n-d
    unfolding full-def by blast+
  ultimately have no-step cdcl_NOT-merged-bj-learn (fst T)
    apply –
    apply (rule no-step-cdcl_NOT-restart-no-step-cdcl_NOT[of - A])
       using full unfolding full-def apply simp
      apply simp
    using fin apply simp
    done
  moreover have all-decomposition-implies-m (clauses_NOT (fst T))
    (get-all-marked-decomposition (trail (fst T)))
    using rtranclp-cdcl_NOT-restart-all-decomposition-implies-m[of S T] inv n-d decomp
    full unfolding full-def by auto
  ultimately have unsatisfiable (set-mset (clauses_NOT (fst T)))
    ∨ trail (fst T) ⊨asm clauses_NOT (fst T) ∧ satisfiable (set-mset (clauses_NOT (fst T)))
    apply –
    apply (rule cdcl_NOT-merged-bj-learn-final-state)
    using atms-cls-T atms-trail-T fin n-d-T fin inv-T by blast+
  then consider
      (unsat) unsatisfiable (set-mset (clauses_NOT (fst T)))
    | (sat) trail (fst T) ⊨asm clauses_NOT (fst T) and satisfiable (set-mset (clauses_NOT (fst T)))
    by auto
  then show unsatisfiable (set-mset (clauses_NOT (fst S)))
    ∨ lits-of-l (trail (fst T)) ⊨sextm clauses_NOT (fst S) ∧ satisfiable (set-mset (clauses_NOT (fst S)))
    proof cases
      case unsat
      then have unsatisfiable (set-mset (clauses_NOT (fst S)))
        unfolding satisfiable-def apply auto
        using rtranclp-cdcl_NOT-restart-eq-sat-iff[of S T ] full inv n-d
        consistent-true-clss-ext-satisfiable true-clss-imp-true-cls-ext
        unfolding satisfiable-def full-def by blast
      then show ?thesis by blast
    next
      case sat
      then have lits-of-l (trail (fst T)) ⊨sextm clauses_NOT (fst T)
        using true-clss-imp-true-cls-ext by (auto simp: true-annots-true-cls)
      then have lits-of-l (trail (fst T)) ⊨sextm clauses_NOT (fst S)
        using rtranclp-cdcl_NOT-restart-eq-sat-iff[of S T] full inv n-d unfolding full-def by blast
      moreover then have satisfiable (set-mset (clauses_NOT (fst S)))
        using consistent-true-clss-ext-satisfiable distinct-consistent-interp n-d-T by fast
      ultimately show ?thesis by fast
    qed
qed

corollary full-cdcl_NOT-restart-normal-form-init-state:
  assumes
    init-state: trail S = [] clauses_NOT S = N and
    full: full cdcl_NOT-restart (S, 0) T and
    inv: inv S
  shows unsatisfiable (set-mset N)
    ∨ lits-of-l (trail (fst T)) ⊨sextm N ∧ satisfiable (set-mset N)
  using  full-cdcl_NOT-restart-normal-form[of (S, 0) T] assms by auto

end

end
theory DPLL-NOT
imports CDCL-NOT
begin
```

17 DPLL as an instance of NOT

17.1 DPLL with simple backtrack

We are using a concrete couple instead of an abstract state.

```
locale dpll-with-backtrack
begin
inductive backtrack :: ('v, unit, unit) marked-lit list × 'v clauses
  ⇒ ('v, unit, unit) marked-lit list × 'v clauses ⇒ bool where
backtrack-split (fst S)  = (M', L # M) ⟹ is-marked L ⟹ D ∈# snd S
  ⟹ fst S ⊨as CNot D ⟹ backtrack S (Propagated (− (lit-of L)) () # M, snd S)

inductive-cases backtrackE[elim]: backtrack (M, N) (M', N')
lemma backtrack-is-backjump:
  fixes M M' :: ('v, unit, unit) marked-lit list
  assumes
    backtrack: backtrack (M, N) (M', N') and
    no-dup: (no-dup ∘ fst) (M, N) and
    decomp: all-decomposition-implies-m N (get-all-marked-decomposition M)
    shows
      ∃ C F' K F L l C'.
        M = F' @ Marked K () # F ∧
        M' = Propagated L l # F ∧ N = N' ∧ C ∈# N ∧ F' @ Marked K d # F ⊨as CNot C ∧
        undefined-lit F L ∧ atm-of L ∈ atms-of-mm N ∪ atm-of ‘ lits-of-l (F' @ Marked K d # F) ∧
        N ⊨pm C' + {#L#} ∧ F ⊨as CNot C'
proof −
  let ?S = (M, N)
let ?T = (M', N')
  obtain F F' P L D where
    b-sp: backtrack-split M = (F', L # F)  and
    is-marked L and
    D ∈# snd ?S and
    M ⊨as CNot D and
    bt: backtrack ?S (Propagated (− (lit-of L)) P # F, N) and
    M': M' = Propagated (− (lit-of L)) P # F and
    [simp]: N' = N
  using backtrackE[OF backtrack] by (metis backtrack fstI sndI)
  let ?K = lit-of L
  let ?C = image-mset lit-of {#K∈#mset M. is-marked K ∧ K≠L#} :: 'v literal multiset
  let ?C' = set-mset (image-mset single (?C+{#?K#}))
  obtain K where L: L = Marked K () using ‹is-marked L› by (cases L) auto

  have M: M = F' @ Marked K () # F
    using b-sp  by (metis L backtrack-split-list-eq fst-conv snd-conv)
  moreover have F' @ Marked K () # F ⊨as CNot D
    using ‹M⊨as CNot D› unfolding M .
  moreover have undefined-lit F (−?K)
    using no-dup unfolding M L by (simp add: defined-lit-map)
  moreover have atm-of (−K) ∈ atms-of-mm N ∪ atm-of ‘ lits-of-l (F' @ Marked K d # F)
    by auto
  moreover
    have set-mset N ∪ ?C' ⊨ps {{#}}
      proof −
        have A: set-mset N ∪ ?C' ∪ unmark-l M  =
        set-mset  N ∪ unmark-l M
        unfolding M L by auto
        have set-mset  N ∪ {{#lit-of L#} |L. is-marked L ∧ L ∈ set M}
            ⊨ps unmark-l M
        using all-decomposition-implies-propagated-lits-are-implied[OF decomp] .
        moreover have C': ?C' = {{#lit-of L#} |L. is-marked L ∧ L ∈ set M}
        unfolding M L apply standard
            apply force
        using IntI by auto
        ultimately have N-C-M: set-mset N ∪ ?C' ⊨ps unmark-l M
        by auto
        have set-mset N ∪ (λL. {#lit-of L#}) ‘ (set M) ⊨ps {{#}}
        unfolding true-clss-clss-def
        proof (intro allI impI, goal-cases)
            case (1 I) note tot = this(1) and cons = this(2) and I-N-M = this(3)
            have I ⊨ D
              using I-N-M ‹D ∈# snd ?S› unfolding true-clss-def by auto
            moreover have I ⊨s CNot D
              using ‹M ⊨as CNot D› unfolding M by (metis 1(3) ‹M ⊨as CNot D›
                true-annots-true-cls true-cls-mono-set-mset-l true-clss-def
                true-clss-singleton-lit-of-implies-incl true-clss-union)
            ultimately show ?case using cons consistent-CNot-not by blast
        qed
        then show ?thesis
        using true-clss-clss-left-right[OF N-C-M, of {{#}}] unfolding A by auto
      qed
    have N ⊨pm image-mset uminus ?C + {#−?K#}
      unfolding true-clss-cls-def true-clss-clss-def total-over-m-def
```

```
      proof (intro allI impI)
        fix I
        assume
          tot: total-over-set I (atms-of-ms (set-mset N ∪ {image-mset uminus ?C + {#− ?K#}})) and
           cons: consistent-interp I and
           I ⊨sm N
        have (K ∈ I ∧ −K ∉ I) ∨ (−K ∈ I ∧ K ∉ I)
          using cons tot unfolding consistent-interp-def L by (cases K) auto
        have {a ∈ set M. is-marked a ∧ a ≠ Marked K ()} =
          set M ∩ {L. is-marked L ∧ L ≠ Marked K ()}
          by auto
        then have
          tI: total-over-set I (atm-of ‘ lit-of ‘ (set M ∩ {L. is-marked L ∧ L ≠ Marked K d}))
          using tot by (auto simp add: L atms-of-uminus-lit-atm-of-lit-of)

        then have H: ⋀x.
            lit-of x ∉ I ⟹ x ∈ set M ⟹is-marked x
            ⟹ x ≠ Marked K d ⟹ −lit-of x ∈ I
          proof −
            fix x :: ('v, unit, unit) marked-lit
            assume a1: x ≠ Marked K d
            assume a2: is-marked x
            assume a3: x ∈ set M
            assume a4: lit-of x ∉ I
            have atm-of (lit-of x) ∈ atm-of ‘ lit-of ‘
              (set M ∩ {m. is-marked m ∧ m ≠ Marked K d})
              using a3 a2 a1 by blast
            then have Pos (atm-of (lit-of x)) ∈ I ∨ Neg (atm-of (lit-of x)) ∈ I
              using tI unfolding total-over-set-def by blast
            then show − lit-of x ∈ I
              using a4 by (metis (no-types) atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
                literal.sel(1,2))
          qed
        have ¬I ⊨s ?C'
          using ‹set-mset N ∪ ?C' ⊨ps {{#}}› tot cons ‹I ⊨sm N›
          unfolding true-clss-clss-def total-over-m-def
          by (simp add: atms-of-uminus-lit-atm-of-lit-of atms-of-ms-single-image-atm-of-lit-of)
        then show I ⊨ image-mset uminus ?C + {#− lit-of L#}
          unfolding true-clss-def true-cls-def
          using ‹(K ∈ I ∧ −K ∉ I) ∨ (−K ∈ I ∧ K ∉ I)›
          unfolding L by (auto dest!: H)
      qed
  moreover
    have set F' ∩ {K. is-marked K ∧ K ≠ L} = {}
      using backtrack-split-fst-not-marked[of - M] b-sp by auto
    then have F ⊨as CNot (image-mset uminus ?C)
       unfolding M CNot-def true-annots-def by (auto simp add: L lits-of-def)
  ultimately show ?thesis
    using M' ‹D ∈# snd ?S› L by force
qed

lemma backtrack-is-backjump':
  fixes M M' :: ('v, unit, unit) marked-lit list
  assumes
    backtrack: backtrack S T and
```

no-dup: (*no-dup* ∘ *fst*) *S* **and**
decomp: *all-decomposition-implies-m* (*snd S*) (*get-all-marked-decomposition* (*fst S*))
shows
∃ *C F′ K F L l C′*.
fst S = *F′* @ *Marked K* () # *F* ∧
T = (*Propagated L l* # *F*, *snd S*) ∧ *C* ∈# *snd S* ∧ *fst S* ⊨*as CNot C*
∧ *undefined-lit F L* ∧ *atm-of L* ∈ *atms-of-mm* (*snd S*) ∪ *atm-of* ‘ *lits-of-l* (*fst S*) ∧
snd S ⊨*pm C′* + {#*L*#} ∧ *F* ⊨*as CNot C′*
apply (*cases S*, *cases T*)
using *backtrack-is-backjump*[*of fst S snd S fst T snd T*] *assms* **by** *fastforce*

sublocale *dpll-state*
id λ*L C*. *C* + {#*L*#} *remove1-mset*
id *op* + *op* ∈# λ*L C*. *C* + {#*L*#} *remove1-mset*
fst snd λ*L* (*M*, *N*). (*L* # *M*, *N*) λ(*M*, *N*). (*tl M*, *N*)
λ*C* (*M*, *N*). (*M*, {#*C*#} + *N*) λ*C* (*M*, *N*). (*M*, *removeAll-mset C N*)
by *unfold-locales* (*auto simp*: *ac-simps*)

sublocale *backjumping-ops*
id λ*L C*. *C* + {#*L*#} *remove1-mset*
id *op* + *op* ∈# λ*L C*. *C* + {#*L*#} *remove1-mset*
fst snd λ*L* (*M*, *N*). (*L* # *M*, *N*) λ(*M*, *N*). (*tl M*, *N*)
λ*C* (*M*, *N*). (*M*, {#*C*#} + *N*) λ*C* (*M*, *N*). (*M*, *removeAll-mset C N*) λ- - - *S T*. *backtrack S T*
by *unfold-locales*

lemma *reduce-trail-to*$_{NOT}$*-snd*:
snd (*reduce-trail-to*$_{NOT}$ *F S*) = *snd S*
apply (*induction F S rule*: *reduce-trail-to*$_{NOT}$*.induct*)
by (*cases S*, *rename-tac F Sa*, *case-tac Sa*)
(*simp add*: *less-imp-diff-less reduce-trail-to*$_{NOT}$*.simps*)

lemma *reduce-trail-to*$_{NOT}$:
reduce-trail-to$_{NOT}$ *F S* =
(*if length* (*fst S*) ≥ *length F*
then drop (*length* (*fst S*) − *length F*) (*fst S*)
else [],
snd S) (**is** *?R* = *?C*)
proof −
have *?R* = (*fst ?R*, *snd ?R*)
by *auto*
also have (*fst ?R*, *snd ?R*) = *?C*
by (*auto simp*: *trail-reduce-trail-to*$_{NOT}$*-drop reduce-trail-to*$_{NOT}$*-snd*)
finally show *?thesis* **.**
qed

lemma *backtrack-is-backjump′′*:
fixes *M M′* :: (*′v*, *unit*, *unit*) *marked-lit list*
assumes
backtrack: *backtrack S T* **and**
no-dup: (*no-dup* ∘ *fst*) *S* **and**
decomp: *all-decomposition-implies-m* (*snd S*) (*get-all-marked-decomposition* (*fst S*))
shows *backjump S T*
proof −
obtain *C F′ K F L l C′* **where**
1: *fst S* = *F′* @ *Marked K* () # *F* **and**

```
    2: T = (Propagated L l # F, snd S) and
    3: C ∈# snd S and
    4: fst S ⊨as CNot C and
    5: undefined-lit F L and
    6: atm-of L ∈ atms-of-mm (snd S) ∪ atm-of ‘ lits-of-l (fst S) and
    7: snd S ⊨pm C′ + {#L#} and
    8: F ⊨as CNot C′
  using backtrack-is-backjump′[OF assms] by force
  show ?thesis
    apply (cases S)
    using backjump.intros[OF 1 - - 4 5 - - 8, of T] 2 backtrack 1 5 3 6 7
    by (auto simp: state-eq_NOT-def trail-reduce-trail-to_NOT-drop
      reduce-trail-to_NOT simp del: state-simp_NOT)
qed

lemma can-do-bt-step:
   assumes
     M: fst S = F′ @ Marked K d # F and
     C ∈# snd S and
     C: fst S ⊨as CNot C
   shows ¬ no-step backtrack S
proof −
  obtain L G′ G where
    backtrack-split (fst S) = (G′, L # G)
    unfolding M by (induction F′ rule: marked-lit-list-induct) auto
  moreover then have is-marked L
     by (metis backtrack-split-snd-hd-marked list.distinct(1) list.sel(1) snd-conv)
  ultimately show ?thesis
     using backtrack.intros[of S G′ L G C] ‹C ∈# snd S› C unfolding M by auto
qed

end

sublocale dpll-with-backtrack ⊆ dpll-with-backjumping-ops
    id λL C. C + {#L#} remove1-mset
    id op + op ∈# λL C. C + {#L#} remove1-mset
    fst snd λL (M, N). (L # M, N)
  λ(M, N). (tl M, N) λC (M, N). (M, {#C#} + N) λC (M, N). (M, removeAll-mset C N)
  λ(M, N). no-dup M ∧ all-decomposition-implies-m N (get-all-marked-decomposition M)
  λ- - - S T. backtrack S T
  λ- -. True
  apply unfold-locales
  by (metis (mono-tags, lifting) case-prod-beta comp-def dpll-with-backtrack.backtrack-is-backjump′′
    dpll-with-backtrack.can-do-bt-step id-apply)

sublocale dpll-with-backtrack ⊆ dpll-with-backjumping
    id λL C. C + {#L#} remove1-mset
    id op + op ∈# λL C. C + {#L#} remove1-mset
    fst snd λL (M, N). (L # M, N)
  λ(M, N). (tl M, N) λC (M, N). (M, {#C#} + N) λC (M, N). (M, removeAll-mset C N)
  λ(M, N). no-dup M ∧ all-decomposition-implies-m N (get-all-marked-decomposition M)
  λ- - - S T. backtrack S T
  λ- -. True
  apply unfold-locales
  using dpll-bj-no-dup dpll-bj-all-decomposition-implies-inv apply fastforce
  done

context dpll-with-backtrack
begin
term learn
end


context dpll-with-backtrack
begin
lemma wf-tranclp-dpll-inital-state:
  assumes fin: finite A
  shows wf {((M'::('v, unit, unit) marked-lits, N'::'v clauses), ([], N))|M' N' N.
    dpll-bj⁺⁺ ([], N) (M', N') ∧ atms-of-mm N ⊆ atms-of-ms A}
  using wf-tranclp-dpll-bj[OF assms(1)] by (rule wf-subset) auto

corollary full-dpll-final-state-conclusive:
  fixes M M' :: ('v, unit, unit) marked-lit list
  assumes
    full: full dpll-bj ([], N) (M', N')
  shows unsatisfiable (set-mset N) ∨ (M' ⊨asm N ∧ satisfiable (set-mset N))
  using assms full-dpll-backjump-final-state[of ([],N) (M', N') set-mset N] by auto

corollary full-dpll-normal-form-from-init-state:
  fixes M M' :: ('v, unit, unit) marked-lit list
  assumes
    full: full dpll-bj ([], N) (M', N')
  shows M' ⊨asm N ⟷ satisfiable (set-mset N)
proof −
  have no-dup M'
    using rtranclp-dpll-bj-no-dup[of ([], N) (M', N')]
    full unfolding full-def by auto
  then have M' ⊨asm N ⟹ satisfiable (set-mset N)
    using distinct-consistent-interp satisfiable-carac' true-annots-true-cls by blast
  then show ?thesis
  using full-dpll-final-state-conclusive[OF full] by auto
qed

interpretation conflict-driven-clause-learning-ops
    id λL C. C + {#L#} remove1-mset
    id op + op ∈# λL C. C + {#L#} remove1-mset
    fst snd λL (M, N). (L # M, N)
  λ(M, N). (tl M, N) λC (M, N). (M, {#C#} + N) λC (M, N). (M, removeAll-mset C N)
  λ(M, N). no-dup M ∧ all-decomposition-implies-m N (get-all-marked-decomposition M)
  λ- - - S T. backtrack S T
  λ- -. True λ- -. False λ- -. False
  by unfold-locales

interpretation conflict-driven-clause-learning
    id λL C. C + {#L#} remove1-mset
    id op + op ∈# λL C. C + {#L#} remove1-mset
    fst snd λL (M, N). (L # M, N)
  λ(M, N). (tl M, N) λC (M, N). (M, {#C#} + N) λC (M, N). (M, removeAll-mset C N)
  λ(M, N). no-dup M ∧ all-decomposition-implies-m N (get-all-marked-decomposition M)
  λ- - - S T. backtrack S T
  λ- -. True λ- -. False λ- -. False
  apply unfold-locales
  using cdcl_NOT-all-decomposition-implies cdcl_NOT-no-dup by fastforce

lemma cdcl_NOT-is-dpll:
  cdcl_NOT S T ⟷ dpll-bj S T
  by (auto simp: cdcl_NOT.simps learn.simps forget_NOT.simps)
```

Another proof of termination:

```
lemma wf {(T, S). dpll-bj S T ∧ cdcl_NOT-NOT-all-inv A S}
  unfolding cdcl_NOT-is-dpll[symmetric]
  by (rule wf-cdcl_NOT-no-learn-and-forget-infinite-chain)
  (auto simp: learn.simps forget_NOT.simps)
end
```

17.2 Adding restarts

This was mainly a test whether it was possible to instantiate the assumption of the locale.

```
locale dpll-withbacktrack-and-restarts =
  dpll-with-backtrack +
  fixes f :: nat ⇒ nat
  assumes unbounded: unbounded f and f-ge-1:⋀n. n≥ 1 ⟹ f n ≥ 1
begin
  sublocale cdcl_NOT-increasing-restarts
    id λL C. C + {#L#} remove1-mset
    id op + op ∈# λL C. C + {#L#} remove1-mset
  fst snd λL (M, N). (L # M, N) λ(M, N). (tl M, N)
    λC (M, N). (M, {#C#} + N) λC (M, N). (M, removeAll-mset C N) f λ(-, N) S. S = ([], N)
  λA (M, N). atms-of-mm N ⊆ atms-of-ms A ∧ atm-of ` lits-of-l M ⊆ atms-of-ms A ∧ finite A
    ∧ all-decomposition-implies-m N (get-all-marked-decomposition M)
  λA T. (2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A))
              − μ_C (1+card (atms-of-ms A)) (2+card (atms-of-ms A)) (trail-weight T) dpll-bj
  λ(M, N). no-dup M ∧ all-decomposition-implies-m N (get-all-marked-decomposition M)
  λA -.  (2+card (atms-of-ms A)) ^ (1+card (atms-of-ms A))
  apply unfold-locales
          apply (rule unbounded)
         using f-ge-1 apply fastforce
        apply (smt dpll-bj-all-decomposition-implies-inv dpll-bj-atms-in-trail-in-set
          dpll-bj-clauses id-apply prod.case-eq-if)
       apply (rule dpll-bj-trail-mes-decreasing-prop; auto)
      apply (rename-tac A T U, case-tac T, simp)
     apply (rename-tac A T U, case-tac U, simp)
    using dpll-bj-clauses dpll-bj-all-decomposition-implies-inv dpll-bj-no-dup by fastforce+
end

end
theory DPLL-W
imports Main Partial-Clausal-Logic Partial-Annotated-Clausal-Logic List-More Wellfounded-More
  DPLL-NOT
begin
```

18 DPLL

18.1 Rules

type-synonym *'a* $dpll_W$*-marked-lit* = (*'a*, *unit*, *unit*) *marked-lit*
type-synonym *'a* $dpll_W$*-marked-lits* = (*'a*, *unit*, *unit*) *marked-lits*
type-synonym *'v* $dpll_W$*-state* = *'v* $dpll_W$*-marked-lits* × *'v clauses*

abbreviation *trail* :: *'v* $dpll_W$*-state* ⇒ *'v* $dpll_W$*-marked-lits* **where**
trail ≡ *fst*
abbreviation *clauses* :: *'v* $dpll_W$*-state* ⇒ *'v clauses* **where**
clauses ≡ *snd*

The definition of DPLL is given in figure 2.13 page 70 of CW.

inductive $dpll_W$ :: *'v* $dpll_W$*-state* ⇒ *'v* $dpll_W$*-state* ⇒ *bool* **where**
propagate: *C* + {#*L*#} ∈# *clauses S* ⟹ *trail S* ⊨*as CNot C* ⟹ *undefined-lit* (*trail S*) *L*
  ⟹ $dpll_W$ *S* (*Propagated L* () # *trail S*, *clauses S*) |
decided: *undefined-lit* (*trail S*) *L* ⟹ *atm-of L* ∈ *atms-of-mm* (*clauses S*)
  ⟹ $dpll_W$ *S* (*Marked L* () # *trail S*, *clauses S*) |
backtrack: *backtrack-split* (*trail S*) = (*M'*, *L* # *M*) ⟹ *is-marked L* ⟹ *D* ∈# *clauses S*
  ⟹ *trail S* ⊨*as CNot D* ⟹ $dpll_W$ *S* (*Propagated* (− (*lit-of L*)) () # *M*, *clauses S*)

18.2 Invariants

lemma $dpll_W$*-distinct-inv*:
  assumes $dpll_W$ *S S'*
  and *no-dup* (*trail S*)
  shows *no-dup* (*trail S'*)
  using *assms*
proof (*induct rule*: $dpll_W$*.induct*)
  case (*decided L S*)
  then show *?case* **using** *defined-lit-map* **by** *force*
next
  case (*propagate C L S*)
  then show *?case* **using** *defined-lit-map* **by** *force*
next
  case (*backtrack S M' L M D*) **note** *extracted* = *this*(*1*) **and** *no-dup* = *this*(*5*)
  show *?case*
    using *no-dup backtrack-split-list-eq*[*of trail S*, *symmetric*] **unfolding** *extracted* **by** *auto*
qed

lemma $dpll_W$*-consistent-interp-inv*:
  assumes $dpll_W$ *S S'*
  and *consistent-interp* (*lits-of-l* (*trail S*))
  and *no-dup* (*trail S*)
  shows *consistent-interp* (*lits-of-l* (*trail S'*))
  using *assms*
proof (*induct rule*: $dpll_W$*.induct*)
  case (*backtrack S M' L M D*) **note** *extracted* = *this*(*1*) **and** *marked* = *this*(*2*) **and** *D* = *this*(*4*) **and** *cons* = *this*(*5*) **and** *no-dup* = *this*(*6*)
  have *no-dup'*: *no-dup M*
    by (*metis* (*no-types*) *backtrack-split-list-eq distinct.simps*(*2*) *distinct-append extracted list.simps*(*9*) *map-append no-dup snd-conv*)
  then have *insert* (*lit-of L*) (*lits-of-l M*) ⊆ *lits-of-l* (*trail S*)
    using *backtrack-split-list-eq*[*of trail S*, *symmetric*] **unfolding** *extracted* **by** *auto*
  then have *cons*: *consistent-interp* (*insert* (*lit-of L*) (*lits-of-l M*))

```
    using consistent-interp-subset cons by blast
  moreover
    have lit-of L ∉ lits-of-l M
      using no-dup backtrack-split-list-eq[of trail S, symmetric] extracted
      unfolding lits-of-def by force
  moreover
    have atm-of (−lit-of L) ∉ (λm. atm-of (lit-of m)) ‘ set M
      using no-dup backtrack-split-list-eq[of trail S, symmetric] unfolding extracted by force
    then have −lit-of L ∉ lits-of-l M
      unfolding lits-of-def by force
  ultimately show ?case by simp
qed (auto intro: consistent-add-undefined-lit-consistent)

lemma dpll_W-vars-in-snd-inv:
  assumes dpll_W S S′
  and atm-of ‘ (lits-of-l (trail S)) ⊆ atms-of-mm (clauses S)
  shows atm-of ‘ (lits-of-l (trail S′)) ⊆ atms-of-mm (clauses S′)
  using assms
proof (induct rule: dpll_W.induct)
  case (backtrack S M′ L M D)
  then have atm-of (lit-of L) ∈ atms-of-mm (clauses S)
    using backtrack-split-list-eq[of trail S, symmetric] by auto
  moreover
    have atm-of ‘ lits-of-l (trail S) ⊆ atms-of-mm (clauses S)
      using backtrack(5) by simp
    then have ⋀xb. xb ∈ set M ⟹ atm-of (lit-of xb) ∈ atms-of-mm (clauses S)
      using backtrack-split-list-eq[symmetric, of trail S] backtrack.hyps(1)
      unfolding lits-of-def by auto
  ultimately show ?case by (auto simp : lits-of-def)
qed (auto simp: in-plus-implies-atm-of-on-atms-of-ms)

lemma atms-of-ms-lit-of-atms-of: atms-of-ms ((λa. {#lit-of a#}) ‘ c) = atm-of ‘ lit-of ‘ c
  unfolding atms-of-ms-def using image-iff by force
```

Lemma theorem 2.8.2 page 71 of CW

```
lemma dpll_W-propagate-is-conclusion:
  assumes dpll_W S S′
  and all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S))
  and atm-of ‘ lits-of-l (trail S) ⊆ atms-of-mm (clauses S)
  shows all-decomposition-implies-m (clauses S′) (get-all-marked-decomposition (trail S′))
  using assms
proof (induct rule: dpll_W.induct)
  case (decided L S)
  then show ?case unfolding all-decomposition-implies-def by simp
next
   case (propagate C L S) note inS = this(1) and cnot = this(2) and IH = this(4) and undef =
this(3) and atms-incl = this(5)
  let ?I = set (map (λa. {#lit-of a#}) (trail S)) ∪ set-mset (clauses S)
  have ?I ⊨p C + {#L#} by (auto simp add: inS)
  moreover have ?I ⊨ps CNot C using true-annots-true-clss-cls cnot by fastforce
  ultimately have ?I ⊨p {#L#} using true-clss-cls-plus-CNot[of ?I C L] inS by blast
  {
    assume get-all-marked-decomposition (trail S) = []
    then have ?case by blast
  }
  moreover {
    assume n: get-all-marked-decomposition (trail S) ≠ []
    have 1: ⋀a b. (a, b) ∈ set (tl (get-all-marked-decomposition (trail S)))
      ⟹ (unmark-l a ∪ set-mset (clauses S)) ⊨ps unmark-l b
      using IH unfolding all-decomposition-implies-def by (fastforce simp add: list.set-sel(2) n)
    moreover have 2: ⋀a c. hd (get-all-marked-decomposition (trail S)) = (a, c)
      ⟹ (unmark-l a ∪ set-mset (clauses S)) ⊨ps (unmark-l c)
      by (metis IH all-decomposition-implies-cons-pair all-decomposition-implies-single
        list.collapse n)
    moreover have 3: ⋀a c. hd (get-all-marked-decomposition (trail S)) = (a, c)
      ⟹ (unmark-l a ∪ set-mset (clauses S)) ⊨p {#L#}
      proof –
        fix a c
        assume h: hd (get-all-marked-decomposition (trail S)) = (a, c)
        have h': trail S = c @ a using get-all-marked-decomposition-decomp h by blast
        have I: set (map (λa. {#lit-of a#})  a) ∪ set-mset (clauses S)
          ∪ unmark-l c ⊨ps CNot C
          using ‹?I ⊨ps CNot C› unfolding h' by (simp add: Un-commute Un-left-commute)
        have
          atms-of-ms (CNot C) ⊆ atms-of-ms (set (map (λa. {#lit-of a#}) a) ∪ set-mset (clauses S))
            and
          atms-of-ms (unmark-l c) ⊆ atms-of-ms (set (map (λa. {#lit-of a#}) a)
            ∪ set-mset (clauses S))
            apply (metis CNot-plus Un-subset-iff atms-of-atms-of-ms-mono atms-of-ms-CNot-atms-of
              atms-of-ms-union inS sup.coboundedI2)
          using inS atms-of-atms-of-ms-mono atms-incl by (fastforce simp: h')

        then have unmark-l a ∪ set-mset (clauses S) ⊨ps CNot C
          using true-clss-clss-left-right[OF - I] h 2 by auto
        then show unmark-l a ∪ set-mset (clauses S) ⊨p {#L#}
          by (metis (no-types) Un-insert-right inS insertI1 mk-disjoint-insert inS
            true-clss-cls-in true-clss-cls-plus-CNot)
      qed
    ultimately have ?case
      by (cases hd (get-all-marked-decomposition (trail S)))
        (auto simp: all-decomposition-implies-def)
  }
  ultimately show ?case by auto
next
  case (backtrack S M' L M D) note extracted = this(1) and marked = this(2) and D = this(3) and
    cnot = this(4) and cons = this(4) and IH = this(5) and atms-incl = this(6)
  have S: trail S = M' @ L # M
    using backtrack-split-list-eq[of trail S] unfolding extracted by auto
  have M': ∀ l ∈ set M'. ¬is-marked l
    using extracted backtrack-split-fst-not-marked[of - trail S] by simp
  have n: get-all-marked-decomposition (trail S) ≠ [] by auto
  then have all-decomposition-implies-m (clauses S) ((L # M, M')
          # tl (get-all-marked-decomposition (trail S)))
    by (metis (no-types) IH extracted get-all-marked-decomposition-backtrack-split list.exhaust-sel)
  then have 1: unmark-l (L # M) ∪ set-mset (clauses S) ⊨ps(λa.{#lit-of a#}) ‘ set M'
    by simp
  moreover
    have unmark-l (L # M) ∪ unmark-l M' ⊨ps CNot D
      by (metis (mono-tags, lifting) S Un-commute cons image-Un set-append
        true-annots-true-clss-clss)
    then have 2: unmark-l (L # M) ∪ set-mset (clauses S) ∪ unmark-l M'
        ⊨ps CNot D
      by (metis (no-types, lifting) Un-assoc Un-left-commute true-clss-clss-union-l-r)
  ultimately
    have set (map (λa. {#lit-of a#}) (L # M)) ∪ set-mset (clauses S) ⊨ps CNot D
      using true-clss-clss-left-right by fastforce
    then have set (map (λa. {#lit-of a#}) (L # M)) ∪ set-mset (clauses S) ⊨p {#}
      by (metis (mono-tags, lifting) D Un-def mem-Collect-eq
        true-clss-clss-contradiction-true-clss-cls-false)
    then have IL: unmark-l M ∪ set-mset (clauses S) ⊨p {#−lit-of L#}
      using true-clss-clss-false-left-right by auto
  show ?case unfolding S all-decomposition-implies-def
    proof
      fix x P level
      assume x: x ∈ set (get-all-marked-decomposition
        (fst (Propagated (− lit-of L) P # M, clauses S)))
      let ?M' = Propagated (− lit-of L) P # M
      let ?hd = hd (get-all-marked-decomposition ?M')
      let ?tl = tl (get-all-marked-decomposition ?M')
      have x = ?hd ∨ x ∈ set ?tl
        using x
        by (cases get-all-marked-decomposition ?M')
          auto
      moreover {
        assume x': x ∈ set ?tl
        have L': Marked (lit-of L) () = L using marked by (cases L, auto)
        have x ∈ set (get-all-marked-decomposition (M' @ L # M))
          using x' get-all-marked-decomposition-except-last-choice-equal[of M' lit-of L P M]
          L' by (metis (no-types) M' list.set-sel(2) tl-Nil)
        then have case x of (Ls, seen) ⇒ unmark-l Ls ∪ set-mset (clauses S)
          ⊨ps unmark-l seen
          using marked IH by (cases L) (auto simp add: S all-decomposition-implies-def)
      }
      moreover {
        assume x': x = ?hd
        have tl: tl (get-all-marked-decomposition (M' @ L # M)) ≠ []
          proof −
            have f1: ⋀ms. length (get-all-marked-decomposition (M' @ ms))
              = length (get-all-marked-decomposition ms)
              by (simp add: M' get-all-marked-decomposition-remove-unmark-ssed-length)
            have Suc (length (get-all-marked-decomposition M)) ≠ Suc 0
              by blast
            then show ?thesis
              using f1 marked by (metis (no-types) get-all-marked-decomposition.simps(1) length-tl
                list.sel(3) list.size(3) marked-lit.collapse(1))
          qed
        obtain M0' M0 where
          L0: hd (tl (get-all-marked-decomposition (M' @ L # M))) = (M0, M0')
          by (cases hd (tl (get-all-marked-decomposition (M' @ L # M))))
        have x'': x = (M0, Propagated (−lit-of L) P # M0')
          unfolding x' using get-all-marked-decomposition-last-choice tl M' L0
          by (metis marked marked-lit.collapse(1))
        obtain l-get-all-marked-decomposition where
          get-all-marked-decomposition (trail S) = (L # M, M') # (M0, M0') #
            l-get-all-marked-decomposition
          using get-all-marked-decomposition-backtrack-split extracted by (metis (no-types) L0 S
            hd-Cons-tl n tl)
        then have M = M0' @ M0 using get-all-marked-decomposition-hd-hd by fastforce
        then have IL': unmark-l M0 ∪ set-mset (clauses S)
          ∪ unmark-l M0' ⊨ps {{#− lit-of L#}}
          using IL by (simp add: Un-commute Un-left-commute image-Un)
        moreover have H: unmark-l M0 ∪ set-mset (clauses S)
          ⊨ps unmark-l M0'
          using IH x'' unfolding all-decomposition-implies-def by (metis (no-types, lifting) L0 S
            list.set-sel(1) list.set-sel(2) old.prod.case tl tl-Nil)
        ultimately have case x of (Ls, seen) ⇒ unmark-l Ls ∪ set-mset (clauses S)
          ⊨ps unmark-l seen
          using true-clss-clss-left-right unfolding x'' by auto
      }
      ultimately show case x of (Ls, seen) ⇒
        unmark-l Ls ∪ set-mset (snd (?M', clauses S))
          ⊨ps unmark-l seen
        unfolding snd-conv by blast
    qed
  qed
```

Lemma theorem 2.8.3 page 72 of CW

```
theorem dpll_W-propagate-is-conclusion-of-decided:
  assumes dpll_W S S'
  and all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S))
  and atm-of ` lits-of-l (trail S) ⊆ atms-of-mm (clauses S)
  shows set-mset (clauses S') ∪ {{#lit-of L#} |L. is-marked L ∧ L ∈ set (trail S')}
    ⊨ps (λa. {#lit-of a#}) ` ⋃(set ` snd ` set (get-all-marked-decomposition (trail S')))
  using all-decomposition-implies-trail-is-implied[OF dpll_W-propagate-is-conclusion[OF assms]] .
```

Lemma theorem 2.8.4 page 72 of CW

```
lemma only-propagated-vars-unsat:
  assumes marked: ∀x ∈ set M. ¬ is-marked x
  and DN: D ∈ N and D: M ⊨as CNot D
  and inv: all-decomposition-implies N (get-all-marked-decomposition M)
  and atm-incl: atm-of ` lits-of-l M  ⊆ atms-of-ms N
  shows unsatisfiable N
proof (rule ccontr)
  assume ¬ unsatisfiable N
  then obtain I where
    I: I ⊨s N and
    cons: consistent-interp I and
    tot: total-over-m I N
    unfolding satisfiable-def by auto
  then have I-D: I ⊨ D
    using DN unfolding true-clss-def by auto

  have l0: {{#lit-of L#} |L. is-marked L ∧ L ∈ set M} = {} using marked by auto
  have atms-of-ms (N ∪ unmark-l M) = atms-of-ms N
    using atm-incl unfolding atms-of-ms-def lits-of-def by auto

  then have total-over-m I (N ∪ (λa. {#lit-of a#}) ` (set M))
    using tot unfolding total-over-m-def by auto
  then have I ⊨s (λa. {#lit-of a#}) ` (set M)
    using all-decomposition-implies-propagated-lits-are-implied[OF inv] cons I
    unfolding true-clss-clss-def l0 by auto
  then have IM: I ⊨s unmark-l M by auto
  {
    fix K
    assume K ∈# D
    then have −K ∈ lits-of-l M
      by (auto split: if-split-asm
        intro: allE[OF D[unfolded true-annots-def Ball-def], of {#−K#}])
    then have −K ∈ I using IM true-clss-singleton-lit-of-implies-incl by fastforce
  }
  then have ¬ I ⊨ D using cons unfolding true-cls-def consistent-interp-def by auto
  then show False using I-D by blast
qed

lemma dpll_W-same-clauses:
  assumes dpll_W S S'
  shows clauses S = clauses S'
  using assms by (induct rule: dpll_W.induct, auto)

lemma rtranclp-dpll_W-inv:
  assumes rtranclp dpll_W S S'
  and inv: all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S))
  and atm-incl: atm-of ‘ lits-of-l (trail S) ⊆ atms-of-mm (clauses S)
  and consistent-interp (lits-of-l (trail S))
  and no-dup (trail S)
  shows all-decomposition-implies-m (clauses S') (get-all-marked-decomposition (trail S'))
  and atm-of ‘ lits-of-l (trail S') ⊆ atms-of-mm (clauses S')
  and clauses S = clauses S'
  and consistent-interp (lits-of-l (trail S'))
  and no-dup (trail S')
  using assms
proof (induct rule: rtranclp-induct)
  case base
  show
    all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S)) and
    atm-of ‘ lits-of-l (trail S) ⊆ atms-of-mm (clauses S) and
    clauses S = clauses S and
    consistent-interp (lits-of-l (trail S)) and
    no-dup (trail S) using assms by auto
next
  case (step S' S'') note dpll_WStar = this(1) and IH = this(3,4,5,6,7) and
    dpll_W = this(2)
  moreover
    assume
      inv: all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S)) and
      atm-incl: atm-of ‘ lits-of-l (trail S) ⊆ atms-of-mm (clauses S) and
      cons: consistent-interp (lits-of-l (trail S)) and
      no-dup (trail S)
  ultimately have decomp: all-decomposition-implies-m (clauses S')
    (get-all-marked-decomposition (trail S')) and
    atm-incl': atm-of ‘ lits-of-l (trail S') ⊆ atms-of-mm (clauses S') and
    snd: clauses S = clauses S' and
    cons': consistent-interp (lits-of-l (trail S')) and
    no-dup': no-dup (trail S') by blast+
  show clauses S = clauses S'' using dpll_W-same-clauses[OF dpll_W] snd by metis
  show all-decomposition-implies-m (clauses S'') (get-all-marked-decomposition (trail S''))
    using dpll_W-propagate-is-conclusion[OF dpll_W] decomp atm-incl' by auto
  show atm-of ` lits-of-l (trail S'') ⊆ atms-of-mm (clauses S'')
    using dpll_W-vars-in-snd-inv[OF dpll_W]  atm-incl atm-incl' by auto
  show no-dup (trail S'') using dpll_W-distinct-inv[OF dpll_W] no-dup' dpll_W by auto
  show consistent-interp (lits-of-l (trail S''))
    using cons' no-dup' dpll_W-consistent-interp-inv[OF dpll_W] by auto
qed

definition dpll_W-all-inv S ≡
  (all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S))
  ∧ atm-of ` lits-of-l (trail S)  ⊆ atms-of-mm (clauses S)
  ∧ consistent-interp (lits-of-l (trail S))
  ∧ no-dup (trail S))

lemma dpll_W-all-inv-dest[dest]:
  assumes dpll_W-all-inv S
  shows all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S))
  and atm-of ` lits-of-l (trail S)  ⊆ atms-of-mm (clauses S)
  and consistent-interp (lits-of-l (trail S)) ∧ no-dup (trail S)
  using assms unfolding dpll_W-all-inv-def lits-of-def by auto

lemma rtranclp-dpll_W-all-inv:
  assumes rtranclp dpll_W  S S'
  and dpll_W-all-inv S
  shows dpll_W-all-inv S'
  using assms rtranclp-dpll_W-inv[OF assms(1)] unfolding dpll_W-all-inv-def lits-of-def by blast

lemma dpll_W-all-inv:
  assumes dpll_W  S S'
  and dpll_W-all-inv S
  shows dpll_W-all-inv S'
  using assms rtranclp-dpll_W-all-inv by blast

lemma rtranclp-dpll_W-inv-starting-from-0:
  assumes rtranclp dpll_W  S S'
  and inv: trail S = []
  shows dpll_W-all-inv S'
proof −
  have dpll_W-all-inv S
    using assms unfolding all-decomposition-implies-def dpll_W-all-inv-def by auto
  then show ?thesis using rtranclp-dpll_W-all-inv[OF assms(1)] by blast
qed

lemma dpll_W-can-do-step:
  assumes consistent-interp (set M)
  and distinct M
  and atm-of ` (set M) ⊆ atms-of-mm N
  shows rtranclp dpll_W ([], N) (map (λM. Marked M ()) M, N)
  using assms
proof (induct M)
  case Nil
  then show ?case by auto
next
```

```
  case (Cons L M)
  then have undefined-lit (map (λM. Marked M ()) M) L
    unfolding defined-lit-def consistent-interp-def by auto
  moreover have atm-of L ∈ atms-of-mm N using Cons.prems(3) by auto
  ultimately have dpll_W (map (λM. Marked M ()) M, N) (map (λM. Marked M ()) (L # M), N)
    using dpll_W.decided by auto
  moreover have consistent-interp (set M) and distinct M and atm-of ‘ set M ⊆ atms-of-mm N
    using Cons.prems unfolding consistent-interp-def by auto
  ultimately show ?case using Cons.hyps by auto
qed

definition conclusive-dpll_W-state (S:: 'v dpll_W-state) ⟷
  (trail S ⊨asm clauses S ∨ ((∀ L ∈ set (trail S). ¬is-marked L)
  ∧ (∃ C ∈# clauses S. trail S ⊨as CNot C)))


lemma dpll_W-strong-completeness:
  assumes set M ⊨sm N
  and consistent-interp (set M)
  and distinct M
  and atm-of ‘ (set M) ⊆ atms-of-mm N
  shows dpll_W** ([], N) (map (λM. Marked M ()) M, N)
  and conclusive-dpll_W-state (map (λM. Marked M ())  M, N)
proof –
  show rtranclp dpll_W ([], N) (map (λM. Marked M ()) M, N) using dpll_W-can-do-step assms by auto
  have map (λM. Marked M ()) M ⊨asm N using assms(1) true-annots-marked-true-cls by auto
  then show conclusive-dpll_W-state (map (λM. Marked M ()) M, N)
    unfolding conclusive-dpll_W-state-def by auto
qed


lemma dpll_W-sound:
  assumes
    rtranclp dpll_W ([], N) (M, N) and
    ∀ S. ¬dpll_W (M, N) S
  shows M ⊨asm N ⟷ satisfiable (set-mset N) (is ?A ⟷ ?B)
proof
  let ?M'= lits-of-l M
  assume ?A
  then have ?M' ⊨sm N by (simp add: true-annots-true-cls)
  moreover have consistent-interp ?M'
    using rtranclp-dpll_W-inv-starting-from-0[OF assms(1)] by auto
  ultimately show ?B by auto
next
  assume ?B
  show ?A
    proof (rule ccontr)
      assume n: ¬ ?A
      have (∃ L. undefined-lit M L ∧ atm-of L ∈ atms-of-mm N) ∨ (∃ D∈#N. M ⊨as CNot D)
        proof –
          obtain D :: 'a clause where D: D ∈# N and ¬ M ⊨a D
            using n unfolding true-annots-def Ball-def by auto
          then have (∃ L. undefined-lit M L ∧ atm-of L ∈ atms-of D) ∨ M ⊨as CNot D
             unfolding true-annots-def Ball-def CNot-def true-annot-def
             using atm-of-lit-in-atms-of true-annot-iff-marked-or-true-lit true-cls-def by blast
          then show ?thesis
            by (metis Bex-def D atms-of-atms-of-ms-mono rev-subsetD)
        qed
      moreover {
        assume ∃L. undefined-lit M L ∧ atm-of L ∈ atms-of-mm N
        then have False using assms(2) decided by fastforce
      }
      moreover {
        assume ∃D∈#N. M ⊨as CNot D
        then obtain D where DN: D ∈# N and MD: M ⊨as CNot D by auto
        {
          assume ∀l ∈ set M. ¬ is-marked l
          moreover have dpll_W-all-inv ([], N)
            using assms unfolding all-decomposition-implies-def dpll_W-all-inv-def by auto
          ultimately have unsatisfiable (set-mset N)
            using only-propagated-vars-unsat[of M D set-mset N] DN MD
            rtranclp-dpll_W-all-inv[OF assms(1)] by force
          then have False using ‹?B› by blast
        }
        moreover {
          assume l: ∃l ∈ set M. is-marked l
          then have False
            using backtrack[of (M, N) - - - D ] DN MD assms(2)
              backtrack-split-some-is-marked-then-snd-has-hd[OF l]
            by (metis backtrack-split-snd-hd-marked fst-conv list.distinct(1) list.sel(1) snd-conv)
        }
        ultimately have False by blast
      }
      ultimately show False by blast
    qed
qed
```

18.3 Termination

```
definition dpll_W-mes M n =
  map (λl. if is-marked l then 2 else (1::nat)) (rev M) @ replicate (n − length M) 3

lemma length-dpll_W-mes:
  assumes length M ≤ n
  shows length (dpll_W-mes M n) = n
  using assms unfolding dpll_W-mes-def by auto

lemma distinctcard-atm-of-lit-of-eq-length:
  assumes no-dup S
  shows card (atm-of ‘ lits-of-l S) = length S
  using assms by (induct S) (auto simp add: image-image lits-of-def)

lemma dpll_W-card-decrease:
  assumes dpll: dpll_W S S′ and length (trail S′) ≤ card vars
  and length (trail S) ≤ card vars
  shows (dpll_W-mes (trail S′) (card vars), dpll_W-mes (trail S) (card vars))
    ∈ lexn {(a, b). a < b} (card vars)
  using assms
proof (induct rule: dpll_W.induct)
  case (propagate C L S)
  have m: map (λl. if is-marked l then 2 else 1) (rev (trail S))
```

```
      @ replicate (card vars − length (trail S)) 3
    = map (λl. if is-marked l then 2 else 1) (rev (trail S)) @ 3
        # replicate (card vars − Suc (length (trail S))) 3
    using propagate.prems[simplified] using Suc-diff-le by fastforce
  then show ?case
    using propagate.prems(1) unfolding dpll_W-mes-def by (fastforce simp add: lexn-conv assms(2))
next
  case (decided S L)
  have m: map (λl. if is-marked l then 2 else 1) (rev (trail S))
      @ replicate (card vars − length (trail S)) 3
    = map (λl. if is-marked l then 2 else 1) (rev (trail S)) @ 3
      # replicate (card vars − Suc (length (trail S))) 3
    using decided.prems[simplified] using Suc-diff-le by fastforce
  then show ?case
    using decided.prems unfolding dpll_W-mes-def by (force simp add: lexn-conv assms(2))
next
  case (backtrack S M′ L M D)
  have L: is-marked L using backtrack.hyps(2) by auto
  have S: trail S = M′ @ L # M
    using backtrack.hyps(1) backtrack-split-list-eq[of trail S] by auto
  show ?case
    using backtrack.prems L unfolding dpll_W-mes-def S by (fastforce simp add: lexn-conv assms(2))
qed
```

Proposition theorem 2.8.7 page 73 of CW

```
lemma dpll_W-card-decrease′:
  assumes dpll: dpll_W S S′
  and atm-incl: atm-of ‘ lits-of-l (trail S) ⊆ atms-of-mm (clauses S)
  and no-dup: no-dup (trail S)
  shows (dpll_W-mes (trail S′) (card (atms-of-mm (clauses S′))),
         dpll_W-mes (trail S) (card (atms-of-mm (clauses S)))) ∈ lex {(a, b). a < b}
proof −
  have finite (atms-of-mm (clauses S)) unfolding atms-of-ms-def by auto
  then have 1: length (trail S) ≤ card (atms-of-mm (clauses S))
    using distinctcard-atm-of-lit-of-eq-length[OF no-dup] atm-incl card-mono by metis

  moreover
    have no-dup′: no-dup (trail S′) using dpll dpll_W-distinct-inv no-dup by blast
    have SS′: clauses S′ = clauses S using dpll by (auto dest!: dpll_W-same-clauses)
    have atm-incl′: atm-of ‘ lits-of-l (trail S′) ⊆ atms-of-mm (clauses S′)
      using atm-incl dpll dpll_W-vars-in-snd-inv[OF dpll] by force
    have finite (atms-of-mm (clauses S′))
      unfolding atms-of-ms-def by auto
    then have 2: length (trail S′) ≤ card (atms-of-mm (clauses S))
      using distinctcard-atm-of-lit-of-eq-length[OF no-dup′] atm-incl′ card-mono SS′ by metis

  ultimately have (dpll_W-mes (trail S′) (card (atms-of-mm (clauses S))),
      dpll_W-mes (trail S) (card (atms-of-mm (clauses S))))
    ∈ lexn {(a, b). a < b} (card (atms-of-mm (clauses S)))
    using dpll_W-card-decrease[OF assms(1), of atms-of-mm (clauses S)] by blast
  then have (dpll_W-mes (trail S′) (card (atms-of-mm (clauses S))),
         dpll_W-mes (trail S) (card (atms-of-mm (clauses S)))) ∈ lex {(a, b). a < b}
    unfolding lex-def by auto
  then show (dpll_W-mes (trail S′) (card (atms-of-mm (clauses S′))),
        dpll_W-mes (trail S) (card (atms-of-mm (clauses S)))) ∈ lex {(a, b). a < b}
    using dpll_W-same-clauses[OF assms(1)]  by auto
qed

lemma wf-lexn: wf (lexn {(a, b). (a::nat) < b} (card (atms-of-mm (clauses S))))
proof -
  have m: {(a, b). a < b} = measure id by auto
  show ?thesis apply (rule wf-lexn) unfolding m by auto
qed

lemma dpll_W-wf:
  wf {(S', S). dpll_W-all-inv S ∧ dpll_W S S'}
  apply (rule wf-wf-if-measure'[OF wf-lex-less, of - -
        λS. dpll_W-mes (trail S) (card (atms-of-mm (clauses S)))])
  using dpll_W-card-decrease' by fast


lemma dpll_W-tranclp-star-commute:
  {(S', S). dpll_W-all-inv S ∧ dpll_W S S'}^+ = {(S', S). dpll_W-all-inv S ∧ tranclp dpll_W S S'}
    (is ?A = ?B)
proof
  { fix S S'
    assume (S, S') ∈ ?A
    then have (S, S') ∈ ?B
      by (induct rule: trancl.induct, auto)
  }
  then show ?A ⊆ ?B by blast
  { fix S S'
    assume (S, S') ∈ ?B
    then have dpll_W^++ S' S and dpll_W-all-inv S' by auto
    then have (S, S') ∈ ?A
      proof (induct rule: tranclp.induct)
        case r-into-trancl
        then show ?case by (simp-all add: r-into-trancl')
      next
        case (trancl-into-trancl S S' S'')
        then have (S', S) ∈ {a. case a of (S', S) ⇒ dpll_W-all-inv S ∧ dpll_W S S'}^+ by blast
        moreover have dpll_W-all-inv S'
          using rtranclp-dpll_W-all-inv[OF tranclp-into-rtranclp[OF trancl-into-trancl.hyps(1)]]
          trancl-into-trancl.prems by auto
        ultimately have (S'', S') ∈ {(pa, p). dpll_W-all-inv p ∧ dpll_W p pa}^+
          using ‹dpll_W-all-inv S'› trancl-into-trancl.hyps(3) by blast
        then show ?case
          using ‹(S', S) ∈ {a. case a of (S', S) ⇒ dpll_W-all-inv S ∧ dpll_W S S'}^+› by auto
      qed
  }
  then show ?B ⊆ ?A by blast
qed

lemma dpll_W-wf-tranclp: wf {(S', S). dpll_W-all-inv S ∧ dpll_W^++ S S'}
  unfolding dpll_W-tranclp-star-commute[symmetric] by (simp add: dpll_W-wf wf-trancl)

lemma dpll_W-wf-plus:
  shows wf {(S', ([], N))| S'. dpll_W^++ ([], N) S'}  (is wf ?P)
  apply (rule wf-subset[OF dpll_W-wf-tranclp, of ?P])
  using assms unfolding dpll_W-all-inv-def by auto
```

18.4 Final States

```
lemma dpll_W-no-more-step-is-a-conclusive-state:
  assumes ∀ S′. ¬dpll_W S S′
  shows conclusive-dpll_W-state S
proof −
  have vars: ∀ s ∈ atms-of-mm (clauses S). s ∈ atm-of ‘ lits-of-l (trail S)
    proof (rule ccontr)
      assume ¬ (∀ s∈atms-of-mm (clauses S). s ∈ atm-of ‘ lits-of-l (trail S))
      then obtain L where
        L-in-atms: L ∈ atms-of-mm (clauses S) and
        L-notin-trail: L ∉ atm-of ‘ lits-of-l (trail S) by metis
      obtain L′ where L′: atm-of L′ = L by (meson literal.sel(2))
      then have undefined-lit (trail S) L′
        unfolding Marked-Propagated-in-iff-in-lits-of-l by (metis L-notin-trail atm-of-uminus imageI)
      then show False using dpll_W.decided assms(1) L-in-atms L′ by blast
    qed
  show ?thesis
    proof (rule ccontr)
      assume not-final: ¬ ?thesis
      then have
        ¬ trail S ⊨asm clauses S and
        (∃ L∈set (trail S). is-marked L) ∨ (∀ C∈#clauses S. ¬trail S ⊨as CNot C)
        unfolding conclusive-dpll_W-state-def by auto
      moreover {
        assume ∃ L∈set (trail S). is-marked L
        then obtain L M′ M where L: backtrack-split (trail S) = (M′, L # M)
          using backtrack-split-some-is-marked-then-snd-has-hd by blast
        obtain D where D ∈# clauses S and ¬ trail S ⊨a D
          using ‹¬ trail S ⊨asm clauses S› unfolding true-annots-def by auto
        then have ∀ s∈atms-of-ms {D}. s ∈ atm-of ‘ lits-of-l (trail S)
          using vars unfolding atms-of-ms-def by auto
        then have trail S ⊨as CNot D
          using all-variables-defined-not-imply-cnot[of D] ‹¬ trail S ⊨a D› by auto
        moreover have is-marked L
          using L by (metis backtrack-split-snd-hd-marked list.distinct(1) list.sel(1) snd-conv)
        ultimately have False
          using assms(1) dpll_W.backtrack L ‹D ∈# clauses S› ‹trail S ⊨as CNot D› by blast
      }
      moreover {
        assume tr: ∀ C∈#clauses S. ¬trail S ⊨as CNot C
        obtain C where C-in-cls: C ∈# clauses S and trC: ¬ trail S ⊨a C
          using ‹¬ trail S ⊨asm clauses S› unfolding true-annots-def by auto
        have ∀ s∈atms-of-ms {C}. s ∈ atm-of ‘ lits-of-l (trail S)
          using vars ‹C ∈# clauses S› unfolding atms-of-ms-def by auto
        then have trail S ⊨as CNot C
          by (meson C-in-cls tr trC all-variables-defined-not-imply-cnot)
        then have False using tr C-in-cls by auto
      }
      ultimately show False by blast
    qed
qed

lemma dpll_W-conclusive-state-correct:
  assumes dpll_W** ([], N) (M, N) and conclusive-dpll_W-state (M, N)
  shows M ⊨asm N ⟷ satisfiable (set-mset N) (is ?A ⟷ ?B)
```

```
proof
  let ?M'= lits-of-l M
  assume ?A
  then have ?M' ⊨sm N by (simp add: true-annots-true-cls)
  moreover have consistent-interp ?M'
    using rtranclp-dpll_W-inv-starting-from-0[OF assms(1)] by auto
  ultimately show ?B by auto
next
  assume ?B
  show ?A
    proof (rule ccontr)
      assume n: ¬ ?A
      have no-mark: ∀ L∈set M. ¬ is-marked L  ∃ C ∈# N. M ⊨as CNot C
        using n assms(2) unfolding conclusive-dpll_W-state-def by auto
      moreover obtain D where DN: D ∈# N and MD: M ⊨as CNot D using no-mark by auto
      ultimately have unsatisfiable (set-mset N)
        using only-propagated-vars-unsat rtranclp-dpll_W-all-inv[OF assms(1)]
        unfolding dpll_W-all-inv-def by force
      then show False using ‹?B› by blast
    qed
qed
```

18.5 Link with NOT's DPLL

```
interpretation dpll_W-_NOT: dpll-with-backtrack .

declare dpll_W-_NOT.state-simp_NOT[simp del]
lemma state-eq_NOT-iff-eq[iff, simp]: dpll_W-_NOT.state-eq_NOT S T ⟷ S = T
  unfolding dpll_W-_NOT.state-eq_NOT-def by (cases S, cases T) auto
lemma dpll_W-dpll_W-bj:
  assumes inv: dpll_W-all-inv S and dpll: dpll_W S T
  shows dpll_W-_NOT.dpll-bj S T
  using dpll inv
  apply (induction rule: dpll_W.induct)
    apply (rule dpll_W-_NOT.bj-propagate_NOT)
    apply (rule dpll_W-_NOT.propagate_NOT.propagate_NOT; simp?)
    apply fastforce
   apply (rule dpll_W-_NOT.bj-decide_NOT)
   apply (rule dpll_W-_NOT.decide_NOT.decide_NOT; simp?)
   apply fastforce
  apply (frule dpll_W-_NOT.backtrack.intros[of - -  - - -], simp-all)
  apply (rule dpll_W-_NOT.dpll-bj.bj-backjump)
  apply (rule dpll_W-_NOT.backtrack-is-backjump'',
    simp-all add: dpll_W-all-inv-def)
  done

lemma dpll_W-bj-dpll:
  assumes inv: dpll_W-all-inv S and dpll: dpll_W-_NOT.dpll-bj S T
  shows dpll_W S T
  using dpll
  apply (induction rule: dpll_W-_NOT.dpll-bj.induct)
    apply (elim dpll_W-_NOT.decide_NOTE, cases S)
    apply (frule decided; simp)

   apply (elim dpll_W-_NOT.propagate_NOTE, cases S)
   apply (auto intro!: propagate[of - - (-, -), simplified])[]
  apply (elim dpll_W-_NOT.backjumpE, cases S)
  by (simp add: dpll_W.simps dpll-with-backtrack.backtrack.simps)

lemma rtranclp-dpll_W-rtranclp-dpll_W-_NOT:
  assumes dpll_W^** S T and dpll_W-all-inv S
  shows dpll_W-_NOT.dpll-bj^** S T
  using assms apply (induction)
   apply simp
  by (auto intro: rtranclp-dpll_W-all-inv dpll_W-dpll_W-bj rtranclp.rtrancl-into-rtrancl)

lemma rtranclp-dpll-rtranclp-dpll_W:
  assumes dpll_W-_NOT.dpll-bj^** S T and dpll_W-all-inv S
  shows dpll_W^** S T
  using assms apply (induction)
   apply simp
  by (auto intro: dpll_W-bj-dpll rtranclp.rtrancl-into-rtrancl rtranclp-dpll_W-all-inv)

lemma dpll-conclusive-state-correctness:
  assumes dpll_W-_NOT.dpll-bj^** ([], N) (M, N) and conclusive-dpll_W-state (M, N)
  shows M ⊨asm N ⟷ satisfiable (set-mset N)
proof −
  have dpll_W-all-inv ([], N)
    unfolding dpll_W-all-inv-def by auto
  show ?thesis
    apply (rule dpll_W-conclusive-state-correct)
      apply (simp add: ‹dpll_W-all-inv ([], N)› assms(1) rtranclp-dpll-rtranclp-dpll_W)
    using assms(2) by simp
qed

end
theory CDCL-W-Level
imports Partial-Annotated-Clausal-Logic
begin
```

18.5.1 Level of literals and clauses

Getting the level of a variable, implies that the list has to be reversed. Here is the function after reversing.

```
fun get-rev-level :: ('v, nat, 'a) marked-lits ⇒ nat ⇒ 'v literal ⇒ nat where
get-rev-level [] - - = 0 |
get-rev-level (Marked l level # Ls) n L =
  (if atm-of l = atm-of L then level else get-rev-level Ls level L) |
get-rev-level (Propagated l - # Ls) n L =
  (if atm-of l = atm-of L then n else get-rev-level Ls n L)

abbreviation get-level M L ≡ get-rev-level (rev M) 0 L

lemma get-rev-level-uminus[simp]: get-rev-level M n(−L) = get-rev-level M n L
  by (induct arbitrary: n rule: get-rev-level.induct) auto

lemma atm-of-notin-get-rev-level-eq-0:
  assumes atm-of L ∉ atm-of ‘ lits-of-l M
  shows get-rev-level M n L = 0
  using assms by (induct M arbitrary: n rule: marked-lit-list-induct) auto
```

```
lemma get-rev-level-ge-0-atm-of-in:
  assumes  get-rev-level M n L > n
  shows atm-of L ∈ atm-of ‘ lits-of-l M
  using assms by (induct M arbitrary: n rule: marked-lit-list-induct)
  (fastforce simp: atm-of-notin-get-rev-level-eq-0)+
```

In *get-rev-level* (resp. *get-level*), the beginning (resp. the end) can be skipped if the literal is not in the beginning (resp. the end).

```
lemma get-rev-level-skip[simp]:
  assumes  atm-of L ∉ atm-of ‘ lits-of-l M
  shows get-rev-level (M @ Marked K i # M') n L = get-rev-level (Marked K i # M') i L
  using assms by (induct M arbitrary: n i rule: marked-lit-list-induct) auto

lemma get-rev-level-notin-end[simp]:
  assumes  atm-of L ∉ atm-of ‘ lits-of-l M'
  shows get-rev-level (M @ M') n L = get-rev-level M n L
  using assms by (induct M arbitrary: n rule: marked-lit-list-induct)
  (auto simp: atm-of-notin-get-rev-level-eq-0)
```

If the literal is at the beginning, then the end can be skipped

```
lemma get-rev-level-skip-end[simp]:
  assumes  atm-of L ∈ atm-of ‘ lits-of-l M
  shows get-rev-level (M @ M') n L = get-rev-level M n L
  using assms by (induct arbitrary: n rule: marked-lit-list-induct) auto

lemma get-level-skip-beginning:
  assumes atm-of L' ≠ atm-of (lit-of K)
  shows get-level (K # M) L' = get-level M L'
  using assms by auto

lemma get-level-skip-beginning-not-marked-rev:
  assumes atm-of L ∉ atm-of ‘ lit-of ‘(set S)
  and ∀ s∈set S. ¬is-marked s
  shows get-level (M @ rev S) L = get-level M L
  using assms by (induction S rule: marked-lit-list-induct) auto

lemma get-level-skip-beginning-not-marked[simp]:
  assumes atm-of L ∉ atm-of ‘ lit-of ‘(set S)
  and ∀ s∈set S. ¬is-marked s
  shows get-level (M @ S) L = get-level M L
  using get-level-skip-beginning-not-marked-rev[of L rev S M] assms by auto

lemma get-rev-level-skip-beginning-not-marked[simp]:
  assumes atm-of L ∉ atm-of ‘ lit-of ‘(set S)
  and ∀ s∈set S. ¬is-marked s
  shows get-rev-level (rev S @ rev M) 0 L = get-level M L
  using get-level-skip-beginning-not-marked-rev[of L rev S M] assms by auto

lemma get-level-skip-in-all-not-marked:
  fixes M :: ('a, nat, 'b) marked-lit list and L :: 'a literal
  assumes ∀ m∈set M. ¬ is-marked m
  and atm-of L ∈ atm-of ‘ lit-of ‘ (set M)
  shows get-rev-level M n L = n
  using assms by (induction M rule: marked-lit-list-induct) auto
```

```
lemma get-level-skip-all-not-marked[simp]:
  fixes M
  defines M' ≡ rev M
  assumes ∀ m∈set M. ¬ is-marked m
  shows get-level M L = 0
proof –
  have M: M = rev M'
    unfolding M'-def by auto
  show ?thesis
    using assms unfolding M by (induction M' rule: marked-lit-list-induct) auto
qed

abbreviation MMax M ≡ Max (set-mset M)
```

the {#0::'a#} is there to ensures that the set is not empty.

```
definition get-maximum-level :: ('a, nat, 'b) marked-lit list ⇒ 'a literal multiset ⇒ nat
  where
get-maximum-level M D = MMax ({#0#} + image-mset (get-level M) D)

lemma get-maximum-level-ge-get-level:
  L ∈# D ⟹ get-maximum-level M D ≥ get-level M L
  unfolding get-maximum-level-def by auto

lemma get-maximum-level-empty[simp]:
  get-maximum-level M {#} = 0
  unfolding get-maximum-level-def by auto

lemma get-maximum-level-exists-lit-of-max-level:
  D ≠ {#} ⟹ ∃ L∈# D. get-level M L = get-maximum-level M D
  unfolding get-maximum-level-def
  apply (induct D)
   apply simp
  by (rename-tac D x, case-tac D = {#}) (auto simp add: max-def)


lemma get-maximum-level-empty-list[simp]:
  get-maximum-level [] D = 0
  unfolding get-maximum-level-def by (simp add: image-constant-conv)

lemma get-maximum-level-single[simp]:
  get-maximum-level M {#L#} = get-level M L
  unfolding get-maximum-level-def by simp

lemma get-maximum-level-plus:
  get-maximum-level M (D + D') = max (get-maximum-level M D) (get-maximum-level M D')
  by (induct D) (auto simp add: get-maximum-level-def)

lemma get-maximum-level-exists-lit:
  assumes n: n > 0
  and max: get-maximum-level M D = n
  shows ∃ L ∈#D. get-level M L = n
proof –
  have f: finite (insert 0 ((λL. get-level M L) ‘ set-mset D)) by auto
  then have n ∈ ((λL. get-level M L) ‘ set-mset D)
    using n max Max-in[OF f] unfolding get-maximum-level-def by simp
  then show ∃ L ∈# D. get-level M L = n by auto
qed

lemma get-maximum-level-skip-first[simp]:
  assumes atm-of L ∉ atms-of D
  shows get-maximum-level (Propagated L C # M) D = get-maximum-level M D
  using assms unfolding get-maximum-level-def atms-of-def
    atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
  by (smt atm-of-in-atm-of-set-in-uminus get-level-skip-beginning image-iff marked-lit.sel(2)
    multiset.map-cong0)

lemma get-maximum-level-skip-beginning:
  assumes DH: atms-of D ⊆ atm-of ‘lits-of-l H
  shows get-maximum-level (c @ Marked Kh i # H) D = get-maximum-level H D
proof –
  have (get-rev-level (rev H @ Marked Kh i # rev c) 0) ‘ set-mset D
      = (get-rev-level (rev H) 0) ‘ set-mset D
    using DH unfolding atms-of-def
    by (metis (no-types, lifting) get-rev-level-skip-end image-cong image-subset-iff set-rev)
  then show ?thesis using DH unfolding get-maximum-level-def by auto
qed

lemma get-maximum-level-D-single-propagated:
  get-maximum-level [Propagated x21 x22] D = 0
proof –
  have A: insert 0 ((λL. 0) ‘ (set-mset D ∩ {L. atm-of x21 = atm-of L})
      ∪ (λL. 0) ‘ (set-mset D ∩ {L. atm-of x21 ≠ atm-of L})) = {0}
    by auto
  show ?thesis unfolding get-maximum-level-def by (simp add: A)
qed

lemma get-maximum-level-skip-notin:
  assumes D: ∀ L∈#D. atm-of L ∈ atm-of ‘lits-of-l M
  shows get-maximum-level M D = get-maximum-level (Propagated x21 x22 # M) D
proof –
  have A: (get-rev-level (rev M @ [Propagated x21 x22]) 0) ‘ set-mset D
      = (get-rev-level (rev M) 0) ‘ set-mset D
    using D by (auto intro!: image-cong simp add:  lits-of-def)
  show ?thesis unfolding get-maximum-level-def by (auto simp: A)
qed

lemma get-maximum-level-skip-un-marked-not-present:
  assumes ∀ L∈#D. atm-of L ∈ atm-of ‘ lits-of-l aa and
  ∀ m∈set M. ¬ is-marked m
  shows get-maximum-level aa D = get-maximum-level (M @ aa) D
  using assms by (induction M rule: marked-lit-list-induct)
  (auto intro!: get-maximum-level-skip-notin[of D - @ aa] simp add: image-Un)

lemma get-maximum-level-union-mset:
  get-maximum-level M (A #∪ B) = get-maximum-level M (A + B)
  unfolding get-maximum-level-def by (auto simp: image-Un)

fun get-maximum-possible-level:: ('b, nat, 'c) marked-lit list ⇒ nat   where
get-maximum-possible-level [] = 0 |
get-maximum-possible-level (Marked K i # l) = max i (get-maximum-possible-level l) |
get-maximum-possible-level (Propagated - - # l) = get-maximum-possible-level l

lemma get-maximum-possible-level-append[simp]:
  get-maximum-possible-level (M@M')
    = max (get-maximum-possible-level M) (get-maximum-possible-level M')
  by (induct M rule: marked-lit-list-induct) auto

lemma get-maximum-possible-level-rev[simp]:
  get-maximum-possible-level (rev M) = get-maximum-possible-level M
  by (induct M rule: marked-lit-list-induct) auto

lemma get-maximum-possible-level-ge-get-rev-level:
  max (get-maximum-possible-level M) i ≥ get-rev-level M i L
  by (induct M arbitrary: i rule: marked-lit-list-induct) (auto simp add: le-max-iff-disj)

lemma get-maximum-possible-level-ge-get-level[simp]:
  get-maximum-possible-level M ≥ get-level M L
  using get-maximum-possible-level-ge-get-rev-level[of rev - 0] by auto

lemma get-maximum-possible-level-ge-get-maximum-level[simp]:
  get-maximum-possible-level M ≥ get-maximum-level M D
  using get-maximum-level-exists-lit-of-max-level unfolding Bex-def
  by (metis get-maximum-level-empty get-maximum-possible-level-ge-get-level le0)

fun get-all-mark-of-propagated where
get-all-mark-of-propagated [] = [] |
get-all-mark-of-propagated (Marked - - # L) = get-all-mark-of-propagated L |
get-all-mark-of-propagated (Propagated - mark # L) = mark # get-all-mark-of-propagated L

lemma get-all-mark-of-propagated-append[simp]:
  get-all-mark-of-propagated (A @ B) = get-all-mark-of-propagated A @ get-all-mark-of-propagated B
  by (induct A rule: marked-lit-list-induct) auto
```

18.5.2 Properties about the levels

```
fun get-all-levels-of-marked :: ('b, 'a, 'c) marked-lit list ⇒ 'a list  where
get-all-levels-of-marked [] = [] |
get-all-levels-of-marked (Marked l level # Ls) = level # get-all-levels-of-marked Ls |
get-all-levels-of-marked (Propagated - - # Ls) = get-all-levels-of-marked Ls

lemma get-all-levels-of-marked-nil-iff-not-is-marked:
  get-all-levels-of-marked xs = [] ⟷ (∀ x ∈ set xs. ¬is-marked x)
  using assms by (induction xs rule: marked-lit-list-induct) auto

lemma get-all-levels-of-marked-cons:
  get-all-levels-of-marked (a # b) =
    (if is-marked a then [level-of a] else []) @ get-all-levels-of-marked b
  by (cases a) simp-all

lemma get-all-levels-of-marked-append[simp]:
  get-all-levels-of-marked (a @ b) = get-all-levels-of-marked a @ get-all-levels-of-marked b
  by (induct a) (simp-all add: get-all-levels-of-marked-cons)

lemma in-get-all-levels-of-marked-iff-decomp:
  i ∈ set (get-all-levels-of-marked M) ⟷ (∃ c K c'. M = c @ Marked K i # c') (is ?A ⟷ ?B)
proof
```

```
  assume ?B
  then show ?A by auto
next
  assume ?A
  then show ?B
    apply (induction M rule: marked-lit-list-induct)
      apply auto[]
     apply (metis append-Cons append-Nil get-all-levels-of-marked.simps(2) set-ConsD)
    by (metis append-Cons get-all-levels-of-marked.simps(3))
qed

lemma get-rev-level-less-max-get-all-levels-of-marked:
  get-rev-level M n L ≤ Max (set (n # get-all-levels-of-marked M))
  by (induct M arbitrary: n rule: get-all-levels-of-marked.induct)
     (simp-all add: max.coboundedI2)

lemma get-rev-level-ge-min-get-all-levels-of-marked:
  assumes atm-of L ∈ atm-of ` lits-of-l M
  shows get-rev-level M n L ≥ Min (set (n # get-all-levels-of-marked M))
  using assms by (induct M arbitrary: n rule: get-all-levels-of-marked.induct)
    (auto simp add: min-le-iff-disj)

lemma get-all-levels-of-marked-rev-eq-rev-get-all-levels-of-marked[simp]:
  get-all-levels-of-marked (rev M) = rev (get-all-levels-of-marked M)
  by (induct M rule: get-all-levels-of-marked.induct)
     (simp-all add: max.coboundedI2)

lemma get-maximum-possible-level-max-get-all-levels-of-marked:
  get-maximum-possible-level M = Max (insert 0 (set (get-all-levels-of-marked M)))
  by (induct M rule: marked-lit-list-induct) (auto simp: insert-commute)

lemma get-rev-level-in-levels-of-marked:
  get-rev-level M n L ∈ {0, n} ∪ set (get-all-levels-of-marked M)
  by (induction M arbitrary: n rule: marked-lit-list-induct) (force simp add: atm-of-eq-atm-of)+

lemma get-rev-level-in-atms-in-levels-of-marked:
  atm-of L ∈ atm-of ` (lits-of-l M) ⟹
    get-rev-level M n L ∈ {n} ∪ set (get-all-levels-of-marked M)
  by (induction M arbitrary: n rule: marked-lit-list-induct) (auto simp add: atm-of-eq-atm-of)

lemma get-all-levels-of-marked-no-marked:
  (∀ l∈set Ls. ¬ is-marked l) ⟷ get-all-levels-of-marked Ls = []
  by (induction Ls) (auto simp add: get-all-levels-of-marked-cons)

lemma get-level-in-levels-of-marked:
  get-level M L ∈ {0} ∪ set (get-all-levels-of-marked M)
  using get-rev-level-in-levels-of-marked[of rev M 0 L] by auto
```

The zero is here to avoid empty-list issues with *last*:

```
lemma get-level-get-rev-level-get-all-levels-of-marked:
  assumes atm-of L ∉ atm-of ` (lits-of-l M)
  shows
    get-level (K @ M) L = get-rev-level (rev K) (last (0 # get-all-levels-of-marked (rev M))) L
  using assms
proof (induct M arbitrary: K)
```

```
  case Nil
  then show ?case by auto
next
  case (Cons a M)
  then have H: ⋀K. get-level (K @ M) L
    = get-rev-level (rev K) (last (0 # get-all-levels-of-marked (rev M))) L
    by auto
  have get-level ((K @ [a])@ M) L
    = get-rev-level (a # rev K) (last (0 # get-all-levels-of-marked (rev M))) L
    using H[of K @ [a]] by simp
  then show ?case using Cons(2) by (cases a) auto
qed

lemma get-rev-level-can-skip-correctly-ordered:
  assumes
    no-dup M and
    atm-of L ∉ atm-of ‘ (lits-of-l M) and
    get-all-levels-of-marked M = rev [Suc 0..<Suc (length (get-all-levels-of-marked M))]
  shows get-rev-level (rev M @ K) 0 L = get-rev-level K (length (get-all-levels-of-marked M)) L
  using assms
proof (induct M arbitrary: K rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
  case (marked L' i M K)
  then have
    i: i = Suc (length (get-all-levels-of-marked M)) and
    get-all-levels-of-marked M = rev [Suc 0..<Suc (length (get-all-levels-of-marked M))]
    by auto
  then have get-rev-level (rev M @ (Marked L' i # K)) 0 L
    = get-rev-level (Marked L' i # K) (length (get-all-levels-of-marked M)) L
    using marked by auto
  then show ?case using marked unfolding i by auto
next
  case (proped L' D M K)
  then have get-all-levels-of-marked M = rev [Suc 0..<Suc (length (get-all-levels-of-marked M))]
    by auto
  then have get-rev-level (rev M @ (Propagated L' D # K)) 0 L
    = get-rev-level (Propagated L' D # K) (length (get-all-levels-of-marked M)) L
    using proped by auto
  then show ?case using proped by auto
qed

lemma get-level-skip-beginning-hd-get-all-levels-of-marked:
  assumes atm-of L ∉ atm-of ‘ lits-of-l S and get-all-levels-of-marked S ≠ []
  shows get-level (M@ S) L = get-rev-level (rev M) (hd (get-all-levels-of-marked S)) L
  using assms
proof (induction S arbitrary: M rule: marked-lit-list-induct)
  case nil
  then show ?case by (auto simp add: lits-of-def)
next
  case (marked K m) note notin = this(2)
  then show ?case by (auto simp add: lits-of-def)
next
  case (proped L l) note IH = this(1) and L = this(2) and neq = this(3)
  show ?case using IH[of M@[Propagated L l]] L neq by (auto simp add: atm-of-eq-atm-of)
qed

end
theory CDCL-W
imports CDCL-Abstract-Clause-Representation List-More CDCL-W-Level Wellfounded-More

begin
```

19 Weidenbach's CDCL

```
declare upt.simps(2)[simp del]
```

19.1 The State

We will abstract the representation of clause and clauses via two locales. We expect our representation to behave like multiset, but the internal representation can be done using list or whatever other representation.

```
locale state_W-ops =
  raw-clss mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    +
  raw-ccls-union mset-ccls union-ccls insert-ccls remove-clit
  for
    — Clause
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and

    — Multiset of Clauses
    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss  and

    mset-ccls:: 'ccls ⇒ 'v clause and
    union-ccls :: 'ccls ⇒ 'ccls ⇒ 'ccls and
    insert-ccls :: 'v literal ⇒ 'ccls ⇒ 'ccls and
    remove-clit :: 'v literal ⇒ 'ccls ⇒ 'ccls
    +
  fixes
    ccls-of-cls :: 'cls ⇒ 'ccls and
    cls-of-ccls :: 'ccls ⇒ 'cls and

    trail :: 'st ⇒ ('v, nat, 'v clause) marked-lits and
    hd-raw-trail :: 'st ⇒ ('v, nat, 'cls) marked-lit and
    raw-init-clss :: 'st ⇒ 'clss and
    raw-learned-clss :: 'st ⇒ 'clss and
    backtrack-lvl :: 'st ⇒ nat and
    raw-conflicting :: 'st ⇒ 'ccls option and

    cons-trail :: ('v, nat, 'cls) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒ 'st and
    add-init-cls :: 'cls ⇒ 'st ⇒ 'st and
    add-learned-cls :: 'cls ⇒ 'st ⇒ 'st and
    remove-cls :: 'cls ⇒ 'st ⇒ 'st and
    update-backtrack-lvl :: nat ⇒ 'st ⇒ 'st and
    update-conflicting :: 'ccls option ⇒ 'st ⇒ 'st and

    init-state :: 'clss ⇒ 'st and
    restart-state :: 'st ⇒ 'st
  assumes
    mset-ccls-ccls-of-cls[simp]:
      mset-ccls (ccls-of-cls C) = mset-cls C and
    mset-cls-cls-of-ccls[simp]:
      mset-cls (cls-of-ccls D) = mset-ccls D and
    ex-mset-cls: ∃ a. mset-cls a = E
begin
fun mmset-of-mlit :: ('a, 'b, 'cls) marked-lit ⇒ ('a, 'b, 'v clause) marked-lit
  where
mmset-of-mlit (Propagated L C) = Propagated L (mset-cls C) |
mmset-of-mlit (Marked L i) = Marked L i

lemma lit-of-mmset-of-mlit[simp]:
  lit-of (mmset-of-mlit a) = lit-of a
  by (cases a) auto

lemma lit-of-mmset-of-mlit-set-lit-of-l[simp]:
  lit-of ‘ mmset-of-mlit ‘ set M' = lits-of-l M'
  by (induction M') auto

lemma map-mmset-of-mlit-true-annots-true-cls[simp]:
  map mmset-of-mlit M' ⊨as C ⟷ M' ⊨as C
  by (simp add: true-annots-true-cls lits-of-def)

abbreviation init-clss ≡ λS. mset-clss (raw-init-clss S)
abbreviation learned-clss ≡ λS. mset-clss (raw-learned-clss S)
abbreviation conflicting ≡ λS. map-option mset-ccls (raw-conflicting S)

notation insert-cls (infix !++ 50)

notation in-clss (infix !∈! 50)
notation union-clss (infix ⊕ 50)
notation insert-clss (infix !++! 50)

notation union-ccls (infix !∪ 50)

definition raw-clauses :: 'st ⇒ 'clss where
raw-clauses S = union-clss (raw-init-clss S) (raw-learned-clss S)

abbreviation clauses :: 'st ⇒ 'v clauses where
clauses S ≡ mset-clss (raw-clauses S)

end

locale state_W =
  state_W-ops
    — functions for clauses:
```

mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss

— functions for the conflicting clause:
mset-ccls union-ccls insert-ccls remove-clit

— Conversion between conflicting and non-conflicting
ccls-of-cls cls-of-ccls

— functions for the state:
— access functions:
trail hd-raw-trail raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting
— changing state:
cons-trail tl-trail add-init-cls add-learned-cls remove-cls update-backtrack-lvl
update-conflicting

— get state:
init-state
restart-state
for
mset-cls:: *'cls* ⇒ *'v clause* **and**
insert-cls :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**
remove-lit :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**

mset-clss:: *'clss* ⇒ *'v clauses* **and**
union-clss :: *'clss* ⇒ *'clss* ⇒ *'clss* **and**
in-clss :: *'cls* ⇒ *'clss* ⇒ *bool* **and**
insert-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**
remove-from-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**

mset-ccls:: *'ccls* ⇒ *'v clause* **and**
union-ccls :: *'ccls* ⇒ *'ccls* ⇒ *'ccls* **and**
insert-ccls :: *'v literal* ⇒ *'ccls* ⇒ *'ccls* **and**
remove-clit :: *'v literal* ⇒ *'ccls* ⇒ *'ccls* **and**

ccls-of-cls :: *'cls* ⇒ *'ccls* **and**
cls-of-ccls :: *'ccls* ⇒ *'cls* **and**

trail :: *'st* ⇒ (*'v*, *nat*, *'v clause*) *marked-lits* **and**
hd-raw-trail :: *'st* ⇒ (*'v*, *nat*, *'cls*) *marked-lit* **and**
raw-init-clss :: *'st* ⇒ *'clss* **and**
raw-learned-clss :: *'st* ⇒ *'clss* **and**
backtrack-lvl :: *'st* ⇒ *nat* **and**
raw-conflicting :: *'st* ⇒ *'ccls option* **and**

cons-trail :: (*'v*, *nat*, *'cls*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-init-cls :: *'cls* ⇒ *'st* ⇒ *'st* **and**
add-learned-cls :: *'cls* ⇒ *'st* ⇒ *'st* **and**
remove-cls :: *'cls* ⇒ *'st* ⇒ *'st* **and**
update-backtrack-lvl :: *nat* ⇒ *'st* ⇒ *'st* **and**
update-conflicting :: *'ccls option* ⇒ *'st* ⇒ *'st* **and**

init-state :: *'clss* ⇒ *'st* **and**
restart-state :: *'st* ⇒ *'st* +

assumes
hd-raw-trail: *trail S* $\neq$ [] $\Longrightarrow$ *mmset-of-mlit* (*hd-raw-trail S*) = *hd* (*trail S*) **and**
trail-cons-trail[*simp*]:
$\bigwedge$*L st. undefined-lit* (*trail st*) (*lit-of L*) $\Longrightarrow$
trail (*cons-trail L st*) = *mmset-of-mlit L* # *trail st* **and**
trail-tl-trail[*simp*]: $\bigwedge$*st. trail* (*tl-trail st*) = *tl* (*trail st*) **and**
trail-add-init-cls[*simp*]:
$\bigwedge$*st C. no-dup* (*trail st*) $\Longrightarrow$ *trail* (*add-init-cls C st*) = *trail st* **and**
trail-add-learned-cls[*simp*]:
$\bigwedge$*C st. no-dup* (*trail st*) $\Longrightarrow$ *trail* (*add-learned-cls C st*) = *trail st* **and**
trail-remove-cls[*simp*]:
$\bigwedge$*C st. trail* (*remove-cls C st*) = *trail st* **and**
trail-update-backtrack-lvl[*simp*]: $\bigwedge$*st C. trail* (*update-backtrack-lvl C st*) = *trail st* **and**
trail-update-conflicting[*simp*]: $\bigwedge$*C st. trail* (*update-conflicting C st*) = *trail st* **and**

init-clss-cons-trail[*simp*]:
$\bigwedge$*M st. undefined-lit* (*trail st*) (*lit-of M*) $\Longrightarrow$
init-clss (*cons-trail M st*) = *init-clss st*
and
init-clss-tl-trail[*simp*]:
$\bigwedge$*st. init-clss* (*tl-trail st*) = *init-clss st* **and**
init-clss-add-init-cls[*simp*]:
$\bigwedge$*st C. no-dup* (*trail st*) $\Longrightarrow$ *init-clss* (*add-init-cls C st*) = {#*mset-cls C*#} + *init-clss st*
and
init-clss-add-learned-cls[*simp*]:
$\bigwedge$*C st. no-dup* (*trail st*) $\Longrightarrow$ *init-clss* (*add-learned-cls C st*) = *init-clss st* **and**
init-clss-remove-cls[*simp*]:
$\bigwedge$*C st. init-clss* (*remove-cls C st*) = *removeAll-mset* (*mset-cls C*) (*init-clss st*) **and**
init-clss-update-backtrack-lvl[*simp*]:
$\bigwedge$*st C. init-clss* (*update-backtrack-lvl C st*) = *init-clss st* **and**
init-clss-update-conflicting[*simp*]:
$\bigwedge$*C st. init-clss* (*update-conflicting C st*) = *init-clss st* **and**

learned-clss-cons-trail[*simp*]:
$\bigwedge$*M st. undefined-lit* (*trail st*) (*lit-of M*) $\Longrightarrow$
learned-clss (*cons-trail M st*) = *learned-clss st* **and**
learned-clss-tl-trail[*simp*]:
$\bigwedge$*st. learned-clss* (*tl-trail st*) = *learned-clss st* **and**
learned-clss-add-init-cls[*simp*]:
$\bigwedge$*st C. no-dup* (*trail st*) $\Longrightarrow$ *learned-clss* (*add-init-cls C st*) = *learned-clss st* **and**
learned-clss-add-learned-cls[*simp*]:
$\bigwedge$*C st. no-dup* (*trail st*) $\Longrightarrow$
learned-clss (*add-learned-cls C st*) = {#*mset-cls C*#} + *learned-clss st* **and**
learned-clss-remove-cls[*simp*]:
$\bigwedge$*C st. learned-clss* (*remove-cls C st*) = *removeAll-mset* (*mset-cls C*) (*learned-clss st*) **and**
learned-clss-update-backtrack-lvl[*simp*]:
$\bigwedge$*st C. learned-clss* (*update-backtrack-lvl C st*) = *learned-clss st* **and**
learned-clss-update-conflicting[*simp*]:
$\bigwedge$*C st. learned-clss* (*update-conflicting C st*) = *learned-clss st* **and**

backtrack-lvl-cons-trail[*simp*]:
$\bigwedge$*M st. undefined-lit* (*trail st*) (*lit-of M*) $\Longrightarrow$
backtrack-lvl (*cons-trail M st*) = *backtrack-lvl st* **and**
backtrack-lvl-tl-trail[*simp*]:
$\bigwedge$*st. backtrack-lvl* (*tl-trail st*) = *backtrack-lvl st* **and**

backtrack-lvl-add-init-cls[*simp*]:
⋀*st C*. *no-dup* (*trail st*) ⟹ *backtrack-lvl* (*add-init-cls C st*) = *backtrack-lvl st* **and**
backtrack-lvl-add-learned-cls[*simp*]:
⋀*C st*. *no-dup* (*trail st*) ⟹ *backtrack-lvl* (*add-learned-cls C st*) = *backtrack-lvl st* **and**
backtrack-lvl-remove-cls[*simp*]:
⋀*C st*. *backtrack-lvl* (*remove-cls C st*) = *backtrack-lvl st* **and**
backtrack-lvl-update-backtrack-lvl[*simp*]:
⋀*st k*. *backtrack-lvl* (*update-backtrack-lvl k st*) = *k* **and**
backtrack-lvl-update-conflicting[*simp*]:
⋀*C st*. *backtrack-lvl* (*update-conflicting C st*) = *backtrack-lvl st* **and**

conflicting-cons-trail[*simp*]:
⋀*M st*. *undefined-lit* (*trail st*) (*lit-of M*) ⟹
conflicting (*cons-trail M st*) = *conflicting st* **and**
conflicting-tl-trail[*simp*]:
⋀*st*. *conflicting* (*tl-trail st*) = *conflicting st* **and**
conflicting-add-init-cls[*simp*]:
⋀*st C*. *no-dup* (*trail st*) ⟹ *conflicting* (*add-init-cls C st*) = *conflicting st* **and**
conflicting-add-learned-cls[*simp*]:
⋀*C st*. *no-dup* (*trail st*) ⟹ *conflicting* (*add-learned-cls C st*) = *conflicting st*
and
conflicting-remove-cls[*simp*]:
⋀*C st*. *conflicting* (*remove-cls C st*) = *conflicting st* **and**
conflicting-update-backtrack-lvl[*simp*]:
⋀*st C*. *conflicting* (*update-backtrack-lvl C st*) = *conflicting st* **and**
conflicting-update-conflicting[*simp*]:
⋀*C st*. *raw-conflicting* (*update-conflicting C st*) = *C* **and**

init-state-trail[*simp*]: ⋀*N*. *trail* (*init-state N*) = [] **and**
init-state-clss[*simp*]: ⋀*N*. (*init-clss* (*init-state N*)) = *mset-clss N* **and**
init-state-learned-clss[*simp*]: ⋀*N*. *learned-clss* (*init-state N*) = {#} **and**
init-state-backtrack-lvl[*simp*]: ⋀*N*. *backtrack-lvl* (*init-state N*) = *0* **and**
init-state-conflicting[*simp*]: ⋀*N*. *conflicting* (*init-state N*) = *None* **and**

trail-restart-state[*simp*]: *trail* (*restart-state S*) = [] **and**
init-clss-restart-state[*simp*]: *init-clss* (*restart-state S*) = *init-clss S* **and**
learned-clss-restart-state[*intro*]:
learned-clss (*restart-state S*) ⊆# *learned-clss S* **and**
backtrack-lvl-restart-state[*simp*]: *backtrack-lvl* (*restart-state S*) = *0* **and**
conflicting-restart-state[*simp*]: *conflicting* (*restart-state S*) = *None*
begin

lemma
shows
clauses-cons-trail[*simp*]:
undefined-lit (*trail S*) (*lit-of M*) ⟹ *clauses* (*cons-trail M S*) = *clauses S* **and**

clss-tl-trail[*simp*]: *clauses* (*tl-trail S*) = *clauses S* **and**
clauses-add-learned-cls-unfolded:
no-dup (*trail S*) ⟹ *clauses* (*add-learned-cls U S*) =
{#*mset-cls U*#} + *learned-clss S* + *init-clss S*
and
clauses-add-init-cls[*simp*]:
no-dup (*trail S*) ⟹
clauses (*add-init-cls N S*) = {#*mset-cls N*#} + *init-clss S* + *learned-clss S* **and**

clauses-update-backtrack-lvl[*simp*]: *clauses* (*update-backtrack-lvl k S*) = *clauses S* **and**
clauses-update-conflicting[*simp*]: *clauses* (*update-conflicting D S*) = *clauses S* **and**
clauses-remove-cls[*simp*]:
clauses (*remove-cls C S*) = *removeAll-mset* (*mset-cls C*) (*clauses S*) **and**
clauses-add-learned-cls[*simp*]:
no-dup (*trail S*) ⟹ *clauses* (*add-learned-cls C S*) = {#*mset-cls C*#} + *clauses S* **and**
clauses-restart[*simp*]: *clauses* (*restart-state S*) ⊆# *clauses S* **and**
clauses-init-state[*simp*]: ⋀*N*. *clauses* (*init-state N*) = *mset-clss N*
prefer *9* **using** *raw-clauses-def learned-clss-restart-state* **apply** *fastforce*
by (*auto simp*: *ac-simps replicate-mset-plus raw-clauses-def intro*: *multiset-eqI*)

abbreviation *state* :: *'st* ⇒ (*'v*, *nat*, *'v clause*) *marked-lit list* × *'v clauses* × *'v clauses*
× *nat* × *'v clause option* **where**
state S ≡ (*trail S*, *init-clss S*, *learned-clss S*, *backtrack-lvl S*, *conflicting S*)

abbreviation *incr-lvl* :: *'st* ⇒ *'st* **where**
incr-lvl S ≡ *update-backtrack-lvl* (*backtrack-lvl S* + *1*) *S*

definition *state-eq* :: *'st* ⇒ *'st* ⇒ *bool* (**infix** ∼ *50*) **where**
S ∼ *T* ⟷ *state S* = *state T*

lemma *state-eq-ref*[*simp*, *intro*]:
S ∼ *S*
unfolding *state-eq-def* **by** *auto*

lemma *state-eq-sym*:
S ∼ *T* ⟷ *T* ∼ *S*
unfolding *state-eq-def* **by** *auto*

lemma *state-eq-trans*:
S ∼ *T* ⟹ *T* ∼ *U* ⟹ *S* ∼ *U*
unfolding *state-eq-def* **by** *auto*

lemma
shows
state-eq-trail: *S* ∼ *T* ⟹ *trail S* = *trail T* **and**
state-eq-init-clss: *S* ∼ *T* ⟹ *init-clss S* = *init-clss T* **and**
state-eq-learned-clss: *S* ∼ *T* ⟹ *learned-clss S* = *learned-clss T* **and**
state-eq-backtrack-lvl: *S* ∼ *T* ⟹ *backtrack-lvl S* = *backtrack-lvl T* **and**
state-eq-conflicting: *S* ∼ *T* ⟹ *conflicting S* = *conflicting T* **and**
state-eq-clauses: *S* ∼ *T* ⟹ *clauses S* = *clauses T* **and**
state-eq-undefined-lit: *S* ∼ *T* ⟹ *undefined-lit* (*trail S*) *L* = *undefined-lit* (*trail T*) *L*
unfolding *state-eq-def raw-clauses-def* **by** *auto*

lemma *state-eq-raw-conflicting-None*:
S ∼ *T* ⟹ *conflicting T* = *None* ⟹ *raw-conflicting S* = *None*
unfolding *state-eq-def raw-clauses-def* **by** *auto*

lemmas *state-simp*[*simp*] = *state-eq-trail state-eq-init-clss state-eq-learned-clss*
state-eq-backtrack-lvl state-eq-conflicting state-eq-clauses state-eq-undefined-lit
state-eq-raw-conflicting-None

lemma *atms-of-ms-learned-clss-restart-state-in-atms-of-ms-learned-clssI*[*intro*]:
x ∈ *atms-of-mm* (*learned-clss* (*restart-state S*)) ⟹ *x* ∈ *atms-of-mm* (*learned-clss S*)
by (*meson atms-of-ms-mono learned-clss-restart-state set-mset-mono subsetCE*)

```
function reduce-trail-to :: 'a list ⇒ 'st ⇒ 'st where
reduce-trail-to F S =
  (if length (trail S) = length F ∨ trail S = [] then S else reduce-trail-to F (tl-trail S))
by fast+
termination
  by (relation measure (λ(-, S). length (trail S))) simp-all

declare reduce-trail-to.simps[simp del]

lemma
  shows
    reduce-trail-to-nil[simp]: trail S = [] ⟹ reduce-trail-to F S = S and
    reduce-trail-to-eq-length[simp]: length (trail S) = length F ⟹ reduce-trail-to F S = S
  by (auto simp: reduce-trail-to.simps)

lemma reduce-trail-to-length-ne:
  length (trail S) ≠ length F ⟹ trail S ≠ [] ⟹
    reduce-trail-to F S = reduce-trail-to F (tl-trail S)
  by (auto simp: reduce-trail-to.simps)

lemma trail-reduce-trail-to-length-le:
  assumes length F > length (trail S)
  shows trail (reduce-trail-to F S) = []
  using assms apply (induction F S rule: reduce-trail-to.induct)
  by (metis (no-types, hide-lams) length-tl less-imp-diff-less less-irrefl trail-tl-trail
    reduce-trail-to.simps)

lemma trail-reduce-trail-to-nil[simp]:
  trail (reduce-trail-to [] S) = []
  apply (induction []::('v, nat, 'v clause) marked-lits S rule: reduce-trail-to.induct)
  by (metis length-0-conv reduce-trail-to-length-ne reduce-trail-to-nil)

lemma clauses-reduce-trail-to-nil:
  clauses (reduce-trail-to [] S) = clauses S
proof (induction [] S rule: reduce-trail-to.induct)
  case (1 Sa)
  then have clauses (reduce-trail-to ([]::'a list) (tl-trail Sa)) = clauses (tl-trail Sa)
    ∨ trail Sa = []
    by fastforce
  then show clauses (reduce-trail-to ([]::'a list) Sa) = clauses Sa
    by (metis (no-types) length-0-conv reduce-trail-to-eq-length clss-tl-trail
      reduce-trail-to-length-ne)
qed

lemma reduce-trail-to-skip-beginning:
  assumes trail S = F' @ F
  shows trail (reduce-trail-to F S) = F
  using assms by (induction F' arbitrary: S) (auto simp: reduce-trail-to-length-ne)

lemma clauses-reduce-trail-to[simp]:
  clauses (reduce-trail-to F S) = clauses S
  apply (induction F S rule: reduce-trail-to.induct)
  by (metis clss-tl-trail reduce-trail-to.simps)
```

```
lemma conflicting-update-trail[simp]:
  conflicting (reduce-trail-to F S) = conflicting S
  apply (induction F S rule: reduce-trail-to.induct)
  by (metis conflicting-tl-trail reduce-trail-to.simps)

lemma backtrack-lvl-update-trail[simp]:
  backtrack-lvl (reduce-trail-to F S) = backtrack-lvl S
  apply (induction F S rule: reduce-trail-to.induct)
  by (metis backtrack-lvl-tl-trail reduce-trail-to.simps)

lemma init-clss-update-trail[simp]:
  init-clss (reduce-trail-to F S) = init-clss S
  apply (induction F S rule: reduce-trail-to.induct)
  by (metis init-clss-tl-trail reduce-trail-to.simps)

lemma learned-clss-update-trail[simp]:
  learned-clss (reduce-trail-to F S) = learned-clss S
  apply (induction F S rule: reduce-trail-to.induct)
  by (metis learned-clss-tl-trail reduce-trail-to.simps)


lemma raw-conflicting-reduce-trail-to[simp]:
  raw-conflicting (reduce-trail-to F S) = None ⟷ raw-conflicting S = None
  apply (induction F S rule: reduce-trail-to.induct)
  by (metis conflicting-update-trail map-option-is-None)

lemma trail-eq-reduce-trail-to-eq:
  trail S = trail T ⟹ trail (reduce-trail-to F S) = trail (reduce-trail-to F T)
  apply (induction F S arbitrary: T rule: reduce-trail-to.induct)
  by (metis trail-tl-trail reduce-trail-to.simps)

lemma reduce-trail-to-state-eq_NOT-compatible:
  assumes ST: S ∼ T
  shows reduce-trail-to F S ∼ reduce-trail-to F T
proof –
  have trail (reduce-trail-to F S) = trail (reduce-trail-to F T)
    using trail-eq-reduce-trail-to-eq[of S T F] ST by auto
  then show ?thesis using ST by (auto simp del: state-simp simp: state-eq-def)
qed

lemma reduce-trail-to-trail-tl-trail-decomp[simp]:
  trail S = F' @ Marked K d # F ⟹ (trail (reduce-trail-to F S)) = F
  apply (rule reduce-trail-to-skip-beginning[of - F' @ Marked K d # []])
  by (cases F') (auto simp add:tl-append reduce-trail-to-skip-beginning)

lemma reduce-trail-to-add-learned-cls[simp]:
  no-dup (trail S) ⟹
    trail (reduce-trail-to F (add-learned-cls C S)) = trail (reduce-trail-to F S)
  by (rule trail-eq-reduce-trail-to-eq) auto

lemma reduce-trail-to-add-init-cls[simp]:
  no-dup (trail S) ⟹
    trail (reduce-trail-to F (add-init-cls C S)) = trail (reduce-trail-to F S)
  by (rule trail-eq-reduce-trail-to-eq) auto
```

```
lemma reduce-trail-to-remove-learned-cls[simp]:
  trail (reduce-trail-to F (remove-cls C S)) = trail (reduce-trail-to F S)
  by (rule trail-eq-reduce-trail-to-eq) auto

lemma reduce-trail-to-update-conflicting[simp]:
  trail (reduce-trail-to F (update-conflicting C S)) = trail (reduce-trail-to F S)
  by (rule trail-eq-reduce-trail-to-eq) auto

lemma reduce-trail-to-update-backtrack-lvl[simp]:
  trail (reduce-trail-to F (update-backtrack-lvl C S)) = trail (reduce-trail-to F S)
  by (rule trail-eq-reduce-trail-to-eq) auto

lemma in-get-all-marked-decomposition-marked-or-empty:
  assumes (a, b) ∈ set (get-all-marked-decomposition M)
  shows a = [] ∨ (is-marked (hd a))
  using assms
proof (induct M arbitrary: a b)
  case Nil then show ?case by simp
next
  case (Cons m M)
  show ?case
    proof (cases m)
      case (Marked l mark)
      then show ?thesis using Cons by auto
    next
      case (Propagated l mark)
      then show ?thesis using Cons by (cases get-all-marked-decomposition M) force+
    qed
qed

lemma reduce-trail-to-length:
  length M = length M' ⟹ reduce-trail-to M S = reduce-trail-to M' S
  apply (induction M S arbitrary:  rule: reduce-trail-to.induct)
  by (simp add: reduce-trail-to.simps)

lemma trail-reduce-trail-to-drop:
  trail (reduce-trail-to F S) =
    (if length (trail S) ≥ length F
    then drop (length (trail S) − length F) (trail S)
    else [])
  apply (induction F S rule: reduce-trail-to.induct)
  apply (rename-tac F S, case-tac trail S)
   apply auto[]
  apply (rename-tac list, case-tac Suc (length list) > length F)
   prefer 2 apply (metis diff-is-0-eq drop-Cons' length-Cons nat-le-linear nat-less-le
     reduce-trail-to-eq-length trail-reduce-trail-to-length-le)
  apply (subgoal-tac Suc (length list) − length F = Suc (length list − length F))
  by (auto simp add: reduce-trail-to-length-ne)

lemma in-get-all-marked-decomposition-trail-update-trail[simp]:
  assumes  H: (L # M1, M2) ∈ set (get-all-marked-decomposition (trail S))
  shows trail (reduce-trail-to M1 S) = M1
proof −
  obtain K mark where
    L: L = Marked K mark
    using H by (cases L) (auto dest!: in-get-all-marked-decomposition-marked-or-empty)
  obtain c where
    tr-S: trail S = c @ M2 @ L # M1
    using H by auto
  show ?thesis
    by (rule reduce-trail-to-trail-tl-trail-decomp[of - c @ M2 K mark])
     (auto simp: tr-S L)
qed

lemma raw-conflicting-cons-trail[simp]:
  assumes undefined-lit (trail S) (lit-of L)
  shows
    raw-conflicting (cons-trail L S) = None ⟷ raw-conflicting S = None
  using assms conflicting-cons-trail[of S L] map-option-is-None by fastforce+

lemma raw-conflicting-add-init-cls[simp]:
  no-dup (trail S) ⟹
    raw-conflicting (add-init-cls C S) = None ⟷ raw-conflicting S = None
  using map-option-is-None conflicting-add-init-cls[of S C] by fastforce+

lemma raw-conflicting-add-learned-cls[simp]:
  no-dup (trail S) ⟹
    raw-conflicting (add-learned-cls C S) = None ⟷ raw-conflicting S = None
  using map-option-is-None conflicting-add-learned-cls[of S C] by fastforce+

lemma raw-conflicting-update-backtracl-lvl[simp]:
  raw-conflicting (update-backtrack-lvl k S) = None ⟷ raw-conflicting S = None
  using map-option-is-None conflicting-update-backtrack-lvl[of k S] by fastforce+

end — end of state_W locale
```

19.2 CDCL Rules

Because of the strategy we will later use, we distinguish propagate, conflict from the other rules

```
locale conflict-driven-clause-learning_W =
  state_W
    — functions for clauses:
    mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss

    — functions for the conflicting clause:
    mset-ccls union-ccls insert-ccls remove-clit

    — conversion
    ccls-of-cls cls-of-ccls

    — functions for the state:
      — access functions:
    trail hd-raw-trail raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting
      — changing state:
    cons-trail tl-trail add-init-cls add-learned-cls remove-cls update-backtrack-lvl
    update-conflicting

      — get state:
    init-state
    restart-state
  for
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and

    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss  and

    mset-ccls:: 'ccls ⇒ 'v clause and
    union-ccls :: 'ccls ⇒ 'ccls ⇒ 'ccls and
    insert-ccls :: 'v literal ⇒ 'ccls ⇒ 'ccls and
    remove-clit :: 'v literal ⇒ 'ccls ⇒ 'ccls and

    ccls-of-cls :: 'cls ⇒ 'ccls and
    cls-of-ccls :: 'ccls ⇒ 'cls and

    trail :: 'st ⇒ ('v, nat, 'v clause) marked-lits and
    hd-raw-trail :: 'st ⇒ ('v, nat, 'cls) marked-lit and
    raw-init-clss :: 'st ⇒ 'clss and
    raw-learned-clss :: 'st ⇒ 'clss and
    backtrack-lvl :: 'st ⇒ nat and
    raw-conflicting :: 'st ⇒ 'ccls option and

    cons-trail :: ('v, nat, 'cls) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒ 'st and
    add-init-cls :: 'cls ⇒ 'st ⇒ 'st and
    add-learned-cls :: 'cls ⇒ 'st ⇒ 'st and
    remove-cls :: 'cls ⇒ 'st ⇒ 'st and
    update-backtrack-lvl :: nat ⇒ 'st ⇒ 'st and
    update-conflicting :: 'ccls option ⇒ 'st ⇒ 'st and

    init-state :: 'clss ⇒ 'st and
    restart-state :: 'st ⇒ 'st
begin

inductive propagate :: 'st ⇒ 'st ⇒ bool for S :: 'st where
propagate-rule: conflicting S = None ⟹
  E !∈! raw-clauses S ⟹
  L ∈# mset-cls E ⟹
  trail S ⊨as CNot (mset-cls (remove-lit L E)) ⟹
  undefined-lit (trail S) L ⟹
  T ∼ cons-trail (Propagated L E) S ⟹
  propagate S T

inductive-cases propagateE: propagate S T

inductive conflict ::  'st ⇒ 'st ⇒ bool for S :: 'st where
conflict-rule:
  conflicting S = None ⟹
  D !∈! raw-clauses S ⟹
  trail S ⊨as CNot (mset-cls D) ⟹
  T ∼ update-conflicting (Some (ccls-of-cls D)) S ⟹
  conflict S T
```

inductive-cases *conflictE*: *conflict S T*

```
inductive backtrack :: 'st ⇒ 'st ⇒ bool for S :: 'st where
backtrack-rule:
  raw-conflicting S = Some D ⟹
  L ∈# mset-ccls D ⟹
  (Marked K (i+1) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) ⟹
  get-level (trail S) L = backtrack-lvl S ⟹
  get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) ⟹
  get-maximum-level (trail S) (mset-ccls (remove-clit L D)) ≡ i ⟹
  T ∼ cons-trail (Propagated L (cls-of-ccls D))
          (reduce-trail-to M1
            (add-learned-cls (cls-of-ccls D)
              (update-backtrack-lvl i
                (update-conflicting None S)))) ⟹
  backtrack S T
```

inductive-cases *backtrackE*: *backtrack S T*
thm *backtrackE*

```
inductive decide :: 'st ⇒ 'st ⇒ bool for S :: 'st where
decide-rule:
  conflicting S = None ⟹
  undefined-lit (trail S) L ⟹
  atm-of L ∈ atms-of-mm (init-clss S) ⟹
  T ∼ cons-trail (Marked L (backtrack-lvl S +1)) (incr-lvl S) ⟹
  decide S T
```

inductive-cases *decideE*: *decide S T*

```
inductive skip :: 'st ⇒ 'st ⇒ bool for S :: 'st where
skip-rule:
  trail S = Propagated L C' # M ⟹
   raw-conflicting S = Some E ⟹
   −L ∉# mset-ccls E ⟹
   mset-ccls E ≠ {#} ⟹
   T ∼ tl-trail S ⟹
   skip S T
```

inductive-cases *skipE*: *skip S T*

get-maximum-level (*Propagated L* (*C* + {#*L*#}) # *M*) *D* = *k* ∨ *k* = *0* (that was in a previous version of the book) is equivalent to *get-maximum-level* (*Propagated L* (*C* + {#*L*#}) # *M*) *D* = *k*, when the structural invariants holds.

```
inductive resolve :: 'st ⇒ 'st ⇒ bool for S :: 'st where
resolve-rule: trail S ≠ [] ⟹
  hd-raw-trail S  = Propagated L E ⟹
  L ∈# mset-cls E ⟹
  raw-conflicting S = Some D' ⟹
  −L ∈# mset-ccls D' ⟹
  get-maximum-level (trail S) (mset-ccls (remove-clit (−L) D')) = backtrack-lvl S ⟹
  T ∼ update-conflicting (Some (union-ccls (remove-clit (−L) D') (ccls-of-cls (remove-lit L E))))
```

(*tl-trail S*) ⟹
resolve S T

inductive-cases *resolveE*: *resolve S T*

inductive *restart* :: *'st* ⇒ *'st* ⇒ *bool* **for** *S* :: *'st* **where**
restart: *state S* = (*M*, *N*, *U*, *k*, *None*) ⟹ ¬*M* ⊨*asm clauses S*
⟹ *T* ∼ *restart-state S*
⟹ *restart S T*

inductive-cases *restartE*: *restart S T*

We add the condition *C* ∉# *init-clss S*, to maintain consistency even without the strategy.

inductive *forget*:: *'st* ⇒ *'st* ⇒ *bool* **where**
forget-rule:
conflicting S = *None* ⟹
C !∈! *raw-learned-clss S* ⟹
¬(*trail S*) ⊨*asm clauses S* ⟹
mset-cls C ∉ *set* (*get-all-mark-of-propagated* (*trail S*)) ⟹
mset-cls C ∉# *init-clss S* ⟹
T ∼ *remove-cls C S* ⟹
forget S T

inductive-cases *forgetE*: *forget S T*

inductive $cdcl_W$*-rf* :: *'st* ⇒ *'st* ⇒ *bool* **for** *S* :: *'st* **where**
restart: *restart S T* ⟹ $cdcl_W$*-rf S T* |
forget: *forget S T* ⟹ $cdcl_W$*-rf S T*

inductive $cdcl_W$*-bj* :: *'st* ⇒ *'st* ⇒ *bool* **where**
skip: *skip S S′* ⟹ $cdcl_W$*-bj S S′* |
resolve: *resolve S S′* ⟹ $cdcl_W$*-bj S S′* |
backtrack: *backtrack S S′* ⟹ $cdcl_W$*-bj S S′*

inductive-cases $cdcl_W$*-bjE*: $cdcl_W$*-bj S T*

inductive $cdcl_W$*-o*:: *'st* ⇒ *'st* ⇒ *bool* **for** *S* :: *'st* **where**
decide: *decide S S′* ⟹ $cdcl_W$*-o S S′* |
bj: $cdcl_W$*-bj S S′* ⟹ $cdcl_W$*-o S S′*

inductive $cdcl_W$ :: *'st* ⇒ *'st* ⇒ *bool* **for** *S* :: *'st* **where**
propagate: *propagate S S′* ⟹ $cdcl_W$ *S S′* |
conflict: *conflict S S′* ⟹ $cdcl_W$ *S S′* |
other: $cdcl_W$*-o S S′* ⟹ $cdcl_W$ *S S′*|
rf: $cdcl_W$*-rf S S′* ⟹ $cdcl_W$ *S S′*

lemma *rtranclp-propagate-is-rtranclp-*$cdcl_W$:
$propagate^{**}$ *S S′* ⟹ $cdcl_W{}^{**}$ *S S′*
apply (*induction rule*: *rtranclp-induct*)
apply *simp*
apply (*frule propagate*)
using *rtranclp-trans*[*of* $cdcl_W$] **by** *blast*

lemma $cdcl_W$*-all-rules-induct*[*consumes 1*, *case-names propagate conflict forget restart decide skip resolve backtrack*]:

```
  fixes S :: 'st
  assumes
    cdcl_W: cdcl_W S S' and
    propagate: ⋀T. propagate S T ⟹ P S T and
    conflict: ⋀T. conflict S T ⟹ P S T and
    forget: ⋀T. forget S T ⟹ P S T and
    restart: ⋀T. restart S T ⟹ P S T and
    decide: ⋀T. decide S T ⟹ P S T and
    skip: ⋀T. skip S T ⟹ P S T and
    resolve: ⋀T. resolve S T ⟹ P S T and
    backtrack: ⋀T. backtrack S T ⟹ P S T
  shows P S S'
  using assms(1)
proof (induct S' rule: cdcl_W.induct)
  case (propagate S') note propagate = this(1)
  then show ?case using assms(2) by auto
next
  case (conflict S')
  then show ?case using assms(3) by auto
next
  case (other S')
  then show ?case
    proof (induct rule: cdcl_W-o.induct)
      case (decide U)
      then show ?case using assms(6) by auto
    next
      case (bj S')
      then show ?case using assms(7−9) by (induction rule: cdcl_W-bj.induct) auto
    qed
next
  case (rf S')
  then show ?case
    by (induct rule: cdcl_W-rf.induct) (fast dest: forget restart)+
qed

lemma cdcl_W-all-induct[consumes 1, case-names propagate conflict forget restart decide skip
    resolve backtrack]:
  fixes S :: 'st
  assumes
    cdcl_W: cdcl_W S S' and
    propagateH: ⋀C L T. conflicting S = None ⟹
       C !∈! raw-clauses S ⟹
       L ∈# mset-cls C ⟹
       trail S ⊨as CNot (remove1-mset L (mset-cls C)) ⟹
       undefined-lit (trail S) L ⟹
       T ∼ cons-trail (Propagated L C) S ⟹
       P S T and
    conflictH: ⋀D T. conflicting S = None ⟹
       D !∈! raw-clauses S ⟹
       trail S ⊨as CNot (mset-cls D) ⟹
       T ∼ update-conflicting (Some (ccls-of-cls D)) S ⟹
       P S T and
    forgetH: ⋀C U T. conflicting S = None ⟹
      C !∈! raw-learned-clss S ⟹
      ¬(trail S) ⊨asm clauses S ⟹
      mset-cls C ∉ set (get-all-mark-of-propagated (trail S)) ⟹
      mset-cls C ∉# init-clss S ⟹
      T ∼ remove-cls C S ⟹
      P S T and
    restartH: ⋀T. ¬trail S ⊨asm clauses S ⟹
      conflicting S = None ⟹
      T ∼ restart-state S ⟹
      P S T and
    decideH: ⋀L T. conflicting S = None ⟹
      undefined-lit (trail S) L ⟹
      atm-of L ∈ atms-of-mm (init-clss S) ⟹
      T ∼ cons-trail (Marked L (backtrack-lvl S +1)) (incr-lvl S) ⟹
      P S T and
    skipH: ⋀L C' M E T.
      trail S = Propagated L C' # M ⟹
      raw-conflicting S = Some E ⟹
      −L ∉# mset-ccls E ⟹  mset-ccls E ≠ {#} ⟹
      T ∼ tl-trail S ⟹
      P S T and
    resolveH: ⋀L E M D T.
      trail S = Propagated L (mset-cls E) # M ⟹
      L ∈# mset-cls E ⟹
      hd-raw-trail S  = Propagated L E ⟹
      raw-conflicting S = Some D ⟹
      −L ∈# mset-ccls D ⟹
      get-maximum-level (trail S) (mset-ccls (remove-clit (−L) D)) = backtrack-lvl S ⟹
      T ∼ update-conflicting
        (Some (union-ccls (remove-clit (−L) D) (ccls-of-cls (remove-lit L E)))) (tl-trail S) ⟹
      P S T and
    backtrackH: ⋀L D K i M1 M2 T.
      raw-conflicting S = Some D ⟹
      L ∈# mset-ccls D ⟹
      (Marked K (i+1) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) ⟹
      get-level (trail S) L = backtrack-lvl S ⟹
      get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) ⟹
      get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≡ i ⟹
      T ∼ cons-trail (Propagated L (cls-of-ccls D))
                (reduce-trail-to M1
                  (add-learned-cls (cls-of-ccls D)
                    (update-backtrack-lvl i
                      (update-conflicting None S)))) ⟹
       P S T
  shows P S S'
  using cdcl_W
proof (induct S S' rule: cdcl_W-all-rules-induct)
  case (propagate S')
  then show ?case
    by (auto elim!: propagateE intro!: propagateH)
next
  case (conflict S')
  then show ?case
    by (auto elim!: conflictE intro!: conflictH)
next
  case (restart S')
  then show ?case
```

```
    by (auto elim!: restartE intro!: restartH)
next
  case (decide T)
  then show ?case
    by (auto elim!: decideE intro!: decideH)
next
  case (backtrack S')
  then show ?case by (auto elim!: backtrackE intro!: backtrackH
    simp del: state-simp simp add: state-eq-def)
next
  case (forget S')
  then show ?case by (auto elim!: forgetE intro!: forgetH)
next
  case (skip S')
  then show ?case by (auto elim!: skipE intro!: skipH)
next
  case (resolve S')
  then show ?case
    using hd-raw-trail[of S] by (cases trail S) (auto elim!: resolveE intro!: resolveH)
qed

lemma cdcl_W-o-induct[consumes 1, case-names decide skip resolve backtrack]:
  fixes S :: 'st
  assumes cdcl_W: cdcl_W-o S T and
    decideH: ⋀L T. conflicting S = None ⟹ undefined-lit (trail S) L
      ⟹ atm-of L ∈ atms-of-mm (init-clss S)
      ⟹ T ∼ cons-trail (Marked L (backtrack-lvl S +1)) (incr-lvl S)
      ⟹ P S T and
    skipH: ⋀L C' M E T.
      trail S = Propagated L C' # M ⟹
      raw-conflicting S = Some E ⟹
      −L ∉# mset-ccls E ⟹ mset-ccls E ≠ {#} ⟹
      T ∼ tl-trail S ⟹
      P S T and
    resolveH: ⋀L E M D T.
      trail S = Propagated L (mset-cls E) # M ⟹
      L ∈# mset-cls E ⟹
      hd-raw-trail S = Propagated L E ⟹
      raw-conflicting S = Some D ⟹
      −L ∈# mset-ccls D ⟹
      get-maximum-level (trail S) (mset-ccls (remove-clit (−L) D)) = backtrack-lvl S ⟹
      T ∼ update-conflicting
        (Some (union-ccls (remove-clit (−L) D) (ccls-of-cls (remove-lit L E)))) (tl-trail S) ⟹
      P S T and
    backtrackH: ⋀L D K i M1 M2 T.
      raw-conflicting S = Some D ⟹
      L ∈# mset-ccls D ⟹
      (Marked K (i+1) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) ⟹
      get-level (trail S) L = backtrack-lvl S ⟹
      get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) ⟹
      get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≡ i ⟹
      T ∼ cons-trail (Propagated L (cls-of-ccls D))
              (reduce-trail-to M1
                (add-learned-cls (cls-of-ccls D)
                  (update-backtrack-lvl i
                    (update-conflicting None S)))) ⟹
        P S T
  shows P S T
  using cdcl_W apply (induct T rule: cdcl_W-o.induct)
   using assms(2) apply (auto elim: decideE)[1]
  apply (elim cdcl_W-bjE skipE resolveE backtrackE)
    apply (frule skipH; simp)
    using hd-raw-trail[of S] apply (cases trail S; auto elim!: resolveE intro!: resolveH)
  apply (frule backtrackH; simp-all del: state-simp add: state-eq-def)
  done

thm  cdcl_W-o.induct
lemma cdcl_W-o-all-rules-induct[consumes 1, case-names decide backtrack skip resolve]:
  fixes S T :: 'st
  assumes
    cdcl_W-o S T and
    ⋀T. decide S T ⟹ P S T and
    ⋀T. backtrack S T ⟹ P S T and
    ⋀T. skip S T ⟹ P S T and
    ⋀T. resolve S T ⟹ P S T
  shows P S T
  using assms by (induct T rule: cdcl_W-o.induct) (auto simp: cdcl_W-bj.simps)

lemma cdcl_W-o-rule-cases[consumes 1, case-names decide backtrack skip resolve]:
  fixes S T :: 'st
  assumes
    cdcl_W-o S T and
    decide S T ⟹ P and
    backtrack S T ⟹ P and
    skip S T ⟹ P and
    resolve S T ⟹ P
  shows P
  using assms by (auto simp: cdcl_W-o.simps cdcl_W-bj.simps)
```

19.3 Invariants

19.3.1 Properties of the trail

We here establish that: * the marks are exactly 1..k where k is the level * the consistency of the trail * the fact that there is no duplicate in the trail.

```
lemma backtrack-lit-skiped:
  assumes
    L: get-level (trail S) L = backtrack-lvl S and
    M1: (Marked K (i + 1) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) and
    no-dup: no-dup (trail S) and
    bt-l: backtrack-lvl S = length (get-all-levels-of-marked (trail S)) and
    order: get-all-levels-of-marked (trail S)
    = rev [1..<(1+length (get-all-levels-of-marked (trail S)))]
  shows atm-of L ∉ atm-of ` lits-of-l M1
proof (rule ccontr)
  let ?M = trail S
  assume L-in-M1: ¬atm-of L ∉ atm-of ` lits-of-l M1
  obtain c where
    Mc: trail S = c @ M2 @ Marked K (i + 1) # M1
    using M1 by blast
  have atm-of L ∉ atm-of ‘ lits-of-l c
    using L-in-M1 no-dup unfolding Mc lits-of-def by force
  have g-M-eq-g-M1: get-level ?M L = get-level M1 L
    using L-in-M1 unfolding Mc by auto
  have g: get-all-levels-of-marked M1 = rev [1..<Suc i]
    using order unfolding Mc by (auto simp del: upt-simps simp: rev-swap[symmetric]
      dest: append-cons-eq-upt-length-i)
  then have Max (set (0 # get-all-levels-of-marked (rev M1))) < Suc i by auto
  then have get-level M1 L < Suc i
    using get-rev-level-less-max-get-all-levels-of-marked[of rev M1 0 L] by linarith
  moreover have Suc i ≤ backtrack-lvl S using bt-l by (simp add: Mc g)
  ultimately show False using L g-M-eq-g-M1 by auto
qed

lemma cdcl_W-distinctinv-1:
  assumes
    cdcl_W S S' and
    no-dup (trail S) and
    backtrack-lvl S = length (get-all-levels-of-marked (trail S)) and
    get-all-levels-of-marked (trail S) = rev [1..<1+length (get-all-levels-of-marked (trail S))]
  shows no-dup (trail S')
  using assms
proof (induct rule: cdcl_W-all-induct)
  case (backtrack L D K i M1 M2 T) note decomp = this(3) and L = this(4) and T = this(7) and
    n-d = this(8)
  obtain c where Mc: trail S = c @ M2 @ Marked K (i + 1) # M1
    using decomp by auto
  have no-dup (M2 @ Marked K (i + 1) # M1)
    using Mc n-d by fastforce
  moreover have atm-of L ∉ (λl. atm-of (lit-of l)) ‘ set M1
    using backtrack-lit-skiped[of S L K i M1 M2] L decomp backtrack.prems
    by (fastforce simp: lits-of-def)
  moreover then have undefined-lit M1 L
     by (simp add: defined-lit-map)
  ultimately show ?case using decomp T n-d by simp
qed (auto simp: defined-lit-map)

lemma cdcl_W-consistent-inv-2:
  assumes
    cdcl_W S S' and
    no-dup (trail S) and
    backtrack-lvl S = length (get-all-levels-of-marked (trail S)) and
    get-all-levels-of-marked (trail S) = rev [1..<1+length (get-all-levels-of-marked (trail S))]
  shows consistent-interp (lits-of-l (trail S'))
  using cdcl_W-distinctinv-1[OF assms] distinct-consistent-interp by fast

lemma cdcl_W-o-bt:
  assumes
    cdcl_W-o S S' and
    backtrack-lvl S = length (get-all-levels-of-marked (trail S)) and
    get-all-levels-of-marked (trail S) =
      rev ([1..<(1+length (get-all-levels-of-marked (trail S)))]) and
    n-d[simp]: no-dup (trail S)
  shows backtrack-lvl S' = length (get-all-levels-of-marked (trail S'))
  using assms
```

```
proof (induct rule: cdcl_W-o-induct)
  case (backtrack L D K i M1 M2 T) note decomp = this(3) and T = this(7) and level = this(9)
  have [simp]: trail (reduce-trail-to M1 S) = M1
    using decomp by auto
  obtain c where M: trail S = c @ M2 @ Marked K (i + 1) # M1 using decomp by auto
  have rev (get-all-levels-of-marked (trail S))
    = [1..<1+ (length (get-all-levels-of-marked (trail S)))]
    using level by (auto simp: rev-swap[symmetric])
  moreover have atm-of L ∉ (λl. atm-of (lit-of l)) ‘ set M1
    using backtrack-lit-skiped[of S L K i M1 M2] backtrack(4,8,9) decomp
    by (fastforce simp add: lits-of-def)
  moreover then have undefined-lit M1 L
     by (simp add: defined-lit-map)
  moreover then have no-dup (trail T)
    using T decomp n-d by (auto simp: defined-lit-map M)
  ultimately show ?case
    using T n-d unfolding M by (auto dest!: append-cons-eq-upt-length simp del: upt-simps)
qed auto

lemma cdcl_W-rf-bt:
  assumes
    cdcl_W-rf S S' and
    backtrack-lvl S = length (get-all-levels-of-marked (trail S)) and
    get-all-levels-of-marked (trail S) = rev [1..<(1+length (get-all-levels-of-marked (trail S)))]
  shows backtrack-lvl S' = length (get-all-levels-of-marked (trail S'))
  using assms by (induct rule: cdcl_W-rf.induct) (auto elim: restartE forgetE)

lemma cdcl_W-bt:
  assumes
    cdcl_W S S' and
    backtrack-lvl S = length (get-all-levels-of-marked (trail S)) and
    get-all-levels-of-marked (trail S)
    = rev ([1..<(1+length (get-all-levels-of-marked (trail S)))]) and
    no-dup (trail S)
  shows backtrack-lvl S' = length (get-all-levels-of-marked (trail S'))
  using assms by (induct rule: cdcl_W.induct) (auto simp add: cdcl_W-o-bt cdcl_W-rf-bt
    elim: conflictE propagateE)

lemma cdcl_W-bt-level':
  assumes
    cdcl_W S S' and
    backtrack-lvl S = length (get-all-levels-of-marked (trail S)) and
    get-all-levels-of-marked (trail S)
      = rev ([1..<(1+length (get-all-levels-of-marked (trail S)))]) and
    n-d: no-dup (trail S)
  shows get-all-levels-of-marked (trail S')
    = rev [1..<1+length (get-all-levels-of-marked (trail S'))]
  using assms
proof (induct rule: cdcl_W-all-induct)
  case (decide L T) note undef = this(2) and T = this(4)
  let ?k = backtrack-lvl S
  let ?M = trail S
  let ?M' = Marked L (?k + 1) # trail S
  have H: get-all-levels-of-marked ?M = rev [Suc 0..<1+length (get-all-levels-of-marked ?M)]
    using decide.prems by simp
  have k: ?k = length (get-all-levels-of-marked ?M)
    using decide.prems by auto
  have get-all-levels-of-marked ?M' = Suc ?k # get-all-levels-of-marked ?M by simp
  then have get-all-levels-of-marked ?M' = Suc ?k #
      rev [Suc 0..<1+length (get-all-levels-of-marked ?M)]
    using H by auto
  moreover have ... = rev [Suc 0..< Suc (1+length (get-all-levels-of-marked ?M))]
    unfolding k by simp
  finally show ?case using T undef by (auto simp add: defined-lit-map)
next
  case (backtrack L D K i M1 M2 T) note decomp = this(3) and confli = this(1) and T = this(7)
and
    all-marked = this(9) and bt-lvl = this(8)
  have atm-of L ∉  atm-of ` lits-of-l M1
    using backtrack-lit-skiped[of S L K i M1 M2] backtrack(4,8−10) decomp
    by (fastforce simp add: lits-of-def)
  moreover then have undefined-lit M1 L
     by (auto simp: defined-lit-map lits-of-def)
  then have [simp]: trail T = Propagated L (mset-ccls D) # M1
    using T decomp n-d by auto
  obtain c where M: trail S = c @ M2 @ Marked K (i + 1) # M1 using decomp by auto
  have get-all-levels-of-marked (rev (trail S))
    = [Suc 0..<2+length (get-all-levels-of-marked c) + (length (get-all-levels-of-marked M2)
              + length (get-all-levels-of-marked M1))]
    using all-marked bt-lvl unfolding M by (auto simp: rev-swap[symmetric] simp del: upt-simps)
  then show ?case
    using T by (auto simp: rev-swap M simp del: upt-simps dest!: append-cons-eq-upt(1))
qed auto
```

We write *1 + length (get-all-levels-of-marked (trail S))* instead of *backtrack-lvl S* to avoid non termination of rewriting.

```
definition cdcl_W-M-level-inv :: 'st ⇒ bool where
cdcl_W-M-level-inv S ⟷
  consistent-interp (lits-of-l (trail S))
  ∧ no-dup (trail S)
  ∧ backtrack-lvl S = length (get-all-levels-of-marked (trail S))
  ∧ get-all-levels-of-marked (trail S)
      = rev [1..<1+length (get-all-levels-of-marked (trail S))]

lemma cdcl_W-M-level-inv-decomp:
  assumes cdcl_W-M-level-inv S
  shows
    consistent-interp (lits-of-l (trail S)) and
    no-dup (trail S)
  using assms unfolding cdcl_W-M-level-inv-def by fastforce+

lemma cdcl_W-consistent-inv:
  fixes S S' :: 'st
  assumes
    cdcl_W S S' and
    cdcl_W-M-level-inv S
  shows cdcl_W-M-level-inv S'
  using assms cdcl_W-consistent-inv-2 cdcl_W-distinctinv-1 cdcl_W-bt cdcl_W-bt-level'
  unfolding cdcl_W-M-level-inv-def by meson+
lemma rtranclp-cdcl_W-consistent-inv:
  assumes
    cdcl_W** S S' and
    cdcl_W-M-level-inv S
  shows cdcl_W-M-level-inv S'
  using assms by (induct rule: rtranclp-induct) (auto intro: cdcl_W-consistent-inv)

lemma tranclp-cdcl_W-consistent-inv:
  assumes
    cdcl_W++ S S' and
    cdcl_W-M-level-inv S
  shows cdcl_W-M-level-inv S'
  using assms by (induct rule: tranclp-induct)
  (auto intro: cdcl_W-consistent-inv)

lemma cdcl_W-M-level-inv-S0-cdcl_W[simp]:
  cdcl_W-M-level-inv (init-state N)
  unfolding cdcl_W-M-level-inv-def by auto

lemma cdcl_W-M-level-inv-get-level-le-backtrack-lvl:
  assumes inv: cdcl_W-M-level-inv S
  shows get-level (trail S) L ≤ backtrack-lvl S
proof -
  have get-all-levels-of-marked (trail S) = rev [1..<1 + backtrack-lvl S]
    using inv unfolding cdcl_W-M-level-inv-def by auto
  then show ?thesis
    using get-rev-level-less-max-get-all-levels-of-marked[of rev (trail S) 0 L]
    by (auto simp: Max-n-upt)
qed

lemma backtrack-ex-decomp:
  assumes
    M-l: cdcl_W-M-level-inv S and
    i-S: i < backtrack-lvl S
  shows ∃ K M1 M2. (Marked K (i + 1) # M1, M2) ∈ set (get-all-marked-decomposition (trail S))
proof -
  let ?M = trail S
  have
    g: get-all-levels-of-marked (trail S) = rev [Suc 0..<Suc (backtrack-lvl S)]
    using M-l unfolding cdcl_W-M-level-inv-def by simp-all
  then have i+1 ∈ set (get-all-levels-of-marked (trail S))
    using i-S by auto

  then obtain c K c' where tr-S: trail S = c @ Marked K (i + 1) # c'
    using in-get-all-levels-of-marked-iff-decomp[of i+1 trail S] by auto

  obtain M1 M2 where (Marked K (i + 1) # M1, M2) ∈ set (get-all-marked-decomposition (trail S))
    using Marked-cons-in-get-all-marked-decomposition-append-Marked-cons unfolding tr-S by fast
  then show ?thesis by blast
qed
```

19.3.2 Better-Suited Induction Principle

We generalise the induction principle defined previously: the induction case for *backtrack* now includes the assumption that *undefined-lit M1 L*. This helps the simplifier and thus the automation.

```
lemma backtrack-induction-lev[consumes 1, case-names M-devel-inv backtrack]:
  assumes
    bt: backtrack S T and
    inv: cdcl_W-M-level-inv S and
    backtrackH: ⋀K i M1 M2 L D T.
      raw-conflicting S = Some D ⟹
      L ∈# mset-ccls D ⟹
      (Marked K (Suc i) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) ⟹
      get-level (trail S) L = backtrack-lvl S ⟹
      get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) ⟹
      get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≡ i ⟹
      undefined-lit M1 L ⟹
      T ∼ cons-trail (Propagated L (cls-of-ccls D))
              (reduce-trail-to M1
                (add-learned-cls (cls-of-ccls D)
                  (update-backtrack-lvl i
                    (update-conflicting None S)))) ⟹
      P S T
  shows P S T
proof –
  obtain K i M1 M2 L D where
    decomp: (Marked K (Suc i) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) and
    L: get-level (trail S) L = backtrack-lvl S and
    confl: raw-conflicting S = Some D and
    LD: L ∈# mset-ccls D and
    lev-L: get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) and
    lev-D: get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≡ i and
    T: T ∼ cons-trail (Propagated L (cls-of-ccls D))
              (reduce-trail-to M1
                (add-learned-cls (cls-of-ccls D)
                  (update-backtrack-lvl i
                    (update-conflicting None S))))
    using bt by (elim backtrackE) metis

  have atm-of L ∉ atm-of ‘ lits-of-l M1
    using backtrack-lit-skiped[of S L K i M1 M2] L decomp bt confl lev-L lev-D inv
    unfolding cdcl_W-M-level-inv-def by force
  then have undefined-lit M1 L
    by (auto simp: defined-lit-map lits-of-def)
  then show ?thesis
    using backtrackH[OF confl LD decomp L lev-L lev-D - T] by simp
qed

lemmas backtrack-induction-lev2 = backtrack-induction-lev[consumes 2, case-names backtrack]

lemma cdcl_W-all-induct-lev-full:
  fixes S :: 'st
  assumes
    cdcl_W: cdcl_W S S' and
    inv[simp]: cdcl_W-M-level-inv S and
    propagateH: ⋀C L T. conflicting S = None ⟹
       C !∈! raw-clauses S ⟹
       L ∈# mset-cls C ⟹
       trail S ⊨as CNot (remove1-mset L (mset-cls C)) ⟹
       undefined-lit (trail S) L ⟹
       T ∼ cons-trail (Propagated L C) S ⟹
       P S T and
     conflictH: ⋀D T.  conflicting S = None ⟹
       D !∈! raw-clauses S ⟹
       trail S ⊨as CNot (mset-cls D) ⟹
       T ∼ update-conflicting (Some (ccls-of-cls D)) S ⟹
       P S T and
     forgetH: ⋀C T. conflicting S = None ⟹
      C !∈! raw-learned-clss S ⟹
      ¬(trail S) ⊨asm clauses S ⟹
      mset-cls C ∉ set (get-all-mark-of-propagated (trail S)) ⟹
      mset-cls C ∉# init-clss S ⟹
      T ∼ remove-cls C S ⟹
      P S T and
     restartH: ⋀T. ¬trail S ⊨asm clauses S ⟹
      conflicting S = None ⟹
      T ∼ restart-state S ⟹
      P S T and
     decideH: ⋀L T. conflicting S = None ⟹
      undefined-lit (trail S) L ⟹
      atm-of L ∈ atms-of-mm (init-clss S) ⟹
      T ∼ cons-trail (Marked L (backtrack-lvl S +1)) (incr-lvl S) ⟹
      P S T and
     skipH: ⋀L C′ M E T.
      trail S = Propagated L C′ # M ⟹
      raw-conflicting S = Some E ⟹
      −L ∉# mset-ccls E ⟹  mset-ccls E ≠ {#} ⟹
      T ∼ tl-trail S ⟹
      P S T and
     resolveH: ⋀L E M D T.
      trail S = Propagated L (mset-cls E) # M ⟹
      L ∈# mset-cls E ⟹
      hd-raw-trail S  = Propagated L E ⟹
      raw-conflicting S = Some D ⟹
      −L ∈# mset-ccls D ⟹
      get-maximum-level (trail S) (mset-ccls (remove-clit (−L) D)) = backtrack-lvl S ⟹
      T ∼ update-conflicting
        (Some (union-ccls (remove-clit (−L) D) (ccls-of-cls (remove-lit L E)))) (tl-trail S) ⟹
      P S T and
    backtrackH: ⋀K i M1 M2 L D T.
      raw-conflicting S = Some D ⟹
      L ∈# mset-ccls D ⟹
      (Marked K (Suc i) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) ⟹
      get-level (trail S) L = backtrack-lvl S ⟹
      get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) ⟹
      get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≡ i ⟹
      undefined-lit M1 L ⟹
      T ∼ cons-trail (Propagated L (cls-of-ccls D))
              (reduce-trail-to M1
                (add-learned-cls (cls-of-ccls D)
                  (update-backtrack-lvl i
                    (update-conflicting None S)))) ⟹
      P S T
  shows P S S′
  using cdcl_W
proof (induct S' rule: cdcl_W-all-rules-induct)
  case (propagate S')
  then show ?case
    by (auto elim!: propagateE intro!: propagateH)
next
  case (conflict S')
  then show ?case
    by (auto elim!: conflictE intro!: conflictH)
next
  case (restart S')
  then show ?case
    by (auto elim!: restartE intro!: restartH)
next
  case (decide T)
  then show ?case
    by (auto elim!: decideE intro!: decideH)
next
  case (backtrack S')
  then show ?case
    apply (induction rule: backtrack-induction-lev)
     apply (rule inv)
    by (rule backtrackH;
      fastforce simp del: state-simp simp add: state-eq-def dest!: HOL.meta-eq-to-obj-eq)
next
  case (forget S')
  then show ?case by (auto elim!: forgetE intro!: forgetH)
next
  case (skip S')
  then show ?case by (auto elim!: skipE intro!: skipH)
next
  case (resolve S')
  then show ?case
    using hd-raw-trail[of S] by (cases trail S) (auto elim!: resolveE intro!: resolveH)
qed

lemmas cdcl_W-all-induct-lev2 = cdcl_W-all-induct-lev-full[consumes 2, case-names propagate conflict
  forget restart decide skip resolve backtrack]

lemmas cdcl_W-all-induct-lev = cdcl_W-all-induct-lev-full[consumes 1, case-names lev-inv propagate
  conflict forget restart decide skip resolve backtrack]

thm cdcl_W-o-induct
lemma cdcl_W-o-induct-lev[consumes 1, case-names M-lev decide skip resolve backtrack]:
  fixes S :: 'st
  assumes
    cdcl_W: cdcl_W-o S T and
    inv[simp]: cdcl_W-M-level-inv S and
    decideH: ⋀L T. conflicting S = None ⟹
      undefined-lit (trail S) L ⟹
      atm-of L ∈ atms-of-mm (init-clss S) ⟹
      T ∼ cons-trail (Marked L (backtrack-lvl S +1)) (incr-lvl S) ⟹
      P S T and
    skipH: ⋀L C' M E T.
      trail S = Propagated L C' # M ⟹
      raw-conflicting S = Some E ⟹
      −L ∉# mset-ccls E ⟹  mset-ccls E ≠ {#} ⟹
      T ∼ tl-trail S ⟹
      P S T and
    resolveH: ⋀L E M D T.
      trail S = Propagated L (mset-cls E) # M ⟹
      L ∈# mset-cls E ⟹
      hd-raw-trail S  = Propagated L E ⟹
      raw-conflicting S = Some D ⟹
      −L ∈# mset-ccls D ⟹
      get-maximum-level (trail S) (mset-ccls (remove-clit (−L) D)) = backtrack-lvl S ⟹
      T ∼ update-conflicting
        (Some (union-ccls (remove-clit (−L) D) (ccls-of-cls (remove-lit L E)))) (tl-trail S) ⟹
      P S T and
    backtrackH: ⋀K i M1 M2 L D T.
      raw-conflicting S = Some D ⟹
      L ∈# mset-ccls D ⟹
      (Marked K (Suc i) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) ⟹
      get-level (trail S) L = backtrack-lvl S ⟹
      get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) ⟹
      get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≡ i ⟹
      undefined-lit M1 L ⟹
      T ∼ cons-trail (Propagated L (cls-of-ccls D))
                (reduce-trail-to M1
                  (add-learned-cls (cls-of-ccls D)
                    (update-backtrack-lvl i
                      (update-conflicting None S)))) ⟹
      P S T
  shows P S T
  using cdcl_W
proof (induct S T rule: cdcl_W-o-all-rules-induct)
  case (decide T)
  then show ?case
    by (auto elim!: decideE intro!: decideH)
next
  case (backtrack S')
  then show ?case
    apply (induction rule: backtrack-induction-lev)
     apply (rule inv)
    by (rule backtrackH;
      fastforce simp del: state-simp simp add: state-eq-def dest!: HOL.meta-eq-to-obj-eq)
next
  case (skip S')
  then show ?case by (auto elim!: skipE intro!: skipH)
next
  case (resolve S')
  then show ?case
    using hd-raw-trail[of S] by (cases trail S) (auto elim!: resolveE intro!: resolveH)
qed

lemmas cdcl_W-o-induct-lev2 = cdcl_W-o-induct-lev[consumes 2, case-names decide skip resolve
  backtrack]
```

19.3.3 Compatibility with *op* ∼

lemma *propagate-state-eq-compatible*:

```
  assumes
    propa: propagate S T and
    SS': S ∼ S' and
    TT': T ∼ T'
  shows propagate S' T'
proof –
  obtain C L where
    conf: conflicting S = None and
    C: C !∈! raw-clauses S and
    L: L ∈# mset-cls C and
    tr: trail S ⊨as CNot (remove1-mset L (mset-cls C)) and
    undef: undefined-lit (trail S) L and
    T: T ∼ cons-trail (Propagated L C) S
  using propa by (elim propagateE) auto

  obtain C' where
    CC': mset-cls C' = mset-cls C and
    C': C' !∈! raw-clauses S'
    using SS' C
    in-mset-clss-exists-preimage[of mset-cls C raw-learned-clss S']
    in-mset-clss-exists-preimage[of mset-cls C raw-init-clss S']
    apply –
    apply (frule in-clss-mset-clss)
    by (auto simp: state-eq-def raw-clauses-def simp del: state-simp dest: in-clss-mset-clss)

  show ?thesis
    apply (rule propagate-rule[of - C'])
    using state-eq-sym[of S S'] SS' conf C' CC' L tr undef TT' T
    by (auto simp: state-eq-def simp del: state-simp)
qed

lemma conflict-state-eq-compatible:
  assumes
    confl: conflict S T and
    TT': T ∼ T' and
    SS': S ∼ S'
  shows conflict S' T'
proof –
  obtain D where
    conf: conflicting S = None and
    D: D !∈! raw-clauses S and
    tr:  trail S ⊨as CNot (mset-cls D) and
    T: T ∼ update-conflicting (Some (ccls-of-cls D)) S
  using confl by (elim conflictE) auto

  obtain D' where
    DD': mset-cls D' = mset-cls D and
    D': D' !∈! raw-clauses S'
    using D SS' in-mset-clss-exists-preimage by fastforce

  show ?thesis
    apply (rule conflict-rule[of - D'])
    using state-eq-sym[of S S'] SS' conf D' DD' tr TT' T
    by (auto simp: state-eq-def simp del: state-simp)
qed
```

```
lemma backtrack-levE[consumes 2]:
  backtrack S S' ⟹ cdcl_W-M-level-inv S ⟹
  (⋀K i M1 M2 L D.
      raw-conflicting S = Some D ⟹
      L ∈# mset-ccls D ⟹
      (Marked K (Suc i) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) ⟹
      get-level (trail S) L = backtrack-lvl S ⟹
      get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) ⟹
      get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≡ i ⟹
      undefined-lit M1 L ⟹
      S' ∼ cons-trail (Propagated L (cls-of-ccls D))
              (reduce-trail-to M1
                (add-learned-cls (cls-of-ccls D)
                  (update-backtrack-lvl i
                    (update-conflicting None S)))) ⟹ P) ⟹
  P
  using assms by (induction rule: backtrack-induction-lev2) metis
thm allI

lemma backtrack-state-eq-compatible:
  assumes
    bt: backtrack S T and
    SS': S ∼ S' and
    TT': T ∼ T' and
    inv: cdcl_W-M-level-inv S
  shows backtrack S' T'
proof −
  obtain D L K i M1 M2 where
    conf: raw-conflicting S = Some D and
    L: L ∈# mset-ccls D and
    decomp: (Marked K (Suc i) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) and
    lev: get-level (trail S) L = backtrack-lvl S and
    max: get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) and
    max-D: get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≡ i and
    undef: undefined-lit M1 L and
    T: T ∼ cons-trail (Propagated L (cls-of-ccls D))
              (reduce-trail-to M1
                (add-learned-cls (cls-of-ccls D)
                  (update-backtrack-lvl i
                    (update-conflicting None S))))
  using bt inv by (elim backtrack-levE) metis
  obtain D' where
    D': raw-conflicting S' = Some D'
    using SS' conf by (cases raw-conflicting S') auto
  have [simp]: mset-ccls D = mset-ccls D'
    using SS' D' conf by (auto simp: state-eq-def simp del: state-simp)[]

  have T': T' ∼ cons-trail (Propagated L (cls-of-ccls D'))
    (reduce-trail-to M1 (add-learned-cls (cls-of-ccls D')
    (update-backtrack-lvl i (update-conflicting None S'))))
    using TT' unfolding state-eq-def
    using decomp undef inv SS' T  by (auto simp add: cdcl_W-M-level-inv-def)

  show ?thesis
```

```
    apply (rule backtrack-rule[of - D'])
        apply (rule D')
      using state-eq-sym[of S S'] TT' SS' D' conf L decomp lev max max-D undef T
      apply (auto simp: state-eq-def simp del: state-simp)[]
     using decomp SS' lev SS' max-D max T' by (auto simp: state-eq-def simp del: state-simp)
qed


lemma decide-state-eq-compatible:
  assumes
    decide S T and
    S ∼ S' and
    T ∼ T'
  shows decide S' T'
  using assms apply (elim decideE)
  by (rule decide-rule) (auto simp: state-eq-def raw-clauses-def simp del: state-simp)

lemma skip-state-eq-compatible:
  assumes
    skip: skip S T and
    SS': S ∼ S' and
    TT': T ∼ T'
  shows skip S' T'
proof -
  obtain L C' M E where
    tr: trail S = Propagated L C' # M and
    raw: raw-conflicting S = Some E and
    L: -L ∉# mset-ccls E and
    E: mset-ccls E ≠ {#} and
    T: T ∼ tl-trail S
  using skip by (elim skipE) simp
  obtain E' where E': raw-conflicting S' = Some E'
    using SS' raw by (cases raw-conflicting S') (auto simp: state-eq-def simp del: state-simp)
  show ?thesis
    apply (rule skip-rule)
       using tr raw L E T SS' apply (auto simp:  simp del: )[]
      using E' apply simp
     using E' SS' L raw E apply (auto simp: state-eq-def simp del: state-simp)[2]
    using T TT' SS' by (auto simp: state-eq-def simp del: state-simp)
qed

lemma resolve-state-eq-compatible:
  assumes
    res: resolve S T and
    TT': T ∼ T' and
    SS': S ∼ S'
  shows resolve S' T'
proof -
  obtain E D L where
    tr: trail S ≠ [] and
    hd: hd-raw-trail S  = Propagated L E and
    L: L ∈# mset-cls E and
    raw: raw-conflicting S = Some D and
    LD: -L ∈# mset-ccls D and
    i: get-maximum-level (trail S) (mset-ccls (remove-clit (-L) D)) = backtrack-lvl S and
    T: T ∼ update-conflicting (Some (union-ccls (remove-clit (−L) D)
      (ccls-of-cls (remove-lit L E)))) (tl-trail S)
  using assms by (elim resolveE) simp

  obtain E′ where
    E′: hd-raw-trail S′ = Propagated L E′
    using SS′ hd by (metis ‹trail S ≠ []› hd-raw-trail is-proped-def marked-lit.disc(3)
      marked-lit.inject(2) mmset-of-mlit.elims state-eq-trail)
  have [simp]: mset-cls E = mset-cls E′
    using hd-raw-trail[of S] tr hd-raw-trail[of S′] tr SS′ hd E′
    by (metis marked-lit.inject(2) mmset-of-mlit.simps(1) state-eq-trail)
  obtain D′ where
    D′: raw-conflicting S′ = Some D′
    using SS′ raw by fastforce
  have [simp]: mset-ccls D = mset-ccls D′
    using D′ SS′ raw state-simp(5) by fastforce
  have T′T: T′ ∼ T
    using TT′ state-eq-sym by auto
  show ?thesis
    apply (rule resolve-rule)
           using tr SS′ apply simp
          using E′ apply simp
         using L apply simp
        using D′ apply simp
       using D′ SS′ raw LD apply (auto simp add: state-eq-def simp del: state-simp)[]
      using D′ SS′ raw LD apply (auto simp add: state-eq-def simp del: state-simp)[]
     using raw SS′ i apply (auto simp add: state-eq-def simp del: state-simp)[]
    using T T′T  SS′ by (auto simp: state-eq-def simp del: state-simp )
qed

lemma forget-state-eq-compatible:
  assumes
    forget: forget S T and
    SS′: S ∼ S′ and
    TT′: T ∼ T′
  shows forget S′ T′
proof −
  obtain C where
    conf: conflicting S = None and
    C !∈! raw-learned-clss S and
    tr: ¬(trail S) ⊨asm clauses S and
    C1: mset-cls C ∉ set (get-all-mark-of-propagated (trail S)) and
    C2: mset-cls C ∉# init-clss S and
    T: T ∼ remove-cls C S
    using forget by (elim forgetE) simp

  obtain C′ where
    C′: C′ !∈! raw-learned-clss S′ and
    [simp]: mset-cls C′ = mset-cls C
    using ‹C !∈! raw-learned-clss S› SS′ in-mset-clss-exists-preimage by fastforce
  show ?thesis
    apply (rule forget-rule)
         using SS′ conf apply simp
        using C′ apply simp
       using SS′ tr apply simp
      using SS' C1 apply simp
     using SS' C2 apply simp
    using T TT' SS' by (auto simp: state-eq-def  simp del: state-simp)
qed

lemma cdcl_W-state-eq-compatible:
  assumes
    cdcl_W S T and ¬restart S T and
    S ∼ S'
    T ∼ T' and
    cdcl_W-M-level-inv S
  shows cdcl_W S' T'
  using assms by (meson backtrack backtrack-state-eq-compatible bj cdcl_W.simps cdcl_W-o-rule-cases
    cdcl_W-rf.cases conflict-state-eq-compatible decide decide-state-eq-compatible forget
    forget-state-eq-compatible propagate-state-eq-compatible resolve resolve-state-eq-compatible
    skip skip-state-eq-compatible state-eq-ref)

lemma cdcl_W-bj-state-eq-compatible:
  assumes
    cdcl_W-bj S T and cdcl_W-M-level-inv S
    T ∼ T'
  shows cdcl_W-bj S T'
  using assms by (meson backtrack backtrack-state-eq-compatible cdcl_W-bjE resolve
    resolve-state-eq-compatible skip skip-state-eq-compatible state-eq-ref)

lemma tranclp-cdcl_W-bj-state-eq-compatible:
  assumes
    cdcl_W-bj⁺⁺ S T and  inv: cdcl_W-M-level-inv S and
    S ∼ S' and
    T ∼ T'
  shows cdcl_W-bj⁺⁺ S' T'
  using assms
proof (induction arbitrary: S' T')
  case base
  then show ?case
    unfolding tranclp-unfold-end by (meson backtrack-state-eq-compatible cdcl_W-bj.simps
      resolve-state-eq-compatible  rtranclp-unfold skip-state-eq-compatible)
next
  case (step T U) note IH = this(3)[OF this(4−5)]
  have cdcl_W⁺⁺ S T
    using tranclp-mono[of cdcl_W-bj cdcl_W] step.hyps(1)  cdcl_W.other cdcl_W-o.bj by blast
  then have cdcl_W-M-level-inv T
    using inv tranclp-cdcl_W-consistent-inv by blast
  then have cdcl_W-bj⁺⁺ T T'
    using ‹U ∼ T'› cdcl_W-bj-state-eq-compatible[of T U] ‹cdcl_W-bj T U› by auto
  then show ?case
    using IH[of T] by auto
qed
```

19.3.4 Conservation of some Properties

```
lemma cdcl_W-o-no-more-init-clss:
  assumes
    cdcl_W-o S S' and
    inv: cdcl_W-M-level-inv S
  shows init-clss S = init-clss S'
```

using *assms* **by** (*induct rule*: cdcl_W*-o-induct-lev2*) (*auto simp*: *inv* cdcl_W*-M-level-inv-decomp*)

lemma *tranclp-*cdcl_W*-o-no-more-init-clss*:
assumes
$\mathit{cdcl}_W\text{-}o^{++}\ S\ S'$ **and**
inv: cdcl_W*-M-level-inv S*
shows *init-clss S* = *init-clss S'*
using *assms* **apply** (*induct rule*: *tranclp.induct*)
by (*auto dest*: cdcl_W*-o-no-more-init-clss*
dest!: *tranclp-*cdcl_W*-consistent-inv dest*: *tranclp-mono-explicit*[*of* cdcl_W*-o - -* cdcl_W]
simp: *other*)

lemma *rtranclp-*cdcl_W*-o-no-more-init-clss*:
assumes
$\mathit{cdcl}_W\text{-}o^{**}\ S\ S'$ **and**
inv: cdcl_W*-M-level-inv S*
shows *init-clss S* = *init-clss S'*
using *assms* **unfolding** *rtranclp-unfold* **by** (*auto intro*: *tranclp-*cdcl_W*-o-no-more-init-clss*)

lemma cdcl_W*-init-clss*:
assumes
cdcl_W *S T* **and**
inv: cdcl_W*-M-level-inv S*
shows *init-clss S* = *init-clss T*
using *assms* **by** (*induct rule*: cdcl_W*-all-induct-lev2*)
(*auto simp*: *inv* cdcl_W*-M-level-inv-decomp not-in-iff*)

lemma *rtranclp-*cdcl_W*-init-clss*:
${\mathit{cdcl}_W}^{**}\ S\ T \Longrightarrow$ cdcl_W*-M-level-inv S* $\Longrightarrow$ *init-clss S* = *init-clss T*
by (*induct rule*: *rtranclp-induct*) (*auto dest*: cdcl_W*-init-clss rtranclp-*cdcl_W*-consistent-inv*)

lemma *tranclp-*cdcl_W*-init-clss*:
${\mathit{cdcl}_W}^{++}\ S\ T \Longrightarrow$ cdcl_W*-M-level-inv S* $\Longrightarrow$ *init-clss S* = *init-clss T*
using *rtranclp-*cdcl_W*-init-clss*[*of S T*] **unfolding** *rtranclp-unfold* **by** *auto*

19.3.5 Learned Clause

This invariant shows that:

- the learned clauses are entailed by the initial set of clauses.
- the conflicting clause is entailed by the initial set of clauses.
- the marks are entailed by the clauses. A more precise version would be to show that either these marked are learned or are in the set of clauses

definition cdcl_W*-learned-clause* (*S*:: *'st*) $\longleftrightarrow$
(*init-clss S* ⊨*psm learned-clss S*
∧ (∀ *T*. *conflicting S* = *Some T* ⟶ *init-clss S* ⊨*pm T*)
∧ *set* (*get-all-mark-of-propagated* (*trail S*)) ⊆ *set-mset* (*clauses S*))

lemma cdcl_W*-learned-clause-S0-*cdcl_W[*simp*]:
cdcl_W*-learned-clause* (*init-state N*)
unfolding cdcl_W*-learned-clause-def* **by** *auto*

```
lemma cdcl_W-learned-clss:
  assumes
    cdcl_W S S' and
    learned: cdcl_W-learned-clause S and
    lev-inv: cdcl_W-M-level-inv S
  shows cdcl_W-learned-clause S'
  using assms(1) lev-inv learned
proof (induct rule: cdcl_W-all-induct-lev2)
  case (backtrack K i M1 M2 L D T) note decomp = this(3) and confl = this(1) and undef = this(7)
  and T = this(8)
  show ?case
    using decomp confl learned undef T unfolding cdcl_W-learned-clause-def
    by (auto dest!: get-all-marked-decomposition-exists-prepend
      simp: raw-clauses-def lev-inv cdcl_W-M-level-inv-decomp dest: true-clss-clss-left-right)
next
  case (resolve L C M D) note trail = this(1) and CL = this(2) and confl = this(4) and DL = this(5)
    and lvl = this(6) and T = this(7)
  moreover
    have init-clss S ⊨psm learned-clss S
      using learned trail unfolding cdcl_W-learned-clause-def raw-clauses-def by auto
    then have init-clss S ⊨pm mset-cls C + {#L#}
      using trail learned  unfolding cdcl_W-learned-clause-def raw-clauses-def
      by (auto dest: true-clss-clss-in-imp-true-clss-cls)
  moreover have remove1-mset (− L) (mset-ccls D) + {#− L#} = mset-ccls D
    using DL by (auto simp: multiset-eq-iff)
  moreover have remove1-mset L (mset-cls C) + {#L#} = mset-cls C
    using CL by (auto simp: multiset-eq-iff)
  ultimately show ?case
    using learned T
    by (auto dest: mk-disjoint-insert
      simp add: cdcl_W-learned-clause-def raw-clauses-def
      intro!: true-clss-cls-union-mset-true-clss-cls-or-not-true-clss-cls-or[of - - L])
next
  case (restart T)
  then show ?case
    using learned learned-clss-restart-state[of T]
    by (auto
      simp: raw-clauses-def state-eq-def cdcl_W-learned-clause-def
      simp del: state-simp
      dest: true-clss-clssm-subsetE)
next
  case propagate
  then show ?case using learned by (auto simp: cdcl_W-learned-clause-def)
next
  case conflict
  then show ?case using learned
    by (fastforce simp: cdcl_W-learned-clause-def raw-clauses-def
      true-clss-clss-in-imp-true-clss-cls)
next
  case (forget U)
  then show ?case using learned
    by (auto simp: cdcl_W-learned-clause-def raw-clauses-def split: if-split-asm)
qed (auto simp: cdcl_W-learned-clause-def raw-clauses-def)
```

```
lemma rtranclp-cdcl_W-learned-clss:
  assumes
    cdcl_W** S S' and
    cdcl_W-M-level-inv S
    cdcl_W-learned-clause S
  shows cdcl_W-learned-clause S'
  using assms by induction (auto dest: cdcl_W-learned-clss intro: rtranclp-cdcl_W-consistent-inv)
```

19.3.6 No alien atom in the state

This invariant means that all the literals are in the set of clauses.

```
definition no-strange-atm S' ⟷ (
    (∀ T. conflicting S' = Some T ⟶ atms-of T ⊆ atms-of-mm (init-clss S'))
  ∧ (∀ L mark. Propagated L mark ∈ set (trail S')
      ⟶ atms-of ( mark) ⊆ atms-of-mm (init-clss S'))
  ∧ atms-of-mm (learned-clss S') ⊆ atms-of-mm (init-clss S')
  ∧ atm-of ‘ (lits-of-l (trail S')) ⊆ atms-of-mm (init-clss S'))

lemma no-strange-atm-decomp:
  assumes no-strange-atm S
  shows conflicting S = Some T ⟹ atms-of T ⊆ atms-of-mm (init-clss S)
  and (∀ L mark. Propagated L mark ∈ set (trail S)
    ⟶ atms-of ( mark) ⊆ atms-of-mm (init-clss S))
  and atms-of-mm (learned-clss S) ⊆ atms-of-mm (init-clss S)
  and atm-of ‘ (lits-of-l (trail S)) ⊆ atms-of-mm (init-clss S)
  using assms unfolding no-strange-atm-def by blast+

lemma no-strange-atm-S0 [simp]: no-strange-atm (init-state N)
  unfolding no-strange-atm-def by auto

lemma in-atms-of-implies-atm-of-on-atms-of-ms:
  C + {#L#} ∈# A ⟹ x ∈ atms-of C ⟹ x ∈ atms-of-mm A
  using multi-member-split by fastforce

lemma propagate-no-strange-atm-inv:
  assumes
    propagate S T and
    alien: no-strange-atm S
  shows no-strange-atm T
  using assms(1)
proof (induction)
  case (propagate-rule C L T) note confl = this(1) and C = this(2) and C-L = this(3) and
    tr = this(4) and undef = this(5) and T = this(6)
  have atm-CL: atms-of (mset-cls C) ⊆ atms-of-mm (init-clss S)
    using C alien unfolding no-strange-atm-def
    by (auto simp: raw-clauses-def atms-of-ms-def dest!:in-clss-mset-clss)
  show ?case
    unfolding no-strange-atm-def
    proof (intro conjI allI impI, goal-cases)
      case 1
      then show ?case
        using confl T undef by auto
    next
      case (2 L' mark')
      then show ?case
```

```
        using C-L T alien undef atm-CL

        unfolding no-strange-atm-def raw-clauses-def apply auto by blast
    next
      case (3)
      show ?case using T alien undef unfolding no-strange-atm-def by auto
    next
      case (4)
      show ?case
        using T alien undef C-L atm-CL unfolding no-strange-atm-def by (auto simp: atms-of-def)
    qed
qed

lemma in-atms-of-remove1-mset-in-atms-of:
  x ∈ atms-of (remove1-mset L C) ⟹  x ∈ atms-of C
  using in-diffD unfolding atms-of-def by fastforce

lemma cdcl_W-no-strange-atm-explicit:
  assumes
    cdcl_W S S' and
    lev: cdcl_W-M-level-inv S and
    conf: ∀ T. conflicting S = Some T ⟶ atms-of T ⊆ atms-of-mm (init-clss S) and
    marked: ∀ L mark. Propagated L mark ∈ set (trail S)
      ⟶ atms-of mark ⊆ atms-of-mm (init-clss S) and
    learned: atms-of-mm (learned-clss S) ⊆ atms-of-mm (init-clss S) and
    trail: atm-of ‘ (lits-of-l (trail S)) ⊆ atms-of-mm (init-clss S)
  shows
    (∀ T. conflicting S' = Some T ⟶ atms-of T ⊆ atms-of-mm (init-clss S')) ∧
    (∀ L mark. Propagated L mark ∈ set (trail S')
      ⟶ atms-of ( mark) ⊆ atms-of-mm (init-clss S')) ∧
    atms-of-mm (learned-clss S') ⊆ atms-of-mm (init-clss S') ∧
    atm-of ‘ (lits-of-l (trail S')) ⊆ atms-of-mm (init-clss S')
    (is ?C S' ∧ ?M S' ∧ ?U S' ∧ ?V S')
  using assms(1,2)
proof (induct rule: cdcl_W-all-induct-lev2)
   case (propagate C L T) note confl = this(1) and C-L = this(2) and tr = this(3) and undef =
this(4)
  and T = this(5)
  show ?case
    using propagate-rule[OF propagate.hyps(1−3) - propagate.hyps(5,6), simplified]
    propagate.hyps(4) propagate-no-strange-atm-inv[of S T]
    conf marked learned trail unfolding no-strange-atm-def by presburger
next
  case (decide L)
  then show ?case using learned marked conf trail unfolding raw-clauses-def by auto
next
  case (skip L C M D)
  then show ?case using learned marked conf trail by auto
next
  case (conflict D T) note D-S = this(2) and T = this(4)
  have D: atm-of ‘ set-mset (mset-cls D) ⊆ ⋃(atms-of ‘ (set-mset (clauses S)))
    using D-S by (auto simp add: atms-of-def atms-of-ms-def)
  moreover {
    fix xa :: 'v literal
    assume a1: atm-of ‘ set-mset (mset-cls D) ⊆ (⋃x∈set-mset (init-clss S). atms-of x)
```

```
      ∪ (⋃x∈set-mset (learned-clss S). atms-of x)
    assume a2:
      (⋃x∈set-mset (learned-clss S). atms-of x) ⊆ (⋃x∈set-mset (init-clss S). atms-of x)
    assume xa ∈# mset-cls D
    then have atm-of xa ∈ UNION (set-mset (init-clss S)) atms-of
      using a2 a1 by (metis (no-types) Un-iff atm-of-lit-in-atms-of atms-of-def subset-Un-eq)
    then have ∃ m∈set-mset (init-clss S). atm-of xa ∈ atms-of m
      by blast
    } note H = this
  ultimately show ?case using conflict.prems T learned marked conf trail
    unfolding atms-of-def atms-of-ms-def raw-clauses-def
    by (auto simp add: H)
next
  case (restart T)
  then show ?case using learned marked conf trail by auto
next
  case (forget C T) note confl = this(1) and C = this(4) and C-le = this(5) and
    T = this(6)
  have H: ⋀L mark. Propagated L mark ∈ set (trail S) ⟹ atms-of mark ⊆ atms-of-mm (init-clss S)
    using marked by simp
  show ?case unfolding raw-clauses-def apply (intro conjI)
       using conf confl T trail C unfolding raw-clauses-def apply (auto dest!: H)[]
      using T trail C C-le apply (auto dest!: H)[]
     using T learned C-le atms-of-ms-remove-subset[of set-mset (learned-clss S)] apply auto[]
   using T trail C-le apply (auto simp: raw-clauses-def lits-of-def)[]
   done
next
  case (backtrack K i M1 M2 L D T) note confl = this(1) and LD = this(2) and decomp = this(3)
and
    undef = this(7) and T = this(8)
  have ?C T
    using conf T decomp undef lev by (auto simp: cdcl_W-M-level-inv-decomp)
  moreover have set M1 ⊆ set (trail S)
    using decomp by auto
  then have M: ?M T
    using marked conf undef confl T decomp lev
    by (auto simp: image-subset-iff raw-clauses-def cdcl_W-M-level-inv-decomp)
  moreover have ?U T
    using learned decomp conf confl T undef lev unfolding raw-clauses-def
    by (auto simp: cdcl_W-M-level-inv-decomp)
  moreover have ?V T
    using M conf confl trail T undef decomp lev LD
    by (auto simp: cdcl_W-M-level-inv-decomp atms-of-def
      dest!: get-all-marked-decomposition-exists-prepend)
  ultimately show ?case by blast
next
  case (resolve L C M D T) note trail-S = this(1) and confl = this(4) and T = this(7)
  let ?T = update-conflicting (Some ((remove-clit (−L) D) !∪ ccls-of-cls ((remove-lit L C))))
    (tl-trail S)
  have ?C ?T
    using confl trail-S conf marked by (auto dest!: in-atms-of-remove1-mset-in-atms-of)
  moreover have  ?M ?T
    using confl trail-S conf marked by auto
  moreover have ?U ?T
    using trail learned by auto
```

moreover have *?V ?T*
using *confl trail-S trail* **by** *auto*
ultimately show *?case* **using** *T* **by** *simp*
qed

lemma *cdcl*$_W$*-no-strange-atm-inv*:
assumes *cdcl*$_W$ *S S'* **and** *no-strange-atm S* **and** *cdcl*$_W$*-M-level-inv S*
shows *no-strange-atm S'*
using *cdcl*$_W$*-no-strange-atm-explicit*[*OF assms*(*1*)] *assms*(*2*,*3*) **unfolding** *no-strange-atm-def* **by** *fast*

lemma *rtranclp-cdcl*$_W$*-no-strange-atm-inv*:
assumes *cdcl*$_W$** *S S'* **and** *no-strange-atm S* **and** *cdcl*$_W$*-M-level-inv S*
shows *no-strange-atm S'*
using *assms* **by** *induction* (*auto intro*: *cdcl*$_W$*-no-strange-atm-inv rtranclp-cdcl*$_W$*-consistent-inv*)

19.3.7 No duplicates all around

This invariant shows that there is no duplicate (no literal appearing twice in the formula). The last part could be proven using the previous invariant moreover.

definition *distinct-cdcl*$_W$*-state* (*S*::*'st*)
⟷ ((∀ *T*. *conflicting S* = *Some T* ⟶ *distinct-mset T*)
∧ *distinct-mset-mset* (*learned-clss S*)
∧ *distinct-mset-mset* (*init-clss S*)
∧ (∀ *L mark*. (*Propagated L mark* ∈ *set* (*trail S*) ⟶ *distinct-mset* (*mark*))))

lemma *distinct-cdcl*$_W$*-state-decomp*:
assumes *distinct-cdcl*$_W$*-state* (*S*::*'st*)
shows ∀ *T*. *conflicting S* = *Some T* ⟶ *distinct-mset T*
and *distinct-mset-mset* (*learned-clss S*)
and *distinct-mset-mset* (*init-clss S*)
and ∀ *L mark*. (*Propagated L mark* ∈ *set* (*trail S*) ⟶ *distinct-mset* (*mark*))
using *assms* **unfolding** *distinct-cdcl*$_W$*-state-def* **by** *blast+*

lemma *distinct-cdcl*$_W$*-state-decomp-2*:
assumes *distinct-cdcl*$_W$*-state* (*S*::*'st*)
shows *conflicting S* = *Some T* ⟹ *distinct-mset T*
using *assms* **unfolding** *distinct-cdcl*$_W$*-state-def* **by** *auto*

lemma *distinct-cdcl*$_W$*-state-S0-cdcl*$_W$[*simp*]:
distinct-mset-mset (*mset-clss N*) ⟹ *distinct-cdcl*$_W$*-state* (*init-state N*)
unfolding *distinct-cdcl*$_W$*-state-def* **by** *auto*

lemma *distinct-cdcl*$_W$*-state-inv*:
assumes
cdcl$_W$ *S S'* **and**
lev-inv: *cdcl*$_W$*-M-level-inv S* **and**
distinct-cdcl$_W$*-state S*
shows *distinct-cdcl*$_W$*-state S'*
using *assms*(*1*,*2*,*2*,*3*)
proof (*induct rule*: *cdcl*$_W$*-all-induct-lev2*)
case (*backtrack L D K i M1 M2*)
then show *?case*
using *lev-inv* **unfolding** *distinct-cdcl*$_W$*-state-def*
by (*auto dest*: *get-all-marked-decomposition-incl simp*: *cdcl*$_W$*-M-level-inv-decomp*)
next

```
  case restart
  then show ?case
    unfolding distinct-cdcl_W-state-def distinct-mset-set-def raw-clauses-def
    using learned-clss-restart-state[of S] by auto
next
  case resolve
  then show ?case
    by (auto simp add: distinct-cdcl_W-state-def distinct-mset-set-def raw-clauses-def
      distinct-mset-single-add
      intro!: distinct-mset-union-mset)
qed (auto simp: distinct-cdcl_W-state-def distinct-mset-set-def raw-clauses-def
  dest!: in-clss-mset-clss in-diffD)

lemma rtanclp-distinct-cdcl_W-state-inv:
  assumes
    cdcl_W** S S' and
    cdcl_W-M-level-inv S and
    distinct-cdcl_W-state S
  shows distinct-cdcl_W-state S'
  using assms apply (induct rule: rtranclp-induct)
  using distinct-cdcl_W-state-inv rtranclp-cdcl_W-consistent-inv by blast+
```

19.3.8 Conflicts and co

This invariant shows that each mark contains a contradiction only related to the previously defined variable.

```
abbreviation every-mark-is-a-conflict :: 'st ⇒ bool where
every-mark-is-a-conflict S ≡
 ∀L mark a b. a @ Propagated L mark # b = (trail S)
   ⟶ (b ⊨as CNot ( mark - {#L#}) ∧ L ∈#  mark)

definition cdcl_W-conflicting S ≡
  (∀T. conflicting S = Some T ⟶ trail S ⊨as CNot T)
  ∧ every-mark-is-a-conflict S

lemma backtrack-atms-of-D-in-M1:
  fixes M1 :: ('v, nat, 'v clause) marked-lits
  assumes
    inv: cdcl_W-M-level-inv S and
    undef: undefined-lit M1 L and
    i: get-maximum-level (trail S) (mset-ccls (remove-clit L D)) ≡ i and
    decomp: (Marked K (Suc i) # M1, M2)
      ∈ set (get-all-marked-decomposition (trail S)) and
    S-lvl: backtrack-lvl S = get-maximum-level (trail S) (mset-ccls D) and
    S-confl: raw-conflicting S = Some D and
    undef: undefined-lit M1 L and
    T: T ∼ cons-trail (Propagated L (cls-of-ccls D))
               (reduce-trail-to M1
                 (add-learned-cls (cls-of-ccls D)
                   (update-backtrack-lvl i
                     (update-conflicting None S)))) and
    confl: ∀T. conflicting S = Some T ⟶ trail S ⊨as CNot T
  shows atms-of (mset-ccls (remove-clit L D)) ⊆ atm-of ` lits-of-l (tl (trail T))
proof (rule ccontr)
  let ?k = get-maximum-level (trail S) (mset-ccls D)
  let ?D = mset-ccls D
  let ?D′ = mset-ccls (remove-clit L D)
  have trail S ⊨as CNot ?D using confl S-confl by auto
  then have vars-of-D: atms-of ?D ⊆ atm-of ‘ lits-of-l (trail S) unfolding atms-of-def
    by (meson image-subsetI true-annots-CNot-all-atms-defined)

  obtain M0 where M: trail S = M0 @ M2 @ Marked K (Suc i) # M1
    using decomp by auto

  have max: ?k = length (get-all-levels-of-marked (M0 @ M2 @ Marked K (Suc i) # M1))
    using inv unfolding cdcl_W-M-level-inv-def S-lvl M by simp
  assume a: ¬ ?thesis
  then obtain L′ where
    L′: L′ ∈ atms-of ?D′ and
    L′-notin-M1: L′ ∉ atm-of ‘ lits-of-l M1
    using T undef decomp inv by (auto simp: cdcl_W-M-level-inv-decomp)
  then have L′-in: L′ ∈ atm-of ‘ lits-of-l (M0 @ M2 @ Marked K (i + 1) # [])
    using vars-of-D unfolding M by (auto dest: in-atms-of-remove1-mset-in-atms-of)
  then obtain L′′ where
    L′′ ∈# ?D′ and
    L′′: L′ = atm-of L′′
    using L′ L′-notin-M1 unfolding atms-of-def by auto
  have lev-L′′:
    get-level (trail S) L′′ = get-rev-level (Marked K (Suc i) # rev M2 @ rev M0) (Suc i) L′′
    using L′-notin-M1 L′′ M by (auto simp del: get-rev-level.simps)
  have get-all-levels-of-marked (trail S) = rev [1..<1+?k]
    using inv S-lvl unfolding cdcl_W-M-level-inv-def by auto
  then have get-all-levels-of-marked (M0 @ M2) = rev [Suc (Suc i)..<Suc ?k]
    unfolding M by (auto simp:rev-swap[symmetric] dest!: append-cons-eq-upt-length-i-end)

  then have M: get-all-levels-of-marked M0 @ get-all-levels-of-marked M2
   = rev [Suc (Suc i)..<Suc (length (get-all-levels-of-marked (M0 @ M2 @ Marked K (Suc i) # M1)))]
    unfolding max unfolding M by simp

  have get-rev-level (Marked K (Suc i) # rev (M0 @ M2)) (Suc i) L′′
    ≥ Min (set ((Suc i) # get-all-levels-of-marked (Marked K (Suc i) # rev (M0 @ M2))))
    using get-rev-level-ge-min-get-all-levels-of-marked[of L′′
      rev (M0 @ M2 @ [Marked K (Suc i)]) Suc i] L′-in
    unfolding L′′ by (fastforce simp add: lits-of-def)
  also have Min (set ((Suc i) # get-all-levels-of-marked (Marked K (Suc i) # rev (M0 @ M2))))
    = Min (set ((Suc i) # get-all-levels-of-marked (rev (M0 @ M2)))) by auto
  also have ... = Min (set ((Suc i) # get-all-levels-of-marked M0 @ get-all-levels-of-marked M2))
    by (simp add: Un-commute)
  also have ... =  Min (set ((Suc i) # [Suc (Suc i)..<2 + length (get-all-levels-of-marked M0)
    + (length (get-all-levels-of-marked M2) + length (get-all-levels-of-marked M1))]))
    unfolding M by (auto simp add: Un-commute)
  also have ... = Suc i by (auto intro: Min-eqI)
  finally have get-rev-level (Marked K (Suc i) # rev (M0 @ M2)) (Suc i) L′′ ≥ Suc i .
  then have get-level (trail S) L′′ ≥ i + 1
    using lev-L′′ by simp
  then have get-maximum-level (trail S) ?D′ ≥ i + 1
    using get-maximum-level-ge-get-level[OF ‹L′′ ∈# ?D′›, of trail S] by auto
  then show False using i by auto
qed
```

```
lemma distinct-atms-of-incl-not-in-other:
  assumes
    a1: no-dup (M @ M′) and
    a2: atms-of D ⊆ atm-of ‘ lits-of-l M′ and
    a3: x ∈ atms-of D
  shows x ∉ atm-of ‘ lits-of-l M
proof –
  have ff1: ⋀l ms. undefined-lit ms l ∨ atm-of l
    ∈ set (map (λm. atm-of (lit-of (m::(′a, ′b, ′c) marked-lit))) ms)
    by (simp add: defined-lit-map)
  have ff2: ⋀a. a ∉ atms-of D ∨ a ∈ atm-of ‘ lits-of-l M′
    using a2 by (meson subsetCE)
  have ff3: ⋀a. a ∉ set (map (λm. atm-of (lit-of m)) M′)
    ∨ a ∉ set (map (λm. atm-of (lit-of m)) M)
    using a1 by (metis (lifting) IntI distinct-append empty-iff map-append)
  have ∀ L a f. ∃ l. ((a::′a) ∉ f ‘ L ∨ (l::′a literal) ∈ L) ∧ (a ∉ f ‘ L ∨ f l = a)
    by blast
  then show x ∉ atm-of ‘ lits-of-l M
    using ff3 ff2 ff1 a3 by (metis (no-types) Marked-Propagated-in-iff-in-lits-of-l)
qed

lemma cdcl_W-propagate-is-conclusion:
  assumes
    cdcl_W S S′ and
    inv: cdcl_W-M-level-inv S and
    decomp: all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S)) and
    learned: cdcl_W-learned-clause S and
    confl: ∀ T. conflicting S = Some T ⟶ trail S ⊨as CNot T and
    alien: no-strange-atm S
  shows all-decomposition-implies-m (init-clss S′) (get-all-marked-decomposition (trail S′))
  using assms(1,2)
proof (induct rule: cdcl_W-all-induct-lev2)
  case restart
  then show ?case by auto
next
  case forget
  then show ?case using decomp by auto
next
  case conflict
  then show ?case using decomp by auto
next
  case (resolve L C M D) note tr = this(1) and T = this(7)
  let ?decomp = get-all-marked-decomposition M
  have M: set ?decomp = insert (hd ?decomp) (set (tl ?decomp))
    by (cases ?decomp) auto
  show ?case
    using decomp tr T unfolding all-decomposition-implies-def
    by (cases hd (get-all-marked-decomposition M))
       (auto simp: M)
next
  case (skip L C′ M D) note tr = this(1) and T = this(5)
  have M: set (get-all-marked-decomposition M)
    = insert (hd (get-all-marked-decomposition M)) (set (tl (get-all-marked-decomposition M)))
    by (cases get-all-marked-decomposition M) auto
  show ?case
```

```
    using decomp tr T unfolding all-decomposition-implies-def
    by (cases hd (get-all-marked-decomposition M))
      (auto simp add: M)
next
  case decide note S = this(1) and undef = this(2) and T = this(4)
  show ?case using decomp T undef unfolding S all-decomposition-implies-def by auto
next
  case (propagate C L T) note propa = this(2) and L = this(3) and undef = this(5) and T = this(6)
  obtain a y where ay: hd (get-all-marked-decomposition (trail S)) = (a, y)
    by (cases hd (get-all-marked-decomposition (trail S)))
  then have M: trail S = y @ a using get-all-marked-decomposition-decomp by blast
  have M': set (get-all-marked-decomposition (trail S))
    = insert (a, y) (set (tl (get-all-marked-decomposition (trail S))))
    using ay by (cases get-all-marked-decomposition (trail S)) auto
  have unmark-l a ∪ set-mset (init-clss S) ⊨ps unmark-l y
    using decomp ay unfolding all-decomposition-implies-def
    by (cases get-all-marked-decomposition (trail S)) fastforce+
  then have a-Un-N-M: unmark-l a ∪ set-mset (init-clss S)
    ⊨ps unmark-l (trail S)
    unfolding M by (auto simp add: all-in-true-clss-clss image-Un)

  have unmark-l a ∪ set-mset (init-clss S) ⊨p {#L#} (is ?I ⊨p -)
    proof (rule true-clss-cls-plus-CNot)
      show ?I ⊨p remove1-mset L (mset-cls C) + {#L#}
        apply (rule true-clss-clss-in-imp-true-clss-cls[of -
            set-mset (init-clss S) ∪ set-mset (learned-clss S)])
        using learned propa L by (auto simp: raw-clauses-def cdcl_W-learned-clause-def
          true-annot-CNot-diff)
    next
      have unmark-l (trail S) ⊨ps CNot (remove1-mset L (mset-cls C))
        using ‹(trail S) ⊨as CNot (remove1-mset L (mset-cls C))› true-annots-true-clss-clss
        by blast
      then show ?I ⊨ps CNot (remove1-mset L (mset-cls C))
        using a-Un-N-M true-clss-clss-left-right true-clss-clss-union-l-r by blast
    qed
  moreover have ⋀aa b.
      ∀ (Ls, seen)∈set (get-all-marked-decomposition (y @ a)).
        unmark-l Ls ∪ set-mset (init-clss S) ⊨ps unmark-l seen
    ⟹ (aa, b) ∈ set (tl (get-all-marked-decomposition (y @ a)))
    ⟹ unmark-l aa ∪ set-mset (init-clss S) ⊨ps unmark-l b
    by (metis (no-types, lifting) case-prod-conv get-all-marked-decomposition-never-empty-sym
      list.collapse list.set-intros(2))

  ultimately show ?case
    using decomp T undef unfolding ay all-decomposition-implies-def
    using M ‹unmark-l a ∪ set-mset (init-clss S) ⊨ps unmark-l y›
     ay by auto
next
  case (backtrack K i M1 M2 L D  T) note conf = this(1) and LD = this(2) and decomp' = this(3)
and
    lev-L = this(4) and undef = this(7) and T = this(8)
  let ?D = mset-ccls D
  let ?D' = mset-ccls (remove-clit L D)
  have ∀ l ∈ set M2. ¬is-marked l
    using get-all-marked-decomposition-snd-not-marked decomp' by blast
```

```
obtain M0 where M: trail S = M0 @ M2 @ Marked K (i + 1) # M1
  using decomp' by auto
show ?case unfolding all-decomposition-implies-def
  proof
    fix x
    assume x ∈ set (get-all-marked-decomposition (trail T))
    then have x: x ∈ set (get-all-marked-decomposition (Propagated L ?D # M1))
      using T decomp' undef inv by (simp add: cdcl_W-M-level-inv-decomp)
    let ?m = get-all-marked-decomposition (Propagated L ?D # M1)
    let ?hd = hd ?m
    let ?tl = tl ?m
    consider
        (hd) x = ?hd
      | (tl) x ∈ set ?tl
      using x by (cases ?m) auto
    then show case x of (Ls, seen) ⇒ unmark-l Ls ∪ set-mset (init-clss T)
      ⊨ps unmark-l seen
      proof cases
        case tl
        then have x ∈ set (get-all-marked-decomposition (trail S))
          using tl-get-all-marked-decomposition-skip-some[of x] by (simp add: list.set-sel(2) M)
        then show ?thesis
          using decomp learned decomp confl alien inv T undef M
          unfolding all-decomposition-implies-def cdcl_W-M-level-inv-def
          by auto
      next
        case hd
        obtain M1' M1'' where M1: hd (get-all-marked-decomposition M1) = (M1', M1'')
          by (cases hd (get-all-marked-decomposition M1))
        then have x': x = (M1', Propagated L ?D # M1'')
          using ‹x = ?hd› by auto
        have (M1', M1'') ∈ set (get-all-marked-decomposition (trail S))
          using M1[symmetric] hd-get-all-marked-decomposition-skip-some[OF M1[symmetric],
            of M0 @ M2 - i+1] unfolding M by fastforce
        then have 1: unmark-l M1' ∪ set-mset (init-clss S) ⊨ps unmark-l M1''
          using decomp unfolding all-decomposition-implies-def by auto

        moreover
          have vars-of-D: atms-of ?D' ⊆ atm-of ‘ lits-of-l M1
            using backtrack-atms-of-D-in-M1[of S M1 L D i K M2 T] backtrack.hyps inv conf confl
            by (auto simp: cdcl_W-M-level-inv-decomp)
          have no-dup (trail S) using inv by (auto simp: cdcl_W-M-level-inv-decomp)
          then have vars-in-M1:
            ∀x ∈ atms-of ?D'. x ∉ atm-of ‘ lits-of-l (M0 @ M2 @ Marked K (i + 1) # [])
            using vars-of-D distinct-atms-of-incl-not-in-other[of
              M0 @M2 @ Marked K (i + 1) # [] M1] unfolding M by auto
          have trail S ⊨as CNot (remove1-mset L (mset-ccls D))
            using conf confl LD unfolding M true-annots-true-cls-def-iff-negation-in-model
            by (auto dest!: Multiset.in-diffD)
          then have M1 ⊨as CNot ?D'
            using vars-in-M1 true-annots-remove-if-notin-vars[of M0 @ M2 @ Marked K (i + 1) # []
              M1 CNot ?D'] conf confl unfolding M lits-of-def by simp
          have M1 = M1'' @ M1' by (simp add: M1 get-all-marked-decomposition-decomp)
          have TT: unmark-l M1' ∪ set-mset (init-clss S) ⊨ps CNot ?D'
            using true-annots-true-clss-cls[OF ‹M1 ⊨as CNot ?D'›] true-clss-clss-left-right[OF 1]
          unfolding ‹M1 = M1'' @ M1'› by (auto simp add: inf-sup-aci(5,7))
        have init-clss S ⊨pm ?D' + {#L#}
          using conf learned confl LD unfolding cdcl_W-learned-clause-def by auto
        then have T': unmark-l M1' ∪ set-mset (init-clss S) ⊨p ?D' + {#L#} by auto
        have atms-of (?D' + {#L#}) ⊆ atms-of-mm (clauses S)
          using alien conf LD unfolding no-strange-atm-def raw-clauses-def by auto
        then have unmark-l M1' ∪ set-mset (init-clss S) ⊨p {#L#}
          using true-clss-cls-plus-CNot[OF T' TT] by auto

      ultimately show ?thesis
          using T' T decomp' undef inv unfolding x' by (simp add: cdcl_W-M-level-inv-decomp)
    qed
  qed
qed

lemma cdcl_W-propagate-is-false:
  assumes
    cdcl_W S S' and
    lev: cdcl_W-M-level-inv S and
    learned: cdcl_W-learned-clause S and
    decomp: all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S)) and
    confl: ∀ T. conflicting S = Some T ⟶ trail S ⊨as CNot T and
    alien: no-strange-atm S and
    mark-confl: every-mark-is-a-conflict S
  shows every-mark-is-a-conflict S'
  using assms(1,2)
proof (induct rule: cdcl_W-all-induct-lev2)
  case (propagate C L T) note LC = this(3) and confl = this(4) and undef = this(5) and T =
this(6)
  show ?case
    proof (intro allI impI)
      fix L' mark a b
      assume a @ Propagated L' mark # b = trail T
      then consider
          (hd) a = [] and L = L' and mark = mset-cls C and b = trail S
        | (tl) tl a @ Propagated L' mark # b = trail S
        using T undef by (cases a) fastforce+
      then show b ⊨as CNot (mark − {#L'#}) ∧ L' ∈#  mark
        using mark-confl confl LC by cases auto
    qed
next
  case (decide L) note undef[simp] = this(2) and T = this(4)
  have ⋀a La mark b. a @ Propagated La mark # b = Marked L (backtrack-lvl S+1) # trail S
    ⟹ tl a @ Propagated La mark # b = trail S by (case-tac a) auto
  then show ?case using mark-confl T unfolding decide.hyps(1) by fastforce
next
  case (skip L C' M D T) note tr = this(1) and T = this(5)
  show ?case
    proof (intro allI impI)
      fix L' mark a b
      assume a @ Propagated L' mark # b = trail T
      then have a @ Propagated L' mark # b = M using tr T by simp
      then have (Propagated L C' # a) @ Propagated L' mark # b = Propagated L C' # M by auto
      moreover have ∀ La mark a b. a @ Propagated La mark # b = Propagated L C' # M
        ⟶ b ⊨as CNot ( mark − {#La#}) ∧ La ∈#  mark
        using mark-confl unfolding skip.hyps(1) by simp
      ultimately show b ⊨as CNot (mark − {#L'#}) ∧ L' ∈# mark by blast
    qed
next
  case (conflict D)
  then show ?case using mark-confl by simp
next
  case (resolve L C M D T) note tr-S = this(1) and T = this(7)
  show ?case unfolding resolve.hyps(1)
    proof (intro allI impI)
      fix L' mark a b
      assume a @ Propagated L' mark # b = trail T
      then have (Propagated L (mset-cls (L !++ C)) # a) @ Propagated L' mark # b
        = Propagated L (mset-cls (L !++ C)) # M
        using T tr-S by auto
      then show b ⊨as CNot (mark − {#L'#}) ∧ L' ∈# mark
        using mark-confl unfolding tr-S by (metis Cons-eq-appendI list.sel(3))
    qed
next
  case restart
  then show ?case by auto
next
  case forget
  then show ?case using mark-confl by auto
next
  case (backtrack K i M1 M2 L D T) note conf = this(1) and LD = this(2) and decomp = this(3)
and
    undef = this(7) and T = this(8)
  have ∀ l ∈ set M2. ¬is-marked l
    using get-all-marked-decomposition-snd-not-marked decomp by blast
  obtain M0 where M: trail S = M0 @ M2 @ Marked K (i + 1) # M1
    using decomp by auto
  have [simp]: trail (reduce-trail-to M1 (add-learned-cls (cls-of-ccls (insert-ccls L D))
    (update-backtrack-lvl i (update-conflicting None S)))) = M1
    using decomp lev by (auto simp: cdcl_W-M-level-inv-decomp)
  let ?D = mset-ccls D
  let ?D' = mset-ccls (remove-clit L D)
  show ?case
    proof (intro allI impI)
      fix La :: 'v literal and mark :: 'v literal multiset and
        a b :: ('v, nat, 'v literal multiset) marked-lit list
      assume a @ Propagated La mark # b = trail T
      then consider
          (hd-tr) a = [] and
            (Propagated La mark :: ('v, nat, 'v literal multiset) marked-lit)
              = Propagated L ?D and
            b = M1
        | (tl-tr) tl a @ Propagated La mark # b = M1
        using M T decomp undef lev by (cases a) (auto simp: cdcl_W-M-level-inv-def)
      then show b ⊨as CNot (mark − {#La#}) ∧ La ∈# mark
        proof cases
          case hd-tr note A = this(1) and P = this(2) and b = this(3)
          have trail S ⊨as CNot ?D using conf confl by auto
          then have vars-of-D: atms-of ?D ⊆ atm-of ‘ lits-of-l (trail S)
            unfolding atms-of-def
            by (meson image-subsetI true-annots-CNot-all-atms-defined)
          have vars-of-D: atms-of ?D' ⊆ atm-of ‘ lits-of-l M1
            using backtrack-atms-of-D-in-M1[of S M1 L D i K M2 T] T backtrack lev confl
            by (auto simp: cdcl_W-M-level-inv-decomp)
          have no-dup (trail S) using lev by (auto simp: cdcl_W-M-level-inv-decomp)
          then have ∀ x ∈ atms-of ?D'. x ∉ atm-of ‘ lits-of-l (M0 @ M2 @ Marked K (i + 1) # [])
            using vars-of-D distinct-atms-of-incl-not-in-other[of
              M0 @ M2 @ Marked K (i + 1) # [] M1] unfolding M by auto
          then have M1 ⊨as CNot ?D'
            using true-annots-remove-if-notin-vars[of M0 @ M2 @ Marked K (i + 1) # []
              M1 CNot ?D'] ‹trail S ⊨as CNot ?D› unfolding M lits-of-def
            by (simp add: true-annot-CNot-diff)
          then show b ⊨as CNot (mark − {#La#}) ∧ La ∈# mark
            using P LD b by auto
        next
          case tl-tr
          then obtain c' where c' @ Propagated La mark # b = trail S
            unfolding M by auto
          then show b ⊨as CNot (mark − {#La#}) ∧ La ∈# mark
            using mark-confl by auto
        qed
    qed
qed

lemma cdcl_W-conflicting-is-false:
  assumes
    cdcl_W S S' and
    M-lev: cdcl_W-M-level-inv S and
    confl-inv: ∀ T. conflicting S = Some T ⟶ trail S ⊨as CNot T and
    marked-confl: ∀ L mark a b. a @ Propagated L mark # b = (trail S)
      ⟶ (b ⊨as CNot (mark − {#L#}) ∧ L ∈# mark) and
    dist: distinct-cdcl_W-state S
  shows ∀ T. conflicting S' = Some T ⟶ trail S' ⊨as CNot T
  using assms(1,2)
proof (induct rule: cdcl_W-all-induct-lev2)
  case (skip L C' M D T) note tr-S = this(1) and confl = this(2) and L-D = this(3) and T =
this(5)
  let ?D = mset-ccls D
  have D: Propagated L C' # M ⊨as CNot (mset-ccls D) using assms skip by auto
  moreover
    have L ∉# ?D
      proof (rule ccontr)
        assume ¬ ?thesis
        then have − L ∈ lits-of-l M
          using in-CNot-implies-uminus(2)[of L ?D Propagated L C' # M]
          ‹Propagated L C' # M ⊨as CNot ?D› by simp
        then show False
          by (metis (no-types, hide-lams) M-lev cdcl_W-M-level-inv-decomp(1) consistent-interp-def
            image-insert insert-iff list.set(2) lits-of-def marked-lit.sel(2) tr-S)
      qed
  ultimately show ?case
    using tr-S confl L-D T unfolding cdcl_W-M-level-inv-def
    by (auto intro: true-annots-CNot-lit-of-notin-skip)
next
  case (resolve L C M D T) note tr = this(1) and LC = this(2) and confl = this(4) and LD =
this(5)
  and T = this(7)
  let ?C = remove1-mset L (mset-cls C)
  let ?D = remove1-mset (−L) (mset-ccls D)
  show ?case
    proof (intro allI impI)
      fix T′
      have tl (trail S) ⊨as CNot ?C using tr marked-confl by fastforce
      moreover
        have distinct-mset (?D + {#− L#}) using confl dist LD
          unfolding distinct-cdcl_W-state-def by auto
        then have −L ∉# ?D unfolding distinct-mset-def
          by (meson ‹distinct-mset (?D + {#− L#})› distinct-mset-single-add)
        have M ⊨as CNot ?D
          proof −
            have Propagated L (?C + {#L#}) # M ⊨as CNot ?D ∪ CNot {#− L#}
              using confl tr confl-inv LC by (metis CNot-plus LD insert-DiffM2 option.simps(9))
            then show ?thesis
              using M-lev ‹− L ∉# ?D› tr true-annots-lit-of-notin-skip
              unfolding cdcl_W-M-level-inv-def by force
          qed
      moreover assume conflicting T = Some T′
      ultimately
        show trail T ⊨as CNot T′
        using tr T by auto
    qed
qed (auto simp: M-lev cdcl_W-M-level-inv-decomp)

lemma cdcl_W-conflicting-decomp:
  assumes cdcl_W-conflicting S
  shows ∀ T. conflicting S = Some T ⟶ trail S ⊨as CNot T
  and ∀ L mark a b. a @ Propagated L mark # b = (trail S)
    ⟶ (b ⊨as CNot ( mark − {#L#}) ∧ L ∈# mark)
  using assms unfolding cdcl_W-conflicting-def by blast+

lemma cdcl_W-conflicting-decomp2:
  assumes cdcl_W-conflicting S and conflicting S = Some T
  shows trail S ⊨as CNot T
  using assms unfolding cdcl_W-conflicting-def by blast+

lemma cdcl_W-conflicting-S0-cdcl_W[simp]:
  cdcl_W-conflicting (init-state N)
  unfolding cdcl_W-conflicting-def by auto
```

19.3.9 Putting all the invariants together

```
lemma cdcl_W-all-inv:
  assumes
    cdcl_W: cdcl_W S S′ and
    1: all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S)) and
    2: cdcl_W-learned-clause S and
    4: cdcl_W-M-level-inv S and
    5: no-strange-atm S and
    7: distinct-cdcl_W-state S and
    8: cdcl_W-conflicting S
  shows
```

```
    all-decomposition-implies-m (init-clss S') (get-all-marked-decomposition (trail S')) and
    cdcl_W-learned-clause S' and
    cdcl_W-M-level-inv S' and
    no-strange-atm S' and
    distinct-cdcl_W-state S' and
    cdcl_W-conflicting S'
proof –
  show S1: all-decomposition-implies-m (init-clss S') (get-all-marked-decomposition (trail S'))
    using cdcl_W-propagate-is-conclusion[OF cdcl_W 4 1 2 - 5] 8 unfolding cdcl_W-conflicting-def
    by blast
  show S2: cdcl_W-learned-clause S' using cdcl_W-learned-clss[OF cdcl_W 2 4] .
  show S4: cdcl_W-M-level-inv S' using cdcl_W-consistent-inv[OF cdcl_W 4] .
  show S5: no-strange-atm S'  using  cdcl_W-no-strange-atm-inv[OF cdcl_W 5 4] .
  show S7: distinct-cdcl_W-state S' using distinct-cdcl_W-state-inv[OF cdcl_W 4 7] .
  show S8: cdcl_W-conflicting S'
    using cdcl_W-conflicting-is-false[OF cdcl_W 4 - - 7]  8 cdcl_W-propagate-is-false[OF cdcl_W 4 2 1 -
      5]
    unfolding cdcl_W-conflicting-def by fast
qed

lemma rtranclp-cdcl_W-all-inv:
  assumes
    cdcl_W: rtranclp cdcl_W S S' and
    1: all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S)) and
    2: cdcl_W-learned-clause S and
    4: cdcl_W-M-level-inv S and
    5: no-strange-atm S and
    7: distinct-cdcl_W-state S and
    8: cdcl_W-conflicting S
  shows
    all-decomposition-implies-m (init-clss S') (get-all-marked-decomposition (trail S')) and
    cdcl_W-learned-clause S' and
    cdcl_W-M-level-inv S' and
    no-strange-atm S' and
    distinct-cdcl_W-state S' and
    cdcl_W-conflicting S'
   using assms
proof (induct rule: rtranclp-induct)
  case base
    case 1 then show ?case by blast
    case 2 then show ?case by blast
    case 3 then show ?case by blast
    case 4 then show ?case by blast
    case 5 then show ?case by blast
    case 6 then show ?case by blast
next
  case (step S' S'') note H = this
    case 1 with H(3−7)[OF this(1−6)] show ?case using cdcl_W-all-inv[OF H(2)]
        H by presburger
    case 2 with H(3−7)[OF this(1−6)] show ?case using cdcl_W-all-inv[OF H(2)]
        H by presburger
    case 3 with H(3−7)[OF this(1−6)] show ?case using cdcl_W-all-inv[OF H(2)]
        H by presburger
    case 4 with H(3−7)[OF this(1−6)] show ?case using cdcl_W-all-inv[OF H(2)]
        H by presburger
    case 5 with H(3−7)[OF this(1−6)] show ?case using cdcl_W-all-inv[OF H(2)]
      H by presburger
    case 6 with H(3−7)[OF this(1−6)] show ?case using cdcl_W-all-inv[OF H(2)]
      H by presburger
qed

lemma all-invariant-S0-cdcl_W:
  assumes distinct-mset-mset (mset-clss N)
  shows
    all-decomposition-implies-m (init-clss (init-state N))
                                    (get-all-marked-decomposition (trail (init-state N))) and
    cdcl_W-learned-clause (init-state N) and
    ∀ T. conflicting (init-state N) = Some T ⟶ (trail (init-state N))⊨as CNot T and
    no-strange-atm (init-state N) and
    consistent-interp (lits-of-l (trail (init-state N))) and
    ∀ L mark a b. a @ Propagated L mark # b =  trail (init-state N) ⟶
     (b ⊨as CNot ( mark − {#L#}) ∧ L ∈#  mark) and
     distinct-cdcl_W-state (init-state N)
  using assms by auto


lemma cdcl_W-only-propagated-vars-unsat:
  assumes
    marked: ∀ x ∈ set M. ¬ is-marked x and
    DN: D ∈# clauses S and
    D: M ⊨as CNot D and
    inv: all-decomposition-implies-m N (get-all-marked-decomposition M) and
    state: state S = (M, N, U, k, C) and
    learned-cl: cdcl_W-learned-clause S and
    atm-incl: no-strange-atm S
  shows unsatisfiable (set-mset N)
proof (rule ccontr)
  assume ¬ unsatisfiable (set-mset N)
  then obtain I where
    I: I ⊨s set-mset N and
    cons: consistent-interp I and
    tot: total-over-m I (set-mset N)
    unfolding satisfiable-def by auto
  have atms-of-mm N ∪ atms-of-mm U = atms-of-mm N
    using atm-incl state unfolding total-over-m-def no-strange-atm-def
     by (auto simp add: raw-clauses-def)
  then have total-over-m I (set-mset N) using tot unfolding total-over-m-def by auto
  moreover then have total-over-m I (set-mset (learned-clss S))
    using atm-incl state unfolding no-strange-atm-def total-over-m-def total-over-set-def
    by auto
  moreover have N ⊨psm U using learned-cl state unfolding cdcl_W-learned-clause-def by auto
  ultimately have I-D: I ⊨ D
    using I DN cons state unfolding true-clss-clss-def true-clss-def Ball-def
    by (auto simp add: raw-clauses-def)

  have l0: {unmark L |L. is-marked L ∧ L ∈ set M} = {} using marked by auto
  have atms-of-ms (set-mset N ∪ unmark-l M) = atms-of-mm N
    using atm-incl state unfolding no-strange-atm-def by auto
  then have total-over-m I (set-mset N ∪ unmark-l M)
    using tot unfolding total-over-m-def by auto
  then have I ⊨s unmark-l M
    using all-decomposition-implies-propagated-lits-are-implied[OF inv] cons I
    unfolding true-clss-clss-def l0 by auto
  then have IM: I ⊨s unmark-l M by auto
  {
    fix K
    assume K ∈# D
    then have −K ∈ lits-of-l M
      using D unfolding true-annots-def Ball-def CNot-def true-annot-def true-cls-def true-lit-def
      Bex-def by force
    then have −K ∈ I using IM true-clss-singleton-lit-of-implies-incl lits-of-def by fastforce }
  then have ¬ I ⊨ D using cons unfolding true-cls-def true-lit-def consistent-interp-def by auto
  then show False using I-D by blast
qed
```

We have actually a much stronger theorem, namely *all-decomposition-implies ?N* (*get-all-marked-decomposition ?M*) $\Longrightarrow$ *?N* ∪ {*unmark L* |*L. is-marked L* ∧ *L* ∈ *set ?M*} ⊨*ps unmark-l ?M*, that show that the only choices we made are marked in the formula

```
lemma
  assumes all-decomposition-implies-m N (get-all-marked-decomposition M)
  and ∀ m ∈ set M. ¬is-marked m
  shows set-mset N ⊨ps unmark-l M
proof −
  have T: {unmark L |L. is-marked L ∧ L ∈ set M} = {} using assms(2) by auto
  then show ?thesis
    using all-decomposition-implies-propagated-lits-are-implied[OF assms(1)] unfolding T by simp
qed
```

```
lemma conflict-with-false-implies-unsat:
  assumes
    cdcl_W: cdcl_W S S' and
    lev: cdcl_W-M-level-inv S and
    [simp]: conflicting S' = Some {#} and
    learned: cdcl_W-learned-clause S
  shows unsatisfiable (set-mset (init-clss S))
  using assms
proof −
  have cdcl_W-learned-clause S' using cdcl_W-learned-clss cdcl_W learned lev by auto
  then have init-clss S' ⊨pm {#} using assms(3) unfolding cdcl_W-learned-clause-def by auto
  then have init-clss S ⊨pm {#}
    using cdcl_W-init-clss[OF assms(1) lev] by auto
  then show ?thesis unfolding satisfiable-def true-clss-cls-def by auto
qed
```

```
lemma conflict-with-false-implies-terminated:
  assumes cdcl_W S S'
  and conflicting S = Some {#}
  shows False
  using assms by (induct rule: cdcl_W-all-induct) auto
```

19.3.10 No tautology is learned

This is a simple consequence of all we have shown previously. It is not strictly necessary, but helps finding a better bound on the number of learned clauses.

```
lemma learned-clss-are-not-tautologies:
  assumes
    cdcl_W S S' and
    lev: cdcl_W-M-level-inv S and
    conflicting: cdcl_W-conflicting S and
    no-tauto: ∀ s ∈# learned-clss S. ¬tautology s
  shows ∀ s ∈# learned-clss S'. ¬tautology s
  using assms
proof (induct rule: cdcl_W-all-induct-lev2)
  case (backtrack K i M1 M2 L D T) note confl = this(1)
  have consistent-interp (lits-of-l (trail S)) using lev by (auto simp: cdcl_W-M-level-inv-decomp)
  moreover
    have trail S ⊨as CNot (mset-ccls D)
      using conflicting confl unfolding cdcl_W-conflicting-def by auto
    then have lits-of-l (trail S) ⊨s CNot (mset-ccls D)
      using true-annots-true-cls by blast
  ultimately have ¬tautology (mset-ccls D) using consistent-CNot-not-tautology by blast
  then show ?case using backtrack no-tauto lev
    by (auto simp: cdcl_W-M-level-inv-decomp split: if-split-asm)
next
  case restart
  then show ?case using learned-clss-restart-state state-eq-learned-clss no-tauto
    by (metis (no-types, lifting) set-mset-mono subsetCE)
qed (auto dest!: in-diffD)

definition final-cdcl_W-state (S:: 'st)
  ⟷ (trail S ⊨asm init-clss S
    ∨ ((∀ L ∈ set (trail S). ¬is-marked L) ∧
       (∃ C ∈# init-clss S. trail S ⊨as CNot C)))

definition termination-cdcl_W-state (S:: 'st)
   ⟷ (trail S ⊨asm init-clss S
     ∨ ((∀ L ∈ atms-of-mm (init-clss S). L ∈ atm-of ` lits-of-l (trail S))
        ∧ (∃ C ∈# init-clss S. trail S ⊨as CNot C)))
```

19.4 CDCL Strong Completeness

```
fun mapi :: ('a ⇒ nat ⇒ 'b) ⇒ nat ⇒ 'a list ⇒ 'b list where
mapi - - [] = [] |
mapi f n (x # xs) = f x n # mapi f (n − 1) xs

lemma mark-not-in-set-mapi[simp]: L ∉ set M ⟹ Marked L k ∉ set (mapi Marked i M)
  by (induct M arbitrary: i) auto

lemma propagated-not-in-set-mapi[simp]: L ∉ set M ⟹ Propagated L k ∉ set (mapi Marked i M)
  by (induct M arbitrary: i) auto

lemma image-set-mapi:
  f ` set (mapi g i M) = set (mapi (λx i. f (g x i)) i M)
  by (induction M arbitrary: i) auto

lemma mapi-map-convert:
  ∀ x i j. f x i = f x j ⟹ mapi f i M = map (λx. f x 0) M
  by (induction M arbitrary: i) auto

lemma defined-lit-mapi: defined-lit (mapi Marked i M) L ⟷ atm-of L ∈ atm-of ` set M
```

```
  by (induction M) (auto simp: defined-lit-map image-set-mapi mapi-map-convert)

lemma cdcl_W-can-do-step:
  assumes
    consistent-interp (set M) and
    distinct M and
    atm-of ‘ (set M) ⊆ atms-of-mm (mset-clss N)
  shows ∃S. rtranclp cdcl_W (init-state N) S
    ∧ state S = (mapi Marked (length M) M, mset-clss N, {#}, length M, None)
  using assms
proof (induct M)
  case Nil
  then show ?case apply − by (rule exI[of - init-state N]) auto
next
  case (Cons L M) note IH = this(1)
  have consistent-interp (set M) and distinct M and atm-of ‘ set M ⊆ atms-of-mm (mset-clss N)
    using Cons.prems(1−3) unfolding consistent-interp-def by auto
  then obtain S where
    st: cdcl_W** (init-state N) S and
    S: state S = (mapi Marked (length M) M, mset-clss N, {#}, length M, None)
    using IH by blast
  let ?S_0 = incr-lvl (cons-trail (Marked L (length M +1)) S)
  have undefined-lit (mapi Marked (length M) M) L
    using Cons.prems(1,2) unfolding defined-lit-def consistent-interp-def by fastforce
  moreover have init-clss S = mset-clss N
    using S by blast
  moreover have atm-of L ∈ atms-of-mm (mset-clss N) using Cons.prems(3) by auto
  moreover have undef: undefined-lit (trail S) L
    using S ‹distinct (L#M)› calculation(1) by (auto simp: defined-lit-mapi defined-lit-map)
  ultimately have cdcl_W S ?S_0
    using cdcl_W.other[OF cdcl_W-o.decide[OF decide-rule[of S L ?S_0]]] S
    by (auto simp: state-eq-def simp del: state-simp)
  then have cdcl_W** (init-state N) ?S_0
    using st by auto
  then show ?case
    using S undef by (auto intro!: exI[of - ?S_0] del: simp del: )
qed

lemma cdcl_W-strong-completeness:
  assumes
    MN: set M ⊨sm mset-clss N and
    cons: consistent-interp (set M) and
    dist: distinct M and
    atm: atm-of ‘ (set M) ⊆ atms-of-mm (mset-clss N)
  obtains S where
    state S = (mapi Marked (length M) M, mset-clss N, {#}, length M, None) and
    rtranclp cdcl_W (init-state N) S and
    final-cdcl_W-state S
proof −
  obtain S where
    st: rtranclp cdcl_W (init-state N) S and
    S: state S = (mapi Marked (length M) M, mset-clss N, {#}, length M, None)
    using cdcl_W-can-do-step[OF cons dist atm] by auto
  have lits-of-l (mapi Marked (length M) M) = set M
    by (induct M, auto)
```

then have *mapi Marked* (*length M*) *M* $\models$*asm mset-clss N* **using** *MN true-annots-true-cls* **by** *metis*
then have *final-cdcl*$_W$*-state S*
using *S* **unfolding** *final-cdcl*$_W$*-state-def* **by** *auto*
then show *?thesis* **using** *that st S* **by** *blast*
qed

19.5 Higher level strategy

The rules described previously do not lead to a conclusive state. We have to add a strategy.

19.5.1 Definition

lemma *tranclp-conflict*:
tranclp conflict S S$'$ $\Longrightarrow$ *conflict S S*$'$
apply (*induct rule*: *tranclp.induct*)
apply *simp*
by (*metis conflictE conflicting-update-conflicting option.distinct*(*1*) *option.simps*(*8*,*9*)
state-eq-conflicting)

lemma *tranclp-conflict-iff* [*iff*]:
full1 conflict S S$'$ $\longleftrightarrow$ *conflict S S*$'$
proof −
have *tranclp conflict S S*$'$ $\Longrightarrow$ *conflict S S*$'$ **by** (*meson tranclp-conflict rtranclpD*)
then show *?thesis* **unfolding** *full1-def*
by (*metis conflict.simps conflicting-update-conflicting option.distinct*(*1*) *option.simps*(*9*)
state-eq-conflicting tranclp.intros(*1*))
qed

inductive *cdcl*$_W$*-cp* :: $'st \Rightarrow 'st \Rightarrow$ *bool* **where**
conflict$'$[*intro*]: *conflict S S*$'$ $\Longrightarrow$ *cdcl*$_W$*-cp S S*$'$ |
propagate$'$: *propagate S S*$'$ $\Longrightarrow$ *cdcl*$_W$*-cp S S*$'$

lemma *rtranclp-cdcl*$_W$*-cp-rtranclp-cdcl*$_W$:
cdcl$_W$*-cp*** *S T* $\Longrightarrow$ *cdcl*$_W{}^{**}$ *S T*
by (*induction rule*: *rtranclp-induct*) (*auto simp*: *cdcl*$_W$*-cp.simps dest*: *cdcl*$_W$*.intros*)

lemma *cdcl*$_W$*-cp-state-eq-compatible*:
assumes
cdcl$_W$*-cp S T* **and**
$S \sim S'$ **and**
$T \sim T'$
shows *cdcl*$_W$*-cp S*$'$ *T*$'$
using *assms*
apply (*induction*)
using *conflict-state-eq-compatible* **apply** *auto*[*1*]
using *propagate*$'$ *propagate-state-eq-compatible* **by** *auto*

lemma *tranclp-cdcl*$_W$*-cp-state-eq-compatible*:
assumes
cdcl$_W$*-cp*$^{++}$ *S T* **and**
$S \sim S'$ **and**
$T \sim T'$
shows *cdcl*$_W$*-cp*$^{++}$ *S*$'$ *T*$'$
using *assms*
proof *induction*

```
  case base
  then show ?case
    using cdcl_W-cp-state-eq-compatible by blast
next
  case (step U V)
  obtain ss :: 'st where
    cdcl_W-cp S ss ∧ cdcl_W-cp^** ss U
    by (metis (no-types) step(1) tranclpD)
  then show ?case
    by (meson cdcl_W-cp-state-eq-compatible rtranclp.rtrancl-into-rtrancl rtranclp-into-tranclp2
      state-eq-ref step(2) step(4) step(5))
qed

lemma option-full-cdcl_W-cp:
  conflicting S ≠ None ⟹ full cdcl_W-cp S S
  unfolding full-def rtranclp-unfold tranclp-unfold
  by (auto simp add: cdcl_W-cp.simps elim: conflictE propagateE)

lemma skip-unique:
  skip S T ⟹ skip S T' ⟹ T ∼ T'
  by (fastforce simp: state-eq-def simp del: state-simp elim: skipE)

lemma resolve-unique:
  resolve S T ⟹ resolve S T' ⟹ T ∼ T'
  by (fastforce simp: state-eq-def simp del: state-simp elim: resolveE)

lemma cdcl_W-cp-no-more-clauses:
  assumes cdcl_W-cp S S'
  shows clauses S = clauses S'
  using assms by (induct rule: cdcl_W-cp.induct) (auto elim!: conflictE propagateE)

lemma tranclp-cdcl_W-cp-no-more-clauses:
  assumes cdcl_W-cp^++ S S'
  shows clauses S = clauses S'
  using assms by (induct rule: tranclp.induct) (auto dest: cdcl_W-cp-no-more-clauses)

lemma rtranclp-cdcl_W-cp-no-more-clauses:
  assumes cdcl_W-cp^** S S'
  shows clauses S = clauses S'
  using assms by (induct rule: rtranclp-induct) (fastforce dest: cdcl_W-cp-no-more-clauses)+

lemma no-conflict-after-conflict:
  conflict S T ⟹ ¬conflict T U
  by (metis None-eq-map-option-iff conflictE conflicting-update-conflicting option.distinct(1)
    state-simp(5))

lemma no-propagate-after-conflict:
  conflict S T ⟹ ¬propagate T U
  by (metis conflictE conflicting-update-conflicting map-option-is-None option.distinct(1)
    propagate.cases state-eq-conflicting)

lemma tranclp-cdcl_W-cp-propagate-with-conflict-or-not:
  assumes cdcl_W-cp^++ S U
  shows (propagate^++ S U ∧ conflicting U = None)
    ∨ (∃ T D. propagate^** S T ∧ conflict T U ∧ conflicting U = Some D)
```

proof −
  have *propagate*$^{++}$ *S U* $\lor$ ($\exists$ *T*. *propagate*** *S T* $\land$ *conflict T U*)
    using *assms* **by** *induction*
    (*force simp*: *cdcl*$_W$*-cp.simps tranclp-into-rtranclp dest*: *no-conflict-after-conflict*
      no-propagate-after-conflict)+
  moreover
    have *propagate*$^{++}$ *S U* $\Longrightarrow$ *conflicting U* = *None*
      unfolding *tranclp-unfold-end* **by** (*auto elim*!: *propagateE*)
  moreover
    have $\bigwedge$*T*. *conflict T U* $\Longrightarrow$ $\exists$ *D*. *conflicting U* = *Some D*
      by (*auto elim*!: *conflictE simp*: *state-eq-def simp del*: *state-simp*)
  ultimately show *?thesis* **by** *meson*
qed

lemma *cdcl*$_W$*-cp-conflicting-not-empty*[*simp*]: *conflicting S* = *Some D* $\Longrightarrow$ $\neg$*cdcl*$_W$*-cp S S*$'$
proof
  assume *cdcl*$_W$*-cp S S*$'$ **and** *conflicting S* = *Some D*
  then show *False* **by** (*induct rule*: *cdcl*$_W$*-cp.induct*)
  (*auto elim*: *conflictE propagateE simp*: *state-eq-def simp del*: *state-simp*)
qed

lemma *no-step-cdcl*$_W$*-cp-no-conflict-no-propagate*:
  assumes *no-step cdcl*$_W$*-cp S*
  shows *no-step conflict S* **and** *no-step propagate S*
  using *assms conflict*$'$ **apply** *blast*
  by (*meson assms conflict*$'$ *propagate*$'$)

CDCL with the reasonable strategy: we fully propagate the conflict and propagate, then we apply any other possible rule *cdcl*$_W$*-o S S*$'$ and re-apply conflict and propagate *full cdcl*$_W$*-cp S*$'$ *S*$''$

inductive *cdcl*$_W$*-stgy* :: $'$*st* $\Rightarrow$ $'$*st* $\Rightarrow$ *bool* **for** *S* :: $'$*st* **where**
conflict$'$: *full1 cdcl*$_W$*-cp S S*$'$ $\Longrightarrow$ *cdcl*$_W$*-stgy S S*$'$ |
other$'$: *cdcl*$_W$*-o S S*$'$ $\Longrightarrow$ *no-step cdcl*$_W$*-cp S* $\Longrightarrow$ *full cdcl*$_W$*-cp S*$'$ *S*$''$ $\Longrightarrow$ *cdcl*$_W$*-stgy S S*$''$

19.5.2 Invariants

These are the same invariants as before, but lifted

lemma *cdcl*$_W$*-cp-learned-clause-inv*:
  assumes *cdcl*$_W$*-cp S S*$'$
  shows *learned-clss S* = *learned-clss S*$'$
  using *assms* **by** (*induct rule*: *cdcl*$_W$*-cp.induct*) (*fastforce elim*: *conflictE propagateE*)+

lemma *rtranclp-cdcl*$_W$*-cp-learned-clause-inv*:
  assumes *cdcl*$_W$*-cp*** *S S*$'$
  shows *learned-clss S* = *learned-clss S*$'$
  using *assms* **by** (*induct rule*: *rtranclp-induct*) (*fastforce dest*: *cdcl*$_W$*-cp-learned-clause-inv*)+

lemma *tranclp-cdcl*$_W$*-cp-learned-clause-inv*:
  assumes *cdcl*$_W$*-cp*$^{++}$ *S S*$'$
  shows *learned-clss S* = *learned-clss S*$'$
  using *assms* **by** (*simp add*: *rtranclp-cdcl*$_W$*-cp-learned-clause-inv tranclp-into-rtranclp*)

lemma *cdcl*$_W$*-cp-backtrack-lvl*:
  assumes *cdcl*$_W$*-cp S S*$'$
  shows *backtrack-lvl S* = *backtrack-lvl S*$'$

```
  using assms by (induct rule: cdcl_W-cp.induct) (fastforce elim: conflictE propagateE)+

lemma rtranclp-cdcl_W-cp-backtrack-lvl:
  assumes cdcl_W-cp** S S'
  shows backtrack-lvl S = backtrack-lvl S'
  using assms by (induct rule: rtranclp-induct) (fastforce dest: cdcl_W-cp-backtrack-lvl)+

lemma cdcl_W-cp-consistent-inv:
  assumes cdcl_W-cp S S'
  and cdcl_W-M-level-inv S
  shows cdcl_W-M-level-inv S'
  using assms
proof (induct rule: cdcl_W-cp.induct)
  case (conflict')
  then show ?case using cdcl_W-consistent-inv cdcl_W.conflict by blast
next
  case (propagate' S S')
  have cdcl_W S S'
    using propagate'.hyps(1) propagate by blast
  then show cdcl_W-M-level-inv S'
    using propagate'.prems(1) cdcl_W-consistent-inv propagate by blast
qed

lemma full1-cdcl_W-cp-consistent-inv:
  assumes full1 cdcl_W-cp S S'
  and cdcl_W-M-level-inv S
  shows cdcl_W-M-level-inv S'
  using assms unfolding full1-def
  by (metis rtranclp-cdcl_W-cp-rtranclp-cdcl_W rtranclp-unfold tranclp-cdcl_W-consistent-inv)

lemma rtranclp-cdcl_W-cp-consistent-inv:
  assumes rtranclp cdcl_W-cp S S'
  and cdcl_W-M-level-inv S
  shows cdcl_W-M-level-inv S'
  using assms unfolding full1-def
  by (induction rule: rtranclp-induct) (blast intro: cdcl_W-cp-consistent-inv)+

lemma cdcl_W-stgy-consistent-inv:
  assumes cdcl_W-stgy S S'
  and cdcl_W-M-level-inv S
  shows cdcl_W-M-level-inv S'
  using assms apply (induct rule: cdcl_W-stgy.induct)
  unfolding full-unfold by (blast intro: cdcl_W-consistent-inv full1-cdcl_W-cp-consistent-inv
    cdcl_W.other)+

lemma rtranclp-cdcl_W-stgy-consistent-inv:
  assumes cdcl_W-stgy** S S'
  and cdcl_W-M-level-inv S
  shows cdcl_W-M-level-inv S'
  using assms by induction (auto dest!: cdcl_W-stgy-consistent-inv)

lemma cdcl_W-cp-no-more-init-clss:
  assumes cdcl_W-cp S S'
  shows init-clss S = init-clss S'
  using assms by (induct rule: cdcl_W-cp.induct) (auto elim: conflictE propagateE)
```

lemma *tranclp-cdcl*$_W$*-cp-no-more-init-clss*:
assumes *cdcl*$_W$*-cp*$^{++}$ *S S'*
shows *init-clss S* = *init-clss S'*
using *assms* **by** (*induct rule*: *tranclp.induct*) (*auto dest*: *cdcl*$_W$*-cp-no-more-init-clss*)

lemma *cdcl*$_W$*-stgy-no-more-init-clss*:
assumes *cdcl*$_W$*-stgy S S'* **and** *cdcl*$_W$*-M-level-inv S*
shows *init-clss S* = *init-clss S'*
using *assms*
apply (*induct rule*: *cdcl*$_W$*-stgy.induct*)
unfolding *full1-def full-def* **apply** (*blast dest*: *tranclp-cdcl*$_W$*-cp-no-more-init-clss*
tranclp-cdcl$_W$*-o-no-more-init-clss*)
by (*metis cdcl*$_W$*-o-no-more-init-clss rtranclp-unfold tranclp-cdcl*$_W$*-cp-no-more-init-clss*)

lemma *rtranclp-cdcl*$_W$*-stgy-no-more-init-clss*:
assumes *cdcl*$_W$*-stgy*** *S S'* **and** *cdcl*$_W$*-M-level-inv S*
shows *init-clss S* = *init-clss S'*
using *assms*
apply (*induct rule*: *rtranclp-induct*, *simp*)
using *cdcl*$_W$*-stgy-no-more-init-clss* **by** (*simp add*: *rtranclp-cdcl*$_W$*-stgy-consistent-inv*)

lemma *cdcl*$_W$*-cp-dropWhile-trail'*:
assumes *cdcl*$_W$*-cp S S'*
obtains *M* **where** *trail S'* = *M* @ *trail S* **and** (∀ *l* ∈ *set M*. ¬*is-marked l*)
using *assms* **by** *induction* (*fastforce elim*: *conflictE propagateE*)+

lemma *rtranclp-cdcl*$_W$*-cp-dropWhile-trail'*:
assumes *cdcl*$_W$*-cp*** *S S'*
obtains *M* :: (*'v*, *nat*, *'v clause*) *marked-lit list* **where**
trail S' = *M* @ *trail S* **and** ∀ *l* ∈ *set M*. ¬*is-marked l*
using *assms* **by** *induction* (*fastforce dest!*: *cdcl*$_W$*-cp-dropWhile-trail'*)+

lemma *cdcl*$_W$*-cp-dropWhile-trail*:
assumes *cdcl*$_W$*-cp S S'*
shows ∃ *M*. *trail S'* = *M* @ *trail S* ∧ (∀ *l* ∈ *set M*. ¬*is-marked l*)
using *assms* **by** *induction* (*fastforce elim*: *conflictE propagateE*)+

lemma *rtranclp-cdcl*$_W$*-cp-dropWhile-trail*:
assumes *cdcl*$_W$*-cp*** *S S'*
shows ∃ *M*. *trail S'* = *M* @ *trail S* ∧ (∀ *l* ∈ *set M*. ¬*is-marked l*)
using *assms* **by** *induction* (*fastforce dest*: *cdcl*$_W$*-cp-dropWhile-trail*)+

This theorem can be seen a a termination theorem for *cdcl*$_W$*-cp*.

lemma *length-model-le-vars*:
assumes
no-strange-atm S **and**
no-d: *no-dup* (*trail S*) **and**
finite (*atms-of-mm* (*init-clss S*))
shows *length* (*trail S*) ≤ *card* (*atms-of-mm* (*init-clss S*))
proof −
obtain *M N U k D* **where** *S*: *state S* = (*M*, *N*, *U*, *k*, *D*) **by** (*cases state S*, *auto*)
have *finite* (*atm-of* ‘ *lits-of-l* (*trail S*))
using *assms*(*1*,*3*) **unfolding** *S* **by** (*auto simp add*: *finite-subset*)
have *length* (*trail S*) = *card* (*atm-of* ‘ *lits-of-l* (*trail S*))

```
    using no-dup-length-eq-card-atm-of-lits-of-l no-d by blast
  then show ?thesis using assms(1) unfolding no-strange-atm-def
  by (auto simp add: assms(3) card-mono)
qed

lemma cdcl_W-cp-decreasing-measure:
  assumes
    cdcl_W: cdcl_W-cp S T and
    M-lev: cdcl_W-M-level-inv S and
    alien: no-strange-atm S
  shows (λS. card (atms-of-mm (init-clss S)) − length (trail S)
      + (if conflicting S = None then 1 else 0)) S
    > (λS. card (atms-of-mm (init-clss S)) − length (trail S)
      + (if conflicting S = None then 1 else 0)) T
  using assms
proof −
  have length (trail T) ≤ card (atms-of-mm (init-clss T))
    apply (rule length-model-le-vars)
       using cdcl_W-no-strange-atm-inv alien M-lev apply (meson cdcl_W cdcl_W.simps cdcl_W-cp.cases)
      using M-lev cdcl_W cdcl_W-cp-consistent-inv cdcl_W-M-level-inv-def apply blast
      using cdcl_W by (auto simp: cdcl_W-cp.simps)
  with assms
  show ?thesis by induction (auto elim!: conflictE propagateE
    simp del: state-simp simp: state-eq-def)+
qed

lemma cdcl_W-cp-wf: wf {(b,a). (cdcl_W-M-level-inv a ∧ no-strange-atm a)
  ∧ cdcl_W-cp a b}
  apply (rule wf-wf-if-measure'[of less-than - -
      (λS. card (atms-of-mm (init-clss S)) − length (trail S)
        + (if conflicting S = None then 1 else 0))])
    apply simp
  using cdcl_W-cp-decreasing-measure unfolding less-than-iff by blast

lemma rtranclp-cdcl_W-all-struct-inv-cdcl_W-cp-iff-rtranclp-cdcl_W-cp:
  assumes
    lev: cdcl_W-M-level-inv S and
    alien: no-strange-atm S
  shows (λa b. (cdcl_W-M-level-inv a ∧ no-strange-atm a) ∧ cdcl_W-cp a b)** S T
  ⟷ cdcl_W-cp** S T
  (is ?I S T ⟷ ?C S T)
proof
  assume
    ?I S T
  then show ?C S T by induction auto
next
  assume
    ?C S T
  then show ?I S T
    proof induction
      case base
      then show ?case by simp
    next
      case (step T U) note st = this(1) and cp = this(2) and IH = this(3)
      have cdcl_W** S T
```

```
        by (metis rtranclp-unfold cdcl_W-cp-conflicting-not-empty cp st
          rtranclp-propagate-is-rtranclp-cdcl_W tranclp-cdcl_W-cp-propagate-with-conflict-or-not)
      then have
        cdcl_W-M-level-inv T and
        no-strange-atm T
         using ‹cdcl_W^** S T› apply (simp add: assms(1) rtranclp-cdcl_W-consistent-inv)
        using ‹cdcl_W^** S T› alien rtranclp-cdcl_W-no-strange-atm-inv lev by blast
      then have (λa b. (cdcl_W-M-level-inv a ∧ no-strange-atm a)
        ∧ cdcl_W-cp a b)^** T U
        using cp by auto
      then show ?case using IH by auto
    qed
qed

lemma cdcl_W-cp-normalized-element:
  assumes
    lev: cdcl_W-M-level-inv S and
    no-strange-atm S
  obtains T where full cdcl_W-cp S T
proof −
  let ?inv = λa. (cdcl_W-M-level-inv a ∧ no-strange-atm a)
  obtain T where T: full (λa b. ?inv a ∧ cdcl_W-cp a b) S T
    using cdcl_W-cp-wf wf-exists-normal-form[of λa b. ?inv a ∧ cdcl_W-cp a b]
    unfolding full-def by blast
    then have cdcl_W-cp^** S T
      using rtranclp-cdcl_W-all-struct-inv-cdcl_W-cp-iff-rtranclp-cdcl_W-cp assms unfolding full-def
      by blast
    moreover
      then have cdcl_W^** S T
        using rtranclp-cdcl_W-cp-rtranclp-cdcl_W by blast
      then have
        cdcl_W-M-level-inv T and
        no-strange-atm T
         using ‹cdcl_W^** S T› apply (simp add: assms(1) rtranclp-cdcl_W-consistent-inv)
        using ‹cdcl_W^** S T› assms(2) rtranclp-cdcl_W-no-strange-atm-inv lev by blast
      then have no-step cdcl_W-cp T
        using T unfolding full-def by auto
    ultimately show thesis using that unfolding full-def by blast
qed

lemma always-exists-full-cdcl_W-cp-step:
  assumes no-strange-atm S
  shows ∃S''. full cdcl_W-cp S S''
  using assms
proof (induct card (atms-of-mm (init-clss S) − atm-of ‘lits-of-l (trail S)) arbitrary: S)
  case 0 note card = this(1) and alien = this(2)
  then have atm: atms-of-mm (init-clss S) = atm-of ‘ lits-of-l (trail S)
    unfolding no-strange-atm-def by auto
  { assume a: ∃S'. conflict S S'
    then obtain S' where S': conflict S S' by metis
    then have ∀S''. ¬cdcl_W-cp S' S''
      by (auto simp: cdcl_W-cp.simps elim!: conflictE propagateE
        simp del: state-simp simp: state-eq-def)
    then have ?case using a S' cdcl_W-cp.conflict' unfolding full-def by blast
  }
  moreover {
    assume a: ∃ S'. propagate S S'
    then obtain S' where propagate S S' by blast
    then obtain E L where
      S: conflicting S = None and
      E:  E !∈! raw-clauses S and
      LE: L ∈# mset-cls E and
      tr: trail S ⊨as CNot (mset-cls (remove-lit L E)) and
      undef: undefined-lit (trail S) L and
      S': S' ∼ cons-trail (Propagated L E) S
      by (elim propagateE) simp
    have atms-of-mm (learned-clss S) ⊆ atms-of-mm (init-clss S)
      using alien S unfolding no-strange-atm-def by auto
    then have atm-of L ∈ atms-of-mm (init-clss S)
      using E LE S undef unfolding raw-clauses-def by (force simp: in-implies-atm-of-on-atms-of-ms)
    then have False using undef S unfolding atm unfolding lits-of-def
      by (auto simp add: defined-lit-map)
  }
  ultimately show ?case unfolding full-def by (metis cdcl_W-cp.cases rtranclp.rtrancl-refl)
next
  case (Suc n) note IH = this(1) and card = this(2) and alien = this(3)
  { assume a: ∃ S'. conflict S S'
    then obtain S' where S': conflict S S' by metis
    then have ∀ S''. ¬cdcl_W-cp S' S''
      by (auto simp: cdcl_W-cp.simps elim!: conflictE propagateE
        simp del: state-simp simp: state-eq-def)
    then have ?case  unfolding full-def Ex-def using S' cdcl_W-cp.conflict' by blast
  }
  moreover {
    assume a: ∃ S'. propagate S S'
    then obtain S' where propagate: propagate S S' by blast
    then obtain E L where
      S: conflicting S = None and
      E:  E !∈! raw-clauses S and
      LE: L ∈# mset-cls E and
      tr: trail S ⊨as CNot (mset-cls (remove-lit L E)) and
      undef: undefined-lit (trail S) L and
      S': S' ∼ cons-trail (Propagated L E) S
      by (elim propagateE) simp
    then have atm-of L ∉ atm-of ` lits-of-l (trail S)
      unfolding lits-of-def by (auto simp add: defined-lit-map)
    moreover
      have no-strange-atm S' using alien propagate propagate-no-strange-atm-inv by blast
      then have atm-of L ∈ atms-of-mm (init-clss S)
        using S' LE E undef unfolding no-strange-atm-def
        by (auto simp: raw-clauses-def in-implies-atm-of-on-atms-of-ms)
      then have ⋀A. {atm-of L} ⊆ atms-of-mm (init-clss S) − A ∨ atm-of L ∈ A by force
    moreover have Suc n − card {atm-of L} = n by simp
    moreover have card (atms-of-mm (init-clss S) − atm-of ` lits-of-l (trail S)) = Suc n
     using card S S' by simp
    ultimately
      have card (atms-of-mm (init-clss S) − atm-of ` insert L (lits-of-l (trail S))) = n
        by (metis (no-types) Diff-insert card-Diff-subset finite.emptyI finite.insertI image-insert)
      then have n = card (atms-of-mm (init-clss S') − atm-of ` lits-of-l (trail S'))
        using card S S' undef by simp
    then have a1: Ex (full cdcl_W-cp S') using IH ‹no-strange-atm S'›  by blast
    have ?case
      proof –
        obtain S'' :: 'st where
          ff1: cdcl_W-cp** S' S'' ∧ no-step cdcl_W-cp S''
          using a1 unfolding full-def by blast
        have cdcl_W-cp** S S''
          using ff1 cdcl_W-cp.intros(2)[OF propagate]
          by (metis (no-types) converse-rtranclp-into-rtranclp)
        then have ∃S''. cdcl_W-cp** S S'' ∧ (∀S'''. ¬ cdcl_W-cp S'' S''')
          using ff1 by blast
        then show ?thesis unfolding full-def
          by meson
      qed
    }
  ultimately show ?case unfolding full-def by (metis cdcl_W-cp.cases rtranclp.rtrancl-refl)
qed
```

19.5.3 Literal of highest level in conflicting clauses

One important property of the $cdcl_W$ with strategy is that, whenever a conflict takes place, there is at least a literal of level k involved (except if we have derived the false clause). The reason is that we apply conflicts before a decision is taken.

```
abbreviation no-clause-is-false :: 'st ⇒ bool where
no-clause-is-false ≡
  λS. (conflicting S = None ⟶ (∀D ∈# clauses S. ¬trail S ⊨as CNot D))

abbreviation conflict-is-false-with-level :: 'st ⇒ bool where
conflict-is-false-with-level S ≡ ∀D. conflicting S = Some D ⟶ D ≠ {#}
  ⟶ (∃L ∈# D. get-level (trail S) L = backtrack-lvl S)

lemma not-conflict-not-any-negated-init-clss:
  assumes ∀ S'. ¬conflict S S'
  shows no-clause-is-false S
proof (clarify)
  fix D
  assume D ∈# local.clauses S and raw-conflicting S = None and trail S ⊨as CNot D
  moreover then obtain D' where
    mset-cls D' = D and
    D' !∈! raw-clauses S
    using in-mset-clss-exists-preimage unfolding raw-clauses-def by blast
  ultimately show False
    using conflict-rule[of S D' update-conflicting (Some (ccls-of-cls D')) S] assms
    by auto
qed

lemma full-cdcl_W-cp-not-any-negated-init-clss:
  assumes full cdcl_W-cp S S'
  shows no-clause-is-false S'
  using assms not-conflict-not-any-negated-init-clss unfolding full-def by auto

lemma full1-cdcl_W-cp-not-any-negated-init-clss:
  assumes full1 cdcl_W-cp S S'
  shows no-clause-is-false S'
  using assms not-conflict-not-any-negated-init-clss unfolding full1-def by auto
```

```
lemma cdcl_W-stgy-not-non-negated-init-clss:
  assumes cdcl_W-stgy S S'
  shows no-clause-is-false S'
  using assms apply (induct rule: cdcl_W-stgy.induct)
  using full1-cdcl_W-cp-not-any-negated-init-clss full-cdcl_W-cp-not-any-negated-init-clss by metis+

lemma rtranclp-cdcl_W-stgy-not-non-negated-init-clss:
  assumes cdcl_W-stgy** S S' and no-clause-is-false S
  shows no-clause-is-false S'
  using assms by (induct rule: rtranclp-induct) (auto simp: cdcl_W-stgy-not-non-negated-init-clss)

lemma cdcl_W-stgy-conflict-ex-lit-of-max-level:
  assumes cdcl_W-cp S S'
  and no-clause-is-false S
  and cdcl_W-M-level-inv S
  shows conflict-is-false-with-level S'
  using assms
proof (induct rule: cdcl_W-cp.induct)
  case conflict'
  then show ?case by (auto elim: conflictE)
next
  case propagate'
  then show ?case by (auto elim: propagateE)
qed

lemma no-chained-conflict:
  assumes conflict S S'
  and conflict S' S''
  shows False
  using assms unfolding conflict.simps
  by (metis conflicting-update-conflicting option.distinct(1) option.simps(9) state-eq-conflicting)

lemma rtranclp-cdcl_W-cp-propa-or-propa-confl:
  assumes cdcl_W-cp** S U
  shows propagate** S U ∨ (∃ T. propagate** S T  ∧ conflict T U)
  using assms
proof induction
  case base
  then show ?case by auto
next
  case (step U V) note SU = this(1) and UV = this(2) and IH = this(3)
  consider (confl) T where propagate** S T and conflict T U
    | (propa) propagate** S U using IH by auto
  then show ?case
    proof cases
      case confl
      then have False using UV by (auto elim: conflictE)
      then show ?thesis by fast
    next
      case propa
      also have conflict U V ∨ propagate U V using UV by (auto simp add: cdcl_W-cp.simps)
      ultimately show ?thesis by force
    qed
qed
```

```
lemma rtranclp-cdcl_W-co-conflict-ex-lit-of-max-level:
  assumes full: full cdcl_W-cp S U
  and cls-f: no-clause-is-false S
  and conflict-is-false-with-level S
  and lev: cdcl_W-M-level-inv S
  shows conflict-is-false-with-level U
proof (intro allI impI)
  fix D
  assume
    confl: conflicting U = Some D and
    D: D ≠ {#}
  consider (CT) conflicting S = None | (SD) D' where conflicting S = Some D'
    by (cases conflicting S) auto
  then show ∃L∈#D. get-level (trail U) L = backtrack-lvl U
    proof cases
      case SD
      then have S = U
        by (metis (no-types) assms(1) cdcl_W-cp-conflicting-not-empty full-def rtranclpD tranclpD)
      then show  ?thesis using assms(3) confl D by blast−
    next
      case CT
      have init-clss U = init-clss S and learned-clss U = learned-clss S
        using full unfolding full-def
          apply (metis (no-types) rtranclpD tranclp-cdcl_W-cp-no-more-init-clss)
        by (metis (mono-tags, lifting) full full-def rtranclp-cdcl_W-cp-learned-clause-inv)
      obtain T where propagate** S T and TU: conflict T U
        proof −
          have f5: U ≠ S
            using confl CT by force
          then have cdcl_W-cp++ S U
            by (metis full full-def rtranclpD)
          have ⋀p pa. ¬ propagate p pa ∨ conflicting pa =
            (None::'v clause option)
            by (auto elim: propagateE)
          then show ?thesis
            using f5 that tranclp-cdcl_W-cp-propagate-with-conflict-or-not[OF ‹cdcl_W-cp++ S U›]
            full confl CT unfolding full-def by auto
        qed
      obtain D' where
        raw-conflicting T = None and
        D': D' !∈! raw-clauses T and
        tr: trail T ⊨as CNot (mset-cls D') and
        U: U ∼ update-conflicting (Some (ccls-of-cls D')) T
        using TU by (auto elim!: conflictE)
      have init-clss T = init-clss S and learned-clss T = learned-clss S
        using U ‹init-clss U = init-clss S› ‹learned-clss U = learned-clss S›  by auto
      then have D ∈# clauses S
        using confl U D' by (auto simp: raw-clauses-def)
      then have ¬ trail S ⊨as CNot D
        using cls-f CT by simp

      moreover
        obtain M where tr-U: trail U = M @ trail S and nm: ∀m∈set M. ¬is-marked m
          by (metis (mono-tags, lifting) assms(1) full-def rtranclp-cdcl_W-cp-dropWhile-trail)
        have trail U ⊨as CNot D
          using tr confl U by (auto elim!: conflictE)
      ultimately obtain L where L ∈# D and −L ∈ lits-of-l M
        unfolding tr-U CNot-def true-annots-def Ball-def true-annot-def true-cls-def by force

      moreover have inv-U: cdcl_W-M-level-inv U
        by (metis cdcl_W-stgy.conflict' cdcl_W-stgy-consistent-inv full full-unfold lev)
      moreover
        have backtrack-lvl U = backtrack-lvl S
          using full unfolding full-def by (auto dest: rtranclp-cdcl_W-cp-backtrack-lvl)

      moreover
        have no-dup (trail U)
          using inv-U unfolding cdcl_W-M-level-inv-def by auto
        { fix x :: ('v, nat, 'v clause) marked-lit and
            xb :: ('v, nat, 'v clause) marked-lit
          assume a1: atm-of L = atm-of (lit-of xb)
          moreover assume a2: − L = lit-of x
          moreover assume a3: (λl. atm-of (lit-of l)) ‘ set M
            ∩ (λl. atm-of (lit-of l)) ‘ set (trail S) = {}
          moreover assume a4: x ∈ set M
          moreover assume a5: xb ∈ set (trail S)
          moreover have atm-of (− L) = atm-of L
            by auto
          ultimately have False
            by auto
         }
        then have LS: atm-of L ∉ atm-of ‘ lits-of-l (trail S)
          using ‹−L ∈ lits-of-l M› ‹no-dup (trail U)› unfolding tr-U lits-of-def by auto
      ultimately have get-level (trail U) L = backtrack-lvl U
        proof (cases get-all-levels-of-marked (trail S) ≠ [], goal-cases)
          case 2 note LD = this(1) and LM = this(2) and inv-U = this(3) and US = this(4) and
            LS = this(5) and ne = this(6)
          have backtrack-lvl S = 0
            using lev ne unfolding cdcl_W-M-level-inv-def by auto
          moreover have get-rev-level (rev M) 0 L = 0
            using nm by auto
          ultimately show ?thesis using LS ne US unfolding tr-U
            by (simp add: get-all-levels-of-marked-nil-iff-not-is-marked lits-of-def)
        next
          case 1 note LD = this(1) and LM = this(2) and inv-U = this(3) and US = this(4) and
            LS = this(5) and ne = this(6)

          have hd (get-all-levels-of-marked (trail S)) = backtrack-lvl S
            using ne lev unfolding cdcl_W-M-level-inv-def
            by (cases get-all-levels-of-marked (trail S)) auto
          moreover have atm-of L ∈ atm-of ‘ lits-of-l M
            using ‹−L ∈ lits-of-l M› by (simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
              lits-of-def)
          ultimately show ?thesis
            using nm ne get-level-skip-beginning-hd-get-all-levels-of-marked[OF LS, of M]
              get-level-skip-in-all-not-marked[of rev M L backtrack-lvl S]
            unfolding lits-of-def US tr-U
            by auto
          qed
```

```
      then show ∃L∈#D. get-level (trail U) L = backtrack-lvl U
        using ‹L ∈# D› by blast
    qed
qed
```

19.5.4 Literal of highest level in marked literals

```
definition mark-is-false-with-level :: 'st ⇒ bool where
mark-is-false-with-level S' ≡
  ∀D M1 M2 L.  M1 @ Propagated L D # M2 = trail S' ⟶  D − {#L#} ≠ {#}
    ⟶ (∃L. L ∈#  D ∧ get-level (trail S') L = get-maximum-possible-level M1)

definition no-more-propagation-to-do:: 'st ⇒ bool where
no-more-propagation-to-do S ≡
  ∀D M M' L. D + {#L#} ∈# clauses S ⟶ trail S = M' @ M ⟶ M ⊨as CNot D
    ⟶ undefined-lit M L ⟶ get-maximum-possible-level M < backtrack-lvl S
    ⟶ (∃L. L ∈# D ∧ get-level (trail S) L = get-maximum-possible-level M)

lemma propagate-no-more-propagation-to-do:
  assumes propagate: propagate S S'
  and H: no-more-propagation-to-do S
  and lev-inv: cdcl_W-M-level-inv S
  shows no-more-propagation-to-do S'
  using assms
proof −
  obtain E L where
    S: conflicting S = None and
    E:  E !∈! raw-clauses S and
    LE: L ∈# mset-cls E and
    tr: trail S ⊨as CNot (mset-cls (remove-lit L E)) and
    undefL: undefined-lit (trail S) L and
    S': S' ∼ cons-trail (Propagated L E) S
    using propagate by (elim propagateE) simp
  let ?M' = Propagated L (mset-cls E) # trail S
  show ?thesis unfolding no-more-propagation-to-do-def
    proof (intro allI impI)
      fix D M1 M2 L'
      assume
        D-L: D + {#L'#} ∈# clauses S' and
        trail S' = M2 @ M1 and
        get-max: get-maximum-possible-level M1 < backtrack-lvl S' and
        M1 ⊨as CNot D and
        undef: undefined-lit M1 L'
      have tl M2 @ M1 = trail S ∨ (M2 = [] ∧ M1 = Propagated L (mset-cls E) # trail S)
        using ‹trail S' = M2 @ M1› S' S undefL lev-inv
        by (cases M2) (auto simp:cdcl_W-M-level-inv-decomp)
      moreover {
        assume tl M2 @ M1 = trail S
        moreover have D + {#L'#} ∈# clauses S
          using D-L S S' undefL unfolding raw-clauses-def by auto
        moreover have get-maximum-possible-level M1 < backtrack-lvl S
          using get-max S S' undefL by auto
        ultimately obtain L' where L' ∈# D and
          get-level (trail S) L' = get-maximum-possible-level M1
          using H ‹M1 ⊨as CNot D› undef unfolding no-more-propagation-to-do-def by metis
        moreover
        { have cdcl_W-M-level-inv S'
            using cdcl_W-consistent-inv lev-inv cdcl_W.propagate[OF propagate] by blast
          then have no-dup ?M' using S' undefL unfolding cdcl_W-M-level-inv-def by auto
          moreover
            have atm-of L' ∈ atm-of ‘ (lits-of-l M1)
              using ‹L' ∈# D› ‹M1 ⊨as CNot D› by (metis atm-of-uminus image-eqI
                in-CNot-implies-uminus(2))
            then have atm-of L' ∈ atm-of ‘ (lits-of-l (trail S))
              using ‹tl M2 @ M1 = trail S›[symmetric] S undefL by auto
          ultimately have atm-of L ≠ atm-of L' unfolding lits-of-def by auto
        }
        ultimately have ∃L' ∈# D. get-level (trail S') L' = get-maximum-possible-level M1
          using S S' undefL by auto
      }
      moreover {
        assume M2 = [] and M1: M1 = Propagated L (mset-cls E) # trail S
        have cdcl_W-M-level-inv S'
          using cdcl_W-consistent-inv[OF - lev-inv] cdcl_W.propagate[OF propagate] by blast
        then have get-all-levels-of-marked (trail S') = rev [Suc 0..<(Suc 0+backtrack-lvl S)]
          using S' undefL unfolding cdcl_W-M-level-inv-def by auto
        then have get-maximum-possible-level M1 = backtrack-lvl S'
          using get-maximum-possible-level-max-get-all-levels-of-marked[of M1] S' M1 undefL
          by (auto intro: Max-eqI)
        then have False using get-max by auto
      }
      ultimately show ∃L. L ∈# D ∧ get-level (trail S') L = get-maximum-possible-level M1
        by fast
  qed
qed

lemma conflict-no-more-propagation-to-do:
  assumes
    conflict: conflict S S' and
    H: no-more-propagation-to-do S and
    M: cdcl_W-M-level-inv S
  shows no-more-propagation-to-do S'
  using assms unfolding no-more-propagation-to-do-def by (force elim!: conflictE)

lemma cdcl_W-cp-no-more-propagation-to-do:
  assumes
    conflict: cdcl_W-cp S S' and
    H: no-more-propagation-to-do S and
    M: cdcl_W-M-level-inv S
  shows no-more-propagation-to-do S'
  using assms
  proof (induct rule: cdcl_W-cp.induct)
  case (conflict' S S')
  then show ?case using conflict-no-more-propagation-to-do[of S S'] by blast
next
  case (propagate' S S') note S = this
  show 1: no-more-propagation-to-do S'
    using propagate-no-more-propagation-to-do[of S S'] S by blast
qed

lemma cdcl_W-then-exists-cdcl_W-stgy-step:
  assumes
    o: cdcl_W-o S S' and
    alien: no-strange-atm S and
    lev: cdcl_W-M-level-inv S
  shows ∃S'. cdcl_W-stgy S S'
proof –
  obtain S'' where full cdcl_W-cp S' S''
    using always-exists-full-cdcl_W-cp-step alien cdcl_W-no-strange-atm-inv cdcl_W-o-no-more-init-clss
     o other lev by (meson cdcl_W-consistent-inv)
  then show ?thesis
    using assms by (metis always-exists-full-cdcl_W-cp-step cdcl_W-stgy.conflict' full-unfold other')
qed

lemma backtrack-no-decomp:
  assumes
    S: raw-conflicting S = Some E and
    LE: L ∈# mset-ccls E and
    L: get-level (trail S) L = backtrack-lvl S and
    D: get-maximum-level (trail S) (remove1-mset L (mset-ccls E)) < backtrack-lvl S and
    bt: backtrack-lvl S = get-maximum-level (trail S) (mset-ccls E) and
    M-L: cdcl_W-M-level-inv S
  shows ∃S'. cdcl_W-o S S'
proof –
  have L-D: get-level (trail S) L = get-maximum-level (trail S) (mset-ccls E)
    using L D bt by (simp add: get-maximum-level-plus)
  let ?i = get-maximum-level (trail S) (remove1-mset L (mset-ccls E))
  obtain K M1 M2 where
    K: (Marked K (?i + 1) # M1, M2) ∈ set (get-all-marked-decomposition (trail S))
    using backtrack-ex-decomp[OF M-L, of ?i] D S by auto
  show ?thesis using backtrack-rule[OF S LE K L] bt L bj cdcl_W-bj.simps by auto
qed

lemma cdcl_W-stgy-final-state-conclusive:
  assumes
    termi: ∀S'. ¬cdcl_W-stgy S S' and
    decomp: all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S)) and
    learned: cdcl_W-learned-clause S and
    level-inv: cdcl_W-M-level-inv S and
    alien: no-strange-atm S and
    no-dup: distinct-cdcl_W-state S and
    confl: cdcl_W-conflicting S and
    confl-k: conflict-is-false-with-level S
  shows (conflicting S = Some {#} ∧ unsatisfiable (set-mset (init-clss S)))
        ∨ (conflicting S = None ∧ trail S ⊨as set-mset (init-clss S))
proof –
  let ?M = trail S
  let ?N = init-clss S
  let ?k = backtrack-lvl S
  let ?U = learned-clss S
  consider
      (None) raw-conflicting S = None
    | (Some-Empty) E where raw-conflicting S = Some E and mset-ccls E = {#}
    | (Some) E' where raw-conflicting S = Some E' and
      conflicting S = Some (mset-ccls E') and mset-ccls E' ≠ {#}
    by (cases conflicting S, simp) auto
```

```
then show ?thesis
  proof cases
    case (Some-Empty E)
    then have conflicting S = Some {#} by auto
    then have unsatisfiable (set-mset (init-clss S))
      using assms(3) unfolding cdcl_W-learned-clause-def true-clss-cls-def
      by (metis (no-types, lifting) Un-insert-right atms-of-empty satisfiable-def
         sup-bot.right-neutral total-over-m-insert total-over-set-empty true-cls-empty)
    then show ?thesis using Some-Empty by auto
  next
    case None
    { assume ¬?M ⊨asm ?N
      have atm-of ` (lits-of-l ?M) = atms-of-mm ?N (is ?A = ?B)
        proof
          show ?A ⊆ ?B using alien unfolding no-strange-atm-def by auto
          show ?B ⊆ ?A
            proof (rule ccontr)
              assume ¬?B ⊆ ?A
              then obtain l where l ∈ ?B and l ∉ ?A by auto
              then have undefined-lit ?M (Pos l)
                using ‹l ∉ ?A› unfolding lits-of-def by (auto simp add: defined-lit-map)
              moreover have conflicting S = None
                using None by auto
              ultimately have ∃S'. cdcl_W-o S S'
                using cdcl_W-o.decide decide-rule ‹l ∈ ?B› no-strange-atm-def
                by (metis literal.sel(1) state-eq-def)
              then show False
                using termi cdcl_W-then-exists-cdcl_W-stgy-step[OF - alien]  level-inv by blast
            qed
        qed
      obtain D where ¬ ?M ⊨a D and D ∈# ?N
         using ‹¬?M ⊨asm ?N› unfolding lits-of-def true-annots-def Ball-def by auto
      have atms-of D ⊆ atm-of ` (lits-of-l ?M)
        using ‹D ∈# ?N› unfolding ‹atm-of ` (lits-of-l ?M) = atms-of-mm ?N› atms-of-ms-def
        by (auto simp add: atms-of-def)
      then have a1: atm-of ` set-mset D ⊆ atm-of ` lits-of-l (trail S)
        by (auto simp add: atms-of-def lits-of-def)
      have total-over-m (lits-of-l ?M) {D}
        using ‹atms-of D ⊆ atm-of ` (lits-of-l ?M)›
        atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set by (fastforce simp: total-over-set-def)
      then have ?M ⊨as CNot D
        using total-not-true-cls-true-clss-CNot  ‹¬ trail S ⊨a D› true-annot-def
        true-annots-true-cls by fastforce
      then have False
        proof -
          obtain S' where
            f2: full cdcl_W-cp S S'
            by (meson alien always-exists-full-cdcl_W-cp-step level-inv)
          then have S' = S
            using cdcl_W-stgy.conflict'[of S] by (metis (no-types) full-unfold termi)
          then show ?thesis
            using f2 ‹D ∈# init-clss S› None ‹trail S ⊨as CNot D›
            raw-clauses-def full-cdcl_W-cp-not-any-negated-init-clss by auto
        qed
    }
  then have ?M ⊨asm ?N by blast
  then show ?thesis
    using None by auto
next
  case (Some E′) note raw-conf = this(1) and LD = this(2) and nempty = this(3)
  then obtain L D where
    E′[simp]: mset-ccls E′ = D + {#L#} and
    lev-L: get-level ?M L = ?k
    by (metis (mono-tags) confl-k insert-DiffM2)
  let ?D = D + {#L#}
  have ?D ≠ {#} by auto
  have ?M ⊨as CNot ?D using confl LD unfolding cdcl_W-conflicting-def by auto
  then have ?M ≠ [] unfolding true-annots-def Ball-def true-annot-def true-cls-def by force
  have M: ?M = hd ?M # tl ?M using ‹?M ≠ []› list.collapse by fastforce
  have g-a-l: get-all-levels-of-marked ?M = rev [1..<1 + ?k]
    using level-inv lev-L M unfolding cdcl_W-M-level-inv-def by auto
  have g-k: get-maximum-level (trail S) D ≤ ?k
    using get-maximum-possible-level-ge-get-maximum-level[of ?M]
      get-maximum-possible-level-max-get-all-levels-of-marked[of ?M]
    by (auto simp add: Max-n-upt g-a-l)
  {
    assume marked: is-marked (hd ?M)
    then obtain k′ where k′: k′ + 1 = ?k
      using level-inv M unfolding cdcl_W-M-level-inv-def
      by (cases hd (trail S); cases trail S) auto
    obtain L′ l′ where L′: hd ?M = Marked L′ l′ using marked by (cases hd ?M) auto
    have marked-hd-tl: get-all-levels-of-marked (hd (trail S) # tl (trail S))
      = rev [1..<1 + length (get-all-levels-of-marked ?M)]
      using level-inv lev-L M unfolding cdcl_W-M-level-inv-def M[symmetric]
      by blast
    then have l′-tl: l′ # get-all-levels-of-marked (tl ?M)
      = rev [1..<1 + length (get-all-levels-of-marked ?M)] unfolding L′ by simp
    moreover have ... = length (get-all-levels-of-marked ?M)
      # rev [1..<length (get-all-levels-of-marked ?M)]
      using M Suc-le-mono calculation by (fastforce simp add: upt.simps(2))
    finally have
       l′-cons:  l′ # get-all-levels-of-marked (tl (trail S)) =
         length (get-all-levels-of-marked (trail S))
           # rev [1..<length (get-all-levels-of-marked (trail S))] and
      l′ = ?k and
      g-r: get-all-levels-of-marked (tl (trail S))
        = rev [1..<length (get-all-levels-of-marked (trail S))]
      using level-inv lev-L M unfolding cdcl_W-M-level-inv-def by auto

    have ∗: ⋀list. no-dup list ⟹
          − L ∈ lits-of-l list ⟹ atm-of L ∈ atm-of ‘ lits-of-l list
      by (metis atm-of-uminus imageI)
    have L′-L: L′ = −L
      proof (rule ccontr)
        assume ¬ ?thesis
        moreover have −L ∈ lits-of-l ?M using confl LD unfolding cdcl_W-conflicting-def by auto
        ultimately have get-level (hd (trail S) # tl (trail S)) L = get-level (tl ?M) L
          using cdcl_W-M-level-inv-decomp(1)[OF level-inv] L′ M atm-of-eq-atm-of
          unfolding lits-of-def consistent-interp-def
          by (metis (mono-tags, hide-lams) marked-lit.sel(1) get-level-skip-beginning image-eqI
        list.set-intros(1))
    moreover
      have length (get-all-levels-of-marked (trail S)) = ?k
        using level-inv unfolding cdcl_W-M-level-inv-def by auto
      then have Max (set (0#get-all-levels-of-marked (tl (trail S)))) = ?k − 1
        unfolding g-r by (auto simp add: Max-n-upt)
      then have get-level (tl ?M) L < ?k
        using get-maximum-possible-level-ge-get-level[of tl ?M L]
        by (metis One-nat-def add.right-neutral add-Suc-right diff-add-inverse2
          get-maximum-possible-level-max-get-all-levels-of-marked k' le-imp-less-Suc
          list.simps(15))
    finally show False using lev-L M by auto
  qed
have L: hd ?M = Marked (−L) ?k  using ‹l' = ?k› L'-L L' by auto

have get-maximum-level (trail S) D < ?k
  proof (rule ccontr)
    assume ¬ ?thesis
    then have get-maximum-level (trail S) D = ?k using M g-k unfolding L by auto
    then obtain L'' where L'' ∈# D and L-k: get-level ?M L'' = ?k
      using get-maximum-level-exists-lit[of ?k ?M D] unfolding k'[symmetric] by auto
    have L ≠ L'' using no-dup ‹L'' ∈# D›
      unfolding distinct-cdcl_W-state-def LD
      by (metis E' add.right-neutral add-diff-cancel-right'
        distinct-mem-diff-mset union-commute union-single-eq-member)
    have L'' = −L
      proof (rule ccontr)
        assume ¬ ?thesis
        then have get-level ?M L'' = get-level (tl ?M) L''
          using M ‹L ≠ L''› get-level-skip-beginning[of L'' hd ?M tl ?M] unfolding L
          by (auto simp: atm-of-eq-atm-of)
        then show False
          by (metis L-k Max-n-upt One-nat-def Suc-n-not-le-n ‹l' = backtrack-lvl S›
            add-Suc-right add-implies-diff g-r
            get-all-levels-of-marked-rev-eq-rev-get-all-levels-of-marked list.set(2)
            get-rev-level-less-max-get-all-levels-of-marked k' l'-cons list.sel(1)
            rev-rev-ident semiring-normalization-rules(6) set-upt)
      qed
    then have taut: tautology (D + {#L#})
      using ‹L'' ∈# D› by (metis add.commute mset-leD mset-le-add-left multi-member-this
        tautology-minus)
    have consistent-interp (lits-of-l ?M)
      using level-inv unfolding cdcl_W-M-level-inv-def by auto
    then have ¬?M ⊨as CNot ?D
      using taut by (metis ‹L'' = − L› ‹L'' ∈# D› add.commute consistent-interp-def
        diff-union-cancelR in-CNot-implies-uminus(2) in-diffD multi-member-this)
    moreover have ?M ⊨as CNot ?D
      using confl no-dup LD unfolding cdcl_W-conflicting-def by auto
    ultimately show False by blast
  qed note H = this
have get-maximum-level (trail S) D < get-maximum-level (trail S) (D + {#L#})
  using H by (auto simp: get-maximum-level-plus lev-L max-def)
moreover have backtrack-lvl S = get-maximum-level (trail S) (D + {#L#})
  using H by (auto simp: get-maximum-level-plus lev-L max-def)
ultimately have False
```

```
    using backtrack-no-decomp[OF raw-conf - lev-L] level-inv termi
    cdcl_W-then-exists-cdcl_W-stgy-step[of S] alien unfolding E'
    by (auto simp add: lev-L max-def)
} note not-is-marked = this

moreover {
  let ?D = D + {#L#}
  have ?D ≠ {#} by auto
  have ?M ⊨as CNot ?D using confl LD unfolding cdcl_W-conflicting-def by auto
  then have ?M ≠ [] unfolding true-annots-def Ball-def true-annot-def true-cls-def by force
  assume nm: ¬is-marked (hd ?M)
  then obtain L' C where L'C: hd-raw-trail S = Propagated L' C
    by (metis ‹trail S ≠ []› hd-raw-trail is-marked-def mmset-of-mlit.elims)
  then have hd ?M = Propagated L' (mset-cls C)
    using ‹trail S ≠ []› hd-raw-trail mmset-of-mlit.simps(1) by fastforce
  then have M: ?M = Propagated L' (mset-cls C) # tl ?M
    using ‹?M ≠ []› list.collapse by fastforce
  then obtain C' where C': mset-cls C = C' + {#L'#}
    using confl unfolding cdcl_W-conflicting-def by (metis append-Nil diff-single-eq-union)
  { assume −L' ∉# ?D
    then have Ex (skip S)
      using skip-rule[OF M raw-conf] unfolding E' by auto
    then have False
      using cdcl_W-then-exists-cdcl_W-stgy-step[of S] alien level-inv termi
      by (auto dest: cdcl_W-o.intros cdcl_W-bj.intros)
  }
  moreover {
    assume L'D: −L' ∈# ?D
    then obtain D' where D': ?D = D' + {#−L'#} by (metis insert-DiffM2)
    have g-r: get-all-levels-of-marked (Propagated L' (mset-cls C) # tl (trail S))
      = rev [Suc 0..<Suc (length (get-all-levels-of-marked (trail S)))]
      using level-inv M unfolding cdcl_W-M-level-inv-def by auto
    have Max (insert 0
        (set (get-all-levels-of-marked (Propagated L' (mset-cls C) # tl (trail S))))) = ?k
      using level-inv M unfolding g-r cdcl_W-M-level-inv-def set-rev
      by (auto simp add:Max-n-upt)
    then have get-maximum-level (trail S) D' ≤ ?k
      using get-maximum-possible-level-ge-get-maximum-level[of
        Propagated L' (mset-cls C) # tl ?M] M
      unfolding get-maximum-possible-level-max-get-all-levels-of-marked by auto
    then have get-maximum-level (trail S) D' = ?k
      ∨ get-maximum-level (trail S) D' < ?k
      using le-neq-implies-less by blast
    moreover {
      assume g-D'-k: get-maximum-level (trail S) D' = ?k
      then have f1: get-maximum-level (trail S) D' = backtrack-lvl S
        using M by auto
      then have Ex (cdcl_W-o S)
        using f1 resolve-rule[of S L' C , OF ‹trail S ≠ []› - - raw-conf] raw-conf g-D'-k
        L'C L'D unfolding C' D' E'
        by (fastforce simp add: D' intro: cdcl_W-o.intros cdcl_W-bj.intros)
      then have False
        by (meson alien cdcl_W-then-exists-cdcl_W-stgy-step termi level-inv)
    }
    moreover {
          assume a1: get-maximum-level (trail S) D′ < ?k
          then have f3: get-maximum-level (trail S) D′ < get-level (trail S) (−L′)
            using a1 lev-L by (metis D′ get-maximum-level-ge-get-level insert-noteq-member
              not-less)
          moreover have backtrack-lvl S = get-level (trail S) L′
            apply (subst M)
            unfolding rev.simps
            apply (subst get-rev-level-can-skip-correctly-ordered)
            using level-inv unfolding cdcl_W-M-level-inv-def
            apply (subst (asm) (2) M) apply (simp add: cdcl_W-M-level-inv-decomp)
            using level-inv unfolding cdcl_W-M-level-inv-def
            apply (subst (asm) (2) M) apply (auto simp: cdcl_W-M-level-inv-decomp lits-of-def)[]
            using level-inv unfolding cdcl_W-M-level-inv-def
            apply (subst (asm) (4) M) apply (auto simp add: cdcl_W-M-level-inv-decomp)[]
            using level-inv unfolding cdcl_W-M-level-inv-def
            apply (subst (asm) (4) M) by (auto simp add: cdcl_W-M-level-inv-decomp)[]
          moreover
            then have get-level (trail S) L′ = get-maximum-level (trail S) (D′ + {#− L′#})
              using a1 by (auto simp add: get-maximum-level-plus max-def)
          ultimately have False
            using M backtrack-no-decomp[of S - −L′, OF raw-conf]
            cdcl_W-then-exists-cdcl_W-stgy-step L′D level-inv termi alien
            unfolding D′ E′ by auto
        }
        ultimately have False by blast
      }
      ultimately have False by blast
    }
    ultimately show ?thesis by blast
  qed
qed

lemma cdcl_W-cp-tranclp-cdcl_W:
  cdcl_W-cp S S′ ⟹ cdcl_W^++ S S′
  apply (induct rule: cdcl_W-cp.induct)
  by (meson cdcl_W.conflict cdcl_W.propagate tranclp.r-into-trancl tranclp.trancl-into-trancl)+

lemma tranclp-cdcl_W-cp-tranclp-cdcl_W:
  cdcl_W-cp^++ S S′ ⟹ cdcl_W^++ S S′
  apply (induct rule: tranclp.induct)
   apply (simp add: cdcl_W-cp-tranclp-cdcl_W)
  by (meson cdcl_W-cp-tranclp-cdcl_W tranclp-trans)

lemma cdcl_W-stgy-tranclp-cdcl_W:
   cdcl_W-stgy S S′ ⟹ cdcl_W^++ S S′
proof (induct rule: cdcl_W-stgy.induct)
  case conflict′
  then show ?case
   unfolding full1-def by (simp add: tranclp-cdcl_W-cp-tranclp-cdcl_W)
next
  case (other′ S′ S″)
  then have S′ = S″ ∨ cdcl_W-cp^++ S′ S″
    by (simp add: rtranclp-unfold full-def)
  then show ?case
    using other′ by (meson cdcl_W.other tranclp.r-into-trancl
    tranclp-cdcl_W-cp-tranclp-cdcl_W tranclp-trans)
qed

lemma tranclp-cdcl_W-stgy-tranclp-cdcl_W:
  cdcl_W-stgy^++ S S' ⟹ cdcl_W^++ S S'
  apply (induct rule: tranclp.induct)
   using cdcl_W-stgy-tranclp-cdcl_W apply blast
  by (meson cdcl_W-stgy-tranclp-cdcl_W tranclp-trans)

lemma rtranclp-cdcl_W-stgy-rtranclp-cdcl_W:
   cdcl_W-stgy^** S S' ⟹ cdcl_W^** S S'
  using rtranclp-unfold[of cdcl_W-stgy S S'] tranclp-cdcl_W-stgy-tranclp-cdcl_W[of S S'] by auto

lemma not-empty-get-maximum-level-exists-lit:
  assumes n: D ≠ {#}
  and max: get-maximum-level M D = n
  shows ∃ L ∈#D. get-level M L = n
proof –
  have f: finite (insert 0 ((λL. get-level M L) ` set-mset D)) by auto
  then have n ∈ ((λL. get-level M L) ` set-mset D)
    using n max get-maximum-level-exists-lit-of-max-level image-iff
    unfolding get-maximum-level-def by force
  then show ∃ L ∈# D. get-level M L = n by auto
qed

lemma cdcl_W-o-conflict-is-false-with-level-inv:
  assumes
    cdcl_W-o S S' and
    lev: cdcl_W-M-level-inv S and
    confl-inv: conflict-is-false-with-level S and
    n-d: distinct-cdcl_W-state S and
    conflicting: cdcl_W-conflicting S
  shows conflict-is-false-with-level S'
  using assms(1,2)
proof (induct rule: cdcl_W-o-induct-lev2)
   case (resolve L C M D T) note tr-S = this(1) and confl = this(4) and LD = this(5) and T =
this(7)
  have uL-not-D: −L ∉# remove1-mset (−L) (mset-ccls D)
    using n-d confl unfolding distinct-cdcl_W-state-def distinct-mset-def
    by (metis distinct-cdcl_W-state-def distinct-mem-diff-mset multi-member-last n-d option.simps(9))
  moreover have L-not-D: L ∉# remove1-mset (−L) (mset-ccls D)
    proof (rule ccontr)
      assume ¬ ?thesis
      then have L ∈# mset-ccls D
        by (auto simp: in-remove1-mset-neq)
      moreover have Propagated L (mset-cls C) # M ⊨as CNot (mset-ccls D)
        using conflicting confl tr-S unfolding cdcl_W-conflicting-def by auto
      ultimately have −L ∈ lits-of-l (Propagated L (mset-cls C) # M)
        using in-CNot-implies-uminus(2) by blast
      moreover have no-dup (Propagated L (mset-cls C) # M)
        using lev tr-S unfolding cdcl_W-M-level-inv-def by auto
      ultimately show False unfolding lits-of-def by (metis consistent-interp-def image-eqI
        list.set-intros(1) lits-of-def marked-lit.sel(2) distinct-consistent-interp)
    qed
  ultimately
    have g-D: get-maximum-level (Propagated L (mset-cls C) # M) (remove1-mset (−L) (mset-ccls D))
      = get-maximum-level M (remove1-mset (−L) (mset-ccls D))
    proof −
      have ∀ a f L. ((a::'v) ∈ f ` L) = (∃ l. (l::'v literal) ∈ L ∧ a = f l)
        by blast
      then show ?thesis
        using get-maximum-level-skip-first[of L remove1-mset (−L) (mset-ccls D) mset-cls C M]
        unfolding atms-of-def
        by (metis (no-types) uL-not-D L-not-D atm-of-eq-atm-of)
    qed
  have lev-L[simp]: get-level M L = 0
    apply (rule atm-of-notin-get-rev-level-eq-0)
    using lev unfolding cdcl_W-M-level-inv-def tr-S by (auto simp: lits-of-def)

  have D: get-maximum-level M (remove1-mset (−L) (mset-ccls D)) = backtrack-lvl S
    using resolve.hyps(6) LD unfolding tr-S by (auto simp: get-maximum-level-plus max-def g-D)
  have get-all-levels-of-marked M = rev [Suc 0..<Suc (backtrack-lvl S)]
    using lev unfolding tr-S cdcl_W-M-level-inv-def by auto
  then have get-maximum-level M (remove1-mset L (mset-cls C)) ≤ backtrack-lvl S
    using get-maximum-possible-level-ge-get-maximum-level[of M]
    get-maximum-possible-level-max-get-all-levels-of-marked[of M] by (auto simp: Max-n-upt)
  then have
    get-maximum-level M (remove1-mset (− L) (mset-ccls D) #∪ remove1-mset L (mset-cls C)) =
      backtrack-lvl S
    by (auto simp: get-maximum-level-union-mset get-maximum-level-plus max-def D)
  then show ?case
    using tr-S not-empty-get-maximum-level-exists-lit[of
      remove1-mset (− L) (mset-ccls D) #∪ remove1-mset L (mset-cls C) M] T
    by auto
next
  case (skip L C' M D T) note tr-S = this(1) and D = this(2) and T = this(5)
  then obtain La where
    La ∈# mset-ccls D and
    get-level (Propagated L C' # M) La = backtrack-lvl S
    using skip confl-inv by auto
  moreover
    have atm-of La ≠ atm-of L
      proof (rule ccontr)
        assume ¬ ?thesis
        then have La: La = L using ‹La ∈# mset-ccls D› ‹− L ∉# mset-ccls D›
          by (auto simp add: atm-of-eq-atm-of)
        have Propagated L C' # M ⊨as CNot (mset-ccls D)
          using conflicting tr-S D unfolding cdcl_W-conflicting-def by auto
        then have −L ∈ lits-of-l M
          using ‹La ∈# mset-ccls D› in-CNot-implies-uminus(2)[of L mset-ccls D
            Propagated L C' # M] unfolding La
          by auto
        then show False using lev tr-S unfolding cdcl_W-M-level-inv-def consistent-interp-def by auto
      qed
    then have get-level (Propagated L C' # M) La = get-level M La  by auto
  ultimately show ?case using D tr-S T by auto
next
  case backtrack
  then show ?case
```

by (*auto split*: *if-split-asm simp*: $cdcl_W$*-M-level-inv-decomp lev*)
qed *auto*

19.5.5 Strong completeness

lemma $cdcl_W$*-cp-propagate-confl*:
assumes $cdcl_W$*-cp S T*
shows $propagate^{**}\ S\ T \lor (\exists S'.\ propagate^{**}\ S\ S' \land conflict\ S'\ T)$
using *assms* **by** *induction blast+*

lemma *rtranclp-*$cdcl_W$*-cp-propagate-confl*:
assumes $cdcl_W\text{-}cp^{**}\ S\ T$
shows $propagate^{**}\ S\ T \lor (\exists S'.\ propagate^{**}\ S\ S' \land conflict\ S'\ T)$
by (*simp add*: *assms rtranclp-*$cdcl_W$*-cp-propa-or-propa-confl*)

lemma *propagate-high-levelE*:
assumes *propagate S T*
obtains *M′ N′ U k L C* **where**
state S = (*M′*, *N′*, *U*, *k*, *None*) **and**
state T = (*Propagated L* (*C* + {#*L*#}) # *M′*, *N′*, *U*, *k*, *None*) **and**
C + {#*L*#} ∈# *local.clauses S* **and**
M′ ⊨*as CNot C* **and**
undefined-lit (*trail S*) *L*
proof −
obtain *E L* **where**
conf: *conflicting S* = *None* **and**
E: *E* !∈! *raw-clauses S* **and**
LE: *L* ∈# *mset-cls E* **and**
tr: *trail S* ⊨*as CNot* (*mset-cls* (*remove-lit L E*)) **and**
undef: *undefined-lit* (*trail S*) *L* **and**
T: *T* ∼ *cons-trail* (*Propagated L E*) *S*
using *assms* **by** (*elim propagateE*) *simp*
obtain *M N U k* **where**
S: *state S* = (*M*, *N*, *U*, *k*, *None*)
using *conf* **by** *auto*
show *thesis*
using *that*[*of M N U k L remove1-mset L* (*mset-cls E*)] *S T LE E tr undef*
by *auto*
qed

lemma $cdcl_W$*-cp-propagate-completeness*:
assumes *MN*: *set M* ⊨*s set-mset N* **and**
cons: *consistent-interp* (*set M*) **and**
tot: *total-over-m* (*set M*) (*set-mset N*) **and**
lits-of-l (*trail S*) ⊆ *set M* **and**
init-clss S = *N* **and**
$propagate^{**}\ S\ S'$ **and**
learned-clss S = {#}
shows *length* (*trail S*) ≤ *length* (*trail S′*) ∧ *lits-of-l* (*trail S′*) ⊆ *set M*
using *assms*(*6*,*4*,*5*,*7*)
proof (*induction rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *auto*
next
case (*step Y Z*)
note *st* = *this*(*1*) **and** *propa* = *this*(*2*) **and** *IH* = *this*(*3*) **and** *lits′* = *this*(*4*) **and** *NS* = *this*(*5*) **and**

```
    learned = this(6)
  then have len: length (trail S) ≤ length (trail Y) and LM: lits-of-l (trail Y) ⊆ set M
     by blast+

  obtain M' N' U k C L where
    Y: state Y = (M', N', U, k, None) and
    Z: state Z = (Propagated L (C + {#L#}) # M', N', U, k, None) and
    C: C + {#L#} ∈# clauses Y and
    M'-C: M' ⊨as CNot C and
    undefined-lit (trail Y) L
    using propa by (auto elim: propagate-high-levelE)
  have init-clss S = init-clss Y
    using st by induction (auto elim: propagateE)
  then have [simp]: N' = N using NS Y Z by simp
  have learned-clss Y = {#}
    using st learned by induction (auto elim: propagateE)
  then have [simp]: U = {#} using Y by auto
  have set M ⊨s CNot C
    using M'-C LM Y unfolding true-annots-def Ball-def true-annot-def true-clss-def true-cls-def
    by force
  moreover
    have set M ⊨ C + {#L#}
      using MN C learned Y NS ‹init-clss S = init-clss Y› ‹learned-clss Y = {#}›
      unfolding true-clss-def raw-clauses-def by fastforce
  ultimately have L ∈ set M by (simp add: cons consistent-CNot-not)
  then show ?case using LM len Y Z by auto
qed

lemma
  assumes propagate^** S X
  shows
    rtranclp-propagate-init-clss: init-clss X = init-clss S and
    rtranclp-propagate-learned-clss: learned-clss X = learned-clss S
  using assms by (induction rule: rtranclp-induct) (auto elim: propagateE)

lemma completeness-is-a-full1-propagation:
  fixes S :: 'st and M :: 'v literal list
  assumes MN: set M ⊨s set-mset N
  and cons: consistent-interp (set M)
  and tot: total-over-m (set M) (set-mset N)
  and alien: no-strange-atm S
  and learned: learned-clss S = {#}
  and clsS[simp]: init-clss S = N
  and lits: lits-of-l (trail S) ⊆ set M
  shows ∃S'. propagate^** S S' ∧ full cdcl_W-cp S S'
proof -
  obtain S' where full: full cdcl_W-cp S S'
    using always-exists-full-cdcl_W-cp-step alien by blast
  then consider (propa) propagate^** S S'
    | (confl) ∃X. propagate^** S X ∧ conflict X S'
    using rtranclp-cdcl_W-cp-propagate-confl unfolding full-def by blast
  then show ?thesis
    proof cases
      case propa then show ?thesis using full by blast
    next
```

```
      case confl
      then obtain X where
        X: propagate** S X and
        Xconf: conflict X S'
      by blast
      have clsX: init-clss X = init-clss S
        using X by (blast dest: rtranclp-propagate-init-clss)
      have learnedX: learned-clss X = {#}
        using X learned by (auto dest: rtranclp-propagate-learned-clss)
      obtain E where
        E: E ∈# init-clss X + learned-clss X and
        Not-E: trail X ⊨as CNot E
        using Xconf by (auto simp add: raw-clauses-def elim!: conflictE)
      have lits-of-l (trail X) ⊆ set M
        using cdcl_W-cp-propagate-completeness[OF assms(1−3) lits - X learned] learned by auto
      then have MNE: set M ⊨s CNot E
        using Not-E
        by (fastforce simp add: true-annots-def true-annot-def true-clss-def true-cls-def)
      have ¬ set M ⊨s set-mset N
         using E consistent-CNot-not[OF cons MNE]
         unfolding learnedX true-clss-def unfolding clsX clsS by auto
      then show ?thesis using MN by blast
    qed
qed
```

See also $cdcl_W$-cp^{**} ?S ?S′ ⟹ ∃ M. trail ?S′ = M @ trail ?S ∧ (∀ l∈set M. ¬ is-marked l)

```
lemma rtranclp-propagate-is-trail-append:
  propagate** S T ⟹ ∃ c. trail T = c @ trail S
  by (induction rule: rtranclp-induct) (auto elim: propagateE)

lemma rtranclp-propagate-is-update-trail:
  propagate** S T ⟹ cdcl_W-M-level-inv S ⟹
    init-clss S = init-clss T ∧ learned-clss S = learned-clss T ∧ backtrack-lvl S = backtrack-lvl T
    ∧ conflicting S = conflicting T
proof (induction rule: rtranclp-induct)
  case base
  then show ?case unfolding state-eq-def by (auto simp: cdcl_W-M-level-inv-decomp)
next
  case (step T U) note IH = this(3)[OF this(4)]
  moreover have cdcl_W-M-level-inv U
    using rtranclp-cdcl_W-consistent-inv ‹propagate** S T› ‹propagate T U›
    rtranclp-mono[of propagate cdcl_W] cdcl_W-cp-consistent-inv propagate'
    rtranclp-propagate-is-rtranclp-cdcl_W step.prems by blast
    then have no-dup (trail U) unfolding cdcl_W-M-level-inv-def by auto
  ultimately show ?case using ‹propagate T U› unfolding state-eq-def
    by (fastforce simp: elim: propagateE)
qed

lemma cdcl_W-stgy-strong-completeness-n:
  assumes
    MN: set M ⊨s set-mset (mset-clss N) and
    cons: consistent-interp (set M) and
    tot: total-over-m (set M) (set-mset (mset-clss N)) and
    atm-incl: atm-of ‘ (set M) ⊆ atms-of-mm (mset-clss N) and
    distM: distinct M and
    length: n ≤ length M
  shows
    ∃ M' k S. length M' ≥ n ∧
      lits-of-l M' ⊆ set M ∧
      no-dup M' ∧
      state S = (M', mset-clss N, {#}, k, None) ∧
      cdcl_W-stgy** (init-state N) S
  using length
proof (induction n)
  case 0
  have state (init-state N) = ([], mset-clss N, {#}, 0, None)
    by (auto simp: state-eq-def simp del: state-simp)
  moreover have
    0 ≤ length [] and
    lits-of-l [] ⊆ set M and
    cdcl_W-stgy** (init-state N) (init-state N)
    and no-dup []
    by (auto simp: state-eq-def simp del: state-simp)
  ultimately show ?case using state-eq-sym by blast
next
  case (Suc n) note IH = this(1) and n = this(2)
  then obtain M' k S where
    l-M': length M' ≥ n and
    M': lits-of-l M' ⊆ set M and
    n-d[simp]: no-dup M' and
    S: state S = (M', mset-clss N, {#}, k, None) and
    st: cdcl_W-stgy** (init-state N) S
    by auto
  have
    M: cdcl_W-M-level-inv S and
    alien: no-strange-atm S
      using cdcl_W-M-level-inv-S0-cdcl_W rtranclp-cdcl_W-stgy-consistent-inv st apply blast
    using cdcl_W-M-level-inv-S0-cdcl_W no-strange-atm-S0 rtranclp-cdcl_W-no-strange-atm-inv
    rtranclp-cdcl_W-stgy-rtranclp-cdcl_W st by blast

  { assume no-step: ¬no-step propagate S
    obtain S' where S': propagate** S S' and full: full cdcl_W-cp S S'
      using completeness-is-a-full1-propagation[OF assms(1−3), of S] alien M' S
      by (auto simp: comp-def)
    have lev: cdcl_W-M-level-inv S'
      using M S' rtranclp-cdcl_W-consistent-inv rtranclp-propagate-is-rtranclp-cdcl_W by blast
    then have n-d'[simp]: no-dup (trail S')
      unfolding cdcl_W-M-level-inv-def by auto
    have length (trail S) ≤ length (trail S') ∧ lits-of-l (trail S') ⊆ set M
      using S' full cdcl_W-cp-propagate-completeness[OF assms(1−3), of S] M' S
      by (auto simp: comp-def)
    moreover
      have full: full1 cdcl_W-cp S S'
        using full no-step no-step-cdcl_W-cp-no-conflict-no-propagate(2) unfolding full1-def full-def
        rtranclp-unfold by blast
      then have cdcl_W-stgy S S' by (simp add: cdcl_W-stgy.conflict')
    moreover
      have propa: propagate++ S S' using S' full unfolding full1-def by (metis rtranclpD tranclpD)
      have trail S = M'
        using S by (auto simp: comp-def rev-map)
      with propa have length (trail S') > n
        using l-M' propa by (induction rule: tranclp.induct) (auto elim: propagateE)
    moreover
      have stS': cdcl_W-stgy** (init-state N) S'
        using st cdcl_W-stgy.conflict'[OF full] by auto
      then have init-clss S' = mset-clss N
        using stS' rtranclp-cdcl_W-stgy-no-more-init-clss by fastforce
    moreover
      have
        [simp]:learned-clss S' = {#} and
        [simp]: init-clss S' = init-clss S and
        [simp]: conflicting S' = None
        using tranclp-into-rtranclp[OF ‹propagate++ S S'›] S
        rtranclp-propagate-is-update-trail[of S S'] S M unfolding state-eq-def
        by (auto simp: comp-def)
      have S-S': state S' = (trail S', mset-clss N, {#}, backtrack-lvl S', None)
        using S by auto
      have cdcl_W-stgy** (init-state N) S'
        apply (rule rtranclp.rtrancl-into-rtrancl)
         using st apply simp
        using ‹cdcl_W-stgy S S'› by simp
    ultimately have ?case
      apply -
      apply (rule exI[of - trail S'], rule exI[of - backtrack-lvl S'],  rule exI[of - S'])
      using S-S' by (auto simp: state-eq-def simp del: state-simp)
  }
  moreover {
    assume no-step: no-step propagate S
    have ?case
      proof (cases length M' ≥ Suc n)
        case True
        then show ?thesis using l-M' M' st M alien S n-d by blast
      next
        case False
        then have n': length M' = n using l-M' by auto
        have no-confl: no-step conflict S
          proof -
            { fix D
              assume D ∈# mset-clss N and M' ⊨as CNot D
              then have set M ⊨ D using MN unfolding true-clss-def by auto
              moreover have set M ⊨s CNot D
                using ‹M' ⊨as CNot D› M'
                by (metis le-iff-sup true-annots-true-cls true-clss-union-increase)
              ultimately have False using cons consistent-CNot-not by blast
            }
            then show ?thesis
              using S by (auto simp: true-clss-def comp-def rev-map
                raw-clauses-def dest!: in-clss-mset-clss elim!: conflictE)
          qed
        have lenM: length M = card (set M) using distM by (induction M) auto
        have no-dup M' using S M unfolding cdcl_W-M-level-inv-def by auto
        then have card (lits-of-l M') = length M'
          by (induction M') (auto simp add: lits-of-def card-insert-if)
        then have lits-of-l M' ⊂ set M
          using n M' n' lenM by auto
        then obtain m where m: m ∈ set M and undef-m: m ∉ lits-of-l M' by auto
        moreover have undef: undefined-lit M' m
          using M' Marked-Propagated-in-iff-in-lits-of-l calculation(1,2) cons
          consistent-interp-def by (metis (no-types, lifting) subset-eq)
        moreover have atm-of m ∈ atms-of-mm (init-clss S)
          using atm-incl calculation S by auto
        ultimately
          have dec: decide S (cons-trail (Marked m (k+1)) (incr-lvl S))
            using decide-rule[of S -
              cons-trail (Marked m (k + 1)) (incr-lvl S)] S
            by auto
        let ?S' = cons-trail (Marked m (k+1)) (incr-lvl S)
        have lits-of-l (trail ?S') ⊆ set M using m M' S undef by auto
        moreover have no-strange-atm ?S'
          using alien dec M by (meson cdcl_W-no-strange-atm-inv decide other)
        ultimately obtain S'' where S'': propagate** ?S' S'' and full: full cdcl_W-cp ?S' S''
          using completeness-is-a-full1-propagation[OF assms(1−3), of ?S'] S undef
          by auto
        have cdcl_W-M-level-inv ?S'
          using M dec rtranclp-mono[of decide cdcl_W] by (meson cdcl_W-consistent-inv decide other)
        then have lev'': cdcl_W-M-level-inv S''
          using S'' rtranclp-cdcl_W-consistent-inv rtranclp-propagate-is-rtranclp-cdcl_W by blast
        then have n-d'': no-dup (trail S'')
          unfolding cdcl_W-M-level-inv-def by auto
        have length (trail ?S') ≤ length (trail S'') ∧ lits-of-l (trail S'') ⊆ set M
          using S'' full cdcl_W-cp-propagate-completeness[OF assms(1−3), of ?S' S''] m M' S undef
          by simp
        then have Suc n ≤ length (trail S'') ∧ lits-of-l (trail S'') ⊆ set M
          using l-M' S undef by auto
        moreover
          have cdcl_W-M-level-inv (cons-trail (Marked m (Suc (backtrack-lvl S)))
            (update-backtrack-lvl (Suc (backtrack-lvl S)) S))
            using S ‹cdcl_W-M-level-inv (cons-trail (Marked m (k + 1)) (incr-lvl S))› by auto
          then have S'':
            state S'' = (trail S'', mset-clss N, {#}, backtrack-lvl S'', None)
            using rtranclp-propagate-is-update-trail[OF S''] S undef n-d'' lev''
            by auto
          then have cdcl_W-stgy** (init-state N) S''
            using cdcl_W-stgy.intros(2)[OF decide[OF dec] - full] no-step no-confl st
            by (auto simp: cdcl_W-cp.simps)
        ultimately show ?thesis using S'' n-d'' by blast
      qed
  }
  ultimately show ?case by blast
qed

lemma cdcl_W-stgy-strong-completeness:
  assumes
    MN: set M ⊨s set-mset (mset-clss N) and
    cons: consistent-interp (set M) and
    tot: total-over-m (set M) (set-mset (mset-clss N)) and
    atm-incl: atm-of ‘ (set M) ⊆ atms-of-mm (mset-clss N) and
    distM: distinct M
  shows
    ∃M' k S.
    lits-of-l M′ = set M ∧
    state S = (M′, mset-clss N, {#}, k, None) ∧
    cdcl_W-stgy^** (init-state N) S ∧
    final-cdcl_W-state S
proof –
  from cdcl_W-stgy-strong-completeness-n[OF assms, of length M]
  obtain M′ k T where
    l: length M ≤ length M′ and
    M′-M: lits-of-l M′ ⊆ set M and
    no-dup: no-dup M′ and
    T: state T = (M′, mset-clss N, {#}, k, None) and
    st: cdcl_W-stgy^** (init-state N) T
    by auto
  have card (set M) = length M using distM by (simp add: distinct-card)
  moreover
    have cdcl_W-M-level-inv T
      using rtranclp-cdcl_W-stgy-consistent-inv[OF st] T by auto
    then have card (set ((map (λl. atm-of (lit-of l)) M′))) = length M′
      using distinct-card no-dup by fastforce
  moreover have card (lits-of-l M′) = card (set ((map (λl. atm-of (lit-of l)) M′)))
    using no-dup unfolding lits-of-def apply (induction M′) by (auto simp add: card-insert-if)
  ultimately have card (set M) ≤ card (lits-of-l M′) using l unfolding lits-of-def by auto
  then have set M = lits-of-l M′
    using M′-M  card-seteq by blast
  moreover
    then have M′ ⊨asm mset-clss N
      using MN unfolding true-annots-def Ball-def true-annot-def true-clss-def by auto
    then have final-cdcl_W-state T
      using T no-dup unfolding final-cdcl_W-state-def by auto
  ultimately show ?thesis using st T by blast
qed
```

19.5.6 No conflict with only variables of level less than backtrack level

This invariant is stronger than the previous argument in the sense that it is a property about all possible conflicts.

```
definition no-smaller-confl (S::′st) ≡
  (∀ M K i M′ D. M′ @ Marked K i # M = trail S ⟶ D ∈# clauses S
    ⟶ ¬M ⊨as CNot D)

lemma no-smaller-confl-init-sate[simp]:
  no-smaller-confl (init-state N) unfolding no-smaller-confl-def by auto

lemma cdcl_W-o-no-smaller-confl-inv:
  fixes S S′ :: ′st
  assumes
    cdcl_W-o S S′ and
    lev: cdcl_W-M-level-inv S and
    max-lev: conflict-is-false-with-level S and
    smaller: no-smaller-confl S and
    no-f: no-clause-is-false S
  shows no-smaller-confl S′
  using assms(1,2) unfolding no-smaller-confl-def
proof (induct rule: cdcl_W-o-induct-lev2)
  case (decide L T) note confl = this(1) and undef = this(2) and T = this(4)
  have [simp]: clauses T = clauses S
    using T undef by auto
  show ?case
    proof (intro allI impI)
      fix M'' K i M' Da
      assume M'' @ Marked K i # M' = trail T
      and D: Da ∈# local.clauses T
      then have tl M'' @ Marked K i # M' = trail S
        ∨ (M'' = [] ∧ Marked K i # M' = Marked L (backtrack-lvl S + 1) # trail S)
        using T undef by (cases M'') auto
      moreover {
        assume tl M'' @ Marked K i # M' = trail S
        then have ¬M' ⊨as CNot Da
          using D T undef no-f confl smaller unfolding no-smaller-confl-def smaller by fastforce
      }
      moreover {
        assume Marked K i # M' = Marked L (backtrack-lvl S + 1) # trail S
        then have ¬M' ⊨as CNot Da using no-f D confl T by auto
      }
      ultimately show ¬M' ⊨as CNot Da by fast
   qed
next
  case resolve
  then show ?case using smaller no-f max-lev unfolding no-smaller-confl-def by auto
next
  case skip
  then show ?case using smaller no-f max-lev unfolding no-smaller-confl-def by auto
next
  case (backtrack K i M1 M2 L D T) note confl = this(1) and LD = this(2) and decomp = this(3)
and
    undef = this(7) and T = this(8)
  obtain c where M: trail S = c @ M2 @ Marked K (i+1) # M1
    using decomp by auto

  show ?case
    proof (intro allI impI)
      fix M ia K' M' Da

      assume M' @ Marked K' ia # M = trail T
      then have tl M' @ Marked K' ia # M = M1
        using T decomp undef lev by (cases M') (auto simp: cdcl_W-M-level-inv-decomp)
      let ?S' = (cons-trail (Propagated L (cls-of-ccls D))
                (reduce-trail-to M1 (add-learned-cls (cls-of-ccls D)
                (update-backtrack-lvl i (update-conflicting None S)))))
      assume D: Da ∈# clauses T
      moreover{
        assume Da ∈# clauses S
        then have ¬M ⊨as CNot Da using ‹tl M' @ Marked K' ia # M = M1› M confl undef smaller
          unfolding no-smaller-confl-def by auto
      }
      moreover {
        assume Da: Da = mset-ccls D
        have ¬M ⊨as CNot Da
          proof (rule ccontr)
            assume ¬ ?thesis
          then have −L ∈ lits-of-l M
            using LD unfolding Da by (simp add: in-CNot-implies-uminus(2))
          then have −L ∈ lits-of-l (Propagated L (mset-ccls D) # M1)
            using UnI2 ‹tl M' @ Marked K' ia # M = M1›
            by auto
          moreover
            have backtrack S ?S'
              using backtrack-rule[of S] backtrack.hyps
              by (force simp: state-eq-def simp del: state-simp)
            then have cdcl_W-M-level-inv ?S'
              using cdcl_W-consistent-inv[OF - lev] other[OF bj] by (auto intro: cdcl_W-bj.intros)
            then have no-dup (Propagated L (mset-ccls D) # M1)
              using decomp undef lev unfolding cdcl_W-M-level-inv-def by auto
          ultimately show False
             using undef by (auto simp: Marked-Propagated-in-iff-in-lits-of-l)
        qed
    }
    ultimately show ¬M ⊨as CNot Da
      using T undef decomp lev unfolding cdcl_W-M-level-inv-def by fastforce
  qed
qed

lemma conflict-no-smaller-confl-inv:
  assumes conflict S S'
  and no-smaller-confl S
  shows no-smaller-confl S'
  using assms unfolding no-smaller-confl-def by (fastforce elim: conflictE)

lemma propagate-no-smaller-confl-inv:
  assumes propagate: propagate S S'
  and n-l: no-smaller-confl S
  shows no-smaller-confl S'
  unfolding no-smaller-confl-def
proof (intro allI impI)
  fix M' K i M'' D
  assume M': M'' @ Marked K i # M' = trail S'
  and D ∈# clauses S'
  obtain M N U k C L where
    S: state S = (M, N, U, k, None) and
    S': state S' = (Propagated L (C + {#L#}) # M, N, U, k, None) and
    C + {#L#} ∈# clauses S and
    M ⊨as CNot C and
    undefined-lit M L
    using propagate by (auto elim: propagate-high-levelE)
  have tl M'' @ Marked K i # M' = trail S using M' S S'
    by (metis Pair-inject list.inject list.sel(3) marked-lit.distinct(1) self-append-conv2
      tl-append2)
  then have ¬M' ⊨as CNot D
    using ‹D ∈# clauses S'› n-l S S' raw-clauses-def unfolding no-smaller-confl-def by auto
  then show ¬M' ⊨as CNot D by auto
qed

lemma cdcl_W-cp-no-smaller-confl-inv:
  assumes propagate: cdcl_W-cp S S'
  and n-l: no-smaller-confl S
  shows no-smaller-confl S'
  using assms
proof (induct rule: cdcl_W-cp.induct)
  case (conflict' S S')
  then show ?case using conflict-no-smaller-confl-inv[of S S'] by blast
next
  case (propagate' S S')
  then show ?case using propagate-no-smaller-confl-inv[of S S'] by fastforce
qed

lemma rtrancp-cdcl_W-cp-no-smaller-confl-inv:
  assumes propagate: cdcl_W-cp** S S'
  and n-l: no-smaller-confl S
  shows no-smaller-confl S'
  using assms
proof (induct rule: rtranclp-induct)
  case base
  then show ?case by simp
next
  case (step S' S'')
  then show ?case using cdcl_W-cp-no-smaller-confl-inv[of S' S''] by fast
qed

lemma trancp-cdcl_W-cp-no-smaller-confl-inv:
  assumes propagate: cdcl_W-cp++ S S'
  and n-l: no-smaller-confl S
  shows no-smaller-confl S'
  using assms
proof (induct rule: tranclp.induct)
  case (r-into-trancl S S')
  then show ?case using cdcl_W-cp-no-smaller-confl-inv[of S S'] by blast
next
  case (trancl-into-trancl S S' S'')
  then show ?case using cdcl_W-cp-no-smaller-confl-inv[of S' S''] by fast
qed

lemma full-cdcl_W-cp-no-smaller-confl-inv:
  assumes full cdcl_W-cp S S'
  and n-l: no-smaller-confl S
  shows no-smaller-confl S'
  using assms unfolding full-def
  using rtrancp-cdcl_W-cp-no-smaller-confl-inv[of S S'] by blast

lemma full1-cdcl_W-cp-no-smaller-confl-inv:
  assumes full1 cdcl_W-cp S S'
  and n-l: no-smaller-confl S
  shows no-smaller-confl S'
  using assms unfolding full1-def
  using trancp-cdcl_W-cp-no-smaller-confl-inv[of S S'] by blast

lemma cdcl_W-stgy-no-smaller-confl-inv:
  assumes cdcl_W-stgy S S'
  and n-l: no-smaller-confl S
  and conflict-is-false-with-level S
  and cdcl_W-M-level-inv S
  shows no-smaller-confl S′
  using assms
proof (induct rule: cdcl_W-stgy.induct)
  case (conflict′ S′)
  then show ?case using full1-cdcl_W-cp-no-smaller-confl-inv[of S S′] by blast
next
  case (other′ S′ S″)
  have no-smaller-confl S′
    using cdcl_W-o-no-smaller-confl-inv[OF other′.hyps(1) other′.prems(3,2,1)]
    not-conflict-not-any-negated-init-clss other′.hyps(2) cdcl_W-cp.simps by auto
  then show ?case using full-cdcl_W-cp-no-smaller-confl-inv[of S′ S″] other′.hyps by blast
qed

lemma is-conflicting-exists-conflict:
  assumes ¬(∀ D∈#init-clss S′ + learned-clss S′. ¬ trail S′ ⊨as CNot D)
  and conflicting S′ = None
  shows ∃ S″. conflict S′ S″
  using assms raw-clauses-def not-conflict-not-any-negated-init-clss by fastforce

lemma cdcl_W-o-conflict-is-no-clause-is-false:
  fixes S S′ :: ′st
  assumes
    cdcl_W-o S S′ and
    lev: cdcl_W-M-level-inv S and
    max-lev: conflict-is-false-with-level S and
    no-f: no-clause-is-false S and
    no-l: no-smaller-confl S
  shows no-clause-is-false S′
    ∨ (conflicting S′ = None
        ⟶ (∀ D ∈# clauses S′. trail S′ ⊨as CNot D
             ⟶ (∃ L. L ∈# D ∧ get-level (trail S′) L = backtrack-lvl S′)))
  using assms(1,2)
proof (induct rule: cdcl_W-o-induct-lev2)
  case (decide L T) note S = this(1) and undef = this(2) and T = this(4)
  show ?case
    proof (rule HOL.disjI2, clarify)
      fix D
      assume D: D ∈# clauses T and M-D: trail T ⊨as CNot D
      let ?M = trail S
      let ?M′ = trail T
      let ?k = backtrack-lvl S
      have ¬?M ⊨as CNot D
          using no-f D S T undef by auto
      have −L ∈# D
        proof (rule ccontr)
          assume ¬ ?thesis
          have ?M ⊨as CNot D
            unfolding true-annots-def Ball-def true-annot-def CNot-def true-cls-def
            proof (intro allI impI)
              fix x
              assume x: x ∈ {{#− L#} |L. L ∈# D}

              then obtain L′ where L′: x = {#−L′#} L′ ∈# D by auto
              obtain L″ where L″ ∈# x and lits-of-l (Marked L (?k + 1) # ?M) ⊨l L″
                using M-D x T undef unfolding true-annots-def Ball-def true-annot-def CNot-def
                true-cls-def Bex-def by auto
              show ∃L ∈# x. lits-of-l ?M ⊨l L unfolding Bex-def
                using L'(1) L'(2) ⟨− L ∉# D⟩ ⟨L'' ∈# x⟩
                ⟨lits-of-l (Marked L (backtrack-lvl S + 1) # trail S) ⊨l L''⟩ by auto
            qed
          then show False using ⟨¬ ?M ⊨as CNot D⟩ by auto
        qed
      have atm-of L ∉ atm-of ‘ (lits-of-l ?M)
        using undef defined-lit-map unfolding lits-of-def by fastforce
      then have get-level (Marked L (?k + 1) # ?M) (−L) = ?k + 1 by simp
      then show ∃La. La ∈# D ∧ get-level ?M' La = backtrack-lvl T
        using ⟨−L ∈# D⟩ T undef by auto
    qed
next
  case resolve
  then show ?case by auto
next
  case skip
  then show ?case by auto
next
  case (backtrack K i M1 M2 L D T) note decomp = this(3) and undef = this(7) and T = this(8)
  show ?case
    proof (rule HOL.disjI2, clarify)
      fix Da
      assume Da: Da ∈# clauses T
      and M-D: trail T ⊨as CNot Da
      obtain c where M: trail S = c @ M2 @ Marked K (i + 1) # M1
        using decomp by auto
      have tr-T: trail T = Propagated L (mset-ccls D) # M1
        using T decomp undef lev by (auto simp: cdcl_W-M-level-inv-decomp)
      have backtrack S T
        using backtrack-rule[of S] backtrack.hyps T
        by (force simp del: state-simp simp: state-eq-def)
      then have lev': cdcl_W-M-level-inv T
        using cdcl_W-consistent-inv lev other cdcl_W-bj.backtrack cdcl_W-o.bj by blast
      then have − L ∉ lits-of-l M1
        using lev cdcl_W-M-level-inv-def Marked-Propagated-in-iff-in-lits-of-l undef by blast
      { assume Da ∈# clauses S
        then have ¬M1 ⊨as CNot Da using no-l M unfolding no-smaller-confl-def by auto
      }
      moreover {
        assume Da: Da = mset-ccls D
        have ¬M1 ⊨as CNot Da using ⟨− L ∉ lits-of-l M1⟩ unfolding Da
          using backtrack.hyps(2) in-CNot-implies-uminus(2) by auto
      }
      ultimately have ¬M1 ⊨as CNot Da
        using Da T undef decomp lev by (fastforce simp: cdcl_W-M-level-inv-decomp)
      then have −L ∈# Da
        using M-D ⟨− L ∉ lits-of-l M1⟩  T unfolding tr-T true-annots-true-cls true-clss-def
        by (auto simp: uminus-lit-swap)
      have g-M1: get-all-levels-of-marked M1 = rev [1..<i+1]
        using lev lev' T decomp undef unfolding cdcl_W-M-level-inv-def by auto
      have no-dup (Propagated L (mset-ccls D) # M1)
        using lev lev' T decomp undef unfolding cdcl_W-M-level-inv-def by auto
      then have L: atm-of L ∉ atm-of ‘ lits-of-l M1 unfolding lits-of-def by auto
      have get-level (Propagated L (mset-ccls D) # M1) (−L) = i
        using get-level-get-rev-level-get-all-levels-of-marked[OF L,
          of [Propagated L (mset-ccls D)]]
        by (simp add: g-M1 split: if-splits)
      then show ∃La. La ∈# Da ∧ get-level (trail T) La = backtrack-lvl T
        using ‹−L ∈# Da› T decomp undef lev by (auto simp: cdcl_W-M-level-inv-def)
    qed
qed

lemma full1-cdcl_W-cp-exists-conflict-decompose:
  assumes
    confl: ∃D∈#clauses S. trail S |=as CNot D and
    full: full cdcl_W-cp S U and
    no-confl: conflicting S = None and
    lev: cdcl_W-M-level-inv S
  shows ∃T. propagate** S T ∧ conflict T U
proof −
  consider (propa) propagate** S U
        | (confl) T where propagate** S T and conflict T U
   using full unfolding full-def by (blast dest:rtranclp-cdcl_W-cp-propa-or-propa-confl)
  then show ?thesis
    proof cases
      case confl
      then show ?thesis by blast
    next
      case propa
      then have conflicting U = None and
        [simp]: learned-clss U = learned-clss S and
        [simp]: init-clss U = init-clss S
        using no-confl rtranclp-propagate-is-update-trail lev by auto
      moreover
        obtain D where D: D∈#clauses U and
          trS: trail S |=as CNot D
          using confl raw-clauses-def by auto
        obtain M where M: trail U = M @ trail S
          using full rtranclp-cdcl_W-cp-dropWhile-trail unfolding full-def by meson
        have tr-U: trail U |=as CNot D
          apply (rule true-annots-mono)
          using trS unfolding M by simp-all
      have ∃V. conflict U V
        using ‹conflicting U = None› D raw-clauses-def not-conflict-not-any-negated-init-clss tr-U
        by meson
      then have False using full cdcl_W-cp.conflict' unfolding full-def by blast
      then show ?thesis by fast
    qed
qed

lemma full1-cdcl_W-cp-exists-conflict-full1-decompose:
  assumes
    confl: ∃D∈#clauses S. trail S |=as CNot D and
    full: full cdcl_W-cp S U and
    no-confl: conflicting S = Noneand
    lev: cdcl_W-M-level-inv S
  shows ∃T D. propagate** S T ∧ conflict T U
    ∧ trail T |=as CNot D ∧ conflicting U = Some D ∧ D ∈# clauses S
```

```
proof –
  obtain T where propa: propagate** S T and conf: conflict T U
    using full1-cdcl_W-cp-exists-conflict-decompose[OF assms] by blast
  have p: learned-clss T = learned-clss S init-clss T = init-clss S
     using propa lev rtranclp-propagate-is-update-trail by auto
  have c: learned-clss U = learned-clss T init-clss U = init-clss T
     using conf by (auto elim: conflictE)
  obtain D where trail T ⊨as CNot D ∧ conflicting U = Some D ∧ D ∈# clauses S
    using conf p c by (fastforce simp: raw-clauses-def elim!: conflictE)
  then show ?thesis
    using propa conf by blast
qed

lemma cdcl_W-stgy-no-smaller-confl:
  assumes
    cdcl_W-stgy S S' and
    n-l: no-smaller-confl S and
    conflict-is-false-with-level S and
    cdcl_W-M-level-inv S and
    no-clause-is-false S and
    distinct-cdcl_W-state S and
    cdcl_W-conflicting S
  shows no-smaller-confl S'
  using assms
proof (induct rule: cdcl_W-stgy.induct)
  case (conflict' S')
  show no-smaller-confl S'
    using conflict'.hyps conflict'.prems(1) full1-cdcl_W-cp-no-smaller-confl-inv by blast
next
  case (other' S' S'')
  have lev': cdcl_W-M-level-inv S'
    using cdcl_W-consistent-inv other other'.hyps(1) other'.prems(3) by blast
  show no-smaller-confl S''
    using cdcl_W-stgy-no-smaller-confl-inv[OF cdcl_W-stgy.other'[OF other'.hyps(1–3)]]
    other'.prems(1–3) by blast
qed

lemma cdcl_W-stgy-ex-lit-of-max-level:
  assumes
    cdcl_W-stgy S S' and
    n-l: no-smaller-confl S and
    conflict-is-false-with-level S and
    cdcl_W-M-level-inv S and
    no-clause-is-false S and
    distinct-cdcl_W-state S and
    cdcl_W-conflicting S
  shows conflict-is-false-with-level S'
  using assms
proof (induct rule: cdcl_W-stgy.induct)
  case (conflict' S')
  have no-smaller-confl S'
    using conflict'.hyps conflict'.prems(1) full1-cdcl_W-cp-no-smaller-confl-inv by blast
  moreover have conflict-is-false-with-level S'
    using conflict'.hyps conflict'.prems(2–4)
    rtranclp-cdcl_W-co-conflict-ex-lit-of-max-level[of S S']
    unfolding full-def full1-def rtranclp-unfold by presburger
  then show ?case by blast
next
  case (other' S' S'')
  have lev': cdcl_W-M-level-inv S'
    using cdcl_W-consistent-inv other other'.hyps(1) other'.prems(3) by blast
  moreover
    have no-clause-is-false S'
      ∨ (conflicting S' = None ⟶ (∀ D∈#clauses S'. trail S' ⊨as CNot D
          ⟶ (∃ L. L ∈# D ∧ get-level (trail S') L = backtrack-lvl S')))
      using cdcl_W-o-conflict-is-no-clause-is-false[of S S'] other'.hyps(1) other'.prems(1−4) by fast
  moreover {
    assume no-clause-is-false S'
    {
      assume conflicting S' = None
      then have conflict-is-false-with-level S' by auto
      moreover have full cdcl_W-cp S' S''
        by (metis (no-types) other'.hyps(3))
      ultimately have conflict-is-false-with-level S''
        using rtranclp-cdcl_W-co-conflict-ex-lit-of-max-level[of S' S''] lev' ‹no-clause-is-false S'›
        by blast
    }
    moreover
    {
      assume c: conflicting S' ≠ None
      have conflicting S ≠ None using other'.hyps(1) c
        by (induct rule: cdcl_W-o-induct) auto
      then have conflict-is-false-with-level S'
        using cdcl_W-o-conflict-is-false-with-level-inv[OF other'.hyps(1)]
        other'.prems(3,5,6,2) by blast
      moreover have cdcl_W-cp** S' S'' using other'.hyps(3) unfolding full-def by auto
      then have S' = S'' using c
        by (induct rule: rtranclp-induct)
           (fastforce intro: option.exhaust)+
      ultimately have conflict-is-false-with-level S'' by auto
    }
    ultimately have conflict-is-false-with-level S'' by blast
  }
  moreover {
     assume
       confl: conflicting S' = None and
       D-L: ∀ D ∈# clauses S'. trail S' ⊨as CNot D
         ⟶ (∃ L. L ∈# D ∧ get-level (trail S') L = backtrack-lvl S')
     { assume ∀ D∈#clauses S'. ¬ trail S' ⊨as CNot D
       then have no-clause-is-false S' using confl by simp
       then have conflict-is-false-with-level S'' using calculation(3) by presburger
     }
     moreover {
       assume ¬(∀ D∈#clauses S'. ¬ trail S' ⊨as CNot D)
       then obtain T D where
         propagate** S' T and
         conflict T S'' and
         D: D ∈# clauses S' and
         trail S'' ⊨as CNot D and
         conflicting S'' = Some D
```

```
 using full1-cdcl_W-cp-exists-conflict-full1-decompose[OF - - confl]
 other'(3) lev' by (metis (mono-tags, lifting) conflictE state-eq-trail
   trail-update-conflicting)
obtain M where M: trail S'' = M @ trail S' and nm: ∀ m∈set M. ¬is-marked m
 using rtranclp-cdcl_W-cp-dropWhile-trail other'(3) unfolding full-def by meson
have btS: backtrack-lvl S'' = backtrack-lvl S'
 using other'.hyps(3) unfolding full-def by (metis rtranclp-cdcl_W-cp-backtrack-lvl)
have inv: cdcl_W-M-level-inv S''
 by (metis (no-types) cdcl_W-stgy.conflict' cdcl_W-stgy-consistent-inv full-unfold lev'
   other'.hyps(3))
then have nd: no-dup (trail S'')
 by (metis (no-types) cdcl_W-M-level-inv-decomp(2))
have conflict-is-false-with-level S''
 proof cases
   assume trail S' ⊨as CNot D
   moreover then obtain L where
     L ∈# D and
     lev-L: get-level (trail S') L = backtrack-lvl S'
     using D-L D by blast
   moreover
     have LS': −L ∈ lits-of-l (trail S')
       using ‹trail S' ⊨as CNot D› ‹L ∈# D› in-CNot-implies-uminus(2) by blast
     {  fix x :: ('v, nat, 'v clause) marked-lit and
         xb :: ('v, nat, 'v clause) marked-lit
       assume a1: x ∈ set (trail S') and
         a2: xb ∈ set M and
         a3: (λl. atm-of (lit-of l)) ` set M ∩ (λl. atm-of (lit-of l)) ` set (trail S')
           = {} and
          a4: − L = lit-of x and
          a5: atm-of L = atm-of (lit-of xb)
       moreover have atm-of (lit-of x) = atm-of L
         using a4 by (metis (no-types) atm-of-uminus)
       ultimately have False
         using a5 a3 a2 a1 by auto
     }
     then have atm-of L ∉ atm-of ` lits-of-l M
       using nd LS' unfolding M by (auto simp add: lits-of-def)
     then have get-level (trail S'') L = get-level (trail S') L
       unfolding M by (simp add: lits-of-def)
   ultimately show ?thesis using btS ‹conflicting S'' = Some D› by auto
 next
   assume ¬trail S' ⊨as CNot D
   then obtain L where L ∈# D and LM: −L ∈ lits-of-l M
     using ‹trail S'' ⊨as CNot D› unfolding M
       by (auto simp add: true-cls-def M true-annots-def true-annot-def
             split: if-split-asm)
   { fix x :: ('v, nat, 'v clause) marked-lit and
       xb :: ('v, nat, 'v clause) marked-lit
     assume a1: xb ∈ set (trail S') and
       a2: x ∈ set M and
       a3: atm-of L = atm-of (lit-of xb) and
       a4: − L = lit-of x and
       a5: (λl. atm-of (lit-of l)) ` set M ∩ (λl. atm-of (lit-of l)) ` set (trail S')
         = {}
     moreover have atm-of (lit-of xb) = atm-of (− L)
```

```
            using a3 by simp
          ultimately have False
            by auto }
        then have LS': atm-of L ∉ atm-of ‘ lits-of-l (trail S')
          using nd ‹L ∈# D› LM unfolding M by (auto simp add: lits-of-def)
        show ?thesis
          proof cases
            assume ne: get-all-levels-of-marked (trail S') = []
            have backtrack-lvl S'' = 0
              using inv ne nm unfolding cdcl_W-M-level-inv-def M
              by (simp add: get-all-levels-of-marked-nil-iff-not-is-marked)
            moreover
              have a1: get-level M L = 0
                using nm by auto
              then have get-level (M @ trail S') L = 0
                by (metis LS' get-all-levels-of-marked-nil-iff-not-is-marked
                  get-level-skip-beginning-not-marked lits-of-def ne)
            ultimately show ?thesis using ‹conflicting S'' = Some D› ‹L ∈# D› unfolding M
              by auto
          next
            assume ne: get-all-levels-of-marked (trail S') ≠ []
            have hd (get-all-levels-of-marked (trail S')) = backtrack-lvl S'
              using ne lev' M nm unfolding cdcl_W-M-level-inv-def
              by (cases get-all-levels-of-marked (trail S'))
              (simp-all add: get-all-levels-of-marked-nil-iff-not-is-marked[symmetric])
            moreover have atm-of L ∈ atm-of ‘ lits-of-l M
               using ‹−L ∈ lits-of-l M›
               by (simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set lits-of-def)
            ultimately show ?thesis
              using nm ne ‹L∈#D› ‹conflicting S'' = Some D›
                get-level-skip-beginning-hd-get-all-levels-of-marked[OF LS', of M]
                get-level-skip-in-all-not-marked[of rev M L backtrack-lvl S']
              unfolding lits-of-def btS M
              by auto
          qed
      qed
    }
    ultimately have conflict-is-false-with-level S'' by blast
  }
  moreover
  {
    assume conflicting S' ≠ None
    have no-clause-is-false S' using ‹conflicting S' ≠ None›  by auto
    then have conflict-is-false-with-level S'' using calculation(3) by presburger
  }
  ultimately show ?case by fast
qed

lemma rtranclp-cdcl_W-stgy-no-smaller-confl-inv:
  assumes
    cdcl_W-stgy** S S' and
    n-l: no-smaller-confl S and
    cls-false: conflict-is-false-with-level S and
    lev: cdcl_W-M-level-inv S and
    no-f: no-clause-is-false S and
    dist: distinct-cdcl_W-state S and
    conflicting: cdcl_W-conflicting S and
    decomp: all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S)) and
    learned: cdcl_W-learned-clause S and
    alien: no-strange-atm S
  shows no-smaller-confl S′ ∧ conflict-is-false-with-level S′
  using assms(1)
proof (induct rule: rtranclp-induct)
  case base
  then show ?case using n-l cls-false by auto
next
  case (step S′ S′′) note st = this(1) and cdcl = this(2) and IH = this(3)
  have no-smaller-confl S′ and conflict-is-false-with-level S′
    using IH by blast+
  moreover have cdcl_W-M-level-inv S′
    using  st lev rtranclp-cdcl_W-stgy-rtranclp-cdcl_W
    by (blast intro: rtranclp-cdcl_W-consistent-inv)+
  moreover have no-clause-is-false S′
    using st no-f rtranclp-cdcl_W-stgy-not-non-negated-init-clss by presburger
  moreover have distinct-cdcl_W-state S′
    using rtanclp-distinct-cdcl_W-state-inv[of S S′] lev rtranclp-cdcl_W-stgy-rtranclp-cdcl_W[OF st]
    dist by auto
  moreover have cdcl_W-conflicting S′
    using rtranclp-cdcl_W-all-inv(6)[of S S′] st  alien conflicting decomp dist learned lev
    rtranclp-cdcl_W-stgy-rtranclp-cdcl_W by blast
  ultimately show ?case
    using cdcl_W-stgy-no-smaller-confl[OF cdcl] cdcl_W-stgy-ex-lit-of-max-level[OF cdcl] by fast
qed
```

19.5.7 Final States are Conclusive

```
lemma full-cdcl_W-stgy-final-state-conclusive-non-false:
  fixes S′ :: ′st
  assumes full: full cdcl_W-stgy (init-state N) S′
  and no-d: distinct-mset-mset (mset-clss N)
  and no-empty: ∀D∈#mset-clss N. D ≠ {#}
  shows (conflicting S′ = Some {#} ∧ unsatisfiable (set-mset (init-clss S′)))
    ∨ (conflicting S′ = None ∧ trail S′ ⊨asm init-clss S′)
proof −
  let ?S = init-state N
  have
    termi: ∀S′′. ¬cdcl_W-stgy S′ S′′ and
    step: cdcl_W-stgy** ?S S′ using full unfolding full-def by auto
  moreover have
    learned: cdcl_W-learned-clause S′ and
    level-inv: cdcl_W-M-level-inv S′ and
    alien: no-strange-atm S′ and
    no-dup: distinct-cdcl_W-state S′ and
    confl: cdcl_W-conflicting S′ and
    decomp: all-decomposition-implies-m (init-clss S′) (get-all-marked-decomposition (trail S′))
    using no-d tranclp-cdcl_W-stgy-tranclp-cdcl_W[of ?S S′] step rtranclp-cdcl_W-all-inv(1−6)[of ?S S′]
    unfolding rtranclp-unfold by auto
  moreover
    have ∀D∈#mset-clss N. ¬ [] ⊨as CNot D using no-empty by auto
    then have confl-k: conflict-is-false-with-level S′
      using rtranclp-cdcl_W-stgy-no-smaller-confl-inv[OF step] no-d by auto
  show ?thesis
    using cdcl_W-stgy-final-state-conclusive[OF termi decomp learned level-inv alien no-dup confl
      confl-k] .
qed


lemma conflict-is-full1-cdcl_W-cp:
  assumes cp: conflict S S′
  shows full1 cdcl_W-cp S S′
proof −
  have cdcl_W-cp S S′ and conflicting S′ ≠ None
    using cp cdcl_W-cp.intros by (auto elim!: conflictE simp: state-eq-def simp del: state-simp)
  then have cdcl_W-cp⁺⁺ S S′ by blast
  moreover have no-step cdcl_W-cp S′
    using ‹conflicting S′ ≠ None› by (metis cdcl_W-cp-conflicting-not-empty
      option.exhaust)
  ultimately show full1 cdcl_W-cp S S′ unfolding full1-def by blast+
qed

lemma cdcl_W-cp-fst-empty-conflicting-false:
  assumes
    cdcl_W-cp S S′ and
    trail S = [] and
    conflicting S ≠ None
  shows False
  using assms by (induct rule: cdcl_W-cp.induct) (auto elim: propagateE conflictE)

lemma cdcl_W-o-fst-empty-conflicting-false:
  assumes cdcl_W-o S S′
  and trail S = []
  and conflicting S ≠ None
  shows False
  using assms by (induct rule: cdcl_W-o-induct) auto

lemma cdcl_W-stgy-fst-empty-conflicting-false:
  assumes cdcl_W-stgy S S′
  and trail S = []
  and conflicting S ≠ None
  shows False
  using assms apply (induct rule: cdcl_W-stgy.induct)
  using tranclpD cdcl_W-cp-fst-empty-conflicting-false unfolding full1-def apply metis
  using cdcl_W-o-fst-empty-conflicting-false by blast
thm cdcl_W-cp.induct[split-format(complete)]

lemma cdcl_W-cp-conflicting-is-false:
  cdcl_W-cp S S′ ⟹ conflicting S = Some {#} ⟹ False
  by (induction rule: cdcl_W-cp.induct) (auto elim: propagateE conflictE)

lemma rtranclp-cdcl_W-cp-conflicting-is-false:
  cdcl_W-cp⁺⁺ S S′ ⟹ conflicting S = Some {#} ⟹ False
  apply (induction rule: tranclp.induct)
  by (auto dest: cdcl_W-cp-conflicting-is-false)

lemma cdcl_W-o-conflicting-is-false:
  cdcl_W-o S S′ ⟹ conflicting S = Some {#} ⟹ False
  by (induction rule: cdcl_W-o-induct) auto

lemma cdcl_W-stgy-conflicting-is-false:
  cdcl_W-stgy S S' ⟹ conflicting S = Some {#} ⟹ False
  apply (induction rule: cdcl_W-stgy.induct)
    unfolding full1-def apply (metis (no-types) cdcl_W-cp-conflicting-not-empty tranclpD)
  unfolding full-def by (metis conflict-with-false-implies-terminated other)

lemma rtranclp-cdcl_W-stgy-conflicting-is-false:
  cdcl_W-stgy** S S' ⟹ conflicting S = Some {#} ⟹ S' = S
  apply (induction rule: rtranclp-induct)
    apply simp
  using cdcl_W-stgy-conflicting-is-false by blast

lemma full-cdcl_W-init-clss-with-false-normal-form:
  assumes
    ∀ m∈ set M. ¬is-marked m and
    E = Some D and
    state S = (M, N, U, 0, E)
    full cdcl_W-stgy S S' and
    all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S))
    cdcl_W-learned-clause S
    cdcl_W-M-level-inv S
    no-strange-atm S
    distinct-cdcl_W-state S
    cdcl_W-conflicting S
  shows ∃ M''. state S' = (M'', N, U, 0, Some {#})
  using assms(10,9,8,7,6,5,4,3,2,1)
proof (induction M arbitrary: E D S)
  case Nil
  then show ?case
    using rtranclp-cdcl_W-stgy-conflicting-is-false unfolding full-def cdcl_W-conflicting-def
    by fastforce
next
  case (Cons L M) note IH = this(1) and full = this(8) and E = this(10) and inv = this(2−7) and
    S = this(9) and nm = this(11)
  obtain K p where K: L = Propagated K p
    using nm by (cases L) auto
  have every-mark-is-a-conflict S using inv unfolding cdcl_W-conflicting-def by auto
  then have MpK: M ⊨as CNot ( p − {#K#}) and Kp: K ∈# p
    using S unfolding K by fastforce+
  then have p: p = (p − {#K#}) + {#K#}
    by (auto simp add: multiset-eq-iff)
  then have K': L = Propagated K ((p − {#K#}) + {#K#})
    using K by auto
  obtain p' where
    p': hd-raw-trail S = Propagated K p' and
    pp': mset-cls p' = p
    using hd-raw-trail[of S] S K by (cases hd-raw-trail S) auto
  obtain raw-D where
    raw-D: raw-conflicting S = Some raw-D
    using S E by (cases raw-conflicting S) auto
  then have raw-DD: mset-ccls raw-D = D
    using S E by auto
  consider (D) D = {#} | (D') D ≠ {#} by blast
```

```
then show ?case
  proof cases
    case D
    then show ?thesis
      using full rtranclp-cdcl_W-stgy-conflicting-is-false S unfolding full-def E D by auto
  next
    case D'
    then have no-p: no-step propagate S and no-c: no-step conflict S
      using S E by (auto elim: propagateE conflictE)
    then have no-step cdcl_W-cp S by (auto simp: cdcl_W-cp.simps)
    have res-skip: ∃ T. (resolve S T ∧ no-step skip S ∧ full cdcl_W-cp T T)
      ∨ (skip S T ∧ no-step resolve S ∧ full cdcl_W-cp T T)
      proof cases
        assume −lit-of L ∉# D
        then obtain T where sk: skip S T
          using S D' K skip-rule unfolding E by fastforce
        then have res: no-step resolve S
          using ‹−lit-of L ∉# D› S D' K hd-raw-trail[of S] unfolding E
          by (auto elim!: skipE resolveE)
        have full cdcl_W-cp T T
          using sk by (auto intro!: option-full-cdcl_W-cp elim: skipE)
        then show ?thesis
          using sk res by blast
      next
        assume LD: ¬−lit-of L ∉# D
        then have D: Some D = Some ((D − {#−lit-of L#}) + {#−lit-of L#})
          by (auto simp add: multiset-eq-iff)

        have ⋀L. get-level M L = 0
          by (simp add: nm)
          then have get-maximum-level (Propagated K (p − {#K#} + {#K#}) # M) (D − {#−
K#}) = 0
          using  LD get-maximum-level-exists-lit-of-max-level
          proof −
            obtain L' where get-level (L#M) L' = get-maximum-level (L#M) D
              using  LD get-maximum-level-exists-lit-of-max-level[of D L#M] by fastforce
            then show ?thesis by (metis (mono-tags) K' get-level-skip-all-not-marked
              get-maximum-level-exists-lit nm not-gr0)
          qed
        then obtain T where sk: resolve S T
          using resolve-rule[of S K p' raw-D] S p' ‹K ∈# p› raw-D LD
          unfolding K' D E  pp' raw-DD by auto
        then have res: no-step skip S
          using LD S D' K hd-raw-trail[of S] unfolding E
          by (auto elim!: skipE resolveE)
        have full cdcl_W-cp T T
          using sk by (auto simp: option-full-cdcl_W-cp elim: resolveE)
        then show ?thesis
         using sk res by blast
      qed
    then have step-s: ∃ T. cdcl_W-stgy S T
      using ‹no-step cdcl_W-cp S› other' by (meson bj resolve skip)
    have get-all-marked-decomposition (L # M) = [([], L#M)]
      using nm unfolding K apply (induction M rule: marked-lit-list-induct, simp)
        by (rename-tac L l xs, case-tac hd (get-all-marked-decomposition xs), auto)+
      then have no-b: no-step backtrack S
        using nm S by (auto elim: backtrackE)
      have no-d: no-step decide S
        using S E by (auto elim: decideE)

      have full-S-S: full cdcl_W-cp S S
        using S E by (auto simp add: option-full-cdcl_W-cp)
      then have no-f: no-step (full1 cdcl_W-cp) S
        unfolding full-def full1-def rtranclp-unfold by (meson tranclpD)
      obtain T where
        s: cdcl_W-stgy S T and st: cdcl_W-stgy** T S'
        using full step-s full unfolding full-def by (metis rtranclp-unfold tranclpD)
      have resolve S T ∨ skip S T
        using s no-b no-d res-skip full-S-S cdcl_W-cp-state-eq-compatible resolve-unique
        skip-unique unfolding cdcl_W-stgy.simps cdcl_W-o.simps full-unfold
        full1-def by (blast dest!: tranclpD elim!: cdcl_W-bj.cases)+
      then obtain D' where T: state T = (M, N, U, 0, Some D')
        using S E by (auto elim!: skipE resolveE simp: state-eq-def simp del: state-simp)

      have st-c: cdcl_W** S T
        using E T rtranclp-cdcl_W-stgy-rtranclp-cdcl_W s by blast
      have cdcl_W-conflicting T
        using rtranclp-cdcl_W-all-inv(6)[OF st-c inv(6,5,4,3,2,1)] .
      show ?thesis
        apply (rule IH[of T])
                  using rtranclp-cdcl_W-all-inv(6)[OF st-c inv(6,5,4,3,2,1)] apply blast
                 using rtranclp-cdcl_W-all-inv(5)[OF st-c inv(6,5,4,3,2,1)] apply blast
                using rtranclp-cdcl_W-all-inv(4)[OF st-c inv(6,5,4,3,2,1)] apply blast
               using rtranclp-cdcl_W-all-inv(3)[OF st-c inv(6,5,4,3,2,1)] apply blast
              using rtranclp-cdcl_W-all-inv(2)[OF st-c inv(6,5,4,3,2,1)] apply blast
             using rtranclp-cdcl_W-all-inv(1)[OF st-c inv(6,5,4,3,2,1)] apply blast
            apply (metis full-def st full)
           using T E apply blast
          apply auto[]
        using nm by simp
    qed
qed

lemma full-cdcl_W-stgy-final-state-conclusive-is-one-false:
  fixes S' :: 'st
  assumes full: full cdcl_W-stgy (init-state N) S'
  and no-d: distinct-mset-mset (mset-clss N)
  and empty: {#} ∈# (mset-clss N)
  shows conflicting S' = Some {#} ∧ unsatisfiable (set-mset (init-clss S'))
proof −
  let ?S = init-state N
  have cdcl_W-stgy** ?S S' and no-step cdcl_W-stgy S' using full unfolding full-def by auto
  then have plus-or-eq: cdcl_W-stgy++ ?S S' ∨ S' = ?S unfolding rtranclp-unfold by auto
  have ∃ S''. conflict ?S S''
    using empty not-conflict-not-any-negated-init-clss[of init-state N] by auto


  then have cdcl_W-stgy: ∃ S'. cdcl_W-stgy ?S S'
    using cdcl_W-cp.conflict'[of ?S] conflict-is-full1-cdcl_W-cp cdcl_W-stgy.intros(1) by metis
  have S' ≠ ?S using ‹no-step cdcl_W-stgy S'› cdcl_W-stgy by blast
```

```
then obtain St:: 'st where St: cdcl_W-stgy ?S St and cdcl_W-stgy** St S'
  using plus-or-eq by (metis (no-types) ⟨cdcl_W-stgy** ?S S'⟩ converse-rtranclpE)
have st: cdcl_W** ?S St
  by (simp add: rtranclp-unfold ⟨cdcl_W-stgy ?S St⟩ cdcl_W-stgy-tranclp-cdcl_W)

have ∃ T. conflict ?S T
  using empty not-conflict-not-any-negated-init-clss[of ?S] by force
then have fullSt: full1 cdcl_W-cp ?S St
  using St unfolding cdcl_W-stgy.simps by blast
then have bt: backtrack-lvl St = (0::nat)
  using rtranclp-cdcl_W-cp-backtrack-lvl unfolding full1-def
  by (fastforce dest!: tranclp-into-rtranclp)
have cls-St: init-clss St = mset-clss N
  using fullSt  cdcl_W-stgy-no-more-init-clss[OF St] by auto
have conflicting St ≠ None
  proof (rule ccontr)
    assume conf: ¬ ?thesis
    obtain E where
      ES: E !∈! raw-init-clss St and
      E: mset-cls E = {#}
      using empty cls-St by (metis in-mset-clss-exists-preimage)
    then have ∃ T. conflict St T
      using empty cls-St conflict-rule[of St E] ES conf unfolding E
      by (auto simp: raw-clauses-def dest: in-mset-clss-exists-preimage)
    then show False using fullSt unfolding full1-def by blast
  qed

have 1: ∀ m∈set (trail St). ¬ is-marked m
  using fullSt unfolding full1-def by (auto dest!: tranclp-into-rtranclp
    rtranclp-cdcl_W-cp-dropWhile-trail)
have 2: full cdcl_W-stgy St S'
  using ⟨cdcl_W-stgy** St S'⟩ ⟨no-step cdcl_W-stgy S'⟩ bt unfolding full-def by auto
have 3: all-decomposition-implies-m
    (init-clss St)
    (get-all-marked-decomposition
       (trail St))
 using rtranclp-cdcl_W-all-inv(1)[OF st] no-d bt by simp
have 4: cdcl_W-learned-clause St
  using rtranclp-cdcl_W-all-inv(2)[OF st] no-d bt bt by simp
have 5: cdcl_W-M-level-inv St
  using rtranclp-cdcl_W-all-inv(3)[OF st] no-d bt by simp
have 6: no-strange-atm St
  using rtranclp-cdcl_W-all-inv(4)[OF st] no-d bt by simp
have 7: distinct-cdcl_W-state St
  using rtranclp-cdcl_W-all-inv(5)[OF st] no-d bt by simp
have 8: cdcl_W-conflicting St
  using rtranclp-cdcl_W-all-inv(6)[OF st] no-d bt by simp
have init-clss S' = init-clss St and conflicting S' = Some {#}
   using ⟨conflicting St ≠ None⟩ full-cdcl_W-init-clss-with-false-normal-form[OF 1, of - - St]
   2 3 4 5 6 7 8 St apply (metis ⟨cdcl_W-stgy** St S'⟩ rtranclp-cdcl_W-stgy-no-more-init-clss)
  using ⟨conflicting St ≠ None⟩ full-cdcl_W-init-clss-with-false-normal-form[OF 1, of - - St - -
    S'] 2 3 4 5 6 7 8 by (metis bt option.exhaust prod.inject)

moreover have init-clss S' = mset-clss N
```

```
    using ‹cdcl_W-stgy** (init-state N) S'› rtranclp-cdcl_W-stgy-no-more-init-clss by fastforce
  moreover have unsatisfiable (set-mset (mset-clss N))
    by (meson empty satisfiable-def true-cls-empty true-clss-def)
  ultimately show ?thesis by auto
qed


lemma full-cdcl_W-stgy-final-state-conclusive:
  fixes S' :: 'st
  assumes full: full cdcl_W-stgy (init-state N) S' and no-d: distinct-mset-mset (mset-clss N)
  shows (conflicting S' = Some {#} ∧ unsatisfiable (set-mset (init-clss S')))
    ∨ (conflicting S' = None ∧ trail S' ⊨asm init-clss S')
  using assms full-cdcl_W-stgy-final-state-conclusive-is-one-false
  full-cdcl_W-stgy-final-state-conclusive-non-false by blast

lemma full-cdcl_W-stgy-final-state-conclusive-from-init-state:
  fixes S' :: 'st
  assumes full: full cdcl_W-stgy (init-state N) S'
  and no-d: distinct-mset-mset (mset-clss N)
  shows (conflicting S' = Some {#} ∧ unsatisfiable (set-mset (mset-clss N)))
   ∨ (conflicting S' = None ∧ trail S' ⊨asm (mset-clss N) ∧ satisfiable (set-mset (mset-clss N)))
proof –
  have N: init-clss S' = (mset-clss N)
    using full unfolding full-def by (auto dest: rtranclp-cdcl_W-stgy-no-more-init-clss)
  consider
      (confl) conflicting S' = Some {#} and unsatisfiable (set-mset (init-clss S'))
    | (sat) conflicting S' = None and trail S' ⊨asm init-clss S'
    using full-cdcl_W-stgy-final-state-conclusive[OF assms] by auto
  then show ?thesis
    proof cases
      case confl
      then show ?thesis by (auto simp: N)
    next
      case sat
      have cdcl_W-M-level-inv (init-state N) by auto
      then have cdcl_W-M-level-inv S'
        using full rtranclp-cdcl_W-stgy-consistent-inv unfolding full-def by blast
      then have consistent-interp (lits-of-l (trail S')) unfolding cdcl_W-M-level-inv-def by blast
      moreover have lits-of-l (trail S') ⊨s set-mset (init-clss S')
        using sat(2) by (auto simp add: true-annots-def true-annot-def true-clss-def)
      ultimately have satisfiable (set-mset (init-clss S')) by simp
      then show ?thesis using sat unfolding N by blast
    qed
qed
end
end
theory CDCL-W-Termination
imports CDCL-W
begin

context conflict-driven-clause-learning_W
begin
```

19.6 Termination

The condition that no learned clause is a tautology is overkill (in the sense that the no-duplicate condition is enough), but we can reuse *simple-clss*.

The invariant contains all the structural invariants that holds,

```
definition cdcl_W-all-struct-inv where
  cdcl_W-all-struct-inv S ⟷
    no-strange-atm S  ∧
    cdcl_W-M-level-inv S ∧
    (∀ s ∈# learned-clss S. ¬tautology s) ∧
    distinct-cdcl_W-state S  ∧
    cdcl_W-conflicting S ∧
    all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S)) ∧
    cdcl_W-learned-clause S

lemma cdcl_W-all-struct-inv-inv:
  assumes cdcl_W  S S' and cdcl_W-all-struct-inv S
  shows cdcl_W-all-struct-inv S'
  unfolding cdcl_W-all-struct-inv-def
proof (intro HOL.conjI)
  show no-strange-atm S'
    using cdcl_W-all-inv[OF assms(1)] assms(2) unfolding cdcl_W-all-struct-inv-def by auto
  show cdcl_W-M-level-inv S'
    using cdcl_W-all-inv[OF assms(1)] assms(2) unfolding cdcl_W-all-struct-inv-def by fast
  show distinct-cdcl_W-state S'
     using cdcl_W-all-inv[OF assms(1)] assms(2) unfolding cdcl_W-all-struct-inv-def by fast
  show cdcl_W-conflicting S'
     using cdcl_W-all-inv[OF assms(1)] assms(2) unfolding cdcl_W-all-struct-inv-def by fast
  show all-decomposition-implies-m (init-clss S') (get-all-marked-decomposition (trail S'))
     using cdcl_W-all-inv[OF assms(1)] assms(2) unfolding cdcl_W-all-struct-inv-def by fast
  show cdcl_W-learned-clause S'
     using cdcl_W-all-inv[OF assms(1)] assms(2) unfolding cdcl_W-all-struct-inv-def by fast

  show ∀ s∈#learned-clss S'. ¬ tautology s
    using assms(1)[THEN learned-clss-are-not-tautologies] assms(2)
    unfolding cdcl_W-all-struct-inv-def by fast
qed

lemma rtranclp-cdcl_W-all-struct-inv-inv:
  assumes cdcl_W** S S' and cdcl_W-all-struct-inv S
  shows cdcl_W-all-struct-inv S'
  using assms by induction (auto intro: cdcl_W-all-struct-inv-inv)

lemma cdcl_W-stgy-cdcl_W-all-struct-inv:
  cdcl_W-stgy S T ⟹ cdcl_W-all-struct-inv S ⟹ cdcl_W-all-struct-inv T
  by (meson cdcl_W-stgy-tranclp-cdcl_W rtranclp-cdcl_W-all-struct-inv-inv rtranclp-unfold)

lemma rtranclp-cdcl_W-stgy-cdcl_W-all-struct-inv:
  cdcl_W-stgy** S T ⟹ cdcl_W-all-struct-inv S ⟹ cdcl_W-all-struct-inv T
  by (induction rule: rtranclp-induct) (auto intro: cdcl_W-stgy-cdcl_W-all-struct-inv)
```

19.7 No Relearning of a clause

```
lemma cdcl_W-o-new-clause-learned-is-backtrack-step:
  assumes learned: D ∈# learned-clss T and
  new: D ∉# learned-clss S and
  cdcl_W: cdcl_W-o S T and
  lev: cdcl_W-M-level-inv S
  shows backtrack S T ∧ conflicting S = Some D
  using cdcl_W lev learned new
proof (induction rule: cdcl_W-o-induct-lev2)
  case (backtrack K i M1 M2 L C T) note decomp = this(3) and undef = this(6) and andef = this(7)
and
    T = this(8) and D-T = this(9) and D-S = this(10)
  then have D = mset-ccls C
    using not-gr0 lev by (auto simp: cdcl_W-M-level-inv-decomp)
  then show ?case
    using T backtrack.hyps(1−5) backtrack.intros[OF backtrack.hyps(1,2)] backtrack.hyps(3−6)
    by auto
qed auto

lemma cdcl_W-cp-new-clause-learned-has-backtrack-step:
  assumes learned: D ∈# learned-clss T and
  new: D ∉# learned-clss S and
  cdcl_W: cdcl_W-stgy S T and
  lev: cdcl_W-M-level-inv S
  shows ∃S'. backtrack S S' ∧ cdcl_W-stgy** S' T ∧ conflicting S = Some D
  using cdcl_W learned new
proof (induction rule: cdcl_W-stgy.induct)
  case (conflict' S')
  then show ?case
    unfolding full1-def by (metis (mono-tags, lifting) rtranclp-cdcl_W-cp-learned-clause-inv
      tranclp-into-rtranclp)
next
  case (other' S' S'')
  then have D ∈# learned-clss S'
    unfolding full-def by (auto dest: rtranclp-cdcl_W-cp-learned-clause-inv)
  then show ?case
    using  cdcl_W-o-new-clause-learned-is-backtrack-step[OF - ⟨D ∉# learned-clss S⟩ ⟨cdcl_W-o S S'⟩]
    ⟨full cdcl_W-cp S' S''⟩ lev by (metis cdcl_W-stgy.conflict' full-unfold r-into-rtranclp
      rtranclp.rtrancl-refl)
qed

lemma rtranclp-cdcl_W-cp-new-clause-learned-has-backtrack-step:
  assumes learned: D ∈# learned-clss T and
  new: D ∉# learned-clss S and
  cdcl_W: cdcl_W-stgy** S T and
  lev: cdcl_W-M-level-inv S
  shows ∃S' S''. cdcl_W-stgy** S S' ∧ backtrack S' S'' ∧ conflicting S' = Some D ∧
    cdcl_W-stgy** S'' T
  using cdcl_W learned new
proof (induction rule: rtranclp-induct)
  case base
  then show ?case by blast
next
  case (step T U) note st = this(1) and o = this(2) and IH = this(3) and
    D-U = this(4) and D-S = this(5)
  show ?case
    proof (cases D ∈# learned-clss T)
      case True
      then obtain S' S'' where
        st': cdcl_W-stgy** S S' and
        bt: backtrack S' S'' and
        confl: conflicting S' = Some D and
        st'': cdcl_W-stgy** S'' T
        using IH D-S by metis
      have cdcl_W-stgy++ S'' U
        using st'' o by force
      then show ?thesis
        by (meson bt confl rtranclp-unfold st')
    next
      case False
      have cdcl_W-M-level-inv T
        using lev rtranclp-cdcl_W-stgy-consistent-inv st by blast
      then obtain S' where
        bt: backtrack T S' and
        st': cdcl_W-stgy** S' U and
        confl: conflicting T = Some D
        using cdcl_W-cp-new-clause-learned-has-backtrack-step[OF D-U False o]
         by metis
      then have cdcl_W-stgy** S T and
        backtrack T S' and
        conflicting T = Some D and
        cdcl_W-stgy** S' U
        using o st by auto
      then show ?thesis by blast
    qed
qed

lemma propagate-no-more-Marked-lit:
  assumes propagate S S'
  shows Marked K i ∈ set (trail S) ⟷ Marked K i ∈ set (trail S')
  using assms by (auto elim: propagateE)

lemma conflict-no-more-Marked-lit:
  assumes conflict S S'
  shows Marked K i ∈ set (trail S) ⟷ Marked K i ∈ set (trail S')
  using assms by (auto elim: conflictE)

lemma cdcl_W-cp-no-more-Marked-lit:
  assumes cdcl_W-cp S S'
  shows Marked K i ∈ set (trail S) ⟷ Marked K i ∈ set (trail S')
  using assms apply (induct rule: cdcl_W-cp.induct)
  using conflict-no-more-Marked-lit propagate-no-more-Marked-lit by auto

lemma rtranclp-cdcl_W-cp-no-more-Marked-lit:
  assumes cdcl_W-cp** S S'
  shows Marked K i ∈ set (trail S) ⟷ Marked K i ∈ set (trail S')
  using assms apply (induct rule: rtranclp-induct)
  using cdcl_W-cp-no-more-Marked-lit by blast+

lemma cdcl_W-o-no-more-Marked-lit:
  assumes cdcl_W-o S S' and lev: cdcl_W-M-level-inv S and ¬decide S S'
  shows Marked K i ∈ set (trail S') ⟶ Marked K i ∈ set (trail S)
  using assms
```

```
proof (induct rule: cdcl_W-o-induct-lev2)
  case backtrack note decomp = this(3) and undef = this(7) and T = this(8)
  then show ?case using lev by (auto simp: cdcl_W-M-level-inv-decomp)
next
  case (decide L T)
  then show ?case using decide-rule[OF decide.hyps] by blast
qed auto

lemma cdcl_W-new-marked-at-beginning-is-decide:
  assumes cdcl_W-stgy S S' and
  lev: cdcl_W-M-level-inv S and
  trail S' = M' @ Marked L i # M and
  trail S = M
  shows ∃ T. decide S T ∧ no-step cdcl_W-cp S
  using assms
proof (induct rule: cdcl_W-stgy.induct)
  case (conflict' S') note st = this(1) and no-dup = this(2) and S' = this(3) and S = this(4)
  have cdcl_W-M-level-inv S'
    using full1-cdcl_W-cp-consistent-inv no-dup st by blast
  then have Marked L i ∈ set (trail S') and Marked L i ∉ set (trail S)
    using no-dup unfolding S S' cdcl_W-M-level-inv-def by (auto simp add: rev-image-eqI)
  then have False
    using st rtranclp-cdcl_W-cp-no-more-Marked-lit[of S S']
    unfolding full1-def rtranclp-unfold by blast
  then show ?case by fast
next
  case (other' T U) note o = this(1) and ns = this(2) and st = this(3) and no-dup = this(4) and
    S' = this(5) and S = this(6)
  have cdcl_W-M-level-inv U
    by (metis (full-types) lev cdcl_W.simps cdcl_W-consistent-inv full-def o
      other'.hyps(3) rtranclp-cdcl_W-cp-consistent-inv)
  then have Marked L i ∈ set (trail U) and Marked L i ∉ set (trail S)
    using no-dup unfolding S S' cdcl_W-M-level-inv-def by (auto simp add: rev-image-eqI)
  then have Marked L i ∈ set (trail T)
    using st rtranclp-cdcl_W-cp-no-more-Marked-lit unfolding full-def by blast
  then show ?case
    using cdcl_W-o-no-more-Marked-lit[OF o] ‹Marked L i ∉ set (trail S)› ns lev by meson
qed

lemma cdcl_W-o-is-decide:
  assumes cdcl_W-o S T and lev: cdcl_W-M-level-inv S
  trail T = drop (length M_0) M' @ Marked L i # H @ Mand
  ¬ (∃ M'. trail S = M' @ Marked L i # H @ M)
  shows decide S T
  using assms
proof (induction rule:cdcl_W-o-induct-lev2)
  case (backtrack K i M1 M2 L D T)
  then obtain c where trail S = c @ M2 @ Marked K (Suc i) # M1
    by auto
  show ?case
    using backtrack lev
    apply (cases drop (length M_0) M')
     apply (auto simp: cdcl_W-M-level-inv-decomp)
    using ‹trail S = c @ M2 @ Marked K (Suc i) # M1›
    by (auto simp: cdcl_W-M-level-inv-decomp)
```

```
next
  case decide
  show ?case using decide-rule[of S] decide(1−4) by auto
qed auto

lemma rtranclp-cdcl_W-new-marked-at-beginning-is-decide:
  assumes cdcl_W-stgy** R U and
  trail U = M' @ Marked L i # H @ M and
  trail R = M and
  cdcl_W-M-level-inv R
  shows
    ∃S T T'. cdcl_W-stgy** R S ∧ decide S T ∧ cdcl_W-stgy** T U ∧ cdcl_W-stgy** S U ∧
      no-step cdcl_W-cp S ∧ trail T = Marked L i # H @ M ∧ trail S = H @ M ∧ cdcl_W-stgy S T' ∧
      cdcl_W-stgy** T' U
  using assms
proof (induct arbitrary: M H M' i rule: rtranclp-induct)
  case base
  then show ?case by auto
next
  case (step T U) note st = this(1) and IH = this(3) and s = this(2) and
    U = this(4) and S = this(5) and lev = this(6)
  show ?case
    proof (cases ∃M'. trail T = M' @ Marked L i # H @ M)
      case False
      with s show ?thesis using U s st S
        proof induction
          case (conflict' W) note cp = this(1) and  nd = this(2) and W = this(3)
          then obtain M_0 where trail W = M_0 @ trail T and nmarked: ∀l∈set M_0. ¬ is-marked l
            using rtranclp-cdcl_W-cp-dropWhile-trail unfolding full1-def rtranclp-unfold by meson
          then have MV: M' @ Marked L i # H @ M = M_0 @ trail T unfolding W by simp
          then have V: trail T = drop (length M_0) (M' @ Marked L i # H @ M)
            by auto
          have takeWhile (Not o is-marked) M' = M_0  @ takeWhile (Not ∘ is-marked) (trail T)
            using arg-cong[OF MV, of takeWhile (Not o is-marked)] nmarked
            by (simp add: takeWhile-tail)
          from arg-cong[OF this, of length] have length M_0 ≤ length M'
            unfolding length-append by (metis (no-types, lifting) Nat.le-trans le-add1
              length-takeWhile-le)
          then have False using nd V by auto
          then show ?case by fast
        next
          case (other' T' U) note o = this(1) and ns = this(2) and cp = this(3) and nd = this(4)
            and U = this(5) and st = this(6)
          obtain M_0 where trail U = M_0 @ trail T' and nmarked: ∀l∈set M_0. ¬ is-marked l
            using rtranclp-cdcl_W-cp-dropWhile-trail cp unfolding full-def by meson
          then have MV: M' @ Marked L i # H @ M = M_0 @ trail T' unfolding U by simp
          then have V: trail T' = drop (length M_0) (M' @ Marked L i # H @ M)
            by auto
          have takeWhile (Not o is-marked) M' = M_0  @ takeWhile (Not ∘ is-marked) (trail T')
            using arg-cong[OF MV, of takeWhile (Not o is-marked)] nmarked
            by (simp add: takeWhile-tail)
          from arg-cong[OF this, of length]  have length M_0 ≤ length M'
            unfolding length-append by (metis (no-types, lifting) Nat.le-trans le-add1
              length-takeWhile-le)
          then have tr-T': trail T' = drop (length M_0) M' @ Marked L i # H @ M using V by auto
          then have LT': Marked L i ∈ set (trail T') by auto
          moreover
            have cdcl_W-M-level-inv T
              using lev rtranclp-cdcl_W-stgy-consistent-inv step.hyps(1) by blast
            then have decide T T' using o nd tr-T' cdcl_W-o-is-decide by metis
          ultimately have decide T T' using cdcl_W-o-no-more-Marked-lit[OF o] by blast
          then have 1: cdcl_W-stgy** R T and 2: decide T T' and 3: cdcl_W-stgy** T' U
            using st other'.prems(4)
            by (metis cdcl_W-stgy.conflict' cp full-unfold r-into-rtranclp rtranclp.rtrancl-refl)+
          have [simp]: drop (length M_0) M' = []
            using ‹decide T T'› ‹Marked L i ∈ set (trail T')› nd tr-T'
            by (auto simp add: Cons-eq-append-conv elim: decideE)
          have T': drop (length M_0) M' @ Marked L i # H @ M = Marked L i # trail T
            using ‹decide T T'› ‹Marked L i ∈ set (trail T')› nd tr-T'
            by (auto elim: decideE)
          have trail T' = Marked L i # trail T
            using ‹decide T T'› ‹Marked L i ∈ set (trail T')› tr-T'
            by (auto elim: decideE)
          then have 5: trail T' = Marked L i # H @ M
              using append.simps(1) list.sel(3) local.other'(5) tl-append2 by (simp add: tr-T')
          have 6: trail T = H @ M
            by (metis (no-types) ‹trail T' = Marked L i # trail T›
              ‹trail T' = drop (length M_0) M' @ Marked L i # H @ M› append-Nil list.sel(3) nd
              tl-append2)
          have 7: cdcl_W-stgy** T U using other'.prems(4) st by auto
          have 8: cdcl_W-stgy T U cdcl_W-stgy** U U
            using cdcl_W-stgy.other'[OF other'(1–3)] by simp-all
          show ?case apply (rule exI[of - T], rule exI[of - T'], rule exI[of - U])
            using ns 1 2 3 5 6 7 8 by fast
        qed
    next
      case True
      then obtain M' where T: trail T = M' @ Marked L i # H @ M by metis
      from IH[OF this S lev] obtain S' S'' S''' where
        1: cdcl_W-stgy** R S' and
        2: decide S' S'' and
        3: cdcl_W-stgy** S'' T and
        4: no-step cdcl_W-cp S' and
        6: trail S'' = Marked L i # H @ M and
        7: trail S' = H @ M and
        8: cdcl_W-stgy** S' T and
        9: cdcl_W-stgy S' S''' and
        10: cdcl_W-stgy** S''' T
          by blast
      have cdcl_W-stgy** S'' U using s ‹cdcl_W-stgy** S'' T › by auto
      moreover have cdcl_W-stgy** S' U using 8 s by auto
      moreover have cdcl_W-stgy** S''' U using 10 s by auto
      ultimately show ?thesis apply – apply (rule exI[of - S'], rule exI[of - S''])
        using 1 2 4 6 7 8 9 by blast
    qed
qed

lemma rtranclp-cdcl_W-new-marked-at-beginning-is-decide':
  assumes cdcl_W-stgy** R U and
  trail U = M' @ Marked L i # H @ M and
  trail R = M and
  cdcl_W-M-level-inv R
  shows ∃ y y'. cdcl_W-stgy** R y ∧ cdcl_W-stgy y y' ∧ ¬ (∃ c. trail y = c @ Marked L i # H @ M)
    ∧ (λa b. cdcl_W-stgy a b ∧ (∃ c. trail a = c @ Marked L i # H @ M))** y' U
proof –
  fix T'
  obtain S' T T' where
    st: cdcl_W-stgy** R S' and
    decide S' T and
    TU: cdcl_W-stgy** T U and
    no-step cdcl_W-cp S' and
    trT: trail T = Marked L i # H @ M and
    trS': trail S' = H @ M and
    S'U: cdcl_W-stgy** S' U and
    S'T': cdcl_W-stgy S' T' and
    T'U: cdcl_W-stgy** T' U
    using rtranclp-cdcl_W-new-marked-at-beginning-is-decide[OF assms] by blast
  have n: ¬ (∃ c. trail S' = c @ Marked L i # H @ M) using trS' by auto
  show ?thesis
    using rtranclp-trans[OF st] rtranclp-exists-last-with-prop[of cdcl_W-stgy S' T' -
        λa -. ¬(∃ c. trail a = c @ Marked L i # H @ M), OF S'T' T'U n]
    by meson
qed

lemma beginning-not-marked-invert:
  assumes A: M @ A = M' @ Marked K i # H and
  nm: ∀ m∈set M. ¬is-marked m
  shows ∃ M. A = M @ Marked K i # H
proof –
  have A = drop (length M) (M' @ Marked K i # H)
    using arg-cong[OF A, of drop (length M)] by auto
  moreover have drop (length M) (M' @ Marked K i # H) = drop (length M) M' @ Marked K i # H
    using nm by (metis (no-types, lifting) A drop-Cons' drop-append marked-lit.disc(1) not-gr0
      nth-append nth-append-length nth-mem zero-less-diff)
  finally show ?thesis by fast
qed

lemma cdcl_W-stgy-trail-has-new-marked-is-decide-step:
  assumes cdcl_W-stgy S T
  ¬ (∃ c. trail S = c @ Marked L i # H @ M) and
  (λa b. cdcl_W-stgy a b ∧ (∃ c. trail a = c @ Marked L i # H @ M))** T U and
  ∃ M'. trail U = M' @ Marked L i # H @ M and
  lev: cdcl_W-M-level-inv S
  shows ∃ S'. decide S S' ∧ full cdcl_W-cp S' T ∧ no-step cdcl_W-cp S
  using assms(3,1,2,4,5)
proof induction
  case (step T U)
  then show ?case by fastforce
next
  case base
  then show ?case
    proof (induction rule: cdcl_W-stgy.induct)
      case (conflict' T) note cp = this(1) and nd = this(2) and M' = this(3) and no-dup = this(3)
      then obtain M' where M': trail T = M' @ Marked L i # H @ M by metis
      obtain M'' where M'': trail T = M'' @ trail S and nm: ∀ m∈ set M''. ¬is-marked m
        using cp unfolding full1-def
        by (metis rtranclp-cdcl_W-cp-dropWhile-trail' tranclp-into-rtranclp)
      have False
        using beginning-not-marked-invert[of M'' trail S M' L i H @ M] M' nm nd unfolding M''
        by fast
      then show ?case by fast
    next
      case (other' T U') note o = this(1) and ns = this(2) and cp = this(3) and nd = this(4)
        and trU' = this(5)
      have cdcl_W-cp^** T U' using cp unfolding full-def by blast
      from rtranclp-cdcl_W-cp-dropWhile-trail[OF this]
      have ∃M'. trail T = M' @ Marked L i # H @ M
        using  trU' beginning-not-marked-invert[of - trail T - L i H @ M] by metis
      then obtain M' where M': trail T = M' @ Marked L i # H @ M
        by auto
      with o lev nd cp ns
      show ?case
        proof (induction rule: cdcl_W-o-induct-lev2)
          case (decide L) note dec = this(1) and cp = this(5) and ns = this(4)
          then have decide S (cons-trail (Marked L (backtrack-lvl S +1)) (incr-lvl S))
            using decide.hyps decide.intros[of S] by force
          then show ?case using cp decide.prems by (meson decide-state-eq-compatible ns state-eq-ref
            state-eq-sym)
        next
          case (backtrack K j M1 M2 L' D T) note decomp = this(3)  and undef = this(7) and
            T = this(8) and trT = this(12)
          obtain MS3 where MS3: trail S = MS3 @ M2 @ Marked K (Suc j) # M1
            using get-all-marked-decomposition-exists-prepend[OF decomp] by metis
          have tl (M' @ Marked L i # H @ M) = tl M' @ Marked L i # H @ M
            using lev trT T lev undef decomp by (cases M') (auto simp: cdcl_W-M-level-inv-decomp)
          then have M'': M1 = tl M' @ Marked L i # H @ M
            using arg-cong[OF trT[simplified], of tl] T decomp undef lev
            by (simp add: cdcl_W-M-level-inv-decomp)
          have False using nd MS3 T undef decomp unfolding M'' by auto
          then show ?case by fast
        qed auto
      qed
qed

lemma rtranclp-cdcl_W-stgy-with-trail-end-has-trail-end:
  assumes (λa b. cdcl_W-stgy a b ∧ (∃c. trail a = c @ Marked L i # H @ M))^** T U and
  ∃M'. trail U = M' @ Marked L i # H @ M
  shows ∃M'. trail T = M' @ Marked L i # H @ M
  using assms by (induction rule: rtranclp-induct) auto

lemma remove1-mset-eq-remove1-mset-same:
  remove1-mset L D = remove1-mset L' D ⟹ L ∈# D ⟹ L = L'
  by (metis diff-single-trivial insert-DiffM multi-drop-mem-not-eq single-eq-single
    union-right-cancel)

lemma cdcl_W-o-cannot-learn:
  assumes
    cdcl_W-o y z and
    lev: cdcl_W-M-level-inv y and
    trM: trail y = c @ Marked Kh i # H and
    DL: D ∉# learned-clss y and
    LD: L ∈# D and
    DH: atms-of (remove1-mset L D) ⊆ atm-of ‘lits-of-l H and
    LH: atm-of L ∉ atm-of ‘lits-of-l H and
    learned: ∀ T. conflicting y = Some T ⟶ trail y ⊨as CNot T and
    z: trail z = c' @ Marked Kh i # H
  shows D ∉# learned-clss z
  using assms(1−2) trM DL DH LH learned z
proof (induction rule: cdcl_W-o-induct-lev2)
  case (backtrack K j M1 M2 L' D' T) note confl = this(1) and LD' = this(2) and decomp = this(3)
    and levL = this(4) and levD = this(5) and j = this(6) and undef = this(7) and T = this(8) and
    z = this(14)

  obtain M3 where M3: trail y = M3 @ M2 @ Marked K (Suc j) # M1
    using decomp get-all-marked-decomposition-exists-prepend by metis
  have M: trail y = c @ Marked Kh i # H using trM by simp
  have H: get-all-levels-of-marked (trail y) = rev [1..<1 + backtrack-lvl y]
    using lev unfolding cdcl_W-M-level-inv-def by auto
  have c' @ Marked Kh i # H = Propagated L' (mset-ccls D') # trail (reduce-trail-to M1 y)
    using z decomp undef T lev by (force simp: cdcl_W-M-level-inv-def)
  then obtain d where d: M1 = d @ Marked Kh i # H
    by (metis (no-types) decomp in-get-all-marked-decomposition-trail-update-trail list.inject
      list.sel(3) marked-lit.distinct(1) self-append-conv2 tl-append2)
  have i ∈ set (get-all-levels-of-marked (M3 @ M2 @ Marked K (Suc j) # d @ Marked Kh i # H))
    by auto
  then have i > 0 unfolding H[unfolded M3 d] by auto
  show ?case
    proof
      assume D ∈# learned-clss T
      then have DLD': D = mset-ccls D'
        using DL T neq0-conv undef decomp lev by (fastforce simp: cdcl_W-M-level-inv-def)
      have L-cKh: atm-of L ∈ atm-of ‘ lits-of-l (c @ [Marked Kh i])
        using LH learned  M DLD'[symmetric] confl LD' LD

        apply (auto simp add: image-iff dest!: in-CNot-implies-uminus)
        apply (metis atm-of-uminus)+ done
      have get-all-levels-of-marked (M3 @ M2 @ Marked K (j + 1) # M1)
        = rev [1..<1 + backtrack-lvl y]
        using lev unfolding cdcl_W-M-level-inv-def M3 by auto
      from arg-cong[OF this, of λa. (Suc j) ∈ set a] have backtrack-lvl y ≥ j by auto

      have DD'[simp]: remove1-mset L D = mset-ccls D' − {#L'#}
        proof (rule ccontr)
          assume DD': ¬ ?thesis
          then have L' ∈# remove1-mset L D using DLD' LD by (metis LD' in-remove1-mset-neq)
          then have get-level (trail y) L' ≤ get-maximum-level (trail y) (remove1-mset L D)
            using get-maximum-level-ge-get-level by blast
          moreover {
            have get-maximum-level (trail y) (remove1-mset L D) =
              get-maximum-level H (remove1-mset L D)
              using DH unfolding M by (simp add: get-maximum-level-skip-beginning)
            moreover
              have get-all-levels-of-marked (trail y) = rev [1..<1 + backtrack-lvl y]
                using lev unfolding cdcl_W-M-level-inv-def by auto
              then have get-all-levels-of-marked H = rev [1..< i]
                unfolding M by (auto dest: append-cons-eq-upt-length-i
                  simp add: rev-swap[symmetric])
              then have get-maximum-possible-level H < i
                using get-maximum-possible-level-max-get-all-levels-of-marked[of H] ‹i > 0› by auto
            ultimately have get-maximum-level (trail y) (remove1-mset L D) < i
              by (metis (full-types) dual-order.strict-trans nat-neq-iff not-le
                get-maximum-possible-level-ge-get-maximum-level) }
          moreover
            have L ∈# remove1-mset L' (mset-ccls D')
              using DLD'[symmetric] DD' LD by (metis in-remove1-mset-neq)
            then have get-maximum-level (trail y) (remove1-mset L' (mset-ccls D')) ≥
              get-level (trail y) L
              using get-maximum-level-ge-get-level by blast
          moreover {
            have get-all-levels-of-marked (c @ [Marked Kh i]) = rev [i..< backtrack-lvl y+1]
              using append-cons-eq-upt-length-i-end[of  rev (get-all-levels-of-marked H) i
                rev (get-all-levels-of-marked c) Suc 0 Suc (backtrack-lvl y)] H
              unfolding M apply (auto simp add: rev-swap[symmetric])
                by (metis (no-types, hide-lams) Nil-is-append-conv Suc-le-eq less-Suc-eq list.sel(1)
                  rev.simps(2) rev-rev-ident upt-Suc upt-rec)
            have get-level (trail y) L  = get-level (c @ [Marked Kh i]) L
              using L-cKh LH unfolding M by simp
            have get-level (c @ [Marked Kh i]) L ≥ i
              using L-cKh levL
                ‹get-all-levels-of-marked (c @ [Marked Kh i]) = rev [i..<backtrack-lvl y + 1]›
               calculation(1,2) by auto
            then have get-level (trail y) L ≥ i
              using M ‹get-level (trail y) L = get-level (c @ [Marked Kh i]) L› by auto }
          moreover
            have get-maximum-level (trail y) (remove1-mset L' (mset-ccls D'))
               < get-level (trail y) L
            using ‹j ≤ backtrack-lvl y› levL j calculation(1−4) by linarith
          ultimately show False using backtrack.hyps(4) by linarith
        qed
      then have LL': L = L'
        using LD LD' remove1-mset-eq-remove1-mset-same unfolding DLD'[symmetric] by fast
      have nd: no-dup (trail y) using lev unfolding cdcl_W-M-level-inv-def by auto

      { assume D: remove1-mset L (mset-ccls D') = {#}
        then have j: j = 0 using levD j by (simp add: LL')
        have ∀ m ∈ set M1. ¬is-marked m
          using H unfolding M3 j
          by (auto simp add: rev-swap[symmetric] get-all-levels-of-marked-no-marked
            dest!: append-cons-eq-upt-length-i)
        then have False using d by auto
      }
      moreover {
        assume D[simp]: remove1-mset L (mset-ccls D') ≠ {#}
        have i ≤ j
          using H unfolding M3 d by (auto simp add: rev-swap[symmetric]
            dest: upt-decomp-lt)
        have j > 0 apply (rule ccontr)
          using H ‹ i > 0› unfolding M3 d
          by (auto simp add: rev-swap[symmetric] dest!: upt-decomp-lt)
        obtain L'' where
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          L''∈# remove1-mset L (mset-ccls D') and
          L''D': get-level (trail y) L'' = get-maximum-level (trail y)
            (remove1-mset L (mset-ccls D'))
          using get-maximum-level-exists-lit-of-max-level[OF D, of trail y] by auto
        have L''M: atm-of L'' ∈ atm-of ‘ lits-of-l (trail y)
          using get-rev-level-ge-0-atm-of-in[of 0 rev (trail y) L''] ‹j>0› levD L''D'
          ‹j ≤ backtrack-lvl y› levL by (simp add: LL' j)
        then have L'' ∈ lits-of-l (Marked Kh i # d)
          proof −
            {
              assume L''H: atm-of L'' ∈ atm-of ‘ lits-of-l H
              have get-all-levels-of-marked H = rev [1..<i]
                using H unfolding M
                by (auto simp add: rev-swap[symmetric] dest!: append-cons-eq-upt-length-i)
              moreover have get-level (trail y) L'' = get-level H L''
                using L''H unfolding M by simp
              ultimately have False
                using levD ‹j>0› get-rev-level-in-levels-of-marked[of rev H 0 L''] ‹i ≤ j›
                unfolding L''D'[symmetric] nd
                by (metis L''D' LL' Max-n-upt Nat.le-trans One-nat-def Suc-pred ‹0 < i›
                  get-all-levels-of-marked-rev-eq-rev-get-all-levels-of-marked
                  get-rev-level-less-max-get-all-levels-of-marked j lessI list.simps(15)
                  not-less rev-rev-ident set-upt)
            }
            moreover
              have atm-of L'' ∈ atm-of ‘ lits-of-l H
                using DD' DH ‹L'' ∈# remove1-mset L (mset-ccls D')› atm-of-lit-in-atms-of LL' LD
                LD' by fastforce
            ultimately show ?thesis
              using DD' DH ‹L'' ∈# remove1-mset L (mset-ccls D')› atm-of-lit-in-atms-of
              by auto
          qed
        moreover
          have atm-of L'' ∈ atms-of ( remove1-mset L (mset-ccls D'))
            using ‹L''∈# remove1-mset L (mset-ccls D')› by (auto simp: atms-of-def)

          then have atm-of L'' ∈ atm-of ‘ lits-of-l H
            using DH unfolding DD' unfolding LL' by blast
        ultimately have False
          using nd unfolding M3 d LL' by (auto simp: lits-of-def)
      }
      ultimately show False by blast
    qed
qed auto

lemma cdcl_W-stgy-with-trail-end-has-not-been-learned:
  assumes
    cdcl_W-stgy y z and
    cdcl_W-M-level-inv y and
    trail y = c @ Marked Kh i # H and
    D ∉# learned-clss y and
    LD: L ∈# D and
    DH: atms-of (remove1-mset L D) ⊆ atm-of ‘ lits-of-l H and
    LH: atm-of L ∉ atm-of ‘ lits-of-l H and
    ∀ T. conflicting y = Some T ⟶ trail y ⊨as CNot T and
    trail z = c' @ Marked Kh i # H
  shows D ∉# learned-clss z
  using assms
proof induction
  case conflict'
  then show ?case
    unfolding full1-def using tranclp-cdcl_W-cp-learned-clause-inv by auto
next
  case (other' T U) note o = this(1) and cp = this(3) and lev = this(4) and trY = this(5) and
    notin = this(6) and LD = this(7) and DH = this(8) and LH = this(9) and confl = this(10) and
    trU = this(11)
  obtain c' where c': trail T = c' @ Marked Kh i # H
    using cp beginning-not-marked-invert[of - trail T c' Kh i H]
      rtranclp-cdcl_W-cp-dropWhile-trail[of T U] unfolding trU full-def by fastforce
  show ?case
    using cdcl_W-o-cannot-learn[OF o lev trY notin LD DH LH  confl c']
      rtranclp-cdcl_W-cp-learned-clause-inv cp unfolding full-def by auto
qed

lemma rtranclp-cdcl_W-stgy-with-trail-end-has-not-been-learned:
  assumes
    (λa b. cdcl_W-stgy a b ∧ (∃ c. trail a = c @ Marked K i # H @ []))** S z and
    cdcl_W-all-struct-inv S and
    trail S = c @ Marked K i # H and
    D ∉# learned-clss S and
    LD: L ∈# D and
    DH: atms-of (remove1-mset L D) ⊆ atm-of ‘ lits-of-l H and
    LH: atm-of L ∉ atm-of ‘ lits-of-l H and
    ∃ c'. trail z = c' @ Marked K i # H
  shows D ∉# learned-clss z
  using assms(1−4,8)
proof (induction rule: rtranclp-induct)
  case base
  then show ?case by auto[1]
next
  case (step T U) note st = this(1) and s = this(2) and IH = this(3)[OF this(4−6)]
    and lev = this(4) and trS = this(5) and DL-S = this(6) and trU = this(7)
  obtain c where c: trail T = c @ Marked K i # H using s by auto
  obtain c' where c': trail U = c' @ Marked K i # H using trU by blast
  have cdcl_W** S T
    proof −
      have ∀p pa. ∃ s sa. ∀ sb sc sd se. (¬ p** (sb::'st) sc ∨ p s sa ∨ pa** sb sc)
        ∧ (¬ pa s sa ∨ ¬ p** sd se ∨ pa** sd se)
        by (metis (no-types) mono-rtranclp)
      then have cdcl_W-stgy** S T
        using st by blast
      then show ?thesis
        using rtranclp-cdcl_W-stgy-rtranclp-cdcl_W by blast
    qed
  then have lev': cdcl_W-all-struct-inv T
    using rtranclp-cdcl_W-all-struct-inv-inv[of S T] lev by auto
  then have confl': ∀ Ta. conflicting T = Some Ta ⟶ trail T ⊨as CNot Ta
    unfolding cdcl_W-all-struct-inv-def cdcl_W-conflicting-def by blast
  show ?case
    apply (rule cdcl_W-stgy-with-trail-end-has-not-been-learned[OF - - c - LD DH LH confl' c'])
    using s lev' IH c unfolding cdcl_W-all-struct-inv-def by blast+
qed

lemma cdcl_W-stgy-new-learned-clause:
  assumes cdcl_W-stgy S T and
    lev: cdcl_W-M-level-inv S and
    E ∉# learned-clss S and
    E ∈# learned-clss T
  shows ∃S'. backtrack S S' ∧ conflicting S = Some E ∧ full cdcl_W-cp S' T
  using assms
proof induction
  case conflict'
  then show ?case unfolding full1-def by (auto dest: tranclp-cdcl_W-cp-learned-clause-inv)
next
  case (other' T U) note o = this(1) and cp = this(3) and not-yet = this(5) and learned = this(6)
  have E ∈# learned-clss T
    using learned cp rtranclp-cdcl_W-cp-learned-clause-inv unfolding full-def by auto
  then have backtrack S T and conflicting S = Some E
    using cdcl_W-o-new-clause-learned-is-backtrack-step[OF - not-yet o] lev by blast+
  then show ?case using cp by blast
qed

lemma cdcl_W-stgy-no-relearned-clause:
  assumes
    invR: cdcl_W-all-struct-inv R and
    st': cdcl_W-stgy** R S and
    bt: backtrack S T and
    confl: raw-conflicting S = Some E and
    already-learned: mset-ccls E ∈# clauses S and
    R: trail R = []
  shows False
proof -
  have M-lev: cdcl_W-M-level-inv R
    using invR unfolding cdcl_W-all-struct-inv-def by auto
  have cdcl_W-M-level-inv S
    using M-lev assms(2) rtranclp-cdcl_W-stgy-consistent-inv by blast
  with bt obtain L M1 M2-loc K i where
     T: T ∼ cons-trail (Propagated L (cls-of-ccls E))
       (reduce-trail-to M1 (add-learned-cls (cls-of-ccls E)
         (update-backtrack-lvl i (update-conflicting None S))))
      and
    decomp: (Marked K (Suc i) # M1, M2-loc) ∈
                set (get-all-marked-decomposition (trail S)) and
    LD: L ∈# mset-ccls E and
    k: get-level (trail S) L = backtrack-lvl S and
    level: get-level (trail S) L = get-maximum-level (trail S) (mset-ccls E) and
    confl-S: raw-conflicting S = Some E and
    i: i = get-maximum-level (trail S) (remove1-mset L (mset-ccls E)) and
    undef: undefined-lit M1 L
    using confl by (induction rule: backtrack-induction-lev2) fastforce
  obtain M2 where
    M: trail S = M2 @ Marked K (Suc i) # M1
   using get-all-marked-decomposition-exists-prepend[OF decomp] unfolding i by (metis append-assoc)
  let ?E = mset-ccls E
  let ?E' = remove1-mset L ?E
```

```
have invS: cdcl_W-all-struct-inv S
  using invR rtranclp-cdcl_W-all-struct-inv-inv rtranclp-cdcl_W-stgy-rtranclp-cdcl_W st' by blast
then have conf: cdcl_W-conflicting S unfolding cdcl_W-all-struct-inv-def by blast
then have trail S ⊨as CNot ?E unfolding cdcl_W-conflicting-def confl-S by auto
then have MD: trail S ⊨as CNot ?E by auto
then have MD': trail S ⊨as CNot ?E' using true-annot-CNot-diff by blast
have lev': cdcl_W-M-level-inv S using invS  unfolding cdcl_W-all-struct-inv-def by blast

have get-lvls-M: get-all-levels-of-marked (trail S) = rev [1..<Suc (backtrack-lvl S)]
  using lev' unfolding cdcl_W-M-level-inv-def by auto

have lev: cdcl_W-M-level-inv R using invR unfolding cdcl_W-all-struct-inv-def by blast
then have vars-of-D: atms-of ?E' ⊆ atm-of ‘ lits-of-l M1
  using backtrack-atms-of-D-in-M1[OF lev' undef - decomp - - - T] confl-S conf T decomp k level
  lev' i undef unfolding cdcl_W-conflicting-def by (auto simp: cdcl_W-M-level-inv-decomp)
have no-dup (trail S) using lev' by (auto simp: cdcl_W-M-level-inv-decomp)
have vars-in-M1:
  ∀ x ∈ atms-of ?E'. x ∉ atm-of ‘ lits-of-l (M2 @ [Marked K (i + 1)])
  unfolding Set.Ball-def apply (intro impI allI)
    apply (rule vars-of-D distinct-atms-of-incl-not-in-other[of
    M2 @ Marked K (i + 1) # [] M1 ?E'])
    using ‹no-dup (trail S)› M vars-of-D by simp-all
have M1-D: M1 ⊨as CNot ?E'
  using vars-in-M1 true-annots-remove-if-notin-vars[of M2 @ Marked K (i + 1) # [] M1 CNot ?E']
  MD' M by simp

have get-lvls-M: get-all-levels-of-marked (trail S) = rev [1..<Suc (backtrack-lvl S)]
  using lev' unfolding cdcl_W-M-level-inv-def by auto
then have backtrack-lvl S > 0 unfolding M by (auto split: if-split-asm simp add: upt.simps(2))

obtain M1' K' Ls where
  M': trail S = Ls @ Marked K' (backtrack-lvl S) # M1' and
  Ls: ∀ l ∈ set Ls. ¬ is-marked l and
  set M1 ⊆ set M1'
  proof -
    let ?Ls = takeWhile (Not o is-marked) (trail S)
    have MLs: trail S = ?Ls @ dropWhile (Not o is-marked) (trail S)
      by auto
    have dropWhile (Not o is-marked) (trail S) ≠ [] unfolding M by auto
    moreover
      from hd-dropWhile[OF this] have is-marked(hd (dropWhile (Not o is-marked) (trail S)))
        by simp
    ultimately
      obtain K' K'k where
        K'k: dropWhile (Not o is-marked) (trail S)
          = Marked K' K'k # tl (dropWhile (Not o is-marked) (trail S))
        by (cases dropWhile (Not ∘ is-marked) (trail S);
            cases hd (dropWhile (Not ∘ is-marked) (trail S)))
          simp-all
    moreover have ∀ l ∈ set ?Ls. ¬is-marked l using set-takeWhileD by force
    moreover
      have get-all-levels-of-marked (trail S)
              = K'k # get-all-levels-of-marked(tl (dropWhile (Not ∘ is-marked) (trail S)))
        apply (subst MLs, subst K'k)
        using calculation(2) by (auto simp add: get-all-levels-of-marked-no-marked)
      then have K'k =  backtrack-lvl S
      using calculation(2) by (auto split: if-split-asm simp add: get-lvls-M upt.simps(2))
    moreover have set M1 ⊆ set (tl (dropWhile (Not o is-marked) (trail S)))
      unfolding M by (induction M2) auto
    ultimately show ?thesis using that MLs by metis
  qed

have get-lvls-M: get-all-levels-of-marked (trail S) = rev [1..<Suc (backtrack-lvl S)]
  using lev' unfolding cdcl_W-M-level-inv-def by auto
then have backtrack-lvl S > 0 unfolding M by (auto split: if-split-asm simp add: upt.simps(2) i)

have M1'-D: M1' ⊨as CNot ?E' using M1-D ‹set M1 ⊆ set M1'› by (auto intro: true-annots-mono)
have −L ∈ lits-of-l (trail S) using conf confl-S LD unfolding cdcl_W-conflicting-def
  by (auto simp: in-CNot-implies-uminus)
have lvls-M1': get-all-levels-of-marked M1' = rev [1..<backtrack-lvl S]
  using get-lvls-M Ls by (auto simp add: get-all-levels-of-marked-no-marked M' upt.simps(2)
    split: if-split-asm)
have L-notin: atm-of L ∈ atm-of ` lits-of-l Ls ∨ atm-of L = atm-of K'
  proof (rule ccontr)
    assume ¬ ?thesis
    then have atm-of L ∉ atm-of ` lits-of-l (Marked K' (backtrack-lvl S) # rev Ls) by simp
    then have get-level (trail S) L = get-level M1' L
      unfolding M' by auto
    then show False using get-level-in-levels-of-marked[of M1' L] ‹backtrack-lvl S > 0›
    unfolding k lvls-M1' by auto
  qed
obtain Y Z where
  RY: cdcl_W-stgy** R Y and
  YZ: cdcl_W-stgy Y Z and
  nt: ¬ (∃ c. trail Y = c @ Marked K' (backtrack-lvl S) # M1' @ []) and
  Z: (λa b. cdcl_W-stgy a b ∧ (∃ c. trail a = c @ Marked K' (backtrack-lvl S) # M1' @ []))** Z S
  using rtranclp-cdcl_W-new-marked-at-beginning-is-decide'[OF  st' - - lev, of Ls K'
    backtrack-lvl S M1' []] unfolding R M' by auto
have [simp]: cdcl_W-M-level-inv Y
  using RY lev rtranclp-cdcl_W-stgy-consistent-inv by blast
obtain M' where trZ: trail Z = M' @ Marked K' (backtrack-lvl S) # M1'
  using rtranclp-cdcl_W-stgy-with-trail-end-has-trail-end[OF Z] M' by auto
have no-dup (trail Y)
  using RY lev rtranclp-cdcl_W-stgy-consistent-inv unfolding cdcl_W-M-level-inv-def by blast
then obtain Y' where
  dec: decide Y Y' and
  Y'Z: full cdcl_W-cp Y' Z and
  no-step cdcl_W-cp Y
  using cdcl_W-stgy-trail-has-new-marked-is-decide-step[OF YZ nt Z] M' by auto
have trY: trail Y = M1'
  proof −
    obtain M' where M: trail Z = M' @ Marked K' (backtrack-lvl S) # M1'
      using rtranclp-cdcl_W-stgy-with-trail-end-has-trail-end[OF Z] M' by auto
    obtain M'' where M'': trail Z = M'' @ trail Y' and ∀ m∈set M''. ¬is-marked m
      using Y'Z rtranclp-cdcl_W-cp-dropWhile-trail' unfolding full-def by blast
    obtain M''' where trail Y' = M''' @ Marked K' (backtrack-lvl S) # M1'
      using M'' unfolding M
      by (metis (no-types, lifting) ‹∀ m∈set M''. ¬ is-marked m› beginning-not-marked-invert)
    then show ?thesis using dec nt  by (induction M''') (auto elim: decideE)
  qed
  have Y-CT: conflicting Y = None using ‹decide Y Y'› by (auto elim: decideE)
  have cdcl_W** R Y by (simp add: RY rtranclp-cdcl_W-stgy-rtranclp-cdcl_W)
  then have init-clss Y = init-clss R using rtranclp-cdcl_W-init-clss[of R Y] M-lev by auto
  { assume DL: mset-ccls E ∈# clauses Y
    have atm-of L ∉ atm-of ‘ lits-of-l M1
      apply (rule backtrack-lit-skiped[of S])
      using decomp i k lev' unfolding cdcl_W-M-level-inv-def by auto
    then have LM1: undefined-lit M1 L
      by (metis Marked-Propagated-in-iff-in-lits-of-l atm-of-uminus image-eqI)
    have L-trY: undefined-lit (trail Y) L
      using  L-notin ‹no-dup (trail S)› unfolding defined-lit-map trY M'
      by (auto simp add: image-iff lits-of-def)
    obtain E' where
      E': E' !∈! raw-clauses Y and
      EE': mset-cls E' = mset-ccls E
      using DL in-mset-clss-exists-preimage by blast
    have Ex (propagate Y)
      using propagate-rule[of Y E' L] DL M1'-D L-trY Y-CT trY LD E'
      by (auto simp: EE')
    then have False using ‹no-step cdcl_W-cp Y›  propagate' by blast
  }
  moreover {
    assume DL: mset-ccls E ∉# clauses Y
    have lY-lZ: learned-clss Y = learned-clss Z
      using dec Y'Z rtranclp-cdcl_W-cp-learned-clause-inv[of Y' Z] unfolding full-def
      by (auto elim: decideE)
    have invZ: cdcl_W-all-struct-inv Z
      by (meson RY YZ invR r-into-rtranclp rtranclp-cdcl_W-all-struct-inv-inv
        rtranclp-cdcl_W-stgy-rtranclp-cdcl_W)
    have n: mset-ccls E ∉# learned-clss Z
       using DL lY-lZ YZ unfolding raw-clauses-def by auto
    have ?E ∉#learned-clss S
      apply (rule rtranclp-cdcl_W-stgy-with-trail-end-has-not-been-learned[OF Z invZ trZ])
          apply (simp add: n)
         using LD apply simp
        apply (metis (no-types, lifting) ‹set M1 ⊆ set M1'› image-mono order-trans
          vars-of-D lits-of-def)
       using L-notin ‹no-dup (trail S)› unfolding M' by (auto simp add: image-iff lits-of-def)
    then have False
      using already-learned DL confl st' M-lev rtranclp-cdcl_W-stgy-no-more-init-clss[of R S]
      unfolding M'
      by (simp add: ‹init-clss Y = init-clss R› raw-clauses-def confl-S
        rtranclp-cdcl_W-stgy-no-more-init-clss)
  }
  ultimately show False by blast
qed

lemma rtranclp-cdcl_W-stgy-distinct-mset-clauses:
  assumes
    invR: cdcl_W-all-struct-inv R and
    st: cdcl_W-stgy** R S and
    dist: distinct-mset (clauses R) and
    R: trail R = []
  shows distinct-mset (clauses S)
  using st
```

```
proof (induction)
  case base
  then show ?case using dist by simp
next
  case (step S T) note st = this(1) and s = this(2) and IH = this(3)
  from s show ?case
    proof (cases rule: cdcl_W-stgy.cases)
      case conflict'
      then show ?thesis
        using IH unfolding full1-def by (auto dest: tranclp-cdcl_W-cp-no-more-clauses)
    next
      case (other' S') note o = this(1) and full = this(3)
      have [simp]: clauses T = clauses S'
        using full unfolding full-def by (auto dest: rtranclp-cdcl_W-cp-no-more-clauses)
      show ?thesis
        using o IH
        proof (cases rule: cdcl_W-o-rule-cases)
          case backtrack
          moreover
            have cdcl_W-all-struct-inv S
              using invR rtranclp-cdcl_W-stgy-cdcl_W-all-struct-inv st by blast
            then have cdcl_W-M-level-inv S
              unfolding cdcl_W-all-struct-inv-def by auto
          ultimately obtain E where
            conflicting S = Some E and
            cls-S': clauses S' = {#E#} + clauses S
            using ‹cdcl_W-M-level-inv S›
            by (induction rule: backtrack-induction-lev2) (auto simp: cdcl_W-M-level-inv-decomp)
          then have E ∉# clauses S
            using cdcl_W-stgy-no-relearned-clause R invR local.backtrack st by blast
          then show ?thesis using IH by (simp add: distinct-mset-add-single cls-S')
        qed (auto elim: decideE skipE resolveE)
    qed
qed

lemma cdcl_W-stgy-distinct-mset-clauses:
  assumes
    st: cdcl_W-stgy** (init-state N) S and
    no-duplicate-clause: distinct-mset (mset-clss N) and
    no-duplicate-in-clause: distinct-mset-mset (mset-clss N)
  shows distinct-mset (clauses S)
  using rtranclp-cdcl_W-stgy-distinct-mset-clauses[OF - st] assms
  by (auto simp: cdcl_W-all-struct-inv-def distinct-cdcl_W-state-def)
```

19.8 Decrease of a measure

```
fun cdcl_W-measure where
cdcl_W-measure S =
  [(3::nat) ^ (card (atms-of-mm (init-clss S))) − card (set-mset (learned-clss S)),
    if conflicting S = None then 1 else 0,
    if conflicting S = None then card (atms-of-mm (init-clss S)) − length (trail S)
    else length (trail S)
    ]

lemma length-model-le-vars-all-inv:
  assumes cdcl_W-all-struct-inv S
  shows length (trail S) ≤ card (atms-of-mm (init-clss S))
  using assms length-model-le-vars[of S] unfolding cdcl_W-all-struct-inv-def
  by (auto simp: cdcl_W-M-level-inv-decomp)
end

context conflict-driven-clause-learning_W
begin

lemma learned-clss-less-upper-bound:
  fixes S :: 'st
  assumes
    distinct-cdcl_W-state S and
    ∀ s ∈# learned-clss S. ¬tautology s
  shows card(set-mset (learned-clss S)) ≤ 3 ^ card (atms-of-mm (learned-clss S))
proof –
  have set-mset (learned-clss S) ⊆ simple-clss (atms-of-mm (learned-clss S))
    apply (rule simplified-in-simple-clss)
    using assms unfolding distinct-cdcl_W-state-def by auto
  then have card(set-mset (learned-clss S))
    ≤ card (simple-clss (atms-of-mm (learned-clss S)))
    by (simp add: simple-clss-finite card-mono)
  then show ?thesis
    by (meson atms-of-ms-finite simple-clss-card finite-set-mset order-trans)
qed


lemma cdcl_W-measure-decreasing:
  fixes S :: 'st
  assumes
    cdcl_W S S' and
    no-restart:
      ¬(learned-clss S ⊆# learned-clss S' ∧ [] = trail S' ∧ conflicting S' = None)
     and
    no-forget: learned-clss S ⊆# learned-clss S'  and
    no-relearn: ⋀S'. backtrack S S' ⟹ ∀ T. conflicting S = Some T ⟶ T ∉# learned-clss S
      and
    alien: no-strange-atm S and
    M-level: cdcl_W-M-level-inv S and
    no-taut: ∀ s ∈# learned-clss S. ¬tautology s and
    no-dup: distinct-cdcl_W-state S and
    confl: cdcl_W-conflicting S
  shows (cdcl_W-measure S', cdcl_W-measure S) ∈ lexn {(a, b). a < b} 3
  using assms(1) M-level assms(2,3)
proof (induct rule: cdcl_W-all-induct-lev2)
  case (propagate C L) note conf = this(1) and undef = this(5) and T = this(6)
  have propa: propagate S (cons-trail (Propagated L C) S)
    using propagate-rule[OF propagate.hyps(1,2)] propagate.hyps by auto
  then have no-dup': no-dup (Propagated L (mset-cls C) # trail S)
    using M-level cdcl_W-M-level-inv-decomp(2) undef defined-lit-map by auto

  let ?N = init-clss S
  have no-strange-atm (cons-trail (Propagated L C) S)
    using alien cdcl_W.propagate cdcl_W-no-strange-atm-inv propa M-level by blast
  then have  atm-of ` lits-of-l (Propagated L (mset-cls C) # trail S)
    ⊆ atms-of-mm (init-clss S)
    using undef unfolding no-strange-atm-def by auto
  then have card (atm-of ` lits-of-l (Propagated L (mset-cls C) # trail S))
    ≤ card (atms-of-mm (init-clss S))
    by (meson atms-of-ms-finite card-mono finite-set-mset)
  then have length (Propagated L (mset-cls C) # trail S) ≤ card (atms-of-mm ?N)
    using no-dup-length-eq-card-atm-of-lits-of-l no-dup' by fastforce
  then have H: card (atms-of-mm (init-clss S)) − length (trail S)
    = Suc (card (atms-of-mm (init-clss S)) − Suc (length (trail S)))
    by simp
  show ?case using conf T undef by (auto simp: H lexn3-conv)
next
  case (decide L) note conf = this(1) and undef = this(2) and T = this(4)
  moreover
    have dec: decide S (cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S))
      using decide-rule decide.hyps by force
    then have cdcl_W:cdcl_W S (cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S))
      using cdcl_W.simps cdcl_W-o.intros by blast
  moreover
    have lev: cdcl_W-M-level-inv (cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S))
      using cdcl_W M-level cdcl_W-consistent-inv[OF cdcl_W] by auto
    then have no-dup: no-dup (Marked L (backtrack-lvl S + 1) # trail S)
      using undef unfolding cdcl_W-M-level-inv-def by auto
    have no-strange-atm (cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S))
      using M-level alien calculation(4) cdcl_W-no-strange-atm-inv by blast
    then have length (Marked L ((backtrack-lvl S) + 1) # (trail S))
      ≤ card (atms-of-mm (init-clss S))
      using no-dup undef
      length-model-le-vars[of cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S)]
      by fastforce
  ultimately show ?case using conf by (simp add: lexn3-conv)
next
  case (skip L C' M D) note tr = this(1) and conf = this(2) and T = this(5)
  show ?case using conf T by (simp add: tr lexn3-conv)
next
  case conflict
  then show ?case by (simp add: lexn3-conv)
next
  case resolve
  then show ?case using finite by (simp add: lexn3-conv)
next
  case (backtrack K i M1 M2 L D T) note conf = this(1) and decomp = this(3) and undef = this(7)
and
    T = this(8) and lev = this(9)
  let ?S' = T
  have bt: backtrack S ?S'
    using backtrack.hyps backtrack.intros[of S D L K i] by auto
  have mset-ccls D ∉# learned-clss S
    using no-relearn conf bt by auto
  then have card-T:
    card (set-mset ({#mset-ccls D#} + learned-clss S)) = Suc (card (set-mset (learned-clss S)))
    by simp
  have distinct-cdcl_W-state ?S'
    using bt M-level distinct-cdcl_W-state-inv no-dup other cdcl_W-o.intros cdcl_W-bj.intros by blast
  moreover have ∀ s∈#learned-clss ?S'. ¬ tautology s
    using learned-clss-are-not-tautologies[OF cdcl_W.other[OF cdcl_W-o.bj[OF
```

```
      cdcl_W-bj.backtrack[OF bt]]]] M-level no-taut confl by auto
  ultimately have card (set-mset (learned-clss T)) ≤ 3 ^ card (atms-of-mm (learned-clss T))
      by (auto simp: learned-clss-less-upper-bound)
    then have H: card (set-mset ({#mset-ccls D#} + learned-clss S))
      ≤ 3 ^ card (atms-of-mm ({#mset-ccls D#} + learned-clss S))
      using T undef decomp M-level by (simp add: cdcl_W-M-level-inv-decomp)
  moreover
    have atms-of-mm ({#mset-ccls D#} + learned-clss S) ⊆ atms-of-mm (init-clss S)
      using alien conf unfolding no-strange-atm-def by auto
    then have card-f: card (atms-of-mm ({#mset-ccls D#} + learned-clss S))
      ≤ card (atms-of-mm (init-clss S))
      by (meson atms-of-ms-finite card-mono finite-set-mset)
    then have (3::nat) ^ card (atms-of-mm ({#mset-ccls D#} + learned-clss S))
      ≤ 3 ^ card (atms-of-mm (init-clss S)) by simp
  ultimately have (3::nat) ^ card (atms-of-mm (init-clss S))
    ≥ card (set-mset ({#mset-ccls D#} + learned-clss S))
    using le-trans by blast
  then show ?case using decomp undef diff-less-mono2 card-T T M-level
    by (auto simp: cdcl_W-M-level-inv-decomp lexn3-conv)
next
  case restart
  then show ?case using alien by (auto simp: state-eq-def simp del: state-simp)
next
  case (forget C T) note no-forget = this(8)
  then have mset-cls C ∈# learned-clss S and mset-cls C ∉# learned-clss T
    using forget.hyps by auto
  then show ?case using no-forget by (auto simp add: mset-leD)
qed

lemma propagate-measure-decreasing:
  fixes S :: 'st
  assumes propagate S S' and cdcl_W-all-struct-inv S
  shows (cdcl_W-measure S', cdcl_W-measure S) ∈ lexn {(a, b). a < b} 3
  apply (rule cdcl_W-measure-decreasing)
  using assms(1) propagate apply blast
          using assms(1) apply (auto simp add: propagate.simps)[3]
       using assms(2) apply (auto simp add: cdcl_W-all-struct-inv-def)
  done

lemma conflict-measure-decreasing:
  fixes S :: 'st
  assumes conflict S S' and cdcl_W-all-struct-inv S
  shows (cdcl_W-measure S', cdcl_W-measure S) ∈ lexn {(a, b). a < b} 3
  apply (rule cdcl_W-measure-decreasing)
  using assms(1) conflict apply blast
           using assms(1) apply (auto simp: state-eq-def simp del: state-simp elim!: conflictE)[3]
        using assms(2) apply (auto simp add: cdcl_W-all-struct-inv-def elim: conflictE)
  done

lemma decide-measure-decreasing:
  fixes S :: 'st
  assumes decide S S' and cdcl_W-all-struct-inv S
  shows (cdcl_W-measure S', cdcl_W-measure S) ∈ lexn {(a, b). a < b} 3
  apply (rule cdcl_W-measure-decreasing)
  using assms(1) decide other apply blast
```

```
      using assms(1) apply (auto simp: state-eq-def simp del: state-simp elim!: decideE)[3]
    using assms(2) apply (auto simp add: cdcl_W-all-struct-inv-def elim: decideE)
  done

lemma trans-le:
  trans {(a, (b::nat)). a < b}
  unfolding trans-def by auto

lemma cdcl_W-cp-measure-decreasing:
  fixes S :: 'st
  assumes cdcl_W-cp S S' and cdcl_W-all-struct-inv S
  shows (cdcl_W-measure S', cdcl_W-measure S) ∈ lexn {(a, b). a < b} 3
  using assms
proof induction
  case conflict'
  then show ?case using conflict-measure-decreasing by blast
next
  case propagate'
  then show ?case using propagate-measure-decreasing by blast
qed

lemma tranclp-cdcl_W-cp-measure-decreasing:
  fixes S :: 'st
  assumes cdcl_W-cp⁺⁺ S S' and cdcl_W-all-struct-inv S
  shows (cdcl_W-measure S', cdcl_W-measure S) ∈ lexn {(a, b). a < b} 3
  using assms
proof induction
  case base
  then show ?case using cdcl_W-cp-measure-decreasing by blast
next
  case (step T U) note st = this(1) and step = this(2) and IH = this(3) and inv = this(4)
  then have (cdcl_W-measure T, cdcl_W-measure S) ∈ lexn {a. case a of (a, b) ⇒ a < b} 3 by blast

  moreover have (cdcl_W-measure U, cdcl_W-measure T) ∈ lexn {a. case a of (a, b) ⇒ a < b} 3
    using cdcl_W-cp-measure-decreasing[OF step] rtranclp-cdcl_W-all-struct-inv-inv inv
    tranclp-cdcl_W-cp-tranclp-cdcl_W[OF st]
    unfolding trans-def rtranclp-unfold
    by blast
  ultimately show ?case using lexn-transI[OF trans-le] unfolding trans-def by blast
qed

lemma cdcl_W-stgy-step-decreasing:
  fixes R S T :: 'st
  assumes cdcl_W-stgy S T and
  cdcl_W-stgy** R S
  trail R = [] and
  cdcl_W-all-struct-inv R
  shows (cdcl_W-measure T, cdcl_W-measure S) ∈ lexn {(a, b). a < b} 3
proof –
  have cdcl_W-all-struct-inv S
    using assms
    by (metis rtranclp-unfold rtranclp-cdcl_W-all-struct-inv-inv tranclp-cdcl_W-stgy-tranclp-cdcl_W)
  with assms show ?thesis
    proof induction
      case (conflict' V) note cp = this(1) and inv = this(5)
      show ?case
        using tranclp-cdcl_W-cp-measure-decreasing[OF HOL.conjunct1[OF cp[unfolded full1-def]] inv]
        .
    next
      case (other' T U) note st = this(1) and H = this(4,5,6,7) and cp = this(3)
      have cdcl_W-all-struct-inv T
        using cdcl_W-all-struct-inv-inv other other'.hyps(1) other'.prems(4) by blast
      from tranclp-cdcl_W-cp-measure-decreasing[OF - this]
      have le-or-eq: (cdcl_W-measure U, cdcl_W-measure T) ∈ lexn {a. case a of (a, b) ⇒ a < b} 3 ∨
        cdcl_W-measure U = cdcl_W-measure T
        using cp unfolding full-def rtranclp-unfold by blast
      moreover
        have cdcl_W-M-level-inv S
          using cdcl_W-all-struct-inv-def other'.prems(4) by blast
        with st have (cdcl_W-measure T, cdcl_W-measure S) ∈ lexn {a. case a of (a, b) ⇒ a < b} 3
        proof (induction rule:cdcl_W-o-induct-lev2)
          case (decide T)
          then show ?case using decide-measure-decreasing H decide.intros[OF decide.hyps] by blast
        next
          case (backtrack K i M1 M2 L D T) note conf = this(1) and decomp = this(3) and
            undef = this(7) and T = this(8)
          have bt: backtrack S T
            apply (rule backtrack-rule)
            using backtrack.hyps by auto
          then have no-relearn: ∀ T. conflicting S = Some T ⟶ T ∉# learned-clss S
            using cdcl_W-stgy-no-relearned-clause[of R S T] H conf
            unfolding cdcl_W-all-struct-inv-def raw-clauses-def by auto
          have inv: cdcl_W-all-struct-inv S
            using ‹cdcl_W-all-struct-inv S› by blast
          show ?case
            apply (rule cdcl_W-measure-decreasing)
                    using bt cdcl_W-bj.backtrack cdcl_W-o.bj other apply simp
                   using bt T undef decomp inv unfolding cdcl_W-all-struct-inv-def
                   cdcl_W-M-level-inv-def apply auto[]
                  using bt T undef decomp inv unfolding cdcl_W-all-struct-inv-def
                   cdcl_W-M-level-inv-def apply auto[]
                 using bt no-relearn apply auto[]
                using inv unfolding cdcl_W-all-struct-inv-def apply simp
               using inv unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def apply simp
              using inv unfolding cdcl_W-all-struct-inv-def apply simp
             using inv unfolding cdcl_W-all-struct-inv-def apply simp
            using inv unfolding cdcl_W-all-struct-inv-def by simp
        next
          case skip
          then show ?case by (auto simp: lexn3-conv)
        next
          case resolve
          then show ?case by (auto simp: lexn3-conv)
        qed
      ultimately show ?case
        by (metis (full-types) lexn-transI transD trans-le)
    qed
qed

lemma tranclp-cdcl_W-stgy-decreasing:
  fixes R S T :: 'st
  assumes cdcl_W-stgy^++ R S
  trail R = [] and
  cdcl_W-all-struct-inv R
  shows (cdcl_W-measure S, cdcl_W-measure R) ∈ lexn {(a, b). a < b} 3
  using assms
  apply induction
   using cdcl_W-stgy-step-decreasing[of R - R] apply blast
  using cdcl_W-stgy-step-decreasing[of - - R]  tranclp-into-rtranclp[of cdcl_W-stgy R]
  lexn-transI[OF trans-le, of 3] unfolding trans-def by blast

lemma tranclp-cdcl_W-stgy-S0-decreasing:
  fixes R S T :: 'st
  assumes
    pl: cdcl_W-stgy^++ (init-state N) S and
    no-dup: distinct-mset-mset (mset-clss N)
  shows (cdcl_W-measure S, cdcl_W-measure (init-state N)) ∈ lexn {(a, b). a < b} 3
proof -
  have cdcl_W-all-struct-inv (init-state N)
    using no-dup unfolding cdcl_W-all-struct-inv-def by auto
  then show ?thesis using pl tranclp-cdcl_W-stgy-decreasing init-state-trail by blast
qed

lemma wf-tranclp-cdcl_W-stgy:
  wf {(S::'st, init-state N)|
     S N. distinct-mset-mset (mset-clss N) ∧ cdcl_W-stgy^++ (init-state N) S}
  apply (rule wf-wf-if-measure'-notation2[of lexn {(a, b). a < b} 3 - - cdcl_W-measure])
   apply (simp add: wf wf-lexn)
  using tranclp-cdcl_W-stgy-S0-decreasing by blast

lemma cdcl_W-cp-wf-all-inv:
  wf {(S′, S). cdcl_W-all-struct-inv S ∧ cdcl_W-cp S S′}
  (is wf ?R)
proof (rule wf-bounded-measure[of -
    λS. card (atms-of-mm (init-clss S))+1
    λS. length (trail S) + (if conflicting S = None then 0 else 1)], goal-cases)
  case (1 S S′)
  then have cdcl_W-all-struct-inv S and cdcl_W-cp S S′ by auto
  moreover then have cdcl_W-all-struct-inv S′
    using cdcl_W-cp.simps  cdcl_W-all-struct-inv-inv conflict cdcl_W.intros cdcl_W-all-struct-inv-inv
    by blast+
  ultimately show ?case
    by (auto simp:cdcl_W-cp.simps state-eq-def simp del: state-simp elim!: conflictE propagateE
      dest: length-model-le-vars-all-inv)
qed

end

end
theory DPLL-CDCL-W-Implementation
imports Partial-Annotated-Clausal-Logic
begin
```

20 Simple Implementation of the DPLL and CDCL

20.1 Common Rules

20.1.1 Propagation

The following theorem holds:

```
lemma lits-of-l-unfold[iff]:
  (∀ c ∈ set C. −c ∈ lits-of-l Ms) ⟷ Ms ⊨as CNot (mset C)
  unfolding true-annots-def Ball-def true-annot-def CNot-def by auto
```

The right-hand version is written at a high-level, but only the left-hand side is executable.

```
definition is-unit-clause :: 'a literal list ⇒ ('a, 'b, 'c) marked-lit list ⇒ 'a literal option
 where
 is-unit-clause l M =
   (case List.filter (λa. atm-of a ∉ atm-of ‘ lits-of-l M) l of
     a # [] ⇒ if M ⊨as CNot (mset l − {#a#}) then Some a else None
   | - ⇒ None)

definition is-unit-clause-code :: 'a literal list ⇒ ('a, 'b, 'c) marked-lit list
  ⇒ 'a literal option where
 is-unit-clause-code l M =
   (case List.filter (λa. atm-of a ∉ atm-of ‘ lits-of-l M) l of
     a # [] ⇒ if (∀ c ∈set (remove1 a l). −c ∈ lits-of-l M) then Some a else None
   | - ⇒ None)

lemma is-unit-clause-is-unit-clause-code[code]:
  is-unit-clause l M = is-unit-clause-code l M
proof −
  have 1: ⋀a. (∀ c∈set (remove1 a l). − c ∈ lits-of-l M) ⟷ M ⊨as CNot (mset l − {#a#})
    using lits-of-l-unfold[of remove1 - l, of - M] by simp
  thus ?thesis
    unfolding is-unit-clause-code-def is-unit-clause-def 1 by blast
qed

lemma is-unit-clause-some-undef:
  assumes is-unit-clause l M = Some a
  shows undefined-lit M a
proof −
  have (case [a←l . atm-of a ∉ atm-of ‘ lits-of-l M] of [] ⇒ None
          | [a] ⇒ if M ⊨as CNot (mset l − {#a#}) then Some a else None
          | a # ab # xa ⇒ Map.empty xa) = Some a
    using assms unfolding is-unit-clause-def .
  hence a ∈ set [a←l . atm-of a ∉ atm-of ‘ lits-of-l M]
    apply (cases [a←l . atm-of a ∉ atm-of ‘ lits-of-l M])
      apply simp
    apply (rename-tac aa list; case-tac list) by (auto split: if-split-asm)
  hence atm-of a ∉ atm-of ‘ lits-of-l M by auto
  thus ?thesis
    by (simp add: Marked-Propagated-in-iff-in-lits-of-l
      atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set )
qed

lemma is-unit-clause-some-CNot: is-unit-clause l M = Some a ⟹ M ⊨as CNot (mset l − {#a#})
  unfolding is-unit-clause-def
proof −
```

```
  assume (case [a←l . atm-of a ∉ atm-of ‘ lits-of-l M] of [] ⇒ None
        | [a] ⇒ if M ⊨as CNot (mset l − {#a#}) then Some a else None
        | a # ab # xa ⇒ Map.empty xa) = Some a
  thus ?thesis
    apply (cases [a←l . atm-of a ∉ atm-of ‘ lits-of-l M], simp)
      apply simp
    apply (rename-tac aa list, case-tac list) by (auto split: if-split-asm)
qed

lemma is-unit-clause-some-in: is-unit-clause l M = Some a ⟹ a ∈ set l
  unfolding is-unit-clause-def
proof −
  assume (case [a←l . atm-of a ∉ atm-of ‘ lits-of-l M] of [] ⇒ None
        | [a] ⇒ if M ⊨as CNot (mset l − {#a#}) then Some a else None
        | a # ab # xa ⇒ Map.empty xa) = Some a
  thus a ∈ set l
    by (cases [a←l . atm-of a ∉ atm-of ‘ lits-of-l M])
      (fastforce dest: filter-eq-ConsD split: if-split-asm  split: list.splits)+
qed

lemma is-unit-clause-nil[simp]: is-unit-clause [] M = None
  unfolding is-unit-clause-def by auto
```

20.1.2 Unit propagation for all clauses

Finding the first clause to propagate

```
fun find-first-unit-clause :: 'a literal list list ⇒ ('a, 'b, 'c) marked-lit list
  ⇒ ('a literal × 'a literal list) option  where
find-first-unit-clause (a # l) M =
  (case is-unit-clause a M of
    None ⇒ find-first-unit-clause l M
  | Some L ⇒ Some (L, a)) |
find-first-unit-clause [] - = None

lemma find-first-unit-clause-some:
  find-first-unit-clause l M = Some (a, c)
  ⟹ c ∈ set l ∧  M ⊨as CNot (mset c − {#a#}) ∧ undefined-lit M a ∧ a ∈ set c
  apply (induction l)
    apply simp
  by (auto split: option.splits dest: is-unit-clause-some-in is-unit-clause-some-CNot
        is-unit-clause-some-undef)

lemma propagate-is-unit-clause-not-None:
  assumes dist: distinct c and
  M: M ⊨as CNot (mset c − {#a#}) and
  undef: undefined-lit M a and
  ac: a ∈ set c
  shows is-unit-clause c M ≠ None
proof −
  have [a←c . atm-of a ∉ atm-of ‘ lits-of-l M] = [a]
    using assms
    proof (induction c)
      case Nil thus ?case by simp
    next
      case (Cons ac c)
      show ?case
        proof (cases a = ac)
          case True
          thus ?thesis using Cons
            by (auto simp del: lits-of-l-unfold
                simp add: lits-of-l-unfold[symmetric] Marked-Propagated-in-iff-in-lits-of-l
                  atm-of-eq-atm-of atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)
        next
          case False
          hence T: mset c + {#ac#} − {#a#} = mset c − {#a#} + {#ac#}
            by (auto simp add: multiset-eq-iff)
          show ?thesis using False Cons
            by (auto simp add: T atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)
        qed
    qed
  thus ?thesis
    using M unfolding is-unit-clause-def by auto
qed

lemma find-first-unit-clause-none:
  distinct c ⟹ c ∈ set l ⟹  M ⊨as CNot (mset c − {#a#}) ⟹ undefined-lit M a ⟹ a ∈ set c
  ⟹ find-first-unit-clause l M ≠ None
  by (induction l)
    (auto split: option.split simp add: propagate-is-unit-clause-not-None)
```

20.1.3 Decide

```
fun find-first-unused-var :: 'a literal list list ⇒ 'a literal set ⇒ 'a literal option  where
find-first-unused-var (a # l) M =
  (case List.find (λlit. lit ∉ M ∧ −lit ∉ M) a of
    None ⇒ find-first-unused-var l M
  | Some a ⇒ Some a) |
find-first-unused-var [] - = None

lemma find-none[iff]:
  List.find (λlit. lit ∉ M ∧ −lit ∉ M) a = None ⟷  atm-of ‘ set a ⊆ atm-of ‘  M
  apply (induct a)
  using atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
    by (force simp add:  atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)+

lemma find-some: List.find (λlit. lit ∉ M ∧ −lit ∉ M) a = Some b ⟹ b ∈ set a ∧ b ∉ M ∧ −b ∉ M
  unfolding find-Some-iff by (metis nth-mem)

lemma find-first-unused-var-None[iff]:
  find-first-unused-var l M = None ⟷ (∀a ∈ set l. atm-of ‘ set a ⊆ atm-of ‘  M)
  by (induct l)
    (auto split: option.splits dest!: find-some
      simp add: image-subset-iff atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)

lemma find-first-unused-var-Some-not-all-incl:
  assumes find-first-unused-var l M = Some c
  shows  ¬(∀a ∈ set l. atm-of ‘ set a ⊆ atm-of ‘  M)
proof −
  have find-first-unused-var l M ≠ None
    using assms by (cases find-first-unused-var l M) auto
  thus ¬(∀a ∈ set l. atm-of ‘ set a ⊆ atm-of ‘  M) by auto
```

```
qed

lemma find-first-unused-var-Some:
  find-first-unused-var l M = Some a ⟹ (∃ m ∈ set l. a ∈ set m ∧ a ∉ M ∧ −a ∉ M)
  by (induct l) (auto split: option.splits dest: find-some)

lemma find-first-unused-var-undefined:
  find-first-unused-var l (lits-of-l Ms) = Some a ⟹ undefined-lit Ms a
  using find-first-unused-var-Some[of l lits-of-l Ms a] Marked-Propagated-in-iff-in-lits-of-l
  by blast

end
theory DPLL-W-Implementation
imports DPLL-CDCL-W-Implementation DPLL-W ~~/src/HOL/Library/Code-Target-Numeral
begin
```

20.2 Simple Implementation of DPLL

20.2.1 Combining the propagate and decide: a DPLL step

```
definition DPLL-step :: int dpll_W-marked-lits × int literal list list
  ⇒ int dpll_W-marked-lits × int literal list list  where
DPLL-step = (λ(Ms, N).
  (case find-first-unit-clause N Ms of
    Some (L, -) ⇒ (Propagated L () # Ms, N)
  | - ⇒
    if ∃ C ∈ set N. (∀ c ∈ set C. −c ∈ lits-of-l Ms)
    then
      (case backtrack-split Ms of
        (-, L # M) ⇒ (Propagated (− (lit-of L)) () # M, N)
      | (-, -) ⇒ (Ms, N)
      )
    else
    (case find-first-unused-var N (lits-of-l Ms) of
        Some a ⇒ (Marked a () # Ms, N)
      | None ⇒ (Ms, N))))
```

Example of propagation:

```
value DPLL-step ([Marked (Neg 1) ()], [[Pos (1::int), Neg 2]])
```

We define the conversion function between the states as defined in *Prop-DPLL* (with multisets) and here (with lists).

```
abbreviation toS ≡ λ(Ms::(int, unit, unit) marked-lit list)
                    (N:: int literal list list). (Ms, mset (map mset N))
abbreviation toS' ≡ λ(Ms::(int, unit, unit) marked-lit list,
                      N:: int literal list list). (Ms, mset (map mset N))
```

Proof of correctness of *DPLL-step*

```
lemma DPLL-step-is-a-dpll_W-step:
  assumes step: (Ms', N') = DPLL-step (Ms, N)
  and neq: (Ms, N) ≠ (Ms', N')
  shows dpll_W (toS Ms N) (toS Ms' N')
proof −
  let ?S = (Ms, mset (map mset N))
  { fix L E
```

```
    assume unit: find-first-unit-clause N Ms = Some (L, E)
    hence Ms'N: (Ms', N') = (Propagated L () # Ms, N)
      using step unfolding DPLL-step-def by auto
    obtain C where
      C: C ∈ set N and
      Ms: Ms ⊨as CNot (mset C − {#L#}) and
      undef: undefined-lit Ms L and
      L ∈ set C using find-first-unit-clause-some[OF unit] by metis
    have dpll_W (Ms, mset (map mset N))
        (Propagated L () # fst (Ms, mset (map mset N)), snd (Ms, mset (map mset N)))
      apply (rule dpll_W.propagate)
      using Ms undef C ‹L ∈ set C› by (auto simp add: C)
    hence ?thesis using Ms'N by auto
  }
  moreover
  { assume unit: find-first-unit-clause N Ms = None
    assume exC: ∃ C ∈ set N. Ms ⊨as CNot (mset C)
    then obtain C where C: C ∈ set N and Ms: Ms ⊨as CNot (mset C) by auto
    then obtain L M M' where bt: backtrack-split Ms  = (M', L # M)
      using step exC neq unfolding DPLL-step-def prod.case unit
      by (cases backtrack-split Ms, rename-tac b, case-tac b) auto
    hence is-marked L using backtrack-split-snd-hd-marked[of Ms] by auto
    have 1: dpll_W (Ms, mset (map mset N))
                (Propagated (− lit-of L) () # M, snd (Ms, mset (map mset N)))
      apply (rule dpll_W.backtrack[OF - ‹is-marked L›, of ])
      using C Ms bt by auto
    moreover have (Ms', N') = (Propagated (− (lit-of L)) () # M, N)
      using step exC unfolding DPLL-step-def bt prod.case unit by auto
    ultimately have ?thesis by auto
  }
  moreover
  { assume unit: find-first-unit-clause N Ms = None
    assume exC: ¬ (∃ C ∈ set N. Ms ⊨as CNot (mset C))
    obtain L where unused: find-first-unused-var N (lits-of-l Ms) = Some L
      using step exC neq unfolding DPLL-step-def prod.case unit
      by (cases find-first-unused-var N (lits-of-l Ms)) auto
    have dpll_W (Ms, mset (map mset N))
              (Marked L () # fst (Ms, mset (map mset N)), snd (Ms, mset (map mset N)))
      apply (rule dpll_W.decided[of ?S L])
      using find-first-unused-var-Some[OF unused]
      by (auto simp add: Marked-Propagated-in-iff-in-lits-of-l atms-of-ms-def)
    moreover have (Ms', N') = (Marked L () # Ms, N)
      using step exC unfolding DPLL-step-def unused prod.case unit by auto
    ultimately have ?thesis by auto
  }
  ultimately show ?thesis by (cases find-first-unit-clause N Ms) auto
qed

lemma DPLL-step-stuck-final-state:
  assumes step: (Ms, N) = DPLL-step (Ms, N)
  shows conclusive-dpll_W-state (toS Ms N)
proof −
  have unit: find-first-unit-clause N Ms = None
    using step unfolding DPLL-step-def by (auto split:option.splits)
```

```
{ assume n: ∃ C ∈ set N. Ms ⊨as CNot (mset C)
  hence Ms: (Ms, N) = (case backtrack-split Ms of (x, []) ⇒ (Ms, N)
                    | (x, L # M) ⇒ (Propagated (− lit-of L) () # M, N))
    using step unfolding DPLL-step-def by (simp add:unit)

have snd (backtrack-split Ms) = []
  proof (cases backtrack-split Ms, cases snd (backtrack-split Ms))
    fix a b
    assume backtrack-split Ms = (a, b) and snd (backtrack-split Ms) = []
    thus snd (backtrack-split Ms) = [] by blast
  next
    fix a b aa list
    assume
      bt: backtrack-split Ms = (a, b) and
      bt': snd (backtrack-split Ms) = aa # list
    hence Ms: Ms = Propagated (− lit-of aa) () # list using Ms by auto
    have is-marked aa using backtrack-split-snd-hd-marked[of Ms] bt bt' by auto
    moreover have fst (backtrack-split Ms) @ aa # list = Ms
      using backtrack-split-list-eq[of Ms] bt' by auto
    ultimately have False unfolding Ms by auto
    thus snd (backtrack-split Ms) = [] by blast
  qed

  hence ?thesis
    using n backtrack-snd-empty-not-marked[of Ms] unfolding conclusive-dpll_W-state-def
    by (cases backtrack-split Ms) auto
}
moreover {
  assume n: ¬ (∃ C ∈ set N. Ms ⊨as CNot (mset C))
  hence find-first-unused-var N (lits-of-l Ms) = None
    using step unfolding DPLL-step-def by (simp add: unit split: option.splits)
  hence a: ∀ a ∈ set N. atm-of ‘ set a ⊆ atm-of ‘ (lits-of-l Ms) by auto
  have fst (toS Ms N) ⊨asm snd (toS Ms N) unfolding true-annots-def CNot-def Ball-def
    proof clarify
      fix x
      assume x: x ∈ set-mset (clauses (toS Ms N))
      hence ¬Ms ⊨as CNot  x using n unfolding true-annots-def CNot-def Ball-def by auto
      moreover have total-over-m (lits-of-l Ms) {x}
        using a x image-iff in-mono atms-of-s-def
        unfolding total-over-m-def total-over-set-def lits-of-def by fastforce
      ultimately show fst (toS Ms N) ⊨a x
        using total-not-CNot[of lits-of-l Ms x] by (simp add: true-annot-def true-annots-true-cls)
    qed
  hence ?thesis unfolding conclusive-dpll_W-state-def by blast
}
ultimately show ?thesis by blast
qed
```

20.2.2 Adding invariants

Invariant tested in the function **function** *DPLL-ci* :: *int dpll*$_W$*-marked-lits* ⇒ *int literal list list*

```
  ⇒ int dpll_W-marked-lits × int literal list list where
DPLL-ci Ms N =
  (if ¬dpll_W-all-inv (Ms, mset (map mset N))
  then (Ms, N)
  else
   let (Ms', N') = DPLL-step (Ms, N) in
   if (Ms', N') = (Ms, N) then (Ms, N) else DPLL-ci Ms' N)
  by fast+
termination
proof (relation {(S', S). (toS' S', toS' S) ∈ {(S', S). dpll_W-all-inv S ∧ dpll_W S S'}})
  show wf {(S', S).(toS' S', toS' S) ∈ {(S', S). dpll_W-all-inv S ∧ dpll_W S S'}}
    using wf-if-measure-f[OF dpll_W-wf, of toS'] by auto
next
  fix Ms :: int dpll_W-marked-lits and N x xa y
  assume¬ ¬ dpll_W-all-inv (toS Ms N)
  and step: x = DPLL-step (Ms, N)
  and x: (xa, y) = x
  and (xa, y) ≠ (Ms, N)
  thus ((xa, N), Ms, N) ∈ {(S', S). (toS' S', toS' S) ∈ {(S', S). dpll_W-all-inv S ∧ dpll_W S S'}}
    using DPLL-step-is-a-dpll_W-step dpll_W-same-clauses split-conv by fastforce
qed
```

No invariant tested

```
function (domintros) DPLL-part:: int dpll_W-marked-lits ⇒ int literal list list
⇒
  int dpll_W-marked-lits × int literal list list where
DPLL-part Ms N =
  (let (Ms', N') = DPLL-step (Ms, N) in
   if (Ms', N') = (Ms, N) then (Ms, N) else DPLL-part Ms' N)
  by fast+

lemma snd-DPLL-step[simp]:
  snd (DPLL-step (Ms, N)) = N
  unfolding DPLL-step-def by (auto split: if-split option.splits prod.splits list.splits)

lemma dpll_W-all-inv-implieS-2-eq3-and-dom:
  assumes dpll_W-all-inv (Ms, mset (map mset N))
  shows DPLL-ci Ms N = DPLL-part Ms N ∧ DPLL-part-dom (Ms, N)
  using assms
proof (induct rule: DPLL-ci.induct)
  case (1 Ms N)
  have snd (DPLL-step (Ms, N)) = N by auto
  then obtain Ms' where Ms': DPLL-step (Ms, N) = (Ms', N) by (cases DPLL-step (Ms, N)) auto
  have inv': dpll_W-all-inv (toS Ms' N) by (metis (mono-tags) 1.prems DPLL-step-is-a-dpll_W-step
    Ms' dpll_W-all-inv old.prod.inject)
  { assume (Ms', N) ≠ (Ms, N)
    hence DPLL-ci Ms' N = DPLL-part Ms' N ∧ DPLL-part-dom (Ms', N) using 1(1)[of - Ms' N]
Ms'
      1(2) inv' by auto
    hence DPLL-part-dom (Ms, N) using DPLL-part.domintros Ms' by fastforce
    moreover have DPLL-ci Ms N = DPLL-part Ms N using 1.prems DPLL-part.psimps Ms'
      ‹DPLL-ci Ms' N = DPLL-part Ms' N ∧ DPLL-part-dom (Ms', N)› ‹DPLL-part-dom (Ms, N)› by
auto
    ultimately have ?case by blast
  }
  moreover {
    assume (Ms', N) = (Ms, N)
    hence ?case using DPLL-part.domintros DPLL-part.psimps Ms' by fastforce
  }
  ultimately show ?case by blast
```

```
qed

lemma DPLL-ci-dpll_W-rtranclp:
  assumes DPLL-ci Ms N = (Ms', N')
  shows dpll_W** (toS Ms N) (toS Ms' N)
  using assms
proof (induct Ms N arbitrary: Ms' N' rule: DPLL-ci.induct)
  case (1 Ms N Ms' N') note IH = this(1) and step = this(2)
  obtain S_1 S_2 where S: (S_1, S_2) = DPLL-step (Ms, N) by (cases DPLL-step (Ms, N)) auto

  { assume ¬dpll_W-all-inv (toS Ms N)
    hence (Ms, N) = (Ms', N) using step by auto
    hence ?case by auto
  }
  moreover
  { assume dpll_W-all-inv (toS Ms N)
    and (S_1, S_2) = (Ms, N)
    hence ?case using S step by auto
  }
  moreover
  { assume dpll_W-all-inv (toS Ms N)
    and (S_1, S_2) ≠ (Ms, N)
    moreover obtain S_1' S_2' where DPLL-ci S_1 N = (S_1', S_2') by (cases DPLL-ci S_1 N) auto
    moreover have DPLL-ci Ms N = DPLL-ci S_1 N using DPLL-ci.simps[of Ms N] calculation
      proof –
        have (case (S_1, S_2) of (ms, lss) ⇒
          if (ms, lss) = (Ms, N) then (Ms, N) else DPLL-ci ms N) = DPLL-ci Ms N
          using S DPLL-ci.simps[of Ms N] calculation by presburger
        hence (if (S_1, S_2) = (Ms, N) then (Ms, N) else DPLL-ci S_1 N) = DPLL-ci Ms N
          by fastforce
        thus ?thesis
          using calculation(2) by presburger
      qed
    ultimately have dpll_W** (toS S_1' N) (toS Ms' N) using IH[of (S_1, S_2) S_1 S_2] S step by simp

    moreover have dpll_W (toS Ms N) (toS S_1 N)
      by (metis DPLL-step-is-a-dpll_W-step S ‹(S_1, S_2) ≠ (Ms, N)› prod.sel(2) snd-DPLL-step)
    ultimately have ?case by (metis (mono-tags, hide-lams) IH S ‹(S_1, S_2) ≠ (Ms, N)›
      ‹DPLL-ci Ms N = DPLL-ci S_1 N› ‹dpll_W-all-inv (toS Ms N)› converse-rtranclp-into-rtranclp
      local.step)
  }
  ultimately show ?case by blast
qed

lemma dpll_W-all-inv-dpll_W-tranclp-irrefl:
  assumes dpll_W-all-inv (Ms, N)
  and dpll_W++ (Ms, N) (Ms, N)
  shows False
proof –
  have 1: wf {(S', S). dpll_W-all-inv S ∧ dpll_W++ S S'} using dpll_W-wf-tranclp by auto
  have ((Ms, N), (Ms, N)) ∈ {(S', S). dpll_W-all-inv S ∧ dpll_W++ S S'} using assms by auto
  thus False using wf-not-refl[OF 1] by blast
qed

lemma DPLL-ci-final-state:
  assumes step: DPLL-ci Ms N = (Ms, N)
  and inv: dpll_W-all-inv (toS Ms N)
  shows conclusive-dpll_W-state (toS Ms N)
proof –
  have st: dpll_W^** (toS Ms N) (toS Ms N) using DPLL-ci-dpll_W-rtranclp[OF step] .
  have DPLL-step (Ms, N) = (Ms, N)
    proof (rule ccontr)
      obtain Ms' N' where Ms'N: (Ms', N') = DPLL-step (Ms, N)
        by (cases DPLL-step (Ms, N)) auto
      assume ¬ ?thesis
      hence DPLL-ci Ms' N = (Ms, N) using step inv st Ms'N[symmetric] by fastforce
      hence dpll_W^++ (toS Ms N) (toS Ms N)
       by (metis DPLL-ci-dpll_W-rtranclp DPLL-step-is-a-dpll_W-step Ms'N ‹DPLL-step (Ms, N) ≠ (Ms,
N)›
         prod.sel(2) rtranclp-into-tranclp2 snd-DPLL-step)
      thus False using dpll_W-all-inv-dpll_W-tranclp-irrefl inv by auto
    qed
  thus ?thesis using DPLL-step-stuck-final-state[of Ms N] by simp
qed

lemma DPLL-step-obtains:
  obtains Ms' where (Ms', N) = DPLL-step (Ms, N)
  unfolding DPLL-step-def by (metis (no-types, lifting) DPLL-step-def prod.collapse snd-DPLL-step)

lemma DPLL-ci-obtains:
  obtains Ms' where (Ms', N) = DPLL-ci Ms N
proof (induct rule: DPLL-ci.induct)
  case (1 Ms N) note IH = this(1) and that = this(2)
  obtain S where SN: (S, N) = DPLL-step (Ms, N) using DPLL-step-obtains by metis
  { assume ¬ dpll_W-all-inv (toS Ms N)
    hence ?case using that by auto
  }
  moreover {
    assume n: (S, N) ≠ (Ms, N)
    and inv: dpll_W-all-inv (toS Ms N)
    have ∃ ms. DPLL-step (Ms, N) = (ms, N)
      by (metis ‹⋀thesisa. (⋀S. (S, N) = DPLL-step (Ms, N) ⟹ thesisa) ⟹ thesisa›)
    hence ?thesis
      using IH that by fastforce
  }
  moreover {
    assume n: (S, N) = (Ms, N)
    hence ?case using SN that by fastforce
 }
  ultimately show ?case by blast
qed


lemma DPLL-ci-no-more-step:
  assumes step: DPLL-ci Ms N = (Ms', N')
  shows DPLL-ci Ms' N' = (Ms', N')
  using assms
proof (induct arbitrary: Ms' N' rule: DPLL-ci.induct)
  case (1 Ms N Ms' N') note IH = this(1) and step = this(2)
  obtain S_1 where S: (S_1, N) = DPLL-step (Ms, N) using DPLL-step-obtains by auto
  { assume ¬dpll_W-all-inv (toS Ms N)
    hence ?case using step by auto
  }
  moreover {
    assume dpll_W-all-inv (toS Ms N)
    and (S_1, N) = (Ms, N)
    hence ?case using S step by auto
  }
  moreover
  { assume inv: dpll_W-all-inv (toS Ms N)
    assume n: (S_1, N) ≠ (Ms, N)
    obtain S_1' where SS: (S_1', N) = DPLL-ci S_1 N using DPLL-ci-obtains by blast
    moreover have DPLL-ci Ms N = DPLL-ci S_1 N
      proof -
        have (case (S_1, N) of (ms, lss) ⇒ if (ms, lss) = (Ms, N) then (Ms, N) else DPLL-ci ms N)
          = DPLL-ci Ms N
          using S DPLL-ci.simps[of Ms N] calculation inv by presburger
        hence (if (S_1, N) = (Ms, N) then (Ms, N) else DPLL-ci S_1 N) = DPLL-ci Ms N
          by fastforce
        thus ?thesis
          using calculation n by presburger
      qed
    moreover
      have DPLL-ci S_1' N = (S_1', N) using step IH[OF - - S n SS[symmetric]] inv by blast
    ultimately have ?case using step by fastforce
  }
  ultimately show ?case by blast
qed
```

```
lemma DPLL-part-dpll_W-all-inv-final:
  fixes M Ms':: (int, unit, unit) marked-lit list and
    N :: int literal list list
  assumes inv: dpll_W-all-inv (Ms, mset (map mset N))
  and MsN: DPLL-part Ms N = (Ms', N)
  shows conclusive-dpll_W-state (toS Ms' N) ∧ dpll_W** (toS Ms N) (toS Ms' N)
proof -
  have 2: DPLL-ci Ms N = DPLL-part Ms N using inv dpll_W-all-inv-implieS-2-eq3-and-dom by blast
   hence star: dpll_W** (toS Ms N) (toS Ms' N) unfolding MsN using DPLL-ci-dpll_W-rtranclp by
blast
  hence inv': dpll_W-all-inv (toS Ms' N) using inv rtranclp-dpll_W-all-inv by blast
  show ?thesis using star DPLL-ci-final-state[OF DPLL-ci-no-more-step inv'] 2 unfolding MsN by
blast
qed
```

Embedding the invariant into the type

Defining the type

```
typedef dpll_W-state =
    {(M::(int, unit, unit) marked-lit list, N::int literal list list).
        dpll_W-all-inv (toS M N)}
  morphisms rough-state-of state-of
proof
    show ([],[]) ∈ {(M, N). dpll_W-all-inv (toS M N)} by (auto simp add: dpll_W-all-inv-def)
qed
```

lemma
DPLL-part-dom ([], *N*)
using *assms dpll$_W$-all-inv-implieS-2-eq3-and-dom*[*of* [] *N*] **by** (*simp add*: *dpll$_W$-all-inv-def*)

Some type classes **instantiation** *dpll$_W$-state* :: *equal*
begin
definition *equal-dpll$_W$-state* :: *dpll$_W$-state* ⇒ *dpll$_W$-state* ⇒ *bool* **where**
equal-dpll$_W$-state S S′ = (*rough-state-of S* = *rough-state-of S′*)
instance
by *standard* (*simp add*: *rough-state-of-inject equal-dpll$_W$-state-def*)
end

DPLL **definition** *DPLL-step′* :: *dpll$_W$-state* ⇒ *dpll$_W$-state* **where**
DPLL-step′ S = *state-of* (*DPLL-step* (*rough-state-of S*))

declare *rough-state-of-inverse*[*simp*]

lemma *DPLL-step-dpll$_W$-conc-inv*:
DPLL-step (*rough-state-of S*) ∈ {(*M*, *N*). *dpll$_W$-all-inv* (*toS M N*)}
by (*smt DPLL-ci.simps DPLL-ci-dpll$_W$-rtranclp case-prodE case-prodI2 rough-state-of mem-Collect-eq old.prod.case prod.sel*(*2*) *rtranclp-dpll$_W$-all-inv snd-DPLL-step*)

lemma *rough-state-of-DPLL-step′-DPLL-step*[*simp*]:
rough-state-of (*DPLL-step′ S*) = *DPLL-step* (*rough-state-of S*)
using *DPLL-step-dpll$_W$-conc-inv DPLL-step′-def state-of-inverse* **by** *auto*

function *DPLL-tot*:: *dpll$_W$-state* ⇒ *dpll$_W$-state* **where**
DPLL-tot S =
(*let S′* = *DPLL-step′ S in*
if S′ = *S then S else DPLL-tot S′*)
by *fast+*
termination
proof (*relation* {(*T′*, *T*).
(*rough-state-of T′*, *rough-state-of T*)
∈ {(*S′*, *S*). (*toS′ S′*, *toS′ S*)
∈ {(*S′*, *S*). *dpll$_W$-all-inv S* ∧ *dpll$_W$ S S′*}}})
show *wf* {(*b*, *a*).
(*rough-state-of b*, *rough-state-of a*)
∈ {(*b*, *a*). (*toS′ b*, *toS′ a*)
∈ {(*b*, *a*). *dpll$_W$-all-inv a* ∧ *dpll$_W$ a b*}}}
using *wf-if-measure-f*[*OF wf-if-measure-f*[*OF dpll$_W$-wf*, *of toS′*], *of rough-state-of*] **.**
next
fix *S x*
assume *x*: *x* = *DPLL-step′ S*
and *x* ≠ *S*
have *dpll$_W$-all-inv* (*case rough-state-of S of* (*Ms*, *N*) ⇒ (*Ms*, *mset* (*map mset N*)))
by (*metis* (*no-types*, *lifting*) *case-prodE mem-Collect-eq old.prod.case rough-state-of*)
moreover have *dpll$_W$* (*case rough-state-of S of* (*Ms*, *N*) ⇒ (*Ms*, *mset* (*map mset N*)))
(*case rough-state-of* (*DPLL-step′ S*) *of* (*Ms*, *N*) ⇒ (*Ms*, *mset* (*map mset N*)))
proof −
obtain *Ms N* **where** *Ms*: (*Ms*, *N*) = *rough-state-of S* **by** (*cases rough-state-of S*) *auto*
have *dpll$_W$-all-inv* (*toS′* (*Ms*, *N*)) **using** *calculation* **unfolding** *Ms* **by** *blast*
moreover obtain *Ms′ N′* **where** *Ms′*: (*Ms′*, *N′*) = *rough-state-of* (*DPLL-step′ S*)
by (*cases rough-state-of* (*DPLL-step′ S*)) *auto*
ultimately have *dpll$_W$-all-inv* (*toS′* (*Ms′*, *N′*)) **unfolding** *Ms′*

```
        by (metis (no-types, lifting) case-prod-unfold mem-Collect-eq rough-state-of)

      have dpll_W (toS Ms N) (toS Ms' N')
        apply (rule DPLL-step-is-a-dpll_W-step[of Ms' N' Ms N])
        unfolding Ms Ms' using ‹x ≠ S› rough-state-of-inject x by fastforce+
      thus ?thesis unfolding Ms[symmetric] Ms'[symmetric] by auto
    qed
  ultimately show (x, S) ∈ {(T', T). (rough-state-of T', rough-state-of T)
    ∈ {(S', S). (toS' S', toS' S) ∈ {(S', S). dpll_W-all-inv S ∧ dpll_W S S'}}}
    by (auto simp add: x)
qed

lemma [code]:
DPLL-tot S =
  (let S' = DPLL-step' S in
   if S' = S then S else DPLL-tot S') by auto

lemma DPLL-tot-DPLL-step-DPLL-tot[simp]: DPLL-tot (DPLL-step' S) = DPLL-tot S
  apply (cases DPLL-step' S = S)
  apply simp
  unfolding DPLL-tot.simps[of S] by (simp del: DPLL-tot.simps)

lemma DOPLL-step'-DPLL-tot[simp]:
  DPLL-step' (DPLL-tot S) = DPLL-tot S
  by (rule  DPLL-tot.induct[of λS. DPLL-step' (DPLL-tot S) = DPLL-tot S S])
     (metis (full-types) DPLL-tot.simps)



lemma DPLL-tot-final-state:
  assumes DPLL-tot S = S
  shows conclusive-dpll_W-state (toS' (rough-state-of S))
proof –
  have DPLL-step' S =  S using assms[symmetric] DOPLL-step'-DPLL-tot by metis
  hence DPLL-step (rough-state-of S) =  (rough-state-of S)
    unfolding DPLL-step'-def using DPLL-step-dpll_W-conc-inv rough-state-of-inverse
    by (metis rough-state-of-DPLL-step'-DPLL-step)
  thus ?thesis
    by (metis (mono-tags, lifting) DPLL-step-stuck-final-state old.prod.exhaust split-conv)
qed

lemma DPLL-tot-star:
  assumes rough-state-of (DPLL-tot S) = S'
  shows dpll_W** (toS' (rough-state-of S)) (toS' S')
  using assms
proof (induction arbitrary: S' rule: DPLL-tot.induct)
  case (1 S S')
  let ?x = DPLL-step' S
  { assume ?x = S
    then have ?case using 1(2) by simp
  }
  moreover {
    assume S: ?x ≠ S
    have ?case
      apply (cases DPLL-step' S  = S)
      using S apply blast
    by (smt 1.IH 1.prems DPLL-step-is-a-dpll_W-step DPLL-tot.simps case-prodE2
      rough-state-of-DPLL-step'-DPLL-step rtranclp.rtrancl-into-rtrancl rtranclp.rtrancl-refl
      rtranclp-idemp split-conv)
  }
  ultimately show ?case by auto
qed

lemma rough-state-of-rough-state-of-nil[simp]:
  rough-state-of (state-of ([], N)) = ([], N)
  apply (rule DPLL-W-Implementation.dpll_W-state.state-of-inverse)
  unfolding dpll_W-all-inv-def by auto
```

Theorem of correctness

```
lemma DPLL-tot-correct:
  assumes rough-state-of (DPLL-tot (state-of (([], N)))) = (M, N')
  and (M', N'') =  toS' (M, N')
  shows M' ⊨asm N'' ⟷ satisfiable (set-mset N'')
proof –
  have dpll_W** (toS' ([], N)) (toS' (M, N')) using DPLL-tot-star[OF assms(1)] by auto
  moreover have conclusive-dpll_W-state (toS' (M, N'))
    using DPLL-tot-final-state by (metis (mono-tags, lifting) DOPLL-step'-DPLL-tot DPLL-tot.simps
      assms(1))
  ultimately show ?thesis using dpll_W-conclusive-state-correct by (smt DPLL-ci.simps
    DPLL-ci-dpll_W-rtranclp assms(2) dpll_W-all-inv-def prod.case prod.sel(1) prod.sel(2)
    rtranclp-dpll_W-inv(3) rtranclp-dpll_W-inv-starting-from-0)
qed
```

20.2.3 Code export

A conversion to *DPLL-W-Implementation.dpll*$_W$*-state*

```
definition Con :: (int, unit, unit) marked-lit list × int literal list list
                    ⇒ dpll_W-state where
  Con xs = state-of (if dpll_W-all-inv (toS (fst xs) (snd xs)) then xs else ([], []))
lemma [code abstype]:
  Con (rough-state-of S) = S
  using rough-state-of[of S] unfolding Con-def by auto

  declare rough-state-of-DPLL-step'-DPLL-step[code abstract]

lemma Con-DPLL-step-rough-state-of-state-of[simp]:
  Con (DPLL-step (rough-state-of s)) = state-of (DPLL-step (rough-state-of s))
  unfolding Con-def by (metis (mono-tags, lifting) DPLL-step-dpll_W-conc-inv mem-Collect-eq
    prod.case-eq-if)
```

A slightly different version of *DPLL-tot* where the returned boolean indicates the result.

```
definition DPLL-tot-rep where
DPLL-tot-rep S =
  (let (M, N) = (rough-state-of (DPLL-tot S)) in (∀ A ∈ set N. (∃ a∈set A. a ∈ lits-of-l (M)), M))
```

One version of the generated SML code is here, but not included in the generated document. The only differences are:

- export *'a literal* from the SML Module *Clausal-Logic*;
- export the constructor *Con* from *DPLL-W-Implementation*;

- export the *int* constructor from *Arith*.

  All these allows to test on the code on some examples.

end
theory *CDCL-W-Implementation*
imports *DPLL-CDCL-W-Implementation CDCL-W-Termination*
begin

notation *image-mset* (**infixr** *‘# 90*)

type-synonym $'a$ $cdcl_W$*-mark* = $'a$ *literal list*
type-synonym $cdcl_W$*-marked-level* = *nat*

type-synonym $'v$ $cdcl_W$*-marked-lit* = ($'v$, $cdcl_W$*-marked-level*, $'v$ $cdcl_W$*-mark*) *marked-lit*
type-synonym $'v$ $cdcl_W$*-marked-lits* = ($'v$, $cdcl_W$*-marked-level*, $'v$ $cdcl_W$*-mark*) *marked-lits*
type-synonym $'v$ $cdcl_W$*-state* =
  $'v$ $cdcl_W$*-marked-lits* × $'v$ *literal list list* × $'v$ *literal list list* × *nat* ×
  $'v$ *literal list option*

abbreviation *raw-trail* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'a$ **where**
raw-trail ≡ (λ(M, -). M)

abbreviation *raw-cons-trail* :: $'a \Rightarrow 'a$ *list* $\times 'b \times 'c \times 'd \times 'e \Rightarrow 'a$ *list* $\times 'b \times 'c \times 'd \times 'e$
  where
raw-cons-trail ≡ (λL (M, S). ($L\#M$, S))

abbreviation *raw-tl-trail* :: $'a$ *list* $\times 'b \times 'c \times 'd \times 'e \Rightarrow 'a$ *list* $\times 'b \times 'c \times 'd \times 'e$ **where**
raw-tl-trail ≡ (λ(M, S). (*tl* M, S))

abbreviation *raw-init-clss* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'b$ **where**
raw-init-clss ≡ λ(M, N, -). N

abbreviation *raw-learned-clss* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'c$ **where**
raw-learned-clss ≡ λ(M, N, U, -). U

abbreviation *raw-backtrack-lvl* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'd$ **where**
raw-backtrack-lvl ≡ λ(M, N, U, k, -). k

abbreviation *raw-update-backtrack-lvl* :: $'d \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e$
  where
raw-update-backtrack-lvl ≡ λk (M, N, U, -, S). (M, N, U, k, S)

abbreviation *raw-conflicting* :: $'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'e$ **where**
raw-conflicting ≡ λ(M, N, U, k, D). D

abbreviation *raw-update-conflicting* :: $'e \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e \Rightarrow 'a \times 'b \times 'c \times 'd \times 'e$
  where
raw-update-conflicting ≡ λS (M, N, U, k, -). (M, N, U, k, S)

abbreviation *raw-add-learned-cls* **where**
raw-add-learned-cls ≡ λC (M, N, U, S). (M, N, {#C#} + U, S)

abbreviation *raw-remove-cls* **where**
raw-remove-cls ≡ λC (M, N, U, S). (M, *removeAll-mset* C N, *removeAll-mset* C U, S)

type-synonym *'v cdcl*$_W$*-state-inv-st* = (*'v*, *nat*, *'v literal list*) *marked-lit list* ×
'v literal list list × *'v literal list list* × *nat* × *'v literal list option*

abbreviation *raw-S0-cdcl*$_W$ *N* ≡ (([], *N*, [], *0*, *None*):: *'v cdcl*$_W$*-state-inv-st*)

fun *mmset-of-mlit'* :: (*'v*, *nat*, *'v literal list*) *marked-lit* ⇒ (*'v*, *nat*, *'v clause*) *marked-lit*
where
mmset-of-mlit' (*Propagated L C*) = *Propagated L* (*mset C*) |
mmset-of-mlit' (*Marked L i*) = *Marked L i*

lemma *lit-of-mmset-of-mlit'*[*simp*]:
lit-of (*mmset-of-mlit' xa*) = *lit-of xa*
by (*induction xa*) *auto*

abbreviation *trail* **where**
trail S ≡ *map mmset-of-mlit'* (*raw-trail S*)

abbreviation *clauses-of-l* **where**
clauses-of-l ≡ λ*L*. *mset* (*map mset L*)

global-interpretation *state*$_W$*-ops*
mset::*'v literal list* ⇒ *'v clause*
op # *remove1*

clauses-of-l op @ λ*L C*. *L* ∈ *set C op* # λ*C*. *remove1-cond* (λ*L*. *mset L* = *mset C*)

mset λ*xs ys*. *case-prod append* (*fold* (λ*x* (*ys*, *zs*). (*remove1 x ys*, *x* # *zs*)) *xs* (*ys*, []))
op # *remove1*

id id

λ(*M*, -). *map mmset-of-mlit' M* λ(*M*, -). *hd M*
λ(-, *N*, -). *N*
λ(-, -, *U*, -). *U*
λ(-, -, -, *k*, -). *k*
λ(-, -, -, -, *C*). *C*

λ*L* (*M*, *S*). (*L* # *M*, *S*)
λ(*M*, *S*). (*tl M*, *S*)
λ*C* (*M*, *N*, *S*). (*M*, *C* # *N*, *S*)
λ*C* (*M*, *N*, *U*, *S*). (*M*, *N*, *C* # *U*, *S*)
λ*C* (*M*, *N*, *U*, *S*). (*M*, *filter* (λ*L*. *mset L* ≠ *mset C*) *N*, *filter* (λ*L*. *mset L* ≠ *mset C*) *U*, *S*)
λ(*k*::*nat*) (*M*, *N*, *U*, -, *D*). (*M*, *N*, *U*, *k*, *D*)
λ*D* (*M*, *N*, *U*, *k*, -). (*M*, *N*, *U*, *k*, *D*)
λ*N*. ([], *N*, [], *0*, *None*)
λ(-, *N*, *U*, -). ([], *N*, *U*, *0*, *None*)
apply *unfold-locales* **by** (*auto simp*: *hd-map comp-def map-tl ac-simps*
union-mset-list mset-map-mset-remove1-cond ex-mset)

lemma *mmset-of-mlit'-mmset-of-mlit*: *mmset-of-mlit' l* = *mmset-of-mlit l*
apply (*induct l*)
apply *auto*
done

lemma *clauses-of-l-filter-removeAll*:

```
  clauses-of-l [L←a . mset L ≠ mset C] = mset (removeAll (mset C) (map mset a))
  by (induct a) auto

interpretation state_W
  mset::'v literal list ⇒ 'v clause
  op # remove1

  clauses-of-l op @ λL C. L ∈ set C op # λC. remove1-cond (λL. mset L = mset C)

  mset λxs ys. case-prod append (fold (λx (ys, zs). (remove1 x ys, x # zs)) xs (ys, []))
  op # remove1

  id id

  λ(M, -). map mmset-of-mlit' M λ(M, -). hd M
  λ(-, N, -). N
  λ(-, -, U, -). U
  λ(-, -, -, k, -). k
  λ(-, -, -, -, C). C

  λL (M, S). (L # M, S)
  λ(M, S). (tl M, S)
  λC (M, N, S). (M, C # N, S)
  λC (M, N, U, S). (M, N, C # U, S)
  λC (M, N, U, S). (M, filter (λL. mset L ≠ mset C) N, filter (λL. mset L ≠ mset C) U, S)
  λ(k::nat) (M, N, U, -, D). (M, N, U, k, D)
  λD (M, N, U, k, -). (M, N, U, k, D)
  λN. ([], N, [], 0, None)
  λ(-, N, U, -). ([], N, U, 0, None)
  apply unfold-locales
  apply (rename-tac S, case-tac S)
  by (auto simp: hd-map comp-def map-tl ac-simps clauses-of-l-filter-removeAll
    mmset-of-mlit'-mmset-of-mlit)

global-interpretation conflict-driven-clause-learning_W
  mset::'v literal list ⇒ 'v clause
  op # remove1

  clauses-of-l op @ λL C. L ∈ set C op # λC. remove1-cond (λL. mset L = mset C)

  mset λxs ys. case-prod append (fold (λx (ys, zs). (remove1 x ys, x # zs)) xs (ys, []))
  op # remove1

  id id

  λ(M, -). map mmset-of-mlit' M λ(M, -). hd M
  λ(-, N, -). N
  λ(-, -, U, -). U
  λ(-, -, -, k, -). k
  λ(-, -, -, -, C). C

  λL (M, S). (L # M, S)
  λ(M, S). (tl M, S)
  λC (M, N, S). (M, C # N, S)
  λC (M, N, U, S). (M, N, C # U, S)
```

λC (*M*, *N*, *U*, *S*). (*M*, *filter* (*λL. mset L* ≠ *mset C*) *N*, *filter* (*λL. mset L* ≠ *mset C*) *U*, *S*)
λ(*k*::*nat*) (*M*, *N*, *U*, -, *D*). (*M*, *N*, *U*, *k*, *D*)
λD (*M*, *N*, *U*, *k*, -). (*M*, *N*, *U*, *k*, *D*)
λN. ([], *N*, [], *0*, *None*)
λ(-, *N*, *U*, -). ([], *N*, *U*, *0*, *None*)
by *intro-locales*

declare *state-simp*[*simp del*] *raw-clauses-def*[*simp*] *state-eq-def*[*simp*]
notation *state-eq* (**infix** ∼ *50*)
term *reduce-trail-to*

lemma *reduce-trail-to-map*[*simp*]:
reduce-trail-to (*map f M1*) = *reduce-trail-to M1*
by (*rule ext*) (*auto intro*: *reduce-trail-to-length*)

20.3 CDCL Implementation

20.3.1 Definition of the rules

Types **lemma** *true-clss-remdups*[*simp*]:
I ⊨*s* (*mset* ∘ *remdups*) ‘ *N* ⟷ *I* ⊨*s mset* ‘ *N*
by (*simp add*: *true-clss-def*)

lemma *satisfiable-mset-remdups*[*simp*]:
satisfiable ((*mset* ∘ *remdups*) ‘ *N*) ⟷ *satisfiable* (*mset* ‘ *N*)
unfolding *satisfiable-carac*[*symmetric*] **by** *simp*

We need some functions to convert between our abstract state *nat* $cdcl_W$*-state* and the concrete state *'v* $cdcl_W$*-state-inv-st*.

abbreviation *convertC* :: *'a list option* ⇒ *'a multiset option* **where**
convertC ≡ *map-option mset*

lemma *convert-Propagated*[*elim*!]:
mmset-of-mlit' z = *Propagated L C* ⟹ (∃ *C'*. *z* = *Propagated L C'* ∧ *C* = *mset C'*)
by (*cases z*) *auto*

lemma *get-rev-level-map-convert*:
get-rev-level (*map mmset-of-mlit' M*) *n x* = *get-rev-level M n x*
by (*induction M arbitrary*: *n rule*: *marked-lit-list-induct*) *auto*

lemma *get-level-map-convert*[*simp*]:
get-level (*map mmset-of-mlit' M*) = *get-level M*
using *get-rev-level-map-convert*[*of rev M*] **by** (*simp add*: *rev-map*)

lemma *get-rev-level-map-mmsetof-mlit*[*simp*]:
get-rev-level (*map mmset-of-mlit M*) = *get-rev-level M*
by (*induction M rule*: *marked-lit-list-induct*) (*auto intro*!: *ext*)

lemma *get-level-map-mmsetof-mlit*[*simp*]:
get-level (*map mmset-of-mlit M*) = *get-level M*
using *get-rev-level-map-mmsetof-mlit*[*of rev M*] **unfolding** *rev-map* **by** *simp*

lemma *get-maximum-level-map-convert*[*simp*]:
get-maximum-level (*map mmset-of-mlit' M*) *D* = *get-maximum-level M D*
by (*induction D*) (*auto simp add*: *get-maximum-level-plus*)

```
lemma get-all-levels-of-marked-map-convert[simp]:
  get-all-levels-of-marked (map mmset-of-mlit' M) = (get-all-levels-of-marked M)
  by (induction M rule: marked-lit-list-induct) auto

lemma reduce-trail-to-empty-trail[simp]:
  reduce-trail-to F ([], aa, ab, ac, b) = ([], aa, ab, ac, b)
  using reduce-trail-to.simps by auto

lemma raw-trail-reduce-trail-to-length-le:
  assumes length F > length (raw-trail S)
  shows raw-trail (reduce-trail-to F S) = []
  using assms trail-reduce-trail-to-length-le[of S F]
  by (cases S, cases reduce-trail-to F S) auto

lemma reduce-trail-to:
  reduce-trail-to F S =
    ((if length (raw-trail S) ≥ length F
    then drop (length (raw-trail S) − length F) (raw-trail S)
    else []), raw-init-clss S, raw-learned-clss S, raw-backtrack-lvl S, raw-conflicting S)
    (is ?S = -)
proof (induction F S rule: reduce-trail-to.induct)
  case (1 F S) note IH = this
  show ?case
    proof (cases raw-trail S)
      case Nil
      then show ?thesis using IH by (cases S) auto
    next
      case (Cons L M)
      then show ?thesis
        apply (cases Suc (length M) > length F)
         prefer 2 using IH reduce-trail-to-length-ne[of S F] apply (cases S) apply auto[]
        apply (subgoal-tac Suc (length M) − length F = Suc (length M − length F))
        using reduce-trail-to-length-ne[of S F] IH by (cases S) (auto simp add:)
    qed
qed
```

Definition an abstract type

```
typedef 'v  cdcl_W-state-inv =  {S::'v cdcl_W-state-inv-st. cdcl_W-all-struct-inv S}
  morphisms rough-state-of state-of
proof
  show ([],[], [], 0, None) ∈ {S. cdcl_W-all-struct-inv S}
    by (auto simp add: cdcl_W-all-struct-inv-def)
qed

instantiation cdcl_W-state-inv :: (type) equal
begin
definition equal-cdcl_W-state-inv :: 'v cdcl_W-state-inv ⇒ 'v cdcl_W-state-inv ⇒ bool where
 equal-cdcl_W-state-inv S S' = (rough-state-of S = rough-state-of S')
instance
  by standard (simp add: rough-state-of-inject equal-cdcl_W-state-inv-def)
end

lemma lits-of-map-convert[simp]: lits-of-l (map mmset-of-mlit' M) = lits-of-l M
  by (induction M rule: marked-lit-list-induct) simp-all
```

```
lemma undefined-lit-map-convert[iff]:
  undefined-lit (map mmset-of-mlit' M) L ⟷ undefined-lit M L
  by (auto simp add: defined-lit-map image-image mmset-of-mlit'-mmset-of-mlit)

lemma true-annot-map-convert[simp]: map mmset-of-mlit' M ⊨a N ⟷ M ⊨a N
  by (induction M rule: marked-lit-list-induct) (simp-all add: true-annot-def
    mmset-of-mlit'-mmset-of-mlit lits-of-def)

lemma true-annots-map-convert[simp]: map mmset-of-mlit' M ⊨as N ⟷ M ⊨as N
  unfolding true-annots-def by auto

lemmas propagateE
lemma find-first-unit-clause-some-is-propagate:
  assumes H: find-first-unit-clause (N @ U) M = Some (L, C)
  shows propagate (M, N, U, k, None) (Propagated L C # M, N, U, k, None)
  using assms
  by (auto dest!: find-first-unit-clause-some intro!: propagate-rule)
```

20.3.2 The Transitions

Propagate **definition** *do-propagate-step* **where**

```
do-propagate-step S =
  (case S of
    (M, N, U, k, None) ⇒
      (case find-first-unit-clause (N @ U) M of
        Some (L, C) ⇒ (Propagated L C # M, N, U, k, None)
      | None ⇒ (M, N, U, k, None))
  | S ⇒ S)

lemma do-propgate-step:
  do-propagate-step S ≠ S ⟹ propagate S (do-propagate-step S)
  apply (cases S, cases conflicting S)
  using find-first-unit-clause-some-is-propagate[of raw-init-clss S raw-learned-clss S]
  by (auto simp add: do-propagate-step-def split: option.splits)

lemma do-propagate-step-option[simp]:
  conflicting S ≠ None ⟹ do-propagate-step S = S
  unfolding do-propagate-step-def by (cases S, cases conflicting S) auto
thm prod-cases

lemma do-propagate-step-no-step:
  assumes dist: ∀c∈set (raw-clauses S). distinct c and
  prop-step: do-propagate-step S = S
  shows no-step propagate S
proof (standard, standard)
  fix T
  assume propagate S T
  then obtain C L where
    toSS: conflicting S = None and
    C: C ∈ set (raw-clauses S) and
    L: L ∈ set C and
    MC: raw-trail S ⊨as CNot (mset (remove1 L C)) and
    T:  T ∼ raw-cons-trail (Propagated L C) S and
    undef: undefined-lit (raw-trail S) L
    apply (cases S rule: prod-cases5)
    by (elim propagateE) simp
  let ?M = raw-trail S
  let ?N = raw-init-clss S
  let ?U = raw-learned-clss S
  let ?k = raw-backtrack-lvl S
  let ?D = None
  have S: S = (?M, ?N, ?U, ?k, ?D)
    using toSS by (cases S, cases conflicting S) simp-all

  have find-first-unit-clause (?N @ ?U) ?M ≠ None
    apply (rule dist find-first-unit-clause-none[of C ?N @ ?U ?M L, OF -])
        using C dist apply auto[]
       using C apply auto[1]
      using MC apply auto[1]
     using undef apply auto[1]
    using L by auto
  then show False using prop-step S unfolding do-propagate-step-def by (cases S) auto
qed
```

Conflict **fun** *find-conflict* **where**

```
find-conflict M [] = None |
find-conflict M (N # Ns) = (if (∀ c ∈ set N. −c ∈ lits-of-l M) then Some N else find-conflict M Ns)

lemma find-conflict-Some:
  find-conflict M Ns = Some N ⟹ N ∈ set Ns ∧ M ⊨as CNot (mset N)
  by (induction Ns rule: find-conflict.induct)
    (auto split: if-split-asm)

lemma find-conflict-None:
  find-conflict M Ns = None ⟷ (∀ N ∈ set Ns. ¬M ⊨as CNot (mset N))
  by (induction Ns) auto

lemma find-conflict-None-no-confl:
  find-conflict M (N@U) = None ⟷ no-step conflict (M, N, U, k, None)
  by (auto simp add: find-conflict-None conflict.simps)

definition do-conflict-step where
do-conflict-step S =
  (case S of
    (M, N, U, k, None) ⇒
      (case find-conflict M (N @ U) of
        Some a ⇒ (M, N, U, k, Some a)
      | None ⇒ (M, N, U, k, None))
  | S ⇒ S)

lemma do-conflict-step:
  do-conflict-step S ≠ S ⟹ conflict S (do-conflict-step S)
  apply (cases S, cases conflicting S)
  unfolding conflict.simps do-conflict-step-def
  by (auto dest!:find-conflict-Some split: option.splits simp: state-eq-def)

lemma do-conflict-step-no-step:
  do-conflict-step S = S ⟹ no-step conflict S
  apply (cases S, cases conflicting S)
  unfolding do-conflict-step-def
  using find-conflict-None-no-confl[of raw-trail S raw-init-clss S raw-learned-clss S
```

```
    raw-backtrack-lvl S]
  by (auto split: option.split elim: conflictE)

lemma do-conflict-step-option[simp]:
  conflicting S ≠ None ⟹ do-conflict-step S = S
  unfolding do-conflict-step-def by (cases S, cases conflicting S) auto

lemma do-conflict-step-conflicting[dest]:
  do-conflict-step S ≠ S ⟹ conflicting (do-conflict-step S) ≠ None
  unfolding do-conflict-step-def by (cases S, cases conflicting S) (auto split: option.splits)

definition do-cp-step where
do-cp-step S =
  (do-propagate-step o do-conflict-step) S

lemma cp-step-is-cdcl_W-cp:
  assumes H: do-cp-step S ≠ S
  shows cdcl_W-cp S (do-cp-step S)
proof –
  show ?thesis
  proof (cases do-conflict-step S ≠ S)
    case True
    then have do-propagate-step (do-conflict-step S) = do-conflict-step S
      by auto
    then show ?thesis
      by (auto simp add: do-conflict-step do-conflict-step-conflicting do-cp-step-def True)
  next
    case False
    then have confl[simp]: do-conflict-step S = S by simp
    show ?thesis
      proof (cases do-propagate-step S = S)
        case True
        then show ?thesis
        using H by (simp add: do-cp-step-def)
      next
        case False
        let ?S = S
        let ?T = (do-propagate-step S)
        let ?U = (do-conflict-step (do-propagate-step S))
        have propa: propagate S ?T using False do-propgate-step by blast
        moreover have ns: no-step conflict S using confl do-conflict-step-no-step by blast
        ultimately show ?thesis
          using cdcl_W-cp.intros(2)[of ?S ?T] confl unfolding do-cp-step-def by auto
      qed
  qed
qed

lemma do-cp-step-eq-no-prop-no-confl:
  do-cp-step S = S ⟹ do-conflict-step S = S ∧ do-propagate-step S = S
  by (cases S, cases raw-conflicting S)
    (auto simp add: do-conflict-step-def do-propagate-step-def do-cp-step-def split: option.splits)

lemma no-cdcl_W-cp-iff-no-propagate-no-conflict:
  no-step cdcl_W-cp S ⟷ no-step propagate S ∧ no-step conflict S
  by (auto simp: cdcl_W-cp.simps)
```

lemma *do-cp-step-eq-no-step*:
  assumes
    H: *do-cp-step S* = *S* **and**
    ∀ *c* ∈ *set* (*raw-init-clss S* @ *raw-learned-clss S*). *distinct c*
  shows *no-step cdcl*$_W$*-cp S*
  unfolding *no-cdcl*$_W$*-cp-iff-no-propagate-no-conflict*
  using *assms* **apply** (*cases S*, *cases conflicting S*)
  using *do-propagate-step-no-step*[*of S*]
  by (*auto dest*!: *do-cp-step-eq-no-prop-no-confl*[*simplified*] *do-conflict-step-no-step*
    split: *option.splits*)

lemma *cdcl*$_W$*-cp-cdcl*$_W$*-st*: *cdcl*$_W$*-cp S S'* ⟹ *cdcl*$_W$** *S S'*
  by (*simp add*: *cdcl*$_W$*-cp-tranclp-cdcl*$_W$ *tranclp-into-rtranclp*)

lemma *cdcl*$_W$*-all-struct-inv-rough-state*[*simp*]: *cdcl*$_W$*-all-struct-inv* (*rough-state-of S*)
  using *rough-state-of* **by** *auto*

lemma [*simp*]: *cdcl*$_W$*-all-struct-inv S* ⟹ *rough-state-of* (*state-of S*) = *S*
  by (*simp add*: *state-of-inverse*)

lemma *rough-state-of-state-of-do-cp-step*[*simp*]:
  rough-state-of (*state-of* (*do-cp-step* (*rough-state-of S*))) = *do-cp-step* (*rough-state-of S*)
proof −
  have *cdcl*$_W$*-all-struct-inv* (*do-cp-step* (*rough-state-of S*))
    apply (*cases do-cp-step* (*rough-state-of S*) = (*rough-state-of S*))
      apply *simp*
    using *cp-step-is-cdcl*$_W$*-cp*[*of rough-state-of S*] *cdcl*$_W$*-all-struct-inv-rough-state*[*of S*]
    cdcl$_W$*-cp-cdcl*$_W$*-st rtranclp-cdcl*$_W$*-all-struct-inv-inv* **by** *blast*
  then show *?thesis* **by** *auto*
qed

Skip **fun** *do-skip-step* :: *'v cdcl*$_W$*-state-inv-st* ⇒ *'v cdcl*$_W$*-state-inv-st* **where**
do-skip-step (*Propagated L C* # *Ls*,*N*,*U*,*k*, *Some D*) =
  (*if* −*L* ∉ *set D* ∧ *D* ≠ []
  then (*Ls*, *N*, *U*, *k*, *Some D*)
  else (*Propagated L C* #*Ls*, *N*, *U*, *k*, *Some D*)) |
do-skip-step S = *S*

lemma *do-skip-step*:
  do-skip-step S ≠ *S* ⟹ *skip S* (*do-skip-step S*)
  apply (*induction S rule*: *do-skip-step.induct*)
  by (*auto simp add*: *skip.simps*)

lemma *do-skip-step-no*:
  do-skip-step S = *S* ⟹ *no-step skip S*
  by (*induction S rule*: *do-skip-step.induct*)
    (*auto simp add*: *other split*: *if-split-asm elim*!: *skipE*)

lemma *do-skip-step-trail-is-None*[*iff*]:
  do-skip-step S = (*a*, *b*, *c*, *d*, *None*) ⟷ *S* = (*a*, *b*, *c*, *d*, *None*)
  by (*cases S rule*: *do-skip-step.cases*) *auto*

Resolve **fun** *maximum-level-code*:: *'a literal list* ⇒ (*'a*, *nat*, *'a literal list*) *marked-lit list* ⇒ *nat*
  where

```
maximum-level-code [] - = 0 |
maximum-level-code (L # Ls) M = max (get-level M L) (maximum-level-code Ls M)

lemma maximum-level-code-eq-get-maximum-level[code, simp]:
  maximum-level-code D M = get-maximum-level M (mset D)
  by (induction D) (auto simp add: get-maximum-level-plus)

fun do-resolve-step :: 'v cdcl_W-state-inv-st ⇒ 'v cdcl_W-state-inv-st where
do-resolve-step (Propagated L C # Ls, N, U, k, Some D) =
  (if −L ∈ set D ∧ maximum-level-code (remove1 (−L) D) (Propagated L C # Ls) = k
  then (Ls, N, U, k, Some (remdups (remove1 L C @ remove1 (−L) D)))
  else (Propagated L C # Ls, N, U, k, Some D)) |
do-resolve-step S = S

lemma do-resolve-step:
  cdcl_W-all-struct-inv S ⟹ do-resolve-step S ≠ S
  ⟹ resolve S (do-resolve-step S)
proof (induction S rule: do-resolve-step.induct)
  case (1 L C M N U k D)
  then have
    LD: − L ∈ set D and
    M: maximum-level-code (remove1 (−L) D) (Propagated L C # M) = k
    by (cases mset D − {#− L#} = {#},
        auto dest!: get-maximum-level-exists-lit-of-max-level[of - Propagated L C # M]
        split: if-split-asm)+
  have every-mark-is-a-conflict (Propagated L C # M, N, U, k, Some D)
    using 1(1) unfolding cdcl_W-all-struct-inv-def cdcl_W-conflicting-def by fast
  then have LC: L ∈ set C by fastforce
  then obtain C' where C: mset C = C' + {#L#}
    by (metis add.commute in-multiset-in-set insert-DiffM)
  obtain D' where D: mset D = D' + {#−L#}
    using ‹− L ∈ set D› by (metis add.commute in-multiset-in-set insert-DiffM)
  have D'L:  D' + {#− L#} − {#−L#} = D' by (auto simp add: multiset-eq-iff)

  have CL: mset C − {#L#} + {#L#} = mset C using ‹L ∈ set C› by (auto simp add: multiset-eq-iff)
  have max: get-maximum-level (Propagated L (C' + {#L#}) # map mmset-of-mlit' M) D' = k
    using M[simplified] unfolding maximum-level-code-eq-get-maximum-level C[symmetric] CL
    by (metis D D'L get-maximum-level-map-convert list.simps(9) mmset-of-mlit'.simps(1))
  have distinct-mset (mset C) and distinct-mset (mset D)
    using ‹cdcl_W-all-struct-inv (Propagated L C # M, N, U, k, Some D)›
    unfolding cdcl_W-all-struct-inv-def distinct-cdcl_W-state-def
    by auto
  then have conf: (mset C − {#L#}) #∪ (mset D − {#− L#}) =
    remdups-mset (mset C − {#L#} + (mset D − {#− L#}))
    by (auto simp: distinct-mset-rempdups-union-mset)
  show ?case
    apply (rule resolve-rule)
    using LC LD max M conf C D by (auto simp: subset-mset.sup.commute)
qed auto

lemma do-resolve-step-no:
  do-resolve-step S = S ⟹ no-step resolve S
  apply (cases S; cases (raw-trail S); cases raw-conflicting S)
  by (auto
    elim!: resolveE  split: if-split-asm
    dest!: union-single-eq-member
    simp del: in-multiset-in-set get-maximum-level-map-convert
    simp: get-maximum-level-map-convert[symmetric] do-resolve-step)

lemma rough-state-of-state-of-resolve[simp]:
  cdcl_W-all-struct-inv S ⟹ rough-state-of (state-of (do-resolve-step S)) = do-resolve-step S
  apply (rule state-of-inverse)
  apply (cases do-resolve-step S = S)
   apply simp
  by (blast dest: other resolve bj do-resolve-step cdcl_W-all-struct-inv-inv)

lemma do-resolve-step-trail-is-None[iff]:
  do-resolve-step S = (a, b, c, d, None) ⟷ S = (a, b, c, d, None)
  by (cases S rule: do-resolve-step.cases) auto
```

Backjumping

```
fun find-level-decomp where
find-level-decomp M [] D k = None |
find-level-decomp M (L # Ls) D k =
  (case (get-level M L, maximum-level-code (D @ Ls) M) of
    (i, j) ⇒ if i = k ∧ j < i then Some (L, j) else find-level-decomp M Ls (L#D) k
  )

lemma find-level-decomp-some:
  assumes find-level-decomp M Ls D k = Some (L, j)
  shows L ∈ set Ls ∧ get-maximum-level M (mset (remove1 L (Ls @ D))) = j ∧ get-level M L = k
  using assms
proof (induction Ls arbitrary: D)
  case Nil
  then show ?case by simp
next
  case (Cons L' Ls) note IH = this(1) and H = this(2)

  def find ≡ (if get-level M L' ≠ k ∨ ¬ get-maximum-level M (mset D + mset Ls) < get-level M L'
    then find-level-decomp M Ls (L' # D) k
    else Some (L', get-maximum-level M (mset D + mset Ls)))
  have a1: ⋀D. find-level-decomp M Ls D k = Some (L, j) ⟹
     L ∈ set Ls ∧ get-maximum-level M (mset Ls + mset D − {#L#}) = j ∧ get-level M L = k
    using IH by simp
  have a2: find = Some (L, j)
    using H unfolding find-def by (auto split: if-split-asm)
  { assume Some (L', get-maximum-level M (mset D + mset Ls)) ≠ find
    then have f3: L ∈ set Ls and get-maximum-level M (mset Ls + mset (L' # D) − {#L#}) = j
      using a1 IH a2 unfolding find-def by meson+
    moreover then have mset Ls + mset D − {#L#} + {#L'#} = {#L'#} + mset D + (mset Ls
− {#L#})
      by (auto simp: ac-simps multiset-eq-iff Suc-leI)
    ultimately have f4: get-maximum-level M (mset Ls + mset D − {#L#} + {#L'#}) = j
      by (metis add.commute diff-union-single-conv in-multiset-in-set mset.simps(2))
  } note f4 = this
  have {#L'#} + (mset Ls + mset D) = mset Ls + (mset D + {#L'#})
      by (auto simp: ac-simps)
  then have
    (L = L' ⟶ get-maximum-level M (mset Ls + mset D) = j ∧ get-level M L' = k) and
    (L ≠ L' ⟶ L ∈ set Ls ∧ get-maximum-level M (mset Ls + mset D − {#L#} + {#L'#}) = j ∧
      get-level M L = k)
    using f4 a2 a1[of L' # D] unfolding find-def by (metis (no-types) add-diff-cancel-left'
      mset.simps(2) option.inject prod.inject union-commute)+
  then show ?case by simp
qed

lemma find-level-decomp-none:
  assumes find-level-decomp M Ls E k = None and mset (L#D) = mset (Ls @ E)
  shows ¬(L ∈ set Ls ∧ get-maximum-level M (mset D) < k ∧ k = get-level M L)
  using assms
proof (induction Ls arbitrary: E L D)
  case Nil
  then show ?case by simp
next
  case (Cons L' Ls) note IH = this(1) and find-none = this(2) and LD = this(3)
  have mset D + {#L'#} = mset E + (mset Ls + {#L'#})  ⟹ mset D = mset E + mset Ls
    by (metis add-right-imp-eq union-assoc)
  then show ?case
    using find-none IH[of L' # E L D] LD by (auto simp add: ac-simps split: if-split-asm)
qed

fun bt-cut where
bt-cut i (Propagated - - # Ls) = bt-cut i Ls |
bt-cut i (Marked K k # Ls) = (if k = Suc i then Some (Marked K k # Ls) else bt-cut i Ls) |
bt-cut i [] = None

lemma bt-cut-some-decomp:
  bt-cut i M = Some M' ⟹ ∃K M2 M1. M = M2 @ M' ∧ M' = Marked K (i+1) # M1
  by (induction i M rule: bt-cut.induct) (auto split: if-split-asm)

lemma bt-cut-not-none: M = M2 @ Marked K (Suc i) # M' ⟹ bt-cut i M ≠ None
  by (induction M2 arbitrary: M rule: marked-lit-list-induct) auto

lemma get-all-marked-decomposition-ex:
  ∃N. (Marked K (Suc i) # M', N) ∈ set (get-all-marked-decomposition (M2@Marked K (Suc i) #
M'))
  apply (induction M2 rule: marked-lit-list-induct)
    apply auto[2]
  by (rename-tac L m xs,  case-tac get-all-marked-decomposition (xs @ Marked K (Suc i) # M'))
  auto

lemma bt-cut-in-get-all-marked-decomposition:
  bt-cut i M = Some M' ⟹ ∃M2. (M', M2) ∈ set (get-all-marked-decomposition M)
  by (auto dest!: bt-cut-some-decomp simp add: get-all-marked-decomposition-ex)

fun do-backtrack-step where
do-backtrack-step (M, N, U, k, Some D) =
  (case find-level-decomp M D [] k of
    None ⇒ (M, N, U, k, Some D)
  | Some (L, j) ⇒
    (case bt-cut j M of
      Some (Marked - - # Ls) ⇒ (Propagated L D # Ls, N, D # U, j, None)
    | - ⇒ (M, N, U, k, Some D))
  ) |
do-backtrack-step S = S
```

```
lemma get-all-marked-decomposition-map-convert:
  (get-all-marked-decomposition (map mmset-of-mlit' M)) =
    map (λ(a, b). (map mmset-of-mlit' a, map mmset-of-mlit' b)) (get-all-marked-decomposition M)
  apply (induction M rule: marked-lit-list-induct)
    apply simp
  by (rename-tac L l xs, case-tac get-all-marked-decomposition xs; auto)+

lemma do-backtrack-step:
  assumes
    db: do-backtrack-step S ≠ S and
    inv: cdcl_W-all-struct-inv S
  shows backtrack S (do-backtrack-step S)
  proof (cases S, cases raw-conflicting S, goal-cases)
    case (1 M N U k E)
    then show ?case using db by auto
  next
    case (2 M N U k E C) note S = this(1) and confl = this(2)
    have E: E = Some C using S confl by auto

    obtain L j where fd: find-level-decomp M C [] k = Some (L, j)
      using db unfolding S E  by (cases C) (auto split: if-split-asm option.splits)
    have
      L ∈ set C and
      j: get-maximum-level M (mset (remove1 L C)) = j and
      levL: get-level M L = k
      using find-level-decomp-some[OF fd] by auto
    obtain C' where C: mset C = mset C' + {#L#}
      using ‹L ∈ set C› by (metis add.commute ex-mset in-multiset-in-set insert-DiffM)
    obtain M_2 where M_2: bt-cut j M = Some M_2
      using db fd unfolding S E by (auto split: option.splits)
    obtain M1 K where M1: M_2 = Marked K (Suc j) # M1
      using bt-cut-some-decomp[OF M_2] by (cases M_2) auto
    obtain c where c: M = c @ Marked K (Suc j) # M1
       using bt-cut-in-get-all-marked-decomposition[OF M_2]
       unfolding M1 by fastforce
    have get-all-levels-of-marked (map mmset-of-mlit' M) = rev [1..<Suc k]
      using inv unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def S by auto
    from arg-cong[OF this, of λa. Suc j ∈ set a] have j ≤ k unfolding c by auto
    have max-l-j: maximum-level-code C' M = j
      using db fd M_2 C unfolding S E by (auto
          split: option.splits list.splits marked-lit.splits
          dest!: find-level-decomp-some)[1]
    have get-maximum-level M (mset C) ≥ k
      using ‹L ∈ set C› levL get-maximum-level-ge-get-level by (metis set-mset-mset)
    moreover have get-maximum-level M (mset C) ≤ k
      using get-maximum-level-exists-lit-of-max-level[of mset C M] inv
        cdcl_W-M-level-inv-get-level-le-backtrack-lvl[of S]
      unfolding C cdcl_W-all-struct-inv-def S by (auto dest: sym[of get-level - -])
    ultimately have get-maximum-level M (mset C) = k by auto

    obtain M2 where M2: (M_2, M2) ∈ set (get-all-marked-decomposition M)
      using bt-cut-in-get-all-marked-decomposition[OF M_2] by metis
    have decomp:
     (Marked K (Suc (get-maximum-level M (remove1-mset L (mset C)))) # (map mmset-of-mlit' M1),
      (map mmset-of-mlit' M2)) ∈
      set (get-all-marked-decomposition (map mmset-of-mlit' M))
      using imageI[of - - λ(a, b). (map mmset-of-mlit' a, map mmset-of-mlit' b), OF M2] j
      unfolding S E M1 by (auto simp add: get-all-marked-decomposition-map-convert)
    have red: (reduce-trail-to (map mmset-of-mlit' M1)
      (M, N, C # U, get-maximum-level M (remove1-mset L (mset C)), None))
      = (M1, N, C # U, get-maximum-level M (remove1-mset L (mset C)), None)
     using  M2 M1 by (auto simp: reduce-trail-to)
    show ?case
      apply (rule backtrack-rule)
      using M_2 fd confl ‹L ∈ set C› j decomp levL ‹get-maximum-level M (mset C) = k›
      unfolding S E M1 apply (auto simp: mset-map)[6]
      unfolding CDCL-W-Implementation.state-eq-def
      using M_2 fd confl ‹L ∈ set C› j decomp levL ‹get-maximum-level M (mset C) = k› red
      unfolding S E M1
      by auto
qed

lemma map-eq-list-length:
  map f L = L' ⟹ length L = length L'
  by auto

lemma map-mmset-of-mlit-eq-cons:
  assumes map mmset-of-mlit' M = a @ c
  obtains a' c' where
     M = a' @  c' and
     a = map mmset-of-mlit' a' and
     c = map mmset-of-mlit' c'
  using that[of take (length a) M drop (length a) M]
  assms by (metis append-eq-conv-conj append-take-drop-id drop-map take-map)

lemma do-backtrack-step-no:
  assumes
    db: do-backtrack-step S = S and
    inv: cdcl_W-all-struct-inv S
  shows no-step backtrack S
proof (rule ccontr, cases S, cases conflicting S, goal-cases)
  case 1
  then show ?case using db by (auto split: option.splits elim: backtrackE)
next
  case (2 M N U k E C) note bt = this(1) and S = this(2) and confl = this(3)
  obtain K j M1 M2 L D where
    CE: raw-conflicting S = Some D and
    LD: L ∈# mset D and
    decomp: (Marked K (Suc j) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) and
    levL: get-level (raw-trail S) L = raw-backtrack-lvl S  and
    k: get-level (raw-trail S) L = get-maximum-level (raw-trail S) (mset D) and
    j: get-maximum-level (raw-trail S) (remove1-mset L (mset D)) ≡ j and
    undef: undefined-lit M1 L
    using bt apply clarsimp
    apply (elim backtrack-levE)
      using inv unfolding  cdcl_W-all-struct-inv-def apply fast
    apply (cases S)
    by (auto simp add: get-all-marked-decomposition-map-convert)

  obtain c where c: trail S = c @ M2 @ Marked K (Suc j) # M1
```

```
    using decomp by blast
  have get-all-levels-of-marked (trail S) = rev [1..<Suc k]
    using inv unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def S by auto
  from arg-cong[OF this, of λa. Suc j ∈ set a] have k > j
    unfolding c by (auto simp: get-all-marked-decomposition-map-convert)
  have [simp]: L ∈ set D
    using LD by auto
  have CD: C = mset D
    using CE confl by auto
  obtain D′ where
    E: E = Some D and
    DD′: mset D = {#L#} + mset D′
    using that[of remove1 L D]
    using S CE confl LD by (auto simp add: insert-DiffM)
  have find-level-decomp M D [] k ≠ None
    apply rule
    apply (drule find-level-decomp-none[of - - - - L D′])
    using DD′ ‹k > j› mset-eq-setD S levL unfolding k[symmetric] j[symmetric]
    by (auto simp: ac-simps)
  then obtain L′ j′ where fd-some: find-level-decomp M D [] k = Some (L′, j′)
    by (cases find-level-decomp M D [] k) auto
  have L′: L′ = L
    proof (rule ccontr)
      assume ¬ ?thesis
      then have L′ ∈# mset (remove1 L D)
        by (metis fd-some find-level-decomp-some in-set-remove1 set-mset-mset)
      then have get-level M L′ ≤ get-maximum-level M (mset (remove1 L D))
        using get-maximum-level-ge-get-level by blast
      then show False using ‹k > j› j find-level-decomp-some[OF fd-some] S DD′ by auto
    qed
  then have j′: j′ = j using find-level-decomp-some[OF fd-some] j S DD′ by auto

  obtain c′ M1′ where cM: M = c′ @ Marked K (Suc j) # M1′
    apply (rule map-mmset-of-mlit-eq-cons[of M c @ M2 Marked K (Suc j) # M1])
      using c S apply simp
    apply (rule map-mmset-of-mlit-eq-cons[of - [Marked K (Suc j)]  M1])
     apply auto[]
    apply (rename-tac a b′ aa b, case-tac aa)
     apply auto[]
    apply (rename-tac a b′ aa b, case-tac aa)
    by auto
  have btc-none: bt-cut j M ≠ None
    apply (rule bt-cut-not-none[of M ])
    using cM by simp
  show ?case using db unfolding S  E
    by (auto split: option.splits list.splits marked-lit.splits
      simp add: fd-some  L′ j′ btc-none
      dest: bt-cut-some-decomp)
qed

lemma rough-state-of-state-of-backtrack[simp]:
  assumes inv: cdcl_W-all-struct-inv S
  shows rough-state-of (state-of (do-backtrack-step S))= do-backtrack-step S
proof (rule state-of-inverse)
  have f2: backtrack S (do-backtrack-step S) ∨ do-backtrack-step S = S
```

```
    using do-backtrack-step inv by blast
  have ⋀p. ¬ cdcl_W-o S p ∨ cdcl_W-all-struct-inv p
    using inv cdcl_W-all-struct-inv-inv other by blast
  then have do-backtrack-step S = S ∨ cdcl_W-all-struct-inv (do-backtrack-step S)
    using f2 inv cdcl_W-o.intros cdcl_W-bj.intros by blast
  then show do-backtrack-step S ∈ {S. cdcl_W-all-struct-inv S}
    using inv by fastforce
qed
```

Decide **fun** *do-decide-step* **where**

```
do-decide-step (M, N, U, k, None) =
  (case find-first-unused-var N (lits-of-l M) of
    None ⇒ (M, N, U, k, None)
  | Some L ⇒ (Marked L (Suc k) # M, N, U, k+1, None)) |
do-decide-step S = S

lemma do-decide-step:
  fixes S :: 'v cdcl_W-state-inv-st
  assumes do-decide-step S ≠ S
  shows decide S (do-decide-step S)
  using assms
  apply (cases S, cases conflicting S)
  defer
  apply (auto split: option.splits simp add: decide.simps Marked-Propagated-in-iff-in-lits-of-l
          dest: find-first-unused-var-undefined find-first-unused-var-Some
          intro:)[1]
proof –
  fix a :: ('v, nat, 'v literal list) marked-lit list and
      b :: 'v literal list list and c :: 'v literal list list and
      d :: nat and e :: 'v literal list option
  {
    fix a :: ('v, nat, 'v literal list) marked-lit list and
        b :: 'v literal list list and c :: 'v literal list list and
        d :: nat and x2 :: 'v literal and m :: 'v literal list
    assume a1: m ∈ set b
    assume x2 ∈ set m
    then have f2: atm-of x2 ∈ atms-of (mset m)
      by simp
    have ⋀f. (f m::'v clause) ∈ f ‘ set b
      using a1 by blast
    then have ⋀f. (atms-of (f m)::'v set) ⊆ atms-of-ms (f ‘ set b)
      by simp
    then have ⋀n f. (n::'v) ∈ atms-of-ms (f ‘ set b) ∨ n ∉ atms-of (f m)
      by (meson contra-subsetD)
    then have atm-of x2 ∈ atms-of-ms (mset ‘ set b)
      using f2 by blast
  } note H = this
  {
    fix m :: 'v literal list and x2
    have m ∈ set b ⟹ x2 ∈ set m ⟹ x2 ∉ lits-of-l a ⟹ – x2 ∉ lits-of-l a ⟹
      ∃ aa∈set b. ¬ atm-of ‘ set aa ⊆ atm-of ‘ lits-of-l a
      by (meson atm-of-in-atm-of-set-in-uminus contra-subsetD rev-image-eqI)
  } note H' = this

  assume do-decide-step S ≠ S and
```

```
    S = (a, b, c, d, e) and
    conflicting S = None
  then show decide S (do-decide-step S)
    using H H' by (auto split: option.splits simp: lits-of-def decide.simps
      Marked-Propagated-in-iff-in-lits-of-l
      dest!: find-first-unused-var-Some)
qed

lemma mmset-of-mlit'-eq-Marked[iff]: mmset-of-mlit' z = Marked x k ⟷ z = Marked x k
  by (cases z) auto

lemma do-decide-step-no:
  do-decide-step S = S ⟹ no-step decide S
  apply (cases S, cases conflicting S)

  apply (auto simp: atms-of-ms-mset-unfold  Marked-Propagated-in-iff-in-lits-of-l lits-of-def
      dest!: atm-of-in-atm-of-set-in-uminus
      elim!: decideE
      split: option.splits)+
using atm-of-eq-atm-of by blast

lemma rough-state-of-state-of-do-decide-step[simp]:
  cdcl_W-all-struct-inv S ⟹ rough-state-of (state-of (do-decide-step S)) = do-decide-step S
proof (subst state-of-inverse, goal-cases)
  case 1
  then show ?case
    by (cases do-decide-step S = S)
      (auto dest: do-decide-step decide other intro: cdcl_W-all-struct-inv-inv)
qed simp

lemma rough-state-of-state-of-do-skip-step[simp]:
  cdcl_W-all-struct-inv S ⟹ rough-state-of (state-of (do-skip-step S)) = do-skip-step S
  apply (subst state-of-inverse, cases do-skip-step S = S)
   apply simp
  by (blast dest: other skip bj do-skip-step cdcl_W-all-struct-inv-inv)+
```

20.3.3 Code generation

Type definition There are two invariants: one while applying conflict and propagate and one for the other rules

```
declare rough-state-of-inverse[simp add]
definition Con where
  Con xs = state-of (if cdcl_W-all-struct-inv xs then xs else ([], [], [], 0, None))

lemma [code abstype]:
 Con (rough-state-of S) = S
  using rough-state-of[of S] unfolding Con-def by simp

definition do-cp-step' where
do-cp-step' S = state-of (do-cp-step (rough-state-of S))

typedef 'v cdcl_W-state-inv-from-init-state = {S::'v cdcl_W-state-inv-st. cdcl_W-all-struct-inv S
  ∧ cdcl_W-stgy** (raw-S0-cdcl_W (raw-init-clss S)) S}
  morphisms rough-state-from-init-state-of state-from-init-state-of
proof
```

```
  show ([],[], [], 0, None) ∈ {S. cdcl_W-all-struct-inv S
    ∧ cdcl_W-stgy** (raw-S0-cdcl_W (raw-init-clss S)) S}
    by (auto simp add: cdcl_W-all-struct-inv-def)
qed

instantiation cdcl_W-state-inv-from-init-state :: (type) equal
begin
definition equal-cdcl_W-state-inv-from-init-state :: 'v cdcl_W-state-inv-from-init-state ⇒
  'v cdcl_W-state-inv-from-init-state ⇒ bool where
 equal-cdcl_W-state-inv-from-init-state S S' ⟷
   (rough-state-from-init-state-of S = rough-state-from-init-state-of S')
instance
  by standard (simp add: rough-state-from-init-state-of-inject
    equal-cdcl_W-state-inv-from-init-state-def)
end

definition ConI where
  ConI S = state-from-init-state-of (if cdcl_W-all-struct-inv S
    ∧ cdcl_W-stgy** (raw-S0-cdcl_W (raw-init-clss S)) S then S else ([], [], [], 0, None))

lemma [code abstype]:
  ConI (rough-state-from-init-state-of S) = S
  using rough-state-from-init-state-of[of S] unfolding ConI-def
  by (simp add: rough-state-from-init-state-of-inverse)

definition id-of-I-to:: 'v cdcl_W-state-inv-from-init-state ⇒ 'v cdcl_W-state-inv where
id-of-I-to S = state-of (rough-state-from-init-state-of S)

lemma [code abstract]:
  rough-state-of (id-of-I-to S) = rough-state-from-init-state-of S
  unfolding id-of-I-to-def using rough-state-from-init-state-of[of S] by auto
```

Conflict and Propagate

```
function do-full1-cp-step :: 'v cdcl_W-state-inv ⇒ 'v cdcl_W-state-inv
where
do-full1-cp-step S =
  (let S' = do-cp-step' S in
   if S = S' then S else do-full1-cp-step S')
by auto
termination
proof (relation {(T', T). (rough-state-of T', rough-state-of T) ∈ {(S', S).
  (S', S) ∈ {(S', S). cdcl_W-all-struct-inv S ∧ cdcl_W-cp S S'}}}, goal-cases)
  case 1
  show ?case
    using wf-if-measure-f[OF wf-if-measure-f[OF cdcl_W-cp-wf-all-inv, of ], of rough-state-of] .
next
  case (2 S' S)
  then show ?case
    unfolding do-cp-step'-def
    apply simp
    by (metis cp-step-is-cdcl_W-cp rough-state-of-inverse)
qed

lemma do-full1-cp-step-fix-point-of-do-full1-cp-step:
  do-cp-step(rough-state-of (do-full1-cp-step S)) = rough-state-of (do-full1-cp-step S)
  by (rule do-full1-cp-step.induct[of λS. do-cp-step(rough-state-of (do-full1-cp-step S))
      = rough-state-of (do-full1-cp-step S)])
    (metis (full-types) do-full1-cp-step.elims rough-state-of-state-of-do-cp-step do-cp-step'-def)

lemma in-clauses-rough-state-of-is-distinct:
  c∈set (raw-init-clss (rough-state-of S) @ raw-learned-clss (rough-state-of S)) ⟹ distinct c
  apply (cases rough-state-of S)
  using rough-state-of[of S] by (auto simp add: distinct-mset-set-distinct cdcl_W-all-struct-inv-def
    distinct-cdcl_W-state-def)

lemma do-full1-cp-step-full:
  full cdcl_W-cp (rough-state-of S)
    (rough-state-of (do-full1-cp-step S))
  unfolding full-def
proof (rule conjI, induction S rule: do-full1-cp-step.induct)
  case (1 S)
  then have f1:
      cdcl_W-cp** ((do-cp-step (rough-state-of S))) (
        (rough-state-of (do-full1-cp-step (state-of (do-cp-step (rough-state-of S))))))
      ∨ state-of (do-cp-step (rough-state-of S)) = S
    using rough-state-of-state-of-do-cp-step[of S] unfolding do-cp-step'-def by fastforce
  have f2: ⋀c. (if c = state-of (do-cp-step (rough-state-of c))
       then c else do-full1-cp-step (state-of (do-cp-step (rough-state-of c))))
     = do-full1-cp-step c
    by (metis (full-types) do-cp-step'-def do-full1-cp-step.simps)
  have f3: ¬ cdcl_W-cp (rough-state-of S) (do-cp-step (rough-state-of S))
    ∨ state-of (do-cp-step (rough-state-of S)) = S
    ∨ cdcl_W-cp++ (rough-state-of S)
       (rough-state-of (do-full1-cp-step (state-of (do-cp-step (rough-state-of S)))))
    using f1 by (meson rtranclp-into-tranclp2)
  { assume do-full1-cp-step S ≠ S
    then have do-cp-step (rough-state-of S) = rough-state-of S
        ⟶ cdcl_W-cp** (rough-state-of S) (rough-state-of (do-full1-cp-step S))
      ∨ do-cp-step (rough-state-of S) ≠ rough-state-of S
        ∧ state-of (do-cp-step (rough-state-of S)) ≠ S
      using f2 f1 by (metis (no-types))
    then have do-cp-step (rough-state-of S) ≠ rough-state-of S
        ∧ state-of (do-cp-step (rough-state-of S)) ≠ S
      ∨ cdcl_W-cp** (rough-state-of S) (rough-state-of (do-full1-cp-step S))
      by (metis rough-state-of-state-of-do-cp-step)
    then have cdcl_W-cp** (rough-state-of S) (rough-state-of (do-full1-cp-step S))
      using f3 f2 by (metis (no-types) cp-step-is-cdcl_W-cp tranclp-into-rtranclp) }
  then show ?case
    by fastforce
next
  show no-step cdcl_W-cp (rough-state-of (do-full1-cp-step S))
    apply (rule do-cp-step-eq-no-step[OF do-full1-cp-step-fix-point-of-do-full1-cp-step[of S]])
    using in-clauses-rough-state-of-is-distinct unfolding do-cp-step'-def by blast
qed

lemma [code abstract]:
 rough-state-of (do-cp-step' S) = do-cp-step (rough-state-of S)
 unfolding do-cp-step'-def by auto
```

The other rules

```
fun do-other-step where
do-other-step S =
  (let T = do-skip-step S in
    if T ≠ S
    then T
    else
      (let U = do-resolve-step T in
      if U ≠ T
      then U else
      (let V = do-backtrack-step U in
      if V ≠ U then V else do-decide-step V)))

lemma do-other-step:
  assumes inv: cdcl_W-all-struct-inv S and
  st: do-other-step S ≠ S
  shows cdcl_W-o S (do-other-step S)
  using st inv by (auto split: if-split-asm
    simp add: Let-def
    intro: do-skip-step do-resolve-step do-backtrack-step do-decide-step
     cdcl_W-o.intros cdcl_W-bj.intros)

lemma do-other-step-no:
  assumes inv: cdcl_W-all-struct-inv S and
  st: do-other-step S = S
  shows no-step cdcl_W-o S
  using st inv by (auto split: if-split-asm elim: cdcl_W-bjE
    simp add: Let-def cdcl_W-bj.simps elim!: cdcl_W-o.cases
    dest!: do-skip-step-no do-resolve-step-no do-backtrack-step-no do-decide-step-no)

lemma rough-state-of-state-of-do-other-step[simp]:
  rough-state-of (state-of (do-other-step (rough-state-of S))) = do-other-step (rough-state-of S)
proof (cases do-other-step (rough-state-of S) = rough-state-of S)
  case True
  then show ?thesis by simp
next
  case False
  have cdcl_W-o (rough-state-of S) (do-other-step (rough-state-of S))
    by (metis False cdcl_W-all-struct-inv-rough-state do-other-step[of rough-state-of S])
  then have cdcl_W-all-struct-inv (do-other-step (rough-state-of S))
    using cdcl_W-all-struct-inv-inv cdcl_W-all-struct-inv-rough-state other by blast
  then show ?thesis
    by (simp add: CollectI state-of-inverse)
qed

definition do-other-step' where
do-other-step' S =
  state-of (do-other-step (rough-state-of S))

lemma rough-state-of-do-other-step'[code abstract]:
 rough-state-of (do-other-step' S) = do-other-step (rough-state-of S)
 apply (cases do-other-step (rough-state-of S) = rough-state-of S)
   unfolding do-other-step'-def apply simp
 using do-other-step[of rough-state-of S] by (auto intro: cdcl_W-all-struct-inv-inv
   cdcl_W-all-struct-inv-rough-state other state-of-inverse)

definition do-cdcl_W-stgy-step where
do-cdcl_W-stgy-step S =
  (let T = do-full1-cp-step S in
    if T ≠ S
    then T
    else
      (let U = (do-other-step' T) in
       (do-full1-cp-step U)))

definition do-cdcl_W-stgy-step' where
do-cdcl_W-stgy-step' S = state-from-init-state-of (rough-state-of (do-cdcl_W-stgy-step (id-of-I-to S)))

lemma toS-do-full1-cp-step-not-eq: do-full1-cp-step S ≠ S ⟹
    rough-state-of S ≠ rough-state-of (do-full1-cp-step S)
proof −
  assume a1: do-full1-cp-step S ≠ S
  then have S ≠ do-cp-step' S
    by fastforce
  then show ?thesis
    by (metis (no-types) do-cp-step'-def  do-full1-cp-step-fix-point-of-do-full1-cp-step
      rough-state-of-inverse)
qed
```

do-full1-cp-step should not be unfolded anymore:

```
declare do-full1-cp-step.simps[simp del]
```

Correction of the transformation

```
lemma do-cdcl_W-stgy-step:
  assumes do-cdcl_W-stgy-step S ≠ S
  shows cdcl_W-stgy (rough-state-of S) (rough-state-of (do-cdcl_W-stgy-step S))
proof (cases do-full1-cp-step S = S)
  case False
  then show ?thesis
    using assms do-full1-cp-step-full[of S] unfolding full-unfold do-cdcl_W-stgy-step-def
    by (auto intro!: cdcl_W-stgy.intros dest: toS-do-full1-cp-step-not-eq)
next
  case True
  have cdcl_W-o (rough-state-of S) (rough-state-of (do-other-step' S))
    by (smt True assms cdcl_W-all-struct-inv-rough-state do-cdcl_W-stgy-step-def do-other-step
      rough-state-of-do-other-step' rough-state-of-inverse)
  moreover
    have
      np: no-step propagate (rough-state-of S) and
      nc: no-step conflict (rough-state-of S)
        apply (metis True cdcl_W-cp.simps do-cp-step-eq-no-step
          do-full1-cp-step-fix-point-of-do-full1-cp-step in-clauses-rough-state-of-is-distinct)
      by (metis True do-conflict-step-no-step do-cp-step-eq-no-prop-no-confl
        do-full1-cp-step-fix-point-of-do-full1-cp-step)
    then have no-step cdcl_W-cp (rough-state-of S)
      by (simp add: cdcl_W-cp.simps)
  moreover have full cdcl_W-cp (rough-state-of (do-other-step' S))
    (rough-state-of (do-full1-cp-step (do-other-step' S)))
    using do-full1-cp-step-full by auto
  ultimately show ?thesis
    using assms True unfolding do-cdcl_W-stgy-step-def
    by (auto intro!: cdcl_W-stgy.other' dest: toS-do-full1-cp-step-not-eq)
qed
```

```
lemma do-skip-step-trail-changed-or-conflict:
  assumes d: do-other-step S ≠ S
  and inv: cdcl_W-all-struct-inv S
  shows trail S ≠ trail (do-other-step S)
proof –
  have M: ⋀M K M1 c. M = c @ K # M1 ⟹ Suc (length M1) ≤ length M
    by auto
  have cdcl_W-M-level-inv S
    using inv unfolding cdcl_W-all-struct-inv-def by auto
  have cdcl_W-o S (do-other-step S) using do-other-step[OF inv d] .
  then show ?thesis
    using ‹cdcl_W-M-level-inv S›
    proof (induction do-other-step S rule: cdcl_W-o-induct-lev2)
      case decide
      then show ?thesis
        apply (cases S)
        apply (auto dest!: find-first-unused-var-Some
          simp: split: option.splits)
        by (meson atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set contra-subsetD)
    next
    case (skip)
    then show ?case
      by (cases S; cases do-other-step S) force
    next
      case (resolve)
      then show ?case
         by (cases S, cases do-other-step S) force
    next
       case (backtrack K i M1 M2 L D) note decomp = this(1) and confl-S = this(3) and undef =
this(6)
        and U = this(7)
      then show ?case
        apply (cases do-other-step S)
        apply (auto split: if-split-asm simp: Let-def)
            apply (cases S rule: do-skip-step.cases; auto split: if-split-asm)
           apply (cases S rule: do-skip-step.cases; auto split: if-split-asm)

          apply (cases S rule: do-backtrack-step.cases;
            auto split: if-split-asm option.splits list.splits marked-lit.splits
              dest!: bt-cut-some-decomp simp: Let-def)
        using d apply (cases S rule: do-decide-step.cases; auto split: option.splits)[]
        done
    qed
qed

lemma do-full1-cp-step-induct:
  (⋀S. (S ≠  do-cp-step' S ⟹ P (do-cp-step' S)) ⟹ P S) ⟹ P a0
  using do-full1-cp-step.induct by metis

lemma do-cp-step-neq-trail-increase:
  ∃ c. raw-trail (do-cp-step S) = c @ raw-trail S ∧(∀ m ∈ set c. ¬ is-marked m)
  by (cases S, cases raw-conflicting S)
     (auto simp add: do-cp-step-def do-conflict-step-def do-propagate-step-def split: option.splits)

lemma do-full1-cp-step-neq-trail-increase:
  ∃ c. raw-trail (rough-state-of (do-full1-cp-step S)) = c @ raw-trail (rough-state-of S)
    ∧ (∀ m ∈ set c. ¬ is-marked m)
  apply (induction rule: do-full1-cp-step-induct)
  apply (rename-tac S, case-tac do-cp-step' S = S)
    apply (simp add: do-full1-cp-step.simps)
  by (smt Un-iff append-assoc do-cp-step'-def do-cp-step-neq-trail-increase do-full1-cp-step.simps
    rough-state-of-state-of-do-cp-step set-append)

lemma do-cp-step-conflicting:
  conflicting (rough-state-of S) ≠ None ⟹ do-cp-step' S = S
  unfolding do-cp-step'-def do-cp-step-def by simp

lemma do-full1-cp-step-conflicting:
  conflicting (rough-state-of S) ≠ None ⟹ do-full1-cp-step S = S
  unfolding do-cp-step'-def do-cp-step-def
  apply (induction rule: do-full1-cp-step-induct)
  by (rename-tac S, case-tac S ≠ do-cp-step' S)
   (auto simp add: do-full1-cp-step.simps do-cp-step-conflicting)

lemma do-decide-step-not-conflicting-one-more-decide:
  assumes
    conflicting S = None and
    do-decide-step S ≠ S
  shows Suc (length (filter is-marked (raw-trail S)))
    = length (filter is-marked (raw-trail (do-decide-step S)))
  using assms unfolding do-other-step'-def
  by (cases S) (force simp: Let-def split: if-split-asm option.splits
     dest!: find-first-unused-var-Some-not-all-incl)

lemma do-decide-step-not-conflicting-one-more-decide-bt:
  assumes conflicting S ≠ None and
  do-decide-step S ≠ S
  shows length (filter is-marked (raw-trail S)) <
    length (filter is-marked (raw-trail (do-decide-step S)))
  using assms unfolding do-other-step'-def by (cases S, cases conflicting S)
    (auto simp add: Let-def split: if-split-asm option.splits)

lemma do-other-step-not-conflicting-one-more-decide-bt:
  assumes
    conflicting (rough-state-of S) ≠ None and
    conflicting (rough-state-of (do-other-step' S)) = None and
    do-other-step' S ≠ S
  shows length (filter is-marked (raw-trail (rough-state-of S)))
    > length (filter is-marked (raw-trail (rough-state-of (do-other-step' S))))
proof (cases S, goal-cases)
  case (1 y) note S = this(1) and inv = this(2)
  obtain M N U k E where y: y = (M, N, U, k, Some E)
    using assms(1) S inv by (cases y, cases conflicting y) auto
  have M: rough-state-of (state-of (M, N, U, k, Some E)) = (M, N, U, k, Some E)
    using inv y by (auto simp add: state-of-inverse)
  have bt: do-other-step' S = state-of (do-backtrack-step (rough-state-of S))
    proof (cases rough-state-of S rule: do-decide-step.cases)
      case 1
      then show ?thesis
        using assms(1,2) by auto[]
    next
      case (2 v vb vd vf vh)
      have f3: ⋀c. (if do-skip-step (rough-state-of c) ≠ rough-state-of c
        then do-skip-step (rough-state-of c)
        else if do-resolve-step (do-skip-step (rough-state-of c)) ≠ do-skip-step (rough-state-of c)
             then do-resolve-step (do-skip-step (rough-state-of c))
             else if do-backtrack-step (do-resolve-step (do-skip-step (rough-state-of c)))
               ≠ do-resolve-step (do-skip-step (rough-state-of c))
             then do-backtrack-step (do-resolve-step (do-skip-step (rough-state-of c)))
             else do-decide-step (do-backtrack-step (do-resolve-step
               (do-skip-step (rough-state-of c)))))
        = rough-state-of (do-other-step' c)
        by (simp add: rough-state-of-do-other-step')
      have
        (raw-trail (rough-state-of (do-other-step' S)),
        raw-init-clss (rough-state-of (do-other-step' S)),
          raw-learned-clss (rough-state-of (do-other-step' S)),
          raw-backtrack-lvl (rough-state-of (do-other-step' S)), None)
        = rough-state-of (do-other-step' S)
        using assms(2) by (cases do-other-step' S) auto
      then show ?thesis
        using f3 2 by (metis (no-types) do-decide-step.simps(2) do-resolve-step-trail-is-None
          do-skip-step-trail-is-None rough-state-of-inverse)
    qed
  show ?case
    using assms(2) S unfolding bt y inv
    apply simp
    by (auto simp add: M bt-cut-not-none
          split: option.splits
          dest!: bt-cut-some-decomp)
qed

lemma do-other-step-not-conflicting-one-more-decide:
  assumes conflicting (rough-state-of S) = None and
  do-other-step' S ≠ S
  shows 1 + length (filter is-marked (raw-trail (rough-state-of S)))
     = length (filter is-marked (raw-trail (rough-state-of (do-other-step' S))))
proof (cases S, goal-cases)
  case (1 y) note S = this(1) and inv = this(2)
  obtain M N U k where y: y = (M, N, U, k, None) using assms(1) S inv by (cases y) auto
  have M: rough-state-of (state-of (M, N, U, k, None)) = (M, N, U, k, None)
    using inv y by (auto simp add: state-of-inverse)
  have state-of (do-decide-step (M, N, U, k, None)) ≠ state-of (M, N, U, k, None)
    using assms(2) unfolding do-other-step'-def y inv S by (auto simp add: M)
  then have f4: do-skip-step (rough-state-of S) = rough-state-of S
    unfolding S M y by (metis (full-types) do-skip-step.simps(4))
  have f5: do-resolve-step (rough-state-of S) = rough-state-of S
    unfolding S M y by (metis (no-types) do-resolve-step.simps(4))
  have f6: do-backtrack-step (rough-state-of S) = rough-state-of S
    unfolding S M y by (metis (no-types) do-backtrack-step.simps(2))
  have do-other-step (rough-state-of S) ≠ rough-state-of S
    using assms(2) unfolding S M y do-other-step'-def by (metis (no-types))
  then show ?case
    using f6 f5 f4 by (simp add: assms(1) do-decide-step-not-conflicting-one-more-decide
      do-other-step'-def)
```

qed

lemma *rough-state-of-state-of-do-skip-step-rough-state-of*[*simp*]:
  rough-state-of (*state-of* (*do-skip-step* (*rough-state-of S*))) = *do-skip-step* (*rough-state-of S*)
  by (*smt do-other-step.simps rough-state-of-inverse rough-state-of-state-of-do-other-step*)

lemma *conflicting-do-resolve-step-iff*[*iff*]:
  conflicting (*do-resolve-step S*) = *None* ⟷ *conflicting S* = *None*
  by (*cases S rule*: *do-resolve-step.cases*)
   (*auto simp add*: *Let-def split*: *option.splits*)

lemma *conflicting-do-skip-step-iff*[*iff*]:
  conflicting (*do-skip-step S*) = *None* ⟷ *conflicting S* = *None*
  by (*cases S rule*: *do-skip-step.cases*)
    (*auto simp add*: *Let-def split*: *option.splits*)

lemma *conflicting-do-decide-step-iff*[*iff*]:
  conflicting (*do-decide-step S*) = *None* ⟷ *conflicting S* = *None*
  by (*cases S rule*: *do-decide-step.cases*)
    (*auto simp add*: *Let-def split*: *option.splits*)

lemma *conflicting-do-backtrack-step-imp*[*simp*]:
  do-backtrack-step S ≠ *S* ⟹ *conflicting* (*do-backtrack-step S*) = *None*
  by (*cases S rule*: *do-backtrack-step.cases*)
    (*auto simp add*: *Let-def split*: *list.splits option.splits marked-lit.splits*)

lemma *do-skip-step-eq-iff-trail-eq*:
  do-skip-step S = *S* ⟷ *trail* (*do-skip-step S*) = *trail S*
  by (*cases S rule*: *do-skip-step.cases*) *auto*

lemma *do-decide-step-eq-iff-trail-eq*:
  do-decide-step S = *S* ⟷ *trail* (*do-decide-step S*) = *trail S*
  by (*cases S rule*: *do-decide-step.cases*) (*auto split*: *option.split*)

lemma *do-backtrack-step-eq-iff-trail-eq*:
  do-backtrack-step S = *S* ⟷ *raw-trail* (*do-backtrack-step S*) = *raw-trail S*
  by (*cases S rule*: *do-backtrack-step.cases*)
    (*auto split*: *option.split list.splits marked-lit.splits*
      dest!: *bt-cut-in-get-all-marked-decomposition*)

lemma *do-resolve-step-eq-iff-trail-eq*:
  do-resolve-step S = *S* ⟷ *trail* (*do-resolve-step S*) = *trail S*
  by (*cases S rule*: *do-resolve-step.cases*) *auto*

lemma *do-other-step-eq-iff-trail-eq*:
  do-other-step S = *S* ⟷ *raw-trail* (*do-other-step S*) = *raw-trail S*

  apply
  (*auto simp add*: *Let-def do-skip-step-eq-iff-trail-eq*
   do-decide-step-eq-iff-trail-eq do-backtrack-step-eq-iff-trail-eq
   do-resolve-step-eq-iff-trail-eq
)
  apply (*simp add*: *do-resolve-step-eq-iff-trail-eq*[*symmetric*]
    do-skip-step-eq-iff-trail-eq[*symmetric*])
  apply (*simp add*: *do-skip-step-eq-iff-trail-eq*[*symmetric*]

```
    do-decide-step-eq-iff-trail-eq  do-backtrack-step-eq-iff-trail-eq[symmetric]
    do-resolve-step-eq-iff-trail-eq[symmetric]
    )
  done

lemma do-full1-cp-step-do-other-step'-normal-form[dest!]:
  assumes H: do-full1-cp-step (do-other-step' S) = S
  shows do-other-step' S = S ∧ do-full1-cp-step S = S
proof –
  let ?T = do-other-step' S
  { assume confl: conflicting (rough-state-of ?T) ≠ None
    then have tr: trail (rough-state-of (do-full1-cp-step ?T)) = trail (rough-state-of ?T)
      using do-full1-cp-step-conflicting by fastforce
    have raw-trail (rough-state-of (do-full1-cp-step (do-other-step' S))) =
      raw-trail (rough-state-of S)
      using arg-cong[OF H, of λS. raw-trail (rough-state-of S)] .
    then have raw-trail (rough-state-of (do-other-step' S)) = raw-trail (rough-state-of S)
       using confl by (auto simp add: do-full1-cp-step-conflicting)
    then have do-other-step' S = S
      by (simp add: do-other-step-eq-iff-trail-eq[symmetric] do-other-step'-def
        del: do-other-step.simps)
  }
  moreover {
    assume eq[simp]: do-other-step' S = S
    obtain c where c: raw-trail (rough-state-of (do-full1-cp-step S)) =
      c @ raw-trail (rough-state-of S)
      using do-full1-cp-step-neq-trail-increase by auto

    moreover have raw-trail (rough-state-of (do-full1-cp-step S)) = raw-trail (rough-state-of S)
      using arg-cong[OF H, of λS. raw-trail (rough-state-of S)] by simp
    finally have c = [] by blast
    then have do-full1-cp-step S = S using assms by auto
    }
  moreover {
    assume confl: conflicting (rough-state-of ?T) = None and neq: do-other-step' S ≠ S
    obtain c where
      c: raw-trail (rough-state-of (do-full1-cp-step ?T)) = c @ raw-trail (rough-state-of ?T) and
      nm: ∀ m∈set c. ¬ is-marked m
      using do-full1-cp-step-neq-trail-increase by auto
    have length (filter is-marked (raw-trail (rough-state-of (do-full1-cp-step ?T))))
       = length (filter is-marked (raw-trail (rough-state-of ?T)))
      using nm unfolding c by force
    moreover have length (filter is-marked (raw-trail (rough-state-of S)))
       ≠ length (filter is-marked (raw-trail (rough-state-of ?T)))
      using do-other-step-not-conflicting-one-more-decide[OF - neq]
      do-other-step-not-conflicting-one-more-decide-bt[of S, OF - confl neq]
      by linarith
    finally have False unfolding H by blast
  }
  ultimately show ?thesis by blast
qed

lemma do-cdcl_W-stgy-step-no:
  assumes S: do-cdcl_W-stgy-step S = S
  shows no-step cdcl_W-stgy (rough-state-of S)
```

```
proof -
  {
    fix S'
    assume full1 cdcl_W-cp (rough-state-of S) S'
    then have False
      using do-full1-cp-step-full[of S] unfolding full-def S rtranclp-unfold full1-def
      by (smt assms do-cdcl_W-stgy-step-def tranclpD)
  }
  moreover {
    fix S' S''
    assume  cdcl_W-o (rough-state-of S) S' and
     no-step propagate (rough-state-of S) and
     no-step conflict (rough-state-of S) and
     full cdcl_W-cp S' S''
    then have False
      using assms unfolding do-cdcl_W-stgy-step-def
      by (smt cdcl_W-all-struct-inv-rough-state do-full1-cp-step-do-other-step'-normal-form
        do-other-step-no rough-state-of-do-other-step')
  }
  ultimately show ?thesis using assms by (force simp: cdcl_W-cp.simps cdcl_W-stgy.simps)
qed

lemma toS-rough-state-of-state-of-rough-state-from-init-state-of[simp]:
  rough-state-of (state-of (rough-state-from-init-state-of S))
    = rough-state-from-init-state-of S
  using rough-state-from-init-state-of[of S] by (auto simp add: state-of-inverse)

lemma cdcl_W-cp-is-rtranclp-cdcl_W: cdcl_W-cp S T ⟹ cdcl_W** S T
  apply (induction rule: cdcl_W-cp.induct)
   using conflict apply blast
  using propagate by blast

lemma rtranclp-cdcl_W-cp-is-rtranclp-cdcl_W: cdcl_W-cp** S T ⟹ cdcl_W** S T
  apply (induction rule: rtranclp-induct)
    apply simp
  by (fastforce dest!: cdcl_W-cp-is-rtranclp-cdcl_W)

lemma cdcl_W-stgy-is-rtranclp-cdcl_W:
  cdcl_W-stgy S T ⟹ cdcl_W** S T
  apply (induction rule: cdcl_W-stgy.induct)
   using cdcl_W-stgy.conflict' rtranclp-cdcl_W-stgy-rtranclp-cdcl_W apply blast
  unfolding full-def by (fastforce dest!:other rtranclp-cdcl_W-cp-is-rtranclp-cdcl_W)

lemma cdcl_W-stgy-init-clss: cdcl_W-stgy S T ⟹ cdcl_W-M-level-inv S ⟹ init-clss S = init-clss T
  using rtranclp-cdcl_W-init-clss cdcl_W-stgy-is-rtranclp-cdcl_W by fast

lemma clauses-toS-rough-state-of-do-cdcl_W-stgy-step[simp]:
  init-clss (rough-state-of (do-cdcl_W-stgy-step (state-of (rough-state-from-init-state-of S))))
    = init-clss (rough-state-from-init-state-of S) (is - = init-clss ?S)
proof (cases do-cdcl_W-stgy-step (state-of ?S) = state-of ?S)
  case True
  then show ?thesis by simp
next
  case False
  have ⋀c. cdcl_W-M-level-inv (rough-state-of c)
```

using *cdcl*$_W$*-all-struct-inv-def cdcl*$_W$*-all-struct-inv-rough-state* **by** *blast*
then have $\bigwedge$*c. init-clss* (*rough-state-of c*) = *init-clss* (*rough-state-of* (*do-cdcl*$_W$*-stgy-step c*))
$\vee$ *do-cdcl*$_W$*-stgy-step c* = *c*
using *cdcl*$_W$*-stgy-no-more-init-clss do-cdcl*$_W$*-stgy-step* **by** *blast*
then show *?thesis*
using *False* **by** *force*
qed

lemma *raw-init-clss-do-cp-step*[*simp*]:
raw-init-clss (*do-cp-step S*) = *raw-init-clss S*
by (*cases S*) (*auto simp*: *do-cp-step-def do-propagate-step-def do-conflict-step-def*
split: *option.splits*)
lemma *raw-init-clss-do-cp-step'*[*simp*]:
raw-init-clss (*rough-state-of* (*do-cp-step' S*)) = *raw-init-clss* (*rough-state-of S*)
by (*simp add*: *do-cp-step'-def*)

lemma *raw-init-clss-rough-state-of-do-full1-cp-step*[*simp*]:
raw-init-clss (*rough-state-of* (*do-full1-cp-step S*))
= *raw-init-clss* (*rough-state-of S*)
apply (*rule do-full1-cp-step.induct*[*of* λ*S.*
raw-init-clss (*rough-state-of* (*do-full1-cp-step S*))
= *raw-init-clss* (*rough-state-of S*)])
by (*metis* (*mono-tags*, *lifting*) *do-full1-cp-step.simps raw-init-clss-do-cp-step'*)

lemma *raw-init-clss-do-skip-def*[*simp*]:
raw-init-clss (*do-skip-step S*) = *raw-init-clss S*
by (*cases S rule*: *do-skip-step.cases*) (*auto simp*: *do-other-step'-def Let-def*
split: *option.splits*)

lemma *raw-init-clss-do-resolve-def*[*simp*]:
raw-init-clss (*do-resolve-step S*) = *raw-init-clss S*
by (*cases S rule*: *do-resolve-step.cases*) (*auto simp*: *do-other-step'-def Let-def*
split: *option.splits*)

lemma *raw-init-clss-do-backtrack-def*[*simp*]:
raw-init-clss (*do-backtrack-step S*) = *raw-init-clss S*
by (*cases S rule*: *do-backtrack-step.cases*) (*auto simp*: *do-other-step'-def Let-def*
split: *option.splits list.splits marked-lit.splits*)

lemma *raw-init-clss-do-decide-def*[*simp*]:
raw-init-clss (*do-decide-step S*) = *raw-init-clss S*
by (*cases S rule*: *do-decide-step.cases*) (*auto simp*: *do-other-step'-def Let-def*
split: *option.splits*)

lemma *raw-init-clss-rough-state-of-do-other-step'*[*simp*]:
raw-init-clss (*rough-state-of* (*do-other-step' S*))
= *raw-init-clss* (*rough-state-of S*)
by (*cases S*) (*auto simp*: *do-other-step'-def Let-def do-skip-step.cases*
split: *option.splits*)

lemma [*simp*]:
raw-init-clss (*rough-state-of* (*do-cdcl*$_W$*-stgy-step* (*state-of* (*rough-state-from-init-state-of S*))))
=
raw-init-clss (*rough-state-from-init-state-of S*)

```
  unfolding do-cdcl_W-stgy-step-def by (auto simp: Let-def)

lemma rough-state-from-init-state-of-do-cdcl_W-stgy-step'[code abstract]:
 rough-state-from-init-state-of (do-cdcl_W-stgy-step' S) =
   rough-state-of (do-cdcl_W-stgy-step (id-of-I-to S))
proof –
  let ?S = (rough-state-from-init-state-of S)
  have cdcl_W-stgy** (raw-S0-cdcl_W (raw-init-clss (rough-state-from-init-state-of S)))
    (rough-state-from-init-state-of S)
    using rough-state-from-init-state-of[of S] by auto
  moreover have cdcl_W-stgy**
                (rough-state-from-init-state-of S)
                (rough-state-of (do-cdcl_W-stgy-step
                  (state-of (rough-state-from-init-state-of S))))
     using do-cdcl_W-stgy-step[of state-of ?S]
     by (cases do-cdcl_W-stgy-step (state-of ?S) = state-of ?S) auto
  ultimately show ?thesis
    unfolding do-cdcl_W-stgy-step'-def id-of-I-to-def
    by (auto intro: state-from-init-state-of-inverse)
qed
```

All rules together

```
function do-all-cdcl_W-stgy where
do-all-cdcl_W-stgy S =
  (let T = do-cdcl_W-stgy-step' S in
  if T = S then S else do-all-cdcl_W-stgy T)
by fast+
termination
proof (relation {(T, S).
    (cdcl_W-measure (rough-state-from-init-state-of T),
    cdcl_W-measure (rough-state-from-init-state-of S))
      ∈ lexn {(a, b). a < b} 3}, goal-cases)
  case 1
  show ?case by (rule wf-if-measure-f) (auto intro!: wf-lexn wf-less)
next
  case (2 S T) note T = this(1) and ST = this(2)
  let ?S = rough-state-from-init-state-of S
  have S: cdcl_W-stgy** (raw-S0-cdcl_W (raw-init-clss ?S)) ?S
    using rough-state-from-init-state-of[of S] by auto
  moreover have cdcl_W-stgy (rough-state-from-init-state-of S)
    (rough-state-from-init-state-of T)

    proof –
      have ⋀c. rough-state-of (state-of (rough-state-from-init-state-of c)) =
        rough-state-from-init-state-of c
        using rough-state-from-init-state-of by force
      then have do-cdcl_W-stgy-step (state-of (rough-state-from-init-state-of S))
        ≠ state-of (rough-state-from-init-state-of S)
        using ST T  rough-state-from-init-state-of-inverse
        unfolding id-of-I-to-def do-cdcl_W-stgy-step'-def
        by fastforce
      from do-cdcl_W-stgy-step[OF this] show ?thesis
        by (simp add: T id-of-I-to-def rough-state-from-init-state-of-do-cdcl_W-stgy-step')
    qed
  moreover
    have cdcl_W-all-struct-inv (rough-state-from-init-state-of S)
```

```
      using rough-state-from-init-state-of[of S] by auto
    then have cdcl_W-all-struct-inv (raw-S0-cdcl_W (raw-init-clss (rough-state-from-init-state-of S)))
      by (cases rough-state-from-init-state-of S)
        (auto simp add: cdcl_W-all-struct-inv-def distinct-cdcl_W-state-def)
  ultimately show ?case
    by (auto intro!: cdcl_W-stgy-step-decreasing[of - - raw-S0-cdcl_W (raw-init-clss ?S)]
      simp del: cdcl_W-measure.simps)
qed

thm do-all-cdcl_W-stgy.induct
lemma do-all-cdcl_W-stgy-induct:
  (⋀S. (do-cdcl_W-stgy-step' S ≠ S ⟹ P (do-cdcl_W-stgy-step' S)) ⟹ P S) ⟹ P a0
 using do-all-cdcl_W-stgy.induct by metis

lemma [simp]: raw-init-clss (rough-state-from-init-state-of (do-all-cdcl_W-stgy S)) =
  raw-init-clss (rough-state-from-init-state-of S)
  apply (induction rule: do-all-cdcl_W-stgy-induct)
  by (smt do-all-cdcl_W-stgy.simps do-cdcl_W-stgy-step-def id-of-I-to-def
    raw-init-clss-rough-state-of-do-full1-cp-step raw-init-clss-rough-state-of-do-other-step'
    rough-state-from-init-state-of-do-cdcl_W-stgy-step'
    toS-rough-state-of-state-of-rough-state-from-init-state-of)

lemma no-step-cdcl_W-stgy-cdcl_W-all:
  fixes S :: 'a cdcl_W-state-inv-from-init-state
  shows no-step cdcl_W-stgy (rough-state-from-init-state-of (do-all-cdcl_W-stgy S))
  apply (induction S rule:do-all-cdcl_W-stgy-induct)
  apply (rename-tac S, case-tac do-cdcl_W-stgy-step' S ≠ S)
proof -
  fix Sa :: 'a cdcl_W-state-inv-from-init-state
  assume a1: ¬ do-cdcl_W-stgy-step' Sa ≠ Sa
  { fix pp
    have (if True then Sa else do-all-cdcl_W-stgy Sa) = do-all-cdcl_W-stgy Sa
      using a1 by auto
    then have ¬ cdcl_W-stgy (rough-state-from-init-state-of (do-all-cdcl_W-stgy Sa)) pp
      using a1 by (smt do-cdcl_W-stgy-step-no id-of-I-to-def
        rough-state-from-init-state-of-do-cdcl_W-stgy-step' rough-state-of-inverse) }
  then show no-step cdcl_W-stgy (rough-state-from-init-state-of (do-all-cdcl_W-stgy Sa))
    by fastforce
next
  fix Sa :: 'a cdcl_W-state-inv-from-init-state
  assume a1: do-cdcl_W-stgy-step' Sa ≠ Sa
    ⟹ no-step cdcl_W-stgy (rough-state-from-init-state-of
      (do-all-cdcl_W-stgy (do-cdcl_W-stgy-step' Sa)))
  assume a2: do-cdcl_W-stgy-step' Sa ≠ Sa
  have do-all-cdcl_W-stgy Sa = do-all-cdcl_W-stgy (do-cdcl_W-stgy-step' Sa)
    by (metis (full-types) do-all-cdcl_W-stgy.simps)
  then show no-step cdcl_W-stgy (rough-state-from-init-state-of (do-all-cdcl_W-stgy Sa))
    using a2 a1 by presburger
qed

lemma do-all-cdcl_W-stgy-is-rtranclp-cdcl_W-stgy:
  cdcl_W-stgy** (rough-state-from-init-state-of S)
    (rough-state-from-init-state-of (do-all-cdcl_W-stgy S))
proof (induction S rule: do-all-cdcl_W-stgy-induct)
  case (1 S) note IH = this(1)
  show ?case
    proof (cases do-cdcl_W-stgy-step' S = S)
      case True
      then show ?thesis by simp
    next
      case False
      have f2: do-cdcl_W-stgy-step (id-of-I-to S) = id-of-I-to S ⟶
        rough-state-from-init-state-of (do-cdcl_W-stgy-step' S)
        = rough-state-of (state-of (rough-state-from-init-state-of S))
        unfolding  rough-state-from-init-state-of-do-cdcl_W-stgy-step'
        id-of-I-to-def by presburger
      have f3: do-all-cdcl_W-stgy S = do-all-cdcl_W-stgy (do-cdcl_W-stgy-step' S)
        by (metis (full-types) do-all-cdcl_W-stgy.simps)
      have cdcl_W-stgy (rough-state-from-init-state-of S)
          (rough-state-from-init-state-of (do-cdcl_W-stgy-step' S))
        = cdcl_W-stgy (rough-state-of (id-of-I-to S))
          (rough-state-of (do-cdcl_W-stgy-step (id-of-I-to S)))
        unfolding id-of-I-to-def rough-state-from-init-state-of-do-cdcl_W-stgy-step'
        toS-rough-state-of-state-of-rough-state-from-init-state-of by presburger
      then show ?thesis
        using f3 f2 IH do-cdcl_W-stgy-step
        by (smt False toS-rough-state-of-state-of-rough-state-from-init-state-of tranclp.intros(1)
          tranclp-into-rtranclp transitive-closurep-trans'(2))
    qed
qed
```

Final theorem:

```
lemma consistent-interp-mmset-of-mlit[simp]:
  consistent-interp (lit-of ` mmset-of-mlit' ` set M') ⟷
   consistent-interp (lit-of ` set M')
  by (auto simp: image-image)


lemma DPLL-tot-correct:
  assumes
    r: rough-state-from-init-state-of (do-all-cdcl_W-stgy (state-from-init-state-of
      (([], map remdups N, [], 0, None)))) = S and
    S: (M', N', U', k, E) = S
  shows (E ≠ Some [] ∧ satisfiable (set (map mset N)))
    ∨ (E = Some [] ∧ unsatisfiable (set (map mset N)))
proof -
  let ?N = map remdups N
  have inv: cdcl_W-all-struct-inv ([], map remdups N, [], 0, None)
    unfolding cdcl_W-all-struct-inv-def distinct-cdcl_W-state-def distinct-mset-set-def by auto
  then have S0: rough-state-of (state-of ([], map remdups N, [], 0, None))
    = ([], map remdups N, [], 0, None) by simp
  have 1: full cdcl_W-stgy ([], ?N, [], 0, None) S
    unfolding full-def apply rule
      using do-all-cdcl_W-stgy-is-rtranclp-cdcl_W-stgy[of
        state-from-init-state-of ([], map remdups N, [], 0, None)] inv
        by (auto simp del: do-all-cdcl_W-stgy.simps simp: state-from-init-state-of-inverse
          r[symmetric] no-step-cdcl_W-stgy-cdcl_W-all)+
  moreover have 2: finite (set (map mset ?N)) by auto
  moreover have 3: distinct-mset-set (set (map mset ?N))
     unfolding distinct-mset-set-def by auto
  moreover
```

```
    have cdcl_W-all-struct-inv S
      by (metis (no-types) cdcl_W-all-struct-inv-rough-state r
        toS-rough-state-of-state-of-rough-state-from-init-state-of)
    then have cons: consistent-interp (lits-of-l M')
      unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def S[symmetric]
      by (auto simp: lits-of-def)
  moreover
    have [simp]:
      rough-state-from-init-state-of (state-from-init-state-of (raw-S0-cdcl_W (map remdups N)))
      = raw-S0-cdcl_W (map remdups N)
      apply (rule cdcl_W-state-inv-from-init-state.state-from-init-state-of-inverse)
      using 3 by (auto simp: cdcl_W-all-struct-inv-def distinct-cdcl_W-state-def
        image-image comp-def)
    have raw-init-clss ([], ?N, [], 0, None) = raw-init-clss S
      using arg-cong[OF r, of raw-init-clss] unfolding S[symmetric]
      by (simp del: do-all-cdcl_W-stgy.simps)
    then have N': N' =map remdups N
      using S[symmetric] by auto

  have conflicting S = Some {#} ∧ unsatisfiable (set-mset (init-clss S)) ∨
    conflicting S = None ∧ (case S of (M, uu-) ⇒ map mmset-of-mlit' M) ⊨asm init-clss S
    apply (rule full-cdcl_W-stgy-final-state-conclusive)
        using 1 apply simp
       using 2 apply simp
      using 3 by simp
  then have (E ≠ Some [] ∧ satisfiable (set (map mset ?N)))
    ∨ (E = Some [] ∧ unsatisfiable (set (map mset ?N)))
    using cons unfolding S[symmetric] N' apply (auto simp: comp-def)
    by (simp add: true-annots-true-cls)
  then show ?thesis by auto
qed
```

The Code The SML code is skipped in the documentation, but stays to ensure that some version of the exported code is working. The only difference between the generated code and the one used here is the export of the constructor ConI.

```
end
theory CDCL-W-Merge
imports CDCL-W-Termination
begin
```

21 Link between Weidenbach's and NOT's CDCL

21.1 Inclusion of the states

```
context conflict-driven-clause-learning_W
begin
declare cdcl_W.intros[intro] cdcl_W-bj.intros[intro] cdcl_W-o.intros[intro]

lemma backtrack-no-cdcl_W-bj:
  assumes cdcl: cdcl_W-bj T U and inv: cdcl_W-M-level-inv V
  shows ¬backtrack V T
  using cdcl inv
  apply (induction rule: cdcl_W-bj.induct)
    apply (elim skipE, force elim!: backtrack-levE[OF - inv] simp: cdcl_W-M-level-inv-def)
```

```
   apply (elim resolveE, force elim!: backtrack-levE[OF - inv] simp: cdcl_W-M-level-inv-def)
  apply standard
  apply (elim backtrack-levE[OF - inv], elim backtrackE)
  apply (force simp del: state-simp simp add: state-eq-def cdcl_W-M-level-inv-decomp)
  done


inductive skip-or-resolve :: 'st ⇒ 'st ⇒ bool where
s-or-r-skip[intro]: skip S T ⟹ skip-or-resolve S T |
s-or-r-resolve[intro]: resolve S T ⟹ skip-or-resolve S T

lemma rtranclp-cdcl_W-bj-skip-or-resolve-backtrack:
  assumes cdcl_W-bj** S U and inv: cdcl_W-M-level-inv S
  shows skip-or-resolve** S U ∨ (∃ T. skip-or-resolve** S T ∧ backtrack T U)
  using assms
proof (induction)
  case base
  then show ?case by simp
next
  case (step U V) note st = this(1) and bj = this(2) and IH = this(3)[OF this(4)]
  consider
      (SU) S = U
    | (SUp) cdcl_W-bj++ S U
    using st unfolding rtranclp-unfold by blast
  then show ?case
    proof cases
      case SUp
      have ⋀T. skip-or-resolve** S T ⟹ cdcl_W** S T
        using mono-rtranclp[of skip-or-resolve cdcl_W]
        by (blast intro: skip-or-resolve.cases)
      then have skip-or-resolve** S U
        using bj IH inv backtrack-no-cdcl_W-bj rtranclp-cdcl_W-consistent-inv[OF - inv] by meson
      then show ?thesis
        using bj by (auto simp: cdcl_W-bj.simps dest!: skip-or-resolve.intros)
    next
      case SU
      then show ?thesis
        using bj by (auto simp: cdcl_W-bj.simps dest!: skip-or-resolve.intros)
    qed
qed

lemma rtranclp-skip-or-resolve-rtranclp-cdcl_W:
  skip-or-resolve** S T ⟹ cdcl_W** S T
  by (induction rule: rtranclp-induct)
  (auto dest!: cdcl_W-bj.intros cdcl_W.intros cdcl_W-o.intros simp: skip-or-resolve.simps)

definition backjump-l-cond :: 'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool where
backjump-l-cond ≡ λC C' L' S T. True

definition inv_NOT :: 'st ⇒ bool where
inv_NOT ≡ λS. no-dup (trail S)

declare inv_NOT-def[simp]
end
```

```
context conflict-driven-clause-learning_W
begin
```

21.2 More lemmas conflict–propagate and backjumping

21.2.1 Termination

```
lemma cdcl_W-cp-normalized-element-all-inv:
  assumes inv: cdcl_W-all-struct-inv S
  obtains T where full cdcl_W-cp S T
  using assms cdcl_W-cp-normalized-element unfolding cdcl_W-all-struct-inv-def by blast
thm backtrackE

lemma cdcl_W-bj-measure:
  assumes cdcl_W-bj S T and cdcl_W-M-level-inv S
  shows length (trail S) + (if conflicting S = None then 0 else 1)
    > length (trail T) +  (if conflicting T = None then 0 else 1)
  using assms by (induction rule: cdcl_W-bj.induct)
  (force dest:arg-cong[of - - length]
    intro: get-all-marked-decomposition-exists-prepend
    elim!: backtrack-levE skipE resolveE
    simp: cdcl_W-M-level-inv-def)+

lemma wf-cdcl_W-bj:
  wf {(b,a). cdcl_W-bj a b ∧ cdcl_W-M-level-inv a}
  apply (rule wfP-if-measure[of λ-. True
      - λT. length (trail T) + (if conflicting T = None then 0 else 1), simplified])
  using cdcl_W-bj-measure by simp

lemma cdcl_W-bj-exists-normal-form:
  assumes lev: cdcl_W-M-level-inv S
  shows ∃ T. full cdcl_W-bj S T
proof −
  obtain T where T: full (λa b. cdcl_W-bj a b ∧ cdcl_W-M-level-inv a) S T
    using wf-exists-normal-form-full[OF wf-cdcl_W-bj] by auto
  then have cdcl_W-bj** S T
     by (auto dest: rtranclp-and-rtranclp-left simp: full-def)
  moreover
    then have cdcl_W** S T
      using mono-rtranclp[of cdcl_W-bj cdcl_W] by blast
    then have cdcl_W-M-level-inv T
      using rtranclp-cdcl_W-consistent-inv lev by auto
  ultimately show ?thesis using T unfolding full-def by auto
qed
lemma rtranclp-skip-state-decomp:
  assumes skip** S T and no-dup (trail S)
  shows
    ∃ M. trail S = M @ trail T ∧ (∀ m∈set M. ¬is-marked m)
    init-clss S = init-clss T
    learned-clss S = learned-clss T
    backtrack-lvl S = backtrack-lvl T
    conflicting S = conflicting T
  using assms by (induction rule: rtranclp-induct)
  (auto simp del: state-simp simp: state-eq-def elim!: skipE)
```

21.2.2 More backjumping

Backjumping after skipping or jump directly

```
lemma rtranclp-skip-backtrack-backtrack:
  assumes
    skip^** S T and
    backtrack T W and
    cdcl_W-all-struct-inv S
  shows backtrack S W
  using assms
proof induction
  case base
  then show ?case by simp
next
  case (step T V) note st = this(1) and skip = this(2) and IH = this(3) and bt = this(4) and
    inv = this(5)
  have skip^** S V
    using st skip by auto
  then have cdcl_W-all-struct-inv V
    using rtranclp-mono[of skip cdcl_W] assms(3) rtranclp-cdcl_W-all-struct-inv-inv mono-rtranclp
    by (auto dest!: bj other cdcl_W-bj.skip)
  then have cdcl_W-M-level-inv V
    unfolding cdcl_W-all-struct-inv-def by auto
  then obtain K i M1 M2 L D where
    conf: raw-conflicting V = Some D and
    LD: L ∈# mset-ccls D and
    decomp: (Marked K (Suc i) # M1, M2) ∈ set (get-all-marked-decomposition (trail V)) and
    lev-L: get-level (trail V) L = backtrack-lvl V and
    max: get-level (trail V) L = get-maximum-level (trail V) (mset-ccls D) and
    max-D: get-maximum-level (trail V) (remove1-mset L (mset-ccls D)) ≡ i and
    undef: undefined-lit M1 L and
    W: W ∼ cons-trail (Propagated L (cls-of-ccls D))
              (reduce-trail-to M1
                (add-learned-cls (cls-of-ccls D)
                  (update-backtrack-lvl i
                    (update-conflicting None V))))
  using bt inv by (elim backtrack-levE) metis+
  obtain L' C' M E where
    tr: trail T = Propagated L' C' # M and
    raw: raw-conflicting T = Some E and
    LE: −L' ∉# mset-ccls E and
    E: mset-ccls E ≠ {#} and
    V: V ∼ tl-trail T
    using skip by (elim skipE) metis
  let ?M = Propagated L' C' # trail V
  have tr-M: trail T = ?M
    using tr V by auto
  have MT: M = tl (trail T) and MV: M = trail V
    using tr V by auto
  have DE[simp]: mset-ccls D = mset-ccls E
    using V conf raw by (auto simp add: state-eq-def simp del: state-simp)
  have cdcl_W^** S T using bj cdcl_W-bj.skip mono-rtranclp[of skip cdcl_W S T] other st by meson
  then have inv': cdcl_W-all-struct-inv T
    using rtranclp-cdcl_W-all-struct-inv-inv inv by blast
  have M-lev: cdcl_W-M-level-inv T using inv' unfolding cdcl_W-all-struct-inv-def by auto
  then have n-d': no-dup ?M
    using tr-M unfolding cdcl_W-M-level-inv-def by auto
```

```
let ?k = backtrack-lvl T
have [simp]:
  backtrack-lvl V = ?k
  using V by simp
have ?k > 0
  using decomp M-lev V tr unfolding cdcl_W-M-level-inv-def by auto
then have atm-of L ∈ atm-of ‘ lits-of-l (trail V)
  using lev-L get-rev-level-ge-0-atm-of-in[of 0 rev (trail V) L]  by auto
then have L-L': atm-of L ≠ atm-of L'
  using n-d' unfolding lits-of-def by auto
have L'-M: atm-of L' ∉ atm-of ‘ lits-of-l (trail V)
  using n-d' unfolding lits-of-def by auto
have ?M ⊨as CNot (mset-ccls D)
  using inv' raw unfolding cdcl_W-conflicting-def cdcl_W-all-struct-inv-def tr-M by auto
then have L' ∉# mset-ccls (remove-clit L D)
  using L-L' L'-M ‹Propagated L' C' # trail V ⊨as CNot (mset-ccls D)›
  unfolding true-annots-true-cls true-clss-def
  by (auto simp: uminus-lit-swap atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set dest!: in-diffD)
have [simp]: trail (reduce-trail-to M1 T) = M1
  using decomp undef tr W V by auto
have skip** S V
  using st skip by auto
have no-dup (trail S)
  using inv unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def by auto
then have [simp]: init-clss S = init-clss V and [simp]: learned-clss S = learned-clss V
  using rtranclp-skip-state-decomp[OF ‹skip** S V›] V
  by (auto simp del: state-simp simp: state-eq-def)
then have
  W-S: W ∼ cons-trail (Propagated L (cls-of-ccls E)) (reduce-trail-to M1
   (add-learned-cls (cls-of-ccls E) (update-backtrack-lvl i (update-conflicting None T))))
  using W V undef M-lev decomp tr
  by (auto simp del: state-simp simp: state-eq-def cdcl_W-M-level-inv-def)

obtain M2' where
  decomp': (Marked K (i+1) # M1, M2') ∈ set (get-all-marked-decomposition (trail T))
  using decomp V unfolding tr-M by (cases hd (get-all-marked-decomposition (trail V)),
    cases get-all-marked-decomposition (trail V)) auto
moreover
  from L-L' have get-level ?M L = ?k
    using lev-L V by (auto split: if-split-asm)
moreover
  have atm-of L' ∉ atms-of (mset-ccls D)
    by (metis DE LE L-L' ‹L' ∉# mset-ccls (remove-clit L D)›  in-remove1-mset-neq remove-clit
      atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set atms-of-def)
  then have get-level ?M L = get-maximum-level ?M (mset-ccls D)
    using calculation(2) lev-L max by auto
moreover
  have atm-of L' ∉ atms-of (mset-ccls (remove-clit L D))
    by (metis DE LE ‹L' ∉# mset-ccls (remove-clit L D)›
      atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set atms-of-def in-remove1-mset-neq remove-clit
      in-atms-of-remove1-mset-in-atms-of)
  have i = get-maximum-level ?M (mset-ccls (remove-clit L D))
    using max-D  ‹atm-of L' ∉ atms-of (mset-ccls (remove-clit L D))› by auto

ultimately have backtrack T W
```

```
    apply -
    apply (rule backtrack-rule[of T - L K i M1 M2' W, OF raw])
    unfolding tr-M[symmetric]
         using LD apply simp
        apply simp
       apply simp
      apply simp
     apply auto[]
    using W-S by auto
  then show ?thesis using IH inv by blast
qed

lemma fst-get-all-marked-decomposition-prepend-not-marked:
  assumes ∀ m∈set MS. ¬ is-marked m
  shows set (map fst (get-all-marked-decomposition M))
    = set (map fst (get-all-marked-decomposition (MS @ M)))
    using assms apply (induction MS rule: marked-lit-list-induct)
    apply auto[2]
    by (rename-tac L m xs; case-tac get-all-marked-decomposition (xs @ M)) simp-all
```

See also $[\![$*skip*** *?S ?T*; *backtrack ?T ?W*; *cdcl*$_W$*-all-struct-inv ?S*$]\!] \Longrightarrow$ *backtrack ?S ?W*

```
lemma rtranclp-skip-backtrack-backtrack-end:
  assumes
    skip: skip** S T and
    bt: backtrack S W and
    inv: cdcl_W-all-struct-inv S
  shows backtrack T W
  using assms
proof -
  have M-lev: cdcl_W-M-level-inv S
    using bt inv unfolding cdcl_W-all-struct-inv-def by (auto elim!: backtrack-levE)
  then obtain K i M1 M2 L D where
    raw-S: raw-conflicting S = Some D and
    LD: L ∈# mset-ccls D and
    decomp: (Marked K (Suc i) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) and
    lev-l: get-level (trail S) L = backtrack-lvl S and
    lev-l-D: get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) and
    i: get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≡ i and
    undef: undefined-lit M1 L and
    W: W ∼ cons-trail (Propagated L (cls-of-ccls D))
             (reduce-trail-to M1
               (add-learned-cls (cls-of-ccls D)
                 (update-backtrack-lvl i
                   (update-conflicting None S))))

    using bt by (elim backtrack-levE)
    (simp-all add: cdcl_W-M-level-inv-decomp state-eq-def del: state-simp)
  let ?D = remove1-mset L (mset-ccls D)

  have [simp]: no-dup (trail S)
    using M-lev by (auto simp: cdcl_W-M-level-inv-decomp)
  have cdcl_W-all-struct-inv T
    using mono-rtranclp[of skip cdcl_W] by (smt bj cdcl_W-bj.skip inv local.skip other
      rtranclp-cdcl_W-all-struct-inv-inv)
  then have [simp]: no-dup (trail T)
```

unfolding *$cdcl_W$-all-struct-inv-def $cdcl_W$-M-level-inv-def* **by** *auto*

obtain *MS M_T* **where** *M*: *trail S* = *MS* @ *M_T* **and** *M_T*: *M_T* = *trail T* **and** *nm*: ∀ *m*∈*set MS*. ¬*is-marked m*
using *rtranclp-skip-state-decomp*(*1*)[*OF skip*] *raw-S M-lev* **by** *auto*
have *T*: *state T* = (*M_T*, *init-clss S*, *learned-clss S*, *backtrack-lvl S*, *Some* (*mset-ccls D*))
using *M_T rtranclp-skip-state-decomp*[*of S T*] *skip raw-S*
by (*auto simp del*: *state-simp simp*: *state-eq-def*)

have *$cdcl_W$-all-struct-inv T*
apply (*rule rtranclp-$cdcl_W$-all-struct-inv-inv*[*OF - inv*])
using *bj $cdcl_W$-bj.skip local.skip other rtranclp-mono*[*of skip $cdcl_W$*] **by** *blast*
then have *M_T* ⊨*as CNot* (*mset-ccls D*)
unfolding *$cdcl_W$-all-struct-inv-def $cdcl_W$-conflicting-def* **using** *T* **by** *blast*
then have ∀ *L*∈#*mset-ccls D*. *atm-of L* ∈ *atm-of* ‘ *lits-of-l M_T*
by (*meson atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set true-annots-true-cls-def-iff-negation-in-model*)
moreover have *no-dup* (*trail S*)
using *inv* **unfolding** *$cdcl_W$-all-struct-inv-def $cdcl_W$-M-level-inv-def* **by** *auto*
ultimately have ∀ *L*∈#*mset-ccls D*. *atm-of L* ∉ *atm-of* ‘ *lits-of-l MS*
unfolding *M* **unfolding** *lits-of-def* **by** *auto*
then have *H*: ⋀*L*. *L*∈#*mset-ccls D* ⟹ *get-level* (*trail S*) *L* = *get-level M_T L*
unfolding *M* **by** (*fastforce simp*: *lits-of-def*)
have [*simp*]: *get-maximum-level* (*trail S*) (*mset-ccls D*) = *get-maximum-level M_T* (*mset-ccls D*)
using ‹*M_T* ⊨*as CNot* (*mset-ccls D*)› *M nm* **by** (*metis true-annots-CNot-all-atms-defined get-maximum-level-skip-un-marked-not-present*)

have *lev-l′*: *get-level M_T L* = *backtrack-lvl S*
using *lev-l LD* **by** (*auto simp*: *H*)
have [*simp*]: *trail* (*reduce-trail-to M1 T*) = *M1*
using *T decomp M nm* **by** (*smt M_T append-assoc beginning-not-marked-invert get-all-marked-decomposition-exists-prepend reduce-trail-to-trail-tl-trail-decomp*)
have *W*: *W* ∼ *cons-trail* (*Propagated L* (*cls-of-ccls D*)) (*reduce-trail-to M1* (*add-learned-cls* (*cls-of-ccls D*) (*update-backtrack-lvl i* (*update-conflicting None T*))))
using *W T i decomp undef* **by** (*auto simp del*: *state-simp simp*: *state-eq-def*)
have *lev-l-D′*: *get-level M_T L* = *get-maximum-level M_T* (*mset-ccls D*)
using *lev-l-D LD* **by** (*auto simp*: *H*)
have [*simp*]: *get-maximum-level* (*trail S*) *?D* = *get-maximum-level M_T ?D*
by (*smt H get-maximum-level-exists-lit get-maximum-level-ge-get-level in-diffD le-antisym not-gr0 not-less*)
then have *i′*: *i* = *get-maximum-level M_T ?D*
using *i* **by** *auto*
have *Marked K* (*i* + *1*) # *M1* ∈ *set* (*map fst* (*get-all-marked-decomposition* (*trail S*)))
using *Set.imageI*[*OF decomp*, *of fst*] **by** *auto*
then have *Marked K* (*i* + *1*) # *M1* ∈ *set* (*map fst* (*get-all-marked-decomposition M_T*))
using *fst-get-all-marked-decomposition-prepend-not-marked*[*OF nm*] **unfolding** *M* **by** *auto*
then obtain *M2′* **where** *decomp′*:(*Marked K* (*i*+*1*) # *M1*, *M2′*) ∈ *set* (*get-all-marked-decomposition M_T*)
by *auto*
then show *backtrack T W*
using *T decomp′ lev-l′ lev-l-D′ i′ W LD undef*
by (*force intro*!: *backtrack.intros simp del*: *state-simp simp*: *state-eq-def*)
qed

lemma *$cdcl_W$-bj-decomp-resolve-skip-and-bj*:

```
  assumes cdcl_W-bj** S T and inv: cdcl_W-M-level-inv S
  shows (skip-or-resolve** S T
    ∨ (∃ U. skip-or-resolve** S U ∧ backtrack U T))
  using assms
proof induction
  case base
  then show ?case by simp
next
  case (step T U) note st = this(1) and bj = this(2) and IH = this(3)
  have IH: skip-or-resolve** S T
    proof −
      { assume (∃ U. skip-or-resolve** S U ∧ backtrack U T)
        then obtain V where
          bt: backtrack V T and
          skip-or-resolve** S V
          by blast
        have cdcl_W** S V
          using ‹skip-or-resolve** S V› rtranclp-skip-or-resolve-rtranclp-cdcl_W by blast
        then have cdcl_W-M-level-inv V and cdcl_W-M-level-inv S
          using rtranclp-cdcl_W-consistent-inv inv by blast+
        with bj bt have False using backtrack-no-cdcl_W-bj by simp
      }
      then show ?thesis using IH inv by blast
    qed
  show ?case
    using bj
    proof (cases rule: cdcl_W-bj.cases)
      case backtrack
      then show ?thesis using IH by blast
    qed (metis (no-types, lifting) IH rtranclp.simps skip-or-resolve.simps)+
qed

lemma resolve-skip-deterministic:
  resolve S T ⟹ skip S U ⟹ False
  by (auto elim!: skipE resolveE dest: hd-raw-trail)

lemma backtrack-unique:
  assumes
    bt-T: backtrack S T and
    bt-U: backtrack S U and
    inv: cdcl_W-all-struct-inv S
  shows T ∼ U
proof −
  have lev: cdcl_W-M-level-inv S
    using inv unfolding cdcl_W-all-struct-inv-def by auto
  then obtain K i M1 M2 L D where
    raw-S: raw-conflicting S = Some D and
    LD: L ∈# mset-ccls D and
    decomp: (Marked K (Suc i) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) and
    lev-l: get-level (trail S) L = backtrack-lvl S and
    lev-l-D: get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) and
    i: get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≡ i and
    undef: undefined-lit M1 L and
    T: T ∼ cons-trail (Propagated L (cls-of-ccls D))
              (reduce-trail-to M1
                (add-learned-cls (cls-of-ccls D)
                  (update-backtrack-lvl i
                    (update-conflicting None S))))
    using bt-T by (elim backtrack-levE) (force simp: cdcl_W-M-level-inv-def)+


  obtain K' i' M1' M2' L' D' where
    raw-S': raw-conflicting S = Some D' and
    LD': L' ∈# mset-ccls D' and
    decomp': (Marked K' (Suc i') # M1', M2') ∈ set (get-all-marked-decomposition (trail S)) and
    lev-l: get-level (trail S) L' = backtrack-lvl S and
    lev-l-D: get-level (trail S) L' = get-maximum-level (trail S) (mset-ccls D') and
    i': get-maximum-level (trail S) (remove1-mset L' (mset-ccls D')) ≡ i' and
    undef': undefined-lit M1' L' and
    U: U ∼ cons-trail (Propagated L' (cls-of-ccls D'))
              (reduce-trail-to M1'
                (add-learned-cls (cls-of-ccls D')
                  (update-backtrack-lvl i'
                    (update-conflicting None S))))
    using bt-U lev by (elim backtrack-levE) (force simp: cdcl_W-M-level-inv-def)+
  obtain c where M: trail S = c @ M2 @ Marked K (i + 1) # M1
    using decomp by auto
  obtain c' where M': trail S = c' @ M2' @ Marked K' (i' + 1) # M1'
    using decomp' by auto
  have marked: get-all-levels-of-marked (trail S) = rev [1..<1+backtrack-lvl S]
    using inv unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def by auto
  then have i < backtrack-lvl S
    unfolding M by (force simp add: rev-swap[symmetric] dest!: arg-cong[of - - set])

  have [simp]: L' = L
    proof (rule ccontr)
      assume ¬ ?thesis
      then have L' ∈# remove1-mset L (mset-ccls D)
        using raw-S raw-S' LD LD' by (simp add: in-remove1-mset-neq)
      then have get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≥ backtrack-lvl S
        using ‹get-level (trail S) L' = backtrack-lvl S› get-maximum-level-ge-get-level
        by metis
      then show False using i' i ‹i < backtrack-lvl S› by auto
    qed
  then have [simp]: mset-ccls D' = mset-ccls D
    using raw-S raw-S' by auto
  have [simp]: i' = i
    using i i' by auto
```

Automation in a step later...

```
  have H: ⋀a A B. insert a A = B ⟹ a : B
    by blast
  have get-all-levels-of-marked (c@M2) = rev [i+2..<1+backtrack-lvl S] and
    get-all-levels-of-marked (c'@M2') = rev [i+2..<1+backtrack-lvl S]
    using marked unfolding M
    using marked unfolding M'
    unfolding rev-swap[symmetric] by (auto dest: append-cons-eq-upt-length-i-end)
  from arg-cong[OF this(1), of set] arg-cong[OF this(2), of set]
  have
    dropWhile (λL. ¬is-marked L ∨ level-of L ≠ Suc i) (c @ M2) = [] and
```

```
      dropWhile (λL. ¬is-marked L ∨ level-of L ≠ Suc i) (c' @ M2') = []
        unfolding dropWhile-eq-Nil-conv Ball-def
        by (intro allI; rename-tac x; case-tac x; auto dest!: H simp add: in-set-conv-decomp)+

  then have [simp]: M1' = M1
    using arg-cong[OF M, of dropWhile (λL. ¬is-marked L ∨ level-of L ≠ Suc i)]
    unfolding M' by auto
  show ?thesis using T U undef inv decomp by (auto simp del: state-simp simp: state-eq-def
    cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-decomp)
qed

lemma if-can-apply-backtrack-no-more-resolve:
  assumes
    skip: skip** S U and
    bt: backtrack S T and
    inv: cdcl_W-all-struct-inv S
  shows ¬resolve U V
proof (rule ccontr)
  assume resolve: ¬¬resolve U V

  obtain L E D where
    U: trail U ≠ [] and
    tr-U: hd-raw-trail U  = Propagated L E and
    LE: L ∈# mset-cls E and
    raw-U: raw-conflicting U = Some D and
    LD: −L ∈# mset-ccls D and
    get-maximum-level (trail U) (mset-ccls (remove-clit (−L) D)) = backtrack-lvl U and
    V: V ∼ update-conflicting (Some (union-ccls (remove-clit (−L) D)
      (ccls-of-cls (remove-lit L E))))
      (tl-trail U)
    using resolve by (auto elim!: resolveE)
  have cdcl_W-all-struct-inv U
    using mono-rtranclp[of skip cdcl_W] by (meson bj cdcl_W-bj.skip inv local.skip other
      rtranclp-cdcl_W-all-struct-inv-inv)
  then have [iff]: no-dup (trail S) cdcl_W-M-level-inv S and [iff]: no-dup (trail U)
    using inv unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def by blast+
  then have
    S: init-clss U = init-clss S
       learned-clss U = learned-clss S
       backtrack-lvl U = backtrack-lvl S
       conflicting S = Some (mset-ccls D)
    using rtranclp-skip-state-decomp[OF skip] U raw-U
    by (auto simp del: state-simp simp: state-eq-def)
  obtain M_0 where
    tr-S: trail S = M_0 @ trail U and
    nm: ∀m∈set M_0. ¬is-marked m
    using rtranclp-skip-state-decomp[OF skip] by blast

  obtain K' i' M1' M2' L' D' where
    raw-S': raw-conflicting S = Some D' and
    LD': L' ∈# mset-ccls D' and
    decomp': (Marked K' (Suc i') # M1', M2') ∈ set (get-all-marked-decomposition (trail S)) and
    lev-l: get-level (trail S) L' = backtrack-lvl S and
    lev-l-D: get-level (trail S) L' = get-maximum-level (trail S) (mset-ccls D') and
    i': get-maximum-level (trail S) (remove1-mset L' (mset-ccls D')) ≡ i' and
    undef': undefined-lit M1' L' and
    R: T ∼ cons-trail (Propagated L' (cls-of-ccls D'))
              (reduce-trail-to M1'
                (add-learned-cls (cls-of-ccls D')
                  (update-backtrack-lvl i'
                    (update-conflicting None S))))
    using bt by (elim backtrack-levE) (fastforce simp: S state-eq-def simp del:state-simp)+
  obtain c where M: trail S = c @ M2' @ Marked K' (i' + 1) # M1'
    using get-all-marked-decomposition-exists-prepend[OF decomp'] by auto
  have marked: get-all-levels-of-marked (trail S) = rev [1..<1+backtrack-lvl S]
    using inv unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def by auto
  then have i' < backtrack-lvl S
    unfolding M by (force simp add: rev-swap[symmetric] dest!: arg-cong[of - - set])

  have U: trail U = Propagated L (mset-cls E) # trail V
   using tr-S hd-raw-trail[OF U] U S V tr-U by (auto simp: lits-of-def)
  have DD'[simp]: mset-ccls D' = mset-ccls D
    using raw-U raw-S' S by auto
  have [simp]: L' = −L
    proof (rule ccontr)
      assume ¬ ?thesis
      then have −L ∈# remove1-mset L' (mset-ccls D')
        using DD' LD' LD by (simp add: in-remove1-mset-neq)
      moreover
        have M': trail S = M_0 @ Propagated L (mset-cls E) # trail V
          using tr-S unfolding U by auto
        have no-dup (trail S)
           using inv U unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def by auto
        then have atm-L-notin-M: atm-of L ∉ atm-of ` (lits-of-l (trail V))
          using M' U S by (auto simp: lits-of-def)
        have get-all-levels-of-marked (trail S) = rev [1..<1+backtrack-lvl S]
          using inv U unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def by auto
        then have get-all-levels-of-marked (trail U) = rev [1..<1+backtrack-lvl S]
          using nm M' U by (simp add: get-all-levels-of-marked-no-marked)
        then have get-lev-L:
          get-level(Propagated L (mset-cls E) # trail V) L = backtrack-lvl S
          using get-level-get-rev-level-get-all-levels-of-marked[OF atm-L-notin-M,
            of [Propagated L (mset-cls E)]] U by auto
        have atm-of L ∉ atm-of ` (lits-of-l (rev M_0))
          using ‹no-dup (trail S)› M' by (auto simp: lits-of-def)
        then have get-level (trail S) L = backtrack-lvl S
          by (metis M' get-lev-L get-rev-level-notin-end rev-append)
      ultimately
        have get-maximum-level (trail S) (remove1-mset L' (mset-ccls D')) ≥ backtrack-lvl S
          by (metis get-maximum-level-ge-get-level get-rev-level-uminus)
      then show False
        using ‹i' < backtrack-lvl S› i' by auto
    qed
  have cdcl_W** S U
    using bj cdcl_W-bj.skip local.skip mono-rtranclp[of skip cdcl_W S U] other by meson
  then have cdcl_W-all-struct-inv U
    using inv rtranclp-cdcl_W-all-struct-inv-inv by blast
  then have Propagated L (mset-cls E) # trail V ⊨as CNot (mset-ccls D')
    using cdcl_W-all-struct-inv-def cdcl_W-conflicting-def raw-U U by auto
   then have ∀L'∈# (remove1-mset L' (mset-ccls D')) . atm-of L' ∈ atm-of ` lits-of-l (Propagated L
  (mset-cls E) # trail U)
      using U atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set in-CNot-implies-uminus(2)
      by (fastforce dest: in-diffD)
    then have get-maximum-level (trail S) (remove1-mset L′ (mset-ccls D′)) = backtrack-lvl S
       using get-maximum-level-skip-un-marked-not-present[of remove1-mset L′ (mset-ccls D′)
           trail U M_0] tr-S nm U
       ⟨get-maximum-level (trail U) (mset-ccls (remove-clit (− L) D)) = backtrack-lvl U⟩
       by (auto simp: S)
    then show False
      using i′ ⟨i′ < backtrack-lvl S⟩ by auto
  qed

  lemma if-can-apply-resolve-no-more-backtrack:
    assumes
      skip: skip** S U and
      resolve: resolve S T and
      inv: cdcl_W-all-struct-inv S
    shows ¬backtrack U V
    using assms
    by (meson if-can-apply-backtrack-no-more-resolve rtranclp.rtrancl-refl
      rtranclp-skip-backtrack-backtrack)

  lemma if-can-apply-backtrack-skip-or-resolve-is-skip:
    assumes
      bt: backtrack S T and
      skip: skip-or-resolve** S U and
      inv: cdcl_W-all-struct-inv S
    shows skip** S U
    using assms(2,3,1)
    by induction (simp-all add: if-can-apply-backtrack-no-more-resolve skip-or-resolve.simps)

  lemma cdcl_W-bj-bj-decomp:
    assumes cdcl_W-bj** S W and cdcl_W-all-struct-inv S
    shows
      (∃ T U V. (λS T. skip-or-resolve S T ∧ no-step backtrack S)** S T
          ∧ (λT U. resolve T U ∧ no-step backtrack T) T U
          ∧ skip** U V  ∧ backtrack V W)
      ∨ (∃ T U. (λS T. skip-or-resolve S T ∧ no-step backtrack S)** S T
          ∧ (λT U. resolve T U ∧ no-step backtrack T) T U ∧ skip** U W)
      ∨ (∃ T. skip** S T  ∧ backtrack T W)
      ∨ skip** S W (is ?RB S W ∨ ?R S W ∨ ?SB S W ∨ ?S S W)
    using assms
  proof induction
    case base
    then show ?case by simp
  next
    case (step W X) note st = this(1) and bj = this(2) and IH = this(3)[OF this(4)] and inv = this(4)

    have ¬?RB S W and ¬?SB S W
      proof (clarify, goal-cases)
        case (1 T U V)
        have skip-or-resolve** S T
          using 1(1) by (auto dest!: rtranclp-and-rtranclp-left)
        then show False
          by (metis (no-types, lifting) 1(2) 1(4) 1(5) backtrack-no-cdcl_W-bj
          cdcl_W-all-struct-inv-def cdcl_W-all-struct-inv-inv cdcl_W-o.bj local.bj other
          resolve rtranclp-cdcl_W-all-struct-inv-inv rtranclp-skip-backtrack-backtrack
          rtranclp-skip-or-resolve-rtranclp-cdcl_W step.prems)
    next
      case 2
      then show ?case by (meson assms(2) cdcl_W-all-struct-inv-def backtrack-no-cdcl_W-bj
        local.bj rtranclp-skip-backtrack-backtrack)
    qed
  then have IH: ?R S W ∨ ?S S W using IH by blast

  have cdcl_W** S W using mono-rtranclp[of cdcl_W-bj cdcl_W] st by blast
  then have inv-W: cdcl_W-all-struct-inv W by (simp add: rtranclp-cdcl_W-all-struct-inv-inv
    step.prems)
  consider
      (BT) X′ where backtrack W X′
    | (skip) no-step backtrack W and skip W X
    | (resolve) no-step backtrack W and resolve W X
    using bj cdcl_W-bj.cases by meson
  then show ?case
    proof cases
      case (BT X′)
      then consider
          (bt) backtrack W X
        | (sk) skip W X
        using bj if-can-apply-backtrack-no-more-resolve[of W W X′ X] inv-W cdcl_W-bj.cases by fast
      then show ?thesis
        proof cases
          case bt
          then show ?thesis using IH by auto
        next
          case sk
          then show ?thesis using IH by (meson rtranclp-trans r-into-rtranclp)
        qed
    next
      case skip
      then show ?thesis using IH by (meson rtranclp.rtrancl-into-rtrancl)
    next
      case resolve note no-bt = this(1) and res = this(2)
      consider
          (RS) T U where
            (λS T. skip-or-resolve S T ∧ no-step backtrack S)** S T and
            resolve T U and
            no-step backtrack T and
            skip** U W
        | (S) skip** S W
        using IH by auto
      then show ?thesis
        proof cases
          case (RS T U)
          have cdcl_W** S T
            using RS(1) cdcl_W-bj.resolve cdcl_W-o.bj other skip
            mono-rtranclp[of (λS T. skip-or-resolve S T ∧ no-step backtrack S) cdcl_W S T]
            by (meson skip-or-resolve.cases)
          then have cdcl_W-all-struct-inv U
            by (meson RS(2) cdcl_W-all-struct-inv-inv cdcl_W-bj.resolve cdcl_W-o.bj other
              rtranclp-cdcl_W-all-struct-inv-inv step.prems)
          { fix U'
            assume skip** U U' and skip** U' W
            have cdcl_W-all-struct-inv U'
              using ‹cdcl_W-all-struct-inv U› ‹skip** U U'› rtranclp-cdcl_W-all-struct-inv-inv
                 cdcl_W-o.bj rtranclp-mono[of skip cdcl_W] other skip by blast
            then have no-step backtrack U'
              using if-can-apply-backtrack-no-more-resolve[OF ‹skip** U' W› ] res by blast
          }
          with ‹skip** U W›
          have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** U W
             proof induction
               case base
               then show ?case by simp
             next
               case (step V W) note st = this(1) and skip = this(2) and IH = this(3) and H = this(4)
               have ⋀U'. skip** U' V ⟹ skip** U' W
                 using skip by auto
               then have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** U V
                 using IH H by blast
               moreover have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** V W

                 by (simp add: local.skip r-into-rtranclp st step.prems skip-or-resolve.intros)
               ultimately show ?case by simp
             qed
          then show ?thesis
            proof -
              have f1: ∀p pa pb pc. ¬ p (pa) pb ∨ ¬ p** pb pc ∨ p** pa pc
                by (meson converse-rtranclp-into-rtranclp)
              have skip-or-resolve T U ∧ no-step backtrack T
                using RS(2) RS(3) by force
              then have (λp pa. skip-or-resolve p pa ∧ no-step backtrack p)** T W
                proof -
                  have (∃ vr19 vr16 vr17 vr18. vr19 (vr16::'st) vr17 ∧ vr19** vr17 vr18
                       ∧ ¬ vr19** vr16 vr18)
                    ∨ ¬ (skip-or-resolve T U ∧ no-step backtrack T)
                    ∨ ¬ (λuu uua. skip-or-resolve uu uua ∧ no-step backtrack uu)** U W
                    ∨ (λuu uua. skip-or-resolve uu uua ∧ no-step backtrack uu)** T W
                    by force
                  then show ?thesis
                    by (metis (no-types) ‹(λS T. skip-or-resolve S T ∧ no-step backtrack S)** U W›
                      ‹skip-or-resolve T U ∧ no-step backtrack T› f1)
                qed
              then have (λp pa. skip-or-resolve p pa ∧ no-step backtrack p)** S W
                using RS(1) by force
              then show ?thesis
                using no-bt res by blast
            qed
        next
          case S
          { fix U'
            assume skip** S U' and skip** U' W
            then have cdcl_W** S U'
              using mono-rtranclp[of skip cdcl_W S U'] by (simp add: cdcl_W-o.bj other skip)
            then have cdcl_W-all-struct-inv U'
```

```
          by (metis (no-types, hide-lams) ‹cdcl_W-all-struct-inv S›
            rtranclp-cdcl_W-all-struct-inv-inv)
        then have no-step backtrack U'
          using if-can-apply-backtrack-no-more-resolve[OF ‹skip** U' W› ] res by blast
      }
      with S
      have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** S W
        proof induction
          case base
          then show ?case by simp
        next
          case (step V W) note st = this(1) and skip = this(2) and IH = this(3) and H = this(4)
          have ⋀U'. skip** U' V ⟹ skip** U' W
            using skip by auto
          then have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** S V
            using IH H by blast
          moreover have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** V W

            by (simp add: local.skip r-into-rtranclp st step.prems skip-or-resolve.intros)
          ultimately show ?case by simp
        qed
      then show ?thesis using res no-bt by blast
    qed
  qed
qed
```

The case distinction is needed, since $T \sim V$ does not imply that $R^{**}\ T\ V$.

```
lemma cdcl_W-bj-strongly-confluent:
  assumes
    cdcl_W-bj** S V and
    cdcl_W-bj** S T and
    n-s: no-step cdcl_W-bj V and
    inv: cdcl_W-all-struct-inv S
  shows T ∼ V ∨ cdcl_W-bj** T V
  using assms(2)
proof induction
  case base
  then show ?case by (simp add: assms(1))
next
  case (step T U) note st = this(1) and s-o-r = this(2) and IH = this(3)
  have cdcl_W** S T
    using st mono-rtranclp[of cdcl_W-bj cdcl_W] other by blast
  then have lev-T: cdcl_W-M-level-inv T
    using inv rtranclp-cdcl_W-consistent-inv[of S T]
    unfolding cdcl_W-all-struct-inv-def by auto

  consider
      (TV) T ∼ V
    | (bj-TV) cdcl_W-bj** T V
    using IH by blast
  then show ?case
    proof cases
      case TV
      have no-step cdcl_W-bj T
        using ‹cdcl_W-M-level-inv T› n-s cdcl_W-bj-state-eq-compatible[of T - V] TV
    by (meson backtrack-state-eq-compatible cdcl_W-bj.simps resolve-state-eq-compatible
      skip-state-eq-compatible state-eq-ref)
  then show ?thesis
    using s-o-r by auto
next
  case bj-TV
  then obtain U' where
    T-U': cdcl_W-bj T U' and
    cdcl_W-bj** U' V
    using IH n-s s-o-r by (metis rtranclp-unfold tranclpD)
  have cdcl_W** S T
    by (metis (no-types, hide-lams) bj mono-rtranclp[of cdcl_W-bj cdcl_W] other st)
  then have inv-T: cdcl_W-all-struct-inv T
    by (metis (no-types, hide-lams) inv rtranclp-cdcl_W-all-struct-inv-inv)

  have lev-U: cdcl_W-M-level-inv U
    using s-o-r cdcl_W-consistent-inv lev-T other by blast
  show ?thesis
    using s-o-r
    proof cases
      case backtrack
      then obtain V0 where skip** T V0 and backtrack V0 V
        using IH if-can-apply-backtrack-skip-or-resolve-is-skip[OF backtrack - inv-T]
         cdcl_W-bj-decomp-resolve-skip-and-bj
         by (meson bj-TV cdcl_W-bj.backtrack inv-T lev-T n-s
           rtranclp-skip-backtrack-backtrack-end)
      then have cdcl_W-bj** T V0 and cdcl_W-bj V0 V
        using rtranclp-mono[of skip cdcl_W-bj] by blast+
      then show ?thesis
        using ‹backtrack V0 V› ‹skip** T V0› backtrack-unique inv-T local.backtrack
        rtranclp-skip-backtrack-backtrack by auto
    next
      case resolve
      then have U ∼ U'
        by (meson T-U' cdcl_W-bj.simps if-can-apply-backtrack-no-more-resolve inv-T
          resolve-skip-deterministic resolve-unique rtranclp.rtrancl-refl)
      then show ?thesis
        using ‹cdcl_W-bj** U' V› unfolding rtranclp-unfold
        by (meson T-U' bj cdcl_W-consistent-inv lev-T other state-eq-ref state-eq-sym
          tranclp-cdcl_W-bj-state-eq-compatible)
    next
      case skip
      consider
          (sk) skip T U'
        | (bt) backtrack T U'
        using T-U' by (meson cdcl_W-bj.cases local.skip resolve-skip-deterministic)
      then show ?thesis
        proof cases
          case sk
          then show ?thesis
            using ‹cdcl_W-bj** U' V› unfolding rtranclp-unfold
            by (meson T-U' bj cdcl_W-all-inv(3) cdcl_W-all-struct-inv-def inv-T local.skip other
              tranclp-cdcl_W-bj-state-eq-compatible skip-unique state-eq-ref)
        next
          case bt
```

```
            have skip^++ T U
              using local.skip by blast
            have cdcl_W-bj U U'
              by (meson ‹skip^++ T U› backtrack bt inv-T rtranclp-skip-backtrack-backtrack-end
                tranclp-into-rtranclp)
            then have cdcl_W-bj^++ U V
              using ‹cdcl_W-bj^** U' V› by auto
            then show ?thesis
              by (meson tranclp-into-rtranclp)
          qed
        qed
    qed
qed


lemma cdcl_W-bj-unique-normal-form:
  assumes
    ST: cdcl_W-bj^** S T and SU: cdcl_W-bj^** S U and
    n-s-U: no-step cdcl_W-bj U and
    n-s-T: no-step cdcl_W-bj T and
    inv: cdcl_W-all-struct-inv S
  shows T ∼ U
proof −
  have T ∼ U ∨ cdcl_W-bj^** T U
    using ST SU cdcl_W-bj-strongly-confluent inv n-s-U by blast
  then show ?thesis
    by (metis (no-types) n-s-T rtranclp-unfold state-eq-ref tranclp-unfold-begin)
qed

lemma full-cdcl_W-bj-unique-normal-form:
 assumes full cdcl_W-bj S T and full cdcl_W-bj S U and
   inv: cdcl_W-all-struct-inv S
 shows T ∼ U
   using cdcl_W-bj-unique-normal-form assms unfolding full-def by blast
```

21.3 CDCL FW

```
inductive cdcl_W-merge-restart :: 'st ⇒ 'st ⇒ bool where
fw-r-propagate: propagate S S' ⟹ cdcl_W-merge-restart S S' |
fw-r-conflict: conflict S T ⟹ full cdcl_W-bj T U ⟹ cdcl_W-merge-restart S U |
fw-r-decide: decide S S' ⟹ cdcl_W-merge-restart S S'|
fw-r-rf: cdcl_W-rf S S' ⟹ cdcl_W-merge-restart S S'

lemma rtranclp-cdcl_W-bj-rtranclp-cdcl_W:
  cdcl_W-bj^** S T ⟹ cdcl_W^** S T
  using mono-rtranclp[of cdcl_W-bj cdcl_W] by blast

lemma cdcl_W-merge-restart-cdcl_W:
  assumes cdcl_W-merge-restart S T
  shows cdcl_W^** S T
  using assms
proof induction
  case (fw-r-conflict S T U) note confl = this(1) and bj = this(2)
  have cdcl_W S T using confl by (simp add: cdcl_W.intros r-into-rtranclp)
  moreover
    have cdcl_W-bj^** T U using bj unfolding full-def by auto
```

then have *cdcl*$_W$** *T U* **using** *rtranclp-cdcl*$_W$*-bj-rtranclp-cdcl*$_W$ **by** *blast*
ultimately show *?case* **by** *auto*
qed (*simp-all add*: *cdcl*$_W$*-o.intros cdcl*$_W$*.intros r-into-rtranclp*)

lemma *cdcl*$_W$*-merge-restart-conflicting-true-or-no-step*:
assumes *cdcl*$_W$*-merge-restart S T*
shows *conflicting T* = *None* ∨ *no-step cdcl*$_W$ *T*
using *assms*
proof *induction*
case (*fw-r-conflict S T U*) **note** *confl* = *this*(*1*) **and** *n-s* = *this*(*2*)
{ fix *D V*
assume *cdcl*$_W$ *U V* **and** *conflicting U* = *Some D*
then have *False*
using *n-s* **unfolding** *full-def*
by (*induction rule*: *cdcl*$_W$*-all-rules-induct*)
(*auto dest*!: *cdcl*$_W$*-bj.intros elim*: *decideE propagateE conflictE forgetE restartE*)
}
then show *?case* **by** (*cases conflicting U*) *fastforce+*
qed (*auto simp add*: *cdcl*$_W$*-rf.simps elim*: *propagateE decideE restartE forgetE*)

inductive *cdcl*$_W$*-merge* :: *'st* ⇒ *'st* ⇒ *bool* **where**
fw-propagate: *propagate S S'* ⟹ *cdcl*$_W$*-merge S S'* |
fw-conflict: *conflict S T* ⟹ *full cdcl*$_W$*-bj T U* ⟹ *cdcl*$_W$*-merge S U* |
fw-decide: *decide S S'* ⟹ *cdcl*$_W$*-merge S S'*|
fw-forget: *forget S S'* ⟹ *cdcl*$_W$*-merge S S'*

lemma *cdcl*$_W$*-merge-cdcl*$_W$*-merge-restart*:
cdcl$_W$*-merge S T* ⟹ *cdcl*$_W$*-merge-restart S T*
by (*meson cdcl*$_W$*-merge.cases cdcl*$_W$*-merge-restart.simps forget*)

lemma *rtranclp-cdcl*$_W$*-merge-tranclp-cdcl*$_W$*-merge-restart*:
cdcl$_W$*-merge*** *S T* ⟹ *cdcl*$_W$*-merge-restart*** *S T*
using *rtranclp-mono*[*of cdcl*$_W$*-merge cdcl*$_W$*-merge-restart*] *cdcl*$_W$*-merge-cdcl*$_W$*-merge-restart* **by** *blast*

lemma *cdcl*$_W$*-merge-rtranclp-cdcl*$_W$:
cdcl$_W$*-merge S T* ⟹ *cdcl*$_W$** *S T*
using *cdcl*$_W$*-merge-cdcl*$_W$*-merge-restart cdcl*$_W$*-merge-restart-cdcl*$_W$ **by** *blast*

lemma *rtranclp-cdcl*$_W$*-merge-rtranclp-cdcl*$_W$:
cdcl$_W$*-merge*** *S T* ⟹ *cdcl*$_W$** *S T*
using *rtranclp-mono*[*of cdcl*$_W$*-merge cdcl*$_W$**] *cdcl*$_W$*-merge-rtranclp-cdcl*$_W$ **by** *auto*

lemmas *rulesE* =
skipE resolveE backtrackE propagateE conflictE decideE restartE forgetE

lemma *cdcl*$_W$*-all-struct-inv-tranclp-cdcl*$_W$*-merge-tranclp-cdcl*$_W$*-merge-cdcl*$_W$*-all-struct-inv*:
assumes
inv: *cdcl*$_W$*-all-struct-inv b*
cdcl$_W$*-merge*$^{++}$ *b a*
shows (λ*S T*. *cdcl*$_W$*-all-struct-inv S* ∧ *cdcl*$_W$*-merge S T*)$^{++}$ *b a*
using *assms*(*2*)
proof *induction*
case *base*
then show *?case* **using** *inv* **by** *auto*
next

```
  case (step c d) note st = this(1) and fw = this(2) and IH = this(3)
  have cdcl_W-all-struct-inv c
    using tranclp-into-rtranclp[OF st] cdcl_W-merge-rtranclp-cdcl_W
    assms(1) rtranclp-cdcl_W-all-struct-inv-inv rtranclp-mono[of cdcl_W-merge cdcl_W^**] by fastforce
  then have (λS T. cdcl_W-all-struct-inv S ∧ cdcl_W-merge S T)^++ c d
    using fw by auto
  then show ?case using IH by auto
qed

lemma backtrack-is-full1-cdcl_W-bj:
  assumes bt: backtrack S T and inv: cdcl_W-M-level-inv S
  shows full1 cdcl_W-bj S T
   using bt inv backtrack-no-cdcl_W-bj unfolding full1-def by blast

lemma rtrancl-cdcl_W-conflicting-true-cdcl_W-merge-restart:
  assumes cdcl_W^** S V and inv: cdcl_W-M-level-inv S and conflicting S = None
  shows (cdcl_W-merge-restart^** S V ∧ conflicting V = None)
    ∨ (∃ T U. cdcl_W-merge-restart^** S T ∧ conflicting V ≠ None ∧ conflict T U ∧ cdcl_W-bj^** U V)
  using assms
proof induction
  case base
  then show ?case by simp
next
  case (step U V) note st = this(1) and cdcl_W = this(2) and IH = this(3)[OF this(4−)] and
    confl[simp] = this(5) and inv = this(4)
  from cdcl_W
  show ?case
    proof (cases)
      case propagate
      moreover then have conflicting U = None and conflicting V = None
        by (auto elim: propagateE)
      ultimately show ?thesis using IH cdcl_W-merge-restart.fw-r-propagate[of U V] by auto
    next
      case conflict
      moreover then have conflicting U = None and conflicting V ≠ None
        by (auto elim!: conflictE simp del: state-simp simp: state-eq-def)
      ultimately show ?thesis using IH by auto
    next
      case other
      then show ?thesis
        proof cases
          case decide
          then show ?thesis using IH cdcl_W-merge-restart.fw-r-decide[of U V] by (auto elim: decideE)
        next
          case bj
          moreover {
            assume skip-or-resolve U V
            have f1: cdcl_W-bj^++ U V
              by (simp add: local.bj tranclp.r-into-trancl)
            obtain T T' :: 'st where
              f2: cdcl_W-merge-restart^** S U
                ∨ cdcl_W-merge-restart^** S T ∧ conflicting U ≠ None
                  ∧ conflict T T' ∧ cdcl_W-bj^** T' U
              using IH confl by blast
            have conflicting V ≠ None ∧ conflicting U ≠ None
              using ‹skip-or-resolve U V›
              by (auto simp: skip-or-resolve.simps state-eq-def elim!: skipE resolveE
                simp del: state-simp)
            then have ?thesis
              by (metis (full-types) IH f1 rtranclp-trans tranclp-into-rtranclp)
          }
          moreover {
            assume backtrack U V
            then have conflicting U ≠ None by (auto elim: backtrackE)
            then obtain T T' where
              cdcl_W-merge-restart** S T and
              conflicting U ≠ None and
              conflict T T' and
              cdcl_W-bj** T' U
              using IH confl by meson
            have invU: cdcl_W-M-level-inv U
              using inv rtranclp-cdcl_W-consistent-inv step.hyps(1) by blast
            then have conflicting V = None
              using ‹backtrack U V› inv by (auto elim: backtrack-levE
                simp: cdcl_W-M-level-inv-decomp)
            have full cdcl_W-bj T' V
              apply (rule rtranclp-fullI[of cdcl_W-bj T' U V])
                using ‹cdcl_W-bj** T' U› apply fast
              using ‹backtrack U V› backtrack-is-full1-cdcl_W-bj invU unfolding full1-def full-def
              by blast
            then have ?thesis
              using cdcl_W-merge-restart.fw-r-conflict[of T T' V] ‹conflict T T'›
              ‹cdcl_W-merge-restart** S T› ‹conflicting V = None› by auto
          }
          ultimately show ?thesis by (auto simp: cdcl_W-bj.simps)
      qed
    next
      case rf
      moreover then have conflicting U = None and conflicting V = None
        by (auto simp: cdcl_W-rf.simps elim: restartE forgetE)
      ultimately show ?thesis using IH cdcl_W-merge-restart.fw-r-rf[of U V] by auto
    qed
qed

lemma no-step-cdcl_W-no-step-cdcl_W-merge-restart: no-step cdcl_W S ⟹ no-step cdcl_W-merge-restart
S
  by (auto simp: cdcl_W.simps cdcl_W-merge-restart.simps cdcl_W-o.simps cdcl_W-bj.simps)

lemma no-step-cdcl_W-merge-restart-no-step-cdcl_W:
  assumes
    conflicting S = None and
    cdcl_W-M-level-inv S and
    no-step cdcl_W-merge-restart S
  shows no-step cdcl_W S
proof -
  { fix S'
    assume conflict S S'
    then have cdcl_W S S' using cdcl_W.conflict by auto
    then have cdcl_W-M-level-inv S'
      using assms(2) cdcl_W-consistent-inv by blast
    then obtain S'' where full cdcl_W-bj S' S''
      using cdcl_W-bj-exists-normal-form[of S'] by auto
    then have False
      using ‹conflict S S'› assms(3) fw-r-conflict by blast
  }
  then show ?thesis
    using assms unfolding cdcl_W.simps cdcl_W-merge-restart.simps cdcl_W-o.simps cdcl_W-bj.simps
    by (auto elim: skipE resolveE backtrackE conflictE decideE restartE)
qed

lemma cdcl_W-merge-restart-no-step-cdcl_W-bj:
  assumes
    cdcl_W-merge-restart S T
  shows no-step cdcl_W-bj T
  using assms
  by (induction rule: cdcl_W-merge-restart.induct)
   (force simp: cdcl_W-bj.simps cdcl_W-rf.simps cdcl_W-merge-restart.simps full-def
     elim!: rulesE)+

lemma rtranclp-cdcl_W-merge-restart-no-step-cdcl_W-bj:
  assumes
    cdcl_W-merge-restart** S T and
    conflicting S = None
  shows no-step cdcl_W-bj T
  using assms unfolding rtranclp-unfold
  apply (elim disjE)
   apply (force simp: cdcl_W-bj.simps cdcl_W-rf.simps elim!: rulesE)
  by (auto simp: tranclp-unfold-end simp: cdcl_W-merge-restart-no-step-cdcl_W-bj)
```

If *conflicting S ≠ None*, we cannot say anything.

Remark that this theorem does not say anything about well-foundedness: even if you know that one relation is well-founded, it only states that the normal forms are shared.

```
lemma conflicting-true-full-cdcl_W-iff-full-cdcl_W-merge:
  assumes confl: conflicting  S = None and lev: cdcl_W-M-level-inv S
  shows full cdcl_W  S V ⟷ full cdcl_W-merge-restart S V
proof
  assume full: full cdcl_W-merge-restart S V
  then have st: cdcl_W** S V
    using rtranclp-mono[of cdcl_W-merge-restart cdcl_W**] cdcl_W-merge-restart-cdcl_W
    unfolding full-def by auto

  have n-s: no-step cdcl_W-merge-restart V
    using full unfolding full-def by auto
  have n-s-bj: no-step cdcl_W-bj V
    using rtranclp-cdcl_W-merge-restart-no-step-cdcl_W-bj confl full unfolding full-def by auto
  have ⋀S'. conflict V S' ⟹ cdcl_W-M-level-inv S'
    using cdcl_W.conflict cdcl_W-consistent-inv lev rtranclp-cdcl_W-consistent-inv st by blast
  then have ⋀S'. conflict V S' ⟹ False
    using n-s n-s-bj cdcl_W-bj-exists-normal-form cdcl_W-merge-restart.simps by meson
  then have n-s-cdcl_W: no-step cdcl_W  V
    using n-s n-s-bj by (auto simp: cdcl_W.simps cdcl_W-o.simps cdcl_W-merge-restart.simps)
  then show full cdcl_W  S V using st unfolding full-def by auto
next
  assume full: full cdcl_W  S V
  have no-step cdcl_W-merge-restart V
```

```
    using full no-step-cdcl_W-no-step-cdcl_W-merge-restart unfolding full-def by blast
  moreover
    consider
        (fw) cdcl_W-merge-restart** S V and conflicting V = None
      | (bj) T U where
        cdcl_W-merge-restart** S T and
        conflicting V ≠ None and
        conflict T U and
        cdcl_W-bj** U V
      using full rtrancl-cdcl_W-conflicting-true-cdcl_W-merge-restart confl lev unfolding full-def
      by meson
    then have cdcl_W-merge-restart** S V
      proof cases
        case fw
        then show ?thesis by fast
      next
        case (bj T U)
        have no-step cdcl_W-bj V
          using full unfolding full-def by (meson cdcl_W-o.bj other)
        then have full cdcl_W-bj U V
          using ⟨ cdcl_W-bj** U V⟩ unfolding full-def by auto
        then have cdcl_W-merge-restart T V
          using ⟨conflict T U⟩ cdcl_W-merge-restart.fw-r-conflict by blast
        then show ?thesis using ⟨cdcl_W-merge-restart** S T⟩ by auto
      qed
  ultimately show full cdcl_W-merge-restart S V unfolding full-def by fast
qed

lemma init-state-true-full-cdcl_W-iff-full-cdcl_W-merge:
  shows full cdcl_W (init-state N) V ⟷ full cdcl_W-merge-restart (init-state N) V
  by (rule conflicting-true-full-cdcl_W-iff-full-cdcl_W-merge) auto
```

21.4 FW with strategy

21.4.1 The intermediate step

```
inductive cdcl_W-s' :: 'st ⇒ 'st ⇒ bool where
conflict': full1 cdcl_W-cp S S' ⟹ cdcl_W-s' S S' |
decide': decide S S' ⟹ no-step cdcl_W-cp S ⟹ full cdcl_W-cp S' S'' ⟹ cdcl_W-s' S S'' |
bj': full1 cdcl_W-bj S S' ⟹ no-step cdcl_W-cp S ⟹ full cdcl_W-cp S' S'' ⟹ cdcl_W-s' S S''

inductive-cases cdcl_W-s'E: cdcl_W-s' S T

lemma rtranclp-cdcl_W-bj-full1-cdclp-cdcl_W-stgy:
  cdcl_W-bj** S S' ⟹ full cdcl_W-cp S' S'' ⟹ cdcl_W-stgy** S S''
proof (induction rule: converse-rtranclp-induct)
  case base
  then show ?case by (metis cdcl_W-stgy.conflict' full-unfold rtranclp.simps)
next
  case (step T U) note st = this(2) and bj = this(1) and IH = this(3)[OF this(4)]
  have no-step cdcl_W-cp T
    using bj by (auto simp add: cdcl_W-bj.simps cdcl_W-cp.simps elim!: rulesE)
  consider
      (U) U = S'
    | (U') U' where cdcl_W-bj U U' and cdcl_W-bj** U' S'
    using st by (metis converse-rtranclpE)
  then show ?case
    proof cases
      case U
      then show ?thesis
        using ‹no-step cdcl_W-cp T› cdcl_W-o.bj local.bj other' step.prems by (meson r-into-rtranclp)
    next
      case U' note U' = this(1)
      have no-step cdcl_W-cp U
        using U' by (fastforce simp: cdcl_W-cp.simps cdcl_W-bj.simps elim: rulesE)
      then have full cdcl_W-cp U U
        by (simp add: full-unfold)
      then have cdcl_W-stgy T U
        using ‹no-step cdcl_W-cp T› cdcl_W-stgy.simps local.bj cdcl_W-o.bj by meson
      then show ?thesis using IH by auto
    qed
qed

lemma cdcl_W-s'-is-rtranclp-cdcl_W-stgy:
  cdcl_W-s' S T ⟹ cdcl_W-stgy^** S T
  apply (induction rule: cdcl_W-s'.induct)
    apply (auto intro: cdcl_W-stgy.intros)[]
   apply (meson decide other' r-into-rtranclp)
  by (metis full1-def rtranclp-cdcl_W-bj-full1-cdclp-cdcl_W-stgy tranclp-into-rtranclp)

lemma cdcl_W-cp-cdcl_W-bj-bissimulation:
  assumes
    full cdcl_W-cp T U and
    cdcl_W-bj^** T T' and
    cdcl_W-all-struct-inv T and
    no-step cdcl_W-bj T'
  shows full cdcl_W-cp T' U
    ∨ (∃ U' U''. full cdcl_W-cp T' U'' ∧ full1 cdcl_W-bj U U' ∧ full cdcl_W-cp U' U''
      ∧ cdcl_W-s'^** U U'')
  using assms(2,1,3,4)
proof (induction rule: rtranclp-induct)
  case base
  then show ?case by blast
next
  case (step T' T'') note st = this(1) and bj = this(2) and IH = this(3)[OF this(4,5)] and
    full = this(4) and inv = this(5)
  have cdcl_W-bj^** T T''
    using local.bj st by auto
  then have cdcl_W^** T T''
    using rtranclp-cdcl_W-bj-rtranclp-cdcl_W by blast
  then have inv-T'': cdcl_W-all-struct-inv T''
    using inv rtranclp-cdcl_W-all-struct-inv-inv by blast
  have cdcl_W-bj^++ T T''
    using local.bj st by auto
  have full1 cdcl_W-bj T T''
    by (metis ‹cdcl_W-bj^++ T T''› full1-def step.prems(3))
  then have T = U
    proof –
      obtain Z where cdcl_W-bj T Z
        using ‹cdcl_W-bj^++ T T''› by (blast dest: tranclpD)
      { assume cdcl_W-cp^++ T U
```

```
        then obtain Z' where cdcl_W-cp T Z'
          by (meson tranclpD)
        then have False
          using ⟨cdcl_W-bj T Z⟩ by (fastforce simp: cdcl_W-bj.simps cdcl_W-cp.simps
            elim: rulesE)
      }
      then show ?thesis
        using full unfolding full-def rtranclp-unfold by blast
    qed
  obtain U'' where full cdcl_W-cp T'' U''
    using cdcl_W-cp-normalized-element-all-inv inv-T'' by blast
  moreover then have cdcl_W-stgy** U U''
    by (metis ⟨T = U⟩ ⟨cdcl_W-bj++ T T''⟩ rtranclp-cdcl_W-bj-full1-cdclp-cdcl_W-stgy rtranclp-unfold)
  moreover have cdcl_W-s'** U U''
    proof -
      obtain ss :: 'st ⇒ 'st where
        f1: ∀x2. (∃v3. cdcl_W-cp x2 v3) = cdcl_W-cp x2 (ss x2)
        by moura
      have ¬ cdcl_W-cp U (ss U)
        by (meson full full-def)
      then show ?thesis
        using f1 by (metis (no-types) ⟨T = U⟩ ⟨full1 cdcl_W-bj T T''⟩ bj' calculation(1)
          r-into-rtranclp)
    qed
  ultimately show ?case
    using ⟨full1 cdcl_W-bj T T''⟩ ⟨full cdcl_W-cp T'' U''⟩ unfolding ⟨T = U⟩ by blast
qed

lemma cdcl_W-cp-cdcl_W-bj-bissimulation':
  assumes
    full cdcl_W-cp T U and
    cdcl_W-bj** T T' and
    cdcl_W-all-struct-inv T and
    no-step cdcl_W-bj T'
  shows full cdcl_W-cp T' U
    ∨ (∃U'. full1 cdcl_W-bj U U' ∧ (∀U''. full cdcl_W-cp U' U'' ⟶ full cdcl_W-cp T' U''
      ∧ cdcl_W-s'** U U''))
  using assms(2,1,3,4)
proof (induction rule: rtranclp-induct)
  case base
  then show ?case by blast
next
  case (step T' T'') note st = this(1) and bj = this(2) and IH = this(3)[OF this(4,5)] and
    full = this(4) and inv = this(5)
  have cdcl_W** T T''
    by (metis local.bj rtranclp.simps rtranclp-cdcl_W-bj-rtranclp-cdcl_W st)
  then have inv-T'': cdcl_W-all-struct-inv T''
    using inv rtranclp-cdcl_W-all-struct-inv-inv by blast
  have cdcl_W-bj++ T T''
    using local.bj st by auto
  have full1 cdcl_W-bj T T''
    by (metis ⟨cdcl_W-bj++ T T''⟩ full1-def step.prems(3))
  then have T = U
    proof -
      obtain Z where cdcl_W-bj T Z
```

```
        using ‹cdcl_W-bj^++ T T''› by (blast dest: tranclpD)
      { assume cdcl_W-cp^++ T U
        then obtain Z' where cdcl_W-cp T Z'
          by (meson tranclpD)
        then have False
          using ‹cdcl_W-bj T Z› by (fastforce simp: cdcl_W-bj.simps cdcl_W-cp.simps elim: rulesE)
      }
      then show ?thesis
        using full unfolding full-def rtranclp-unfold by blast
    qed
  { fix U''
    assume full cdcl_W-cp T'' U''
    moreover then have cdcl_W-stgy^** U U''
      by (metis ‹T = U› ‹cdcl_W-bj^++ T T''› rtranclp-cdcl_W-bj-full1-cdclp-cdcl_W-stgy rtranclp-unfold)
    moreover have cdcl_W-s'^** U U''
      proof –
        obtain ss :: 'st ⇒ 'st where
          f1: ∀x2. (∃v3. cdcl_W-cp x2 v3) = cdcl_W-cp x2 (ss x2)
          by moura
        have ¬ cdcl_W-cp U (ss U)
          by (meson assms(1) full-def)
        then show ?thesis
          using f1 by (metis (no-types) ‹T = U› ‹full1 cdcl_W-bj T T''› bj' calculation(1)
            r-into-rtranclp)
      qed
    ultimately have full1 cdcl_W-bj U T'' and  cdcl_W-s'^** T'' U''
      using ‹full1 cdcl_W-bj T T''› ‹full cdcl_W-cp T'' U''› unfolding ‹T = U›
        apply blast
      by (metis ‹full cdcl_W-cp T'' U''› cdcl_W-s'.simps full-unfold rtranclp.simps)
    }
  then show ?case
    using ‹full1 cdcl_W-bj T T''› full bj' unfolding ‹T = U› full-def by (metis r-into-rtranclp)
qed

lemma cdcl_W-stgy-cdcl_W-s'-connected:
  assumes cdcl_W-stgy S U and cdcl_W-all-struct-inv S
  shows cdcl_W-s' S U
    ∨ (∃U'. full1 cdcl_W-bj U U' ∧ (∀U''. full cdcl_W-cp U' U'' ⟶ cdcl_W-s' S U'' ))
  using assms
proof (induction rule: cdcl_W-stgy.induct)
  case (conflict' T)
  then have cdcl_W-s' S T
    using cdcl_W-s'.conflict' by blast
  then show ?case
    by blast
next
  case (other' T U) note o = this(1) and n-s = this(2) and full = this(3) and inv = this(4)
  show ?case
    using o
    proof cases
      case decide
      then show ?thesis using cdcl_W-s'.simps full n-s by blast
    next
      case bj
      have inv-T: cdcl_W-all-struct-inv T
```

```
        using cdcl_W-all-struct-inv-inv o other other'.prems by blast
      consider
          (cp) full cdcl_W-cp T U and no-step cdcl_W-bj T
        | (fbj) T' where full1 cdcl_W-bj T T'
        apply (cases no-step cdcl_W-bj T)
         using full apply blast
        using cdcl_W-bj-exists-normal-form[of T] inv-T unfolding cdcl_W-all-struct-inv-def
        by (metis full-unfold)
      then show ?thesis
        proof cases
          case cp
          then show ?thesis
            proof -
              obtain ss :: 'st ⇒ 'st where
                f1: ∀ s sa sb. (¬ full1 cdcl_W-bj s sa ∨ cdcl_W-cp s (ss s) ∨ ¬ full cdcl_W-cp sa sb)
                  ∨ cdcl_W-s' s sb
                using bj' by moura
              have full1 cdcl_W-bj S T
                by (simp add: cp(2) full1-def local.bj tranclp.r-into-trancl)
              then show ?thesis
                using f1 full n-s by blast
            qed
        next
          case (fbj U')
          then have full1 cdcl_W-bj S U'
            using bj unfolding full1-def by auto
          moreover have no-step cdcl_W-cp S
            using n-s by blast
          moreover have T = U
            using full fbj unfolding full1-def full-def rtranclp-unfold
            by (force dest!: tranclpD simp:cdcl_W-bj.simps elim: rulesE)
          ultimately show ?thesis using cdcl_W-s'.bj'[of S U'] using fbj by blast
        qed
    qed
qed

lemma cdcl_W-stgy-cdcl_W-s'-connected':
  assumes cdcl_W-stgy S U and cdcl_W-all-struct-inv S
  shows cdcl_W-s' S U
    ∨ (∃ U' U''. cdcl_W-s' S U'' ∧ full1 cdcl_W-bj U U' ∧ full cdcl_W-cp U' U'')
  using assms
proof (induction rule: cdcl_W-stgy.induct)
  case (conflict' T)
  then have cdcl_W-s' S T
    using cdcl_W-s'.conflict' by blast
  then show ?case
    by blast
next
  case (other' T U) note o = this(1) and n-s = this(2) and full = this(3) and inv = this(4)
  show ?case
    using o
    proof cases
      case decide
      then show ?thesis using cdcl_W-s'.simps full n-s by blast
    next
```

```
      case bj
      have cdcl_W-all-struct-inv T
        using cdcl_W-all-struct-inv-inv o other other'.prems by blast
      then obtain T' where T': full cdcl_W-bj T T'
        using cdcl_W-bj-exists-normal-form unfolding full-def cdcl_W-all-struct-inv-def by metis
      then have full cdcl_W-bj S T'
        proof –
          have f1: cdcl_W-bj** T T' ∧ no-step cdcl_W-bj T'
            by (metis (no-types) T' full-def)
          then have cdcl_W-bj** S T'
            by (meson converse-rtranclp-into-rtranclp local.bj)
          then show ?thesis
            using f1 by (simp add: full-def)
        qed
      have cdcl_W-bj** T T'
        using T' unfolding full-def by simp
      have cdcl_W-all-struct-inv T
        using cdcl_W-all-struct-inv-inv o other other'.prems by blast
      then consider
          (T'U) full cdcl_W-cp T' U
        | (U) U' U'' where
            full cdcl_W-cp T' U'' and
            full1 cdcl_W-bj U U' and
            full cdcl_W-cp U' U'' and
            cdcl_W-s'** U U''
        using cdcl_W-cp-cdcl_W-bj-bissimulation[OF full ‹cdcl_W-bj** T T'›] T' unfolding full-def
        by blast
      then show ?thesis by (metis T' cdcl_W-s'.simps full-fullI local.bj n-s)
    qed
qed

lemma cdcl_W-stgy-cdcl_W-s'-no-step:
  assumes cdcl_W-stgy S U and cdcl_W-all-struct-inv S and no-step cdcl_W-bj U
  shows cdcl_W-s' S U
  using cdcl_W-stgy-cdcl_W-s'-connected[OF assms(1,2)] assms(3)
  by (metis (no-types, lifting) full1-def tranclpD)

lemma rtranclp-cdcl_W-stgy-connected-to-rtranclp-cdcl_W-s':
  assumes cdcl_W-stgy** S U and inv: cdcl_W-M-level-inv S
  shows cdcl_W-s'** S U ∨ (∃ T. cdcl_W-s'** S T ∧ cdcl_W-bj++ T U ∧ conflicting U ≠ None)
  using assms(1)
proof induction
  case base
  then show ?case by simp
next
  case (step T V) note st = this(1) and o = this(2) and IH = this(3)
  from o show ?case
    proof cases
      case conflict'
      then have f2: cdcl_W-s' T V
        using cdcl_W-s'.conflict' by blast
      obtain ss :: 'st where
        f3: S = T ∨ cdcl_W-stgy** S ss ∧ cdcl_W-stgy ss T
        by (metis (full-types) rtranclp.simps st)
      obtain ssa :: 'st where
        ssa: cdcl_W-cp T ssa
        using conflict′ by (metis (no-types) full1-def tranclpD)
      have ∀ s. ¬ full cdcl_W-cp s T
        by (meson ssa full-def)
      then have S = T
        by (metis (full-types) f3 ssa cdcl_W-stgy.cases full1-def)
      then show ?thesis
        using f2 by blast
    next
      case (other′ U) note o = this(1) and n-s = this(2) and full = this(3)
      then show ?thesis
        using o
        proof (cases rule: cdcl_W-o-rule-cases)
          case decide
          then have cdcl_W-s′** S T
            using IH by (auto elim: rulesE)
          then show ?thesis
            by (meson decide decide′ full n-s rtranclp.rtrancl-into-rtrancl)
        next
          case backtrack
          consider
              (s′) cdcl_W-s′** S T
            | (bj) S′ where cdcl_W-s′** S S′ and cdcl_W-bj++ S′ T and conflicting T ≠ None
            using IH by blast
          then show ?thesis
            proof cases
              case s′
              moreover
                have cdcl_W-M-level-inv T
                  using inv local.step(1) rtranclp-cdcl_W-stgy-consistent-inv by auto
                then have full1 cdcl_W-bj T U
                  using backtrack-is-full1-cdcl_W-bj backtrack by blast
                then have cdcl_W-s′ T V
                 using full bj′ n-s by blast
              ultimately show ?thesis by auto
            next
              case (bj S′) note S-S′ = this(1) and bj-T = this(2)
              have no-step cdcl_W-cp S′
                using bj-T by (fastforce simp: cdcl_W-cp.simps cdcl_W-bj.simps dest!: tranclpD
                  elim: rulesE)
              moreover
                have cdcl_W-M-level-inv T
                  using inv local.step(1) rtranclp-cdcl_W-stgy-consistent-inv by auto
                then have full1 cdcl_W-bj T U
                  using backtrack-is-full1-cdcl_W-bj backtrack by blast
                then have full1 cdcl_W-bj S′ U
                  using bj-T unfolding full1-def by fastforce
              ultimately have cdcl_W-s′ S′ V using full by (simp add: bj′)
              then show ?thesis using S-S′ by auto
            qed
        next
          case skip
          then have [simp]: U = V
            using full converse-rtranclpE unfolding full-def by (fastforce elim: rulesE)
          then have confl-V: conflicting V ≠ None
```

```
            using skip by (auto elim!: rulesE simp del: state-simp simp: state-eq-def)
          consider
              (s') cdcl_W-s'** S T
            | (bj) S' where cdcl_W-s'** S S' and cdcl_W-bj++ S' T and conflicting T ≠ None
            using IH by blast
          then show ?thesis
            proof cases
              case s'
              show ?thesis using s' confl-V skip by force
            next
              case (bj S') note S-S' = this(1) and bj-T = this(2)
              have cdcl_W-bj++ S' V
                using skip bj-T by (metis ‹U = V› cdcl_W-bj.skip tranclp.simps)
              then show ?thesis using S-S' confl-V by auto
            qed
        next
          case resolve
          then have [simp]: U = V
            using full unfolding full-def rtranclp-unfold
            by (auto elim!: rulesE dest!: tranclpD
              simp del: state-simp simp: state-eq-def cdcl_W-cp.simps)
          have confl-V: conflicting V ≠ None
            using resolve by (auto elim!: rulesE simp del: state-simp simp: state-eq-def)

          consider
              (s') cdcl_W-s'** S T
            | (bj) S' where cdcl_W-s'** S S' and cdcl_W-bj++ S' T and conflicting T ≠ None
            using IH by blast
          then show ?thesis
            proof cases
              case s'
              have cdcl_W-bj++ T V
                using resolve by force
              then show ?thesis using s' confl-V by auto
            next
              case (bj S') note S-S' = this(1) and bj-T = this(2)
              have cdcl_W-bj++ S' V
                using resolve bj-T by (metis ‹U = V› cdcl_W-bj.resolve tranclp.simps)
              then show ?thesis using confl-V S-S' by auto
            qed
        qed
    qed
qed

lemma n-step-cdcl_W-stgy-iff-no-step-cdcl_W-cl-cdcl_W-o:
  assumes inv: cdcl_W-all-struct-inv S
  shows no-step cdcl_W-s' S ⟷ no-step cdcl_W-cp S ∧ no-step cdcl_W-o S (is ?S' S ⟷ ?C S ∧ ?O S)
proof
  assume ?C S ∧ ?O S
  then show ?S' S
    by (auto simp: cdcl_W-s'.simps full1-def tranclp-unfold-begin)
next
  assume n-s: ?S' S
  have ?C S
    proof (rule ccontr)
```

```
      assume ¬ ?thesis
      then obtain S' where cdcl_W-cp S S'
        by auto
      then obtain T where full1 cdcl_W-cp S T
        using cdcl_W-cp-normalized-element-all-inv inv by (metis (no-types, lifting) full-unfold)
      then show False using n-s cdcl_W-s'.conflict' by blast
    qed
  moreover have ?O S
    proof (rule ccontr)
      assume ¬ ?thesis
      then obtain S' where cdcl_W-o S S'
        by auto
      then obtain T where full1 cdcl_W-cp S' T
        using cdcl_W-cp-normalized-element-all-inv inv
        by (meson cdcl_W-all-struct-inv-def n-s
          cdcl_W-stgy-cdcl_W-s'-connected' cdcl_W-then-exists-cdcl_W-stgy-step )
      then show False using n-s by (meson ‹cdcl_W-o S S'› cdcl_W-all-struct-inv-def
        cdcl_W-stgy-cdcl_W-s'-connected' cdcl_W-then-exists-cdcl_W-stgy-step inv)
    qed
  ultimately show ?C S ∧ ?O S by auto
qed

lemma cdcl_W-s'-tranclp-cdcl_W:
   cdcl_W-s' S S' ⟹ cdcl_W^++ S S'
proof (induct rule: cdcl_W-s'.induct)
  case conflict'
  then show ?case
    by (simp add: full1-def tranclp-cdcl_W-cp-tranclp-cdcl_W)
next
  case decide'
  then show ?case
    using cdcl_W-stgy.simps cdcl_W-stgy-tranclp-cdcl_W by (meson cdcl_W-o.simps)
next
  case (bj' Sa S'a S'') note a2 = this(1) and a1 = this(2) and n-s = this(3)
  obtain ss :: 'st ⇒ 'st ⇒ ('st ⇒ 'st ⇒ bool) ⇒ 'st where
    ∀x0 x1 x2. (∃v3. x2 x1 v3 ∧ x2^** v3 x0) = (x2 x1 (ss x0 x1 x2) ∧ x2^** (ss x0 x1 x2) x0)
    by moura
  then have f3: ∀p s sa. ¬ p^++ s sa ∨ p s (ss sa s p) ∧ p^** (ss sa s p) sa
    by (metis (full-types) tranclpD)
  have cdcl_W-bj^++ Sa S'a ∧ no-step cdcl_W-bj S'a
    using a2 by (simp add: full1-def)
  then have cdcl_W-bj Sa (ss S'a Sa cdcl_W-bj) ∧ cdcl_W-bj^** (ss S'a Sa cdcl_W-bj) S'a
    using f3 by auto
  then show cdcl_W^++ Sa S''
    using a1 n-s by (meson bj other rtranclp-cdcl_W-bj-full1-cdclp-cdcl_W-stgy
      rtranclp-cdcl_W-stgy-rtranclp-cdcl_W rtranclp-into-tranclp2)
qed

lemma tranclp-cdcl_W-s'-tranclp-cdcl_W:
  cdcl_W-s'^++ S S' ⟹ cdcl_W^++ S S'
  apply (induct rule: tranclp.induct)
   using cdcl_W-s'-tranclp-cdcl_W apply blast
  by (meson cdcl_W-s'-tranclp-cdcl_W tranclp-trans)

lemma rtranclp-cdcl_W-s'-rtranclp-cdcl_W:
    cdcl_W-s'** S S' ⟹ cdcl_W** S S'
  using rtranclp-unfold[of cdcl_W-s' S S'] tranclp-cdcl_W-s'-tranclp-cdcl_W[of S S'] by auto

lemma full-cdcl_W-stgy-iff-full-cdcl_W-s':
  assumes inv: cdcl_W-all-struct-inv S
  shows full cdcl_W-stgy S T ⟷ full cdcl_W-s' S T (is ?S ⟷ ?S')
proof
  assume ?S'
  then have cdcl_W** S T
    using rtranclp-cdcl_W-s'-rtranclp-cdcl_W[of S T] unfolding full-def by blast
  then have inv': cdcl_W-all-struct-inv T
    using rtranclp-cdcl_W-all-struct-inv-inv inv by blast
  have cdcl_W-stgy** S T
    using ‹?S'› unfolding full-def
      using cdcl_W-s'-is-rtranclp-cdcl_W-stgy rtranclp-mono[of cdcl_W-s' cdcl_W-stgy**] by auto
  then show ?S
    using ‹?S'› inv' cdcl_W-stgy-cdcl_W-s'-connected' unfolding full-def by blast
next
  assume ?S
  then have inv-T:cdcl_W-all-struct-inv T
    by (metis assms full-def rtranclp-cdcl_W-all-struct-inv-inv rtranclp-cdcl_W-stgy-rtranclp-cdcl_W)

  consider
      (s') cdcl_W-s'** S T
    | (st) S' where cdcl_W-s'** S S' and cdcl_W-bj++ S' T and conflicting T ≠ None
    using rtranclp-cdcl_W-stgy-connected-to-rtranclp-cdcl_W-s'[of S T] inv ‹?S›
    unfolding full-def cdcl_W-all-struct-inv-def
    by blast
  then show ?S'
    proof cases
      case s'
      have no-step cdcl_W-s' T
        using ‹full cdcl_W-stgy S T› unfolding full-def
        by (meson cdcl_W-all-struct-inv-def cdcl_W-s'E cdcl_W-stgy.conflict'
          cdcl_W-then-exists-cdcl_W-stgy-step inv-T n-step-cdcl_W-stgy-iff-no-step-cdcl_W-cl-cdcl_W-o)
      then show ?thesis
        using s' unfolding full-def by blast
    next
      case (st S')
      have full cdcl_W-cp T T
        using option-full-cdcl_W-cp st(3) by blast
      moreover
        have n-s: no-step cdcl_W-bj T
          by (metis ‹full cdcl_W-stgy S T› bj inv-T cdcl_W-all-struct-inv-def
            cdcl_W-then-exists-cdcl_W-stgy-step full-def)
        then have full1 cdcl_W-bj S' T
          using st(2) unfolding full1-def by blast
      moreover have no-step cdcl_W-cp S'
        using st(2) by (fastforce dest!: tranclpD simp: cdcl_W-cp.simps cdcl_W-bj.simps
          elim: rulesE)
      ultimately have cdcl_W-s' S' T
        using cdcl_W-s'.bj'[of S' T T] by blast
      then have cdcl_W-s'** S T
        using st(1) by auto
      moreover have no-step cdcl_W-s' T
```

```
        using inv-T ‹full cdcl_W-cp T T› ‹full cdcl_W-stgy S T› unfolding full-def
        by (metis cdcl_W-all-struct-inv-def cdcl_W-then-exists-cdcl_W-stgy-step
          n-step-cdcl_W-stgy-iff-no-step-cdcl_W-cl-cdcl_W-o)
      ultimately show ?thesis
        unfolding full-def by blast
    qed
qed

lemma conflict-step-cdcl_W-stgy-step:
  assumes
    conflict S T
    cdcl_W-all-struct-inv S
  shows ∃ T. cdcl_W-stgy S T
proof –
  obtain U where full cdcl_W-cp S U
    using cdcl_W-cp-normalized-element-all-inv assms by blast
  then have full1 cdcl_W-cp S U
    by (metis cdcl_W-cp.conflict' assms(1) full-unfold)
  then show ?thesis using cdcl_W-stgy.conflict' by blast
qed

lemma decide-step-cdcl_W-stgy-step:
  assumes
    decide S T
    cdcl_W-all-struct-inv S
  shows ∃ T. cdcl_W-stgy S T
proof –
  obtain U where full cdcl_W-cp T U
    using cdcl_W-cp-normalized-element-all-inv by (meson assms(1) assms(2) cdcl_W-all-struct-inv-inv
      cdcl_W-cp-normalized-element-all-inv decide other)
  then show ?thesis
    by (metis assms cdcl_W-cp-normalized-element-all-inv cdcl_W-stgy.conflict' decide full-unfold
      other')
qed

lemma rtranclp-cdcl_W-cp-conflicting-Some:
  cdcl_W-cp^** S T ⟹ conflicting S = Some D ⟹ S = T
  using rtranclpD tranclpD by fastforce

inductive cdcl_W-merge-cp :: 'st ⇒ 'st ⇒ bool where
conflict': conflict S T ⟹ full cdcl_W-bj T U ⟹ cdcl_W-merge-cp S U |
propagate': propagate^++ S S' ⟹ cdcl_W-merge-cp S S'

lemma cdcl_W-merge-restart-cases[consumes 1, case-names conflict propagate]:
  assumes
    cdcl_W-merge-cp S U and
    ⋀T. conflict S T ⟹ full cdcl_W-bj T U ⟹ P and
    propagate^++ S U ⟹ P
  shows P
  using assms unfolding cdcl_W-merge-cp.simps by auto

lemma cdcl_W-merge-cp-tranclp-cdcl_W-merge:
  cdcl_W-merge-cp S T ⟹ cdcl_W-merge^++ S T
  apply (induction rule: cdcl_W-merge-cp.induct)
    using cdcl_W-merge.simps apply auto[1]
```

using *tranclp-mono*[*of propagate cdcl*$_W$*-merge*] *fw-propagate* **by** *blast*

lemma *rtranclp-cdcl*$_W$*-merge-cp-rtranclp-cdcl*$_W$:
cdcl$_W$*-merge-cp*** *S T* $\Longrightarrow$ *cdcl*$_W$** *S T*
apply (*induction rule*: *rtranclp-induct*)
apply *simp*
unfolding *cdcl*$_W$*-merge-cp.simps* **by** (*meson cdcl*$_W$*-merge-restart-cdcl*$_W$ *fw-r-conflict*
rtranclp-propagate-is-rtranclp-cdcl$_W$ *rtranclp-trans tranclp-into-rtranclp*)

lemma *full1-cdcl*$_W$*-bj-no-step-cdcl*$_W$*-bj*:
full1 cdcl$_W$*-bj S T* $\Longrightarrow$ *no-step cdcl*$_W$*-cp S*
unfolding *full1-def* **by** (*metis rtranclp-unfold cdcl*$_W$*-cp-conflicting-not-empty option.exhaust*
rtranclp-cdcl$_W$*-merge-restart-no-step-cdcl*$_W$*-bj tranclpD*)

inductive *cdcl*$_W$*-s'-without-decide* **where**
conflict'-without-decide[*intro*]: *full1 cdcl*$_W$*-cp S S'* $\Longrightarrow$ *cdcl*$_W$*-s'-without-decide S S'* |
bj'-without-decide[*intro*]: *full1 cdcl*$_W$*-bj S S'* $\Longrightarrow$ *no-step cdcl*$_W$*-cp S* $\Longrightarrow$ *full cdcl*$_W$*-cp S' S''*
$\Longrightarrow$ *cdcl*$_W$*-s'-without-decide S S''*

lemma *rtranclp-cdcl*$_W$*-s'-without-decide-rtranclp-cdcl*$_W$:
cdcl$_W$*-s'-without-decide*** *S T* $\Longrightarrow$ *cdcl*$_W$** *S T*
apply (*induction rule*: *rtranclp-induct*)
apply *simp*
by (*meson cdcl*$_W$*-s'.simps cdcl*$_W$*-s'-tranclp-cdcl*$_W$ *cdcl*$_W$*-s'-without-decide.simps*
rtranclp-tranclp-tranclp tranclp-into-rtranclp)

lemma *rtranclp-cdcl*$_W$*-s'-without-decide-rtranclp-cdcl*$_W$*-s'*:
cdcl$_W$*-s'-without-decide*** *S T* $\Longrightarrow$ *cdcl*$_W$*-s'*** *S T*
proof (*induction rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *simp*
next
case (*step y z*) **note** *a2* = *this*(*2*) **and** *a1* = *this*(*3*)
have *cdcl*$_W$*-s' y z*
using *a2* **by** (*metis* (*no-types*) *bj' cdcl*$_W$*-s'.conflict' cdcl*$_W$*-s'-without-decide.cases*)
then show *cdcl*$_W$*-s'*** *S z*
using *a1* **by** (*meson r-into-rtranclp rtranclp-trans*)
qed

lemma *rtranclp-cdcl*$_W$*-merge-cp-is-rtranclp-cdcl*$_W$*-s'-without-decide*:
assumes
cdcl$_W$*-merge-cp*** *S V*
conflicting S = *None*
shows
(*cdcl*$_W$*-s'-without-decide*** *S V*)
$\vee$ ($\exists$ *T*. *cdcl*$_W$*-s'-without-decide*** *S T* $\wedge$ *propagate*$^{++}$ *T V*)
$\vee$ ($\exists$ *T U*. *cdcl*$_W$*-s'-without-decide*** *S T* $\wedge$ *full1 cdcl*$_W$*-bj T U* $\wedge$ *propagate*** *U V*)
using *assms*
proof (*induction rule*: *rtranclp-induct*)
case *base*
then show *?case* **by** *simp*
next
case (*step U V*) **note** *st* = *this*(*1*) **and** *cp* = *this*(*2*) **and** *IH* = *this*(*3*)[*OF this*(*4*)]
from *cp* **show** *?case*
proof (*cases rule*: *cdcl*$_W$*-merge-restart-cases*)

```
      case propagate
      then show ?thesis using IH by (meson rtranclp-tranclp-tranclp tranclp-into-rtranclp)
    next
      case (conflict U') note confl = this(1) and bj = this(2)
      have full1-U-U': full1 cdcl_W-cp U U'
        by (simp add: conflict-is-full1-cdcl_W-cp local.conflict(1))
      consider
          (s') cdcl_W-s'-without-decide** S U
        | (propa) T' where cdcl_W-s'-without-decide** S T' and propagate++ T' U
        | (bj-prop) T' T'' where
            cdcl_W-s'-without-decide** S T'  and
            full1 cdcl_W-bj T' T'' and
            propagate** T'' U
        using IH by blast
      then show ?thesis
        proof cases
          case s'
          have cdcl_W-s'-without-decide U U'
           using full1-U-U' conflict'-without-decide by blast
          then have cdcl_W-s'-without-decide** S U'
            using ‹cdcl_W-s'-without-decide** S U› by auto
          moreover have U' = V ∨ full1 cdcl_W-bj U' V
            using bj by (meson full-unfold)
          ultimately show ?thesis by blast
        next
          case propa note s' = this(1) and T'-U = this(2)
          have full1 cdcl_W-cp T' U'
            using rtranclp-mono[of propagate cdcl_W-cp] T'-U cdcl_W-cp.propagate' full1-U-U'
            rtranclp-full1I[of cdcl_W-cp T'] by (metis (full-types) predicate2D predicate2I
              tranclp-into-rtranclp)
          have cdcl_W-s'-without-decide** S U'
            using ‹full1 cdcl_W-cp T' U'› conflict'-without-decide s' by force
          have full1 cdcl_W-bj U' V ∨ V = U' using bj unfolding full-unfold by blast
          then show ?thesis
            using ‹cdcl_W-s'-without-decide** S U'› by blast
        next
          case bj-prop note s' = this(1) and bj-T' = this(2) and T''-U = this(3)
          have no-step cdcl_W-cp T'
            using bj-T' full1-cdcl_W-bj-no-step-cdcl_W-bj by blast
          moreover have full1 cdcl_W-cp T'' U'
            using rtranclp-mono[of propagate cdcl_W-cp] T''-U cdcl_W-cp.propagate' full1-U-U'
            rtranclp-full1I[of cdcl_W-cp T''] by blast
          ultimately have cdcl_W-s'-without-decide T' U'
            using bj'-without-decide[of T' T'' U'] bj-T' by (simp add: full-unfold)
          then have cdcl_W-s'-without-decide** S U'
            using s' rtranclp.intros(2)[of - S T' U'] by blast
          then show ?thesis
            using local.bj unfolding full-unfold by blast
        qed
    qed
qed

lemma rtranclp-cdcl_W-s'-without-decide-is-rtranclp-cdcl_W-merge-cp:
  assumes
    cdcl_W-s'-without-decide** S V and
    confl: conflicting S = None
  shows
    (cdcl_W-merge-cp** S V ∧ conflicting V = None)
    ∨ (cdcl_W-merge-cp** S V ∧ conflicting V ≠ None ∧ no-step cdcl_W-cp V ∧ no-step cdcl_W-bj V)
    ∨ (∃ T. cdcl_W-merge-cp** S T ∧ conflict T V)
  using assms(1)
proof (induction)
  case base
  then show ?case using confl by auto
next
  case (step U V) note st = this(1) and s = this(2) and IH = this(3)
  from s show ?case
    proof (cases rule: cdcl_W-s'-without-decide.cases)
      case conflict'-without-decide
      then have rt: cdcl_W-cp++ U V  unfolding full1-def by fast
      then have conflicting U = None
        using tranclp-cdcl_W-cp-propagate-with-conflict-or-not[of U V]
        conflict by (auto dest!: tranclpD simp: rtranclp-unfold elim: rulesE)
      then have cdcl_W-merge-cp** S U using IH by (auto elim: rulesE
        simp del: state-simp simp: state-eq-def)
      consider
          (propa) propagate++ U V
         | (confl') conflict U V
         | (propa-confl') U' where propagate++ U U' conflict U' V
        using tranclp-cdcl_W-cp-propagate-with-conflict-or-not[OF rt] unfolding rtranclp-unfold
        by fastforce
      then show ?thesis
        proof cases
          case propa
          then have cdcl_W-merge-cp U V
            by (auto intro: cdcl_W-merge-cp.intros)
          moreover have conflicting V = None
            using propa unfolding tranclp-unfold-end by (auto elim: rulesE)
          ultimately show ?thesis using ‹cdcl_W-merge-cp** S U› by (auto elim!: rulesE
            simp del: state-simp simp: state-eq-def)
        next
          case confl'
          then show ?thesis using ‹cdcl_W-merge-cp** S U› by auto
        next
          case propa-confl' note propa = this(1) and confl' = this(2)
          then have cdcl_W-merge-cp U U' by (auto intro: cdcl_W-merge-cp.intros)
          then have cdcl_W-merge-cp** S U' using ‹cdcl_W-merge-cp** S U› by auto
          then show ?thesis using ‹cdcl_W-merge-cp** S U› confl' by auto
        qed
    next
      case (bj'-without-decide U') note full-bj = this(1) and cp = this(3)
      then have conflicting U ≠ None
        using full-bj unfolding full1-def by (fastforce dest!: tranclpD simp: cdcl_W-bj.simps
          elim: rulesE)
      with IH obtain T where
        S-T: cdcl_W-merge-cp** S T and T-U: conflict T U
        using full-bj unfolding full1-def by (blast dest: tranclpD)
      then have cdcl_W-merge-cp T U'
        using cdcl_W-merge-cp.conflict'[of T U U'] full-bj by (simp add: full-unfold)
      then have S-U': cdcl_W-merge-cp** S U' using S-T by auto
      consider
          (n-s) U' = V
         | (propa) propagate^++ U' V
         | (confl') conflict U' V
         | (propa-confl') U'' where propagate^++ U' U'' conflict U'' V
        using tranclp-cdcl_W-cp-propagate-with-conflict-or-not cp
        unfolding rtranclp-unfold full-def by metis
      then show ?thesis
        proof cases
          case propa
          then have cdcl_W-merge-cp U' V by (blast intro: cdcl_W-merge-cp.intros)
          moreover have conflicting V = None
            using propa unfolding tranclp-unfold-end by (auto elim: rulesE)
          ultimately show ?thesis using S-U' by (auto elim: rulesE
            simp del: state-simp simp: state-eq-def)
        next
          case confl'
          then show ?thesis using S-U' by auto
        next
          case propa-confl' note propa = this(1) and confl = this(2)
          have cdcl_W-merge-cp U' U'' using propa by (blast intro: cdcl_W-merge-cp.intros)
          then show ?thesis using S-U' confl by (meson rtranclp.rtrancl-into-rtrancl)
        next
          case n-s
          then show ?thesis
            using S-U' apply (cases conflicting V = None)
             using full-bj apply simp
            by (metis cp full-def full-unfold full-bj)
        qed
    qed
qed

lemma no-step-cdcl_W-s'-no-ste-cdcl_W-merge-cp:
  assumes
    cdcl_W-all-struct-inv S
    conflicting S = None
    no-step cdcl_W-s' S
  shows no-step cdcl_W-merge-cp S
  using assms apply (auto simp: cdcl_W-s'.simps cdcl_W-merge-cp.simps)
    using conflict-is-full1-cdcl_W-cp apply blast
  using cdcl_W-cp-normalized-element-all-inv cdcl_W-cp.propagate' by (metis cdcl_W-cp.propagate'
    full-unfold tranclpD)
```

The *no-step decide S* is needed, since *cdcl*$_W$*-merge-cp* is *cdcl*$_W$*-s'* without *decide*.

```
lemma conflicting-true-no-step-cdcl_W-merge-cp-no-step-s'-without-decide:
  assumes
    confl: conflicting S = None and
    inv: cdcl_W-M-level-inv S and
    n-s: no-step cdcl_W-merge-cp S
  shows no-step cdcl_W-s'-without-decide S
proof (rule ccontr)
  assume ¬ no-step cdcl_W-s'-without-decide S
  then obtain T where
    cdcl_W: cdcl_W-s'-without-decide S T
    by auto
  then have inv-T: cdcl_W-M-level-inv T
    using rtranclp-cdcl_W-s'-without-decide-rtranclp-cdcl_W[of S T]
    rtranclp-cdcl_W-consistent-inv inv by blast
  from cdcl_W show False
    proof cases
      case conflict'-without-decide
      have no-step propagate S
        using n-s by (blast intro: cdcl_W-merge-cp.intros)
      then have conflict S T
        using local.conflict' tranclp-cdcl_W-cp-propagate-with-conflict-or-not[of S T]
        local.conflict'-without-decide unfolding full1-def rtranclp-unfold
        by (metis tranclp-unfold-begin)
      moreover
        then obtain T' where full cdcl_W-bj T T'
          using cdcl_W-bj-exists-normal-form inv-T by blast
      ultimately show False using cdcl_W-merge-cp.conflict' n-s by meson
    next
      case (bj'-without-decide S')
      then show ?thesis
        using confl unfolding full1-def by (fastforce simp: cdcl_W-bj.simps dest: tranclpD
          elim: rulesE)
    qed
qed

lemma conflicting-true-no-step-s'-without-decide-no-step-cdcl_W-merge-cp:
  assumes
    inv: cdcl_W-all-struct-inv S and
    n-s: no-step cdcl_W-s'-without-decide S
  shows no-step cdcl_W-merge-cp S
proof (rule ccontr)
  assume ¬ ?thesis
  then obtain T where cdcl_W-merge-cp S T
    by auto
  then show False
    proof cases
      case (conflict' S')
      then show False using n-s conflict'-without-decide conflict-is-full1-cdcl_W-cp by blast
    next
      case propagate'
      moreover
        have cdcl_W-all-struct-inv T
          using inv by (meson local.propagate' rtranclp-cdcl_W-all-struct-inv-inv
            rtranclp-propagate-is-rtranclp-cdcl_W tranclp-into-rtranclp)
        then obtain U where full cdcl_W-cp T U
          using cdcl_W-cp-normalized-element-all-inv by auto
      ultimately have full1 cdcl_W-cp S U
        using tranclp-full-full1I[of cdcl_W-cp S T U] cdcl_W-cp.propagate'
        tranclp-mono[of propagate cdcl_W-cp] by blast
      then show False using conflict'-without-decide n-s by blast
    qed
qed

lemma no-step-cdcl_W-merge-cp-no-step-cdcl_W-cp:
  no-step cdcl_W-merge-cp S ⟹ cdcl_W-M-level-inv S ⟹ no-step cdcl_W-cp S
  using cdcl_W-bj-exists-normal-form cdcl_W-consistent-inv[OF cdcl_W.conflict, of S]
  by (metis cdcl_W-cp.cases cdcl_W-merge-cp.simps tranclp.intros(1))

lemma conflicting-not-true-rtranclp-cdcl_W-merge-cp-no-step-cdcl_W-bj:
  assumes
    conflicting S = None and
    cdcl_W-merge-cp^** S T
  shows no-step cdcl_W-bj T
  using assms(2,1) by (induction)
  (fastforce simp: cdcl_W-merge-cp.simps full-def tranclp-unfold-end cdcl_W-bj.simps
    elim: rulesE)+

lemma conflicting-true-full-cdcl_W-merge-cp-iff-full-cdcl_W-s'-without-decode:
  assumes
    confl: conflicting S = None and
    inv: cdcl_W-all-struct-inv S
  shows
    full cdcl_W-merge-cp S V ⟷ full cdcl_W-s'-without-decide S V (is ?fw ⟷ ?s')
proof
  assume ?fw
  then have st: cdcl_W-merge-cp^** S V and n-s: no-step cdcl_W-merge-cp V
    unfolding full-def by blast+
  have inv-V: cdcl_W-all-struct-inv V
    using rtranclp-cdcl_W-merge-cp-rtranclp-cdcl_W[of S V] ‹?fw› unfolding full-def
    by (simp add: inv rtranclp-cdcl_W-all-struct-inv-inv)
  consider
      (s') cdcl_W-s'-without-decide^** S V
    | (propa) T where cdcl_W-s'-without-decide^** S T and propagate^++ T V
    | (bj) T U where cdcl_W-s'-without-decide^** S T and full1 cdcl_W-bj T U and propagate^** U V
    using rtranclp-cdcl_W-merge-cp-is-rtranclp-cdcl_W-s'-without-decide confl st n-s by metis
  then have cdcl_W-s'-without-decide^** S V
    proof cases
      case s'
      then show ?thesis .
    next
      case propa note s' = this(1) and propa = this(2)
      have no-step cdcl_W-cp V
        using no-step-cdcl_W-merge-cp-no-step-cdcl_W-cp n-s inv-V
        unfolding cdcl_W-all-struct-inv-def by blast
      then have full1 cdcl_W-cp T V
        using propa tranclp-mono[of propagate cdcl_W-cp] cdcl_W-cp.propagate' unfolding full1-def
        by blast
      then have cdcl_W-s'-without-decide T V
        using conflict'-without-decide by blast
      then show ?thesis using s' by auto
    next
      case bj note s' = this(1) and bj = this(2) and propa = this(3)
      have no-step cdcl_W-cp V
        using no-step-cdcl_W-merge-cp-no-step-cdcl_W-cp n-s inv-V
        unfolding cdcl_W-all-struct-inv-def by blast
      then have full cdcl_W-cp U V
        using propa rtranclp-mono[of propagate cdcl_W-cp] cdcl_W-cp.propagate' unfolding full-def
        by blast
      moreover have no-step cdcl_W-cp T
        using bj unfolding full1-def by (fastforce dest!: tranclpD simp:cdcl_W-bj.simps elim: rulesE)
      ultimately have cdcl_W-s'-without-decide T V
```

```
        using bj'-without-decide[of T U V] bj by blast
      then show ?thesis using s' by auto
    qed
  moreover have no-step cdcl_W-s'-without-decide V
    proof (cases conflicting V = None)
      case False
      { fix ss :: 'st
        have ff1: ∀s sa. ¬ cdcl_W-s' s sa ∨ full1 cdcl_W-cp s sa
          ∨ (∃sb. decide s sb ∧ no-step cdcl_W-cp s ∧ full cdcl_W-cp sb sa)
          ∨ (∃sb. full1 cdcl_W-bj s sb ∧ no-step cdcl_W-cp s ∧ full cdcl_W-cp sb sa)
          by (metis cdcl_W-s'.cases)
        have ff2: (∀p s sa. ¬ full1 p (s::'st) sa ∨ p^++ s sa ∧ no-step p sa)
          ∧ (∀p s sa. (¬ p^++ (s::'st) sa ∨ (∃s. p sa s)) ∨ full1 p s sa)
          by (meson full1-def)
        obtain ssa :: ('st ⇒ 'st ⇒ bool) ⇒ 'st ⇒ 'st ⇒ 'st where
          ff3: ∀p s sa. ¬ p^++ s sa ∨ p s (ssa p s sa) ∧ p^** (ssa p s sa) sa
          by (metis (no-types) tranclpD)
        then have a3: ¬ cdcl_W-cp^++ V ss
          using False by (metis option-full-cdcl_W-cp full-def)
        have ⋀s. ¬ cdcl_W-bj^++ V s
          using ff3 False by (metis confl st
            conflicting-not-true-rtranclp-cdcl_W-merge-cp-no-step-cdcl_W-bj)
        then have ¬ cdcl_W-s'-without-decide V ss
          using ff1 a3 ff2 by (metis cdcl_W-s'-without-decide.cases)
      }
      then show ?thesis
        by fastforce
      next
        case True
        then show ?thesis
          using conflicting-true-no-step-cdcl_W-merge-cp-no-step-s'-without-decide n-s inv-V
          unfolding cdcl_W-all-struct-inv-def by simp
    qed
  ultimately show ?s' unfolding full-def by blast
next
  assume s': ?s'
  then have st: cdcl_W-s'-without-decide^** S V and n-s: no-step cdcl_W-s'-without-decide V
    unfolding full-def by auto
  then have cdcl_W^** S V
    using rtranclp-cdcl_W-s'-without-decide-rtranclp-cdcl_W st by blast
  then have inv-V: cdcl_W-all-struct-inv V using inv rtranclp-cdcl_W-all-struct-inv-inv by blast
  then have n-s-cp-V: no-step cdcl_W-cp V
    using cdcl_W-cp-normalized-element-all-inv[of V] full-fullI[of cdcl_W-cp V] n-s
    conflict'-without-decide conflicting-true-no-step-s'-without-decide-no-step-cdcl_W-merge-cp
    no-step-cdcl_W-merge-cp-no-step-cdcl_W-cp
    unfolding cdcl_W-all-struct-inv-def by presburger
  have n-s-bj: no-step cdcl_W-bj V
    proof (rule ccontr)
      assume ¬ ?thesis
      then obtain W where W: cdcl_W-bj V W by blast
      have cdcl_W-all-struct-inv W
        using W cdcl_W.simps cdcl_W-all-struct-inv-inv inv-V by blast
      then obtain W' where full1 cdcl_W-bj V W'
        using cdcl_W-bj-exists-normal-form[of W] full-fullI[of cdcl_W-bj V W]  W
        unfolding cdcl_W-all-struct-inv-def
        by blast
      moreover
        then have cdcl_W^++ V W'
          using tranclp-mono[of cdcl_W-bj cdcl_W] cdcl_W.other cdcl_W-o.bj unfolding full1-def by blast
        then have cdcl_W-all-struct-inv W'
          by (meson inv-V rtranclp-cdcl_W-all-struct-inv-inv tranclp-into-rtranclp)
        then obtain X where full cdcl_W-cp W' X
          using cdcl_W-cp-normalized-element-all-inv by blast
      ultimately show False
        using bj'-without-decide n-s-cp-V n-s by blast
    qed
from s' consider
      (cp-true) cdcl_W-merge-cp** S V and conflicting V = None
    | (cp-false) cdcl_W-merge-cp** S V and conflicting V ≠ None and no-step cdcl_W-cp V and
        no-step cdcl_W-bj V
    | (cp-confl) T where cdcl_W-merge-cp** S T conflict T V
    using rtranclp-cdcl_W-s'-without-decide-is-rtranclp-cdcl_W-merge-cp[of S V] confl
    unfolding full-def by meson
  then have cdcl_W-merge-cp** S V
    proof cases
      case cp-confl note S-T = this(1) and conf-V = this(2)
      have full cdcl_W-bj V V
        using conf-V n-s-bj unfolding full-def by fast
      then have cdcl_W-merge-cp T V
        using cdcl_W-merge-cp.conflict' conf-V by auto
      then show ?thesis using S-T by auto
    qed fast+
  moreover
    then have cdcl_W** S V using rtranclp-cdcl_W-merge-cp-rtranclp-cdcl_W by blast
    then have cdcl_W-all-struct-inv V
      using inv rtranclp-cdcl_W-all-struct-inv-inv by blast
    then have no-step cdcl_W-merge-cp V
      using conflicting-true-no-step-s'-without-decide-no-step-cdcl_W-merge-cp s'
      unfolding full-def by blast
  ultimately show ?fw unfolding full-def by auto
qed

lemma conflicting-true-full1-cdcl_W-merge-cp-iff-full1-cdcl_W-s'-without-decode:
  assumes
    confl: conflicting S = None and
    inv: cdcl_W-all-struct-inv S
  shows
    full1 cdcl_W-merge-cp S V ⟷ full1 cdcl_W-s'-without-decide S V
proof -
  have full cdcl_W-merge-cp S V = full cdcl_W-s'-without-decide S V
    using confl conflicting-true-full-cdcl_W-merge-cp-iff-full-cdcl_W-s'-without-decode inv
    by simp
  then show ?thesis unfolding full-unfold full1-def tranclp-unfold-begin by blast
qed

lemma conflicting-true-full1-cdcl_W-merge-cp-imp-full1-cdcl_W-s'-without-decode:
  assumes
    fw: full1 cdcl_W-merge-cp S V and
    inv: cdcl_W-all-struct-inv S
  shows
```

```
    full1 cdcl_W-s'-without-decide S V
proof -
  have conflicting S = None
    using fw unfolding full1-def by (auto dest!: tranclpD simp: cdcl_W-merge-cp.simps elim: rulesE)
  then show ?thesis
    using conflicting-true-full1-cdcl_W-merge-cp-iff-full1-cdcl_W-s'-without-decode fw inv by simp
qed

inductive cdcl_W-merge-stgy where
fw-s-cp[intro]: full1 cdcl_W-merge-cp S T ⟹ cdcl_W-merge-stgy S T |
fw-s-decide[intro]: decide S T ⟹ no-step cdcl_W-merge-cp S ⟹ full cdcl_W-merge-cp T U
  ⟹ cdcl_W-merge-stgy S U

lemma cdcl_W-merge-stgy-tranclp-cdcl_W-merge:
  assumes fw: cdcl_W-merge-stgy S T
  shows cdcl_W-merge^++ S T
proof -
  { fix S T
    assume full1 cdcl_W-merge-cp S T
    then have cdcl_W-merge^++ S T
      using tranclp-mono[of cdcl_W-merge-cp cdcl_W-merge^++] cdcl_W-merge-cp-tranclp-cdcl_W-merge
      unfolding full1-def
      by auto
  } note full1-cdcl_W-merge-cp-cdcl_W-merge = this
  show ?thesis
    using fw
    apply (induction rule: cdcl_W-merge-stgy.induct)
      using full1-cdcl_W-merge-cp-cdcl_W-merge apply simp
    unfolding full-unfold by (auto dest!: full1-cdcl_W-merge-cp-cdcl_W-merge fw-decide)
qed

lemma rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W-merge:
  assumes fw: cdcl_W-merge-stgy^** S T
  shows cdcl_W-merge^** S T
  using fw cdcl_W-merge-stgy-tranclp-cdcl_W-merge rtranclp-mono[of cdcl_W-merge-stgy cdcl_W-merge^++]
  unfolding tranclp-rtranclp-rtranclp by blast

lemma cdcl_W-merge-stgy-rtranclp-cdcl_W:
  cdcl_W-merge-stgy S T ⟹ cdcl_W^** S T
  apply (induction rule: cdcl_W-merge-stgy.induct)
    using rtranclp-cdcl_W-merge-cp-rtranclp-cdcl_W unfolding full1-def
    apply (simp add: tranclp-into-rtranclp)
  using rtranclp-cdcl_W-merge-cp-rtranclp-cdcl_W cdcl_W-o.decide cdcl_W.other unfolding full-def
  by (meson r-into-rtranclp rtranclp-trans)

lemma rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W:
  cdcl_W-merge-stgy^** S T ⟹ cdcl_W^** S T
  using rtranclp-mono[of cdcl_W-merge-stgy cdcl_W^**] cdcl_W-merge-stgy-rtranclp-cdcl_W by auto

lemma cdcl_W-merge-stgy-cases[consumes 1, case-names fw-s-cp fw-s-decide]:
  assumes
    cdcl_W-merge-stgy S U
    full1 cdcl_W-merge-cp S U ⟹ P
    ⋀T. decide S T ⟹ no-step cdcl_W-merge-cp S ⟹ full cdcl_W-merge-cp T U ⟹ P
  shows P
  using assms by (auto simp: cdcl_W-merge-stgy.simps)

inductive cdcl_W-s'-w :: 'st ⇒ 'st ⇒ bool where
conflict': full1 cdcl_W-s'-without-decide S S' ⟹ cdcl_W-s'-w S S' |
decide': decide S S' ⟹ no-step cdcl_W-s'-without-decide S ⟹ full cdcl_W-s'-without-decide S' S''
  ⟹ cdcl_W-s'-w S S''

lemma cdcl_W-s'-w-rtranclp-cdcl_W:
  cdcl_W-s'-w S T ⟹ cdcl_W** S T
  apply (induction rule: cdcl_W-s'-w.induct)
    using rtranclp-cdcl_W-s'-without-decide-rtranclp-cdcl_W unfolding full1-def
    apply (simp add: tranclp-into-rtranclp)
  using rtranclp-cdcl_W-s'-without-decide-rtranclp-cdcl_W unfolding full-def
  by (meson decide other rtranclp-into-tranclp2 tranclp-into-rtranclp)

lemma rtranclp-cdcl_W-s'-w-rtranclp-cdcl_W:
  cdcl_W-s'-w** S T ⟹ cdcl_W** S T
  using rtranclp-mono[of cdcl_W-s'-w cdcl_W**] cdcl_W-s'-w-rtranclp-cdcl_W by auto

lemma no-step-cdcl_W-cp-no-step-cdcl_W-s'-without-decide:
  assumes no-step cdcl_W-cp S and conflicting S = None and inv: cdcl_W-M-level-inv S
  shows no-step cdcl_W-s'-without-decide S
  by (metis assms cdcl_W-cp.conflict' cdcl_W-cp.propagate' cdcl_W-merge-restart-cases tranclpD
    conflicting-true-no-step-cdcl_W-merge-cp-no-step-s'-without-decide)

lemma no-step-cdcl_W-cp-no-step-cdcl_W-merge-restart:
  assumes no-step cdcl_W-cp S and conflicting S = None
  shows no-step cdcl_W-merge-cp S
  by (metis assms(1) cdcl_W-cp.conflict' cdcl_W-cp.propagate' cdcl_W-merge-restart-cases tranclpD)
lemma after-cdcl_W-s'-without-decide-no-step-cdcl_W-cp:
  assumes cdcl_W-s'-without-decide S T
  shows no-step cdcl_W-cp T
  using assms by (induction rule: cdcl_W-s'-without-decide.induct) (auto simp: full1-def full-def)

lemma no-step-cdcl_W-s'-without-decide-no-step-cdcl_W-cp:
  cdcl_W-all-struct-inv S ⟹ no-step cdcl_W-s'-without-decide S ⟹ no-step cdcl_W-cp S
  by (simp add: conflicting-true-no-step-s'-without-decide-no-step-cdcl_W-merge-cp
    no-step-cdcl_W-merge-cp-no-step-cdcl_W-cp cdcl_W-all-struct-inv-def)

lemma after-cdcl_W-s'-w-no-step-cdcl_W-cp:
  assumes cdcl_W-s'-w S T and cdcl_W-all-struct-inv S
  shows no-step cdcl_W-cp T
  using assms
proof (induction rule: cdcl_W-s'-w.induct)
  case conflict'
  then show ?case
    by (auto simp: full1-def tranclp-unfold-end after-cdcl_W-s'-without-decide-no-step-cdcl_W-cp)
next
  case (decide' S T U)
  moreover
    then have cdcl_W** S U
      using rtranclp-cdcl_W-s'-without-decide-rtranclp-cdcl_W[of T U] cdcl_W.other[of S T]
      cdcl_W-o.decide unfolding full-def by auto
    then have cdcl_W-all-struct-inv U
      using decide'.prems rtranclp-cdcl_W-all-struct-inv-inv by blast
  ultimately show ?case
    using no-step-cdcl_W-s'-without-decide-no-step-cdcl_W-cp unfolding full-def by blast
qed

lemma rtranclp-cdcl_W-s'-w-no-step-cdcl_W-cp-or-eq:
  assumes cdcl_W-s'-w** S T and cdcl_W-all-struct-inv S
  shows S = T ∨ no-step cdcl_W-cp T
  using assms
proof (induction rule: rtranclp-induct)
  case base
  then show ?case by simp
next
  case (step T U)
  moreover have cdcl_W-all-struct-inv T
    using rtranclp-cdcl_W-s'-w-rtranclp-cdcl_W[of S U] assms(2) rtranclp-cdcl_W-all-struct-inv-inv
    rtranclp-cdcl_W-s'-w-rtranclp-cdcl_W step.hyps(1) by blast
  ultimately show ?case using after-cdcl_W-s'-w-no-step-cdcl_W-cp by fast
qed

lemma rtranclp-cdcl_W-merge-stgy'-no-step-cdcl_W-cp-or-eq:
  assumes cdcl_W-merge-stgy** S T and inv: cdcl_W-all-struct-inv S
  shows S = T ∨ no-step cdcl_W-cp T
  using assms
proof (induction rule: rtranclp-induct)
  case base
  then show ?case by simp
next
  case (step T U)
  moreover have cdcl_W-all-struct-inv T
    using rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W[of S U] assms(2) rtranclp-cdcl_W-all-struct-inv-inv
    rtranclp-cdcl_W-s'-w-rtranclp-cdcl_W step.hyps(1)
    by (meson rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W)
  ultimately show ?case
    using after-cdcl_W-s'-w-no-step-cdcl_W-cp inv unfolding cdcl_W-all-struct-inv-def
    by (metis cdcl_W-all-struct-inv-def cdcl_W-merge-stgy.simps full1-def full-def
      no-step-cdcl_W-merge-cp-no-step-cdcl_W-cp rtranclp-cdcl_W-all-struct-inv-inv
      rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W tranclp.intros(1) tranclp-into-rtranclp)
qed

lemma no-step-cdcl_W-s'-without-decide-no-step-cdcl_W-bj:
  assumes no-step cdcl_W-s'-without-decide S and inv: cdcl_W-all-struct-inv S
  shows no-step cdcl_W-bj S
proof (rule ccontr)
  assume ¬ ?thesis
  then obtain T where S-T: cdcl_W-bj S T
    by auto
  have cdcl_W-all-struct-inv T
    using S-T cdcl_W-all-struct-inv-inv inv other by blast
  then obtain T' where full1 cdcl_W-bj S T'
    using cdcl_W-bj-exists-normal-form[of T] full-fullI S-T unfolding cdcl_W-all-struct-inv-def
    by metis
  moreover
    then have cdcl_W** S T'
      using rtranclp-mono[of cdcl_W-bj cdcl_W] cdcl_W.other cdcl_W-o.bj tranclp-into-rtranclp[of cdcl_W-bj]
      unfolding full1-def by blast
    then have cdcl_W-all-struct-inv T'
      using inv  rtranclp-cdcl_W-all-struct-inv-inv by blast
    then obtain U where full cdcl_W-cp T' U
      using cdcl_W-cp-normalized-element-all-inv by blast
  moreover have no-step cdcl_W-cp S
    using S-T by (auto simp: cdcl_W-bj.simps elim: rulesE)
  ultimately show False
  using assms cdcl_W-s'-without-decide.intros(2)[of S T' U] by fast
qed

lemma cdcl_W-s'-w-no-step-cdcl_W-bj:
  assumes cdcl_W-s'-w S T and cdcl_W-all-struct-inv S
  shows no-step cdcl_W-bj T
  using assms apply induction
    using rtranclp-cdcl_W-s'-without-decide-rtranclp-cdcl_W rtranclp-cdcl_W-all-struct-inv-inv
    no-step-cdcl_W-s'-without-decide-no-step-cdcl_W-bj unfolding full1-def
    apply (meson tranclp-into-rtranclp)
  using rtranclp-cdcl_W-s'-without-decide-rtranclp-cdcl_W rtranclp-cdcl_W-all-struct-inv-inv
    no-step-cdcl_W-s'-without-decide-no-step-cdcl_W-bj unfolding full-def
  by (meson cdcl_W-merge-restart-cdcl_W fw-r-decide)

lemma rtranclp-cdcl_W-s'-w-no-step-cdcl_W-bj-or-eq:
  assumes cdcl_W-s'-w** S T and cdcl_W-all-struct-inv S
  shows S = T ∨ no-step cdcl_W-bj T
  using assms apply induction
    apply simp
  using rtranclp-cdcl_W-s'-w-rtranclp-cdcl_W rtranclp-cdcl_W-all-struct-inv-inv
    cdcl_W-s'-w-no-step-cdcl_W-bj by meson

lemma rtranclp-cdcl_W-s'-no-step-cdcl_W-s'-without-decide-decomp-into-cdcl_W-merge:
  assumes
    cdcl_W-s'** R V and
    conflicting R = None and
    inv: cdcl_W-all-struct-inv R
  shows (cdcl_W-merge-stgy** R V ∧ conflicting V = None)
  ∨ (cdcl_W-merge-stgy** R V ∧ conflicting V ≠ None ∧ no-step cdcl_W-bj V)
  ∨ (∃S T U. cdcl_W-merge-stgy** R S ∧ no-step cdcl_W-merge-cp S ∧ decide S T
    ∧ cdcl_W-merge-cp** T U ∧ conflict U V)
  ∨ (∃S T. cdcl_W-merge-stgy** R S ∧ no-step cdcl_W-merge-cp S ∧ decide S T
    ∧ cdcl_W-merge-cp** T V
      ∧ conflicting V = None)
  ∨ (cdcl_W-merge-cp** R V ∧ conflicting V = None)
  ∨ (∃U. cdcl_W-merge-cp** R U ∧ conflict U V)
  using assms(1,2)
proof induction
  case base
  then show ?case by simp
next
  case (step V W) note st = this(1) and s' = this(2) and IH = this(3)[OF this(4)] and
    n-s-R = this(4)
  from s'
  show ?case
    proof cases
      case conflict'
      consider
  (s') cdcl_W-merge-stgy** R V
 | (dec-confl) S T U where cdcl_W-merge-stgy** R S and no-step cdcl_W-merge-cp S and
    decide S T and cdcl_W-merge-cp** T U and conflict U V
 | (dec) S T where cdcl_W-merge-stgy** R S and no-step cdcl_W-merge-cp S and decide S T
    and cdcl_W-merge-cp** T V and conflicting V = None
 | (cp) cdcl_W-merge-cp** R V
 | (cp-confl) U where cdcl_W-merge-cp** R U and conflict U V
 using IH by meson
then show ?thesis
 proof cases
 next
  case s'
  then have R = V
   by (metis full1-def inv local.conflict' tranclp-unfold-begin
    rtranclp-cdcl_W-merge-stgy'-no-step-cdcl_W-cp-or-eq)
  consider
     (V-W) V = W
   | (propa) propagate++ V W and conflicting W = None
   | (propa-confl) V' where propagate** V V' and conflict V' W
   using tranclp-cdcl_W-cp-propagate-with-conflict-or-not[of V W] conflict'
   unfolding full-unfold full1-def by meson
  then show ?thesis
   proof cases
    case V-W
    then show ?thesis using ⟨R = V⟩ n-s-R by simp
   next
    case propa
    then show ?thesis using ⟨R = V⟩ by (auto intro: cdcl_W-merge-cp.intros)
   next
    case propa-confl
    moreover
     then have cdcl_W-merge-cp** V V'
      by (metis rtranclp-unfold cdcl_W-merge-cp.propagate' r-into-rtranclp)
    ultimately show ?thesis using s' ⟨R = V⟩ by blast
   qed
 next
  case dec-confl note - = this(5)
  then have False using conflict' unfolding full1-def by (auto dest!: tranclpD elim: rulesE)
  then show ?thesis by fast
 next
  case dec note T-V = this(4)
  consider
     (propa) propagate++ V W and conflicting W = None
   | (propa-confl) V' where propagate** V V' and conflict V' W
   using tranclp-cdcl_W-cp-propagate-with-conflict-or-not[of V W] conflict'
   unfolding full1-def by meson
  then show ?thesis
   proof cases
    case propa
    then show ?thesis
     by (meson T-V cdcl_W-merge-cp.propagate' dec rtranclp.rtrancl-into-rtrancl)
   next
    case propa-confl
    then have cdcl_W-merge-cp** T V'
     using T-V by (metis rtranclp-unfold cdcl_W-merge-cp.propagate' rtranclp.simps)
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          then show ?thesis using dec propa-confl(2) by metis
        qed
    next
      case cp
      consider
          (propa) propagate⁺⁺ V W and conflicting W = None
        | (propa-confl) V' where propagate** V V' and conflict V' W
        using tranclp-cdcl_W-cp-propagate-with-conflict-or-not[of V W] conflict'
        unfolding full1-def by meson
      then show ?thesis
        proof cases
          case propa
          then show ?thesis by (meson cdcl_W-merge-cp.propagate' cp
            rtranclp.rtrancl-into-rtrancl)
        next
          case propa-confl
          then show ?thesis
            using propa-confl(2) cp
            by (metis (full-types) cdcl_W-merge-cp.propagate' rtranclp.rtrancl-into-rtrancl
              rtranclp-unfold)
        qed
    next
      case cp-confl
      then show ?thesis using conflict' unfolding full1-def by (fastforce dest!: tranclpD
        elim!: rulesE)
    qed
next
  case (decide' V')
  then have conf-V: conflicting V = None
    by (auto elim: rulesE)
  consider
     (s') cdcl_W-merge-stgy** R V
    | (dec-confl) S T U where cdcl_W-merge-stgy** R S and no-step cdcl_W-merge-cp S and
        decide S T and cdcl_W-merge-cp** T U and conflict U V
    | (dec) S T where cdcl_W-merge-stgy** R S and no-step cdcl_W-merge-cp S and decide S T
         and cdcl_W-merge-cp** T V and conflicting V = None
    | (cp) cdcl_W-merge-cp** R V
    | (cp-confl) U where cdcl_W-merge-cp** R U and conflict U V
    using IH by meson
  then show ?thesis
    proof cases
      case s'
      have confl-V': conflicting V' = None using decide'(1) by (auto elim: rulesE)
      have full: full1 cdcl_W-cp V' W ∨ (V' = W ∧ no-step cdcl_W-cp W)
        using decide'(3) unfolding full-unfold by blast
      consider
          (V'-W) V' = W
        | (propa) propagate⁺⁺ V' W and conflicting W = None
        | (propa-confl) V'' where propagate** V' V'' and conflict V'' W
        using tranclp-cdcl_W-cp-propagate-with-conflict-or-not[of V W] decide'
        ⟨full1 cdcl_W-cp V' W ∨ V' = W ∧ no-step cdcl_W-cp W⟩ unfolding full1-def
        by (metis tranclp-cdcl_W-cp-propagate-with-conflict-or-not)
      then show ?thesis
        proof cases
          case V'-W
          then show ?thesis
            using confl-V' local.decide'(1,2) s' conf-V
            no-step-cdcl_W-cp-no-step-cdcl_W-merge-restart[of V]
            by auto
        next
          case propa
          then show ?thesis using local.decide'(1,2) s' by (metis cdcl_W-merge-cp.simps conf-V
            no-step-cdcl_W-cp-no-step-cdcl_W-merge-restart r-into-rtranclp)
        next
          case propa-confl
          then have cdcl_W-merge-cp** V' V''
            by (metis rtranclp-unfold cdcl_W-merge-cp.propagate' r-into-rtranclp)
          then show ?thesis
            using local.decide'(1,2) propa-confl(2) s' conf-V
            no-step-cdcl_W-cp-no-step-cdcl_W-merge-restart
            by metis
        qed
    next
      case (dec) note s' = this(1) and dec = this(2) and cp = this(3) and ns-cp-T = this(4)
      have full cdcl_W-merge-cp T V
        unfolding full-def by (simp add: conf-V local.decide'(2)
          no-step-cdcl_W-cp-no-step-cdcl_W-merge-restart ns-cp-T)
      moreover have no-step cdcl_W-merge-cp V
         by (simp add: conf-V local.decide'(2) no-step-cdcl_W-cp-no-step-cdcl_W-merge-restart)
      moreover have no-step cdcl_W-merge-cp S
        by (metis dec)
      ultimately have cdcl_W-merge-stgy S V
        using cp by blast
      then have cdcl_W-merge-stgy** R V using s' by auto
      consider
          (V'-W) V' = W
        | (propa) propagate++ V' W and conflicting W = None
        | (propa-confl) V'' where propagate** V' V'' and conflict V'' W
        using tranclp-cdcl_W-cp-propagate-with-conflict-or-not[of V' W] decide'
        unfolding full-unfold full1-def by meson
      then show ?thesis
        proof cases
          case V'-W
          moreover have conflicting V' = None
            using decide'(1) by (auto elim: rulesE)
          ultimately show ?thesis
            using ‹cdcl_W-merge-stgy** R V› decide' ‹no-step cdcl_W-merge-cp V› by blast
        next
          case propa
          moreover then have cdcl_W-merge-cp V' W by (blast intro: cdcl_W-merge-cp.intros)
          ultimately show ?thesis
            using ‹cdcl_W-merge-stgy** R V› decide' ‹no-step cdcl_W-merge-cp V›
            by (meson r-into-rtranclp)
        next
          case propa-confl
          moreover then have cdcl_W-merge-cp** V' V''
            by (metis cdcl_W-merge-cp.propagate' rtranclp-unfold tranclp-unfold-end)
          ultimately show ?thesis using ‹cdcl_W-merge-stgy** R V› decide'
            ‹no-step cdcl_W-merge-cp V› by (meson r-into-rtranclp)
        qed
    next
      case cp
      have no-step cdcl_W-merge-cp V
        using conf-V local.decide'(2) no-step-cdcl_W-cp-no-step-cdcl_W-merge-restart by auto
      then have full cdcl_W-merge-cp R V
        unfolding full-def using cp by fast
      then have cdcl_W-merge-stgy** R V
        unfolding full-unfold by auto
      have full1 cdcl_W-cp V' W ∨ (V' = W ∧ no-step cdcl_W-cp W)
        using decide'(3) unfolding full-unfold by blast

      consider
          (V'-W) V' = W
        | (propa) propagate++ V' W and conflicting W = None
        | (propa-confl) V'' where propagate** V' V'' and conflict V'' W
        using tranclp-cdcl_W-cp-propagate-with-conflict-or-not[of V' W] decide'
        unfolding full-unfold full1-def by meson
      then show ?thesis

        proof cases
          case V'-W
          moreover have conflicting V' = None
            using decide'(1) by (auto elim: rulesE)
          ultimately show ?thesis
            using ⟨cdcl_W-merge-stgy** R V⟩ decide' ⟨no-step cdcl_W-merge-cp V⟩ by blast
        next
          case propa
          moreover then have cdcl_W-merge-cp V' W
            by (blast intro: cdcl_W-merge-cp.intros)
          ultimately show ?thesis using ⟨cdcl_W-merge-stgy** R V⟩ decide'
            ⟨no-step cdcl_W-merge-cp V⟩ by (meson r-into-rtranclp)
        next
          case propa-confl
          moreover then have cdcl_W-merge-cp** V' V''
            by (metis cdcl_W-merge-cp.propagate' rtranclp-unfold tranclp-unfold-end)
          ultimately show ?thesis using ⟨cdcl_W-merge-stgy** R V⟩ decide'
            ⟨no-step cdcl_W-merge-cp V⟩ by (meson r-into-rtranclp)
        qed
    next
      case (dec-confl)
      show ?thesis using conf-V dec-confl(5) by (auto elim!: rulesE
        simp del: state-simp simp: state-eq-def)
    next
      case cp-confl
      then show ?thesis using decide' apply - by (intro HOL.disjI2) (fastforce elim: rulesE
        simp del: state-simp simp: state-eq-def)
  qed
next
  case (bj' V')
  then have ¬no-step cdcl_W-bj V
    by (auto dest: tranclpD simp: full1-def)
  then consider
      (s') cdcl_W-merge-stgy** R V and conflicting V = None
    | (dec-confl) S T U where cdcl_W-merge-stgy** R S and no-step cdcl_W-merge-cp S and
        decide S T and cdcl_W-merge-cp** T U and conflict U V
    | (dec) S T where cdcl_W-merge-stgy** R S and no-step cdcl_W-merge-cp S and decide S T
      and cdcl_W-merge-cp** T V and conflicting V = None
    | (cp) cdcl_W-merge-cp** R V and conflicting V = None
    | (cp-confl) U where cdcl_W-merge-cp** R U and conflict U V
    using IH by meson
  then show ?thesis
    proof cases
      case s' note - = this(2)
      then have False
        using bj'(1) unfolding full1-def by (force dest!: tranclpD simp: cdcl_W-bj.simps
          elim: rulesE)
      then show ?thesis by fast
    next
      case dec note - = this(5)
      then have False
        using bj'(1) unfolding full1-def by (force dest!: tranclpD simp: cdcl_W-bj.simps
          elim: rulesE)
      then show ?thesis by fast
    next
      case dec-confl
      then have cdcl_W-merge-cp U V'
        using bj' cdcl_W-merge-cp.intros(1)[of U V V'] by (simp add: full-unfold)
      then have cdcl_W-merge-cp** T V'
        using dec-confl(4) by simp
      consider
          (V'-W) V' = W
        | (propa) propagate++ V' W and conflicting W = None
        | (propa-confl) V'' where propagate** V' V'' and conflict V'' W
        using tranclp-cdcl_W-cp-propagate-with-conflict-or-not[of V' W] bj'(3)
        unfolding full-unfold full1-def by meson
      then show ?thesis
        proof cases
          case V'-W
          then have no-step cdcl_W-cp V'
            using bj'(3) unfolding full-def by auto
          then have no-step cdcl_W-merge-cp V'
            by (metis cdcl_W-cp.propagate' cdcl_W-merge-cp.cases tranclpD
              no-step-cdcl_W-cp-no-conflict-no-propagate(1) )
          then have full1 cdcl_W-merge-cp T V'
            unfolding full1-def using ‹cdcl_W-merge-cp U V'› dec-confl(4) by auto
          then have full  cdcl_W-merge-cp T V'
            by (simp add: full-unfold)
          then have cdcl_W-merge-stgy S V'
            using dec-confl(3) cdcl_W-merge-stgy.fw-s-decide ‹no-step cdcl_W-merge-cp S› by blast
          then have cdcl_W-merge-stgy** R V'
            using ‹cdcl_W-merge-stgy** R S› by auto
          show ?thesis
            proof cases
              assume conflicting W = None
              then show ?thesis using ‹cdcl_W-merge-stgy** R V'› ‹V' = W› by auto
            next
              assume conflicting W ≠ None
              then show ?thesis
                using ‹cdcl_W-merge-stgy** R V'› ‹V' = W› by (metis ‹cdcl_W-merge-cp U V'›
                  conflictE conflicting-not-true-rtranclp-cdcl_W-merge-cp-no-step-cdcl_W-bj
          dec-confl(5) map-option-is-None r-into-rtranclp)
      qed
    next
      case propa
      moreover then have cdcl_W-merge-cp V' W by (blast intro: cdcl_W-merge-cp.intros)
     ultimately show ?thesis using decide' by (meson ‹cdcl_W-merge-cp** T V'› dec-confl(1−3)
        rtranclp.rtrancl-into-rtrancl)
    next
      case propa-confl
      moreover then have cdcl_W-merge-cp** V' V''
        by (metis cdcl_W-merge-cp.propagate' rtranclp-unfold tranclp-unfold-end)
     ultimately show ?thesis by (meson ‹cdcl_W-merge-cp** T V'› dec-confl(1−3) rtranclp-trans)
    qed
next
  case cp note - = this(2)
  then show ?thesis using bj'(1) ‹¬ no-step cdcl_W-bj V›
    conflicting-not-true-rtranclp-cdcl_W-merge-cp-no-step-cdcl_W-bj by auto
next
  case cp-confl
  then have cdcl_W-merge-cp U V' by (simp add: cdcl_W-merge-cp.conflict' full-unfold
    local.bj'(1))
  consider
      (V'-W) V' = W
    | (propa) propagate++ V' W and conflicting W = None
    | (propa-confl) V'' where propagate** V' V'' and conflict V'' W
    using tranclp-cdcl_W-cp-propagate-with-conflict-or-not[of V' W] bj'
    unfolding full-unfold full1-def by meson
  then show ?thesis

    proof cases
      case V'-W
      show ?thesis
        proof cases
          assume conflicting V' = None
          then show ?thesis
            using V'-W ‹cdcl_W-merge-cp U V'› cp-confl(1) by force
        next
          assume confl: conflicting V' ≠ None
          then have no-step cdcl_W-merge-stgy V'
            by (fastforce simp: cdcl_W-merge-stgy.simps full1-def full-def
              cdcl_W-merge-cp.simps dest!: tranclpD elim: rulesE)
          have no-step cdcl_W-merge-cp V'
            using confl by (auto simp: full1-def full-def cdcl_W-merge-cp.simps
            dest!: tranclpD elim: rulesE)
          moreover have cdcl_W-merge-cp U W
            using V'-W ‹cdcl_W-merge-cp U V'› by blast
          ultimately have full1 cdcl_W-merge-cp R V'
            using cp-confl(1) V'-W unfolding full1-def by auto
          then have cdcl_W-merge-stgy R V'
            by auto
          moreover have no-step cdcl_W-merge-stgy V'
            using confl ‹no-step cdcl_W-merge-cp V'› by (auto simp: cdcl_W-merge-stgy.simps
              full1-def dest!: tranclpD elim: rulesE)
          ultimately have cdcl_W-merge-stgy** R V' by auto
          { fix ss :: 'st
```

```
                    have cdcl_W-merge-cp U W
                      using V'-W ‹cdcl_W-merge-cp U V'› by blast
                    then have ¬ cdcl_W-bj W ss
                      by (meson conflicting-not-true-rtranclp-cdcl_W-merge-cp-no-step-cdcl_W-bj
                        cp-confl(1) rtranclp.rtrancl-into-rtrancl step.prems)
                    then have cdcl_W-merge-stgy^** R W ∧ conflicting W = None ∨
                      cdcl_W-merge-stgy^** R W ∧ ¬ cdcl_W-bj W ss
                      using V'-W ‹cdcl_W-merge-stgy^** R V'› by presburger }
                  then show ?thesis
                    by presburger
              qed
            next
              case propa
              moreover then have cdcl_W-merge-cp V' W
                by (blast intro: cdcl_W-merge-cp.intros)
              ultimately show ?thesis using ‹cdcl_W-merge-cp U V'› cp-confl(1) by force
            next
              case propa-confl
              moreover then have cdcl_W-merge-cp^** V' V''
                by (metis cdcl_W-merge-cp.propagate' rtranclp-unfold tranclp-unfold-end)
              ultimately show ?thesis
                using ‹cdcl_W-merge-cp U V'› cp-confl(1) by (metis rtranclp.rtrancl-into-rtrancl
                  rtranclp-trans)
            qed
        qed
    qed
qed

lemma decide-rtranclp-cdcl_W-s'-rtranclp-cdcl_W-s':
  assumes
    dec: decide S T and
    cdcl_W-s'^** T U and
    n-s-S: no-step cdcl_W-cp S and
    no-step cdcl_W-cp U
  shows cdcl_W-s'^** S U
  using assms(2,4)
proof induction
  case (step U V) note st = this(1) and s' = this(2) and IH = this(3) and n-s = this(4)
  consider
      (TU) T = U
    | (s'-st) T' where cdcl_W-s' T T' and cdcl_W-s'^** T' U
    using st[unfolded rtranclp-unfold] by (auto dest!: tranclpD)
  then show ?case
    proof cases
      case TU
      then show ?thesis
        proof -
          assume a1: T = U
          then have f2: cdcl_W-s' T V
            using s' by force
          obtain ss :: 'st where
            ss: cdcl_W-s'^** S T ∨ cdcl_W-cp T ss
            using a1 step.IH by blast−
          obtain ssa :: 'st ⇒ 'st where
            f3: ∀s sa sb. (¬ decide s sa ∨ cdcl_W-cp s (ssa s) ∨ ¬ full cdcl_W-cp sa sb)
```

```
              ∨ cdcl_W-s' s sb
            using cdcl_W-s'.decide' by moura
          have ∀ s sa. ¬ cdcl_W-s' s sa ∨ full1 cdcl_W-cp s sa ∨
            (∃ sb. decide s sb ∧ no-step cdcl_W-cp s ∧ full cdcl_W-cp sb sa) ∨
            (∃ sb. full1 cdcl_W-bj s sb ∧ no-step cdcl_W-cp s ∧ full cdcl_W-cp sb sa)
            by (metis cdcl_W-s'E)
          then have ∃ s. cdcl_W-s'** S s ∧ cdcl_W-s' s V
            using f3 ss f2 by (metis dec full1-is-full n-s-S rtranclp-unfold)
          then show ?thesis
            by force
        qed
    next
      case (s'-st T') note s'-T' = this(1) and st = this(2)
      have cdcl_W-s'** S T'
        using s'-T'
        proof cases
          case conflict'
          then have cdcl_W-s' S T'
             using dec cdcl_W-s'.decide' n-s-S by (simp add: full-unfold)
          then show ?thesis
             using st by auto
        next
          case (decide' T'')
          then have cdcl_W-s' S T
             using dec cdcl_W-s'.decide' n-s-S by (simp add: full-unfold)
          then show ?thesis using decide' s'-T' by auto
        next
          case bj'
          then have False
            using dec unfolding full1-def by (fastforce dest!: tranclpD simp: cdcl_W-bj.simps
              elim: rulesE)
          then show ?thesis by fast
        qed
      then show ?thesis using s' st by auto
    qed
next
  case base
  then have full cdcl_W-cp T T
    by (simp add: full-unfold)
  then show ?case
    using cdcl_W-s'.simps dec n-s-S by auto
qed

lemma rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W-s':
  assumes
    cdcl_W-merge-stgy** R V and
    inv: cdcl_W-all-struct-inv R
  shows cdcl_W-s'** R V
  using assms(1)
proof induction
  case base
  then show ?case by simp
next
  case (step S T) note st = this(1) and fw = this(2) and IH = this(3)
  have cdcl_W-all-struct-inv S
```

```
    using inv rtranclp-cdcl_W-all-struct-inv-inv rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W st by blast
  from fw show ?case
    proof (cases rule: cdcl_W-merge-stgy-cases)
      case fw-s-cp
      have ⋀s. ¬ full cdcl_W-merge-cp s S
        using fw-s-cp unfolding full-def full1-def by (metis tranclp-unfold-begin)
      then have S = R
        using fw-s-cp unfolding full1-def by (metis cdcl_W-cp.conflict' cdcl_W-cp.propagate'
          cdcl_W-merge-cp.cases tranclp-unfold-begin inv st
          rtranclp-cdcl_W-merge-stgy'-no-step-cdcl_W-cp-or-eq)
      then have full1 cdcl_W-s'-without-decide R T
        using inv local.fw-s-cp
        by (blast intro: conflicting-true-full1-cdcl_W-merge-cp-imp-full1-cdcl_W-s'-without-decode)
      then show ?thesis unfolding full1-def
        by (metis (no-types) rtranclp-cdcl_W-s'-without-decide-rtranclp-cdcl_W-s' rtranclp-unfold)
    next
      case (fw-s-decide S') note dec = this(1) and n-S = this(2) and full = this(3)
      moreover then have conflicting S' = None
        by (auto elim: rulesE)
      ultimately have full cdcl_W-s'-without-decide S' T
        by (meson ‹cdcl_W-all-struct-inv S› cdcl_W-merge-restart-cdcl_W fw-r-decide
          rtranclp-cdcl_W-all-struct-inv-inv
          conflicting-true-full-cdcl_W-merge-cp-iff-full-cdcl_W-s'-without-decode)
      then have a1: cdcl_W-s'** S' T
        unfolding full-def by (metis (full-types) rtranclp-cdcl_W-s'-without-decide-rtranclp-cdcl_W-s')
      have cdcl_W-merge-stgy** S T
        using fw by blast
      then have cdcl_W-s'** S T
        using decide-rtranclp-cdcl_W-s'-rtranclp-cdcl_W-s' a1 by (metis ‹cdcl_W-all-struct-inv S› dec
          n-S no-step-cdcl_W-merge-cp-no-step-cdcl_W-cp cdcl_W-all-struct-inv-def
          rtranclp-cdcl_W-merge-stgy'-no-step-cdcl_W-cp-or-eq)
      then show ?thesis using IH by auto
    qed
qed

lemma rtranclp-cdcl_W-merge-stgy-distinct-mset-clauses:
  assumes invR: cdcl_W-all-struct-inv R and
  st: cdcl_W-merge-stgy** R S and
  dist: distinct-mset (clauses R) and
  R: trail R = []
  shows distinct-mset (clauses S)
  using rtranclp-cdcl_W-stgy-distinct-mset-clauses[OF invR - dist R]
  invR st rtranclp-mono[of cdcl_W-s' cdcl_W-stgy**] cdcl_W-s'-is-rtranclp-cdcl_W-stgy
  by (auto dest!: cdcl_W-s'-is-rtranclp-cdcl_W-stgy rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W-s')

lemma no-step-cdcl_W-s'-no-step-cdcl_W-merge-stgy:
  assumes
    inv: cdcl_W-all-struct-inv R and s': no-step cdcl_W-s' R
  shows no-step cdcl_W-merge-stgy R
proof –
  { fix ss :: 'st
    obtain ssa :: 'st ⇒ 'st ⇒ 'st where
      ff1: ⋀s sa. ¬ cdcl_W-merge-stgy s sa ∨ full1 cdcl_W-merge-cp s sa ∨ decide s (ssa s sa)
      using cdcl_W-merge-stgy.cases by moura
    obtain ssb :: ('st ⇒ 'st ⇒ bool) ⇒ 'st ⇒ 'st ⇒ 'st where
```

```
      ff2: ⋀p s sa. ¬ p^++ s sa ∨ p s (ssb p s sa)
      by (meson tranclp-unfold-begin)
    obtain ssc :: 'st ⇒ 'st where
      ff3: ⋀s sa sb. (¬ cdcl_W-all-struct-inv s ∨ ¬ cdcl_W-cp s sa ∨ cdcl_W-s' s (ssc s))
        ∧ (¬ cdcl_W-all-struct-inv s ∨ ¬ cdcl_W-o s sb ∨ cdcl_W-s' s (ssc s))
      using n-step-cdcl_W-stgy-iff-no-step-cdcl_W-cl-cdcl_W-o by moura
    then have ff4: ⋀s. ¬ cdcl_W-o R s
      using s' inv by blast
    have ff5: ⋀s. ¬ cdcl_W-cp^++ R s
      using ff3 ff2 s' by (metis inv)
    have ⋀s. ¬ cdcl_W-bj^++ R s
      using ff4 ff2 by (metis bj)
    then have ⋀s. ¬ cdcl_W-s'-without-decide R s
      using ff5 by (simp add: cdcl_W-s'-without-decide.simps full1-def)
    then have ¬ cdcl_W-s'-without-decide^++ R ss
      using ff2 by blast
    then have ¬ full1 cdcl_W-s'-without-decide R ss
      by (simp add: full1-def)
    then have ¬ cdcl_W-merge-stgy R ss
      using ff4 ff1 conflicting-true-full1-cdcl_W-merge-cp-imp-full1-cdcl_W-s'-without-decode inv
      by blast }
  then show ?thesis
    by fastforce
qed
end
```

We will discharge the assumption later.

```
locale conflict-driven-clause-learning_W-termination =
  conflict-driven-clause-learning_W +
  assumes wf-cdcl_W-merge-inv: wf {(T, S). cdcl_W-all-struct-inv S ∧ cdcl_W-merge S T}
begin

lemma wf-tranclp-cdcl_W-merge: wf {(T, S). cdcl_W-all-struct-inv S ∧ cdcl_W-merge^++ S T}
  using wf-trancl[OF wf-cdcl_W-merge-inv]
  apply (rule wf-subset)
  by (auto simp: trancl-set-tranclp
    cdcl_W-all-struct-inv-tranclp-cdcl_W-merge-tranclp-cdcl_W-merge-cdcl_W-all-struct-inv)

lemma wf-cdcl_W-merge-cp:
  wf{(T, S). cdcl_W-all-struct-inv S ∧ cdcl_W-merge-cp S T}
  using wf-tranclp-cdcl_W-merge by (rule wf-subset) (auto simp: cdcl_W-merge-cp-tranclp-cdcl_W-merge)

lemma wf-cdcl_W-merge-stgy:
  wf{(T, S). cdcl_W-all-struct-inv S ∧ cdcl_W-merge-stgy S T}
  using wf-tranclp-cdcl_W-merge by (rule wf-subset)
  (auto simp add: cdcl_W-merge-stgy-tranclp-cdcl_W-merge)

lemma cdcl_W-merge-cp-obtain-normal-form:
  assumes inv: cdcl_W-all-struct-inv R
  obtains S where full cdcl_W-merge-cp R S
proof –
  obtain S where full (λS T. cdcl_W-all-struct-inv S ∧ cdcl_W-merge-cp S T) R S
    using wf-exists-normal-form-full[OF wf-cdcl_W-merge-cp] by blast
  then have
    st: (λS T. cdcl_W-all-struct-inv S ∧ cdcl_W-merge-cp S T)^** R S and
    n-s: no-step (λS T. cdcl_W-all-struct-inv S ∧ cdcl_W-merge-cp S T) S
    unfolding full-def by blast+
  have cdcl_W-merge-cp** R S
    using st by induction auto
  moreover
    have cdcl_W-all-struct-inv S
      using st inv
      apply (induction rule: rtranclp-induct)
        apply simp
      by (meson r-into-rtranclp rtranclp-cdcl_W-all-struct-inv-inv
        rtranclp-cdcl_W-merge-cp-rtranclp-cdcl_W)
    then have no-step cdcl_W-merge-cp S
      using n-s by auto
  ultimately show ?thesis
    using that unfolding full-def by blast
qed

lemma no-step-cdcl_W-merge-stgy-no-step-cdcl_W-s':
  assumes
    inv: cdcl_W-all-struct-inv R and
    confl: conflicting R = None and
    n-s: no-step cdcl_W-merge-stgy R
  shows no-step cdcl_W-s' R
proof (rule ccontr)
  assume ¬ ?thesis
  then obtain S where cdcl_W-s' R S by auto
  then show False
    proof cases
      case conflict'
      then obtain S' where full1 cdcl_W-merge-cp R S'
        proof –
          obtain R' :: 'e where
            cdcl_W-merge-cp R R'
            using inv unfolding cdcl_W-all-struct-inv-def by (meson confl
              cdcl_W-s'-without-decide.simps conflict'
              conflicting-true-no-step-cdcl_W-merge-cp-no-step-s'-without-decide)
          then show ?thesis
            using that by (metis cdcl_W-merge-cp-obtain-normal-form full-unfold inv)
        qed
      then show False using n-s by blast
    next
      case (decide' R')
      then have cdcl_W-all-struct-inv R'
        using inv cdcl_W-all-struct-inv-inv cdcl_W.other cdcl_W-o.decide by meson
      then obtain R'' where full cdcl_W-merge-cp R' R''
        using cdcl_W-merge-cp-obtain-normal-form by blast
      moreover have no-step cdcl_W-merge-cp R
        by (simp add: confl local.decide'(2) no-step-cdcl_W-cp-no-step-cdcl_W-merge-restart)
      ultimately show False using n-s cdcl_W-merge-stgy.intros local.decide'(1) by blast
    next
      case (bj' R')
      then show False
        using confl no-step-cdcl_W-cp-no-step-cdcl_W-s'-without-decide inv
        unfolding cdcl_W-all-struct-inv-def by auto
    qed
```

```
qed

lemma rtranclp-cdcl_W-merge-cp-no-step-cdcl_W-bj:
  assumes conflicting R = None and cdcl_W-merge-cp** R S
  shows no-step cdcl_W-bj S
  using assms conflicting-not-true-rtranclp-cdcl_W-merge-cp-no-step-cdcl_W-bj by auto

lemma rtranclp-cdcl_W-merge-stgy-no-step-cdcl_W-bj:
  assumes confl: conflicting R = None and cdcl_W-merge-stgy** R S
  shows no-step cdcl_W-bj S
  using assms(2)
proof induction
  case base
  then show ?case
    using confl by (auto simp: cdcl_W-bj.simps elim: rulesE)
next
  case (step S T) note st = this(1) and fw = this(2) and IH = this(3)
  have confl-S: conflicting S = None
    using fw apply cases
    by (auto simp: full1-def cdcl_W-merge-cp.simps dest!: tranclpD elim: rulesE)
  from fw show ?case
    proof cases
      case fw-s-cp
      then show ?thesis
        using rtranclp-cdcl_W-merge-cp-no-step-cdcl_W-bj confl-S
        by (simp add: full1-def tranclp-into-rtranclp)
    next
      case (fw-s-decide S')
      moreover then have conflicting S' = None by (auto elim: rulesE)
      ultimately show ?thesis
        using conflicting-not-true-rtranclp-cdcl_W-merge-cp-no-step-cdcl_W-bj
        unfolding full-def by meson
    qed
qed

end

end
theory CDCL-W-Restart
imports CDCL-W-Merge
begin
```

21.5 Adding Restarts

```
locale cdcl_W-restart =
  conflict-driven-clause-learning_W
    — functions for clauses:
    mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss

    — functions for the conflicting clause:
    mset-ccls union-ccls insert-ccls remove-clit

    — conversion
    ccls-of-cls cls-of-ccls
```

— functions for the state:
— access functions:
trail hd-raw-trail raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting
— changing state:
cons-trail tl-trail add-init-cls add-learned-cls remove-cls update-backtrack-lvl update-conflicting

— get state:
init-state
restart-state
for
mset-cls:: *'cls* ⇒ *'v clause* **and**
insert-cls :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**
remove-lit :: *'v literal* ⇒ *'cls* ⇒ *'cls* **and**

mset-clss:: *'clss* ⇒ *'v clauses* **and**
union-clss :: *'clss* ⇒ *'clss* ⇒ *'clss* **and**
in-clss :: *'cls* ⇒ *'clss* ⇒ *bool* **and**
insert-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**
remove-from-clss :: *'cls* ⇒ *'clss* ⇒ *'clss* **and**

mset-ccls:: *'ccls* ⇒ *'v clause* **and**
union-ccls :: *'ccls* ⇒ *'ccls* ⇒ *'ccls* **and**
insert-ccls :: *'v literal* ⇒ *'ccls* ⇒ *'ccls* **and**
remove-clit :: *'v literal* ⇒ *'ccls* ⇒ *'ccls* **and**

ccls-of-cls :: *'cls* ⇒ *'ccls* **and**
cls-of-ccls :: *'ccls* ⇒ *'cls* **and**

trail :: *'st* ⇒ (*'v*, *nat*, *'v clause*) *marked-lits* **and**
hd-raw-trail :: *'st* ⇒ (*'v*, *nat*, *'cls*) *marked-lit* **and**
raw-init-clss :: *'st* ⇒ *'clss* **and**
raw-learned-clss :: *'st* ⇒ *'clss* **and**
backtrack-lvl :: *'st* ⇒ *nat* **and**
raw-conflicting :: *'st* ⇒ *'ccls option* **and**

cons-trail :: (*'v*, *nat*, *'cls*) *marked-lit* ⇒ *'st* ⇒ *'st* **and**
tl-trail :: *'st* ⇒ *'st* **and**
add-init-cls :: *'cls* ⇒ *'st* ⇒ *'st* **and**
add-learned-cls :: *'cls* ⇒ *'st* ⇒ *'st* **and**
remove-cls :: *'cls* ⇒ *'st* ⇒ *'st* **and**
update-backtrack-lvl :: *nat* ⇒ *'st* ⇒ *'st* **and**
update-conflicting :: *'ccls option* ⇒ *'st* ⇒ *'st* **and**

init-state :: *'clss* ⇒ *'st* **and**
restart-state :: *'st* ⇒ *'st* +
fixes *f* :: *nat* ⇒ *nat*
assumes *f*: *unbounded f*
begin

The condition of the differences of cardinality has to be strict. Otherwise, you could be in a strange state, where nothing remains to do, but a restart is done. See the proof of well-foundedness.

inductive *$cdcl_W$-merge-with-restart* **where**
restart-step:

```
  (cdcl_W-merge-stgy^^(card (set-mset (learned-clss T)) − card (set-mset (learned-clss S)))) S T
  ⟹ card (set-mset (learned-clss T)) − card (set-mset (learned-clss S)) > f n
  ⟹ restart T U ⟹ cdcl_W-merge-with-restart (S, n) (U, Suc n) |
restart-full: full1 cdcl_W-merge-stgy S T ⟹ cdcl_W-merge-with-restart (S, n) (T, Suc n)

lemma cdcl_W-merge-with-restart S T ⟹ cdcl_W-merge-restart** (fst S) (fst T)
  by (induction rule: cdcl_W-merge-with-restart.induct)
  (auto dest!: relpowp-imp-rtranclp cdcl_W-merge-stgy-tranclp-cdcl_W-merge tranclp-into-rtranclp
     rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W-merge rtranclp-cdcl_W-merge-tranclp-cdcl_W-merge-restart
     fw-r-rf cdcl_W-rf.restart
    simp: full1-def)

lemma cdcl_W-merge-with-restart-rtranclp-cdcl_W:
  cdcl_W-merge-with-restart S T ⟹ cdcl_W** (fst S) (fst T)
  by (induction rule: cdcl_W-merge-with-restart.induct)
  (auto dest!: relpowp-imp-rtranclp rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W cdcl_W.rf
    cdcl_W-rf.restart tranclp-into-rtranclp simp: full1-def)

lemma cdcl_W-merge-with-restart-increasing-number:
  cdcl_W-merge-with-restart S T ⟹ snd T = 1 + snd S
  by (induction rule: cdcl_W-merge-with-restart.induct) auto

lemma full1 cdcl_W-merge-stgy S T ⟹ cdcl_W-merge-with-restart (S, n) (T, Suc n)
  using restart-full by blast

lemma cdcl_W-all-struct-inv-learned-clss-bound:
  assumes inv: cdcl_W-all-struct-inv S
  shows set-mset (learned-clss S) ⊆ simple-clss (atms-of-mm (init-clss S))
proof
  fix C
  assume C: C ∈ set-mset (learned-clss S)
  have distinct-mset C
    using C inv unfolding cdcl_W-all-struct-inv-def distinct-cdcl_W-state-def distinct-mset-set-def
    by auto
  moreover have ¬tautology C
    using C inv unfolding cdcl_W-all-struct-inv-def cdcl_W-learned-clause-def by auto
  moreover
    have atms-of C ⊆ atms-of-mm (learned-clss S)
      using C by auto
    then have atms-of C ⊆ atms-of-mm (init-clss S)
    using inv  unfolding cdcl_W-all-struct-inv-def no-strange-atm-def by force
  moreover have finite (atms-of-mm (init-clss S))
    using inv unfolding cdcl_W-all-struct-inv-def by auto
  ultimately show C ∈ simple-clss (atms-of-mm (init-clss S))
    using distinct-mset-not-tautology-implies-in-simple-clss simple-clss-mono
    by blast
qed

lemma cdcl_W-merge-with-restart-init-clss:
  cdcl_W-merge-with-restart S T ⟹ cdcl_W-M-level-inv (fst S) ⟹
  init-clss (fst S) = init-clss (fst T)
  using cdcl_W-merge-with-restart-rtranclp-cdcl_W rtranclp-cdcl_W-init-clss by blast

lemma
  wf {(T, S). cdcl_W-all-struct-inv (fst S) ∧ cdcl_W-merge-with-restart S T}
```

```
proof (rule ccontr)
  assume ¬ ?thesis
    then obtain g where
    g: ⋀i. cdcl_W-merge-with-restart (g i) (g (Suc i)) and
    inv: ⋀i. cdcl_W-all-struct-inv (fst (g i))
    unfolding wf-iff-no-infinite-down-chain by fast
  { fix i
    have init-clss (fst (g i)) = init-clss (fst (g 0))
      apply (induction i)
        apply simp
      using g inv unfolding cdcl_W-all-struct-inv-def by (metis cdcl_W-merge-with-restart-init-clss)
    } note init-g = this
  let ?S = g 0
  have finite (atms-of-mm (init-clss (fst ?S)))
    using inv unfolding cdcl_W-all-struct-inv-def by auto
  have snd-g: ⋀i. snd (g i) = i + snd (g 0)
    apply (induct-tac i)
      apply simp
    by (metis Suc-eq-plus1-left add-Suc cdcl_W-merge-with-restart-increasing-number g)
  then have snd-g-0: ⋀i. i > 0 ⟹ snd (g i) = i + snd (g 0)
    by blast
  have unbounded-f-g: unbounded (λi. f (snd (g i)))
    using f unfolding bounded-def by (metis add.commute f less-or-eq-imp-le snd-g
      not-bounded-nat-exists-larger not-le le-iff-add)

  obtain k where
    f-g-k: f (snd (g k)) > card (simple-clss (atms-of-mm (init-clss (fst ?S)))) and
    k > card (simple-clss (atms-of-mm (init-clss (fst ?S))))
    using not-bounded-nat-exists-larger[OF unbounded-f-g] by blast
```

The following does not hold anymore with the non-strict version of cardinality in the definition.

```
  { fix i
    assume no-step cdcl_W-merge-stgy (fst (g i))
    with g[of i]
    have False
      proof (induction rule: cdcl_W-merge-with-restart.induct)
        case (restart-step T S n) note H = this(1) and c = this(2) and n-s = this(4)
        obtain S' where cdcl_W-merge-stgy S S'
          using H c by (metis gr-implies-not0 relpowp-E2)
        then show False using n-s by auto
      next
        case (restart-full S T)
        then show False unfolding full1-def by (auto dest: tranclpD)
      qed
    } note H = this
  obtain m T where
    m: m = card (set-mset (learned-clss T)) − card (set-mset (learned-clss (fst (g k)))) and
    m > f (snd (g k)) and
    restart T (fst (g (k+1))) and
    cdcl_W-merge-stgy: (cdcl_W-merge-stgy ^^ m) (fst (g k)) T
    using g[of k] H[of Suc k] by (force simp: cdcl_W-merge-with-restart.simps full1-def)
  have cdcl_W-merge-stgy** (fst (g k)) T
    using cdcl_W-merge-stgy relpowp-imp-rtranclp by metis
  then have cdcl_W-all-struct-inv T
    using inv[of k]  rtranclp-cdcl_W-all-struct-inv-inv rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W
```

by *blast*
moreover have *card* (*set-mset* (*learned-clss* *T*)) − *card* (*set-mset* (*learned-clss* (*fst* (*g* *k*))))
> *card* (*simple-clss* (*atms-of-mm* (*init-clss* (*fst* *?S*))))
unfolding *m*[*symmetric*] **using** ⟨*m* > *f* (*snd* (*g* *k*))⟩ *f-g-k* **by** *linarith*
then have *card* (*set-mset* (*learned-clss* *T*))
> *card* (*simple-clss* (*atms-of-mm* (*init-clss* (*fst* *?S*))))
by *linarith*
moreover
have *init-clss* (*fst* (*g* *k*)) = *init-clss* *T*
using ⟨*cdcl*$_W$*-merge-stgy*** (*fst* (*g* *k*)) *T*⟩ *rtranclp-cdcl*$_W$*-merge-stgy-rtranclp-cdcl*$_W$
rtranclp-cdcl$_W$*-init-clss* *inv* **unfolding** *cdcl*$_W$*-all-struct-inv-def* **by** *blast*
then have *init-clss* (*fst* *?S*) = *init-clss* *T*
using *init-g*[*of* *k*] **by** *auto*
ultimately show *False*
using *cdcl*$_W$*-all-struct-inv-learned-clss-bound*
by (*simp add*: ⟨*finite* (*atms-of-mm* (*init-clss* (*fst* (*g* *0*))))⟩ *simple-clss-finite*
card-mono *leD*)
qed

lemma *cdcl*$_W$*-merge-with-restart-distinct-mset-clauses*:
assumes *invR*: *cdcl*$_W$*-all-struct-inv* (*fst* *R*) **and**
st: *cdcl*$_W$*-merge-with-restart* *R* *S* **and**
dist: *distinct-mset* (*clauses* (*fst* *R*)) **and**
R: *trail* (*fst* *R*) = []
shows *distinct-mset* (*clauses* (*fst* *S*))
using *assms*(*2*,*1*,*3*,*4*)
proof (*induction*)
case (*restart-full* *S* *T*)
then show *?case* **using** *rtranclp-cdcl*$_W$*-merge-stgy-distinct-mset-clauses*[*of* *S* *T*] **unfolding** *full1-def*
by (*auto dest*: *tranclp-into-rtranclp*)
next
case (*restart-step* *T* *S* *n* *U*)
then have *distinct-mset* (*clauses* *T*)
using *rtranclp-cdcl*$_W$*-merge-stgy-distinct-mset-clauses*[*of* *S* *T*] **unfolding** *full1-def*
by (*auto dest*: *relpowp-imp-rtranclp*)
then show *?case* **using** ⟨*restart* *T* *U*⟩ **by** (*metis* *clauses-restart* *distinct-mset-union* *fstI*
mset-le-exists-conv *restart.cases* *state-eq-clauses*)
qed

inductive *cdcl*$_W$*-with-restart* **where**
restart-step:
(*cdcl*$_W$*-stgy*$^{\wedge\wedge}$(*card* (*set-mset* (*learned-clss* *T*)) − *card* (*set-mset* (*learned-clss* *S*)))) *S* *T* ⟹
card (*set-mset* (*learned-clss* *T*)) − *card* (*set-mset* (*learned-clss* *S*)) > *f* *n* ⟹
restart *T* *U* ⟹
cdcl$_W$*-with-restart* (*S*, *n*) (*U*, *Suc* *n*) |
restart-full: *full1* *cdcl*$_W$*-stgy* *S* *T* ⟹ *cdcl*$_W$*-with-restart* (*S*, *n*) (*T*, *Suc* *n*)

lemma *cdcl*$_W$*-with-restart-rtranclp-cdcl*$_W$:
cdcl$_W$*-with-restart* *S* *T* ⟹ *cdcl*$_W$** (*fst* *S*) (*fst* *T*)
apply (*induction rule*: *cdcl*$_W$*-with-restart.induct*)
by (*auto dest*!: *relpowp-imp-rtranclp* *tranclp-into-rtranclp* *fw-r-rf*
cdcl$_W$*-rf.restart* *rtranclp-cdcl*$_W$*-stgy-rtranclp-cdcl*$_W$ *cdcl*$_W$*-merge-restart-cdcl*$_W$
simp: *full1-def*)

lemma *cdcl*$_W$*-with-restart-increasing-number*:

```
  cdcl_W-with-restart S T ⟹ snd T = 1 + snd S
  by (induction rule: cdcl_W-with-restart.induct) auto

lemma full1 cdcl_W-stgy S T ⟹ cdcl_W-with-restart (S, n) (T, Suc n)
  using restart-full by blast

lemma cdcl_W-with-restart-init-clss:
  cdcl_W-with-restart S T ⟹  cdcl_W-M-level-inv (fst S) ⟹ init-clss (fst S) = init-clss (fst T)
  using cdcl_W-with-restart-rtranclp-cdcl_W rtranclp-cdcl_W-init-clss by blast

lemma
  wf {(T, S). cdcl_W-all-struct-inv (fst S) ∧ cdcl_W-with-restart S T}
proof (rule ccontr)
  assume ¬ ?thesis
    then obtain g where
    g: ⋀i. cdcl_W-with-restart (g i) (g (Suc i)) and
    inv: ⋀i. cdcl_W-all-struct-inv (fst (g i))
    unfolding wf-iff-no-infinite-down-chain by fast
  { fix i
    have init-clss (fst (g i)) = init-clss (fst (g 0))
      apply (induction i)
        apply simp
      using g inv unfolding cdcl_W-all-struct-inv-def by (metis cdcl_W-with-restart-init-clss)
    } note init-g = this
  let ?S = g 0
  have finite (atms-of-mm (init-clss (fst ?S)))
    using inv unfolding cdcl_W-all-struct-inv-def by auto
  have snd-g: ⋀i. snd (g i) = i + snd (g 0)
    apply (induct-tac i)
      apply simp
    by (metis Suc-eq-plus1-left add-Suc cdcl_W-with-restart-increasing-number g)
  then have snd-g-0: ⋀i. i > 0 ⟹ snd (g i) = i + snd (g 0)
    by blast
  have unbounded-f-g: unbounded (λi. f (snd (g i)))
    using f unfolding bounded-def by (metis add.commute f less-or-eq-imp-le snd-g
      not-bounded-nat-exists-larger not-le le-iff-add)

  obtain k where
    f-g-k: f (snd (g k)) > card (simple-clss (atms-of-mm (init-clss (fst ?S)))) and
    k > card (simple-clss (atms-of-mm (init-clss (fst ?S))))
    using not-bounded-nat-exists-larger[OF unbounded-f-g] by blast
```

The following does not hold anymore with the non-strict version of cardinality in the definition.

```
  { fix i
    assume no-step cdcl_W-stgy (fst (g i))
    with g[of i]
    have False
      proof (induction rule: cdcl_W-with-restart.induct)
        case (restart-step T S n) note H = this(1) and c = this(2) and n-s = this(4)
        obtain S' where cdcl_W-stgy S S'
          using H c by (metis gr-implies-not0 relpowp-E2)
        then show False using n-s by auto
      next
        case (restart-full S T)
        then show False unfolding full1-def by (auto dest: tranclpD)
      qed
    } note H = this
  obtain m T where
    m: m = card (set-mset (learned-clss T)) − card (set-mset (learned-clss (fst (g k)))) and
    m > f (snd (g k)) and
    restart T (fst (g (k+1))) and
    cdcl_W-merge-stgy: (cdcl_W-stgy ^^ m) (fst (g k)) T
    using g[of k] H[of Suc k] by (force simp: cdcl_W-with-restart.simps full1-def)
  have cdcl_W-stgy^** (fst (g k)) T
    using cdcl_W-merge-stgy relpowp-imp-rtranclp by metis
  then have cdcl_W-all-struct-inv T
    using inv[of k]  rtranclp-cdcl_W-all-struct-inv-inv rtranclp-cdcl_W-stgy-rtranclp-cdcl_W by blast
  moreover have card (set-mset (learned-clss T)) − card (set-mset (learned-clss (fst (g k))))
      > card (simple-clss (atms-of-mm (init-clss (fst ?S))))
      unfolding m[symmetric] using ‹m > f (snd (g k))› f-g-k by linarith
    then have card (set-mset (learned-clss T))
      > card (simple-clss (atms-of-mm (init-clss (fst ?S))))
      by linarith
  moreover
    have init-clss (fst (g k)) = init-clss T
      using ‹cdcl_W-stgy^** (fst (g k)) T› rtranclp-cdcl_W-stgy-rtranclp-cdcl_W rtranclp-cdcl_W-init-clss
      inv unfolding cdcl_W-all-struct-inv-def
      by blast
    then have init-clss (fst ?S) = init-clss T
      using init-g[of k] by auto
  ultimately show False
    using cdcl_W-all-struct-inv-learned-clss-bound
    by (simp add: ‹finite (atms-of-mm (init-clss (fst (g 0))))› simple-clss-finite
      card-mono leD)
qed

lemma cdcl_W-with-restart-distinct-mset-clauses:
  assumes invR: cdcl_W-all-struct-inv (fst R) and
  st: cdcl_W-with-restart R S and
  dist: distinct-mset (clauses (fst R)) and
  R: trail (fst R) = []
  shows distinct-mset (clauses (fst S))
  using assms(2,1,3,4)
proof (induction)
  case (restart-full S T)
  then show ?case using rtranclp-cdcl_W-stgy-distinct-mset-clauses[of S T] unfolding full1-def
    by (auto dest: tranclp-into-rtranclp)
next
  case (restart-step T S n U)
  then have distinct-mset (clauses T) using rtranclp-cdcl_W-stgy-distinct-mset-clauses[of S T]
    unfolding full1-def by (auto dest: relpowp-imp-rtranclp)
  then show ?case using ‹restart T U› by (metis clauses-restart distinct-mset-union fstI
    mset-le-exists-conv restart.cases state-eq-clauses)
qed
end

locale luby-sequence =
  fixes ur :: nat
  assumes ur > 0
begin
```

```
lemma exists-luby-decomp:
  fixes i ::nat
  shows ∃k::nat. (2 ^ (k − 1) ≤ i ∧ i < 2 ^ k − 1) ∨ i = 2 ^ k − 1
proof (induction i)
  case 0
  then show ?case
    by (rule exI[of - 0], simp)
next
  case (Suc n)
  then obtain k where 2 ^ (k − 1) ≤ n ∧ n < 2 ^ k − 1 ∨ n = 2 ^ k − 1
    by blast
  then consider
      (st-interv) 2 ^ (k − 1) ≤ n and  n ≤ 2 ^ k − 2
    | (end-interv) 2 ^ (k − 1) ≤ n and n = 2 ^ k − 2
    | (pow2) n = 2 ^ k − 1
    by linarith
  then show ?case
    proof cases
      case st-interv
      then show ?thesis apply − apply (rule exI[of - k])
        by (metis (no-types, lifting) One-nat-def Suc-diff-Suc Suc-lessI
          ‹2 ^ (k − 1) ≤ n ∧ n < 2 ^ k − 1 ∨ n = 2 ^ k − 1› diff-self-eq-0
          dual-order.trans le-SucI le-imp-less-Suc numeral-2-eq-2 one-le-numeral
          one-le-power zero-less-numeral zero-less-power)
    next
      case end-interv
      then show ?thesis apply − apply (rule exI[of - k]) by auto
    next
      case pow2
      then show ?thesis apply − apply (rule exI[of - k+1]) by auto
    qed
qed
```

Luby sequences are defined by:

- $2^k - 1$, if $i = (2::'a)^k - (1::'a)$
- $luby\text{-}sequence\text{-}core\ (i - 2^{k - 1} + 1)$, if $(2::'a)^{k - 1} \leq i$ and $i \leq (2::'a)^k - (1::'a)$

Then the sequence is then scaled by a constant unit run (called *ur* here), strictly positive.

```
function luby-sequence-core :: nat ⇒ nat where
luby-sequence-core i =
  (if ∃k. i = 2^k − 1
  then 2^((SOME k. i = 2^k − 1) − 1)
  else luby-sequence-core (i − 2^((SOME k. 2^(k−1)≤ i ∧ i < 2^k − 1) − 1) + 1))
by auto
termination
proof (relation less-than, goal-cases)
  case 1
  then show ?case by auto
next
  case (2 i)
  let ?k = (SOME k. 2 ^ (k − 1) ≤ i ∧ i < 2 ^ k − 1)
  have 2 ^ (?k − 1) ≤ i ∧ i < 2 ^ ?k − 1
```

```
    apply (rule someI-ex)
    using 2 exists-luby-decomp by blast
  then show ?case

    proof −
      have ∀ n na. ¬ (1::nat) ≤ n ∨ 1 ≤ n ^ na
        by (meson one-le-power)
      then have f1: (1::nat) ≤ 2 ^ (?k − 1)
        using one-le-numeral by blast
      have f2: i − 2 ^ (?k − 1) + 2 ^ (?k − 1) = i
        using ‹2 ^ (?k − 1) ≤ i ∧ i < 2 ^ ?k − 1› le-add-diff-inverse2 by blast
      have f3: 2 ^ ?k − 1 ≠ Suc 0
        using f1 ‹2 ^ (?k − 1) ≤ i ∧ i < 2 ^ ?k − 1› by linarith
      have 2 ^ ?k − (1::nat) ≠ 0
        using ‹2 ^ (?k − 1) ≤ i ∧ i < 2 ^ ?k − 1› gr-implies-not0 by blast
      then have f4: 2 ^ ?k ≠ (1::nat)
        by linarith
      have f5: ∀ n na. if na = 0 then (n::nat) ^ na = 1 else n ^ na = n ∗ n ^ (na − 1)
        by (simp add: power-eq-if)
      then have ?k ≠ 0
        using f4 by meson
      then have 2 ^ (?k − 1) ≠ Suc 0
        using f5 f3 by presburger
      then have Suc 0 < 2 ^ (?k − 1)
        using f1 by linarith
      then show ?thesis
        using f2 less-than-iff by presburger
    qed
qed

function natlog2 :: nat ⇒ nat where
natlog2 n = (if n = 0 then 0 else 1 + natlog2 (n div 2))
  using not0-implies-Suc by auto
termination by (relation measure (λn. n)) auto

declare natlog2.simps[simp del]

declare luby-sequence-core.simps[simp del]

lemma two-pover-n-eq-two-power-n′-eq:
  assumes H: (2::nat) ^ (k::nat) − 1 = 2 ^ k′ − 1
  shows k′ = k
proof −
  have (2::nat) ^ (k::nat) = 2 ^ k′
    using H by (metis One-nat-def Suc-pred zero-less-numeral zero-less-power)
  then show ?thesis by simp
qed

lemma luby-sequence-core-two-power-minus-one:
  luby-sequence-core (2^k − 1) = 2^(k−1) (is ?L = ?K)
proof −
  have decomp: ∃ ka. 2 ^ k − 1 = 2 ^ ka − 1
    by auto
  have ?L = 2^((SOME k′. (2::nat)^k − 1 = 2^k′ − 1) − 1)
    apply (subst luby-sequence-core.simps, subst decomp)
    by simp
  moreover have (SOME k'. (2::nat)^k − 1 = 2^k' − 1) = k
    apply (rule some-equality)
      apply simp
      using two-pover-n-eq-two-power-n'-eq by blast
  ultimately show ?thesis by presburger
qed

lemma different-luby-decomposition-false:
  assumes
    H: 2 ^ (k − Suc 0) ≤ i and
    k': i < 2 ^ k' − Suc 0 and
    k-k': k > k'
  shows False
proof −
  have 2 ^ k' − Suc 0 < 2 ^ (k − Suc 0)
    using k-k' less-eq-Suc-le by auto
  then show ?thesis
    using H k' by linarith
qed

lemma luby-sequence-core-not-two-power-minus-one:
  assumes
    k-i: 2 ^ (k − 1) ≤ i and
    i-k: i < 2^ k − 1
  shows luby-sequence-core i = luby-sequence-core (i − 2 ^ (k − 1) + 1)
proof −
  have H: ¬ (∃ka. i = 2 ^ ka − 1)
    proof (rule ccontr)
      assume ¬ ?thesis
      then obtain k'::nat where k': i = 2 ^ k' − 1 by blast
      have (2::nat) ^ k' − 1 < 2 ^ k − 1
        using i-k unfolding k' .
      then have (2::nat) ^ k' < 2 ^ k
        by linarith
      then have k' < k
        by simp
      have 2 ^ (k − 1) ≤ 2 ^ k' − (1::nat)
        using k-i unfolding k' .
      then have (2::nat) ^ (k−1) < 2 ^ k'
        by (metis Suc-diff-1 not-le not-less-eq zero-less-numeral zero-less-power)
      then have k−1 < k'
        by simp

      show False using ‹k' < k› ‹k−1 < k'› by linarith
    qed
  have ⋀k k'. 2 ^ (k − Suc 0) ≤ i ⟹ i < 2 ^ k − Suc 0 ⟹ 2 ^ (k' − Suc 0) ≤ i ⟹
    i < 2 ^ k' − Suc 0 ⟹ k = k'
    by (meson different-luby-decomposition-false linorder-neqE-nat)
  then have k: (SOME k. 2 ^ (k − Suc 0) ≤ i ∧ i < 2 ^ k − Suc 0) = k
    using k-i i-k by auto
  show ?thesis
    apply (subst luby-sequence-core.simps[of i], subst H)
    by (simp add: k)
qed
```

```
lemma unbounded-luby-sequence-core: unbounded luby-sequence-core
  unfolding bounded-def
proof
  assume ∃ b. ∀ n. luby-sequence-core n ≤ b
  then obtain b where b: ⋀n. luby-sequence-core n ≤ b
    by metis
  have luby-sequence-core (2^(b+1) − 1) = 2^b
    using luby-sequence-core-two-power-minus-one[of b+1] by simp
  moreover have (2::nat)^b > b
    by (induction b) auto
  ultimately show False using b[of 2^(b+1) − 1] by linarith
qed

abbreviation luby-sequence :: nat ⇒ nat where
luby-sequence n ≡  ur ∗ luby-sequence-core n

lemma bounded-luby-sequence: unbounded luby-sequence
  using bounded-const-product[of ur] luby-sequence-axioms
  luby-sequence-def unbounded-luby-sequence-core by blast

lemma luby-sequence-core-0: luby-sequence-core 0 = 1
proof −
  have 0: (0::nat) = 2^0−1
    by auto
  show ?thesis
    by (subst 0, subst luby-sequence-core-two-power-minus-one) simp
qed

lemma luby-sequence-core n ≥ 1
proof (induction n rule: nat-less-induct-case)
  case 0
  then show ?case by (simp add: luby-sequence-core-0)
next
  case (Suc n) note IH = this

  consider
      (interv) k where 2 ^ (k − 1) ≤ Suc n and Suc n < 2 ^ k − 1
    | (pow2)  k where Suc n = 2 ^ k − Suc 0
    using exists-luby-decomp[of Suc n] by auto

  then show ?case
     proof cases
       case pow2
       show ?thesis
         using luby-sequence-core-two-power-minus-one pow2 by auto
     next
       case interv
       have n: Suc n − 2 ^ (k − 1) + 1 < Suc n
         by (metis Suc-1 Suc-eq-plus1 add.commute add-diff-cancel-left' add-less-mono1 gr0I
           interv(1) interv(2) le-add-diff-inverse2 less-Suc-eq not-le power-0 power-one-right
           power-strict-increasing-iff)
       show ?thesis
         apply (subst luby-sequence-core-not-two-power-minus-one[OF interv])
         using IH n by auto
    qed
qed
end

locale luby-sequence-restart =
  luby-sequence ur +
  conflict-driven-clause-learning_W  — functions for clauses:
    mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss

    — functions for the conflicting clause:
    mset-ccls union-ccls insert-ccls remove-clit

    — conversion
    ccls-of-cls cls-of-ccls

    — functions for the state:
      — access functions:
    trail hd-raw-trail raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting
      — changing state:
    cons-trail tl-trail add-init-cls add-learned-cls remove-cls update-backtrack-lvl
    update-conflicting

      — get state:
    init-state
    restart-state
  for
    ur :: nat and
    mset-cls:: 'cls ⇒ 'v clause and
    insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
    remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and

    mset-clss:: 'clss ⇒ 'v clauses and
    union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
    in-clss :: 'cls ⇒ 'clss ⇒ bool and
    insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
    remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss  and

    mset-ccls:: 'ccls ⇒ 'v clause and
    union-ccls :: 'ccls ⇒ 'ccls ⇒ 'ccls and
    insert-ccls :: 'v literal ⇒ 'ccls ⇒ 'ccls and
    remove-clit :: 'v literal ⇒ 'ccls ⇒ 'ccls and

    ccls-of-cls :: 'cls ⇒ 'ccls and
    cls-of-ccls :: 'ccls ⇒ 'cls and

    trail :: 'st ⇒ ('v, nat, 'v clause) marked-lits and
    hd-raw-trail :: 'st ⇒ ('v, nat, 'cls) marked-lit and
    raw-init-clss :: 'st ⇒ 'clss and
    raw-learned-clss :: 'st ⇒ 'clss and
    backtrack-lvl :: 'st ⇒ nat and
    raw-conflicting :: 'st ⇒ 'ccls option and

    cons-trail :: ('v, nat, 'cls) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒ 'st and
    add-init-cls :: 'cls ⇒ 'st ⇒ 'st and
    add-learned-cls :: 'cls ⇒ 'st ⇒ 'st and
    remove-cls :: 'cls ⇒ 'st ⇒ 'st and
    update-backtrack-lvl :: nat ⇒ 'st ⇒ 'st and
    update-conflicting :: 'ccls option ⇒ 'st ⇒ 'st and

    init-state :: 'clss ⇒ 'st and
    restart-state :: 'st ⇒ 'st
begin

sublocale cdcl_W-restart - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - luby-sequence
  apply unfold-locales
  using bounded-luby-sequence by blast

end
end
theory CDCL-WNOT
imports CDCL-NOT CDCL-W-Termination CDCL-W-Merge
begin
```

22 Link between Weidenbach's and NOT's CDCL

22.1 Inclusion of the states

```
declare upt.simps(2)[simp del]

fun convert-marked-lit-from-W where
convert-marked-lit-from-W (Propagated L -) = Propagated L () |
convert-marked-lit-from-W (Marked L -) = Marked L ()

abbreviation convert-trail-from-W ::
  ('v,  'lvl, 'a) marked-lit list
    ⇒ ('v, unit, unit) marked-lit list  where
convert-trail-from-W ≡ map convert-marked-lit-from-W

lemma lits-of-l-convert-trail-from-W[simp]:
  lits-of-l (convert-trail-from-W M) = lits-of-l M
  by (induction rule: marked-lit-list-induct) simp-all

lemma lit-of-convert-trail-from-W[simp]:
  lit-of (convert-marked-lit-from-W L) = lit-of L
  by (cases L) auto

lemma no-dup-convert-from-W[simp]:
  no-dup (convert-trail-from-W M) ⟷ no-dup M
  by (auto simp: comp-def)

lemma convert-trail-from-W-true-annots[simp]:
  convert-trail-from-W M ⊨as C ⟷ M ⊨as C
  by (auto simp: true-annots-true-cls image-image lits-of-def)

lemma defined-lit-convert-trail-from-W[simp]:
  defined-lit (convert-trail-from-W S) L ⟷ defined-lit S L
  by (auto simp: defined-lit-map image-comp)
```

The values *0* and {#} are dummy values.

consts *dummy-cls* :: *'cls*
fun *convert-marked-lit-from-NOT*
  :: (*'a*, *'e*, *'b*) *marked-lit* ⇒ (*'a*, *nat*, *'cls*) *marked-lit* **where**
convert-marked-lit-from-NOT (*Propagated L -*) = *Propagated L dummy-cls* |
convert-marked-lit-from-NOT (*Marked L -*) = *Marked L 0*

abbreviation *convert-trail-from-NOT* **where**
convert-trail-from-NOT ≡ *map convert-marked-lit-from-NOT*

lemma *undefined-lit-convert-trail-from-NOT*[*simp*]:
  undefined-lit (*convert-trail-from-NOT F*) *L* ⟷ *undefined-lit F L*
  by (*induction F rule*: *marked-lit-list-induct*) (*auto simp*: *defined-lit-map*)

lemma *lits-of-l-convert-trail-from-NOT*:
  lits-of-l (*convert-trail-from-NOT F*) = *lits-of-l F*
  by (*induction F rule*: *marked-lit-list-induct*) *auto*

lemma *convert-trail-from-W-from-NOT*[*simp*]:
  convert-trail-from-W (*convert-trail-from-NOT M*) = *M*
  by (*induction rule*: *marked-lit-list-induct*) *auto*

lemma *convert-trail-from-W-convert-lit-from-NOT*[*simp*]:
  convert-marked-lit-from-W (*convert-marked-lit-from-NOT L*) = *L*
  by (*cases L*) *auto*

abbreviation $trail_{NOT}$ **where**
$trail_{NOT}$ *S* ≡ *convert-trail-from-W* (*fst S*)

lemma *undefined-lit-convert-trail-from-W*[*iff*]:
  undefined-lit (*convert-trail-from-W M*) *L* ⟷ *undefined-lit M L*
  by (*auto simp*: *defined-lit-map image-comp*)

lemma *lit-of-convert-marked-lit-from-NOT*[*iff*]:
  lit-of (*convert-marked-lit-from-NOT L*) = *lit-of L*
  by (*cases L*) *auto*

sublocale $state_W$ ⊆ *dpll-state-ops*
   mset-cls insert-cls remove-lit
   mset-clss union-clss in-clss insert-clss remove-from-clss
   λS. convert-trail-from-W (*trail S*)
   raw-clauses
   λL S. cons-trail (*convert-marked-lit-from-NOT L*) *S*
   λS. tl-trail S
   λC S. add-learned-cls C S
   λC S. remove-cls C S
   by *unfold-locales*

context $state_W$
begin
lemma *convert-marked-lit-from-W-convert-marked-lit-from-NOT*[*simp*]:
  convert-marked-lit-from-W (*mmset-of-mlit* (*convert-marked-lit-from-NOT L*)) = *L*
  by (*cases L*) *auto*
end

sublocale $state_W$ ⊆ *dpll-state*

mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
λS. convert-trail-from-W (*trail S*)
raw-clauses
λL S. cons-trail (*convert-marked-lit-from-NOT L*) *S*
λS. tl-trail S
λC S. add-learned-cls C S
λC S. remove-cls C S
by *unfold-locales* (*auto simp*: *map-tl o-def*)

context *state*$_W$
begin
declare *state-simp*$_{NOT}$[*simp del*]

end

sublocale *conflict-driven-clause-learning*$_W$ ⊆ *cdcl*$_{NOT}$*-merge-bj-learn-ops*
mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
λS. convert-trail-from-W (*trail S*)
raw-clauses
λL S. cons-trail (*convert-marked-lit-from-NOT L*) *S*
λS. tl-trail S
λC S. add-learned-cls C S
λC S. remove-cls C S
λ- -. True
λ- S. raw-conflicting S = *None*
λC C′ L′ S T. backjump-l-cond C C′ L′ S T
∧ *distinct-mset* (*C′* + {#*L′*#}) ∧ ¬*tautology* (*C′* + {#*L′*#})
by *unfold-locales*

thm *cdcl*$_{NOT}$*-merge-bj-learn-proxy.axioms*
sublocale *conflict-driven-clause-learning*$_W$ ⊆ *cdcl*$_{NOT}$*-merge-bj-learn-proxy*
mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
λS. convert-trail-from-W (*trail S*)
raw-clauses
λL S. cons-trail (*convert-marked-lit-from-NOT L*) *S*
λS. tl-trail S
λC S. add-learned-cls C S
λC S. remove-cls C S

λ- -. True
λ- S. raw-conflicting S = *None*
backjump-l-cond
inv$_{NOT}$
proof (*unfold-locales*, *goal-cases*)
case *2*
then show *?case* **using** *cdcl*$_{NOT}$*-merged-bj-learn-no-dup-inv* **by** (*auto simp*: *comp-def*)
next
case (*1 C′ S C F′ K F L*)
moreover
let *?C′* = *remdups-mset C′*

```
    have L ∉# C'
      using ‹F ⊨as CNot C'› ‹undefined-lit F L› Marked-Propagated-in-iff-in-lits-of-l
      in-CNot-implies-uminus(2) by fast
    then have distinct-mset (?C' + {#L#})
      by (simp add: distinct-mset-single-add)
  moreover
    have no-dup F
      using ‹inv_NOT S› ‹convert-trail-from-W (trail S) = F' @ Marked K () # F›
      unfolding inv_NOT-def
      by (smt comp-apply distinct.simps(2) distinct-append list.simps(9) map-append
        no-dup-convert-from-W)
    then have consistent-interp (lits-of-l F)
      using distinct-consistent-interp by blast
    then have ¬ tautology (C')
      using ‹F ⊨as CNot C'› consistent-CNot-not-tautology true-annots-true-cls by blast
    then have ¬ tautology (?C' + {#L#})
      using ‹F ⊨as CNot C'› ‹undefined-lit F L› by (metis  CNot-remdups-mset
        Marked-Propagated-in-iff-in-lits-of-l add.commute in-CNot-uminus tautology-add-single
        tautology-remdups-mset true-annot-singleton true-annots-def)
  show ?case
    proof –
      have f2: no-dup (convert-trail-from-W (trail S))
        using ‹inv_NOT S› unfolding inv_NOT-def by (simp add: o-def)
      have f3: atm-of L ∈ atms-of-mm (clauses S)
        ∪ atm-of ‘ lits-of-l (convert-trail-from-W (trail S))
        using ‹convert-trail-from-W (trail S) = F' @ Marked K () # F›
        ‹atm-of L ∈ atms-of-mm (clauses S) ∪ atm-of ‘ lits-of-l (F' @ Marked K () # F)› by auto
      have f4: clauses S ⊨pm remdups-mset C' + {#L#}
        by (metis (no-types) ‹L ∉# C'› ‹clauses S ⊨pm C' + {#L#}› remdups-mset-singleton-sum(2)
          true-clss-cls-remdups-mset union-commute)
      have F ⊨as CNot (remdups-mset C')
        by (simp add: ‹F ⊨as CNot C'›)
      obtain D where D: mset-cls D = remdups-mset C' + {#L#}
        using ex-mset-cls by blast
      have Ex (backjump-l S)
        apply standard
        apply (rule backjump-l.intros[OF  - f2, of - - - ])
        using f4 f3 f2 ‹¬ tautology (remdups-mset C' + {#L#})›
        calculation(2−5,9) ‹F ⊨as CNot (remdups-mset C')›
        state-eq_NOT-ref D unfolding backjump-l-cond-def by blast+
      then show ?thesis
        by blast
    qed
qed

sublocale conflict-driven-clause-learning_W ⊆ cdcl_NOT-merge-bj-learn-proxy2 - - - - - - - -
  λS. convert-trail-from-W (trail S)
  raw-clauses
  λL S. cons-trail (convert-marked-lit-from-NOT L) S
  λS. tl-trail S
  λC S. add-learned-cls C S
  λC S. remove-cls C S
  λ- -. True
  λ- S. raw-conflicting S = None  backjump-l-cond inv_NOT
  by unfold-locales
```

```
sublocale conflict-driven-clause-learning_W ⊆ cdcl_NOT-merge-bj-learn - - - - - - - -
  λS. convert-trail-from-W (trail S)
  raw-clauses
  λL S. cons-trail (convert-marked-lit-from-NOT L) S
  λS. tl-trail S
  λC S. add-learned-cls C S
  λC S. remove-cls C S
  backjump-l-cond
  λ- -. True
  λ- S. raw-conflicting S = None  inv_NOT
  apply unfold-locales
   using dpll-bj-no-dup apply (simp add: comp-def)
  using cdcl_NOT.simps cdcl_NOT-no-dup no-dup-convert-from-W  unfolding inv_NOT-def by blast

context conflict-driven-clause-learning_W
begin
```

Notations are lost while proving locale inclusion:

```
notation state-eq_NOT (infix ∼_NOT 50)
```

22.2 Additional Lemmas between NOT and W states

```
lemma trail_W-eq-reduce-trail-to_NOT-eq:
  trail S = trail T ⟹ trail (reduce-trail-to_NOT F S) = trail (reduce-trail-to_NOT F T)
proof (induction F S arbitrary: T rule: reduce-trail-to_NOT.induct)
  case (1 F S T) note IH = this(1) and tr = this(2)
  then have [] = convert-trail-from-W (trail S)
    ∨ length F = length (convert-trail-from-W (trail S))
    ∨ trail (reduce-trail-to_NOT F (tl-trail S)) = trail (reduce-trail-to_NOT F (tl-trail T))
    using IH by (metis (no-types) trail-tl-trail)
  then show trail (reduce-trail-to_NOT F S) = trail (reduce-trail-to_NOT F T)
    using tr by (metis (no-types) reduce-trail-to_NOT.elims)
qed

lemma trail-reduce-trail-to_NOT-add-learned-cls:
no-dup (trail S) ⟹
  trail (reduce-trail-to_NOT M (add-learned-cls D S)) = trail (reduce-trail-to_NOT M S)
 by (rule trail_W-eq-reduce-trail-to_NOT-eq) simp

lemma reduce-trail-to_NOT-reduce-trail-convert:
  reduce-trail-to_NOT C S = reduce-trail-to (convert-trail-from-NOT C) S
  apply (induction C S rule: reduce-trail-to_NOT.induct)
  apply (subst reduce-trail-to_NOT.simps, subst reduce-trail-to.simps)
  by auto

lemma reduce-trail-to-map[simp]:
  reduce-trail-to (map f M) S = reduce-trail-to M S
  by (rule reduce-trail-to-length) simp

lemma reduce-trail-to_NOT-map[simp]:
  reduce-trail-to_NOT (map f M) S = reduce-trail-to_NOT M S
  by (rule reduce-trail-to_NOT-length) simp

lemma skip-or-resolve-state-change:
  assumes skip-or-resolve** S T
  shows
    ∃M. trail S = M @ trail T ∧ (∀m ∈ set M. ¬is-marked m)
    clauses S = clauses T
    backtrack-lvl S = backtrack-lvl T
  using assms
proof (induction rule: rtranclp-induct)
  case base
  case 1 show ?case by simp
  case 2 show ?case by simp
  case 3 show ?case by simp
next
  case (step T U) note st = this(1) and s-o-r = this(2) and IH = this(3) and IH' = this(3−5)

  case 2 show ?case using IH' s-o-r by (auto elim!: rulesE simp: skip-or-resolve.simps)
  case 3 show ?case using IH' s-o-r by (auto elim!: rulesE simp: skip-or-resolve.simps)
  case 1 show ?case
    using s-o-r
    proof cases
      case s-or-r-skip
      then show ?thesis using IH by (auto elim!: rulesE simp: skip-or-resolve.simps)
    next
      case s-or-r-resolve
      then show ?thesis
        using IH by (cases trail T) (auto elim!: rulesE simp: skip-or-resolve.simps dest!:
        hd-raw-trail)
    qed
qed
```

22.3 More lemmas conflict–propagate and backjumping

22.4 CDCL FW

```
lemma cdcl_W-merge-is-cdcl_NOT-merged-bj-learn:
  assumes
    inv: cdcl_W-all-struct-inv S and
    cdcl_W:cdcl_W-merge S T
  shows cdcl_NOT-merged-bj-learn S T
    ∨ (no-step cdcl_W-merge T ∧ conflicting T ≠ None)
  using cdcl_W inv
proof induction
  case (fw-propagate S T) note propa = this(1)
  then obtain M N U k L C where
    H: state S = (M, N, U, k, None) and
    CL: C + {#L#} ∈# clauses S and
    M-C: M ⊨as CNot C and
    undef: undefined-lit (trail S) L and
    T: state T = (Propagated L (C + {#L#}) # M, N, U, k, None)
    by (auto elim: propagate-high-levelE)
  have propagate_NOT S T
    using H CL T undef M-C by (auto simp: state-eq_NOT-def state-eq-def raw-clauses-def
      simp del: state-simp)
  then show ?case
    using cdcl_NOT-merged-bj-learn.intros(2) by blast
next
  case (fw-decide S T) note dec = this(1) and inv = this(2)
  then obtain L where
```

```
      undef-L: undefined-lit (trail S) L and
      atm-L: atm-of L ∈ atms-of-mm (init-clss S) and
      T: T ∼ cons-trail (Marked L (Suc (backtrack-lvl S)))
        (update-backtrack-lvl (Suc (backtrack-lvl S)) S)
      by (auto elim: decideE)
    have decide$_{NOT}$ S T
      apply (rule decide$_{NOT}$.decide$_{NOT}$)
          using undef-L apply simp
        using atm-L inv unfolding cdcl$_W$-all-struct-inv-def no-strange-atm-def raw-clauses-def
          apply auto[]
      using T undef-L unfolding state-eq-def state-eq$_{NOT}$-def by (auto simp: raw-clauses-def)
    then show ?case using cdcl$_{NOT}$-merged-bj-learn-decide$_{NOT}$ by blast
  next
    case (fw-forget S T) note rf = this(1) and inv = this(2)
    then obtain C where
        S: conflicting S = None and
        C-le: C !∈! raw-learned-clss S and
        ¬(trail S) ⊨asm clauses S and
        mset-cls C ∉ set (get-all-mark-of-propagated (trail S)) and
        C-init: mset-cls C ∉# init-clss S and
        T: T ∼ remove-cls C S
      by (auto elim: forgetE)
    have init-clss S ⊨pm mset-cls C
      using inv C-le unfolding cdcl$_W$-all-struct-inv-def cdcl$_W$-learned-clause-def raw-clauses-def
      by (meson in-clss-mset-clss true-clss-clss-in-imp-true-clss-cls)
    then have S-C: removeAll-mset (mset-cls C) (clauses S) ⊨pm mset-cls C
      using C-init C-le unfolding raw-clauses-def by (auto simp add: Un-Diff ac-simps)
    have forget$_{NOT}$ S T
      apply (rule forget$_{NOT}$.forget$_{NOT}$)
          using S-C apply blast
        using S apply simp
        using C-init C-le apply (simp add: raw-clauses-def)
      using T C-le C-init by (auto
        simp: state-eq-def Un-Diff state-eq$_{NOT}$-def raw-clauses-def ac-simps
        simp del: state-simp)
    then show ?case using cdcl$_{NOT}$-merged-bj-learn-forget$_{NOT}$ by blast
  next
    case (fw-conflict S T U) note confl = this(1) and bj = this(2) and inv = this(3)
    obtain $C_S$ CT where
      confl-T: raw-conflicting T = Some CT and
      CT: mset-ccls CT = mset-cls $C_S$ and
      $C_S$: $C_S$ !∈! raw-clauses S and
      tr-S-$C_S$: trail S ⊨as CNot (mset-cls $C_S$)
      using confl by (elim conflictE) (auto simp del: state-simp simp: state-eq-def)
    have cdcl$_W$-all-struct-inv T
      using cdcl$_W$.simps cdcl$_W$-all-struct-inv-inv confl inv by blast
    then have cdcl$_W$-M-level-inv T
      unfolding cdcl$_W$-all-struct-inv-def by auto
    then consider
        (no-bt) skip-or-resolve$^{**}$ T U
      | (bt) T' where skip-or-resolve$^{**}$ T T' and backtrack T' U
      using bj rtranclp-cdcl$_W$-bj-skip-or-resolve-backtrack unfolding full-def by meson
    then show ?case
      proof cases
        case no-bt
  then have conflicting U ≠ None
    using confl by (induction rule: rtranclp-induct)
    (auto simp del: state-simp simp: skip-or-resolve.simps state-eq-def elim!: rulesE)
  moreover then have no-step cdcl_W-merge U
    by (auto simp: cdcl_W-merge.simps elim: rulesE)
  ultimately show ?thesis by blast
next
  case bt note s-or-r = this(1) and bt = this(2)
  have cdcl_W** T T'
    using s-or-r mono-rtranclp[of skip-or-resolve cdcl_W] rtranclp-skip-or-resolve-rtranclp-cdcl_W
    by blast
  then have cdcl_W-M-level-inv T'
    using rtranclp-cdcl_W-consistent-inv ‹cdcl_W-M-level-inv T› by blast
  then obtain M1 M2 i D L K where
    confl-T': raw-conflicting T' = Some D and
    LD: L ∈# mset-ccls D and
    M1-M2:(Marked K (i+1) # M1, M2) ∈ set (get-all-marked-decomposition (trail T')) and
    get-level (trail T') L = backtrack-lvl T' and
    get-level (trail T') L = get-maximum-level (trail T') (mset-ccls D) and
    get-maximum-level (trail T') (mset-ccls (remove-clit L D)) = i and
    undef-L: undefined-lit M1 L and
    U: U ∼ cons-trail (Propagated L (cls-of-ccls D))
            (reduce-trail-to M1
                (add-learned-cls (cls-of-ccls D)
                  (update-backtrack-lvl i
                    (update-conflicting None T'))))
    using bt by (auto elim: backtrack-levE)
  have [simp]: clauses S = clauses T
    using confl by (auto elim: rulesE)
  have [simp]: clauses T = clauses T'
    using s-or-r
    proof (induction)
      case base
      then show ?case by simp
    next
      case (step U V) note st = this(1) and s-o-r = this(2) and IH = this(3)
      have clauses U = clauses V
        using s-o-r by (auto simp: skip-or-resolve.simps elim: rulesE)
      then show ?case using IH by auto
    qed
  have inv-T: cdcl_W-all-struct-inv T
    by (meson cdcl_W-cp.simps confl inv r-into-rtranclp rtranclp-cdcl_W-all-struct-inv-inv
      rtranclp-cdcl_W-cp-rtranclp-cdcl_W)
  have cdcl_W** T T'
    using rtranclp-skip-or-resolve-rtranclp-cdcl_W s-or-r by blast
  have inv-T': cdcl_W-all-struct-inv T'
    using ‹cdcl_W** T T'› inv-T rtranclp-cdcl_W-all-struct-inv-inv by blast
  have inv-U: cdcl_W-all-struct-inv U
    using cdcl_W-merge-restart-cdcl_W confl fw-r-conflict inv local.bj
    rtranclp-cdcl_W-all-struct-inv-inv by blast

  have [simp]: init-clss S = init-clss T'
    using ‹cdcl_W** T T'› cdcl_W-init-clss confl cdcl_W-all-struct-inv-def conflict inv
    by (metis ‹cdcl_W-M-level-inv T› rtranclp-cdcl_W-init-clss)
  then have atm-L: atm-of L ∈ atms-of-mm (clauses S)
```

```
    using inv-T' confl-T' LD unfolding cdcl_W-all-struct-inv-def no-strange-atm-def
    raw-clauses-def
    by (simp add: atms-of-def image-subset-iff)
  obtain M where tr-T: trail T = M @ trail T'
    using s-or-r skip-or-resolve-state-change by meson
  obtain M' where
    tr-T': trail T' = M' @  Marked K (i+1) # tl (trail U) and
    tr-U: trail U = Propagated L (mset-ccls D) # tl (trail U)
    using U M1-M2 undef-L inv-T' unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def
    by fastforce
  def M'' ≡ M @ M'
  have tr-T: trail S = M'' @  Marked K (i+1) # tl (trail U)
    using tr-T tr-T' confl unfolding M''-def by (auto elim: rulesE)
  have init-clss T' + learned-clss S ⊨pm mset-ccls D
    using inv-T' confl-T' unfolding cdcl_W-all-struct-inv-def cdcl_W-learned-clause-def
    raw-clauses-def by simp
  have reduce-trail-to (convert-trail-from-NOT (convert-trail-from-W M1)) S =
    reduce-trail-to M1 S
    by (rule reduce-trail-to-length) simp
  moreover have trail (reduce-trail-to M1 S) = M1
    apply (rule reduce-trail-to-skip-beginning[of - M @ - @ M2 @ [Marked K (Suc i)]])
    using confl M1-M2 ‹trail T = M @ trail T'›
      apply (auto dest!: get-all-marked-decomposition-exists-prepend
        elim!: conflictE)
      by (rule sym) auto
  ultimately have [simp]: trail (reduce-trail-to_NOT M1 S) = M1
    using M1-M2 confl by (subst reduce-trail-to_NOT-reduce-trail-convert)
    (auto simp: comp-def elim: rulesE)
  have every-mark-is-a-conflict U
    using inv-U unfolding cdcl_W-all-struct-inv-def cdcl_W-conflicting-def by simp
  then have U-D: tl (trail U) ⊨as CNot (remove1-mset L (mset-ccls D))
    by (metis append-self-conv2 tr-U)
  thm  backjump-l[of - - - - - L cls-of-ccls D - remove1-mset L (mset-ccls D)]
  have backjump-l S U
    apply (rule backjump-l[of - - - - - L cls-of-ccls D - remove1-mset L (mset-ccls D)])
             using tr-T apply simp
            using inv unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def
            apply (simp add: comp-def)
         using U M1-M2 confl undef-L M1-M2 inv-T' inv undef-L unfolding cdcl_W-all-struct-inv-def
           cdcl_W-M-level-inv-def apply (auto simp: state-eq_NOT-def
             trail-reduce-trail-to_NOT-add-learned-cls)[]
          using C_S apply auto[]
         using tr-S-C_S apply simp

        using U undef-L M1-M2  inv-T' inv unfolding cdcl_W-all-struct-inv-def
        cdcl_W-M-level-inv-def apply auto[]
       using undef-L atm-L apply (simp add: trail-reduce-trail-to_NOT-add-learned-cls)
      using ‹init-clss T' + learned-clss S ⊨pm mset-ccls D› LD unfolding raw-clauses-def
      apply simp
     using LD apply simp
    apply (metis U-D convert-trail-from-W-true-annots)
    using  inv-T' inv-U U confl-T' undef-L M1-M2 LD unfolding cdcl_W-all-struct-inv-def
    distinct-cdcl_W-state-def by (simp add: cdcl_W-M-level-inv-decomp backjump-l-cond-def)
  then show ?thesis using cdcl_NOT-merged-bj-learn-backjump-l by fast
qed
```

qed

abbreviation $cdcl_{NOT}$*-restart* **where**
$cdcl_{NOT}$*-restart* ≡ *restart-ops.*$cdcl_{NOT}$*-raw-restart* $cdcl_{NOT}$ *restart*

lemma $cdcl_W$*-merge-restart-is-*$cdcl_{NOT}$*-merged-bj-learn-restart-no-step*:
  assumes
    inv: $cdcl_W$*-all-struct-inv S* **and**
    $cdcl_W$:$cdcl_W$*-merge-restart S T*
  shows $cdcl_{NOT}$*-restart*** *S T* ∨ (*no-step* $cdcl_W$*-merge T* ∧ *conflicting T* ≠ *None*)
proof –
  consider
      (*fw*) $cdcl_W$*-merge S T*
    | (*fw-r*) *restart S T*
    using $cdcl_W$ **by** (*meson* $cdcl_W$*-merge-restart.simps* $cdcl_W$*-rf.cases fw-conflict fw-decide fw-forget fw-propagate*)
  then show *?thesis*
    proof *cases*
      case *fw*
      then have *IH*: $cdcl_{NOT}$*-merged-bj-learn S T* ∨ (*no-step* $cdcl_W$*-merge T* ∧ *conflicting T* ≠ *None*)
        using *inv* $cdcl_W$*-merge-is-*$cdcl_{NOT}$*-merged-bj-learn* **by** *blast*
      have *invS*: inv_{NOT} *S*
        using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* $cdcl_W$*-M-level-inv-def* **by** *auto*
      have *ff2*: $cdcl_{NOT}$$^{++}$ *S T* ⟶ $cdcl_{NOT}$** *S T*
          by (*meson tranclp-into-rtranclp*)
      have *ff3*: *no-dup* (*convert-trail-from-W* (*trail S*))
        using *invS* **by** (*simp add*: *comp-def*)
      have $cdcl_{NOT}$ ≤ $cdcl_{NOT}$*-restart*
        by (*auto simp*: *restart-ops.*$cdcl_{NOT}$*-raw-restart.simps*)
      then show *?thesis*
        using *ff3 ff2 IH* $cdcl_{NOT}$*-merged-bj-learn-is-tranclp-*$cdcl_{NOT}$
        rtranclp-mono[*of* $cdcl_{NOT}$ $cdcl_{NOT}$*-restart*] *invS predicate2D* **by** *blast*
    next
      case *fw-r*
      then show *?thesis* **by** (*blast intro*: *restart-ops.*$cdcl_{NOT}$*-raw-restart.intros*)
    qed
qed

abbreviation μ_{FW} :: *'st* ⇒ *nat* **where**
μ_{FW} *S* ≡ (*if no-step* $cdcl_W$*-merge S then 0 else 1+*μ_{CDCL}*'-merged* (*set-mset* (*init-clss S*)) *S*)

lemma $cdcl_W$*-merge-*μ_{FW}*-decreasing*:
  assumes
    inv: $cdcl_W$*-all-struct-inv S* **and**
    fw: $cdcl_W$*-merge S T*
  shows μ_{FW} *T* < μ_{FW} *S*
proof –
  let *?A* = *init-clss S*
  have *atm-clauses*: *atms-of-mm* (*clauses S*) ⊆ *atms-of-mm ?A*
    using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def no-strange-atm-def raw-clauses-def* **by** *auto*
  have *atm-trail*: *atm-of* ' *lits-of-l* (*trail S*) ⊆ *atms-of-mm ?A*
    using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def no-strange-atm-def raw-clauses-def* **by** *auto*
  have *n-d*: *no-dup* (*trail S*)
    using *inv* **unfolding** $cdcl_W$*-all-struct-inv-def* **by** (*auto simp*: $cdcl_W$*-M-level-inv-decomp*)
  have [*simp*]: ¬ *no-step* $cdcl_W$*-merge S*

```
    using fw by auto
  have [simp]: init-clss S = init-clss T
    using cdcl_W-merge-restart-cdcl_W[of S T] inv rtranclp-cdcl_W-init-clss
    unfolding cdcl_W-all-struct-inv-def
    by (meson cdcl_W-merge.simps cdcl_W-merge-restart.simps  cdcl_W-rf.simps fw)
  consider
      (merged) cdcl_NOT-merged-bj-learn S T
    | (n-s) no-step cdcl_W-merge T
    using cdcl_W-merge-is-cdcl_NOT-merged-bj-learn inv fw by blast
  then show ?thesis
    proof cases
      case merged
      then show ?thesis
        using cdcl_NOT-decreasing-measure'[OF - - atm-clauses, of T] atm-trail n-d
        by (auto split: if-split simp: comp-def image-image lits-of-def)
    next
      case n-s
      then show ?thesis by simp
    qed
qed

lemma wf-cdcl_W-merge: wf {(T, S). cdcl_W-all-struct-inv S ∧ cdcl_W-merge S T}
  apply (rule wfP-if-measure[of - - μ_FW])
  using cdcl_W-merge-μ_FW-decreasing by blast

sublocale conflict-driven-clause-learning_W-termination
  by unfold-locales (simp add: wf-cdcl_W-merge)

lemma full-cdcl_W-s'-full-cdcl_W-merge-restart:
  assumes
    conflicting R = None and
    inv: cdcl_W-all-struct-inv R
  shows full cdcl_W-s' R V ⟷ full cdcl_W-merge-stgy R V (is ?s' ⟷ ?fw)
proof
  assume ?s'
  then have cdcl_W-s'** R V unfolding full-def by blast
  have cdcl_W-all-struct-inv V
    using ‹cdcl_W-s'** R V› inv rtranclp-cdcl_W-all-struct-inv-inv rtranclp-cdcl_W-s'-rtranclp-cdcl_W
    by blast
  then have n-s: no-step cdcl_W-merge-stgy V
    using no-step-cdcl_W-s'-no-step-cdcl_W-merge-stgy by (meson ‹full cdcl_W-s' R V› full-def)
  have n-s-bj: no-step cdcl_W-bj V
    by (metis ‹cdcl_W-all-struct-inv V› ‹full cdcl_W-s' R V› bj full-def
      n-step-cdcl_W-stgy-iff-no-step-cdcl_W-cl-cdcl_W-o)
  have n-s-cp: no-step cdcl_W-merge-cp V
    proof –
      { fix ss :: 'st
        obtain ssa :: 'st ⇒ 'st where
          ff1: ∀ s. ¬ cdcl_W-all-struct-inv s ∨ cdcl_W-s'-without-decide s (ssa s)
            ∨ no-step cdcl_W-merge-cp s
          using conflicting-true-no-step-s'-without-decide-no-step-cdcl_W-merge-cp by moura
        have (∀ p s sa. ¬ full p (s::'st) sa ∨ p** s sa ∧ no-step p sa) and
          (∀ p s sa. (¬ p** (s::'st) sa ∨ (∃ s. p sa s)) ∨ full p s sa)
          by (meson full-def)+
        then have ¬ cdcl_W-merge-cp V ss
        using ff1 by (metis (no-types) ⟨cdcl_W-all-struct-inv V⟩ ⟨full cdcl_W-s' R V⟩ cdcl_W-s'.simps
          cdcl_W-s'-without-decide.cases) }
      then show ?thesis
        by blast
    qed
  consider
      (fw-no-confl) cdcl_W-merge-stgy^** R V and conflicting V = None
    | (fw-confl) cdcl_W-merge-stgy^** R V and conflicting V ≠ None and no-step cdcl_W-bj V
    | (fw-dec-confl) S T U where cdcl_W-merge-stgy^** R S and no-step cdcl_W-merge-cp S and
        decide S T and cdcl_W-merge-cp^** T U and conflict U V
    | (fw-dec-no-confl) S T where cdcl_W-merge-stgy^** R S and no-step cdcl_W-merge-cp S and
        decide S T and cdcl_W-merge-cp^** T V and conflicting V = None
    | (cp-no-confl) cdcl_W-merge-cp^** R V and conflicting V = None
    | (cp-confl) U where cdcl_W-merge-cp^** R U and conflict U V
    using rtranclp-cdcl_W-s'-no-step-cdcl_W-s'-without-decide-decomp-into-cdcl_W-merge[OF
      ⟨cdcl_W-s'^** R V⟩ assms] by auto
  then show ?fw
    proof cases
      case fw-no-confl
      then show ?thesis using n-s unfolding full-def by blast
    next
      case fw-confl
      then show ?thesis using n-s unfolding full-def by blast
    next
      case fw-dec-confl
      have cdcl_W-merge-cp U V
        using n-s-bj by (metis cdcl_W-merge-cp.simps full-unfold fw-dec-confl(5))
      then have full1 cdcl_W-merge-cp T V
        unfolding full1-def by (metis fw-dec-confl(4) n-s-cp tranclp-unfold-end)
      then have cdcl_W-merge-stgy S V using ⟨decide S T⟩ ⟨no-step cdcl_W-merge-cp S⟩ by auto
      then show ?thesis using n-s ⟨ cdcl_W-merge-stgy^** R S⟩ unfolding full-def by auto
    next
      case fw-dec-no-confl
      then have full cdcl_W-merge-cp T V
        using n-s-cp unfolding full-def by blast
      then have cdcl_W-merge-stgy S V using ⟨decide S T⟩ ⟨no-step cdcl_W-merge-cp S⟩ by auto
      then show ?thesis using n-s ⟨ cdcl_W-merge-stgy^** R S⟩ unfolding full-def by auto
    next
      case cp-no-confl
      then have full cdcl_W-merge-cp R V
        by (simp add: full-def n-s-cp)
      then have R = V ∨ cdcl_W-merge-stgy^++ R V
        using fw-s-cp unfolding full-unfold fw-s-cp
        by (metis (no-types) rtranclp-unfold tranclp-unfold-end)
      then show ?thesis
        by (simp add: full-def n-s rtranclp-unfold)
    next
      case cp-confl
      have full cdcl_W-bj V V
        using n-s-bj unfolding full-def by blast
      then have full1 cdcl_W-merge-cp R V
        unfolding full1-def by (meson cdcl_W-merge-cp.conflict' cp-confl(1,2) n-s-cp
          rtranclp-into-tranclp1)
      then show ?thesis using n-s unfolding full-def by auto
    qed
```

```
next
  assume ?fw
  then have cdcl_W** R V using rtranclp-mono[of cdcl_W-merge-stgy cdcl_W**]
    cdcl_W-merge-stgy-rtranclp-cdcl_W unfolding full-def by auto
  then have inv': cdcl_W-all-struct-inv V using inv rtranclp-cdcl_W-all-struct-inv-inv by blast
  have cdcl_W-s'** R V
    using ‹?fw› by (simp add: full-def inv rtranclp-cdcl_W-merge-stgy-rtranclp-cdcl_W-s')
  moreover have no-step cdcl_W-s' V
    proof cases
      assume conflicting V = None
      then show ?thesis
        by (metis inv' ‹full cdcl_W-merge-stgy R V› full-def
          no-step-cdcl_W-merge-stgy-no-step-cdcl_W-s')
    next
      assume confl-V: conflicting V ≠ None
      then have no-step cdcl_W-bj V
      using rtranclp-cdcl_W-merge-stgy-no-step-cdcl_W-bj by (meson ‹full cdcl_W-merge-stgy R V›
        assms(1) full-def)
      then show ?thesis using confl-V by (fastforce simp: cdcl_W-s'.simps full1-def cdcl_W-cp.simps
        dest!: tranclpD elim: rulesE)
    qed
  ultimately show ?s' unfolding full-def by blast
qed

lemma full-cdcl_W-stgy-full-cdcl_W-merge:
  assumes
    conflicting R = None and
    inv: cdcl_W-all-struct-inv R
  shows full cdcl_W-stgy R V ⟷ full cdcl_W-merge-stgy R V
  by (simp add: assms(1) full-cdcl_W-s'-full-cdcl_W-merge-restart full-cdcl_W-stgy-iff-full-cdcl_W-s'
    inv)

lemma full-cdcl_W-merge-stgy-final-state-conclusive':
  fixes S' :: 'st
  assumes full: full cdcl_W-merge-stgy (init-state N) S'
  and no-d: distinct-mset-mset (mset-clss N)
  shows (conflicting S' = Some {#} ∧ unsatisfiable (set-mset (mset-clss N)))
    ∨ (conflicting S' = None ∧ trail S' ⊨asm mset-clss N ∧ satisfiable (set-mset (mset-clss N)))
proof -
  have cdcl_W-all-struct-inv (init-state N)
    using no-d unfolding cdcl_W-all-struct-inv-def by auto
  moreover have conflicting (init-state N) = None
    by auto
  ultimately show ?thesis
    using full full-cdcl_W-stgy-final-state-conclusive-from-init-state
    full-cdcl_W-stgy-full-cdcl_W-merge no-d by presburger
qed
end

end
theory CDCL-W-Incremental
imports CDCL-W-Termination
begin
```

23 Incremental SAT solving

```
context conflict-driven-clause-learning_W
begin
```

This invariant holds all the invariant related to the strategy. See the structural invariant in *$cdcl_W$-all-struct-inv*

```
definition cdcl_W-stgy-invariant where
cdcl_W-stgy-invariant S ⟷
  conflict-is-false-with-level S
  ∧ no-clause-is-false S
  ∧ no-smaller-confl S
  ∧ no-clause-is-false S

lemma cdcl_W-stgy-cdcl_W-stgy-invariant:
  assumes
   cdcl_W: cdcl_W-stgy S T and
   inv-s: cdcl_W-stgy-invariant S and
   inv: cdcl_W-all-struct-inv S
  shows
    cdcl_W-stgy-invariant T
  unfolding cdcl_W-stgy-invariant-def cdcl_W-all-struct-inv-def apply (intro conjI)
    apply (rule cdcl_W-stgy-ex-lit-of-max-level[of S])
    using assms unfolding cdcl_W-stgy-invariant-def cdcl_W-all-struct-inv-def apply auto[7]
    using cdcl_W cdcl_W-stgy-not-non-negated-init-clss apply simp
  apply (rule cdcl_W-stgy-no-smaller-confl-inv)
   using assms unfolding cdcl_W-stgy-invariant-def cdcl_W-all-struct-inv-def apply auto[4]
  using cdcl_W cdcl_W-stgy-not-non-negated-init-clss by auto

lemma rtranclp-cdcl_W-stgy-cdcl_W-stgy-invariant:
  assumes
   cdcl_W: cdcl_W-stgy** S T and
   inv-s: cdcl_W-stgy-invariant S and
   inv: cdcl_W-all-struct-inv S
  shows
    cdcl_W-stgy-invariant T
  using assms apply (induction)
    apply simp
  using cdcl_W-stgy-cdcl_W-stgy-invariant rtranclp-cdcl_W-all-struct-inv-inv
  rtranclp-cdcl_W-stgy-rtranclp-cdcl_W by blast

abbreviation decr-bt-lvl where
decr-bt-lvl S ≡ update-backtrack-lvl (backtrack-lvl S − 1) S
```

When we add a new clause, we reduce the trail until we get to tho first literal included in C. Then we can mark the conflict.

```
fun cut-trail-wrt-clause where
cut-trail-wrt-clause C [] S = S |
cut-trail-wrt-clause C (Marked L - # M) S =
  (if −L ∈# C then S
    else cut-trail-wrt-clause C M (decr-bt-lvl (tl-trail S))) |
cut-trail-wrt-clause C (Propagated L - # M) S =
  (if −L ∈# C then S
    else cut-trail-wrt-clause C M (tl-trail S))
```

```
definition add-new-clause-and-update :: 'ccls ⇒ 'st ⇒ 'st where
add-new-clause-and-update C S =
  (if trail S ⊨as CNot (mset-ccls C)
  then update-conflicting (Some C) (add-init-cls (cls-of-ccls C)
    (cut-trail-wrt-clause (mset-ccls C) (trail S) S))
  else add-init-cls (cls-of-ccls C) S)

thm cut-trail-wrt-clause.induct
lemma init-clss-cut-trail-wrt-clause[simp]:
  init-clss (cut-trail-wrt-clause C M S) = init-clss S
  by (induction rule: cut-trail-wrt-clause.induct) auto

lemma learned-clss-cut-trail-wrt-clause[simp]:
  learned-clss (cut-trail-wrt-clause C M S) = learned-clss S
  by (induction rule: cut-trail-wrt-clause.induct) auto

lemma conflicting-clss-cut-trail-wrt-clause[simp]:
  conflicting (cut-trail-wrt-clause C M S) = conflicting S
  by (induction rule: cut-trail-wrt-clause.induct) auto

lemma trail-cut-trail-wrt-clause:
  ∃ M.  trail S = M @ trail (cut-trail-wrt-clause C (trail S) S)
proof (induction trail S arbitrary: S rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
  case (marked L l M) note IH = this(1)[of decr-bt-lvl (tl-trail S)] and M = this(2)[symmetric]
  then show ?case using Cons-eq-appendI by fastforce+
next
  case (proped L l M) note IH = this(1)[of tl-trail S] and M = this(2)[symmetric]
  then show ?case using Cons-eq-appendI by fastforce+
qed

lemma n-dup-no-dup-trail-cut-trail-wrt-clause[simp]:
  assumes n-d: no-dup (trail T)
  shows no-dup (trail (cut-trail-wrt-clause C (trail T) T))
proof –
  obtain M where
    M:  trail T = M @ trail (cut-trail-wrt-clause C (trail T) T)
    using trail-cut-trail-wrt-clause[of T C] by auto
  show ?thesis
    using n-d unfolding arg-cong[OF M, of no-dup] by auto
qed

lemma cut-trail-wrt-clause-backtrack-lvl-length-marked:
  assumes
     backtrack-lvl T = length (get-all-levels-of-marked (trail T))
  shows
  backtrack-lvl (cut-trail-wrt-clause C (trail T) T) =
     length (get-all-levels-of-marked (trail (cut-trail-wrt-clause C (trail T) T)))
  using assms
proof (induction trail T arbitrary:T rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
```

```
  case (marked L l M) note IH = this(1)[of decr-bt-lvl (tl-trail T)] and M = this(2)[symmetric]
    and bt = this(3)
  then show ?case by auto
next
  case (proped L l M) note IH = this(1)[of tl-trail T] and M = this(2)[symmetric] and bt = this(3)
  then show ?case by auto
qed

lemma cut-trail-wrt-clause-get-all-levels-of-marked:
  assumes get-all-levels-of-marked (trail T) = rev [Suc 0..<
    Suc (length (get-all-levels-of-marked (trail T)))]
  shows
    get-all-levels-of-marked (trail ((cut-trail-wrt-clause C (trail T) T))) = rev [Suc 0..<
    Suc (length (get-all-levels-of-marked (trail ((cut-trail-wrt-clause C (trail T) T)))))]
  using assms
proof (induction trail T arbitrary:T rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
  case (marked L l M) note IH = this(1)[of decr-bt-lvl (tl-trail T)] and M = this(2)[symmetric]
    and bt = this(3)
  then show ?case by (cases count C L = 0) auto
next
  case (proped L l M) note IH = this(1)[of tl-trail T] and M = this(2)[symmetric] and bt = this(3)
  then show ?case by (cases count C L = 0) auto
qed

lemma cut-trail-wrt-clause-CNot-trail:
  assumes trail T ⊨as CNot C
  shows
    (trail ((cut-trail-wrt-clause C (trail T) T))) ⊨as CNot C
  using assms
proof (induction trail T arbitrary:T rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
  case (marked L l M) note IH = this(1)[of decr-bt-lvl (tl-trail T)] and M = this(2)[symmetric]
    and bt = this(3)
  show ?case
    proof (cases count C (−L) = 0)
      case False
      then show ?thesis
        using IH M bt by (auto simp: true-annots-true-cls)
    next
      case True
      obtain mma :: 'v literal multiset where
        f6: (mma ∈ {{#− l#} |l. l ∈# C} ⟶ M ⊨a mma) ⟶ M ⊨as {{#− l#} |l. l ∈# C}
        using true-annots-def by blast
      have mma ∈ {{#− l#} |l. l ∈# C} ⟶ trail T ⊨a mma
        using CNot-def M bt by (metis (no-types) true-annots-def)
      then have M ⊨as {{#− l#} |l. l ∈# C}
        using f6 True M bt by (force simp: count-eq-zero-iff)
      then show ?thesis
        using IH true-annots-true-cls M by (auto simp: CNot-def)
    qed
```

```
next
  case (proped L l M) note IH = this(1)[of tl-trail T] and M = this(2)[symmetric] and bt = this(3)
  show ?case
    proof (cases count C (−L) = 0)
      case False
      then show ?thesis
        using IH M bt by (auto simp: true-annots-true-cls)
    next
      case True
      obtain mma :: 'v literal multiset where
        f6: (mma ∈ {{#− l#} |l. l ∈# C} ⟶ M ⊨a mma) ⟶ M ⊨as {{#− l#} |l. l ∈# C}
        using true-annots-def by blast
      have mma ∈ {{#− l#} |l. l ∈# C} ⟶ trail T ⊨a mma
        using CNot-def M bt by (metis (no-types) true-annots-def)
      then have M ⊨as {{#− l#} |l. l ∈# C}
        using f6 True M bt by (force simp: count-eq-zero-iff)
      then show ?thesis
        using IH true-annots-true-cls M by (auto simp: CNot-def)
    qed
qed

lemma cut-trail-wrt-clause-hd-trail-in-or-empty-trail:
  ((∀L ∈#C. −L ∉ lits-of-l (trail T)) ∧ trail (cut-trail-wrt-clause C (trail T) T) = [])
    ∨ (−lit-of (hd (trail (cut-trail-wrt-clause C (trail T) T))) ∈# C
       ∧ length (trail (cut-trail-wrt-clause C (trail T) T)) ≥ 1)
  using assms
proof (induction trail T arbitrary:T rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
  case (marked L l M) note IH = this(1)[of decr-bt-lvl (tl-trail T)] and M = this(2)[symmetric]
  then show ?case by simp force
next
  case (proped L l M) note IH = this(1)[of tl-trail T] and M = this(2)[symmetric]
  then show ?case by simp force
qed
```

We can fully run *cdcl_W-s* or add a clause. Remark that we use *cdcl_W-s* to avoid an explicit *skip*, *resolve*, and *backtrack* normalisation to get rid of the conflict *C* if possible.

```
inductive incremental-cdcl_W :: 'st ⇒ 'st ⇒ bool for S where
add-confl:
  trail S ⊨asm init-clss S ⟹ distinct-mset (mset-ccls C) ⟹ conflicting S = None ⟹
   trail S ⊨as CNot (mset-ccls C) ⟹
   full cdcl_W-stgy
     (update-conflicting (Some C)
       (add-init-cls (cls-of-ccls C) (cut-trail-wrt-clause (mset-ccls C) (trail S) S))) T ⟹
   incremental-cdcl_W S T |
add-no-confl:
  trail S ⊨asm init-clss S ⟹ distinct-mset (mset-ccls C) ⟹ conflicting S = None ⟹
   ¬trail S ⊨as CNot (mset-ccls C) ⟹
   full cdcl_W-stgy (add-init-cls (cls-of-ccls C) S) T ⟹
   incremental-cdcl_W S T

lemma cdcl_W-all-struct-inv-add-new-clause-and-update-cdcl_W-all-struct-inv:
  assumes
```

```
    inv-T: cdcl_W-all-struct-inv T and
    tr-T-N[simp]: trail T ⊨asm N and
    tr-C[simp]: trail T ⊨as CNot (mset-ccls C) and
    [simp]: distinct-mset (mset-ccls C)
  shows cdcl_W-all-struct-inv (add-new-clause-and-update C T) (is cdcl_W-all-struct-inv ?T')
proof –
  let ?T = update-conflicting (Some C)
    (add-init-cls (cls-of-ccls C) (cut-trail-wrt-clause (mset-ccls C) (trail T) T))
  obtain M where
    M: trail T = M @ trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)
      using trail-cut-trail-wrt-clause[of T mset-ccls C] by blast
  have H[dest]: ⋀x. x ∈ lits-of-l (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)) ⟹
    x ∈ lits-of-l (trail T)
    using inv-T arg-cong[OF M, of lits-of-l] by auto
  have H'[dest]: ⋀x. x ∈ set (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)) ⟹
    x ∈ set (trail T)
    using inv-T arg-cong[OF M, of set] by auto

  have H-proped:⋀x. x ∈ set (get-all-mark-of-propagated (trail (cut-trail-wrt-clause (mset-ccls C)
   (trail T) T))) ⟹ x ∈ set (get-all-mark-of-propagated (trail T))
  using inv-T arg-cong[OF M, of get-all-mark-of-propagated] by auto

  have [simp]: no-strange-atm ?T
    using inv-T unfolding cdcl_W-all-struct-inv-def no-strange-atm-def add-new-clause-and-update-def
    cdcl_W-M-level-inv-def by (auto 20 1)
  have M-lev: cdcl_W-M-level-inv T
    using inv-T unfolding cdcl_W-all-struct-inv-def by blast
  then have no-dup (M @ trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))
    unfolding cdcl_W-M-level-inv-def unfolding M[symmetric] by auto
  then have [simp]: no-dup (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))
    by auto

  have consistent-interp (lits-of-l (M @ trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
    using M-lev unfolding cdcl_W-M-level-inv-def unfolding M[symmetric] by auto
  then have [simp]: consistent-interp (lits-of-l (trail (cut-trail-wrt-clause (mset-ccls C)
   (trail T) T)))
    unfolding consistent-interp-def by auto

  have [simp]: cdcl_W-M-level-inv ?T
    using M-lev cut-trail-wrt-clause-get-all-levels-of-marked[of T mset-ccls C]
    unfolding cdcl_W-M-level-inv-def by (auto dest: H H'
      simp: M-lev cdcl_W-M-level-inv-def cut-trail-wrt-clause-backtrack-lvl-length-marked)

  have [simp]: ⋀s. s ∈# learned-clss T ⟹ ¬tautology s
    using inv-T unfolding cdcl_W-all-struct-inv-def by auto

  have distinct-cdcl_W-state T
    using inv-T unfolding cdcl_W-all-struct-inv-def by auto
  then have [simp]: distinct-cdcl_W-state ?T
    unfolding distinct-cdcl_W-state-def by auto

  have  cdcl_W-conflicting T
    using inv-T unfolding cdcl_W-all-struct-inv-def by auto
  have trail ?T ⊨as CNot (mset-ccls C)
     by (simp add: cut-trail-wrt-clause-CNot-trail)
  then have [simp]: cdcl_W-conflicting ?T
    unfolding cdcl_W-conflicting-def apply simp
    by (metis M ‹cdcl_W-conflicting T› append-assoc cdcl_W-conflicting-decomp(2))

  have
    decomp-T: all-decomposition-implies-m (init-clss T) (get-all-marked-decomposition (trail T))
    using inv-T unfolding cdcl_W-all-struct-inv-def by auto
  have  all-decomposition-implies-m  (init-clss ?T)
    (get-all-marked-decomposition (trail ?T))
    unfolding all-decomposition-implies-def
    proof clarify
      fix a b
      assume (a, b) ∈ set (get-all-marked-decomposition (trail ?T))
      from in-get-all-marked-decomposition-in-get-all-marked-decomposition-prepend[OF this, of M]
      obtain b' where
        (a, b' @ b) ∈ set (get-all-marked-decomposition (trail T))
        using M by auto
      then have unmark-l a ∪ set-mset (init-clss T) ⊨ps unmark-l (b' @ b)
        using decomp-T unfolding all-decomposition-implies-def by fastforce
      then have unmark-l a ∪ set-mset (init-clss ?T) ⊨ps unmark-l (b @ b')
        by (simp add: Un-commute)
      then show unmark-l a ∪ set-mset (init-clss ?T) ⊨ps unmark-l b
        by (auto simp: image-Un)
    qed

  have [simp]: cdcl_W-learned-clause ?T
    using inv-T unfolding cdcl_W-all-struct-inv-def cdcl_W-learned-clause-def
    by (auto dest!: H-proped simp: raw-clauses-def)
  show ?thesis
    using ‹all-decomposition-implies-m  (init-clss ?T)
    (get-all-marked-decomposition (trail ?T))›
    unfolding cdcl_W-all-struct-inv-def by (auto simp: add-new-clause-and-update-def)
qed

lemma cdcl_W-all-struct-inv-add-new-clause-and-update-cdcl_W-stgy-inv:
  assumes
    inv-s: cdcl_W-stgy-invariant T and
    inv: cdcl_W-all-struct-inv T and
    tr-T-N[simp]: trail T ⊨asm N and
    tr-C[simp]: trail T ⊨as CNot (mset-ccls C) and
    [simp]: distinct-mset (mset-ccls C)
  shows cdcl_W-stgy-invariant (add-new-clause-and-update C T)
    (is cdcl_W-stgy-invariant ?T')
proof −
  have cdcl_W-all-struct-inv ?T'
    using cdcl_W-all-struct-inv-add-new-clause-and-update-cdcl_W-all-struct-inv assms by blast
  then have
    no-dup-cut-T[simp]: no-dup (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)) and
    n-d[simp]: no-dup (trail T)
    using cdcl_W-M-level-inv-decomp(2) cdcl_W-all-struct-inv-def inv
    n-dup-no-dup-trail-cut-trail-wrt-clause by blast+
  then have trail (add-new-clause-and-update C T) ⊨as CNot (mset-ccls C)
    by (simp add: add-new-clause-and-update-def cut-trail-wrt-clause-CNot-trail
      cdcl_W-M-level-inv-def cdcl_W-all-struct-inv-def)
  obtain MT where
  MT: trail T = MT @  trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)
  using trail-cut-trail-wrt-clause by blast
consider
    (false) ∀ L∈#mset-ccls C. − L ∉ lits-of-l (trail T) and
      trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T) = []
  | (not-false)
    − lit-of (hd (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))) ∈# (mset-ccls C) and
    1 ≤ length (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))
  using cut-trail-wrt-clause-hd-trail-in-or-empty-trail[of mset-ccls C T] by auto
then show ?thesis
  proof cases
    case false note C = this(1) and empty-tr = this(2)
    then have [simp]: mset-ccls C = {#}
      by (simp add: in-CNot-implies-uminus(2) multiset-eqI)
    show ?thesis
      using empty-tr unfolding cdcl_W-stgy-invariant-def no-smaller-confl-def
      cdcl_W-all-struct-inv-def by (auto simp: add-new-clause-and-update-def)
  next
    case not-false note C = this(1) and l = this(2)
    let ?L = − lit-of (hd (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
    have get-all-levels-of-marked (trail (add-new-clause-and-update C T)) =
      rev [1..<1 + length (get-all-levels-of-marked (trail (add-new-clause-and-update C T)))]
      using ‹cdcl_W-all-struct-inv ?T'› unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def
      by blast
    moreover
      have backtrack-lvl (cut-trail-wrt-clause (mset-ccls C) (trail T) T) =
        length (get-all-levels-of-marked (trail (add-new-clause-and-update C T)))
        using ‹cdcl_W-all-struct-inv ?T'› unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def
        by (auto simp:add-new-clause-and-update-def)
    moreover
      have no-dup (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))
        using ‹cdcl_W-all-struct-inv ?T'› unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def
        by (auto simp:add-new-clause-and-update-def)
      then have atm-of ?L ∉ atm-of ‘ lits-of-l
        (tl (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
        by (cases trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))
        (auto simp: lits-of-def)

    ultimately have L: get-level (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)) (−?L)
      = length (get-all-levels-of-marked (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
      using get-level-get-rev-level-get-all-levels-of-marked[OF
        ‹atm-of ?L ∉ atm-of ‘ lits-of-l (tl (trail (cut-trail-wrt-clause (mset-ccls C)
          (trail T) T)))›,
        of [hd (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))]]

        apply (cases trail (add-init-cls (cls-of-ccls C)
            (cut-trail-wrt-clause (mset-ccls C) (trail T) T));
         cases hd (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
        using l by (auto split: if-split-asm
          simp:rev-swap[symmetric] add-new-clause-and-update-def)

    have L': length (get-all-levels-of-marked (trail (cut-trail-wrt-clause (mset-ccls C)
      (trail T) T)))
      = backtrack-lvl (cut-trail-wrt-clause (mset-ccls C) (trail T) T)
      using ‹cdcl_W-all-struct-inv ?T'› unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def
```

```
        by (auto simp:add-new-clause-and-update-def)

      have [simp]: no-smaller-confl (update-conflicting (Some C)
        (add-init-cls (cls-of-ccls C) (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
        unfolding no-smaller-confl-def
      proof (clarify, goal-cases)
        case (1 M K i M' D)
        then consider
            (DC) D = mset-ccls C
          | (D-T) D ∈# clauses T
          by (auto simp: raw-clauses-def split: if-split-asm)
        then show False
          proof cases
            case D-T
            have no-smaller-confl T
              using inv-s unfolding cdcl_W-stgy-invariant-def by auto
            have (MT @ M') @ Marked K i # M = trail T
              using MT 1(1) by auto
            thus False using D-T ‹no-smaller-confl T› 1(3) unfolding no-smaller-confl-def by blast
          next
            case DC note -[simp] = this
            then have atm-of (−?L) ∈ atm-of ‘ (lits-of-l M)
              using 1(3) C in-CNot-implies-uminus(2) by blast
            moreover
              have lit-of (hd (M' @ Marked K i # [])) = −?L
                using l 1(1)[symmetric] inv
                by (cases trail (add-init-cls (cls-of-ccls C)
                    (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
                (auto dest!: arg-cong[of - # - - hd] simp: hd-append cdcl_W-all-struct-inv-def
                  cdcl_W-M-level-inv-def)
              from arg-cong[OF this, of atm-of]
              have atm-of (−?L) ∈ atm-of ‘ (lits-of-l (M' @ Marked K i # []))
                by (cases (M' @ Marked K i # [])) auto
            moreover have no-dup (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))
              using ‹cdcl_W-all-struct-inv ?T'› unfolding cdcl_W-all-struct-inv-def
              cdcl_W-M-level-inv-def by (auto simp: add-new-clause-and-update-def)
            ultimately show False
              unfolding 1(1)[symmetric, simplified] by (auto simp: lits-of-def)
        qed
      qed
      show ?thesis using L L' C
        unfolding cdcl_W-stgy-invariant-def
        unfolding cdcl_W-all-struct-inv-def by (auto simp: add-new-clause-and-update-def)
    qed
qed

lemma full-cdcl_W-stgy-inv-normal-form:
  assumes
    full: full cdcl_W-stgy S T and
    inv-s: cdcl_W-stgy-invariant S and
    inv: cdcl_W-all-struct-inv S
  shows conflicting T = Some {#} ∧ unsatisfiable (set-mset (init-clss S))
    ∨ conflicting T = None ∧ trail T ⊨asm init-clss S ∧ satisfiable (set-mset (init-clss S))
proof −
  have no-step cdcl_W-stgy T
```

```
    using full unfolding full-def by blast
  moreover have cdcl_W-all-struct-inv T and inv-s: cdcl_W-stgy-invariant T
    apply (metis rtranclp-cdcl_W-stgy-rtranclp-cdcl_W full full-def inv
      rtranclp-cdcl_W-all-struct-inv-inv)
    by (metis full full-def inv inv-s rtranclp-cdcl_W-stgy-cdcl_W-stgy-invariant)
  ultimately have conflicting T = Some {#} ∧ unsatisfiable (set-mset (init-clss T))
    ∨ conflicting T = None ∧ trail T ⊨asm init-clss T
    using cdcl_W-stgy-final-state-conclusive[of T] full
    unfolding cdcl_W-all-struct-inv-def cdcl_W-stgy-invariant-def full-def by fast
  moreover have consistent-interp (lits-of-l (trail T))
    using ‹cdcl_W-all-struct-inv T› unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def
    by auto
  moreover have init-clss S = init-clss T
    using inv unfolding cdcl_W-all-struct-inv-def
    by (metis rtranclp-cdcl_W-stgy-no-more-init-clss full full-def)
  ultimately show ?thesis
    by (metis satisfiable-carac' true-annot-def true-annots-def true-clss-def)
qed

lemma incremental-cdcl_W-inv:
  assumes
    inc: incremental-cdcl_W S T and
    inv: cdcl_W-all-struct-inv S and
    s-inv: cdcl_W-stgy-invariant S
  shows
    cdcl_W-all-struct-inv T and
    cdcl_W-stgy-invariant T
  using inc
proof (induction)
  case (add-confl C T)
  let ?T = (update-conflicting (Some C) (add-init-cls (cls-of-ccls C)
    (cut-trail-wrt-clause (mset-ccls C) (trail S) S)))
  have cdcl_W-all-struct-inv ?T and inv-s-T: cdcl_W-stgy-invariant ?T
    using add-confl.hyps(1,2,4) add-new-clause-and-update-def
    cdcl_W-all-struct-inv-add-new-clause-and-update-cdcl_W-all-struct-inv inv apply auto[1]
    using add-confl.hyps(1,2,4) add-new-clause-and-update-def
    cdcl_W-all-struct-inv-add-new-clause-and-update-cdcl_W-stgy-inv inv s-inv by auto
  case 1 show ?case
     by (metis add-confl.hyps(1,2,4,5) add-new-clause-and-update-def
       cdcl_W-all-struct-inv-add-new-clause-and-update-cdcl_W-all-struct-inv
       rtranclp-cdcl_W-all-struct-inv-inv rtranclp-cdcl_W-stgy-rtranclp-cdcl_W full-def inv)

  case 2  show ?case
    by (metis inv-s-T add-confl.hyps(1,2,4,5) add-new-clause-and-update-def
      cdcl_W-all-struct-inv-add-new-clause-and-update-cdcl_W-all-struct-inv full-def inv
      rtranclp-cdcl_W-stgy-cdcl_W-stgy-invariant)
next
  case (add-no-confl C T)
  case 1
  have cdcl_W-all-struct-inv (add-init-cls (cls-of-ccls C) S)
    using inv ‹distinct-mset (mset-ccls C)› unfolding cdcl_W-all-struct-inv-def no-strange-atm-def
    cdcl_W-M-level-inv-def distinct-cdcl_W-state-def cdcl_W-conflicting-def cdcl_W-learned-clause-def
    by (auto 9 1 simp: all-decomposition-implies-insert-single raw-clauses-def)

  then show ?case
```

```
    using add-no-confl(5) unfolding full-def by (auto intro: rtranclp-cdcl_W-stgy-cdcl_W-all-struct-inv)
  case 2
  have nc: ∀M. (∃K i M'. trail S = M' @ Marked K i # M) ⟶ ¬ M ⊨as CNot (mset-ccls C)
    using ‹¬ trail S ⊨as CNot (mset-ccls C)›
    by (auto simp: true-annots-true-cls-def-iff-negation-in-model)

  have cdcl_W-stgy-invariant (add-init-cls (cls-of-ccls C) S)
    using s-inv ‹¬ trail S ⊨as CNot (mset-ccls C)› inv unfolding cdcl_W-stgy-invariant-def
    no-smaller-confl-def eq-commute[of - trail -] cdcl_W-M-level-inv-def cdcl_W-all-struct-inv-def
    by (auto simp: raw-clauses-def nc)
  then show ?case
    by (metis ‹cdcl_W-all-struct-inv (add-init-cls (cls-of-ccls C) S)› add-no-confl.hyps(5) full-def
      rtranclp-cdcl_W-stgy-cdcl_W-stgy-invariant)
qed

lemma rtranclp-incremental-cdcl_W-inv:
  assumes
    inc: incremental-cdcl_W** S T and
    inv: cdcl_W-all-struct-inv S and
    s-inv: cdcl_W-stgy-invariant S
  shows
    cdcl_W-all-struct-inv T and
    cdcl_W-stgy-invariant T
     using inc apply induction
    using inv apply simp
   using s-inv apply simp
  using incremental-cdcl_W-inv by blast+

lemma incremental-conclusive-state:
  assumes
    inc: incremental-cdcl_W S T and
    inv: cdcl_W-all-struct-inv S and
    s-inv: cdcl_W-stgy-invariant S
  shows conflicting T = Some {#} ∧ unsatisfiable (set-mset (init-clss T))
    ∨ conflicting T = None ∧ trail T ⊨asm init-clss T ∧ satisfiable (set-mset (init-clss T))
  using inc
proof induction
  print-cases
  case (add-confl C T) note tr = this(1) and dist = this(2) and conf = this(3) and C = this(4) and
  full = this(5)

  have full cdcl_W-stgy T T
    using full unfolding full-def by auto
  then show ?case
    using full C conf dist tr
    by (metis full-cdcl_W-stgy-inv-normal-form incremental-cdcl_W.simps incremental-cdcl_W-inv(1)
      incremental-cdcl_W-inv(2) inv s-inv)
next
  case (add-no-confl C T) note tr = this(1) and dist = this(2) and conf = this(3) and C = this(4)
    and  full = this(5)

  have full cdcl_W-stgy T T
    using full unfolding full-def by auto
  then show ?case
     by (meson C conf dist full full-cdcl_W-stgy-inv-normal-form incremental-cdcl_W.add-no-confl
    incremental-cdcl_W-inv(1) incremental-cdcl_W-inv(2) inv s-inv tr)
qed

lemma tranclp-incremental-correct:
  assumes
    inc: incremental-cdcl_W^++ S T and
    inv: cdcl_W-all-struct-inv S and
    s-inv: cdcl_W-stgy-invariant S
  shows conflicting T = Some {#} ∧ unsatisfiable (set-mset (init-clss T))
    ∨ conflicting T = None ∧ trail T ⊨asm init-clss T ∧ satisfiable (set-mset (init-clss T))
  using inc apply induction
   using assms incremental-conclusive-state apply blast
  by (meson incremental-conclusive-state inv rtranclp-incremental-cdcl_W-inv s-inv
    tranclp-into-rtranclp)

end

end
```

24 2-Watched-Literal

```
theory CDCL-Two-Watched-Literals
imports CDCL-WNOT
begin
```

We will directly on the two-watched literals datastructure with lists: it could be also seen as a state over some abstract clause representation we would later refine as lists. However, as we need a way to select element from a clause, working on lists is better.

24.1 Datastructure and Access Functions

Only the 2-watched literals have to be verified here: the backtrack level and the trail that appear in the state are not related to the 2-watched algoritm.

```
datatype 'v twl-clause =
  TWL-Clause (watched: 'v literal list) (unwatched: 'v literal list)

datatype 'v twl-state =
  TWL-State (raw-trail: ('v, nat, 'v twl-clause) marked-lit list)
    (raw-init-clss: 'v twl-clause list)
    (raw-learned-clss: 'v twl-clause list) (backtrack-lvl: nat)
    (raw-conflicting: 'v literal list option)

fun mmset-of-mlit' :: ('v, nat, 'v twl-clause) marked-lit ⇒ ('v, nat, 'v clause) marked-lit
  where
mmset-of-mlit' (Propagated L C) = Propagated L (mset (watched C @ unwatched C)) |
mmset-of-mlit' (Marked L i) = Marked L i

lemma lit-of-mmset-of-mlit'[simp]: lit-of (mmset-of-mlit' x) = lit-of x
  by (cases x) auto

lemma lits-of-mmset-of-mlit'[simp]: lits-of (mmset-of-mlit' ` S) = lits-of S
  by (auto simp: lits-of-def image-image)

abbreviation trail where
```

```
trail S ≡ map mmset-of-mlit′ (raw-trail S)

abbreviation clauses-of-l where
  clauses-of-l ≡ λL. mset (map mset L)

definition raw-clause :: ′v twl-clause ⇒ ′v literal list where
  raw-clause C ≡ watched C @ unwatched C

abbreviation raw-clss :: ′v twl-state ⇒ ′v clauses where
  raw-clss S ≡ clauses-of-l (map raw-clause (raw-init-clss S @ raw-learned-clss S))

interpretation raw-cls
  λC. mset (raw-clause C)
  λL C. TWL-Clause (watched C) (L # unwatched C)
  λL C. TWL-Clause [] (remove1 L (raw-clause C))
  apply (unfold-locales)
  by (auto simp:hd-map comp-def map-tl ac-simps
    mset-map-mset-remove1-cond ex-mset raw-clause-def
    simp del: )

lemma XXX:
  mset (map (λx. mset (unwatched x) + mset (watched x))
    (remove1-cond (λD. mset (raw-clause D) = mset (raw-clause a)) Cs)) =
   remove1-mset (mset (raw-clause a)) (mset (map (λx. mset (raw-clause x)) Cs))
   apply (induction Cs)
     apply simp
   by (auto simp: ac-simps remove1-mset-single-add raw-clause-def)

interpretation raw-clss
  λC. mset (raw-clause C)

  λL C. TWL-Clause (watched C) (L # unwatched C)
  λL C. TWL-Clause [] (remove1 L (raw-clause C))
  λC. clauses-of-l (map raw-clause C) op @
  λL C. L ∈ set C op # λC. remove1-cond (λD. mset (raw-clause D) = mset (raw-clause C))
  apply (unfold-locales)
  using XXX by (auto simp:hd-map comp-def map-tl ac-simps raw-clause-def
    union-mset-list mset-map-mset-remove1-cond ex-mset
    simp del: )

lemma ex-mset-unwatched-watched:
  ∃ a. mset (unwatched a) + mset (watched a) = E
proof −
  obtain e where mset e = E
    using ex-mset by blast
  then have mset (unwatched (TWL-Clause [] e)) + mset (watched (TWL-Clause [] e)) = E
    by auto
  then show ?thesis by fast
qed

thm CDCL-Two-Watched-Literals.raw-cls-axioms
interpretation twl: state_W-ops
  λC. mset (raw-clause C)

  λL C. TWL-Clause (watched C) (L # unwatched C)
  λL C. TWL-Clause [] (remove1 L (raw-clause C))
  λC. clauses-of-l (map raw-clause C) op @
  λL C. L ∈ set C op # λC. remove1-cond (λD. mset (raw-clause D) = mset (raw-clause C))

  mset λxs ys. case-prod append (fold (λx (ys, zs). (remove1 x ys, x # zs)) xs (ys, []))
  op # remove1

  raw-clause λC. TWL-Clause [] C
  trail λS. hd (raw-trail S)
  raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting

  apply unfold-locales apply (auto simp: hd-map comp-def map-tl ac-simps raw-clause-def
    union-mset-list mset-map-mset-remove1-cond ex-mset-unwatched-watched)
  done
declare CDCL-Two-Watched-Literals.twl.mset-ccls-ccls-of-cls[simp del]

lemma mmset-of-mlit'-mmset-of-mlit[simp]:
  twl.mmset-of-mlit L = mmset-of-mlit' L
  by (metis mmset-of-mlit'.simps(1) mmset-of-mlit'.simps(2)  twl.mmset-of-mlit.elims raw-clause-def)

definition
  candidates-propagate :: 'v twl-state ⇒ ('v literal × 'v twl-clause) set
where
  candidates-propagate S =
   {(L, C) | L C.
     C ∈ set (twl.raw-clauses S)  ∧
     set (watched C) − (uminus ‘ lits-of-l (trail S)) = {L} ∧
     undefined-lit (trail S) L}

definition candidates-conflict :: 'v twl-state ⇒ 'v twl-clause set where
  candidates-conflict S =
   {C. C ∈ set (twl.raw-clauses S) ∧
     set (watched C) ⊆ uminus ‘ lits-of-l (trail S)}

primrec (nonexhaustive) index :: 'a list ⇒'a ⇒ nat where
index (a # l) c = (if a = c then 0 else 1+index l c)

lemma index-nth:
  a ∈ set l ⟹ l ! (index l a) = a
  by (induction l) auto
```

24.2 Invariants

We need the following property about updates: if there is a literal L with $-L$ in the trail, and L is not watched, then it stays unwatched; i.e., while updating with *rewatch* it does not get swap with a watched literal L' such that $-L'$ is in the trail.

```
primrec watched-decided-most-recently :: ('v, 'lvl, 'mark) marked-lit list ⇒
  'v twl-clause ⇒ bool
  where
watched-decided-most-recently M (TWL-Clause W UW) ⟷
  (∀ L'∈ set W. ∀ L∈ set UW.
    −L' ∈ lits-of-l M ⟶ −L ∈ lits-of-l M ⟶ L ∉# mset W ⟶
      index (map lit-of M) (−L') ≤ index (map lit-of M) (−L))
```

Here are the invariant strictly related to the 2-WL data structure.

```
primrec wf-twl-cls :: ('v, 'lvl, 'mark) marked-lit list ⇒ 'v twl-clause ⇒ bool where
  wf-twl-cls M (TWL-Clause W UW) ⟷
   distinct W ∧ length W ≤ 2 ∧ (length W < 2 ⟶ set UW ⊆ set W) ∧
   (∀ L ∈ set W. −L ∈ lits-of-l M ⟶ (∀ L' ∈ set UW. L' ∉ set W ⟶ −L' ∈ lits-of-l M)) ∧
   watched-decided-most-recently M (TWL-Clause W UW)

lemma size-mset-2: size x1 = 2 ⟷ (∃ a b. x1 = {#a, b#})
  apply (cases x1)
   apply simp
  by (metis (no-types, hide-lams) Suc-eq-plus1 one-add-one size-1-singleton-mset
  size-Diff-singleton size-Suc-Diff1 size-single union-single-eq-diff union-single-eq-member)

lemma distinct-mset-size-2: distinct-mset {#a, b#} ⟷ a ≠ b
  unfolding distinct-mset-def by auto

lemma wf-twl-cls-annotation-independant:
  assumes M: map lit-of M = map lit-of M'
  shows wf-twl-cls M (TWL-Clause W UW) ⟷ wf-twl-cls M' (TWL-Clause W UW)
proof −
  have lits-of-l M = lits-of-l M'
    using arg-cong[OF M, of set] by (simp add: lits-of-def)
  then show ?thesis
    by (simp add: lits-of-def M)
qed

lemma wf-twl-cls-wf-twl-cls-tl:
  assumes wf: wf-twl-cls M C and n-d: no-dup M
  shows wf-twl-cls (tl M) C
proof (cases M)
  case Nil
  then show ?thesis using wf
    by (cases C) (simp add: wf-twl-cls.simps[of tl -])
next
  case (Cons l M') note M = this(1)
  obtain W UW where C: C = TWL-Clause W UW
    by (cases C)
  { fix L L'
    assume
      LW: L ∈ set W and
      LM: − L ∈ lits-of-l M' and
      L'UW: L' ∈ set UW and
      L'∉ set W
    then have
      L'M: − L' ∈ lits-of-l M
      using wf by (auto simp: C M)
    have watched-decided-most-recently M C
      using wf by (auto simp: C)
    then have
      index (map lit-of M) (−L) ≤ index (map lit-of M) (−L')
      using LM L'M L'UW LW ‹L'∉ set W› C M unfolding lits-of-def
      by (fastforce simp: lits-of-def)
    then have − L' ∈ lits-of-l M'
      using ‹L'∉ set W› LW L'M by (auto simp: C M split: if-split-asm)
  }
  moreover
```

```
    {
      fix L′ L
      assume
        L′ ∈ set W and
        L ∈ set UW and
        L′M: − L′ ∈ lits-of-l M′ and
        − L ∈ lits-of-l M′ and
        L ∉ set W
      moreover
        have lit-of l ≠ − L′
        using n-d unfolding M
          by (metis (no-types) L′M M Marked-Propagated-in-iff-in-lits-of-l defined-lit-map
            distinct.simps(2) list.simps(9) set-map)
      moreover have watched-decided-most-recently M C
        using wf by (auto simp: C)
      ultimately have index (map lit-of M′) (− L′) ≤ index (map lit-of M′) (− L)
        by (fastforce simp: M C split: if-split-asm)
    }
  moreover have distinct W and length W ≤ 2 and (length W < 2 ⟶ set UW ⊆ set W)
    using wf by (auto simp: C M)
  ultimately show ?thesis by (auto simp add: M C)
qed

definition wf-twl-state :: ′v twl-state ⇒ bool where
  wf-twl-state S ⟷ (∀ C ∈ set (twl.raw-clauses S). wf-twl-cls (trail S) C) ∧ no-dup (trail S)

lemma wf-candidates-propagate-sound:
  assumes wf: wf-twl-state S and
    cand: (L, C) ∈ candidates-propagate S
  shows trail S ⊨as CNot (mset (removeAll L (raw-clause C))) ∧ undefined-lit (trail S) L
    (is ?Not ∧ ?undef)
proof
  def M ≡ trail S
  def N ≡ raw-init-clss S
  def U ≡ raw-learned-clss S

  note MNU-defs [simp] = M-def N-def U-def

  have cw:
    C ∈ set (N @ U)
    set (watched C) − uminus ‘ lits-of-l M = {L}
    undefined-lit M L
    using cand unfolding candidates-propagate-def MNU-defs twl.raw-clauses-def by auto

  obtain W UW where cw-eq: C = TWL-Clause W UW
    by (cases C)

  have l-w: L ∈ set W
    using cw(2) cw-eq by auto

  have wf-c: wf-twl-cls M C
    using wf cw(1) unfolding wf-twl-state-def by (simp add: twl.raw-clauses-def)

  have w-nw:
    distinct W
    length W < 2 ⟹ set UW ⊆ set W
    ⋀L L'. L ∈ set W ⟹ −L ∈ lits-of-l M ⟹ L' ∈ set UW ⟹ L' ∉ set W ⟹ −L' ∈ lits-of-l M
   using wf-c unfolding cw-eq by (auto simp: image-image)

  have ∀ L' ∈ set (raw-clause C) − {L}. −L' ∈ lits-of-l M
  proof (cases length W < 2)
    case True
    moreover have size W ≠ 0
      using cw(2) cw-eq by auto
    ultimately have size W = 1
      by linarith
    then have w: W = [L]
      using l-w by (auto simp: length-list-Suc-0)
    from True have set UW ⊆ set W
      using w-nw(2) by blast
    then show ?thesis
      using w cw(1) cw-eq by (auto simp: raw-clause-def)
  next
    case sz2: False
    show ?thesis
    proof
      fix L'
      assume l': L' ∈ set (raw-clause C) − {L}
      have ex-la: ∃ La. La ≠ L ∧ La ∈ set W
      proof (cases W)
        case w: Nil
        thus ?thesis
          using l-w by auto
      next
        case lb: (Cons Lb W')
        show ?thesis
        proof (cases W')
          case Nil
          thus ?thesis
            using lb sz2 by simp
        next
          case lc: (Cons Lc W'')
          thus ?thesis
            by (metis distinct-length-2-or-more lb list.set-intros(1) list.set-intros(2) w-nw(1))
        qed
      qed
      then obtain La where la: La ≠ L La ∈ set W
        by blast
      then have La ∈ uminus ‘ lits-of-l M
        using cw(2)[unfolded cw-eq, simplified, folded M-def] ‹La ∈ set W› ‹La ≠ L› by auto
      then have nla: −La ∈ lits-of-l M
        by (auto simp: image-image)
      then show −L' ∈ lits-of-l M

      proof −
        have f1: L' ∈ set (raw-clause C)
          using l' by blast
        have f2: L' ∉ {L}
          using l' by fastforce
        have ⋀l L. − (l::'a literal) ∈ L ∨ l ∉ uminus ‘ L
```

```
        by force
      then show ?thesis
        using cw(1) cw-eq w-nw(3) raw-clause-def by (metis DiffI Un-iff cw(2) f1 f2 la(2) nla
          set-append twl-clause.sel(1) twl-clause.sel(2))
    qed
  qed
 qed
 then show ?Not
   unfolding true-annots-def by (auto simp: image-image Ball-def CNot-def)

 show ?undef
   using cw(3) M-def by blast
qed

lemma wf-candidates-propagate-complete:
 assumes wf: wf-twl-state S and
   c-mem: C ∈ set (twl.raw-clauses S) and
   l-mem: L ∈ set (raw-clause C) and
   unsat: trail S ⊨as CNot (mset-set (set (raw-clause C) − {L})) and
   undef: undefined-lit (trail S) L
 shows (L, C) ∈ candidates-propagate S
proof −
 def M ≡ trail S
 def N ≡ raw-init-clss S
 def U ≡ raw-learned-clss S

 note MNU-defs [simp] = M-def N-def U-def

 obtain W UW where cw-eq: C = TWL-Clause W UW
   by (cases C, blast)

 have wf-c: wf-twl-cls M C
   using wf c-mem unfolding wf-twl-state-def by simp

 have w-nw:
   distinct W
   length W < 2 ⟹ set UW ⊆ set W
   ⋀L L'. L ∈ set W ⟹ −L ∈ lits-of-l M ⟹ L' ∈ set UW ⟹ L' ∉ set W ⟹ −L' ∈ lits-of-l M
  using wf-c unfolding cw-eq by (auto simp: image-image)

 have unit-set: set W −(uminus ‘ lits-of-l M) = {L} (is ?W = ?L)
 proof
   show ?W ⊆ {L}
   proof
     fix L'
     assume l': L' ∈ ?W
     hence l'-mem-w: L' ∈ set W
       by (simp add: in-diffD)
     have L' ∉ uminus ‘ lits-of-l M
       using l' by blast
     then have ¬ M ⊨a {#−L'#}
       by (auto simp: lits-of-def uminus-lit-swap image-image)
     moreover have L' ∈ set (raw-clause C)
       using c-mem cw-eq l'-mem-w by (auto simp: raw-clause-def)
     ultimately have L' = L
```

```
        using unsat[unfolded CNot-def true-annots-def, simplified]
        unfolding M-def by fastforce
      then show L′ ∈ {L}
        by simp
    qed
  next
    show {L} ⊆ ?W
    proof clarify
      have L ∈ set W
      proof (cases W)
        case Nil
        thus ?thesis
          using w-nw(2) cw-eq l-mem by (auto simp: raw-clause-def)
      next
        case (Cons La W′)
        thus ?thesis
        proof (cases La = L)
          case True
          thus ?thesis
            using Cons by simp
        next
          case False
          have −La ∈ lits-of-l M
            using False Cons cw-eq unsat[unfolded CNot-def true-annots-def, simplified]
            by (fastforce simp: raw-clause-def)
          then show ?thesis
            using Cons cw-eq l-mem undef w-nw(3)
            by (auto simp: Marked-Propagated-in-iff-in-lits-of-l raw-clause-def)
        qed
      qed
      moreover have L ∉# mset-set (uminus ‘ lits-of-l M)
        using undef by (auto simp: Marked-Propagated-in-iff-in-lits-of-l image-image)
      ultimately show L ∈ ?W
        by simp
    qed
  qed

  show ?thesis
    unfolding candidates-propagate-def using unit-set undef c-mem unfolding cw-eq M-def
    by (auto simp: image-image cw-eq intro!: exI[of - C])
qed

lemma wf-candidates-conflict-sound:
  assumes wf: wf-twl-state S and
    cand: C ∈ candidates-conflict S
  shows trail S ⊨as CNot (mset (raw-clause C)) ∧ C ∈ set (twl.raw-clauses S)
proof
  def M ≡ trail S
  def N ≡ raw-init-clss S
  def U ≡ raw-learned-clss S

  note MNU-defs [simp] = M-def N-def U-def

  have cw:
    C ∈ set (N @ U)
    set (watched C) ⊆ uminus ‘ lits-of-l (trail S)
    using cand[unfolded candidates-conflict-def, simplified] unfolding twl.raw-clauses-def by auto

  obtain W UW where cw-eq: C = TWL-Clause W UW
    by (cases C, blast)

  have wf-c: wf-twl-cls M C
    using wf cw(1) unfolding wf-twl-state-def by (simp add: comp-def twl.raw-clauses-def)

  have w-nw:
    distinct W
    length W < 2 ⟹ set UW ⊆ set W
    ⋀L L'. L ∈ set W ⟹ −L ∈ lits-of-l M ⟹ L' ∈ set UW ⟹ L' ∉ set W ⟹ −L' ∈ lits-of-l M
   using wf-c unfolding cw-eq by (auto simp: image-image)

  have ∀L ∈ set (raw-clause C). −L ∈ lits-of-l M
  proof (cases W)
    case Nil
    then have raw-clause C = []
      using cw(1) cw-eq w-nw(2) by (auto simp: raw-clause-def)
    then show ?thesis
      by simp
  next
    case (Cons La W') note W' = this(1)
    show ?thesis
    proof
      fix L
      assume l: L ∈ set (raw-clause C)
      show −L ∈ lits-of-l M
      proof (cases L ∈ set W)
        case True
        thus ?thesis
          using cw(2) cw-eq by fastforce
      next
        case False
        thus ?thesis
          by (metis (no-types, hide-lams) M-def UnE W' contra-subsetD cw(2) cw-eq imageE
            insertI1 l list.set(2) set-append twl-clause.sel(1) twl-clause.sel(2)
            uminus-of-uminus-id w-nw(3) raw-clause-def)
      qed
    qed
  qed
  then show trail S ⊨as CNot (mset (raw-clause C))
    unfolding CNot-def true-annots-def by auto

  show C ∈ set (twl.raw-clauses S)
    using cw unfolding twl.raw-clauses-def by auto
qed

lemma wf-candidates-conflict-complete:
  assumes wf: wf-twl-state S and
    c-mem: C ∈ set (twl.raw-clauses S) and
    unsat: trail S ⊨as CNot (mset (raw-clause C))
  shows C ∈ candidates-conflict S
proof −
```

```
  def M ≡ trail S
  def N ≡ twl.init-clss S
  def U ≡ twl.learned-clss S

  note MNU-defs [simp] = M-def N-def U-def

  obtain W UW where cw-eq: C = TWL-Clause W UW
    by (cases C, blast)

  have wf-c: wf-twl-cls M C
    using wf c-mem unfolding wf-twl-state-def by simp

  have w-nw:
    distinct W
    length W < 2 ⟹ set UW ⊆ set W
    ⋀L L'. L ∈ set W ⟹ −L ∈ lits-of-l M ⟹ L' ∈ set UW ⟹ L' ∉ set W ⟹ −L' ∈ lits-of-l M
   using wf-c unfolding cw-eq by (auto simp: image-image)

  have ⋀L. L ∈ set (raw-clause C) ⟹ −L ∈ lits-of-l M
    unfolding M-def using unsat[unfolded CNot-def true-annots-def, simplified] by auto
  then have set (raw-clause C) ⊆ uminus ‘ lits-of-l M
    by (metis imageI subsetI uminus-of-uminus-id)
  then have set W ⊆ uminus ‘ lits-of-l M
    using cw-eq by (auto simp: raw-clause-def)
  then have subset: set W ⊆ uminus ‘ lits-of-l M
    by (simp add: w-nw(1))

  have W = watched C
    using cw-eq twl-clause.sel(1) by simp
  then show ?thesis
    using MNU-defs c-mem subset candidates-conflict-def by blast
qed

typedef 'v wf-twl = {S::'v twl-state. wf-twl-state S}
morphisms rough-state-of-twl twl-of-rough-state
proof −
  have TWL-State ([]::('v, nat, 'v twl-clause) marked-lits)
    [] [] 0 None ∈ {S:: 'v twl-state. wf-twl-state S}
    by (auto simp: wf-twl-state-def twl.raw-clauses-def)
  then show ?thesis by auto
qed

lemma [code abstype]:
  twl-of-rough-state (rough-state-of-twl S) = S
  by (fact CDCL-Two-Watched-Literals.wf-twl.rough-state-of-twl-inverse)

lemma wf-twl-state-rough-state-of-twl[simp]: wf-twl-state (rough-state-of-twl S)
  using rough-state-of-twl by auto

abbreviation candidates-conflict-twl :: 'v wf-twl ⇒ 'v twl-clause set where
candidates-conflict-twl S ≡ candidates-conflict (rough-state-of-twl S)

abbreviation candidates-propagate-twl :: 'v wf-twl ⇒ ('v literal × 'v twl-clause) set where
candidates-propagate-twl S ≡ candidates-propagate (rough-state-of-twl S)
```

abbreviation *raw-trail-twl* :: *'a wf-twl* ⇒ (*'a*, *nat*, *'a twl-clause*) *marked-lit list* **where**
raw-trail-twl S ≡ *raw-trail* (*rough-state-of-twl S*)

abbreviation *trail-twl* :: *'a wf-twl* ⇒ (*'a*, *nat*, *'a literal multiset*) *marked-lit list* **where**
trail-twl S ≡ *trail* (*rough-state-of-twl S*)

abbreviation *raw-clauses-twl* :: *'a wf-twl* ⇒ *'a twl-clause list* **where**
raw-clauses-twl S ≡ *twl.raw-clauses* (*rough-state-of-twl S*)

abbreviation *raw-init-clss-twl* :: *'a wf-twl* ⇒ *'a twl-clause list* **where**
raw-init-clss-twl S ≡ *raw-init-clss* (*rough-state-of-twl S*)

abbreviation *raw-learned-clss-twl* :: *'a wf-twl* ⇒ *'a twl-clause list* **where**
raw-learned-clss-twl S ≡ *raw-learned-clss* (*rough-state-of-twl S*)

abbreviation *backtrack-lvl-twl* **where**
backtrack-lvl-twl S ≡ *backtrack-lvl* (*rough-state-of-twl S*)

abbreviation *raw-conflicting-twl* **where**
raw-conflicting-twl S ≡ *raw-conflicting* (*rough-state-of-twl S*)

lemma *wf-candidates-twl-conflict-complete*:
  assumes
    c-mem: *C* ∈ *set* (*raw-clauses-twl S*) **and**
    unsat: *trail-twl S* ⊨*as CNot* (*mset* (*raw-clause C*))
  shows *C* ∈ *candidates-conflict-twl S*
  using *c-mem unsat wf-candidates-conflict-complete wf-twl-state-rough-state-of-twl* **by** *blast*

abbreviation *update-backtrack-lvl* **where**
  update-backtrack-lvl k S ≡
   TWL-State (*raw-trail S*) (*raw-init-clss S*) (*raw-learned-clss S*) *k* (*raw-conflicting S*)

abbreviation *update-conflicting* **where**
  update-conflicting C S ≡
    TWL-State (*raw-trail S*) (*raw-init-clss S*) (*raw-learned-clss S*) (*backtrack-lvl S*) *C*

24.3 Abstract 2-WL

definition *tl-trail* **where**
  tl-trail S =
   TWL-State (*tl* (*raw-trail S*)) (*raw-init-clss S*) (*raw-learned-clss S*) (*backtrack-lvl S*)
   (*raw-conflicting S*)

locale *abstract-twl* =
  fixes
    watch :: *'v twl-state* ⇒ *'v literal list* ⇒ *'v twl-clause* **and**
    rewatch :: *'v literal* ⇒ *'v twl-state* ⇒
      'v twl-clause ⇒ *'v twl-clause* **and**
    restart-learned :: *'v twl-state* ⇒ *'v twl-clause list*
  assumes
    clause-watch: *no-dup* (*trail S*) ⟹ *mset* (*raw-clause* (*watch S C*)) = *mset C* **and**
    wf-watch: *no-dup* (*trail S*) ⟹ *wf-twl-cls* (*trail S*) (*watch S C*) **and**
    clause-rewatch: *mset* (*raw-clause* (*rewatch L' S C'*)) = *mset* (*raw-clause C'*) **and**
    wf-rewatch:
      no-dup (*trail S*) ⟹ *undefined-lit* (*trail S*) (*lit-of L*) ⟹ *wf-twl-cls* (*trail S*) *C'* ⟹
        wf-twl-cls (*L* # *trail S*) (*rewatch* (*lit-of L*) *S C'*)

```
    and
   restart-learned: mset (restart-learned S) ⊆# mset (raw-learned-clss S) — We need mset and not set
to take care of duplicates.
begin

definition
  cons-trail :: ('v, nat, 'v twl-clause) marked-lit ⇒ 'v twl-state ⇒ 'v twl-state
where
  cons-trail L S =
   TWL-State (L # raw-trail S) (map (rewatch (lit-of L) S) (raw-init-clss S))
     (map (rewatch (lit-of L) S) (raw-learned-clss S)) (backtrack-lvl S) (raw-conflicting S)

definition
  add-init-cls :: 'v literal list ⇒ 'v twl-state ⇒ 'v twl-state
where
  add-init-cls C S =
   TWL-State (raw-trail S) (watch S C # raw-init-clss S) (raw-learned-clss S) (backtrack-lvl S)
     (raw-conflicting S)

definition
  add-learned-cls :: 'v literal list ⇒ 'v twl-state ⇒ 'v twl-state
where
  add-learned-cls C S =
   TWL-State (raw-trail S) (raw-init-clss S) (watch S C # raw-learned-clss S) (backtrack-lvl S)
     (raw-conflicting S)

definition
  remove-cls :: 'v literal list ⇒ 'v twl-state ⇒ 'v twl-state
where
  remove-cls C S =
   TWL-State (raw-trail S)
     (removeAll-cond (λD. mset (raw-clause D) = mset C) (raw-init-clss S))
     (removeAll-cond (λD. mset (raw-clause D) = mset C) (raw-learned-clss S))
     (backtrack-lvl S)
     (raw-conflicting S)

definition init-state :: 'v literal list list ⇒ 'v twl-state where
  init-state N = fold add-init-cls N (TWL-State [] [] [] 0 None)

lemma unchanged-fold-add-init-cls:
  raw-trail (fold add-init-cls Cs (TWL-State M N U k C)) = M
  raw-learned-clss (fold add-init-cls Cs (TWL-State M N U k C)) = U
  backtrack-lvl (fold add-init-cls Cs (TWL-State M N U k C)) = k
  raw-conflicting (fold add-init-cls Cs (TWL-State M N U k C)) = C
  by (induct Cs arbitrary: N) (auto simp: add-init-cls-def)

lemma unchanged-init-state[simp]:
  raw-trail (init-state N) = []
  raw-learned-clss (init-state N) = []
  backtrack-lvl (init-state N) = 0
  raw-conflicting (init-state N) = None
  unfolding init-state-def by (rule unchanged-fold-add-init-cls)+

lemma clauses-init-fold-add-init:
  no-dup M ⟹
   twl.init-clss (fold add-init-cls Cs (TWL-State M N U k C)) =
   clauses-of-l Cs + clauses-of-l (map raw-clause N)
  by (induct Cs arbitrary: N) (auto simp: add-init-cls-def clause-watch comp-def ac-simps)

lemma init-clss-init-state[simp]: twl.init-clss (init-state N) = clauses-of-l N
  unfolding init-state-def by (subst clauses-init-fold-add-init) simp-all

definition restart' where
  restart' S = TWL-State [] (raw-init-clss S) (restart-learned S) 0 None

end
```

24.4 Instanciation of the previous locale

```
definition watch-nat :: 'v twl-state ⇒ 'v literal list ⇒ 'v twl-clause where
 watch-nat S C =
  (let
    C' = remdups C;
    neg-not-assigned = filter (λL. −L ∉ lits-of-l (raw-trail S)) C';
    neg-assigned-sorted-by-trail = filter (λL. L ∈ set C) (map (λL. −lit-of L) (raw-trail S));
    W = take 2 (neg-not-assigned @ neg-assigned-sorted-by-trail);
    UW = foldr remove1 W C
   in TWL-Clause W UW)

lemma list-cases2:
 fixes l :: 'a list
 assumes
   l = [] ⟹ P and
   ⋀x. l = [x] ⟹ P and
   ⋀x y xs. l = x # y # xs ⟹ P
 shows P
 by (metis assms list.collapse)

lemma filter-in-list-prop-verifiedD:
 assumes [L←P . Q L] = l
 shows ∀x ∈ set l. x ∈ set P ∧ Q x
 using assms by auto

lemma no-dup-filter-diff:
 assumes n-d: no-dup M and H: [L←map (λL. − lit-of L) M. L ∈ set C] = l
 shows distinct l
 unfolding H[symmetric]
 apply (rule distinct-filter)
 using n-d by (induction M) auto

lemma watch-nat-lists-disjointD:
 assumes
   l: [L←remdups C. − L ∉ lits-of-l (raw-trail S)] = l and
   l': [L←map (λL. − lit-of L) (raw-trail S) . L ∈ set C] = l'
 shows ∀x ∈ set l. ∀y ∈ set l'. x ≠ y
 by (auto simp: l[symmetric] l'[symmetric] lits-of-def image-image)

lemma watch-nat-list-cases-witness[consumes 2, case-names nil-nil nil-single nil-other
 single-nil single-other other]:
 fixes
   C :: 'v literal list and
   S :: 'v twl-state
 defines
   xs ≡ [L←remdups C. − L ∉ lits-of-l (raw-trail S)] and
   ys ≡ [L←map (λL. − lit-of L) (raw-trail S) . L ∈ set C]
 assumes
   n-d: no-dup (raw-trail S) and
   nil-nil: xs = [] ⟹ ys = [] ⟹ P and
   nil-single:
    ⋀a. xs = [] ⟹ ys = [a] ⟹ a ∈ set C ⟹ P and
   nil-other: ⋀a b ys'. xs = [] ⟹ ys = a # b # ys' ⟹ a ≠ b ⟹ P and
   single-nil: ⋀a. xs = [a] ⟹ ys = [] ⟹ P and
   single-other: ⋀a b ys'. xs = [a] ⟹ ys = b # ys' ⟹ a ≠ b ⟹ P and
   other: ⋀a b xs'. xs = a # b # xs' ⟹ a ≠ b ⟹ P
 shows P
proof −
 note xs-def[simp] and ys-def[simp]
 have dist: ⋀P. distinct [L←remdups C . P L]
   by auto
 then have H: ⋀a b P xs. [L←remdups C . P L] = a # b # xs ⟹ a ≠ b
   by (metis distinct-length-2-or-more)
 show ?thesis
 apply (cases [L←remdups C. − L ∉ lits-of-l (raw-trail S)]
      rule: list-cases2;
    cases [L←map (λL. − lit-of L) (raw-trail S) . L ∈ set C] rule: list-cases2)
        using nil-nil apply simp
       using nil-single apply (force dest: filter-in-list-prop-verifiedD)
      using nil-other no-dup-filter-diff[OF n-d, of C]
      apply fastforce
     using single-nil apply simp
    using single-other xs-def ys-def apply (metis list.set-intros(1) watch-nat-lists-disjointD)
    using single-other unfolding xs-def ys-def apply (metis list.set-intros(1)
     watch-nat-lists-disjointD)
   using other xs-def ys-def by (metis H)+
qed

lemma watch-nat-list-cases [consumes 1, case-names nil-nil nil-single nil-other single-nil
 single-other other]:
 fixes
   C :: 'v literal list and
   S :: 'v twl-state
 defines
   xs ≡ [L←remdups C . − L ∉ lits-of-l (raw-trail S)] and
   ys ≡ [L←map (λL. − lit-of L) (raw-trail S) . L ∈ set C]
 assumes
   n-d: no-dup (raw-trail S) and
   nil-nil: xs = [] ⟹ ys = [] ⟹ P and
   nil-single:
    ⋀a. xs = [] ⟹ ys = [a] ⟹  a ∈ set C ⟹ P and
   nil-other: ⋀a b ys'. xs = [] ⟹ ys = a # b # ys' ⟹ a ≠ b ⟹ P and
   single-nil: ⋀a. xs = [a] ⟹ ys = [] ⟹ P and
   single-other: ⋀a b ys'. xs = [a] ⟹ ys = b # ys' ⟹ a ≠ b ⟹ P and
   other: ⋀a b xs'. xs = a # b # xs' ⟹ a ≠ b ⟹ P
 shows P
 using watch-nat-list-cases-witness[OF n-d, of C P]
 nil-nil nil-single nil-other single-nil single-other other
```

```
  unfolding xs-def[symmetric] ys-def[symmetric] by auto

lemma watch-nat-lists-set-union-witness:
  fixes
    C :: 'v literal list and
    S :: 'v twl-state
  defines
    xs ≡ [L←remdups C. − L ∉ lits-of-l (raw-trail S)] and
    ys ≡ [L←map (λL. − lit-of L) (raw-trail S) . L ∈ set C]
  assumes n-d: no-dup (raw-trail S)
  shows set C = set xs ∪ set ys
  using n-d unfolding xs-def ys-def by (auto simp: lits-of-def comp-def uminus-lit-swap)

lemma mset-intersection-inclusion: A + (B − A) = B ⟷ A ⊆# B
  apply (rule iffI)
   apply (metis mset-le-add-left)
  by (auto simp: ac-simps multiset-eq-iff subseteq-mset-def)

lemma clause-watch-nat:
  assumes no-dup (raw-trail S)
  shows mset (raw-clause (watch-nat S C)) = mset C
  using assms
  apply (cases rule: watch-nat-list-cases[OF assms(1), of C])
  by (auto dest: filter-in-list-prop-verifiedD simp: watch-nat-def multiset-eq-iff raw-clause-def)

lemma index-uminus-index-map-uminus:
  −a ∈ set L ⟹ index L (−a) = index (map uminus L) (a::'a literal)
  by (induction L) auto

lemma index-filter:
  a ∈ set L ⟹ b ∈ set L ⟹ P a ⟹ P b ⟹
   index L a ≤ index L b ⟷ index (filter P L) a ≤ index (filter P L) b
  by (induction L) auto

lemma foldr-remove1-W-Nil[simp]: foldr remove1 W [] = []
  by (induct W) auto

lemma image-lit-of-mmset-of-mlit'[simp]:
  lit-of ` mmset-of-mlit' ` A = lit-of ` A
  by (auto simp: image-image comp-def)

lemma distinct-filter-eq:
  assumes distinct xs
  shows [L← xs. L = a] = (if a ∈ set xs then [a] else [])
  using assms by (induction xs) auto

lemma no-dup-distinct-map-uminus-lit-of:
  no-dup xs ⟹ distinct (map (λL. − lit-of L) xs)
  by (induction xs) auto

lemma wf-watch-witness:
   fixes C :: 'v literal list and
     S :: 'v twl-state
   defines
     ass: neg-not-assigned ≡ filter (λL. −L ∉ lits-of-l (raw-trail S)) (remdups C) and
     tr: neg-assigned-sorted-by-trail ≡ filter (λL. L ∈ set C) (map (λL. −lit-of L) (raw-trail S))
    defines
      W: W ≡ take 2 (neg-not-assigned @ neg-assigned-sorted-by-trail)
  assumes
    n-d[simp]: no-dup (raw-trail S)
  shows wf-twl-cls (trail S) (TWL-Clause W (foldr remove1 W C))
  unfolding wf-twl-cls.simps
proof (intro conjI, goal-cases)
  case 1
  then show ?case using n-d W unfolding ass tr
    apply (cases rule: watch-nat-list-cases-witness[of S C, OF n-d])
    by (auto simp: distinct-mset-add-single)
next
  case 2
  then show ?case unfolding W by simp
next
  case 3
  show ?case using n-d
    proof (cases rule: watch-nat-list-cases-witness[of S C])
      case nil-nil
      then have set C = set [] ∪ set []
        using watch-nat-lists-set-union-witness n-d by metis
      then show ?thesis
        by simp
    next
      case (nil-single a)
      moreover have ⋀x. set C = {a} ⟹ − a ∈ lits-of-l (trail S) ⟹ x ∈ set (remove1 a C) ⟹
        x = a
        using notin-set-remove1 by auto
      ultimately show ?thesis
        using watch-nat-lists-set-union-witness[of S C] 3 by (auto simp: W ass tr comp-def)
    next
      case nil-other
      then show ?thesis
       using 3 by (auto simp: W ass tr)
    next
      case (single-nil a)
      show ?thesis
        using watch-nat-lists-set-union-witness[of S C] 3
        by (fastforce simp add: W ass tr single-nil comp-def distinct-filter-eq
          no-dup-distinct-map-uminus-lit-of min-def)
    next
      case single-other
      then show ?thesis
        using 3 by (auto simp: W ass tr)
    next
      case other
      then show ?thesis
        using 3 by (auto simp: W ass tr)
    qed
next
  case 4 note -[simp] = this
  show ?case
    using n-d apply (cases rule: watch-nat-list-cases-witness[of S C])
      apply (auto dest: filter-in-list-prop-verifiedD
        simp: W ass tr lits-of-def  filter-empty-conv)[4]
    using watch-nat-lists-set-union-witness[of S C]
    by (force dest: filter-in-list-prop-verifiedD simp: W ass tr lits-of-def)+
next
  case 5
  from n-d show ?case
    proof (cases rule: watch-nat-list-cases-witness[of S C])
      case nil-nil
      then show ?thesis by (auto simp:  W ass tr)
    next
      case nil-single
      then show ?thesis
        using watch-nat-lists-set-union-witness[of S C] tr by (fastforce simp: W ass)
    next
      case nil-other
      then show ?thesis
        unfolding watched-decided-most-recently.simps Ball-def
        apply (intro allI impI)
        apply (subst index-uminus-index-map-uminus,
          simp add: index-uminus-index-map-uminus lits-of-def o-def)
        apply (subst index-uminus-index-map-uminus,
          simp add: index-uminus-index-map-uminus lits-of-def o-def)

        apply (subst index-filter[of - - - λL. L ∈ set C])
        by (auto dest: filter-in-list-prop-verifiedD
          simp: uminus-lit-swap lits-of-def o-def W ass tr dest: in-diffD)
    next
      case single-nil
      then show ?thesis
         using watch-nat-lists-set-union-witness[of S C] tr by (fastforce simp: W ass)
    next
      case single-other
      then show ?thesis
        unfolding watched-decided-most-recently.simps Ball-def
        apply (clarify)
        apply (subst index-uminus-index-map-uminus,
          simp add: index-uminus-index-map-uminus lits-of-def image-image o-def)
        apply (subst index-uminus-index-map-uminus,
          simp add: index-uminus-index-map-uminus lits-of-def o-def)

        apply (subst index-filter[of - - - λL. L ∈ set C])
        by (auto dest: filter-in-list-prop-verifiedD
          simp: W ass tr uminus-lit-swap lits-of-def o-def dest: in-diffD)
    next
      case other
      then show ?thesis
        unfolding watched-decided-most-recently.simps
        apply clarify
        apply (subst index-uminus-index-map-uminus,
          simp add: index-uminus-index-map-uminus lits-of-def o-def)[1]
        apply (subst index-uminus-index-map-uminus,
          simp add: index-uminus-index-map-uminus lits-of-def o-def)[1]

        apply (subst index-filter[of - - - λL. L ∈ set C])
        by (auto dest: filter-in-list-prop-verifiedD
        simp: index-uminus-index-map-uminus lits-of-def o-def uminus-lit-swap
         W ass tr)
  qed
qed

lemma wf-watch-nat: no-dup (raw-trail S) ⟹ wf-twl-cls (trail S) (watch-nat S C)
  using wf-watch-witness[of S C] watch-nat-def by metis

definition
  rewatch-nat ::
  'v literal ⇒ 'v twl-state ⇒ 'v twl-clause ⇒ 'v twl-clause
where
  rewatch-nat L S C =
   (if − L ∈ set (watched C) then
      case filter (λL'. L' ∉ set (watched C) ∧ − L' ∉ insert L (lits-of-l (trail S)))
         (unwatched C) of
        [] ⇒ C
      | L' # - ⇒
         TWL-Clause (L' # remove1 (−L) (watched C)) (−L # remove1 L' (unwatched C))
    else
      C)

lemma clause-rewatch-nat:
  fixes UW :: 'v literal list and
    S :: 'v twl-state and
    L :: 'v literal and C :: 'v twl-clause
  shows mset (raw-clause (rewatch-nat L S C)) = mset (raw-clause C)
  using List.set-remove1-subset[of −L watched C]
  apply (cases C)
  by (auto simp: raw-clause-def rewatch-nat-def ac-simps multiset-eq-iff
    split: list.split
    dest: filter-in-list-prop-verifiedD)

lemma filter-sorted-list-of-multiset-Nil:
  [x ← sorted-list-of-multiset M. p x] = [] ⟷ (∀ x ∈# M. ¬ p x)
  by auto (metis empty-iff filter-set list.set(1) member-filter set-sorted-list-of-multiset)

lemma filter-sorted-list-of-multiset-ConsD:
  [x ← sorted-list-of-multiset M. p x] = x # xs ⟹ p x
  by (metis filter-set insert-iff list.set(2) member-filter)

lemma mset-minus-single-eq-mempty:
  a − {#b#} = {#} ⟷ a = {#b#} ∨ a = {#}
  by (metis Multiset.diff-cancel add.right-neutral diff-single-eq-union
    diff-single-trivial zero-diff)

lemma size-mset-le-2-cases:
  assumes size W ≤ 2
  shows W = {#} ∨ (∃ a. W = {#a#}) ∨ (∃ a b. W = {#a,b#})
  by (metis One-nat-def Suc-1 Suc-eq-plus1-left assms linorder-not-less nat-less-le
    not-less-eq-eq le-iff-add size-1-singleton-mset
    size-eq-0-iff-empty size-mset-2)

lemma filter-sorted-list-of-multiset-eqD:
  assumes [x ← sorted-list-of-multiset A. p x] = x # xs (is ?comp = -)
  shows x ∈# A
proof −
  have x ∈ set ?comp
    using assms by simp
  then have x ∈ set (sorted-list-of-multiset A)
    by simp
  then show x ∈# A
    by simp
qed

lemma clause-rewatch-witness':
  assumes
    wf: wf-twl-cls (trail S) C and
    n-d: no-dup (raw-trail S) and
    undef: undefined-lit (trail S) (lit-of L)
  shows wf-twl-cls (L # trail S) (rewatch-nat (lit-of L) S C)
proof (cases − lit-of L ∈ set (watched C))
  case False
  then show ?thesis
    apply (cases C)
    using wf n-d undef unfolding rewatch-nat-def
    by (auto simp: uminus-lit-swap Marked-Propagated-in-iff-in-lits-of-l comp-def)
next
  case falsified: True

  let ?unwatched-nonfalsified =
    [L'← unwatched C. L' ∉ set (watched C) ∧ − L' ∉ insert (lit-of L) (lits-of-l (trail S))]
  obtain W UW where C: C = TWL-Clause W UW
    by (cases C)

  show ?thesis
  proof (cases ?unwatched-nonfalsified)
    case Nil
    show ?thesis
      using falsified Nil
      apply (simp only: wf-twl-cls.simps if-True list.cases C rewatch-nat-def)
      apply (intro conjI)
      proof goal-cases
        case 1
        then show ?case using wf C by simp
      next
        case 2
        then show ?case using wf C by simp
      next
        case 3
        then show ?case using wf C by simp
      next
        case 4
        have ⋀p l. filter p (unwatched C) ≠ [] ∨ l ∉ set UW ∨ ¬ p l
          unfolding C by (metis (no-types) filter-empty-conv twl-clause.sel(2))
        then show ?case
          using 4(2) C by auto
      next
        case 5
        then show ?case
          using wf by (fastforce simp add: C comp-def uminus-lit-swap)
      qed
  next
    case (Cons L' Ls)
    show ?thesis
      unfolding rewatch-nat-def
      using falsified Cons
      apply (simp only: wf-twl-cls.simps if-True list.cases C)
      apply (intro conjI)
      proof goal-cases
        case 1
        have distinct (watched (TWL-Clause W UW))
          using wf unfolding C by auto
        moreover have L' ∉ set (remove1 (−lit-of L) (watched (TWL-Clause W UW)))
          using 1(2) not-gr0 by (fastforce dest: filter-in-list-prop-verifiedD in-diffD)
        ultimately show ?case
          by (auto simp: distinct-mset-single-add)
      next
        case 2
        have f2: [l←unwatched (TWL-Clause W UW) . l ∉ set (watched (TWL-Clause W UW))
          ∧ − l ∉ insert (lit-of L) (lits-of-l (trail S))] ≠ []
          using 2(2) by simp
        then have ¬ set UW ⊆ set W
           using 2 by (auto simp add: filter-empty-conv)
        then show ?case
          using wf C 2(1) by (auto simp: length-remove1)
      next
        case 3
        have W: length W ≤ Suc 0 ⟷ length W = 0 ∨ length W = Suc 0
          by linarith
        show ?case
          using wf C 3 by (auto simp: length-remove1 W length-list-Suc-0 dest!: subset-singletonD)
      next
        case 4
        have H: ∀ L∈ set W. − L ∈ lits-of-l (trail S) ⟶
          (∀ L'∈ set UW. L' ∉ set W ⟶ − L' ∈ lits-of-l (trail S))
          using wf by (auto simp: C)
        have W: length W ≤ 2 and W-UW: length W < 2 ⟶ set UW ⊆ set W
          using wf by (auto simp: C)
        have distinct: distinct W
          using wf by (auto simp: C)
        show ?case
          using 4
          unfolding C watched-decided-most-recently.simps Ball-def twl-clause.sel
          apply (intro allI impI)
          apply (rename-tac xW xUW)
          apply (case-tac − lit-of L = xW; case-tac xW = xUW; case-tac L' = xW)
                  apply (auto simp: uminus-lit-swap)[2]
                apply (force dest: filter-in-list-prop-verifiedD)
               using H distinct apply (fastforce split: if-split-asm)
             using distinct apply (fastforce split: if-split-asm)
            using distinct apply (fastforce split: if-split-asm)
           apply (force dest: filter-in-list-prop-verifiedD)
          using H by (auto simp: uminus-lit-swap)
      next
```

```
        case 5
        have H: ∀x. x ∈ set W ⟶ − x ∈ lits-of-l (trail S) ⟶ (∀x. x ∈ set UW ⟶ x ∉ set W
          ⟶ − x ∈ lits-of-l (trail S))
          using wf by (auto simp: C)
        show ?case
          unfolding C watched-decided-most-recently.simps Ball-def
          proof (intro allI impI conjI, goal-cases)
            case (1 xW x)
            show ?case
              proof (cases − lit-of L = xW)
                case True
                then show ?thesis
                  by (cases xW = x) (auto simp: uminus-lit-swap)
              next
                case False note LxW = this
                have f9: L′ ∈ set [l←unwatched C. l ∉ set (watched (TWL-Clause W UW))
                    ∧ − l ∉ lits-of-l (L # trail S)]
                  using 1(2) 5 C by auto
                moreover then have f11: − xW ∈ lits-of-l (trail S)
                  using 1(3) LxW by (auto simp: uminus-lit-swap)
                moreover then have xW ∉ set W
                  using f9 1(2) H by (auto simp: C)
                ultimately have False
                  using 1 by auto
                then show ?thesis
                  by fast
              qed
          qed
      qed
  qed
qed


interpretation twl: abstract-twl watch-nat rewatch-nat raw-learned-clss
  apply unfold-locales
  apply (rule clause-watch-nat; simp add: image-image comp-def)
  apply (rule wf-watch-nat; simp add: image-image comp-def)
  apply (rule clause-rewatch-nat)
  apply (rule clause-rewatch-witness′; simp add: image-image comp-def)
  apply (simp)
  done

interpretation twl2: abstract-twl watch-nat rewatch-nat λ-. []
  apply unfold-locales
  apply (rule clause-watch-nat; simp add: image-image comp-def)
  apply (rule wf-watch-nat; simp add: image-image comp-def)
  apply (rule clause-rewatch-nat)
  apply (rule clause-rewatch-witness′; simp add: image-image comp-def)
  apply (simp)
  done

end
```

25 Invariants for 2 Watched-Literals

```
theory CDCL-Two-Watched-Literals-Invariant
imports CDCL-Two-Watched-Literals DPLL-CDCL-W-Implementation
begin
```

25.1 Interpretation for *conflict-driven-clause-learning*$_W$.*cdcl*$_W$

We define here the 2-WL with the invariant and show the role of the candidates.

```
context abstract-twl
begin
```

25.1.1 Direct Interpretation

```
lemma mset-map-removeAll-cond:
  mset (map (λx. mset (raw-clause x))
    (removeAll-cond (λD. mset (raw-clause D) = mset (raw-clause C)) N))
  = mset (removeAll (mset (raw-clause C)) (map (λx. mset (raw-clause x)) N))
  by (induction N) auto

lemma mset-raw-init-clss-init-state:
  mset (map (λx. mset (raw-clause x)) (raw-init-clss (init-state (map raw-clause N))))
  = mset (map (λx. mset (raw-clause x)) N)
  by (metis (no-types, lifting) init-clss-init-state map-eq-conv map-map o-def)

interpretation rough-cdcl: state_W
  λC. mset (raw-clause C)

  λL C. TWL-Clause (watched C) (L # unwatched C)
  λL C. TWL-Clause [] (remove1 L (raw-clause C))
  λC. clauses-of-l (map raw-clause C) op @
  λL C. L ∈ set C op # λC. remove1-cond (λD. mset (raw-clause D) = mset (raw-clause C))

  mset λxs ys. case-prod append (fold (λx (ys, zs). (remove1 x ys, x # zs)) xs (ys, []))
  op # remove1

  raw-clause λC. TWL-Clause [] C
  trail λS. hd (raw-trail S)
  raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting
  cons-trail tl-trail λC. add-init-cls (raw-clause C) λC. add-learned-cls (raw-clause C)
  λC. remove-cls (raw-clause C)
  update-backtrack-lvl
  update-conflicting λN. init-state (map raw-clause N) restart'
  apply unfold-locales
  apply (case-tac raw-trail S)
  apply (simp-all add: add-init-cls-def add-learned-cls-def clause-rewatch clause-watch
    cons-trail-def remove-cls-def restart'-def tl-trail-def map-tl comp-def
    ac-simps mset-map-removeAll-cond mset-raw-init-clss-init-state)

  apply (auto simp: mset-map image-mset-subseteq-mono[OF restart-learned] )
  done

interpretation rough-cdcl: conflict-driven-clause-learning_W
  λC. mset (raw-clause C)
```

λL C. TWL-Clause (watched C) (L # unwatched C)
λL C. TWL-Clause [] (remove1 L (raw-clause C))
λC. clauses-of-l (map raw-clause C) op @
λL C. L ∈ set C op # λC. remove1-cond (λD. mset (raw-clause D) = mset (raw-clause C))

mset λxs ys. case-prod append (fold (λx (ys, zs). (remove1 x ys, x # zs)) xs (ys, []))
op # remove1

λC. raw-clause C λC. TWL-Clause [] C
trail λS. hd (raw-trail S)
raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting
cons-trail tl-trail λC. add-init-cls (raw-clause C) λC. add-learned-cls (raw-clause C)
λC. remove-cls (raw-clause C)
update-backtrack-lvl
update-conflicting λN. init-state (map raw-clause N) restart′
by *unfold-locales*

declare *local.rough-cdcl.mset-ccls-ccls-of-cls[simp del]*

25.1.2 Opaque Type with Invariant

declare *rough-cdcl.state-simp[simp del]*

definition *cons-trail-twl* :: *('v, nat, 'v twl-clause) marked-lit ⇒ 'v wf-twl ⇒ 'v wf-twl*
where
cons-trail-twl L S ≡ twl-of-rough-state (cons-trail L (rough-state-of-twl S))

lemma *wf-twl-state-cons-trail*:
assumes
undef: *undefined-lit (trail S) (lit-of L)* **and**
wf: *wf-twl-state S*
shows *wf-twl-state (cons-trail L S)*
using *undef wf wf-rewatch[of S mmset-of-mlit′ L]* **unfolding** *wf-twl-state-def Ball-def*
by *(auto simp: cons-trail-def defined-lit-map comp-def image-def twl.raw-clauses-def)*

lemma *rough-state-of-twl-cons-trail*:
undefined-lit (trail-twl S) (lit-of L) ⟹
rough-state-of-twl (cons-trail-twl L S) = cons-trail L (rough-state-of-twl S)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-cons-trail*
unfolding *cons-trail-twl-def* **by** *blast*

abbreviation *add-init-cls-twl* **where**
add-init-cls-twl C S ≡ twl-of-rough-state (add-init-cls C (rough-state-of-twl S))

lemma *wf-twl-add-init-cls*: *wf-twl-state S ⟹ wf-twl-state (add-init-cls L S)*
unfolding *wf-twl-state-def* **by** *(auto simp: wf-watch add-init-cls-def comp-def twl.raw-clauses-def split: if-split-asm)*

lemma *rough-state-of-twl-add-init-cls*:
rough-state-of-twl (add-init-cls-twl L S) = add-init-cls L (rough-state-of-twl S)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-add-init-cls* **by** *blast*

abbreviation *add-learned-cls-twl* **where**
add-learned-cls-twl C S ≡ twl-of-rough-state (add-learned-cls C (rough-state-of-twl S))

lemma *wf-twl-add-learned-cls*: *wf-twl-state S ⟹ wf-twl-state (add-learned-cls L S)*

unfolding *wf-twl-state-def* **by** (*auto simp*: *wf-watch add-learned-cls-def twl.raw-clauses-def*
split: *if-split-asm*)

lemma *rough-state-of-twl-add-learned-cls*:
rough-state-of-twl (*add-learned-cls-twl L S*) = *add-learned-cls L* (*rough-state-of-twl S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-add-learned-cls* **by** *blast*

abbreviation *remove-cls-twl* **where**
remove-cls-twl C S ≡ *twl-of-rough-state* (*remove-cls C* (*rough-state-of-twl S*))

lemma *set-removeAll-condD*: *x* ∈ *set* (*removeAll-cond f xs*) ⟹ *x* ∈ *set xs*
by (*induction xs*) (*auto split*: *if-split-asm*)

lemma *wf-twl-remove-cls*: *wf-twl-state S* ⟹ *wf-twl-state* (*remove-cls L S*)
unfolding *wf-twl-state-def* **by** (*auto simp*: *wf-watch remove-cls-def twl.raw-clauses-def comp-def*
split: *if-split-asm dest*: *set-removeAll-condD*)

lemma *rough-state-of-twl-remove-cls*:
rough-state-of-twl (*remove-cls-twl L S*) = *remove-cls L* (*rough-state-of-twl S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-remove-cls* **by** *blast*

abbreviation *init-state-twl* **where**
init-state-twl N ≡ *twl-of-rough-state* (*init-state N*)

lemma *wf-twl-state-wf-twl-state-fold-add-init-cls*:
assumes *wf-twl-state S*
shows *wf-twl-state* (*fold add-init-cls N S*)
using *assms* **apply** (*induction N arbitrary*: *S*)
apply (*auto simp*: *wf-twl-state-def*)[]
by (*simp add*: *wf-twl-add-init-cls*)

lemma *wf-twl-state-epsilon-state*[*simp*]:
wf-twl-state (*TWL-State* [] [] [] *0 None*)
by (*auto simp*: *wf-twl-state-def twl.raw-clauses-def*)

lemma *wf-twl-init-state*: *wf-twl-state* (*init-state N*)
unfolding *init-state-def* **by** (*auto intro*!: *wf-twl-state-wf-twl-state-fold-add-init-cls*)

lemma *rough-state-of-twl-init-state*:
rough-state-of-twl (*init-state-twl N*) = *init-state N*
by (*simp add*: *twl-of-rough-state-inverse wf-twl-init-state*)

abbreviation *tl-trail-twl* **where**
tl-trail-twl S ≡ *twl-of-rough-state* (*tl-trail* (*rough-state-of-twl S*))

lemma *wf-twl-state-tl-trail*: *wf-twl-state S* ⟹ *wf-twl-state* (*tl-trail S*)
by (*auto simp add*: *twl-of-rough-state-inverse wf-twl-init-state wf-twl-cls-wf-twl-cls-tl*
tl-trail-def wf-twl-state-def distinct-tl map-tl comp-def twl.raw-clauses-def)

lemma *rough-state-of-twl-tl-trail*:
rough-state-of-twl (*tl-trail-twl S*) = *tl-trail* (*rough-state-of-twl S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-tl-trail* **by** *blast*

abbreviation *update-backtrack-lvl-twl* **where**
update-backtrack-lvl-twl k S ≡ *twl-of-rough-state* (*update-backtrack-lvl k* (*rough-state-of-twl S*))

lemma *wf-twl-state-update-backtrack-lvl*:
wf-twl-state S ⟹ *wf-twl-state* (*update-backtrack-lvl k S*)
unfolding *wf-twl-state-def* **by** (*auto simp*: *comp-def twl.raw-clauses-def*)

lemma *rough-state-of-twl-update-backtrack-lvl*:
rough-state-of-twl (*update-backtrack-lvl-twl k S*) = *update-backtrack-lvl k*
(*rough-state-of-twl S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-update-backtrack-lvl* **by** *fast*

abbreviation *update-conflicting-twl* **where**
update-conflicting-twl k S ≡ *twl-of-rough-state* (*update-conflicting k* (*rough-state-of-twl S*))

lemma *wf-twl-state-update-conflicting*:
wf-twl-state S ⟹ *wf-twl-state* (*update-conflicting k S*)
unfolding *wf-twl-state-def* **by** (*auto simp*: *twl.raw-clauses-def comp-def*)

lemma *rough-state-of-twl-update-conflicting*:
rough-state-of-twl (*update-conflicting-twl k S*) = *update-conflicting k*
(*rough-state-of-twl S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-update-conflicting* **by** *fast*

abbreviation *raw-clauses-twl* **where**
raw-clauses-twl S ≡ *twl.raw-clauses* (*rough-state-of-twl S*)

abbreviation *restart-twl* **where**
restart-twl S ≡ *twl-of-rough-state* (*restart′* (*rough-state-of-twl S*))
lemma *mset-union-mset-setD*:
mset A ⊆# *mset B* ⟹ *set A* ⊆ *set B*
by *auto*

lemma *wf-wf-restart′*: *wf-twl-state S* ⟹ *wf-twl-state* (*restart′ S*)
unfolding *restart′-def wf-twl-state-def* **apply** *standard*
apply *clarify*
apply (*rename-tac x*)
apply (*subgoal-tac wf-twl-cls* (*trail S*) *x*)
apply (*case-tac x*)
using *restart-learned* **by** (*auto simp*: *twl.raw-clauses-def comp-def dest*: *mset-union-mset-setD*)

lemma *rough-state-of-twl-restart-twl*:
rough-state-of-twl (*restart-twl S*) = *restart′* (*rough-state-of-twl S*)
by (*simp add*: *twl-of-rough-state-inverse wf-wf-restart′*)

sublocale *conflict-driven-clause-learning*$_W$
λC. mset (*raw-clause C*)

λL C. TWL-Clause (*watched C*) (*L* # *unwatched C*)
λL C. TWL-Clause [] (*remove1 L* (*raw-clause C*))
λC. clauses-of-l (*map raw-clause C*) *op* @
λL C. L ∈ *set C op* # *λC. remove1-cond* (*λD. mset* (*raw-clause D*) = *mset* (*raw-clause C*))

mset λxs ys. case-prod append (*fold* (*λx* (*ys*, *zs*). (*remove1 x ys*, *x* # *zs*)) *xs* (*ys*, []))
op # *remove1*

λC. raw-clause C λC. TWL-Clause [] *C*

```
  trail-twl λS. hd (raw-trail-twl S)
  raw-init-clss-twl
  raw-learned-clss-twl
  backtrack-lvl-twl
  raw-conflicting-twl
  cons-trail-twl
  tl-trail-twl
  λC. add-init-cls-twl (raw-clause C)
  λC. add-learned-cls-twl (raw-clause C)
  λC. remove-cls-twl (raw-clause C)
  update-backtrack-lvl-twl
  update-conflicting-twl
  λN. init-state-twl (map raw-clause N)
  restart-twl
  apply unfold-locales
          using rough-cdcl.hd-raw-trail apply blast
        apply (simp-all add: rough-state-of-twl-cons-trail rough-state-of-twl-tl-trail
          rough-state-of-twl-add-init-cls rough-state-of-twl-add-learned-cls
          rough-state-of-twl-remove-cls rough-state-of-twl-update-backtrack-lvl
          rough-state-of-twl-update-conflicting)[7]
      using rough-cdcl.init-clss-cons-trail rough-cdcl.init-clss-tl-trail
      rough-cdcl.init-clss-add-init-cls rough-cdcl.init-clss-remove-cls
      rough-cdcl.init-clss-add-learned-cls
      rough-cdcl.init-clss-update-backtrack-lvl
      rough-cdcl.init-clss-update-conflicting
      apply (auto simp add: rough-state-of-twl-cons-trail rough-state-of-twl-tl-trail
        rough-state-of-twl-add-init-cls rough-state-of-twl-add-learned-cls
        rough-state-of-twl-remove-cls rough-state-of-twl-update-backtrack-lvl
        rough-state-of-twl-update-conflicting comp-def)[7]
      using rough-cdcl.learned-clss-cons-trail rough-cdcl.learned-clss-tl-trail
      rough-cdcl.learned-clss-add-init-cls rough-cdcl.learned-clss-remove-cls
      rough-cdcl.learned-clss-add-learned-cls
      rough-cdcl.learned-clss-update-backtrack-lvl
      rough-cdcl.learned-clss-update-conflicting
      apply (auto simp add: rough-state-of-twl-cons-trail rough-state-of-twl-tl-trail
        rough-state-of-twl-add-init-cls rough-state-of-twl-add-learned-cls
        rough-state-of-twl-remove-cls rough-state-of-twl-update-backtrack-lvl
        rough-state-of-twl-update-conflicting comp-def)[7]
     apply (auto simp add: rough-state-of-twl-cons-trail rough-state-of-twl-tl-trail
       rough-state-of-twl-add-init-cls rough-state-of-twl-add-learned-cls
       rough-state-of-twl-remove-cls rough-state-of-twl-update-backtrack-lvl
       rough-state-of-twl-update-conflicting comp-def)[14]
   using init-clss-init-state apply (auto simp: rough-state-of-twl-init-state)[5]
  using rough-cdcl.init-clss-restart-state rough-cdcl.learned-clss-restart-state
  apply (auto simp: rough-state-of-twl-restart-twl)[5]
  done

declare local.rough-cdcl.mset-ccls-ccls-of-cls[simp del]
abbreviation state-eq-twl (infix ∼TWL 51) where
state-eq-twl S S′ ≡ rough-cdcl.state-eq (rough-state-of-twl S) (rough-state-of-twl S′)
notation state-eq (infix ∼ 51)
declare state-simp[simp del]
```

To avoid ambiguities:

```
no-notation state-eq-twl (infix ∼ 51)
```

```
inductive propagate-twl :: 'v wf-twl ⇒ 'v wf-twl ⇒ bool where
propagate-twl-rule: (L, C) ∈ candidates-propagate-twl S ⟹
  S' ∼ cons-trail-twl (Propagated L C) S ⟹
  raw-conflicting-twl S = None ⟹
  propagate-twl S S'

inductive-cases propagate-twlE: propagate-twl S T
thm propagateE

lemma distinct-filter-eq-if:
  distinct C ⟹ length (filter (op = L) C) = (if L ∈ set C then 1 else 0)
  by (induction C) auto

lemma distinct-mset-remove1-All:
  distinct-mset C ⟹ remove1-mset L C = removeAll-mset L C
  by (auto simp: multiset-eq-iff distinct-mset-count-less-1)

lemma propagate-twl-iff-propagate:
  assumes inv: cdcl_W-all-struct-inv S
  shows propagate S T ⟷ propagate-twl S T (is ?P ⟷ ?T)
proof
  assume ?P
  then obtain L E where
    raw-conflicting-twl S = None and
    CL-Clauses: E ∈ set (twl.raw-clauses S) and
    LE: L ∈# mset (raw-clause E) and
    tr-CNot: trail-twl S ⊨as CNot (remove1-mset L (mset (raw-clause E))) and
    undef-lot[simp]: undefined-lit (trail-twl S) L and
    T ∼ cons-trail-twl (Propagated L E) S
     by (blast elim: propagateE)
  have distinct (raw-clause E)
    using inv CL-Clauses unfolding cdcl_W-all-struct-inv-def distinct-mset-set-def
    distinct-cdcl_W-state-def raw-clauses-def by auto
  then have X: remove1-mset L (mset (raw-clause E)) = mset-set (set (raw-clause E) − {L})
    by (auto simp: multiset-eq-iff raw-clause-def count-mset distinct-filter-eq-if)
  have (L, E) ∈ candidates-propagate-twl S
    apply (rule wf-candidates-propagate-complete)
         using rough-state-of-twl apply auto[]
        using CL-Clauses unfolding raw-clauses-def twl.raw-clauses-def
        apply auto[]
       using LE apply simp
      using tr-CNot X apply simp
     using undef-lot apply blast
     done
  show ?T
    apply (rule propagate-twl-rule)
       apply (rule ‹(L, E) ∈ candidates-propagate-twl S›)
      using ‹T ∼ cons-trail-twl (Propagated L E) S›
      apply (auto simp: ‹raw-conflicting-twl S = None› twl.state-eq-def)
    done
next
  assume ?T
  then obtain L C where
    LC: (L, C) ∈ candidates-propagate-twl S and
    T: T ∼ cons-trail-twl (Propagated L C) S and
    confl: raw-conflicting-twl S = None
    by (auto elim: propagate-twlE)
  have
    C'S: C ∈ set (raw-clauses-twl S) and
    L: set (watched C) − uminus ‘ lits-of-l (trail-twl S) = {L} and
    undef: undefined-lit (trail-twl S) L
    using LC unfolding candidates-propagate-def raw-clauses-def by auto
  have dist: distinct (raw-clause C)
    using inv C'S unfolding cdcl_W-all-struct-inv-def distinct-cdcl_W-state-def
     distinct-mset-set-def twl.raw-clauses-def by fastforce
  then have C-L-L: mset-set (set (raw-clause C) − {L}) = mset (raw-clause C) − {#L#}
    by (metis distinct-mset-distinct distinct-mset-minus distinct-mset-set-mset-ident mset-remove1
      set-mset-mset set-remove1-eq)

  show ?P
    apply (rule propagate-rule[of S C L])
        using confl apply auto[]
       using C'S unfolding twl.raw-clauses-def apply (simp add: raw-clauses-def)
       using L unfolding candidates-propagate-def apply (auto simp: raw-clause-def)[]
      using wf-candidates-propagate-sound[OF - LC] rough-state-of-twl dist
      apply (simp add: distinct-mset-remove1-All)
     using undef apply simp
    using T undef by (smt backtrack-lvl-cons-trail confl init-clss-cons-trail
      learned-clss-cons-trail marked-lit.sel(2) raw-conflicting-cons-trail state-eq-def
      trail-cons-trail twl2.mmset-of-mlit.simps(1) twl2.mset-cls-cls-of-ccls)
qed

no-notation twl.state-eq-twl (infix ∼TWL 51)

inductive conflict-twl where
conflict-twl-rule:
C ∈ candidates-conflict-twl S ⟹
  S' ∼ update-conflicting-twl (Some (raw-clause C)) S ⟹
  raw-conflicting-twl S = None ⟹
  conflict-twl S S'

inductive-cases conflict-twlE: conflict-twl S T

lemma conflict-twl-iff-conflict:
  shows conflict S T ⟷ conflict-twl S T (is ?C ⟷ ?T)
proof
  assume ?C
  then obtain D where
    S: raw-conflicting-twl S = None and
    D: D ∈ set (raw-clauses S) and
    MD: trail-twl S ⊨as CNot (mset (raw-clause D)) and
    T: T ∼ update-conflicting-twl (Some (raw-clause D)) S
    by (elim conflictE)

  have D ∈ candidates-conflict-twl S
    apply (rule wf-candidates-conflict-complete)
       apply simp
      using D apply (auto simp: raw-clauses-def twl.raw-clauses-def)[]
    using MD S by auto
  moreover have T ∼ twl-of-rough-state (update-conflicting (Some (raw-clause D))
  (rough-state-of-twl S))
    using T unfolding rough-cdcl.state-eq-def state-eq-def by auto
  ultimately show ?T
    using S by (auto intro: conflict-twl-rule)
next
  assume ?T
  then obtain C where
    C: C ∈ candidates-conflict-twl S and
    T: T ∼ update-conflicting-twl (Some (raw-clause C)) S and
    confl: raw-conflicting-twl S = None
    by (auto elim: conflict-twlE)
  have
    C ∈ set (raw-clauses S)
    using C unfolding candidates-conflict-def raw-clauses-def twl.raw-clauses-def by auto
 moreover have trail-twl S |=as CNot (mset (raw-clause C))
    using wf-candidates-conflict-sound[OF - C] by auto
 ultimately show ?C apply −
   apply (rule conflict.conflict-rule[of - C])
   using confl T unfolding rough-cdcl.state-eq-def by (auto simp del: map-map)
qed

inductive cdcl_W-twl :: 'v wf-twl ⇒ 'v wf-twl ⇒ bool for S :: 'v wf-twl where
propagate: propagate-twl S S' ⟹ cdcl_W-twl S S' |
conflict: conflict-twl S S' ⟹ cdcl_W-twl S S' |
other: cdcl_W-o S S' ⟹ cdcl_W-twl S S'|
rf: cdcl_W-rf S S' ⟹ cdcl_W-twl S S'

lemma cdcl_W-twl-iff-cdcl_W:
  assumes cdcl_W-all-struct-inv S
  shows cdcl_W-twl S T ⟷ cdcl_W S T
  by (simp add: assms cdcl_W.simps cdcl_W-twl.simps conflict-twl-iff-conflict
    propagate-twl-iff-propagate del: map-map)

lemma rtranclp-cdcl_W-twl-all-struct-inv-inv:
  assumes cdcl_W-twl** S T and cdcl_W-all-struct-inv S
  shows cdcl_W-all-struct-inv T
  using assms by (induction rule: rtranclp-induct)
  (simp-all add: cdcl_W-twl-iff-cdcl_W cdcl_W-all-struct-inv-inv del: map-map)

lemma rtranclp-cdcl_W-twl-iff-rtranclp-cdcl_W:
  assumes cdcl_W-all-struct-inv S
  shows cdcl_W-twl** S T ⟷ cdcl_W** S T (is ?T ⟷ ?W)
proof
  assume ?W
  then show ?T
    proof (induction rule: rtranclp-induct)
      case base
      then show ?case by simp
    next
      case (step T U) note st = this(1) and cdcl = this(2) and IH = this(3)
      have cdcl_W-twl T U
        using assms st cdcl rtranclp-cdcl_W-all-struct-inv-inv cdcl_W-twl-iff-cdcl_W
        by blast
      then show ?case using IH by auto
    qed
next
  assume ?T
  then show ?W
    proof (induction rule: rtranclp-induct)
      case base
      then show ?case by simp
    next
      case (step T U) note st = this(1) and cdcl = this(2) and IH = this(3)
      have cdcl_W T U
        using assms st cdcl rtranclp-cdcl_W-twl-all-struct-inv-inv cdcl_W-twl-iff-cdcl_W
        by blast
      then show ?case using IH by auto
    qed
qed

end

end
theory Prop-Superposition
imports Partial-Clausal-Logic ../lib/Herbrand-Interpretation
begin
```

26 Superposition

```
no-notation Herbrand-Interpretation.true-cls (infix ⊨ 50)
notation Herbrand-Interpretation.true-cls (infix ⊨h 50)

no-notation Herbrand-Interpretation.true-clss (infix ⊨s 50)
notation Herbrand-Interpretation.true-clss (infix ⊨hs 50)

lemma herbrand-interp-iff-partial-interp-cls:
  S ⊨h C ⟷ {Pos P|P. P∈S} ∪ {Neg P|P. P∉S} ⊨ C
  unfolding Herbrand-Interpretation.true-cls-def Partial-Clausal-Logic.true-cls-def
  by auto

lemma herbrand-consistent-interp:
  consistent-interp ({Pos P|P. P∈S} ∪ {Neg P|P. P∉S})
  unfolding consistent-interp-def by auto

lemma herbrand-total-over-set:
  total-over-set ({Pos P|P. P∈S} ∪ {Neg P|P. P∉S}) T
  unfolding total-over-set-def by auto

lemma herbrand-total-over-m:
  total-over-m ({Pos P|P. P∈S} ∪ {Neg P|P. P∉S}) T
  unfolding total-over-m-def by (auto simp add: herbrand-total-over-set)

lemma herbrand-interp-iff-partial-interp-clss:
  S ⊨hs C ⟷ {Pos P|P. P∈S} ∪ {Neg P|P. P∉S} ⊨s C
  unfolding true-clss-def Ball-def herbrand-interp-iff-partial-interp-cls
  Partial-Clausal-Logic.true-clss-def by auto

definition clss-lt :: 'a::wellorder clauses ⇒ 'a clause ⇒ 'a clauses where
clss-lt N C = {D ∈ N. D #⊂# C}
```

```
notation (latex output)
 clss-lt (-<^bsup>-<^esup>)

locale selection =
  fixes S :: 'a clause ⇒ 'a clause
  assumes
    S-selects-subseteq: ⋀C. S C ≤# C and
    S-selects-neg-lits: ⋀C L. L ∈# S C ⟹ is-neg L

locale ground-resolution-with-selection =
  selection S for S :: ('a :: wellorder) clause ⇒ 'a clause
begin

context
  fixes N :: 'a clause set
begin
```

We do not create an equivalent of δ, but we directly defined N_C by inlining the definition.

```
function
  production :: 'a clause ⇒ 'a interp
where
  production C =
   {A. C ∈ N ∧ C ≠ {#} ∧ Max (set-mset C) = Pos A ∧ count C (Pos A) ≤ 1
     ∧ ¬ (⋃D ∈ {D. D #⊂# C}. production D) ⊨h C ∧ S C = {#}}
  by auto
termination by (relation {(D, C). D #⊂# C}) (auto simp: wf-less-multiset)

declare production.simps[simp del]

definition interp :: 'a clause ⇒ 'a interp where
  interp C = (⋃D ∈ {D. D #⊂# C}. production D)

lemma production-unfold:
  production C = {A. C ∈ N ∧ C ≠ {#} ∧ Max (set-mset C) = Pos A∧ count C (Pos A) ≤ 1 ∧ ¬
interp C ⊨h C ∧ S C = {#}}
  unfolding interp-def by (rule production.simps)

abbreviation productive A ≡ (production A ≠ {})

abbreviation produces :: 'a clause ⇒ 'a ⇒ bool where
  produces C A ≡ production C = {A}

lemma producesD:
  produces C A ⟹ C ∈ N ∧ C ≠ {#} ∧ Pos A = Max (set-mset C) ∧ count C (Pos A) ≤ 1 ∧
    ¬ interp C ⊨h C ∧ S C = {#}
  unfolding production-unfold by auto

lemma produces C A ⟹ Pos A ∈# C
  by (simp add: Max-in-lits producesD)

lemma interp'-def-in-set:
  interp C = (⋃D ∈ {D ∈ N. D #⊂# C}. production D)
  unfolding interp-def apply auto
  unfolding production-unfold apply auto
  done

lemma production-iff-produces:
  produces D A ⟷ A ∈ production D
  unfolding production-unfold by auto

definition Interp :: 'a clause ⇒ 'a interp where
  Interp C = interp C ∪ production C

lemma
  assumes produces C P
  shows Interp C ⊨h C
  unfolding Interp-def assms using producesD[OF assms]
  by (metis Max-in-lits Un-insert-right insertI1 pos-literal-in-imp-true-cls)

definition INTERP :: 'a interp where
INTERP = (⋃ D ∈N. production D)


lemma interp-subseteq-Interp[simp]: interp C ⊆ Interp C
  unfolding Interp-def by simp

lemma Interp-as-UNION: Interp C = (⋃ D ∈ {D. D #⊆# C}. production D)
  unfolding Interp-def interp-def le-multiset-def by fast

lemma productive-not-empty: productive C ⟹ C ≠ {#}
  unfolding production-unfold by auto

lemma productive-imp-produces-Max-literal: productive C ⟹ produces C (atm-of (Max (set-mset C)))
  unfolding production-unfold by (auto simp del: atm-of-Max-lit)

lemma productive-imp-produces-Max-atom: productive C ⟹ produces C (Max (atms-of C))
  unfolding atms-of-def Max-atm-of-set-mset-commute[OF productive-not-empty]
  by (rule productive-imp-produces-Max-literal)

lemma produces-imp-Max-literal: produces C A ⟹ A = atm-of (Max (set-mset C))
  by (metis Max-singleton insert-not-empty productive-imp-produces-Max-literal)

lemma produces-imp-Max-atom: produces C A ⟹ A = Max (atms-of C)
  by (metis Max-singleton insert-not-empty productive-imp-produces-Max-atom)

lemma produces-imp-Pos-in-lits: produces C A ⟹ Pos A ∈# C
  by (auto intro: Max-in-lits dest!: producesD)

lemma productive-in-N: productive C ⟹ C ∈ N
  unfolding production-unfold by auto

lemma produces-imp-atms-leq: produces C A ⟹ B ∈ atms-of C ⟹ B ≤ A
  by (metis Max-ge finite-atms-of insert-not-empty productive-imp-produces-Max-atom
    singleton-inject)

lemma produces-imp-neg-notin-lits: produces C A ⟹ ¬ Neg A ∈# C
  by (rule pos-Max-imp-neg-notin) (auto dest: producesD)

lemma less-eq-imp-interp-subseteq-interp: C #⊆# D ⟹ interp C ⊆ interp D
```

```
  unfolding interp-def by auto (metis multiset-order.order.strict-trans2)

lemma less-eq-imp-interp-subseteq-Interp: C #⊆# D ⟹ interp C ⊆ Interp D
  unfolding Interp-def using less-eq-imp-interp-subseteq-interp by blast

lemma less-imp-production-subseteq-interp: C #⊂# D ⟹ production C ⊆ interp D
  unfolding interp-def by fast

lemma less-eq-imp-production-subseteq-Interp: C #⊆# D ⟹ production C ⊆ Interp D
  unfolding Interp-def using less-imp-production-subseteq-interp
  by (metis multiset-order.le-imp-less-or-eq le-supI1 sup-ge2)

lemma less-imp-Interp-subseteq-interp: C #⊂# D ⟹ Interp C ⊆ interp D
  unfolding Interp-def
  by (auto simp: less-eq-imp-interp-subseteq-interp less-imp-production-subseteq-interp)

lemma less-eq-imp-Interp-subseteq-Interp: C #⊆# D ⟹ Interp C ⊆ Interp D
  using less-imp-Interp-subseteq-interp
  unfolding Interp-def by (metis multiset-order.le-imp-less-or-eq le-supI2 subset-refl sup-commute)

lemma false-Interp-to-true-interp-imp-less-multiset: A ∉ Interp C ⟹ A ∈ interp D ⟹ C #⊂# D
  using less-eq-imp-interp-subseteq-Interp multiset-linorder.not-less by blast

lemma false-interp-to-true-interp-imp-less-multiset: A ∉ interp C ⟹ A ∈ interp D ⟹ C #⊂# D
  using less-eq-imp-interp-subseteq-interp multiset-linorder.not-less by blast

lemma false-Interp-to-true-Interp-imp-less-multiset: A ∉ Interp C ⟹ A ∈ Interp D ⟹ C #⊂# D
  using less-eq-imp-Interp-subseteq-Interp multiset-linorder.not-less by blast

lemma false-interp-to-true-Interp-imp-le-multiset: A ∉ interp C ⟹ A ∈ Interp D ⟹ C #⊆# D
  using less-imp-Interp-subseteq-interp multiset-linorder.not-less by blast

lemma interp-subseteq-INTERP: interp C ⊆ INTERP
  unfolding interp-def INTERP-def by (auto simp: production-unfold)

lemma production-subseteq-INTERP: production C ⊆ INTERP
  unfolding INTERP-def using production-unfold by blast

lemma Interp-subseteq-INTERP: Interp C ⊆ INTERP
  unfolding Interp-def by (auto intro!: interp-subseteq-INTERP production-subseteq-INTERP)
```

This lemma corresponds to theorem 2.7.6 page 66 of CW.

```
lemma produces-imp-in-interp:
  assumes a-in-c: Neg A ∈# C and d: produces D A
  shows A ∈ interp C
proof −
  from d have Max (set-mset D) = Pos A
    using production-unfold by blast
  hence D #⊂# {#Neg A#}
    by (auto intro: Max-pos-neg-less-multiset)
  moreover have {#Neg A#} #⊆# C
    by (rule less-eq-imp-le-multiset) (rule mset-le-single[OF a-in-c])
  ultimately show ?thesis
    using d by (blast dest: less-eq-imp-interp-subseteq-interp less-imp-production-subseteq-interp)
qed
```

```
lemma neg-notin-Interp-not-produce: Neg A ∈# C ⟹ A ∉ Interp D ⟹ C #⊆# D ⟹ ¬ produces
D'' A
  by (auto dest: produces-imp-in-interp less-eq-imp-interp-subseteq-Interp)

lemma in-production-imp-produces: A ∈ production C ⟹ produces C A
  by (metis insert-absorb productive-imp-produces-Max-atom singleton-insert-inj-eq')

lemma not-produces-imp-notin-production: ¬ produces C A ⟹ A ∉ production C
  by (metis in-production-imp-produces)

lemma not-produces-imp-notin-interp: (⋀D. ¬ produces D A) ⟹ A ∉ interp C
  unfolding interp-def by (fast intro!: in-production-imp-produces)
```

The results below corresponds to Lemma 3.4.

Nitpicking: If $D = D'$ and D is productive, $I^D \subseteq I_{D'}$ does not hold.

```
lemma true-Interp-imp-general:
  assumes
    c-le-d: C #⊆# D and
    d-lt-d': D #⊂# D' and
    c-at-d: Interp D ⊨h C and
    subs: interp D' ⊆ (⋃C ∈ CC. production C)
  shows (⋃C ∈ CC. production C) ⊨h C
proof (cases ∃A. Pos A ∈# C ∧ A ∈ Interp D)
  case True
  then obtain A where a-in-c: Pos A ∈# C and a-at-d: A ∈ Interp D
    by blast
  from a-at-d have A ∈ interp D'
    using d-lt-d' less-imp-Interp-subseteq-interp by blast
  thus ?thesis
    using subs a-in-c by (blast dest: contra-subsetD)
next
  case False
  then obtain A where a-in-c: Neg A ∈# C and A ∉ Interp D
    using c-at-d unfolding true-cls-def by blast
  hence ⋀D''. ¬ produces D'' A
    using c-le-d neg-notin-Interp-not-produce by simp
  thus ?thesis
    using a-in-c subs not-produces-imp-notin-production by auto
qed

lemma true-Interp-imp-interp: C #⊆# D ⟹ D #⊂# D' ⟹ Interp D ⊨h C ⟹ interp D' ⊨h C
  using interp-def true-Interp-imp-general by simp

lemma true-Interp-imp-Interp: C #⊆# D ⟹ D #⊂# D' ⟹ Interp D ⊨h C ⟹ Interp D' ⊨h C
  using Interp-as-UNION interp-subseteq-Interp true-Interp-imp-general by simp

lemma true-Interp-imp-INTERP: C #⊆# D ⟹ Interp D ⊨h C ⟹ INTERP ⊨h C
  using INTERP-def interp-subseteq-INTERP
    true-Interp-imp-general[OF - less-multiset-right-total]
  by simp

lemma true-interp-imp-general:
  assumes
    c-le-d: C #⊆# D and
    d-lt-d′: D #⊂# D′ and
    c-at-d: interp D ⊨h C and
    subs: interp D′ ⊆ (⋃C ∈ CC. production C)
  shows (⋃C ∈ CC. production C) ⊨h C
proof (cases ∃A. Pos A ∈# C ∧ A ∈ interp D)
  case True
  then obtain A where a-in-c: Pos A ∈# C and a-at-d: A ∈ interp D
    by blast
  from a-at-d have A ∈ interp D′
    using d-lt-d′ less-eq-imp-interp-subseteq-interp[OF multiset-order.less-imp-le] by blast
  thus ?thesis
    using subs a-in-c by (blast dest: contra-subsetD)
next
  case False
  then obtain A where a-in-c: Neg A ∈# C and A ∉ interp D
    using c-at-d unfolding true-cls-def by blast
  hence ⋀D″. ¬ produces D″ A
    using c-le-d by (auto dest: produces-imp-in-interp less-eq-imp-interp-subseteq-interp)
  thus ?thesis
    using a-in-c subs not-produces-imp-notin-production by auto
qed
```

This lemma corresponds to theorem 2.7.6 page 66 of CW. Here the strict maximality is important

```
lemma true-interp-imp-interp: C #⊆# D ⟹ D #⊂# D′ ⟹ interp D ⊨h C ⟹ interp D′ ⊨h C
  using interp-def true-interp-imp-general by simp

lemma true-interp-imp-Interp: C #⊆# D ⟹ D #⊂# D′ ⟹ interp D ⊨h C ⟹ Interp D′ ⊨h C
  using Interp-as-UNION interp-subseteq-Interp[of D′] true-interp-imp-general by simp

lemma true-interp-imp-INTERP: C #⊆# D ⟹ interp D ⊨h C ⟹ INTERP ⊨h C
  using INTERP-def interp-subseteq-INTERP
    true-interp-imp-general[OF - less-multiset-right-total]
  by simp

lemma productive-imp-false-interp: productive C ⟹ ¬ interp C ⊨h C
  unfolding production-unfold by auto
```

This lemma corresponds to theorem 2.7.6 page 66 of CW. Here the strict maximality is important

```
lemma cls-gt-double-pos-no-production:
  assumes D: {#Pos P, Pos P#} #⊂# C
  shows ¬produces C P
proof –
  let ?D = {#Pos P, Pos P#}
  note D′ = D[unfolded less-multiset_HO]
  consider
    (P) count C (Pos P) ≥ 2
  | (Q) Q where Q > Pos P and Q ∈# C
    using HOL.spec[OF HOL.conjunct2[OF D′], of Pos P] by (auto split: if-split-asm)
  thus ?thesis
    proof cases
      case Q
      have Q ∈ set-mset C
        using Q(2) by (auto split: if-split-asm)
      then have Max (set-mset C) > Pos P
        using Q(1) Max-gr-iff by blast
      thus ?thesis
        unfolding production-unfold by auto
    next
      case P
      thus ?thesis
        unfolding production-unfold by auto
    qed
qed
```

This lemma corresponds to theorem 2.7.6 page 66 of CW.

```
lemma
  assumes D: C+{#Neg P#} #⊂# D
  shows production D ≠ {P}
proof −
  note D′ = D[unfolded less-multiset_HO]
  consider
    (P) Neg P ∈# D
  | (Q) Q where Q > Neg P and count D Q > count (C + {#Neg P#}) Q
    using HOL.spec[OF HOL.conjunct2[OF D′], of Neg P] count-greater-zero-iff by fastforce
  thus ?thesis
    proof cases
      case Q
      have Q ∈ set-mset D
        using Q(2) gr-implies-not0 by fastforce
      then have Max (set-mset D) > Neg P
        using Q(1) Max-gr-iff by blast
      hence Max (set-mset D) > Pos P
        using less-trans[of Pos P Neg P Max (set-mset D)] by auto
      thus ?thesis
        unfolding production-unfold by auto
    next
      case P
      hence Max (set-mset D) > Pos P
        by (meson Max-ge finite-set-mset le-less-trans linorder-not-le pos-less-neg)
      thus ?thesis
        unfolding production-unfold by auto
    qed
qed

lemma in-interp-is-produced:
  assumes P ∈ INTERP
  shows ∃ D. D +{#Pos P#} ∈ N ∧ produces (D +{#Pos P#}) P
  using assms unfolding INTERP-def UN-iff production-iff-produces Ball-def
  by (metis ground-resolution-with-selection.produces-imp-Pos-in-lits insert-DiffM2
    ground-resolution-with-selection-axioms not-produces-imp-notin-production)


end
end

abbreviation MMax M ≡ Max (set-mset M)
```

26.1 We can now define the rules of the calculus

```
inductive superposition-rules :: ′a clause ⇒ ′a clause ⇒ ′a clause ⇒ bool where
factoring: superposition-rules (C + {#Pos P#} + {#Pos P#}) B (C + {#Pos P#}) |
superposition-l: superposition-rules (C₁ + {#Pos P#}) (C₂ + {#Neg P#}) (C₁+ C₂)

inductive superposition :: 'a clauses ⇒ 'a clauses ⇒ bool where
superposition: A ∈ N ⟹ B ∈ N ⟹ superposition-rules A B C
  ⟹ superposition N (N ∪ {C})

definition abstract-red :: 'a::wellorder clause ⇒ 'a clauses ⇒ bool where
abstract-red C N = (clss-lt N C ⊨p C)

lemma less-multiset[iff]: M < N ⟷ M #⊂# N
  unfolding less-multiset-def by auto

lemma less-eq-multiset[iff]: M ≤ N ⟷ M #⊆# N
  unfolding less-eq-multiset-def by auto

lemma herbrand-true-clss-true-clss-cls-herbrand-true-clss:
  assumes
    AB: A ⊨hs B and
    BC: B ⊨p C
  shows A ⊨h C
proof –
  let ?I = {Pos P |P. P ∈ A} ∪ {Neg P |P. P ∉ A}
  have B: ?I ⊨s B using AB
    by (auto simp add: herbrand-interp-iff-partial-interp-clss)

  have IH: ⋀I. total-over-set I (atms-of C) ⟹ total-over-m I B ⟹ consistent-interp I
    ⟹ I ⊨s B ⟹ I ⊨ C using BC
    by (auto simp add: true-clss-cls-def)
  show ?thesis
    unfolding herbrand-interp-iff-partial-interp-cls
    by (auto intro: IH[of ?I] simp add: herbrand-total-over-set herbrand-total-over-m
      herbrand-consistent-interp B)
qed

lemma abstract-red-subset-mset-abstract-red:
  assumes
    abstr: abstract-red C N and
    c-lt-d: C ⊆# D
  shows abstract-red D N
proof –
  have {D ∈ N. D #⊂# C} ⊆ {D' ∈ N. D' #⊂# D}
    using c-lt-d less-eq-imp-le-multiset by fastforce
  thus ?thesis
    using abstr unfolding abstract-red-def clss-lt-def
    by (metis (no-types, lifting) c-lt-d subset-mset.diff-add true-clss-cls-mono-r'
      true-clss-cls-subset)
qed


lemma true-clss-cls-extended:
  assumes
    A ⊨p B and
    tot: total-over-m I (A) and
    cons: consistent-interp I and
    I-A: I ⊨s A
  shows I ⊨ B
proof −
  let ?I = I ∪ {Pos P|P. P ∈ atms-of B ∧ P ∉ atms-of-s I}
  have consistent-interp ?I
    using cons unfolding consistent-interp-def atms-of-s-def atms-of-def
      apply (auto 1 5 simp add: image-iff)
    by (metis atm-of-uminus literal.sel(1))
  moreover have total-over-m ?I (A ∪ {B})
    proof −
      obtain aa :: 'a set ⇒ 'a literal set ⇒ 'a where
        f2: ∀ x0 x1. (∃ v2. v2 ∈ x0 ∧ Pos v2 ∉ x1 ∧ Neg v2 ∉ x1)
           ⟷ (aa x0 x1 ∈ x0 ∧ Pos (aa x0 x1) ∉ x1 ∧ Neg (aa x0 x1) ∉ x1)
        by moura
      have ∀ a. a ∉ atms-of-ms A ∨ Pos a ∈ I ∨ Neg a ∈ I
        using tot by (simp add: total-over-m-def total-over-set-def)
      hence aa (atms-of-ms A ∪ atms-of-ms {B}) (I ∪ {Pos a |a. a ∈ atms-of B ∧ a ∉ atms-of-s I})
        ∉ atms-of-ms A ∪ atms-of-ms {B} ∨ Pos (aa (atms-of-ms A ∪ atms-of-ms {B})
          (I ∪ {Pos a |a. a ∈ atms-of B ∧ a ∉ atms-of-s I})) ∈ I
            ∪ {Pos a |a. a ∈ atms-of B ∧ a ∉ atms-of-s I}
          ∨ Neg (aa (atms-of-ms A ∪ atms-of-ms {B})
            (I ∪ {Pos a |a. a ∈ atms-of B ∧ a ∉ atms-of-s I})) ∈ I
            ∪ {Pos a |a. a ∈ atms-of B ∧ a ∉ atms-of-s I}
        by auto
      hence total-over-set (I ∪ {Pos a |a. a ∈ atms-of B ∧ a ∉ atms-of-s I})
        (atms-of-ms A ∪ atms-of-ms {B})
        using f2 by (meson total-over-set-def)
      thus ?thesis
        by (simp add: total-over-m-def)
    qed
  moreover have ?I ⊨s A
    using I-A by auto
  ultimately have ?I ⊨ B
    using ‹A⊨pB› unfolding true-clss-cls-def by auto
  thus ?thesis
oops
lemma
  assumes
    CP: ¬ clss-lt N ({#C#} + {#E#}) ⊨p {#C#} + {#Neg P#} and
     clss-lt N ({#C#} + {#E#}) ⊨p {#E#} + {#Pos P#} ∨ clss-lt N ({#C#} + {#E#}) ⊨p
{#C#} + {#Neg P#}
  shows clss-lt N ({#C#} + {#E#}) ⊨p {#E#} + {#Pos P#}

oops

locale ground-ordered-resolution-with-redundancy =
  ground-resolution-with-selection +
  fixes redundant :: 'a::wellorder clause ⇒ 'a clauses ⇒ bool
  assumes
    redundant-iff-abstract: redundant A N ⟷ abstract-red A N
begin
definition saturated :: 'a clauses ⇒ bool where
saturated N ⟷ (∀ A B C. A ∈ N ⟶ B ∈ N ⟶ ¬redundant A N ⟶ ¬redundant B N
  ⟶ superposition-rules A B C ⟶ redundant C N ∨ C ∈ N)

lemma
```

```
  assumes
    saturated: saturated N and
    finite: finite N and
    empty: {#} ∉ N
  shows INTERP N ⊨hs N
proof (rule ccontr)
  let ?N_ℐ = INTERP N
  assume ¬ ?thesis
  hence not-empty: {E∈N. ¬?N_ℐ ⊨h E} ≠ {}
    unfolding true-clss-def Ball-def by auto
  def D ≡ Min {E∈N. ¬?N_ℐ ⊨h E}
  have [simp]: D ∈ N
    unfolding D-def
    by (metis (mono-tags, lifting) Min-in not-empty finite mem-Collect-eq rev-finite-subset subsetI)
  have not-d-interp: ¬?N_ℐ ⊨h D
    unfolding D-def
    by (metis (mono-tags, lifting) Min-in finite mem-Collect-eq not-empty rev-finite-subset subsetI)
  have cls-not-D: ⋀E. E ∈ N ⟹ E ≠ D ⟹ ¬?N_ℐ ⊨h E ⟹ D ≤ E
    using finite D-def by (auto simp del: less-eq-multiset)
  obtain C L where D: D = C + {#L#} and LSD: L ∈# S D ∨ (S D = {#} ∧ Max (set-mset D)
= L)
    proof (cases S D = {#})
      case False
      then obtain L where L ∈# S D
        using Max-in-lits by blast
      moreover
        hence L ∈# D
          using S-selects-subseteq[of D] by auto
        hence D = (D − {#L#}) + {#L#}
          by auto
      ultimately show ?thesis using that by blast
    next
      let ?L = MMax D
      case True
      moreover
        have ?L ∈# D
          by (metis (no-types, lifting) Max-in-lits ‹D ∈ N› empty)
        hence D = (D − {#?L#}) + {#?L#}
          by auto
      ultimately show ?thesis using that by blast
    qed
  have red: ¬redundant D N
    proof (rule ccontr)
      assume red[simplified]: ~~redundant D N
      have ∀E < D. E ∈ N ⟶ ?N_ℐ ⊨h E
        using cls-not-D not-le by fastforce
      hence ?N_ℐ ⊨hs clss-lt N D
        unfolding clss-lt-def true-clss-def Ball-def by blast
      thus False
        using  red not-d-interp unfolding abstract-red-def redundant-iff-abstract
        using herbrand-true-clss-true-clss-cls-herbrand-true-clss by fast
    qed

  consider
    (L) P where L = Pos P and S D = {#} and Max (set-mset D) = Pos P
```

```
| (Lneg) P where L = Neg P
  using LSD S-selects-neg-lits[of L D] by (cases L) auto
thus False
  proof cases
    case L note P = this(1) and S = this(2) and max = this(3)
    have count D L > 1
      proof (rule ccontr)
        assume ~ ?thesis
        hence count: count D L = 1
          unfolding D by (auto simp: not-in-iff)
        have ¬?N_ℐ⊨h D
          using not-d-interp true-interp-imp-INTERP ground-resolution-with-selection-axioms
            by blast
        hence produces N D P
          using not-empty empty finite ‹D ∈ N› count L
            true-interp-imp-INTERP unfolding production-iff-produces unfolding production-unfold
          by (auto simp add: max not-empty)
        hence INTERP N ⊨h D
          unfolding D
          by (metis pos-literal-in-imp-true-cls produces-imp-Pos-in-lits
            production-subseteq-INTERP singletonI subsetCE)
        thus False
          using not-d-interp by blast
      qed
    then have Pos P ∈# C
      by (simp add: P D)
    then obtain C' where C':D = C' + {#Pos P#} + {#Pos P#}
      unfolding D by (metis (full-types) P insert-DiffM2)
    have sup: superposition-rules D D (D − {#L#})
      unfolding C' L by (auto simp add: superposition-rules.simps)
    have C' + {#Pos P#}  #⊂# C' + {#Pos P#} + {#Pos P#}
      by auto
    moreover have ¬?N_ℐ ⊨h (D − {#L#})
      using not-d-interp unfolding C' L by auto
    ultimately have C' + {#Pos P#} ∉ N
      by (metis (no-types, lifting) C' P add-diff-cancel-right' cls-not-D less-multiset
        multi-self-add-other-not-self not-le)
    have D − {#L#} #⊂# D
      unfolding C' L by auto
    have c'-p-p: C' + {#Pos P#} + {#Pos P#} − {#Pos P#} = C' + {#Pos P#}
      by auto
    have redundant (C' + {#Pos P#}) N
      using saturated red sup ‹D ∈ N›‹C' + {#Pos P#} ∉ N›  unfolding saturated-def C' L c'-p-p
      by blast
    moreover have C' + {#Pos P#}  ⊆# C' + {#Pos P#} + {#Pos P#}
      by auto
    ultimately show False
      using red unfolding C' redundant-iff-abstract by (blast dest:
        abstract-red-subset-mset-abstract-red)
  next
    case Lneg note L = this(1)
    have P ∈ ?N_ℐ
      using not-d-interp unfolding D true-cls-def L by (auto split: if-split-asm)
    then obtain E where
      DPN: E + {#Pos P#} ∈ N and
```

```
 prod: production N (E + {#Pos P#}) = {P}
 using in-interp-is-produced by blast
have sup-EC: superposition-rules (E + {#Pos P#}) (C + {#Neg P#}) (E + C)
 using superposition-l by fast
hence superposition N (N ∪ {E+C})
 using DPN ‹D ∈ N› unfolding D L by (auto simp add: superposition.simps)
have
 PMax: Pos P = MMax (E + {#Pos P#}) and
 count (E + {#Pos P#}) (Pos P) ≤ 1 and
 S (E + {#Pos P#}) = {#} and
 ¬interp N (E + {#Pos P#}) ⊨h E + {#Pos P#}
 using prod unfolding production-unfold by auto
have Neg P ∉# E
 using prod produces-imp-neg-notin-lits by force
hence ⋀y. y ∈# (E + {#Pos P#})
 ⟹ count (E + {#Pos P#}) (Neg P) < count (C + {#Neg P#}) (Neg P)
 using count-greater-zero-iff by fastforce
moreover have  ⋀y. y ∈# (E + {#Pos P#}) ⟹ y < Neg P
 using PMax by (metis DPN Max-less-iff empty finite-set-mset pos-less-neg
   set-mset-eq-empty-iff)
moreover have E + {#Pos P#} ≠ C + {#Neg P#}
 using prod produces-imp-neg-notin-lits by force
ultimately have E + {#Pos P#} #⊂# C + {#Neg P#}
 unfolding less-multiset_HO by (metis count-greater-zero-iff less-iff-Suc-add zero-less-Suc)
have ce-lt-d: C + E #⊂# D
unfolding D L by (simp add: ‹⋀y. y ∈# E + {#Pos P#} ⟹ y < Neg P› ex-gt-imp-less-multiset)
have ?N_ℐ ⊨h E + {#Pos P#}
 using ‹P ∈ ?N_ℐ› by blast
have ?N_ℐ ⊨h C+E ∨ C+E ∉ N
 using ce-lt-d cls-not-D unfolding D-def by fastforce
have Pos P ∉# C+E
 using D ‹P ∈ ground-resolution-with-selection.INTERP S N›
   ‹count (E + {#Pos P#}) (Pos P) ≤ 1› multi-member-skip not-d-interp
   by (auto simp: not-in-iff)
hence ⋀y. y ∈# C+E
 ⟹ count (C+E) (Pos P) < count (E + {#Pos P#}) (Pos P)
 using set-mset-def by fastforce

have ¬redundant (C + E) N
 proof (rule ccontr)
   assume red'[simplified]: ¬ ?thesis
   have abs: clss-lt N (C + E) ⊨p C + E
     using redundant-iff-abstract red' unfolding abstract-red-def by auto
   have clss-lt N (C + E) ⊨p E + {#Pos P#} ∨ clss-lt N (C + E) ⊨p C + {#Neg P#}
     proof clarify
       assume CP: ¬ clss-lt N (C + E) ⊨p C + {#Neg P#}
       { fix I
         assume
           total-over-m I (clss-lt N (C + E) ∪ {E + {#Pos P#}}) and
           consistent-interp I and
           I ⊨s clss-lt N (C + E)
           hence I ⊨ C + E
             using  abs sorry
           moreover have ¬ I ⊨ C + {#Neg P#}
             using CP unfolding true-clss-cls-def
```

```
                sorry
              ultimately have  I ⊨ E + {#Pos P#} by auto
            }
            then show clss-lt N (C + E) ⊨p E + {#Pos P#}
              unfolding true-clss-cls-def by auto
          qed
        moreover have clss-lt N (C + E) ⊆ clss-lt N (C + {#Neg P#})
          using ce-lt-d mult-less-trans unfolding clss-lt-def D L by force
        ultimately have redundant (C + {#Neg P#}) N ∨ clss-lt N (C + E) ⊨p E + {#Pos P#}
          unfolding redundant-iff-abstract abstract-red-def using true-clss-cls-subset by blast
        show False sorry
      qed
    moreover have ¬ redundant (E + {#Pos P#}) N
      sorry
    ultimately have CEN: C + E ∈ N
      using ‹D∈N› ‹E + {#Pos P#}∈N› saturated sup-EC red unfolding saturated-def D L
      by (metis union-commute)
    have CED: C + E ≠ D
      using D ce-lt-d by auto
    have interp: ¬ INTERP N ⊨h C + E
    sorry
    show False
        using cls-not-D[OF CEN CED interp] ce-lt-d unfolding INTERP-def less-eq-multiset-def by
auto
  qed
qed

end

lemma tautology-is-redundant:
  assumes tautology C
  shows abstract-red C N
  using assms unfolding abstract-red-def true-clss-cls-def tautology-def by auto

lemma subsumed-is-redundant:
  assumes AB: A ⊂# B
  and AN: A ∈ N
  shows abstract-red B N
proof −
  have A ∈ clss-lt N B using AN AB unfolding clss-lt-def
    by (auto dest: less-eq-imp-le-multiset simp add: multiset-order.dual-order.order-iff-strict)
  thus ?thesis
    using AB unfolding abstract-red-def true-clss-cls-def Partial-Clausal-Logic.true-clss-def
    by blast
qed

inductive redundant :: 'a clause ⇒ 'a clauses ⇒ bool where
subsumption: A ∈ N ⟹ A ⊂# B ⟹ redundant B N

lemma redundant-is-redundancy-criterion:
  fixes A :: 'a :: wellorder clause and N :: 'a :: wellorder clauses
  assumes redundant A N
  shows  abstract-red A N
  using assms
proof (induction rule: redundant.induct)
```

```
  case (subsumption A B N)
  thus ?case
    using subsumed-is-redundant[of A N B] unfolding abstract-red-def clss-lt-def by auto
qed

lemma redundant-mono:
  redundant A N ⟹ A ⊆# B ⟹  redundant B N
  apply (induction rule: redundant.induct)
  by (meson subset-mset.less-le-trans subsumption)

locale truc =
    selection S for S :: nat clause ⇒ nat clause
begin

end

end
theory Weidenbach-Book
imports
  Prop-Normalisation

  Prop-Resolution

  Prop-Superposition

  CDCL-NOT DPLL-NOT DPLL-W-Implementation CDCL-W-Implementation CDCL-W-Incremental
  CDCL-WNOT

begin

end
```