

Formalisation of Ground Resolution and CDCL in Isabelle/HOL

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theory *Wellfounded-More*

imports *Main*

begin

1 Transitions

This theory contains more facts about closure, the definition of full transformations, and well-foundedness.

1.1 More theorems about Closures

This is the equivalent of $?r \leq ?s \implies ?r^{**} \leq ?s^{**}$ for *tranclp*

lemma *tranclp-mono-explicit*:

$r^{++} a b \implies r \leq s \implies s^{++} a b$

using *rtranclp-mono* **by** (*auto dest!*: *tranclpD intro: rtranclp-into-tranclp2*)

lemma *tranclp-mono*:

assumes *mono*: $r \leq s$

shows $r^{++} \leq s^{++}$

using *rtranclp-mono[OF mono]* *mono* **by** (*auto dest!*: *tranclpD intro: rtranclp-into-tranclp2*)

lemma *tranclp-idemp-rel*:

$R^{++++} a b \iff R^{++} a b$

apply (*rule iffI*)

prefer 2 **apply** *blast*

by (*induction rule: tranclp-induct*) *auto*

Equivalent of $?r^{****} = ?r^{**}$

lemma *trancl-idemp*: $(r^+)^+ = r^+$

by *simp*

lemmas *tranclp-idemp[simp] = trancl-idemp[to-pred]*

This theorem already exists as $?r^{**} ?a ?b \equiv ?a = ?b \vee ?r^{++} ?a ?b$ (and sledgehammer uses it), but it makes sense to duplicate it, because it is unclear how stable the lemmas in Nitpick are.

lemma *rtranclp-unfold*: $rtranclp r a b \iff (a = b \vee tranclp r a b)$

by (*meson rtranclp.simps rtranclpD tranclp-into-rtranclp*)

lemma *tranclp-unfold-end*: $\text{tranclp } r \ a \ b \longleftrightarrow (\exists a'. \text{rtranclp } r \ a \ a' \wedge r \ a' \ b)$
by (*metis rtranclp.rtrancl-refl rtranclp-into-tranclp1 tranclp.cases tranclp-into-rtranclp*)

lemma *tranclp-unfold-begin*: $\text{tranclp } r \ a \ b \longleftrightarrow (\exists a'. r \ a \ a' \wedge \text{rtranclp } r \ a' \ b)$
by (*meson rtranclp-into-tranclp2 tranclpD*)

lemma *trancl-set-tranclp*: $(a, b) \in \{(b, a). P \ a \ b\}^+ \longleftrightarrow P^{++} \ b \ a$
apply (*rule iffI*)
apply (*induction rule: trancl-induct; simp*)
apply (*induction rule: tranclp-induct; auto simp: trancl-into-trancl2*)
done

lemma *tranclp-rtranclp-rtranclp-rel*: $R^{+++} \ a \ b \longleftrightarrow R^{**} \ a \ b$
by (*simp add: rtranclp-unfold*)

lemma *tranclp-rtranclp-rtranclp[simp]*: $R^{+++} = R^{**}$
by (*fastforce simp: rtranclp-unfold*)

lemma *rtranclp-exists-last-with-prop*:
assumes $R \ x \ z$
and $R^{**} \ z \ z'$ **and** $P \ x \ z$
shows $\exists y \ y'. R^{**} \ x \ y \wedge R \ y \ y' \wedge P \ y \ y' \wedge (\lambda a \ b. R \ a \ b \wedge \neg P \ a \ b)^{**} \ y' \ z'$
using *assms(2,1,3)*
proof (*induction arbitrary:*)
case *base*
then show *?case* **by** *auto*
next
case (*step* $z' \ z''$) **note** $z = \text{this}(2)$ **and** $IH = \text{this}(3)[OF \ \text{this}(4-5)]$
show *?case*
apply (*cases* $P \ z' \ z''$)
apply (*rule exI[of - z'], rule exI[of - z'']*)
using $z \ \text{assms}(1) \ \text{step.hyps}(1) \ \text{step.prem}(2)$ **apply** *auto[1]*
using $IH \ z \ \text{rtranclp.rtrancl-into-rtrancl}$ **by** *fastforce*
qed

lemma *rtranclp-and-rtranclp-left*: $(\lambda a \ b. P \ a \ b \wedge Q \ a \ b)^{**} \ S \ T \Longrightarrow P^{**} \ S \ T$
by (*induction rule: rtranclp-induct*) *auto*

1.2 Full Transitions

We define here properties to define properties after all possible transitions.

abbreviation *no-step step* $S \equiv (\forall S'. \neg \text{step } S \ S')$

definition *full1* :: $('a \Rightarrow 'a \Rightarrow \text{bool}) \Rightarrow 'a \Rightarrow 'a \Rightarrow \text{bool}$ **where**
 $\text{full1 } \text{transf} = (\lambda S \ S'. \text{tranclp } \text{transf } S \ S' \wedge (\forall S''. \neg \text{transf } S' \ S''))$

definition *full* :: $('a \Rightarrow 'a \Rightarrow \text{bool}) \Rightarrow 'a \Rightarrow 'a \Rightarrow \text{bool}$ **where**
 $\text{full } \text{transf} = (\lambda S \ S'. \text{rtranclp } \text{transf } S \ S' \wedge (\forall S''. \neg \text{transf } S' \ S''))$

lemma *rtranclp-full1I*:
 $R^{**} \ a \ b \Longrightarrow \text{full1 } R \ b \ c \Longrightarrow \text{full1 } R \ a \ c$
unfolding *full1-def* **by** *auto*

lemma *tranclp-full1I*:
 $R^{++} \ a \ b \Longrightarrow \text{full1 } R \ b \ c \Longrightarrow \text{full1 } R \ a \ c$

```

unfolding full1-def by auto

lemma rtrancpl-fullI:
   $R^{**} a b \implies \text{full } R b c \implies \text{full } R a c$ 
  unfolding full-def by auto

lemma trancpl-full-fullII:
   $R^{++} a b \implies \text{full } R b c \implies \text{full1 } R a c$ 
  unfolding full-def full1-def by auto

lemma full-fullI:
   $R a b \implies \text{full } R b c \implies \text{full1 } R a c$ 
  unfolding full-def full1-def by auto

lemma full-unfold:
   $\text{full } r S S' \iff ((S = S' \wedge \text{no-step } r S') \vee \text{full1 } r S S')$ 
  unfolding full-def full1-def by (auto simp add: rtrancpl-unfold)

lemma full1-is-full[intro]:  $\text{full1 } R S T \implies \text{full } R S T$ 
  by (simp add: full-unfold)

lemma not-full1-rtrancpl-relation:  $\neg \text{full1 } R^{**} a b$ 
  by (meson full1-def rtrancpl.rtrancpl-refl)

lemma not-full-rtrancpl-relation:  $\neg \text{full } R^{**} a b$ 
  by (meson full-fullI not-full1-rtrancpl-relation rtrancpl.rtrancpl-refl)

lemma full1-trancpl-relation-full:
   $\text{full1 } R^{++} a b \iff \text{full1 } R a b$ 
  by (metis converse-trancplE full1-def reflclp-trancpl rtrancplD rtrancpl-idemp rtrancpl-reflclp
    trancpl.r-into-trancpl trancpl-into-rtrancpl)

lemma full-trancpl-relation-full:
   $\text{full } R^{++} a b \iff \text{full } R a b$ 
  by (metis full-unfold full1-trancpl-relation-full trancpl.r-into-trancpl trancplD)

lemma rtrancpl-full1-eq-or-full1:
   $(\text{full1 } R)^{**} a b \iff (a = b \vee \text{full1 } R a b)$ 
proof -
  have  $\forall p a aa. \neg p^{**} (a::'a) aa \vee a = aa \vee (\exists ab. p^{**} a ab \wedge p ab aa)$ 
  by (metis rtrancpl.cases)
  then obtain  $aa :: ('a \Rightarrow 'a \Rightarrow \text{bool}) \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a$  where
     $f1: \forall p a ab. \neg p^{**} a ab \vee a = ab \vee p^{**} a (aa p a ab) \wedge p (aa p a ab) ab$ 
  by mouna
  { assume  $a \neq b$ 
    { assume  $\neg \text{full1 } R a b \wedge a \neq b$ 
      then have  $a \neq b \wedge a \neq b \wedge \neg \text{full1 } R (aa (\text{full1 } R) a b) b \vee \neg (\text{full1 } R)^{**} a b \wedge a \neq b$ 
        using f1 by (metis (no-types) full1-def full1-trancpl-relation-full)
      then have ?thesis
        using f1 by blast }
    then have ?thesis
      by auto }
  then show ?thesis
    by fastforce
qed

```

lemma *tranclp-full1-full1*:
 $(full1\ R)^{++}\ a\ b \longleftrightarrow full1\ R\ a\ b$
by (*metis full1-def rtranclp-full1-eq-or-full1 tranclp-unfold-begin*)

1.3 Well-Foundedness and Full Transitions

lemma *wf-exists-normal-form*:
assumes $wf:wf\ \{(x, y). R\ y\ x\}$
shows $\exists b. R^{**}\ a\ b \wedge no\text{-}step\ R\ b$
proof (*rule ccontr*)
assume $\neg\ ?thesis$
then have $H: \bigwedge b. \neg R^{**}\ a\ b \vee \neg no\text{-}step\ R\ b$
by *blast*
def $F \equiv rec\text{-}nat\ a\ (\lambda i\ b. SOME\ c. R\ b\ c)$
have [*simp*]: $F\ 0 = a$
unfolding *F-def* **by** *auto*
have [*simp*]: $\bigwedge i. F\ (Suc\ i) = (SOME\ b. R\ (F\ i)\ b)$
using *F-def* **by** *simp*
{ fix i
have $\forall j < i. R\ (F\ j)\ (F\ (Suc\ j))$
proof (*induction i*)
case 0
then show *?case* **by** *auto*
next
case $(Suc\ i)$
then have $R^{**}\ a\ (F\ i)$
by (*induction i*) *auto*
then have $R\ (F\ i)\ (SOME\ b. R\ (F\ i)\ b)$
using *H* **by** (*simp add: someI-ex*)
then have $\forall j < Suc\ i. R\ (F\ j)\ (F\ (Suc\ j))$
using *H Suc* **by** (*simp add: less-Suc-eq*)
then show *?case* **by** *fast*
qed
}
then have $\forall j. R\ (F\ j)\ (F\ (Suc\ j))$ **by** *blast*
then show *False*
using *wf* **unfolding** *wfP-def wf-iff-no-infinite-down-chain* **by** *blast*
qed

lemma *wf-exists-normal-form-full*:
assumes $wf:wf\ \{(x, y). R\ y\ x\}$
shows $\exists b. full\ R\ a\ b$
using *wf-exists-normal-form[OF assms]* **unfolding** *full-def* **by** *blast*

1.4 More Well-Foundedness

A little list of theorems that could be useful, but are hidden:

- link between *wf* and infinite chains: $wf\ ?r = (\nexists f. \forall i. (f\ (Suc\ i), f\ i) \in ?r), \llbracket wf\ ?r; \bigwedge k. (?f\ (Suc\ k), ?f\ k) \notin ?r \implies ?thesis \rrbracket \implies ?thesis$

lemma *wf-if-measure-in-wf*:
 $wf\ R \implies (\bigwedge a\ b. (a, b) \in S \implies (\nu\ a, \nu\ b) \in R) \implies wf\ S$
by (*metis in-inv-image wfE-min wfI-min wf-inv-image*)


```

lemma wfP-if-measure: fixes f :: 'a ⇒ nat
shows (⋀x y. P x ⇒ g x y ⇒ f y < f x) ⇒ wf {(y,x). P x ∧ g x y}
  apply(insert wf-measure[of f])
  apply(simp only: measure-def inv-image-def less-than-def less-eq)
  apply(erule wf-subset)
  apply auto
done

```

```

lemma wf-if-measure-f:
assumes wf r
shows wf {(b, a). (f b, f a) ∈ r}
  using assms by (metis inv-image-def wf-inv-image)

```

```

lemma wf-wf-if-measure':
assumes wf r and H: (⋀x y. P x ⇒ g x y ⇒ (f y, f x) ∈ r)
shows wf {(y,x). P x ∧ g x y}
proof -
  have wf {(b, a). (f b, f a) ∈ r} using assms(1) wf-if-measure-f by auto
  then have wf {(b, a). P a ∧ g a b ∧ (f b, f a) ∈ r}
    using wf-subset[of - {(b, a). P a ∧ g a b ∧ (f b, f a) ∈ r}] by auto
  moreover have {(b, a). P a ∧ g a b ∧ (f b, f a) ∈ r} ⊆ {(b, a). (f b, f a) ∈ r} by auto
  moreover have {(b, a). P a ∧ g a b ∧ (f b, f a) ∈ r} = {(b, a). P a ∧ g a b} using H by auto
  ultimately show ?thesis using wf-subset by simp
qed

```

```

lemma wf-lex-less: wf (lex {(a, b). (a::nat) < b})
proof -
  have m: {(a, b). a < b} = measure id by auto
  show ?thesis apply (rule wf-lex) unfolding m by auto
qed

```

```

lemma wfP-if-measure2: fixes f :: 'a ⇒ nat
shows (⋀x y. P x y ⇒ g x y ⇒ f x < f y) ⇒ wf {(x,y). P x y ∧ g x y}
  apply(insert wf-measure[of f])
  apply(simp only: measure-def inv-image-def less-than-def less-eq)
  apply(erule wf-subset)
  apply auto
done

```

```

lemma lexord-on-finite-set-is-wf:
assumes
  P-finite: ⋀U. P U ⟶ U ∈ A and
  finite: finite A and
  wf: wf R and
  trans: trans R
shows wf {(T, S). (P S ∧ P T) ∧ (T, S) ∈ lexord R}
proof (rule wfP-if-measure2)
  fix T S
  assume P: P S ∧ P T and
  s-le-t: (T, S) ∈ lexord R
  let ?f = λS. {U. (U, S) ∈ lexord R ∧ P U ∧ P S}
  have ?f T ⊆ ?f S
    using s-le-t P lexord-trans trans by auto
  moreover have T ∈ ?f S

```

```

    using s-le-t P by auto
  moreover have  $T \notin ?f\ T$ 
    using s-le-t by (auto simp add: lexord-irreflexive local.wf)
  ultimately have  $\{U. (U, T) \in \text{lexord } R \wedge P\ U \wedge P\ T\} \subset \{U. (U, S) \in \text{lexord } R \wedge P\ U \wedge P\ S\}$ 
    by auto
  moreover have finite  $\{U. (U, S) \in \text{lexord } R \wedge P\ U \wedge P\ S\}$ 
    using finite by (metis (no-types, lifting) P-finite finite-subset mem-Collect-eq subsetI)
  ultimately show  $\text{card } (?f\ T) < \text{card } (?f\ S)$  by (simp add: psubset-card-mono)
qed

```

```

lemma wf-fst-wf-pair:
  assumes wf  $\{(M', M). R\ M'\ M\}$ 
  shows wf  $\{((M', N'), (M, N)). R\ M'\ M\}$ 
proof -
  have wf  $\{(M', M). R\ M'\ M\} <*\text{lex*}> \{\}$ 
    using assms by auto
  then show ?thesis
    by (rule wf-subset) auto
qed

```

```

lemma wf-snd-wf-pair:
  assumes wf  $\{(M', M). R\ M'\ M\}$ 
  shows wf  $\{((M', N'), (M, N)). R\ N'\ N\}$ 
proof -
  have wf: wf  $\{((M', N'), (M, N)). R\ M'\ M\}$ 
    using assms wf-fst-wf-pair by auto
  then have wf:  $\bigwedge P. (\forall x. (\forall y. (y, x) \in \{((M', N'), M, N). R\ M'\ M\} \longrightarrow P\ y) \longrightarrow P\ x) \Longrightarrow \text{All } P$ 
    unfolding wf-def by auto
  show ?thesis
    unfolding wf-def
  proof (intro allI impI)
    fix  $P :: 'c \times 'a \Rightarrow \text{bool}$  and  $x :: 'c \times 'a$ 
    assume  $H: \forall x. (\forall y. (y, x) \in \{((M', N'), M, y). R\ N'\ y\} \longrightarrow P\ y) \longrightarrow P\ x$ 
    obtain  $a\ b$  where  $x = (a, b)$  by (cases x)
    have  $P: P\ x = (P \circ (\lambda(a, b). (b, a))) (b, a)$ 
      unfolding  $x$  by auto
    show  $P\ x$ 
      using wf [of  $P \circ (\lambda(a, b). (b, a))$ ] apply rule
      using  $H$  apply simp
      unfolding  $P$  by blast
  qed
qed

```

```

lemma wf-if-measure-f-notation2:
  assumes wf  $r$ 
  shows wf  $\{(b, h\ a)|b\ a. (f\ b, f\ (h\ a)) \in r\}$ 
  apply (rule wf-subset)
  using wf-if-measure-f [OF assms, of f] by auto

```

```

lemma wf-wf-if-measure'-notation2:
  assumes wf  $r$  and  $H: (\bigwedge x\ y. P\ x \Longrightarrow g\ x\ y \Longrightarrow (f\ y, f\ (h\ x)) \in r)$ 
  shows wf  $\{(y, h\ x)|\ y\ x. P\ x \wedge g\ x\ y\}$ 
proof -
  have wf  $\{(b, h\ a)|b\ a. (f\ b, f\ (h\ a)) \in r\}$  using assms (1) wf-if-measure-f-notation2 by auto

```

```

then have wf  $\{(b, h\ a)|b\ a.\ P\ a \wedge g\ a\ b \wedge (f\ b, f\ (h\ a)) \in r\}$ 
  using wf-subset[of -  $\{(b, h\ a)|b\ a.\ P\ a \wedge g\ a\ b \wedge (f\ b, f\ (h\ a)) \in r\}$ ] by auto
moreover have  $\{(b, h\ a)|b\ a.\ P\ a \wedge g\ a\ b \wedge (f\ b, f\ (h\ a)) \in r\}$ 
   $\subseteq \{(b, h\ a)|b\ a.\ (f\ b, f\ (h\ a)) \in r\}$  by auto
moreover have  $\{(b, h\ a)|b\ a.\ P\ a \wedge g\ a\ b \wedge (f\ b, f\ (h\ a)) \in r\} = \{(b, h\ a)|b\ a.\ P\ a \wedge g\ a\ b\}$ 
  using H by auto
ultimately show ?thesis using wf-subset by simp
qed

```

```

end
theory List-More
imports Main ../lib/Multiset-More
begin

```

Sledgehammer parameters

```
sledgehammer-params[debug]
```

2 Various Lemmas

Close to $(\bigwedge n. \forall m < n. ?P\ m \implies ?P\ n) \implies ?P\ ?n$, but with a separation between zero and non-zero, and case names.

thm *nat-less-induct*

lemma *nat-less-induct-case*[*case-names 0 Suc*]:

assumes

$P\ 0$ and

$\bigwedge n. (\forall m < Suc\ n. P\ m) \implies P\ (Suc\ n)$

shows $P\ n$

apply (*induction rule: nat-less-induct*)

by (*rename-tac n, case-tac n*) (*auto intro: assms*)

This is only proved in simple cases by auto. In assumptions, nothing happens, and $?P$ (*if ?Q then ?x else ?y*) = $(\neg (?Q \wedge \neg ?P\ ?x \vee \neg ?Q \wedge \neg ?P\ ?y))$ can blow up goals (because of other if expression).

lemma *if-0-1-ge-0*[*simp*]:

$0 < (if\ P\ then\ a\ else\ (0::nat)) \longleftrightarrow P \wedge 0 < a$

by auto

Bounded function have not been defined in Isabelle.

definition *bounded* **where**

bounded f $\longleftrightarrow (\exists b. \forall n. f\ n \leq b)$

abbreviation *unbounded* :: $('a \Rightarrow 'b::ord) \Rightarrow bool$ **where**

unbounded f $\equiv \neg bounded\ f$

lemma *not-bounded-nat-exists-larger*:

fixes $f :: nat \Rightarrow nat$

assumes *unbound*: *unbounded f*

shows $\exists n. f\ n > m \wedge n > n_0$

proof (*rule ccontr*)

assume *H*: $\neg ?thesis$

have *finite* $\{f\ n | n. n \leq n_0\}$

by auto

have $\bigwedge n. f\ n \leq Max\ (\{f\ n | n. n \leq n_0\} \cup \{m\})$

```

apply (case-tac  $n \leq n_0$ )
apply (metis (mono-tags, lifting) Max-ge Un-insert-right ⟨finite {f n | n. n ≤ n0}⟩
  finite-insert insertCI mem-Collect-eq sup-bot.right-neutral)
by (metis (no-types, lifting) H Max-less-iff Un-insert-right ⟨finite {f n | n. n ≤ n0}⟩
  finite-insert insertI1 insert-not-empty leI sup-bot.right-neutral)
then show False
  using unbound unfolding bounded-def by auto
qed

```

```

lemma bounded-const-product:
  fixes k :: nat and f :: nat ⇒ nat
  assumes k > 0
  shows bounded f ⟷ bounded (λi. k * f i)
  unfolding bounded-def apply (rule iffI)
  using mult-le-mono2 apply blast
  by (meson assms le-less-trans less-or-eq-imp-le nat-mult-less-cancel-disj split-div-lemma)

```

This lemma is not used, but here to show that a property that can be expected from *bounded* holds.

```

lemma bounded-finite-linorder:
  fixes f :: 'a ⇒ 'a :: {finite, linorder}
  shows bounded f
proof –
  have ∧x. f x ≤ Max {f x | x. True}
    by (metis (mono-tags) Max-ge finite mem-Collect-eq)
  then show ?thesis
    unfolding bounded-def by blast
qed

```

3 More List

3.1 upt

The simplification rules are not very handy, because $[?i..<Suc\ ?j] = (if\ ?i \leq\ ?j\ then\ [?i..<?j]\ @\ [?j]\ else\ [])$ leads to a case distinction, that we do not want if the condition is not in the context.

```

lemma upt-Suc-le-append:  $\neg i \leq j \implies [i..<Suc\ j] = []$ 
  by auto

```

```

lemmas upt-simps[simp] = upt-Suc-append upt-Suc-le-append

```

```

declare upt.simps(2)[simp del]

```

```

lemma
  assumes  $i \leq n - m$ 
  shows take i [m.. $n$ ] = [m.. $m+i$ ]
  by (metis Nat.le-diff-conv2 add commute assms diff-is-0-eq' linear take-upt upt-conv-Nil)

```

The counterpart for this lemma when $n - m < i$ is $length\ ?xs \leq ?n \implies take\ ?n\ ?xs = ?xs$. It is close to $?i + ?m \leq ?n \implies take\ ?m\ [?i..<?n] = [?i..<?i + ?m]$, but seems more general.

```

lemma take-upt-bound-minus[simp]:
  assumes  $i \leq n - m$ 
  shows take i [m.. $n$ ] = [m.. $m+i$ ]

```

using *assms* **by** (*induction i*) *auto*

lemma *append-cons-eq-upt*:

assumes $A @ B = [m..<n]$

shows $A = [m ..<m+length\ A]$ **and** $B = [m + length\ A..<n]$

proof –

have *take* (*length A*) ($A @ B$) = A **by** *auto*

moreover

have $length\ A \leq n - m$ **using** *assms linear calculation* **by** *fastforce*

then have *take* (*length A*) $[m..<n] = [m ..<m+length\ A]$ **by** *auto*

ultimately show $A = [m ..<m+length\ A]$ **using** *assms* **by** *auto*

show $B = [m + length\ A..<n]$ **using** *assms* **by** (*metis append-eq-conv-conj drop-upt*)

qed

The converse of $?A @ ?B = [?m..<?n] \implies ?A = [?m..<?m + length\ ?A]$

$?A @ ?B = [?m..<?n] \implies ?B = [?m + length\ ?A..<?n]$ does not hold, for example if B is empty and A is $[0::'a]$:

lemma $A @ B = [m..<n] \longleftrightarrow A = [m ..<m+length\ A] \wedge B = [m + length\ A..<n]$

oops

A more restrictive version holds:

lemma $B \neq [] \implies A @ B = [m..<n] \longleftrightarrow A = [m ..<m+length\ A] \wedge B = [m + length\ A..<n]$
(**is** $?P \implies ?A = ?B$)

proof

assume $?A$ **then show** $?B$ **by** (*auto simp add: append-cons-eq-upt*)

next

assume $?P$ **and** $?B$

then show $?A$ **using** *append-eq-conv-conj* **by** *fastforce*

qed

lemma *append-cons-eq-upt-length-i*:

assumes $A @ i \# B = [m..<n]$

shows $A = [m ..<i]$

proof –

have $A = [m ..<m + length\ A]$ **using** *assms append-cons-eq-upt* **by** *auto*

have $(A @ i \# B) ! (length\ A) = i$ **by** *auto*

moreover have $n - m = length\ (A @ i \# B)$

using *assms length-upt* **by** *presburger*

then have $[m..<n] ! (length\ A) = m + length\ A$ **by** *simp*

ultimately have $i = m + length\ A$ **using** *assms* **by** *auto*

then show *thesis* **using** $\langle A = [m ..<m + length\ A] \rangle$ **by** *auto*

qed

lemma *append-cons-eq-upt-length*:

assumes $A @ i \# B = [m..<n]$

shows $length\ A = i - m$

using *assms*

proof (*induction A arbitrary: m*)

case *Nil*

then show *?case* **by** (*metis append-Nil diff-is-0-eq list.size(3) order-refl upt-eq-Cons-conv*)

next

case (*Cons a A*)

then have $A : A @ i \# B = [m + 1..<n]$ **by** (*metis append-Cons upt-eq-Cons-conv*)

then have $m < i$ **by** (*metis Cons.premis append-cons-eq-upt-length-i upt-eq-Cons-conv*)

with *Cons.IH*[*OF A*] **show** ?*case* **by** *auto*
qed

lemma *append-cons-eq-upt-length-i-end*:

assumes $A @ i \# B = [m..<n]$

shows $B = [Suc\ i \ ..<n]$

proof –

have $B = [Suc\ m + length\ A..<n]$ **using** *assms append-cons-eq-upt[of A @ [i] B m n]* **by** *auto*

have $(A @ i \# B) ! (length\ A) = i$ **by** *auto*

moreover **have** $n - m = length\ (A @ i \# B)$

using *assms length-upt* **by** *auto*

then **have** $[m..<n] ! (length\ A) = m + length\ A$ **by** *simp*

ultimately **have** $i = m + length\ A$ **using** *assms* **by** *auto*

then **show** ?*thesis* **using** $B = [Suc\ m + length\ A..<n]$ **by** *auto*

qed

lemma *Max-n-upt*: $Max\ (insert\ 0\ \{Suc\ 0..<n\}) = n - Suc\ 0$

proof (*induct n*)

case 0

then **show** ?*case* **by** *simp*

next

case (*Suc n*) **note** *IH = this*

have $i: insert\ 0\ \{Suc\ 0..<Suc\ n\} = insert\ 0\ \{Suc\ 0..<n\} \cup \{n\}$ **by** *auto*

show ?*case* **using** *IH* **unfolding** *i* **by** *auto*

qed

lemma *upt-decomp-lt*:

assumes $H: xs @ i \# ys @ j \# zs = [m \ ..< n]$

shows $i < j$

proof –

have $xs: xs = [m \ ..< i]$ **and** $ys: ys = [Suc\ i \ ..< j]$ **and** $zs: zs = [Suc\ j \ ..< n]$

using *H* **by** (*auto dest: append-cons-eq-upt-length-i append-cons-eq-upt-length-i-end*)

show ?*thesis*

by (*metis append-cons-eq-upt-length-i-end assms lessI less-trans self-append-conv2*
upt-eq-Cons-conv upt-rec ys)

qed

3.2 Lexicographic ordering

We are working a lot on lexicographic ordering over pairs.

lemma *list-length2-append-cons*:

$[c, d] = ys @ y \# ys' \longleftrightarrow (ys = [] \wedge y = c \wedge ys' = [d]) \vee (ys = [c] \wedge y = d \wedge ys' = [])$

by (*cases ys; cases ys'*) *auto*

lemma *lexn2-conv*:

$([a, b], [c, d]) \in lexn\ r\ 2 \longleftrightarrow (a, c) \in r \vee (a = c \wedge (b, d) \in r)$

unfolding *lexn-conv* **by** (*auto simp add: list-length2-append-cons*)

3.3 Remove and Multiset equality

lemma *remove1-mset-single-add*:

$a \neq b \implies remove1\ mset\ a\ (\{\#b\# \} + C) = \{\#b\# \} + remove1\ mset\ a\ C$

$remove1\ mset\ a\ (\{\#a\# \} + C) = C$

by (*auto simp: multiset-eq-iff*)

This is the same as *remove1* under the assumptions of non-duplication inside a clause.

```

fun remove1-cond where
  remove1-cond f [] = [] |
  remove1-cond f (C' # L) = (if f C' then L else C' # remove1-cond f L)

```

```

lemma mset-map-mset-remove1-cond:
  mset (map mset (remove1-cond (λL. mset L = mset a) C)) =
    remove1-mset (mset a) (mset (map mset C))
  by (induction C) (auto simp: ac-simps remove1-mset-single-add)

```

```

fun removeAll-cond where
  removeAll-cond f [] = [] |
  removeAll-cond f (C' # L) =
    (if f C' then removeAll-cond f L else C' # removeAll-cond f L)

```

```

lemma mset-map-mset-removeAll-cond:
  mset (map mset (removeAll-cond (λb. mset b = mset a) C))
    = removeAll-mset (mset a) (mset (map mset C))
  by (induction C) (auto simp: ac-simps mset-less-eqI multiset-diff-union-assoc)

```

```

abbreviation union-mset-list where
  union-mset-list xs ys ≡ case-prod append (fold (λx (ys, zs). (remove1 x ys, x # zs)) xs (ys, []))

```

```

lemma union-mset-list:
  mset xs #∪ mset ys = mset (union-mset-list xs ys)
proof –
  have ∧zs. mset (case-prod append (fold (λx (ys, zs). (remove1 x ys, x # zs)) xs (ys, zs))) =
    (mset xs #∪ mset ys) + mset zs
  by (induct xs arbitrary: ys) (simp-all add: multiset-eq-iff)
  then show ?thesis by simp
qed
end
theory Prop-Logic

```

```

imports Main

```

```

begin

```

4 Logics

In this section we define the syntax of the formula and an abstraction over it to have simpler proofs. After that we define some properties like subformula and rewriting.

4.1 Definition and abstraction

The propositional logic is defined inductively. The type parameter is the type of the variables.

```

datatype 'v propo =
  FT | FF | FVar 'v | FNot 'v propo | FAnd 'v propo 'v propo | FOr 'v propo 'v propo
  | FImp 'v propo 'v propo | FEq 'v propo 'v propo

```

We do not define any notation for the formula, to distinguish properly between the formulas and Isabelle's logic.

To ease the proofs, we will write the the formula on a homogeneous manner, namely a connecting argument and a list of arguments.

datatype *'v connective* = *CT* | *CF* | *CVar 'v* | *CNot* | *CAnd* | *COr* | *CImp* | *CEq*

abbreviation *nullary-connective* $\equiv \{CF\} \cup \{CT\} \cup \{CVar\ x \mid x. \text{True}\}$

definition *binary-connectives* $\equiv \{CAnd, COr, CImp, CEq\}$

We define our own induction principal: instead of distinguishing every constructor, we group them by arity.

lemma *propo-induct-arity*[*case-names nullary unary binary*]:

fixes $\varphi\ \psi :: 'v\ propo$
assumes *nullary*: $(\bigwedge \varphi\ x. \varphi = FF \vee \varphi = FT \vee \varphi = FVar\ x \implies P\ \varphi)$
and *unary*: $(\bigwedge \psi. P\ \psi \implies P\ (FNot\ \psi))$
and *binary*: $(\bigwedge \varphi\ \psi1\ \psi2. P\ \psi1 \implies P\ \psi2 \implies \varphi = FAnd\ \psi1\ \psi2 \vee \varphi = FOr\ \psi1\ \psi2 \vee \varphi = FImp\ \psi1\ \psi2 \vee \varphi = FEq\ \psi1\ \psi2 \implies P\ \varphi)$
shows $P\ \psi$
apply (*induct rule: propo.induct*)
using *assms* **by** *metis+*

The function *conn* is the interpretation of our representation (connective and list of arguments). We define any thing that has no sense to be false

fun *conn* :: *'v connective* \Rightarrow *'v propo list* \Rightarrow *'v propo* **where**
conn CT [] = *FT* |
conn CF [] = *FF* |
conn (CVar v) [] = *FVar v* |
conn CNot [φ] = *FNot* φ |
conn CAnd (φ # [ψ]) = *FAnd* $\varphi\ \psi$ |
conn COr (φ # [ψ]) = *FOr* $\varphi\ \psi$ |
conn CImp (φ # [ψ]) = *FImp* $\varphi\ \psi$ |
conn CEq (φ # [ψ]) = *FEq* $\varphi\ \psi$ |
conn - - = *FF*

We will often use case distinction, based on the arity of the *'v connective*, thus we define our own splitting principle.

lemma *connective-cases-arity*[*case-names nullary binary unary*]:

assumes *nullary*: $\bigwedge x. c = CT \vee c = CF \vee c = CVar\ x \implies P$
and *binary*: $c \in \text{binary-connectives} \implies P$
and *unary*: $c = CNot \implies P$
shows P
using *assms* **by** (*cases c*) (*auto simp: binary-connectives-def*)

lemma *connective-cases-arity-2*[*case-names nullary unary binary*]:

assumes *nullary*: $c \in \text{nullary-connective} \implies P$
and *unary*: $c = CNot \implies P$
and *binary*: $c \in \text{binary-connectives} \implies P$
shows P
using *assms* **by** (*cases c, auto simp add: binary-connectives-def*)

Our previous definition is not necessary correct (connective and list of arguments) , so we define an inductive predicate.

inductive *wf-conn* :: *'v connective* \Rightarrow *'v propo list* \Rightarrow *bool* **for** $c :: 'v\ connective$ **where**

wf-conn-nullary[*simp*]: $(c = CT \vee c = CF \vee c = CVar\ v) \implies \text{wf-conn } c\ []$ |

wf-conn-unary[*simp*]: $c = CNot \implies \text{wf-conn } c\ [\psi]$ |

wf-conn-binary[*simp*]: $c \in \text{binary-connectives} \implies \text{wf-conn } c\ (\psi\ \# \psi'\ \# [])$

thm *wf-conn.induct*

lemma *wf-conn-induct*[consumes 1, case-names *CT CF CVar CNot COr CAnd CImp CEq*]:

assumes *wf-conn c x* **and**

$(\bigwedge v. c = CT \implies P [])$ **and**

$(\bigwedge v. c = CF \implies P [])$ **and**

$(\bigwedge v. c = CVar\ v \implies P [])$ **and**

$(\bigwedge \psi. c = CNot \implies P [\psi])$ **and**

$(\bigwedge \psi\ \psi'. c = COr \implies P [\psi, \psi'])$ **and**

$(\bigwedge \psi\ \psi'. c = CAnd \implies P [\psi, \psi'])$ **and**

$(\bigwedge \psi\ \psi'. c = CImp \implies P [\psi, \psi'])$ **and**

$(\bigwedge \psi\ \psi'. c = CEq \implies P [\psi, \psi'])$

shows $P\ x$

using *assms* **by** *induction (auto simp: binary-connectives-def)*

4.2 properties of the abstraction

First we can define simplification rules.

lemma *wf-conn-conn*[*simp*]:

wf-conn CT l \implies *conn CT l* = *FT*

wf-conn CF l \implies *conn CF l* = *FF*

wf-conn (CVar x) l \implies *conn (CVar x) l* = *FVar x*

apply (*simp-all add: wf-conn.simps*)

unfolding *binary-connectives-def* **by** *simp-all*

lemma *wf-conn-list-decomp*[*simp*]:

wf-conn CT l $\longleftrightarrow l = []$

wf-conn CF l $\longleftrightarrow l = []$

wf-conn (CVar x) l $\longleftrightarrow l = []$

wf-conn CNot ($\xi @ \varphi \# \xi'$) $\longleftrightarrow \xi = [] \wedge \xi' = []$

apply (*simp-all add: wf-conn.simps*)

unfolding *binary-connectives-def* **apply** *simp-all*

by (*metis append-Nil append-is-Nil-conv list.distinct(1) list.sel(3) tl-append2*)

lemma *wf-conn-list*:

wf-conn c l \implies *conn c l* = *FT* $\longleftrightarrow (c = CT \wedge l = [])$

wf-conn c l \implies *conn c l* = *FF* $\longleftrightarrow (c = CF \wedge l = [])$

wf-conn c l \implies *conn c l* = *FVar x* $\longleftrightarrow (c = CVar\ x \wedge l = [])$

wf-conn c l \implies *conn c l* = *FAnd a b* $\longleftrightarrow (c = CAnd \wedge l = a \# b \# [])$

wf-conn c l \implies *conn c l* = *FOr a b* $\longleftrightarrow (c = COr \wedge l = a \# b \# [])$

wf-conn c l \implies *conn c l* = *FEq a b* $\longleftrightarrow (c = CEq \wedge l = a \# b \# [])$

wf-conn c l \implies *conn c l* = *FImp a b* $\longleftrightarrow (c = CImp \wedge l = a \# b \# [])$

wf-conn c l \implies *conn c l* = *FNot a* $\longleftrightarrow (c = CNot \wedge l = a \# [])$

apply (*induct l rule: wf-conn.induct*)

unfolding *binary-connectives-def* **by** *auto*

In the binary connective cases, we will often decompose the list of arguments (of length 2) into two elements.

lemma *list-length2-decomp*: $\text{length } l = 2 \implies (\exists\ a\ b. l = a \# b \# [])$

apply (*induct l, auto*)

by (*rename-tac l, case-tac l, auto*)

wf-conn for binary operators means that there are two arguments.

lemma *wf-conn-bin-list-length*:

```

fixes  $l :: 'v \text{ propo list}$ 
assumes  $\text{conn}: c \in \text{binary-connectives}$ 
shows  $\text{length } l = 2 \longleftrightarrow \text{wf-conn } c \ l$ 
proof
  assume  $\text{length } l = 2$ 
  then show  $\text{wf-conn } c \ l$  using  $\text{wf-conn-binary list-length2-decomp}$  using  $\text{conn}$  by  $\text{metis}$ 
next
  assume  $\text{wf-conn } c \ l$ 
  then show  $\text{length } l = 2$  (is ?P l)
  proof  $(\text{cases rule: wf-conn.induct})$ 
    case  $\text{wf-conn-nullary}$ 
    then show  $?P []$  using  $\text{conn binary-connectives-def}$ 
    using  $\text{connective.distinct(11) connective.distinct(13) connective.distinct(9)}$  by  $\text{blast}$ 
  next
    fix  $\psi :: 'v \text{ propo}$ 
    case  $\text{wf-conn-unary}$ 
    then show  $?P [\psi]$  using  $\text{conn binary-connectives-def}$ 
    using  $\text{connective.distinct}$  by  $\text{blast}$ 
  next
    fix  $\psi \ \psi' :: 'v \text{ propo}$ 
    show  $?P [\psi, \psi']$  by  $\text{auto}$ 
  qed
qed

```

```

lemma  $\text{wf-conn-not-list-length[iff]}:$ 
  fixes  $l :: 'v \text{ propo list}$ 
  shows  $\text{wf-conn } \text{CNot } l \longleftrightarrow \text{length } l = 1$ 
  apply  $\text{auto}$ 
  apply  $(\text{metis append-Nil connective.distinct(5,17,27) length-Cons list.size(3) wf-conn.simps wf-conn-list-decomp(4)})$ 
  by  $(\text{simp add: length-Suc-conv wf-conn.simps})$ 

```

Decomposing the Not into an element is moreover very useful.

```

lemma  $\text{wf-conn-Not-decomp}:$ 
  fixes  $l :: 'v \text{ propo list}$  and  $a :: 'v$ 
  assumes  $\text{corr}: \text{wf-conn } \text{CNot } l$ 
  shows  $\exists a. l = [a]$ 
  by  $(\text{metis (no-types, lifting) One-nat-def Suc-length-conv corr length-0-conv wf-conn-not-list-length})$ 

```

The wf-conn remains correct if the length of list does not change. This lemma is very useful when we do one rewriting step

```

lemma  $\text{wf-conn-no-arity-change}:$ 
   $\text{length } l = \text{length } l' \implies \text{wf-conn } c \ l \longleftrightarrow \text{wf-conn } c \ l'$ 
proof –
  {
    fix  $l \ l'$ 
    have  $\text{length } l = \text{length } l' \implies \text{wf-conn } c \ l \implies \text{wf-conn } c \ l'$ 
    apply  $(\text{cases } c \ l \text{ rule: wf-conn.induct, auto})$ 
    by  $(\text{metis wf-conn-bin-list-length})$ 
  }
  then show  $\text{length } l = \text{length } l' \implies \text{wf-conn } c \ l = \text{wf-conn } c \ l'$  by  $\text{metis}$ 
qed

```

```

lemma  $\text{wf-conn-no-arity-change-helper}:$ 

```

```
length (ξ @ φ # ξ') = length (ξ @ φ' # ξ')
by auto
```

The injectivity of *conn* is useful to prove equality of the connectives and the lists.

```
lemma conn-inj-not:
  assumes correct: wf-conn c l
  and conn: conn c l = FNot ψ
  shows c = CNot and l = [ψ]
  apply (cases c l rule: wf-conn.cases)
  using correct conn unfolding binary-connectives-def apply auto
  apply (cases c l rule: wf-conn.cases)
  using correct conn unfolding binary-connectives-def by auto
```

```
lemma conn-inj:
  fixes c ca :: 'v connective and l ψs :: 'v propo list
  assumes corr: wf-conn ca l
  and corr': wf-conn c ψs
  and eq: conn ca l = conn c ψs
  shows ca = c ∧ ψs = l
  using corr
proof (cases ca l rule: wf-conn.cases)
  case (wf-conn-nullary v)
  then show ca = c ∧ ψs = l using assms
    by (metis conn.simps(1) conn.simps(2) conn.simps(3) wf-conn-list(1-3))
next
  case (wf-conn-unary ψ')
  then have *: FNot ψ' = conn c ψs using conn-inj-not eq assms by auto
  then have c = ca by (metis conn-inj-not(1) corr' wf-conn-unary(2))
  moreover have ψs = l using * conn-inj-not(2) corr' wf-conn-unary(1) by force
  ultimately show ca = c ∧ ψs = l by auto
next
  case (wf-conn-binary ψ' ψ'')
  then show ca = c ∧ ψs = l
    using eq corr' unfolding binary-connectives-def apply (cases ca, auto simp add: wf-conn-list)
    using wf-conn-list(4-7) corr' by metis+
qed
```

4.3 Subformulas and properties

A characterization using sub-formulas is interesting for rewriting: we will define our relation on the sub-term level, and then lift the rewriting on the term-level. So the rewriting takes place on a subformula.

```
inductive subformula :: 'v propo ⇒ 'v propo ⇒ bool (infix ≤ 45) for φ where
  subformula-refl[simp]: φ ≤ φ |
  subformula-into-subformula: ψ ∈ set l ⇒ wf-conn c l ⇒ φ ≤ ψ ⇒ φ ≤ conn c l
```

On the *subformula-into-subformula*, we can see why we use our *conn* representation: one case is enough to express the subformulas property instead of listing all the cases.

This is an example of a property related to subformulas.

```
lemma subformula-in-subformula-not:
  shows b: FNot φ ≤ ψ ⇒ φ ≤ ψ
  apply (induct rule: subformula.induct)
```

using *subformula-into-subformula wf-conn-unary subformula-refl list.set-intros(1) subformula-refl*
by (*fastforce intro: subformula-into-subformula*)**+**

lemma *subformula-in-binary-conn:*

assumes *conn: c ∈ binary-connectives*

shows $f \preceq \text{conn } c [f, g]$

and $g \preceq \text{conn } c [f, g]$

proof $-$

have *a: wf-conn c (f# [g])* **using** *conn wf-conn-binary binary-connectives-def* **by** *auto*

moreover **have** *b: f ≼ f* **using** *subformula-refl* **by** *auto*

ultimately **show** $f \preceq \text{conn } c [f, g]$

by (*metis append-Nil in-set-conv-decomp subformula-into-subformula*)

next

have *a: wf-conn c ([f] @ [g])* **using** *conn wf-conn-binary binary-connectives-def* **by** *auto*

moreover **have** *b: g ≼ g* **using** *subformula-refl* **by** *auto*

ultimately **show** $g \preceq \text{conn } c [f, g]$ **using** *subformula-into-subformula* **by** *force*

qed

lemma *subformula-trans:*

$\psi \preceq \psi' \implies \varphi \preceq \psi \implies \varphi \preceq \psi'$

apply (*induct ψ' rule: subformula.inducts*)

by (*auto simp: subformula-into-subformula*)

lemma *subformula-leaf:*

fixes $\varphi \psi :: 'v \text{ propo}$

assumes *incl: φ ≼ ψ*

and *simple: ψ = FT ∨ ψ = FF ∨ ψ = FVar x*

shows $\varphi = \psi$

using *incl simple*

by (*induct rule: subformula.induct, auto simp: wf-conn-list*)

lemma *subformula-not-incl-eq:*

assumes $\varphi \preceq \text{conn } c l$

and *wf-conn c l*

and $\forall \psi. \psi \in \text{set } l \longrightarrow \neg \varphi \preceq \psi$

shows $\varphi = \text{conn } c l$

using *assms* **apply** (*induction conn c l rule: subformula.induct, auto*)

using *conn-inj* **by** *blast*

lemma *wf-subformula-conn-cases:*

$\text{wf-conn } c l \implies \varphi \preceq \text{conn } c l \longleftrightarrow (\varphi = \text{conn } c l \vee (\exists \psi. \psi \in \text{set } l \wedge \varphi \preceq \psi))$

apply *standard*

using *subformula-not-incl-eq* **apply** *metis*

by (*auto simp add: subformula-into-subformula*)

lemma *subformula-decomp-explicit[simp]:*

$\varphi \preceq \text{FAnd } \psi \psi' \longleftrightarrow (\varphi = \text{FAnd } \psi \psi' \vee \varphi \preceq \psi \vee \varphi \preceq \psi')$ (**is** *?P FAnd*)

$\varphi \preceq \text{FOr } \psi \psi' \longleftrightarrow (\varphi = \text{FOr } \psi \psi' \vee \varphi \preceq \psi \vee \varphi \preceq \psi')$

$\varphi \preceq \text{FEq } \psi \psi' \longleftrightarrow (\varphi = \text{FEq } \psi \psi' \vee \varphi \preceq \psi \vee \varphi \preceq \psi')$

$\varphi \preceq \text{FImp } \psi \psi' \longleftrightarrow (\varphi = \text{FImp } \psi \psi' \vee \varphi \preceq \psi \vee \varphi \preceq \psi')$

proof $-$

have *wf-conn CAnd [ψ, ψ']* **by** (*simp add: binary-connectives-def*)

then **have** $\varphi \preceq \text{conn } \text{CAnd } [\psi, \psi'] \longleftrightarrow$

$(\varphi = \text{conn } \text{CAnd } [\psi, \psi'] \vee (\exists \psi''. \psi'' \in \text{set } [\psi, \psi'] \wedge \varphi \preceq \psi''))$

using *wf-subformula-conn-cases* **by** *metis*

```

    then show ?P FAnd by auto
next
have wf-conn COr [ $\psi$ ,  $\psi'$ ] by (simp add: binary-connectives-def)
then have  $\varphi \preceq \text{conn } COr [\psi, \psi'] \longleftrightarrow$ 
  ( $\varphi = \text{conn } COr [\psi, \psi'] \vee (\exists \psi''. \psi'' \in \text{set } [\psi, \psi'] \wedge \varphi \preceq \psi'')$ )
  using wf-subformula-conn-cases by metis
then show ?P FOr by auto
next
have wf-conn CEq [ $\psi$ ,  $\psi'$ ] by (simp add: binary-connectives-def)
then have  $\varphi \preceq \text{conn } CEq [\psi, \psi'] \longleftrightarrow$ 
  ( $\varphi = \text{conn } CEq [\psi, \psi'] \vee (\exists \psi''. \psi'' \in \text{set } [\psi, \psi'] \wedge \varphi \preceq \psi'')$ )
  using wf-subformula-conn-cases by metis
then show ?P FEq by auto
next
have wf-conn CImp [ $\psi$ ,  $\psi'$ ] by (simp add: binary-connectives-def)
then have  $\varphi \preceq \text{conn } CImp [\psi, \psi'] \longleftrightarrow$ 
  ( $\varphi = \text{conn } CImp [\psi, \psi'] \vee (\exists \psi''. \psi'' \in \text{set } [\psi, \psi'] \wedge \varphi \preceq \psi'')$ )
  using wf-subformula-conn-cases by metis
then show ?P FImp by auto
qed

lemma wf-conn-helper-facts[iff]:
  wf-conn CNot [ $\varphi$ ]
  wf-conn CT []
  wf-conn CF []
  wf-conn (CVar  $x$ ) []
  wf-conn CAnd [ $\varphi$ ,  $\psi$ ]
  wf-conn COr [ $\varphi$ ,  $\psi$ ]
  wf-conn CImp [ $\varphi$ ,  $\psi$ ]
  wf-conn CEq [ $\varphi$ ,  $\psi$ ]
  using wf-conn.intros unfolding binary-connectives-def by fastforce+

lemma exists-c-conn:  $\exists c l. \varphi = \text{conn } c l \wedge \text{wf-conn } c l$ 
  by (cases  $\varphi$ ) force+

lemma subformula-conn-decomp[simp]:
  assumes wf: wf-conn  $c l$ 
  shows  $\varphi \preceq \text{conn } c l \longleftrightarrow (\varphi = \text{conn } c l \vee (\exists \psi \in \text{set } l. \varphi \preceq \psi))$  (is ?A  $\longleftrightarrow$  ?B)
proof (rule iffI)
{
  fix  $\xi$ 
  have  $\varphi \preceq \xi \implies \xi = \text{conn } c l \implies \text{wf-conn } c l \implies \forall x::'a \text{ propo} \in \text{set } l. \neg \varphi \preceq x \implies \varphi = \text{conn } c l$ 
  apply (induct rule: subformula.induct)
  apply simp
  using conn-inj by blast
}
moreover assume ?A
ultimately show ?B using wf by metis
next
assume ?B
then show  $\varphi \preceq \text{conn } c l$  using wf wf-subformula-conn-cases by blast
qed

lemma subformula-leaf-explicit[simp]:
 $\varphi \preceq FT \longleftrightarrow \varphi = FT$ 

```

```

 $\varphi \preceq FF \longleftrightarrow \varphi = FF$ 
 $\varphi \preceq FVar\ x \longleftrightarrow \varphi = FVar\ x$ 
apply auto
using subformula-leaf by metis +

```

The variables inside the formula gives precisely the variables that are needed for the formula.

```

primrec vars-of-prop:: 'v propo  $\Rightarrow$  'v set where
vars-of-prop FT = {} |
vars-of-prop FF = {} |
vars-of-prop (FVar x) = {x} |
vars-of-prop (FNot  $\varphi$ ) = vars-of-prop  $\varphi$  |
vars-of-prop (FAnd  $\varphi\ \psi$ ) = vars-of-prop  $\varphi \cup$  vars-of-prop  $\psi$  |
vars-of-prop (FOr  $\varphi\ \psi$ ) = vars-of-prop  $\varphi \cup$  vars-of-prop  $\psi$  |
vars-of-prop (FImp  $\varphi\ \psi$ ) = vars-of-prop  $\varphi \cup$  vars-of-prop  $\psi$  |
vars-of-prop (FEq  $\varphi\ \psi$ ) = vars-of-prop  $\varphi \cup$  vars-of-prop  $\psi$ 

lemma vars-of-prop-incl-conn:
  fixes  $\xi\ \xi' :: 'v\ propo\ list$  and  $\psi :: 'v\ propo$  and  $c :: 'v\ connective$ 
  assumes corr: wf-conn c l and incl:  $\psi \in set\ l$ 
  shows vars-of-prop  $\psi \subseteq vars-of-prop\ (conn\ c\ l)$ 
proof (cases c rule: connective-cases-arity-2)
  case nullary
  then have False using corr incl by auto
  then show vars-of-prop  $\psi \subseteq vars-of-prop\ (conn\ c\ l)$  by blast
next
  case binary note c = this
  then obtain a b where ab: l = [a, b]
  using wf-conn-bin-list-length list-length2-decomp corr by metis
  then have  $\psi = a \vee \psi = b$  using incl by auto
  then show vars-of-prop  $\psi \subseteq vars-of-prop\ (conn\ c\ l)$ 
  using ab c unfolding binary-connectives-def by auto
next
  case unary note c = this
  fix  $\varphi :: 'v\ propo$ 
  have l = [ $\psi$ ] using corr c incl split-list by force
  then show vars-of-prop  $\psi \subseteq vars-of-prop\ (conn\ c\ l)$  using c by auto
qed

```

The set of variables is compatible with the subformula order.

```

lemma subformula-vars-of-prop:
 $\varphi \preceq \psi \implies vars-of-prop\ \varphi \subseteq vars-of-prop\ \psi$ 
apply (induct rule: subformula.induct)
apply simp
using vars-of-prop-incl-conn by blast

```

4.4 Positions

Instead of 1 or 2 we use *L* or *R*

```
datatype sign = L | R
```

We use *nil* instead of ε .

```

fun pos :: 'v propo  $\Rightarrow$  sign list set where
pos FF = {} |
pos FT = {} |

```

$pos (FVar x) = \{\emptyset\} \mid$
 $pos (FAnd \varphi \psi) = \{\emptyset\} \cup \{L \# p \mid p. p \in pos \varphi\} \cup \{R \# p \mid p. p \in pos \psi\} \mid$
 $pos (FOr \varphi \psi) = \{\emptyset\} \cup \{L \# p \mid p. p \in pos \varphi\} \cup \{R \# p \mid p. p \in pos \psi\} \mid$
 $pos (FEq \varphi \psi) = \{\emptyset\} \cup \{L \# p \mid p. p \in pos \varphi\} \cup \{R \# p \mid p. p \in pos \psi\} \mid$
 $pos (FImp \varphi \psi) = \{\emptyset\} \cup \{L \# p \mid p. p \in pos \varphi\} \cup \{R \# p \mid p. p \in pos \psi\} \mid$
 $pos (FNot \varphi) = \{\emptyset\} \cup \{L \# p \mid p. p \in pos \varphi\}$

lemma *finite-pos*: *finite* (*pos* φ)
by (*induct* φ , *auto*)

lemma *finite-inj-comp-set*:
fixes $s :: 'v \text{ set}$
assumes *finite*: *finite* s
and *inj*: *inj* f
shows $card (\{f p \mid p. p \in s\}) = card s$
using *finite*

proof (*induct* s *rule*: *finite-induct*)
show $card \{f p \mid p. p \in \{\}\} = card \{\}$ **by** *auto*
next

fix $x :: 'v$ **and** $s :: 'v \text{ set}$
assume f : *finite* s **and** *notin*: $x \notin s$
and *IH*: $card \{f p \mid p. p \in s\} = card s$
have f' : *finite* $\{f p \mid p. p \in insert x s\}$ **using** f **by** *auto*
have *notin'*: $f x \notin \{f p \mid p. p \in s\}$ **using** *notin* *inj* *injD* **by** *fastforce*
have $\{f p \mid p. p \in insert x s\} = insert (f x) \{f p \mid p. p \in s\}$ **by** *auto*
then have $card \{f p \mid p. p \in insert x s\} = 1 + card \{f p \mid p. p \in s\}$
using *finite* *card-insert-disjoint* f' *notin'* **by** *auto*
moreover have $\dots = card (insert x s)$ **using** *notin* f *IH* **by** *auto*
finally show $card \{f p \mid p. p \in insert x s\} = card (insert x s)$.

qed

lemma *cons-inject*:
inj (*op* $\# s$)
by (*meson* *injI* *list.inject*)

lemma *finite-insert-nil-cons*:
 $finite s \implies card (insert \emptyset \{L \# p \mid p. p \in s\}) = 1 + card \{L \# p \mid p. p \in s\}$
using *card-insert-disjoint* **by** *auto*

lemma *cord-not[simp]*:
 $card (pos (FNot \varphi)) = 1 + card (pos \varphi)$
by (*simp* *add*: *cons-inject* *finite-inj-comp-set* *finite-pos*)

lemma *card-seperate*:
assumes *finite* $s1$ **and** *finite* $s2$
shows $card (\{L \# p \mid p. p \in s1\} \cup \{R \# p \mid p. p \in s2\}) = card (\{L \# p \mid p. p \in s1\})$
 $+ card (\{R \# p \mid p. p \in s2\})$ (**is** $card (?L \cup ?R) = card ?L + card ?R$)

proof –
have *finite* $?L$ **using** *assms* **by** *auto*
moreover have *finite* $?R$ **using** *assms* **by** *auto*
moreover have $?L \cap ?R = \{\}$ **by** *blast*
ultimately show *thesis* **using** *assms* *card-Un-disjoint* **by** *blast*
qed

definition *prop-size* **where** *prop-size* $\varphi = \text{card } (\text{pos } \varphi)$

lemma *prop-size-vars-of-prop*:

fixes $\varphi :: 'v \text{ propo}$

shows $\text{card } (\text{vars-of-prop } \varphi) \leq \text{prop-size } \varphi$

unfolding *prop-size-def* **apply** (*induct* φ , *auto simp add: cons-inject finite-inj-comp-set finite-pos*)

proof –

fix $\varphi 1 \ \varphi 2 :: 'v \text{ propo}$

assume *IH1*: $\text{card } (\text{vars-of-prop } \varphi 1) \leq \text{card } (\text{pos } \varphi 1)$

and *IH2*: $\text{card } (\text{vars-of-prop } \varphi 2) \leq \text{card } (\text{pos } \varphi 2)$

let $?L = \{L \# p \mid p. p \in \text{pos } \varphi 1\}$

let $?R = \{R \# p \mid p. p \in \text{pos } \varphi 2\}$

have $\text{card } (?L \cup ?R) = \text{card } ?L + \text{card } ?R$

using *card-seperate finite-pos* **by** *blast*

moreover have $\dots = \text{card } (\text{pos } \varphi 1) + \text{card } (\text{pos } \varphi 2)$

by (*simp add: cons-inject finite-inj-comp-set finite-pos*)

moreover have $\dots \geq \text{card } (\text{vars-of-prop } \varphi 1) + \text{card } (\text{vars-of-prop } \varphi 2)$ **using** *IH1 IH2* **by** *arith*

then have $\dots \geq \text{card } (\text{vars-of-prop } \varphi 1 \cup \text{vars-of-prop } \varphi 2)$ **using** *card-Un-le le-trans* **by** *blast*

ultimately

show $\text{card } (\text{vars-of-prop } \varphi 1 \cup \text{vars-of-prop } \varphi 2) \leq \text{Suc } (\text{card } (?L \cup ?R))$

$\text{card } (\text{vars-of-prop } \varphi 1 \cup \text{vars-of-prop } \varphi 2) \leq \text{Suc } (\text{card } (?L \cup ?R))$

$\text{card } (\text{vars-of-prop } \varphi 1 \cup \text{vars-of-prop } \varphi 2) \leq \text{Suc } (\text{card } (?L \cup ?R))$

$\text{card } (\text{vars-of-prop } \varphi 1 \cup \text{vars-of-prop } \varphi 2) \leq \text{Suc } (\text{card } (?L \cup ?R))$

by *auto*

qed

value *pos* (*FImp* (*FAnd* (*FVar* *P*) (*FVar* *Q*)) (*FOr* (*FVar* *P*) (*FVar* *Q*)))

inductive *path-to* $:: \text{sign list} \Rightarrow 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ **where**

path-to-refl[*intro*]: *path-to* $[] \ \varphi \ \varphi \mid$

path-to-l: $c \in \text{binary-connectives} \vee c = \text{CNot} \implies \text{wf-conn } c \ (\varphi \# l) \implies \text{path-to } p \ \varphi \ \varphi' \implies$

path-to (*L* # *p*) (*conn* *c* ($\varphi \# l$)) $\varphi' \mid$

path-to-r: $c \in \text{binary-connectives} \implies \text{wf-conn } c \ (\psi \# \varphi \# []) \implies \text{path-to } p \ \varphi \ \varphi' \implies$

path-to (*R* # *p*) (*conn* *c* ($\psi \# \varphi \# []$)) φ'

There is a deep link between subformulas and pathes: a (correct) path leads to a subformula and a subformula is associated to a given path.

lemma *path-to-subformula*:

path-to $p \ \varphi \ \varphi' \implies \varphi' \preceq \varphi$

apply (*induct rule: path-to.induct*)

apply *simp*

apply (*metis list.set-intros*(1) *subformula-into-subformula*)

using *subformula-trans subformula-in-binary-conn*(2) **by** *metis*

lemma *subformula-path-exists*:

fixes $\varphi \ \varphi' :: 'v \text{ propo}$

shows $\varphi' \preceq \varphi \implies \exists p. \text{path-to } p \ \varphi \ \varphi'$

proof (*induct rule: subformula.induct*)

case *subformula-refl*

have *path-to* $[] \ \varphi' \ \varphi'$ **by** *auto*

then show $\exists p. \text{path-to } p \ \varphi' \ \varphi'$ **by** *metis*

next

case (*subformula-into-subformula* $\psi \ l \ c$)

note *wf* = *this*(2) **and** *IH* = *this*(4) **and** $\psi = \text{this}(1)$


```

then obtain  $p$  where  $p$ : path-to  $p$   $\psi$   $\varphi'$  by metis
{
  fix  $x :: 'v$ 
  assume  $c = CT \vee c = CF \vee c = CVar\ x$ 
  then have False using subformula-into-subformula by auto
  then have  $\exists p$ . path-to  $p$  (conn  $c$   $l$ )  $\varphi'$  by blast
}
moreover {
  assume  $c$ :  $c = CNot$ 
  then have  $l = [\psi]$  using wf  $\psi$  wf-conn-Not-decomp by fastforce
  then have path-to ( $L \# p$ ) (conn  $c$   $l$ )  $\varphi'$  by (metis  $c$  wf-conn-unary  $p$  path-to- $l$ )
  then have  $\exists p$ . path-to  $p$  (conn  $c$   $l$ )  $\varphi'$  by blast
}
moreover {
  assume  $c$ :  $c \in \text{binary-connectives}$ 
  obtain  $a\ b$  where  $ab$ :  $[a, b] = l$  using subformula-into-subformula  $c$  wf-conn-bin-list-length
    list-length2-decomp by metis
  then have  $a = \psi \vee b = \psi$  using  $\psi$  by auto
  then have path-to ( $L \# p$ ) (conn  $c$   $l$ )  $\varphi' \vee$  path-to ( $R \# p$ ) (conn  $c$   $l$ )  $\varphi'$  using  $c$  path-to- $l$ 
    path-to-r  $p$   $ab$  by (metis wf-conn-binary)
  then have  $\exists p$ . path-to  $p$  (conn  $c$   $l$ )  $\varphi'$  by blast
}
ultimately show  $\exists p$ . path-to  $p$  (conn  $c$   $l$ )  $\varphi'$  using connective-cases-arity by metis
qed

```

```

fun replace-at :: sign list  $\Rightarrow$   $'v$  propo  $\Rightarrow$   $'v$  propo  $\Rightarrow$   $'v$  propo where
replace-at [] -  $\psi = \psi$  |
replace-at ( $L \# l$ ) (FAnd  $\varphi$   $\varphi'$ )  $\psi = FAnd$  (replace-at  $l$   $\varphi$   $\psi$ )  $\varphi'$  |
replace-at ( $R \# l$ ) (FAnd  $\varphi$   $\varphi'$ )  $\psi = FAnd$   $\varphi$  (replace-at  $l$   $\varphi'$   $\psi$ ) |
replace-at ( $L \# l$ ) (FOr  $\varphi$   $\varphi'$ )  $\psi = FOr$  (replace-at  $l$   $\varphi$   $\psi$ )  $\varphi'$  |
replace-at ( $R \# l$ ) (FOr  $\varphi$   $\varphi'$ )  $\psi = FOr$   $\varphi$  (replace-at  $l$   $\varphi'$   $\psi$ ) |
replace-at ( $L \# l$ ) (FEq  $\varphi$   $\varphi'$ )  $\psi = FEq$  (replace-at  $l$   $\varphi$   $\psi$ )  $\varphi'$  |
replace-at ( $R \# l$ ) (FEq  $\varphi$   $\varphi'$ )  $\psi = FEq$   $\varphi$  (replace-at  $l$   $\varphi'$   $\psi$ ) |
replace-at ( $L \# l$ ) (FImp  $\varphi$   $\varphi'$ )  $\psi = FImp$  (replace-at  $l$   $\varphi$   $\psi$ )  $\varphi'$  |
replace-at ( $R \# l$ ) (FImp  $\varphi$   $\varphi'$ )  $\psi = FImp$   $\varphi$  (replace-at  $l$   $\varphi'$   $\psi$ ) |
replace-at ( $L \# l$ ) (FNot  $\varphi$ )  $\psi = FNot$  (replace-at  $l$   $\varphi$   $\psi$ )

```

5 Semantics over the syntax

Given the syntax defined above, we define a semantics, by defining an evaluation function *eval*. This function is the bridge between the logic as we define it here and the built-in logic of Isabelle.

```

fun eval :: ( $'v \Rightarrow \text{bool}$ )  $\Rightarrow$   $'v$  propo  $\Rightarrow$  bool (infix  $\models$  50) where
 $\mathcal{A} \models FT = \text{True}$  |
 $\mathcal{A} \models FF = \text{False}$  |
 $\mathcal{A} \models FVar\ v = (\mathcal{A}\ v)$  |
 $\mathcal{A} \models FNot\ \varphi = (\neg(\mathcal{A} \models \varphi))$  |
 $\mathcal{A} \models FAnd\ \varphi_1\ \varphi_2 = (\mathcal{A} \models \varphi_1 \wedge \mathcal{A} \models \varphi_2)$  |
 $\mathcal{A} \models FOr\ \varphi_1\ \varphi_2 = (\mathcal{A} \models \varphi_1 \vee \mathcal{A} \models \varphi_2)$  |
 $\mathcal{A} \models FImp\ \varphi_1\ \varphi_2 = (\mathcal{A} \models \varphi_1 \longrightarrow \mathcal{A} \models \varphi_2)$  |
 $\mathcal{A} \models FEq\ \varphi_1\ \varphi_2 = (\mathcal{A} \models \varphi_1 \longleftrightarrow \mathcal{A} \models \varphi_2)$ 

```

definition evalf (infix \models_f 50) where
 $evalf\ \varphi\ \psi = (\forall A. A \models \varphi \longrightarrow A \models \psi)$

The deduction rule is in the book. And the proof looks like to the one of the book.

theorem *deduction-theorem*:

$(\varphi \models_f \psi) \longleftrightarrow (\forall A. (A \models FImp \varphi \psi))$

proof

assume $H: \varphi \models_f \psi$

{

fix A

have $A \models FImp \varphi \psi$

proof (cases $A \models \varphi$)

case *True*

then have $A \models \psi$ using H unfolding *evalf-def* by *metis*

then show $A \models FImp \varphi \psi$ by *auto*

next

case *False*

then show $A \models FImp \varphi \psi$ by *auto*

qed

}

then show $\forall A. A \models FImp \varphi \psi$ by *blast*

next

assume $A: \forall A. A \models FImp \varphi \psi$

show $\varphi \models_f \psi$

proof (rule *ccontr*)

assume $\neg \varphi \models_f \psi$

then obtain A where $A \models \varphi$ and $\neg A \models \psi$ using *evalf-def* by *metis*

then have $\neg A \models FImp \varphi \psi$ by *auto*

then show *False* using A by *blast*

qed

qed

A shorter proof:

lemma $\varphi \models_f \psi \longleftrightarrow (\forall A. A \models FImp \varphi \psi)$

by (*simp add: evalf-def*)

definition *same-over-set*:: $('v \Rightarrow bool) \Rightarrow ('v \Rightarrow bool) \Rightarrow 'v \text{ set} \Rightarrow bool$ **where**

same-over-set $A B S = (\forall c \in S. A c = B c)$

If two mapping A and B have the same value over the variables, then the same formula are satisfiable.

lemma *same-over-set-eval*:

assumes *same-over-set* $A B$ (*vars-of-prop* φ)

shows $A \models \varphi \longleftrightarrow B \models \varphi$

using *assms* unfolding *same-over-set-def* by (*induct* φ , *auto*)

end

theory *Prop-Abstract-Transformation*

imports *Main Prop-Logic Wellfounded-More*

begin

This file is devoted to abstract properties of the transformations, like consistency preservation and lifting from terms to proposition.

6 Rewrite systems and properties

6.1 Lifting of rewrite rules

We can lift a rewrite relation r over a full formula: the relation r works on terms, while $propo\text{-}rew\text{-}step$ works on formulas.

```
inductive propo-rew-step :: ('v propo  $\Rightarrow$  'v propo  $\Rightarrow$  bool)  $\Rightarrow$  'v propo  $\Rightarrow$  'v propo  $\Rightarrow$  bool
  for r :: 'v propo  $\Rightarrow$  'v propo  $\Rightarrow$  bool where
    global-rel: r  $\varphi$   $\psi \implies$  propo-rew-step r  $\varphi$   $\psi$  |
    propo-rew-one-step-lift: propo-rew-step r  $\varphi$   $\varphi' \implies$  wf-conn c ( $\psi$ s @  $\varphi$  #  $\psi$ s')
       $\implies$  propo-rew-step r (conn c ( $\psi$ s @  $\varphi$  #  $\psi$ s')) (conn c ( $\psi$ s @  $\varphi'$  #  $\psi$ s'))
```

Here is a more precise link between the lifting and the subformulas: if a rewriting takes place between φ and φ' , then there are two subformulas ψ in φ and ψ' in φ' , ψ' is the result of the rewriting of r on ψ .

This lemma is only a health condition:

```
lemma propo-rew-step-subformula-imp:
shows propo-rew-step r  $\varphi$   $\varphi' \implies \exists \psi \psi'. \psi \preceq \varphi \wedge \psi' \preceq \varphi' \wedge r \psi \psi'$ 
apply (induct rule: propo-rew-step.induct)
using subformula.simps subformula-into-subformula apply blast
using wf-conn-no-arity-change subformula-into-subformula wf-conn-no-arity-change-helper
in-set-conv-decomp by metis
```

The converse is moreover true: if there is a ψ and ψ' , then every formula φ containing ψ , can be rewritten into a formula φ' , such that it contains φ' .

```
lemma propo-rew-step-subformula-rec:
fixes  $\psi \psi' \varphi ::$  'v propo
shows  $\psi \preceq \varphi \implies r \psi \psi' \implies (\exists \varphi'. \psi' \preceq \varphi' \wedge propo\text{-}rew\text{-}step r \varphi \varphi')$ 
proof (induct  $\varphi$  rule: subformula.induct)
case subformula-refl
hence propo-rew-step r  $\psi \psi'$  using propo-rew-step.intros by auto
moreover have  $\psi' \preceq \psi'$  using Prop-Logic.subformula-refl by auto
ultimately show  $\exists \varphi'. \psi' \preceq \varphi' \wedge propo\text{-}rew\text{-}step r \psi \varphi'$  by fastforce
next
case (subformula-into-subformula  $\psi''$  l c)
note IH = this(4) and r = this(5) and  $\psi'' =$  this(1) and wf = this(2) and incl = this(3)
then obtain  $\varphi'$  where *:  $\psi' \preceq \varphi' \wedge propo\text{-}rew\text{-}step r \psi'' \varphi'$  by metis
moreover obtain  $\xi \xi' ::$  'v propo list where
  l: l =  $\xi$  @  $\psi''$  #  $\xi'$  using List.split-list  $\psi''$  by metis
ultimately have propo-rew-step r (conn c l) (conn c ( $\xi$  @  $\varphi'$  #  $\xi'$ ))
using propo-rew-step.intros(2) wf by metis
moreover have  $\psi' \preceq$  conn c ( $\xi$  @  $\varphi'$  #  $\xi'$ )
using wf * wf-conn-no-arity-change Prop-Logic.subformula-into-subformula
by (metis (no-types) in-set-conv-decomp l wf-conn-no-arity-change-helper)
ultimately show  $\exists \varphi'. \psi' \preceq \varphi' \wedge propo\text{-}rew\text{-}step r$  (conn c l)  $\varphi'$  by metis
qed
```

```
lemma propo-rew-step-subformula:
  ( $\exists \psi \psi'. \psi \preceq \varphi \wedge r \psi \psi'$ )  $\longleftrightarrow$  ( $\exists \varphi'. propo\text{-}rew\text{-}step r \varphi \varphi'$ )
using propo-rew-step-subformula-imp propo-rew-step-subformula-rec by metis+
```

```
lemma consistency-decompose-into-list:
assumes wf: wf-conn c l and wf': wf-conn c l'
and same:  $\forall n. (A \models l ! n \longleftrightarrow (A \models l' ! n))$ 
```

```

shows (A ⊨ conn c l) = (A ⊨ conn c l')
proof (cases c rule: connective-cases-arity-2)
  case nullary
  thus (A ⊨ conn c l) ⟷ (A ⊨ conn c l') using wf wf' by auto
next
case unary note c = this
then obtain a where l: l = [a] using wf-conn-Not-decomp wf by metis
obtain a' where l': l' = [a'] using wf-conn-Not-decomp wf' c by metis
have A ⊨ a ⟷ A ⊨ a' using l l' by (metis nth-Cons-0 same)
thus A ⊨ conn c l ⟷ A ⊨ conn c l' using l l' c by auto
next
case binary note c = this
then obtain a b where l: l = [a, b]
  using wf-conn-bin-list-length list-length2-decomp wf by metis
obtain a' b' where l': l' = [a', b']
  using wf-conn-bin-list-length list-length2-decomp wf' c by metis

have p: A ⊨ a ⟷ A ⊨ a' A ⊨ b ⟷ A ⊨ b'
  using l l' same by (metis diff-Suc-1 nth-Cons' nat.distinct(2))+
show A ⊨ conn c l ⟷ A ⊨ conn c l'
  using wf c p unfolding binary-connectives-def l l' by auto
qed

```

Relation between *propo-rew-step* and the rewriting we have seen before: *propo-rew-step* $r \varphi \varphi'$ means that we rewrite ψ inside φ (ie at a path p) into ψ' .

lemma *propo-rew-step-rewrite*:

```

fixes  $\varphi \varphi' :: 'v \text{ propo}$  and  $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ 
assumes propo-rew-step  $r \varphi \varphi'$ 
shows  $\exists \psi \psi' p. r \psi \psi' \wedge \text{path-to } p \varphi \psi \wedge \text{replace-at } p \varphi \psi' = \varphi'$ 
using assms
proof (induct rule: propo-rew-step.induct)
  case (global-rel  $\varphi \psi$ )
  moreover have path-to []  $\varphi \varphi$  by auto
  moreover have replace-at []  $\varphi \psi = \psi$  by auto
  ultimately show ?case by metis
next
case (propo-rew-one-step-lift  $\varphi \varphi' c \xi \xi'$ ) note rel = this(1) and IH0 = this(2) and corr = this(3)
obtain  $\psi \psi' p$  where IH:  $r \psi \psi' \wedge \text{path-to } p \varphi \psi \wedge \text{replace-at } p \varphi \psi' = \varphi'$  using IH0 by metis

{
  fix  $x :: 'v$ 
  assume  $c = CT \vee c = CF \vee c = CVar x$ 
  hence False using corr by auto
  hence  $\exists \psi \psi' p. r \psi \psi' \wedge \text{path-to } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi$ 
     $\wedge \text{replace-at } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi' = \text{conn } c (\xi @ (\varphi' \# \xi'))$ 
    by fast
}
moreover {
  assume  $c: c = CNot$ 
  hence empty:  $\xi = [] \ \xi' = []$  using corr by auto
  have path-to ( $L \# p$ ) ( $\text{conn } c (\xi @ (\varphi \# \xi'))$ )  $\psi$ 
    using c empty IH wf-conn-unary path-to-l by fastforce
  moreover have replace-at ( $L \# p$ ) ( $\text{conn } c (\xi @ (\varphi \# \xi'))$ )  $\psi' = \text{conn } c (\xi @ (\varphi' \# \xi'))$ 
    using c empty IH by auto
  ultimately have  $\exists \psi \psi' p. r \psi \psi' \wedge \text{path-to } p (\text{conn } c (\xi @ (\varphi \# \xi'))) \psi$ 

```

$\wedge \text{ replace-at } p \text{ (conn } c \text{ (}\xi @ (\varphi \# \xi')\text{)) } \psi' = \text{conn } c \text{ (}\xi @ (\varphi' \# \xi')\text{)}$

```

using IH by metis
}
moreover {
  assume c: c ∈ binary-connectives
  have length (ξ @ φ # ξ') = 2 using wf-conn-bin-list-length corr c by metis
  hence length ξ + length ξ' = 1 by auto
  hence ld: (length ξ = 1 ∧ length ξ' = 0) ∨ (length ξ = 0 ∧ length ξ' = 1) by arith
  obtain a b where ab: (ξ = [] ∧ ξ' = [b]) ∨ (ξ = [a] ∧ ξ' = [])
  using ld by (case-tac ξ, case-tac ξ', auto)
  {
    assume φ: ξ = [] ∧ ξ' = [b]
    have path-to (L#p) (conn c (ξ @ (φ # ξ'))) ψ
      using φ c IH ab corr by (simp add: path-to-l)
    moreover have replace-at (L#p) (conn c (ξ @ (φ # ξ'))) ψ' = conn c (ξ @ (φ' # ξ'))
      using c IH ab φ unfolding binary-connectives-def by auto
    ultimately have ∃ ψ ψ' p. r ψ ψ' ∧ path-to p (conn c (ξ @ (φ # ξ'))) ψ
      ∧ replace-at p (conn c (ξ @ (φ # ξ'))) ψ' = conn c (ξ @ (φ' # ξ'))
      using IH by metis
  }
  moreover {
    assume φ: ξ = [a] ∧ ξ' = []
    hence path-to (R#p) (conn c (ξ @ (φ # ξ'))) ψ
      using c IH corr path-to-r corr φ by (simp add: path-to-r)
    moreover have replace-at (R#p) (conn c (ξ @ (φ # ξ'))) ψ' = conn c (ξ @ (φ' # ξ'))
      using c IH ab φ unfolding binary-connectives-def by auto
    ultimately have ?case using IH by metis
  }
  ultimately have ?case using ab by blast
}
ultimately show ?case using connective-cases-arity by blast
qed

```

6.2 Consistency preservation

We define *preserves-un-sat*: it means that a relation preserves consistency.

definition *preserves-un-sat* **where**

preserves-un-sat $r \longleftrightarrow (\forall \varphi \psi. r \varphi \psi \longrightarrow (\forall A. A \models \varphi \longleftrightarrow A \models \psi))$

lemma *propo-rew-step-preservers-val-explicit*:

propo-rew-step $r \varphi \psi \implies \text{preserves-un-sat } r \implies \text{propo-rew-step } r \varphi \psi \implies (\forall A. A \models \varphi \longleftrightarrow A \models \psi)$

unfolding *preserves-un-sat-def*

proof (*induction rule: propo-rew-step.induct*)

case *global-rel*

thus ?case **by** *simp*

next

case (*propo-rew-one-step-lift* $\varphi \varphi' c \xi \xi'$) **note** $\text{rel} = \text{this}(1)$ **and** $\text{wf} = \text{this}(2)$

and $\text{IH} = \text{this}(3)[\text{OF } \text{this}(4) \text{ this}(1)]$ **and** $\text{consistent} = \text{this}(4)$

{

fix A

from IH **have** $\forall n. (A \models (\xi @ \varphi \# \xi') ! n) = (A \models (\xi @ \varphi' \# \xi') ! n)$

by (*metis* (*mono-tags*, *hide-lams*) *list-update-length nth-Cons-0 nth-append-length-plus*
nth-list-update-neg)

hence $(A \models \text{conn } c \text{ (}\xi @ \varphi \# \xi')\text{)}) = (A \models \text{conn } c \text{ (}\xi @ \varphi' \# \xi')\text{)})$

```

    by (meson consistency-decompose-into-list wf wf-conn-no-arity-change-helper
        wf-conn-no-arity-change)
  }
  thus  $\forall A. A \models \text{conn } c (\xi @ \varphi \# \xi') \longleftrightarrow A \models \text{conn } c (\xi @ \varphi' \# \xi')$  by auto
qed

```

```

lemma propo-rew-step-preservers-val':
  assumes preserves-un-sat r
  shows preserves-un-sat (propo-rew-step r)
  using assms by (simp add: preserves-un-sat-def propo-rew-step-preservers-val-explicit)

```

```

lemma preserves-un-sat-OO[intro]:
  preserves-un-sat f  $\implies$  preserves-un-sat g  $\implies$  preserves-un-sat (f OO g)
  unfolding preserves-un-sat-def by auto

```

```

lemma star-consistency-preservation-explicit:
  assumes (propo-rew-step r)**  $\varphi \psi$  and preserves-un-sat r
  shows  $\forall A. A \models \varphi \longleftrightarrow A \models \psi$ 
  using assms by (induct rule: rtranclp-induct)
  (auto simp add: propo-rew-step-preservers-val-explicit)

```

```

lemma star-consistency-preservation:
  preserves-un-sat r  $\implies$  preserves-un-sat (propo-rew-step r)**
  by (simp add: star-consistency-preservation-explicit preserves-un-sat-def)

```

6.3 Full Lifting

In the previous a relation was lifted to a formula, now we define the relation such it is applied as long as possible. The definition is thus simply: it can be derived and nothing more can be derived.

```

lemma full-ropo-rew-step-preservers-val[simp]:
  preserves-un-sat r  $\implies$  preserves-un-sat (full (propo-rew-step r))
  by (metis full-def preserves-un-sat-def star-consistency-preservation)

```

```

lemma full-propo-rew-step-subformula:
  full (propo-rew-step r)  $\varphi' \varphi \implies \neg(\exists \psi \psi'. \psi \preceq \varphi \wedge r \psi \psi')$ 
  unfolding full-def using propo-rew-step-subformula-rec by metis

```

7 Transformation testing

7.1 Definition and first properties

To prove correctness of our transformation, we create a *all-subformula-st* predicate. It tests recursively all subformulas. At each step, the actual formula is tested. The aim of this *test-symb* function is to test locally some properties of the formulas (i.e. at the level of the connective or at first level). This allows a clause description between the rewrite relation and the *test-symb*

definition *all-subformula-st* :: $('a \text{ propo} \Rightarrow \text{bool}) \Rightarrow 'a \text{ propo} \Rightarrow \text{bool}$ **where**
all-subformula-st test-symb $\varphi \equiv \forall \psi. \psi \preceq \varphi \longrightarrow \text{test-symb } \psi$

lemma *test-symb-imp-all-subformula-st[simp]*:
test-symb FT \implies all-subformula-st test-symb FT
test-symb FF \implies all-subformula-st test-symb FF
test-symb (FVar x) \implies all-subformula-st test-symb (FVar x)
unfolding *all-subformula-st-def* **using** *subformula-leaf* **by** *metis+*

lemma *all-subformula-st-test-symb-true-phi*:
all-subformula-st test-symb $\varphi \implies$ test-symb φ
unfolding *all-subformula-st-def* **by** *auto*

lemma *all-subformula-st-decomp-imp*:
wf-conn c l \implies (test-symb (conn c l) \wedge ($\forall \varphi \in$ set l. all-subformula-st test-symb φ))
 \implies all-subformula-st test-symb (conn c l)
unfolding *all-subformula-st-def* **by** *auto*

To ease the finding of proofs, we give some explicit theorem about the decomposition.

lemma *all-subformula-st-decomp-rec*:
all-subformula-st test-symb (conn c l) \implies wf-conn c l
 \implies (test-symb (conn c l) \wedge ($\forall \varphi \in$ set l. all-subformula-st test-symb φ))
unfolding *all-subformula-st-def* **by** *auto*

lemma *all-subformula-st-decomp*:
fixes *c :: 'v connective and l :: 'v propo list*
assumes *wf-conn c l*
shows *all-subformula-st test-symb (conn c l)*
 \longleftrightarrow (test-symb (conn c l) \wedge ($\forall \varphi \in$ set l. all-subformula-st test-symb φ))
using *assms all-subformula-st-decomp-rec all-subformula-st-decomp-imp* **by** *metis*

lemma *helper-fact: c \in binary-connectives \longleftrightarrow (c = COr \vee c = CAnd \vee c = CEq \vee c = CImp)*
unfolding *binary-connectives-def* **by** *auto*

lemma *all-subformula-st-decomp-explicit[simp]*:
fixes *$\varphi \psi :: 'v propo$*
shows *all-subformula-st test-symb (FAnd $\varphi \psi$)*
 \longleftrightarrow (test-symb (FAnd $\varphi \psi$) \wedge all-subformula-st test-symb $\varphi \wedge$ all-subformula-st test-symb ψ)
and *all-subformula-st test-symb (FOr $\varphi \psi$)*
 \longleftrightarrow (test-symb (FOr $\varphi \psi$) \wedge all-subformula-st test-symb $\varphi \wedge$ all-subformula-st test-symb ψ)
and *all-subformula-st test-symb (FNot φ)*
 \longleftrightarrow (test-symb (FNot φ) \wedge all-subformula-st test-symb φ)
and *all-subformula-st test-symb (FEq $\varphi \psi$)*
 \longleftrightarrow (test-symb (FEq $\varphi \psi$) \wedge all-subformula-st test-symb $\varphi \wedge$ all-subformula-st test-symb ψ)
and *all-subformula-st test-symb (FImp $\varphi \psi$)*
 \longleftrightarrow (test-symb (FImp $\varphi \psi$) \wedge all-subformula-st test-symb $\varphi \wedge$ all-subformula-st test-symb ψ)

proof –

have *all-subformula-st test-symb (FAnd $\varphi \psi$) \longleftrightarrow all-subformula-st test-symb (conn CAnd [φ , ψ])*
by *auto*
moreover have $\dots \longleftrightarrow$ *test-symb (conn CAnd [φ , ψ]) \wedge ($\forall \xi \in$ set [φ , ψ]. all-subformula-st test-symb ξ)*
using *all-subformula-st-decomp wf-conn-helper-facts(5)* **by** *metis*
finally show *all-subformula-st test-symb (FAnd $\varphi \psi$)*
 \longleftrightarrow (test-symb (FAnd $\varphi \psi$) \wedge all-subformula-st test-symb $\varphi \wedge$ all-subformula-st test-symb ψ)
by *simp*
have *all-subformula-st test-symb (FOr $\varphi \psi$) \longleftrightarrow all-subformula-st test-symb (conn COr [φ , ψ])*
by *auto*

```

moreover have ...  $\longleftrightarrow$ 
  (test-symb (conn COr  $[\varphi, \psi]$ )  $\wedge$  ( $\forall \xi \in \text{set } [\varphi, \psi]. \text{all-subformula-st test-symb } \xi$ ))
  using all-subformula-st-decomp wf-conn-helper-facts(6) by metis
finally show all-subformula-st test-symb (FOr  $\varphi \psi$ )
 $\longleftrightarrow$  (test-symb (FOr  $\varphi \psi$ )  $\wedge$  all-subformula-st test-symb  $\varphi \wedge$  all-subformula-st test-symb  $\psi$ )
by simp

have all-subformula-st test-symb (FEq  $\varphi \psi$ )  $\longleftrightarrow$  all-subformula-st test-symb (conn CEq  $[\varphi, \psi]$ )
by auto
moreover have ...
 $\longleftrightarrow$  (test-symb (conn CEq  $[\varphi, \psi]$ )  $\wedge$  ( $\forall \xi \in \text{set } [\varphi, \psi]. \text{all-subformula-st test-symb } \xi$ ))
  using all-subformula-st-decomp wf-conn-helper-facts(8) by metis
finally show all-subformula-st test-symb (FEq  $\varphi \psi$ )
 $\longleftrightarrow$  (test-symb (FEq  $\varphi \psi$ )  $\wedge$  all-subformula-st test-symb  $\varphi \wedge$  all-subformula-st test-symb  $\psi$ )
by simp

have all-subformula-st test-symb (FImp  $\varphi \psi$ )  $\longleftrightarrow$  all-subformula-st test-symb (conn CImp  $[\varphi, \psi]$ )
by auto
moreover have ...
 $\longleftrightarrow$  (test-symb (conn CImp  $[\varphi, \psi]$ )  $\wedge$  ( $\forall \xi \in \text{set } [\varphi, \psi]. \text{all-subformula-st test-symb } \xi$ ))
  using all-subformula-st-decomp wf-conn-helper-facts(7) by metis
finally show all-subformula-st test-symb (FImp  $\varphi \psi$ )
 $\longleftrightarrow$  (test-symb (FImp  $\varphi \psi$ )  $\wedge$  all-subformula-st test-symb  $\varphi \wedge$  all-subformula-st test-symb  $\psi$ )
by simp

have all-subformula-st test-symb (FNot  $\varphi$ )  $\longleftrightarrow$  all-subformula-st test-symb (conn CNot  $[\varphi]$ )
by auto
moreover have ... = (test-symb (conn CNot  $[\varphi]$ )  $\wedge$  ( $\forall \xi \in \text{set } [\varphi]. \text{all-subformula-st test-symb } \xi$ ))
  using all-subformula-st-decomp wf-conn-helper-facts(1) by metis
finally show all-subformula-st test-symb (FNot  $\varphi$ )
 $\longleftrightarrow$  (test-symb (FNot  $\varphi$ )  $\wedge$  all-subformula-st test-symb  $\varphi$ ) by simp
qed

```

As *all-subformula-st* tests recursively, the function is true on every subformula.

lemma *subformula-all-subformula-st*:

```

 $\psi \preceq \varphi \implies \text{all-subformula-st test-symb } \varphi \implies \text{all-subformula-st test-symb } \psi$ 
by (induct rule: subformula.induct, auto simp add: all-subformula-st-decomp)

```

The following theorem *no-test-symb-step-exists* shows the link between the *test-symb* function and the corresponding rewrite relation *r*: if we assume that if every time *test-symb* is true, then a *r* can be applied, finally as long as $\neg \text{all-subformula-st test-symb } \varphi$, then something can be rewritten in φ .

lemma *no-test-symb-step-exists*:

```

fixes r:: 'v propo  $\Rightarrow$  'v propo  $\Rightarrow$  bool and test-symb:: 'v propo  $\Rightarrow$  bool and x :: 'v
and  $\varphi$  :: 'v propo
assumes test-symb-false-nullary:  $\forall x. \text{test-symb } FF \wedge \text{test-symb } FT \wedge \text{test-symb } (FVar\ x)$ 
and  $\forall \varphi'. \varphi' \preceq \varphi \longrightarrow (\neg \text{test-symb } \varphi') \longrightarrow (\exists \psi. r\ \varphi'\ \psi)$  and
 $\neg \text{all-subformula-st test-symb } \varphi$ 
shows  $(\exists \psi\ \psi'. \psi \preceq \varphi \wedge r\ \psi\ \psi')$ 
using assms
proof (induct  $\varphi$  rule: propo-induct-arity)
case (nullary  $\varphi\ x$ )
thus  $\exists \psi\ \psi'. \psi \preceq \varphi \wedge r\ \psi\ \psi'$ 
  using wf-conn-nullary test-symb-false-nullary by fastforce
next

```



```

  case (unary  $\varphi$ ) note  $IH = this(1)[OF\ this(2)]$  and  $r = this(2)$  and  $nst = this(3)$  and  $subf = this(4)$ 
  from  $r\ IH\ nst$  have  $H: \neg all\text{-}subformula\text{-}st\ test\text{-}symb\ \varphi \implies \exists \psi. \psi \preceq \varphi \wedge (\exists \psi'. r\ \psi\ \psi')$ 
    by (metis subformula-in-subformula-not subformula-refl subformula-trans)
  {
    assume  $n: \neg test\text{-}symb\ (FNot\ \varphi)$ 
    obtain  $\psi$  where  $r\ (FNot\ \varphi)\ \psi$  using subformula-refl  $r\ n\ nst$  by blast
    moreover have  $FNot\ \varphi \preceq FNot\ \varphi$  using subformula-refl by auto
    ultimately have  $\exists \psi\ \psi'. \psi \preceq FNot\ \varphi \wedge r\ \psi\ \psi'$  by metis
  }
  moreover {
    assume  $n: test\text{-}symb\ (FNot\ \varphi)$ 
    hence  $\neg all\text{-}subformula\text{-}st\ test\text{-}symb\ \varphi$ 
      using all-subformula-st-decomp-explicit(3)  $nst\ subf$  by blast
    hence  $\exists \psi\ \psi'. \psi \preceq FNot\ \varphi \wedge r\ \psi\ \psi'$ 
      using  $H\ subformula\text{-}in\text{-}subformula\text{-}not\ subformula\text{-}refl\ subformula\text{-}trans$  by blast
  }
  ultimately show  $\exists \psi\ \psi'. \psi \preceq FNot\ \varphi \wedge r\ \psi\ \psi'$  by blast
next
case (binary  $\varphi\ \varphi1\ \varphi2$ )
note  $IH\varphi1\text{-}0 = this(1)[OF\ this(4)]$  and  $IH\varphi2\text{-}0 = this(2)[OF\ this(4)]$  and  $r = this(4)$ 
  and  $\varphi = this(3)$  and  $le = this(5)$  and  $nst = this(6)$ 

obtain  $c :: 'v\ connective$  where
   $c: (c = CAnd \vee c = COr \vee c = CImp \vee c = CEq) \wedge conn\ c\ [\varphi1, \varphi2] = \varphi$ 
  using  $\varphi$  by fastforce

hence  $corr: wf\text{-}conn\ c\ [\varphi1, \varphi2]$  using  $wf\text{-}conn.simps$  unfolding binary-connectives-def by auto
have  $inc: \varphi1 \preceq \varphi\ \varphi2 \preceq \varphi$  using binary-connectives-def  $c\ subformula\text{-}in\text{-}binary\text{-}conn$  by blast+
from  $r\ IH\varphi1\text{-}0$  have  $IH\varphi1: \neg all\text{-}subformula\text{-}st\ test\text{-}symb\ \varphi1 \implies \exists \psi\ \psi'. \psi \preceq \varphi1 \wedge r\ \psi\ \psi'$ 
  using  $inc(1)\ subformula\text{-}trans\ le$  by blast
from  $r\ IH\varphi2\text{-}0$  have  $IH\varphi2: \neg all\text{-}subformula\text{-}st\ test\text{-}symb\ \varphi2 \implies \exists \psi. \psi \preceq \varphi2 \wedge (\exists \psi'. r\ \psi\ \psi')$ 
  using  $inc(2)\ subformula\text{-}trans\ le$  by blast
have cases:  $\neg test\text{-}symb\ \varphi \vee \neg all\text{-}subformula\text{-}st\ test\text{-}symb\ \varphi1 \vee \neg all\text{-}subformula\text{-}st\ test\text{-}symb\ \varphi2$ 
  using  $c\ nst$  by auto
show  $\exists \psi\ \psi'. \psi \preceq \varphi \wedge r\ \psi\ \psi'$ 
  using  $IH\varphi1\ IH\varphi2\ subformula\text{-}trans\ inc\ subformula\text{-}refl\ cases\ le$  by blast
qed

```

7.2 Invariant conservation

If two rewrite relation are independant (or at least independant enough), then the property characterizing the first relation *all-subformula-st test-symb* remains true. The next show the same property, with changes in the assumptions.

The assumption $\forall \varphi' \psi. \varphi' \preceq \Phi \longrightarrow r\ \varphi' \psi \longrightarrow all\text{-}subformula\text{-}st\ test\text{-}symb\ \varphi' \longrightarrow all\text{-}subformula\text{-}st\ test\text{-}symb\ \psi$ means that rewriting with r does not mess up the property we want to preserve locally.

The previous assumption is not enough to go from r to *propo-rew-step* r : we have to add the assumption that rewriting inside does not mess up the term: $\forall c\ \xi\ \varphi\ \xi'\ \varphi'. \varphi \preceq \Phi \longrightarrow propo\text{-}rew\text{-}step\ r\ \varphi\ \varphi' \longrightarrow wf\text{-}conn\ c\ (\xi\ @\ \varphi\ \# \xi') \longrightarrow test\text{-}symb\ (conn\ c\ (\xi\ @\ \varphi\ \# \xi')) \longrightarrow test\text{-}symb\ \varphi' \longrightarrow test\text{-}symb\ (conn\ c\ (\xi\ @\ \varphi'\ \# \xi'))$

7.2.1 Invariant while lifting of the rewriting relation

The condition $\varphi \preceq \Phi$ (that will be used with $\Phi = \varphi$ most of the time) is here to ensure that the recursive conditions on Φ will moreover hold for the subterm we are rewriting. For example if there is no equivalence symbol in Φ , we do not have to care about equivalence symbols in the two previous assumptions.

lemma *propo-rew-step-inv-stay*:

```

fixes  $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$  and  $\text{test-symb} :: 'v \text{ propo} \Rightarrow \text{bool}$  and  $x :: 'v$ 
and  $\varphi \ \psi \ \Phi :: 'v \text{ propo}$ 
assumes  $H: \forall \varphi' \psi. \varphi' \preceq \Phi \longrightarrow r \ \varphi' \ \psi \longrightarrow \text{all-subformula-st test-symb } \varphi'$ 
 $\longrightarrow \text{all-subformula-st test-symb } \psi$ 
and  $H': \forall (c :: 'v \text{ connective}) \ \xi \ \varphi \ \xi' \ \varphi'. \varphi \preceq \Phi \longrightarrow \text{propo-rew-step } r \ \varphi \ \varphi'$ 
 $\longrightarrow \text{wf-conn } c \ (\xi @ \varphi \# \xi') \longrightarrow \text{test-symb } (\text{conn } c \ (\xi @ \varphi \# \xi')) \longrightarrow \text{test-symb } \varphi'$ 
 $\longrightarrow \text{test-symb } (\text{conn } c \ (\xi @ \varphi' \# \xi'))$  and
 $\text{propo-rew-step } r \ \varphi \ \psi$  and
 $\varphi \preceq \Phi$  and
 $\text{all-subformula-st test-symb } \varphi$ 
shows  $\text{all-subformula-st test-symb } \psi$ 
using assms(3-5)
proof (induct rule: propo-rew-step.induct)
case global-rel
thus ?case using  $H$  by simp
next
case (propo-rew-one-step-lift  $\varphi \ \varphi' \ c \ \xi \ \xi'$ )
note  $\text{rel} = \text{this}(1)$  and  $\varphi = \text{this}(2)$  and  $\text{corr} = \text{this}(3)$  and  $\Phi = \text{this}(4)$  and  $\text{nst} = \text{this}(5)$ 
have  $\text{sq}: \varphi \preceq \Phi$ 
using  $\Phi \ \text{corr} \ \text{subformula-into-subformula} \ \text{subformula-refl} \ \text{subformula-trans}$ 
by (metis in-set-conv-decomp)
from  $\text{corr}$  have  $\forall \psi. \psi \in \text{set } (\xi @ \varphi \# \xi') \longrightarrow \text{all-subformula-st test-symb } \psi$ 
using  $\text{all-subformula-st-decomp} \ \text{nst}$  by blast
hence  $*$ :  $\forall \psi. \psi \in \text{set } (\xi @ \varphi' \# \xi') \longrightarrow \text{all-subformula-st test-symb } \psi$  using  $\varphi \ \text{sq}$  by fastforce
hence  $\text{test-symb } \varphi'$  using  $\text{all-subformula-st-test-symb-true-phi}$  by auto
moreover from  $\text{corr} \ \text{nst}$  have  $\text{test-symb } (\text{conn } c \ (\xi @ \varphi \# \xi'))$ 
using  $\text{all-subformula-st-decomp}$  by blast
ultimately have  $\text{test-symb}: \text{test-symb } (\text{conn } c \ (\xi @ \varphi' \# \xi'))$  using  $H' \ \text{sq} \ \text{corr} \ \text{rel}$  by blast

have  $\text{wf-conn } c \ (\xi @ \varphi' \# \xi')$ 
by (metis wf-conn-no-arity-change-helper corr wf-conn-no-arity-change)
thus  $\text{all-subformula-st test-symb } (\text{conn } c \ (\xi @ \varphi' \# \xi'))$ 
using  $*$   $\text{test-symb}$  by (metis all-subformula-st-decomp)
qed

```

The need for $\varphi \preceq \Phi$ is not always necessary, hence we moreover have a version without inclusion.

lemma *propo-rew-step-inv-stay*:

```

fixes  $r :: 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$  and  $\text{test-symb} :: 'v \text{ propo} \Rightarrow \text{bool}$  and  $x :: 'v$ 
and  $\varphi \ \psi :: 'v \text{ propo}$ 
assumes
 $H: \forall \varphi' \psi. r \ \varphi' \ \psi \longrightarrow \text{all-subformula-st test-symb } \varphi' \longrightarrow \text{all-subformula-st test-symb } \psi$  and
 $H': \forall (c :: 'v \text{ connective}) \ \xi \ \varphi \ \xi' \ \varphi'. \text{wf-conn } c \ (\xi @ \varphi \# \xi') \longrightarrow \text{test-symb } (\text{conn } c \ (\xi @ \varphi \# \xi'))$ 
 $\longrightarrow \text{test-symb } \varphi' \longrightarrow \text{test-symb } (\text{conn } c \ (\xi @ \varphi' \# \xi'))$  and
 $\text{propo-rew-step } r \ \varphi \ \psi$  and
 $\text{all-subformula-st test-symb } \varphi$ 
shows  $\text{all-subformula-st test-symb } \psi$ 
using propo-rew-step-inv-stay [of  $\varphi \ r \ \text{test-symb } \varphi \ \psi$ ] assms subformula-refl by metis

```

The lemmas can be lifted to *full* (*propo-rew-step* *r*) instead of *propo-rew-step*

7.2.2 Invariant after all rewriting

lemma *full-propo-rew-step-inv-stay-with-inc*:

fixes *r*:: '*v* *propo* \Rightarrow '*v* *propo* \Rightarrow *bool* **and** *test-symb*:: '*v* *propo* \Rightarrow *bool* **and** *x* :: '*v*
and φ ψ :: '*v* *propo*

assumes

H: $\forall \varphi \psi. \text{propo-rew-step } r \varphi \psi \longrightarrow \text{all-subformula-st test-symb } \varphi$
 $\longrightarrow \text{all-subformula-st test-symb } \psi$ **and**

H': $\forall (c:: 'v \text{ connective}) \xi \varphi \xi' \varphi'. \varphi \preceq \Phi \longrightarrow \text{propo-rew-step } r \varphi \varphi'$
 $\longrightarrow \text{wf-conn } c (\xi @ \varphi \# \xi') \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi')) \longrightarrow \text{test-symb } \varphi'$
 $\longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$ **and**
 $\varphi \preceq \Phi$ **and**

full: *full* (*propo-rew-step* *r*) $\varphi \psi$ **and**

init: *all-subformula-st test-symb* φ

shows *all-subformula-st test-symb* ψ

using *assms unfolding full-def*

proof –

have *rel*: (*propo-rew-step* *r*)** $\varphi \psi$

using *full unfolding full-def by auto*

thus *all-subformula-st test-symb* ψ

using *init*

proof (*induct rule*: *rtrancpl-induct*)

case *base*

then show *all-subformula-st test-symb* φ **by** *blast*

next

case (*step b c*) **note** *star* = *this*(1) **and** *IH* = *this*(3) **and** *one* = *this*(2) **and** *all* = *this*(4)

then have *all-subformula-st test-symb* *b* **by** *metis*

then show *all-subformula-st test-symb* *c* **using** *propo-rew-step-inv-stay' H H' rel one* **by** *auto*

qed

qed

lemma *full-propo-rew-step-inv-stay'*:

fixes *r*:: '*v* *propo* \Rightarrow '*v* *propo* \Rightarrow *bool* **and** *test-symb*:: '*v* *propo* \Rightarrow *bool* **and** *x* :: '*v*
and $\varphi \psi$:: '*v* *propo*

assumes

H: $\forall \varphi \psi. \text{propo-rew-step } r \varphi \psi \longrightarrow \text{all-subformula-st test-symb } \varphi$
 $\longrightarrow \text{all-subformula-st test-symb } \psi$ **and**

H': $\forall (c:: 'v \text{ connective}) \xi \varphi \xi' \varphi'. \text{propo-rew-step } r \varphi \varphi' \longrightarrow \text{wf-conn } c (\xi @ \varphi \# \xi')$
 $\longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi')) \longrightarrow \text{test-symb } \varphi' \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$ **and**

full: *full* (*propo-rew-step* *r*) $\varphi \psi$ **and**

init: *all-subformula-st test-symb* φ

shows *all-subformula-st test-symb* ψ

using *full-propo-rew-step-inv-stay-with-inc*[*of r test-symb* φ] *assms subformula-refl* **by** *metis*

lemma *full-propo-rew-step-inv-stay*:

fixes *r*:: '*v* *propo* \Rightarrow '*v* *propo* \Rightarrow *bool* **and** *test-symb*:: '*v* *propo* \Rightarrow *bool* **and** *x* :: '*v*
and $\varphi \psi$:: '*v* *propo*

assumes

H: $\forall \varphi \psi. r \varphi \psi \longrightarrow \text{all-subformula-st test-symb } \varphi \longrightarrow \text{all-subformula-st test-symb } \psi$ **and**

H': $\forall (c:: 'v \text{ connective}) \xi \varphi \xi' \varphi'. \text{wf-conn } c (\xi @ \varphi \# \xi') \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi'))$
 $\longrightarrow \text{test-symb } \varphi' \longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$ **and**

full: *full* (*propo-rew-step* *r*) $\varphi \psi$ **and**

init: *all-subformula-st test-symb* φ

```

shows all-subformula-st test-symb  $\psi$ 
unfolding full-def
proof -
  have rel: (propo-rew-step r)**  $\varphi \psi$ 
    using full unfolding full-def by auto
  thus all-subformula-st test-symb  $\psi$ 
    using init
  proof (induct rule: rtrancplp-induct)
    case base
      thus all-subformula-st test-symb  $\varphi$  by blast
    next
      case (step b c)
      note star = this(1) and IH = this(3) and one = this(2) and all = this(4)
      hence all-subformula-st test-symb b by metis
      thus all-subformula-st test-symb c
        using propo-rew-step-inv-stay subformula-refl H H' rel one by auto
    qed
  qed

lemma full-propo-rew-step-inv-stay-conn:
  fixes r:: 'v propo  $\Rightarrow$  'v propo  $\Rightarrow$  bool and test-symb:: 'v propo  $\Rightarrow$  bool and x:: 'v
  and  $\varphi \psi$  :: 'v propo
  assumes
    H:  $\forall \varphi \psi. r \varphi \psi \longrightarrow \text{all-subformula-st test-symb } \varphi \longrightarrow \text{all-subformula-st test-symb } \psi$  and
    H':  $\forall (c:: 'v \text{ connective}) l l'. \text{wf-conn } c l \longrightarrow \text{wf-conn } c l' \longrightarrow (\text{test-symb } (\text{conn } c l) \longleftrightarrow \text{test-symb } (\text{conn } c l'))$  and
    full: full (propo-rew-step r)  $\varphi \psi$  and
    init: all-subformula-st test-symb  $\varphi$ 
  shows all-subformula-st test-symb  $\psi$ 
proof -
  have  $\bigwedge (c:: 'v \text{ connective}) \xi \varphi \xi' \varphi'. \text{wf-conn } c (\xi @ \varphi \# \xi') \Longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi \# \xi')) \Longrightarrow \text{test-symb } \varphi' \Longrightarrow \text{test-symb } (\text{conn } c (\xi @ \varphi' \# \xi'))$ 
    using H' by (metis wf-conn-no-arity-change-helper wf-conn-no-arity-change)
  thus all-subformula-st test-symb  $\psi$ 
    using H full init full-propo-rew-step-inv-stay by blast
qed

end
theory Prop-Normalisation
imports Main Prop-Logic Prop-Abstract-Transformation
begin

```

Given the previous definition about abstract rewriting and theorem about them, we now have the detailed rule making the transformation into CNF/DNF.

8 Rewrite Rules

The idea of Christoph Weidenbach's book is to remove gradually the operators: first equivalencies, then implication, after that the unused true/false and finally the reorganizing the or/and. We will prove each transformation separately.

8.1 Elimination of the equivalences

The first transformation consists in removing every equivalence symbol.

inductive *elim-equiv* :: 'v propo \Rightarrow 'v propo \Rightarrow bool **where**
elim-equiv[simp]: *elim-equiv* (FEq φ ψ) (FAnd (FImp φ ψ) (FImp ψ φ))

lemma *elim-equiv-transformation-consistent*:

$A \models \text{FEq } \varphi \ \psi \longleftrightarrow A \models \text{FAnd } (\text{FImp } \varphi \ \psi) \ (\text{FImp } \psi \ \varphi)$
by *auto*

lemma *elim-equiv-explicit*: *elim-equiv* $\varphi \ \psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$
by (*induct rule: elim-equiv.induct, auto*)

lemma *elim-equiv-consistent*: *preserves-un-sat elim-equiv*

unfolding *preserves-un-sat-def* **by** (*simp add: elim-equiv-explicit*)

lemma *elimEquiv-lifted-consistant*:

preserves-un-sat (*full* (*propo-rew-step elim-equiv*))
by (*simp add: elim-equiv-consistent*)

This function ensures that there is no equivalencies left in the formula tested by *no-equiv-symb*.

fun *no-equiv-symb* :: 'v propo \Rightarrow bool **where**

no-equiv-symb (FEq -) = False |

no-equiv-symb - = True

Given the definition of *no-equiv-symb*, it does not depend on the formula, but only on the connective used.

lemma *no-equiv-symb-conn-characterization*[simp]:

fixes *c* :: 'v connective **and** *l* :: 'v propo list

assumes *wf*: *wf-conn c l*

shows *no-equiv-symb* (*conn c l*) $\longleftrightarrow c \neq \text{CEq}$

by (*metis connective.distinct(13,25,35,43) wf no-equiv-symb.elims(3) no-equiv-symb.simps(1) wf-conn.cases wf-conn-list(6)*)

definition *no-equiv* **where** *no-equiv* = *all-subformula-st no-equiv-symb*

lemma *no-equiv-eq*[simp]:

fixes $\varphi \ \psi$:: 'v propo

shows

$\neg \text{no-equiv } (\text{FEq } \varphi \ \psi)$

no-equiv FT

no-equiv FF

using *no-equiv-symb.simps(1) all-subformula-st-test-symb-true-phi* **unfolding** *no-equiv-def* **by** *auto*

The following lemma helps to reconstruct *no-equiv* expressions: this representation is easier to use than the set definition.

lemma *all-subformula-st-decomp-explicit-no-equiv*[iff]:

fixes $\varphi \ \psi$:: 'v propo

shows

no-equiv (FNot φ) $\longleftrightarrow \text{no-equiv } \varphi$

no-equiv (FAnd $\varphi \ \psi$) $\longleftrightarrow (\text{no-equiv } \varphi \wedge \text{no-equiv } \psi)$

no-equiv (FOr $\varphi \ \psi$) $\longleftrightarrow (\text{no-equiv } \varphi \wedge \text{no-equiv } \psi)$

no-equiv (FImp $\varphi \ \psi$) $\longleftrightarrow (\text{no-equiv } \varphi \wedge \text{no-equiv } \psi)$

by (*auto simp: no-equiv-def*)

A theorem to show the link between the rewrite relation *elim-equiv* and the function *no-equiv-symb*. This theorem is one of the assumption we need to characterize the transformation.

lemma *no-equiv-elim-equiv-step*:

fixes $\varphi :: 'v \text{ propo}$

assumes *no-equiv*: $\neg \text{no-equiv } \varphi$

shows $\exists \psi \psi'. \psi \preceq \varphi \wedge \text{elim-equiv } \psi \psi'$

proof –

have *test-symb-false-nullary*:

$\forall x::'v. \text{no-equiv-symb } FF \wedge \text{no-equiv-symb } FT \wedge \text{no-equiv-symb } (FVar\ x)$

unfolding *no-equiv-def* **by** *auto*

moreover {

fix $c::'v \text{ connective}$ **and** $l::'v \text{ propo list}$ **and** $\psi::'v \text{ propo}$

assume *a1*: $\text{elim-equiv } (\text{conn } c\ l)\ \psi$

have $\bigwedge p\ pa. \neg \text{elim-equiv } (p::'v \text{ propo})\ pa \vee \neg \text{no-equiv-symb } p$

using *elim-equiv.cases no-equiv-symb.simps(1)* **by** *blast*

then have $\text{elim-equiv } (\text{conn } c\ l)\ \psi \implies \neg \text{no-equiv-symb } (\text{conn } c\ l)$ **using** *a1* **by** *metis*

}

moreover have $H': \forall \psi. \neg \text{elim-equiv } FT\ \psi \vee \psi. \neg \text{elim-equiv } FF\ \psi \vee \psi\ x. \neg \text{elim-equiv } (FVar\ x)\ \psi$

using *elim-equiv.cases* **by** *auto*

moreover have $\bigwedge \varphi. \neg \text{no-equiv-symb } \varphi \implies \exists \psi. \text{elim-equiv } \varphi\ \psi$

by (*case-tac* φ , *auto simp: elim-equiv.simps*)

then have $\bigwedge \varphi'. \varphi' \preceq \varphi \implies \neg \text{no-equiv-symb } \varphi' \implies \exists \psi. \text{elim-equiv } \varphi'\ \psi$ **by** *force*

ultimately show *?thesis*

using *no-test-symb-step-exists no-equiv test-symb-false-nullary unfolding no-equiv-def* **by** *blast*

qed

Given all the previous theorem and the characterization, once we have rewritten everything, there is no equivalence symbol any more.

lemma *no-equiv-full-propo-rew-step-elim-equiv*:

full (propo-rew-step elim-equiv) $\varphi\ \psi \implies \text{no-equiv } \psi$

using *full-propo-rew-step-subformula no-equiv-elim-equiv-step* **by** *blast*

8.2 Eliminate Implication

After that, we can eliminate the implication symbols.

inductive *elim-imp* :: $'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$ **where**

[*simp*]: *elim-imp* (*FImp* $\varphi\ \psi$) (*FOr* (*FNot* φ) ψ)

lemma *elim-imp-transformation-consistent*:

$A \models \text{FImp } \varphi\ \psi \longleftrightarrow A \models \text{FOr } (\text{FNot } \varphi)\ \psi$

by *auto*

lemma *elim-imp-explicit*: $\text{elim-imp } \varphi\ \psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$

by (*induct* $\varphi\ \psi$ *rule: elim-imp.induct, auto*)

lemma *elim-imp-consistent*: *preserves-un-sat elim-imp*

unfolding *preserves-un-sat-def* **by** (*simp add: elim-imp-explicit*)

lemma *elim-imp-lifted-consistant*:

preserves-un-sat (full (propo-rew-step elim-imp))

by (*simp add: elim-imp-consistent*)

fun *no-imp-symb* **where**

no-imp-symb (*FImp* -) = *False* |

no-imp-symb - = *True*

lemma *no-imp-symb-conn-characterization*:

wf-conn c l \implies *no-imp-symb* (*conn c l*) \longleftrightarrow *c* \neq *CImp*

by (*induction rule*: *wf-conn-induct*) *auto*

definition *no-imp* **where** *no-imp* \equiv *all-subformula-st no-imp-symb*

declare *no-imp-def*[*simp*]

lemma *no-imp-Imp*[*simp*]:

\neg *no-imp* (*FImp* φ ψ)

no-imp FT

no-imp FF

unfolding *no-imp-def* **by** *auto*

lemma *all-subformula-st-decomp-explicit-imp*[*simp*]:

fixes $\varphi \psi :: 'v \text{ propo}$

shows

no-imp (*FNot* φ) \longleftrightarrow *no-imp* φ

no-imp (*FAnd* $\varphi \psi$) \longleftrightarrow (*no-imp* $\varphi \wedge$ *no-imp* ψ)

no-imp (*FOr* $\varphi \psi$) \longleftrightarrow (*no-imp* $\varphi \wedge$ *no-imp* ψ)

by *auto*

Invariant of the *elim-imp* transformation

lemma *elim-imp-no-equiv*:

elim-imp $\varphi \psi \implies$ *no-equiv* $\varphi \implies$ *no-equiv* ψ

by (*induct* $\varphi \psi$ *rule*: *elim-imp.induct*, *auto*)

lemma *elim-imp-inv*:

fixes $\varphi \psi :: 'v \text{ propo}$

assumes *full* (*propo-rew-step elim-imp*) $\varphi \psi$ **and** *no-equiv* φ

shows *no-equiv* ψ

using *full-propo-rew-step-inv-stay-conn*[*of elim-imp no-equiv-symb* $\varphi \psi$] *assms elim-imp-no-equiv*

no-equiv-symb-conn-characterization **unfolding** *no-equiv-def* **by** *metis*

lemma *no-no-imp-elim-imp-step-exists*:

fixes $\varphi :: 'v \text{ propo}$

assumes *no-equiv*: \neg *no-imp* φ

shows $\exists \psi \psi'. \psi \preceq \varphi \wedge$ *elim-imp* $\psi \psi'$

proof –

have *test-symb-false-nullary*: $\forall x. \text{no-imp-symb } FF \wedge \text{no-imp-symb } FT \wedge \text{no-imp-symb } (FVar (x:: 'v))$

by *auto*

moreover {

fix *c*: $'v \text{ connective}$ **and** *l*: $'v \text{ propo list}$ **and** $\psi :: 'v \text{ propo}$

have *H*: *elim-imp* (*conn c l*) $\psi \implies$ \neg *no-imp-symb* (*conn c l*)

by (*auto elim*: *elim-imp.cases*)

}

moreover

have *H'*: $\forall \psi. \neg \text{elim-imp } FT \psi \forall \psi. \neg \text{elim-imp } FF \psi \forall \psi x. \neg \text{elim-imp } (FVar x) \psi$

by (*auto elim*: *elim-imp.cases*)+

moreover

have $\bigwedge \varphi. \neg \text{no-imp-symb } \varphi \implies \exists \psi. \text{elim-imp } \varphi \psi$

by (*case-tac* φ) (*force simp*: *elim-imp.simps*)+

then have ($\bigwedge \varphi'. \varphi' \preceq \varphi \implies \neg \text{no-imp-symb } \varphi' \implies \exists \psi. \text{elim-imp } \varphi' \psi$) **by** *force*

ultimately show *?thesis*

using *no-test-symb-step-exists no-equiv test-symb-false-nullary* **unfolding** *no-imp-def* **by** *blast*
qed

lemma *no-imp-full-propo-rew-step-elim-imp: full (propo-rew-step elim-imp) $\varphi \psi \implies \text{no-imp } \psi$*
using *full-propo-rew-step-subformula no-no-imp-elim-imp-step-exists* **by** *blast*

8.3 Eliminate all the True and False in the formula

Contrary to the book, we have to give the transformation and the “commutative” transformation. The latter is implicit in the book.

inductive *elimTB* **where**

ElimTB1: elimTB (FAnd φ FT) φ |

ElimTB1': elimTB (FAnd FT φ) φ |

ElimTB2: elimTB (FAnd φ FF) FF |

ElimTB2': elimTB (FAnd FF φ) FF |

ElimTB3: elimTB (FOr φ FT) FT |

ElimTB3': elimTB (FOr FT φ) FT |

ElimTB4: elimTB (FOr φ FF) φ |

ElimTB4': elimTB (FOr FF φ) φ |

ElimTB5: elimTB (FNot FT) FF |

ElimTB6: elimTB (FNot FF) FT

lemma *elimTB-consistent: preserves-un-sat elimTB*

proof –

{
fix $\varphi \psi :: 'b \text{ propo}$
have *elimTB $\varphi \psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$* **by** (*induction rule: elimTB.inducts*) *auto*
}
then show *?thesis* **using** *preserves-un-sat-def* **by** *auto*
qed

inductive *no-T-F-symb :: 'v propo \Rightarrow bool* **where**

no-T-F-symb-comp: $c \neq CF \implies c \neq CT \implies \text{wf-conn } c \ l \implies (\forall \varphi \in \text{set } l. \varphi \neq FT \wedge \varphi \neq FF)$
 $\implies \text{no-T-F-symb } (\text{conn } c \ l)$

lemma *wf-conn-no-T-F-symb-iff[simp]:*

wf-conn c ψ s \implies

no-T-F-symb (conn c ψ s) $\longleftrightarrow (c \neq CF \wedge c \neq CT \wedge (\forall \psi \in \text{set } \psi\text{s}. \psi \neq FF \wedge \psi \neq FT))$

unfolding *no-T-F-symb.simps* **apply** (*cases c*)

using *wf-conn-list(1)* **apply** *fastforce*

using *wf-conn-list(2)* **apply** *fastforce*

using *wf-conn-list(3)* **apply** *fastforce*

apply (*metis (no-types, hide-lams) conn-inj connective.distinct(5,17)*)

using *conn-inj* **apply** *blast+*

done

lemma *wf-conn-no-T-F-symb-iff-explicit[simp]:*

no-T-F-symb (FAnd $\varphi \psi$) $\longleftrightarrow (\forall \chi \in \text{set } [\varphi, \psi]. \chi \neq FF \wedge \chi \neq FT)$

no-T-F-symb (FOr $\varphi \psi$) $\longleftrightarrow (\forall \chi \in \text{set } [\varphi, \psi]. \chi \neq FF \wedge \chi \neq FT)$


```

no-T-F-symb (FEq  $\varphi$   $\psi$ )  $\longleftrightarrow$  ( $\forall \chi \in \text{set } [\varphi, \psi]. \chi \neq FF \wedge \chi \neq FT$ )
no-T-F-symb (FImp  $\varphi$   $\psi$ )  $\longleftrightarrow$  ( $\forall \chi \in \text{set } [\varphi, \psi]. \chi \neq FF \wedge \chi \neq FT$ )
  apply (metis conn.simps(36) conn.simps(37) conn.simps(5) propo.distinct(19)
    wf-conn-helper-facts(5) wf-conn-no-T-F-symb-iff)
  apply (metis conn.simps(36) conn.simps(37) conn.simps(6) propo.distinct(22)
    wf-conn-helper-facts(6) wf-conn-no-T-F-symb-iff)
  using wf-conn-no-T-F-symb-iff apply fastforce
by (metis conn.simps(36) conn.simps(37) conn.simps(7) propo.distinct(23) wf-conn-helper-facts(7)
  wf-conn-no-T-F-symb-iff)

```

```

lemma no-T-F-symb-false[simp]:
  fixes c :: 'v connective
  shows
     $\neg$ no-T-F-symb (FT :: 'v propo)
     $\neg$ no-T-F-symb (FF :: 'v propo)
  by (metis (no-types) conn.simps(1,2) wf-conn-no-T-F-symb-iff wf-conn-nullary)+

```

```

lemma no-T-F-symb-bool[simp]:
  fixes x :: 'v
  shows no-T-F-symb (FVar x)
  using no-T-F-symb-comp wf-conn-nullary by (metis connective.distinct(3, 15) conn.simps(3)
    empty-iff list.set(1))

```

```

lemma no-T-F-symb-fnot-imp:
   $\neg$ no-T-F-symb (FNot  $\varphi$ )  $\implies \varphi = FT \vee \varphi = FF$ 
proof (rule ccontr)
  assume n:  $\neg$  no-T-F-symb (FNot  $\varphi$ )
  assume  $\neg (\varphi = FT \vee \varphi = FF)$ 
  then have  $\forall \varphi' \in \text{set } [\varphi]. \varphi' \neq FT \wedge \varphi' \neq FF$  by auto
  moreover have wf-conn CNot  $[\varphi]$  by simp
  ultimately have no-T-F-symb (FNot  $\varphi$ )
    using no-T-F-symb.intros by (metis conn.simps(4) connective.distinct(5,17))
  then show False using n by blast
qed

```

```

lemma no-T-F-symb-fnot[simp]:
  no-T-F-symb (FNot  $\varphi$ )  $\longleftrightarrow \neg(\varphi = FT \vee \varphi = FF)$ 
  using no-T-F-symb.simps no-T-F-symb-fnot-imp by (metis conn-inj-not(2) list.set-intros(1))

```

Actually it is not possible to remove every FT and FF : if the formula is equal to true or false, we can not remove it.

```

inductive no-T-F-symb-except-toplevel where
  no-T-F-symb-except-toplevel-true[simp]: no-T-F-symb-except-toplevel FT |
  no-T-F-symb-except-toplevel-false[simp]: no-T-F-symb-except-toplevel FF |
  noTrue-no-T-F-symb-except-toplevel[simp]: no-T-F-symb  $\varphi \implies$  no-T-F-symb-except-toplevel  $\varphi$ 

```

```

lemma no-T-F-symb-except-toplevel-bool:
  fixes x :: 'v
  shows no-T-F-symb-except-toplevel (FVar x)
  by simp

```

```

lemma no-T-F-symb-except-toplevel-not-decom:
   $\varphi \neq FT \implies \varphi \neq FF \implies$  no-T-F-symb-except-toplevel (FNot  $\varphi$ )

```

by *simp*

lemma *no-T-F-symb-except-toplevel-bin-decom*:

fixes $\varphi \ \psi :: 'v \text{ propo}$

assumes $\varphi \neq FT$ **and** $\varphi \neq FF$ **and** $\psi \neq FT$ **and** $\psi \neq FF$

and $c :: \text{binary-connectives}$

shows *no-T-F-symb-except-toplevel* (*conn* c $[\varphi, \psi]$)

by (*metis* (*no-types*, *lifting*) *assms* c *conn.simps*(4) *list.discI* *noTrue-no-T-F-symb-except-toplevel* *wf-conn-no-T-F-symb-iff* *no-T-F-symb-fnot* *set.ConsD* *wf-conn-binary* *wf-conn-helper-facts*(1) *wf-conn-list-decomp*(1,2))

lemma *no-T-F-symb-except-toplevel-if-is-a-true-false*:

fixes $l :: 'v \text{ propo list}$ **and** $c :: 'v \text{ connective}$

assumes *corr*: *wf-conn* c l

and $FT \in \text{set } l \vee FF \in \text{set } l$

shows $\neg \text{no-T-F-symb-except-toplevel} (\text{conn } c \ l)$

by (*metis* *assms* *empty-iff* *no-T-F-symb-except-toplevel.simps* *wf-conn-no-T-F-symb-iff* *set-empty* *wf-conn-list*(1,2))

lemma *no-T-F-symb-except-top-level-false-example*[*simp*]:

fixes $\varphi \ \psi :: 'v \text{ propo}$

assumes $\varphi = FT \vee \psi = FT \vee \varphi = FF \vee \psi = FF$

shows

$\neg \text{no-T-F-symb-except-toplevel} (FAnd \ \varphi \ \psi)$

$\neg \text{no-T-F-symb-except-toplevel} (FOr \ \varphi \ \psi)$

$\neg \text{no-T-F-symb-except-toplevel} (FImp \ \varphi \ \psi)$

$\neg \text{no-T-F-symb-except-toplevel} (FEq \ \varphi \ \psi)$

using *assms* *no-T-F-symb-except-toplevel-if-is-a-true-false* **unfolding** *binary-connectives-def*

by (*metis* (*no-types*) *conn.simps*(5–8) *insert-iff* *list.simps*(14–15) *wf-conn-helper-facts*(5–8))+

lemma *no-T-F-symb-except-top-level-false-not*[*simp*]:

fixes $\varphi \ \psi :: 'v \text{ propo}$

assumes $\varphi = FT \vee \varphi = FF$

shows

$\neg \text{no-T-F-symb-except-toplevel} (FNot \ \varphi)$

by (*simp* *add*: *assms* *no-T-F-symb-except-toplevel.simps*)

This is the local extension of *no-T-F-symb-except-toplevel*.

definition *no-T-F-except-top-level* **where**

no-T-F-except-top-level $\equiv \text{all-subformula-st } \text{no-T-F-symb-except-toplevel}$

This is another property we will use. While this version might seem to be the one we want to prove, it is not since *FT* can not be reduced.

definition *no-T-F* **where**

no-T-F $\equiv \text{all-subformula-st } \text{no-T-F-symb}$

lemma *no-T-F-except-top-level-false*:

fixes $l :: 'v \text{ propo list}$ **and** $c :: 'v \text{ connective}$

assumes *wf-conn* c l

and $FT \in \text{set } l \vee FF \in \text{set } l$

shows $\neg \text{no-T-F-except-top-level} (\text{conn } c \ l)$

by (*simp* *add*: *all-subformula-st-decomp* *assms* *no-T-F-except-top-level-def* *no-T-F-symb-except-toplevel-if-is-a-true-false*)

lemma *no-T-F-except-top-level-false-example[simp]*:
fixes $\varphi \ \psi :: 'v \text{ propo}$
assumes $\varphi = FT \vee \psi = FT \vee \varphi = FF \vee \psi = FF$
shows
 $\neg \text{no-T-F-except-top-level } (FAnd \ \varphi \ \psi)$
 $\neg \text{no-T-F-except-top-level } (FOr \ \varphi \ \psi)$
 $\neg \text{no-T-F-except-top-level } (FEq \ \varphi \ \psi)$
 $\neg \text{no-T-F-except-top-level } (FImp \ \varphi \ \psi)$
by (*metis all-subformula-st-test-symb-true-phi assms no-T-F-except-top-level-def*
no-T-F-symb-except-top-level-false-example)**+**

lemma *no-T-F-symb-except-toplevel-no-T-F-symb*:
 $\text{no-T-F-symb-except-toplevel } \varphi \implies \varphi \neq FF \implies \varphi \neq FT \implies \text{no-T-F-symb } \varphi$
by (*induct rule: no-T-F-symb-except-toplevel.induct, auto*)

The two following lemmas give the precise link between the two definitions.

lemma *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb*:
 $\text{no-T-F-except-top-level } \varphi \implies \varphi \neq FF \implies \varphi \neq FT \implies \text{no-T-F } \varphi$
unfolding *no-T-F-except-top-level-def no-T-F-def* **apply** (*induct* φ)
using *no-T-F-symb-fnot* **by** *fastforce***+**

lemma *no-T-F-no-T-F-except-top-level*:
 $\text{no-T-F } \varphi \implies \text{no-T-F-except-top-level } \varphi$
unfolding *no-T-F-except-top-level-def no-T-F-def*
unfolding *all-subformula-st-def* **by** *auto*

lemma *no-T-F-except-top-level-simp[simp]*: $\text{no-T-F-except-top-level } FF \ \text{no-T-F-except-top-level } FT$
unfolding *no-T-F-except-top-level-def* **by** *auto*

lemma *no-T-F-no-T-F-except-top-level'[simp]*:
 $\text{no-T-F-except-top-level } \varphi \longleftrightarrow (\varphi = FF \vee \varphi = FT \vee \text{no-T-F } \varphi)$
using *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb no-T-F-no-T-F-except-top-level*
by *auto*

lemma *no-T-F-bin-decomp[simp]*:
assumes $c: c \in \text{binary-connectives}$
shows $\text{no-T-F } (\text{conn } c \ [\varphi, \psi]) \longleftrightarrow (\text{no-T-F } \varphi \wedge \text{no-T-F } \psi)$
proof –
have *wf*: $\text{wf-conn } c \ [\varphi, \psi]$ **using** c **by** *auto*
then have $\text{no-T-F } (\text{conn } c \ [\varphi, \psi]) \longleftrightarrow (\text{no-T-F-symb } (\text{conn } c \ [\varphi, \psi]) \wedge \text{no-T-F } \varphi \wedge \text{no-T-F } \psi)$
by (*simp add: all-subformula-st-decomp no-T-F-def*)
then show $\text{no-T-F } (\text{conn } c \ [\varphi, \psi]) \longleftrightarrow (\text{no-T-F } \varphi \wedge \text{no-T-F } \psi)$
using c *wf all-subformula-st-decomp list.discI no-T-F-def no-T-F-symb-except-toplevel-bin-decom*
 $\text{no-T-F-symb-except-toplevel-no-T-F-symb no-T-F-symb-false}(1,2) \ \text{wf-conn-helper-facts}(2,3)$
 $\text{wf-conn-list}(1,2)$ **by** *metis*

qed

lemma *no-T-F-bin-decomp-expanded[simp]*:
assumes $c: c = CAnd \vee c = COr \vee c = CEq \vee c = CImp$
shows $\text{no-T-F } (\text{conn } c \ [\varphi, \psi]) \longleftrightarrow (\text{no-T-F } \varphi \wedge \text{no-T-F } \psi)$
using *no-T-F-bin-decomp assms unfolding binary-connectives-def* **by** *blast*

lemma *no-T-F-comp-expanded-explicit[simp]*:
fixes $\varphi \ \psi :: 'v \text{ propo}$

```

shows
   $no\text{-}T\text{-}F (FAnd\ \varphi\ \psi) \longleftrightarrow (no\text{-}T\text{-}F\ \varphi \wedge no\text{-}T\text{-}F\ \psi)$ 
   $no\text{-}T\text{-}F (FOr\ \varphi\ \psi) \longleftrightarrow (no\text{-}T\text{-}F\ \varphi \wedge no\text{-}T\text{-}F\ \psi)$ 
   $no\text{-}T\text{-}F (FEq\ \varphi\ \psi) \longleftrightarrow (no\text{-}T\text{-}F\ \varphi \wedge no\text{-}T\text{-}F\ \psi)$ 
   $no\text{-}T\text{-}F (FImp\ \varphi\ \psi) \longleftrightarrow (no\text{-}T\text{-}F\ \varphi \wedge no\text{-}T\text{-}F\ \psi)$ 
using assms conn.simps(5-8) no-T-F-bin-decomp-expanded by (metis (no-types))+

lemma no-T-F-comp-not[simp]:
  fixes  $\varphi\ \psi :: 'v\ propo$ 
  shows  $no\text{-}T\text{-}F (FNot\ \varphi) \longleftrightarrow no\text{-}T\text{-}F\ \varphi$ 
  by (metis all-subformula-st-decomp-explicit(3) all-subformula-st-test-symb-true-phi no-T-F-def
    no-T-F-symb-false(1,2) no-T-F-symb-fnot-imp)

lemma no-T-F-decomp:
  fixes  $\varphi\ \psi :: 'v\ propo$ 
  assumes  $\varphi: no\text{-}T\text{-}F (FAnd\ \varphi\ \psi) \vee no\text{-}T\text{-}F (FOr\ \varphi\ \psi) \vee no\text{-}T\text{-}F (FEq\ \varphi\ \psi) \vee no\text{-}T\text{-}F (FImp\ \varphi\ \psi)$ 
  shows  $no\text{-}T\text{-}F\ \psi$  and  $no\text{-}T\text{-}F\ \varphi$ 
  using assms by auto

lemma no-T-F-decomp-not:
  fixes  $\varphi :: 'v\ propo$ 
  assumes  $\varphi: no\text{-}T\text{-}F (FNot\ \varphi)$ 
  shows  $no\text{-}T\text{-}F\ \varphi$ 
  using assms by auto

lemma no-T-F-symb-except-toplevel-step-exists:
  fixes  $\varphi\ \psi :: 'v\ propo$ 
  assumes  $no\text{-}equiv\ \varphi$  and  $no\text{-}imp\ \varphi$ 
  shows  $\psi \preceq \varphi \implies \neg no\text{-}T\text{-}F\text{-}symb\text{-}except\text{-}toplevel\ \psi \implies \exists \psi'. elimTB\ \psi\ \psi'$ 
proof (induct  $\psi$  rule: propo-induct-arity)
  case (nullary  $\varphi'\ x$ )
  then have False using no-T-F-symb-except-toplevel-true no-T-F-symb-except-toplevel-false by auto
  then show ?case by blast
next
  case (unary  $\psi$ )
  then have  $\psi = FF \vee \psi = FT$  using no-T-F-symb-except-toplevel-not-decom by blast
  then show ?case using ElimTB5 ElimTB6 by blast
next
  case (binary  $\varphi'\ \psi1\ \psi2$ )
  note  $IH1 = this(1)$  and  $IH2 = this(2)$  and  $\varphi' = this(3)$  and  $F\varphi = this(4)$  and  $n = this(5)$ 
  {
    assume  $\varphi' = FImp\ \psi1\ \psi2 \vee \varphi' = FEq\ \psi1\ \psi2$ 
    then have False using  $n\ F\varphi\ subformula\text{-}all\text{-}subformula\text{-}st\ assms$ 
    by (metis (no-types) no-equiv-eq(1) no-equiv-def no-imp-Imp(1) no-imp-def)
    then have ?case by blast
  }
  moreover {
    assume  $\varphi': \varphi' = FAnd\ \psi1\ \psi2 \vee \varphi' = FOr\ \psi1\ \psi2$ 
    then have  $\psi1 = FT \vee \psi2 = FT \vee \psi1 = FF \vee \psi2 = FF$ 
    using no-T-F-symb-except-toplevel-bin-decom conn.simps(5,6) n unfolding binary-connectives-def
    by fastforce+
    then have ?case using elimTB.intros  $\varphi'$  by blast
  }
  ultimately show ?case using  $\varphi'$  by blast
qed

```

lemma *no-T-F-except-top-level-rew*:
fixes $\varphi :: 'v \text{ propo}$
assumes *noTB*: $\neg \text{no-T-F-except-top-level } \varphi$ **and** *no-equiv*: *no-equiv* φ **and** *no-imp*: *no-imp* φ
shows $\exists \psi \psi'. \psi \preceq \varphi \wedge \text{elimTB } \psi \psi'$
proof –
have *test-symb-false-nullary*: $\forall x. \text{no-T-F-symb-except-toplevel } (FF :: 'v \text{ propo})$
 $\wedge \text{no-T-F-symb-except-toplevel } FT \wedge \text{no-T-F-symb-except-toplevel } (FVar (x :: 'v))$ **by** *auto*
moreover {
fix $c :: 'v \text{ connective}$ **and** $l :: 'v \text{ propo list}$ **and** $\psi :: 'v \text{ propo}$
have $H: \text{elimTB } (\text{conn } c \ l) \ \psi \implies \neg \text{no-T-F-symb-except-toplevel } (\text{conn } c \ l)$
by (*cases* (*conn* $c \ l$) *rule*: *elimTB.cases*, *auto*)
}
moreover {
fix $x :: 'v$
have $H': \text{no-T-F-except-top-level } FT \ \text{no-T-F-except-top-level } FF$
 $\text{no-T-F-except-top-level } (FVar \ x)$
by (*auto simp*: *no-T-F-except-top-level-def test-symb-false-nullary*)
}
moreover {
fix ψ
have $\psi \preceq \varphi \implies \neg \text{no-T-F-symb-except-toplevel } \psi \implies \exists \psi'. \text{elimTB } \psi \psi'$
using *no-T-F-symb-except-toplevel-step-exists no-equiv no-imp* **by** *auto*
}
ultimately show *?thesis*
using *no-test-symb-step-exists noTB unfolding no-T-F-except-top-level-def* **by** *blast*
qed

lemma *elimTB-inv*:
fixes $\varphi \ \psi :: 'v \text{ propo}$
assumes *full* (*propo-rew-step elimTB*) $\varphi \ \psi$
and *no-equiv* φ **and** *no-imp* φ
shows *no-equiv* ψ **and** *no-imp* ψ
proof –
{
fix $\varphi \ \psi :: 'v \text{ propo}$
have $H: \text{elimTB } \varphi \ \psi \implies \text{no-equiv } \varphi \implies \text{no-equiv } \psi$
by (*induct* $\varphi \ \psi$ *rule*: *elimTB.induct*, *auto*)
}
then show *no-equiv* ψ
using *full-propo-rew-step-inv-stay-conn*[*of elimTB no-equiv-symb* $\varphi \ \psi$]
no-equiv-symb-conn-characterization assms **unfolding** *no-equiv-def* **by** *metis*
next
{
fix $\varphi \ \psi :: 'v \text{ propo}$
have $H: \text{elimTB } \varphi \ \psi \implies \text{no-imp } \varphi \implies \text{no-imp } \psi$
by (*induct* $\varphi \ \psi$ *rule*: *elimTB.induct*, *auto*)
}
then show *no-imp* ψ
using *full-propo-rew-step-inv-stay-conn*[*of elimTB no-imp-symb* $\varphi \ \psi$] *assms*
no-imp-symb-conn-characterization **unfolding** *no-imp-def* **by** *metis*
qed

lemma *elimTB-full-propo-rew-step*:
fixes $\varphi \ \psi :: 'v \text{ propo}$

assumes *no-equiv* φ **and** *no-imp* φ **and** *full (propo-rew-step elimTB)* φ ψ
shows *no-T-F-except-top-level* ψ
using *full-propo-rew-step-subformula no-T-F-except-top-level-rew assms elimTB-inv* **by** *fastforce*

8.4 PushNeg

Push the negation inside the formula, until the litteral.

inductive *pushNeg* **where**

PushNeg1[simp]: *pushNeg* (*FNot* (*FAnd* φ ψ)) (*FOR* (*FNot* φ) (*FNot* ψ)) |
PushNeg2[simp]: *pushNeg* (*FNot* (*FOR* φ ψ)) (*FAnd* (*FNot* φ) (*FNot* ψ)) |
PushNeg3[simp]: *pushNeg* (*FNot* (*FNot* φ)) φ

lemma *pushNeg-transformation-consistent*:

$A \models \text{FNot } (\text{FAnd } \varphi \ \psi) \longleftrightarrow A \models (\text{FOR } (\text{FNot } \varphi) \ (\text{FNot } \psi))$
 $A \models \text{FNot } (\text{FOR } \varphi \ \psi) \longleftrightarrow A \models (\text{FAnd } (\text{FNot } \varphi) \ (\text{FNot } \psi))$
 $A \models \text{FNot } (\text{FNot } \varphi) \longleftrightarrow A \models \varphi$
by *auto*

lemma *pushNeg-explicit*: *pushNeg* φ $\psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$
by (*induct* φ ψ *rule: pushNeg.induct, auto*)

lemma *pushNeg-consistent*: *preserves-un-sat pushNeg*
unfolding *preserves-un-sat-def* **by** (*simp add: pushNeg-explicit*)

lemma *pushNeg-lifted-consistant*:

preserves-un-sat (full (propo-rew-step pushNeg))
by (*simp add: pushNeg-consistent*)

fun *simple* **where**

simple FT = *True* |
simple FF = *True* |
simple (FVar -) = *True* |
simple - = *False*

lemma *simple-decomp*:

simple $\varphi \longleftrightarrow (\varphi = \text{FT} \vee \varphi = \text{FF} \vee (\exists x. \varphi = \text{FVar } x))$
by (*cases* φ) *auto*

lemma *subformula-conn-decomp-simple*:

fixes $\varphi \ \psi :: 'v \text{ propo}$
assumes *s*: *simple* ψ
shows $\varphi \preceq \text{FNot } \psi \longleftrightarrow (\varphi = \text{FNot } \psi \vee \varphi = \psi)$

proof –

have $\varphi \preceq \text{conn } \text{CNot } [\psi] \longleftrightarrow (\varphi = \text{conn } \text{CNot } [\psi] \vee (\exists \psi \in \text{set } [\psi]. \varphi \preceq \psi))$
using *subformula-conn-decomp wf-conn-helper-facts(1)* **by** *metis*
then show $\varphi \preceq \text{FNot } \psi \longleftrightarrow (\varphi = \text{FNot } \psi \vee \varphi = \psi)$ **using** *s* **by** (*auto simp: simple-decomp*)

qed

lemma *subformula-conn-decomp-explicit[simp]*:

fixes $\varphi :: 'v \text{ propo}$ **and** $x :: 'v$
shows
 $\varphi \preceq \text{FNot } \text{FT} \longleftrightarrow (\varphi = \text{FNot } \text{FT} \vee \varphi = \text{FT})$

```

 $\varphi \preceq FNot\ FF \longleftrightarrow (\varphi = FNot\ FF \vee \varphi = FF)$ 
 $\varphi \preceq FNot\ (FVar\ x) \longleftrightarrow (\varphi = FNot\ (FVar\ x) \vee \varphi = FVar\ x)$ 
by (auto simp: subformula-conn-decomp-simple)

```

```

fun simple-not-symb where
  simple-not-symb (FNot  $\varphi$ ) = (simple  $\varphi$ ) |
  simple-not-symb - = True

```

```

definition simple-not where
  simple-not = all-subformula-st simple-not-symb
declare simple-not-def[simp]

```

```

lemma simple-not-Not[simp]:
   $\neg$  simple-not (FNot (FAnd  $\varphi\ \psi$ ))
   $\neg$  simple-not (FNot (FOr  $\varphi\ \psi$ ))
by auto

```

```

lemma simple-not-step-exists:
  fixes  $\varphi\ \psi :: 'v\ propo$ 
  assumes no-equiv  $\varphi$  and no-imp  $\varphi$ 
  shows  $\psi \preceq \varphi \implies \neg$  simple-not-symb  $\psi \implies \exists \psi'. \text{pushNeg } \psi\ \psi'$ 
  apply (induct  $\psi$ , auto)
  apply (rename-tac  $\psi$ , case-tac  $\psi$ , auto intro: pushNeg.intros)
  by (metis assms(1,2) no-imp-Imp(1) no-equiv-eq(1) no-imp-def no-equiv-def
    subformula-in-subformula-not subformula-all-subformula-st)+

```

```

lemma simple-not-rew:
  fixes  $\varphi :: 'v\ propo$ 
  assumes noTB:  $\neg$  simple-not  $\varphi$  and no-equiv: no-equiv  $\varphi$  and no-imp: no-imp  $\varphi$ 
  shows  $\exists \psi\ \psi'. \psi \preceq \varphi \wedge \text{pushNeg } \psi\ \psi'$ 

```

```

proof -
  have  $\forall x. \text{simple-not-symb } (FF :: 'v\ propo) \wedge \text{simple-not-symb } FT \wedge \text{simple-not-symb } (FVar\ (x :: 'v))$ 
  by auto
  moreover {
    fix  $c :: 'v\ connective$  and  $l :: 'v\ propo\ list$  and  $\psi :: 'v\ propo$ 
    have  $H: \text{pushNeg } (\text{conn } c\ l)\ \psi \implies \neg \text{simple-not-symb } (\text{conn } c\ l)$ 
    by (cases (conn  $c\ l$ ) rule: pushNeg.cases) auto
  }
  moreover {
    fix  $x :: 'v$ 
    have  $H': \text{simple-not } FT\ \text{simple-not } FF\ \text{simple-not } (FVar\ x)$ 
    by simp-all
  }
  moreover {
    fix  $\psi :: 'v\ propo$ 
    have  $\psi \preceq \varphi \implies \neg \text{simple-not-symb } \psi \implies \exists \psi'. \text{pushNeg } \psi\ \psi'$ 
    using simple-not-step-exists no-equiv no-imp by blast
  }
  ultimately show ?thesis using no-test-symb-step-exists noTB unfolding simple-not-def by blast
qed

```

```

lemma no-T-F-except-top-level-pushNeg1:
  no-T-F-except-top-level (FNot (FAnd  $\varphi\ \psi$ ))  $\implies$  no-T-F-except-top-level (FOr (FNot  $\varphi$ ) (FNot  $\psi$ ))
  using no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb no-T-F-comp-not no-T-F-decomp(1)

```

no-T-F-decomp(2) *no-T-F-no-T-F-except-top-level* **by** (*metis no-T-F-comp-expanded-explicit*(2) *propo.distinct*(5,17))

lemma *no-T-F-except-top-level-pushNeg2*:

no-T-F-except-top-level (*FNot* (*FOr* φ ψ)) \implies *no-T-F-except-top-level* (*FAnd* (*FNot* φ) (*FNot* ψ))
by *auto*

lemma *no-T-F-symb-pushNeg*:

no-T-F-symb (*FOr* (*FNot* φ') (*FNot* ψ'))
no-T-F-symb (*FAnd* (*FNot* φ') (*FNot* ψ'))
no-T-F-symb (*FNot* (*FNot* φ'))
by *auto*

lemma *propo-rew-step-pushNeg-no-T-F-symb*:

propo-rew-step pushNeg φ $\psi \implies$ *no-T-F-except-top-level* $\varphi \implies$ *no-T-F-symb* $\varphi \implies$ *no-T-F-symb* ψ
apply (*induct rule: propo-rew-step.induct*)
apply (*cases rule: pushNeg.cases*)
apply *simp-all*
apply (*metis no-T-F-symb-pushNeg*(1))
apply (*metis no-T-F-symb-pushNeg*(2))
apply (*simp, metis all-subformula-st-test-symb-true-phi no-T-F-def*)

proof –

fix $\varphi \varphi'$: 'a *propo* **and** c :: 'a *connective* **and** $\xi \xi'$:: 'a *propo list*
assume *rel: propo-rew-step pushNeg* φ φ'
and *IH: no-T-F* $\varphi \implies$ *no-T-F-symb* $\varphi \implies$ *no-T-F-symb* φ'
and *wf: wf-conn* c ($\xi @ \varphi \# \xi'$)
and n : *conn* c ($\xi @ \varphi \# \xi'$) = *FF* \vee *conn* c ($\xi @ \varphi \# \xi'$) = *FT* \vee *no-T-F* (*conn* c ($\xi @ \varphi \# \xi'$))
and x : $c \neq CF \wedge c \neq CT \wedge \varphi \neq FF \wedge \varphi \neq FT \wedge (\forall \psi \in \text{set } \xi \cup \text{set } \xi'. \psi \neq FF \wedge \psi \neq FT)$
then have $c \neq CF \wedge c \neq CT \wedge \text{wf-conn } c$ ($\xi @ \varphi' \# \xi'$)
using *wf-conn-no-arity-change-helper wf-conn-no-arity-change* **by** *metis*
moreover have n' : *no-T-F* (*conn* c ($\xi @ \varphi \# \xi'$)) **using** n **by** (*simp add: wf wf-conn-list*(1,2))
moreover
{
have *no-T-F* φ
by (*metis Un-iff all-subformula-st-decomp list.set-intros*(1) n' *wf no-T-F-def set-append*)
moreover then have *no-T-F-symb* φ
by (*simp add: all-subformula-st-test-symb-true-phi no-T-F-def*)
ultimately have $\varphi' \neq FF \wedge \varphi' \neq FT$
using *IH no-T-F-symb-false*(1) *no-T-F-symb-false*(2) **by** *blast*
then have $\forall \psi \in \text{set } (\xi @ \varphi' \# \xi'). \psi \neq FF \wedge \psi \neq FT$ **using** x **by** *auto*
}
ultimately show *no-T-F-symb* (*conn* c ($\xi @ \varphi' \# \xi'$)) **by** (*simp add: x*)
qed

lemma *propo-rew-step-pushNeg-no-T-F*:

propo-rew-step pushNeg φ $\psi \implies$ *no-T-F* $\varphi \implies$ *no-T-F* ψ

proof (*induct rule: propo-rew-step.induct*)

case *global-rel*

then show ?*case*

by (*metis* (*no-types, lifting*) *no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb*
no-T-F-def no-T-F-except-top-level-pushNeg1 no-T-F-except-top-level-pushNeg2
no-T-F-no-T-F-except-top-level all-subformula-st-decomp-explicit(3) *pushNeg.simps*
simple.simps(1,2,5,6))

next

case (*propo-rew-one-step-lift* φ φ' c $\xi \xi'$)


```

note  $rel = this(1)$  and  $IH = this(2)$  and  $wf = this(3)$  and  $no-T-F = this(4)$ 
moreover have  $wf'$ :  $wf\text{-conn } c \ (\xi @ \varphi' \# \xi')$ 
  using  $wf\text{-conn-no-arity-change } wf\text{-conn-no-arity-change-helper } wf$  by  $metis$ 
ultimately show  $no-T-F \ (conn \ c \ (\xi @ \varphi' \# \xi'))$ 
  using  $all\text{-subformula-st-test-symb-true-phi}$ 
  by  $(fastforce \ simp: no-T-F\text{-def } all\text{-subformula-st-decomp } wf \ wf')$ 
qed

```

lemma $pushNeg\text{-inv}$:

```

fixes  $\varphi \ \psi :: 'v \text{ propo}$ 
assumes  $full \ (propo\text{-rew-step } pushNeg) \ \varphi \ \psi$ 
and  $no\text{-equiv } \varphi$  and  $no\text{-imp } \varphi$  and  $no-T-F\text{-except-top-level } \varphi$ 
shows  $no\text{-equiv } \psi$  and  $no\text{-imp } \psi$  and  $no-T-F\text{-except-top-level } \psi$ 
proof -
{
  fix  $\varphi \ \psi :: 'v \text{ propo}$ 
  assume  $rel$ :  $propo\text{-rew-step } pushNeg \ \varphi \ \psi$ 
  and  $no$ :  $no-T-F\text{-except-top-level } \varphi$ 
  then have  $no-T-F\text{-except-top-level } \psi$ 
  proof -
  {
    assume  $\varphi = FT \vee \varphi = FF$ 
    from  $rel$  this have  $False$ 
    apply  $(induct \ rule: propo\text{-rew-step.induct})$ 
    using  $pushNeg.cases$  apply  $blast$ 
    using  $wf\text{-conn-list}(1) \ wf\text{-conn-list}(2)$  by  $auto$ 
    then have  $no-T-F\text{-except-top-level } \psi$  by  $blast$ 
  }
  moreover {
    assume  $\varphi \neq FT \wedge \varphi \neq FF$ 
    then have  $no-T-F \ \varphi$ 
    by  $(metis \ no \ no-T-F\text{-symb-except-toplevel-all-subformula-st-no-T-F-symb})$ 
    then have  $no-T-F \ \psi$ 
    using  $propo\text{-rew-step-pushNeg-no-T-F } rel$  by  $auto$ 
    then have  $no-T-F\text{-except-top-level } \psi$  by  $(simp \ add: no-T-F\text{-no-T-F-except-top-level})$ 
  }
  ultimately show  $no-T-F\text{-except-top-level } \psi$  by  $metis$ 
  qed
}
moreover {
  fix  $c :: 'v \text{ connective}$  and  $\xi \ \xi' :: 'v \text{ propo list}$  and  $\zeta \ \zeta' :: 'v \text{ propo}$ 
  assume  $rel$ :  $propo\text{-rew-step } pushNeg \ \zeta \ \zeta'$ 
  and  $incl$ :  $\zeta \preceq \varphi$ 
  and  $corr$ :  $wf\text{-conn } c \ (\xi @ \zeta \# \xi')$ 
  and  $no-T-F$ :  $no-T-F\text{-symb-except-toplevel } (conn \ c \ (\xi @ \zeta \# \xi'))$ 
  and  $n$ :  $no-T-F\text{-symb-except-toplevel } \zeta'$ 
  have  $no-T-F\text{-symb-except-toplevel } (conn \ c \ (\xi @ \zeta' \# \xi'))$ 
  proof
    have  $p$ :  $no-T-F\text{-symb } (conn \ c \ (\xi @ \zeta \# \xi'))$ 
    using  $corr \ wf\text{-conn-list}(1) \ wf\text{-conn-list}(2) \ no-T-F\text{-symb-except-toplevel-no-T-F-symb } no-T-F$ 
    by  $blast$ 
    have  $l$ :  $\forall \varphi \in set \ (\xi @ \zeta \# \xi'). \ \varphi \neq FT \wedge \varphi \neq FF$ 
    using  $corr \ wf\text{-conn-no-T-F-symb-iff } p$  by  $blast$ 
    from  $rel \ incl$  have  $\zeta' \neq FT \wedge \zeta' \neq FF$ 
  }

```

```

    apply (induction  $\zeta \ \zeta'$  rule: propo-rew-step.induct)
    apply (cases rule: pushNeg.cases, auto)
    by (metis assms(4) no-T-F-symb-except-top-level-false-not no-T-F-except-top-level-def
        all-subformula-st-test-symb-true-phi subformula-in-subformula-not
        subformula-all-subformula-st append-is-Nil-conv list.distinct(1)
        wf-conn-no-arity-change-helper wf-conn-list(1,2) wf-conn-no-arity-change)+
    then have  $\forall \varphi \in \text{set } (\xi @ \zeta' \# \xi'). \varphi \neq FT \wedge \varphi \neq FF$  using l by auto
    moreover have  $c \neq CT \wedge c \neq CF$  using corr by auto
    ultimately show no-T-F-symb (conn c ( $\xi @ \zeta' \# \xi'$ ))
    by (metis corr no-T-F-symb-comp wf-conn-no-arity-change wf-conn-no-arity-change-helper)
  qed
}
ultimately show no-T-F-except-top-level  $\psi$ 
using full-propo-rew-step-inv-stay-with-inc[of pushNeg no-T-F-symb-except-toplevel  $\varphi$ ] assms
subformula-refl unfolding no-T-F-except-top-level-def full-unfold by metis
next
{
  fix  $\varphi \ \psi :: 'v \text{ propo}$ 
  have  $H: \text{pushNeg } \varphi \ \psi \implies \text{no-equiv } \varphi \implies \text{no-equiv } \psi$ 
  by (induct  $\varphi \ \psi$  rule: pushNeg.induct, auto)
}
then show no-equiv  $\psi$ 
using full-propo-rew-step-inv-stay-conn[of pushNeg no-equiv-symb  $\varphi \ \psi$ ]
no-equiv-symb-conn-characterization assms unfolding no-equiv-def full-unfold by metis
next
{
  fix  $\varphi \ \psi :: 'v \text{ propo}$ 
  have  $H: \text{pushNeg } \varphi \ \psi \implies \text{no-imp } \varphi \implies \text{no-imp } \psi$ 
  by (induct  $\varphi \ \psi$  rule: pushNeg.induct, auto)
}
then show no-imp  $\psi$ 
using full-propo-rew-step-inv-stay-conn[of pushNeg no-imp-symb  $\varphi \ \psi$ ] assms
no-imp-symb-conn-characterization unfolding no-imp-def full-unfold by metis
qed

```

lemma *pushNeg-full-propo-rew-step:*
fixes $\varphi \ \psi :: 'v \text{ propo}$
assumes
 $\text{no-equiv } \varphi$ **and**
 $\text{no-imp } \varphi$ **and**
 $\text{full } (\text{propo-rew-step } \text{pushNeg}) \ \varphi \ \psi$ **and**
 $\text{no-T-F-except-top-level } \varphi$
shows $\text{simple-not } \psi$
using $\text{assms full-propo-rew-step-subformula pushNeg-inv}(1,2) \text{ simple-not-rew}$ **by** blast

8.5 Push inside

inductive *push-conn-inside* :: $'v \text{ connective} \Rightarrow 'v \text{ connective} \Rightarrow 'v \text{ propo} \Rightarrow 'v \text{ propo} \Rightarrow \text{bool}$
for $c :: 'v \text{ connective}$ **where**
push-conn-inside-l[simp]: $c = CAnd \vee c = COr \implies c' = CAnd \vee c' = COr$
 $\implies \text{push-conn-inside } c \ c' \ (\text{conn } c \ [\text{conn } c' \ [\varphi 1, \varphi 2], \psi])$
 $\quad (\text{conn } c' \ [\text{conn } c \ [\varphi 1, \psi], \text{conn } c \ [\varphi 2, \psi]]) \mid$
push-conn-inside-r[simp]: $c = CAnd \vee c = COr \implies c' = CAnd \vee c' = COr$
 $\implies \text{push-conn-inside } c \ c' \ (\text{conn } c \ [\psi, \text{conn } c' \ [\varphi 1, \varphi 2]])$
 $\quad (\text{conn } c' \ [\text{conn } c \ [\psi, \varphi 1], \text{conn } c \ [\psi, \varphi 2]])$

lemma *push-conn-inside-explicit*: $\text{push-conn-inside } c \ c' \ \varphi \ \psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$
by (*induct* $\varphi \ \psi$ *rule*: *push-conn-inside.induct*, *auto*)

lemma *push-conn-inside-consistent*: *preserves-un-sat* (*push-conn-inside* $c \ c'$)
unfolding *preserves-un-sat-def* **by** (*simp* *add*: *push-conn-inside-explicit*)

lemma *propo-rew-step-push-conn-inside*[*simp*]:
 $\neg \text{propo-rew-step } (\text{push-conn-inside } c \ c') \ FT \ \psi \ \neg \text{propo-rew-step } (\text{push-conn-inside } c \ c') \ FF \ \psi$
proof –
{
{
fix $\varphi \ \psi$
have $\text{push-conn-inside } c \ c' \ \varphi \ \psi \implies \varphi = FT \vee \varphi = FF \implies \text{False}$
by (*induct* *rule*: *push-conn-inside.induct*, *auto*)
} **note** $H = \text{this}$
fix φ
have $\text{propo-rew-step } (\text{push-conn-inside } c \ c') \ \varphi \ \psi \implies \varphi = FT \vee \varphi = FF \implies \text{False}$
apply (*induct* *rule*: *propo-rew-step.induct*, *auto* *simp*: *wf-conn-list*(1) *wf-conn-list*(2))
using H **by** *blast+*
}
then show
 $\neg \text{propo-rew-step } (\text{push-conn-inside } c \ c') \ FT \ \psi$
 $\neg \text{propo-rew-step } (\text{push-conn-inside } c \ c') \ FF \ \psi$ **by** *blast+*
qed

inductive *not-c-in-c'-symb*:: '*v* *connective* \Rightarrow '*v* *connective* \Rightarrow '*v* *propo* \Rightarrow *bool* **for** $c \ c'$ **where**
not-c-in-c'-symb-l[*simp*]: $\text{wf-conn } c \ [\text{conn } c' \ [\varphi, \varphi'], \psi] \implies \text{wf-conn } c' \ [\varphi, \varphi']$
 $\implies \text{not-c-in-c'-symb } c \ c' \ (\text{conn } c \ [\text{conn } c' \ [\varphi, \varphi'], \psi]) \mid$
not-c-in-c'-symb-r[*simp*]: $\text{wf-conn } c \ [\psi, \text{conn } c' \ [\varphi, \varphi']] \implies \text{wf-conn } c' \ [\varphi, \varphi']$
 $\implies \text{not-c-in-c'-symb } c \ c' \ (\text{conn } c \ [\psi, \text{conn } c' \ [\varphi, \varphi']])$

abbreviation *c-in-c'-symb* $c \ c' \ \varphi \equiv \neg \text{not-c-in-c'-symb } c \ c' \ \varphi$

lemma *c-in-c'-symb-simp*:
 $\text{not-c-in-c'-symb } c \ c' \ \xi \implies \xi = FF \vee \xi = FT \vee \xi = FVar \ x \vee \xi = FNot \ FF \vee \xi = FNot \ FT$
 $\vee \xi = FNot \ (FVar \ x) \implies \text{False}$
apply (*induct* *rule*: *not-c-in-c'-symb.induct*, *auto* *simp*: *wf-conn.simps* *wf-conn-list*(1–3))
using *conn-inj-not*(2) *wf-conn-binary* **unfolding** *binary-connectives-def* **by** *fastforce+*

lemma *c-in-c'-symb-simp'*[*simp*]:
 $\neg \text{not-c-in-c'-symb } c \ c' \ FF$
 $\neg \text{not-c-in-c'-symb } c \ c' \ FT$
 $\neg \text{not-c-in-c'-symb } c \ c' \ (FVar \ x)$
 $\neg \text{not-c-in-c'-symb } c \ c' \ (FNot \ FF)$
 $\neg \text{not-c-in-c'-symb } c \ c' \ (FNot \ FT)$
 $\neg \text{not-c-in-c'-symb } c \ c' \ (FNot \ (FVar \ x))$
using *c-in-c'-symb-simp* **by** *metis+*

definition *c-in-c'-only* **where**
 $c\text{-in-c'-only } c \ c' \equiv \text{all-subformula-st } (c\text{-in-c'-symb } c \ c')$

lemma *c-in-c'-only-simp*[simp]:

c-in-c'-only *c* *c'* *FF*
c-in-c'-only *c* *c'* *FT*
c-in-c'-only *c* *c'* (*FVar* *x*)
c-in-c'-only *c* *c'* (*FNot* *FF*)
c-in-c'-only *c* *c'* (*FNot* *FT*)
c-in-c'-only *c* *c'* (*FNot* (*FVar* *x*))
unfolding *c-in-c'-only-def* **by** *auto*

lemma *not-c-in-c'-symb-commute*:

not-c-in-c'-symb *c* *c'* $\xi \implies \text{wf-conn } c \ [\varphi, \psi] \implies \xi = \text{conn } c \ [\varphi, \psi]$
 $\implies \text{not-c-in-c'-symb } c \ c' \ (\text{conn } c \ [\psi, \varphi])$

proof (*induct rule: not-c-in-c'-symb.induct*)

case (*not-c-in-c'-symb-r* $\varphi' \ \varphi'' \ \psi'$) **note** *H = this*
then have $\psi = \text{conn } c' \ [\varphi'', \psi']$ **using** *conn-inj* **by** *auto*
have $\text{wf-conn } c \ [\text{conn } c' \ [\varphi'', \psi'], \varphi]$
using *H(1)* *wf-conn-no-arity-change length-Cons* **by** *metis*
then show *not-c-in-c'-symb* *c* *c'* ($\text{conn } c \ [\psi, \varphi]$)
unfolding ψ **using** *not-c-in-c'-symb.intros(1)* *H* **by** *auto*

next

case (*not-c-in-c'-symb-l* $\varphi' \ \varphi'' \ \psi'$) **note** *H = this*
then have $\varphi = \text{conn } c' \ [\varphi', \varphi']$ **using** *conn-inj* **by** *auto*
moreover have $\text{wf-conn } c \ [\psi', \text{conn } c' \ [\varphi', \varphi']]$
using *H(1)* *wf-conn-no-arity-change length-Cons* **by** *metis*
ultimately show *not-c-in-c'-symb* *c* *c'* ($\text{conn } c \ [\psi, \varphi]$)
using *not-c-in-c'-symb.intros(2)* *conn-inj not-c-in-c'-symb-l.hyps*
not-c-in-c'-symb-l.prem(1,2) **by** *blast*

qed

lemma *not-c-in-c'-symb-commute'*:

$\text{wf-conn } c \ [\varphi, \psi] \implies \text{c-in-c'-symb } c \ c' \ (\text{conn } c \ [\varphi, \psi]) \longleftrightarrow \text{c-in-c'-symb } c \ c' \ (\text{conn } c \ [\psi, \varphi])$
using *not-c-in-c'-symb-commute* *wf-conn-no-arity-change* **by** (*metis length-Cons*)

lemma *not-c-in-c'-comm*:

assumes *wf*: $\text{wf-conn } c \ [\varphi, \psi]$
shows *c-in-c'-only* *c* *c'* ($\text{conn } c \ [\varphi, \psi]$) $\longleftrightarrow \text{c-in-c'-only } c \ c' \ (\text{conn } c \ [\psi, \varphi])$ (**is** $?A \longleftrightarrow ?B$)

proof –

have $?A \longleftrightarrow (\text{c-in-c'-symb } c \ c' \ (\text{conn } c \ [\varphi, \psi])$
 $\wedge (\forall \xi \in \text{set } [\varphi, \psi]. \text{all-subformula-st } (\text{c-in-c'-symb } c \ c') \ \xi))$
using *all-subformula-st-decomp wf* **unfolding** *c-in-c'-only-def* **by** *fastforce*
also have $\dots \longleftrightarrow (\text{c-in-c'-symb } c \ c' \ (\text{conn } c \ [\psi, \varphi])$
 $\wedge (\forall \xi \in \text{set } [\psi, \varphi]. \text{all-subformula-st } (\text{c-in-c'-symb } c \ c') \ \xi))$
using *not-c-in-c'-symb-commute' wf* **by** *auto*

also

have $\text{wf-conn } c \ [\psi, \varphi]$ **using** *wf-conn-no-arity-change wf* **by** (*metis length-Cons*)
then have $(\text{c-in-c'-symb } c \ c' \ (\text{conn } c \ [\psi, \varphi])$
 $\wedge (\forall \xi \in \text{set } [\psi, \varphi]. \text{all-subformula-st } (\text{c-in-c'-symb } c \ c') \ \xi))$
 $\longleftrightarrow ?B$

using *all-subformula-st-decomp* **unfolding** *c-in-c'-only-def* **by** *fastforce*

finally show *?thesis* .

qed

lemma *not-c-in-c'-simp*[simp]:

fixes $\varphi_1 \ \varphi_2 \ \psi :: 'v \ \text{propo}$ **and** $x :: 'v$

```

shows
  c-in-c'-symb c c' FT
  c-in-c'-symb c c' FF
  c-in-c'-symb c c' (FVar x)
  wf-conn c [conn c' [φ1, φ2], ψ] ⇒ wf-conn c' [φ1, φ2]
    ⇒ ¬ c-in-c'-only c c' (conn c [conn c' [φ1, φ2], ψ])
  apply (simp-all add: c-in-c'-only-def)
  using all-subformula-st-test-symb-true-phi not-c-in-c'-symb-l by blast

lemma c-in-c'-symb-not[simp]:
  fixes c c' :: 'v connective and ψ :: 'v propo
  shows c-in-c'-symb c c' (FNot ψ)
proof -
  {
    fix ξ :: 'v propo
    have not-c-in-c'-symb c c' (FNot ψ) ⇒ False
      apply (induct FNot ψ rule: not-c-in-c'-symb.induct)
      using conn-inj-not(2) by blast+
  }
  then show ?thesis by auto
qed

lemma c-in-c'-symb-step-exists:
  fixes φ :: 'v propo
  assumes c: c = CAnd ∨ c = COr and c': c' = CAnd ∨ c' = COr
  shows ψ ≤ φ ⇒ ¬ c-in-c'-symb c c' ψ ⇒ ∃ ψ'. push-conn-inside c c' ψ ψ'
  apply (induct ψ rule: propo-induct-arity)
  apply auto[2]
proof -
  fix ψ1 ψ2 φ': 'v propo
  assume IHψ1: ψ1 ≤ φ ⇒ ¬ c-in-c'-symb c c' ψ1 ⇒ Ex (push-conn-inside c c' ψ1)
  and IHψ2: ψ1 ≤ φ ⇒ ¬ c-in-c'-symb c c' ψ1 ⇒ Ex (push-conn-inside c c' ψ1)
  and φ': φ' = FAnd ψ1 ψ2 ∨ φ' = FOr ψ1 ψ2 ∨ φ' = FImp ψ1 ψ2 ∨ φ' = FEq ψ1 ψ2
  and inφ: φ' ≤ φ and n0: ¬ c-in-c'-symb c c' φ'
  then have n: not-c-in-c'-symb c c' φ' by auto
  {
    assume φ': φ' = conn c [ψ1, ψ2]
    obtain a b where ψ1 = conn c' [a, b] ∨ ψ2 = conn c' [a, b]
      using n φ' apply (induct rule: not-c-in-c'-symb.induct)
      using c by force+
    then have Ex (push-conn-inside c c' φ')
      unfolding φ' apply auto
      using push-conn-inside.intros(1) c c' apply blast
      using push-conn-inside.intros(2) c c' by blast
  }
  moreover {
    assume φ': φ' ≠ conn c [ψ1, ψ2]
    have ∀ φ c ca. ∃ φ1 ψ1 ψ2 ψ1' ψ2' φ2'. conn (c::'v connective) [φ1, conn ca [ψ1, ψ2]] = φ
      ∨ conn c [conn ca [ψ1', ψ2'], φ2'] = φ ∨ c-in-c'-symb c ca φ
      by (metis not-c-in-c'-symb.cases)
    then have Ex (push-conn-inside c c' φ')
      by (metis (no-types) c c' n push-conn-inside-l push-conn-inside-r)
  }
  ultimately show Ex (push-conn-inside c c' φ') by blast
qed

```

```

lemma c-in-c'-symb-rew:
  fixes  $\varphi :: 'v \text{ propo}$ 
  assumes noTB:  $\neg c\text{-in-c'-only } c \ c' \ \varphi$ 
  and  $c: c = CAnd \vee c = COr$  and  $c': c' = CAnd \vee c' = COr$ 
  shows  $\exists \psi \ \psi'. \ \psi \preceq \varphi \wedge \text{push-conn-inside } c \ c' \ \psi \ \psi'$ 
proof -
  have test-symb-false-nullary:
     $\forall x. \ c\text{-in-c'-symb } c \ c' \ (FF:: 'v \text{ propo}) \wedge c\text{-in-c'-symb } c \ c' \ FT$ 
     $\wedge c\text{-in-c'-symb } c \ c' \ (FVar \ (x:: 'v))$ 
  by auto
  moreover {
    fix  $x :: 'v$ 
    have  $H': c\text{-in-c'-symb } c \ c' \ FT \ c\text{-in-c'-symb } c \ c' \ FF \ c\text{-in-c'-symb } c \ c' \ (FVar \ x)$ 
    by simp+
  }
  moreover {
    fix  $\psi :: 'v \text{ propo}$ 
    have  $\psi \preceq \varphi \implies \neg c\text{-in-c'-symb } c \ c' \ \psi \implies \exists \psi'. \ \text{push-conn-inside } c \ c' \ \psi \ \psi'$ 
    by (auto simp: assms(2) c' c-in-c'-symb-step-exists)
  }
  ultimately show ?thesis using noTB no-test-symb-step-exists[of c-in-c'-symb c c']
  unfolding c-in-c'-only-def by metis
qed

```

```

lemma push-conn-insidec-in-c'-symb-no-T-F:
  fixes  $\varphi \ \psi :: 'v \text{ propo}$ 
  shows propo-rew-step (push-conn-inside  $c \ c'$ )  $\varphi \ \psi \implies \text{no-T-F } \varphi \implies \text{no-T-F } \psi$ 
proof (induct rule: propo-rew-step.induct)
  case (global-rel  $\varphi \ \psi$ )
  then show no-T-F  $\psi$ 
    by (cases rule: push-conn-inside.cases, auto)
next
  case (propo-rew-one-step-lift  $\varphi \ \varphi' \ c \ \xi \ \xi'$ )
  note rel = this(1) and IH = this(2) and wf = this(3) and no-T-F = this(4)
  have no-T-F  $\varphi$ 
    using wf no-T-F no-T-F-def subformula-into-subformula subformula-all-subformula-st
    subformula-refl by (metis (no-types) in-set-conv-decomp)
  then have  $\varphi': \text{no-T-F } \varphi'$  using IH by blast

  have  $\forall \zeta \in \text{set } (\xi @ \varphi \# \xi'). \ \text{no-T-F } \zeta$  by (metis wf no-T-F no-T-F-def all-subformula-st-decomp)
  then have  $n: \forall \zeta \in \text{set } (\xi @ \varphi' \# \xi'). \ \text{no-T-F } \zeta$  using  $\varphi'$  by auto
  then have  $n': \forall \zeta \in \text{set } (\xi @ \varphi' \# \xi'). \ \zeta \neq FF \wedge \zeta \neq FT$ 
    using  $\varphi'$  by (metis no-T-F-symb-false(1) no-T-F-symb-false(2) no-T-F-def
    all-subformula-st-test-symb-true-phi)

  have  $wf': \text{wf-conn } c \ (\xi @ \varphi' \# \xi')$ 
    using wf wf-conn-no-arity-change by (metis wf-conn-no-arity-change-helper)
  {
    fix  $x :: 'v$ 
    assume  $c = CT \vee c = CF \vee c = CVar \ x$ 
    then have False using wf by auto
    then have no-T-F (conn  $c \ (\xi @ \varphi' \# \xi')$ ) by blast
  }

```

```

moreover {
  assume  $c: c = CNot$ 
  then have  $\xi = [] \ \xi' = []$  using  $wf$  by  $auto$ 
  then have  $no-T-F \ (conn \ c \ (\xi @ \varphi' \# \xi'))$ 
    using  $c$  by ( $metis \ \varphi' \ conn.simps(4) \ no-T-F-symb-false(1,2) \ no-T-F-symb-fnot \ no-T-F-def$ 
       $all-subformula-st-decomp-explicit(3) \ all-subformula-st-test-symb-true-phi \ self-append-conv2$ )
}
moreover {
  assume  $c: c \in binary-connectives$ 
  then have  $no-T-F-symb \ (conn \ c \ (\xi @ \varphi' \# \xi'))$  using  $wf' \ n' \ no-T-F-symb.simps$  by  $fastforce$ 
  then have  $no-T-F \ (conn \ c \ (\xi @ \varphi' \# \xi'))$ 
    by ( $metis \ all-subformula-st-decomp-imp \ wf' \ n \ no-T-F-def$ )
}
ultimately show  $no-T-F \ (conn \ c \ (\xi @ \varphi' \# \xi'))$  using  $connective-cases-arity$  by  $auto$ 
qed

```

lemma *simple-propo-rew-step-push-conn-inside-inv*:

```

propo-rew-step (push-conn-inside  $c \ c'$ )  $\varphi \ \psi \implies simple \ \varphi \implies simple \ \psi$ 
apply (induct rule: propo-rew-step.induct)
apply (rename-tac  $\varphi$ , case-tac  $\varphi$ , auto simp: push-conn-inside.simps)[]
by (metis append-is-Nil-conv list.distinct(1) simple.elims(2) wf-conn-list(1-3))

```

lemma *simple-propo-rew-step-inv-push-conn-inside-simple-not*:

```

fixes  $c \ c' :: 'v \ connective$  and  $\varphi \ \psi :: 'v \ propo$ 
shows propo-rew-step (push-conn-inside  $c \ c'$ )  $\varphi \ \psi \implies simple-not \ \varphi \implies simple-not \ \psi$ 
proof (induct rule: propo-rew-step.induct)
  case (global-rel  $\varphi \ \psi$ )
    then show  $?case$  by (cases  $\varphi$ , auto simp: push-conn-inside.simps)
next
  case (propo-rew-one-step-lift  $\varphi \ \varphi' \ ca \ \xi \ \xi'$ ) note  $rew = this(1)$  and  $IH = this(2)$  and  $wf = this(3)$ 
    and  $simple = this(4)$ 
  show  $?case$ 
    proof (cases  $ca$  rule: connective-cases-arity)
      case nullary
        then show  $?thesis$  using propo-rew-one-step-lift by  $auto$ 
      next
        case binary note  $ca = this$ 
        obtain  $a \ b$  where  $ab: \xi @ \varphi' \# \xi' = [a, b]$ 
          using  $wf \ ca \ list-length2-decomp \ wf-conn-bin-list-length$ 
          by (metis (no-types) wf-conn-no-arity-change-helper)
        have  $\forall \zeta \in set \ (\xi @ \varphi \# \xi'). \ simple-not \ \zeta$ 
          by (metis  $wf \ all-subformula-st-decomp \ simple \ simple-not-def$ )
        then have  $\forall \zeta \in set \ (\xi @ \varphi' \# \xi'). \ simple-not \ \zeta$  using  $IH$  by simp
        moreover have  $simple-not-symb \ (conn \ ca \ (\xi @ \varphi' \# \xi'))$  using  $ca$ 
          by (metis  $ab \ conn.simps(5-8) \ helper-fact \ simple-not-symb.simps(5) \ simple-not-symb.simps(6)$ 
             $simple-not-symb.simps(7) \ simple-not-symb.simps(8)$ )
        ultimately show  $?thesis$ 
          by (simp add: ab all-subformula-st-decomp ca)
      next
        case unary
        then show  $?thesis$ 
          using  $rew \ simple-propo-rew-step-push-conn-inside-inv[OF \ rew] \ IH \ local.wf \ simple$  by  $auto$ 
    qed

```

qed

lemma *propo-rew-step-push-conn-inside-simple-not:*

fixes $\varphi \varphi' :: 'v \text{ propo}$ **and** $\xi \xi' :: 'v \text{ propo list}$ **and** $c :: 'v \text{ connective}$

assumes

propo-rew-step (*push-conn-inside* $c \ c'$) $\varphi \ \varphi'$ **and**

wf-conn $c \ (\xi \ @ \ \varphi \ \# \ \xi')$ **and**

simple-not-symb (*conn* $c \ (\xi \ @ \ \varphi \ \# \ \xi')$) **and**

simple-not-symb φ'

shows *simple-not-symb* (*conn* $c \ (\xi \ @ \ \varphi' \ \# \ \xi')$)

using *assms*

proof (*induction rule: propo-rew-step.induct*)

print-cases

case (*global-rel*)

then show ?*case*

by (*metis conn.simps*(12,17) *list.discI push-conn-inside.cases simple-not-symb.elims*(3)

wf-conn-helper-facts(5) *wf-conn-list*(2) *wf-conn-list*(8) *wf-conn-no-arity-change*

wf-conn-no-arity-change-helper)

next

case (*propo-rew-one-step-lift* $\varphi \ \varphi' \ c' \ \chi s \ \chi s'$) **note** *tel* = *this*(1) **and** *wf* = *this*(2) **and**

IH = *this*(3) **and** *wf'* = *this*(4) **and** *simple'* = *this*(5) **and** *simple* = *this*(6)

then show ?*case*

proof (*cases c' rule: connective-cases-arity*)

case *nullary*

then show ?*thesis* **using** *wf simple simple'* **by** *auto*

next

case *binary* **note** $c = \text{this}(1)$

have *corr'*: *wf-conn* $c \ (\xi \ @ \ \text{conn } c' \ (\chi s \ @ \ \varphi' \ \# \ \chi s') \ \# \ \xi')$

using *wf wf-conn-no-arity-change*

by (*metis wf' wf-conn-no-arity-change-helper*)

then show ?*thesis*

using $c \ \text{propo-rew-one-step-lift } wf$

by (*metis conn.simps*(17) *connective.distinct*(37) *propo-rew-step-subformula-imp*
push-conn-inside.cases simple-not-symb.elims(3) *tel wf-conn.simps wf-conn-list*(2,8))

next

case *unary*

then have *empty*: $\chi s = [] \ \chi s' = []$ **using** *wf* **by** *auto*

then show ?*thesis* **using** *simple unary simple' wf wf'*

by (*metis connective.distinct*(37) *connective.distinct*(39) *propo-rew-step-subformula-imp*
push-conn-inside.cases simple-not-symb.elims(3) *tel wf-conn-list*(8)

wf-conn-no-arity-change wf-conn-no-arity-change-helper)

qed

qed

lemma *push-conn-inside-not-true-false:*

push-conn-inside $c \ c' \ \varphi \ \psi \implies \psi \neq FT \wedge \psi \neq FF$

by (*induct rule: push-conn-inside.induct, auto*)

lemma *push-conn-inside-inv:*

fixes $\varphi \ \psi :: 'v \text{ propo}$

assumes *full* (*propo-rew-step* (*push-conn-inside* $c \ c'$)) $\varphi \ \psi$

and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ **and** *simple-not* φ

shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ **and** *simple-not* ψ

proof –

{


```

{
  fix  $\varphi \psi :: 'v \text{ propo}$ 
  have  $H: \text{push-conn-inside } c \ c' \ \varphi \ \psi \implies \text{all-subformula-st simple-not-symb } \varphi$ 
     $\implies \text{all-subformula-st simple-not-symb } \psi$ 
    by (induct  $\varphi \ \psi$  rule: push-conn-inside.induct, auto)
} note  $H = \text{this}$ 

fix  $\varphi \psi :: 'v \text{ propo}$ 
have  $H: \text{propo-rew-step } (\text{push-conn-inside } c \ c') \ \varphi \ \psi \implies \text{all-subformula-st simple-not-symb } \varphi$ 
   $\implies \text{all-subformula-st simple-not-symb } \psi$ 
  apply (induct  $\varphi \ \psi$  rule: propo-rew-step.induct)
  using  $H$  apply simp
proof (rename-tac  $\varphi \ \varphi' \text{ ca } \psi s \ \psi s'$ , case-tac  $\text{ca}$  rule: connective-cases-arity)
  fix  $\varphi \ \varphi' :: 'v \text{ propo}$  and  $c :: 'v \text{ connective}$  and  $\xi \ \xi' :: 'v \text{ propo list}$ 
  and  $x :: 'v$ 
  assume  $\text{wf-conn } c \ (\xi @ \varphi \# \xi')$ 
  and  $c = CT \vee c = CF \vee c = CVar \ x$ 
  then have  $\xi @ \varphi \# \xi' = []$  by auto
  then have False by auto
  then show  $\text{all-subformula-st simple-not-symb } (\text{conn } c \ (\xi @ \varphi' \# \xi'))$  by blast
next
  fix  $\varphi \ \varphi' :: 'v \text{ propo}$  and  $\text{ca} :: 'v \text{ connective}$  and  $\xi \ \xi' :: 'v \text{ propo list}$ 
  and  $x :: 'v$ 
  assume  $\text{rel: propo-rew-step } (\text{push-conn-inside } c \ c') \ \varphi \ \varphi'$ 
  and  $\varphi\text{-}\varphi': \text{all-subformula-st simple-not-symb } \varphi \implies \text{all-subformula-st simple-not-symb } \varphi'$ 
  and  $\text{corr: wf-conn } \text{ca} \ (\xi @ \varphi \# \xi')$ 
  and  $n: \text{all-subformula-st simple-not-symb } (\text{conn } \text{ca} \ (\xi @ \varphi \# \xi'))$ 
  and  $c: \text{ca} = CNot$ 

  have empty:  $\xi = [] \ \xi' = []$  using  $c \ \text{corr}$  by auto
  then have  $\text{simple-not:all-subformula-st simple-not-symb } (FNot \ \varphi)$  using  $\text{corr } c \ n$  by auto
  then have simple  $\varphi$ 
    using  $\text{all-subformula-st-test-symb-true-phi simple-not-symb.simps(1)}$  by blast
  then have simple  $\varphi'$ 
    using  $\text{rel simple-propo-rew-step-push-conn-inside-inv}$  by blast
  then show  $\text{all-subformula-st simple-not-symb } (\text{conn } \text{ca} \ (\xi @ \varphi' \# \xi'))$  using  $c \ \text{empty}$ 
    by (metis simple-not  $\varphi\text{-}\varphi'$  append-Nil  $\text{conn.simps(4)}$  all-subformula-st-decomp-explicit(3)
      simple-not-symb.simps(1))
next
  fix  $\varphi \ \varphi' :: 'v \text{ propo}$  and  $\text{ca} :: 'v \text{ connective}$  and  $\xi \ \xi' :: 'v \text{ propo list}$ 
  and  $x :: 'v$ 
  assume  $\text{rel: propo-rew-step } (\text{push-conn-inside } c \ c') \ \varphi \ \varphi'$ 
  and  $n\varphi: \text{all-subformula-st simple-not-symb } \varphi \implies \text{all-subformula-st simple-not-symb } \varphi'$ 
  and  $\text{corr: wf-conn } \text{ca} \ (\xi @ \varphi \# \xi')$ 
  and  $n: \text{all-subformula-st simple-not-symb } (\text{conn } \text{ca} \ (\xi @ \varphi \# \xi'))$ 
  and  $c: \text{ca} \in \text{binary-connectives}$ 

  have  $\text{all-subformula-st simple-not-symb } \varphi$ 
    using  $n \ c \ \text{corr}$  all-subformula-st-decomp by fastforce
  then have  $\varphi': \text{all-subformula-st simple-not-symb } \varphi'$  using  $n\varphi$  by blast
  obtain  $a \ b$  where  $\text{ab: } [a, b] = (\xi @ \varphi \# \xi')$ 
    using  $\text{corr } c \ \text{list-length2-decomp wf-conn-bin-list-length}$  by metis
  then have  $\xi @ \varphi' \# \xi' = [a, \varphi'] \vee (\xi @ \varphi' \# \xi') = [\varphi', b]$ 
    using  $\text{ab}$  by (metis (no-types, hide-lams) append-Cons append-Nil append-Nil2
      append-is-Nil-conv butlast.simps(2) butlast-append list.sel(3) tl-append2)

```

```

moreover
{
  fix  $\chi :: 'v \text{ propo}$ 
  have  $wf'$ :  $wf\text{-conn } ca [a, b]$ 
    using  $ab \text{ corr}$  by  $presburger$ 
  have  $all\text{-subformula-st simple-not-symb } (conn \text{ } ca [a, b])$ 
    using  $ab \text{ } n$  by  $presburger$ 
  then have  $all\text{-subformula-st simple-not-symb } \chi \vee \chi \notin set (\xi @ \varphi' \# \xi')$ 
    using  $wf'$  by ( $metis (no\text{-types}) \varphi' all\text{-subformula-st-decomp calculation insert-iff list.set(2)$ )
}
then have  $\forall \varphi. \varphi \in set (\xi @ \varphi' \# \xi') \longrightarrow all\text{-subformula-st simple-not-symb } \varphi$ 
  by ( $metis (no\text{-types})$ )

moreover have  $simple\text{-not-symb } (conn \text{ } ca (\xi @ \varphi' \# \xi'))$ 
  using  $ab \text{ conn-inj-not}(1) \text{ corr } wf\text{-conn-list-decomp}(4) \text{ wf-conn-no-arity-change}$ 
   $not\text{-Cons-self2 self-append-conv2 simple-not-symb.elims}(3)$  by ( $metis (no\text{-types}) c$ 
   $calculation(1) wf\text{-conn-binary}$ )
moreover have  $wf\text{-conn } ca (\xi @ \varphi' \# \xi')$  using  $c \text{ calculation}(1)$  by  $auto$ 
ultimately show  $all\text{-subformula-st simple-not-symb } (conn \text{ } ca (\xi @ \varphi' \# \xi'))$ 
  by ( $metis all\text{-subformula-st-decomp-imp}$ )
qed
}
moreover {
  fix  $ca :: 'v \text{ connective}$  and  $\xi \xi' :: 'v \text{ propo list}$  and  $\varphi \varphi' :: 'v \text{ propo}$ 
  have  $propo\text{-rew-step } (push\text{-conn-inside } c \text{ } c') \varphi \varphi' \Longrightarrow wf\text{-conn } ca (\xi @ \varphi \# \xi')$ 
     $\Longrightarrow simple\text{-not-symb } (conn \text{ } ca (\xi @ \varphi \# \xi')) \Longrightarrow simple\text{-not-symb } \varphi'$ 
     $\Longrightarrow simple\text{-not-symb } (conn \text{ } ca (\xi @ \varphi' \# \xi'))$ 
  by ( $metis append\text{-self-conv2 conn.simps}(4) \text{ conn-inj-not}(1) \text{ simple-not-symb.elims}(3)$ 
   $simple\text{-not-symb.simps}(1) \text{ simple-propo-rew-step-push-conn-inside-inv}$ 
   $wf\text{-conn-no-arity-change-helper wf-conn-list-decomp}(4) \text{ wf-conn-no-arity-change}$ )
}
ultimately show  $simple\text{-not } \psi$ 
  using  $full\text{-propo-rew-step-inv-stay'}$  [of  $push\text{-conn-inside } c \text{ } c' \text{ simple-not-symb}$ ]  $assms$ 
  unfolding  $no\text{-T-F-except-top-level-def simple-not-def full-unfold}$  by  $metis$ 
next
{
  fix  $\varphi \psi :: 'v \text{ propo}$ 
  have  $H$ :  $propo\text{-rew-step } (push\text{-conn-inside } c \text{ } c') \varphi \psi \Longrightarrow no\text{-T-F-except-top-level } \varphi$ 
     $\Longrightarrow no\text{-T-F-except-top-level } \psi$ 
  proof –
    assume  $rel$ :  $propo\text{-rew-step } (push\text{-conn-inside } c \text{ } c') \varphi \psi$ 
    and  $no\text{-T-F-except-top-level } \varphi$ 
    then have  $no\text{-T-F } \varphi \vee \varphi = FF \vee \varphi = FT$ 
      by ( $metis no\text{-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb}$ )
    moreover {
      assume  $\varphi = FF \vee \varphi = FT$ 
      then have  $False$  using  $rel \text{ propo-rew-step-push-conn-inside}$  by  $blast$ 
      then have  $no\text{-T-F-except-top-level } \psi$  by  $blast$ 
    }
    moreover {
      assume  $no\text{-T-F } \varphi \wedge \varphi \neq FF \wedge \varphi \neq FT$ 
      then have  $no\text{-T-F } \psi$  using  $rel \text{ push-conn-insidec-in-c'-symb-no-T-F}$  by  $blast$ 
      then have  $no\text{-T-F-except-top-level } \psi$  using  $no\text{-T-F-no-T-F-except-top-level}$  by  $blast$ 
    }
  }
}

```

```

    ultimately show no-T-F-except-top-level  $\psi$  by blast
  qed
}
moreover {
  fix  $ca :: 'v$  connective and  $\xi \xi' :: 'v$  propo list and  $\varphi \varphi' :: 'v$  propo
  assume rel: propo-rew-step (push-conn-inside  $c$   $c'$ )  $\varphi \varphi'$ 
  assume corr: wf-conn  $ca$  ( $\xi @ \varphi \# \xi'$ )
  then have  $c: ca \neq CT \wedge ca \neq CF$  by auto
  assume no-T-F: no-T-F-symb-except-toplevel (conn  $ca$  ( $\xi @ \varphi \# \xi'$ ))
  have no-T-F-symb-except-toplevel (conn  $ca$  ( $\xi @ \varphi' \# \xi'$ ))
  proof
    have  $c: ca \neq CT \wedge ca \neq CF$  using corr by auto
    have  $\zeta: \forall \zeta \in \text{set } (\xi @ \varphi \# \xi'). \zeta \neq FT \wedge \zeta \neq FF$ 
      using corr no-T-F no-T-F-symb-except-toplevel-if-is-a-true-false by blast
    then have  $\varphi \neq FT \wedge \varphi \neq FF$  by auto
    from rel this have  $\varphi' \neq FT \wedge \varphi' \neq FF$ 
    apply (induct rule: propo-rew-step.induct)
    by (metis append-is-Nil-conv conn.simps(2) conn-inj list.distinct(1)
      wf-conn-helper-facts(3) wf-conn-list(1) wf-conn-no-arity-change
      wf-conn-no-arity-change-helper push-conn-inside-not-true-false) +
    then have  $\forall \zeta \in \text{set } (\xi @ \varphi' \# \xi'). \zeta \neq FT \wedge \zeta \neq FF$  using  $\zeta$  by auto
    moreover have wf-conn  $ca$  ( $\xi @ \varphi' \# \xi'$ )
      using corr wf-conn-no-arity-change by (metis wf-conn-no-arity-change-helper)
    ultimately show no-T-F-symb (conn  $ca$  ( $\xi @ \varphi' \# \xi'$ )) using no-T-F-symb.intros  $c$  by metis
  qed
}
ultimately show no-T-F-except-top-level  $\psi$ 
using full-propo-rew-step-inv-stay[of push-conn-inside  $c$   $c'$  no-T-F-symb-except-toplevel]
assms unfolding no-T-F-except-top-level-def full-unfold by metis

next
{
  fix  $\varphi \psi :: 'v$  propo
  have  $H: \text{push-conn-inside } c \ c' \ \varphi \ \psi \implies \text{no-equiv } \varphi \implies \text{no-equiv } \psi$ 
    by (induct  $\varphi \ \psi$  rule: push-conn-inside.induct, auto)
}
then show no-equiv  $\psi$ 
using full-propo-rew-step-inv-stay-conn[of push-conn-inside  $c$   $c'$  no-equiv-symb] assms
no-equiv-symb-conn-characterization unfolding no-equiv-def by metis

next
{
  fix  $\varphi \psi :: 'v$  propo
  have  $H: \text{push-conn-inside } c \ c' \ \varphi \ \psi \implies \text{no-imp } \varphi \implies \text{no-imp } \psi$ 
    by (induct  $\varphi \ \psi$  rule: push-conn-inside.induct, auto)
}
then show no-imp  $\psi$ 
using full-propo-rew-step-inv-stay-conn[of push-conn-inside  $c$   $c'$  no-imp-symb] assms
no-imp-symb-conn-characterization unfolding no-imp-def by metis
qed

lemma push-conn-inside-full-propo-rew-step:
  fixes  $\varphi \psi :: 'v$  propo
  assumes

```

no-equiv φ **and**
no-imp φ **and**
full (*propo-rew-step* (*push-conn-inside* c c')) φ ψ **and**
no-T-F-except-top-level φ **and**
simple-not φ **and**
 $c = CAnd \vee c = COr$ **and**
 $c' = CAnd \vee c' = COr$
shows *c-in-c'-only* c c' ψ
using *c-in-c'-symb-rew* *assms* *full-propo-rew-step-subformula* **by** *blast*

8.5.1 Only one type of connective in the formula (+ not)

inductive *only-c-inside-symb* :: '*v* connective \Rightarrow '*v* propo \Rightarrow bool **for** *c*:: '*v* connective **where**
simple-only-c-inside[*simp*]: *simple* $\varphi \Rightarrow$ *only-c-inside-symb* c φ |
simple-cnot-only-c-inside[*simp*]: *simple* $\varphi \Rightarrow$ *only-c-inside-symb* c (*FNot* φ) |
only-c-inside-into-only-c-inside: *wf-conn* c $l \Rightarrow$ *only-c-inside-symb* c (*conn* c l)

lemma *only-c-inside-symb-simp*[*simp*]:
only-c-inside-symb c *FF* *only-c-inside-symb* c *FT* *only-c-inside-symb* c (*FVar* x) **by** *auto*

definition *only-c-inside* **where** *only-c-inside* $c =$ *all-subformula-st* (*only-c-inside-symb* c)

lemma *only-c-inside-symb-decomp*:
only-c-inside-symb c $\psi \longleftrightarrow$ (*simple* ψ
 $\vee (\exists \varphi'. \psi = FNot \varphi' \wedge \text{simple } \varphi')$
 $\vee (\exists l. \psi = \text{conn } c \ l \wedge \text{wf-conn } c \ l))$
by (*auto simp: only-c-inside-symb.intros(3)*) (*induct rule: only-c-inside-symb.induct, auto*)

lemma *only-c-inside-symb-decomp-not*[*simp*]:
fixes $c ::$ '*v* connective
assumes $c: c \neq CNot$
shows *only-c-inside-symb* c (*FNot* ψ) \longleftrightarrow *simple* ψ
apply (*auto simp: only-c-inside-symb.intros(3)*)
by (*induct FNot* ψ *rule: only-c-inside-symb.induct, auto simp: wf-conn-list(8) c*)

lemma *only-c-inside-decomp-not*[*simp*]:
assumes $c: c \neq CNot$
shows *only-c-inside* c (*FNot* ψ) \longleftrightarrow *simple* ψ
by (*metis* (*no-types, hide-lams*) *all-subformula-st-def all-subformula-st-test-symb-true-phi c*
only-c-inside-def only-c-inside-symb-decomp-not simple-only-c-inside
subformula-conn-decomp-simple)

lemma *only-c-inside-decomp*:
only-c-inside c $\varphi \longleftrightarrow$
 $(\forall \psi. \psi \preceq \varphi \longrightarrow (\text{simple } \psi \vee (\exists \varphi'. \psi = FNot \varphi' \wedge \text{simple } \varphi')$
 $\vee (\exists l. \psi = \text{conn } c \ l \wedge \text{wf-conn } c \ l)))$
unfolding *only-c-inside-def* **by** (*auto simp: all-subformula-st-def only-c-inside-symb-decomp*)

lemma *only-c-inside-c-c'-false*:
fixes $c \ c' ::$ '*v* connective **and** $l ::$ '*v* propo list **and** $\varphi ::$ '*v* propo
assumes $cc': c \neq c'$ **and** $c: c = CAnd \vee c = COr$ **and** $c': c' = CAnd \vee c' = COr$
and *only: only-c-inside* c φ **and** *incl: conn* $c' \ l \preceq \varphi$ **and** *wf: wf-conn* $c' \ l$
shows *False*

proof –
 let $?ψ = \text{conn } c' \ l$
 have $\text{simple } ?ψ \vee (\exists \varphi'. ?ψ = \text{FNot } \varphi' \wedge \text{simple } \varphi') \vee (\exists l. ?ψ = \text{conn } c \ l \wedge \text{wf-conn } c \ l)$
 using *only-c-inside-decomp only incl* **by** *blast*
 moreover have $\neg \text{simple } ?ψ$
 using *wf simple-decomp* **by** $(\text{metis } c' \ \text{connective.distinct}(19) \ \text{connective.distinct}(7,9,21,29,31) \ \text{wf-conn-list}(1-3))$
 moreover
 {
 fix φ'
 have $?ψ \neq \text{FNot } \varphi'$ **using** $c' \ \text{conn-inj-not}(1) \ \text{wf}$ **by** *blast*
 }
 ultimately obtain $l :: 'v \ \text{propo list}$ **where** $?ψ = \text{conn } c \ l \wedge \text{wf-conn } c \ l$ **by** *metis*
 then have $c = c'$ **using** *conn-inj wf* **by** *metis*
 then show *False* **using** cc' **by** *auto*
qed

lemma *only-c-inside-implies-c-in-c'-symb*:
 assumes $\delta: c \neq c'$ **and** $c: c = \text{CAnd} \vee c = \text{COr}$ **and** $c': c' = \text{CAnd} \vee c' = \text{COr}$
 shows $\text{only-c-inside } c \ \varphi \implies \text{c-in-c'-symb } c \ c' \ \varphi$
 apply (rule *ccontr*)
 apply (cases rule: *not-c-in-c'-symb.cases, auto*)
 by (metis $\delta \ c \ c' \ \text{connective.distinct}(37,39) \ \text{list.distinct}(1) \ \text{only-c-inside-c-c'-false} \ \text{subformula-in-binary-conn}(1,2) \ \text{wf-conn.simps}$)
 +

lemma *c-in-c'-symb-decomp-level1*:
 fixes $l :: 'v \ \text{propo list}$ **and** $c \ c' \ ca :: 'v \ \text{connective}$
 shows $\text{wf-conn } ca \ l \implies ca \neq c \implies \text{c-in-c'-symb } c \ c' \ (\text{conn } ca \ l)$
proof –
 have $\text{not-c-in-c'-symb } c \ c' \ (\text{conn } ca \ l) \implies \text{wf-conn } ca \ l \implies ca = c$
 by (induct $\text{conn } ca \ l$ rule: *not-c-in-c'-symb.induct, auto simp: conn-inj*)
 then show $\text{wf-conn } ca \ l \implies ca \neq c \implies \text{c-in-c'-symb } c \ c' \ (\text{conn } ca \ l)$ **by** *blast*
qed

lemma *only-c-inside-implies-c-in-c'-only*:
 assumes $\delta: c \neq c'$ **and** $c: c = \text{CAnd} \vee c = \text{COr}$ **and** $c': c' = \text{CAnd} \vee c' = \text{COr}$
 shows $\text{only-c-inside } c \ \varphi \implies \text{c-in-c'-only } c \ c' \ \varphi$
 unfolding *c-in-c'-only-def all-subformula-st-def*
 using *only-c-inside-implies-c-in-c'-symb*
 by (metis *all-subformula-st-def assms(1) c c' only-c-inside-def subformula-trans*)

lemma *c-in-c'-symb-c-implies-only-c-inside*:
 assumes $\delta: c = \text{CAnd} \vee c = \text{COr} \ c' = \text{CAnd} \vee c' = \text{COr} \ c \neq c'$ **and** $\text{wf: wf-conn } c \ [\varphi, \psi]$
 and $\text{inv: no-equiv } (\text{conn } c \ l) \ \text{no-imp } (\text{conn } c \ l) \ \text{simple-not } (\text{conn } c \ l)$
 shows $\text{wf-conn } c \ l \implies \text{c-in-c'-only } c \ c' \ (\text{conn } c \ l) \implies (\forall \psi \in \text{set } l. \text{only-c-inside } c \ \psi)$
 using *inv*
proof (induct $\text{conn } c \ l$ arbitrary: l rule: *propo-induct-arity*)
 case (nullary x)
 then show $?case$ **by** (auto simp: *wf-conn-list assms*)
next
 case (unary $\varphi \ la$)
 then have $c = \text{CNot} \wedge la = [\varphi]$ **by** (metis (no-types) *wf-conn-list(8)*)
 then show $?case$ **using** *assms(2) assms(1)* **by** *blast*

next

```

case (binary  $\varphi_1$   $\varphi_2$ )
note  $IH\varphi_1 = \text{this}(1)$  and  $IH\varphi_2 = \text{this}(2)$  and  $\varphi = \text{this}(3)$  and  $\text{only} = \text{this}(5)$  and  $\text{wf} = \text{this}(4)$ 
  and  $\text{no-equiv} = \text{this}(6)$  and  $\text{no-imp} = \text{this}(7)$  and  $\text{simple-not} = \text{this}(8)$ 
then have  $l: l = [\varphi_1, \varphi_2]$  by (meson wf-conn-list(4-7))
let  $? \varphi = \text{conn } c \ l$ 

obtain  $c_1 \ l_1 \ c_2 \ l_2$  where  $\varphi_1: \varphi_1 = \text{conn } c_1 \ l_1$  and  $\text{wf}\varphi_1: \text{wf-conn } c_1 \ l_1$ 
  and  $\varphi_2: \varphi_2 = \text{conn } c_2 \ l_2$  and  $\text{wf}\varphi_2: \text{wf-conn } c_2 \ l_2$  using exists-c-conn by metis
then have  $c\text{-in-only}\varphi_1: c\text{-in-}c'\text{-only } c \ c' (\text{conn } c_1 \ l_1)$  and  $c\text{-in-}c'\text{-only } c \ c' (\text{conn } c_2 \ l_2)$ 
  using only  $l$  unfolding  $c\text{-in-}c'\text{-only-def}$  using assms(1) by auto
have  $\text{inc}\varphi_1: \varphi_1 \preceq ? \varphi$  and  $\text{inc}\varphi_2: \varphi_2 \preceq ? \varphi$ 
  using  $\varphi_1 \ \varphi_2 \ \varphi \ \text{local.wf}$  by (metis conn.simps(5-8) helper-fact subformula-in-binary-conn(1,2))+

have  $c_1\text{-eq}: c_1 \neq \text{CEq}$  and  $c_2\text{-eq}: c_2 \neq \text{CEq}$ 
  unfolding no-equiv-def using  $\text{inc}\varphi_1 \ \text{inc}\varphi_2$  by (metis  $\varphi_1 \ \varphi_2 \ \text{wf}\varphi_1 \ \text{wf}\varphi_2 \ \text{assms}(1) \ \text{no-equiv}$ 
    no-equiv-eq(1) no-equiv-symb.elims(3) no-equiv-symb-conn-characterization wf-conn-list(4,5)
    no-equiv-def subformula-all-subformula-st)+
have  $c_1\text{-imp}: c_1 \neq \text{CImp}$  and  $c_2\text{-imp}: c_2 \neq \text{CImp}$ 
  using no-imp by (metis  $\varphi_1 \ \varphi_2 \ \text{all-subformula-st-decomp-explicit-imp}(2,3) \ \text{assms}(1)$ 
    conn.simps(5,6)  $l$  no-imp-imp(1) no-imp-symb.elims(3) no-imp-symb-conn-characterization
    wf $\varphi_1 \ \text{wf}\varphi_2 \ \text{all-subformula-st-decomp} \ \text{no-imp-symb-conn-characterization}$ )+
have  $c_1c: c_1 \neq c'$ 
proof
  assume  $c_1c: c_1 = c'$ 
  then obtain  $\xi_1 \ \xi_2$  where  $l_1: l_1 = [\xi_1, \xi_2]$ 
    by (metis assms(2) connective.distinct(37,39) helper-fact wf $\varphi_1 \ \text{wf-conn.simps}$ 
      wf-conn-list-decomp(1-3))
  have  $c\text{-in-}c'\text{-only } c \ c' (\text{conn } c \ [\text{conn } c' \ l_1, \varphi_2])$  using  $c_1c \ l$  only  $\varphi_1$  by auto
  moreover have  $\text{not-}c\text{-in-}c'\text{-symb } c \ c' (\text{conn } c \ [\text{conn } c' \ l_1, \varphi_2])$ 
    using  $l_1 \ \varphi_1 \ c_1c \ l \ \text{local.wf} \ \text{not-}c\text{-in-}c'\text{-symb-}l \ \text{wf}\varphi_1$  by blast
  ultimately show False using  $\varphi_1 \ c_1c \ l \ l_1 \ \text{local.wf} \ \text{not-}c\text{-in-}c'\text{-simp}(4) \ \text{wf}\varphi_1$  by blast
qed
then have  $(\varphi_1 = \text{conn } c \ l_1 \wedge \text{wf-conn } c \ l_1) \vee (\exists \psi_1. \varphi_1 = \text{FNot } \psi_1) \vee \text{simple } \varphi_1$ 
  by (metis  $\varphi_1 \ \text{assms}(1-3) \ c_1\text{-eq} \ c_1\text{-imp} \ \text{simple.elims}(3) \ \text{wf}\varphi_1 \ \text{wf-conn-list}(4) \ \text{wf-conn-list}(5-7))$ 
moreover {
  assume  $\varphi_1 = \text{conn } c \ l_1 \wedge \text{wf-conn } c \ l_1$ 
  then have only-c-inside  $c \ \varphi_1$ 
    by (metis  $IH\varphi_1 \ \varphi_1 \ \text{all-subformula-st-decomp-imp} \ \text{inc}\varphi_1 \ \text{no-equiv} \ \text{no-equiv-def} \ \text{no-imp} \ \text{no-imp-def}$ 
       $c\text{-in-only}\varphi_1 \ \text{only-c-inside-def} \ \text{only-c-inside-into-only-c-inside} \ \text{simple-not} \ \text{simple-not-def}$ 
      subformula-all-subformula-st)
}
moreover {
  assume  $\exists \psi_1. \varphi_1 = \text{FNot } \psi_1$ 
  then obtain  $\psi_1$  where  $\varphi_1 = \text{FNot } \psi_1$  by metis
  then have only-c-inside  $c \ \varphi_1$ 
    by (metis all-subformula-st-def assms(1) connective.distinct(37,39)  $\text{inc}\varphi_1$ 
      only-c-inside-decomp-not simple-not simple-not-def simple-not-symb.simps(1))
}
moreover {
  assume simple  $\varphi_1$ 
  then have only-c-inside  $c \ \varphi_1$ 
    by (metis all-subformula-st-decomp-explicit(3) assms(1) connective.distinct(37,39)
      only-c-inside-decomp-not only-c-inside-def)
}

```

ultimately have *only-c-inside* $\varphi 1$: *only-c-inside* c $\varphi 1$ by *metis*

have *c-in-only* $\varphi 2$: *c-in-c'-only* c c' (*conn* $c2$ $l2$)
 using *only* l $\varphi 2$ *wf* $\varphi 2$ *assms* **unfolding** *c-in-c'-only-def* by *auto*

have *c2c*: $c2 \neq c'$

proof
 assume *c2c*: $c2 = c'$
 then obtain $\xi 1$ $\xi 2$ **where** $l2$: $l2 = [\xi 1, \xi 2]$
 by (*metis* *assms*(2) *wf* $\varphi 2$ *wf-conn.simps* *connective.distinct*(7,9,19,21,29,31,37,39))
 then have *c-in-c'-symb* c c' (*conn* c [$\varphi 1$, *conn* c' $l2$])
 using *c2c* l *only* $\varphi 2$ *all-subformula-st-test-symb-true-phi* **unfolding** *c-in-c'-only-def* by *auto*
 moreover have *not-c-in-c'-symb* c c' (*conn* c [$\varphi 1$, *conn* c' $l2$])
 using *assms*(1) *c2c* $l2$ *not-c-in-c'-symb-r* *wf* $\varphi 2$ *wf-conn-helper-facts*(5,6) by *metis*
 ultimately show *False* by *auto*

qed

then have $(\varphi 2 = \text{conn } c \text{ } l2 \wedge \text{wf-conn } c \text{ } l2) \vee (\exists \psi 2. \varphi 2 = \text{FNot } \psi 2) \vee \text{simple } \varphi 2$
 using *c2-eq* by (*metis* $\varphi 2$ *assms*(1-3) *c2-eq* *c2-imp* *simple.elims*(3) *wf* $\varphi 2$ *wf-conn-list*(4-7))

moreover {
 assume $\varphi 2 = \text{conn } c \text{ } l2 \wedge \text{wf-conn } c \text{ } l2$
 then have *only-c-inside* c $\varphi 2$
 by (*metis* *IH* $\varphi 2$ $\varphi 2$ *all-subformula-st-decomp* *inc* $\varphi 2$ *no-equiv* *no-equiv-def* *no-imp* *no-imp-def* *c-in-only* $\varphi 2$ *only-c-inside-def* *only-c-inside-into-only-c-inside* *simple-not* *simple-not-def* *subformula-all-subformula-st*)
 }
 moreover {
 assume $\exists \psi 2. \varphi 2 = \text{FNot } \psi 2$
 then obtain $\psi 2$ **where** $\varphi 2 = \text{FNot } \psi 2$ by *metis*
 then have *only-c-inside* c $\varphi 2$
 by (*metis* *all-subformula-st-def* *assms*(1-3) *connective.distinct*(38,40) *inc* $\varphi 2$ *only-c-inside-decomp-not* *simple-not* *simple-not-def* *simple-not-symb.simps*(1))
 }
 moreover {
 assume *simple* $\varphi 2$
 then have *only-c-inside* c $\varphi 2$
 by (*metis* *all-subformula-st-decomp-explicit*(3) *assms*(1) *connective.distinct*(37,39) *only-c-inside-decomp-not* *only-c-inside-def*)
 }
 ultimately have *only-c-inside* $\varphi 2$: *only-c-inside* c $\varphi 2$ by *metis*
 show ?case using l *only-c-inside* $\varphi 1$ *only-c-inside* $\varphi 2$ by *auto*

qed

8.5.2 Push Conjunction

definition *pushConj* **where** *pushConj* = *push-conn-inside* *CAnd* *COr*

lemma *pushConj-consistent*: *preserves-un-sat* *pushConj*
unfolding *pushConj-def* by (*simp* *add*: *push-conn-inside-consistent*)

definition *and-in-or-symb* **where** *and-in-or-symb* = *c-in-c'-symb* *CAnd* *COr*

definition *and-in-or-only* **where**
and-in-or-only = *all-subformula-st* (*c-in-c'-symb* *CAnd* *COr*)

lemma *pushConj-inv*:
fixes φ ψ :: '*v* *propo*
assumes *full* (*propo-rew-step* *pushConj*) φ ψ

and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ **and** *simple-not* φ
shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ **and** *simple-not* ψ
using *push-conn-inside-inv* *assms* **unfolding** *pushConj-def* **by** *metis+*

lemma *pushConj-full-propo-rew-step*:
fixes $\varphi \psi :: 'v \text{ propo}$
assumes
no-equiv φ **and**
no-imp φ **and**
full (*propo-rew-step* *pushConj*) $\varphi \psi$ **and**
no-T-F-except-top-level φ **and**
simple-not φ
shows *and-in-or-only* ψ
using *assms* *push-conn-inside-full-propo-rew-step*
unfolding *pushConj-def* *and-in-or-only-def* *c-in-c'-only-def* **by** (*metis* (*no-types*))

8.5.3 Push Disjunction

definition *pushDisj* **where** *pushDisj* = *push-conn-inside* *COr* *CAnd*

lemma *pushDisj-consistent*: *preserves-un-sat* *pushDisj*
unfolding *pushDisj-def* **by** (*simp* *add*: *push-conn-inside-consistent*)

definition *or-in-and-symb* **where** *or-in-and-symb* = *c-in-c'-symb* *COr* *CAnd*

definition *or-in-and-only* **where**
or-in-and-only = *all-subformula-st* (*c-in-c'-symb* *COr* *CAnd*)

lemma *not-or-in-and-only-or-and[simp]*:
 $\sim \text{or-in-and-only } (FOr \ (FAnd \ \psi1 \ \psi2) \ \varphi')$
unfolding *or-in-and-only-def*
by (*metis* *all-subformula-st-test-symb-true-phi* *conn.simps*(5–6) *not-c-in-c'-symb-l*
wf-conn-helper-facts(5) *wf-conn-helper-facts*(6))

lemma *pushDisj-inv*:
fixes $\varphi \psi :: 'v \text{ propo}$
assumes *full* (*propo-rew-step* *pushDisj*) $\varphi \psi$
and *no-equiv* φ **and** *no-imp* φ **and** *no-T-F-except-top-level* φ **and** *simple-not* φ
shows *no-equiv* ψ **and** *no-imp* ψ **and** *no-T-F-except-top-level* ψ **and** *simple-not* ψ
using *push-conn-inside-inv* *assms* **unfolding** *pushDisj-def* **by** *metis+*

lemma *pushDisj-full-propo-rew-step*:
fixes $\varphi \psi :: 'v \text{ propo}$
assumes
no-equiv φ **and**
no-imp φ **and**
full (*propo-rew-step* *pushDisj*) $\varphi \psi$ **and**
no-T-F-except-top-level φ **and**
simple-not φ
shows *or-in-and-only* ψ
using *assms* *push-conn-inside-full-propo-rew-step*
unfolding *pushDisj-def* *or-in-and-only-def* *c-in-c'-only-def* **by** (*metis* (*no-types*))

9 The full transformations

9.1 Abstract Property characterizing that only some connective are inside the others

9.1.1 Definition

The normal is a super group of groups

inductive *grouped-by* :: 'a connective \Rightarrow 'a propo \Rightarrow bool **for** c **where**
simple-is-grouped[simp]: *simple* $\varphi \Rightarrow$ *grouped-by* c φ |
simple-not-is-grouped[simp]: *simple* $\varphi \Rightarrow$ *grouped-by* c (FNot φ) |
connected-is-group[simp]: *grouped-by* c $\varphi \Rightarrow$ *grouped-by* c $\psi \Rightarrow$ wf-conn c [φ , ψ]
 \Rightarrow *grouped-by* c (conn c [φ , ψ])

lemma *simple-clause*[simp]:

grouped-by c FT
grouped-by c FF
grouped-by c (FVar x)
grouped-by c (FNot FT)
grouped-by c (FNot FF)
grouped-by c (FNot (FVar x))
by simp+

lemma *only-c-inside-symb-c-eq-c'*:

only-c-inside-symb c (conn c' [$\varphi 1$, $\varphi 2$]) \Rightarrow $c' = CAnd \vee c' = COr \Rightarrow$ wf-conn c' [$\varphi 1$, $\varphi 2$]
 $\Rightarrow c' = c$
by (induct conn c' [$\varphi 1$, $\varphi 2$] rule: *only-c-inside-symb.induct*, auto simp: *conn-inj*)

lemma *only-c-inside-c-eq-c'*:

only-c-inside c (conn c' [$\varphi 1$, $\varphi 2$]) \Rightarrow $c' = CAnd \vee c' = COr \Rightarrow$ wf-conn c' [$\varphi 1$, $\varphi 2$] \Rightarrow $c = c'$
unfolding *only-c-inside-def* *all-subformula-st-def* **using** *only-c-inside-symb-c-eq-c'* *subformula-refl*
by blast

lemma *only-c-inside-imp-grouped-by*:

assumes c: $c \neq CNot$ **and** c': $c' = CAnd \vee c' = COr$
shows *only-c-inside* c $\varphi \Rightarrow$ *grouped-by* c φ (**is** ?O $\varphi \Rightarrow$?G φ)

proof (induct φ rule: *propo-induct-arity*)

case (*nullary* φ x)

then show ?G φ **by** auto

next

case (*unary* ψ)

then show ?G (FNot ψ) **by** (auto simp: c)

next

case (*binary* φ $\varphi 1$ $\varphi 2$)

note $IH \varphi 1 = this(1)$ **and** $IH \varphi 2 = this(2)$ **and** $\varphi = this(3)$ **and** $only = this(4)$

have φ -conn: $\varphi = conn$ c [$\varphi 1$, $\varphi 2$] **and** wf: wf-conn c [$\varphi 1$, $\varphi 2$]

proof –

obtain c'' l'' **where** φ -c'': $\varphi = conn$ c'' l'' **and** wf: wf-conn c'' l''

using *exists-c-conn* **by** metis

then have l'': l'' = [$\varphi 1$, $\varphi 2$] **using** φ **by** (metis wf-conn-list(4–7))

have *only-c-inside-symb* c (conn c'' [$\varphi 1$, $\varphi 2$])

using *all-subformula-st-test-symb-true-phi*

unfolding *only-c-inside-def* φ -c'' l'' **by** metis

then have $c = c''$

```

    by (metis  $\varphi$   $\varphi$ -c'' conn-inj conn-inj-not(2) l'' list.distinct(1) list.inject wf
        only-c-inside-symb.cases simple.simps(5-8))
  then show  $\varphi = \text{conn } c [\varphi 1, \varphi 2]$  and  $\text{wf-conn } c [\varphi 1, \varphi 2]$  using  $\varphi$ -c'' wf l'' by auto
qed
have grouped-by  $c \varphi 1$  using wf IH $\varphi 1$  IH $\varphi 2$   $\varphi$ -conn only  $\varphi$  unfolding only-c-inside-def by auto
moreover have grouped-by  $c \varphi 2$ 
  using wf  $\varphi$  IH $\varphi 1$  IH $\varphi 2$   $\varphi$ -conn only unfolding only-c-inside-def by auto
ultimately show ?G  $\varphi$  using  $\varphi$ -conn connected-is-group local.wf by blast
qed

```

lemma grouped-by-false:

```

grouped-by  $c (\text{conn } c' [\varphi, \psi]) \implies c \neq c' \implies \text{wf-conn } c' [\varphi, \psi] \implies \text{False}$ 
apply (induct conn  $c' [\varphi, \psi]$  rule: grouped-by.induct)
apply (auto simp: simple-decomp wf-conn-list, auto simp: conn-inj)
by (metis list.distinct(1) list.sel(3) wf-conn-list(8))+

```

Then the CNF form is a conjunction of clauses: every clause is in CNF form and two formulas in CNF form can be related by an and.

inductive super-grouped-by:: 'a connective \Rightarrow 'a connective \Rightarrow 'a propo \Rightarrow bool **for** $c \ c'$ **where**
 grouped-is-super-grouped[simp]: grouped-by $c \varphi \implies \text{super-grouped-by } c \ c' \varphi \mid$
 connected-is-super-group: super-grouped-by $c \ c' \varphi \implies \text{super-grouped-by } c \ c' \psi \implies \text{wf-conn } c [\varphi, \psi]$
 $\implies \text{super-grouped-by } c \ c' (\text{conn } c' [\varphi, \psi])$

lemma simple-cnf[simp]:

```

super-grouped-by  $c \ c' \text{ FT}$ 
super-grouped-by  $c \ c' \text{ FF}$ 
super-grouped-by  $c \ c' (\text{FVar } x)$ 
super-grouped-by  $c \ c' (\text{FNot FT})$ 
super-grouped-by  $c \ c' (\text{FNot FF})$ 
super-grouped-by  $c \ c' (\text{FNot } (\text{FVar } x))$ 
by auto

```

lemma c-in-c'-only-super-grouped-by:

```

assumes  $c: c = \text{CAnd} \vee c = \text{COr}$  and  $c': c' = \text{CAnd} \vee c' = \text{COr}$  and  $cc': c \neq c'$ 
shows no-equiv  $\varphi \implies \text{no-imp } \varphi \implies \text{simple-not } \varphi \implies \text{c-in-c'-only } c \ c' \varphi$ 
 $\implies \text{super-grouped-by } c \ c' \varphi$ 
(is ?NE  $\varphi \implies ?NI \varphi \implies ?SN \varphi \implies ?C \varphi \implies ?S \varphi$ )

```

proof (induct φ rule: propo-induct-arity)

case (nullary $\varphi \ x$)

then show ?S φ **by** auto

next

case (unary φ)

then have simple-not-symb (FNot φ)

using all-subformula-st-test-symb-true-phi **unfolding** simple-not-def **by** blast

then have $\varphi = \text{FT} \vee \varphi = \text{FF} \vee (\exists x. \varphi = \text{FVar } x)$ **by** (cases φ , auto)

then show ?S (FNot φ) **by** auto

next

case (binary $\varphi \varphi 1 \varphi 2$)

note IH $\varphi 1 = \text{this}(1)$ and IH $\varphi 2 = \text{this}(2)$ and no-equiv = this(4) and no-imp = this(5)

and simpleN = this(6) and c-in-c'-only = this(7) and $\varphi' = \text{this}(3)$

{

assume $\varphi = \text{FImp } \varphi 1 \varphi 2 \vee \varphi = \text{FEq } \varphi 1 \varphi 2$

then have False **using** no-equiv no-imp **by** auto

then have ?S φ **by** auto

```

}
moreover {
  assume  $\varphi$ :  $\varphi = \text{conn } c' [\varphi 1, \varphi 2] \wedge \text{wf-conn } c' [\varphi 1, \varphi 2]$ 
  have c-in-c'-only:  $c\text{-in-}c'\text{-only } c \ c' \ \varphi 1 \wedge c\text{-in-}c'\text{-only } c \ c' \ \varphi 2 \wedge c\text{-in-}c'\text{-symb } c \ c' \ \varphi$ 
    using c-in-c'-only  $\varphi'$  unfolding c-in-c'-only-def by auto
  have super-grouped-by  $c \ c' \ \varphi 1$  using  $\varphi \ c' \text{ no-equiv no-imp simpleN IH}\varphi 1 \text{ c-in-}c'\text{-only}$  by auto
  moreover have super-grouped-by  $c \ c' \ \varphi 2$ 
    using  $\varphi \ c' \text{ no-equiv no-imp simpleN IH}\varphi 2 \text{ c-in-}c'\text{-only}$  by auto
  ultimately have  $?S \ \varphi$ 
    using super-grouped-by.intros(2)  $\varphi$  by (metis c wf-conn-helper-facts(5,6))
}
moreover {
  assume  $\varphi$ :  $\varphi = \text{conn } c [\varphi 1, \varphi 2] \wedge \text{wf-conn } c [\varphi 1, \varphi 2]$ 
  then have only-c-inside  $c \ \varphi 1 \wedge \text{only-c-inside } c \ \varphi 2$ 
    using c-in-c'-symb-c-implies-only-c-inside  $c \ c' \text{ c-in-}c'\text{-only list.set-intros}(1)
      wf-conn-helper-facts(5,6) no-equiv no-imp simpleN last-ConsL last-ConsR last-in-set
      list.distinct(1) by (metis (no-types, hide-lams) cc')
  then have only-c-inside  $c \ (\text{conn } c [\varphi 1, \varphi 2])$ 
    unfolding only-c-inside-def using  $\varphi$ 
    by (simp add: only-c-inside-into-only-c-inside all-subformula-st-decomp)
  then have grouped-by  $c \ \varphi$  using  $\varphi \text{ only-c-inside-imp-grouped-by } c$  by blast
  then have  $?S \ \varphi$  using super-grouped-by.intros(1) by metis
}
ultimately show  $?S \ \varphi$  by (metis  $\varphi' \ c \ c' \text{ cc' conn.simps}$ (5,6) wf-conn-helper-facts(5,6))
qed$ 
```

9.2 Conjunctive Normal Form

definition *is-conj-with-TF* **where** *is-conj-with-TF* == *super-grouped-by COr CAnd*

lemma *or-in-and-only-conjunction-in-disj*:

shows $\text{no-equiv } \varphi \implies \text{no-imp } \varphi \implies \text{simple-not } \varphi \implies \text{or-in-and-only } \varphi \implies \text{is-conj-with-TF } \varphi$
using *c-in-c'-only-super-grouped-by*
unfolding *is-conj-with-TF-def or-in-and-only-def c-in-c'-only-def*
by (*simp add: c-in-c'-only-def c-in-c'-only-super-grouped-by*)

definition *is-cnf* **where** *is-cnf* φ == *is-conj-with-TF* $\varphi \wedge \text{no-T-F-except-top-level } \varphi$

9.2.1 Full CNF transformation

The full CNF transformation consists simply in chaining all the transformation defined before.

definition *cnf-rew* **where** *cnf-rew* =
 (*full* (*propo-rew-step elim-equiv*)) *OO*
 (*full* (*propo-rew-step elim-imp*)) *OO*
 (*full* (*propo-rew-step elimTB*)) *OO*
 (*full* (*propo-rew-step pushNeg*)) *OO*
 (*full* (*propo-rew-step pushDisj*))

lemma *cnf-rew-consistent*: *preserves-un-sat* *cnf-rew*

by (*simp add: cnf-rew-def elimEquiv-lifted-consistant elim-imp-lifted-consistant elimTB-consistent*
preserves-un-sat-OO pushDisj-consistent pushNeg-lifted-consistant)

lemma *cnf-rew-is-cnf*: *cnf-rew* $\varphi \ \varphi' \implies \text{is-cnf } \varphi'$
apply (*unfold cnf-rew-def OO-def*)

```

apply auto
proof –
  fix  $\varphi$   $\varphi Eq$   $\varphi Imp$   $\varphi TB$   $\varphi Neg$   $\varphi Disj$  :: 'v propo
  assume Eq: full (propo-rew-step elim-equiv)  $\varphi$   $\varphi Eq$ 
  then have no-equiv: no-equiv  $\varphi Eq$  using no-equiv-full-propo-rew-step-elim-equiv by blast

  assume Imp: full (propo-rew-step elim-imp)  $\varphi Eq$   $\varphi Imp$ 
  then have no-imp: no-imp  $\varphi Imp$  using no-imp-full-propo-rew-step-elim-imp by blast
  have no-imp-inv: no-equiv  $\varphi Imp$  using no-equiv Imp elim-imp-inv by blast

  assume TB: full (propo-rew-step elimTB)  $\varphi Imp$   $\varphi TB$ 
  then have noTB: no-T-F-except-top-level  $\varphi TB$ 
    using no-imp-inv no-imp elimTB-full-propo-rew-step by blast
  have noTB-inv: no-equiv  $\varphi TB$  no-imp  $\varphi TB$  using elimTB-inv TB no-imp no-imp-inv by blast+

  assume Neg: full (propo-rew-step pushNeg)  $\varphi TB$   $\varphi Neg$ 
  then have noNeg: simple-not  $\varphi Neg$ 
    using noTB-inv noTB pushNeg-full-propo-rew-step by blast
  have noNeg-inv: no-equiv  $\varphi Neg$  no-imp  $\varphi Neg$  no-T-F-except-top-level  $\varphi Neg$ 
    using pushNeg-inv Neg noTB noTB-inv by blast+

  assume Disj: full (propo-rew-step pushDisj)  $\varphi Neg$   $\varphi Disj$ 
  then have no-Disj: or-in-and-only  $\varphi Disj$ 
    using noNeg-inv noNeg pushDisj-full-propo-rew-step by blast
  have noDisj-inv: no-equiv  $\varphi Disj$  no-imp  $\varphi Disj$  no-T-F-except-top-level  $\varphi Disj$ 
    simple-not  $\varphi Disj$ 
  using pushDisj-inv Disj noNeg noNeg-inv by blast+

  moreover have is-conj-with-TF  $\varphi Disj$ 
    using or-in-and-only-conjunction-in-disj noDisj-inv no-Disj by blast
  ultimately show is-cnf  $\varphi Disj$  unfolding is-cnf-def by blast
qed

```

9.3 Disjunctive Normal Form

definition *is-disj-with-TF* **where** *is-disj-with-TF* \equiv *super-grouped-by CAnd COr*

lemma *and-in-or-only-conjunction-in-disj*:

shows *no-equiv $\varphi \implies$ no-imp $\varphi \implies$ simple-not $\varphi \implies$ and-in-or-only $\varphi \implies$ is-disj-with-TF φ*
using *c-in-c'-only-super-grouped-by*
unfolding *is-disj-with-TF-def and-in-or-only-def c-in-c'-only-def*
by (*simp add: c-in-c'-only-def c-in-c'-only-super-grouped-by*)

definition *is-dnf* :: '*a propo* \Rightarrow *bool* **where**

is-dnf $\varphi \iff$ is-disj-with-TF $\varphi \wedge$ no-T-F-except-top-level φ

9.3.1 Full DNF transform

The full DNF transformation consists simply in chaining all the transformation defined before.

definition *dnf-rew* **where** *dnf-rew* \equiv
 (*full (propo-rew-step elim-equiv)*) *OO*
 (*full (propo-rew-step elim-imp)*) *OO*
 (*full (propo-rew-step elimTB)*) *OO*
 (*full (propo-rew-step pushNeg)*) *OO*
 (*full (propo-rew-step pushConj)*)

lemma *dnf-rew-consistent: preserves-un-sat dnf-rew*

by (*simp add: dnf-rew-def elimEquiv-lifted-consistant elim-imp-lifted-consistant elimTB-consistent preserves-un-sat-OO pushConj-consistent pushNeg-lifted-consistant*)

theorem *dnf-transformation-correction:*

dnf-rew $\varphi \varphi' \implies is-dnf \varphi'$

apply (*unfold dnf-rew-def OO-def*)

by (*meson and-in-or-only-conjunction-in-disj elimTB-full-propo-rew-step elimTB-inv(1,2) elim-imp-inv is-dnf-def no-equiv-full-propo-rew-step-elim-equiv no-imp-full-propo-rew-step-elim-imp pushConj-full-propo-rew-step pushConj-inv(1-4) pushNeg-full-propo-rew-step pushNeg-inv(1-3)*)

10 More aggressive simplifications: Removing true and false at the beginning

10.1 Transformation

We should remove *FT* and *FF* at the beginning and not in the middle of the algorithm. To do this, we have to use more rules (one for each connective):

inductive *elimTBFull* **where**

ElimTBFull1[simp]: elimTBFull (FAnd φ FT) φ |

ElimTBFull1'[simp]: elimTBFull (FAnd FT φ) φ |

ElimTBFull2[simp]: elimTBFull (FAnd φ FF) FF |

ElimTBFull2'[simp]: elimTBFull (FAnd FF φ) FF |

ElimTBFull3[simp]: elimTBFull (FOr φ FT) FT |

ElimTBFull3'[simp]: elimTBFull (FOr FT φ) FT |

ElimTBFull4[simp]: elimTBFull (FOr φ FF) φ |

ElimTBFull4'[simp]: elimTBFull (FOr FF φ) φ |

ElimTBFull5[simp]: elimTBFull (FNot FT) FF |

ElimTBFull5'[simp]: elimTBFull (FNot FF) FT |

ElimTBFull6-l[simp]: elimTBFull (FImp FT φ) φ |

ElimTBFull6-l'[simp]: elimTBFull (FImp FF φ) FT |

ElimTBFull6-r[simp]: elimTBFull (FImp φ FT) FT |

ElimTBFull6-r'[simp]: elimTBFull (FImp φ FF) (FNot φ) |

ElimTBFull7-l[simp]: elimTBFull (FEq FT φ) φ |

ElimTBFull7-l'[simp]: elimTBFull (FEq FF φ) (FNot φ) |

ElimTBFull7-r[simp]: elimTBFull (FEq φ FT) φ |

ElimTBFull7-r'[simp]: elimTBFull (FEq φ FF) (FNot φ)

The transformation is still consistent.

lemma *elimTBFull-consistent: preserves-un-sat elimTBFull*

proof –

```
{
  fix  $\varphi \psi :: 'b$  propo
  have elimTBFull  $\varphi \psi \implies \forall A. A \models \varphi \longleftrightarrow A \models \psi$ 
    by (induct-tac rule: elimTBFull.inducts, auto)
}
```

then show *?thesis* **using** *preserves-un-sat-def* **by** *auto*
qed

Contrary to the theorem $\llbracket \text{no-equiv } ?\varphi; \text{no-imp } ?\varphi; ?\psi \preceq ?\varphi; \neg \text{no-T-F-symb-except-toplevel } ?\psi \rrbracket \implies \exists \psi'. \text{elimTB } ?\psi \psi'$, we do not need the assumption *no-equiv* φ and *no-imp* φ , since our transformation is more general.

lemma *no-T-F-symb-except-toplevel-step-exists'*:

fixes $\varphi :: 'v \text{ propo}$
shows $\psi \preceq \varphi \implies \neg \text{no-T-F-symb-except-toplevel } \psi \implies \exists \psi'. \text{elimTBFULL } \psi \psi'$
proof (*induct* ψ *rule: propo-induct-arity*)
case (*nullary* φ')
then have *False* **using** *no-T-F-symb-except-toplevel-true no-T-F-symb-except-toplevel-false* **by** *auto*
then show *Ex* (*elimTBFULL* φ') **by** *blast*
next
case (*unary* ψ)
then have $\psi = FF \vee \psi = FT$ **using** *no-T-F-symb-except-toplevel-not-decom* **by** *blast*
then show *Ex* (*elimTBFULL* (*FNot* ψ)) **using** *ElimTBFULL5 ElimTBFULL5'* **by** *blast*
next
case (*binary* $\varphi' \psi1 \psi2$)
then have $\psi1 = FT \vee \psi2 = FT \vee \psi1 = FF \vee \psi2 = FF$
by (*metis binary-connectives-def conn.simps(5-8) insertI1 insert-commute*
no-T-F-symb-except-toplevel-bin-decom binary.hyps(3))
then show *Ex* (*elimTBFULL* φ') **using** *elimTBFULL.intros binary.hyps(3)* **by** *blast*
qed

The same applies here. We do not need the assumption, but the deep link between $\neg \text{no-T-F-except-top-level}$ φ and the existence of a rewriting step, still exists.

lemma *no-T-F-except-top-level-rew'*:

fixes $\varphi :: 'v \text{ propo}$
assumes *noTB*: $\neg \text{no-T-F-except-top-level } \varphi$
shows $\exists \psi \psi'. \psi \preceq \varphi \wedge \text{elimTBFULL } \psi \psi'$
proof –
have *test-symb-false-nullary*:
 $\forall x. \text{no-T-F-symb-except-toplevel } (FF :: 'v \text{ propo}) \wedge \text{no-T-F-symb-except-toplevel } FT$
 $\wedge \text{no-T-F-symb-except-toplevel } (FVar (x :: 'v))$
by *auto*
moreover {
fix $c :: 'v \text{ connective}$ **and** $l :: 'v \text{ propo list}$ **and** $\psi :: 'v \text{ propo}$
have $H: \text{elimTBFULL } (\text{conn } c \ l) \ \psi \implies \neg \text{no-T-F-symb-except-toplevel } (\text{conn } c \ l)$
by (*cases* (*conn* $c \ l$) *rule: elimTBFULL.cases*) *auto*
}
ultimately show *?thesis*
using *no-test-symb-step-exists*[*of no-T-F-symb-except-toplevel* φ *elimTBFULL*] *noTB*
no-T-F-symb-except-toplevel-step-exists' **unfolding** *no-T-F-except-top-level-def* **by** *metis*
qed

lemma *elimTBFULL-full-propo-rew-step*:

fixes $\varphi \psi :: 'v \text{ propo}$
assumes *full* (*propo-rew-step* *elimTBFULL*) $\varphi \ \psi$
shows *no-T-F-except-top-level* ψ
using *full-propo-rew-step-subformula no-T-F-except-top-level-rew' assms* **by** *fastforce*

10.2 More invariants

As the aim is to use the transformation as the first transformation, we have to show some more invariants for *elim-equiv* and *elim-imp*. For the other transformation, we have already proven it.

lemma *propo-rew-step-ElimEquiv-no-T-F*: *propo-rew-step elim-equiv* $\varphi \psi \implies \text{no-T-F } \varphi \implies \text{no-T-F } \psi$

proof (*induct rule: propo-rew-step.induct*)

fix $\varphi' :: 'v \text{ propo}$ **and** $\psi' :: 'v \text{ propo}$

assume $a1: \text{no-T-F } \varphi'$

assume $a2: \text{elim-equiv } \varphi' \psi'$

have $\forall x0 \ x1. (\neg \text{elim-equiv } (x1 :: 'v \text{ propo}) \ x0 \vee (\exists v2 \ v3 \ v4 \ v5 \ v6 \ v7. x1 = \text{FEq } v2 \ v3 \wedge x0 = \text{FAnd } (\text{FImp } v4 \ v5) (\text{FImp } v6 \ v7) \wedge v2 = v4 \wedge v4 = v7 \wedge v3 = v5 \wedge v3 = v6))$
 $= (\neg \text{elim-equiv } x1 \ x0 \vee (\exists v2 \ v3 \ v4 \ v5 \ v6 \ v7. x1 = \text{FEq } v2 \ v3 \wedge x0 = \text{FAnd } (\text{FImp } v4 \ v5) (\text{FImp } v6 \ v7) \wedge v2 = v4 \wedge v4 = v7 \wedge v3 = v5 \wedge v3 = v6))$

by *meson*

then have $\forall p \ pa. \neg \text{elim-equiv } (p :: 'v \text{ propo}) \ pa \vee (\exists pb \ pc \ pd \ pe \ pf \ pg. p = \text{FEq } pb \ pc \wedge pa = \text{FAnd } (\text{FImp } pd \ pe) (\text{FImp } pf \ pg) \wedge pb = pd \wedge pd = pg \wedge pc = pe \wedge pc = pf)$

using *elim-equiv.cases* **by** *force*

then show *no-T-F* ψ' **using** $a1 \ a2$ **by** *fastforce*

next

fix $\varphi \ \varphi' :: 'v \text{ propo}$ **and** $\xi \ \xi' :: 'v \text{ propo list}$ **and** $c :: 'v \text{ connective}$

assume *rel*: *propo-rew-step elim-equiv* $\varphi \varphi'$

and *IH*: *no-T-F* $\varphi \implies \text{no-T-F } \varphi'$

and *corr*: *wf-conn* $c \ (\xi @ \varphi \# \xi')$

and *no-T-F*: *no-T-F* (*conn* $c \ (\xi @ \varphi \# \xi')$)

{

assume $c: c = \text{CNot}$

then have *empty*: $\xi = [] \ \xi' = []$ **using** *corr* **by** *auto*

then have *no-T-F* φ **using** *no-T-F* c *no-T-F-decomp-not* **by** *auto*

then have *no-T-F* (*conn* $c \ (\xi @ \varphi' \# \xi')$) **using** c *empty* *no-T-F-comp-not* *IH* **by** *auto*

}

moreover {

assume $c: c \in \text{binary-connectives}$

obtain $a \ b$ **where** $ab: \xi @ \varphi \# \xi' = [a, b]$

using *corr* *c* *list-length2-decomp* *wf-conn-bin-list-length* **by** *metis*

then have $\varphi: \varphi = a \vee \varphi = b$

by (*metis* *append.simps*(1) *append-is-Nil-conv* *list.distinct*(1) *list.sel*(3) *nth-Cons-0* *tl-append2*)

have $\zeta: \forall \zeta \in \text{set } (\xi @ \varphi \# \xi'). \text{no-T-F } \zeta$

using *no-T-F* *unfolding* *no-T-F-def* **using** *corr* *all-subformula-st-decomp* **by** *blast*

then have $\varphi': \text{no-T-F } \varphi'$ **using** $ab \ IH \ \varphi$ **by** *auto*

have $l': \xi @ \varphi' \# \xi' = [\varphi', b] \vee \xi @ \varphi' \# \xi' = [a, \varphi']$

by (*metis* (*no-types*, *hide-lams*) *ab* *append-Cons* *append-Nil* *append-Nil2* *butlast.simps*(2) *butlast-append* *list.distinct*(1) *list.sel*(3))

then have $\forall \zeta \in \text{set } (\xi @ \varphi' \# \xi'). \text{no-T-F } \zeta$ **using** $\zeta \ \varphi' \ ab$ **by** *fastforce*

moreover

have $\forall \zeta \in \text{set } (\xi @ \varphi \# \xi'). \zeta \neq \text{FT} \wedge \zeta \neq \text{FF}$

using ζ *corr* *no-T-F* *no-T-F-except-top-level-false* *no-T-F-no-T-F-except-top-level* **by** *blast*

then have *no-T-F-symb* (*conn* $c \ (\xi @ \varphi' \# \xi')$)

by (*metis* $\varphi' \ l' \ ab$ *all-subformula-st-test-symb-true-phi* c *list.distinct*(1)

list.set-intros(1,2) *no-T-F-symb-except-toplevel-bin-decom*

no-T-F-symb-except-toplevel-no-T-F-symb *no-T-F-symb-false*(1,2) *no-T-F-def* *wf-conn-binary* *wf-conn-list*(1,2))

ultimately have *no-T-F* (*conn* $c \ (\xi @ \varphi' \# \xi')$)

```

    by (metis l' all-subformula-st-decomp-imp c no-T-F-def wf-conn-binary)
  }
  moreover {
    fix x
    assume c = CVar x  $\vee$  c = CF  $\vee$  c = CT
    then have False using corr by auto
    then have no-T-F (conn c ( $\xi @ \varphi' \# \xi'$ )) by auto
  }
  ultimately show no-T-F (conn c ( $\xi @ \varphi' \# \xi'$ )) using corr wf-conn.cases by metis
qed

lemma elim-equiv-inv':
  fixes  $\varphi \psi :: 'v$  propo
  assumes full (propo-rew-step elim-equiv)  $\varphi \psi$  and no-T-F-except-top-level  $\varphi$ 
  shows no-T-F-except-top-level  $\psi$ 
proof -
  {
    fix  $\varphi \psi :: 'v$  propo
    have propo-rew-step elim-equiv  $\varphi \psi \implies$  no-T-F-except-top-level  $\varphi$ 
       $\implies$  no-T-F-except-top-level  $\psi$ 
    proof -
      assume rel: propo-rew-step elim-equiv  $\varphi \psi$ 
      and no: no-T-F-except-top-level  $\varphi$ 
      {
        assume  $\varphi = FT \vee \varphi = FF$ 
        from rel this have False
        apply (induct rule: propo-rew-step.induct, auto simp: wf-conn-list(1,2))
        using elim-equiv.simps by blast+
        then have no-T-F-except-top-level  $\psi$  by blast
      }
      moreover {
        assume  $\varphi \neq FT \wedge \varphi \neq FF$ 
        then have no-T-F  $\varphi$ 
          by (metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
        then have no-T-F  $\psi$  using propo-rew-step-ElimEquiv-no-T-F rel by blast
        then have no-T-F-except-top-level  $\psi$  by (simp add: no-T-F-no-T-F-except-top-level)
      }
      ultimately show no-T-F-except-top-level  $\psi$  by metis
    qed
  }
  moreover {
    fix c :: 'v connective and  $\xi \xi' :: 'v$  propo list and  $\zeta \zeta' :: 'v$  propo
    assume rel: propo-rew-step elim-equiv  $\zeta \zeta'$ 
    and incl:  $\zeta \preceq \varphi$ 
    and corr: wf-conn c ( $\xi @ \zeta \# \xi'$ )
    and no-T-F: no-T-F-symb-except-toplevel (conn c ( $\xi @ \zeta \# \xi'$ ))
    and n: no-T-F-symb-except-toplevel  $\zeta'$ 
    have no-T-F-symb-except-toplevel (conn c ( $\xi @ \zeta' \# \xi'$ ))
    proof
      have p: no-T-F-symb (conn c ( $\xi @ \zeta \# \xi'$ ))
        using corr wf-conn-list(1) wf-conn-list(2) no-T-F-symb-except-toplevel-no-T-F-symb no-T-F
        by blast
      have l:  $\forall \varphi \in \text{set } (\xi @ \zeta \# \xi'). \varphi \neq FT \wedge \varphi \neq FF$ 
        using corr wf-conn-no-T-F-symb-iff p by blast
      from rel incl have  $\zeta' \neq FT \wedge \zeta' \neq FF$ 

```



```

    apply (induction  $\zeta \ \zeta'$  rule: propo-rew-step.induct)
    apply (cases rule: elim-equiv.cases, auto simp: elim-equiv.simps)
    by (metis append-is-Nil-conv list.distinct wf-conn-list(1,2) wf-conn-no-arity-change
        wf-conn-no-arity-change-helper)+
    then have  $\forall \varphi \in \text{set } (\xi @ \zeta' \# \xi'). \varphi \neq FT \wedge \varphi \neq FF$  using  $l$  by auto
    moreover have  $c \neq CT \wedge c \neq CF$  using corr by auto
    ultimately show no-T-F-symb (conn  $c$  ( $\xi @ \zeta' \# \xi'$ ))
      by (metis corr wf-conn-no-arity-change wf-conn-no-arity-change-helper no-T-F-symb-comp)
  qed
}
ultimately show no-T-F-except-top-level  $\psi$ 
  using full-propo-rew-step-inv-stay-with-inc[of elim-equiv no-T-F-symb-except-toplevel  $\varphi$ ]
  assms subformula-refl unfolding no-T-F-except-top-level-def by metis
qed

```

lemma *propo-rew-step-ElimImp-no-T-F*: *propo-rew-step elim-imp* $\varphi \ \psi \implies \text{no-T-F } \varphi \implies \text{no-T-F } \psi$

```

proof (induct rule: propo-rew-step.induct)
  case (global-rel  $\varphi' \ \psi'$ )
  then show no-T-F  $\psi'$ 
    using elim-imp.cases no-T-F-comp-not no-T-F-decomp(1,2)
    by (metis no-T-F-comp-expanded-explicit(2))
next
  case (propo-rew-one-step-lift  $\varphi \ \varphi' \ c \ \xi \ \xi'$ )
  note rel = this(1) and IH = this(2) and corr = this(3) and no-T-F = this(4)
  {
    assume  $c: c = CNot$ 
    then have empty:  $\xi = [] \ \xi' = []$  using corr by auto
    then have no-T-F  $\varphi$  using no-T-F c no-T-F-decomp-not by auto
    then have no-T-F (conn  $c$  ( $\xi @ \varphi' \# \xi'$ )) using c empty no-T-F-comp-not IH by auto
  }
  moreover {
    assume  $c: c \in \text{binary-connectives}$ 
    then obtain  $a \ b$  where  $ab: \xi @ \varphi \# \xi' = [a, b]$ 
      using corr list-length2-decomp wf-conn-bin-list-length by metis
    then have  $\varphi: \varphi = a \vee \varphi = b$ 
      by (metis append-self-conv2 wf-conn-list-decomp(4) wf-conn-unary list.discI list.sel(3)
          nth-Cons-0 tl-append2)
    have  $\zeta: \forall \zeta \in \text{set } (\xi @ \varphi \# \xi'). \text{no-T-F } \zeta$  using  $ab \ c$  propo-rew-one-step-lift.prems by auto

    then have  $\varphi': \text{no-T-F } \varphi'$ 
      using  $ab \ IH \ \varphi \ \text{corr} \ \text{no-T-F} \ \text{no-T-F-def all-subformula-st-decomp-explicit}$  by auto
    have  $\chi: \xi @ \varphi' \# \xi' = [\varphi', b] \vee \xi @ \varphi' \# \xi' = [a, \varphi']$ 
      by (metis (no-types, hide-lams) ab append-Cons append-Nil append-Nil2 butlast.simps(2)
          butlast-append list.distinct(1) list.sel(3))
    then have  $\forall \zeta \in \text{set } (\xi @ \varphi' \# \xi'). \text{no-T-F } \zeta$  using  $\zeta \ \varphi' \ ab$  by fastforce
    moreover
      have no-T-F (last ( $\xi @ \varphi' \# \xi'$ )) by (simp add: calculation)
      then have no-T-F-symb (conn  $c$  ( $\xi @ \varphi' \# \xi'$ ))
        by (metis  $\chi \ \varphi' \ \zeta \ ab \ \text{all-subformula-st-test-symb-true-phi} \ c \ \text{last.simps list.distinct}(1)
            list.set-intros(1) no-T-F-bin-decomp no-T-F-def)
    ultimately have no-T-F (conn  $c$  ( $\xi @ \varphi' \# \xi'$ )) using  $c \ \chi$  by fastforce
  }
  moreover {
    fix  $x$$ 
```

```

    assume  $c = CVar\ x \vee c = CF \vee c = CT$ 
    then have False using corr by auto
    then have no-T-F (conn  $c$  ( $\xi @ \varphi' \# \xi'$ )) by auto
  }
  ultimately show no-T-F (conn  $c$  ( $\xi @ \varphi' \# \xi'$ )) using corr wf-conn.cases by blast
qed

```

lemma *elim-imp-inv'*:

```

  fixes  $\varphi\ \psi :: 'v\ propo$ 
  assumes full (propo-rew-step elim-imp)  $\varphi\ \psi$  and no-T-F-except-top-level  $\varphi$ 
  shows no-T-F-except-top-level  $\psi$ 
proof -
  {
    {
      fix  $\varphi\ \psi :: 'v\ propo$ 
      have  $H: elim-imp\ \varphi\ \psi \implies no-T-F-except-top-level\ \varphi \implies no-T-F-except-top-level\ \psi$ 
        by (induct  $\varphi\ \psi$  rule: elim-imp.induct, auto)
    } note  $H = this$ 
    fix  $\varphi\ \psi :: 'v\ propo$ 
    have propo-rew-step elim-imp  $\varphi\ \psi \implies no-T-F-except-top-level\ \varphi \implies no-T-F-except-top-level\ \psi$ 
    proof -
      assume rel: propo-rew-step elim-imp  $\varphi\ \psi$ 
      and no: no-T-F-except-top-level  $\varphi$ 
      {
        assume  $\varphi = FT \vee \varphi = FF$ 
        from rel this have False
        apply (induct rule: propo-rew-step.induct)
        by (cases rule: elim-imp.cases, auto simp: wf-conn-list(1,2))
        then have no-T-F-except-top-level  $\psi$  by blast
      }
      moreover {
        assume  $\varphi \neq FT \wedge \varphi \neq FF$ 
        then have no-T-F  $\varphi$ 
        by (metis no no-T-F-symb-except-toplevel-all-subformula-st-no-T-F-symb)
        then have no-T-F  $\psi$ 
        using rel propo-rew-step-ElimImp-no-T-F by blast
        then have no-T-F-except-top-level  $\psi$  by (simp add: no-T-F-no-T-F-except-top-level)
      }
      ultimately show no-T-F-except-top-level  $\psi$  by metis
    qed
  }
  moreover {
    fix  $c :: 'v\ connective$  and  $\xi\ \xi' :: 'v\ propo\ list$  and  $\zeta\ \zeta' :: 'v\ propo$ 
    assume rel: propo-rew-step elim-imp  $\zeta\ \zeta'$ 
    and incl:  $\zeta \preceq \varphi$ 
    and corr: wf-conn  $c$  ( $\xi @ \zeta \# \xi'$ )
    and no-T-F: no-T-F-symb-except-toplevel (conn  $c$  ( $\xi @ \zeta \# \xi'$ ))
    and n: no-T-F-symb-except-toplevel  $\zeta'$ 
    have no-T-F-symb-except-toplevel (conn  $c$  ( $\xi @ \zeta' \# \xi'$ ))
    proof
      have p: no-T-F-symb (conn  $c$  ( $\xi @ \zeta \# \xi'$ ))
      by (simp add: corr no-T-F no-T-F-symb-except-toplevel-no-T-F-symb wf-conn-list(1,2))

      have l:  $\forall \varphi \in set\ (\xi @ \zeta \# \xi').\ \varphi \neq FT \wedge \varphi \neq FF$ 

```

```

    using corr wf-conn-no-T-F-symb-iff p by blast
  from rel incl have  $\zeta' \neq FT \wedge \zeta' \neq FF$ 
  apply (induction  $\zeta \ \zeta'$  rule: propo-rew-step.induct)
  apply (cases rule: elim-imp.cases, auto)
  using wf-conn-list(1,2) wf-conn-no-arity-change wf-conn-no-arity-change-helper
  by (metis append-is-Nil-conv list.distinct(1))+
  then have  $\forall \varphi \in \text{set } (\xi @ \zeta' \# \xi'). \varphi \neq FT \wedge \varphi \neq FF$  using l by auto
  moreover have  $c \neq CT \wedge c \neq CF$  using corr by auto
  ultimately show no-T-F-symb (conn c ( $\xi @ \zeta' \# \xi'$ ))
    using corr wf-conn-no-arity-change no-T-F-symb-comp
    by (metis wf-conn-no-arity-change-helper)
qed
}
ultimately show no-T-F-except-top-level  $\psi$ 
  using full-propo-rew-step-inv-stay-with-inc[of elim-imp no-T-F-symb-except-toplevel  $\varphi$ ]
  assms subformula-refl unfolding no-T-F-except-top-level-def by metis
qed

```

10.3 The new CNF and DNF transformation

The transformation is the same as before, but the order is not the same.

definition $\text{dnf-rew}' :: 'a \text{ propo} \Rightarrow 'a \text{ propo} \Rightarrow \text{bool}$ **where**

$\text{dnf-rew}' =$
 (full (propo-rew-step elimTBFULL)) OO
 (full (propo-rew-step elim-equiv)) OO
 (full (propo-rew-step elim-imp)) OO
 (full (propo-rew-step pushNeg)) OO
 (full (propo-rew-step pushConj))

lemma $\text{dnf-rew}'\text{-consistent}$: preserves-un-sat $\text{dnf-rew}'$

by (simp add: $\text{dnf-rew}'\text{-def}$ elimEquiv-lifted-consistant elim-imp-lifted-consistant
 elimTBFULL-consistent preserves-un-sat-OO pushConj-consistent pushNeg-lifted-consistant)

theorem $\text{cnf-transformation-correction}$:

$\text{dnf-rew}' \varphi \varphi' \Longrightarrow \text{is-dnf } \varphi'$

unfolding $\text{dnf-rew}'\text{-def}$ OO-def

by (meson and-in-or-only-conjunction-in-disj elimTBFULL-full-propo-rew-step elim-equiv-inv'
 elim-imp-inv elim-imp-inv' is-dnf-def no-equiv-full-propo-rew-step-elim-equiv
 no-imp-full-propo-rew-step-elim-imp pushConj-full-propo-rew-step pushConj-inv(1-4)
 pushNeg-full-propo-rew-step pushNeg-inv(1-3))

Given all the lemmas before the CNF transformation is easy to prove:

definition $\text{cnf-rew}' :: 'a \text{ propo} \Rightarrow 'a \text{ propo} \Rightarrow \text{bool}$ **where**

$\text{cnf-rew}' =$
 (full (propo-rew-step elimTBFULL)) OO
 (full (propo-rew-step elim-equiv)) OO
 (full (propo-rew-step elim-imp)) OO
 (full (propo-rew-step pushNeg)) OO
 (full (propo-rew-step pushDisj))

lemma $\text{cnf-rew}'\text{-consistent}$: preserves-un-sat $\text{cnf-rew}'$

by (simp add: $\text{cnf-rew}'\text{-def}$ elimEquiv-lifted-consistant elim-imp-lifted-consistant
 elimTBFULL-consistent preserves-un-sat-OO pushDisj-consistent pushNeg-lifted-consistant)

theorem $\text{cnf}'\text{-transformation-correction}$:

```

cnf-rew'  $\varphi$   $\varphi'$   $\implies$  is-cnff  $\varphi'$ 
unfolding cnf-rew'-def OO-def
by (meson elimTBFull-full-propo-rew-step elim-equiv-inv' elim-imp-inv elim-imp-inv' is-cnff-def
  no-equiv-full-propo-rew-step-elim-equiv no-imp-full-propo-rew-step-elim-imp
  or-in-and-only-conjunction-in-disj pushDisj-full-propo-rew-step pushDisj-inv(1-4)
  pushNeg-full-propo-rew-step pushNeg-inv(1) pushNeg-inv(2) pushNeg-inv(3))

```

end

11 Partial Clausal Logic

```

theory Partial-Clausal-Logic
imports ../lib/Clausal-Logic List-More
begin

```

11.1 Clauses

Clauses are (finite) multisets of literals.

```

type-synonym 'a clause = 'a literal multiset
type-synonym 'v clauses = 'v clause set

```

11.2 Partial Interpretations

```

type-synonym 'a interp = 'a literal set

```

```

definition true-lit :: 'a interp  $\Rightarrow$  'a literal  $\Rightarrow$  bool (infix  $\models_l$  50) where
   $I \models_l L \longleftrightarrow L \in I$ 

```

```

declare true-lit-def[simp]

```

11.2.1 Consistency

```

definition consistent-interp :: 'a literal set  $\Rightarrow$  bool where
  consistent-interp  $I = (\forall L. \neg(L \in I \wedge \neg L \in I))$ 

```

```

lemma consistent-interp-empty[simp]:
  consistent-interp {} unfolding consistent-interp-def by auto

```

```

lemma consistent-interp-single[simp]:
  consistent-interp {L} unfolding consistent-interp-def by auto

```

```

lemma consistent-interp-subset:
  assumes
     $A \subseteq B$  and
    consistent-interp B
  shows consistent-interp A
  using assms unfolding consistent-interp-def by auto

```

```

lemma consistent-interp-change-insert:
   $a \notin A \implies \neg a \notin A \implies \text{consistent-interp} (\text{insert } (\neg a) A) \longleftrightarrow \text{consistent-interp} (\text{insert } a A)$ 
  unfolding consistent-interp-def by fastforce

```

```

lemma consistent-interp-insert-pos[simp]:
   $a \notin A \implies \text{consistent-interp} (\text{insert } a A) \longleftrightarrow \text{consistent-interp } A \wedge \neg a \notin A$ 
  unfolding consistent-interp-def by auto

```

lemma *consistent-interp-insert-not-in*:

consistent-interp $A \implies a \notin A \implies -a \notin A \implies \text{consistent-interp } (\text{insert } a \ A)$

unfolding *consistent-interp-def* **by** *auto*

11.2.2 Atoms

definition *atms-of-ms* :: 'a literal multiset set \Rightarrow 'a set **where**

atms-of-ms $\psi s = \bigcup (\text{atms-of } ' \psi s)$

lemma *atms-of-mmltiset[simp]*:

atms-of (*mset* a) = *atm-of* ' set a

by (*induct* a) *auto*

lemma *atms-of-ms-mset-unfold*:

atms-of-ms (*mset* ' b) = $(\bigcup x \in b. \text{atm-of } ' \text{ set } x)$

unfolding *atms-of-ms-def* **by** *simp*

definition *atms-of-s* :: 'a literal set \Rightarrow 'a set **where**

atms-of-s $C = \text{atm-of } ' C$

lemma *atms-of-ms-empty-set[simp]*:

atms-of-ms $\{\} = \{\}$

unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-mempty[simp]*:

atms-of-ms $\{\{\#\}\} = \{\}$

unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-mono*:

$A \subseteq B \implies \text{atms-of-ms } A \subseteq \text{atms-of-ms } B$

unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-finite[simp]*:

finite $\psi s \implies \text{finite } (\text{atms-of-ms } \psi s)$

unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-union[simp]*:

atms-of-ms ($\psi s \cup \chi s$) = *atms-of-ms* $\psi s \cup \text{atms-of-ms } \chi s$

unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-insert[simp]*:

atms-of-ms (*insert* ψs χs) = *atms-of* $\psi s \cup \text{atms-of-ms } \chi s$

unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-singleton[simp]*: *atms-of-ms* $\{L\} = \text{atms-of } L$

unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-atms-of-ms-mono[simp]*:

$A \in \psi \implies \text{atms-of } A \subseteq \text{atms-of-ms } \psi$

unfolding *atms-of-ms-def* **by** *fastforce*

lemma *atms-of-ms-single-set-mset-atms-of[simp]*:

atms-of-ms (*single* ' set-mset B) = *atms-of* B

unfolding *atms-of-ms-def* *atms-of-def* **by** *auto*

lemma *atms-of-ms-remove-incl*:
shows *atms-of-ms* (*Set.remove a ψ*) \subseteq *atms-of-ms ψ*
unfolding *atms-of-ms-def* **by** *auto*

lemma *atms-of-ms-remove-subset*:
atms-of-ms ($\varphi - \psi$) \subseteq *atms-of-ms* φ
unfolding *atms-of-ms-def* **by** *auto*

lemma *finite-atms-of-ms-remove-subset[simp]*:
finite (*atms-of-ms A*) \implies *finite* (*atms-of-ms* ($A - C$))
using *atms-of-ms-remove-subset[of A C]* *finite-subset* **by** *blast*

lemma *atms-of-ms-empty-iff*:
atms-of-ms A = {} \longleftrightarrow $A = \{\{\#\}\} \vee A = \{\}$
apply (*rule iffI*)
apply (*metis* (*no-types*, *lifting*) *atms-empty-iff-empty* *atms-of-atms-of-ms-mono* *insert-absorb*
singleton-iff *singleton-insert-inj-eq'* *subsetI* *subset-empty*)
apply *auto*[]
done

lemma *in-implies-atm-of-on-atms-of-ms*:
assumes $L \in \# C$ **and** $C \in N$
shows *atm-of L* \in *atms-of-ms N*
using *atms-of-atms-of-ms-mono[of C N]* *assms* **by** (*simp add: atm-of-lit-in-atms-of subset-iff*)

lemma *in-plus-implies-atm-of-on-atms-of-ms*:
assumes $C + \{\#L\# \} \in N$
shows *atm-of L* \in *atms-of-ms N*
using *in-implies-atm-of-on-atms-of-ms[of - C + {\#L\#}]* *assms* **by** *auto*

lemma *in-m-in-literals*:
assumes $\{\#A\# \} + D \in \psi s$
shows *atm-of A* \in *atms-of-ms ψ s*
using *assms* **by** (*auto dest: atms-of-atms-of-ms-mono*)

lemma *atms-of-s-union[simp]*:
atms-of-s ($Ia \cup Ib$) = *atms-of-s Ia* \cup *atms-of-s Ib*
unfolding *atms-of-s-def* **by** *auto*

lemma *atms-of-s-single[simp]*:
atms-of-s {*L*} = {*atm-of L*}
unfolding *atms-of-s-def* **by** *auto*

lemma *atms-of-s-insert[simp]*:
atms-of-s (*insert L Ib*) = {*atm-of L*} \cup *atms-of-s Ib*
unfolding *atms-of-s-def* **by** *auto*

lemma *in-atms-of-s-decomp[iff]*:
 $P \in \text{atms-of-s } I \longleftrightarrow (Pos\ P \in I \vee Neg\ P \in I)$ (**is** $?P \longleftrightarrow ?Q$)
proof
assume $?P$
then show $?Q$ **unfolding** *atms-of-s-def* **by** (*metis image-iff literal.exhaust-sel*)
next
assume $?Q$
then show $?P$ **unfolding** *atms-of-s-def* **by** *force*

qed

lemma *atm-of-in-atm-of-set-in-uminus*:
 atm-of $L' \in \text{atm-of } 'B \implies L' \in B \vee - L' \in B$
 using *atms-of-s-def* **by** (*cases* L') *fastforce+*

11.2.3 Totality

definition *total-over-set* :: $'a \text{ interp} \Rightarrow 'a \text{ set} \Rightarrow \text{bool}$ **where**
total-over-set $I \ S = (\forall l \in S. \text{Pos } l \in I \vee \text{Neg } l \in I)$

definition *total-over-m* :: $'a \text{ literal set} \Rightarrow 'a \text{ clause set} \Rightarrow \text{bool}$ **where**
total-over-m $I \ \psi s = \text{total-over-set } I \ (\text{atms-of-ms } \psi s)$

lemma *total-over-set-empty[simp]*:
 total-over-set $I \ \{\}$
 unfolding *total-over-set-def* **by** *auto*

lemma *total-over-m-empty[simp]*:
 total-over-m $I \ \{\}$
 unfolding *total-over-m-def* **by** *auto*

lemma *total-over-set-single[iff]*:
 total-over-set $I \ \{L\} \longleftrightarrow (\text{Pos } L \in I \vee \text{Neg } L \in I)$
 unfolding *total-over-set-def* **by** *auto*

lemma *total-over-set-insert[iff]*:
 total-over-set $I \ (\text{insert } L \ Ls) \longleftrightarrow ((\text{Pos } L \in I \vee \text{Neg } L \in I) \wedge \text{total-over-set } I \ Ls)$
 unfolding *total-over-set-def* **by** *auto*

lemma *total-over-set-union[iff]*:
 total-over-set $I \ (Ls \cup Ls') \longleftrightarrow (\text{total-over-set } I \ Ls \wedge \text{total-over-set } I \ Ls')$
 unfolding *total-over-set-def* **by** *auto*

lemma *total-over-m-subset*:
 $A \subseteq B \implies \text{total-over-m } I \ B \implies \text{total-over-m } I \ A$
 using *atms-of-ms-mono[of A]* **unfolding** *total-over-m-def* *total-over-set-def* **by** *auto*

lemma *total-over-m-sum[iff]*:
 shows *total-over-m* $I \ \{C + D\} \longleftrightarrow (\text{total-over-m } I \ \{C\} \wedge \text{total-over-m } I \ \{D\})$
 using *assms* **unfolding** *total-over-m-def* *total-over-set-def* **by** *auto*

lemma *total-over-m-union[iff]*:
 total-over-m $I \ (A \cup B) \longleftrightarrow (\text{total-over-m } I \ A \wedge \text{total-over-m } I \ B)$
 unfolding *total-over-m-def* *total-over-set-def* **by** *auto*

lemma *total-over-m-insert[iff]*:
 total-over-m $I \ (\text{insert } a \ A) \longleftrightarrow (\text{total-over-set } I \ (\text{atms-of } a) \wedge \text{total-over-m } I \ A)$
 unfolding *total-over-m-def* *total-over-set-def* **by** *fastforce*

lemma *total-over-m-extension*:
 fixes $I :: 'v \text{ literal set}$ **and** $A :: 'v \text{ clauses}$
 assumes *total*: *total-over-m* $I \ A$
 shows $\exists I'. \text{total-over-m } (I \cup I') \ (A \cup B)$
 $\wedge (\forall x \in I'. \text{atm-of } x \in \text{atms-of-ms } B \wedge \text{atm-of } x \notin \text{atms-of-ms } A)$
proof —

```

let ?I' = {Pos v | v. v ∈ atms-of-ms B ∧ v ∉ atms-of-ms A}
have (∀ x ∈ ?I'. atm-of x ∈ atms-of-ms B ∧ atm-of x ∉ atms-of-ms A) by auto
moreover have total-over-m (I ∪ ?I') (A ∪ B)
  using total unfolding total-over-m-def total-over-set-def by auto
ultimately show ?thesis by blast
qed

lemma total-over-m-consistent-extension:
  fixes I :: 'v literal set and A :: 'v clauses
  assumes total: total-over-m I A
  and cons: consistent-interp I
  shows ∃ I'. total-over-m (I ∪ I') (A ∪ B)
    ∧ (∀ x ∈ I'. atm-of x ∈ atms-of-ms B ∧ atm-of x ∉ atms-of-ms A) ∧ consistent-interp (I ∪ I')
proof -
  let ?I' = {Pos v | v. v ∈ atms-of-ms B ∧ v ∉ atms-of-ms A ∧ Pos v ∉ I ∧ Neg v ∉ I}
  have (∀ x ∈ ?I'. atm-of x ∈ atms-of-ms B ∧ atm-of x ∉ atms-of-ms A) by auto
  moreover have total-over-m (I ∪ ?I') (A ∪ B)
    using total unfolding total-over-m-def total-over-set-def by auto
  moreover have consistent-interp (I ∪ ?I')
    using cons unfolding consistent-interp-def by (intro allI) (rename-tac L, case-tac L, auto)
  ultimately show ?thesis by blast
qed

lemma total-over-set-atms-of-m[simp]:
  total-over-set Ia (atms-of-s Ia)
  unfolding total-over-set-def atms-of-s-def by (metis image-iff literal.exhaust-sel)

lemma total-over-set-literal-defined:
  assumes {#A#} + D ∈ ψs
  and total-over-set I (atms-of-ms ψs)
  shows A ∈ I ∨ ¬A ∈ I
  using assms unfolding total-over-set-def by (metis (no-types) Neg-atm-of-iff in-m-in-literals
    literal.collapse(1) uminus-Neg uminus-Pos)

lemma tot-over-m-remove:
  assumes total-over-m (I ∪ {L}) {ψ}
  and L: ¬ L ∈ # ψ ¬ L ∉ # ψ
  shows total-over-m I {ψ}
  unfolding total-over-m-def total-over-set-def
proof
  fix l
  assume l: l ∈ atms-of-ms {ψ}
  then have Pos l ∈ I ∨ Neg l ∈ I ∨ l = atm-of L
    using assms unfolding total-over-m-def total-over-set-def by auto
  moreover have atm-of L ∉ atms-of-ms {ψ}
  proof (rule ccontr)
    assume ¬ ?thesis
    then have atm-of L ∈ atms-of ψ by auto
    then have Pos (atm-of L) ∈ # ψ ∨ Neg (atm-of L) ∈ # ψ
      using atm-imp-pos-or-neg-lit by metis
    then have L ∈ # ψ ∨ ¬ L ∈ # ψ by (cases L) auto
    then show False using L by auto
  qed
qed
ultimately show Pos l ∈ I ∨ Neg l ∈ I using l by metis
qed

```


lemma *total-union*:
assumes *total-over-m* I ψ
shows *total-over-m* $(I \cup I')$ ψ
using *assms* **unfolding** *total-over-m-def* *total-over-set-def* **by** *auto*

lemma *total-union-2*:
assumes *total-over-m* I ψ
and *total-over-m* I' ψ'
shows *total-over-m* $(I \cup I')$ $(\psi \cup \psi')$
using *assms* **unfolding** *total-over-m-def* *total-over-set-def* **by** *auto*

11.2.4 Interpretations

definition *true-cls* :: 'a *interp* \Rightarrow 'a *clause* \Rightarrow bool (**infix** \models 50) **where**
 $I \models C \longleftrightarrow (\exists L \in \# C. I \models_l L)$

lemma *true-cls-empty*[*iff*]: $\neg I \models \{\#\}$
unfolding *true-cls-def* **by** *auto*

lemma *true-cls-singleton*[*iff*]: $I \models \{\#L\# \} \longleftrightarrow I \models_l L$
unfolding *true-cls-def* **by** (*auto* *split-if-split-asm*)

lemma *true-cls-union*[*iff*]: $I \models C + D \longleftrightarrow I \models C \vee I \models D$
unfolding *true-cls-def* **by** *auto*

lemma *true-cls-mono-set-mset*: $\text{set-mset } C \subseteq \text{set-mset } D \Longrightarrow I \models C \Longrightarrow I \models D$
unfolding *true-cls-def* *subset-eq* *Bex-def* **by** *metis*

lemma *true-cls-mono-leD*[*dest*]: $A \subseteq \# B \Longrightarrow I \models A \Longrightarrow I \models B$
unfolding *true-cls-def* **by** *auto*

lemma
assumes $I \models \psi$
shows *true-cls-union-increase*[*simp*]: $I \cup I' \models \psi$
and *true-cls-union-increase'*[*simp*]: $I' \cup I \models \psi$
using *assms* **unfolding** *true-cls-def* **by** *auto*

lemma *true-cls-mono-set-mset-l*:
assumes $A \models \psi$
and $A \subseteq B$
shows $B \models \psi$
using *assms* **unfolding** *true-cls-def* **by** *auto*

lemma *true-cls-replicate-mset*[*iff*]: $I \models \text{replicate-mset } n L \longleftrightarrow n \neq 0 \wedge I \models_l L$
by (*induct* n) *auto*

lemma *true-cls-empty-entails*[*iff*]: $\neg \{\} \models N$
by (*auto* *simp* *add: true-cls-def*)

lemma *true-cls-not-in-remove*:
assumes $L \notin \# \chi$
and $I \cup \{L\} \models \chi$
shows $I \models \chi$
using *assms* **unfolding** *true-cls-def* **by** *auto*

definition *true-clss* :: 'a interp \Rightarrow 'a clauses \Rightarrow bool (**infix** \models_s 50) **where**
 $I \models_s CC \longleftrightarrow (\forall C \in CC. I \models C)$

lemma *true-clss-empty[simp]*: $I \models_s \{\}$
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-singleton[iff]*: $I \models_s \{C\} \longleftrightarrow I \models C$
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-empty-entails-empty[iff]*: $\{\} \models_s N \longleftrightarrow N = \{\}$
unfolding *true-clss-def* **by** (*auto simp add: true-clss-def*)

lemma *true-clss-insert-l [simp]*:
 $M \models A \implies \text{insert } L \ M \models A$
unfolding *true-clss-def* **by** *auto*

lemma *true-clss-union[iff]*: $I \models_s CC \cup DD \longleftrightarrow I \models_s CC \wedge I \models_s DD$
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-insert[iff]*: $I \models_s \text{insert } C \ DD \longleftrightarrow I \models C \wedge I \models_s DD$
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-mono*: $DD \subseteq CC \implies I \models_s CC \implies I \models_s DD$
unfolding *true-clss-def* **by** *blast*

lemma *true-clss-union-increase[simp]*:
assumes $I \models_s \psi$
shows $I \cup I' \models_s \psi$
using *assms* **unfolding** *true-clss-def* **by** *auto*

lemma *true-clss-union-increase'[simp]*:
assumes $I' \models_s \psi$
shows $I \cup I' \models_s \psi$
using *assms* **by** (*auto simp add: true-clss-def*)

lemma *true-clss-commute-l*:
 $(I \cup I' \models_s \psi) \longleftrightarrow (I' \cup I \models_s \psi)$
by (*simp add: Un-commute*)

lemma *model-remove[simp]*: $I \models_s N \implies I \models_s \text{Set.remove } a \ N$
by (*simp add: true-clss-def*)

lemma *model-remove-minus[simp]*: $I \models_s N \implies I \models_s N - A$
by (*simp add: true-clss-def*)

lemma *notin-vars-union-true-clss-true-clss*:
assumes $\forall x \in I'. \text{atm-of } x \notin \text{atms-of-ms } A$
and $\text{atms-of } L \subseteq \text{atms-of-ms } A$
and $I \cup I' \models L$
shows $I \models L$
using *assms* **unfolding** *true-clss-def true-lit-def Bex-def*
by (*metis Un-iff atm-of-lit-in-atms-of contra-subsetD*)

lemma *notin-vars-union-true-clss-true-clss*:
assumes $\forall x \in I'. \text{atm-of } x \notin \text{atms-of-ms } A$

and $\text{atms-of-ms } L \subseteq \text{atms-of-ms } A$
and $I \cup I' \models_s L$
shows $I \models_s L$
using *assms unfolding true-clss-def true-lit-def Ball-def*
by (*meson atms-of-atms-of-ms-mono notin-vars-union-true-clss-true-clss subset-trans*)

11.2.5 Satisfiability

definition *satisfiable* :: 'a clause set \Rightarrow bool **where**
satisfiable $CC \equiv \exists I. (I \models_s CC \wedge \text{consistent-interp } I \wedge \text{total-over-m } I CC)$

lemma *satisfiable-single[simp]*:
satisfiable $\{\{\#L\#\}\}$
unfolding *satisfiable-def* **by** *fastforce*

abbreviation *unsatisfiable* :: 'a clause set \Rightarrow bool **where**
unsatisfiable $CC \equiv \neg \text{satisfiable } CC$

lemma *satisfiable-decreasing*:
assumes *satisfiable* $(\psi \cup \psi')$
shows *satisfiable* ψ
using *assms total-over-m-union unfolding satisfiable-def* **by** *blast*

lemma *satisfiable-def-min*:
satisfiable CC
 $\longleftrightarrow (\exists I. I \models_s CC \wedge \text{consistent-interp } I \wedge \text{total-over-m } I CC \wedge \text{atm-of } I = \text{atms-of-ms } CC)$
(is ?sat \longleftrightarrow ?B)

proof

assume ?B **then show** ?sat **by** (*auto simp add: satisfiable-def*)

next

assume ?sat

then obtain I **where**

$I \models_s CC$ **and**

cons: *consistent-interp* I **and**

tot: *total-over-m* $I CC$

unfolding *satisfiable-def* **by** *auto*

let $?I = \{P. P \in I \wedge \text{atm-of } P \in \text{atms-of-ms } CC\}$

have $I \models_s CC$

using $I \models_s CC$ *in-implies-atm-of-on-atms-of-ms* **unfolding** *true-clss-def Ball-def true-clss-def Bex-def true-lit-def*

by *blast*

moreover have *cons*: *consistent-interp* $?I$

using *cons* **unfolding** *consistent-interp-def* **by** *auto*

moreover have *total-over-m* $?I CC$

using *tot* **unfolding** *total-over-m-def total-over-set-def* **by** *auto*

moreover

have *atms-CC-incl*: $\text{atms-of-ms } CC \subseteq \text{atm-of } I$

using *tot* **unfolding** *total-over-m-def total-over-set-def atms-of-ms-def*
by (*auto simp add: atms-of-def atms-of-s-def[symmetric]*)

have $\text{atm-of } ?I = \text{atms-of-ms } CC$

using *atms-CC-incl* **unfolding** *atms-of-ms-def* **by** *force*

ultimately show ?B **by** *auto*

qed

11.2.6 Entailment for Multisets of Clauses

definition *true-cls-mset* :: 'a interp \Rightarrow 'a clause multiset \Rightarrow bool (infix \models_m 50) **where**
 $I \models_m CC \longleftrightarrow (\forall C \in \# CC. I \models C)$

lemma *true-cls-mset-empty[simp]*: $I \models_m \{\#\}$
unfolding *true-cls-mset-def* **by** *auto*

lemma *true-cls-mset-singleton[iff]*: $I \models_m \{\# C \#\} \longleftrightarrow I \models C$
unfolding *true-cls-mset-def* **by** (*auto split: if-split-asm*)

lemma *true-cls-mset-union[iff]*: $I \models_m CC + DD \longleftrightarrow I \models_m CC \wedge I \models_m DD$
unfolding *true-cls-mset-def* **by** *fastforce*

lemma *true-cls-mset-image-mset[iff]*: $I \models_m \text{image-mset } f A \longleftrightarrow (\forall x \in \# A. I \models f x)$
unfolding *true-cls-mset-def* **by** *fastforce*

lemma *true-cls-mset-mono*: $\text{set-mset } DD \subseteq \text{set-mset } CC \Longrightarrow I \models_m CC \Longrightarrow I \models_m DD$
unfolding *true-cls-mset-def subset-iff* **by** *auto*

lemma *true-clss-set-mset[iff]*: $I \models_s \text{set-mset } CC \longleftrightarrow I \models_m CC$
unfolding *true-clss-def true-cls-mset-def* **by** *auto*

lemma *true-cls-mset-increasing-r[simp]*:
 $I \models_m CC \Longrightarrow I \cup J \models_m CC$
unfolding *true-cls-mset-def* **by** *auto*

theorem *true-cls-remove-unused*:
assumes $I \models \psi$
shows $\{v \in I. \text{atm-of } v \in \text{atms-of } \psi\} \models \psi$
using *assms unfolding true-cls-def atms-of-def* **by** *auto*

theorem *true-clss-remove-unused*:
assumes $I \models_s \psi$
shows $\{v \in I. \text{atm-of } v \in \text{atms-of-ms } \psi\} \models_s \psi$
unfolding *true-clss-def atms-of-def Ball-def*
proof (*intro allI impI*)
fix x
assume $x \in \psi$
then have $I \models x$
using *assms unfolding true-clss-def atms-of-def Ball-def* **by** *auto*

then have $\{v \in I. \text{atm-of } v \in \text{atms-of } x\} \models x$
by (*simp only: true-cls-remove-unused[of I]*)
moreover have $\{v \in I. \text{atm-of } v \in \text{atms-of } x\} \subseteq \{v \in I. \text{atm-of } v \in \text{atms-of-ms } \psi\}$
using $\langle x \in \psi \rangle$ **by** (*auto simp add: atms-of-ms-def*)
ultimately show $\{v \in I. \text{atm-of } v \in \text{atms-of-ms } \psi\} \models x$
using *true-cls-mono-set-mset-l* **by** *blast*
qed

A simple application of the previous theorem:

lemma *true-clss-union-decrease*:
assumes $II': I \cup I' \models \psi$
and $H: \forall v \in I'. \text{atm-of } v \notin \text{atms-of } \psi$
shows $I \models \psi$
proof –

let $?I = \{v \in I \cup I'. \text{ atm-of } v \in \text{atms-of } \psi\}$
 have $?I \models \psi$ **using** *true-cls-remove-unused II'* **by** *blast*
 moreover have $?I \subseteq I$ **using** *H* **by** *auto*
 ultimately show *?thesis* **using** *true-cls-mono-set-mset-l* **by** *blast*
 qed

lemma *multiset-not-empty*:
 assumes $M \neq \{\#\}$
 and $x \in\# M$
 shows $\exists A. x = \text{Pos } A \vee x = \text{Neg } A$
using *assms literal.exhaust-sel* **by** *blast*

lemma *atms-of-ms-empty*:
 fixes $\psi :: 'v \text{ clauses}$
 assumes $\text{atms-of-ms } \psi = \{\}$
 shows $\psi = \{\} \vee \psi = \{\{\#\}\}$
using *assms* **by** (*auto simp add: atms-of-ms-def*)

lemma *consistent-interp-disjoint*:
 assumes *consI*: *consistent-interp I*
 and *disj*: $\text{atms-of-s } A \cap \text{atms-of-s } I = \{\}$
 and *consA*: *consistent-interp A*
 shows *consistent-interp* $(A \cup I)$
proof (*rule ccontr*)
 assume $\neg ?thesis$
 moreover have $\bigwedge L. \neg (L \in A \wedge \neg L \in I)$
using *disj unfolding atms-of-s-def* **by** (*auto simp add: rev-image-eqI*)
 ultimately show *False*
using *consA consI unfolding consistent-interp-def* **by** (*metis (full-types) Un-iff*
literal.exhaust-sel uminus-Neg uminus-Pos)
 qed

lemma *total-remove-unused*:
 assumes *total-over-m I* ψ
 shows *total-over-m* $\{v \in I. \text{ atm-of } v \in \text{atms-of-ms } \psi\} \psi$
using *assms unfolding total-over-m-def total-over-set-def*
by (*metis (lifting) literal.sel(1,2) mem-Collect-eq*)

lemma *true-cls-remove-hd-if-notin-vars*:
 assumes $\text{insert } a \ M' \models D$
 and $\text{atm-of } a \notin \text{atms-of } D$
 shows $M' \models D$
using *assms* **by** (*auto simp add: atm-of-lit-in-atms-of true-cls-def*)

lemma *total-over-set-atm-of*:
 fixes $I :: 'v \text{ interp}$ and $K :: 'v \text{ set}$
 shows *total-over-set* $I \ K \longleftrightarrow (\forall l \in K. l \in (\text{atm-of } I))$
unfolding *total-over-set-def* **by** (*metis atms-of-s-def in-atms-of-s-decomp*)

11.2.7 Tautologies

definition *tautology* $(\psi :: 'v \text{ clause}) \equiv \forall I. \text{total-over-set } I (\text{atms-of } \psi) \longrightarrow I \models \psi$

lemma *tautology-Pos-Neg[intro]*:
 assumes $\text{Pos } p \in\# A$ and $\text{Neg } p \in\# A$
 shows *tautology A*

using *assms* **unfolding** *tautology-def total-over-set-def true-cls-def Bex-def*
by (*meson atm-iff-pos-or-neg-lit true-lit-def*)

lemma *tautology-minus[simp]*:
assumes $L \in\# A$ **and** $\neg L \in\# A$
shows *tautology* A
by (*metis assms literal.exhaust tautology-Pos-Neg uminus-Neg uminus-Pos*)

lemma *tautology-exists-Pos-Neg*:
assumes *tautology* ψ
shows $\exists p. \text{Pos } p \in\# \psi \wedge \text{Neg } p \in\# \psi$
proof (*rule ccontr*)
assume $p: \neg (\exists p. \text{Pos } p \in\# \psi \wedge \text{Neg } p \in\# \psi)$
let $?I = \{-L \mid L. L \in\# \psi\}$
have *total-over-set* $?I$ (*atms-of* ψ)
unfolding *total-over-set-def* **using** *atm-imp-pos-or-neg-lit* **by** *force*
moreover **have** $\neg ?I \models \psi$
unfolding *true-cls-def true-lit-def Bex-def* **apply** *clarify*
using p **by** (*rename-tac x L, case-tac L*) *fastforce+*
ultimately show *False* **using** *assms* **unfolding** *tautology-def* **by** *auto*
qed

lemma *tautology-decomp*:
tautology $\psi \longleftrightarrow (\exists p. \text{Pos } p \in\# \psi \wedge \text{Neg } p \in\# \psi)$
using *tautology-exists-Pos-Neg* **by** *auto*

lemma *tautology-false[simp]*: $\neg \text{tautology } \{\#\}$
unfolding *tautology-def* **by** *auto*

lemma *tautology-add-single*:
tautology $(\{\#a\# \} + L) \longleftrightarrow \text{tautology } L \vee \neg a \in\# L$
unfolding *tautology-decomp* **by** (*cases a*) *auto*

lemma *minus-interp-tautology*:
assumes $\{-L \mid L. L \in\# \chi\} \models \chi$
shows *tautology* χ
proof $-$
obtain L **where** $L \in\# \chi \wedge \neg L \in\# \chi$
using *assms* **unfolding** *true-cls-def* **by** *auto*
then show *?thesis* **using** *tautology-decomp literal.exhaust uminus-Neg uminus-Pos* **by** *metis*
qed

lemma *remove-literal-in-model-tautology*:
assumes $I \cup \{\text{Pos } P\} \models \varphi$
and $I \cup \{\text{Neg } P\} \models \varphi$
shows $I \models \varphi \vee \text{tautology } \varphi$
using *assms* **unfolding** *true-cls-def* **by** *auto*

lemma *tautology-imp-tautology*:
fixes $\chi \chi' :: 'v \text{ clause}$
assumes $\forall I. \text{total-over-m } I \ \{\chi\} \longrightarrow I \models \chi \longrightarrow I \models \chi'$ **and** *tautology* χ
shows *tautology* χ' **unfolding** *tautology-def*
proof (*intro allI HOL.impI*)
fix $I :: 'v \text{ literal set}$
assume *totI*: *total-over-set* I (*atms-of* χ')

let $?I' = \{Pos\ v \mid v. v \in \text{atms-of } \chi \wedge v \notin \text{atms-of-} s\ I\}$
 have $totI'$: *total-over-m* $(I \cup ?I')\ \{\chi\}$ **unfolding** *total-over-m-def total-over-set-def* **by** *auto*
 then have χ : $I \cup ?I' \models \chi$ **using** *assms(2)* **unfolding** *total-over-m-def tautology-def* **by** *simp*
 then have $I \cup (?I' - I) \models \chi'$ **using** *assms(1)* $totI'$ **by** *auto*
 moreover have $\bigwedge L. L \in \# \chi' \implies L \notin ?I'$
 using $totI$ **unfolding** *total-over-set-def* **by** (*auto dest: pos-lit-in-atms-of*)
 ultimately show $I \models \chi'$ **unfolding** *true-cls-def* **by** *auto*
qed

11.2.8 Entailment for clauses and propositions

definition *true-cls-cls* :: *'a clause* \Rightarrow *'a clause* \Rightarrow *bool* (**infix** \models_f 49) **where**
 $\psi \models_f \chi \iff (\forall I. \text{total-over-m } I\ (\{\psi\} \cup \{\chi\}) \longrightarrow \text{consistent-interp } I \longrightarrow I \models \psi \longrightarrow I \models \chi)$

definition *true-cls-clss* :: *'a clause* \Rightarrow *'a clauses* \Rightarrow *bool* (**infix** \models_{fs} 49) **where**
 $\psi \models_{fs} \chi \iff (\forall I. \text{total-over-m } I\ (\{\psi\} \cup \chi) \longrightarrow \text{consistent-interp } I \longrightarrow I \models \psi \longrightarrow I \models_s \chi)$

definition *true-clss-cls* :: *'a clauses* \Rightarrow *'a clause* \Rightarrow *bool* (**infix** \models_p 49) **where**
 $N \models_p \chi \iff (\forall I. \text{total-over-m } I\ (N \cup \{\chi\}) \longrightarrow \text{consistent-interp } I \longrightarrow I \models_s N \longrightarrow I \models \chi)$

definition *true-clss-clss* :: *'a clauses* \Rightarrow *'a clauses* \Rightarrow *bool* (**infix** \models_{ps} 49) **where**
 $N \models_{ps} N' \iff (\forall I. \text{total-over-m } I\ (N \cup N') \longrightarrow \text{consistent-interp } I \longrightarrow I \models_s N \longrightarrow I \models_s N')$

lemma *true-cls-cls-refl[simp]*:
 $A \models_f A$
unfolding *true-cls-cls-def* **by** *auto*

lemma *true-cls-cls-insert-l[simp]*:
 $a \models_f C \implies \text{insert } a\ A \models_p C$
unfolding *true-cls-cls-def true-clss-cls-def true-clss-def* **by** *fastforce*

lemma *true-cls-clss-empty[iff]*:
 $N \models_{fs} \{\}$
unfolding *true-cls-clss-def* **by** *auto*

lemma *true-prop-true-clause[iff]*:
 $\{\varphi\} \models_p \psi \iff \varphi \models_f \psi$
unfolding *true-cls-cls-def true-clss-cls-def* **by** *auto*

lemma *true-clss-clss-true-clss-cls[iff]*:
 $N \models_{ps} \{\psi\} \iff N \models_p \psi$
unfolding *true-clss-clss-def true-clss-cls-def* **by** *auto*

lemma *true-clss-clss-true-cls-clss[iff]*:
 $\{\chi\} \models_{ps} \psi \iff \chi \models_{fs} \psi$
unfolding *true-clss-clss-def true-cls-clss-def* **by** *auto*

lemma *true-clss-clss-empty[simp]*:
 $N \models_{ps} \{\}$
unfolding *true-clss-clss-def* **by** *auto*

lemma *true-clss-cls-subset*:
 $A \subseteq B \implies A \models_p CC \implies B \models_p CC$
unfolding *true-clss-cls-def total-over-m-union* **by** (*simp add: total-over-m-subset true-clss-mono*)

lemma *true-clss-clss-mono-l[simp]*:

$A \models_p CC \implies A \cup B \models_p CC$
by (*auto intro: true-clss-clss-subset*)

lemma *true-clss-clss-mono-l2[simp]*:
 $B \models_p CC \implies A \cup B \models_p CC$
by (*auto intro: true-clss-clss-subset*)

lemma *true-clss-clss-mono-r[simp]*:
 $A \models_p CC \implies A \models_p CC + CC'$
unfolding *true-clss-clss-def total-over-m-union total-over-m-sum* **by** *blast*

lemma *true-clss-clss-mono-r'[simp]*:
 $A \models_p CC' \implies A \models_p CC + CC'$
unfolding *true-clss-clss-def total-over-m-union total-over-m-sum* **by** *blast*

lemma *true-clss-clss-union-l[simp]*:
 $A \models_{ps} CC \implies A \cup B \models_{ps} CC$
unfolding *true-clss-clss-def total-over-m-union* **by** *fastforce*

lemma *true-clss-clss-union-l-r[simp]*:
 $B \models_{ps} CC \implies A \cup B \models_{ps} CC$
unfolding *true-clss-clss-def total-over-m-union* **by** *fastforce*

lemma *true-clss-clss-in[simp]*:
 $CC \in A \implies A \models_p CC$
unfolding *true-clss-clss-def true-clss-def total-over-m-union* **by** *fastforce*

lemma *true-clss-clss-insert-l[simp]*:
 $A \models_p C \implies \text{insert } a \ A \models_p C$
unfolding *true-clss-clss-def true-clss-def* **using** *total-over-m-union*
by (*metis Un-iff insert-is-Un sup commute*)

lemma *true-clss-clss-insert-l[simp]*:
 $A \models_{ps} C \implies \text{insert } a \ A \models_{ps} C$
unfolding *true-clss-clss-def true-clss-clss-def true-clss-def* **by** *blast*

lemma *true-clss-clss-union-and[iff]*:
 $A \models_{ps} C \cup D \longleftrightarrow (A \models_{ps} C \wedge A \models_{ps} D)$

proof
 {
fix $A \ C \ D :: 'a \ \text{clauses}$
assume $A: A \models_{ps} C \cup D$
have $A \models_{ps} C$
 unfolding *true-clss-clss-def true-clss-clss-def insert-def total-over-m-insert*
proof (*intro allI impI*)
 fix I
 assume
 totAC: total-over-m I (A ∪ C) and
 cons: consistent-interp I and
 I: I ⊨s A
 then have *tot: total-over-m I A and tot': total-over-m I C* **by** *auto*
 obtain I' **where**
 tot': total-over-m (I ∪ I') (A ∪ C ∪ D) and
 cons': consistent-interp (I ∪ I') and
 H: ∀x∈I'. atm-of x ∈ atms-of-ms D ∧ atm-of x ∉ atms-of-ms (A ∪ C)
 }

using *total-over-m-consistent-extension*[*OF - cons, of $A \cup C$*] *tot tot'* **by** *blast*
 moreover have $I \cup I' \models_s A$ using *I* **by** *simp*
 ultimately have $I \cup I' \models_s C \cup D$ using *A* **unfolding** *true-clss-clss-def* **by** *auto*
 then have $I \cup I' \models_s C \cup D$ **by** *auto*
 then show $I \models_s C$ using *notin-vars-union-true-clss-true-clss*[*of I*] *H* **by** *auto*
 qed
 } note $H = \text{this}$
 assume $A \models_{ps} C \cup D$
 then show $A \models_{ps} C \wedge A \models_{ps} D$ using *H*[*of A*] *Un-commute*[*of C D*] **by** *metis*
 next
 assume $A \models_{ps} C \wedge A \models_{ps} D$
 then show $A \models_{ps} C \cup D$
 unfolding *true-clss-clss-def* **by** *auto*
 qed

lemma *true-clss-clss-insert*[*iff*]:
 $A \models_{ps} \text{insert } L \text{ } Ls \longleftrightarrow (A \models_p L \wedge A \models_{ps} Ls)$
 using *true-clss-clss-union-and*[*of A {L} Ls*] **by** *auto*

lemma *true-clss-clss-subset*:
 $A \subseteq B \implies A \models_{ps} CC \implies B \models_{ps} CC$
by (*metis subset-Un-eq true-clss-clss-union-l*)

lemma *union-trus-clss-clss*[*simp*]: $A \cup B \models_{ps} B$
 unfolding *true-clss-clss-def* **by** *auto*

lemma *true-clss-clss-remove*[*simp*]:
 $A \models_{ps} B \implies A \models_{ps} B - C$
by (*metis Un-Diff-Int true-clss-clss-union-and*)

lemma *true-clss-clss-subsetE*:
 $N \models_{ps} B \implies A \subseteq B \implies N \models_{ps} A$
by (*metis sup.orderE true-clss-clss-union-and*)

lemma *true-clss-clss-in-imp-true-clss-clss*:
 assumes $N \models_{ps} U$
 and $A \in U$
 shows $N \models_p A$
 using *assms mk-disjoint-insert* **by** *fastforce*

lemma *all-in-true-clss-clss*: $\forall x \in B. x \in A \implies A \models_{ps} B$
 unfolding *true-clss-clss-def true-clss-def* **by** *auto*

lemma *true-clss-clss-left-right*:
 assumes $A \models_{ps} B$
 and $A \cup B \models_{ps} M$
 shows $A \models_{ps} M \cup B$
 using *assms* **unfolding** *true-clss-clss-def* **by** *auto*

lemma *true-clss-clss-generalise-true-clss-clss*:
 $A \cup C \models_{ps} D \implies B \models_{ps} C \implies A \cup B \models_{ps} D$
proof –
 assume *a1*: $A \cup C \models_{ps} D$
 assume $B \models_{ps} C$
 then have *f2*: $\bigwedge M. M \cup B \models_{ps} C$

```

  by (meson true-clss-clss-union-l-r)
have  $\bigwedge M. C \cup (M \cup A) \models_{ps} D$ 
  using a1 by (simp add: Un-commute sup-left-commute)
then show ?thesis
  using f2 by (metis (no-types) Un-commute true-clss-clss-left-right true-clss-clss-union-and)
qed

```

lemma *true-clss-clss-or-true-clss-clss-or-not-true-clss-clss-or*:

```

  assumes  $D: N \models_p D + \{\#- L\# \}$ 
  and  $C: N \models_p C + \{\#L\# \}$ 
  shows  $N \models_p D + C$ 
  unfolding true-clss-clss-def
proof (intro allI impI)
  fix I
  assume
    tot: total-over-m I (N  $\cup$  {D + C}) and
    consistent-interp I and
    I  $\models_s N$ 
  {
    assume L:  $L \in I \vee -L \in I$ 
    then have total-over-m I {D + {#- L#}}
      using tot by (cases L) auto
    then have  $I \models D + \{\#- L\# \}$  using D  $\langle I \models_s N \rangle$  tot  $\langle$ consistent-interp I $\rangle$ 
      unfolding true-clss-clss-def by auto
    moreover
      have total-over-m I {C + {#L#}}
        using L tot by (cases L) auto
      then have  $I \models C + \{\#L\# \}$ 
        using C  $\langle I \models_s N \rangle$  tot  $\langle$ consistent-interp I $\rangle$  unfolding true-clss-clss-def by auto
      ultimately have  $I \models D + C$  using  $\langle$ consistent-interp I $\rangle$  consistent-interp-def by fastforce
    }
  moreover {
    assume L:  $L \notin I \wedge -L \notin I$ 
    let ?I' =  $I \cup \{L\}$ 
    have consistent-interp ?I' using L  $\langle$ consistent-interp I $\rangle$  by auto
    moreover have total-over-m ?I' {D + {#- L#}}
      using tot unfolding total-over-m-def total-over-set-def by (auto simp add: atms-of-def)
    moreover have total-over-m ?I' N using tot using total-union by blast
    moreover have  $?I' \models_s N$  using  $\langle I \models_s N \rangle$  using true-clss-union-increase by blast
    ultimately have  $?I' \models D + \{\#- L\# \}$ 
      using D unfolding true-clss-clss-def by blast
    then have  $?I' \models D$  using L by auto
    moreover
      have total-over-set I (atms-of (D + C)) using tot by auto
      then have  $L \notin \# D \wedge -L \notin \# D$ 
        using L unfolding total-over-set-def atms-of-def by (cases L) force+
      ultimately have  $I \models D + C$  unfolding true-clss-clss-def by auto
    }
  ultimately show  $I \models D + C$  by blast
qed

```

lemma *true-clss-union-mset[iff]*: $I \models C \# \cup D \longleftrightarrow I \models C \vee I \models D$

unfolding true-clss-def by force

lemma *true-clss-clss-union-mset-true-clss-clss-or-not-true-clss-clss-or*:

```

assumes
   $D: N \models_p D + \{\#- L\# \}$  and
   $C: N \models_p C + \{\#L\# \}$ 
shows  $N \models_p D \# \cup C$ 
unfolding true-clss-cls-def
proof (intro allI impI)
  fix  $I$ 
  assume
     $tot: total-over-m\ I\ (N \cup \{D \# \cup C\})$  and
     $consistent-interp\ I$  and
     $I \models_s N$ 
  {
    assume  $L: L \in I \vee -L \in I$ 
    then have  $total-over-m\ I\ \{D + \{\#- L\# \}\}$ 
      using  $tot$  by (cases L) auto
    then have  $I \models D + \{\#- L\# \}$ 
      using  $D\ \langle I \models_s N \rangle\ tot\ \langle consistent-interp\ I \rangle$  unfolding true-clss-cls-def by auto
    moreover
      have  $total-over-m\ I\ \{C + \{\#L\# \}\}$ 
        using  $L\ tot$  by (cases L) auto
      then have  $I \models C + \{\#L\# \}$ 
        using  $C\ \langle I \models_s N \rangle\ tot\ \langle consistent-interp\ I \rangle$  unfolding true-clss-cls-def by auto
    ultimately have  $I \models D \# \cup C$  using  $\langle consistent-interp\ I \rangle$  unfolding consistent-interp-def
    by auto
  }
  moreover {
    assume  $L: L \notin I \wedge -L \notin I$ 
    let  $?I' = I \cup \{L\}$ 
    have  $consistent-interp\ ?I'$  using  $L\ \langle consistent-interp\ I \rangle$  by auto
    moreover have  $total-over-m\ ?I'\ \{D + \{\#- L\# \}\}$ 
      using  $tot$  unfolding total-over-m-def total-over-set-def by (auto simp add: atms-of-def)
    moreover have  $total-over-m\ ?I'\ N$  using  $tot$  using total-union by blast
    moreover have  $?I' \models_s N$  using  $\langle I \models_s N \rangle$  using true-clss-union-increase by blast
    ultimately have  $?I' \models D + \{\#- L\# \}$ 
      using  $D$  unfolding true-clss-cls-def by blast
    then have  $?I' \models D$  using  $L$  by auto
    moreover
      have  $total-over-set\ I\ (atms-of\ (D + C))$  using  $tot$  by auto
      then have  $L \notin \# D \wedge -L \notin \# D$ 
        using  $L$  unfolding total-over-set-def atms-of-def by (cases L) force+
    ultimately have  $I \models D \# \cup C$  unfolding true-cls-def by auto
  }
  ultimately show  $I \models D \# \cup C$  by blast
qed

```

lemma *satisfiable-carac*[*iff*]:

$(\exists I. consistent-interp\ I \wedge I \models_s \varphi) \longleftrightarrow satisfiable\ \varphi$ (**is** $(\exists I. ?Q\ I) \longleftrightarrow ?S$)

proof

assume $?S$

then show $\exists I. ?Q\ I$ **unfolding** *satisfiable-def* **by** *auto*

next

assume $\exists I. ?Q\ I$

then obtain I **where** *cons: consistent-interp I* **and** $I: I \models_s \varphi$ **by** *metis*

let $?I' = \{Pos\ v \mid v. v \notin atms-of-s\ I \wedge v \in atms-of-ms\ \varphi\}$

have $consistent-interp\ (I \cup ?I')$

```

    using cons unfolding consistent-interp-def by (intro allI) (rename-tac L, case-tac L, auto)
  moreover have total-over-m ( $I \cup ?I'$ )  $\varphi$ 
    unfolding total-over-m-def total-over-set-def by auto
  moreover have  $I \cup ?I' \models_s \varphi$ 
    using I unfolding Ball-def true-clss-def true-clss-def by auto
  ultimately show ?S unfolding satisfiable-def by blast
qed

```

```

lemma satisfiable-carac'[simp]: consistent-interp  $I \implies I \models_s \varphi \implies$  satisfiable  $\varphi$ 
  using satisfiable-carac by metis

```

11.3 Subsumptions

lemma *subsumption-total-over-m*:

```

  assumes  $A \subseteq\# B$ 
  shows total-over-m  $I \{B\} \implies$  total-over-m  $I \{A\}$ 
  using assms unfolding subset-mset-def total-over-m-def total-over-set-def
  by (auto simp add: mset-le-exists-conv)

```

lemma *atms-of-replicate-mset-replicate-mset-uminus[simp]*:

```

  atms-of ( $D - \text{replicate-mset (count } D \text{ } L) \text{ } L - \text{replicate-mset (count } D \text{ } (-L)) (-L)$ )
    = atms-of  $D - \{ \text{atm-of } L \}$ 
  by (fastforce simp: atm-of-eq-atm-of atms-of-def)

```

lemma *subsumption-chained*:

```

  assumes
     $\forall I. \text{total-over-m } I \{D\} \longrightarrow I \models D \longrightarrow I \models \varphi$  and
     $C \subseteq\# D$ 
  shows  $(\forall I. \text{total-over-m } I \{C\} \longrightarrow I \models C \longrightarrow I \models \varphi) \vee$  tautology  $\varphi$ 
  using assms

```

proof (induct card $\{ \text{Pos } v \mid v. v \in \text{atms-of } D \wedge v \notin \text{atms-of } C \}$ arbitrary: D
rule: nat-less-induct-case)

```

  case 0 note n = this(1) and H = this(2) and incl = this(3)
  then have atms-of  $D \subseteq$  atms-of  $C$  by auto
  then have  $\forall I. \text{total-over-m } I \{C\} \longrightarrow \text{total-over-m } I \{D\}$ 
    unfolding total-over-m-def total-over-set-def by auto
  moreover have  $\forall I. I \models C \longrightarrow I \models D$  using incl true-clss-mono-leD by blast
  ultimately show ?case using H by auto

```

next

```

  case (Suc n D) note IH = this(1) and card = this(2) and H = this(3) and incl = this(4)
  let ?atms =  $\{ \text{Pos } v \mid v. v \in \text{atms-of } D \wedge v \notin \text{atms-of } C \}$ 
  have finite ?atms by auto
  then obtain L where L:  $L \in ?atms$ 
    using card by (metis (no-types, lifting) Collect-empty-eq card-0-eq mem-Collect-eq
    nat.simps(3))
  let ?D' =  $D - \text{replicate-mset (count } D \text{ } L) \text{ } L - \text{replicate-mset (count } D \text{ } (-L)) (-L)$ 
  have atms-of-D: atms-of-ms  $\{D\} \subseteq$  atms-of-ms  $\{?D'\} \cup \{ \text{atm-of } L \}$  by auto

```

```

{
  fix I
  assume total-over-m  $I \{?D'\}$ 
  then have tot: total-over-m  $(I \cup \{L\}) \{D\}$ 
    unfolding total-over-m-def total-over-set-def using atms-of-D by auto

```

```

  assume IDL:  $I \models ?D'$ 
  then have  $I \cup \{L\} \models D$  unfolding true-clss-def by force

```

```

then have  $I \cup \{L\} \models \varphi$  using  $H$  tot by auto

moreover
  have  $tot'$ : total-over-m  $(I \cup \{-L\}) \{D\}$ 
    using tot unfolding total-over-m-def total-over-set-def by auto
  have  $I \cup \{-L\} \models D$  using IDL unfolding true-cls-def by force
  then have  $I \cup \{-L\} \models \varphi$  using  $H$  tot' by auto
  ultimately have  $I \models \varphi \vee \text{tautology } \varphi$ 
    using L remove-literal-in-model-tautology by force
} note  $H' = \text{this}$ 

have  $L \notin \# C$  and  $-L \notin \# C$  using L atm-iff-pos-or-neg-lit by force+
then have  $C\text{-in-}D'$ :  $C \subseteq \# ?D'$  using  $\langle C \subseteq \# D \rangle$  by (auto simp: subseteq-mset-def not-in-iff)
have card  $\{Pos\ v \mid v. v \in \text{atms-of } ?D' \wedge v \notin \text{atms-of } C\} <$ 
  card  $\{Pos\ v \mid v. v \in \text{atms-of } D \wedge v \notin \text{atms-of } C\}$ 
  using L by (auto intro!: psubset-card-mono)
then show ?case
  using IH  $C\text{-in-}D'$   $H'$  unfolding card[symmetric] by blast
qed

```

11.4 Removing Duplicates

```

lemma tautology-remdups-mset[iff]:
  tautology (remdups-mset C)  $\longleftrightarrow$  tautology C
  unfolding tautology-decomp by auto

lemma atms-of-remdups-mset[simp]: atms-of (remdups-mset C) = atms-of C
  unfolding atms-of-def by auto

lemma true-cls-remdups-mset[iff]:  $I \models \text{remdups-mset } C \longleftrightarrow I \models C$ 
  unfolding true-cls-def by auto

lemma true-clss-cls-remdups-mset[iff]:  $A \models_p \text{remdups-mset } C \longleftrightarrow A \models_p C$ 
  unfolding true-clss-cls-def total-over-m-def by auto

```

11.5 Set of all Simple Clauses

definition *simple-clss* :: $'v \text{ set} \Rightarrow 'v \text{ clause set}$ **where**
simple-clss atms = $\{C. \text{atms-of } C \subseteq \text{atms} \wedge \neg \text{tautology } C \wedge \text{distinct-mset } C\}$

```

lemma simple-clss-empty[simp]:
  simple-clss {} =  $\{\{\#\}\}$ 
  unfolding simple-clss-def by auto

```

```

lemma simple-clss-insert:
  assumes  $l \notin \text{atms}$ 
  shows simple-clss (insert l atms) =
    (op +  $\{\#Pos\ l\ \#\}$ ) ' (simple-clss atms)
     $\cup$  (op +  $\{\#Neg\ l\ \#\}$ ) ' (simple-clss atms)
     $\cup$  simple-clss atms(is ?I = ?U)

```

```

proof (standard; standard)
  fix C
  assume  $C \in ?I$ 
  then have
    atms: atms-of C  $\subseteq$  insert l atms and
    taut:  $\neg \text{tautology } C$  and

```

```

dist: distinct-mset  $C$ 
unfolding simple-clss-def by auto
have  $H: \bigwedge x. x \in \# C \implies \text{atm-of } x \in \text{insert } l \text{ } \text{atms}$ 
  using atm-of-lit-in-atms-of atms by blast
consider
  (Add)  $L$  where  $L \in \# C$  and  $L = \text{Neg } l \vee L = \text{Pos } l$ 
  | (No)  $\text{Pos } l \notin \# C$   $\text{Neg } l \notin \# C$ 
  by auto
then show  $C \in ?U$ 
proof cases
  case Add
  then have  $L \notin \# C - \{\#L\}$ 
    using dist unfolding distinct-mset-def by (auto simp: not-in-iff)
  moreover have  $-L \notin \# C$ 
    using taut Add by auto
  ultimately have  $\text{atms-of } (C - \{\#L\}) \subseteq \text{atms}$ 
    using atms Add by (smt H atms-of-def imageE in-diffD insertE literal.exhaust-sel
      subset-iff uminus-Neg uminus-Pos)

  moreover have  $\neg \text{tautology } (C - \{\#L\})$ 
    using taut by (metis Add(1) insert-DiffM tautology-add-single)
  moreover have distinct-mset  $(C - \{\#L\})$ 
    using dist by auto
  ultimately have  $(C - \{\#L\}) \in \text{simple-clss atms}$ 
    using Add unfolding simple-clss-def by auto
  moreover have  $C = \{\#L\} + (C - \{\#L\})$ 
    using Add by (auto simp: multiset-eq-iff)
  ultimately show ?thesis using Add by auto
next
  case No
  then have  $C \in \text{simple-clss atms}$ 
    using taut atms dist unfolding simple-clss-def
    by (auto simp: atm-iff-pos-or-neg-lit split: if-split-asm dest!: H)
  then show ?thesis by blast
qed
next
fix  $C$ 
assume  $C \in ?U$ 
then consider
  (Add)  $L$   $C'$  where  $C = \{\#L\} + C'$  and  $C' \in \text{simple-clss atms}$  and
     $L = \text{Pos } l \vee L = \text{Neg } l$ 
  | (No)  $C \in \text{simple-clss atms}$ 
  by auto
then show  $C \in ?I$ 
proof cases
  case No
  then show ?thesis unfolding simple-clss-def by auto
next
  case (Add  $L$   $C'$ ) note  $C' = \text{this}(1)$  and  $C = \text{this}(2)$  and  $L = \text{this}(3)$ 
  then have
    atms:  $\text{atms-of } C' \subseteq \text{atms}$  and
    taut:  $\neg \text{tautology } C'$  and
    dist: distinct-mset  $C'$ 
    unfolding simple-clss-def by auto
  have  $\text{atms-of } C \subseteq \text{insert } l \text{ } \text{atms}$ 

```

```

    using atms C' L by auto
  moreover have  $\neg$  tautology C
    using taut C' L by (metis assms atm-of-lit-in-atms-of atms literal.sel(1,2) subset-eq
      tautology-add-single uminus-Neg uminus-Pos)
  moreover have distinct-mset C
    using dist C' L
    by (metis assms atm-of-lit-in-atms-of atms contra-subsetD distinct-mset-add-single
      literal.sel(1,2))
  ultimately show ?thesis unfolding simple-clss-def by blast
qed
qed

lemma simple-clss-finite:
  fixes atms :: 'v set
  assumes finite atms
  shows finite (simple-clss atms)
  using assms by (induction rule: finite-induct) (auto simp: simple-clss-insert)

lemma simple-clssE:
  assumes
     $x \in \text{simple-clss } \textit{atms}$ 
  shows  $\textit{atms-of } x \subseteq \textit{atms} \wedge \neg \textit{tautology } x \wedge \textit{distinct-mset } x$ 
  using assms unfolding simple-clss-def by auto

lemma cls-in-simple-clss:
  shows  $\{\#\} \in \text{simple-clss } s$ 
  unfolding simple-clss-def by auto

lemma simple-clss-card:
  fixes atms :: 'v set
  assumes finite atms
  shows  $\text{card } (\text{simple-clss } \textit{atms}) \leq (3::\text{nat}) \wedge (\text{card } \textit{atms})$ 
  using assms
proof (induct atms rule: finite-induct)
  case empty
  then show ?case by auto
next
  case (insert l C) note fin = this(1) and l = this(2) and IH = this(3)
  have notin:
     $\bigwedge C'. \{\#\text{Pos } l\# \} + C' \notin \text{simple-clss } C$ 
     $\bigwedge C'. \{\#\text{Neg } l\# \} + C' \notin \text{simple-clss } C$ 
    using l unfolding simple-clss-def by auto
  have H:  $\bigwedge C' D. \{\#\text{Pos } l\# \} + C' = \{\#\text{Neg } l\# \} + D \implies D \in \text{simple-clss } C \implies \text{False}$ 
  proof -
    fix C' D
    assume C'D:  $\{\#\text{Pos } l\# \} + C' = \{\#\text{Neg } l\# \} + D$  and D:  $D \in \text{simple-clss } C$ 
    then have Pos l  $\in \# D$  by (metis insert-noteq-member literal.distinct(1) union-commute)
    then have l  $\in \textit{atms-of } D$ 
      by (simp add: atm-iff-pos-or-neg-lit)
    then show False using D l unfolding simple-clss-def by auto
  qed
  let ?P =  $(\text{op } + \{\#\text{Pos } l\# \}) \cdot (\text{simple-clss } C)$ 
  let ?N =  $(\text{op } + \{\#\text{Neg } l\# \}) \cdot (\text{simple-clss } C)$ 
  let ?O = simple-clss C
  have  $\text{card } (?P \cup ?N \cup ?O) = \text{card } (?P \cup ?N) + \text{card } ?O$ 

```

```

  apply (subst card-Un-disjoint)
  using l fin by (auto simp: simple-clss-finite notin)
moreover have card (?P ∪ ?N) = card ?P + card ?N
  apply (subst card-Un-disjoint)
  using l fin H by (auto simp: simple-clss-finite notin)
moreover
  have card ?P = card ?O
    using inj-on-iff-eq-card[of ?O op + {#Pos l#}]
    by (auto simp: fin simple-clss-finite inj-on-def)
  moreover have card ?N = card ?O
    using inj-on-iff-eq-card[of ?O op + {#Neg l#}]
    by (auto simp: fin simple-clss-finite inj-on-def)
  moreover have (3::nat) ^ card (insert l C) = 3 ^ (card C) + 3 ^ (card C) + 3 ^ (card C)
    using l by (simp add: fin mult-2-right numeral-3-eq-3)
  ultimately show ?case using IH l by (auto simp: simple-clss-insert)
qed

```

```

lemma simple-clss-mono:
  assumes incl: atms ⊆ atms'
  shows simple-clss atms ⊆ simple-clss atms'
  using assms unfolding simple-clss-def by auto

```

```

lemma distinct-mset-not-tautology-implies-in-simple-clss:
  assumes distinct-mset χ and ¬tautology χ
  shows χ ∈ simple-clss (atms-of χ)
  using assms unfolding simple-clss-def by auto

```

```

lemma simplified-in-simple-clss:
  assumes distinct-mset-set ψ and ∀ χ ∈ ψ. ¬tautology χ
  shows ψ ⊆ simple-clss (atms-of-ms ψ)
  using assms unfolding simple-clss-def
  by (auto simp: distinct-mset-set-def atms-of-ms-def)

```

11.6 Experiment: Expressing the Entailments as Locales

```

locale entail =
  fixes entail :: 'a set ⇒ 'b ⇒ bool (infix |=e 50)
  assumes entail-insert[simp]: I ≠ {} ⇒ insert L I |=e x ⟷ {L} |=e x ∨ I |=e x
  assumes entail-union[simp]: I |=e A ⇒ I ∪ I' |=e A
begin

```

```

definition entails :: 'a set ⇒ 'b set ⇒ bool (infix |=es 50) where
  I |=es A ⟷ (∀ a ∈ A. I |=e a)

```

```

lemma entails-empty[simp]:
  I |=es {}
  unfolding entails-def by auto

```

```

lemma entails-single[iff]:
  I |=es {a} ⟷ I |=e a
  unfolding entails-def by auto

```

```

lemma entails-insert-l[simp]:
  M |=es A ⇒ insert L M |=es A
  unfolding entails-def by (metis Un-commute entail-union insert-is-Un)

```


lemma *entails-union*[*iff*]: $I \models_{es} CC \cup DD \longleftrightarrow I \models_{es} CC \wedge I \models_{es} DD$
unfolding *entails-def* **by** *blast*

lemma *entails-insert*[*iff*]: $I \models_{es} insert\ C\ DD \longleftrightarrow I \models_e C \wedge I \models_{es} DD$
unfolding *entails-def* **by** *blast*

lemma *entails-insert-mono*: $DD \subseteq CC \implies I \models_{es} CC \implies I \models_{es} DD$
unfolding *entails-def* **by** *blast*

lemma *entails-union-increase*[*simp*]:
assumes $I \models_{es} \psi$
shows $I \cup I' \models_{es} \psi$
using *assms* **unfolding** *entails-def* **by** *auto*

lemma *true-clss-commute-l*:
 $(I \cup I' \models_{es} \psi) \longleftrightarrow (I' \cup I \models_{es} \psi)$
by (*simp* *add*: *Un-commute*)

lemma *entails-remove*[*simp*]: $I \models_{es} N \implies I \models_{es} Set.remove\ a\ N$
by (*simp* *add*: *entails-def*)

lemma *entails-remove-minus*[*simp*]: $I \models_{es} N \implies I \models_{es} N - A$
by (*simp* *add*: *entails-def*)

end

interpretation *true-cls*: *entail true-cls*
by *standard* (*auto* *simp* *add*: *true-cls-def*)

11.7 Entailment to be extended

definition *true-clss-ext* :: '*a literal set* \Rightarrow '*a literal multiset set* \Rightarrow *bool* (**infix** \models_{sext} 49)
where
 $I \models_{sext} N \longleftrightarrow (\forall J. I \subseteq J \longrightarrow consistent_interp\ J \longrightarrow total_over_m\ J\ N \longrightarrow J \models_s N)$

lemma *true-clss-imp-true-cls-ext*:
 $I \models_s N \implies I \models_{sext} N$
unfolding *true-clss-ext-def* **by** (*metis* *sup.orderE* *true-clss-union-increase*)

lemma *true-clss-ext-decrease-right-remove-r*:
assumes $I \models_{sext} N$
shows $I \models_{sext} N - \{C\}$
unfolding *true-clss-ext-def*
proof (*intro* *allI* *impI*)
fix *J*
assume
 $I \subseteq J$ **and**
cons: *consistent-interp* *J* **and**
tot: *total-over-m* *J* ($N - \{C\}$)
let $?J = J \cup \{Pos\ (atm_of\ P) | P. P \in\# C \wedge atm_of\ P \notin atm_of\ 'J\}$
have $I \subseteq ?J$ **using** ($I \subseteq J$) **by** *auto*
moreover **have** *consistent-interp* $?J$
using *cons* **unfolding** *consistent-interp-def* **apply** (*intro* *allI*)
by (*rename-tac* *L*, *case-tac* *L*) (*fastforce* *simp* *add*: *image-iff*)
moreover **have** *total-over-m* $?J\ N$
using *tot* **unfolding** *total-over-m-def* *total-over-set-def* *atms-of-ms-def*

```

apply clarify
apply (rename-tac l a, case-tac  $a \in N - \{C\}$ )
  apply auto
  using atms-of-s-def atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
  by (fastforce simp: atms-of-def)
ultimately have  $?J \models_s N$ 
  using assms unfolding true-clss-ext-def by blast
then have  $?J \models_s N - \{C\}$  by auto
have  $\{v \in ?J. \text{atm-of } v \in \text{atms-of-ms } (N - \{C\})\} \subseteq J$ 
  using tot unfolding total-over-m-def total-over-set-def
  by (auto intro!: rev-image-eqI)
then show  $J \models_s N - \{C\}$ 
  using true-clss-remove-unused[OF (?J \models_s N - \{C\})] unfolding true-clss-def
  by (meson true-clss-mono-set-mset-l)
qed

```

```

lemma consistent-true-clss-ext-satisfiable:
  assumes consistent-interp I and  $I \models_{\text{sext}} A$ 
  shows satisfiable A
  by (metis Un-empty-left assms satisfiable-carac subset-Un-eq sup.left-idem
    total-over-m-consistent-extension total-over-m-empty true-clss-ext-def)

```

```

lemma not-consistent-true-clss-ext:
  assumes  $\neg \text{consistent-interp } I$ 
  shows  $I \models_{\text{sext}} A$ 
  by (meson assms consistent-interp-subset true-clss-ext-def)
end

```

```

theory Prop-Resolution
imports Partial-Clausal-Logic List-More Wellfounded-More

```

```

begin

```

12 Resolution

12.1 Simplification Rules

inductive *simplify* :: '*v* clauses \Rightarrow '*v* clauses \Rightarrow bool **for** *N* :: '*v* clause set **where**

tautology-deletion:

$(A + \{\#Pos\ P\# \} + \{\#Neg\ P\# \}) \in N \Longrightarrow \text{simplify } N (N - \{A + \{\#Pos\ P\# \} + \{\#Neg\ P\# \}\})$

condensation:

$(A + \{\#L\# \} + \{\#L\# \}) \in N \Longrightarrow \text{simplify } N (N - \{A + \{\#L\# \} + \{\#L\# \}\} \cup \{A + \{\#L\# \}\})$

subsumption:

$A \in N \Longrightarrow A \subset\# B \Longrightarrow B \in N \Longrightarrow \text{simplify } N (N - \{B\})$

lemma *simplify-preserves-un-sat'*:

fixes *N N'* :: '*v* clauses

assumes *simplify N N'*

and *total-over-m I N*

shows $I \models_s N' \longrightarrow I \models_s N$

using *assms*

proof (*induct rule: simplify.induct*)

case (*tautology-deletion A P*)

then have $I \models A + \{\#Pos\ P\# \} + \{\#Neg\ P\# \}$

by (*metis total-over-m-def total-over-set-literal-defined true-clss-singleton true-clss-union*
true-lit-def uminus-Neg union-commute)

```

  then show ?case by (metis Un-Diff-cancel2 true-clss-singleton true-clss-union)
next
case (condensation A P)
then show ?case by (metis Diff-insert-absorb Set.set-insert insertE true-clss-union true-clss-def
  true-clss-singleton true-clss-union)
next
case (subsumption A B)
have  $A \neq B$  using subsumption.hyps(2) by auto
then have  $I \models_s N - \{B\} \implies I \models A$  using  $\langle A \in N \rangle$  by (simp add: true-clss-def)
moreover have  $I \models A \implies I \models B$  using  $\langle A <_{\#} B \rangle$  by auto
ultimately show ?case by (metis insert-Diff-single true-clss-insert)
qed

```

```

lemma simplify-preserves-un-sat:
  fixes  $N N' :: 'v \text{ clauses}$ 
  assumes simplify  $N N'$ 
  and total-over-m  $I N$ 
  shows  $I \models_s N \longrightarrow I \models_s N'$ 
  using assms apply (induct rule: simplify.induct)
  using true-clss-def by fastforce+

```

```

lemma simplify-preserves-un-sat'':
  fixes  $N N' :: 'v \text{ clauses}$ 
  assumes simplify  $N N'$ 
  and total-over-m  $I N'$ 
  shows  $I \models_s N \longrightarrow I \models_s N'$ 
  using assms apply (induct rule: simplify.induct)
  using true-clss-def by fastforce+

```

```

lemma simplify-preserves-un-sat-eq:
  fixes  $N N' :: 'v \text{ clauses}$ 
  assumes simplify  $N N'$ 
  and total-over-m  $I N$ 
  shows  $I \models_s N \longleftrightarrow I \models_s N'$ 
  using simplify-preserves-un-sat simplify-preserves-un-sat' assms by blast

```

```

lemma simplify-preserves-finite:
  assumes simplify  $\psi \psi'$ 
  shows  $\text{finite } \psi \longleftrightarrow \text{finite } \psi'$ 
  using assms by (induct rule: simplify.induct, auto simp add: remove-def)

```

```

lemma rtranclp-simplify-preserves-finite:
  assumes rtranclp simplify  $\psi \psi'$ 
  shows  $\text{finite } \psi \longleftrightarrow \text{finite } \psi'$ 
  using assms by (induct rule: rtranclp-induct) (auto simp add: simplify-preserves-finite)

```

```

lemma simplify-atms-of-ms:
  assumes simplify  $\psi \psi'$ 
  shows  $\text{atms-of-ms } \psi' \subseteq \text{atms-of-ms } \psi$ 
  using assms unfolding atms-of-ms-def
proof (induct rule: simplify.induct)
  case (tautology-deletion A P)
  then show ?case by auto
next
case (condensation A P)

```

moreover have $A + \{\#P\# \} + \{\#P\# \} \in \psi \implies \exists x \in \psi. \text{ atm-of } P \in \text{ atm-of ' set-mset } x$
by (*metis Un-iff atms-of-def atms-of-plus atms-of-singleton insert-iff*)
ultimately show ?case **by** (*auto simp add: atms-of-def*)
next
case (*subsumption A P*)
then show ?case **by** *auto*
qed

lemma *rtranclp-simplify-atms-of-ms*:
assumes *rtranclp simplify* $\psi \psi'$
shows *atms-of-ms* $\psi' \subseteq \text{atms-of-ms } \psi$
using *assms* **apply** (*induct rule: rtranclp-induct*)
apply (*fastforce intro: simplify-atms-of-ms*)
using *simplify-atms-of-ms* **by** *blast*

lemma *factoring-imp-simplify*:
assumes $\{\#L\# \} + \{\#L\# \} + C \in N$
shows $\exists N'. \text{ simplify } N N'$
proof –
have $C + \{\#L\# \} + \{\#L\# \} \in N$ **using** *assms* **by** (*simp add: add.commute union-lcomm*)
from *condensation[OF this]* **show** ?thesis **by** *blast*
qed

12.2 Unconstrained Resolution

type-synonym *'v uncon-state* = *'v clauses*
inductive *uncon-res* :: *'v uncon-state* \Rightarrow *'v uncon-state* \Rightarrow *bool* **where**
resolution:
 $\{\#Pos \ p\# \} + C \in N \implies \{\#Neg \ p\# \} + D \in N \implies (\{\#Pos \ p\# \} + C, \{\#Neg \ p\# \} + D) \notin$
already-used
 $\implies \text{uncon-res } (N) (N \cup \{C + D\}) \mid$
factoring: $\{\#L\# \} + \{\#L\# \} + C \in N \implies \text{uncon-res } N (N \cup \{C + \{\#L\# \}\})$

lemma *uncon-res-increasing*:
assumes *uncon-res* $S S'$ **and** $\psi \in S$
shows $\psi \in S'$
using *assms* **by** (*induct rule: uncon-res.induct*) *auto*

lemma *rtranclp-uncon-inference-increasing*:
assumes *rtranclp uncon-res* $S S'$ **and** $\psi \in S$
shows $\psi \in S'$
using *assms* **by** (*induct rule: rtranclp-induct*) (*auto simp add: uncon-res-increasing*)

12.2.1 Subsumption

definition *subsumes* :: *'a literal multiset* \Rightarrow *'a literal multiset* \Rightarrow *bool* **where**
subsumes $\chi \chi' \longleftrightarrow$
 $(\forall I. \text{total-over-m } I \ \{\chi'\} \longrightarrow \text{total-over-m } I \ \{\chi\})$
 $\wedge (\forall I. \text{total-over-m } I \ \{\chi\} \longrightarrow I \models \chi \longrightarrow I \models \chi')$

lemma *subsumes-refl[simp]*:
subsumes $\chi \chi$
unfolding *subsumes-def* **by** *auto*

lemma *subsumes-subsumption*:

assumes *subsumes* $D \chi$
and $C \subset\# D$ **and** $\neg \text{tautology } \chi$
shows *subsumes* $C \chi$ **unfolding** *subsumes-def*
using *assms* *subsumption-total-over-m* *subsumption-chained* **unfolding** *subsumes-def*
by (*blast intro!*: *subset-mset.less-imp-le*)

lemma *subsumes-tautology*:

assumes *subsumes* $(C + \{\#Pos\ P\# \} + \{\#Neg\ P\# \}) \chi$
shows *tautology* χ
using *assms* **unfolding** *subsumes-def* **by** (*simp add: tautology-def*)

12.3 Inference Rule

type-synonym *'v state* = *'v clauses* \times (*'v clause* \times *'v clause*) *set*

inductive *inference-clause* :: *'v state* \Rightarrow *'v clause* \times (*'v clause* \times *'v clause*) *set* \Rightarrow *bool*

(**infix** \Rightarrow_{Res} 100) **where**

resolution:

$\{\#Pos\ p\# \} + C \in N \implies \{\#Neg\ p\# \} + D \in N \implies (\{\#Pos\ p\# \} + C, \{\#Neg\ p\# \} + D) \notin$
already-used

$\implies \text{inference-clause } (N, \text{already-used}) (C + D, \text{already-used} \cup \{(\{\#Pos\ p\# \} + C, \{\#Neg\ p\# \} + D)\}) \mid$

factoring: $\{\#L\# \} + \{\#L\# \} + C \in N \implies \text{inference-clause } (N, \text{already-used}) (C + \{\#L\# \}, \text{already-used})$

inductive *inference* :: *'v state* \Rightarrow *'v state* \Rightarrow *bool* **where**

inference-step: *inference-clause* S (*clause*, *already-used*)

$\implies \text{inference } S (\text{fst } S \cup \{\text{clause}\}, \text{already-used})$

abbreviation *already-used-inv*

:: *'a literal multiset set* \times (*'a literal multiset* \times *'a literal multiset*) *set* \Rightarrow *bool* **where**

already-used-inv state \equiv

$(\forall (A, B) \in \text{snd state}. \exists p. \text{Pos } p \in\# A \wedge \text{Neg } p \in\# B \wedge$
 $((\exists \chi \in \text{fst state}. \text{subsumes } \chi ((A - \{\#Pos\ p\# \}) + (B - \{\#Neg\ p\# \})))$
 $\vee \text{tautology } ((A - \{\#Pos\ p\# \}) + (B - \{\#Neg\ p\# \}))))$

lemma *inference-clause-preserves-already-used-inv*:

assumes *inference-clause* $S S'$

and *already-used-inv* S

shows *already-used-inv* ($\text{fst } S \cup \{\text{fst } S'\}$, *snd* S')

using *assms* **apply** (*induct rule: inference-clause.induct*)

by *fastforce*+

lemma *inference-preserves-already-used-inv*:

assumes *inference* $S S'$

and *already-used-inv* S

shows *already-used-inv* S'

using *assms*

proof (*induct rule: inference.induct*)

case (*inference-step* S *clause* *already-used*)

then show ?*case*

using *inference-clause-preserves-already-used-inv*[*of* S (*clause*, *already-used*)] **by** *simp*

qed

lemma *rtranclp-inference-preserves-already-used-inv*:

assumes *rtranclp inference* $S S'$

and *already-used-inv* S

```

shows already-used-inv S'
using assms apply (induct rule: rtranclp-induct, simp)
using inference-preserves-already-used-inv unfolding tautology-def by fast

lemma subsumes-condensation:
  assumes subsumes (C + {#L#} + {#L#}) D
  shows subsumes (C + {#L#}) D
  using assms unfolding subsumes-def by simp

lemma simplify-preserves-already-used-inv:
  assumes simplify N N'
  and already-used-inv (N, already-used)
  shows already-used-inv (N', already-used)
  using assms
proof (induct rule: simplify.induct)
  case (condensation C L)
  then show ?case
    using subsumes-condensation by simp fast
next
{
  fix a:: 'a and A :: 'a set and P
  have  $(\exists x \in \text{Set.remove } a \ A. P \ x) \longleftrightarrow (\exists x \in A. x \neq a \wedge P \ x)$  by auto
} note ex-member-remove = this
{
  fix a a0 :: 'v clause and A :: 'v clauses and y
  assume a ∈ A and a0 ⊂# a
  then have  $(\exists x \in A. \text{subsumes } x \ y) \longleftrightarrow (\text{subsumes } a \ y \vee (\exists x \in A. x \neq a \wedge \text{subsumes } x \ y))$ 
    by auto
} note tt2 = this
case (subsumption A B) note A = this(1) and AB = this(2) and B = this(3) and inv = this(4)
show ?case
proof (standard, standard)
  fix x a b
  assume x:  $x \in \text{snd } (N - \{B\}, \text{already-used})$  and [simp]:  $x = (a, b)$ 
  obtain p where p:  $\text{Pos } p \in\# a \wedge \text{Neg } p \in\# b$  and
    q:  $(\exists \chi \in N. \text{subsumes } \chi (a - \{\# \text{Pos } p\# \} + (b - \{\# \text{Neg } p\# \})))$ 
     $\vee \text{tautology } (a - \{\# \text{Pos } p\# \} + (b - \{\# \text{Neg } p\# \}))$ 
  using inv x by fastforce
  consider (taut)  $\text{tautology } (a - \{\# \text{Pos } p\# \} + (b - \{\# \text{Neg } p\# \})) \mid$ 
     $(\chi) \chi$  where  $\chi \in N$   $\text{subsumes } \chi (a - \{\# \text{Pos } p\# \} + (b - \{\# \text{Neg } p\# \}))$ 
     $\neg \text{tautology } (a - \{\# \text{Pos } p\# \} + (b - \{\# \text{Neg } p\# \}))$ 
  using q by auto
  then show
     $\exists p. \text{Pos } p \in\# a \wedge \text{Neg } p \in\# b$ 
     $\wedge ((\exists \chi \in \text{fst } (N - \{B\}, \text{already-used}). \text{subsumes } \chi (a - \{\# \text{Pos } p\# \} + (b - \{\# \text{Neg } p\# \})))$ 
     $\vee \text{tautology } (a - \{\# \text{Pos } p\# \} + (b - \{\# \text{Neg } p\# \})))$ 
  proof cases
    case taut
    then show ?thesis using p by auto
  next
    case  $\chi$  note H = this
    show ?thesis using p A AB B subsumes-subsumption[OF - AB H(3)] H(1,2) by auto
  qed
qed
next

```

case (*tautology-deletion* $C\ P$)
then show ?*case* **apply** *clarify*
proof –
 fix $a\ b$
 assume $C + \{\#Pos\ P\# \} + \{\#Neg\ P\# \} \in N$
 assume *already-used-inv* (N , *already-used*)
 and $(a, b) \in snd\ (N - \{C + \{\#Pos\ P\# \} + \{\#Neg\ P\# \}\}, \textit{already-used})$
 then obtain p **where**
 $Pos\ p \in \# a \wedge Neg\ p \in \# b \wedge$
 $((\exists \chi \in fst\ (N \cup \{C + \{\#Pos\ P\# \} + \{\#Neg\ P\# \}\}, \textit{already-used}).$
 $subsumes\ \chi\ (a - \{\#Pos\ p\# \} + (b - \{\#Neg\ p\# \})))$
 $\vee\ \textit{tautology}\ (a - \{\#Pos\ p\# \} + (b - \{\#Neg\ p\# \})))$
 by *fastforce*
moreover have *tautology* ($C + \{\#Pos\ P\# \} + \{\#Neg\ P\# \}$) **by** *auto*
ultimately show
 $\exists p. Pos\ p \in \# a \wedge Neg\ p \in \# b$
 $\wedge ((\exists \chi \in fst\ (N - \{C + \{\#Pos\ P\# \} + \{\#Neg\ P\# \}\}, \textit{already-used}).$
 $subsumes\ \chi\ (a - \{\#Pos\ p\# \} + (b - \{\#Neg\ p\# \})))$
 $\vee\ \textit{tautology}\ (a - \{\#Pos\ p\# \} + (b - \{\#Neg\ p\# \})))$
 by (*metis* (*no-types*) *Diff-iff* *Un-insert-right* *empty-iff* *fst-conv* *insertE* *subsumes-tautology*
 sup-bot.right-neutral)
qed
qed

lemma

factoring-satisfiable: $I \models \{\#L\# \} + \{\#L\# \} + C \longleftrightarrow I \models \{\#L\# \} + C$ **and**
resolution-satisfiable:
 consistent-interp $I \implies I \models \{\#Pos\ p\# \} + C \implies I \models \{\#Neg\ p\# \} + D \implies I \models C + D$ **and**
 factoring-same-vars: $atms-of\ (\{\#L\# \} + \{\#L\# \} + C) = atms-of\ (\{\#L\# \} + C)$
unfolding *true-cls-def* *consistent-interp-def* **by** (*fastforce* *split*: *if-split-asm*) +

lemma *inference-increasing*:

assumes *inference* $S\ S'$ **and** $\psi \in fst\ S$
shows $\psi \in fst\ S'$
using *assms* **by** (*induct* *rule*: *inference.induct*, *auto*)

lemma *rtranclp-inference-increasing*:

assumes *rtranclp* *inference* $S\ S'$ **and** $\psi \in fst\ S$
shows $\psi \in fst\ S'$
using *assms* **by** (*induct* *rule*: *rtranclp-induct*, *auto* *simp* *add*: *inference-increasing*)

lemma *inference-clause-already-used-increasing*:

assumes *inference-clause* $S\ S'$
shows $snd\ S \subseteq snd\ S'$
using *assms* **by** (*induct* *rule*: *inference-clause.induct*, *auto*)

lemma *inference-already-used-increasing*:

assumes *inference* $S\ S'$
shows $snd\ S \subseteq snd\ S'$
using *assms* **apply** (*induct* *rule*: *inference.induct*)
using *inference-clause-already-used-increasing* **by** *fastforce*

lemma *inference-clause-preserves-un-sat*:

fixes $N\ N' :: 'v\ clauses$
assumes *inference-clause* $T\ T'$
and *total-over-m* $I\ (fst\ T)$
and *consistent: consistent-interp* I
shows $I \models_s fst\ T \longleftrightarrow I \models_s fst\ T \cup \{fst\ T'\}$
using *assms* **apply** (*induct rule: inference-clause.induct*)
unfolding *consistent-interp-def true-clss-def* **by** *auto force+*

lemma *inference-preserves-un-sat:*

fixes $N\ N' :: 'v\ clauses$
assumes *inference* $T\ T'$
and *total-over-m* $I\ (fst\ T)$
and *consistent: consistent-interp* I
shows $I \models_s fst\ T \longleftrightarrow I \models_s fst\ T'$
using *assms* **apply** (*induct rule: inference.induct*)
using *inference-clause-preserves-un-sat* **by** *fastforce*

lemma *inference-clause-preserves-atms-of-ms:*

assumes *inference-clause* $S\ S'$
shows *atms-of-ms* $(fst\ (fst\ S \cup \{fst\ S'\}, snd\ S')) \subseteq atms-of-ms\ (fst\ S)$
using *assms* **apply** (*induct rule: inference-clause.induct*)
apply *auto*
apply (*metis Set.set-insert UnCI atms-of-ms-insert atms-of-plus*)
apply (*metis Set.set-insert UnCI atms-of-ms-insert atms-of-plus*)
apply (*simp add: in-m-in-literals union-assoc*)
unfolding *atms-of-ms-def* **using** *assms* **by** *fastforce*

lemma *inference-preserves-atms-of-ms:*

fixes $N\ N' :: 'v\ clauses$
assumes *inference* $T\ T'$
shows *atms-of-ms* $(fst\ T') \subseteq atms-of-ms\ (fst\ T)$
using *assms* **apply** (*induct rule: inference.induct*)
using *inference-clause-preserves-atms-of-ms* **by** *fastforce*

lemma *inference-preserves-total:*

fixes $N\ N' :: 'v\ clauses$
assumes *inference* $(N, already-used)\ (N', already-used')$
shows *total-over-m* $I\ N \implies total-over-m\ I\ N'$
using *assms* *inference-preserves-atms-of-ms* **unfolding** *total-over-m-def total-over-set-def*
by *fastforce*

lemma *rtranclp-inference-preserves-total:*

assumes *rtranclp inference* $T\ T'$
shows *total-over-m* $I\ (fst\ T) \implies total-over-m\ I\ (fst\ T')$
using *assms* **by** (*induct rule: rtranclp-induct, auto simp add: inference-preserves-total*)

lemma *rtranclp-inference-preserves-un-sat:*

assumes *rtranclp inference* $N\ N'$
and *total-over-m* $I\ (fst\ N)$
and *consistent: consistent-interp* I
shows $I \models_s fst\ N \longleftrightarrow I \models_s fst\ N'$
using *assms* **apply** (*induct rule: rtranclp-induct*)
apply (*simp add: inference-preserves-un-sat*)


```

using inference-preserves-un-sat rtranclp-inference-preserves-total by blast

lemma inference-preserves-finite:
  assumes inference  $\psi$   $\psi'$  and finite (fst  $\psi$ )
  shows finite (fst  $\psi'$ )
  using assms by (induct rule: inference.induct, auto simp add: simplify-preserves-finite)

lemma inference-clause-preserves-finite-snd:
  assumes inference-clause  $\psi$   $\psi'$  and finite (snd  $\psi$ )
  shows finite (snd  $\psi'$ )
  using assms by (induct rule: inference-clause.induct, auto)

lemma inference-preserves-finite-snd:
  assumes inference  $\psi$   $\psi'$  and finite (snd  $\psi$ )
  shows finite (snd  $\psi'$ )
  using assms inference-clause-preserves-finite-snd by (induct rule: inference.induct, fastforce)

lemma rtranclp-inference-preserves-finite:
  assumes rtranclp inference  $\psi$   $\psi'$  and finite (fst  $\psi$ )
  shows finite (fst  $\psi'$ )
  using assms by (induct rule: rtranclp-induct)
  (auto simp add: simplify-preserves-finite inference-preserves-finite)

lemma consistent-interp-insert:
  assumes consistent-interp I
  and atm-of P  $\notin$  atm-of 'I
  shows consistent-interp (insert P I)
proof –
  have P: insert P I = I  $\cup$  {P} by auto
  show ?thesis unfolding P
  apply (rule consistent-interp-disjoint)
  using assms by (auto simp: image-iff)
qed

lemma simplify-clause-preserves-sat:
  assumes simp: simplify  $\psi$   $\psi'$ 
  and satisfiable  $\psi'$ 
  shows satisfiable  $\psi$ 
  using assms
proof induction
  case (tautology-deletion A P) note AP = this(1) and sat = this(2)
  let  $?A' = A + \{\#Pos\ P\# \} + \{\#Neg\ P\# \}$ 
  let  $? \psi' = \psi - \{?A'\}$ 
  obtain I where
    I: I  $\models_s$   $? \psi'$  and
    cons: consistent-interp I and
    tot: total-over-m I  $? \psi'$ 
  using sat unfolding satisfiable-def by auto
  { assume Pos P  $\in$  I  $\vee$  Neg P  $\in$  I
    then have I  $\models$   $?A'$  by auto
    then have I  $\models_s$   $\psi$  using I by (metis insert-Diff tautology-deletion.hyps true-clss-insert)
    then have ?case using cons tot by auto
  }

```

```

}
moreover {
  assume Pos: Pos P  $\notin$  I and Neg: Neg P  $\notin$  I
  then have consistent-interp (I  $\cup$  {Pos P}) using cons by simp
  moreover have I'A: I  $\cup$  {Pos P}  $\models$  ?A' by auto
  have {Pos P}  $\cup$  I  $\models_s \psi - \{A + \{\#Pos\ P\#\} + \{\#Neg\ P\#\}\}$ 
    using  $\langle I \models_s \psi - \{A + \{\#Pos\ P\#\} + \{\#Neg\ P\#\}\} \rangle$  true-clss-union-increase' by blast
  then have I  $\cup$  {Pos P}  $\models_s \psi$ 
    by (metis (no-types) Un-empty-right Un-insert-left Un-insert-right I'A insert-Diff
      sup-bot.left-neutral tautology-deletion.hyps true-clss-insert)
  ultimately have ?case using satisfiable-carac' by blast
}
ultimately show ?case by blast
next
case (condensation A L) note AL = this(1) and sat = this(2)
have f3: simplify  $\psi$  ( $\psi - \{A + \{\#L\#\} + \{\#L\#\}\} \cup \{A + \{\#L\#\}\}$ )
  using AL simplify.condensation by blast
obtain LL :: 'a literal multiset set  $\Rightarrow$  'a literal set where
  f4: LL ( $\psi - \{A + \{\#L\#\} + \{\#L\#\}\} \cup \{A + \{\#L\#\}\}$ )  $\models_s \psi - \{A + \{\#L\#\} + \{\#L\#\}\} \cup \{A$ 
 $+ \{\#L\#\}$ 
     $\wedge$  consistent-interp (LL ( $\psi - \{A + \{\#L\#\} + \{\#L\#\}\} \cup \{A + \{\#L\#\}\}$ ))
     $\wedge$  total-over-m (LL ( $\psi - \{A + \{\#L\#\} + \{\#L\#\}\}$ 
       $\cup \{A + \{\#L\#\}\}$ )) ( $\psi - \{A + \{\#L\#\} + \{\#L\#\}\} \cup \{A + \{\#L\#\}\}$ )
  using sat by (meson satisfiable-def)
have f5: insert (A + {#L#} + {#L#}) ( $\psi - \{A + \{\#L\#\} + \{\#L\#\}\}$ ) =  $\psi$ 
  using AL by fastforce
have atms-of (A + {#L#} + {#L#}) = atms-of ({#L#} + A)
  by simp
then show ?case
  using f5 f4 f3 by (metis (no-types) add.commute satisfiable-def simplify-preserves-un-sat'
    total-over-m-insert total-over-m-union)
next
case (subsumption A B) note A = this(1) and AB = this(2) and B = this(3) and sat = this(4)
let ? $\psi'$  =  $\psi - \{B\}$ 
obtain I where I: I  $\models_s$  ? $\psi'$  and cons: consistent-interp I and tot: total-over-m I ? $\psi'$ 
  using sat unfolding satisfiable-def by auto
have I  $\models$  A using A I by (metis AB Diff-iff subset-mset.less-irrefl singletonD true-clss-def)
then have I  $\models$  B using AB subset-mset.less-imp-le true-clss-mono-leD by blast
then have I  $\models_s \psi$  using I by (metis insert-Diff-single true-clss-insert)
then show ?case using cons satisfiable-carac' by blast
qed

```

lemma simplify-preserves-unsat:

```

assumes inference  $\psi$   $\psi'$ 
shows satisfiable (fst  $\psi'$ )  $\longrightarrow$  satisfiable (fst  $\psi$ )
using assms apply (induct rule: inference.induct)
using satisfiable-decreasing by (metis fst-conv)+

```

lemma inference-preserves-unsat:

```

assumes inference** S S'
shows satisfiable (fst S')  $\longrightarrow$  satisfiable (fst S)
using assms apply (induct rule: rtranclp-induct)
apply simp-all
using simplify-preserves-unsat by blast

```

datatype 'v sem-tree = Node 'v 'v sem-tree 'v sem-tree | Leaf

fun sem-tree-size :: 'v sem-tree \Rightarrow nat **where**

sem-tree-size Leaf = 0 |

sem-tree-size (Node - ag ad) = 1 + sem-tree-size ag + sem-tree-size ad

lemma sem-tree-size[case-names bigger]:

($\bigwedge xs:: 'v \text{ sem-tree. } (\bigwedge ys:: 'v \text{ sem-tree. } \text{sem-tree-size } ys < \text{sem-tree-size } xs \Rightarrow P \text{ } ys) \Rightarrow P \text{ } xs$)
 $\Rightarrow P \text{ } xs$

by (fact Nat.measure-induct-rule)

fun partial-interps :: 'v sem-tree \Rightarrow 'v interp \Rightarrow 'v clauses \Rightarrow bool **where**

partial-interps Leaf I ψ = ($\exists \chi. \neg I \models \chi \wedge \chi \in \psi \wedge \text{total-over-m } I \{ \chi \}$) |

partial-interps (Node v ag ad) I $\psi \longleftrightarrow$

(partial-interps ag (I \cup {Pos v}) $\psi \wedge$ partial-interps ad (I \cup {Neg v}) ψ)

lemma simplify-preserve-partial-leaf:

simplify N N' \Rightarrow partial-interps Leaf I N \Rightarrow partial-interps Leaf I N'

apply (induct rule: simplify.induct)

using union-lcomm **apply** auto[1]

apply (simp, metis atms-of-plus total-over-set-union true-cls-union)

apply simp

by (metis atms-of-ms-singleton mset-le-exists-conv subset-mset-def true-cls-mono-leD
total-over-m-def total-over-m-sum)

lemma simplify-preserve-partial-tree:

assumes simplify N N'

and partial-interps t I N

shows partial-interps t I N'

using assms **apply** (induct t arbitrary: I, simp)

using simplify-preserve-partial-leaf **by** metis

lemma inference-preserve-partial-tree:

assumes inference S S'

and partial-interps t I (fst S)

shows partial-interps t I (fst S')

using assms **apply** (induct t arbitrary: I, simp-all)

by (meson inference-increasing)

lemma rtranclp-inference-preserve-partial-tree:

assumes rtranclp inference N N'

and partial-interps t I (fst N)

shows partial-interps t I (fst N')

using assms **apply** (induct rule: rtranclp-induct, auto)

using inference-preserve-partial-tree **by** force

function build-sem-tree :: 'v :: linorder set \Rightarrow 'v clauses \Rightarrow 'v sem-tree **where**

build-sem-tree atms ψ =

(if atms = {} $\vee \neg$ finite atms

```

then Leaf
else Node (Min atms) (build-sem-tree (Set.remove (Min atms) atms)  $\psi$ )
  (build-sem-tree (Set.remove (Min atms) atms)  $\psi$ )
by auto
termination
  apply (relation measure ( $\lambda(A, -). \text{card } A$ ), simp-all)
  apply (metis Min-in card-Diff1-less remove-def)+
done
declare build-sem-tree.induct[case-names tree]

lemma unsatisfiable-empty[simp]:
   $\neg$ unsatisfiable {}
  unfolding satisfiable-def apply auto
  using consistent-interp-def unfolding total-over-m-def total-over-set-def atms-of-ms-def by blast

lemma partial-interps-build-sem-tree-atms-general:
  fixes  $\psi :: 'v :: \text{linorder}$  clauses and  $p :: 'v$  literal list
  assumes unsat: unsatisfiable  $\psi$  and finite  $\psi$  and consistent-interp  $I$ 
  and finite atms
  and atms-of-ms  $\psi = \text{atms} \cup \text{atms-of-s } I$  and  $\text{atms} \cap \text{atms-of-s } I = \{\}$ 
  shows partial-interps (build-sem-tree atms  $\psi$ )  $I$   $\psi$ 
  using assms
proof (induct arbitrary:  $I$  rule: build-sem-tree.induct)
case (1 atms  $\psi$   $I_a$ ) note IH1 = this(1) and IH2 = this(2) and unsat = this(3) and finite = this(4)
  and cons = this(5) and  $f = \text{this}(6)$  and  $un = \text{this}(7)$  and  $disj = \text{this}(8)$ 
{
  assume atms: atms = {}
  then have atmsIa: atms-of-ms  $\psi = \text{atms-of-s } I_a$  using  $un$  by auto
  then have total-over-m  $I_a$   $\psi$  unfolding total-over-m-def atmsIa by auto
  then have  $\chi: \exists \chi \in \psi. \neg I_a \models \chi$ 
    using unsat cons unfolding true-clss-def satisfiable-def by auto
  then have build-sem-tree atms  $\psi = \text{Leaf}$  using atms by auto
  moreover
    have tot:  $\bigwedge \chi. \chi \in \psi \implies \text{total-over-m } I_a \{\chi\}$ 
    unfolding total-over-m-def total-over-set-def atms-of-ms-def atms-of-s-def
    using atmsIa atms-of-ms-def by fastforce
  have partial-interps Leaf  $I_a$   $\psi$ 
    using  $\chi$  tot by (auto simp add: total-over-m-def total-over-set-def atms-of-ms-def)

  ultimately have ?case by metis
}
moreover {
  assume atms: atms  $\neq \{\}$ 
  have build-sem-tree atms  $\psi = \text{Node } (Min \text{ atms}) (\text{build-sem-tree } (\text{Set.remove } (Min \text{ atms}) \text{ atms}) \psi)$ 
    ( $\text{build-sem-tree } (\text{Set.remove } (Min \text{ atms}) \text{ atms}) \psi$ )
    using build-sem-tree.simps[of atms  $\psi$ ]  $f$  atms by metis

  have consistent-interp ( $I_a \cup \{\text{Pos } (Min \text{ atms})\}$ ) unfolding consistent-interp-def
    by (metis Int-iff Min-in Un-iff atm-of-uminus atms cons consistent-interp-def disj empty-iff
       $f$  in-atms-of-s-decomp insert-iff literal.distinct(1) literal.exhaust-sel literal.sel(2)
      uminus-Neg uminus-Pos)
  moreover have atms-of-ms  $\psi = \text{Set.remove } (Min \text{ atms}) \text{ atms} \cup \text{atms-of-s } (I_a \cup \{\text{Pos } (Min \text{ atms})\})$ 
    using Min-in atms  $f$   $un$  by fastforce
  moreover have  $disj'$ :  $\text{Set.remove } (Min \text{ atms}) \text{ atms} \cap \text{atms-of-s } (I_a \cup \{\text{Pos } (Min \text{ atms})\}) = \{\}$ 
    by simp (metis disj disjoint-iff-not-equal member-remove)
}

```

moreover have *finite* (*Set.remove* (*Min* *atms*) *atms*) **using** *f* **by** (*simp add: remove-def*)
ultimately have *subtree1: partial-interps* (*build-sem-tree* (*Set.remove* (*Min* *atms*) *atms*) ψ)
(*Ia* \cup {*Pos* (*Min* *atms*)}) ψ
using *IH1*[*of Ia* \cup {*Pos* (*Min* (*atms*))}] *atms f unsat finite by metis*

have *consistent-interp* (*Ia* \cup {*Neg* (*Min* *atms*)}) **unfolding** *consistent-interp-def*
by (*metis Int-iff Min-in Un-iff atm-of-uminus atms cons consistent-interp-def disj empty-iff*
f in-atms-of-s-decomp insert-iff literal.distinct(1) literal.exhaust-sel literal.sel(2)
uminus-Neg)
moreover have *atms-of-ms* $\psi = \text{Set.remove } (\text{Min } \text{atms}) \text{ atms} \cup \text{atms-of-s } (\text{Ia} \cup \{\text{Neg } (\text{Min } \text{atms})\})$
using $\langle \text{atms-of-ms } \psi = \text{Set.remove } (\text{Min } \text{atms}) \text{ atms} \cup \text{atms-of-s } (\text{Ia} \cup \{\text{Pos } (\text{Min } \text{atms})\}) \rangle$ **by**
blast

moreover have *disj'*: *Set.remove* (*Min* *atms*) *atms* \cap *atms-of-s* (*Ia* \cup {*Neg* (*Min* *atms*)}) = {}
using *disj* **by** *auto*
moreover have *finite* (*Set.remove* (*Min* *atms*) *atms*) **using** *f* **by** (*simp add: remove-def*)
ultimately have *subtree2: partial-interps* (*build-sem-tree* (*Set.remove* (*Min* *atms*) *atms*) ψ)
(*Ia* \cup {*Neg* (*Min* *atms*)}) ψ
using *IH2*[*of Ia* \cup {*Neg* (*Min* (*atms*))}] *atms f unsat finite by metis*

then have *?case*
using *IH1 subtree1 subtree2 f local.finite unsat atms by simp*
}
ultimately show *?case* **by** *metis*
qed

lemma *partial-interps-build-sem-tree-atms:*

fixes $\psi :: 'v :: \text{linorder clauses}$ **and** $p :: 'v \text{ literal list}$
assumes *unsat: unsatisfiable* ψ **and** *finite: finite* ψ
shows *partial-interps* (*build-sem-tree* (*atms-of-ms* ψ) ψ) {} ψ

proof –

have *consistent-interp* {} **unfolding** *consistent-interp-def* **by** *auto*
moreover have *atms-of-ms* $\psi = \text{atms-of-ms } \psi \cup \text{atms-of-s } \{\}$ **unfolding** *atms-of-s-def* **by** *auto*
moreover have *atms-of-ms* $\psi \cap \text{atms-of-s } \{\} = \{\}$ **unfolding** *atms-of-s-def* **by** *auto*
moreover have *finite* (*atms-of-ms* ψ) **unfolding** *atms-of-ms-def* **using** *finite* **by** *simp*
ultimately show *partial-interps* (*build-sem-tree* (*atms-of-ms* ψ) ψ) {} ψ
using *partial-interps-build-sem-tree-atms-general*[*of* ψ {} *atms-of-ms* ψ] *assms* **by** *metis*

qed

lemma *can-decrease-count:*

fixes $\psi'' :: 'v \text{ clauses} \times ('v \text{ clause} \times 'v \text{ clause} \times 'v) \text{ set}$
assumes *count* $\chi \ L = n$
and $L \in \# \chi$ **and** $\chi \in \text{fst } \psi$
shows $\exists \psi' \chi'. \text{inference}^{**} \psi \psi' \wedge \chi' \in \text{fst } \psi' \wedge (\forall L. L \in \# \chi \longleftrightarrow L \in \# \chi')$
 $\wedge \text{count } \chi' \ L = 1$
 $\wedge (\forall \varphi. \varphi \in \text{fst } \psi \longrightarrow \varphi \in \text{fst } \psi')$
 $\wedge (I \models \chi \longleftrightarrow I \models \chi')$
 $\wedge (\forall I'. \text{total-over-m } I' \{\chi\} \longrightarrow \text{total-over-m } I' \{\chi'\})$

using *assms*

proof (*induct n arbitrary: $\chi \psi$*)

case 0

then show *?case* **by** (*simp add: not-in-iff[symmetric]*)

next

case (*Suc n χ*)

```

note  $IH = this(1)$  and  $count = this(2)$  and  $L = this(3)$  and  $\chi = this(4)$ 
{
  assume  $n = 0$ 
  then have  $inference^{**} \psi \psi$ 
  and  $\chi \in fst \psi$ 
  and  $\forall L. (L \in \# \chi) \longleftrightarrow (L \in \# \chi)$ 
  and  $count \chi L = (1 :: nat)$ 
  and  $\forall \varphi. \varphi \in fst \psi \longrightarrow \varphi \in fst \psi$ 
  by (auto simp add: count L  $\chi$ )
  then have  $?case$  by metis
}
moreover {
  assume  $n > 0$ 
  then have  $\exists C. \chi = C + \{\#L, L\# \}$ 
  by (smt L Suc-eq-plus1-left add.left-commute add-diff-cancel-left' add-diff-cancel-right'
count-greater-zero-iff count-single local.count multi-member-split plus-multiset.rep-eq)
  then obtain  $C$  where  $C: \chi = C + \{\#L, L\# \}$  by metis
  let  $? \chi' = C + \{\#L\# \}$ 
  let  $? \psi' = (fst \psi \cup \{? \chi'\}, snd \psi)$ 
  have  $\varphi: \forall \varphi \in fst \psi. (\varphi \in fst \psi \vee \varphi \neq ? \chi') \longleftrightarrow \varphi \in fst ? \psi'$  unfolding  $C$  by auto
  have  $inf: inference \psi ? \psi'$ 
  using  $C$  factoring  $\chi$  prod.collapse union-commute inference-step by metis
  moreover have  $count': count ? \chi' L = n$  using  $C$  count by auto
  moreover have  $L \chi': L \in \# ? \chi'$  by auto
  moreover have  $\chi' \psi': ? \chi' \in fst ? \psi'$  by auto
  ultimately obtain  $\psi''$  and  $\chi''$ 
  where
     $inference^{**} ? \psi' \psi''$  and
     $\alpha: \chi'' \in fst \psi''$  and
     $\forall La. (La \in \# ? \chi') \longleftrightarrow (La \in \# \chi'')$  and
     $\beta: count \chi'' L = (1 :: nat)$  and
     $\varphi': \forall \varphi. \varphi \in fst ? \psi' \longrightarrow \varphi \in fst \psi''$  and
     $I \chi: I \models ? \chi' \longleftrightarrow I \models \chi''$  and
     $tot: \forall I'. total-over-m I' \{? \chi'\} \longrightarrow total-over-m I' \{\chi''\}$ 
    using  $IH[of ? \chi' ? \psi']$   $count' L \chi' \chi' \psi'$  by blast

  then have  $inference^{**} \psi \psi''$ 
  and  $\forall La. (La \in \# \chi) \longleftrightarrow (La \in \# \chi'')$ 
  using  $inf$  unfolding  $C$  by auto
  moreover have  $\forall \varphi. \varphi \in fst \psi \longrightarrow \varphi \in fst \psi''$  using  $\varphi \varphi'$  by metis
  moreover have  $I \models \chi \longleftrightarrow I \models \chi''$  using  $I \chi$  unfolding true-cls-def C by auto
  moreover have  $\forall I'. total-over-m I' \{\chi\} \longrightarrow total-over-m I' \{\chi''\}$ 
  using  $tot$  unfolding  $C$  total-over-m-def by auto
  ultimately have  $?case$  using  $\varphi \varphi' \alpha \beta$  by metis
}
ultimately show  $?case$  by auto
qed

```

lemma *can-decrease-tree-size:*

```

fixes  $\psi :: 'v$  state and  $tree :: 'v$  sem-tree
assumes finite (fst  $\psi$ ) and already-used-inv  $\psi$ 
and partial-interps tree I (fst  $\psi$ )
shows  $\exists (tree' :: 'v \text{ sem-tree}) \psi'. inference^{**} \psi \psi' \wedge partial-interps tree' I (fst \psi')$ 
   $\wedge (sem-tree-size tree' < sem-tree-size tree \vee sem-tree-size tree = 0)$ 
using assms

```

```

proof (induct arbitrary: I rule: sem-tree-size)
  case (bigger xs I) note  $IH = \text{this}(1)$  and  $\text{finite} = \text{this}(2)$  and  $a-u-i = \text{this}(3)$  and  $\text{part} = \text{this}(4)$ 

  {
    assume sem-tree-size xs = 0
    then have ?case using part by blast
  }

moreover {
  assume sn0: sem-tree-size xs > 0
  obtain ag ad v where xs: xs = Node v ag ad using sn0 by (cases xs, auto)
  {
    assume sem-tree-size ag = 0 and sem-tree-size ad = 0
    then have ag: ag = Leaf and ad: ad = Leaf by (cases ag, auto) (cases ad, auto)

    then obtain  $\chi \chi'$  where
       $\chi: \neg I \cup \{\text{Pos } v\} \models \chi$  and
       $\text{tot}\chi: \text{total-over-m } (I \cup \{\text{Pos } v\}) \{\chi\}$  and
       $\chi\psi: \chi \in \text{fst } \psi$  and
       $\chi': \neg I \cup \{\text{Neg } v\} \models \chi'$  and
       $\text{tot}\chi': \text{total-over-m } (I \cup \{\text{Neg } v\}) \{\chi'\}$  and
       $\chi'\psi: \chi' \in \text{fst } \psi$ 
      using part unfolding xs by auto
    have Posv:  $\neg \text{Pos } v \in \# \chi$  using  $\chi$  unfolding true-cls-def true-lit-def by auto
    have Negv:  $\neg \text{Neg } v \in \# \chi'$  using  $\chi'$  unfolding true-cls-def true-lit-def by auto
    {
      assume Neg $\chi$ :  $\neg \text{Neg } v \in \# \chi$ 
      have  $\neg I \models \chi$  using  $\chi$  Posv unfolding true-cls-def true-lit-def by auto
      moreover have total-over-m I  $\{\chi\}$ 
        using Posv Neg $\chi$  atm-imp-pos-or-neg-lit tot $\chi$  unfolding total-over-m-def total-over-set-def
        by fastforce
      ultimately have partial-interps Leaf I (fst  $\psi$ )
      and sem-tree-size Leaf < sem-tree-size xs
      and inference**  $\psi \psi$ 
      unfolding xs by (auto simp add:  $\chi\psi$ )
    }
  }
  moreover {
    assume Pos $\chi$ :  $\neg \text{Pos } v \in \# \chi'$ 
    then have I $\chi$ :  $\neg I \models \chi'$  using  $\chi'$  Posv unfolding true-cls-def true-lit-def by auto
    moreover have total-over-m I  $\{\chi'\}$ 
      using Negv Pos $\chi$  atm-imp-pos-or-neg-lit tot $\chi'$ 
      unfolding total-over-m-def total-over-set-def by fastforce
    ultimately have partial-interps Leaf I (fst  $\psi$ ) and
      sem-tree-size Leaf < sem-tree-size xs and
      inference**  $\psi \psi$ 
    using  $\chi'\psi$  I $\chi$  unfolding xs by auto
  }
}
moreover {
  assume neg: Neg v  $\in \# \chi$  and pos: Pos v  $\in \# \chi'$ 
  then obtain  $\psi' \chi^2$  where inf: rtrnclp inference  $\psi \psi'$  and  $\chi^2\text{incl}: \chi^2 \in \text{fst } \psi'$ 
    and  $\chi\chi^2\text{-incl}: \forall L. L \in \# \chi \longleftrightarrow L \in \# \chi^2$ 
    and count $\chi^2$ : count  $\chi^2$  (Neg v) = 1
    and  $\varphi: \forall \varphi::'v \text{ literal multiset. } \varphi \in \text{fst } \psi \longrightarrow \varphi \in \text{fst } \psi'$ 
    and I $\chi$ :  $I \models \chi \longleftrightarrow I \models \chi^2$ 
    and tot-imp $\chi$ :  $\forall I'. \text{total-over-m } I' \{\chi\} \longrightarrow \text{total-over-m } I' \{\chi^2\}$ 

```

```

using can-decrease-count[of  $\chi$   $Neg\ v$  count  $\chi\ (Neg\ v)\ \psi\ I$ ]  $\chi\psi\ \chi'\psi$  by auto

have  $\chi' \in fst\ \psi'$  by (simp add:  $\chi'\psi\ \varphi$ )
with pos
obtain  $\psi''\ \chi2'$  where
   $inf'$ : inference**  $\psi'\ \psi''$ 
  and  $\chi2'-incl$ :  $\chi2' \in fst\ \psi''$ 
  and  $\chi'\chi2'-incl$ :  $\forall L::'v\ literal. (L \in\# \chi') = (L \in\# \chi2')$ 
  and  $count\chi2'$ : count  $\chi2'\ (Pos\ v) = (1::nat)$ 
  and  $\varphi'$ :  $\forall \varphi::'v\ literal\ multiset. \varphi \in fst\ \psi' \longrightarrow \varphi \in fst\ \psi''$ 
  and  $I\chi'$ :  $I \models \chi' \longleftrightarrow I \models \chi2'$ 
  and  $tot-imp\chi'$ :  $\forall I'. total-over-m\ I'\ \{\chi'\} \longrightarrow total-over-m\ I'\ \{\chi2'\}$ 
using can-decrease-count[of  $\chi'\ Pos\ v$  count  $\chi'\ (Pos\ v)\ \psi'\ I$ ] by auto

obtain  $C$  where  $\chi2: \chi2 = C + \{\#Neg\ v\# \}$  and  $negC$ :  $Neg\ v \notin\# C$  and  $posC$ :  $Pos\ v \notin\# C$ 
proof -
  have  $\bigwedge m. Suc\ 0 - count\ m\ (Neg\ v) = count\ (\chi2 - m)\ (Neg\ v)$ 
    by (simp add: count $\chi2$ )
  then show ?thesis
    using that by (metis (no-types) One-nat-def Posv Suc-inject Suc-pred  $\chi\chi2'-incl$ 
      count-diff count-single insert-DiffM2 mem-Collect-eq multi-member-skip neg
      not-gr0 set-mset-def union-commute)
qed

obtain  $C'$  where
   $\chi2': \chi2' = C' + \{\#Pos\ v\# \}$  and
   $posC'$ :  $Pos\ v \notin\# C'$  and
   $negC'$ :  $Neg\ v \notin\# C'$ 
proof -
  assume  $a1$ :  $\bigwedge C'. \llbracket \chi2' = C' + \{\#Pos\ v\# \}; Pos\ v \notin\# C'; Neg\ v \notin\# C' \rrbracket \Longrightarrow thesis$ 
  have  $f2$ :  $\bigwedge n. (n::nat) - n = 0$ 
    by simp
  have  $Neg\ v \notin\# \chi2' - \{\#Pos\ v\# \}$ 
    using Negv  $\chi'\chi2'-incl$  by (auto simp: not-in-iff)
  have count  $\{\#Pos\ v\# \}\ (Pos\ v) = 1$ 
    by simp
  then show ?thesis
    by (metis  $\chi'\chi2'-incl\ \langle Neg\ v \notin\# \chi2' - \{\#Pos\ v\# \} \rangle a1\ count\chi2'\ count-diff\ f2$ 
      insert-DiffM2 less-numeral-extra(3) mem-Collect-eq pos set-mset-def)
qed

have already-used-inv  $\psi'$ 
  using rtranclp-inference-preserves-already-used-inv[of  $\psi\ \psi'$ ] a-u-i inf by blast
then have a-u-i- $\psi''$ : already-used-inv  $\psi''$ 
  using rtranclp-inference-preserves-already-used-inv a-u-i inf' unfolding tautology-def
  by simp

have  $totC$ : total-over-m  $I\ \{C\}$ 
  using tot-imp $\chi$  tot $\chi$  tot-over-m-remove[of  $I\ Pos\ v\ C$ ] negC posC unfolding  $\chi2$ 
  by (metis total-over-m-sum uminus-Neg uminus-of-uminus-id)
have  $totC'$ : total-over-m  $I\ \{C'\}$ 
  using tot-imp $\chi'$  tot $\chi'$  total-over-m-sum tot-over-m-remove[of  $I\ Neg\ v\ C'$ ] negC' posC'
  unfolding  $\chi2'$  by (metis total-over-m-sum uminus-Neg)
have  $\neg I \models C + C'$ 
  using  $\chi\ I\chi\ \chi'\ I\chi'$  unfolding  $\chi2\ \chi2'$  true-cls-def by auto

```



```

then have part-I-ψ''': partial-interps Leaf I (fst ψ'' ∪ {C + C'})
  using totC totC' by simp
  (metis  $\lhd I \models C + C'$  atms-of-ms-singleton total-over-m-def total-over-m-sum)
{
  assume ( $\{\#Pos\ v\# \} + C', \{\#Neg\ v\# \} + C \notin \text{snd } \psi''$ 
then have inf'': inference ψ'' (fst ψ'' ∪ {C + C'}, snd ψ'' ∪ {(χ2', χ2)})
  using add.commute φ' χ2incl (χ2' ∈ fst ψ'') unfolding χ2 χ2'
  by (metis prod.collapse inference-step resolution)
have inference** ψ (fst ψ'' ∪ {C + C'}, snd ψ'' ∪ {(χ2', χ2)})
  using inf inf' inf'' rtranclp-trans by auto
moreover have sem-tree-size Leaf < sem-tree-size xs unfolding xs by auto
ultimately have ?case using part-I-ψ''' by (metis fst-conv)
}
moreover {
  assume a: ( $\{\#Pos\ v\# \} + C', \{\#Neg\ v\# \} + C \in \text{snd } \psi''$ 
then have ( $\exists \chi \in \text{fst } \psi''. (\forall I. \text{total-over-m } I \{C+C'\} \longrightarrow \text{total-over-m } I \{\chi\})$ 
     $\wedge (\forall I. \text{total-over-m } I \{\chi\} \longrightarrow I \models \chi \longrightarrow I \models C' + C)$ 
     $\vee \text{tautology } (C' + C)$ 
proof –
  obtain p where p: Pos p ∈# ({#Pos v#} + C') and
  n: Neg p ∈# ({#Neg v#} + C) and
  decomp: ( $(\exists \chi \in \text{fst } \psi''. (\forall I. \text{total-over-m } I \{(\{\#Pos\ v\# \} + C') - \{\#Pos\ p\# \} + ((\{\#Neg\ v\# \} + C) - \{\#Neg\ p\# \})) \longrightarrow \text{total-over-m } I \{\chi\})$ 
     $\wedge (\forall I. \text{total-over-m } I \{\chi\} \longrightarrow I \models \chi \longrightarrow I \models (\{\#Pos\ v\# \} + C') - \{\#Pos\ p\# \} + ((\{\#Neg\ v\# \} + C) - \{\#Neg\ p\# \}))$ 
     $\vee \text{tautology } ((\{\#Pos\ v\# \} + C') - \{\#Pos\ p\# \} + ((\{\#Neg\ v\# \} + C) - \{\#Neg\ p\# \})))$ 
using a by (blast intro: allE[OF a-u-i-ψ''[unfolding subsumes-def Ball-def], of ({#Pos v#} + C', {#Neg v#} + C)])
  { assume p ≠ v
    then have Pos p ∈# C' ∧ Neg p ∈# C using p n by force
    then have ?thesis unfolding Bex-def by auto
  }
moreover {
  assume p = v
  then have ?thesis using decomp by (metis add.commute add-diff-cancel-left')
}
ultimately show ?thesis by auto
qed
moreover {
  assume  $\exists \chi \in \text{fst } \psi''. (\forall I. \text{total-over-m } I \{C+C'\} \longrightarrow \text{total-over-m } I \{\chi\})$ 
   $\wedge (\forall I. \text{total-over-m } I \{\chi\} \longrightarrow I \models \chi \longrightarrow I \models C' + C)$ 
then obtain ∅ where ∅: ∅ ∈ fst ψ'' and
  tot-∅-CC':  $\forall I. \text{total-over-m } I \{C+C'\} \longrightarrow \text{total-over-m } I \{\emptyset\}$  and
  ∅-inv:  $\forall I. \text{total-over-m } I \{\emptyset\} \longrightarrow I \models \emptyset \longrightarrow I \models C' + C$  by blast
have partial-interps Leaf I (fst ψ'')
  using tot-∅-CC' ∅ ∅-inv totC totC' (λ I. I ⊨ C + C') total-over-m-sum by fastforce
moreover have sem-tree-size Leaf < sem-tree-size xs unfolding xs by auto
ultimately have ?case by (metis inf inf' rtranclp-trans)
}
moreover {
  assume tautCC': tautology (C' + C)
  have total-over-m I {C'+C} using totC totC' total-over-m-sum by auto

```

```

    then have  $\neg$ tautology  $(C' + C)$ 
      using  $\langle \neg I \models C + C' \rangle$  unfolding add.commute[ $\text{of } C \ C'$ ] total-over-m-def
      unfolding tautology-def by auto
    then have False using tautCC' unfolding tautology-def by auto
  }
  ultimately have ?case by auto
}
ultimately have ?case by auto
}
ultimately have ?case using part by (metis (no-types) sem-tree-size.simps(1))
}
moreover {
  assume size-ag: sem-tree-size ag > 0
  have sem-tree-size ag < sem-tree-size xs unfolding xs by auto
  moreover have partial-interps ag ( $I \cup \{Pos\ v\}$ ) (fst  $\psi$ )
    and partad: partial-interps ad ( $I \cup \{Neg\ v\}$ ) (fst  $\psi$ )
    using part partial-interps.simps(2) unfolding xs by metis+
  moreover have sem-tree-size ag < sem-tree-size xs  $\longrightarrow$  finite (fst  $\psi$ )  $\longrightarrow$  already-used-inv  $\psi$ 
     $\longrightarrow$  (partial-interps ag ( $I \cup \{Pos\ v\}$ ) (fst  $\psi$ )  $\longrightarrow$ 
      ( $\exists$  tree'  $\psi'$ . inference**  $\psi\ \psi' \wedge$  partial-interps tree' ( $I \cup \{Pos\ v\}$ ) (fst  $\psi'$ )
         $\wedge$  (sem-tree-size tree' < sem-tree-size ag  $\vee$  sem-tree-size ag = 0)))
    using IH by auto
  ultimately obtain  $\psi' :: 'v$  state and tree' :: 'v sem-tree where
    inf: inference**  $\psi\ \psi'$ 
    and part: partial-interps tree' ( $I \cup \{Pos\ v\}$ ) (fst  $\psi'$ )
    and size: sem-tree-size tree' < sem-tree-size ag  $\vee$  sem-tree-size ag = 0
    using finite part rtranclp.rtrancl-refl a-u-i by blast

  have partial-interps ad ( $I \cup \{Neg\ v\}$ ) (fst  $\psi'$ )
    using rtranclp-inference-preserve-partial-tree inf partad by metis
  then have partial-interps (Node v tree' ad) I (fst  $\psi'$ ) using part by auto
  then have ?case using inf size size-ag part unfolding xs by fastforce
}
moreover {
  assume size-ad: sem-tree-size ad > 0
  have sem-tree-size ad < sem-tree-size xs unfolding xs by auto
  moreover have partag: partial-interps ag ( $I \cup \{Pos\ v\}$ ) (fst  $\psi$ ) and
    partial-interps ad ( $I \cup \{Neg\ v\}$ ) (fst  $\psi$ )
    using part partial-interps.simps(2) unfolding xs by metis+
  moreover have sem-tree-size ad < sem-tree-size xs  $\longrightarrow$  finite (fst  $\psi$ )  $\longrightarrow$  already-used-inv  $\psi$ 
     $\longrightarrow$  (partial-interps ad ( $I \cup \{Neg\ v\}$ ) (fst  $\psi$ )
       $\longrightarrow$  ( $\exists$  tree'  $\psi'$ . inference**  $\psi\ \psi' \wedge$  partial-interps tree' ( $I \cup \{Neg\ v\}$ ) (fst  $\psi'$ )
         $\wedge$  (sem-tree-size tree' < sem-tree-size ad  $\vee$  sem-tree-size ad = 0)))
    using IH by auto
  ultimately obtain  $\psi' :: 'v$  state and tree' :: 'v sem-tree where
    inf: inference**  $\psi\ \psi'$ 
    and part: partial-interps tree' ( $I \cup \{Neg\ v\}$ ) (fst  $\psi'$ )
    and size: sem-tree-size tree' < sem-tree-size ad  $\vee$  sem-tree-size ad = 0
    using finite part rtranclp.rtrancl-refl a-u-i by blast

  have partial-interps ag ( $I \cup \{Pos\ v\}$ ) (fst  $\psi'$ )
    using rtranclp-inference-preserve-partial-tree inf partag by metis
  then have partial-interps (Node v ag tree') I (fst  $\psi'$ ) using part by auto
  then have ?case using inf size size-ad unfolding xs by fastforce
}

```

```

    ultimately have ?case by auto
  }
  ultimately show ?case by auto
qed

lemma inference-completeness-inv:
  fixes  $\psi :: 'v :: \text{linorder}$  state
  assumes
    unsat:  $\neg \text{satisfiable (fst } \psi)$  and
    finite:  $\text{finite (fst } \psi)$  and
    a-u-v:  $\text{already-used-inv } \psi$ 
  shows  $\exists \psi'. (\text{inference}^{**} \psi \psi' \wedge \{\#\} \in \text{fst } \psi')$ 
proof -
  obtain tree where  $\text{partial-interps tree } \{\}$   $(\text{fst } \psi)$ 
  using  $\text{partial-interps-build-sem-tree-atms assms}$  by metis
  then show ?thesis
  using unsat finite a-u-v
  proof (induct tree arbitrary:  $\psi$  rule:  $\text{sem-tree-size}$ )
    case (bigger tree  $\psi$ ) note  $H = \text{this}$ 
    {
      fix  $\chi$ 
      assume  $\text{tree: tree} = \text{Leaf}$ 
      obtain  $\chi$  where  $\chi: \neg \{\} \models \chi$  and  $\text{tot}\chi: \text{total-over-m } \{\} \{\chi\}$  and  $\chi\psi: \chi \in \text{fst } \psi$ 
      using  $H$  unfolding tree by auto
      moreover have  $\{\#\} = \chi$ 
      using  $\text{tot}\chi$  unfolding  $\text{total-over-m-def total-over-set-def}$  by fastforce
      moreover have  $\text{inference}^{**} \psi \psi$  by auto
      ultimately have ?case by metis
    }
    moreover {
      fix  $v$  tree1 tree2
      assume  $\text{tree: tree} = \text{Node } v \text{ tree1 tree2}$ 
      obtain
        tree'  $\psi'$  where  $\text{inf: inference}^{**} \psi \psi'$  and
        part':  $\text{partial-interps tree' } \{\}$   $(\text{fst } \psi')$  and
        decrease:  $\text{sem-tree-size tree}' < \text{sem-tree-size tree} \vee \text{sem-tree-size tree} = 0$ 
        using  $\text{can-decrease-tree-size[of } \psi] H(2,4,5)$  unfolding  $\text{tautology-def}$  by meson
        have  $\text{sem-tree-size tree}' < \text{sem-tree-size tree}$  using decrease unfolding tree by auto
        moreover have  $\text{finite (fst } \psi')$  using  $\text{rtranclp-inference-preserves-finite inf } H(4)$  by metis
        moreover have  $\text{unsatisfiable (fst } \psi')$ 
        using  $\text{inference-preserves-unsat inf bigger.prem}(2)$  by blast
        moreover have  $\text{already-used-inv } \psi'$ 
        using  $H(5)$  inf  $\text{rtranclp-inference-preserves-already-used-inv[of } \psi \psi']$  by auto
        ultimately have ?case using inf  $\text{rtranclp-trans part' } H(1)$  by fastforce
    }
  }
  ultimately show ?case by (cases tree, auto)
qed
qed

```

```

lemma inference-completeness:
  fixes  $\psi :: 'v :: \text{linorder}$  state
  assumes unsat:  $\neg \text{satisfiable (fst } \psi)$ 
  and finite:  $\text{finite (fst } \psi)$ 
  and snd  $\psi = \{\}$ 
  shows  $\exists \psi'. (\text{rtranclp inference } \psi \psi' \wedge \{\#\} \in \text{fst } \psi')$ 

```

proof –

have *already-used-inv* ψ **unfolding** *assms* **by** *auto*
 then show *?thesis* **using** *assms inference-completeness-inv* **by** *blast*
qed

lemma *inference-soundness*:

fixes $\psi :: 'v :: \text{linorder state}$
 assumes *rtranclp inference* $\psi \ \psi'$ **and** $\{\#\} \in \text{fst } \psi'$
 shows *unsatisfiable* (*fst* ψ)
 using *assms* **by** (*meson rtranclp-inference-preserves-un-sat satisfiable-def true-cls-empty true-cls-def*)

lemma *inference-soundness-and-completeness*:

fixes $\psi :: 'v :: \text{linorder state}$
 assumes *finite*: *finite* (*fst* ψ)
 and *snd* $\psi = \{\}$
 shows $(\exists \psi'. (\text{inference}^{**} \ \psi \ \psi' \wedge \{\#\} \in \text{fst } \psi')) \longleftrightarrow \text{unsatisfiable } (\text{fst } \psi)$
 using *assms inference-completeness inference-soundness* **by** *metis*

12.4 Lemma about the simplified state

abbreviation *simplified* $\psi \equiv (\text{no-step simplify } \psi)$

lemma *simplified-count*:

assumes *simp*: *simplified* ψ **and** $\chi: \chi \in \psi$
 shows *count* $\chi \ L \leq 1$

proof –

 {
 let $? \chi' = \chi - \{\#L, L\#\}$
 assume *count* $\chi \ L \geq 2$
 then have *f1*: *count* $(\chi - \{\#L, L\#\} + \{\#L, L\\}) \ L = \text{count } \chi \ L$
 by *simp*
 then have $L \in \# \ \chi - \{\#L\#\}$
 by (*metis* (*no-types*) *add.left-neutral add-diff-cancel-left' count-union diff-diff-add diff-single-trivial insert-DiffM mem-Collect-eq multi-member-this not-gr0 set-mset-def*)
 then have $\chi': ? \chi' + \{\#L\#\} + \{\#L\#\} = \chi$
 using *f1* **by** (*metis diff-diff-add diff-single-eq-union in-diffD*)

have $\exists \psi'. \text{simplify } \psi \ \psi'$

by (*metis* (*no-types, hide-lams*) $\chi \ \chi'$ *add.commute factoring-imp-simplify union-assoc*)

then have *False* **using** *simp* **by** *auto*

 }

then show *?thesis* **by** *arith*

qed

lemma *simplified-no-both*:

assumes *simp*: *simplified* ψ **and** $\chi: \chi \in \psi$
 shows $\neg (L \in \# \ \chi \wedge -L \in \# \ \chi)$

proof (*rule ccontr*)

assume $\neg \neg (L \in \# \ \chi \wedge -L \in \# \ \chi)$

then have $L \in \# \ \chi \wedge -L \in \# \ \chi$ **by** *metis*

then obtain χ' **where** $\chi = \chi' + \{\#Pos \ (\text{atm-of } L)\#\} + \{\#Neg \ (\text{atm-of } L)\#\}$

by (*metis Neg-atm-of-iff Pos-atm-of-iff diff-union-swap insert-DiffM2 uminus-Neg uminus-Pos*)

then show *False* **using** χ *simp tautology-deletion* **by** *fastforce*

qed

```

lemma simplified-not-tautology:
  assumes simplified  $\{\psi\}$ 
  shows  $\sim \text{tautology } \psi$ 
proof (rule ccontr)
  assume  $\sim ?thesis$ 
  then obtain  $p$  where  $Pos\ p \in \# \psi \wedge Neg\ p \in \# \psi$  using tautology-decomp by metis
  then obtain  $\chi$  where  $\psi = \chi + \{\#Pos\ p\# \} + \{\#Neg\ p\# \}$ 
    by (metis insert-noteq-member literal.distinct(1) multi-member-split)
  then have  $\sim \text{simplified } \{\psi\}$  by (auto intro: tautology-deletion)
  then show False using assms by auto
qed

```

```

lemma simplified-remove:
  assumes simplified  $\{\psi\}$ 
  shows simplified  $\{\psi - \{\#l\#\}\}$ 
proof (rule ccontr)
  assume  $ns: \neg \text{simplified } \{\psi - \{\#l\#\}\}$ 
  {
    assume  $\neg l \in \# \psi$ 
    then have  $\psi - \{\#l\#\} = \psi$  by simp
    then have False using ns assms by auto
  }
  moreover {
    assume  $l\psi: l \in \# \psi$ 
    have  $A: \bigwedge A. A \in \{\psi - \{\#l\#\}\} \longleftrightarrow A + \{\#l\#\} \in \{\psi\}$  by (auto simp add: lψ)
    obtain  $l'$  where  $l': \text{simplify } \{\psi - \{\#l\#\}\} \ l'$  using ns by metis
    then have  $\exists l'. \text{simplify } \{\psi\} \ l'$ 
    proof (induction rule: simplify.induct)
      case (tautology-deletion A P)
      have  $\{\#Neg\ P\# \} + (\{\#Pos\ P\# \} + (A + \{\#l\#\})) \in \{\psi\}$ 
        by (metis (no-types) A add.commute tautology-deletion.hyps union-lcomm)
      then show ?thesis
        by (metis simplify.tautology-deletion[of A + \{\#l\#\} P \{\psi\}] add.commute)
    next
      case (condensation A L)
      have  $A + \{\#L\#\} + \{\#L\#\} + \{\#l\#\} \in \{\psi\}$ 
        using A condensation.hyps by blast
      then have  $\{\#L, L\#\} + (A + \{\#l\#\}) \in \{\psi\}$ 
        by (metis (no-types) union-assoc union-commute)
      then show ?case
        using factoring-imp-simplify by blast
    next
      case (subsumption A B)
      then show ?case by blast
    qed
  }
  then have False using assms(1) by blast
}
ultimately show False by auto
qed

```

```

lemma in-simplified-simplified:
  assumes simp: simplified  $\psi$  and incl:  $\psi' \subseteq \psi$ 
  shows simplified  $\psi'$ 
proof (rule ccontr)

```

```

assume  $\neg$  ?thesis
then obtain  $\psi''$  where simplify  $\psi' \psi''$  by metis
then have  $\exists l'. \text{simplify } \psi \ l'$ 
  proof (induction rule: simplify.induct)
    case (tautology-deletion A P)
    then show ?thesis using simplify.tautology-deletion[of A P  $\psi$ ] incl by blast
  next
    case (condensation A L)
    then show ?case using simplify.condensation[of A L  $\psi$ ] incl by blast
  next
    case (subsumption A B)
    then show ?case using simplify.subsumption[of A  $\psi$  B] incl by auto
  qed
then show False using assms(1) by blast
qed

```

```

lemma simplified-in:
  assumes simplified  $\psi$ 
  and  $N \in \psi$ 
  shows simplified  $\{N\}$ 
  using assms by (metis Set.set-insert empty-subsetI in-simplified-simplified insert-mono)

```

```

lemma subsumes-imp-formula:
  assumes  $\psi \leq \# \varphi$ 
  shows  $\{\psi\} \models_p \varphi$ 
  unfolding true-clss-cls-def apply auto
  using assms true-cls-mono-leD by blast

```

```

lemma simplified-imp-distinct-mset-tauto:
  assumes simp: simplified  $\psi'$ 
  shows distinct-mset-set  $\psi'$  and  $\forall \chi \in \psi'. \neg \text{tautology } \chi$ 
proof -
  show  $\forall \chi \in \psi'. \neg \text{tautology } \chi$ 
    using simp by (auto simp add: simplified-in simplified-not-tautology)

```

```

show distinct-mset-set  $\psi'$ 
proof (rule ccontr)
  assume  $\neg$ ?thesis
  then obtain  $\chi$  where  $\chi \in \psi'$  and  $\neg \text{distinct-mset } \chi$  unfolding distinct-mset-set-def by auto
  then obtain L where count  $\chi$  L  $\geq 2$ 
    unfolding distinct-mset-def
    by (meson count-greater-eq-one-iff le-antisym simp simplified-count)
  then show False by (metis Suc-1  $\langle \chi \in \psi' \rangle$  not-less-eq-eq simp simplified-count)
qed
qed

```

```

lemma simplified-no-more-full1-simplified:
  assumes simplified  $\psi$ 
  shows  $\neg \text{full1 simplify } \psi \ \psi'$ 
  using assms unfolding full1-def by (meson tranclpD)

```

12.5 Resolution and Invariants

inductive resolution :: $'v \text{ state} \Rightarrow 'v \text{ state} \Rightarrow \text{bool}$ **where**
full1-simp: $\text{full1 simplify } N \ N' \Longrightarrow \text{resolution } (N, \text{already-used}) \ (N', \text{already-used})$ |
inferring: $\text{inference } (N, \text{already-used}) \ (N', \text{already-used}') \Longrightarrow \text{simplified } N$

$\implies \text{full simplify } N' N'' \implies \text{resolution } (N, \text{already-used}) (N'', \text{already-used}')$

12.5.1 Invariants

lemma *resolution-finite*:

assumes *resolution* $\psi \psi'$ **and** *finite* (*fst* ψ)
shows *finite* (*fst* ψ')
using *assms* **by** (*induct* *rule*: *resolution.induct*)
(auto simp add: full1-def full-def rtranclp-simplify-preserves-finite
dest: tranclp-into-rtranclp inference-preserves-finite)

lemma *rtranclp-resolution-finite*:

assumes *resolution*** $\psi \psi'$ **and** *finite* (*fst* ψ)
shows *finite* (*fst* ψ')
using *assms* **by** (*induct* *rule*: *rtranclp-induct*, *auto simp add: resolution-finite*)

lemma *resolution-finite-snd*:

assumes *resolution* $\psi \psi'$ **and** *finite* (*snd* ψ)
shows *finite* (*snd* ψ')
using *assms* **apply** (*induct* *rule*: *resolution.induct*, *auto simp add: inference-preserves-finite-snd*)
using *inference-preserves-finite-snd snd-conv* **by** *metis*

lemma *rtranclp-resolution-finite-snd*:

assumes *resolution*** $\psi \psi'$ **and** *finite* (*snd* ψ)
shows *finite* (*snd* ψ')
using *assms* **by** (*induct* *rule*: *rtranclp-induct*, *auto simp add: resolution-finite-snd*)

lemma *resolution-always-simplified*:

assumes *resolution* $\psi \psi'$
shows *simplified* (*fst* ψ')
using *assms* **by** (*induct* *rule*: *resolution.induct*)
(auto simp add: full1-def full-def)

lemma *tranclp-resolution-always-simplified*:

assumes *tranclp resolution* $\psi \psi'$
shows *simplified* (*fst* ψ')
using *assms* **by** (*induct* *rule*: *tranclp.induct*, *auto simp add: resolution-always-simplified*)

lemma *resolution-atms-of*:

assumes *resolution* $\psi \psi'$ **and** *finite* (*fst* ψ)
shows *atms-of-ms* (*fst* ψ') \subseteq *atms-of-ms* (*fst* ψ)
using *assms* **apply** (*induct* *rule*: *resolution.induct*)
apply (*simp add: rtranclp-simplify-atms-of-ms tranclp-into-rtranclp full1-def*)
by (*metis* (*no-types*, *lifting*) *contra-subsetD fst-conv full-def*
inference-preserves-atms-of-ms rtranclp-simplify-atms-of-ms subsetI)

lemma *rtranclp-resolution-atms-of*:

assumes *resolution*** $\psi \psi'$ **and** *finite* (*fst* ψ)
shows *atms-of-ms* (*fst* ψ') \subseteq *atms-of-ms* (*fst* ψ)
using *assms* **apply** (*induct* *rule*: *rtranclp-induct*)
using *resolution-atms-of rtranclp-resolution-finite* **by** *blast+*

lemma *resolution-include*:

assumes *res*: *resolution* $\psi \psi'$ **and** *finite*: *finite* (*fst* ψ)
shows *fst* $\psi' \subseteq$ *simple-clss* (*atms-of-ms* (*fst* ψ))

proof –

have *finite'*: *finite* (*fst* ψ') **using** *local.finite* *res* *resolution-finite* **by** *blast*
have *simplified* (*fst* ψ') **using** *res* *finite'* *resolution-always-simplified* **by** *blast*
then have *fst* $\psi' \subseteq \text{simple-clss}$ (*atms-of-ms* (*fst* ψ'))
using *simplified-in-simple-clss* *finite'* *simplified-imp-distinct-mset-tauto*[*of* *fst* ψ'] **by** *auto*
moreover have *atms-of-ms* (*fst* ψ') \subseteq *atms-of-ms* (*fst* ψ)
using *res* *finite* *resolution-atms-of*[*of* ψ ψ'] **by** *auto*
ultimately show *?thesis* **by** (*meson* *atms-of-ms-finite* *local.finite* *order.trans* *rev-finite-subset*
simple-clss-mono)
qed

lemma *rtranclp-resolution-include*:
assumes *res*: *trancplp* *resolution* ψ ψ' **and** *finite*: *finite* (*fst* ψ)
shows *fst* $\psi' \subseteq \text{simple-clss}$ (*atms-of-ms* (*fst* ψ))
using *assms* **apply** (*induct* *rule*: *trancplp.induct*)
apply (*simp* *add*: *resolution-include*)
by (*meson* *simple-clss-mono* *order-class.le-trans* *resolution-include*
rtranclp-resolution-atms-of *rtranclp-resolution-finite* *trancplp-into-rtranclp*)

abbreviation *already-used-all-simple*
 $:: ('a \text{ literal multiset} \times 'a \text{ literal multiset}) \text{ set} \Rightarrow 'a \text{ set} \Rightarrow \text{bool}$ **where**
already-used-all-simple *already-used* *vars* \equiv
 $(\forall (A, B) \in \text{already-used}. \text{simplified } \{A\} \wedge \text{simplified } \{B\} \wedge \text{atms-of } A \subseteq \text{vars} \wedge \text{atms-of } B \subseteq \text{vars})$

lemma *already-used-all-simple-vars-incl*:
assumes *vars* \subseteq *vars'*
shows *already-used-all-simple* *a* *vars* \implies *already-used-all-simple* *a* *vars'*
using *assms* **by** *fast*

lemma *inference-clause-preserves-already-used-all-simple*:
assumes *inference-clause* *S* *S'*
and *already-used-all-simple* (*snd* *S*) *vars*
and *simplified* (*fst* *S*)
and *atms-of-ms* (*fst* *S*) \subseteq *vars*
shows *already-used-all-simple* (*snd* (*fst* $S \cup \{\text{fst } S'\}$, *snd* S')) *vars*
using *assms*
proof (*induct* *rule*: *inference-clause.induct*)
case (*factoring* *L* *C* *N* *already-used*)
then show *?case* **by** (*simp* *add*: *simplified-in* *factoring-imp-simplify*)
next
case (*resolution* *P* *C* *N* *D* *already-used*) **note** *H* = *this*
show *?case* **apply** *clarify*
proof –
fix *A* *B* *v*
assume (*A*, *B*) \in *snd* (*fst* (*N*, *already-used*)
 $\cup \{\text{fst } (C + D, \text{already-used} \cup \{(\{\#Pos\ P\# \} + C, \{\#Neg\ P\# \} + D)\}),$
 $\text{snd } (C + D, \text{already-used} \cup \{(\{\#Pos\ P\# \} + C, \{\#Neg\ P\# \} + D)\})\}$)
then have (*A*, *B*) \in *already-used* \vee (*A*, *B*) = ($\{\#Pos\ P\# \} + C$, $\{\#Neg\ P\# \} + D$) **by** *auto*
moreover {
assume (*A*, *B*) \in *already-used*
then have *simplified* $\{A\} \wedge \text{simplified } \{B\} \wedge \text{atms-of } A \subseteq \text{vars} \wedge \text{atms-of } B \subseteq \text{vars}$
using *H*(4) **by** *auto*
}
moreover {
assume *eq*: (*A*, *B*) = ($\{\#Pos\ P\# \} + C$, $\{\#Neg\ P\# \} + D$)
then have *simplified* $\{A\}$ **using** *simplified-in* *H*(1,5) **by** *auto*
}


```

    moreover have simplified  $\{B\}$  using eq simplified-in  $H(2,5)$  by auto
    moreover have atms-of  $A \subseteq \text{atms-of-ms } N$ 
      using eq  $H(1)$ 
      using atms-of-atms-of-ms-mono[of  $A$   $N$ ] by auto
    moreover have atms-of  $B \subseteq \text{atms-of-ms } N$ 
      using eq  $H(2)$  atms-of-atms-of-ms-mono[of  $B$   $N$ ] by auto
    ultimately have simplified  $\{A\} \wedge \text{simplified } \{B\} \wedge \text{atms-of } A \subseteq \text{vars} \wedge \text{atms-of } B \subseteq \text{vars}$ 
      using  $H(6)$  by auto
  }
  ultimately show simplified  $\{A\} \wedge \text{simplified } \{B\} \wedge \text{atms-of } A \subseteq \text{vars} \wedge \text{atms-of } B \subseteq \text{vars}$ 
    by fast
qed

```

```

lemma inference-preserves-already-used-all-simple:
  assumes inference  $S$   $S'$ 
  and already-used-all-simple (snd  $S$ ) vars
  and simplified (fst  $S$ )
  and atms-of-ms (fst  $S$ )  $\subseteq \text{vars}$ 
  shows already-used-all-simple (snd  $S'$ ) vars
  using assms
proof (induct rule: inference.induct)
  case (inference-step  $S$  clause already-used)
  then show ?case
    using inference-clause-preserves-already-used-all-simple[of  $S$  (clause, already-used) vars]
    by auto
qed

```

```

lemma already-used-all-simple-inv:
  assumes resolution  $S$   $S'$ 
  and already-used-all-simple (snd  $S$ ) vars
  and atms-of-ms (fst  $S$ )  $\subseteq \text{vars}$ 
  shows already-used-all-simple (snd  $S'$ ) vars
  using assms
proof (induct rule: resolution.induct)
  case (full1-simp  $N$   $N'$ )
  then show ?case by simp
next
  case (inferring  $N$  already-used  $N'$  already-used'  $N''$ )
  then show already-used-all-simple (snd ( $N''$ , already-used') vars)
    using inference-preserves-already-used-all-simple[of ( $N$ , already-used)] by simp
qed

```

```

lemma rtrancpl-already-used-all-simple-inv:
  assumes resolution**  $S$   $S'$ 
  and already-used-all-simple (snd  $S$ ) vars
  and atms-of-ms (fst  $S$ )  $\subseteq \text{vars}$ 
  and finite (fst  $S$ )
  shows already-used-all-simple (snd  $S'$ ) vars
  using assms
proof (induct rule: rtrancpl-induct)
  case base
  then show ?case by simp
next
  case (step  $S'$   $S''$ ) note infstar = this(1) and IH = this(3) and res = this(2) and

```

$already = this(4)$ **and** $atms = this(5)$ **and** $finite = this(6)$
have $already-used-all-simple\ (snd\ S')$ $vars$ **using** $IH\ already\ atms\ finite$ **by** $simp$
moreover **have** $atms-of-ms\ (fst\ S') \subseteq atms-of-ms\ (fst\ S)$
by $(simp\ add: infstar\ local.finite\ rtranclp-resolution-atms-of)$
then **have** $atms-of-ms\ (fst\ S') \subseteq vars$ **using** $atms$ **by** $auto$
ultimately **show** $?case$
using $already-used-all-simple-inv[OF\ res]$ **by** $simp$
qed

lemma $inference-clause-simplified-already-used-subset$:
assumes $inference-clause\ S\ S'$
and $simplified\ (fst\ S)$
shows $snd\ S \subset snd\ S'$
using $assms$ **apply** $(induct\ rule: inference-clause.induct, auto)$
using $factoring-imp-simplify$ **by** $blast$

lemma $inference-simplified-already-used-subset$:
assumes $inference\ S\ S'$
and $simplified\ (fst\ S)$
shows $snd\ S \subset snd\ S'$
using $assms$ **apply** $(induct\ rule: inference.induct)$
by $(metis\ inference-clause-simplified-already-used-subset\ snd-conv)$

lemma $resolution-simplified-already-used-subset$:
assumes $resolution\ S\ S'$
and $simplified\ (fst\ S)$
shows $snd\ S \subset snd\ S'$
using $assms$ **apply** $(induct\ rule: resolution.induct, simp-all\ add: full1-def)$
apply $(meson\ tranclpD)$
by $(metis\ inference-simplified-already-used-subset\ fst-conv\ snd-conv)$

lemma $tranclp-resolution-simplified-already-used-subset$:
assumes $tranclp\ resolution\ S\ S'$
and $simplified\ (fst\ S)$
shows $snd\ S \subset snd\ S'$
using $assms$ **apply** $(induct\ rule: tranclp.induct)$
using $resolution-simplified-already-used-subset$ **apply** $metis$
by $(meson\ tranclp-resolution-always-simplified\ resolution-simplified-already-used-subset\ less-trans)$

abbreviation $already-used-top\ vars \equiv simple-clss\ vars \times simple-clss\ vars$

lemma $already-used-all-simple-in-already-used-top$:
assumes $already-used-all-simple\ s\ vars$ **and** $finite\ vars$
shows $s \subseteq already-used-top\ vars$
proof
fix x
assume $x-s: x \in s$
obtain $A\ B$ **where** $x: x = (A, B)$ **by** $(cases\ x, auto)$
then **have** $simplified\ \{A\}$ **and** $atms-of\ A \subseteq vars$ **using** $assms(1)\ x-s$ **by** $fastforce+$
then **have** $A: A \in simple-clss\ vars$
using $simple-clss-mono[of\ atms-of\ A\ vars]\ x\ assms(2)$
 $simplified-imp-distinct-mset-tauto[of\ \{A\}]$
 $distinct-mset-not-tautology-implies-in-simple-clss$ **by** $fast$
moreover **have** $simplified\ \{B\}$ **and** $atms-of\ B \subseteq vars$ **using** $assms(1)\ x-s\ x$ **by** $fast+$

then have $B: B \in \text{simple-clss vars}$
using *simplified-imp-distinct-mset-tauto*[of $\{B\}$]
distinct-mset-not-tautology-implies-in-simple-clss
simple-clss-mono[of *atms-of* $B \text{ vars}$] $x \text{ assms}(2)$ **by** *fast*
ultimately show $x \in \text{simple-clss vars} \times \text{simple-clss vars}$
unfolding x **by** *auto*
qed

lemma *already-used-top-finite*:
assumes *finite vars*
shows *finite (already-used-top vars)*
using *simple-clss-finite assms* **by** *auto*

lemma *already-used-top-increasing*:
assumes $\text{var} \subseteq \text{var}'$ **and** *finite var'*
shows *already-used-top var* \subseteq *already-used-top var'*
using *assms simple-clss-mono* **by** *auto*

lemma *already-used-all-simple-finite*:
fixes $s :: ('a \text{ literal multiset} \times 'a \text{ literal multiset}) \text{ set}$ **and** $\text{vars} :: 'a \text{ set}$
assumes *already-used-all-simple s vars* **and** *finite vars*
shows *finite s*
using *assms already-used-all-simple-in-already-used-top*[*OF assms(1)*]
rev-finite-subset[*OF already-used-top-finite*[of *vars*]] **by** *auto*

abbreviation *card-simple vars* $\psi \equiv \text{card (already-used-top vars} - \psi)$

lemma *resolution-card-simple-decreasing*:
assumes *res: resolution $\psi \psi'$*
and *a-u-s: already-used-all-simple (snd ψ) vars*
and *finite-v: finite vars*
and *finite-fst: finite (fst ψ)*
and *finite-snd: finite (snd ψ)*
and *simp: simplified (fst ψ)*
and *atms-of-ms (fst ψ) \subseteq vars*
shows *card-simple vars (snd ψ')* $<$ *card-simple vars (snd ψ)*

proof –
let $?vars = \text{vars}$
let $?top = \text{simple-clss } ?vars \times \text{simple-clss } ?vars$
have 1: *card-simple vars (snd ψ)* $= \text{card } ?top - \text{card (snd } \psi)$
using *card-Diff-subset finite-snd already-used-all-simple-in-already-used-top*[*OF a-u-s*]
finite-v **by** *metis*
have *a-u-s': already-used-all-simple (snd ψ') vars*
using *already-used-all-simple-inv res a-u-s assms(7)* **by** *blast*
have *f: finite (snd ψ')* **using** *already-used-all-simple-finite a-u-s' finite-v* **by** *auto*
have 2: *card-simple vars (snd ψ')* $= \text{card } ?top - \text{card (snd } \psi')$
using *card-Diff-subset*[*OF f*] *already-used-all-simple-in-already-used-top*[*OF a-u-s' finite-v*]
by *auto*
have *card (already-used-top vars)* $\geq \text{card (snd } \psi')$
using *already-used-all-simple-in-already-used-top*[*OF a-u-s' finite-v*]
card-mono[of *already-used-top vars snd ψ'*] *already-used-top-finite*[*OF finite-v*] **by** *metis*
then show *?thesis*
using *psubset-card-mono*[*OF f resolution-simplified-already-used-subset*[*OF res simp*]]
unfolding 1 2 **by** *linarith*
qed

lemma *trancpl-resolution-card-simple-decreasing*:
assumes *trancpl resolution* $\psi \ \psi'$ **and** *finite-fst*: *finite* (*fst* ψ)
and *already-used-all-simple* (*snd* ψ) *vars*
and *atms-of-ms* (*fst* ψ) \subseteq *vars*
and *finite-v*: *finite vars*
and *finite-snd*: *finite* (*snd* ψ)
and *simplified* (*fst* ψ)
shows *card-simple vars* (*snd* ψ') $<$ *card-simple vars* (*snd* ψ)
using *assms*
proof (*induct rule*: *trancpl-induct*)
case (*base* ψ')
then show ?*case* **by** (*simp add*: *resolution-card-simple-decreasing*)
next
case (*step* $\psi' \ \psi''$) **note** *res* = *this*(1) **and** *res'* = *this*(2) **and** *a-u-s* = *this*(5) **and**
atms = *this*(6) **and** *f-v* = *this*(7) **and** *f-fst* = *this*(4) **and** *H* = *this*
then have *card-simple vars* (*snd* ψ') $<$ *card-simple vars* (*snd* ψ) **by** *auto*
moreover have *a-u-s'*: *already-used-all-simple* (*snd* ψ') *vars*
using *rtrancpl-already-used-all-simple-inv*[*OF* *trancpl-into-rtrancpl*[*OF* *res*] *a-u-s atms f-fst*] .
have *finite* (*fst* ψ')
by (*meson finite-fst res rtrancpl-resolution-finite trancpl-into-rtrancpl*)
moreover have *finite* (*snd* ψ') **using** *already-used-all-simple-finite*[*OF* *a-u-s' f-v*] .
moreover have *simplified* (*fst* ψ') **using** *res trancpl-resolution-always-simplified* **by** *blast*
moreover have *atms-of-ms* (*fst* ψ') \subseteq *vars*
by (*meson atms f-fst order.trans res rtrancpl-resolution-atms-of trancpl-into-rtrancpl*)
ultimately show ?*case*
using *resolution-card-simple-decreasing*[*OF* *res' a-u-s' f-v*] *f-v*
less-trans[*of* *card-simple vars* (*snd* ψ'') *card-simple vars* (*snd* ψ')
card-simple vars (*snd* ψ)]
by *blast*
qed

lemma *trancpl-resolution-card-simple-decreasing-2*:
assumes *trancpl resolution* $\psi \ \psi'$
and *finite-fst*: *finite* (*fst* ψ)
and *empty-snd*: *snd* ψ = {}
and *simplified* (*fst* ψ)
shows *card-simple* (*atms-of-ms* (*fst* ψ)) (*snd* ψ') $<$ *card-simple* (*atms-of-ms* (*fst* ψ)) (*snd* ψ)
proof –
let ?*vars* = (*atms-of-ms* (*fst* ψ))
have *already-used-all-simple* (*snd* ψ) ?*vars* **unfolding** *empty-snd* **by** *auto*
moreover have *atms-of-ms* (*fst* ψ) \subseteq ?*vars* **by** *auto*
moreover have *finite-v*: *finite* ?*vars* **using** *finite-fst* **by** *auto*
moreover have *finite-snd*: *finite* (*snd* ψ) **unfolding** *empty-snd* **by** *auto*
ultimately show ?*thesis*
using *assms*(1,2,4) *trancpl-resolution-card-simple-decreasing*[*of* $\psi \ \psi'$] **by** *presburger*
qed

12.5.2 well-foundness if the relation

lemma *wf-simplified-resolution*:
assumes *f-vars*: *finite vars*
shows *wf* {(*y*: '*v*:: *linorder state*, *x*). (*atms-of-ms* (*fst* *x*) \subseteq *vars* \wedge *simplified* (*fst* *x*)
 \wedge *finite* (*snd* *x*) \wedge *finite* (*fst* *x*) \wedge *already-used-all-simple* (*snd* *x*) *vars*) \wedge *resolution* *x y*}

proof –

```

{
  fix a b :: 'v::linorder state
  assume (b, a) ∈ {(y, x). (atms-of-ms (fst x) ⊆ vars ∧ simplified (fst x) ∧ finite (snd x)
    ∧ finite (fst x) ∧ already-used-all-simple (snd x) vars) ∧ resolution x y}
  then have
    atms-of-ms (fst a) ⊆ vars and
    simp: simplified (fst a) and
    finite (snd a) and
    finite (fst a) and
    a-u-v: already-used-all-simple (snd a) vars and
    res: resolution a b by auto
  have finite (already-used-top vars) using f-vars already-used-top-finite by blast
  moreover have already-used-top vars ⊆ already-used-top vars by auto
  moreover have snd b ⊆ already-used-top vars
    using already-used-all-simple-in-already-used-top[of snd b vars]
    a-u-v already-used-all-simple-inv[OF res] ⟨finite (fst a)⟩ ⟨atms-of-ms (fst a) ⊆ vars⟩ f-vars
    by presburger
  moreover have snd a ⊆ snd b using resolution-simplified-already-used-subset[OF res simp] .
  ultimately have finite (already-used-top vars) ∧ already-used-top vars ⊆ already-used-top vars
    ∧ snd b ⊆ already-used-top vars ∧ snd a ⊆ snd b by metis
}
then show ?thesis using wf-bounded-set[of {(y:: 'v:: linorder state, x).
  (atms-of-ms (fst x) ⊆ vars
  ∧ simplified (fst x) ∧ finite (snd x) ∧ finite (fst x) ∧ already-used-all-simple (snd x) vars)
  ∧ resolution x y} λ-. already-used-top vars snd] by auto

```

qed

lemma *wf-simplified-resolution'*:

```

assumes f-vars: finite vars
shows wf {(y:: 'v:: linorder state, x). (atms-of-ms (fst x) ⊆ vars ∧ ¬simplified (fst x)
  ∧ finite (snd x) ∧ finite (fst x) ∧ already-used-all-simple (snd x) vars) ∧ resolution x y}
unfolding wf-def
apply (simp add: resolution-always-simplified)
by (metis (mono-tags, hide-lams) fst-conv resolution-always-simplified)

```

lemma *wf-resolution*:

```

assumes f-vars: finite vars
shows wf {(y:: 'v:: linorder state, x). (atms-of-ms (fst x) ⊆ vars ∧ simplified (fst x)
  ∧ finite (snd x) ∧ finite (fst x) ∧ already-used-all-simple (snd x) vars) ∧ resolution x y}
  ∪ {(y, x). (atms-of-ms (fst x) ⊆ vars ∧ ¬simplified (fst x) ∧ finite (snd x) ∧ finite (fst x)
  ∧ already-used-all-simple (snd x) vars) ∧ resolution x y} (is wf (?R ∪ ?S))

```

proof –

```

have Domain ?R Int Range ?S = {} using resolution-always-simplified by auto blast
then show wf (?R ∪ ?S)
  using wf-simplified-resolution[OF f-vars] wf-simplified-resolution'[OF f-vars] wf-Un[of ?R ?S]
  by fast

```

qed

lemma *rtrancp-simplify-already-used-inv*:

```

assumes simplify** S S'
and already-used-inv (S, N)
shows already-used-inv (S', N)
using assms apply induction
using simplify-preserved-already-used-inv by fast+

```

```

lemma full1-simplify-already-used-inv:
  assumes full1 simplify S S'
  and already-used-inv (S, N)
  shows already-used-inv (S', N)
  using assms tranclp-into-rtranclp[of simplify S S'] rtranclp-simplify-already-used-inv
  unfolding full1-def by fast

lemma full-simplify-already-used-inv:
  assumes full simplify S S'
  and already-used-inv (S, N)
  shows already-used-inv (S', N)
  using assms rtranclp-simplify-already-used-inv unfolding full-def by fast

lemma resolution-already-used-inv:
  assumes resolution S S'
  and already-used-inv S
  shows already-used-inv S'
  using assms
proof induction
  case (full1-simp N N' already-used)
  then show ?case using full1-simplify-already-used-inv by fast
next
  case (inferring N already-used N' already-used' N'')
  note inf = this(1) and full = this(3) and
    a-u-v = this(4)
  then show ?case
    using inference-preserves-already-used-inv[OF inf a-u-v] full-simplify-already-used-inv full
    by fast
qed

lemma rtranclp-resolution-already-used-inv:
  assumes resolution** S S'
  and already-used-inv S
  shows already-used-inv S'
  using assms apply induction
  using resolution-already-used-inv by fast+

lemma rtanclp-simplify-preserves-unsat:
  assumes simplify**  $\psi$   $\psi'$ 
  shows satisfiable  $\psi' \longrightarrow$  satisfiable  $\psi$ 
  using assms apply induction
  using simplify-clause-preserves-sat by blast+

lemma full1-simplify-preserves-unsat:
  assumes full1 simplify  $\psi$   $\psi'$ 
  shows satisfiable  $\psi' \longrightarrow$  satisfiable  $\psi$ 
  using assms rtanclp-simplify-preserves-unsat[of  $\psi$   $\psi'$ ] tranclp-into-rtranclp
  unfolding full1-def by metis

lemma full-simplify-preserves-unsat:
  assumes full simplify  $\psi$   $\psi'$ 
  shows satisfiable  $\psi' \longrightarrow$  satisfiable  $\psi$ 
  using assms rtanclp-simplify-preserves-unsat[of  $\psi$   $\psi'$ ] unfolding full-def by metis

lemma resolution-preserves-unsat:
  assumes resolution  $\psi$   $\psi'$ 

```

shows *satisfiable* (fst ψ') \longrightarrow *satisfiable* (fst ψ)
using *assms* **apply** (*induct rule: resolution.induct*)
using *full1-simplify-preserves-unsat* **apply** (*metis fst-conv*)
using *full-simplify-preserves-unsat* *simplify-preserves-unsat* **by** *fastforce*

lemma *rtrancp-resolution-preserves-unsat*:
assumes *resolution*** ψ ψ'
shows *satisfiable* (fst ψ') \longrightarrow *satisfiable* (fst ψ)
using *assms* **apply** *induction*
using *resolution-preserves-unsat* **by** *fast+*

lemma *rtrancp-simplify-preserve-partial-tree*:
assumes *simplify*** N N'
and *partial-interps* t I N
shows *partial-interps* t I N'
using *assms* **apply** (*induction, simp*)
using *simplify-preserve-partial-tree* **by** *metis*

lemma *full1-simplify-preserve-partial-tree*:
assumes *full1 simplify* N N'
and *partial-interps* t I N
shows *partial-interps* t I N'
using *assms* *rtrancp-simplify-preserve-partial-tree*[of N N' t I] *trancp-into-rtrancp*
unfolding *full1-def* **by** *fast*

lemma *full-simplify-preserve-partial-tree*:
assumes *full simplify* N N'
and *partial-interps* t I N
shows *partial-interps* t I N'
using *assms* *rtrancp-simplify-preserve-partial-tree*[of N N' t I] *trancp-into-rtrancp*
unfolding *full-def* **by** *fast*

lemma *resolution-preserve-partial-tree*:
assumes *resolution* S S'
and *partial-interps* t I (fst S)
shows *partial-interps* t I (fst S')
using *assms* **apply** *induction*
using *full1-simplify-preserve-partial-tree* *fst-conv* **apply** *metis*
using *full-simplify-preserve-partial-tree* *inference-preserve-partial-tree* **by** *fastforce*

lemma *rtrancp-resolution-preserve-partial-tree*:
assumes *resolution*** S S'
and *partial-interps* t I (fst S)
shows *partial-interps* t I (fst S')
using *assms* **apply** *induction*
using *resolution-preserve-partial-tree* **by** *fast+*
thm *nat-less-induct* *nat.induct*

lemma *nat-ge-induct*[*case-names* 0 *Suc*]:
assumes P 0
and $(\bigwedge n. (\bigwedge m. m < \text{Suc } n \implies P m) \implies P (\text{Suc } n))$
shows P n
using *assms* **apply** (*induct rule: nat-less-induct*)
by (*rename-tac* n , *case-tac* n) *auto*

```

lemma wf-always-more-step-False:
  assumes wf R
  shows  $(\forall x. \exists z. (z, x) \in R) \implies \text{False}$ 
  using assms unfolding wf-def by (meson Domain.DomainI assms wfE-min)

lemma finite-finite-mset-element-of-mset[simp]:
  assumes finite N
  shows finite  $\{f \varphi L \mid \varphi L. \varphi \in N \wedge L \in \# \varphi \wedge P \varphi L\}$ 
  using assms
proof (induction N rule: finite-induct)
  case empty
  show ?case by auto
next
  case (insert x N) note finite = this(1) and IH = this(3)
  have  $\{f \varphi L \mid \varphi L. (\varphi = x \vee \varphi \in N) \wedge L \in \# \varphi \wedge P \varphi L\} \subseteq \{f x L \mid L. L \in \# x \wedge P x L\}$ 
     $\cup \{f \varphi L \mid \varphi L. \varphi \in N \wedge L \in \# \varphi \wedge P \varphi L\}$  by auto
  moreover have finite  $\{f x L \mid L. L \in \# x\}$  by auto
  ultimately show ?case using IH finite-subset by fastforce
qed

value card
value filter-mset
value  $\{\# \text{count } \varphi L \mid L \in \# \varphi. 2 \leq \text{count } \varphi L \#\}$ 
value  $(\lambda \varphi. \text{msetsum } \{\# \text{count } \varphi L \mid L \in \# \varphi. 2 \leq \text{count } \varphi L \#\})$ 

syntax
  -comprehension1'-mset :: 'a  $\Rightarrow$  'b  $\Rightarrow$  'b multiset  $\Rightarrow$  'a multiset
    (( $\{\# \cdot / \cdot \cdot : \text{setof } \cdot \#\}$ )))
translations
   $\{\# e. x. \text{setof } M \#\} == \text{CONST set-mset } (\text{CONST image-mset } (\%x. e) M)$ 
value  $\{\# a. a : \text{setof } \{\# 1, 1, 2 :: \text{int}\} \#\} = \{1, 2\}$ 

definition sum-count-ge-2 :: 'a multiset set  $\Rightarrow$  nat ( $\Xi$ ) where
  sum-count-ge-2  $\equiv \text{folding.F } (\lambda \varphi. \text{op} + (\text{msetsum } \{\# \text{count } \varphi L \mid L \in \# \varphi. 2 \leq \text{count } \varphi L \#\})) 0$ 

interpretation sum-count-ge-2:
  folding  $(\lambda \varphi. \text{op} + (\text{msetsum } \{\# \text{count } \varphi L \mid L \in \# \varphi. 2 \leq \text{count } \varphi L \#\})) 0$ 
rewrites
  folding.F  $(\lambda \varphi. \text{op} + (\text{msetsum } \{\# \text{count } \varphi L \mid L \in \# \varphi. 2 \leq \text{count } \varphi L \#\})) 0 = \text{sum-count-ge-2}$ 
proof –
  show folding  $(\lambda \varphi. \text{op} + (\text{msetsum } (\text{image-mset } (\text{count } \varphi) \{\# L \in \# \varphi. 2 \leq \text{count } \varphi L \#\})))$ 
    by standard auto
  then interpret sum-count-ge-2:
    folding  $(\lambda \varphi. \text{op} + (\text{msetsum } \{\# \text{count } \varphi L \mid L \in \# \varphi. 2 \leq \text{count } \varphi L \#\})) 0 .$ 
  show folding.F  $(\lambda \varphi. \text{op} + (\text{msetsum } (\text{image-mset } (\text{count } \varphi) \{\# L \in \# \varphi. 2 \leq \text{count } \varphi L \#\}))) 0$ 
     $= \text{sum-count-ge-2}$  by (auto simp add: sum-count-ge-2-def)
qed

lemma finite-incl-le-setsum:
  finite (B::'a multiset set)  $\implies A \subseteq B \implies \Xi A \leq \Xi B$ 
proof (induction arbitrary:A rule: finite-induct)
  case empty
  then show ?case by simp

```



```

next
case (insert a F) note finite = this(1) and aF = this(2) and IH = this(3) and AF = this(4)
show ?case
proof (cases a ∈ A)
  assume a ∉ A
  then have A ⊆ F using AF by auto
  then show ?case using IH[of A] by (simp add: aF local.finite)
next
assume aA: a ∈ A
then have A - {a} ⊆ F using AF by auto
then have  $\Xi (A - \{a\}) \leq \Xi F$  using IH by blast
then show ?case
proof -
  obtain nn :: nat ⇒ nat ⇒ nat where
     $\forall x0\ x1. (\exists v2. x0 = x1 + v2) = (x0 = x1 + nn\ x0\ x1)$ 
  by moura
  then have  $\Xi F = \Xi (A - \{a\}) + nn (\Xi F) (\Xi (A - \{a\}))$ 
  by (meson  $\langle \Xi (A - \{a\}) \leq \Xi F \rangle$  le-iff-add)
  then show ?thesis
    by (metis (no-types) le-iff-add aA aF add.assoc finite.insertI finite-subset
      insert.prem local.finite sum-count-ge-2.insert sum-count-ge-2.remove)
qed
qed
qed

```

lemma *simplify-finite-measure-decrease:*

simplify N N' ⇒ finite N ⇒ card N' + $\Xi N' < \text{card } N + \Xi N$

proof (*induction rule: simplify.induct*)

case (*tautology-deletion A P*) **note** an = this(1) **and** fin = this(2)

let ?N' = N - {A + {#Pos P#} + {#Neg P#}}

have card ?N' < card N

by (*meson card-Diff1-less tautology-deletion.hyps tautology-deletion.prem*)

moreover have ?N' ⊆ N **by** auto

then have sum-count-ge-2 ?N' ≤ sum-count-ge-2 N **using** finite-incl-le-setsum[OF fin] **by** blast

ultimately show ?case **by** linarith

next

case (*condensation A L*) **note** AN = this(1) **and** fin = this(2)

let ?C' = A + {#L#}

let ?C = A + {#L#} + {#L#}

let ?N' = N - {?C} ∪ {?C'}

have card ?N' ≤ card N

using AN **by** (*metis (no-types, lifting) Diff-subset Un-empty-right Un-insert-right card.remove card-insert-if card-mono fin finite-Diff order-refl*)

moreover have $\Xi \{?C'\} < \Xi \{?C\}$

proof -

have mset-decomp:

$\{\# L_a \in \# A. (L = L_a \longrightarrow L_a \in \# A) \wedge (L \neq L_a \longrightarrow 2 \leq \text{count } A\ L_a)\#\}$
 = $\{\# L_a \in \# A. L \neq L_a \wedge 2 \leq \text{count } A\ L_a\#\} +$
 $\{\# L_a \in \# A. L = L_a \wedge \text{Suc } 0 \leq \text{count } A\ L\#\}$

by (*auto simp: multiset-eq-iff ac-simps*)

have mset-decomp2: $\{\# L_a \in \# A. L \neq L_a \longrightarrow 2 \leq \text{count } A\ L_a\#\} =$

$\{\# L_a \in \# A. L \neq L_a \wedge 2 \leq \text{count } A\ L_a\#\} + \text{replicate-mset } (\text{count } A\ L)\ L$

by (*auto simp: multiset-eq-iff*)

show ?thesis

by (*auto simp: mset-decomp mset-decomp2 filter-mset-eq ac-simps*)

```

qed
have  $\Xi \ ?N' < \Xi \ N$ 
proof cases
  assume  $a1: \ ?C' \in N$ 
  then show  $\ ?thesis$ 
  proof -
    have  $f2: \bigwedge m \ M. \ insert \ (m::'a \ literal \ multiset) \ (M - \{m\}) = M \cup \{m\} \vee m \notin M$ 
      using Un-empty-right insert-Diff by blast
    have  $f3: \bigwedge m \ M \ Ma. \ insert \ (m::'a \ literal \ multiset) \ M - insert \ m \ Ma = M - insert \ m \ Ma$ 
      by simp
    then have  $f4: \bigwedge M \ m. \ M - \{m::'a \ literal \ multiset\} = M \cup \{m\} \vee m \in M$ 
      using Diff-insert-absorb Un-empty-right by fastforce
    have  $f5: insert \ (A + \{\#L\# \} + \{\#L\# \}) \ N = N$ 
      using  $f3 \ f2$  Un-empty-right condensation.hyps insert-iff by fastforce
    have  $\bigwedge m \ M. \ insert \ (m::'a \ literal \ multiset) \ M = M \cup \{m\} \vee m \notin M$ 
      using  $f3 \ f2$  Un-empty-right add.right-neutral insert-iff by fastforce
    then have  $\Xi \ (N - \{A + \{\#L\# \} + \{\#L\# \}\}) < \Xi \ N$ 
      using  $f5 \ f4$  by (metis Un-empty-right  $\langle \Xi \ \{A + \{\#L\# \}\} < \Xi \ \{A + \{\#L\# \} + \{\#L\# \}\} \rangle$ 
        add.right-neutral add-diff-cancel-left' add-gr-0 diff-less fin finite.emptyI not-le
        sum-count-ge-2.empty sum-count-ge-2.insert-remove trans-le-add2)
    then show  $\ ?thesis$ 
      using  $f3 \ f2 \ a1$  by (metis (no-types) Un-empty-right Un-insert-right condensation.hyps
        insert-iff multi-self-add-other-not-self)
  qed
next
assume  $\ ?C' \notin N$ 
have mset-decomp:
   $\{\# \ La \in \# \ A. \ (L = La \longrightarrow Suc \ 0 \leq count \ A \ La) \wedge (L \neq La \longrightarrow 2 \leq count \ A \ La)\#\}$ 
  =  $\{\# \ La \in \# \ A. \ L \neq La \wedge 2 \leq count \ A \ La\#\} +$ 
   $\{\# \ La \in \# \ A. \ L = La \wedge Suc \ 0 \leq count \ A \ L\#\}$ 
  by (auto simp: multiset-eq-iff ac-simps)
have mset-decomp2:  $\{\# \ La \in \# \ A. \ L \neq La \longrightarrow 2 \leq count \ A \ La\#\} =$ 
 $\{\# \ La \in \# \ A. \ L \neq La \wedge 2 \leq count \ A \ La\#\} + replicate\text{-}mset \ (count \ A \ L) \ L$ 
  by (auto simp: multiset-eq-iff)

show  $\ ?thesis$ 
  using  $\langle \Xi \ \{A + \{\#L\# \}\} < \Xi \ \{A + \{\#L\# \} + \{\#L\# \}\} \rangle$  condensation.hyps fin
  sum-count-ge-2.remove[of - A + \{\#L\# \} + \{\#L\# \}] (\ ?C' \notin N)
  by (auto simp: mset-decomp mset-decomp2 filter-mset-eq)
qed
ultimately show  $\ ?case$  by linarith
next
case (subsumption  $A \ B$ ) note  $AN = this(1)$  and  $AB = this(2)$  and  $BN = this(3)$  and  $fin = this(4)$ 
have  $card \ (N - \{B\}) < card \ N$  using  $BN$  by (meson card-Diff1-less subsumption.prems)
moreover have  $\Xi \ (N - \{B\}) \leq \Xi \ N$ 
  by (simp add: Diff-subset finite-incl-le-setsum subsumption.prems)
ultimately show  $\ ?case$  by linarith
qed

lemma simplify-terminates:
  wf  $\{(N', N). \ finite \ N \wedge simplify \ N \ N'\}$ 
  using assms apply (rule wfP-if-measure[of finite simplify  $\lambda N. card \ N + \Xi \ N$ ])
  using simplify-finite-measure-decrease by blast

```

lemma *wf-terminates*:
assumes *wf r*
shows $\exists N'. (N', N) \in r^* \wedge (\forall N''. (N'', N') \notin r)$
proof –
let $?P = \lambda N. (\exists N'. (N', N) \in r^* \wedge (\forall N''. (N'', N') \notin r))$
have $(\forall x. (\forall y. (y, x) \in r \longrightarrow ?P y) \longrightarrow ?P x)$
proof *clarify*
fix *x*
assume $H: \forall y. (y, x) \in r \longrightarrow ?P y$
{ assume $\exists y. (y, x) \in r$
then obtain *y* **where** $y: (y, x) \in r$ **by** *blast*
then have $?P y$ **using** *H* **by** *blast*
then have $?P x$ **using** *y* **by** (*meson rtrancl.rtrancl-into-rtrancl*)
}
moreover {
assume $\neg(\exists y. (y, x) \in r)$
then have $?P x$ **by** *auto*
}
ultimately show $?P x$ **by** *blast*
qed
moreover have $(\forall x. (\forall y. (y, x) \in r \longrightarrow ?P y) \longrightarrow ?P x) \longrightarrow \text{All } ?P$
using *assms unfolding wf-def* **by** (*rule allE*)
ultimately have $\text{All } ?P$ **by** *blast*
then show $?P N$ **by** *blast*
qed

lemma *rtranclp-simplify-terminates*:
assumes *fin: finite N*
shows $\exists N'. \text{simplify}^{**} N N' \wedge \text{simplified } N'$
proof –
have $H: \{(N', N). \text{finite } N \wedge \text{simplify } N N'\} = \{(N', N). \text{simplify } N N' \wedge \text{finite } N\}$ **by** *auto*
then have $\text{wf}: \text{wf } \{(N', N). \text{simplify } N N' \wedge \text{finite } N\}$
using *simplify-terminates* **by** (*simp add: H*)
obtain N' **where** $N': (N', N) \in \{(b, a). \text{simplify } a b \wedge \text{finite } a\}^*$ **and**
more: $(\forall N''. (N'', N') \notin \{(b, a). \text{simplify } a b \wedge \text{finite } a\})$
using *Prop-Resolution.wf-terminates[OF wf, of N]* **by** *blast*
have $1: \text{simplify}^{**} N N'$
using N' **by** (*induction rule: rtrancl.induct*) *auto*
then have *finite N'* **using** *fin rtranclp-simplify-preserves-finite* **by** *blast*
then have $2: \forall N''. \neg \text{simplify } N' N''$ **using** *more* **by** *auto*

show *?thesis* **using** $1\ 2$ **by** *blast*
qed

lemma *finite-simplified-full1-simp*:
assumes *finite N*
shows $\text{simplified } N \vee (\exists N'. \text{full1 simplify } N N')$
using *rtranclp-simplify-terminates[OF assms]* **unfolding** *full1-def*
by (*metis Nitpick.rtranclp-unfold*)

lemma *finite-simplified-full-simp*:
assumes *finite N*
shows $\exists N'. \text{full simplify } N N'$
using *rtranclp-simplify-terminates[OF assms]* **unfolding** *full-def* **by** *metis*

lemma *can-decrease-tree-size-resolution:*

fixes $\psi :: 'v \text{ state}$ **and** $\text{tree} :: 'v \text{ sem-tree}$

assumes *finite* ($\text{fst } \psi$) **and** *already-used-inv* ψ

and *partial-interps* $\text{tree } I$ ($\text{fst } \psi$)

and *simplified* ($\text{fst } \psi$)

shows $\exists (\text{tree}' :: 'v \text{ sem-tree}) \psi'. \text{resolution}^{**} \psi \psi' \wedge \text{partial-interps } \text{tree}' I (\text{fst } \psi')$

$\wedge (\text{sem-tree-size } \text{tree}' < \text{sem-tree-size } \text{tree} \vee \text{sem-tree-size } \text{tree} = 0)$

using *assms*

proof (*induct arbitrary: I rule: sem-tree-size*)

case (*bigger* xs I) **note** $IH = \text{this}(1)$ **and** $\text{finite} = \text{this}(2)$ **and** $a-u-i = \text{this}(3)$ **and** $\text{part} = \text{this}(4)$

and $\text{simp} = \text{this}(5)$

{ **assume** $\text{sem-tree-size } xs = 0$
then have $?case$ **using** part **by** blast
}

moreover {

assume $sn0: \text{sem-tree-size } xs > 0$

obtain $ag \text{ ad } v$ **where** $xs: xs = \text{Node } v \text{ ag ad}$ **using** $sn0$ **by** ($\text{cases } xs, \text{auto}$)

{

assume $\text{sem-tree-size } ag = 0 \wedge \text{sem-tree-size } ad = 0$

then have $ag: ag = \text{Leaf}$ **and** $ad: ad = \text{Leaf}$ **by** ($\text{cases } ag, \text{auto}, \text{cases } ad, \text{auto}$)

then obtain $\chi \chi'$ **where**

$\chi: \neg I \cup \{\text{Pos } v\} \models \chi$ **and**

$\text{tot}\chi: \text{total-over-m } (I \cup \{\text{Pos } v\}) \{\chi\}$ **and**

$\chi\psi: \chi \in \text{fst } \psi$ **and**

$\chi': \neg I \cup \{\text{Neg } v\} \models \chi'$ **and**

$\text{tot}\chi': \text{total-over-m } (I \cup \{\text{Neg } v\}) \{\chi'\}$ **and** $\chi'\psi: \chi' \in \text{fst } \psi$

using part **unfolding** xs **by** auto

have $\text{Pos}v: \text{Pos } v \notin \chi$ **using** χ **unfolding** $\text{true-clb-def true-lit-def}$ **by** auto

have $\text{Neg}v: \text{Neg } v \notin \chi'$ **using** χ' **unfolding** $\text{true-clb-def true-lit-def}$ **by** auto

{

assume $\text{Neg}\chi: \neg \text{Neg } v \in \chi$

then have $\neg I \models \chi$ **using** χ $\text{Pos}v$ **unfolding** $\text{true-clb-def true-lit-def}$ **by** auto

moreover have $\text{total-over-m } I \{\chi\}$

using $\text{Pos}v \text{ Neg}\chi \text{ atm-imp-pos-or-neg-lit tot}\chi$ **unfolding** $\text{total-over-m-def total-over-set-def}$
by fastforce

ultimately have $\text{partial-interps } \text{Leaf } I (\text{fst } \psi)$

and $\text{sem-tree-size } \text{Leaf} < \text{sem-tree-size } xs$

and $\text{resolution}^{**} \psi \psi$

unfolding xs **by** ($\text{auto simp add: } \chi\psi$)

}

moreover {

assume $\text{Pos}\chi: \neg \text{Pos } v \in \chi'$

then have $I\chi: \neg I \models \chi'$ **using** $\chi' \text{ Pos}v$ **unfolding** $\text{true-clb-def true-lit-def}$ **by** auto

moreover have $\text{total-over-m } I \{\chi'\}$

using $\text{Neg}v \text{ Pos}\chi \text{ atm-imp-pos-or-neg-lit tot}\chi'$

unfolding $\text{total-over-m-def total-over-set-def}$ **by** fastforce

ultimately have $\text{partial-interps } \text{Leaf } I (\text{fst } \psi)$

and $\text{sem-tree-size } \text{Leaf} < \text{sem-tree-size } xs$

and $\text{resolution}^{**} \psi \psi$ **using** $\chi'\psi I\chi$ **unfolding** xs **by** auto

}

moreover {

```

assume neg:  $Neg\ v \in \# \chi$  and pos:  $Pos\ v \in \# \chi'$ 
have count  $\chi\ (Neg\ v) = 1$ 
  using simplified-count[OF simp  $\chi\psi$ ] neg
  by (simp add: dual-order.antisym)
have count  $\chi'\ (Pos\ v) = 1$ 
  using simplified-count[OF simp  $\chi'\psi$ ] pos
  by (simp add: dual-order.antisym)

obtain C where  $\chi C$ :  $\chi = C + \{\#Neg\ v\# \}$  and negC:  $Neg\ v \notin \# C$  and posC:  $Pos\ v \notin \# C$ 
  by (metis (no-types, lifting) One-nat-def Posv Suc-eq-plus1-left  $\langle count\ \chi\ (Neg\ v) = 1 \rangle$ 
    add-diff-cancel-left' count-diff count-greater-eq-one-iff count-single insert-DiffM
    insert-DiffM2 less-numeral-extra(3) multi-member-skip not-le not-less-eq-eq)

obtain C' where
   $\chi C'$ :  $\chi' = C' + \{\#Pos\ v\# \}$  and
  posC':  $Pos\ v \notin \# C'$  and
  negC':  $Neg\ v \notin \# C'$ 
  by (metis (no-types, lifting) One-nat-def Negv Suc-eq-plus1-left  $\langle count\ \chi'\ (Pos\ v) = 1 \rangle$ 
    add-diff-cancel-left' count-diff count-greater-eq-one-iff count-single insert-DiffM
    insert-DiffM2 less-numeral-extra(3) multi-member-skip not-le not-less-eq-eq)

have totC: total-over-m I  $\{C\}$ 
  using tot $\chi$  tot-over-m-remove[of I Pos v C] negC posC unfolding  $\chi C$ 
  by (metis total-over-m-sum uminus-Neg uminus-of-uminus-id)
have totC': total-over-m I  $\{C'\}$ 
  using tot $\chi'$  total-over-m-sum tot-over-m-remove[of I Neg v C'] negC' posC'
  unfolding  $\chi C'$  by (metis total-over-m-sum uminus-Neg)
have  $\neg I \models C + C'$ 
  using  $\chi\ \chi'\ \chi C\ \chi C'$  by auto
then have part-I- $\psi'''$ : partial-interps Leaf I (fst  $\psi \cup \{C + C'\}$ )
  using totC totC'  $\neg I \models C + C'$  by (metis Un-insert-right insertI1
    partial-interps.simps(1) total-over-m-sum)
{
  assume  $(\{\#Pos\ v\# \} + C', \{\#Neg\ v\# \} + C) \notin snd\ \psi$ 
  then have inf'': inference  $\psi$  (fst  $\psi \cup \{C + C'\}$ , snd  $\psi \cup \{(\chi', \chi)\}$ )
    by (metis  $\chi'\psi\ \chi C\ \chi C'\ \chi\psi$  add.commute inference-step prod.collapse resolution)
  obtain N' where full: full simplify (fst  $\psi \cup \{C + C'\}$ ) N'
    by (metis finite-simplified-full-simp fst-conv inf'' inference-preserves-finite
      local.finite)
  have resolution  $\psi$  (N', snd  $\psi \cup \{(\chi', \chi)\}$ )
    using resolution.intros(2)[OF - simp full, of snd  $\psi$  snd  $\psi \cup \{(\chi', \chi)\}$ ] inf''
    by (metis surjective-pairing)
  moreover have partial-interps Leaf I N'
    using full-simplify-preserve-partial-tree[OF full part-I- $\psi'''$ ] .
  moreover have sem-tree-size Leaf < sem-tree-size xs unfolding xs by auto
  ultimately have ?case
    by (metis (no-types) prod.sel(1) rtranclp.rtrancl-into-rtrancl rtranclp.rtrancl-refl)
}
moreover {
  assume a:  $(\{\#Pos\ v\# \} + C', \{\#Neg\ v\# \} + C) \in snd\ \psi$ 
  then have  $(\exists \chi \in fst\ \psi. (\forall I. total-over-m\ I\ \{C+C'\} \longrightarrow total-over-m\ I\ \{\chi\})$ 
     $\wedge (\forall I. total-over-m\ I\ \{\chi\} \longrightarrow I \models \chi \longrightarrow I \models C' + C)) \vee tautology\ (C' + C)$ 
  proof -
    obtain p where p:  $Pos\ p \in \# (\{\#Pos\ v\# \} + C') \wedge Neg\ p \in \# (\{\#Neg\ v\# \} + C)$ 
       $\wedge ((\exists \chi \in fst\ \psi. (\forall I. total-over-m\ I\ \{(\{\#Pos\ v\# \} + C') - \{\#Pos\ p\# \} + (\{\#Neg\ v\# \}))$ 

```

```

+ C) - {#Neg p#})} → total-over-m I {χ} ∧ (∀ I. total-over-m I {χ} → I ⊨ χ → I ⊨ ({#Pos
v#} + C') - {#Pos p#} + (({#Neg v#} + C) - {#Neg p#}))) ∨ tautology (({#Pos v#} + C') -
{#Pos p#} + (({#Neg v#} + C) - {#Neg p#})))
  using a by (blast intro: allE[OF a-u-i[unfolded subsumes-def Ball-def],
    of ({#Pos v#} + C', {#Neg v#} + C)])
  { assume p ≠ v
    then have Pos p ∈# C' ∧ Neg p ∈# C using p by force
    then have ?thesis by auto
  }
  moreover {
    assume p = v
    then have ?thesis using p by (metis add.commute add-diff-cancel-left')
  }
  ultimately show ?thesis by auto
qed
moreover {
  assume ∃χ ∈ fst ψ. (∀ I. total-over-m I {C+C'} → total-over-m I {χ})
  ∧ (∀ I. total-over-m I {χ} → I ⊨ χ → I ⊨ C' + C)
  then obtain ϑ where
    ϑ: ϑ ∈ fst ψ and
    tot-ϑ-CC': ∀ I. total-over-m I {C+C'} → total-over-m I {ϑ} and
    ϑ-inv: ∀ I. total-over-m I {ϑ} → I ⊨ ϑ → I ⊨ C' + C by blast
  have partial-interps Leaf I (fst ψ)
    using tot-ϑ-CC' ϑ ϑ-inv totC totC' ⊢ I ⊨ C + C' total-over-m-sum by fastforce
  moreover have sem-tree-size Leaf < sem-tree-size xs unfolding xs by auto
  ultimately have ?case by blast
}
moreover {
  assume tautCC': tautology (C' + C)
  have total-over-m I {C'+C} using totC totC' total-over-m-sum by auto
  then have ¬tautology (C' + C)
    using ⊢ I ⊨ C + C' unfolding add.commute[of C C'] total-over-m-def
    unfolding tautology-def by auto
  then have False using tautCC' unfolding tautology-def by auto
}
ultimately have ?case by auto
}
ultimately have ?case using part by (metis (no-types) sem-tree-size.simps(1))
}
moreover {
  assume size-ag: sem-tree-size ag > 0
  have sem-tree-size ag < sem-tree-size xs unfolding xs by auto
  moreover have partial-interps ag (I ∪ {Pos v}) (fst ψ)
  and partad: partial-interps ad (I ∪ {Neg v}) (fst ψ)
    using part partial-interps.simps(2) unfolding xs by metis+
  moreover
    have sem-tree-size ag < sem-tree-size xs ⇒ finite (fst ψ) ⇒ already-used-inv ψ
    ⇒ partial-interps ag (I ∪ {Pos v}) (fst ψ) ⇒ simplified (fst ψ)
    ⇒ ∃ tree' ψ'. resolution** ψ ψ' ∧ partial-interps tree' (I ∪ {Pos v}) (fst ψ')
    ∧ (sem-tree-size tree' < sem-tree-size ag ∨ sem-tree-size ag = 0)
    using IH[of ag I ∪ {Pos v}] by auto
  ultimately obtain ψ' :: 'v state and tree' :: 'v sem-tree where
    inf: resolution** ψ ψ'

```

```

    and part: partial-interps tree' (I ∪ {Pos v}) (fst ψ')
    and size: sem-tree-size tree' < sem-tree-size ag ∨ sem-tree-size ag = 0
    using finite part rtranclp.rtrancl-refl a-u-i simp by blast

  have partial-interps ad (I ∪ {Neg v}) (fst ψ')
    using rtranclp-resolution-preserve-partial-tree inf partad by fast
  then have partial-interps (Node v tree' ad) I (fst ψ') using part by auto
  then have ?case using inf size size-ag part unfolding xs by fastforce
}
moreover {
  assume size-ad: sem-tree-size ad > 0
  have sem-tree-size ad < sem-tree-size xs unfolding xs by auto
  moreover
    have
      partag: partial-interps ag (I ∪ {Pos v}) (fst ψ) and
      partial-interps ad (I ∪ {Neg v}) (fst ψ)
      using part partial-interps.simps(2) unfolding xs by metis+
  moreover have sem-tree-size ad < sem-tree-size xs ⟶ finite (fst ψ) ⟶ already-used-inv ψ
    ⟶ ( partial-interps ad (I ∪ {Neg v}) (fst ψ) ⟶ simplified (fst ψ)
    ⟶ (∃ tree' ψ'. resolution** ψ ψ' ∧ partial-interps tree' (I ∪ {Neg v}) (fst ψ')
      ∧ (sem-tree-size tree' < sem-tree-size ad ∨ sem-tree-size ad = 0)))
    using IH by blast
  ultimately obtain ψ' :: 'v state and tree' :: 'v sem-tree where
    inf: resolution** ψ ψ'
    and part: partial-interps tree' (I ∪ {Neg v}) (fst ψ')
    and size: sem-tree-size tree' < sem-tree-size ad ∨ sem-tree-size ad = 0
    using finite part rtranclp.rtrancl-refl a-u-i simp by blast

  have partial-interps ag (I ∪ {Pos v}) (fst ψ')
    using rtranclp-resolution-preserve-partial-tree inf partag by fast
  then have partial-interps (Node v ag tree') I (fst ψ') using part by auto
  then have ?case using inf size size-ad unfolding xs by fastforce
}
ultimately have ?case by auto
}
ultimately show ?case by auto
qed

```

lemma resolution-completeness-inv:

```

  fixes ψ :: 'v :: linorder state
  assumes
    unsat: ¬satisfiable (fst ψ) and
    finite: finite (fst ψ) and
    a-u-v: already-used-inv ψ
  shows ∃ ψ'. (resolution** ψ ψ' ∧ {#} ∈ fst ψ')
proof -
  obtain tree where partial-interps tree {} (fst ψ)
    using partial-interps-build-sem-tree-atms assms by metis
  then show ?thesis
    using unsat finite a-u-v
  proof (induct tree arbitrary: ψ rule: sem-tree-size)
    case (bigger tree ψ) note H = this
    {
      fix χ
      assume tree: tree = Leaf
    }
  end

```

```

obtain  $\chi$  where  $\chi: \neg \{\} \models \chi$  and  $\text{tot}\chi: \text{total-over-m } \{\} \{\chi\}$  and  $\chi\psi: \chi \in \text{fst } \psi$ 
  using  $H$  unfolding  $\text{tree}$  by  $\text{auto}$ 
moreover have  $\{\#\} = \chi$ 
  using  $H$   $\text{atms-empty-iff-empty tot}\chi$ 
  unfolding  $\text{true-cls-def total-over-m-def total-over-set-def}$  by  $\text{fastforce}$ 
moreover have  $\text{resolution}^{**} \psi \psi$  by  $\text{auto}$ 
ultimately have  $?case$  by  $\text{metis}$ 
}
moreover {
  fix  $v$   $\text{tree1 tree2}$ 
  assume  $\text{tree}: \text{tree} = \text{Node } v \text{ tree1 tree2}$ 
  obtain  $\psi_0$  where  $\psi_0: \text{resolution}^{**} \psi \psi_0$  and  $\text{simp}: \text{simplified } (\text{fst } \psi_0)$ 
  proof –
    { assume  $\text{simplified } (\text{fst } \psi)$ 
      moreover have  $\text{resolution}^{**} \psi \psi$  by  $\text{auto}$ 
      ultimately have  $\text{thesis}$  using  $\text{that}$  by  $\text{blast}$ 
    }
    moreover {
      assume  $\neg \text{simplified } (\text{fst } \psi)$ 
      then have  $\exists \psi'. \text{full1 simplify } (\text{fst } \psi) \psi'$ 
        by ( $\text{metis Nitpick.rtranclp-unfold bigger.prem}(3) \text{full1-def}$ 
           $\text{rtranclp-simplify-terminates}$ )
      then obtain  $N$  where  $\text{full1 simplify } (\text{fst } \psi) N$  by  $\text{metis}$ 
      then have  $\text{resolution } \psi (N, \text{snd } \psi)$ 
        using  $\text{resolution.intros}(1)[\text{of } \text{fst } \psi N \text{snd } \psi]$  by  $\text{auto}$ 
      moreover have  $\text{simplified } N$ 
        using  $\langle \text{full1 simplify } (\text{fst } \psi) N \rangle \text{unfolding full1-def}$  by  $\text{blast}$ 
      ultimately have  $?thesis$  using  $\text{that}$  by  $\text{force}$ 
    }
    ultimately show  $?thesis$  by  $\text{auto}$ 
  }
qed

have  $p: \text{partial-interps tree } \{\} (\text{fst } \psi_0)$ 
and  $\text{uns}: \text{unsatisfiable } (\text{fst } \psi_0)$ 
and  $f: \text{finite } (\text{fst } \psi_0)$ 
and  $a\text{-}u\text{-}v: \text{already-used-inv } \psi_0$ 
  using  $\psi_0 \text{ bigger.prem}(1) \text{rtranclp-resolution-preserve-partial-tree}$  apply  $\text{blast}$ 
  using  $\psi_0 \text{ bigger.prem}(2) \text{rtranclp-resolution-preserves-unsat}$  apply  $\text{blast}$ 
  using  $\psi_0 \text{ bigger.prem}(3) \text{rtranclp-resolution-finite}$  apply  $\text{blast}$ 
  using  $\text{rtranclp-resolution-already-used-inv}[OF \psi_0 \text{ bigger.prem}(4)]$  by  $\text{blast}$ 
obtain  $\text{tree}' \psi'$  where
   $\text{inf}: \text{resolution}^{**} \psi_0 \psi'$  and
   $\text{part}': \text{partial-interps tree}' \{\} (\text{fst } \psi')$  and
   $\text{decrease}: \text{sem-tree-size tree}' < \text{sem-tree-size tree} \vee \text{sem-tree-size tree} = 0$ 
  using  $\text{can-decrease-tree-size-resolution}[OF f a\text{-}u\text{-}v p \text{simp}]$  unfolding  $\text{tautology-def}$ 
  by  $\text{meson}$ 
have  $s: \text{sem-tree-size tree}' < \text{sem-tree-size tree}$  using  $\text{decrease}$  unfolding  $\text{tree}$  by  $\text{auto}$ 
have  $\text{fin}: \text{finite } (\text{fst } \psi')$ 
  using  $f \text{inf rtranclp-resolution-finite}$  by  $\text{blast}$ 
have  $\text{unsat}: \text{unsatisfiable } (\text{fst } \psi')$ 
  using  $\text{rtranclp-resolution-preserves-unsat inf uns}$  by  $\text{metis}$ 
have  $a\text{-}u\text{-}i': \text{already-used-inv } \psi'$ 
  using  $a\text{-}u\text{-}v \text{inf rtranclp-resolution-already-used-inv}[\text{of } \psi_0 \psi']$  by  $\text{auto}$ 
have  $?case$ 

```



```

      using inf rtrancpl-trans[of resolution]  $H(1)[OF\ s\ part'\ unsat\ fin\ a-u-i']\ \psi_0$  by blast
    }
    ultimately show ?case by (cases tree, auto)
  qed
qed

```

```

lemma resolution-preserves-already-used-inv:
  assumes resolution  $S\ S'$ 
  and already-used-inv  $S$ 
  shows already-used-inv  $S'$ 
  using assms
  apply (induct rule: resolution.induct)
  apply (rule full1-simplify-already-used-inv; simp)
  apply (rule full-simplify-already-used-inv, simp)
  apply (rule inference-preserves-already-used-inv, simp)
  apply blast
done

```

```

lemma rtrancpl-resolution-preserves-already-used-inv:
  assumes resolution**  $S\ S'$ 
  and already-used-inv  $S$ 
  shows already-used-inv  $S'$ 
  using assms
  apply (induct rule: rtrancpl-induct)
  apply simp
  using resolution-preserves-already-used-inv by fast

```

```

lemma resolution-completeness:
  fixes  $\psi :: 'v :: linorder\ state$ 
  assumes unsat:  $\neg\text{satisfiable}\ (fst\ \psi)$ 
  and finite:  $finite\ (fst\ \psi)$ 
  and snd  $\psi = \{\}$ 
  shows  $\exists\ \psi'.\ (resolution**\ \psi\ \psi' \wedge \{\#\} \in fst\ \psi')$ 
proof -
  have already-used-inv  $\psi$  unfolding assms by auto
  then show ?thesis using assms resolution-completeness-inv by blast
qed

```

```

lemma rtrancpl-preserves-sat:
  assumes simplify**  $S\ S'$ 
  and satisfiable  $S$ 
  shows satisfiable  $S'$ 
  using assms apply induction
  apply simp
  by (meson satisfiable-carac satisfiable-def simplify-preserves-un-sat-eq)

```

```

lemma resolution-preserves-sat:
  assumes resolution  $S\ S'$ 
  and satisfiable  $(fst\ S)$ 
  shows satisfiable  $(fst\ S')$ 
  using assms apply (induction rule: resolution.induct)
  using rtrancpl-preserves-sat trancpl-into-rtrancpl unfolding full1-def apply fastforce
  by (metis fst-conv full-def inference-preserves-un-sat rtrancpl-preserves-sat
    satisfiable-carac' satisfiable-def)

```

```

lemma rtrancpl-resolution-preserves-sat:
  assumes resolution**  $S\ S'$ 
  and satisfiable (fst  $S$ )
  shows satisfiable (fst  $S'$ )
  using assms apply (induction rule: rtrancpl-induct)
  apply simp
  using resolution-preserves-sat by blast

lemma resolution-soundness:
  fixes  $\psi :: 'v :: \text{linorder state}$ 
  assumes resolution**  $\psi\ \psi'$  and  $\{\#\} \in \text{fst } \psi'$ 
  shows unsatisfiable (fst  $\psi$ )
  using assms by (meson rtrancpl-resolution-preserves-sat satisfiable-def true-cls-empty
    true-clss-def)

lemma resolution-soundness-and-completeness:
  fixes  $\psi :: 'v :: \text{linorder state}$ 
  assumes finite: finite (fst  $\psi$ )
  and snd: snd  $\psi = \{\}$ 
  shows  $(\exists \psi'. (\text{resolution** } \psi\ \psi' \wedge \{\#\} \in \text{fst } \psi')) \longleftrightarrow \text{unsatisfiable } (\text{fst } \psi)$ 
  using assms resolution-completeness resolution-soundness by metis

lemma simplified-falsity:
  assumes simp: simplified  $\psi$ 
  and  $\{\#\} \in \psi$ 
  shows  $\psi = \{\{\#\}\}$ 
proof (rule ccontr)
  assume  $H: \neg ?thesis$ 
  then obtain  $\chi$  where  $\chi \in \psi$  and  $\chi \neq \{\#\}$  using assms(2) by blast
  then have  $\{\#\} \subsetneq \chi$  by (simp add: mset-less-empty-nonempty)
  then have simplify  $\psi$  ( $\psi - \{\chi\}$ )
    using simplify.subsumption[OF assms(2)  $\langle \{\#\} \subsetneq \chi \rangle \langle \chi \in \psi \rangle$ ] by blast
  then show False using simp by blast
qed

lemma simplify-falsity-in-preserved:
  assumes simplify  $\chi s\ \chi s'$ 
  and  $\{\#\} \in \chi s$ 
  shows  $\{\#\} \in \chi s'$ 
  using assms
  by induction auto

lemma rtrancpl-simplify-falsity-in-preserved:
  assumes simplify**  $\chi s\ \chi s'$ 
  and  $\{\#\} \in \chi s$ 
  shows  $\{\#\} \in \chi s'$ 
  using assms
  by induction (auto intro: simplify-falsity-in-preserved)

lemma resolution-falsity-get-falsity-alone:
  assumes finite (fst  $\psi$ )
  shows  $(\exists \psi'. (\text{resolution** } \psi\ \psi' \wedge \{\#\} \in \text{fst } \psi')) \longleftrightarrow (\exists a-u-v. \text{resolution** } \psi\ (\{\{\#\}\}, a-u-v))$ 
  (is  $?A \longleftrightarrow ?B$ )
proof

```

```

  assume ?B
  then show ?A by auto
next
  assume ?A
  then obtain  $\chi s$   $a-u-v$  where  $\chi s$ : resolution**  $\psi$  ( $\chi s$ ,  $a-u-v$ ) and  $F$ :  $\{\#\} \in \chi s$  by auto
  { assume simplified  $\chi s$ 
    then have ?B using simplified-falsity[ $OF - F$ ]  $\chi s$  by blast
  }
  moreover {
    assume  $\neg$  simplified  $\chi s$ 
    then obtain  $\chi s'$  where full1 simplify  $\chi s$   $\chi s'$ 
      by (metis  $\chi s$  assms finite-simplified-full1-simp fst-conv rtranclp-resolution-finite)
    then have  $\{\#\} \in \chi s'$ 
      unfolding full1-def by (meson  $F$  rtranclp-simplify-falsity-in-preserved
        trancpl-into-rtranclp)
    then have ?B
      by (metis  $\chi s$  (full1 simplify  $\chi s$   $\chi s'$ ) fst-conv full1-simp resolution-always-simplified
        rtranclp.rtrancl-into-rtrancl simplified-falsity)
  }
  ultimately show ?B by blast
qed

lemma resolution-soundness-and-completeness':
  fixes  $\psi :: 'v :: \text{linorder state}$ 
  assumes
    finite: finite (fst  $\psi$ ) and
    snd: snd  $\psi = \{\}$ 
  shows  $(\exists a-u-v. (\text{resolution}^{**} \psi (\{\{\#\}\}, a-u-v))) \longleftrightarrow \text{unsatisfiable} (\text{fst } \psi)$ 
    using assms resolution-completeness resolution-soundness resolution-falsity-get-falsity-alone
    by metis

end

theory Partial-Annotated-Clausal-Logic
imports Partial-Clausal-Logic

begin

```

13 Partial Clausal Logic

We here define marked literals (that will be used in both DPLL and CDCL) and the entailment corresponding to it.

13.1 Marked Literals

13.1.1 Definition

```

datatype ('v, 'wl, 'mark) marked-lit =
  is-marked: Marked (lit-of: 'v literal) (level-of: 'wl) |
  is-proped: Propagated (lit-of: 'v literal) (mark-of: 'mark)

```

```

lemma marked-lit-list-induct[case-names nil marked proped]:
  assumes  $P []$  and
   $\bigwedge L l xs. P xs \implies P (\text{Marked } L l \# xs)$  and
   $\bigwedge L m xs. P xs \implies P (\text{Propagated } L m \# xs)$ 

```

shows $P\ xs$
using *assms* **apply** (*induction xs, simp*)
by (*rename-tac a xs, case-tac a*) *auto*

lemma *is-marked-ex-Marked*:
 $is_marked\ L \implies \exists K\ lvl.\ L = Marked\ K\ lvl$
by (*cases L*) *auto*

type-synonym (*'v, 'l, 'm*) *marked-lits* = (*'v, 'l, 'm*) *marked-lit list*

definition *lits-of* :: (*'a, 'b, 'c*) *marked-lit set* \Rightarrow *'a literal set* **where**
 $lits_of\ Ls = lit_of\ 'Ls$

abbreviation *lits-of-l* :: (*'a, 'b, 'c*) *marked-lit list* \Rightarrow *'a literal set* **where**
 $lits_of_l\ Ls \equiv lits_of\ (set\ Ls)$

lemma *lits-of-l-empty[simp]*:
 $lits_of\ \{\} = \{\}$
unfolding *lits-of-def* **by** *auto*

lemma *lits-of-insert[simp]*:
 $lits_of\ (insert\ L\ Ls) = insert\ (lit_of\ L)\ (lits_of\ Ls)$
unfolding *lits-of-def* **by** *auto*

lemma *lits-of-l-append[simp]*:
 $lits_of\ (l \cup l') = lits_of\ l \cup lits_of\ l'$
unfolding *lits-of-def* **by** *auto*

lemma *finite-lits-of-def[simp]*:
 $finite\ (lits_of_l\ L)$
unfolding *lits-of-def* **by** *auto*

abbreviation *unmark* **where**
 $unmark \equiv (\lambda a.\ \{\#lit_of\ a\# \})$

abbreviation *unmark-s* **where**
 $unmark_s\ M \equiv unmark\ 'M$

abbreviation *unmark-l* **where**
 $unmark_l\ M \equiv unmark_s\ (set\ M)$

lemma *atms-of-ms-lambda-lit-of-is-atm-of-lit-of[simp]*:
 $atms_of_ms\ (unmark_l\ M') = atm_of\ 'lits_of_l\ M'$
unfolding *atms-of-ms-def lits-of-def* **by** *auto*

lemma *lits-of-l-empty-is-empty[iff]*:
 $lits_of_l\ M = \{\} \longleftrightarrow M = []$
by (*induct M*) (*auto simp: lits-of-def*)

13.1.2 Entailment

definition *true-annot* :: (*'a, 'l, 'm*) *marked-lits* \Rightarrow *'a clause* \Rightarrow *bool* (**infix** \models_a 49) **where**
 $I \models_a C \longleftrightarrow (lits_of_l\ I) \models C$

definition *true-annots* :: (*'a, 'l, 'm*) *marked-lits* \Rightarrow *'a clauses* \Rightarrow *bool* (**infix** \models_{as} 49) **where**
 $I \models_{as} CC \longleftrightarrow (\forall C \in CC.\ I \models_a C)$

lemma *true-annot-empty-model*[simp]:
 $\neg[] \models_a \psi$
unfolding *true-annot-def true-cls-def* **by** *simp*

lemma *true-annot-empty*[simp]:
 $\neg I \models_a \{\#\}$
unfolding *true-annot-def true-cls-def* **by** *simp*

lemma *empty-true-annots-def*[iff]:
 $[] \models_{as} \psi \longleftrightarrow \psi = \{\}$
unfolding *true-annots-def* **by** *auto*

lemma *true-annots-empty*[simp]:
 $I \models_{as} \{\}$
unfolding *true-annots-def* **by** *auto*

lemma *true-annots-single-true-annot*[iff]:
 $I \models_{as} \{C\} \longleftrightarrow I \models_a C$
unfolding *true-annots-def* **by** *auto*

lemma *true-annot-insert-l*[simp]:
 $M \models_a A \implies L \# M \models_a A$
unfolding *true-annot-def* **by** *auto*

lemma *true-annots-insert-l* [simp]:
 $M \models_{as} A \implies L \# M \models_{as} A$
unfolding *true-annots-def* **by** *auto*

lemma *true-annots-union*[iff]:
 $M \models_{as} A \cup B \longleftrightarrow (M \models_{as} A \wedge M \models_{as} B)$
unfolding *true-annots-def* **by** *auto*

lemma *true-annots-insert*[iff]:
 $M \models_{as} \text{insert } a \ A \longleftrightarrow (M \models_a a \wedge M \models_{as} A)$
unfolding *true-annots-def* **by** *auto*

Link between \models_{as} and \models_s :

lemma *true-annots-true-cls*:
 $I \models_{as} CC \longleftrightarrow \text{lits-of-l } I \models_s CC$
unfolding *true-annots-def Ball-def true-annot-def true-clss-def* **by** *auto*

lemma *in-lit-of-true-annot*:
 $a \in \text{lits-of-l } M \longleftrightarrow M \models_a \{\#a\#\}$
unfolding *true-annot-def lits-of-def* **by** *auto*

lemma *true-annot-lit-of-notin-skip*:
 $L \# M \models_a A \implies \text{lit-of } L \notin \# A \implies M \models_a A$
unfolding *true-annot-def true-cls-def* **by** *auto*

lemma *true-clss-singleton-lit-of-implies-incl*:
 $I \models_s \text{unmark-l } MLs \implies \text{lits-of-l } MLs \subseteq I$
unfolding *true-clss-def lits-of-def* **by** *auto*

lemma *true-annot-true-clss-clss*:

$MLs \models_a \psi \implies \text{set } (\text{map } (\lambda a. \{\#lit\text{-of } a\#\}) MLs) \models_p \psi$
unfolding *true-annot-def true-clss-clss-def true-clss-def*
by (*auto dest: true-clss-singleton-lit-of-implies-incl*)

lemma *true-annots-true-clss-clss*:

$MLs \models_{as} \psi \implies \text{set } (\text{map } (\lambda a. \{\#lit\text{-of } a\#\}) MLs) \models_{ps} \psi$
by (*auto*
dest: true-clss-singleton-lit-of-implies-incl
simp add: true-clss-def true-annots-def true-annot-def lits-of-def true-clss-def
true-clss-clss-def)

lemma *true-annots-marked-true-clss[iff]*:

$\text{map } (\lambda M. \text{Marked } M \ a) \ M \models_{as} N \iff \text{set } M \models_s N$

proof –

have *: *lit-of* ‘ $(\lambda M. \text{Marked } M \ a)$ ’ *set* $M = \text{set } M$ **unfolding** *lits-of-def* **by** *force*
show ?thesis **by** (*simp add: true-annots-true-clss * lits-of-def*)

qed

lemma *true-annot-singleton[iff]*: $M \models_a \{\#L\#\} \iff L \in \text{lits-of-l } M$

unfolding *true-annot-def lits-of-def* **by** *auto*

lemma *true-annots-true-clss-clss*:

$A \models_{as} \Psi \implies \text{unmark-l } A \models_{ps} \Psi$
unfolding *true-clss-clss-def true-annots-def true-clss-def*
by (*auto*
dest!: true-clss-singleton-lit-of-implies-incl
simp add: lits-of-def true-annot-def true-clss-def)

lemma *true-annot-commute*:

$M @ M' \models_a D \iff M' @ M \models_a D$
unfolding *true-annot-def* **by** (*simp add: Un-commute*)

lemma *true-annots-commute*:

$M @ M' \models_{as} D \iff M' @ M \models_{as} D$
unfolding *true-annots-def* **by** (*auto simp add: true-annot-commute*)

lemma *true-annot-mono[dest]*:

$\text{set } I \subseteq \text{set } I' \implies I \models_a N \implies I' \models_a N$
using *true-clss-mono-set-mset-l* **unfolding** *true-annot-def lits-of-def*
by (*metis (no-types) Un-commute Un-upper1 image-Un sup.orderE*)

lemma *true-annots-mono*:

$\text{set } I \subseteq \text{set } I' \implies I \models_{as} N \implies I' \models_{as} N$
unfolding *true-annots-def* **by** *auto*

13.1.3 Defined and undefined literals

definition *defined-lit* :: $('a, 'l, 'm) \text{ marked-lit list} \Rightarrow 'a \text{ literal} \Rightarrow \text{bool}$

where

$\text{defined-lit } I \ L \iff (\exists l. \text{Marked } L \ l \in \text{set } I) \vee (\exists P. \text{Propagated } L \ P \in \text{set } I)$
 $\vee (\exists l. \text{Marked } (-L) \ l \in \text{set } I) \vee (\exists P. \text{Propagated } (-L) \ P \in \text{set } I)$

abbreviation *undefined-lit* :: $('a, 'l, 'm) \text{ marked-lit list} \Rightarrow 'a \text{ literal} \Rightarrow \text{bool}$

where $\text{undefined-lit } I \ L \equiv \neg \text{defined-lit } I \ L$

lemma *defined-lit-rev[simp]*:
defined-lit (rev M) L \longleftrightarrow defined-lit M L
unfolding *defined-lit-def* **by** *auto*

lemma *atm-imp-marked-or-proped*:
assumes $x \in \text{set } I$
shows
 $(\exists l. \text{Marked } (\neg \text{lit-of } x) l \in \text{set } I)$
 $\vee (\exists l. \text{Marked } (\text{lit-of } x) l \in \text{set } I)$
 $\vee (\exists l. \text{Propagated } (\neg \text{lit-of } x) l \in \text{set } I)$
 $\vee (\exists l. \text{Propagated } (\text{lit-of } x) l \in \text{set } I)$
using *assms marked-lit.exhaust-sel* **by** *metis*

lemma *literal-is-lit-of-marked*:
assumes $L = \text{lit-of } x$
shows $(\exists l. x = \text{Marked } L l) \vee (\exists l'. x = \text{Propagated } L l')$
using *assms* **by** (*cases x*) *auto*

lemma *true-annot-iff-marked-or-true-lit*:
defined-lit I L \longleftrightarrow ((lits-of-l I) \models L \vee (lits-of-l I) \models \neg L)
unfolding *defined-lit-def* **by** (*auto simp add: lits-of-def rev-image-eqI*
dest!: literal-is-lit-of-marked)

lemma *consistent-interp (lits-of-l I) \implies I \models_{as} N \implies satisfiable N*
by (*simp add: true-annots-true-cl*)

lemma *defined-lit-map*:
defined-lit Ls L \longleftrightarrow atm-of L \in ($\lambda l. \text{atm-of } (\text{lit-of } l)$) ' set Ls
unfolding *defined-lit-def* **apply** (*rule iffI*)
using *image-iff* **apply** *fastforce*
by (*fastforce simp add: atm-of-eq-atm-of dest: atm-imp-marked-or-proped*)

lemma *defined-lit-uminus[iff]*:
defined-lit I (\neg L) \longleftrightarrow defined-lit I L
unfolding *defined-lit-def* **by** *auto*

lemma *Marked-Propagated-in-iff-in-lits-of-l*:
defined-lit I L \longleftrightarrow (L \in lits-of-l I \vee \neg L \in lits-of-l I)
unfolding *lits-of-def defined-lit-def*
by (*auto simp: rev-image-eqI*) (*rename-tac x, case-tac x, auto*)⁺

lemma *consistent-add-undefined-lit-consistent[simp]*:
assumes
consistent-interp (lits-of-l Ls) and
undefined-lit Ls L
shows *consistent-interp (insert L (lits-of-l Ls))*
using *assms* **unfolding** *consistent-interp-def* **by** (*auto simp: Marked-Propagated-in-iff-in-lits-of-l*)

lemma *decided-empty[simp]*:
 $\neg \text{defined-lit } [] L$
unfolding *defined-lit-def* **by** *simp*

13.2 Backtracking

fun *backtrack-split* :: (*'v, 'l, 'm*) *marked-lits*
 \Rightarrow (*'v, 'l, 'm*) *marked-lits* \times (*'v, 'l, 'm*) *marked-lits* **where**

$\text{backtrack-split } [] = ([], []) \mid$
 $\text{backtrack-split } (\text{Propagated } L \ P \ \# \ \text{mlits}) = \text{apfst } ((\text{op } \#) (\text{Propagated } L \ P)) (\text{backtrack-split } \text{mlits}) \mid$
 $\text{backtrack-split } (\text{Marked } L \ l \ \# \ \text{mlits}) = ([], \text{Marked } L \ l \ \# \ \text{mlits})$

lemma *backtrack-split-fst-not-marked*: $a \in \text{set } (\text{fst } (\text{backtrack-split } l)) \implies \neg \text{is-marked } a$
by (induct l rule: marked-lit-list-induct) auto

lemma *backtrack-split-snd-hd-marked*:
 $\text{snd } (\text{backtrack-split } l) \neq [] \implies \text{is-marked } (\text{hd } (\text{snd } (\text{backtrack-split } l)))$
by (induct l rule: marked-lit-list-induct) auto

lemma *backtrack-split-list-eq[simp]*:
 $\text{fst } (\text{backtrack-split } l) @ (\text{snd } (\text{backtrack-split } l)) = l$
by (induct l rule: marked-lit-list-induct) auto

lemma *backtrack-snd-empty-not-marked*:
 $\text{backtrack-split } M = (M'', []) \implies \forall l \in \text{set } M. \neg \text{is-marked } l$
by (metis append-Nil2 backtrack-split-fst-not-marked backtrack-split-list-eq snd-conv)

lemma *backtrack-split-some-is-marked-then-snd-has-hd*:
 $\exists l \in \text{set } M. \text{is-marked } l \implies \exists M' \ L' \ M''. \text{backtrack-split } M = (M'', L' \ \# \ M')$
by (metis backtrack-snd-empty-not-marked list.exhaust prod.collapse)

Another characterisation of the result of *backtrack-split*. This view allows some simpler proofs, since *takeWhile* and *dropWhile* are highly automated:

lemma *backtrack-split-takeWhile-dropWhile*:
 $\text{backtrack-split } M = (\text{takeWhile } (\text{Not } o \text{ is-marked}) \ M, \text{dropWhile } (\text{Not } o \text{ is-marked}) \ M)$
proof (induct M)
case Nil show ?case by simp
next
case (Cons L M) then show ?case by (cases L) auto
qed

13.3 Decomposition with respect to the marked literals

The pattern *get-all-marked-decomposition* $[] = [([], [])]$ is necessary otherwise, we can call the *hd* function in the other pattern.

fun *get-all-marked-decomposition* :: $('a, 'l, 'm) \text{ marked-lits}$
 $\implies ((('a, 'l, 'm) \text{ marked-lits} \times ('a, 'l, 'm) \text{ marked-lits}) \text{ list } \textbf{where}$
 $\text{get-all-marked-decomposition } (\text{Marked } L \ l \ \# \ Ls) =$
 $(\text{Marked } L \ l \ \# \ Ls, []) \ \# \ \text{get-all-marked-decomposition } Ls \mid$
 $\text{get-all-marked-decomposition } (\text{Propagated } L \ P \ \# \ Ls) =$
 $(\text{apsnd } ((\text{op } \#) (\text{Propagated } L \ P)) (\text{hd } (\text{get-all-marked-decomposition } Ls)))$
 $\ \# \ \text{tl } (\text{get-all-marked-decomposition } Ls) \mid$
 $\text{get-all-marked-decomposition } [] = [([], [])]$

value *get-all-marked-decomposition* [Propagated A5 B5, Marked C4 D4, Propagated A3 B3,
 Propagated A2 B2, Marked C1 D1, Propagated A0 B0]

lemma *get-all-marked-decomposition-never-empty[iff]*:
 $\text{get-all-marked-decomposition } M = [] \longleftrightarrow \text{False}$
by (induct M, simp) (rename-tac a xs, case-tac a, auto)

lemma *get-all-marked-decomposition-never-empty-sym[iff]*:


```

[] = get-all-marked-decomposition M  $\longleftrightarrow$  False
using get-all-marked-decomposition-never-empty[of M] by presburger

lemma get-all-marked-decomposition-decomp:
  hd (get-all-marked-decomposition S) = (a, c)  $\implies$  S = c @ a
proof (induct S arbitrary: a c)
  case Nil
  then show ?case by simp
next
  case (Cons x A)
  then show ?case by (cases x; cases hd (get-all-marked-decomposition A)) auto
qed

lemma get-all-marked-decomposition-backtrack-split:
  backtrack-split S = (M, M')  $\longleftrightarrow$  hd (get-all-marked-decomposition S) = (M', M)
proof (induction S arbitrary: M M')
  case Nil
  then show ?case by auto
next
  case (Cons a S)
  then show ?case using backtrack-split-takeWhile-dropWhile by (cases a) force+
qed

lemma get-all-marked-decomposition-nil-backtrack-split-snd-nil:
  get-all-marked-decomposition S = [([], A)]  $\implies$  snd (backtrack-split S) = []
  by (simp add: get-all-marked-decomposition-backtrack-split sndI)

lemma get-all-marked-decomposition-length-1-fst-empty-or-length-1:
  assumes get-all-marked-decomposition M = (a, b) # []
  shows a = []  $\vee$  (length a = 1  $\wedge$  is-marked (hd a)  $\wedge$  hd a  $\in$  set M)
  using assms
proof (induct M arbitrary: a b rule: marked-lit-list-induct)
  case nil then show ?case by simp
next
  case (marked L mark M)
  then show ?case by simp
next
  case (proped L mark M)
  then show ?case by (cases get-all-marked-decomposition M) force+
qed

lemma get-all-marked-decomposition-fst-empty-or-hd-in-M:
  assumes get-all-marked-decomposition M = (a, b) # l
  shows a = []  $\vee$  (is-marked (hd a)  $\wedge$  hd a  $\in$  set M)
  using assms apply (induct M arbitrary: a b rule: marked-lit-list-induct)
  apply auto[2]
  by (metis UnCI backtrack-split-snd-hd-marked get-all-marked-decomposition-backtrack-split
    get-all-marked-decomposition-decomp hd-in-set list.sel(1) set-append snd-conv)

lemma get-all-marked-decomposition-snd-not-marked:
  assumes (a, b)  $\in$  set (get-all-marked-decomposition M)
  and L  $\in$  set b
  shows  $\neg$ is-marked L
  using assms apply (induct M arbitrary: a b rule: marked-lit-list-induct, simp)
  by (rename-tac L' l xs a b, case-tac get-all-marked-decomposition xs; fastforce)+

```

```

lemma tl-get-all-marked-decomposition-skip-some:
  assumes  $x \in \text{set } (\text{tl } (\text{get-all-marked-decomposition } M1))$ 
  shows  $x \in \text{set } (\text{tl } (\text{get-all-marked-decomposition } (M0 @ M1)))$ 
  using assms
  by (induct  $M0$  rule: marked-lit-list-induct)
    (auto simp add: list.set-sel(2))

lemma hd-get-all-marked-decomposition-skip-some:
  assumes  $(x, y) = \text{hd } (\text{get-all-marked-decomposition } M1)$ 
  shows  $(x, y) \in \text{set } (\text{get-all-marked-decomposition } (M0 @ \text{Marked } K \ i \ \# \ M1))$ 
  using assms
proof (induct  $M0$ )
  case Nil
  then show ?case by auto
next
  case (Cons  $L \ M0$ )
  then have  $xy: (x, y) \in \text{set } (\text{get-all-marked-decomposition } (M0 @ \text{Marked } K \ i \ \# \ M1))$  by blast
  show ?case
    proof (cases  $L$ )
    case (Marked  $l \ m$ )
    then show ?thesis using  $xy$  by auto
    next
    case (Propagated  $l \ m$ )
    then show ?thesis
      using  $xy$  Cons.prems
      by (cases get-all-marked-decomposition  $(M0 @ \text{Marked } K \ i \ \# \ M1)$ )
        (auto dest!: get-all-marked-decomposition-decomp
          arg-cong[of get-all-marked-decomposition - - hd])
    qed
  qed

lemma get-all-marked-decomposition-snd-union:
   $\text{set } M = \bigcup (\text{set 'snd ' set } (\text{get-all-marked-decomposition } M)) \cup \{L \mid L. \text{is-marked } L \wedge L \in \text{set } M\}$ 
  (is ? $M \ M = ?U \ M \cup ?Ls \ M$ )
proof (induct  $M$  arbitrary:)
  case Nil
  then show ?case by simp
next
  case (Cons  $L \ M$ )
  show ?case
    proof (cases  $L$ )
    case (Marked  $a \ l$ ) note  $L = \text{this}$ 
    then have  $L \in ?Ls \ (L \ \# \ M)$  by auto
    moreover have  $?U \ (L \ \# \ M) = ?U \ M$  unfolding  $L$  by auto
    moreover have  $?M \ M = ?U \ M \cup ?Ls \ M$  using Cons.hyps by auto
    ultimately show ?thesis by auto
    next
    case (Propagated  $a \ P$ )
    then show ?thesis using Cons.hyps by (cases (get-all-marked-decomposition  $M$ )) auto
  qed
qed

lemma in-get-all-marked-decomposition-in-get-all-marked-decomposition-prepend:
   $(a, b) \in \text{set } (\text{get-all-marked-decomposition } M') \implies$ 

```

```

   $\exists b'. (a, b' @ b) \in \text{set } (\text{get-all-marked-decomposition } (M @ M'))$ 
apply (induction M rule: marked-lit-list-induct)
  apply (metis append-Nil)
  apply auto
by (rename-tac L' m xs, case-tac get-all-marked-decomposition (xs @ M') auto)

lemma get-all-marked-decomposition-remove-unmark-ssed-length:
  assumes  $\forall l \in \text{set } M'. \neg \text{is-marked } l$ 
  shows  $\text{length } (\text{get-all-marked-decomposition } (M' @ M''))$ 
     $= \text{length } (\text{get-all-marked-decomposition } M'')$ 
  using assms by (induct M' arbitrary: M'' rule: marked-lit-list-induct) auto

lemma get-all-marked-decomposition-not-is-marked-length:
  assumes  $\forall l \in \text{set } M'. \neg \text{is-marked } l$ 
  shows  $1 + \text{length } (\text{get-all-marked-decomposition } (\text{Propagated } (-L) P \# M))$ 
     $= \text{length } (\text{get-all-marked-decomposition } (M' @ \text{Marked } L l \# M))$ 
  using assms get-all-marked-decomposition-remove-unmark-ssed-length by fastforce

lemma get-all-marked-decomposition-last-choice:
  assumes  $\text{tl } (\text{get-all-marked-decomposition } (M' @ \text{Marked } L l \# M)) \neq []$ 
  and  $\forall l \in \text{set } M'. \neg \text{is-marked } l$ 
  and  $\text{hd } (\text{tl } (\text{get-all-marked-decomposition } (M' @ \text{Marked } L l \# M))) = (M0', M0)$ 
  shows  $\text{hd } (\text{get-all-marked-decomposition } (\text{Propagated } (-L) P \# M)) = (M0', \text{Propagated } (-L) P \# M0)$ 
  using assms by (induct M' rule: marked-lit-list-induct) auto

lemma get-all-marked-decomposition-except-last-choice-equal:
  assumes  $\forall l \in \text{set } M'. \neg \text{is-marked } l$ 
  shows  $\text{tl } (\text{get-all-marked-decomposition } (\text{Propagated } (-L) P \# M))$ 
     $= \text{tl } (\text{tl } (\text{get-all-marked-decomposition } (M' @ \text{Marked } L l \# M)))$ 
  using assms by (induct M' rule: marked-lit-list-induct) auto

lemma get-all-marked-decomposition-hd-hd:
  assumes  $\text{get-all-marked-decomposition } Ls = (M, C) \# (M0, M0') \# l$ 
  shows  $\text{tl } M = M0' @ M0 \wedge \text{is-marked } (\text{hd } M)$ 
  using assms
proof (induct Ls arbitrary: M C M0 M0' l)
  case Nil
  then show ?case by simp
next
  case (Cons a Ls M C M0 M0' l) note IH = this(1) and g = this(2)
  { fix L level
    assume a: a = Marked L level
    have  $Ls = M0' @ M0$ 
    using g a by (force intro: get-all-marked-decomposition-decomp)
    then have  $\text{tl } M = M0' @ M0 \wedge \text{is-marked } (\text{hd } M)$  using g a by auto
  }
moreover {
  fix L P
  assume a: a = Propagated L P
  have  $\text{tl } M = M0' @ M0 \wedge \text{is-marked } (\text{hd } M)$ 
  using IH Cons.premis unfolding a by (cases get-all-marked-decomposition Ls) auto
}
ultimately show ?case by (cases a) auto
qed

```

lemma *get-all-marked-decomposition-exists-prepend*[*dest*]:
assumes $(a, b) \in \text{set } (\text{get-all-marked-decomposition } M)$
shows $\exists c. M = c @ b @ a$
using *assms* **apply** (*induct* *M* *rule*: *marked-lit-list-induct*)
apply *simp*
by (*rename-tac* *L' m xs*, *case-tac* *get-all-marked-decomposition xs*;
auto dest!: *arg-cong*[*of* *get-all-marked-decomposition - - hd*]
get-all-marked-decomposition-decomp)+

lemma *get-all-marked-decomposition-incl*:
assumes $(a, b) \in \text{set } (\text{get-all-marked-decomposition } M)$
shows $\text{set } b \subseteq \text{set } M$ **and** $\text{set } a \subseteq \text{set } M$
using *assms* *get-all-marked-decomposition-exists-prepend* **by** *fastforce*+

lemma *get-all-marked-decomposition-exists-prepend'*:
assumes $(a, b) \in \text{set } (\text{get-all-marked-decomposition } M)$
obtains *c* **where** $M = c @ b @ a$
using *assms* **apply** (*induct* *M* *rule*: *marked-lit-list-induct*)
apply *auto*[1]
by (*rename-tac* *L' m xs*, *case-tac* *hd (get-all-marked-decomposition xs)*,
auto dest!: *get-all-marked-decomposition-decomp simp add: list.set-sel(2)*)+

lemma *union-in-get-all-marked-decomposition-is-subset*:
assumes $(a, b) \in \text{set } (\text{get-all-marked-decomposition } M)$
shows $\text{set } a \cup \text{set } b \subseteq \text{set } M$
using *assms* **by** *force*

lemma *Marked-cons-in-get-all-marked-decomposition-append-Marked-cons*:
 $\exists M1 M2. (\text{Marked } K i \# M1, M2) \in \text{set } (\text{get-all-marked-decomposition } (c @ \text{Marked } K i \# c'))$
apply (*induction* *c* *rule*: *marked-lit-list-induct*)
apply *auto*[2]
apply (*rename-tac* *L m xs*,
case-tac *hd (get-all-marked-decomposition (xs @ Marked K i # c'))*)
apply (*case-tac* *get-all-marked-decomposition (xs @ Marked K i # c')*)
by *auto*

definition *all-decomposition-implies* :: 'a *literal multiset set*
 $\Rightarrow ((\text{'a, 'l, 'm}) \text{ marked-lit list} \times (\text{'a, 'l, 'm}) \text{ marked-lit list}) \text{ list} \Rightarrow \text{bool})$ **where**
all-decomposition-implies *N S*
 $\longleftrightarrow (\forall (Ls, \text{seen}) \in \text{set } S. \text{unmark-l } Ls \cup N \models_{ps} \text{unmark-l seen})$

lemma *all-decomposition-implies-empty*[*iff*]:
all-decomposition-implies *N* [] **unfolding** *all-decomposition-implies-def* **by** *auto*

lemma *all-decomposition-implies-single*[*iff*]:
all-decomposition-implies *N* [(*Ls*, *seen*)]
 $\longleftrightarrow \text{unmark-l } Ls \cup N \models_{ps} \text{unmark-l seen}$
unfolding *all-decomposition-implies-def* **by** *auto*

lemma *all-decomposition-implies-append*[*iff*]:
all-decomposition-implies *N* (*S* @ *S'*)
 $\longleftrightarrow (\text{all-decomposition-implies } N S \wedge \text{all-decomposition-implies } N S')$
unfolding *all-decomposition-implies-def* **by** *auto*

```

lemma all-decomposition-implies-cons-pair[iff]:
  all-decomposition-implies  $N$   $((Ls, seen) \# S')$ 
   $\longleftrightarrow$  (all-decomposition-implies  $N$   $[(Ls, seen)] \wedge$  all-decomposition-implies  $N$   $S'$ )
unfolding all-decomposition-implies-def by auto

lemma all-decomposition-implies-cons-single[iff]:
  all-decomposition-implies  $N$   $(l \# S') \longleftrightarrow$ 
  (unmark-l (fst  $l$ )  $\cup N \models_{ps}$  unmark-l (snd  $l$ )  $\wedge$ 
   all-decomposition-implies  $N$   $S'$ )
unfolding all-decomposition-implies-def by auto

lemma all-decomposition-implies-trail-is-implied:
  assumes all-decomposition-implies  $N$  (get-all-marked-decomposition  $M$ )
  shows  $N \cup \{\{\#lit-of\ L\# \mid L. is-marked\ L \wedge L \in set\ M\}\}$ 
   $\models_{ps}$   $(\lambda a. \{\#lit-of\ a\#\}) \text{ ' } \bigcup (set \text{ ' } snd \text{ ' } set (get-all-marked-decomposition\ M))$ 
using assms
proof (induct length (get-all-marked-decomposition M) arbitrary: M)
  case 0
  then show ?case by auto
next
  case (Suc n) note  $IH = this(1)$  and  $length = this(2)$  and  $decomp = this(3)$ 
  consider
    (le1)  $length (get-all-marked-decomposition\ M) \leq 1$ 
    | (gt1)  $length (get-all-marked-decomposition\ M) > 1$ 
  by arith
  then show ?case
  proof cases
    case le1
    then obtain  $a\ b$  where  $g: get-all-marked-decomposition\ M = (a, b) \# []$ 
    by (cases get-all-marked-decomposition M) auto
    moreover {
      assume  $a = []$ 
      then have ?thesis using Suc.premis g by auto
    }
    moreover {
      assume  $l: length\ a = 1$  and  $m: is-marked\ (hd\ a)$  and  $hd: hd\ a \in set\ M$ 
      then have  $(\lambda a. \{\#lit-of\ a\#\}) (hd\ a) \in \{\{\#lit-of\ L\# \mid L. is-marked\ L \wedge L \in set\ M\}\}$  by auto
      then have  $H: unmark-l\ a \cup N \subseteq N \cup \{\{\#lit-of\ L\# \mid L. is-marked\ L \wedge L \in set\ M\}\}$ 
      using  $l$  by (cases a) auto
      have  $f1: (\lambda m. \{\#lit-of\ m\#\}) \text{ ' } set\ a \cup N \models_{ps} (\lambda m. \{\#lit-of\ m\#\}) \text{ ' } set\ b$ 
      using decomp unfolding all-decomposition-implies-def g by simp
      have ?thesis
      apply (rule true-clss-clss-subset) using  $f1\ H\ g$  by auto
    }
  ultimately show ?thesis
  using get-all-marked-decomposition-length-1-fst-empty-or-length-1 by blast
next
  case gt1
  then obtain  $Ls0\ seen0\ M'$  where
     $Ls0: get-all-marked-decomposition\ M = (Ls0, seen0) \# get-all-marked-decomposition\ M'$  and
     $length': length (get-all-marked-decomposition\ M') = n$  and
     $M'-in-M: set\ M' \subseteq set\ M$ 
  using  $length$  by (induct M rule: marked-lit-list-induct) (auto simp: subset-insertI2)
  let ? $d = \bigcup (set \text{ ' } snd \text{ ' } set (get-all-marked-decomposition\ M'))$ 
  let ? $unM = \{unmark\ L \mid L. is-marked\ L \wedge L \in set\ M\}$ 

```

```

let ?unM' = {unmark L | L. is-marked L ∧ L ∈ set M'}
{
  assume n = 0
  then have get-all-marked-decomposition M' = [] using length' by auto
  then have ?thesis using Suc.premis unfolding all-decomposition-implies-def Ls0 by auto
}
moreover {
  assume n: n > 0
  then obtain Ls1 seen1 l where
    Ls1: get-all-marked-decomposition M' = (Ls1, seen1) # l
    using length' by (induct M' rule: marked-lit-list-induct) auto

  have all-decomposition-implies N (get-all-marked-decomposition M')
    using decomp unfolding Ls0 by auto
  then have N: N ∪ ?unM' ⊨ps unmark-s ?d
    using IH length' by auto
  have l: N ∪ ?unM' ⊆ N ∪ ?unM
    using M'-in-M by auto
  from true-clss-clss-subset[OF this N]
  have ΨN: N ∪ ?unM ⊨ps unmark-s ?d by auto
  have is-marked (hd Ls0) and LS: tl Ls0 = seen1 @ Ls1
    using get-all-marked-decomposition-hd-hd[of M] unfolding Ls0 Ls1 by auto

  have LSM: seen1 @ Ls1 = M' using get-all-marked-decomposition-decomp[of M'] Ls1 by auto
  have M': set M' = ?d ∪ {L | L. is-marked L ∧ L ∈ set M'}
    using get-all-marked-decomposition-snd-union by auto

  {
    assume Ls0 ≠ []
    then have hd Ls0 ∈ set M
      using get-all-marked-decomposition-fst-empty-or-hd-in-M Ls0 by blast
    then have N ∪ ?unM ⊨p unmark (hd Ls0)
      using ⟨is-marked (hd Ls0)⟩ by (metis (mono-tags, lifting) UnCI mem-Collect-eq
        true-clss-clss-in)
  } note hd-Ls0 = this

  have l: unmark ' (?d ∪ {L | L. is-marked L ∧ L ∈ set M'}) = unmark-s ?d ∪ ?unM'
    by auto
  have N ∪ ?unM' ⊨ps unmark ' (?d ∪ {L | L. is-marked L ∧ L ∈ set M'})
    unfolding l using N by (auto simp: all-in-true-clss-clss)
  then have t: N ∪ ?unM' ⊨ps unmark-l (tl Ls0)
    using M' unfolding LS LSM by auto
  then have N ∪ ?unM ⊨ps unmark-l (tl Ls0)
    using M'-in-M true-clss-clss-subset[OF - t, of N ∪ ?unM] by auto
  then have N ∪ ?unM ⊨ps unmark-l Ls0
    using hd-Ls0 by (cases Ls0) auto

  moreover have unmark-l Ls0 ∪ N ⊨ps unmark-l seen0
    using decomp unfolding Ls0 by simp
  moreover have ∧M Ma. (M::'a literal multiset set) ∪ Ma ⊨ps M
    by (simp add: all-in-true-clss-clss)
  ultimately have Ψ: N ∪ ?unM ⊨ps unmark-l seen0
    by (meson true-clss-clss-left-right true-clss-clss-union-and true-clss-clss-union-l-r)

  moreover have unmark ' (set seen0 ∪ ?d) = unmark-l seen0 ∪ unmark-s ?d

```

```

      by auto
    ultimately have ?thesis using  $\Psi N$  unfolding  $Ls0$  by simp
  }
  ultimately show ?thesis by auto
qed
qed

```

lemma *all-decomposition-implies-propagated-lits-are-implied*:

```

  assumes all-decomposition-implies  $N$  (get-all-marked-decomposition  $M$ )
  shows  $N \cup \{\{\#lit\text{-of } L\# \mid L. \text{is-marked } L \wedge L \in \text{set } M\} \mid \vdash_{ps} \text{unmark-}l \text{ } M\}$ 
    (is  $?I \vdash_{ps} ?A$ )
proof -
  have  $?I \vdash_{ps} \text{unmark-}s \{L \mid L. \text{is-marked } L \wedge L \in \text{set } M\}$ 
    by (auto intro: all-in-true-clss-clss)
  moreover have  $?I \vdash_{ps} \text{unmark } ' \bigcup (\text{set } ' \text{snd } ' \text{set } (\text{get-all-marked-decomposition } M))$ 
    using all-decomposition-implies-trail-is-implied assms by blast
  ultimately have  $N \cup \{\text{unmark } m \mid m. \text{is-marked } m \wedge m \in \text{set } M\}$ 
     $\vdash_{ps} \text{unmark } ' \bigcup (\text{set } ' \text{snd } ' \text{set } (\text{get-all-marked-decomposition } M))$ 
     $\cup \text{unmark } ' \{m \mid m. \text{is-marked } m \wedge m \in \text{set } M\}$ 
    by blast
  then show ?thesis
    by (metis (no-types) get-all-marked-decomposition-snd-union[of  $M$ ] image-Un)
qed

```

lemma *all-decomposition-implies-insert-single*:

```

  all-decomposition-implies  $N \ M \implies \text{all-decomposition-implies } (\text{insert } C \ N) \ M$ 
  unfolding all-decomposition-implies-def by auto

```

13.4 Negation of Clauses

definition $CNot :: 'v \text{ clause} \Rightarrow 'v \text{ clauses}$ **where**
 $CNot \ \psi = \{ \{\#-L\# \} \mid L. \ L \in \# \ \psi \}$

lemma *in-CNot-uminus*[*iff*]:
 shows $\{\#L\# \} \in CNot \ \psi \longleftrightarrow -L \in \# \ \psi$
 unfolding *CNot-def* by force

lemma
 shows
 CNot-singleton[*simp*]: $CNot \ \{\#L\# \} = \{\{\#-L\# \}\}$ **and**
 CNot-empty[*simp*]: $CNot \ \{\# \} = \{\}$ **and**
 CNot-plus[*simp*]: $CNot \ (A + B) = CNot \ A \cup CNot \ B$
 unfolding *CNot-def* by auto

lemma *CNot-eq-empty*[*iff*]:
 $CNot \ D = \{\} \longleftrightarrow D = \{\# \}$
 unfolding *CNot-def* by (*auto simp add: multiset-eqI*)

lemma *in-CNot-implies-uminus*:
 assumes $L \in \# \ D$ **and** $M \vdash_{as} CNot \ D$
 shows $M \vdash_a \{\#-L\# \}$ **and** $-L \in \text{lits-of-}l \ M$
 using *assms* by (*auto simp: true-annot-def true-annot-def CNot-def*)

lemma *CNot-remdups-mset*[*simp*]:
 $CNot \ (\text{remdups-mset } A) = CNot \ A$
 unfolding *CNot-def* by auto

lemma *Ball-CNot-Ball-mset[simp]* :
 $(\forall x \in CNot\ D. P\ x) \longleftrightarrow (\forall L \in \#\ D. P\ \{\# - L\# \})$
unfolding *CNot-def* **by** *auto*

lemma *consistent-CNot-not*:
assumes *consistent-interp I*
shows $I \models_s CNot\ \varphi \implies \neg I \models \varphi$
using *assms* **unfolding** *consistent-interp-def true-clss-def true-cls-def* **by** *auto*

lemma *total-not-true-cls-true-clss-CNot*:
assumes *total-over-m I {φ}* **and** $\neg I \models \varphi$
shows $I \models_s CNot\ \varphi$
using *assms* **unfolding** *total-over-m-def total-over-set-def true-clss-def true-cls-def CNot-def*
apply *clarify*
by (*rename-tac x L, case-tac L*) (*force intro: pos-lit-in-atms-of neg-lit-in-atms-of*)**+**

lemma *total-not-CNot*:
assumes *total-over-m I {φ}* **and** $\neg I \models_s CNot\ \varphi$
shows $I \models \varphi$
using *assms* *total-not-true-cls-true-clss-CNot* **by** *auto*

lemma *atms-of-ms-CNot-atms-of[simp]*:
 $atms-of-ms\ (CNot\ C) = atms-of\ C$
unfolding *atms-of-ms-def atms-of-def CNot-def* **by** *fastforce*

lemma *true-clss-clss-contradiction-true-clss-cls-false*:
 $C \in D \implies D \models_{ps} CNot\ C \implies D \models_p \{\#\}$
unfolding *true-clss-clss-def true-clss-cls-def total-over-m-def*
by (*metis Un-commute atms-of-empty atms-of-ms-CNot-atms-of atms-of-ms-insert atms-of-ms-union*
consistent-CNot-not insert-absorb sup-bot.left-neutral true-clss-def)

lemma *true-annots-CNot-all-atms-defined*:
assumes $M \models_{as} CNot\ T$ **and** $a1: L \in \# T$
shows $atm-of\ L \in atm-of\ \text{'lits-of-l}\ M$
by (*metis assms atm-of-uminus image-eqI in-CNot-implies-uminus(1) true-annot-singleton*)

lemma *true-annots-CNot-all-uminus-atms-defined*:
assumes $M \models_{as} CNot\ T$ **and** $a1: -L \in \# T$
shows $atm-of\ L \in atm-of\ \text{'lits-of-l}\ M$
by (*metis assms atm-of-uminus image-eqI in-CNot-implies-uminus(1) true-annot-singleton*)

lemma *true-clss-clss-false-left-right*:
assumes $\{\{\#L\#\}\} \cup B \models_p \{\#\}$
shows $B \models_{ps} CNot\ \{\#L\#\}$
unfolding *true-clss-clss-def true-clss-cls-def*
proof (*intro allI impI*)
fix I
assume
tot: total-over-m I (B \cup CNot {#L#}) **and**
cons: consistent-interp I **and**
 $I \models_s B$
have *total-over-m I ({#L#} \cup B)* **using** *tot* **by** *auto*
then have $\neg I \models_s insert\ \{\#L\#\}\ B$
using *assms cons* **unfolding** *true-clss-cls-def* **by** *simp*

then show $I \models_s CNot \{ \#L\# \}$
using *tot I* **by** (*cases L*) *auto*
qed

lemma *true-annots-true-clss-def-iff-negation-in-model*:
 $M \models_{as} CNot C \longleftrightarrow (\forall L \in \# C. -L \in \text{ lits-of-}l M)$
unfolding *CNot-def true-annots-true-clss true-clss-def* **by** *auto*

lemma *consistent-CNot-not-tautology*:
 $\text{consistent-interp } M \implies M \models_s CNot D \implies \neg \text{tautology } D$
by (*metis atms-of-ms-CNot-atms-of consistent-CNot-not satisfiable-carac' satisfiable-def tautology-def total-over-m-def*)

lemma *atms-of-ms-CNot-atms-of-ms*: $\text{atms-of-ms } (CNot CC) = \text{atms-of-ms } \{CC\}$
by *simp*

lemma *total-over-m-CNot-toal-over-m[simp]*:
 $\text{total-over-m } I (CNot C) = \text{total-over-set } I (\text{atms-of } C)$
unfolding *total-over-m-def total-over-set-def* **by** *auto*

lemma *uminus-lit-swap*: $\neg(a::'a \text{ literal}) = i \longleftrightarrow a = -i$
by *auto*

lemma *true-clss-clss-plus-CNot*:
assumes *CC-L*: $A \models_p CC + \{ \#L\# \}$
and *CNot-CC*: $A \models_{ps} CNot CC$
shows $A \models_p \{ \#L\# \}$
unfolding *true-clss-clss-def true-clss-clss-def CNot-def total-over-m-def*

proof (*intro allI impI*)

fix *I*

assume

tot: $\text{total-over-set } I (\text{atms-of-ms } (A \cup \{ \#L\# \}))$ **and**

cons: *consistent-interp I* **and**

I: $I \models_s A$

let $?I = I \cup \{Pos P | P. P \in \text{atms-of } CC \wedge P \notin \text{atm-of } 'I\}$

have *cons'*: *consistent-interp ?I*

using *cons* **unfolding** *consistent-interp-def*

by (*auto simp: uminus-lit-swap atms-of-def rev-image-eqI*)

have *I'*: $?I \models_s A$

using *I* *true-clss-union-increase* **by** *blast*

have *tot-CNot*: $\text{total-over-m } ?I (A \cup CNot CC)$

using *tot atms-of-s-def* **by** (*fastforce simp: total-over-m-def total-over-set-def*)

then have *tot-I-A-CC-L*: $\text{total-over-m } ?I (A \cup \{CC + \{ \#L\# \})$

using *tot* **unfolding** *total-over-m-def total-over-set-atm-of* **by** *auto*

then have $?I \models CC + \{ \#L\# \}$ **using** *CC-L cons' I'* **unfolding** *true-clss-clss-def* **by** *blast*

moreover

have $?I \models_s CNot CC$ **using** *CNot-CC cons' I'* *tot-CNot* **unfolding** *true-clss-clss-def* **by** *auto*

then have $\neg A \models_p CC$

by (*metis (no-types, lifting) I' atms-of-ms-CNot-atms-of-ms atms-of-ms-union cons' consistent-CNot-not tot-CNot total-over-m-def true-clss-clss-def*)

then have $\neg ?I \models CC$ **using** $\langle ?I \models_s CNot CC \rangle$ *cons'* *consistent-CNot-not* **by** *blast*

ultimately have $?I \models \{ \#L\# \}$ **by** *blast*

then show $I \models \{ \#L\# \}$

by (*metis (no-types, lifting) atms-of-ms-union cons' consistent-CNot-not tot total-not-CNot*)

total-over-m-def total-over-set-union true-clss-union-increase)
qed

lemma *true-annots-CNot-lit-of-notin-skip*:

assumes $LM: L \# M \models_{as} CNot\ A$ **and** $LA: lit\text{-}of\ L \notin\# A \text{ -- } lit\text{-}of\ L \notin\# A$
shows $M \models_{as} CNot\ A$

using *LM unfolding true-annots-def Ball-def*

proof (*intro allI impI*)

fix l

assume $H: \forall x. x \in CNot\ A \longrightarrow L \# M \models_a x$ **and** $l: l \in CNot\ A$

then have $L \# M \models_a l$ **by** *auto*

then show $M \models_a l$ **using** *LA l* **by** (*cases L*) (*auto simp: CNot-def*)

qed

lemma *true-clss-clss-union-false-true-clss-clss-cnot*:

$A \cup \{B\} \models_{ps} \{\{\#\}\} \longleftrightarrow A \models_{ps} CNot\ B$

using *total-not-CNot consistent-CNot-not unfolding total-over-m-def true-clss-clss-def*
by *fastforce*

lemma *true-annot-remove-hd-if-notin-vars*:

assumes $a \# M' \models_a D$ **and** $atm\text{-}of\ (lit\text{-}of\ a) \notin atm\text{-}of\ D$

shows $M' \models_a D$

using *assms true-clss-remove-hd-if-notin-vars unfolding true-annot-def* **by** *auto*

lemma *true-annot-remove-if-notin-vars*:

assumes $M @ M' \models_a D$ **and** $\forall x \in atm\text{-}of\ D. x \notin atm\text{-}of\ \text{' } lits\text{-}of\text{-}l\ M$

shows $M' \models_a D$

using *assms apply (induct M, simp)*

using *true-annot-remove-hd-if-notin-vars* **by** *force+*

lemma *true-annots-remove-if-notin-vars*:

assumes $M @ M' \models_{as} D$ **and** $\forall x \in atm\text{-}of\text{-}ms\ D. x \notin atm\text{-}of\ \text{' } lits\text{-}of\text{-}l\ M$

shows $M' \models_{as} D$ **unfolding** *true-annots-def*

using *assms true-annot-remove-if-notin-vars[of M M']*

unfolding *true-annots-def atm\text{-}of\text{-}ms\text{-}def* **by** *force*

lemma *all-variables-defined-not-imply-cnot*:

assumes

$\forall s \in atm\text{-}of\text{-}ms\ \{B\}. s \in atm\text{-}of\ \text{' } lits\text{-}of\text{-}l\ A$ **and**

$\neg A \models_a B$

shows $A \models_{as} CNot\ B$

unfolding *true-annot-def true-annots-def Ball-def CNot-def true-lit-def*

proof (*clarify, rule ccontr*)

fix L

assume $LB: L \in\# B$ **and** $\neg lits\text{-}of\text{-}l\ A \models_l \neg L$

then have $atm\text{-}of\ L \in atm\text{-}of\ \text{' } lits\text{-}of\text{-}l\ A$

using *assms(1) by (simp add: atm\text{-}of\text{-}lit\text{-}in\text{-}atms\text{-}of\ lits\text{-}of\text{-}def)*

then have $L \in lits\text{-}of\text{-}l\ A \vee \neg L \in lits\text{-}of\text{-}l\ A$

using *atm\text{-}of\text{-}in\text{-}atm\text{-}of\text{-}set\text{-}iff\text{-}in\text{-}set\text{-}or\text{-}uminus\text{-}in\text{-}set* **by** *metis*

then have $L \in lits\text{-}of\text{-}l\ A$ **using** $\langle \neg lits\text{-}of\text{-}l\ A \models_l \neg L \rangle$ **by** *auto*

then show *False*

using *LB assms(2) unfolding true-annot-def true-lit-def true-clss-def Bex-def*

by *blast*

qed

lemma *CNot-union-mset*[simp]:
 $CNot (A \# \cup B) = CNot A \cup CNot B$
unfolding *CNot-def* **by** *auto*

13.5 Other

abbreviation *no-dup* $L \equiv distinct (map (\lambda l. atm-of (lit-of l)) L)$

lemma *no-dup-rev*[simp]:
 $no-dup (rev M) \longleftrightarrow no-dup M$
by (*auto simp: rev-map[symmetric]*)

lemma *no-dup-length-eq-card-atm-of-lits-of-l*:
assumes *no-dup* M
shows $length M = card (atm-of ' lits-of-l M)$
using *assms* **unfolding** *lits-of-def* **by** (*induct M*) (*auto simp add: image-image*)

lemma *distinct-consistent-interp*:
 $no-dup M \implies consistent-interp (lits-of-l M)$

proof (*induct M*)

case *Nil*
show *?case* **by** *auto*

next

case (*Cons L M*)
then have *a1: consistent-interp (lits-of-l M)* **by** *auto*
have *a2: atm-of (lit-of L) $\notin (\lambda l. atm-of (lit-of l)) ' set M$* **using** *Cons.prem*s **by** *auto*
have *undefined-lit M (lit-of L)*
using *a2 image-iff* **unfolding** *defined-lit-def* **by** *fastforce*
then show *?case*
using *a1* **by** *simp*

qed

lemma *distinct-get-all-marked-decomposition-no-dup*:
assumes $(a, b) \in set (get-all-marked-decomposition M)$
and *no-dup* M
shows *no-dup* $(a @ b)$
using *assms* **by** *force*

lemma *true-annots-lit-of-notin-skip*:

assumes $L \# M \models_{as} CNot A$
and $\neg lit-of L \notin \# A$
and *no-dup* $(L \# M)$
shows $M \models_{as} CNot A$

proof $-$

have $\forall l \in \# A. \neg l \in lits-of-l (L \# M)$
using *assms(1) in-CNot-implies-uminus(2)* **by** *blast*

moreover

have $atm-of (lit-of L) \notin atm-of ' lits-of-l M$
using *assms(3)* **unfolding** *lits-of-def* **by** *force*
then have $\neg lit-of L \notin lits-of-l M$ **unfolding** *lits-of-def*
by (*metis (no-types) atm-of-uminus imageI*)

ultimately have $\forall l \in \# A. \neg l \in lits-of-l M$

using *assms(2)* **by** (*metis insert-iff list.simps(15) lits-of-insert uminus-of-uminus-id*)
then show *?thesis* **by** (*auto simp add: true-annots-def*)

qed

abbreviation *true-annots-mset* (**infix** \models_{asm} 50) **where**
 $I \models_{asm} C \equiv I \models_{as} (set-mset\ C)$

abbreviation *true-clss-clss-m:: 'v clause multiset \Rightarrow 'v clause multiset \Rightarrow bool* (**infix** \models_{psm} 50)
where
 $I \models_{psm} C \equiv set-mset\ I \models_{ps} (set-mset\ C)$

Analog of $\llbracket ?N \models_{ps} ?B; ?A \subseteq ?B \rrbracket \Longrightarrow ?N \models_{ps} ?A$

lemma *true-clss-clssm-subsetE*: $N \models_{psm} B \Longrightarrow A \subseteq_{\#} B \Longrightarrow N \models_{psm} A$
using *set-mset-mono true-clss-clss-subsetE* **by** *blast*

abbreviation *true-clss-clss-m:: 'a clause multiset \Rightarrow 'a clause \Rightarrow bool* (**infix** \models_{pm} 50) **where**
 $I \models_{pm} C \equiv set-mset\ I \models_p C$

abbreviation *distinct-mset-mset :: 'a multiset multiset \Rightarrow bool* **where**
 $distinct-mset-mset\ \Sigma \equiv distinct-mset-set\ (set-mset\ \Sigma)$

abbreviation *all-decomposition-implies-m* **where**
 $all-decomposition-implies-m\ A\ B \equiv all-decomposition-implies\ (set-mset\ A)\ B$

abbreviation *atms-of-mm :: 'a literal multiset multiset \Rightarrow 'a set* **where**
 $atms-of-mm\ U \equiv atms-of-ms\ (set-mset\ U)$

Other definition using *Union-mset*

lemma *atms-of-mm* $U \equiv set-mset\ (\bigcup_{\#} image-mset\ (image-mset\ atm-of)\ U)$
unfolding *atms-of-ms-def* **by** (*auto simp: atms-of-def*)

abbreviation *true-clss-m:: 'a interp \Rightarrow 'a clause multiset \Rightarrow bool* (**infix** \models_{sm} 50) **where**
 $I \models_{sm} C \equiv I \models_s set-mset\ C$

abbreviation *true-clss-ext-m* (**infix** \models_{sextm} 49) **where**
 $I \models_{sextm} C \equiv I \models_{sext} set-mset\ C$

end

theory *CDCL-Abstract-Clause-Representation*

imports *Main Partial-Clausal-Logic*

begin

type-synonym *'v clause* = *'v literal multiset*

type-synonym *'v clauses* = *'v clause multiset*

13.6 Abstract Clause Representation

We will abstract the representation of clause and clauses via two locales. We expect our representation to behave like multiset, but the internal representation can be done using list or whatever other representation.

We assume the following:

- there is an equivalent to adding and removing a literal and to taking the union of clauses.

locale *raw-cls* =

fixes

mset-cls:: 'cls \Rightarrow 'v clause **and**

insert-cls :: 'v literal \Rightarrow 'cls \Rightarrow 'cls **and**

```

    remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls
assumes
    insert-cls[simp]: mset-cls (insert-cls L C) = mset-cls C + {#L#} and
    remove-lit[simp]: mset-cls (remove-lit L C) = remove1-mset L (mset-cls C)
begin
end

locale raw-ccls-union =
  fixes
    mset-cls:: 'cls  $\Rightarrow$  'v clause and
    union-cls :: 'cls  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
    insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
    remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls
  assumes
    insert-ccls[simp]: mset-cls (insert-cls L C) = mset-cls C + {#L#} and
    mset-ccls-union-cls[simp]: mset-cls (union-cls C D) = mset-cls C # $\cup$  mset-cls D and
    remove-clit[simp]: mset-cls (remove-lit L C) = remove1-mset L (mset-cls C)
begin
end

```

Instantiation of the previous locale, in an unnamed context to avoid polluting with simp rules

```

context
begin
  interpretation list-cls: raw-cls mset
    op # remove1
    by unfold-locales (auto simp: union-mset-list ex-mset)

  interpretation cls-cls: raw-cls id
     $\lambda L C. C + \{ \#L\# \}$  remove1-mset
    by unfold-locales (auto simp: union-mset-list)

  interpretation list-cls: raw-ccls-union mset
    union-mset-list
    op # remove1
    by unfold-locales (auto simp: union-mset-list ex-mset)

  interpretation cls-cls: raw-ccls-union id
    op # $\cup$   $\lambda L C. C + \{ \#L\# \}$  remove1-mset
    by unfold-locales (auto simp: union-mset-list)
end

```

Over the abstract clauses, we have the following properties:

- We can insert a clause
- We can take the union (used only in proofs for the definition of *clauses*)
- there is an operator indicating whether the abstract clause is contained or not
- if a concrete clause is contained the abstract clauses, then there is an abstract clause

```

locale raw-clss =
  raw-cls mset-cls insert-cls remove-lit
for
  mset-cls:: 'cls  $\Rightarrow$  'v clause and

```

```

insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls +
fixes
mset-clss :: 'clss  $\Rightarrow$  'v clauses and
union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss
assumes
insert-clss[simp]: mset-clss (insert-clss L C) = mset-clss C + {#mset-cls L#} and
union-clss[simp]: mset-clss (union-clss C D) = mset-clss C + mset-clss D and
mset-clss-union-clss[simp]: mset-clss (insert-clss C' D) = {#mset-cls C'#} + mset-clss D and
in-clss-mset-clss[dest]: in-clss a C  $\implies$  mset-cls a  $\in$  # mset-clss C and
in-mset-clss-exists-preimage: b  $\in$  # mset-clss C  $\implies$   $\exists$  b'. in-clss b' C  $\wedge$  mset-cls b' = b and
remove-from-clss-mset-clss[simp]:
  mset-clss (remove-from-clss a C) = mset-clss C - {#mset-cls a#} and
in-clss-union-clss[simp]:
  in-clss a (union-clss C D)  $\longleftrightarrow$  in-clss a C  $\vee$  in-clss a D
begin

end

experiment
begin
  fun remove-first where
    remove-first - [] = [] |
    remove-first C (C' # L) = (if mset C = mset C' then L else C' # remove-first C L)

  lemma mset-map-mset-remove-first:
    mset (map mset (remove-first a C)) = remove1-mset (mset a) (mset (map mset C))
    by (induction C) (auto simp: ac-simps remove1-mset-single-add)

  interpretation clss-clss: raw-clss id  $\lambda$ L C. C + {#L#} remove1-mset
    id op + op  $\in$  #  $\lambda$ L C. C + {#L#} remove1-mset
    by unfold-locales (auto simp: ac-simps)

  interpretation list-clss: raw-clss mset
    op # remove1  $\lambda$ L. mset (map mset L) op @  $\lambda$ L C. L  $\in$  set C op #
    remove-first
    by unfold-locales (auto simp: ac-simps union-mset-list mset-map-mset-remove-first ex-mset)
end

end

theory CDCL-WNOT-Measure
imports Main
begin

```

14 Measure

This measure show the termination of the core of CDCL: each step improves the number of literals we know for sure.

This measure can also be seen as the increasing lexicographic order: it is an order on bounded sequences, when each element is bounded. The proof involves a measure like the one defined here (the same?).

definition $\mu_C :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat list} \Rightarrow \text{nat}$ **where**
 $\mu_C \ s \ b \ M \equiv (\sum i=0..<\text{length } M. M!i * b^\wedge (s+i - \text{length } M))$

lemma $\mu_C\text{-nil}[simp]$:
 $\mu_C \ s \ b \ [] = 0$
unfolding $\mu_C\text{-def}$ **by** *auto*

lemma $\mu_C\text{-single}[simp]$:
 $\mu_C \ s \ b \ [L] = L * b^\wedge (s - \text{Suc } 0)$
unfolding $\mu_C\text{-def}$ **by** *auto*

lemma $\text{set-sum-atLeastLessThan-add}$:
 $(\sum i=k..<k+(b::\text{nat}). f \ i) = (\sum i=0..<b. f \ (k+ i))$
by (*induction b*) *auto*

lemma $\text{set-sum-atLeastLessThan-Suc}$:
 $(\sum i=1..<\text{Suc } j. f \ i) = (\sum i=0..<j. f \ (\text{Suc } i))$
using $\text{set-sum-atLeastLessThan-add}[of \ - \ 1 \ j]$ **by** *force*

lemma $\mu_C\text{-cons}$:
 $\mu_C \ s \ b \ (L \# M) = L * b^\wedge (s - 1 - \text{length } M) + \mu_C \ s \ b \ M$
proof –
have $\mu_C \ s \ b \ (L \# M) = (\sum i=0..<\text{length } (L\#M). (L\#M)!i * b^\wedge (s+i - \text{length } (L\#M)))$
unfolding $\mu_C\text{-def}$ **by** *blast*
also have $\dots = (\sum i=0..<1. (L\#M)!i * b^\wedge (s+i - \text{length } (L\#M)))$
 $+ (\sum i=1..<\text{length } (L\#M). (L\#M)!i * b^\wedge (s+i - \text{length } (L\#M)))$
by (*rule setsum-add-nat-ivl[symmetric]*) *simp-all*
finally have $\mu_C \ s \ b \ (L \# M) = L * b^\wedge (s - 1 - \text{length } M)$
 $+ (\sum i=1..<\text{length } (L\#M). (L\#M)!i * b^\wedge (s+i - \text{length } (L\#M)))$
by *auto*
moreover {
have $(\sum i=1..<\text{length } (L\#M). (L\#M)!i * b^\wedge (s+i - \text{length } (L\#M))) =$
 $(\sum i=0..<\text{length } (M). (L\#M)!(\text{Suc } i) * b^\wedge (s + (\text{Suc } i) - \text{length } (L\#M)))$
unfolding length-Cons $\text{set-sum-atLeastLessThan-Suc}$ **by** *blast*
also have $\dots = (\sum i=0..<\text{length } (M). M!i * b^\wedge (s+i - \text{length } M))$
by *auto*
finally have $(\sum i=1..<\text{length } (L\#M). (L\#M)!i * b^\wedge (s+i - \text{length } (L\#M))) = \mu_C \ s \ b \ M$
unfolding $\mu_C\text{-def}$.
}
ultimately show *?thesis* **by** *presburger*
qed

lemma $\mu_C\text{-append}$:
assumes $s \geq \text{length } (M@M')$
shows $\mu_C \ s \ b \ (M@M') = \mu_C \ (s - \text{length } M') \ b \ M + \mu_C \ s \ b \ M'$
proof –
have $\mu_C \ s \ b \ (M@M') = (\sum i=0..<\text{length } (M@M'). (M@M')!i * b^\wedge (s+i - \text{length } (M@M')))$
unfolding $\mu_C\text{-def}$ **by** *blast*
moreover then have $\dots = (\sum i=0..<\text{length } M. (M@M')!i * b^\wedge (s+i - \text{length } (M@M')))$
 $+ (\sum i=\text{length } M..<\text{length } (M@M'). (M@M')!i * b^\wedge (s+i - \text{length } (M@M')))$
by (*auto intro!: setsum-add-nat-ivl[symmetric]*)
moreover
have $\forall i \in \{0..<\text{length } M\}. (M@M')!i * b^\wedge (s+i - \text{length } (M@M')) = M!i * b^\wedge (s - \text{length } M'$
 $+ i - \text{length } M)$
using $\langle s \geq \text{length } (M@M') \rangle$ **by** (*auto simp add: nth-append ac-simps*)

```

    then have  $\mu_C (s - \text{length } M') \ b \ M = (\sum i=0..< \text{length } M. (M @ M')!i * b^\wedge (s + i - \text{length } (M @ M')))$ 
    unfolding  $\mu_C\text{-def}$  by auto
    ultimately have  $\mu_C \ s \ b \ (M @ M') = \mu_C (s - \text{length } M') \ b \ M$ 
      +  $(\sum i=\text{length } M..<\text{length } (M @ M'). (M @ M')!i * b^\wedge (s + i - \text{length } (M @ M')))$ 
    by auto
  moreover {
    have  $(\sum i=\text{length } M..<\text{length } (M @ M'). (M @ M')!i * b^\wedge (s + i - \text{length } (M @ M')) =$ 
       $(\sum i=0..<\text{length } M'. M!i * b^\wedge (s + i - \text{length } M'))$ 
    unfolding length-append set-sum-atLeastLessThan-add by auto
    then have  $(\sum i=\text{length } M..<\text{length } (M @ M'). (M @ M')!i * b^\wedge (s + i - \text{length } (M @ M')) = \mu_C \ s \ b \ M'$ 
    unfolding  $\mu_C\text{-def}$  .
  }
  ultimately show ?thesis by presburger
qed

```

```

lemma  $\mu_C\text{-cons-non-empty-inf}$ :
  assumes M-ge-1:  $\forall i \in \text{set } M. i \geq 1$  and M:  $M \neq []$ 
  shows  $\mu_C \ s \ b \ M \geq b^\wedge (s - \text{length } M)$ 
  using assms by (cases M) (auto simp: mult-eq-if  $\mu_C\text{-cons}$ )

```

Duplicate of `~~/src/HOL/ex/NatSum.thy` (but generalized to $(0::'a) \leq k$)

```

lemma sum-of-powers:  $0 \leq k \implies (k - 1) * (\sum i=0..<n. k^\wedge i) = k^\wedge n - (1::nat)$ 
  apply (cases k = 0)
  apply (cases n; simp)
  by (induct n) (auto simp: Nat.nat-distrib)

```

In the degenerated cases, we only have the large inequality holds. In the other cases, the following strict inequality holds:

```

lemma  $\mu_C\text{-bounded-non-degenerated}$ :
  fixes b :: nat
  assumes
    b > 0 and
    M  $\neq []$  and
    M-le:  $\forall i < \text{length } M. M!i < b$  and
    s  $\geq \text{length } M$ 
  shows  $\mu_C \ s \ b \ M < b^\wedge s$ 
proof -
  consider (b1)  $b = 1$  | (b)  $b > 1$  using  $\langle b > 0 \rangle$  by (cases b) auto
  then show ?thesis
  proof cases
    case b1
    then have  $\forall i < \text{length } M. M!i = 0$  using M-le by auto
    then have  $\mu_C \ s \ b \ M = 0$  unfolding  $\mu_C\text{-def}$  by auto
    then show ?thesis using  $\langle b > 0 \rangle$  by auto
  next
    case b
    have  $\forall i \in \{0..<\text{length } M\}. M!i * b^\wedge (s + i - \text{length } M) \leq (b-1) * b^\wedge (s + i - \text{length } M)$ 
      using M-le  $\langle b > 1 \rangle$  by auto
    then have  $\mu_C \ s \ b \ M \leq (\sum i=0..<\text{length } M. (b-1) * b^\wedge (s + i - \text{length } M))$ 
      using  $\langle M \neq [] \rangle \ \langle b > 0 \rangle$  unfolding  $\mu_C\text{-def}$  by (auto intro: setsum-mono)
    also
    have  $\forall i \in \{0..<\text{length } M\}. (b-1) * b^\wedge (s + i - \text{length } M) = (b-1) * b^\wedge i * b^\wedge (s - \text{length } M)$ 
      by (metis Nat.add-diff-assoc2 add.commute assms(4) mult.assoc power-add)

```



```

then have  $(\sum_{i=0..<\text{length } M}. (b-1) * b^{\wedge} (s+i - \text{length } M))$ 
   $= (\sum_{i=0..<\text{length } M}. (b-1) * b^{\wedge} i * b^{\wedge} (s - \text{length } M))$ 
  by (auto simp add: ac-simps)
also have  $\dots = (\sum_{i=0..<\text{length } M}. b^{\wedge} i) * b^{\wedge} (s - \text{length } M) * (b-1)$ 
  by (simp add: setsum-left-distrib setsum-right-distrib ac-simps)
finally have  $\mu_C \ s \ b \ M \leq (\sum_{i=0..<\text{length } M}. b^{\wedge} i) * (b-1) * b^{\wedge} (s - \text{length } M)$ 
  by (simp add: ac-simps)

also
  have  $(\sum_{i=0..<\text{length } M}. b^{\wedge} i) * (b-1) = b^{\wedge} (\text{length } M) - 1$ 
    using sum-of-powers[of b length M] <b>1>
    by (auto simp add: ac-simps)
  finally have  $\mu_C \ s \ b \ M \leq (b^{\wedge} (\text{length } M) - 1) * b^{\wedge} (s - \text{length } M)$ 
    by auto
  also have  $\dots < b^{\wedge} (\text{length } M) * b^{\wedge} (s - \text{length } M)$ 
    using <b>1> by auto
  also have  $\dots = b^{\wedge} s$ 
    by (metis assms(4) le-add-diff-inverse power-add)
  finally show ?thesis unfolding  $\mu_C$ -def by (auto simp add: ac-simps)
qed
qed

```

In the degenerate case $b = (0::'a)$, the list M is empty (since the list cannot contain any element).

```

lemma  $\mu_C$ -bounded:
  fixes  $b :: \text{nat}$ 
  assumes
     $M\text{-le}: \forall i < \text{length } M. M!i < b$  and
     $s \geq \text{length } M$ 
     $b > 0$ 
  shows  $\mu_C \ s \ b \ M < b^{\wedge} s$ 
proof -
  consider  $(M0) \ M = [] \mid (M) \ b > 0 \text{ and } M \neq []$ 
    using  $M\text{-le}$  by (cases b, cases M) auto
  then show ?thesis
    proof cases
      case  $M0$ 
        then show ?thesis using  $M\text{-le}$  <b>0> by auto
      next
        case  $M$ 
          show ?thesis using  $\mu_C$ -bounded-non-degenerated[OF M assms(1,2)] by arith
    qed
qed

```

When $b = 0$, we cannot show that the measure is empty, since $0^0 = 1$.

```

lemma  $\mu_C$ -base-0:
  assumes  $\text{length } M \leq s$ 
  shows  $\mu_C \ s \ 0 \ M \leq M!0$ 
proof -
  {
    assume  $s = \text{length } M$ 
    moreover {
      fix  $n$ 
      have  $(\sum_{i=0..<n}. M!i * (0::\text{nat})^{\wedge} i) \leq M!0$ 
        apply (induction n rule: nat-induct)

```

```

    by simp (rename-tac n, case-tac n, auto)
  }
  ultimately have ?thesis unfolding  $\mu_C$ -def by auto
}
moreover
{
  assume length  $M < s$ 
  then have  $\mu_C \ s \ 0 \ M = 0$  unfolding  $\mu_C$ -def by auto}
ultimately show ?thesis using assms unfolding  $\mu_C$ -def by linarith
qed

end
theory CDCL-NOT
imports CDCL-Abstract-Clause-Representation List-More Wellfounded-More CDCL-WNOT-Measure
Partial-Annotated-Clausal-Logic
begin

```

15 NOT's CDCL

15.1 Auxiliary Lemmas and Measure

lemma *no-dup-cannot-not-lit-and-uminus*:
 $no_dup \ M \implies - \ lit_of \ xa = lit_of \ x \implies x \in set \ M \implies xa \notin set \ M$
 by (metis atm-of-uminus distinct-map inj-on-eq-iff uminus-not-id')

lemma *true-clss-single-iff-incl*:
 $I \models s \ single \ 'B \longleftrightarrow B \subseteq I$
 unfolding true-clss-def by auto

lemma *atms-of-ms-single-atm-of[simp]*:
 $atms_of_ms \ \{unmark \ L \mid L. \ P \ L\} = atm_of \ ' \ \{lit_of \ L \mid L. \ P \ L\}$
 unfolding atms-of-ms-def by force

lemma *atms-of-uminus-lit-atm-of-lit-of*:
 $atms_of \ \{\# \ - \ lit_of \ x. \ x \in \# \ A \#\} = atm_of \ ' \ (lit_of \ ' \ (set_mset \ A))$
 unfolding atms-of-def by (auto simp add: Fun.image-comp)

lemma *atms-of-ms-single-image-atm-of-lit-of*:
 $atms_of_ms \ (unmark \ 'A) = atm_of \ ' \ (lit_of \ 'A)$
 unfolding atms-of-ms-def by auto

15.2 Initial definitions

15.2.1 The state

We define here an abstraction over operation on the state we are manipulating.

```

locale dpll-state-ops =
  raw-clss mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
for
  mset-clss:: 'cls  $\Rightarrow$  'v clause and
  insert-clss:: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit:: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss:: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and

```

```

in-clss :: 'cls ⇒ 'clss ⇒ bool and
insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss +
fixes
  trail :: 'st ⇒ ('v, unit, unit) marked-lits and
  raw-clauses :: 'st ⇒ 'clss and
  prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
  tl-trail :: 'st ⇒ 'st and
  add-clNOT :: 'cls ⇒ 'st ⇒ 'st and
  remove-clNOT :: 'cls ⇒ 'st ⇒ 'st
begin

notation insert-cls (infix !++ 50)

notation in-clss (infix !∈! 50)
notation union-clss (infix ⊕ 50)
notation insert-clss (infix !++! 50)

abbreviation clausesNOT where
  clausesNOT S ≡ mset-clss (raw-clauses S)

end

locale dpll-state =
  dpll-state-ops mset-cls insert-cls remove-lit — related to each clause
  mset-clss union-clss in-clss insert-clss remove-from-clss — related to the clauses

  trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT — related to the state
for
  mset-cls :: 'cls ⇒ 'v clause and
  insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
  remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
  mset-clss :: 'clss ⇒ 'v clauses and
  union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
  in-clss :: 'cls ⇒ 'clss ⇒ bool and
  insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
  remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
  trail :: 'st ⇒ ('v, unit, unit) marked-lits and
  raw-clauses :: 'st ⇒ 'clss and
  prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
  tl-trail :: 'st ⇒ 'st and
  add-clNOT :: 'cls ⇒ 'st ⇒ 'st and
  remove-clNOT :: 'cls ⇒ 'st ⇒ 'st +
assumes
  trail-prepend-trail[simp]:
    ∧st L. undefined-lit (trail st) (lit-of L) ⇒ trail (prepend-trail L st) = L # trail st
  and
  tl-trail[simp]: trail (tl-trail S) = tl (trail S) and
  trail-add-clNOT[simp]: ∧st C. no-dup (trail st) ⇒ trail (add-clNOT C st) = trail st and
  trail-remove-clNOT[simp]: ∧st C. trail (remove-clNOT C st) = trail st and

  clauses-prepend-trail[simp]:
    ∧st L. undefined-lit (trail st) (lit-of L) ⇒
      clausesNOT (prepend-trail L st) = clausesNOT st
  and

```

$clauses_tl_trail[simp]: \bigwedge st. clauses_{NOT} (tl_trail\ st) = clauses_{NOT}\ st$ **and**
 $clauses_add_cls_{NOT}[simp]:$
 $\bigwedge st\ C. no_dup\ (trail\ st) \implies clauses_{NOT}\ (add_cls_{NOT}\ C\ st) = \{\#mset_cls\ C\} + clauses_{NOT}\ st$
and
 $clauses_remove_cls_{NOT}[simp]:$
 $\bigwedge st\ C. clauses_{NOT}\ (remove_cls_{NOT}\ C\ st) = removeAll_mset\ (mset_cls\ C)\ (clauses_{NOT}\ st)$
begin

function $reduce_trail_to_{NOT} :: 'a\ list \Rightarrow 'st \Rightarrow 'st$ **where**
 $reduce_trail_to_{NOT}\ F\ S =$
 $(if\ length\ (trail\ S) = length\ F \vee trail\ S = []\ then\ S\ else\ reduce_trail_to_{NOT}\ F\ (tl_trail\ S))$
by $fast+$
termination by $(relation\ measure\ (\lambda(-, S). length\ (trail\ S)))\ auto$
declare $reduce_trail_to_{NOT}.simps[simp\ del]$

lemma
shows
 $reduce_trail_to_{NOT}\ nil[simp]: trail\ S = [] \implies reduce_trail_to_{NOT}\ F\ S = S$ **and**
 $reduce_trail_to_{NOT}\ eq_length[simp]: length\ (trail\ S) = length\ F \implies reduce_trail_to_{NOT}\ F\ S = S$
by $(auto\ simp: reduce_trail_to_{NOT}.simps)$

lemma $reduce_trail_to_{NOT}\ length_ne[simp]:$
 $length\ (trail\ S) \neq length\ F \implies trail\ S \neq [] \implies$
 $reduce_trail_to_{NOT}\ F\ S = reduce_trail_to_{NOT}\ F\ (tl_trail\ S)$
by $(auto\ simp: reduce_trail_to_{NOT}.simps)$

lemma $trail_reduce_trail_to_{NOT}\ length_le:$
assumes $length\ F > length\ (trail\ S)$
shows $trail\ (reduce_trail_to_{NOT}\ F\ S) = []$
using $assms$ **by** $(induction\ F\ S\ rule: reduce_trail_to_{NOT}.induct)$
 $(simp\ add: less_imp_diff_less\ reduce_trail_to_{NOT}.simps)$

lemma $trail_reduce_trail_to_{NOT}\ nil[simp]:$
 $trail\ (reduce_trail_to_{NOT}\ []\ S) = []$
by $(induction\ []\ S\ rule: reduce_trail_to_{NOT}.induct)$
 $(simp\ add: less_imp_diff_less\ reduce_trail_to_{NOT}.simps)$

lemma $clauses_reduce_trail_to_{NOT}\ nil:$
 $clauses_{NOT}\ (reduce_trail_to_{NOT}\ []\ S) = clauses_{NOT}\ S$
by $(induction\ []\ S\ rule: reduce_trail_to_{NOT}.induct)$
 $(simp\ add: less_imp_diff_less\ reduce_trail_to_{NOT}.simps)$

lemma $trail_reduce_trail_to_{NOT}\ drop:$
 $trail\ (reduce_trail_to_{NOT}\ F\ S) =$
 $(if\ length\ (trail\ S) \geq length\ F$
 $then\ drop\ (length\ (trail\ S) - length\ F)\ (trail\ S)$
 $else\ [])$
apply $(induction\ F\ S\ rule: reduce_trail_to_{NOT}.induct)$
apply $(rename_tac\ F\ S, case_tac\ trail\ S)$
apply $auto[]$
apply $(rename_tac\ list, case_tac\ Suc\ (length\ list) > length\ F)$
prefer 2 **apply** $simp$
apply $(subgoal_tac\ Suc\ (length\ list) - length\ F = Suc\ (length\ list - length\ F))$
apply $simp$
apply $simp$

done

lemma *reduce-trail-to_{NOT}-skip-beginning*:
 assumes *trail* $S = F' @ F$
 shows *trail* (*reduce-trail-to_{NOT}* $F S$) = F
 using *assms* **by** (*auto simp: trail-reduce-trail-to_{NOT}-drop*)

lemma *reduce-trail-to_{NOT}-clauses[simp]*:
 clauses_{NOT} (*reduce-trail-to_{NOT}* $F S$) = *clauses_{NOT}* S
 by (*induction F S rule: reduce-trail-to_{NOT}.induct*)
 (*simp add: less-imp-diff-less reduce-trail-to_{NOT}.simps*)

abbreviation *trail-weight* **where**

trail-weight $S \equiv \text{map } ((\lambda l. 1 + \text{length } l) \circ \text{snd}) (\text{get-all-marked-decomposition } (\text{trail } S))$

definition *state-eq_{NOT}* :: '*st* \Rightarrow '*st* \Rightarrow bool (**infix** \sim 50) **where**
 $S \sim T \longleftrightarrow \text{trail } S = \text{trail } T \wedge \text{clauses}_{\text{NOT}} S = \text{clauses}_{\text{NOT}} T$

lemma *state-eq_{NOT}-ref[simp]*:
 $S \sim S$
 unfolding *state-eq_{NOT}-def* **by** *auto*

lemma *state-eq_{NOT}-sym*:
 $S \sim T \longleftrightarrow T \sim S$
 unfolding *state-eq_{NOT}-def* **by** *auto*

lemma *state-eq_{NOT}-trans*:
 $S \sim T \Longrightarrow T \sim U \Longrightarrow S \sim U$
 unfolding *state-eq_{NOT}-def* **by** *auto*

lemma
 shows
 state-eq_{NOT}-trail: $S \sim T \Longrightarrow \text{trail } S = \text{trail } T$ **and**
 state-eq_{NOT}-clauses: $S \sim T \Longrightarrow \text{clauses}_{\text{NOT}} S = \text{clauses}_{\text{NOT}} T$
 unfolding *state-eq_{NOT}-def* **by** *auto*

lemmas *state-simp_{NOT}[simp]* = *state-eq_{NOT}-trail state-eq_{NOT}-clauses*

lemma *trail-eq-reduce-trail-to_{NOT}-eq*:
 $\text{trail } S = \text{trail } T \Longrightarrow \text{trail } (\text{reduce-trail-to}_{\text{NOT}} F S) = \text{trail } (\text{reduce-trail-to}_{\text{NOT}} F T)$
 apply (*induction F S arbitrary: T rule: reduce-trail-to_{NOT}.induct*)
 by (*metis tl-trail reduce-trail-to_{NOT}-eq-length reduce-trail-to_{NOT}-length-ne reduce-trail-to_{NOT}-nil*)

lemma *reduce-trail-to_{NOT}-state-eq_{NOT}-compatible*:
 assumes *ST*: $S \sim T$
 shows *reduce-trail-to_{NOT}* $F S \sim \text{reduce-trail-to}_{\text{NOT}} F T$
proof –
 have *clauses_{NOT}* (*reduce-trail-to_{NOT}* $F S$) = *clauses_{NOT}* (*reduce-trail-to_{NOT}* $F T$)
 using *ST* **by** *auto*
 moreover have *trail* (*reduce-trail-to_{NOT}* $F S$) = *trail* (*reduce-trail-to_{NOT}* $F T$)
 using *trail-eq-reduce-trail-to_{NOT}-eq[of S T F]* *ST* **by** *auto*
 ultimately show ?thesis **by** (*auto simp del: state-simp_{NOT} simp: state-eq_{NOT}-def*)
qed

lemma *trail-reduce-trail-to_{NOT}-add-cl_{NOT}[simp]*:

$no_dup\ (trail\ S) \implies$
 $trail\ (reduce_trail_to_{NOT}\ F\ (add_cls_{NOT}\ C\ S)) = trail\ (reduce_trail_to_{NOT}\ F\ S)$
by (rule trail-eq-reduce-trail-to_{NOT}-eq) simp

lemma reduce-trail-to_{NOT}-trail-tl-trail-decomp[simp]:
 $trail\ S = F' @ Marked\ K\ () \# F \implies$
 $trail\ (reduce_trail_to_{NOT}\ F\ (tl_trail\ S)) = F$
apply (rule reduce-trail-to_{NOT}-skip-beginning[of - tl (F' @ Marked K () # [])])
by (cases F') (auto simp add:tl-append reduce-trail-to_{NOT}-skip-beginning)

lemma reduce-trail-to_{NOT}-length:
 $length\ M = length\ M' \implies reduce_trail_to_{NOT}\ M\ S = reduce_trail_to_{NOT}\ M'\ S$
apply (induction M S arbitrary: rule: reduce-trail-to_{NOT}.induct)
by (simp add: reduce-trail-to_{NOT}.simps)

end

15.2.2 Definition of the operation

locale propagate-ops =
 dpll-state mset-cls insert-cls remove-lit
 mset-clss union-clss in-clss insert-clss remove-from-clss
 trail raw-clauses prepend-trail tl-trail add-cls_{NOT} remove-cls_{NOT}
for
 mset-cls:: 'cls \Rightarrow 'v clause **and**
 insert-cls :: 'v literal \Rightarrow 'cls \Rightarrow 'cls **and**
 remove-lit :: 'v literal \Rightarrow 'cls \Rightarrow 'cls **and**
 mset-clss:: 'clss \Rightarrow 'v clauses **and**
 union-clss :: 'clss \Rightarrow 'clss \Rightarrow 'clss **and**
 in-clss :: 'cls \Rightarrow 'clss \Rightarrow bool **and**
 insert-clss :: 'cls \Rightarrow 'clss \Rightarrow 'clss **and**
 remove-from-clss :: 'cls \Rightarrow 'clss \Rightarrow 'clss **and**
 trail :: 'st \Rightarrow ('v, unit, unit) marked-lits **and**
 raw-clauses :: 'st \Rightarrow 'clss **and**
 prepend-trail :: ('v, unit, unit) marked-lit \Rightarrow 'st \Rightarrow 'st **and**
 tl-trail :: 'st \Rightarrow 'st **and**
 add-cls_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st **and**
 remove-cls_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st +
fixes
 propagate-cond :: ('v, unit, unit) marked-lit \Rightarrow 'st \Rightarrow bool
begin
inductive propagate_{NOT} :: 'st \Rightarrow 'st \Rightarrow bool **where**
 propagate_{NOT}[intro]: C + {#L#} \in # clauses_{NOT} S \implies trail S \models as CNot C
 \implies undefined-lit (trail S) L
 \implies propagate-cond (Propagated L ()) S
 \implies T \sim prepend-trail (Propagated L ()) S
 \implies propagate_{NOT} S T
inductive-cases propagate_{NOT}E[elim]: propagate_{NOT} S T
end

locale decide-ops =
 dpll-state mset-cls insert-cls remove-lit
 mset-clss union-clss in-clss insert-clss remove-from-clss
 trail raw-clauses prepend-trail tl-trail add-cls_{NOT} remove-cls_{NOT}
for

```

mset-cls:: 'cls  $\Rightarrow$  'v clause and
insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
mset-clss:: 'clss  $\Rightarrow$  'v clauses and
union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
raw-clauses :: 'st  $\Rightarrow$  'clss and
prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
tl-trail :: 'st  $\Rightarrow$  'st and
add-clNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
remove-clNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st

begin
inductive decideNOT :: 'st  $\Rightarrow$  'st  $\Rightarrow$  bool where
decideNOT[intro]: undefined-lit (trail S) L  $\Rightarrow$  atm-of L  $\in$  atms-of-mm (clausesNOT S)
 $\Rightarrow$  T  $\sim$  prepend-trail (Marked L ()) S
 $\Rightarrow$  decideNOT S T

inductive-cases decideNOTE[elim]: decideNOT S S'
end

locale backjumping-ops =
  dpll-state mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT
for
  mset-cls:: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
  raw-clauses :: 'st  $\Rightarrow$  'clss and
  prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
  tl-trail :: 'st  $\Rightarrow$  'st and
  add-clNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  remove-clNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st +
fixes
  backjump-conds :: 'v clause  $\Rightarrow$  'v clause  $\Rightarrow$  'v literal  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  bool
begin

inductive backjump where
trail S = F' @ Marked K () # F
 $\Rightarrow$  T  $\sim$  prepend-trail (Propagated L ()) (reduce-trail-toNOT F S)
 $\Rightarrow$  C  $\in$  # clausesNOT S
 $\Rightarrow$  trail S  $\models_{as}$  CNot C
 $\Rightarrow$  undefined-lit F L
 $\Rightarrow$  atm-of L  $\in$  atms-of-mm (clausesNOT S)  $\cup$  atm-of ' (lits-of-l (trail S))
 $\Rightarrow$  clausesNOT S  $\models_{pm}$  C' + {#L#}
 $\Rightarrow$  F  $\models_{as}$  CNot C'

```

$\Rightarrow \text{backjump-conds } C \ C' \ L \ S \ T$
 $\Rightarrow \text{backjump } S \ T$
inductive-cases *backjumpE*: *backjump* *S* *T*

The condition *atm-of* *L* \in *atms-of-mm* (*clauses*_{NOT} *S*) \cup *atm-of* ‘ *lits-of-l* (*trail* *S*) is not implied by the the condition *clauses*_{NOT} *S* \models_{pm} *C'* + {#*L*#} (no negation).

end

15.3 DPLL with backjumping

locale *dpll-with-backjumping-ops* =

propagate-ops *mset-cls* *insert-cls* *remove-lit*
mset-clss *union-clss* *in-clss* *insert-clss* *remove-from-clss*
trail *raw-clauses* *prepend-trail* *tl-trail* *add-cls*_{NOT} *remove-cls*_{NOT} *propagate-conds* +
decide-ops *mset-cls* *insert-cls* *remove-lit*
mset-clss *union-clss* *in-clss* *insert-clss* *remove-from-clss*
trail *raw-clauses* *prepend-trail* *tl-trail* *add-cls*_{NOT} *remove-cls*_{NOT} +
backjumping-ops *mset-cls* *insert-cls* *remove-lit*
mset-clss *union-clss* *in-clss* *insert-clss* *remove-from-clss*
trail *raw-clauses* *prepend-trail* *tl-trail* *add-cls*_{NOT} *remove-cls*_{NOT} *backjump-conds*

for

mset-cls:: '*cls* \Rightarrow '*v* *clause* **and**
insert-cls :: '*v* *literal* \Rightarrow '*cls* \Rightarrow '*cls* **and**
remove-lit :: '*v* *literal* \Rightarrow '*cls* \Rightarrow '*cls* **and**
mset-clss:: '*clss* \Rightarrow '*v* *clauses* **and**
union-clss :: '*clss* \Rightarrow '*clss* \Rightarrow '*clss* **and**
in-clss :: '*cls* \Rightarrow '*clss* \Rightarrow *bool* **and**
insert-clss :: '*cls* \Rightarrow '*clss* \Rightarrow '*clss* **and**
remove-from-clss :: '*cls* \Rightarrow '*clss* \Rightarrow '*clss* **and**
trail :: '*st* \Rightarrow ('*v*, *unit*, *unit*) *marked-lits* **and**
raw-clauses :: '*st* \Rightarrow '*clss* **and**
prepend-trail :: ('*v*, *unit*, *unit*) *marked-lit* \Rightarrow '*st* \Rightarrow '*st* **and**
tl-trail :: '*st* \Rightarrow '*st* **and**
*add-cls*_{NOT} :: '*cls* \Rightarrow '*st* \Rightarrow '*st* **and**
*remove-cls*_{NOT} :: '*cls* \Rightarrow '*st* \Rightarrow '*st* **and**
inv :: '*st* \Rightarrow *bool* **and**
backjump-conds :: '*v* *clause* \Rightarrow '*v* *clause* \Rightarrow '*v* *literal* \Rightarrow '*st* \Rightarrow '*st* \Rightarrow *bool* **and**
propagate-conds :: ('*v*, *unit*, *unit*) *marked-lit* \Rightarrow '*st* \Rightarrow *bool* +

assumes

bj-can-jump:
 $\bigwedge S \ C \ F' \ K \ F \ L.$
inv *S* \Rightarrow
no-dup (*trail* *S*) \Rightarrow
trail *S* = *F'* @ *Marked* *K* () # *F* \Rightarrow
 $C \in \# \text{ clauses}_{NOT} \ S \Rightarrow$
trail *S* \models_{as} *CNot* *C* \Rightarrow
undefined-lit *F* *L* \Rightarrow
 $\text{atm-of } L \in \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \cup \text{atm-of ' (lits-of-l (F' @ Marked K () \# F))} \Rightarrow$
 $\text{clauses}_{NOT} \ S \models_{pm} C' + \{\#L\# \} \Rightarrow$
 $F \models_{as} CNot \ C' \Rightarrow$
 $\neg \text{no-step backjump } S$

begin

We cannot add a like condition *atms-of* *C'* \subseteq *atms-of-ms* *N* because to ensure that we can backjump even if the last decision variable has disappeared.

The part of the condition $atm\text{-}of\ L \in atm\text{-}of\ ' \textit{ lits-of-l } (F' @ \textit{ Marked } K\ () \# F)$ is important, otherwise you are not sure that you can backtrack.

15.3.1 Definition

We define $dpll$ with backjumping:

inductive $dpll\text{-}bj :: 'st \Rightarrow 'st \Rightarrow bool$ **for** $S :: 'st$ **where**
 $bj\text{-}decide_{NOT}:$ $decide_{NOT}\ S\ S' \Longrightarrow dpll\text{-}bj\ S\ S' \mid$
 $bj\text{-}propagate_{NOT}:$ $propagate_{NOT}\ S\ S' \Longrightarrow dpll\text{-}bj\ S\ S' \mid$
 $bj\text{-}backjump:$ $backjump\ S\ S' \Longrightarrow dpll\text{-}bj\ S\ S'$

lemmas $dpll\text{-}bj\text{-}induct = dpll\text{-}bj.induct[split\text{-}format(complete)]$

thm $dpll\text{-}bj\text{-}induct[OF\ dpll\text{-}with\text{-}backjumping\text{-}ops\text{-}axioms]$

lemma $dpll\text{-}bj\text{-}all\text{-}induct[consumes\ 2, case\text{-}names\ decide_{NOT}\ propagate_{NOT}\ backjump]:$

fixes $S\ T :: 'st$

assumes

$dpll\text{-}bj\ S\ T$ **and**

$inv\ S$

$\bigwedge L\ T. \textit{ undefined-lit } (trail\ S)\ L \Longrightarrow atm\text{-}of\ L \in atm\text{-}of\text{-}mm\ (clauses_{NOT}\ S)$

$\Longrightarrow T \sim \textit{ prepend-trail } (\textit{ Marked } L\ ())\ S$

$\Longrightarrow P\ S\ T$ **and**

$\bigwedge C\ L\ T. C + \{\#L\#\} \in \# \textit{ clauses}_{NOT}\ S \Longrightarrow trail\ S \models_{as}\ CNot\ C \Longrightarrow \textit{ undefined-lit } (trail\ S)\ L$

$\Longrightarrow T \sim \textit{ prepend-trail } (\textit{ Propagated } L\ ())\ S$

$\Longrightarrow P\ S\ T$ **and**

$\bigwedge C\ F'\ K\ F\ L\ C'\ T. C \in \# \textit{ clauses}_{NOT}\ S \Longrightarrow F' @ \textit{ Marked } K\ () \# F \models_{as}\ CNot\ C$

$\Longrightarrow trail\ S = F' @ \textit{ Marked } K\ () \# F$

$\Longrightarrow \textit{ undefined-lit } F\ L$

$\Longrightarrow atm\text{-}of\ L \in atm\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \cup atm\text{-}of\ ' \textit{ (lits-of-l } (F' @ \textit{ Marked } K\ () \# F))$

$\Longrightarrow clauses_{NOT}\ S \models_{pm}\ C' + \{\#L\#\}$

$\Longrightarrow F \models_{as}\ CNot\ C'$

$\Longrightarrow T \sim \textit{ prepend-trail } (\textit{ Propagated } L\ ())\ (\textit{ reduce-trail-to}_{NOT}\ F\ S)$

$\Longrightarrow P\ S\ T$

shows $P\ S\ T$

apply ($induct\ T\ rule: dpll\text{-}bj\text{-}induct[OF\ local.dpll\text{-}with\text{-}backjumping\text{-}ops\text{-}axioms]$)

apply ($rule\ assms(1)$)

using $assms(3)$ **apply** $blast$

apply ($elim\ propagate_{NOT}E$) **using** $assms(4)$ **apply** $blast$

apply ($elim\ backjumpE$) **using** $assms(5)$ $\langle inv\ S \rangle$ **by** $simp$

15.3.2 Basic properties

First, some better suited induction principle **lemma** $dpll\text{-}bj\text{-}clauses:$

assumes $dpll\text{-}bj\ S\ T$ **and** $inv\ S$

shows $clauses_{NOT}\ S = clauses_{NOT}\ T$

using $assms$ **by** ($induction\ rule: dpll\text{-}bj\text{-}all\text{-}induct$) $auto$

No duplicates in the trail **lemma** $dpll\text{-}bj\text{-}no\text{-}dup:$

assumes $dpll\text{-}bj\ S\ T$ **and** $inv\ S$

and $no\text{-}dup\ (trail\ S)$

shows $no\text{-}dup\ (trail\ T)$

using $assms$ **by** ($induction\ rule: dpll\text{-}bj\text{-}all\text{-}induct$)

($auto\ simp\ add: \textit{ defined-lit-map } \textit{ reduce-trail-to}_{NOT}\text{-}skip\text{-}beginning$)

Valuations **lemma** $dpll\text{-}bj\text{-}sat\text{-}iff:$

assumes *dpll-bj S T and inv S*
shows $I \models_{sm} clauses_{NOT} S \longleftrightarrow I \models_{sm} clauses_{NOT} T$
using *assms by (induction rule: dpll-bj-all-induct) auto*

Clauses lemma *dpll-bj-atms-of-ms-clauses-inv:*

assumes
dpll-bj S T and
inv S
shows $atms-of-mm (clauses_{NOT} S) = atms-of-mm (clauses_{NOT} T)$
using *assms by (induction rule: dpll-bj-all-induct) auto*

lemma *dpll-bj-atms-in-trail:*

assumes
dpll-bj S T and
inv S and
atm-of ' (lits-of-l (trail S)) \subseteq atms-of-mm (clauses_{NOT} S)
shows $atm-of ' (lits-of-l (trail T)) \subseteq atms-of-mm (clauses_{NOT} S)$
using *assms by (induction rule: dpll-bj-all-induct)*
(auto simp: in-plus-implies-atm-of-on-atms-of-ms reduce-trail-to_{NOT}-skip-beginning)

lemma *dpll-bj-atms-in-trail-in-set:*

assumes *dpll-bj S T and*
inv S and
atms-of-mm (clauses_{NOT} S) \subseteq A and
atm-of ' (lits-of-l (trail S)) \subseteq A
shows $atm-of ' (lits-of-l (trail T)) \subseteq A$
using *assms by (induction rule: dpll-bj-all-induct)*
(auto simp: in-plus-implies-atm-of-on-atms-of-ms)

lemma *dpll-bj-all-decomposition-implies-inv:*

assumes
dpll-bj S T and
inv: inv S and
decomp: all-decomposition-implies-m (clauses_{NOT} S) (get-all-marked-decomposition (trail S))
shows $all-decomposition-implies-m (clauses_{NOT} T) (get-all-marked-decomposition (trail T))$
using *assms(1,2)*

proof *(induction rule: dpll-bj-all-induct)*

case *decide_{NOT}*

then show *?case using decomp by auto*

next

case *(propagate_{NOT} C L T) note propa = this(1) and undef = this(3) and T = this(4)*

let *?M' = trail (prepend-trail (Propagated L ()) S)*

let *?N = clauses_{NOT} S*

obtain *a y l where ay: get-all-marked-decomposition ?M' = (a, y) # l*

by *(cases get-all-marked-decomposition ?M') fastforce+*

then have *M': ?M' = y @ a using get-all-marked-decomposition-decomp[of ?M'] by auto*

have *M: get-all-marked-decomposition (trail S) = (a, tl y) # l*

using *ay undef by (cases get-all-marked-decomposition (trail S)) auto*

have *y₀: y = (Propagated L ()) # (tl y)*

using *ay undef by (auto simp add: M)*

from *arg-cong[OF this, of set] have y[simp]: set y = insert (Propagated L ()) (set (tl y))*

by *simp*

have *tr-S: trail S = tl y @ a*

using *arg-cong[OF M', of tl] y₀ M get-all-marked-decomposition-decomp by force*

have *a-Un-N-M: unmark-l a \cup set-mset ?N \models_{ps} unmark-l (tl y)*

```

using decomp ay unfolding all-decomposition-implies-def by (simp add: M)+

moreover have unmark-l a ∪ set-mset ?N ⊨ps {#L#} (is ?I ⊨ps -)
proof (rule true-clss-clss-plus-CNot)
  show ?I ⊨ps C + {#L#}
    using propa propagateNOT.prems by (auto dest!: true-clss-clss-in-imp-true-clss-clss)
next
  have (λm. {#lit-of m#}) ‘set ?M' ⊨ps CNot C’
    using (trail S ⊨as CNot C) undef by (auto simp add: true-annots-true-clss-clss)
  have a1: (λm. {#lit-of m#}) ‘set a ∪ (λm. {#lit-of m#}) ‘set (tl y) ⊨ps CNot C
    using propagateNOT.hyps(2) tr-S true-annots-true-clss-clss
    by (force simp add: image-Un sup-commute)
  have a2: set-mset (clausesNOT S) ∪ unmark-l a
    ⊨ps unmark-l (tl y)
    using calculation by (auto simp add: sup-commute)
  show (λm. {#lit-of m#}) ‘set a ∪ set-mset (clausesNOT S) ⊨ps CNot C’
    proof –
      have set-mset (clausesNOT S) ∪ (λm. {#lit-of m#}) ‘set a ⊨ps
        (λm. {#lit-of m#}) ‘set a ∪ (λm. {#lit-of m#}) ‘set (tl y)’
        using a2 true-clss-clss-def by blast
      then show (λm. {#lit-of m#}) ‘set a ∪ set-mset (clausesNOT S) ⊨ps CNot C’
        using a1 unfolding sup-commute by (meson true-clss-clss-left-right
          true-clss-clss-union-and true-clss-clss-union-l-r)
    qed
  qed
ultimately have unmark-l a ∪ set-mset ?N ⊨ps unmark-l ?M'
  unfolding M' by (auto simp add: all-in-true-clss-clss image-Un)

then show ?case
  using decomp T M undef unfolding ay all-decomposition-implies-def by (auto simp add: ay)
next
case (backjump C F' K F L D T) note confl = this(2) and tr = this(3) and undef = this(4)
and L = this(5) and N-C = this(6) and vars-D = this(5) and T = this(8)
have decomp: all-decomposition-implies-m (clausesNOT S) (get-all-marked-decomposition F)
using decomp unfolding tr all-decomposition-implies-def
by (metis (no-types, lifting) get-all-marked-decomposition.simps(1)
  get-all-marked-decomposition-never-empty hd-Cons-tl insert-iff list.sel(3) list.set(2)
  tl-get-all-marked-decomposition-skip-some)

moreover have unmark-l (fst (hd (get-all-marked-decomposition F)))
  ∪ set-mset (clausesNOT S)
  ⊨ps unmark-l (snd (hd (get-all-marked-decomposition F)))
by (metis all-decomposition-implies-cons-single decomp get-all-marked-decomposition-never-empty
  hd-Cons-tl)
moreover
  have vars-of-D: atms-of D ⊆ atm-of ‘lits-of-l F
    using (F ⊨as CNot D) unfolding atms-of-def
    by (meson image-subsetI true-annots-CNot-all-atms-defined)

obtain a b li where F: get-all-marked-decomposition F = (a, b) # li
by (cases get-all-marked-decomposition F) auto
have F = b @ a
  using get-all-marked-decomposition-decomp[of F a b] F by auto
have a-N-b: unmark-l a ∪ set-mset (clausesNOT S) ⊨ps unmark-l b
  using decomp unfolding all-decomposition-implies-def by (auto simp add: F)

```

```

have F-D:unmark-l F  $\models_{ps}$  CNot D
  using  $\langle F \models_{as} CNot D \rangle$  by (simp add: true-annots-true-clss-clss)
then have unmark-l a  $\cup$  unmark-l b  $\models_{ps}$  CNot D
  unfolding  $\langle F = b @ a \rangle$  by (simp add: image-Un sup commute)
have a-N-CNot-D: unmark-l a  $\cup$  set-mset (clausesNOT S)
 $\models_{ps}$  CNot D  $\cup$  unmark-l b
  apply (rule true-clss-clss-left-right)
  using a-N-b F-D unfolding  $\langle F = b @ a \rangle$  by (auto simp add: image-Un ac-simps)

have a-N-D-L: unmark-l a  $\cup$  set-mset (clausesNOT S)  $\models_p$  D+{#L#}
  by (simp add: N-C)
have unmark-l a  $\cup$  set-mset (clausesNOT S)  $\models_p$  {#L#}
  using a-N-D-L a-N-CNot-D by (blast intro: true-clss-clss-plus-CNot)
then show ?case
  using decomp T tr undef unfolding all-decomposition-implies-def by (auto simp add: F)
qed

```

15.3.3 Termination

Using a proper measure lemma *length-get-all-marked-decomposition-append-Marked*:
 $length (get-all-marked-decomposition (F' @ Marked K () \# F)) =$
 $length (get-all-marked-decomposition F')$
 $+ length (get-all-marked-decomposition (Marked K () \# F))$
 $- 1$
 by (induction F' rule: marked-lit-list-induct) auto

lemma *take-length-get-all-marked-decomposition-marked-sandwich*:
 $take (length (get-all-marked-decomposition F'))$
 $(map (f o snd) (rev (get-all-marked-decomposition (F' @ Marked K () \# F))))$
 $=$
 $map (f o snd) (rev (get-all-marked-decomposition F'))$

```

proof (induction F' rule: marked-lit-list-induct)
  case nil
  then show ?case by auto
next
  case (marked K)
  then show ?case by (simp add: length-get-all-marked-decomposition-append-Marked)
next
  case (proped L m F') note IH = this(1)
  obtain a b l where F': get-all-marked-decomposition (F' @ Marked K () \# F) = (a, b) \# l
    by (cases get-all-marked-decomposition (F' @ Marked K () \# F)) auto
  have length (get-all-marked-decomposition F) - length l = 0
    using length-get-all-marked-decomposition-append-Marked[of F' K F]
    unfolding F' by (cases get-all-marked-decomposition F') auto
  then show ?case
    using IH by (simp add: F')
qed

```

lemma *length-get-all-marked-decomposition-length*:
 $length (get-all-marked-decomposition M) \leq 1 + length M$
 by (induction M rule: marked-lit-list-induct) auto

lemma *length-in-get-all-marked-decomposition-bounded*:
 assumes $i:i \in set (trail-weight S)$

shows $i \leq \text{Suc } (\text{length } (\text{trail } S))$
proof –
obtain $a \ b$ **where**
 $(a, b) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$ **and**
 $ib: i = \text{Suc } (\text{length } b)$
using i **by** *auto*
then obtain c **where** $\text{trail } S = c @ b @ a$
using *get-all-marked-decomposition-exists-prepend'* **by** *metis*
from *arg-cong[OF this, of length]* **show** *?thesis* **using** $i \ ib$ **by** *auto*
qed

Well-foundedness The bounds are the following:

- $1 + \text{card } (\text{atms-of-ms } A)$: $\text{card } (\text{atms-of-ms } A)$ is an upper bound on the length of the list. As *get-all-marked-decomposition* appends an possibly empty couple at the end, adding one is needed.
- $2 + \text{card } (\text{atms-of-ms } A)$: $\text{card } (\text{atms-of-ms } A)$ is an upper bound on the number of elements, where adding one is necessary for the same reason as for the bound on the list, and one is needed to have a strict bound.

abbreviation *unassigned-lit* :: ' b literal multiset set \Rightarrow ' a list \Rightarrow nat **where**

unassigned-lit $N \ M \equiv \text{card } (\text{atms-of-ms } N) - \text{length } M$

lemma *dpll-bj-trail-mes-increasing-prop*:

fixes $M :: ('v, \text{unit}, \text{unit}) \text{ marked-lits}$ **and** $N :: 'v \text{ clauses}$

assumes

dpll-bj $S \ T$ **and**

inv S **and**

$NA: \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \subseteq \text{atms-of-ms } A$ **and**

$MA: \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A$ **and**

$n-d: \text{no-dup } (\text{trail } S)$ **and**

finite: *finite* A

shows $\mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } T)$

$> \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } S)$

using *assms*(1,2)

proof (*induction rule*: *dpll-bj-all-induct*)

case (*propagate*_{NOT} $C \ L$) **note** $CLN = \text{this}(1)$ **and** $MC = \text{this}(2)$ **and** $\text{undef-L} = \text{this}(3)$ **and** $T = \text{this}(4)$

have *incl*: *atm-of* ' *lits-of-l* (*Propagated* $L \ ()$ # *trail* S) $\subseteq \text{atms-of-ms } A$

using *propagate*_{NOT}.*hyps* *propagate-ops*.*propagate*_{NOT} *dpll-bj-atms-in-trail-in-set* *bj-propagate*_{NOT}

$NA \ MA \ CLN$ **by** (*auto simp: in-plus-implies-atm-of-on-atms-of-ms*)

have *no-dup*: *no-dup* (*Propagated* $L \ ()$ # *trail* S)

using *defined-lit-map* *n-d* *undef-L* **by** *auto*

obtain $a \ b \ l$ **where** $M: \text{get-all-marked-decomposition } (\text{trail } S) = (a, b) \# l$

by (*cases* *get-all-marked-decomposition* (*trail* S)) *auto*

have $b \leq M: \text{length } b \leq \text{length } (\text{trail } S)$

using *get-all-marked-decomposition-decomp*[*of trail S*] **by** (*simp add: M*)

have *finite* (*atms-of-ms* A) **using** *finite* **by** *simp*

then have $\text{length } (\text{Propagated } L \ () \# \text{trail } S) \leq \text{card } (\text{atms-of-ms } A)$

using *incl* *finite* **unfolding** *no-dup-length-eq-card-atm-of-lits-of-l*[*OF no-dup*]

by (*simp add: card-mono*)

then have *latm*: *unassigned-lit* $A \ b = \text{Suc } (\text{unassigned-lit } A \ (\text{Propagated } L \ d \# b))$

```

    using b-le-M by auto
  then show ?case using T undef-L by (auto simp: latm M  $\mu_C$ -cons)
next
case (decideNOT L) note undef-L = this(1) and MC = this(2) and T = this(3)
have incl: atm-of ' lits-of-l (Marked L () # (trail S))  $\subseteq$  atms-of-ms A
  using dpll-bj-atms-in-trail-in-set bj-decideNOT decideNOT.decideNOT[OF decideNOT.hyps] NA MA
MC
  by auto

have no-dup: no-dup (Marked L () # (trail S))
  using defined-lit-map n-d undef-L by auto
obtain a b l where M: get-all-marked-decomposition (trail S) = (a, b) # l
  by (cases get-all-marked-decomposition (trail S)) auto

then have length (Marked L () # (trail S))  $\leq$  card (atms-of-ms A)
  using incl finite unfolding no-dup-length-eq-card-atm-of-lits-of-l[OF no-dup]
  by (simp add: card-mono)
then have latm: unassigned-lit A (trail S) = Suc (unassigned-lit A (Marked L lv # (trail S)))
  by force
show ?case using T undef-L by (simp add: latm  $\mu_C$ -cons)
next
case (backjump C F' K F L C' T) note undef-L = this(4) and MC = this(1) and tr-S = this(3)
and
  L = this(5) and T = this(8)
have incl: atm-of ' lits-of-l (Propagated L () # F)  $\subseteq$  atms-of-ms A
  using dpll-bj-atms-in-trail-in-set NA MA tr-S L by auto

have no-dup: no-dup (Propagated L () # F)
  using defined-lit-map n-d undef-L tr-S by auto
obtain a b l where M: get-all-marked-decomposition (trail S) = (a, b) # l
  by (cases get-all-marked-decomposition (trail S)) auto
have b-le-M: length b  $\leq$  length (trail S)
  using get-all-marked-decomposition-decomp[of trail S] by (simp add: M)
have fin-atms-A: finite (atms-of-ms A) using finite by simp

then have F-le-A: length (Propagated L () # F)  $\leq$  card (atms-of-ms A)
  using incl finite unfolding no-dup-length-eq-card-atm-of-lits-of-l[OF no-dup]
  by (simp add: card-mono)
have tr-S-le-A: length (trail S)  $\leq$  (card (atms-of-ms A))
  using n-d MA by (metis fin-atms-A card-mono no-dup-length-eq-card-atm-of-lits-of-l)
obtain a b l where F: get-all-marked-decomposition F = (a, b) # l
  by (cases get-all-marked-decomposition F) auto
then have F = b @ a
  using get-all-marked-decomposition-decomp[of Propagated L () # F a
    Propagated L () # b] by simp
then have latm: unassigned-lit A b = Suc (unassigned-lit A (Propagated L () # b))
  using F-le-A by simp
obtain rem where
  rem: map ( $\lambda a. \text{Suc} (\text{length} (\text{snd } a))$ ) (rev (get-all-marked-decomposition (F' @ Marked K () # F)))
  = map ( $\lambda a. \text{Suc} (\text{length} (\text{snd } a))$ ) (rev (get-all-marked-decomposition F)) @ rem
  using take-length-get-all-marked-decomposition-marked-sandwich[of F  $\lambda a. \text{Suc} (\text{length } a)$  F' K]
  unfolding o-def by (metis append-take-drop-id)
then have rem: map ( $\lambda a. \text{Suc} (\text{length} (\text{snd } a))$ )
  (get-all-marked-decomposition (F' @ Marked K () # F))
  = rev rem @ map ( $\lambda a. \text{Suc} (\text{length} (\text{snd } a))$ ) ((get-all-marked-decomposition F))

```

```

  by (simp add: rev-map[symmetric] rev-swap)
have length (rev rem @ map (λa. Suc (length (snd a))) (get-all-marked-decomposition F))
  ≤ Suc (card (atms-of-ms A))
  using arg-cong[OF rem, of length] tr-S-le-A
  length-get-all-marked-decomposition-length[of F' @ Marked K () # F] tr-S by auto
moreover
{ fix i :: nat and xs :: 'a list
  have i < length xs ⇒ length xs - Suc i < length xs
    by auto
  then have H: i < length xs ⇒ rev xs ! i ∈ set xs
    using rev-nth[of i xs] unfolding in-set-conv-nth by (force simp add: in-set-conv-nth)
} note H = this
have ∀ i < length rem. rev rem ! i < card (atms-of-ms A) + 2
  using tr-S-le-A length-in-get-all-marked-decomposition-bounded[of - S] unfolding tr-S
  by (force simp add: o-def rem dest!: H intro: length-get-all-marked-decomposition-length)
ultimately show ?case
  using μC-bounded[of rev rem card (atms-of-ms A)+2 unassigned-lit A l] T undef-L
  by (simp add: rem μC-append μC-cons F tr-S)
qed

```

lemma *dpll-bj-trail-mes-decreasing-prop*:

assumes *dpll*: *dpll-bj S T* **and** *inv*: *inv S* **and**
N-A: *atms-of-mm (clauses_{NOT} S) ⊆ atms-of-ms A* **and**
M-A: *atm-of ' lits-of-l (trail S) ⊆ atms-of-ms A* **and**
nd: *no-dup (trail S)* **and**
fin-A: *finite A*

shows $(2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } T)$
 $< (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } S)$

proof –

```

let ?b = 2 + card (atms-of-ms A)
let ?s = 1 + card (atms-of-ms A)
let ?μ = μC ?s ?b
have M'-A: atm-of ' lits-of-l (trail T) ⊆ atms-of-ms A
  by (meson M-A N-A dpll dpll-bj-atms-in-trail-in-set inv)
have nd': no-dup (trail T)
  using ⟨dpll-bj S T⟩ dpll-bj-no-dup nd inv by blast
{ fix i :: nat and xs :: 'a list
  have i < length xs ⇒ length xs - Suc i < length xs
    by auto
  then have H: i < length xs ⇒ xs ! i ∈ set xs
    using rev-nth[of i xs] unfolding in-set-conv-nth by (force simp add: in-set-conv-nth)
} note H = this

```

```

have l-M-A: length (trail S) ≤ card (atms-of-ms A)
  by (simp add: fin-A M-A card-mono no-dup-length-eq-card-atm-of-lits-of-l nd)
have l-M'-A: length (trail T) ≤ card (atms-of-ms A)
  by (simp add: fin-A M'-A card-mono no-dup-length-eq-card-atm-of-lits-of-l nd')
have l-trail-weight-M: length (trail-weight T) ≤ 1 + card (atms-of-ms A)
  using l-M'-A length-get-all-marked-decomposition-length[of trail T] by auto
have bounded-M: ∀ i < length (trail-weight T). (trail-weight T) ! i < card (atms-of-ms A) + 2
  using length-in-get-all-marked-decomposition-bounded[of - T] l-M'-A
  by (metis (no-types, lifting) Nat.le-trans One-nat-def Suc-1 add.right-neutral add-Suc-right
    le-imp-less-Suc less-eq-Suc-le nth-mem)

```

from *dpll-bj-trail-mes-increasing-prop*[*OF dpll inv N-A M-A nd fin-A*]
have $\mu_C \text{ ?s ?b } (trail\text{-}weight\ S) < \mu_C \text{ ?s ?b } (trail\text{-}weight\ T)$ **by** *simp*
moreover from $\mu_C\text{-bounded}$ [*OF bounded-M l-trail-weight-M*]
have $\mu_C \text{ ?s ?b } (trail\text{-}weight\ T) \leq ?b \wedge ?s$ **by** *auto*
ultimately show *?thesis* **by** *linarith*
qed

lemma *wf-dpll-bj*:

assumes *fin*: *finite A*

shows *wf* $\{(T, S). \text{dpll-bj } S\ T$

$\wedge \text{atms-of-mm } (clauses_{NOT} S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of ' lits-of-l } (trail\ S) \subseteq \text{atms-of-ms } A$

$\wedge \text{no-dup } (trail\ S) \wedge \text{inv } S\}$

(**is** *wf ?A*)

proof (*rule wf-bounded-measure*[*of -*

$\lambda\cdot. (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$

$\lambda S. \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (trail\text{-}weight\ S)]$)

fix *a b* :: '*st*

let *?b* = $2 + \text{card } (\text{atms-of-ms } A)$

let *?s* = $1 + \text{card } (\text{atms-of-ms } A)$

let *?μ* = $\mu_C \text{ ?s ?b}$

assume *ab*: $(b, a) \in \{(T, S). \text{dpll-bj } S\ T$

$\wedge \text{atms-of-mm } (clauses_{NOT} S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of ' lits-of-l } (trail\ S) \subseteq \text{atms-of-ms } A$

$\wedge \text{no-dup } (trail\ S) \wedge \text{inv } S\}$

have *fin-A*: *finite* (*atms-of-ms A*)

using *fin* **by** *auto*

have

dpll-bj: *dpll-bj a b* **and**

N-A: $\text{atms-of-mm } (clauses_{NOT} a) \subseteq \text{atms-of-ms } A$ **and**

M-A: $\text{atm-of ' lits-of-l } (trail\ a) \subseteq \text{atms-of-ms } A$ **and**

nd: *no-dup* (*trail a*) **and**

inv: *inv a*

using *ab* **by** *auto*

have *M'-A*: $\text{atm-of ' lits-of-l } (trail\ b) \subseteq \text{atms-of-ms } A$

by (*meson M-A N-A* $\langle \text{dpll-bj } a\ b \rangle \text{dpll-bj-atms-in-trail-in-set inv}$)

have *nd'*: *no-dup* (*trail b*)

using $\langle \text{dpll-bj } a\ b \rangle \text{dpll-bj-no-dup nd inv}$ **by** *blast*

{ **fix** *i* :: *nat* **and** *xs* :: '*a list*

have $i < \text{length } xs \implies \text{length } xs - \text{Suc } i < \text{length } xs$

by *auto*

then have *H*: $i < \text{length } xs \implies xs ! i \in \text{set } xs$

using *rev-nth*[*of i xs*] **unfolding** *in-set-conv-nth* **by** (*force simp add: in-set-conv-nth*)

} **note** *H = this*

have *l-M-A*: $\text{length } (trail\ a) \leq \text{card } (\text{atms-of-ms } A)$

by (*simp add: fin-A M-A card-mono no-dup-length-eq-card-atm-of-lits-of-l nd*)

have *l-M'-A*: $\text{length } (trail\ b) \leq \text{card } (\text{atms-of-ms } A)$

by (*simp add: fin-A M'-A card-mono no-dup-length-eq-card-atm-of-lits-of-l nd'*)

have *l-trail-weight-M*: $\text{length } (trail\text{-}weight\ b) \leq 1 + \text{card } (\text{atms-of-ms } A)$

using *l-M'-A length-get-all-marked-decomposition-length*[*of trail b*] **by** *auto*

have *bounded-M*: $\forall i < \text{length } (trail\text{-}weight\ b). (trail\text{-}weight\ b) ! i < \text{card } (\text{atms-of-ms } A) + 2$

using *length-in-get-all-marked-decomposition-bounded*[*of - b*] *l-M'-A*

by (*metis* (*no-types*, *lifting*) *Nat.le-trans One-nat-def Suc-1 add.right-neutral add-Suc-right*)

le-imp-less-Suc less-eq-Suc-le nth-mem)

from *dpll-bj-trail-mes-increasing-prop*[*OF dpll-bj inv N-A M-A nd fin*]
have $\mu_C \ ?s \ ?b \ (\text{trail-weight } a) < \mu_C \ ?s \ ?b \ (\text{trail-weight } b)$ **by** *simp*
moreover from $\mu_C\text{-bounded}$ [*OF bounded-M l-trail-weight-M*]
have $\mu_C \ ?s \ ?b \ (\text{trail-weight } b) \leq ?b \wedge ?s$ **by** *auto*
ultimately show $?b \wedge ?s \leq ?b \wedge ?s \wedge$
 $\mu_C \ ?s \ ?b \ (\text{trail-weight } b) \leq ?b \wedge ?s \wedge$
 $\mu_C \ ?s \ ?b \ (\text{trail-weight } a) < \mu_C \ ?s \ ?b \ (\text{trail-weight } b)$
by *blast*
qed

15.3.4 Normal Forms

We prove that given a normal form of DPLL, with some invariants, the either N is satisfiable and the built valuation M is a model; or N is unsatisfiable.

Idea of the proof: We have to prove that *satisfiable* N , $\neg M \models_{as} N$ and there is no remaining step is incompatible.

1. The *decide* rules tells us that every variable in N has a value.
2. $\neg M \models_{as} N$ tells us that there is conflict.
3. There is at least one decision in the trail (otherwise, M is a model of N).
4. Now if we build the clause with all the decision literals of the trail, we can apply the *backjump* rule.

The assumption are saying that we have a finite upper bound A for the literals, that we cannot do any step *no-step dpll-bj* S

theorem *dpll-backjump-final-state*:

fixes $A :: 'v \text{ literal multiset set}$ **and** $S \ T :: 'st$

assumes

atms-of-mm (*clauses*_{NOT} S) \subseteq *atms-of-ms* A **and**

atm-of ' *lits-of-l* (*trail* S) \subseteq *atms-of-ms* A **and**

no-dup (*trail* S) **and**

finite A **and**

inv: *inv* S **and**

n-s: *no-step dpll-bj* S **and**

decomp: *all-decomposition-implies-m* (*clauses*_{NOT} S) (*get-all-marked-decomposition* (*trail* S))

shows *unsatisfiable* (*set-mset* (*clauses*_{NOT} S))

\vee (*trail* $S \models_{asm}$ *clauses*_{NOT} $S \wedge$ *satisfiable* (*set-mset* (*clauses*_{NOT} S)))

proof –

let $?N = \text{set-mset} \ (\text{clauses}_{NOT} \ S)$

let $?M = \text{trail } S$

consider

(*sat*) *satisfiable* $?N$ **and** $?M \models_{as} ?N$

| (*sat'*) *satisfiable* $?N$ **and** $\neg ?M \models_{as} ?N$

| (*unsat*) *unsatisfiable* $?N$

by *auto*

then show *?thesis*

proof *cases*

case *sat'* **note** $\text{sat} = \text{this}(1)$ **and** $M = \text{this}(2)$

obtain C **where** $C \in ?N$ **and** $\neg ?M \models_{as} C$ **using** M **unfolding** *true-annots-def* **by** *auto*

```

obtain  $I :: 'v$  literal set where
   $I \models_s ?N$  and
  cons: consistent-interp  $I$  and
  tot: total-over-m  $I$   $?N$  and
  atm-I-N: atm-of ' $I \subseteq$  atms-of-ms  $?N$ 
  using sat unfolding satisfiable-def-min by auto
let  $?I = I \cup \{P \mid P. P \in \text{lits-of-l } ?M \wedge \text{atm-of } P \notin \text{atm-of ' } I\}$ 
let  $?O = \{\{\# \text{lit-of } L \# \} \mid L. \text{is-marked } L \wedge L \in \text{set } ?M \wedge \text{atm-of (lit-of } L) \notin \text{atms-of-ms } ?N\}$ 
have cons-I': consistent-interp  $?I$ 
  using cons using (no-dup ?M) unfolding consistent-interp-def
  by (auto simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set lits-of-def
    dest!: no-dup-cannot-not-lit-and-uminus)
have tot-I': total-over-m  $?I$  ( $?N \cup \text{unmark-l } ?M$ )
  using tot atm-I-N unfolding total-over-m-def total-over-set-def
  by (fastforce simp: image-iff lits-of-def)
have  $\{P \mid P. P \in \text{lits-of-l } ?M \wedge \text{atm-of } P \notin \text{atm-of ' } I\} \models_s ?O$ 
  using  $\langle I \models_s ?N \rangle$  atm-I-N by (auto simp add: atm-of-eq-atm-of true-clss-def lits-of-def)
then have  $I'-N: ?I \models_s ?N \cup ?O$ 
  using  $\langle I \models_s ?N \rangle$  true-clss-union-increase by force
have tot': total-over-m  $?I$  ( $?N \cup ?O$ )
  using atm-I-N tot unfolding total-over-m-def total-over-set-def
  by (force simp: image-iff lits-of-def dest!: is-marked-ex-Marked)

have atms-N-M: atms-of-ms  $?N \subseteq \text{atm-of ' lits-of-l } ?M$ 
proof (rule ccontr)
  assume  $\neg ?thesis$ 
  then obtain  $l :: 'v$  where
     $l-N: l \in \text{atms-of-ms } ?N$  and
     $l-M: l \notin \text{atm-of ' lits-of-l } ?M$ 
  by auto
have undefined-lit  $?M$  (Pos l)
  using  $l-M$  by (metis Marked-Propagated-in-iff-in-lits-of-l
    atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set literal.sel(1))
from bj-decideNOT[OF decideNOT[OF this]] show False
  using  $l-N$  n-s by (metis literal.sel(1) state-eqNOT-ref)
qed
have  $?M \models_{as} CNot\ C$ 
by (metis (no-types, lifting) (C \in set-mset (clausesNOT  $S) \rangle (\neg \text{trail } S \models_a C)
  all-variables-defined-not-imply-cnot atms-N-M atms-of-atms-of-ms-mono
  atms-of-ms-CNot-atms-of atms-of-ms-CNot-atms-of-ms subset-eq)
have  $\exists l \in \text{set } ?M. \text{is-marked } l$ 
proof (rule ccontr)
  let  $?O = \{\{\# \text{lit-of } L \# \} \mid L. \text{is-marked } L \wedge L \in \text{set } ?M \wedge \text{atm-of (lit-of } L) \notin \text{atms-of-ms } ?N\}$ 
  have  $\vartheta[\text{iff}]: \bigwedge I. \text{total-over-m } I \text{ } (?N \cup ?O \cup \text{unmark-l } ?M)$ 
     $\longleftrightarrow \text{total-over-m } I \text{ } (?N \cup \text{unmark-l } ?M)$ 
  unfolding total-over-set-def total-over-m-def atms-of-ms-def by auto
  assume  $\neg ?thesis$ 
  then have  $[\text{simp}]: \{\{\# \text{lit-of } L \# \} \mid L. \text{is-marked } L \wedge L \in \text{set } ?M\}$ 
     $= \{\{\# \text{lit-of } L \# \} \mid L. \text{is-marked } L \wedge L \in \text{set } ?M \wedge \text{atm-of (lit-of } L) \notin \text{atms-of-ms } ?N\}$ 
  by auto
  then have  $?N \cup ?O \models_{ps} \text{unmark-l } ?M$ 
  using all-decomposition-implies-propagated-lits-are-IMPLIED[OF decomp] by auto

then have  $?I \models_s \text{unmark-l } ?M$ 
  using cons-I' I'-N tot-I' ( ?I \models_s ?N \cup ?O ) unfolding  $\vartheta$  true-clss-clss-def by blast$ 
```

```

then have lits-of-l ?M  $\subseteq$  ?I
  unfolding true-clss-def lits-of-def by auto
then have ?M  $\models_{as}$  ?N
  using I'-N  $\langle C \in ?N \rangle \langle \neg ?M \models_a C \rangle$  cons-I' atms-N-M
  by (meson  $\langle \text{trail } S \models_{as} CNot\ C \rangle$  consistent-CNot-not rev-subsetD sup-ge1 true-annot-def
    true-annots-def true-clss-mono-set-mset-l true-clss-def)
then show False using M by fast
qed
from List.split-list-first-propE[OF this] obtain K :: 'v literal and
  F F' :: ('v, unit, unit) marked-lit list where
  M-K: ?M = F' @ Marked K () # F and
  nm:  $\forall f \in \text{set } F'. \neg \text{is-marked } f$ 
  unfolding is-marked-def by (metis (full-types) old.unit.exhaust)
let ?K = Marked K () :: ('v, unit, unit) marked-lit
have ?K  $\in \text{set } ?M$ 
  unfolding M-K by auto
let ?C = image-mset lit-of {#L  $\in$  #mset ?M. is-marked L  $\wedge$  L  $\neq$  ?K#} :: 'v literal multiset
let ?C' = set-mset (image-mset ( $\lambda L::'v \text{ literal. } \{ \#L \# \}$ ) (?C + {#lit-of ?K#}))
have ?N  $\cup \{ \{ \# \text{lit-of } L \# \} \mid L. \text{is-marked } L \wedge L \in \text{set } ?M \} \models_{ps} \text{unmark-l } ?M$ 
  using all-decomposition-implies-propagated-lits-are-implied[OF decomp] .
moreover have C': ?C' = {#lit-of L#}  $\mid L. \text{is-marked } L \wedge L \in \text{set } ?M$ 
  unfolding M-K apply standard
  apply force
  using IntI by auto
ultimately have N-C-M: ?N  $\cup$  ?C'  $\models_{ps} \text{unmark-l } ?M$ 
  by auto
have N-M-False: ?N  $\cup$  ( $\lambda L. \{ \# \text{lit-of } L \# \}$ ) ' (set ?M)  $\models_{ps} \{ \{ \# \} \}$ 
  using M  $\langle ?M \models_{as} CNot\ C \rangle \langle C \in ?N \rangle$  unfolding true-clss-clss-def true-annots-def Ball-def
  true-annot-def by (metis consistent-CNot-not sup.orderE sup-commute true-clss-def
    true-clss-singleton-lit-of-implies-incl true-clss-union true-clss-union-increase)

have undefined-lit F K using  $\langle \text{no-dup } ?M \rangle$  unfolding M-K by (simp add: defined-lit-map)
moreover
  have ?N  $\cup$  ?C'  $\models_{ps} \{ \{ \# \} \}$ 
  proof -
    have A: ?N  $\cup$  ?C'  $\cup \text{unmark-l } ?M =$ 
      ?N  $\cup \text{unmark-l } ?M$ 
    unfolding M-K by auto
    show ?thesis
      using true-clss-clss-left-right[OF N-C-M, of {#}]] N-M-False unfolding A by auto
  qed
have ?N  $\models_p \text{image-mset uminus } ?C + \{ \# - K \# \}$ 
  unfolding true-clss-clss-def true-clss-clss-def total-over-m-def
  proof (intro allI impI)
    fix I
    assume
      tot: total-over-set I (atms-of-ms (?N  $\cup$  {image-mset uminus ?C + {# - K#}})) and
      cons: consistent-interp I and
      I  $\models_s ?N$ 
    have (K  $\in I \wedge -K \notin I$ )  $\vee$  ( $-K \in I \wedge K \notin I$ )
      using cons tot unfolding consistent-interp-def by (cases K) auto
    have {a  $\in \text{set } (\text{trail } S). \text{is-marked } a \wedge a \neq \text{Marked } K ()$ } =
      set (trail S)  $\cap \{ L. \text{is-marked } L \wedge L \neq \text{Marked } K () \}$ 
    by auto
    then have tot': total-over-set I

```

```

    (atm-of 'lit-of ' (set ?M  $\cap$  {L. is-marked L  $\wedge$  L  $\neq$  Marked K ()}))
  using tot by (auto simp add: atms-of-uminus-lit-atm-of-lit-of)
} fix x :: ('v, unit, unit) marked-lit
assume
  a3: lit-of x  $\notin$  I and
  a1: x  $\in$  set ?M and
  a4: is-marked x and
  a5: x  $\neq$  Marked K ()
then have Pos (atm-of (lit-of x))  $\in$  I  $\vee$  Neg (atm-of (lit-of x))  $\in$  I
  using a5 a4 tot' a1 unfolding total-over-set-def atms-of-s-def by blast
moreover have f6: Neg (atm-of (lit-of x)) = - Pos (atm-of (lit-of x))
  by simp
ultimately have - lit-of x  $\in$  I
  using f6 a3 by (metis (no-types) atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
    literal.sel(1))
} note H = this

have  $\neg I \models_s ?C'$ 
  using  $\langle ?N \cup ?C' \models_{ps} \{\{\#\}\} \rangle$  tot cons  $\langle I \models_s ?N \rangle$ 
  unfolding true-clss-clss-def total-over-m-def
  by (simp add: atms-of-uminus-lit-atm-of-lit-of atms-of-ms-single-image-atm-of-lit-of)
then show I  $\models$  image-mset uminus ?C + {#- K#}
  unfolding true-clss-def true-cls-def Bex-def
  using  $\langle (K \in I \wedge -K \notin I) \vee (-K \in I \wedge K \notin I) \rangle$ 
  by (auto dest!: H)
qed
moreover have F  $\models_{as}$  CNot (image-mset uminus ?C)
  using nm unfolding true-annots-def CNot-def M-K by (auto simp add: lits-of-def)
ultimately have False
  using bj-can-jump[of S F' K F C -K
    image-mset uminus (image-mset lit-of {# L :# mset ?M. is-marked L  $\wedge$  L  $\neq$  Marked K ()#})]
     $\langle C \in ?N \rangle$  n-s  $\langle ?M \models_{as} CNot C \rangle$  bj-backjump inv  $\langle$ no-dup (trail S) $\rangle$  unfolding M-K by auto
  then show ?thesis by fast
qed auto
qed
end

locale dpll-with-backjumping =
  dpll-with-backjumping-ops mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clsNOT remove-clsNOT inv backjump-conds
  propagate-conds
for
  mset-cls:: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
  raw-clauses :: 'st  $\Rightarrow$  'clss and
  prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and

```

```

  tl-trail :: 'st  $\Rightarrow$  'st and
  add-clsNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  remove-clsNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  inv :: 'st  $\Rightarrow$  bool and
  backjump-conds :: 'v clause  $\Rightarrow$  'v clause  $\Rightarrow$  'v literal  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  bool and
  propagate-conds :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  bool
+
assumes dpll-bj-inv:  $\bigwedge S T. \text{dpll-bj } S T \Rightarrow \text{inv } S \Rightarrow \text{inv } T$ 
begin

lemma rtrancpl-dpll-bj-inv:
assumes dpll-bj**  $S T$  and inv  $S$ 
shows inv  $T$ 
using assms by (induction rule: rtrancpl-induct)
  (auto simp add: dpll-bj-no-dup intro: dpll-bj-inv)

lemma rtrancpl-dpll-bj-no-dup:
assumes dpll-bj**  $S T$  and inv  $S$ 
and no-dup (trail  $S$ )
shows no-dup (trail  $T$ )
using assms by (induction rule: rtrancpl-induct)
  (auto simp add: dpll-bj-no-dup dest: rtrancpl-dpll-bj-inv dpll-bj-inv)

lemma rtrancpl-dpll-bj-atms-of-ms-clauses-inv:
assumes
  dpll-bj**  $S T$  and inv  $S$ 
shows atms-of-mm (clausesNOT  $S$ ) = atms-of-mm (clausesNOT  $T$ )
using assms by (induction rule: rtrancpl-induct)
  (auto dest: rtrancpl-dpll-bj-inv dpll-bj-atms-of-ms-clauses-inv)

lemma rtrancpl-dpll-bj-atms-in-trail:
assumes
  dpll-bj**  $S T$  and
  inv  $S$  and
  atm-of ' (lits-of-l (trail  $S$ ))  $\subseteq$  atms-of-mm (clausesNOT  $S$ )
shows atm-of ' (lits-of-l (trail  $T$ ))  $\subseteq$  atms-of-mm (clausesNOT  $T$ )
using assms apply (induction rule: rtrancpl-induct)
using dpll-bj-atms-in-trail dpll-bj-atms-of-ms-clauses-inv rtrancpl-dpll-bj-inv by auto

lemma rtrancpl-dpll-bj-sat-iff:
assumes dpll-bj**  $S T$  and inv  $S$ 
shows  $I \models_{sm} \text{clauses}_{NOT} S \longleftrightarrow I \models_{sm} \text{clauses}_{NOT} T$ 
using assms by (induction rule: rtrancpl-induct)
  (auto dest!: dpll-bj-sat-iff simp: rtrancpl-dpll-bj-inv)

lemma rtrancpl-dpll-bj-atms-in-trail-in-set:
assumes
  dpll-bj**  $S T$  and
  inv  $S$ 
  atms-of-mm (clausesNOT  $S$ )  $\subseteq A$  and
  atm-of ' (lits-of-l (trail  $S$ ))  $\subseteq A$ 
shows atm-of ' (lits-of-l (trail  $T$ ))  $\subseteq A$ 
using assms
by (induction rule: rtrancpl-induct)
  (auto dest: rtrancpl-dpll-bj-inv)

```

*simp add: dpll-bj-atms-in-trail-in-set rtranclp-dpll-bj-atms-of-ms-clauses-inv
rtranclp-dpll-bj-inv*

lemma *rtranclp-dpll-bj-all-decomposition-implies-inv:*

assumes
*dpll-bj** S T and*
inv S
all-decomposition-implies-m (clauses_{NOT} S) (get-all-marked-decomposition (trail S))
shows *all-decomposition-implies-m (clauses_{NOT} T) (get-all-marked-decomposition (trail T))*
using *assms by (induction rule: rtranclp-induct)*
(auto intro: dpll-bj-all-decomposition-implies-inv simp: rtranclp-dpll-bj-inv)

lemma *rtranclp-dpll-bj-inv-incl-dpll-bj-inv-trancl:*

{(T, S). dpll-bj⁺⁺ S T
∧ atms-of-mm (clauses_{NOT} S) ⊆ atms-of-ms A ∧ atm-of ' lits-of-l (trail S) ⊆ atms-of-ms A
∧ no-dup (trail S) ∧ inv S}
⊆ {(T, S). dpll-bj S T ∧ atms-of-mm (clauses_{NOT} S) ⊆ atms-of-ms A
∧ atm-of ' lits-of-l (trail S) ⊆ atms-of-ms A ∧ no-dup (trail S) ∧ inv S}⁺
(is ?A ⊆ ?B⁺)

proof *standard*

fix *x*
assume *x-A: x ∈ ?A*
obtain *S T::'st where*
x[simp]: x = (T, S) by (cases x) auto
have
dpll-bj⁺⁺ S T and
atms-of-mm (clauses_{NOT} S) ⊆ atms-of-ms A and
atm-of ' lits-of-l (trail S) ⊆ atms-of-ms A and
no-dup (trail S) and
inv S
using *x-A by auto*
then show *x ∈ ?B⁺ unfolding x*
proof *(induction rule: tranclp-induct)*
case *base*
then show *?case by auto*
next
case *(step T U) note step = this(1) and ST = this(2) and IH = this(3)[OF this(4-7)]*
and *N-A = this(4) and M-A = this(5) and nd = this(6) and inv = this(7)*

have *[simp]: atms-of-mm (clauses_{NOT} S) = atms-of-mm (clauses_{NOT} T)*
using *step rtranclp-dpll-bj-atms-of-ms-clauses-inv tranclp-into-rtranclp inv by fastforce*
have *no-dup (trail T)*
using *local.step nd rtranclp-dpll-bj-no-dup tranclp-into-rtranclp inv by fastforce*
moreover have *atm-of ' (lits-of-l (trail T)) ⊆ atms-of-ms A*
by *(metis inv M-A N-A local.step rtranclp-dpll-bj-atms-in-trail-in-set*
tranclp-into-rtranclp)
moreover have *inv T*
using *inv local.step rtranclp-dpll-bj-inv tranclp-into-rtranclp by fastforce*
ultimately have *(U, T) ∈ ?B using ST N-A M-A inv by auto*
then show *?case using IH by (rule trancl-into-trancl2)*
qed
qed

lemma *wf-tranclp-dpll-bj:*

assumes *fin: finite A*

shows $wf \{(T, S). \text{dpll-bj}^{++} S T$
 $\wedge \text{atms-of-mm} (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of } \text{' lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A$
 $\wedge \text{no-dup } (\text{trail } S) \wedge \text{inv } S\}$
using $wf\text{-tranc1}[OF \text{ wf-dpll-bj}[OF \text{ fin}]] \text{ rtranc1p-dpll-bj-inv-incl-dpll-bj-inv-tranc1}$
by (rule $wf\text{-subset}$)

lemma $\text{dpll-bj-sat-ext-iff}$:

$\text{dpll-bj } S T \implies \text{inv } S \implies I \models_{\text{sextm}} \text{clauses}_{NOT} S \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{NOT} T$
by (simp add: dpll-bj-clauses)

lemma $\text{rtranc1p-dpll-bj-sat-ext-iff}$:

$\text{dpll-bj}^{**} S T \implies \text{inv } S \implies I \models_{\text{sextm}} \text{clauses}_{NOT} S \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{NOT} T$
by (induction rule: rtranc1p-induct) (simp-all add: $\text{rtranc1p-dpll-bj-inv dpll-bj-sat-ext-iff}$)

theorem $\text{full-dpll-backjump-final-state}$:

fixes $A :: \text{'v literal multiset set and } S T :: \text{'st}$

assumes

$\text{full: full dpll-bj } S T$ **and**

$\text{atms-S: atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A$ **and**

$\text{atms-trail: atm-of } \text{' lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A$ **and**

$\text{n-d: no-dup } (\text{trail } S)$ **and**

$\text{finite } A$ **and**

$\text{inv: inv } S$ **and**

$\text{decomp: all-decomposition-implies-m } (\text{clauses}_{NOT} S) (\text{get-all-marked-decomposition } (\text{trail } S))$

shows $\text{unsatisfiable } (\text{set-mset } (\text{clauses}_{NOT} S))$

$\vee (\text{trail } T \models_{\text{asm}} \text{clauses}_{NOT} S \wedge \text{satisfiable } (\text{set-mset } (\text{clauses}_{NOT} S)))$

proof –

have $\text{st: dpll-bj}^{**} S T$ **and** $\text{no-step dpll-bj } T$

using $\text{full unfolding full-def}$ **by** fast+

moreover have $\text{atms-of-mm } (\text{clauses}_{NOT} T) \subseteq \text{atms-of-ms } A$

using $\text{atms-S inv rtranc1p-dpll-bj-atms-of-ms-clauses-inv st}$ **by** blast

moreover have $\text{atm-of } \text{' lits-of-l } (\text{trail } T) \subseteq \text{atms-of-ms } A$

using $\text{atms-S atms-trail inv rtranc1p-dpll-bj-atms-in-trail-in-set st}$ **by** auto

moreover have $\text{no-dup } (\text{trail } T)$

using $\text{n-d inv rtranc1p-dpll-bj-no-dup st}$ **by** blast

moreover have $\text{inv: inv } T$

using $\text{inv rtranc1p-dpll-bj-inv st}$ **by** blast

moreover

have $\text{decomp: all-decomposition-implies-m } (\text{clauses}_{NOT} T) (\text{get-all-marked-decomposition } (\text{trail } T))$

using $\langle \text{inv } S \rangle \text{ decomp rtranc1p-dpll-bj-all-decomposition-implies-inv st}$ **by** blast

ultimately have $\text{unsatisfiable } (\text{set-mset } (\text{clauses}_{NOT} T))$

$\vee (\text{trail } T \models_{\text{asm}} \text{clauses}_{NOT} T \wedge \text{satisfiable } (\text{set-mset } (\text{clauses}_{NOT} T)))$

using $\langle \text{finite } A \rangle \text{ dpll-backjump-final-state}$ **by** force

then show ?thesis

by (meson $\langle \text{inv } S \rangle \text{ rtranc1p-dpll-bj-sat-iff satisfiable-carac st true-annots-true-cls}$)

qed

corollary $\text{full-dpll-backjump-final-state-from-init-state}$:

fixes $A :: \text{'v literal multiset set and } S T :: \text{'st}$

assumes

$\text{full: full dpll-bj } S T$ **and**

$\text{trail } S = []$ **and**

$\text{clauses}_{NOT} S = N$ **and**

$\text{inv } S$

shows $\text{unsatisfiable } (\text{set-mset } N) \vee (\text{trail } T \models_{\text{asm}} N \wedge \text{satisfiable } (\text{set-mset } N))$

```

using assms full-dpll-backjump-final-state[of S T set-mset N] by auto

lemma trancpl-dpll-bj-trail-mes-decreasing-prop:
  assumes dpll: dpll-bj++ S T and inv: inv S and
  N-A: atms-of-mm (clausesNOT S) ⊆ atms-of-ms A and
  M-A: atm-of ' lits-of-l (trail S) ⊆ atms-of-ms A and
  n-d: no-dup (trail S) and
  fin-A: finite A
  shows  $(2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$ 
     $- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } T)$ 
     $< (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$ 
     $- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } S)$ 
  using dpll
proof (induction)
  case base
  then show ?case
    using N-A M-A n-d dpll-bj-trail-mes-decreasing-prop fin-A inv by blast
next
  case (step T U) note st = this(1) and dpll = this(2) and IH = this(3)
  have atms-of-mm (clausesNOT S) = atms-of-mm (clausesNOT T)
    using rtrancpl-dpll-bj-atms-of-ms-clauses-inv by (metis dpll-bj-clauses dpll-bj-inv inv st
      trancplD)
  then have N-A': atms-of-mm (clausesNOT T) ⊆ atms-of-ms A
    using N-A by auto
  moreover have M-A': atm-of ' lits-of-l (trail T) ⊆ atms-of-ms A
    by (meson M-A N-A inv rtrancpl-dpll-bj-atms-in-trail-in-set st dpll
      trancpl.r-into-trancpl trancpl-into-rtrancpl trancpl-trans)
  moreover have nd: no-dup (trail T)
    by (metis inv n-d rtrancpl-dpll-bj-no-dup st trancpl-into-rtrancpl)
  moreover have inv T
    by (meson dpll dpll-bj-inv inv rtrancpl-dpll-bj-inv st trancpl-into-rtrancpl)
  ultimately show ?case
    using IH dpll-bj-trail-mes-decreasing-prop[of T U A] dpll fin-A by linarith
qed

end

```

15.4 CDCL

15.4.1 Learn and Forget

```

locale learn-ops =
  dpll-state mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clssNOT remove-clssNOT
for
  mset-clss:: 'cls ⇒ 'v clause and
  insert-clss:: 'v literal ⇒ 'cls ⇒ 'cls and
  remove-lit:: 'v literal ⇒ 'cls ⇒ 'cls and
  mset-clss:: 'clss ⇒ 'v clauses and
  union-clss:: 'clss ⇒ 'clss ⇒ 'clss and
  in-clss:: 'cls ⇒ 'clss ⇒ bool and
  insert-clss:: 'cls ⇒ 'clss ⇒ 'clss and
  remove-from-clss:: 'cls ⇒ 'clss ⇒ 'clss and
  trail:: 'st ⇒ ('v, unit, unit) marked-lits and
  raw-clauses:: 'st ⇒ 'clss and

```



```

    prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
    tl-trail :: 'st ⇒ 'st and
    add-clsNOT :: 'cls ⇒ 'st ⇒ 'st and
    remove-clsNOT :: 'cls ⇒ 'st ⇒ 'st +
  fixes
    learn-cond :: 'cls ⇒ 'st ⇒ bool
begin
inductive learn :: 'st ⇒ 'st ⇒ bool where
learnNOT-rule: clausesNOT S ⊨pm mset-cls C ⇒
  atms-of (mset-cls C) ⊆ atms-of-mm (clausesNOT S) ∪ atm-of ' (lits-of-l (trail S)) ⇒
  learn-cond C S ⇒
  T ~ add-clsNOT C S ⇒
  learn S T
inductive-cases learnNOTE: learn S T

lemma learn-μC-stable:
  assumes learn S T and no-dup (trail S)
  shows μC A B (trail-weight S) = μC A B (trail-weight T)
  using assms by (auto elim: learnNOTE)
end

locale forget-ops =
  dpll-state mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clsNOT remove-clsNOT
for
  mset-cls:: 'cls ⇒ 'v clause and
  insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
  remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
  mset-clss:: 'clss ⇒ 'v clauses and
  union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
  in-clss :: 'cls ⇒ 'clss ⇒ bool and
  insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
  remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
  trail :: 'st ⇒ ('v, unit, unit) marked-lits and
  raw-clauses :: 'st ⇒ 'clss and
  prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
  tl-trail :: 'st ⇒ 'st and
  add-clsNOT :: 'cls ⇒ 'st ⇒ 'st and
  remove-clsNOT :: 'cls ⇒ 'st ⇒ 'st +
  fixes
    forget-cond :: 'cls ⇒ 'st ⇒ bool
begin
inductive forgetNOT :: 'st ⇒ 'st ⇒ bool where
forgetNOT:
  removeAll-mset (mset-cls C)(clausesNOT S) ⊨pm mset-cls C ⇒
  forget-cond C S ⇒
  C !∈! raw-clauses S ⇒
  T ~ remove-clsNOT C S ⇒
  forgetNOT S T
inductive-cases forgetNOTE: forgetNOT S T

lemma forget-μC-stable:
  assumes forgetNOT S T
  shows μC A B (trail-weight S) = μC A B (trail-weight T)

```

```

using assms by (auto elim!: forgetNOTE)
end

locale learn-and-forgetNOT =
  learn-ops mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT learn-cond +
  forget-ops mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT forget-cond
for
  mset-cls :: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss :: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
  raw-clauses :: 'st  $\Rightarrow$  'clss and
  prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
  tl-trail :: 'st  $\Rightarrow$  'st and
  add-clNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  remove-clNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  learn-cond forget-cond :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  bool
begin
inductive learn-and-forgetNOT :: 'st  $\Rightarrow$  'st  $\Rightarrow$  bool
where
  lf-learn: learn S T  $\Longrightarrow$  learn-and-forgetNOT S T |
  lf-forget: forgetNOT S T  $\Longrightarrow$  learn-and-forgetNOT S T
end

```

15.4.2 Definition of CDCL

```

locale conflict-driven-clause-learning-ops =
  dpll-with-backjumping-ops mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT
    inv backjump-conds propagate-conds +
  learn-and-forgetNOT mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT learn-cond
    forget-cond
for
  mset-cls :: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss :: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
  raw-clauses :: 'st  $\Rightarrow$  'clss and
  prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and

```

$tl_trail :: 'st \Rightarrow 'st$ **and**
 $add_cls_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st$ **and**
 $remove_cls_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st$ **and**
 $inv :: 'st \Rightarrow bool$ **and**
 $backjump_conds :: 'v\ clause \Rightarrow 'v\ clause \Rightarrow 'v\ literal \Rightarrow 'st \Rightarrow 'st \Rightarrow bool$ **and**
 $propagate_conds :: ('v, unit, unit)\ marked_lit \Rightarrow 'st \Rightarrow bool$ **and**
 $learn_cond\ forget_cond :: 'cls \Rightarrow 'st \Rightarrow bool$

begin

inductive $cdcl_{NOT} :: 'st \Rightarrow 'st \Rightarrow bool$ **for** $S :: 'st$ **where**

$c_dpll_bj: dpll_bj\ S\ S' \Longrightarrow cdcl_{NOT}\ S\ S' \mid$

$c_learn: learn\ S\ S' \Longrightarrow cdcl_{NOT}\ S\ S' \mid$

$c_forget_{NOT}: forget_{NOT}\ S\ S' \Longrightarrow cdcl_{NOT}\ S\ S'$

lemma $cdcl_{NOT}\text{-all-induct}[consumes\ 1, case\text{-names}\ dpll_bj\ learn\ forget_{NOT}]$:

fixes $S\ T :: 'st$

assumes $cdcl_{NOT}\ S\ T$ **and**

$dpll: \bigwedge T. dpll_bj\ S\ T \Longrightarrow P\ S\ T$ **and**

learning:

$\bigwedge C\ T. clauses_{NOT}\ S \models_{pm} mset_cls\ C \Longrightarrow$

$atms_of\ (mset_cls\ C) \subseteq atms_of_mm\ (clauses_{NOT}\ S) \cup atm_of\ ' (lits_of_l\ (trail\ S)) \Longrightarrow$

$T \sim add_cls_{NOT}\ C\ S \Longrightarrow$

$P\ S\ T$ **and**

forgetting: $\bigwedge C\ T. removeAll_mset\ (mset_cls\ C)\ (clauses_{NOT}\ S) \models_{pm} mset_cls\ C \Longrightarrow$

$C \notin raw_clauses\ S \Longrightarrow$

$T \sim remove_cls_{NOT}\ C\ S \Longrightarrow$

$P\ S\ T$

shows $P\ S\ T$

using $assms(1)$ **by** (*induction rule:* $cdcl_{NOT}.induct$)

(*auto intro:* $assms(2, 3, 4)$ *elim!:* $learn_{NOT}E\ forget_{NOT}E$)**+**

lemma $cdcl_{NOT}\text{-no-dup}$:

assumes

$cdcl_{NOT}\ S\ T$ **and**

$inv\ S$ **and**

$no_dup\ (trail\ S)$

shows $no_dup\ (trail\ T)$

using $assms$ **by** (*induction rule:* $cdcl_{NOT}\text{-all-induct}$) (*auto intro:* $dpll_bj\text{-no-dup}$)

Consistency of the trail **lemma** $cdcl_{NOT}\text{-consistent}$:

assumes

$cdcl_{NOT}\ S\ T$ **and**

$inv\ S$ **and**

$no_dup\ (trail\ S)$

shows $consistent_interp\ (lits_of_l\ (trail\ T))$

using $cdcl_{NOT}\text{-no-dup}[OF\ assms]$ $distinct_consistent_interp$ **by** *fast*

The subtle problem here is that tautologies can be removed, meaning that some variable can disappear of the problem. It is also possible that some variable of the trail are not in the clauses anymore.

lemma $cdcl_{NOT}\text{-atms-of-ms-clauses-decreasing}$:

assumes $cdcl_{NOT}\ S\ T$ **and** $inv\ S$ **and** $no_dup\ (trail\ S)$

shows $atms_of_mm\ (clauses_{NOT}\ T) \subseteq atms_of_mm\ (clauses_{NOT}\ S) \cup atm_of\ ' (lits_of_l\ (trail\ S))$

using $assms$ **by** (*induction rule:* $cdcl_{NOT}\text{-all-induct}$)

(*auto dest!:* $dpll_bj\text{-atms-of-ms-clauses-inv}\ set_mp\ simp\ add: atms_of_ms_def\ Union_eq$)

lemma *cdcl_{NOT}-atms-in-trail*:

assumes *cdcl_{NOT} S T and inv S and no-dup (trail S)*
and *atm-of ‘ (lits-of-l (trail S)) ⊆ atms-of-mm (clauses_{NOT} S)*
shows *atm-of ‘ (lits-of-l (trail T)) ⊆ atms-of-mm (clauses_{NOT} S)*
using *assms* **by** (*induction rule: cdcl_{NOT}-all-induct*) (*auto simp add: dp11-bj-atms-in-trail*)

lemma *cdcl_{NOT}-atms-in-trail-in-set*:

assumes
cdcl_{NOT} S T and inv S and no-dup (trail S) and
atms-of-mm (clauses_{NOT} S) ⊆ A and
atm-of ‘ (lits-of-l (trail S)) ⊆ A
shows *atm-of ‘ (lits-of-l (trail T)) ⊆ A*
using *assms*
by (*induction rule: cdcl_{NOT}-all-induct*)
(*simp-all add: dp11-bj-atms-in-trail-in-set dp11-bj-atms-of-ms-clauses-inv*)

lemma *cdcl_{NOT}-all-decomposition-implies*:

assumes *cdcl_{NOT} S T and inv S and n-d[simp]: no-dup (trail S) and*
all-decomposition-implies-m (clauses_{NOT} S) (get-all-marked-decomposition (trail S))
shows
all-decomposition-implies-m (clauses_{NOT} T) (get-all-marked-decomposition (trail T))
using *assms(1,2,4)*

proof (*induction rule: cdcl_{NOT}-all-induct*)

case *dp11-bj*

then show *?case*

using *dp11-bj-all-decomposition-implies-inv n-d* **by** *blast*

next

case *learn*

then show *?case* **by** (*auto simp add: all-decomposition-implies-def*)

next

case (*forget_{NOT} C T*) **note** *cls-C = this(1) and C = this(2) and T = this(3) and inv = this(4)*

and

decomp = this(5)

show *?case*

unfolding *all-decomposition-implies-def Ball-def*

proof (*intro allI, clarify*)

fix *a b*

assume *(a, b) ∈ set (get-all-marked-decomposition (trail T))*

then have *unmark-l a ∪ set-mset (clauses_{NOT} S) ⊨_{ps} unmark-l b*

using *decomp T* **by** (*auto simp add: all-decomposition-implies-def*)

moreover

have *mset-cls C ∈ set-mset (clauses_{NOT} S)*

using *C* **by** *blast*

then have *set-mset (clauses_{NOT} T) ⊨_{ps} set-mset (clauses_{NOT} S)*

by (*metis (no-types, lifting) T clauses-remove-cls_{NOT} cls-C insert-Diff order-refl*

set-mset-minus-replicate-mset(1) state-eq_{NOT}-clauses true-clss-clss-def

true-clss-clss-insert)

ultimately show *unmark-l a ∪ set-mset (clauses_{NOT} T)*

⊨_{ps} unmark-l b

using *true-clss-clss-generalise-true-clss-clss* **by** *blast*

qed

qed

Extension of models **lemma** *cdcl_{NOT}-bj-sat-ext-iff*:

```

assumes  $cdcl_{NOT} S$  Tand  $inv S$  and  $n-d: no-dup (trail S)$ 
shows  $I \models_{sextm} clauses_{NOT} S \longleftrightarrow I \models_{sextm} clauses_{NOT} T$ 
using assms
proof (induction rule:cdclNOT-all-induct)
  case dpll-bj
  then show ?case by (simp add: dpll-bj-clauses)
next
  case (learn C T) note  $T = this(3)$ 
  { fix  $J$ 
    assume
       $I \models_{sextm} clauses_{NOT} S$  and
       $I \subseteq J$  and
      tot: total-over-m J (set-mset ({#mset-cls C#} + clausesNOT S)) and
      cons: consistent-interp J
    then have  $J \models_{sm} clauses_{NOT} S$  unfolding true-clss-ext-def by auto

    moreover
      with  $\langle clauses_{NOT} S \models_{pm} mset-cls C \rangle$  have  $J \models mset-cls C$ 
      using tot cons unfolding true-clss-cls-def by auto
      ultimately have  $J \models_{sm} \{ \#mset-cls C \# \} + clauses_{NOT} S$  by auto
    }
  then have  $H: I \models_{sextm} (clauses_{NOT} S) \implies I \models_{sext} insert (mset-cls C) (set-mset (clauses_{NOT} S))$ 
    unfolding true-clss-ext-def by auto
  show ?case
    apply standard
    using  $T$   $n-d$  apply (auto simp add: H)[]
    using  $T$   $n-d$  apply simp
    by (metis Diff-insert-absorb insert-subset subsetI subset-antisym
      true-clss-ext-decrease-right-remove-r)
  next
  case (forgetNOT C T) note  $cls-C = this(1)$  and  $T = this(3)$ 
  { fix  $J$ 
    assume
       $I \models_{sext} set-mset (clauses_{NOT} S) - \{mset-cls C\}$  and
       $I \subseteq J$  and
      tot: total-over-m J (set-mset (clausesNOT S)) and
      cons: consistent-interp J
    then have  $J \models_s set-mset (clauses_{NOT} S) - \{mset-cls C\}$ 
      unfolding true-clss-ext-def by (meson Diff-subset total-over-m-subset)

    moreover
      with  $cls-C$  have  $J \models mset-cls C$ 
      using tot cons unfolding true-clss-cls-def
      by (metis Un-commute forgetNOT.hyps(2) in-clss-mset-clss insert-Diff insert-is-Un order-refl
        set-mset-minus-replicate-mset(1))
      ultimately have  $J \models_{sm} (clauses_{NOT} S)$  by (metis insert-Diff-single true-clss-insert)
    }
  then have  $H: I \models_{sext} set-mset (clauses_{NOT} S) - \{mset-cls C\} \implies I \models_{sextm} (clauses_{NOT} S)$ 
    unfolding true-clss-ext-def by blast
  show ?case using  $T$  by (auto simp: true-clss-ext-decrease-right-remove-r H)
qed

end — end of conflict-driven-clause-learning-ops

```

15.5 CDCL with invariant

```

locale conflict-driven-clause-learning =
  conflict-driven-clause-learning-ops +
  assumes  $cdcl_{NOT}\text{-inv}$ :  $\bigwedge S\ T. cdcl_{NOT}\ S\ T \implies inv\ S \implies inv\ T$ 
begin
sublocale dpll-with-backjumping
  apply unfold-locales
  using  $cdcl_{NOT}.simps\ cdcl_{NOT}\text{-inv}$  by auto

lemma rtracp-cdclNOT-inv:
   $cdcl_{NOT}^{**}\ S\ T \implies inv\ S \implies inv\ T$ 
  by (induction rule: rtracp-induct) (auto simp add:  $cdcl_{NOT}\text{-inv}$ )

lemma rtracp-cdclNOT-no-dup:
  assumes  $cdcl_{NOT}^{**}\ S\ T$  and  $inv\ S$ 
  and no-dup (trail  $S$ )
  shows no-dup (trail  $T$ )
  using assms by (induction rule: rtracp-induct) (auto intro:  $cdcl_{NOT}\text{-no-dup}\ rtracp\text{-}cdcl_{NOT}\text{-inv}$ )

lemma rtracp-cdclNOT-trail-clauses-bound:
  assumes
     $cdcl$ :  $cdcl_{NOT}^{**}\ S\ T$  and
     $inv$ :  $inv\ S$  and
     $n\text{-d}$ : no-dup (trail  $S$ ) and
     $atms\text{-clauses}\text{-}S$ :  $atms\text{-of}\text{-}mm\ (clauses_{NOT}\ S) \subseteq A$  and
     $atms\text{-trail}\text{-}S$ :  $atm\text{-of}\ (lits\text{-of}\text{-}l\ (trail\ S)) \subseteq A$ 
  shows  $atm\text{-of}\ (lits\text{-of}\text{-}l\ (trail\ T)) \subseteq A \wedge atms\text{-of}\text{-}mm\ (clauses_{NOT}\ T) \subseteq A$ 
  using  $cdcl$ 
proof (induction rule: rtracp-induct)
  case base
  then show ?case using  $atms\text{-clauses}\text{-}S\ atms\text{-trail}\text{-}S$  by simp
next
  case (step  $T\ U$ ) note  $st = this(1)$  and  $cdcl_{NOT} = this(2)$  and  $IH = this(3)$ 
  have  $inv\ T$  using  $inv\ st\ rtracp\text{-}cdcl_{NOT}\text{-inv}$  by blast
  have no-dup (trail  $T$ )
    using  $rtracp\text{-}cdcl_{NOT}\text{-no-dup}[of\ S\ T]\ st\ cdcl_{NOT}\ inv\ n\text{-d}$  by blast
  then have  $atms\text{-of}\text{-}mm\ (clauses_{NOT}\ U) \subseteq A$ 
    using  $cdcl_{NOT}\text{-}atms\text{-of}\text{-}ms\text{-clauses}\text{-decreasing}[OF\ cdcl_{NOT}]\ IH\ n\text{-d}\ \langle inv\ T \rangle$  by fast
  moreover
    have  $atm\text{-of}\ (lits\text{-of}\text{-}l\ (trail\ U)) \subseteq A$ 
      using  $cdcl_{NOT}\text{-}atms\text{-in}\text{-}trail\text{-in}\text{-}set[OF\ cdcl_{NOT},\ of\ A]\ \langle no\text{-dup}\ (trail\ T) \rangle$ 
      by (meson  $atms\text{-trail}\text{-}S\ atms\text{-clauses}\text{-}S\ IH\ \langle inv\ T \rangle\ cdcl_{NOT}$ )
    ultimately show ?case by fast
qed

lemma rtracp-cdclNOT-all-decomposition-implies:
  assumes  $cdcl_{NOT}^{**}\ S\ T$  and  $inv\ S$  and no-dup (trail  $S$ ) and
    all-decomposition-implies- $m\ (clauses_{NOT}\ S)\ (get\text{-all}\text{-marked}\text{-decomposition}\ (trail\ S))$ 
  shows
    all-decomposition-implies- $m\ (clauses_{NOT}\ T)\ (get\text{-all}\text{-marked}\text{-decomposition}\ (trail\ T))$ 
  using assms by (induction)
  (auto intro:  $rtracp\text{-}cdcl_{NOT}\text{-inv}\ cdcl_{NOT}\text{-all}\text{-decomposition}\text{-implies}\ rtracp\text{-}cdcl_{NOT}\text{-no-dup}$ )

lemma rtracp-cdclNOT-bj-sat-ext-iff:
  assumes  $cdcl_{NOT}^{**}\ S\ T$  and  $inv\ S$  and no-dup (trail  $S$ )

```

shows $I \models_{\text{sextm}} \text{clauses}_{\text{NOT}} S \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{\text{NOT}} T$
using *assms* **apply** (induction rule: *rtranclp-induct*)
using *cdcl_{NOT}-bj-sat-ext-iff* **by** (auto intro: *rtranclp-cdcl_{NOT}-inv* *rtranclp-cdcl_{NOT}-no-dup*)

definition *cdcl_{NOT}-NOT-all-inv* **where**

$\text{cdcl}_{\text{NOT}}\text{-NOT-all-inv } A \ S \longleftrightarrow (\text{finite } A \wedge \text{inv } S \wedge \text{atms-of-mm } (\text{clauses}_{\text{NOT}} S) \subseteq \text{atms-of-ms } A$
 $\wedge \text{atm-of ' lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \wedge \text{no-dup } (\text{trail } S))$

lemma *cdcl_{NOT}-NOT-all-inv*:

assumes *cdcl_{NOT}*** $S \ T$ **and** *cdcl_{NOT}-NOT-all-inv* $A \ S$

shows *cdcl_{NOT}-NOT-all-inv* $A \ T$

using *assms* **unfolding** *cdcl_{NOT}-NOT-all-inv-def*

by (*simp* add: *rtranclp-cdcl_{NOT}-inv* *rtranclp-cdcl_{NOT}-no-dup* *rtranclp-cdcl_{NOT}-trail-clauses-bound*)

abbreviation *learn-or-forget* **where**

$\text{learn-or-forget } S \ T \equiv (\lambda S \ T. \text{learn } S \ T \vee \text{forget}_{\text{NOT}} S \ T) \ S \ T$

lemma *rtranclp-learn-or-forget-cdcl_{NOT}*:

$\text{learn-or-forget}^{**} S \ T \implies \text{cdcl}_{\text{NOT}}^{**} S \ T$

using *rtranclp-mono*[of *learn-or-forget cdcl_{NOT}*] *cdcl_{NOT}.c-learn* *cdcl_{NOT}.c-forget_{NOT}* **by** *blast*

lemma *learn-or-forget-dpll- μ_C* :

assumes

l-f: *learn-or-forget*** $S \ T$ **and**

dpll: *dpll-bj* $T \ U$ **and**

inv: *cdcl_{NOT}-NOT-all-inv* $A \ S$

shows $(2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$

$- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } U)$

$< (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$

$- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } S)$

(is $?_{\mu} U < ?_{\mu} S$)

proof –

have $?_{\mu} S = ?_{\mu} T$

using *l-f*

proof (*induction*)

case *base*

then show *?case* **by** *simp*

next

case (*step* $T \ U$)

moreover then have *no-dup* (*trail* T)

using *rtranclp-cdcl_{NOT}-no-dup*[of $S \ T$] *cdcl_{NOT}-NOT-all-inv-def* *inv*

rtranclp-learn-or-forget-cdcl_{NOT} **by** *auto*

ultimately show *?case*

using *forget- μ_C -stable* *learn- μ_C -stable* *inv* **unfolding** *cdcl_{NOT}-NOT-all-inv-def* **by** *presburger*

qed

moreover have *cdcl_{NOT}-NOT-all-inv* $A \ T$

using *rtranclp-learn-or-forget-cdcl_{NOT}* *cdcl_{NOT}-NOT-all-inv* *l-f* *inv* **by** *blast*

ultimately show *?thesis*

using *dpll-bj-trail-mes-decreasing-prop*[of $T \ U \ A$, *OF* *dpll*] *finite*

unfolding *cdcl_{NOT}-NOT-all-inv-def* **by** *linarith*

qed

lemma *infinite-cdcl_{NOT}-exists-learn-and-forget-infinite-chain*:

assumes

```

 $\bigwedge i. \text{cdcl}_{NOT} (f\ i) (f(\text{Suc}\ i))$  and
 $\text{inv}: \text{cdcl}_{NOT-NOT-all-inv}\ A\ (f\ 0)$ 
shows  $\exists j. \forall i \geq j. \text{learn-or-forget}\ (f\ i) (f\ (\text{Suc}\ i))$ 
using assms
proof (induction ( $2 + \text{card}\ (\text{atms-of-ms}\ A)) \wedge (1 + \text{card}\ (\text{atms-of-ms}\ A))$ 
   $-\ \mu_C\ (1 + \text{card}\ (\text{atms-of-ms}\ A))\ (2 + \text{card}\ (\text{atms-of-ms}\ A))\ (\text{trail-weight}\ (f\ 0))$ 
  arbitrary: f
  rule: nat-less-induct-case)
case (Suc n) note  $\text{IH} = \text{this}(1)$  and  $\mu = \text{this}(2)$  and  $\text{cdcl}_{NOT} = \text{this}(3)$  and  $\text{inv} = \text{this}(4)$ 
consider
  (dpll-end)  $\exists j. \forall i \geq j. \text{learn-or-forget}\ (f\ i) (f\ (\text{Suc}\ i))$ 
  | (dpll-more)  $\neg(\exists j. \forall i \geq j. \text{learn-or-forget}\ (f\ i) (f\ (\text{Suc}\ i)))$ 
by blast
then show ?case
proof cases
  case dpll-end
  then show ?thesis by auto
next
  case dpll-more
  then have  $j: \exists i. \neg \text{learn}\ (f\ i) (f\ (\text{Suc}\ i)) \wedge \neg \text{forget}_{NOT}\ (f\ i) (f\ (\text{Suc}\ i))$ 
  by blast
obtain i where
   $\neg \text{learn}\ (f\ i) (f\ (\text{Suc}\ i)) \wedge \neg \text{forget}_{NOT}\ (f\ i) (f\ (\text{Suc}\ i))$  and
   $\forall k < i. \text{learn-or-forget}\ (f\ k) (f\ (\text{Suc}\ k))$ 
proof  $-$ 
  obtain  $i_0$  where  $\neg \text{learn}\ (f\ i_0) (f\ (\text{Suc}\ i_0)) \wedge \neg \text{forget}_{NOT}\ (f\ i_0) (f\ (\text{Suc}\ i_0))$ 
  using j by auto
  then have  $\{i. i \leq i_0 \wedge \neg \text{learn}\ (f\ i) (f\ (\text{Suc}\ i)) \wedge \neg \text{forget}_{NOT}\ (f\ i) (f\ (\text{Suc}\ i))\} \neq \{\}$ 
  by auto
  let  $?I = \{i. i \leq i_0 \wedge \neg \text{learn}\ (f\ i) (f\ (\text{Suc}\ i)) \wedge \neg \text{forget}_{NOT}\ (f\ i) (f\ (\text{Suc}\ i))\}$ 
  let  $?i = \text{Min}\ ?I$ 
  have finite ?I
  by auto
  have  $\neg \text{learn}\ (f\ ?i) (f\ (\text{Suc}\ ?i)) \wedge \neg \text{forget}_{NOT}\ (f\ ?i) (f\ (\text{Suc}\ ?i))$ 
  using Min-in[OF (finite ?I) (?I ≠ {})] by auto
  moreover have  $\forall k < ?i. \text{learn-or-forget}\ (f\ k) (f\ (\text{Suc}\ k))$ 
  using Min.coboundedI[of  $\{i. i \leq i_0 \wedge \neg \text{learn}\ (f\ i) (f\ (\text{Suc}\ i)) \wedge \neg \text{forget}_{NOT}\ (f\ i) (f\ (\text{Suc}\ i))\}$ , simplified]
  by (meson  $(\neg \text{learn}\ (f\ i_0) (f\ (\text{Suc}\ i_0)) \wedge \neg \text{forget}_{NOT}\ (f\ i_0) (f\ (\text{Suc}\ i_0)))$  less-imp-le
    dual-order.trans not-le)
  ultimately show ?thesis using that by blast
qed
def  $g \equiv \lambda n. f\ (n + \text{Suc}\ i)$ 
have dpll-bj  $(f\ i) (g\ 0)$ 
  using  $(\neg \text{learn}\ (f\ i) (f\ (\text{Suc}\ i)) \wedge \neg \text{forget}_{NOT}\ (f\ i) (f\ (\text{Suc}\ i)))$  cdclNOT cdclNOT.cases
  g-def by auto
{
  fix j
  assume  $j \leq i$ 
  then have learn-or-forget**  $(f\ 0) (f\ j)$ 
  apply (induction j)
  apply simp
  by (metis (no-types, lifting) Suc-leD Suc-le-lessD rtrancIp.simps
     $(\forall k < i. \text{learn}\ (f\ k) (f\ (\text{Suc}\ k)) \vee \text{forget}_{NOT}\ (f\ k) (f\ (\text{Suc}\ k)))$ )
}

```



```

then have learn-or-forget** (f 0) (f i) by blast
then have (2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A))
  -  $\mu_C$  (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A)) (trail-weight (g 0))
  < (2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A))
  -  $\mu_C$  (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A)) (trail-weight (f 0))
using learn-or-forget-dpll- $\mu_C$ [of f 0 f i g 0 A] inv <dpll-bj (f i) (g 0)>
unfolding cdclNOT-NOT-all-inv-def by linarith

moreover have cdclNOT-i: cdclNOT** (f 0) (g 0)
using rtrancpl-learn-or-forget-cdclNOT[of f 0 f i] <learn-or-forget** (f 0) (f i)>
cdclNOT[of i] unfolding g-def by auto
moreover have  $\bigwedge i.$  cdclNOT (g i) (g (Suc i))
using cdclNOT g-def by auto
moreover have cdclNOT-NOT-all-inv A (g 0)
using inv cdclNOT-i rtrancpl-cdclNOT-trail-clauses-bound g-def cdclNOT-NOT-all-inv by auto
ultimately obtain j where j:  $\bigwedge i.$   $i \geq j \implies \text{learn-or-forget} (g i) (g (Suc i))$ 
using IH unfolding  $\mu$ [symmetric] by presburger
show ?thesis
proof
  {
    fix k
    assume  $k \geq j + \text{Suc } i$ 
    then have learn-or-forget (f k) (f (Suc k))
      using j[of k-Suc i] unfolding g-def by auto
  }
  then show  $\forall k \geq j + \text{Suc } i.$  learn-or-forget (f k) (f (Suc k))
    by auto
qed
qed
next
case 0 note H = this(1) and cdclNOT = this(2) and inv = this(3)
show ?case
proof (rule ccontr)
assume  $\neg ?case$ 
then have j:  $\exists i.$   $\neg \text{learn} (f i) (f (Suc i)) \wedge \neg \text{forget}_{NOT} (f i) (f (Suc i))$ 
by blast
obtain i where
   $\neg \text{learn} (f i) (f (Suc i)) \wedge \neg \text{forget}_{NOT} (f i) (f (Suc i))$  and
   $\forall k < i.$  learn-or-forget (f k) (f (Suc k))
proof -
  obtain i0 where  $\neg \text{learn} (f i_0) (f (Suc i_0)) \wedge \neg \text{forget}_{NOT} (f i_0) (f (Suc i_0))$ 
using j by auto
then have  $\{i. i \leq i_0 \wedge \neg \text{learn} (f i) (f (Suc i)) \wedge \neg \text{forget}_{NOT} (f i) (f (Suc i))\} \neq \{\}$ 
by auto
let ?I =  $\{i. i \leq i_0 \wedge \neg \text{learn} (f i) (f (Suc i)) \wedge \neg \text{forget}_{NOT} (f i) (f (Suc i))\}$ 
let ?i = Min ?I
have finite ?I
by auto
have  $\neg \text{learn} (f ?i) (f (Suc ?i)) \wedge \neg \text{forget}_{NOT} (f ?i) (f (Suc ?i))$ 
using Min-in[OF <finite ?I> <?I ≠ {}>] by auto
moreover have  $\forall k < ?i.$  learn-or-forget (f k) (f (Suc k))
using Min.coboundedI[of  $\{i. i \leq i_0 \wedge \neg \text{learn} (f i) (f (Suc i)) \wedge \neg \text{forget}_{NOT} (f i) (f (Suc i))\}$ , simplified]
by (meson  $\neg \text{learn} (f i_0) (f (Suc i_0)) \wedge \neg \text{forget}_{NOT} (f i_0) (f (Suc i_0))$ ) less-imp-le
dual-order.trans not-le

```

```

    ultimately show ?thesis using that by blast
  qed
have dpll-bj (f i) (f (Suc i))
  using (¬ learn (f i) (f (Suc i)) ∧ ¬ forgetNOT (f i) (f (Suc i))) cdclNOT cdclNOT.cases
  by blast
{
  fix j
  assume j ≤ i
  then have learn-or-forget** (f 0) (f j)
    apply (induction j)
    apply simp
    by (metis (no-types, lifting) Suc-leD Suc-le-lessD rtranclp.simps
      ⟨∀ k < i. learn (f k) (f (Suc k)) ∨ forgetNOT (f k) (f (Suc k))⟩)
}
then have learn-or-forget** (f 0) (f i) by blast

then show False
  using learn-or-forget-dpll-μC[of f 0 f i f (Suc i) A] inv 0
  ⟨dpll-bj (f i) (f (Suc i))⟩ unfolding cdclNOT-NOT-all-inv-def by linarith
qed
qed

lemma wf-cdclNOT-no-learn-and-forget-infinite-chain:
  assumes
    no-infinite-lf: ∧ f j. ¬ (∀ i ≥ j. learn-or-forget (f i) (f (Suc i)))
  shows wf {(T, S). cdclNOT S T ∧ cdclNOT-NOT-all-inv A S} (is wf {(T, S). cdclNOT S T
    ∧ ?inv S})
  unfolding wf-iff-no-infinite-down-chain
proof (rule ccontr)
  assume ¬ ¬ (∃ f. ∀ i. (f (Suc i), f i) ∈ {(T, S). cdclNOT S T ∧ ?inv S})
  then obtain f where
    ∀ i. cdclNOT (f i) (f (Suc i)) ∧ ?inv (f i)
  by fast
  then have ∃ j. ∀ i ≥ j. learn-or-forget (f i) (f (Suc i))
    using infinite-cdclNOT-exists-learn-and-forget-infinite-chain[of f] by meson
  then show False using no-infinite-lf by blast
qed

lemma inv-and-tranclp-cdclNOT-tranclp-cdclNOT-and-inv:
  cdclNOT++ S T ∧ cdclNOT-NOT-all-inv A S ⟷ (λ S T. cdclNOT S T ∧ cdclNOT-NOT-all-inv A
  S)++ S T
  (is ?A ∧ ?I ⟷ ?B)
proof
  assume ?A ∧ ?I
  then have ?A and ?I by blast+
  then show ?B
    apply induction
    apply (simp add: tranclp.r-into-trancl)
    by (metis (no-types, lifting) cdclNOT-NOT-all-inv tranclp.simps tranclp-into-rtranclp)
next
  assume ?B
  then have ?A by induction auto
  moreover have ?I using ⟨?B⟩ tranclpD by fastforce
  ultimately show ?A ∧ ?I by blast
qed

```

lemma *wf-tranclp-cdcl_{NOT}-no-learn-and-forget-infinite-chain*:

assumes

no-infinite-lf: $\bigwedge f j. \neg (\forall i \geq j. \text{learn-or-forget } (f i) (f (Suc i)))$

shows *wf* $\{(T, S). \text{cdcl}_{NOT}^{++} S T \wedge \text{cdcl}_{NOT}\text{-NOT-all-inv } A S\}$

using *wf-tranclp*[*OF wf-cdcl_{NOT}-no-learn-and-forget-infinite-chain*[*OF no-infinite-lf*]]

apply (rule *wf-subset*)

by (auto simp: *trancl-set-tranclp inv-and-tranclp-cdcl_{NOT}-tranclp-cdcl_{NOT}-and-inv*)

lemma *cdcl_{NOT}-final-state*:

assumes

n-s: *no-step cdcl_{NOT} S* **and**

inv: *cdcl_{NOT}-NOT-all-inv A S* **and**

decomp: *all-decomposition-implies-m (clauses_{NOT} S) (get-all-marked-decomposition (trail S))*

shows *unsatisfiable (set-mset (clauses_{NOT} S))*

$\vee (\text{trail } S \models_{asm} \text{clauses}_{NOT} S \wedge \text{satisfiable } (\text{set-mset } (\text{clauses}_{NOT} S)))$

proof –

have *n-s'*: *no-step dpll-bj S*

using *n-s* **by** (auto simp: *cdcl_{NOT}.simps*)

show *?thesis*

apply (rule *dpll-backjump-final-state*[of *S A*])

using *inv decomp n-s'* **unfolding** *cdcl_{NOT}-NOT-all-inv-def* **by** auto

qed

lemma *full-cdcl_{NOT}-final-state*:

assumes

full: *full cdcl_{NOT} S T* **and**

inv: *cdcl_{NOT}-NOT-all-inv A S* **and**

n-d: *no-dup (trail S)* **and**

decomp: *all-decomposition-implies-m (clauses_{NOT} S) (get-all-marked-decomposition (trail S))*

shows *unsatisfiable (set-mset (clauses_{NOT} T))*

$\vee (\text{trail } T \models_{asm} \text{clauses}_{NOT} T \wedge \text{satisfiable } (\text{set-mset } (\text{clauses}_{NOT} T)))$

proof –

have *st*: *cdcl_{NOT}** S T* **and** *n-s*: *no-step cdcl_{NOT} T*

using *full* **unfolding** *full-def* **by** *blast+*

have *n-s'*: *cdcl_{NOT}-NOT-all-inv A T*

using *cdcl_{NOT}-NOT-all-inv inv st* **by** *blast*

moreover have *all-decomposition-implies-m (clauses_{NOT} T) (get-all-marked-decomposition (trail T))*

using *cdcl_{NOT}-NOT-all-inv-def decomp inv rtranclp-cdcl_{NOT}-all-decomposition-implies st* **by** auto

ultimately show *?thesis*

using *cdcl_{NOT}-final-state n-s* **by** *blast*

qed

end — end of *conflict-driven-clause-learning*

15.6 Termination

15.6.1 Restricting learn and forget

locale *conflict-driven-clause-learning-learning-before-backjump-only-distinct-learnt* =

dpll-state mset-cls insert-cls remove-lit

mset-clss union-clss in-clss insert-clss remove-from-clss

trail raw-clauses prepend-trail tl-trail add-cl_{NOT} remove-cl_{NOT} +

conflict-driven-clause-learning mset-cls insert-cls remove-lit

mset-clss union-clss in-clss insert-clss remove-from-clss

trail raw-clauses prepend-trail tl-trail add-cl_{NOT} remove-cl_{NOT}

inv $backjump$ -conds $propagate$ -conds
 $\lambda C S. distinct\text{-}mset (mset\text{-}cls C) \wedge \neg tautology (mset\text{-}cls C) \wedge learn\text{-}restrictions C S \wedge$
 $(\exists F K d F' C' L. trail S = F' @ Marked K () \# F \wedge mset\text{-}cls C = C' + \{\#L\# \} \wedge F \models_{as} CNot$
 C'
 $\wedge C' + \{\#L\# \} \notin \# clauses_{NOT} S)$
 $\lambda C S. \neg(\exists F' F K d L. trail S = F' @ Marked K () \# F \wedge F \models_{as} CNot (remove1\text{-}mset L (mset\text{-}cls$
 $C)))$
 $\wedge forget\text{-}restrictions C S$
for
 $mset\text{-}cls :: 'cls \Rightarrow 'v clause$ **and**
 $insert\text{-}cls :: 'v literal \Rightarrow 'cls \Rightarrow 'cls$ **and**
 $remove\text{-}lit :: 'v literal \Rightarrow 'cls \Rightarrow 'cls$ **and**
 $mset\text{-}clss :: 'clss \Rightarrow 'v clauses$ **and**
 $union\text{-}clss :: 'clss \Rightarrow 'clss \Rightarrow 'clss$ **and**
 $in\text{-}clss :: 'cls \Rightarrow 'clss \Rightarrow bool$ **and**
 $insert\text{-}clss :: 'cls \Rightarrow 'clss \Rightarrow 'clss$ **and**
 $remove\text{-}from\text{-}clss :: 'cls \Rightarrow 'clss \Rightarrow 'clss$ **and**
 $trail :: 'st \Rightarrow ('v, unit, unit) marked\text{-}lits$ **and**
 $raw\text{-}clauses :: 'st \Rightarrow 'clss$ **and**
 $prepend\text{-}trail :: ('v, unit, unit) marked\text{-}lit \Rightarrow 'st \Rightarrow 'st$ **and**
 $tl\text{-}trail :: 'st \Rightarrow 'st$ **and**
 $add\text{-}cls_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st$ **and**
 $remove\text{-}cls_{NOT} :: 'cls \Rightarrow 'st \Rightarrow 'st$ **and**
 $inv :: 'st \Rightarrow bool$ **and**
 $backjump\text{-}conds :: 'v clause \Rightarrow 'v clause \Rightarrow 'v literal \Rightarrow 'st \Rightarrow 'st \Rightarrow bool$ **and**
 $propagate\text{-}conds :: ('v, unit, unit) marked\text{-}lit \Rightarrow 'st \Rightarrow bool$ **and**
 $learn\text{-}restrictions forget\text{-}restrictions :: 'cls \Rightarrow 'st \Rightarrow bool$

begin

lemma $cdcl_{NOT}\text{-}learn\text{-}all\text{-}induct[consumes 1, case\text{-}names dp\text{-}ll\text{-}bj learn forget_{NOT}]$:

fixes $S T :: 'st$

assumes $cdcl_{NOT} S T$ **and**

$dp\text{-}ll: \bigwedge T. dp\text{-}ll\text{-}bj S T \implies P S T$ **and**

learning:

$\bigwedge C F K F' C' L T. clauses_{NOT} S \models_{pm} mset\text{-}cls C \implies$
 $atms\text{-}of (mset\text{-}cls C) \subseteq atms\text{-}of\text{-}mm (clauses_{NOT} S) \cup atm\text{-}of (lits\text{-}of\text{-}l (trail S)) \implies$
 $distinct\text{-}mset (mset\text{-}cls C) \implies$
 $\neg tautology (mset\text{-}cls C) \implies$
 $learn\text{-}restrictions C S \implies$
 $trail S = F' @ Marked K () \# F \implies$
 $mset\text{-}cls C = C' + \{\#L\# \} \implies$
 $F \models_{as} CNot C' \implies$
 $C' + \{\#L\# \} \notin \# clauses_{NOT} S \implies$
 $T \sim add\text{-}cls_{NOT} C S \implies$
 $P S T$ **and**

forgetting: $\bigwedge C T. removeAll\text{-}mset (mset\text{-}cls C) (clauses_{NOT} S) \models_{pm} mset\text{-}cls C \implies$

$C !\in! raw\text{-}clauses S \implies$

$\neg(\exists F' F K L. trail S = F' @ Marked K () \# F \wedge F \models_{as} CNot (mset\text{-}cls C - \{\#L\# \})) \implies$

$T \sim remove\text{-}cls_{NOT} C S \implies$

forget-restrictions $C S \implies$

$P S T$

shows $P S T$

using $assms(1)$

apply (*induction rule*: $cdcl_{NOT}.induct$)

apply (*auto dest*: $assms(2)$ *simp add*: $learn\text{-}ops\text{-}axioms$)[]

apply (*auto elim!*: *learn-ops.learn.cases*[*OF learn-ops-axioms*] *dest*: *assms*(3))[]
apply (*auto elim!*: *forget-ops.forget_{NOT}.cases*[*OF forget-ops-axioms*] *dest!*: *assms*(4))
done

lemma *rtrancpl-cdcl_{NOT}-inv*:

*cdcl_{NOT}** S T \implies inv S \implies inv T*
apply (*induction rule*: *rtrancpl-induct*)
apply *simp*
using *cdcl_{NOT}-inv unfolding conflict-driven-clause-learning-def*
conflict-driven-clause-learning-axioms-def **by** *blast*

lemma *learn-always-simple-clauses*:

assumes
learn: *learn S T* **and**
n-d: *no-dup (trail S)*
shows *set-mset (clauses_{NOT} T - clauses_{NOT} S)*
 \subseteq *simple-clss (atms-of-mm (clauses_{NOT} S) \cup atm-of 'lits-of-l (trail S))*

proof

fix *C* **assume** *C*: *C \in set-mset (clauses_{NOT} T - clauses_{NOT} S)*
have *distinct-mset C \neg tautology C* **using** *learn C n-d* **by** (*elim learn_{NOT}E*; *auto*) +
then have *C \in simple-clss (atms-of C)*
using *distinct-mset-not-tautology-implies-in-simple-clss* **by** *blast*
moreover have *atms-of C \subseteq atms-of-mm (clauses_{NOT} S) \cup atm-of 'lits-of-l (trail S)*
using *learn C n-d* **by** (*elim learn_{NOT}E*) (*auto simp*: *atms-of-ms-def atms-of-def image-Un*
true-annots-CNot-all-atms-defined)
moreover have *finite (atms-of-mm (clauses_{NOT} S) \cup atm-of 'lits-of-l (trail S))*
by *auto*
ultimately show *C \in simple-clss (atms-of-mm (clauses_{NOT} S) \cup atm-of 'lits-of-l (trail S))*
using *simple-clss-mono* **by** (*metis (no-types) insert-subset mk-disjoint-insert*)

qed

definition *conflicting-bj-clss S \equiv*

$\{C + \{\#L\# \} \mid C \text{ L. } C + \{\#L\# \} \in \# \text{ clauses}_{NOT} S \wedge \text{distinct-mset } (C + \{\#L\# \})$
 $\wedge \neg \text{tautology } (C + \{\#L\# \})$
 $\wedge (\exists F' K F. \text{trail } S = F' @ \text{Marked } K () \# F \wedge F \models_{as} C \text{Not } C)\}$

lemma *conflicting-bj-clss-remove-cl_{NOT}[simp]*:

conflicting-bj-clss (remove-cl_{NOT} C S) = conflicting-bj-clss S - {mset-cl_s C}
unfolding *conflicting-bj-clss-def* **by** *fastforce*

lemma *conflicting-bj-clss-remove-cl_{NOT}'[simp]*:

T \sim remove-cl_{NOT} C S \implies conflicting-bj-clss T = conflicting-bj-clss S - {mset-cl_s C}
unfolding *conflicting-bj-clss-def* **by** *fastforce*

lemma *conflicting-bj-clss-add-cl_{NOT}-state-eq*:

T \sim add-cl_{NOT} C' S \implies no-dup (trail S) \implies conflicting-bj-clss T
 $=$ *conflicting-bj-clss S*
 \cup (*if* $\exists C \text{ L. mset-cl } C' = C + \{\#L\# \} \wedge \text{distinct-mset } (C + \{\#L\# \}) \wedge \neg \text{tautology } (C + \{\#L\# \})$
 $\wedge (\exists F' K d F. \text{trail } S = F' @ \text{Marked } K () \# F \wedge F \models_{as} C \text{Not } C)$
then {mset-cl_s C'} else {})
unfolding *conflicting-bj-clss-def* **by** *auto metis+*

lemma *conflicting-bj-clss-add-cl_{NOT}*:

no-dup (trail S) \implies
conflicting-bj-clss (add-cl_{NOT} C' S)

$= \text{conflicting-bj-clss } S$
 $\cup (if \exists C L. \text{mset-cls } C' = C + \{\#L\} \wedge \text{distinct-mset } (C + \{\#L\}) \wedge \neg \text{tautology } (C + \{\#L\})$
 $\wedge (\exists F' K d F. \text{trail } S = F' @ \text{Marked } K () \# F \wedge F \models_{as} CNot C)$
 $\text{then } \{\text{mset-cls } C'\} \text{ else } \{\})$
using *conflicting-bj-clss-add-clss-_{NOT}-state-eq* **by** *auto*

lemma *conflicting-bj-clss-incl-clauses*:
 $\text{conflicting-bj-clss } S \subseteq \text{set-mset } (\text{clauses}_{NOT} S)$
unfolding *conflicting-bj-clss-def* **by** *auto*

lemma *finite-conflicting-bj-clss[simp]*:
 $\text{finite } (\text{conflicting-bj-clss } S)$
using *conflicting-bj-clss-incl-clauses[of S]* *rev-finite-subset* **by** *blast*

lemma *learn-conflicting-increasing*:
 $\text{no-dup } (\text{trail } S) \implies \text{learn } S T \implies \text{conflicting-bj-clss } S \subseteq \text{conflicting-bj-clss } T$
apply (*elim learn_{NOT}E*)
by (*subst conflicting-bj-clss-add-clss-_{NOT}-state-eq[of T]*) *auto*

abbreviation *conflicting-bj-clss-yet b S* \equiv
 $3 \wedge b - \text{card } (\text{conflicting-bj-clss } S)$

abbreviation $\mu_L :: \text{nat} \Rightarrow 'st \Rightarrow \text{nat} \times \text{nat}$ **where**
 $\mu_L b S \equiv (\text{conflicting-bj-clss-yet } b S, \text{card } (\text{set-mset } (\text{clauses}_{NOT} S)))$

lemma *do-not-forget-before-backtrack-rule-clause-learned-clause-untouched*:
assumes *forget_{NOT} S T*
shows $\text{conflicting-bj-clss } S = \text{conflicting-bj-clss } T$
using *assms* **apply** (*elim forget_{NOT}E*)
apply *auto*
unfolding *conflicting-bj-clss-def*
apply *clarify*
using *diff-union-cancelR* **by** (*metis diff-union-cancelR*)

lemma *size-mset-removeAll-mset-le-iff*:
 $\text{size } (\text{removeAll-mset } x M) < \text{size } M \longleftrightarrow x \in \# M$
apply (*rule iffI*)
apply (*force intro: count-inI*)
apply (*rule mset-less-size*)
apply (*auto dest: simp: subset-mset-def multiset-eq-iff*)
done

lemma *forget- μ_L -decrease*:
assumes *forget_{NOT}: forget_{NOT} S T*
shows $(\mu_L b T, \mu_L b S) \in \text{less-than} <*\text{lex}*> \text{less-than}$
proof –
have $\text{card } (\text{set-mset } (\text{clauses}_{NOT} S)) > 0$
using *forget_{NOT}* **by** (*elim forget_{NOT}E*) (*auto simp: size-mset-removeAll-mset-le-iff card-gt-0-iff*)
then have $\text{card } (\text{set-mset } (\text{clauses}_{NOT} T)) < \text{card } (\text{set-mset } (\text{clauses}_{NOT} S))$
using *forget_{NOT}* **by** (*elim forget_{NOT}E*) (*auto simp: size-mset-removeAll-mset-le-iff*)
then show *?thesis*
unfolding *do-not-forget-before-backtrack-rule-clause-learned-clause-untouched[OF forget_{NOT}]*
by *auto*
qed

lemma *set-condition-or-split*:

$\{a. (a = b \vee Q a) \wedge S a\} = (\text{if } S b \text{ then } \{b\} \text{ else } \{\}) \cup \{a. Q a \wedge S a\}$
by *auto*

lemma *set-insert-neg*:

$A \neq \text{insert } a A \longleftrightarrow a \notin A$
by *auto*

lemma *learn- μ_L -decrease*:

assumes *learnST*: *learn* $S T$ **and** *n-d*: *no-dup* (*trail* S) **and**
 A : *atms-of-mm* (*clauses*_{NOT} S) \cup *atm-of* ‘*lits-of-l* (*trail* S)’ $\subseteq A$ **and**
fin-A: *finite* A
shows $(\mu_L (\text{card } A) T, \mu_L (\text{card } A) S) \in \text{less-than} <*\text{lex*}> \text{less-than}$

proof –

have [*simp*]: $(\text{atms-of-mm } (\text{clauses}_{NOT} T) \cup \text{atm-of } ‘\text{lits-of-l } (\text{trail } T))$
 $= (\text{atms-of-mm } (\text{clauses}_{NOT} S) \cup \text{atm-of } ‘\text{lits-of-l } (\text{trail } S))$
using *learnST* *n-d* **by** (*elim learn*_{NOT} E) *auto*

then have $\text{card } (\text{atms-of-mm } (\text{clauses}_{NOT} T) \cup \text{atm-of } ‘\text{lits-of-l } (\text{trail } T))$
 $= \text{card } (\text{atms-of-mm } (\text{clauses}_{NOT} S) \cup \text{atm-of } ‘\text{lits-of-l } (\text{trail } S))$
by (*auto intro!*: *card-mono*)

then have $3: (3::\text{nat}) \wedge \text{card } (\text{atms-of-mm } (\text{clauses}_{NOT} T) \cup \text{atm-of } ‘\text{lits-of-l } (\text{trail } T))$
 $= 3 \wedge \text{card } (\text{atms-of-mm } (\text{clauses}_{NOT} S) \cup \text{atm-of } ‘\text{lits-of-l } (\text{trail } S))$
by (*auto intro*: *power-mono*)

moreover have *conflicting-bj-clss* $S \subseteq \text{conflicting-bj-clss } T$
using *learnST* *n-d* **by** (*simp add*: *learn-conflicting-increasing*)

moreover have *conflicting-bj-clss* $S \neq \text{conflicting-bj-clss } T$
using *learnST*

proof (*elim learn*_{NOT} E , *goal-cases*)

case ($1 C$) **note** *cls-S* = *this*(1) **and** *atms-C* = *this*(2) **and** *inv* = *this*(3) **and** $T = \text{this}(4)$

then obtain $F K F' C' L$ **where**

tr-S: *trail* $S = F' @ \text{Marked } K () \# F$ **and**

C : *mset-cls* $C = C' + \{\#L\# \}$ **and**

F : $F \models_{as} C \text{Not } C'$ **and**

$C-S$: $C' + \{\#L\# \} \notin \text{clauses}_{NOT} S$

by *blast*

moreover have *distinct-mset* (*mset-cls* C) \neg *tautology* (*mset-cls* C) **using** *inv* **by** *blast+*

ultimately have $C' + \{\#L\# \} \in \text{conflicting-bj-clss } T$

using T *n-d* **unfolding** *conflicting-bj-clss-def* **by** *fastforce*

moreover have $C' + \{\#L\# \} \notin \text{conflicting-bj-clss } S$

using $C-S$ **unfolding** *conflicting-bj-clss-def* **by** *auto*

ultimately show *?case* **by** *blast*

qed

moreover have *fin-T*: *finite* (*conflicting-bj-clss* T)

using *learnST* **by** *induction* (*auto simp add*: *conflicting-bj-clss-add-clss*_{NOT})

ultimately have $\text{card } (\text{conflicting-bj-clss } T) \geq \text{card } (\text{conflicting-bj-clss } S)$

using *card-mono* **by** *blast*

moreover

have *fin'*: *finite* (*atms-of-mm* (*clauses*_{NOT} T) \cup *atm-of* ‘*lits-of-l* (*trail* T)’)
by *auto*

have 1 : *atms-of-mm* (*conflicting-bj-clss* T) \subseteq *atms-of-mm* (*clauses*_{NOT} T)
unfolding *conflicting-bj-clss-def* *atms-of-mm-def* **by** *auto*

have 2 : $\bigwedge x. x \in \text{conflicting-bj-clss } T \implies \neg \text{tautology } x \wedge \text{distinct-mset } x$
unfolding *conflicting-bj-clss-def* **by** *auto*

```

have T: conflicting-bj-clss T
  ⊆ simple-clss (atms-of-mm (clausesNOT T) ∪ atm-of ' lits-of-l (trail T))
  by standard (meson 1 2 fin' ⟨finite (conflicting-bj-clss T)⟩ simple-clss-mono
    distinct-mset-set-def simplified-in-simple-clss subsetCE sup.coboundedI1)
moreover
  then have #: 3 ^ card (atms-of-mm (clausesNOT T) ∪ atm-of ' lits-of-l (trail T))
    ≥ card (conflicting-bj-clss T)
    by (meson Nat.le-trans simple-clss-card simple-clss-finite card-mono fin')
  have atms-of-mm (clausesNOT T) ∪ atm-of ' lits-of-l (trail T) ⊆ A
    using learnNOTE[OF learnST] A by simp
  then have 3 ^ (card A) ≥ card (conflicting-bj-clss T)
    using # fin-A by (meson simple-clss-card simple-clss-finite
      simple-clss-mono calculation(2) card-mono dual-order.trans)
ultimately show ?thesis
  using psubset-card-mono[OF fin-T ]
  unfolding less-than-iff lex-prod-def by clarify
  (meson ⟨conflicting-bj-clss S ≠ conflicting-bj-clss T⟩
    ⟨conflicting-bj-clss S ⊆ conflicting-bj-clss T⟩
    diff-less-mono2 le-less-trans not-le psubsetI)
qed

```

We have to assume the following:

- *inv S*: the invariant holds in the initial state.
- *A* is a (finite *finite A*) superset of the literals in the trail *atm-of ' lits-of-l (trail S) ⊆ atms-of-ms A* and in the clauses *atms-of-mm (clauses_{NOT} S) ⊆ atms-of-ms A*. This can be the set of all the literals in the starting set of clauses.
- *no-dup (trail S)*: no duplicate in the trail. This is invariant along the path.

definition μ_{CDCL} where

$$\mu_{CDCL} A T \equiv ((2 + \text{card}(\text{atms-of-ms } A)) \wedge (1 + \text{card}(\text{atms-of-ms } A)) - \mu_C (1 + \text{card}(\text{atms-of-ms } A)) (2 + \text{card}(\text{atms-of-ms } A)) (\text{trail-weight } T), \text{conflicting-bj-clss-yet}(\text{card}(\text{atms-of-ms } A)) T, \text{card}(\text{set-mset}(\text{clauses}_{NOT} T)))$$

lemma *cdcl_{NOT}-decreasing-measure*:

```

assumes
  cdclNOT S T and
  inv: inv S and
  atm-clss: atms-of-mm (clausesNOT S) ⊆ atms-of-ms A and
  atm-lits: atm-of ' lits-of-l (trail S) ⊆ atms-of-ms A and
  n-d: no-dup (trail S) and
  fin-A: finite A
shows (μCDCL A T, μCDCL A S)
  ∈ less-than <*lex*> (less-than <*lex*> less-than)
using assms(1)
proof induction
  case (c-dpll-bj T)
  from dpll-bj-trail-mes-decreasing-prop[OF this(1) inv atm-clss atm-lits n-d fin-A]
  show ?case unfolding μCDCL-def
    by (meson in-lex-prod less-than-iff)
next
  case (c-learn T) note learn = this(1)
  then have S: trail S = trail T
    using inv atm-clss atm-lits n-d fin-A

```



```

  by (elim learnNOTE) auto
show ?case
  using learn- $\mu_L$ -decrease[OF learn n-d, of atms-of-ms A] atm-clss atm-lits fin-A n-d
  unfolding S  $\mu_{CDCL}$ -def by auto
next
case (c-forgetNOT T) note forgetNOT = this(1)
have trail S = trail T using forgetNOT by induction auto
then show ?case
  using forget- $\mu_L$ -decrease[OF forgetNOT] unfolding  $\mu_{CDCL}$ -def by auto
qed

```

lemma wf-cdcl_{NOT}-restricted-learning:

```

  assumes finite A
  shows wf {(T, S).
    (atms-of-mm (clausesNOT S)  $\subseteq$  atms-of-ms A  $\wedge$  atm-of ' lits-of-l (trail S)  $\subseteq$  atms-of-ms A
     $\wedge$  no-dup (trail S)
     $\wedge$  inv S)
     $\wedge$  cdclNOT S T }
  by (rule wf-wf-if-measure'[of less-than <*lex*> (less-than <*lex*> less-than)])
    (auto intro: cdclNOT-decreasing-measure[OF - - - - assms])

```

definition $\mu_C' :: 'v$ literal multiset set \Rightarrow 'st \Rightarrow nat **where**

$\mu_C' A T \equiv \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } T)$

definition $\mu_{CDCL}' :: 'v$ literal multiset set \Rightarrow 'st \Rightarrow nat **where**

```

 $\mu_{CDCL}' A T \equiv$ 
  ((2 + card (atms-of-ms A))  $\wedge$  (1 + card (atms-of-ms A)) -  $\mu_C' A T$ ) * (1 + 3card (atms-of-ms A)) *
  2
  + conflicting-bj-clss-yet (card (atms-of-ms A)) T * 2
  + card (set-mset (clausesNOT T))

```

lemma cdcl_{NOT}-decreasing-measure':

```

  assumes
    cdclNOT S T and
    inv: inv S and
    atms-clss: atms-of-mm (clausesNOT S)  $\subseteq$  atms-of-ms A and
    atms-trail: atm-of ' lits-of-l (trail S)  $\subseteq$  atms-of-ms A and
    n-d: no-dup (trail S) and
    fin-A: finite A
  shows  $\mu_{CDCL}' A T < \mu_{CDCL}' A S$ 
  using assms(1)

```

proof (induction rule: cdcl_{NOT}-learn-all-induct)

```

  case (dpll-bj T)
  then have (2 + card (atms-of-ms A))  $\wedge$  (1 + card (atms-of-ms A)) -  $\mu_C' A T$ 
    < (2 + card (atms-of-ms A))  $\wedge$  (1 + card (atms-of-ms A)) -  $\mu_C' A S$ 
    using dpll-bj-trail-mes-decreasing-prop fin-A inv n-d atms-clss atms-trail
    unfolding  $\mu_C'$ -def by blast
  then have XX: ((2 + card (atms-of-ms A))  $\wedge$  (1 + card (atms-of-ms A)) -  $\mu_C' A T$ ) + 1
     $\leq$  (2 + card (atms-of-ms A))  $\wedge$  (1 + card (atms-of-ms A)) -  $\mu_C' A S$ 
    by auto
  from mult-le-mono1[OF this, of (1 + 3card (atms-of-ms A))]
  have ((2 + card (atms-of-ms A))  $\wedge$  (1 + card (atms-of-ms A)) -  $\mu_C' A T$ ) *
    (1 + 3card (atms-of-ms A)) + (1 + 3card (atms-of-ms A))
     $\leq$  ((2 + card (atms-of-ms A))  $\wedge$  (1 + card (atms-of-ms A)) -  $\mu_C' A S$ )
    * (1 + 3card (atms-of-ms A))

```

```

unfolding Nat.add-mult-distrib
by presburger
moreover
  have cl-T-S: clausesNOT T = clausesNOT S
    using dpll-bj.hyps inv dpll-bj-clauses by auto
  have conflicting-bj-clss-yet (card (atms-of-ms A)) S < 1 + 3 ^ card (atms-of-ms A)
    by simp
ultimately have ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) -  $\mu_C'$  A T)
  * (1 + 3 ^ card (atms-of-ms A)) + conflicting-bj-clss-yet (card (atms-of-ms A)) T
  < ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) -  $\mu_C'$  A S) * (1 + 3 ^ card (atms-of-ms
A))
    by linarith
then have ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) -  $\mu_C'$  A T)
  * (1 + 3 ^ card (atms-of-ms A))
  + conflicting-bj-clss-yet (card (atms-of-ms A)) T
  < ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) -  $\mu_C'$  A S)
  * (1 + 3 ^ card (atms-of-ms A))
  + conflicting-bj-clss-yet (card (atms-of-ms A)) S
    by linarith
then have ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) -  $\mu_C'$  A T)
  * (1 + 3 ^ card (atms-of-ms A)) * 2
  + conflicting-bj-clss-yet (card (atms-of-ms A)) T * 2
  < ((2 + card (atms-of-ms A)) ^ (1 + card (atms-of-ms A)) -  $\mu_C'$  A S)
  * (1 + 3 ^ card (atms-of-ms A)) * 2
  + conflicting-bj-clss-yet (card (atms-of-ms A)) S * 2
    by linarith
then show ?case unfolding  $\mu_{CDCL}'$ -def cl-T-S by presburger
next
case (learn C F' K F C' L T) note clss-S-C = this(1) and atms-C = this(2) and dist = this(3)
and tauto = this(4) and learn-restr = this(5) and tr-S = this(6) and C' = this(7) and
F-C = this(8) and C-new = this(9) and T = this(10)
have insert (mset-cls C) (conflicting-bj-clss S)  $\subseteq$  simple-clss (atms-of-ms A)
proof –
  have mset-cls C  $\in$  simple-clss (atms-of-ms A)
    using C'
    by (metis (no-types, hide-lams) Un-subset-iff simple-clss-mono
      contra-subsetD dist distinct-mset-not-tautology-implies-in-simple-clss
      dual-order.trans atms-C atms-clss atms-trail tauto)
  moreover have conflicting-bj-clss S  $\subseteq$  simple-clss (atms-of-ms A)
    unfolding conflicting-bj-clss-def
    proof
      fix x :: 'v literal multiset
      assume x  $\in$  {C + {#L#} | C L. C + {#L#}  $\in$  # clausesNOT S
         $\wedge$  distinct-mset (C + {#L#})  $\wedge$   $\neg$  tautology (C + {#L#})
         $\wedge$  ( $\exists$  F' K F. trail S = F' @ Marked K () # F  $\wedge$  F  $\models_{as}$  CNot C)}
      then have  $\exists$  m l. x = m + {#l#}  $\wedge$  m + {#l#}  $\in$  # clausesNOT S
         $\wedge$  distinct-mset (m + {#l#})  $\wedge$   $\neg$  tautology (m + {#l#})
         $\wedge$  ( $\exists$  ms l msa. trail S = ms @ Marked l () # msa  $\wedge$  msa  $\models_{as}$  CNot m)
      by blast
      then show x  $\in$  simple-clss (atms-of-ms A)
        by (meson atms-clss atms-of-atms-of-ms-mono atms-of-ms-finite simple-clss-mono
          distinct-mset-not-tautology-implies-in-simple-clss fin-A finite-subset
          set-rev-mp)
    qed
ultimately show ?thesis

```

by auto
 qed
 then have card (insert (mset-cls C) (conflicting-bj-clss S)) $\leq 3 \wedge$ (card (atms-of-ms A))
 by (meson Nat.le-trans atms-of-ms-finite simple-clss-card simple-clss-finite
 card-mono fin-A)
 moreover have [simp]: card (insert (mset-cls C) (conflicting-bj-clss S))
 = Suc (card ((conflicting-bj-clss S)))
 by (metis (no-types) C' C-new card-insert-if conflicting-bj-clss-incl-clauses contra-subsetD
 finite-conflicting-bj-clss)
 moreover have [simp]: conflicting-bj-clss (add-cls_{NOT} C S) = conflicting-bj-clss S \cup {mset-cls C}
 using dist tauto F-C n-d by (subst conflicting-bj-clss-add-cls_{NOT})
 (force simp add: ac-simps C' tr-S)+
 ultimately have [simp]: conflicting-bj-clss-yet (card (atms-of-ms A)) S
 = Suc (conflicting-bj-clss-yet (card (atms-of-ms A)) (add-cls_{NOT} C S))
 by simp
 have 1: clauses_{NOT} T = clauses_{NOT} (add-cls_{NOT} C S) using T by auto
 have 2: conflicting-bj-clss-yet (card (atms-of-ms A)) T
 = conflicting-bj-clss-yet (card (atms-of-ms A)) (add-cls_{NOT} C S)
 using T unfolding conflicting-bj-clss-def by auto
 have 3: $\mu_{C'} A T = \mu_{C'} A$ (add-cls_{NOT} C S)
 using T unfolding $\mu_{C'}$ -def by auto
 have ((2 + card (atms-of-ms A)) \wedge (1 + card (atms-of-ms A)) - $\mu_{C'} A$ (add-cls_{NOT} C S))
 * (1 + 3 \wedge card (atms-of-ms A)) * 2
 = ((2 + card (atms-of-ms A)) \wedge (1 + card (atms-of-ms A)) - $\mu_{C'} A S$)
 * (1 + 3 \wedge card (atms-of-ms A)) * 2
 using n-d unfolding $\mu_{C'}$ -def by auto
 moreover
 have conflicting-bj-clss-yet (card (atms-of-ms A)) (add-cls_{NOT} C S)
 * 2
 + card (set-mset (clauses_{NOT} (add-cls_{NOT} C S)))
 < conflicting-bj-clss-yet (card (atms-of-ms A)) S * 2
 + card (set-mset (clauses_{NOT} S))
 by (simp add: C' C-new n-d)
 ultimately show ?case unfolding μ_{CDCL} '-def 1 2 3 by presburger
 next
 case (forget_{NOT} C T) note T = this(4)
 have [simp]: $\mu_{C'} A$ (remove-cls_{NOT} C S) = $\mu_{C'} A S$
 unfolding $\mu_{C'}$ -def by auto
 have forget_{NOT} S T
 apply (rule forget_{NOT}.intros) using forget_{NOT} by auto
 then have conflicting-bj-clss T = conflicting-bj-clss S
 using do-not-forget-before-backtrack-rule-clause-learned-clause-untouched by blast
 moreover have card (set-mset (clauses_{NOT} T)) < card (set-mset (clauses_{NOT} S))
 by (metis T card-Diff1-less clauses-remove-cls_{NOT} finite-set-mset forget_{NOT}.hyps(2)
 in-clss-mset-clss order-refl set-mset-minus-replicate-mset(1) state-eq_{NOT}-clauses)
 ultimately show ?case unfolding μ_{CDCL} '-def
 by (metis (no-types) T $\mu_{C'} A$ (remove-cls_{NOT} C S) = $\mu_{C'} A S$ add-le-cancel-left
 $\mu_{C'}$ -def not-le state-eq_{NOT}-trail)
 qed

lemma cdcl_{NOT}-clauses-bound:
 assumes
 cdcl_{NOT} S T and
 inv S and
 atms-of-mm (clauses_{NOT} S) \subseteq A and

$atm\text{-}of \text{ '}(lits\text{-}of\text{-}l \text{ (trail } S)) \subseteq A$ and
 $n\text{-}d\text{: } no\text{-}dup \text{ (trail } S)$ and
 $fin\text{-}A[simp]\text{: } finite \text{ } A$
shows $set\text{-}mset \text{ (clauses}_{NOT} \text{ } T) \subseteq set\text{-}mset \text{ (clauses}_{NOT} \text{ } S) \cup simple\text{-}clss \text{ } A$
using *assms*
proof (*induction rule: cdcl_{NOT}-learn-all-induct*)
case *dpll-bj*
then show *?case using dpll-bj-clauses by simp*
next
case *forget_{NOT}*
then show *?case using clauses-remove-cl_{NOT} unfolding state-eq_{NOT}-def by auto*
next
case (*learn C F K d F' C' L*) **note** $atms\text{-}C = this(2)$ and $dist = this(3)$ and $tauto = this(4)$ and
 $T = this(10)$ and $atms\text{-}clss\text{-}S = this(12)$ and $atms\text{-}trail\text{-}S = this(13)$
have $atms\text{-}of \text{ (mset-cl_s } C) \subseteq A$
using $atms\text{-}C \text{ } atms\text{-}clss\text{-}S \text{ } atms\text{-}trail\text{-}S$ **by** *fast*
then have $simple\text{-}clss \text{ (atms-of (mset-cl_s } C)) \subseteq simple\text{-}clss \text{ } A$
by (*simp add: simple-clss-mono*)
then have $mset\text{-}cls \text{ } C \in simple\text{-}clss \text{ } A$
using *finite dist tauto*
by (*auto dest: distinct-mset-not-tautology-implies-in-simple-clss*)
then show *?case using T n-d by auto*
qed

lemma *rtrancpl-cdcl_{NOT}-clauses-bound:*

assumes
 $cdcl_{NOT}^{**} \text{ } S \text{ } T$ and
 $inv \text{ } S$ and
 $atms\text{-}of\text{-}mm \text{ (clauses}_{NOT} \text{ } S) \subseteq A$ and
 $atm\text{-}of \text{ '}(lits\text{-}of\text{-}l \text{ (trail } S)) \subseteq A$ and
 $n\text{-}d\text{: } no\text{-}dup \text{ (trail } S)$ and
 $finite\text{: } finite \text{ } A$
shows $set\text{-}mset \text{ (clauses}_{NOT} \text{ } T) \subseteq set\text{-}mset \text{ (clauses}_{NOT} \text{ } S) \cup simple\text{-}clss \text{ } A$
using *assms(1-5)*
proof *induction*
case *base*
then show *?case by simp*
next
case (*step T U*) **note** $st = this(1)$ and $cdcl_{NOT} = this(2)$ and $IH = this(3)[OF \text{ } this(4-7)]$ and
 $inv = this(4)$ and $atms\text{-}clss\text{-}S = this(5)$ and $atms\text{-}trail\text{-}S = this(6)$ and $finite\text{-}cls\text{-}S = this(7)$
have $inv \text{ } T$
using *rtrancpl-cdcl_{NOT}-inv st inv by blast*
moreover have $atms\text{-}of\text{-}mm \text{ (clauses}_{NOT} \text{ } T) \subseteq A$ and $atm\text{-}of \text{ '}(lits\text{-}of\text{-}l \text{ (trail } T) \subseteq A$
using *rtrancpl-cdcl_{NOT}-trail-clauses-bound[OF st] inv atms-clss-S atms-trail-S n-d by auto*
moreover have $no\text{-}dup \text{ (trail } T)$
using *rtrancpl-cdcl_{NOT}-no-dup[OF st (inv S) n-d] by simp*
ultimately have $set\text{-}mset \text{ (clauses}_{NOT} \text{ } U) \subseteq set\text{-}mset \text{ (clauses}_{NOT} \text{ } T) \cup simple\text{-}clss \text{ } A$
using *cdcl_{NOT} finite n-d by (auto simp: cdcl_{NOT}-clauses-bound)*
then show *?case using IH by auto*
qed

lemma *rtrancpl-cdcl_{NOT}-card-clauses-bound:*

assumes
 $cdcl_{NOT}^{**} \text{ } S \text{ } T$ and
 $inv \text{ } S$ and

$atms\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \subseteq A$ and
 $atm\text{-}of\ ('(lits\text{-}of\text{-}l\ (trail\ S))) \subseteq A$ and
 $n\text{-}d\text{-}no\text{-}dup\ (trail\ S)$ and
 $finite\text{-}finite\ A$
shows $card\ (set\text{-}mset\ (clauses_{NOT}\ T)) \leq card\ (set\text{-}mset\ (clauses_{NOT}\ S)) + 3 \wedge (card\ A)$
using $rtranclp\text{-}cdcl_{NOT}\text{-}clauses\text{-}bound[OF\ assms]$ **finite** **by** $(meson\ Nat.le\text{-}trans\ simple\text{-}clss\text{-}card\ simple\text{-}clss\text{-}finite\ card\text{-}Un\text{-}le\ card\text{-}mono\ finite\text{-}UnI\ finite\text{-}set\text{-}mset\ nat\text{-}add\text{-}left\text{-}cancel\text{-}le)$

lemma $rtranclp\text{-}cdcl_{NOT}\text{-}card\text{-}clauses\text{-}bound'$:

assumes
 $cdcl_{NOT}^{**}\ S\ T$ and
 $inv\ S$ and
 $atms\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \subseteq A$ and
 $atm\text{-}of\ ('(lits\text{-}of\text{-}l\ (trail\ S))) \subseteq A$ and
 $n\text{-}d\text{-}no\text{-}dup\ (trail\ S)$ and
 $finite\text{-}finite\ A$
shows $card\ \{C \mid C \in \# clauses_{NOT}\ T \wedge (tautology\ C \vee \neg distinct\text{-}mset\ C)\}$
 $\leq card\ \{C \mid C \in \# clauses_{NOT}\ S \wedge (tautology\ C \vee \neg distinct\text{-}mset\ C)\} + 3 \wedge (card\ A)$
 $(is\ card\ ?T \leq card\ ?S + -)$
using $rtranclp\text{-}cdcl_{NOT}\text{-}clauses\text{-}bound[OF\ assms]$ **finite**
proof –
have $?T \subseteq ?S \cup simple\text{-}clss\ A$
using $rtranclp\text{-}cdcl_{NOT}\text{-}clauses\text{-}bound[OF\ assms]$ **by** *force*
then have $card\ ?T \leq card\ (?S \cup simple\text{-}clss\ A)$
using $finite$ **by** $(simp\ add\text{-}assms(5)\ simple\text{-}clss\text{-}finite\ card\text{-}mono)$
then show $?thesis$
by $(meson\ le\text{-}trans\ simple\text{-}clss\text{-}card\ card\text{-}Un\text{-}le\ local.finite\ nat\text{-}add\text{-}left\text{-}cancel\text{-}le)$
qed

lemma $rtranclp\text{-}cdcl_{NOT}\text{-}card\text{-}simple\text{-}clauses\text{-}bound$:

assumes
 $cdcl_{NOT}^{**}\ S\ T$ and
 $inv\ S$ and
 $atms\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \subseteq A$ and
 $atm\text{-}of\ ('(lits\text{-}of\text{-}l\ (trail\ S))) \subseteq A$ and
 $n\text{-}d\text{-}no\text{-}dup\ (trail\ S)$ and
 $finite\text{-}finite\ A$
shows $card\ (set\text{-}mset\ (clauses_{NOT}\ T))$
 $\leq card\ \{C. C \in \# clauses_{NOT}\ S \wedge (tautology\ C \vee \neg distinct\text{-}mset\ C)\} + 3 \wedge (card\ A)$
 $(is\ card\ ?T \leq card\ ?S + -)$
using $rtranclp\text{-}cdcl_{NOT}\text{-}clauses\text{-}bound[OF\ assms]$ **finite**
proof –
have $\bigwedge x. x \in \# clauses_{NOT}\ T \implies \neg tautology\ x \implies distinct\text{-}mset\ x \implies x \in simple\text{-}clss\ A$
using $rtranclp\text{-}cdcl_{NOT}\text{-}clauses\text{-}bound[OF\ assms]$ **by** $(metis\ (no\text{-}types,\ hide\text{-}lams)\ Un\text{-}iff\ assms(3)\ atms\text{-}of\text{-}atms\text{-}of\text{-}ms\text{-}mono\ simple\text{-}clss\text{-}mono\ contra\text{-}subsetD\ distinct\text{-}mset\text{-}not\text{-}tautology\text{-}implies\text{-}in\text{-}simple\text{-}clss\ subset\text{-}trans)$
then have $set\text{-}mset\ (clauses_{NOT}\ T) \subseteq ?S \cup simple\text{-}clss\ A$
using $rtranclp\text{-}cdcl_{NOT}\text{-}clauses\text{-}bound[OF\ assms]$ **by** *auto*
then have $card(set\text{-}mset\ (clauses_{NOT}\ T)) \leq card\ (?S \cup simple\text{-}clss\ A)$
using $finite$ **by** $(simp\ add\text{-}assms(5)\ simple\text{-}clss\text{-}finite\ card\text{-}mono)$
then show $?thesis$
by $(meson\ le\text{-}trans\ simple\text{-}clss\text{-}card\ card\text{-}Un\text{-}le\ local.finite\ nat\text{-}add\text{-}left\text{-}cancel\text{-}le)$
qed

definition $\mu_{CDCL}'\text{-bound} :: 'v \text{ literal multiset set} \Rightarrow 'st \Rightarrow nat$ **where**

$\mu_{CDCL}'\text{-bound } A \ S =$
 $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))) * (1 + 3 \wedge \text{card } (\text{atms-of-ms } A)) * 2$
 $+ 2 * 3 \wedge (\text{card } (\text{atms-of-ms } A))$
 $+ \text{card } \{C. C \in \# \text{ clauses}_{NOT} S \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\} + 3 \wedge (\text{card } (\text{atms-of-ms } A))$

lemma $\mu_{CDCL}'\text{-bound-reduce-trail-to}_{NOT}[simp]:$

$\mu_{CDCL}'\text{-bound } A \ (\text{reduce-trail-to}_{NOT} M \ S) = \mu_{CDCL}'\text{-bound } A \ S$

unfolding $\mu_{CDCL}'\text{-bound-def}$ **by** *auto*

lemma $rtrancpl\text{-}cdcl_{NOT}\text{-}\mu_{CDCL}'\text{-bound-reduce-trail-to}_{NOT}$:

assumes

$cdcl_{NOT}^{**} S \ T$ **and**

$inv \ S$ **and**

$\text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A$ **and**

$\text{atm-of } '(\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-ms } A$ **and**

$n\text{-d: no-dup } (\text{trail } S)$ **and**

$\text{finite: finite } (\text{atms-of-ms } A)$ **and**

$U: U \sim \text{reduce-trail-to}_{NOT} M \ T$

shows $\mu_{CDCL}' A \ U \leq \mu_{CDCL}'\text{-bound } A \ S$

proof –

have $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A \ U)$

$\leq (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$

by *auto*

then have $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A \ U)$

$* (1 + 3 \wedge \text{card } (\text{atms-of-ms } A)) * 2$

$\leq (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) * (1 + 3 \wedge \text{card } (\text{atms-of-ms } A)) * 2$

using *mult-le-mono1* **by** *blast*

moreover

have $\text{conflicting-bj-clss-yet } (\text{card } (\text{atms-of-ms } A)) \ T * 2 \leq 2 * 3 \wedge \text{card } (\text{atms-of-ms } A)$

by *linarith*

moreover have $\text{card } (\text{set-mset } (\text{clauses}_{NOT} U))$

$\leq \text{card } \{C. C \in \# \text{ clauses}_{NOT} S \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\} + 3 \wedge \text{card } (\text{atms-of-ms } A)$

using $rtrancpl\text{-}cdcl_{NOT}\text{-card-simple-clauses-bound}[OF \text{ assms}(1-6)] \ U$ **by** *auto*

ultimately show *?thesis*

unfolding $\mu_{CDCL}'\text{-def}$ $\mu_{CDCL}'\text{-bound-def}$ **by** *linarith*

qed

lemma $rtrancpl\text{-}cdcl_{NOT}\text{-}\mu_{CDCL}'\text{-bound}$:

assumes

$cdcl_{NOT}^{**} S \ T$ **and**

$inv \ S$ **and**

$\text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A$ **and**

$\text{atm-of } '(\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-ms } A$ **and**

$n\text{-d: no-dup } (\text{trail } S)$ **and**

$\text{finite: finite } (\text{atms-of-ms } A)$

shows $\mu_{CDCL}' A \ T \leq \mu_{CDCL}'\text{-bound } A \ S$

proof –

have $\mu_{CDCL}' A \ (\text{reduce-trail-to}_{NOT} (\text{trail } T) \ T) = \mu_{CDCL}' A \ T$

unfolding $\mu_{CDCL}'\text{-def}$ $\mu_C'\text{-def}$ $\text{conflicting-bj-clss-def}$ **by** *auto*

then show *?thesis* **using** $rtrancpl\text{-}cdcl_{NOT}\text{-}\mu_{CDCL}'\text{-bound-reduce-trail-to}_{NOT}[OF \text{ assms, of - trail } T]$

state-eq_{NOT}\text{-ref} **by** *fastforce*

qed

lemma $rtrancpl\text{-}\mu_{CDCL}'\text{-bound-decreasing}$:

assumes
*cdcl_{NOT}** S T and*
inv S and
atms-of-mm (clauses_{NOT} S) \subseteq atms-of-ms A and
atm-of '(lits-of-l (trail S)) \subseteq atms-of-ms A and
n-d: no-dup (trail S) and
finite[simp]: finite (atms-of-ms A)
shows $\mu_{CDCL}'\text{-bound } A \ T \leq \mu_{CDCL}'\text{-bound } A \ S$
proof –
have $\{C. C \in \# \text{ clauses}_{NOT} \ T \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\}$
 $\subseteq \{C. C \in \# \text{ clauses}_{NOT} \ S \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\}$ (**is** $?T \subseteq ?S$)
proof (*rule Set.subsetI*)
fix C **assume** $C \in ?T$
then have $C\text{-}T: C \in \# \text{ clauses}_{NOT} \ T$ **and** $t\text{-}d: \text{tautology } C \vee \neg \text{distinct-mset } C$
by *auto*
then have $C \notin \text{simple-clss (atms-of-ms A)}$
by (*auto dest: simple-clssE*)
then show $C \in ?S$
using $C\text{-}T$ *rtrancp-cdcl_{NOT}-clauses-bound[OF assms]* $t\text{-}d$ **by** *force*
qed
then have $\text{card } \{C. C \in \# \text{ clauses}_{NOT} \ T \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\} \leq$
 $\text{card } \{C. C \in \# \text{ clauses}_{NOT} \ S \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\}$
by (*simp add: card-mono*)
then show $?thesis$
unfolding $\mu_{CDCL}'\text{-bound-def}$ **by** *auto*
qed
end — end of *conflict-driven-clause-learning-learning-before-backjump-only-distinct-learnt*

15.7 CDCL with restarts

15.7.1 Definition

locale *restart-ops* =
fixes
 $cdcl_{NOT} :: 'st \Rightarrow 'st \Rightarrow \text{bool}$ **and**
 $restart :: 'st \Rightarrow 'st \Rightarrow \text{bool}$
begin
inductive $cdcl_{NOT}\text{-raw-restart} :: 'st \Rightarrow 'st \Rightarrow \text{bool}$ **where**
 $cdcl_{NOT} \ S \ T \Longrightarrow cdcl_{NOT}\text{-raw-restart } S \ T \mid$
 $restart \ S \ T \Longrightarrow cdcl_{NOT}\text{-raw-restart } S \ T$
end

locale *conflict-driven-clause-learning-with-restarts* =
 $conflict\text{-driven-clause-learning } mset\text{-cls } insert\text{-cls } remove\text{-lit}$
 $mset\text{-clss } union\text{-clss } in\text{-clss } insert\text{-clss } remove\text{-from-clss}$
 $trail \ raw\text{-clauses } prepend\text{-trail } tl\text{-trail } add\text{-cls}_{NOT} \ remove\text{-cls}_{NOT}$
 $inv \ backjump\text{-conds } propagate\text{-conds } learn\text{-cond } forget\text{-cond}$
for
 $mset\text{-cls} :: 'cls \Rightarrow 'v \text{ clause}$ **and**
 $insert\text{-cls} :: 'v \text{ literal} \Rightarrow 'cls \Rightarrow 'cls$ **and**
 $remove\text{-lit} :: 'v \text{ literal} \Rightarrow 'cls \Rightarrow 'cls$ **and**
 $mset\text{-clss} :: 'clss \Rightarrow 'v \text{ clauses}$ **and**
 $union\text{-clss} :: 'clss \Rightarrow 'clss \Rightarrow 'clss$ **and**
 $in\text{-clss} :: 'cls \Rightarrow 'clss \Rightarrow \text{bool}$ **and**

```

insert-cls :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
raw-clauses :: 'st  $\Rightarrow$  'clss and
prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
tl-trail :: 'st  $\Rightarrow$  'st and
add-clsNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
remove-clsNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
inv :: 'st  $\Rightarrow$  bool and
backjump-conds :: 'v clause  $\Rightarrow$  'v clause  $\Rightarrow$  'v literal  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  bool and
propagate-conds :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  bool and
learn-cond forget-cond :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  bool
begin

```

lemma *cdcl_{NOT}-iff-cdcl_{NOT}-raw-restart-no-restarts:*

```

cdclNOT S T  $\longleftrightarrow$  restart-ops.cdclNOT-raw-restart cdclNOT ( $\lambda$ - . False) S T
(is ?C S T  $\longleftrightarrow$  ?R S T)

```

proof

```

fix S T
assume ?C S T
then show ?R S T by (simp add: restart-ops.cdclNOT-raw-restart.intros(1))
next
fix S T
assume ?R S T
then show ?C S T
  apply (cases rule: restart-ops.cdclNOT-raw-restart.cases)
  using (?R S T) by fast+
qed

```

lemma *cdcl_{NOT}-cdcl_{NOT}-raw-restart:*

```

cdclNOT S T  $\implies$  restart-ops.cdclNOT-raw-restart cdclNOT restart S T
by (simp add: restart-ops.cdclNOT-raw-restart.intros(1))
end

```

15.7.2 Increasing restarts

To add restarts we need some assumptions on the predicate (called *cdcl_{NOT}* here):

- a function f that is strictly monotonic. The first step is actually only used as a restart to clean the state (e.g. to ensure that the trail is empty). Then we assume that $(1::'a) \leq f\ n$ for $(1::'a) \leq n$: it means that between two consecutive restarts, at least one step will be done. This is necessary to avoid sequence. like: full – restart – full – ...
- a measure μ : it should decrease under the assumptions *bound-inv*, whenever a *cdcl_{NOT}* or a *restart* is done. A parameter is given to μ : for conflict- driven clause learning, it is an upper-bound of the clauses. We are assuming that such a bound can be found after a restart whenever the invariant holds.
- we also assume that the measure decrease after any *cdcl_{NOT}* step.
- an invariant on the states *cdcl_{NOT}-inv* that also holds after restarts.
- it is *not required* that the measure decrease with respect to restarts, but the measure has to be bound by some function μ -*bound* taking the same parameter as μ and the initial state of the considered *cdcl_{NOT}* chain.


```

locale cdclNOT-increasing-restarts-ops =
  restart-ops cdclNOT restart for
    restart :: 'st  $\Rightarrow$  'st  $\Rightarrow$  bool and
    cdclNOT :: 'st  $\Rightarrow$  'st  $\Rightarrow$  bool +
fixes
  f :: nat  $\Rightarrow$  nat and
  bound-inv :: 'bound  $\Rightarrow$  'st  $\Rightarrow$  bool and
   $\mu$  :: 'bound  $\Rightarrow$  'st  $\Rightarrow$  nat and
  cdclNOT-inv :: 'st  $\Rightarrow$  bool and
   $\mu$ -bound :: 'bound  $\Rightarrow$  'st  $\Rightarrow$  nat
assumes
  f: unbounded f and
  f-ge-1:  $\bigwedge n. n \geq 1 \Rightarrow f\ n \neq 0$  and
  bound-inv:  $\bigwedge A\ S\ T. \text{cdcl}_{NOT}\text{-inv}\ S \Rightarrow \text{bound-inv}\ A\ S \Rightarrow \text{cdcl}_{NOT}\ S\ T \Rightarrow \text{bound-inv}\ A\ T$  and
  cdclNOT-measure:  $\bigwedge A\ S\ T. \text{cdcl}_{NOT}\text{-inv}\ S \Rightarrow \text{bound-inv}\ A\ S \Rightarrow \text{cdcl}_{NOT}\ S\ T \Rightarrow \mu\ A\ T < \mu$ 
A S and
  measure-bound2:  $\bigwedge A\ T\ U. \text{cdcl}_{NOT}\text{-inv}\ T \Rightarrow \text{bound-inv}\ A\ T \Rightarrow \text{cdcl}_{NOT}^{**}\ T\ U$ 
 $\Rightarrow \mu\ A\ U \leq \mu\text{-bound}\ A\ T$  and
  measure-bound4:  $\bigwedge A\ T\ U. \text{cdcl}_{NOT}\text{-inv}\ T \Rightarrow \text{bound-inv}\ A\ T \Rightarrow \text{cdcl}_{NOT}^{**}\ T\ U$ 
 $\Rightarrow \mu\text{-bound}\ A\ U \leq \mu\text{-bound}\ A\ T$  and
  cdclNOT-restart-inv:  $\bigwedge A\ U\ V. \text{cdcl}_{NOT}\text{-inv}\ U \Rightarrow \text{restart}\ U\ V \Rightarrow \text{bound-inv}\ A\ U \Rightarrow \text{bound-inv}$ 
A V
and
  exists-bound:  $\bigwedge R\ S. \text{cdcl}_{NOT}\text{-inv}\ R \Rightarrow \text{restart}\ R\ S \Rightarrow \exists A. \text{bound-inv}\ A\ S$  and
  cdclNOT-inv:  $\bigwedge S\ T. \text{cdcl}_{NOT}\text{-inv}\ S \Rightarrow \text{cdcl}_{NOT}\ S\ T \Rightarrow \text{cdcl}_{NOT}\text{-inv}\ T$  and
  cdclNOT-inv-restart:  $\bigwedge S\ T. \text{cdcl}_{NOT}\text{-inv}\ S \Rightarrow \text{restart}\ S\ T \Rightarrow \text{cdcl}_{NOT}\text{-inv}\ T$ 
begin

lemma cdclNOT-cdclNOT-inv:
assumes
  (cdclNOT  $\widetilde{\sim}$  n) S T and
  cdclNOT-inv S
shows cdclNOT-inv T
using assms by (induction n arbitrary: T) (auto intro: bound-inv cdclNOT-inv)

lemma cdclNOT-bound-inv:
assumes
  (cdclNOT  $\widetilde{\sim}$  n) S T and
  cdclNOT-inv S
  bound-inv A S
shows bound-inv A T
using assms by (induction n arbitrary: T) (auto intro: bound-inv cdclNOT-cdclNOT-inv)

lemma rtrancpl-cdclNOT-cdclNOT-inv:
assumes
  cdclNOT** S T and
  cdclNOT-inv S
shows cdclNOT-inv T
using assms by induction (auto intro: cdclNOT-inv)

lemma rtrancpl-cdclNOT-bound-inv:
assumes
  cdclNOT** S T and
  bound-inv A S and
  cdclNOT-inv S

```

shows $\text{bound-inv } A \ T$
using assms **by** $\text{induction (auto intro:bound-inv rtranclp-cdcl}_{NOT}\text{-cdcl}_{NOT}\text{-inv)}$

lemma $\text{cdcl}_{NOT}\text{-comp-n-le}$:
assumes
 $(\text{cdcl}_{NOT} \sim (Suc \ n)) \ S \ T$ **and**
 $\text{bound-inv } A \ S$
 $\text{cdcl}_{NOT}\text{-inv } S$
shows $\mu \ A \ T < \mu \ A \ S - n$
using assms

proof ($\text{induction } n \text{ arbitrary: } T$)
case 0
then show $?case$ **using** $\text{cdcl}_{NOT}\text{-measure}$ **by** auto

next
case $(Suc \ n)$ **note** $IH = \text{this}(1)[OF - \text{this}(3) \ \text{this}(4)]$ **and** $S-T = \text{this}(2)$ **and** $b\text{-inv} = \text{this}(3)$ **and**
 $c\text{-inv} = \text{this}(4)$
obtain $U :: 'st$ **where** $S-U: (\text{cdcl}_{NOT} \sim (Suc \ n)) \ S \ U$ **and** $U-T: \text{cdcl}_{NOT} \ U \ T$ **using** $S-T$ **by** auto
then have $\mu \ A \ U < \mu \ A \ S - n$ **using** $IH[of \ U]$ **by** simp
moreover
have $\text{bound-inv } A \ U$
using $S-U \ b\text{-inv} \ \text{cdcl}_{NOT}\text{-bound-inv } c\text{-inv}$ **by** blast
then have $\mu \ A \ T < \mu \ A \ U$ **using** $\text{cdcl}_{NOT}\text{-measure}[OF - - \ U-T] \ S-U \ c\text{-inv} \ \text{cdcl}_{NOT}\text{-cdcl}_{NOT}\text{-inv}$
by auto
ultimately show $?case$ **by** linarith

qed

lemma wf-cdcl_{NOT} :
 $\text{wf } \{(T, S). \ \text{cdcl}_{NOT} \ S \ T \wedge \text{cdcl}_{NOT}\text{-inv } S \wedge \text{bound-inv } A \ S\}$ (**is** $\text{wf } ?A$)
apply ($\text{rule } \text{wfP-if-measure2}[of - - \mu \ A]$)
using $\text{cdcl}_{NOT}\text{-comp-n-le}[of \ 0 - - \ A]$ **by** auto

lemma $\text{rtranclp-cdcl}_{NOT}\text{-measure}$:
assumes
 $\text{cdcl}_{NOT}^{**} \ S \ T$ **and**
 $\text{bound-inv } A \ S$ **and**
 $\text{cdcl}_{NOT}\text{-inv } S$
shows $\mu \ A \ T \leq \mu \ A \ S$
using assms

proof ($\text{induction rule: rtranclp-induct}$)
case base
then show $?case$ **by** auto

next
case $(\text{step } T \ U)$ **note** $IH = \text{this}(3)[OF \ \text{this}(4) \ \text{this}(5)]$ **and** $st = \text{this}(1)$ **and** $\text{cdcl}_{NOT} = \text{this}(2)$ **and**
 $b\text{-inv} = \text{this}(4)$ **and** $c\text{-inv} = \text{this}(5)$
have $\text{bound-inv } A \ T$
by ($\text{meson } \text{cdcl}_{NOT}\text{-bound-inv } \text{rtranclp-imp-relpoup } st \ \text{step.prem}$)
moreover have $\text{cdcl}_{NOT}\text{-inv } T$
using $c\text{-inv } \text{rtranclp-cdcl}_{NOT}\text{-cdcl}_{NOT}\text{-inv } st$ **by** blast
ultimately have $\mu \ A \ U < \mu \ A \ T$ **using** $\text{cdcl}_{NOT}\text{-measure}[OF - - \ \text{cdcl}_{NOT}]$ **by** auto
then show $?case$ **using** IH **by** linarith

qed

lemma $\text{cdcl}_{NOT}\text{-comp-bounded}$:
assumes
 $\text{bound-inv } A \ S$ **and** $\text{cdcl}_{NOT}\text{-inv } S$ **and** $m \geq 1 + \mu \ A \ S$

shows $\neg(\text{cdcl}_{NOT} \sim m) S T$
using *assms cdcl_{NOT}-comp-n-le[of m-1 S T A]* **by** *fastforce*

- $f n < m$ ensures that at least one step has been done.

inductive *cdcl_{NOT}-restart* **where**

restart-step: $(\text{cdcl}_{NOT} \sim m) S T \implies m \geq f n \implies \text{restart } T U$
 $\implies \text{cdcl}_{NOT}\text{-restart } (S, n) (U, \text{Suc } n) \mid$

restart-full: $\text{full1 } \text{cdcl}_{NOT} S T \implies \text{cdcl}_{NOT}\text{-restart } (S, n) (T, \text{Suc } n)$

lemmas *cdcl_{NOT}-with-restart-induct* = *cdcl_{NOT}-restart.induct[split-format(complete),*
OF cdcl_{NOT}-increasing-restarts-ops-axioms]

lemma *cdcl_{NOT}-restart-cdcl_{NOT}-raw-restart*:

cdcl_{NOT}-restart $S T \implies \text{cdcl}_{NOT}\text{-raw-restart}^{**} (fst S) (fst T)$

proof (*induction rule: cdcl_{NOT}-restart.induct*)

case (*restart-step* $m S T n U$)

then have *cdcl_{NOT}*** $S T$ **by** (*meson relpowp-imp-rtrancp*)

then have *cdcl_{NOT}-raw-restart*** $S T$ **using** *cdcl_{NOT}-raw-restart.intros(1)*

rtrancp-mono[of cdcl_{NOT} cdcl_{NOT}-raw-restart] **by** *blast*

moreover have *cdcl_{NOT}-raw-restart* $T U$

using (*restart* $T U$) *cdcl_{NOT}-raw-restart.intros(2)* **by** *blast*

ultimately show *?case* **by** *auto*

next

case (*restart-full* $S T$)

then have *cdcl_{NOT}*** $S T$ **unfolding** *full1-def* **by** *auto*

then show *?case* **using** *cdcl_{NOT}-raw-restart.intros(1)*

rtrancp-mono[of cdcl_{NOT} cdcl_{NOT}-raw-restart] **by** *auto*

qed

lemma *cdcl_{NOT}-with-restart-bound-inv*:

assumes

cdcl_{NOT}-restart $S T$ **and**

bound-inv $A (fst S)$ **and**

cdcl_{NOT}-inv $(fst S)$

shows *bound-inv* $A (fst T)$

using *assms* **apply** (*induction rule: cdcl_{NOT}-restart.induct*)

prefer 2 **apply** (*metis rtrancp-unfold fstI full1-def rtrancp-cdcl_{NOT}-bound-inv*)

by (*metis cdcl_{NOT}-bound-inv cdcl_{NOT}-cdcl_{NOT}-inv cdcl_{NOT}-restart-inv fst-conv*)

lemma *cdcl_{NOT}-with-restart-cdcl_{NOT}-inv*:

assumes

cdcl_{NOT}-restart $S T$ **and**

cdcl_{NOT}-inv $(fst S)$

shows *cdcl_{NOT}-inv* $(fst T)$

using *assms* **apply** *induction*

apply (*metis cdcl_{NOT}-cdcl_{NOT}-inv cdcl_{NOT}-inv-restart fst-conv*)

apply (*metis fstI full-def full-unfold rtrancp-cdcl_{NOT}-cdcl_{NOT}-inv*)

done

lemma *rtrancp-cdcl_{NOT}-with-restart-cdcl_{NOT}-inv*:

assumes

*cdcl_{NOT}-restart*** $S T$ **and**

cdcl_{NOT}-inv $(fst S)$

shows *cdcl_{NOT}-inv* $(fst T)$

using *assms* by induction (auto intro: *cdcl_{NOT}-with-restart-cdcl_{NOT}-inv*)

lemma *rtrancpl-cdcl_{NOT}-with-restart-bound-inv*:

assumes

*cdcl_{NOT}-restart*** *S T* **and**

cdcl_{NOT}-inv (*fst S*) **and**

bound-inv *A* (*fst S*)

shows *bound-inv* *A* (*fst T*)

using *assms* **apply** induction

apply (*simp* add: *cdcl_{NOT}-cdcl_{NOT}-inv cdcl_{NOT}-with-restart-bound-inv*)

using *cdcl_{NOT}-with-restart-bound-inv* *rtrancpl-cdcl_{NOT}-with-restart-cdcl_{NOT}-inv* **by** blast

lemma *cdcl_{NOT}-with-restart-increasing-number*:

cdcl_{NOT}-restart *S T* $\implies \text{snd } T = 1 + \text{snd } S$

by (induction rule: *cdcl_{NOT}-restart.induct*) auto

end

locale *cdcl_{NOT}-increasing-restarts* =

cdcl_{NOT}-increasing-restarts-ops *restart cdcl_{NOT} f bound-inv μ cdcl_{NOT}-inv μ -bound +*

dpll-state mset-cls insert-cls remove-lit

mset-clss union-clss in-clss insert-clss remove-from-clss

trail raw-clauses prepend-trail tl-trail add-cl_{NOT} remove-cl_{NOT}

for

mset-cls:: *'cls* \Rightarrow *'v* clause **and**

insert-cls :: *'v* literal \Rightarrow *'cls* \Rightarrow *'cls* **and**

remove-lit :: *'v* literal \Rightarrow *'cls* \Rightarrow *'cls* **and**

mset-clss:: *'clss* \Rightarrow *'v* clauses **and**

union-clss :: *'clss* \Rightarrow *'clss* \Rightarrow *'clss* **and**

in-clss :: *'cls* \Rightarrow *'clss* \Rightarrow bool **and**

insert-clss :: *'cls* \Rightarrow *'clss* \Rightarrow *'clss* **and**

remove-from-clss :: *'cls* \Rightarrow *'clss* \Rightarrow *'clss* **and**

trail :: *'st* \Rightarrow (*'v*, unit, unit) marked-lits **and**

raw-clauses :: *'st* \Rightarrow *'clss* **and**

prepend-trail :: (*'v*, unit, unit) marked-lit \Rightarrow *'st* \Rightarrow *'st* **and**

tl-trail :: *'st* \Rightarrow *'st* **and**

add-cl_{NOT} :: *'cls* \Rightarrow *'st* \Rightarrow *'st* **and**

remove-cl_{NOT} :: *'cls* \Rightarrow *'st* \Rightarrow *'st* **and**

f :: nat \Rightarrow nat **and**

restart :: *'st* \Rightarrow *'st* \Rightarrow bool **and**

bound-inv :: *'bound* \Rightarrow *'st* \Rightarrow bool **and**

μ :: *'bound* \Rightarrow *'st* \Rightarrow nat **and**

cdcl_{NOT} :: *'st* \Rightarrow *'st* \Rightarrow bool **and**

cdcl_{NOT}-inv :: *'st* \Rightarrow bool **and**

μ -bound :: *'bound* \Rightarrow *'st* \Rightarrow nat +

assumes

measure-bound: $\bigwedge A T V n. \text{cdcl}_{NOT}\text{-inv } T \implies \text{bound-inv } A T$

$\implies \text{cdcl}_{NOT}\text{-restart } (T, n) (V, \text{Suc } n) \implies \mu A V \leq \mu\text{-bound } A T$ **and**

cdcl_{NOT}-raw-restart- μ -bound:

$\text{cdcl}_{NOT}\text{-restart } (T, a) (V, b) \implies \text{cdcl}_{NOT}\text{-inv } T \implies \text{bound-inv } A T$

$\implies \mu\text{-bound } A V \leq \mu\text{-bound } A T$

begin

lemma *rtrancpl-cdcl_{NOT}-raw-restart- μ -bound*:

*cdcl_{NOT}-restart*** (*T*, *a*) (*V*, *b*) $\implies \text{cdcl}_{NOT}\text{-inv } T \implies \text{bound-inv } A T$

$\implies \mu\text{-bound } A V \leq \mu\text{-bound } A T$

apply (*induction rule: rtrancp-induct2*)
apply *simp*
by (*metis cdcl_{NOT}-raw-restart-μ-bound dual-order.trans fst-conv*
rtrancp-cdcl_{NOT}-with-restart-bound-inv rtrancp-cdcl_{NOT}-with-restart-cdcl_{NOT}-inv)

lemma *cdcl_{NOT}-raw-restart-measure-bound:*
 $cdcl_{NOT}\text{-restart } (T, a) (V, b) \implies cdcl_{NOT}\text{-inv } T \implies bound\text{-inv } A \ T$
 $\implies \mu \ A \ V \leq \mu\text{-bound } A \ T$
apply (*cases rule: cdcl_{NOT}-restart.cases*)
apply *simp*
using *measure-bound relpowp-imp-rtrancp* **apply** *fastforce*
by (*metis full-def full-unfold measure-bound2 prod.inject*)

lemma *rtrancp-cdcl_{NOT}-raw-restart-measure-bound:*
 $cdcl_{NOT}\text{-restart}^{**} (T, a) (V, b) \implies cdcl_{NOT}\text{-inv } T \implies bound\text{-inv } A \ T$
 $\implies \mu \ A \ V \leq \mu\text{-bound } A \ T$
apply (*induction rule: rtrancp-induct2*)
apply (*simp add: measure-bound2*)
by (*metis dual-order.trans fst-conv measure-bound2 r-into-rtrancp rtrancp.rtrancp-refl*
rtrancp-cdcl_{NOT}-with-restart-bound-inv rtrancp-cdcl_{NOT}-with-restart-cdcl_{NOT}-inv
rtrancp-cdcl_{NOT}-raw-restart-μ-bound)

lemma *wf-cdcl_{NOT}-restart:*
 $wf \{ (T, S). cdcl_{NOT}\text{-restart } S \ T \wedge cdcl_{NOT}\text{-inv } (fst \ S) \}$ (**is** *wf ?A*)
proof (*rule ccontr*)
assume $\neg ?thesis$
then obtain *g* **where**
 $g: \bigwedge i. cdcl_{NOT}\text{-restart } (g \ i) (g \ (Suc \ i))$ **and**
 $cdcl_{NOT}\text{-inv-}g: \bigwedge i. cdcl_{NOT}\text{-inv } (fst \ (g \ i))$
unfolding *wf-iff-no-infinite-down-chain* **by** *fast*

have *snd-g:* $\bigwedge i. snd \ (g \ i) = i + snd \ (g \ 0)$
apply (*induct-tac i*)
apply *simp*
by (*metis Suc-eq-plus1-left add commute add.left-commute*
cdcl_{NOT}-with-restart-increasing-number g)
then have *snd-g-0:* $\bigwedge i. i > 0 \implies snd \ (g \ i) = i + snd \ (g \ 0)$
by *blast*
have *unbounded-f-g:* $unbounded \ (\lambda i. f \ (snd \ (g \ i)))$
using *f* **unfolding** *bounded-def* **by** (*metis add commute f less-or-eq-imp-le snd-g*
not-bounded-nat-exists-larger not-le le-iff-add)

{ fix *i*
have *H:* $\bigwedge T \ Ta \ m. (cdcl_{NOT} \ \sim m) \ T \ Ta \implies no\text{-step } cdcl_{NOT} \ T \implies m = 0$
apply (*case-tac m*) **by** *simp (meson relpowp-E2)*
have $\exists \ T \ m. (cdcl_{NOT} \ \sim m) \ (fst \ (g \ i)) \ T \wedge m \geq f \ (snd \ (g \ i))$
using *g[of i]* **apply** (*cases rule: cdcl_{NOT}-restart.cases*)
apply *auto[]*
using *g[of Suc i] f-ge-1* **apply** (*cases rule: cdcl_{NOT}-restart.cases*)
apply (*auto simp add: full1-def full-def dest: H dest: trancpD*)
using *H Suc-leI leD* **by** *blast*
} **note** *H = this*
obtain *A* **where** $bound\text{-inv } A \ (fst \ (g \ 1))$
using *g[of 0] cdcl_{NOT}-inv-g[of 0]* **apply** (*cases rule: cdcl_{NOT}-restart.cases*)
apply (*metis One-nat-def cdcl_{NOT}-inv exists-bound fst-conv relpowp-imp-rtrancp*)

```

      rtrancpl-induct)
    using  $H[of\ 1]$  unfolding full1-def by (metis One-nat-def Suc-eq-plus1 diff-is-0-eq' diff-zero
      f-ge-1 fst-conv le-add2 relpowp-E2 snd-conv)
  let ?j =  $\mu$ -bound A (fst (g 1)) + 1
  obtain j where
    j: f (snd (g j)) > ?j and j > 1
    using unbounded-f-g not-bounded-nat-exists-larger by blast
  {
    fix i j
    have cdclNOT-with-restart:  $j \geq i \implies \text{cdcl}_{NOT}\text{-restart}^{**} (g\ i) (g\ j)$ 
      apply (induction j)
      apply simp
      by (metis g le-Suc-eq rtrancpl.rtrancpl-into-rtrancpl rtrancpl.rtrancpl-refl)
  } note cdclNOT-restart = this
  have cdclNOT-inv (fst (g (Suc 0)))
    by (simp add: cdclNOT-inv-g)
  have cdclNOT-restart** (fst (g 1), snd (g 1)) (fst (g j), snd (g j))
    using <j > 1> by (simp add: cdclNOT-restart)
  have  $\mu\ A\ (\text{fst}\ (g\ j)) \leq \mu\text{-bound}\ A\ (\text{fst}\ (g\ 1))$ 
    apply (rule rtrancpl-cdclNOT-raw-restart-measure-bound)
    using <cdclNOT-restart** (fst (g 1), snd (g 1)) (fst (g j), snd (g j))> apply blast
    apply (simp add: cdclNOT-inv-g)
    using <bound-inv A (fst (g 1))> apply simp
  done
  then have  $\mu\ A\ (\text{fst}\ (g\ j)) \leq ?j$ 
    by auto
  have inv: bound-inv A (fst (g j))
    using <bound-inv A (fst (g 1))> <cdclNOT-inv (fst (g (Suc 0)))>
    <cdclNOT-restart** (fst (g 1), snd (g 1)) (fst (g j), snd (g j))>
    rtrancpl-cdclNOT-with-restart-bound-inv by auto
  obtain T m where
    cdclNOT-m: (cdclNOT  $\rightsquigarrow$  m) (fst (g j)) T and
    f-m: f (snd (g j))  $\leq$  m
    using  $H[of\ j]$  by blast
  have ?j < m
    using f-m j Nat.le-trans by linarith

  then show False
    using < $\mu\ A\ (\text{fst}\ (g\ j)) \leq \mu\text{-bound}\ A\ (\text{fst}\ (g\ 1))$ >
    cdclNOT-comp-bounded[OF inv cdclNOT-inv-g, of ] cdclNOT-inv-g cdclNOT-m
    <?j < m> by auto
qed

```

lemma cdcl_{NOT}-restart-steps-bigger-than-bound:

```

  assumes
    cdclNOT-restart S T and
    bound-inv A (fst S) and
    cdclNOT-inv (fst S) and
    f (snd S) >  $\mu$ -bound A (fst S)
  shows full1 cdclNOT (fst S) (fst T)
  using assms
  proof (induction rule: cdclNOT-restart.induct)
  case restart-full
  then show ?case by auto
next

```

case (*restart-step* m S T n U) **note** $st = \text{this}(1)$ **and** $f = \text{this}(2)$ **and** $\text{bound-inv} = \text{this}(4)$ **and**
 $\text{cdcl}_{NOT}\text{-inv} = \text{this}(5)$ **and** $\mu = \text{this}(6)$
then obtain m' **where** $m: m = \text{Suc } m'$ **by** (*cases* m) *auto*
have $\mu \ A \ S - m' = 0$
using $f \ \text{bound-inv} \ \text{cdcl}_{NOT}\text{-inv} \ \mu \ m \ \text{rtrancpl-cdcl}_{NOT}\text{-raw-restart-measure-bound}$ **by** *fastforce*
then have *False* **using** $\text{cdcl}_{NOT}\text{-comp-n-le}$ [*of* $m' \ S \ T \ A$] *restart-step* **unfolding** m **by** *simp*
then show *?case* **by** *fast*
qed

lemma *rtrancpl-cdcl_{NOT}-with-inv-inv-rtrancpl-cdcl_{NOT}*:

assumes

inv: $\text{cdcl}_{NOT}\text{-inv} \ S$ **and**

binv: $\text{bound-inv} \ A \ S$

shows $(\lambda S \ T. \ \text{cdcl}_{NOT} \ S \ T \wedge \text{cdcl}_{NOT}\text{-inv} \ S \wedge \text{bound-inv} \ A \ S)^{**} \ S \ T \longleftrightarrow \text{cdcl}_{NOT}^{**} \ S \ T$
(is $?A^{**} \ S \ T \longleftrightarrow ?B^{**} \ S \ T$)

apply (*rule iffI*)

using *rtrancpl-mono*[*of* $?A \ ?B$] **apply** *blast*

apply (*induction rule: rtrancpl-induct*)

using *inv binv* **apply** *simp*

by (*metis* (*mono-tags*, *lifting*) *binv inv rtrancpl.simps rtrancpl-cdcl_{NOT}-bound-inv*
rtrancpl-cdcl_{NOT}-cdcl_{NOT}-inv)

lemma *no-step-cdcl_{NOT}-restart-no-step-cdcl_{NOT}*:

assumes

n-s: *no-step cdcl_{NOT}-restart* S **and**

inv: $\text{cdcl}_{NOT}\text{-inv} \ (\text{fst } S)$ **and**

binv: $\text{bound-inv} \ A \ (\text{fst } S)$

shows *no-step cdcl_{NOT}* $(\text{fst } S)$

proof (*rule ccontr*)

assume $\neg ?thesis$

then obtain T **where** $T: \text{cdcl}_{NOT} \ (\text{fst } S) \ T$

by *blast*

then obtain U **where** $U: \text{full } (\lambda S \ T. \ \text{cdcl}_{NOT} \ S \ T \wedge \text{cdcl}_{NOT}\text{-inv} \ S \wedge \text{bound-inv} \ A \ S) \ T \ U$

using *wf-exists-normal-form-full*[*OF* *wf-cdcl_{NOT}*, *of* $A \ T$] **by** *auto*

moreover have *inv-T*: $\text{cdcl}_{NOT}\text{-inv} \ T$

using $\langle \text{cdcl}_{NOT} \ (\text{fst } S) \ T \rangle \text{cdcl}_{NOT}\text{-inv} \ \text{inv}$ **by** *blast*

moreover have *b-inv-T*: $\text{bound-inv} \ A \ T$

using $\langle \text{cdcl}_{NOT} \ (\text{fst } S) \ T \rangle \text{binv} \ \text{bound-inv} \ \text{inv}$ **by** *blast*

ultimately have *full cdcl_{NOT}* $T \ U$

using *rtrancpl-cdcl_{NOT}-with-inv-inv-rtrancpl-cdcl_{NOT}* *rtrancpl-cdcl_{NOT}-bound-inv*

rtrancpl-cdcl_{NOT}-cdcl_{NOT}-inv **unfolding** *full-def* **by** *blast*

then have *full1 cdcl_{NOT}* $(\text{fst } S) \ U$

using $T \ \text{full-fullI}$ **by** *metis*

then show *False* **by** (*metis* *n-s prod.collapse restart-full*)

qed

end

15.8 Merging backjump and learning

locale *cdcl_{NOT}-merge-bj-learn-ops* =

decide-ops *mset-cls insert-cls remove-lit*

mset-clss union-clss in-clss insert-clss remove-from-clss

trail raw-clauses prepend-trail tl-trail add-cl_{NOT} remove-cl_{NOT} +

forget-ops mset-cls insert-cls remove-lit

mset-clss union-clss in-clss insert-clss remove-from-clss

```

  trail raw-clauses prepend-trail tl-trail add-clsNOT remove-clsNOT forget-cond +
  propagate-ops mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clsNOT remove-clsNOT propagate-conds
for
  mset-cls:: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
  raw-clauses :: 'st  $\Rightarrow$  'clss and
  prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
  tl-trail :: 'st  $\Rightarrow$  'st and
  add-clsNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  remove-clsNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  propagate-conds :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  bool and
  forget-cond :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  bool +
fixes backjump-l-cond :: 'v clause  $\Rightarrow$  'v clause  $\Rightarrow$  'v literal  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  bool
begin
inductive backjump-l where
backjump-l: trail S = F' @ Marked K () # F
 $\Rightarrow$  no-dup (trail S)
 $\Rightarrow$  T  $\sim$  prepend-trail (Propagated L ()) (reduce-trail-toNOT F (add-clsNOT C'' S))
 $\Rightarrow$  C  $\in$  # clausesNOT S
 $\Rightarrow$  trail S  $\models_{as}$  CNot C
 $\Rightarrow$  undefined-lit F L
 $\Rightarrow$  atm-of L  $\in$  atms-of-mm (clausesNOT S)  $\cup$  atm-of ' (lits-of-l (trail S))
 $\Rightarrow$  clausesNOT S  $\models_{pm}$  C' + {#L#}
 $\Rightarrow$  mset-cls C'' = C' + {#L#}
 $\Rightarrow$  F  $\models_{as}$  CNot C'
 $\Rightarrow$  backjump-l-cond C C' L S T
 $\Rightarrow$  backjump-l S T

```

Avoid (meaningless) simplification:

```

declare reduce-trail-toNOT-length-ne[simp del] Set.Un-iff[simp del] Set.insert-iff[simp del]
inductive-cases backjump-lE: backjump-l S T
thm backjump-lE
declare reduce-trail-toNOT-length-ne[simp] Set.Un-iff[simp] Set.insert-iff[simp]

```

```

inductive cdclNOT-merged-bj-learn :: 'st  $\Rightarrow$  'st  $\Rightarrow$  bool for S :: 'st where
cdclNOT-merged-bj-learn-decideNOT: decideNOT S S'  $\Rightarrow$  cdclNOT-merged-bj-learn S S' |
cdclNOT-merged-bj-learn-propagateNOT: propagateNOT S S'  $\Rightarrow$  cdclNOT-merged-bj-learn S S' |
cdclNOT-merged-bj-learn-backjump-l: backjump-l S S'  $\Rightarrow$  cdclNOT-merged-bj-learn S S' |
cdclNOT-merged-bj-learn-forgetNOT: forgetNOT S S'  $\Rightarrow$  cdclNOT-merged-bj-learn S S'

```

lemma cdcl_{NOT}-merged-bj-learn-no-dup-inv:

```

cdclNOT-merged-bj-learn S T  $\Rightarrow$  no-dup (trail S)  $\Rightarrow$  no-dup (trail T)
apply (induction rule: cdclNOT-merged-bj-learn.induct)
  using defined-lit-map apply fastforce
  using defined-lit-map apply fastforce
apply (force simp: defined-lit-map elim!: backjump-lE)[]

```



```

using forgetNOT.simps apply auto[1]
done
end

locale cdclNOT-merge-bj-learn-proxy =
  cdclNOT-merge-bj-learn-ops mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clsNOT remove-clsNOT propagate-conds
  forget-cond
   $\lambda C C' L' S T. \text{backjump-l-cond } C C' L' S T$ 
   $\wedge \text{distinct-mset } (C' + \{\#L'\# \}) \wedge \neg \text{tautology } (C' + \{\#L'\# \})$ 
for
  mset-cls :: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss :: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
  raw-clauses :: 'st  $\Rightarrow$  'clss and
  prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
  tl-trail :: 'st  $\Rightarrow$  'st and
  add-clsNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  remove-clsNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
  propagate-conds :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  bool and
  forget-cond :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  bool and
  backjump-l-cond :: 'v clause  $\Rightarrow$  'v clause  $\Rightarrow$  'v literal  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  bool +
fixes
  inv :: 'st  $\Rightarrow$  bool
assumes
  bj-merge-can-jump:
   $\bigwedge S C F' K F L.$ 
  inv S
   $\Rightarrow \text{trail } S = F' @ \text{Marked } K () \# F$ 
   $\Rightarrow C \in \# \text{clauses}_{\text{NOT}} S$ 
   $\Rightarrow \text{trail } S \models_{\text{as}} C \text{Not } C$ 
   $\Rightarrow \text{undefined-lit } F L$ 
   $\Rightarrow \text{atm-of } L \in \text{atms-of-mm } (\text{clauses}_{\text{NOT}} S) \cup \text{atm-of } ' (\text{lits-of-l } (F' @ \text{Marked } K () \# F))$ 
   $\Rightarrow \text{clauses}_{\text{NOT}} S \models_{\text{pm}} C' + \{\#L'\# \}$ 
   $\Rightarrow F \models_{\text{as}} C \text{Not } C'$ 
   $\Rightarrow \neg \text{no-step backjump-l } S$  and
  cdcl-merged-inv:  $\bigwedge S T. \text{cdcl}_{\text{NOT}}\text{-merged-bj-learn } S T \Rightarrow \text{inv } S \Rightarrow \text{inv } T$ 
begin

abbreviation backjump-conds :: 'v clause  $\Rightarrow$  'v clause  $\Rightarrow$  'v literal  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  bool
where
  backjump-conds  $\equiv \lambda C C' L' S T. \text{distinct-mset } (C' + \{\#L'\# \}) \wedge \neg \text{tautology } (C' + \{\#L'\# \})$ 

```

Without additional knowledge on *backjump-l-cond*, it is impossible to have the same invariant.

```

sublocale dpll-with-backjumping-ops mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clsNOT remove-clsNOT inv
  backjump-conds propagate-conds

```

```

proof (unfold-locales, goal-cases)
  case 1
  { fix S S'
    assume bj: backjump-l S S' and no-dup (trail S)
    then obtain F' K F L C' C D where
      S': S' ~ prepend-trail (Propagated L ()) (reduce-trail-toNOT F (add-clsNOT D S))
      and
      tr-S: trail S = F' @ Marked K () # F and
      C: C ∈ # clausesNOT S and
      tr-S-C: trail S ⊨as CNot C and
      undef-L: undefined-lit F L and
      atm-L:
        atm-of L ∈ insert (atm-of K) (atms-of-mm (clausesNOT S) ∪ atm-of ' (lits-of-l F' ∪ lits-of-l F))
      and
      cls-S-C': clausesNOT S ⊨pm C' + {#L#} and
      F-C': F ⊨as CNot C' and
      dist: distinct-mset (C' + {#L#}) and
      not-tauto: ¬ tautology (C' + {#L#}) and
      cond: backjump-l-cond C C' L S S'
      mset-cls D = C' + {#L#}
      by (elim backjump-lE) metis
    interpret backjumping-ops mset-cls insert-cls remove-lit
    mset-clss union-clss in-clss insert-clss remove-from-clss
    trail raw-clauses prepend-trail tl-trail add-clsNOT remove-clsNOT
    backjump-conds
    by unfold-locales
    have ∃ T. backjump S T
    apply rule
    apply (rule backjump.intros)
      using tr-S apply simp
      apply (rule state-eqNOT-ref)
      using C apply simp
      using tr-S-C apply simp
      using undef-L apply simp
      using atm-L tr-S apply simp
      using cls-S-C' apply simp
      using F-C' apply simp
      using dist not-tauto cond apply simp
    done
  } note H = this(1)
  then show ?case using 1 bj-merge-can-jump by meson
qed

end

locale cdclNOT-merge-bj-learn-proxy2 =
  cdclNOT-merge-bj-learn-proxy mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clsNOT remove-clsNOT
  propagate-conds forget-cond backjump-l-cond inv
for
  mset-cls:: 'cls ⇒ 'v clause and
  insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
  remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
  mset-clss:: 'clss ⇒ 'v clauses and

```

```

union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
in-clss :: 'cls ⇒ 'clss ⇒ bool and
insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
trail :: 'st ⇒ ('v, unit, unit) marked-lits and
raw-clauses :: 'st ⇒ 'clss and
prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
tl-trail :: 'st ⇒ 'st and
add-clNOT :: 'cls ⇒ 'st ⇒ 'st and
remove-clNOT :: 'cls ⇒ 'st ⇒ 'st and
propagate-conds :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ bool and
forget-cond :: 'cls ⇒ 'st ⇒ bool and
backjump-l-cond :: 'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool and
inv :: 'st ⇒ bool
begin

sublocale conflict-driven-clause-learning-ops mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT
  inv backjump-conds propagate-conds
  λC -. distinct-mset (mset-cls C) ∧ ¬tautology (mset-cls C)
  forget-cond
by unfold-locales
end

locale cdclNOT-merge-bj-learn =
  cdclNOT-merge-bj-learn-proxy2 mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT
  propagate-conds forget-cond backjump-l-cond inv
for
  mset-cls:: 'cls ⇒ 'v clause and
  insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
  remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
  mset-clss:: 'clss ⇒ 'v clauses and
  union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
  in-clss :: 'cls ⇒ 'clss ⇒ bool and
  insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
  remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and
  trail :: 'st ⇒ ('v, unit, unit) marked-lits and
  raw-clauses :: 'st ⇒ 'clss and
  prepend-trail :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ 'st and
  tl-trail :: 'st ⇒ 'st and
  add-clNOT :: 'cls ⇒ 'st ⇒ 'st and
  remove-clNOT :: 'cls ⇒ 'st ⇒ 'st and
  backjump-l-cond :: 'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool and
  propagate-conds :: ('v, unit, unit) marked-lit ⇒ 'st ⇒ bool and
  forget-cond :: 'cls ⇒ 'st ⇒ bool and
  inv :: 'st ⇒ bool +
assumes
  dpll-merge-bj-inv: ∧S T. dpll-bj S T ⇒ inv S ⇒ inv T and
  learn-inv: ∧S T. learn S T ⇒ inv S ⇒ inv T
begin

sublocale

```

conflict-driven-clause-learning mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss
trail raw-clauses prepend-trail tl-trail add-cl_{NOT} remove-cl_{NOT}
inv backjump-conds propagate-conds
 $\lambda C \cdot \text{distinct-mset (mset-cl_s } C) \wedge \neg \text{tautology (mset-cl_s } C)$
forget-cond

apply *unfold-locales*

using *cdcl_{NOT}-merged-bj-learn-forget_{NOT} cdcl-merged-inv learn-inv*

by (*auto simp add: cdcl_{NOT}.simps dpll-merge-bj-inv*)

lemma *backjump-l-learn-backjump:*

assumes *bt: backjump-l S T and inv: inv S and n-d: no-dup (trail S)*

shows $\exists C' L D. \text{learn } S \text{ (add-cl_s_{NOT} } D S)$

$\wedge \text{mset-cl_s } D = (C' + \{\#L\# \})$

$\wedge \text{backjump (add-cl_s_{NOT} } D S) T$

$\wedge \text{atms-of } (C' + \{\#L\# \}) \subseteq \text{atms-of-mm (clauses_{NOT} } S) \cup \text{atm-of ' (lits-of-l (trail S))}$

proof –

obtain *C F' K F L l C' D where*

tr-S: trail S = F' @ Marked K () # F and

T: T ~ prepend-trail (Propagated L l) (reduce-trail-to_{NOT} F (add-cl_s_{NOT} D S)) and

C-cl_s-S: C ∈ # clauses_{NOT} S and

tr-S-CNot-C: trail S ⊨_{as} CNot C and

undef: undefined-lit F L and

atm-L: atm-of L ∈ atms-of-mm (clauses_{NOT} S) ∪ atm-of ' (lits-of-l (trail S)) and

clss-C: clauses_{NOT} S ⊨_{pm} mset-cl_s D and

D: mset-cl_s D = C' + {#L#}

F ⊨_{as} CNot C' and

distinct: distinct-mset (mset-cl_s D) and

not-tauto: ¬ tautology (mset-cl_s D)

using *bt inv by (elim backjump-lE) simp*

have *atms-C': atms-of C' ⊆ atm-of ' (lits-of-l F)*

proof –

obtain *ll :: 'v ⇒ ('v literal ⇒ 'v) ⇒ 'v literal set ⇒ 'v literal where*

$\forall v f L. v \notin f \text{ ' } L \vee v = f (ll \ v \ f \ L) \wedge ll \ v \ f \ L \in L$

by *moura*

then show *?thesis unfolding tr-S*

by (*metis (no-types) (F ⊨_{as} CNot C') atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set*

atms-of-def in-CNot-implies-uminus(2) subsetI)

qed

then have *atms-of (C' + {#L#}) ⊆ atms-of-mm (clauses_{NOT} S) ∪ atm-of ' (lits-of-l (trail S))*

using *atm-L tr-S by auto*

moreover have *learn: learn S (add-cl_s_{NOT} D S)*

apply (*rule learn.intros*)

apply (*rule clss-C*)

using *atms-C' atm-L D apply (fastforce simp add: tr-S in-plus-implies-atm-of-on-atms-of-ms)*

apply *standard*

apply (*rule distinct*)

apply (*rule not-tauto*)

apply *simp*

done

moreover have *bj: backjump (add-cl_s_{NOT} D S) T*

apply (*rule backjump.intros*)

using *(F ⊨_{as} CNot C') C-cl_s-S tr-S-CNot-C undef T distinct not-tauto n-d D*

by (*auto simp: tr-S state-eq_{NOT}-def simp del: state-simp_{NOT}*)

ultimately show *?thesis using D by blast*

qed

lemma *cdcl_{NOT}-merged-bj-learn-is-tranclp-cdcl_{NOT}:*

cdcl_{NOT}-merged-bj-learn S T \implies inv S \implies no-dup (trail S) \implies cdcl_{NOT}⁺⁺ S T

proof (*induction rule: cdcl_{NOT}-merged-bj-learn.induct*)

case (*cdcl_{NOT}-merged-bj-learn-decide_{NOT} T*)

then have *cdcl_{NOT} S T*

using *bj-decide_{NOT} cdcl_{NOT}.simps* **by** *fastforce*

then show *?case* **by** *auto*

next

case (*cdcl_{NOT}-merged-bj-learn-propagate_{NOT} T*)

then have *cdcl_{NOT} S T*

using *bj-propagate_{NOT} cdcl_{NOT}.simps* **by** *fastforce*

then show *?case* **by** *auto*

next

case (*cdcl_{NOT}-merged-bj-learn-forget_{NOT} T*)

then have *cdcl_{NOT} S T*

using *c-forget_{NOT}* **by** *blast*

then show *?case* **by** *auto*

next

case (*cdcl_{NOT}-merged-bj-learn-backjump-l T*) **note** *bt = this(1)* **and** *inv = this(2)* **and** *n-d = this(3)*

obtain *C' :: 'v literal multiset* **and** *L :: 'v literal* **and** *D :: 'cls* **where**

f3: learn S (add-cls_{NOT} D S) \wedge

backjump (add-cls_{NOT} D S) T \wedge

atms-of (C' + {#L#}) \subseteq atms-of-mm (clauses_{NOT} S) \cup atm-of ' lits-of-l (trail S) **and**

D: mset-cls D = C' + {#L#}

using *n-d backjump-l-learn-backjump[OF bt inv]* **by** *blast*

then have *f4: cdcl_{NOT} S (add-cls_{NOT} D S)*

using *n-d c-learn* **by** *blast*

have *cdcl_{NOT} (add-cls_{NOT} D S) T*

using *f3 n-d bj-backjump c-dpll-bj* **by** *blast*

then show *?case*

using *f4* **by** (*meson tranclp.r-into-trancl tranclp.trancl-into-trancl*)

qed

lemma *rtranclp-cdcl_{NOT}-merged-bj-learn-is-rtranclp-cdcl_{NOT}-and-inv:*

*cdcl_{NOT}-merged-bj-learn** S T \implies inv S \implies no-dup (trail S) \implies cdcl_{NOT}** S T \wedge inv T*

proof (*induction rule: rtranclp-induct*)

case *base*

then show *?case* **by** *auto*

next

case (*step T U*) **note** *st = this(1)* **and** *cdcl_{NOT} = this(2)* **and** *IH = this(3)[OF this(4-)]* **and** *inv = this(4)* **and** *n-d = this(5)*

have *cdcl_{NOT}** T U*

using *cdcl_{NOT}-merged-bj-learn-is-tranclp-cdcl_{NOT}[OF cdcl_{NOT}] IH*

rtranclp-cdcl_{NOT}-no-dup inv n-d **by** *auto*

then have *cdcl_{NOT}** S U* **using** *IH* **by** *fastforce*

moreover have *inv U* **using** *n-d IH \langle cdcl_{NOT}** T U \rangle rtranclp-cdcl_{NOT}-inv* **by** *blast*

ultimately show *?case* **using** *st* **by** *fast*

qed

lemma *rtranclp-cdcl_{NOT}-merged-bj-learn-is-rtranclp-cdcl_{NOT}:*

*cdcl_{NOT}-merged-bj-learn** S T \implies inv S \implies no-dup (trail S) \implies cdcl_{NOT}** S T*

using *rtranclp-cdcl_{NOT}-merged-bj-learn-is-rtranclp-cdcl_{NOT}-and-inv* **by** *blast*

lemma *rtrancp-cdcl_{NOT}-merged-bj-learn-inv*:

*cdcl_{NOT}-merged-bj-learn** S T \implies inv S \implies no-dup (trail S) \implies inv T*

using *rtrancp-cdcl_{NOT}-merged-bj-learn-is-rtrancp-cdcl_{NOT}-and-inv* **by** *blast*

definition $\mu_C' :: 'v \text{ literal multiset set} \Rightarrow 'st \Rightarrow \text{nat}$ **where**

$\mu_C' A T \equiv \mu_C (1 + \text{card} (\text{atms-of-ms } A)) (2 + \text{card} (\text{atms-of-ms } A)) (\text{trail-weight } T)$

definition $\mu_{CDCL}'\text{-merged} :: 'v \text{ literal multiset set} \Rightarrow 'st \Rightarrow \text{nat}$ **where**

$\mu_{CDCL}'\text{-merged } A T \equiv$

$((2 + \text{card} (\text{atms-of-ms } A)) \wedge (1 + \text{card} (\text{atms-of-ms } A)) - \mu_C' A T) * 2 + \text{card} (\text{set-mset} (\text{clauses}_{NOT} T))$

lemma *cdcl_{NOT}-decreasing-measure'*:

assumes

cdcl_{NOT}-merged-bj-learn S T **and**

inv: inv S **and**

atm-clss: atms-of-mm (clauses_{NOT} S) \subseteq atms-of-ms A **and**

atm-trail: atm-of ' lits-of-l (trail S) \subseteq atms-of-ms A **and**

n-d: no-dup (trail S) **and**

fin-A: finite A

shows $\mu_{CDCL}'\text{-merged } A T < \mu_{CDCL}'\text{-merged } A S$

using *assms(1)*

proof *induction*

case (*cdcl_{NOT}-merged-bj-learn-decide_{NOT} T*)

have *clauses_{NOT} S = clauses_{NOT} T*

using *cdcl_{NOT}-merged-bj-learn-decide_{NOT}.hyps* **by** *auto*

moreover have

$(2 + \text{card} (\text{atms-of-ms } A)) \wedge (1 + \text{card} (\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card} (\text{atms-of-ms } A)) (2 + \text{card} (\text{atms-of-ms } A)) (\text{trail-weight } T)$
 $< (2 + \text{card} (\text{atms-of-ms } A)) \wedge (1 + \text{card} (\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card} (\text{atms-of-ms } A)) (2 + \text{card} (\text{atms-of-ms } A)) (\text{trail-weight } S)$

apply (*rule dp11-bj-trail-mes-decreasing-prop*)

using *cdcl_{NOT}-merged-bj-learn-decide_{NOT} fin-A atm-clss atm-trail n-d inv*

by (*simp-all add: bj-decide_{NOT} cdcl_{NOT}-merged-bj-learn-decide_{NOT}.hyps*)

ultimately show *?case*

unfolding $\mu_{CDCL}'\text{-merged-def } \mu_C'\text{-def}$ **by** *simp*

next

case (*cdcl_{NOT}-merged-bj-learn-propagate_{NOT} T*)

have *clauses_{NOT} S = clauses_{NOT} T*

using *cdcl_{NOT}-merged-bj-learn-propagate_{NOT}.hyps*

by (*simp add: bj-propagate_{NOT} inv dp11-bj-clauses*)

moreover have

$(2 + \text{card} (\text{atms-of-ms } A)) \wedge (1 + \text{card} (\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card} (\text{atms-of-ms } A)) (2 + \text{card} (\text{atms-of-ms } A)) (\text{trail-weight } T)$
 $< (2 + \text{card} (\text{atms-of-ms } A)) \wedge (1 + \text{card} (\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card} (\text{atms-of-ms } A)) (2 + \text{card} (\text{atms-of-ms } A)) (\text{trail-weight } S)$

apply (*rule dp11-bj-trail-mes-decreasing-prop*)

using *inv n-d atm-clss atm-trail fin-A* **by** (*simp-all add: bj-propagate_{NOT}*

cdcl_{NOT}-merged-bj-learn-propagate_{NOT}.hyps)

ultimately show *?case*

unfolding $\mu_{CDCL}'\text{-merged-def } \mu_C'\text{-def}$ **by** *simp*

next

case (*cdcl_{NOT}-merged-bj-learn-forget_{NOT} T*)

have $\text{card} (\text{set-mset} (\text{clauses}_{NOT} T)) < \text{card} (\text{set-mset} (\text{clauses}_{NOT} S))$

using $\langle \text{forget}_{NOT} S T \rangle$ **by** (*metis card-Diff1-less clauses-remove-cls_{NOT} finite-set-mset*
forget_{NOT}.cases in-clss-mset-clss linear set-mset-minus-replicate-mset(1) state-eq_{NOT}-def)
moreover
have $\text{trail } S = \text{trail } T$
using $\langle \text{forget}_{NOT} S T \rangle$ **by** (*auto elim: forget_{NOT}E*)
then have
 $(2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } T)$
 $= (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } S)$
by *auto*
ultimately show *?case*
unfolding $\mu_{CDCL}'\text{-merged-def } \mu_C'\text{-def}$ **by** *simp*
next
case (*cdcl_{NOT}-merged-bj-learn-backjump-l T*) **note** *bj-l = this(1)*
obtain $C' L D$ **where**
learn: learn S (add-cls_{NOT} D S) and
bj: backjump (add-cls_{NOT} D S) T and
atms-C: atms-of (C' + {#L#}) \subseteq atms-of-mm (clauses_{NOT} S) \cup atm-of ' (lits-of-l (trail S)) and
D: mset-cls D = C' + {#L#}
using *bj-l inv backjump-l-learn-backjump n-d atm-clss atm-trail by meson*
have *card-T-S: card (set-mset (clauses_{NOT} T)) \leq 1 + card (set-mset (clauses_{NOT} S))*
using *bj-l inv by (force elim!: backjump-lE simp: card-insert-if)*
have
 $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } T))$
 $< ((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A))$
 $(\text{trail-weight } (\text{add-cls}_{NOT} D S)))$
apply (*rule dp11-bj-trail-mes-decreasing-prop*)
using *bj bj-backjump apply blast*
using *cdcl_{NOT}.c-learn cdcl_{NOT}-inv inv learn apply blast*
using *atms-C atm-clss atm-trail D apply (simp add: n-d) apply fast*
using *atm-trail n-d apply simp*
apply (*simp add: n-d*)
using *fin-A apply simp*
done
then have $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } T))$
 $< ((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } S))$
using *n-d by auto*
then show *?case*
using *card-T-S unfolding $\mu_{CDCL}'\text{-merged-def } \mu_C'\text{-def}$ by linarith*
qed

lemma *wf-cdcl_{NOT}-merged-bj-learn:*
assumes
fin-A: finite A
shows *wf {(T, S).*
 $(\text{inv } S \wedge \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of ' lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A$
 $\wedge \text{no-dup } (\text{trail } S))$
 $\wedge \text{cdcl}_{NOT}\text{-merged-bj-learn } S T\}$
apply (*rule wfP-if-measure[of - - $\mu_{CDCL}'\text{-merged } A]$*)
using *cdcl_{NOT}-decreasing-measure' fin-A by simp*

lemma *trancpl-cdcl_{NOT}-cdcl_{NOT}-trancpl*:

assumes

cdcl_{NOT}-merged-bj-learn⁺⁺ *S T* **and**

inv: *inv S* **and**

atm-clss: *atms-of-mm (clauses_{NOT} S) ⊆ atms-of-ms A* **and**

atm-trail: *atm-of ‘ lits-of-l (trail S) ⊆ atms-of-ms A* **and**

n-d: *no-dup (trail S)* **and**

fin-A[simp]: *finite A*

shows $(T, S) \in \{(T, S).$

$(inv\ S \wedge atms-of-mm\ (clauses_{NOT}\ S) \subseteq atms-of-ms\ A \wedge atm-of\ ‘\ lits-of-l\ (trail\ S) \subseteq atms-of-ms\ A$
 $\wedge no-dup\ (trail\ S))$

$\wedge cdcl_{NOT}\text{-merged-bj-learn}\ S\ T\}^+ \text{ (is - } \in ?P^+)$

using *assms(1)*

proof (*induction rule: trancpl-induct*)

case *base*

then show *?case using n-d atm-clss atm-trail inv by auto*

next

case (*step T U*) **note** *st = this(1)* **and** *cdcl_{NOT} = this(2)* **and** *IH = this(3)*

have *cdcl_{NOT}** S T*

apply (*rule rtrancpl-cdcl_{NOT}-merged-bj-learn-is-rtrancpl-cdcl_{NOT}*)

using *st cdcl_{NOT} inv n-d atm-clss atm-trail inv by auto*

have *inv T*

apply (*rule rtrancpl-cdcl_{NOT}-merged-bj-learn-inv*)

using *inv st cdcl_{NOT} n-d atm-clss atm-trail inv by auto*

moreover have *atms-of-mm (clauses_{NOT} T) ⊆ atms-of-ms A*

using *rtrancpl-cdcl_{NOT}-trail-clauses-bound[OF ⟨cdcl_{NOT}** S T⟩ inv n-d atm-clss atm-trail]*

by fast

moreover have *atm-of ‘ (lits-of-l (trail T)) ⊆ atms-of-ms A*

using *rtrancpl-cdcl_{NOT}-trail-clauses-bound[OF ⟨cdcl_{NOT}** S T⟩ inv n-d atm-clss atm-trail]*

by fast

moreover have *no-dup (trail T)*

using *rtrancpl-cdcl_{NOT}-no-dup[OF ⟨cdcl_{NOT}** S T⟩ inv n-d] by fast*

ultimately have $(U, T) \in ?P$

using *cdcl_{NOT} by auto*

then show *?case using IH by (simp add: trancpl-into-trancpl2)*

qed

lemma *wf-trancpl-cdcl_{NOT}-merged-bj-learn*:

assumes *finite A*

shows *wf {(T, S).*

$(inv\ S \wedge atms-of-mm\ (clauses_{NOT}\ S) \subseteq atms-of-ms\ A \wedge atm-of\ ‘\ lits-of-l\ (trail\ S) \subseteq atms-of-ms\ A$
 $\wedge no-dup\ (trail\ S))$

$\wedge cdcl_{NOT}\text{-merged-bj-learn}^{++}\ S\ T\}$

apply (*rule wf-subset*)

apply (*rule wf-trancpl[OF wf-cdcl_{NOT}-merged-bj-learn]*)

using *assms apply simp*

using *trancpl-cdcl_{NOT}-cdcl_{NOT}-trancpl[OF - - - - ⟨finite A⟩] by auto*

lemma *backjump-no-step-backjump-l*:

backjump S T ⇒ inv S ⇒ ¬no-step backjump-l S

apply (*elim backjumpE*)

apply (*rule bj-merge-can-jump*)

apply *auto[7]*

by blast

lemma *cdcl_{NOT}-merged-bj-learn-final-state*:
fixes $A :: 'v$ literal multiset set **and** $S\ T :: 'st$
assumes
n-s: no-step *cdcl_{NOT}-merged-bj-learn* S **and**
atms-S: *atms-of-mm* (*clauses_{NOT}* S) \subseteq *atms-of-ms* A **and**
atms-trail: *atm-of* ' *lits-of-l* (*trail* S) \subseteq *atms-of-ms* A **and**
n-d: no-dup (*trail* S) **and**
finite A **and**
inv: *inv* S **and**
decomp: *all-decomposition-implies-m* (*clauses_{NOT}* S) (*get-all-marked-decomposition* (*trail* S))
shows *unsatisfiable* (*set-mset* (*clauses_{NOT}* S))
 \vee (*trail* $S \models_{asm}$ *clauses_{NOT}* $S \wedge$ *satisfiable* (*set-mset* (*clauses_{NOT}* S)))

proof –
let $?N = \text{set-mset } (\text{clauses}_{NOT} S)$
let $?M = \text{trail } S$
consider
 (*sat*) *satisfiable* $?N$ **and** $?M \models_{as} ?N$
 | (*sat'*) *satisfiable* $?N$ **and** $\neg ?M \models_{as} ?N$
 | (*unsat*) *unsatisfiable* $?N$
by *auto*
then show *?thesis*
proof *cases*
case *sat'* **note** *sat* = *this*(1) **and** $M = \text{this}(2)$
obtain C **where** $C \in ?N$ **and** $\neg ?M \models_a C$ **using** M **unfolding** *true-annots-def* **by** *auto*
obtain $I :: 'v$ literal set **where**
 $I \models_s ?N$ **and**
cons: *consistent-interp* I **and**
tot: *total-over-m* I $?N$ **and**
atm-I-N: *atm-of* ' $I \subseteq$ *atms-of-ms* $?N$
using *sat* **unfolding** *satisfiable-def-min* **by** *auto*
let $?I = I \cup \{P \mid P. P \in \text{lits-of-l } ?M \wedge \text{atm-of } P \notin \text{atm-of ' } I\}$
let $?O = \{\{\#lit\text{-of } L\# \mid L. \text{is-marked } L \wedge L \in \text{set } ?M \wedge \text{atm-of } (lit\text{-of } L) \notin \text{atms-of-ms } ?N\}\}$
have *cons-I'*: *consistent-interp* $?I$
using *cons* **using** (*no-dup* $?M$) **unfolding** *consistent-interp-def*
by (*auto simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set lits-of-def*
dest!:: no-dup-cannot-not-lit-and-uminus)
have *tot-I'*: *total-over-m* $?I$ ($?N \cup \text{unmark-l } ?M$)
using *tot* *atms-of-s-def* **unfolding** *total-over-m-def total-over-set-def*
by (*fastforce simp: image-iff*)
have $\{P \mid P. P \in \text{lits-of-l } ?M \wedge \text{atm-of } P \notin \text{atm-of ' } I\} \models_s ?O$
using ($I \models_s ?N$) *atm-I-N* **by** (*auto simp add: atm-of-eq-atm-of true-clss-def lits-of-def*)
then have $I' \text{-} N :: ?I \models_s ?N \cup ?O$
using ($I \models_s ?N$) *true-clss-union-increase* **by** *force*
have *tot'*: *total-over-m* $?I$ ($?N \cup ?O$)
using *atm-I-N* *tot* **unfolding** *total-over-m-def total-over-set-def*
by (*force simp: image-iff lits-of-def dest!:: is-marked-ex-Marked*)

have *atms-N-M*: *atms-of-ms* $?N \subseteq$ *atm-of* ' *lits-of-l* $?M$
proof (*rule ccontr*)
assume $\neg ?thesis$
then obtain $l :: 'v$ **where**
 $l \text{-} N :: l \in \text{atms-of-ms } ?N$ **and**
 $l \text{-} M :: l \notin \text{atm-of ' } \text{lits-of-l } ?M$
by *auto*

```

have undefined-lit ?M (Pos l)
  using l-M by (metis Marked-Propagated-in-iff-in-lits-of-l
    atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set literal.sel(1))
have decideNOT S (prepend-trail (Marked (Pos l) ()) S)
  by (metis (undefined-lit ?M (Pos l)) decideNOT.intros l-N literal.sel(1)
    state-eqNOT-ref)
then show False
  using cdclNOT-merged-bj-learn-decideNOT n-s by blast
qed

have ?M  $\models_{as}$  CNot C
  by (metis atms-N-M  $\langle C \in ?N \rangle \hookrightarrow ?M \models_a C$  all-variables-defined-not-imply-cnot
    atms-of-atms-of-ms-mono atms-of-ms-CNot-atms-of atms-of-ms-CNot-atms-of-ms subsetCE)
have  $\exists l \in \text{set } ?M. \text{is-marked } l$ 
  proof (rule ccontr)
    let ?O =  $\{\{\# \text{lit-of } L\# \} \mid L. \text{is-marked } L \wedge L \in \text{set } ?M \wedge \text{atm-of } (\text{lit-of } L) \notin \text{atms-of-ms } ?N\}$ 
    have  $\vartheta[\text{iff}]: \bigwedge I. \text{total-over-m } I \ (\ ?N \cup ?O \cup \text{unmark-l } ?M)$ 
       $\longleftrightarrow \text{total-over-m } I \ (\ ?N \cup \text{unmark-l } ?M)$ 
    unfolding total-over-set-def total-over-m-def atms-of-ms-def by auto
    assume  $\neg ?thesis$ 
    then have [simp]:  $\{\{\# \text{lit-of } L\# \} \mid L. \text{is-marked } L \wedge L \in \text{set } ?M\}$ 
      =  $\{\{\# \text{lit-of } L\# \} \mid L. \text{is-marked } L \wedge L \in \text{set } ?M \wedge \text{atm-of } (\text{lit-of } L) \notin \text{atms-of-ms } ?N\}$ 
    by auto
    then have  $?N \cup ?O \models_{ps} \text{unmark-l } ?M$ 
      using all-decomposition-implies-propagated-lits-are-implied[OF decomp] by auto

    then have ?I  $\models_s \text{unmark-l } ?M$ 
      using cons-I' I'-N tot-I'  $\langle ?I \models_s ?N \cup ?O \rangle$  unfolding  $\vartheta$  true-clss-clss-def by blast
    then have lits-of-l ?M  $\subseteq ?I$ 
      unfolding true-clss-def lits-of-def by auto
    then have ?M  $\models_{as} ?N$ 
      using I'-N  $\langle C \in ?N \rangle \hookrightarrow ?M \models_a C$  cons-I' atms-N-M
      by (meson (trail S  $\models_{as}$  CNot C) consistent-CNot-not rev-subsetD sup-ge1 true-annot-def
        true-annots-def true-clss-mono-set-mset-l true-clss-def)
    then show False using M by fast
  qed

from List.split-list-first-propE[OF this] obtain K :: 'v literal and d :: unit and
  F F' :: ('v, unit, unit) marked-lit list where
  M-K: ?M = F' @ Marked K () # F and
  nm:  $\forall f \in \text{set } F'. \neg \text{is-marked } f$ 
  unfolding is-marked-def by (metis (full-types) old.unit.exhaust)
let ?K = Marked K () :: ('v, unit, unit) marked-lit
have ?K  $\in \text{set } ?M$ 
  unfolding M-K by auto
let ?C = image-mset lit-of  $\{\# L \in \# \text{mset } ?M. \text{is-marked } L \wedge L \neq ?K\# \} :: 'v \text{ literal multiset}$ 
let ?C' = set-mset (image-mset ( $\lambda L. 'v \text{ literal. } \{\# L\# \}$ ) (?C +  $\{\# \text{lit-of } ?K\# \}$ ))
have  $?N \cup \{\{\# \text{lit-of } L\# \} \mid L. \text{is-marked } L \wedge L \in \text{set } ?M\} \models_{ps} \text{unmark-l } ?M$ 
  using all-decomposition-implies-propagated-lits-are-implied[OF decomp] .
moreover have C': ?C' =  $\{\{\# \text{lit-of } L\# \} \mid L. \text{is-marked } L \wedge L \in \text{set } ?M\}$ 
  unfolding M-K apply standard
  apply force
  using IntI by auto
ultimately have N-C-M:  $?N \cup ?C' \models_{ps} \text{unmark-l } ?M$ 
  by auto
have N-M-False:  $?N \cup (\lambda L. \{\# \text{lit-of } L\# \}) \text{ ' } (\text{set } ?M) \models_{ps} \{\{\#\}$ 

```

```

using  $M \langle ?M \models_{as} CNot\ C \rangle \langle C \in ?N \rangle$  unfolding true-clss-clss-def true-annots-def Ball-def
true-annot-def by (metis consistent-CNot-not sup.orderE sup-commute true-clss-def
true-clss-singleton-lit-of-implies-incl true-clss-union true-clss-union-increase)

have undefined-lit F K using  $\langle no\text{-}dup\ ?M \rangle$  unfolding M-K by (simp add: defined-lit-map)
moreover
  have  $?N \cup ?C' \models_{ps} \{\{\#\}\}$ 
  proof –
    have  $A: ?N \cup ?C' \cup unmark\text{-}l\ ?M =$ 
       $?N \cup unmark\text{-}l\ ?M$ 
    unfolding M-K by auto
    show ?thesis
    using true-clss-clss-left-right[OF N-C-M, of \{\{\#\}\} N-M-False] unfolding A by auto
  qed
have  $?N \models_p image\text{-}mset\ minus\ ?C + \{\# - K\}$ 
unfolding true-clss-clss-def true-clss-clss-def total-over-m-def
proof (intro allI impI)
  fix I
  assume
    tot: total-over-set I (atms-of-ms (?N  $\cup$  {image-mset minus ?C + {\# - K}})) and
    cons: consistent-interp I and
     $I \models_s ?N$ 
  have  $(K \in I \wedge -K \notin I) \vee (-K \in I \wedge K \notin I)$ 
    using cons tot unfolding consistent-interp-def by (cases K) auto
  have  $\{a \in set\ (trail\ S). is\text{-}marked\ a \wedge a \neq Marked\ K\ ()\} =$ 
     $set\ (trail\ S) \cap \{L. is\text{-}marked\ L \wedge L \neq Marked\ K\ ()\}$ 
  by auto
  then have tot': total-over-set I
    (atm-of 'lit-of ' (set ?M  $\cap$  {L. is-marked L  $\wedge$  L  $\neq$  Marked K ()}))
    using tot by (auto simp add: atms-of-minus-lit-atm-of-lit-of)
  { fix  $x :: ('v, unit, unit)\ marked\text{-}lit$ 
    assume
      a3: lit-of x  $\notin$  I and
      a1: x  $\in$  set ?M and
      a4: is-marked x and
      a5: x  $\neq$  Marked K ()
    then have  $Pos\ (atm\text{-}of\ (lit\text{-}of\ x)) \in I \vee Neg\ (atm\text{-}of\ (lit\text{-}of\ x)) \in I$ 
      using a5 a4 tot' a1 unfolding total-over-set-def atms-of-s-def by blast
    moreover have  $Neg\ (atm\text{-}of\ (lit\text{-}of\ x)) = -\ Pos\ (atm\text{-}of\ (lit\text{-}of\ x))$ 
      by simp
    ultimately have – lit-of x  $\in$  I
      using f6 a3 by (metis (no-types) atm-of-in-atm-of-set-iff-in-set-or-minus-in-set
        literal.sel(1))
  } note  $H = this$ 

  have  $\neg I \models_s ?C'$ 
    using  $\langle ?N \cup ?C' \models_{ps} \{\{\#\}\} \rangle$  tot cons (I  $\models_s ?N$ )
    unfolding true-clss-clss-def total-over-m-def
    by (simp add: atms-of-minus-lit-atm-of-lit-of atms-of-ms-single-image-atm-of-lit-of)
  then show  $I \models image\text{-}mset\ minus\ ?C + \{\# - K\}$ 
    unfolding true-clss-def true-clss-def Bex-def
    using  $\langle (K \in I \wedge -K \notin I) \vee (-K \in I \wedge K \notin I) \rangle$ 
    by (auto dest!: H)
  qed
moreover have  $F \models_{as} CNot\ (image\text{-}mset\ minus\ ?C)$ 

```

```

    using nm unfolding true-annots-def CNot-def M-K by (auto simp add: lits-of-def)
ultimately have False
using bj-merge-can-jump[of S F' K F C -K
  image-mset uminus (image-mset lit-of {# L :# mset ?M. is-marked L ∧ L ≠ Marked K ()#})]
  (C∈?N) n-s (M ⊨ as CNot C) bj-backjump inv unfolding M-K
  by (auto simp: cdclNOT-merged-bj-learn.simps)
  then show ?thesis by fast
qed auto
qed

lemma full-cdclNOT-merged-bj-learn-final-state:
  fixes A :: 'v literal multiset set and S T :: 'st
  assumes
    full: full cdclNOT-merged-bj-learn S T and
    atms-S: atms-of-mm (clausesNOT S) ⊆ atms-of-ms A and
    atms-trail: atm-of ' lits-of-l (trail S) ⊆ atms-of-ms A and
    n-d: no-dup (trail S) and
    finite A and
    inv: inv S and
    decomp: all-decomposition-implies-m (clausesNOT S) (get-all-marked-decomposition (trail S))
  shows unsatisfiable (set-mset (clausesNOT T))
    ∨ (trail T ⊨ asm clausesNOT T ∧ satisfiable (set-mset (clausesNOT T)))
proof -
  have st: cdclNOT-merged-bj-learn** S T and n-s: no-step cdclNOT-merged-bj-learn T
  using full unfolding full-def by blast+
  then have st: cdclNOT** S T
  using inv rtranclp-cdclNOT-merged-bj-learn-is-rtranclp-cdclNOT-and-inv n-d by auto
  have atms-of-mm (clausesNOT T) ⊆ atms-of-ms A and atm-of ' lits-of-l (trail T) ⊆ atms-of-ms A
  using rtranclp-cdclNOT-trail-clauses-bound[OF st inv n-d atms-S atms-trail] by blast+
  moreover have no-dup (trail T)
  using rtranclp-cdclNOT-no-dup inv n-d st by blast
  moreover have inv T
  using rtranclp-cdclNOT-inv inv st by blast
  moreover have all-decomposition-implies-m (clausesNOT T) (get-all-marked-decomposition (trail T))
  using rtranclp-cdclNOT-all-decomposition-implies inv st decomp n-d by blast
  ultimately show ?thesis
  using cdclNOT-merged-bj-learn-final-state[of T A] (finite A) n-s by fast
qed

end

```

15.8.1 Instantiations

```

locale cdclNOT-with-backtrack-and-restarts =
  conflict-driven-clause-learning-learning-before-backjump-only-distinct-learn
  mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clNOT remove-clNOT
  inv backjump-conds propagate-conds learn-restrictions forget-restrictions
for
  mset-cls:: 'cls ⇒ 'v clause and
  insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
  remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and
  mset-clss:: 'clss ⇒ 'v clauses and
  union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
  in-clss :: 'cls ⇒ 'clss ⇒ bool and

```

```

insert-cls :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and
raw-clauses :: 'st  $\Rightarrow$  'clss and
prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
tl-trail :: 'st  $\Rightarrow$  'st and
add-clsNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
remove-clsNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
inv :: 'st  $\Rightarrow$  bool and
backjump-conds :: 'v clause  $\Rightarrow$  'v clause  $\Rightarrow$  'v literal  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  bool and
propagate-conds :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  bool and
learn-restrictions forget-restrictions :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  bool
+
fixes f :: nat  $\Rightarrow$  nat
assumes
  unbounded: unbounded f and f-ge-1:  $\bigwedge n. n \geq 1 \Rightarrow f\ n \geq 1$  and
  inv-restart:  $\bigwedge S\ T. inv\ S \Rightarrow T \sim reduce\_trail\_to_{NOT} ([::'a\ list)\ S \Rightarrow inv\ T$ 
begin

lemma bound-inv-inv:
  assumes
    inv S and
    n-d: no-dup (trail S) and
    atms-clss-S-A: atms-of-mm (clausesNOT S)  $\subseteq$  atms-of-ms A and
    atms-trail-S-A: atm-of ' lits-of-l (trail S)  $\subseteq$  atms-of-ms A and
    finite A and
    cdclNOT: cdclNOT S T
  shows
    atms-of-mm (clausesNOT T)  $\subseteq$  atms-of-ms A and
    atm-of ' lits-of-l (trail T)  $\subseteq$  atms-of-ms A and
    finite A
  proof -
    have cdclNOT S T
      using  $\langle inv\ S \rangle$  cdclNOT by linarith
    then have atms-of-mm (clausesNOT T)  $\subseteq$  atms-of-mm (clausesNOT S)  $\cup$  atm-of ' lits-of-l (trail S)
      using  $\langle inv\ S \rangle$ 
      by (meson conflict-driven-clause-learning-ops.cdclNOT-atms-of-ms-clauses-decreasing
          conflict-driven-clause-learning-ops-axioms n-d)
    then show atms-of-mm (clausesNOT T)  $\subseteq$  atms-of-ms A
      using atms-clss-S-A atms-trail-S-A by blast
  next
    show atm-of ' lits-of-l (trail T)  $\subseteq$  atms-of-ms A
      by (meson  $\langle inv\ S \rangle$  atms-clss-S-A atms-trail-S-A cdclNOT cdclNOT-atms-in-trail-in-set n-d)
  next
    show finite A
      using  $\langle finite\ A \rangle$  by simp
qed

sublocale cdclNOT-increasing-restarts-ops  $\lambda S\ T. T \sim reduce\_trail\_to_{NOT} ([::'a\ list)\ S\ cdcl_{NOT}\ f$ 
 $\lambda A\ S. atms\_of\_mm\ (clauses_{NOT}\ S) \subseteq atms\_of\_ms\ A \wedge atm\_of\ ' \ lits\_of\_l\ (trail\ S) \subseteq atms\_of\_ms\ A \wedge$ 
finite A
 $\mu_{CDCL}'\ \lambda S. inv\ S \wedge no\_dup\ (trail\ S)$ 
 $\mu_{CDCL}'$ -bound
apply unfold-locales
  apply (simp add: unbounded)

```

```

    using f-ge-1 apply force
    using bound-inv-inv apply meson
    apply (rule cdclNOT-decreasing-measure'; simp)
    apply (rule rtrancpl-cdclNOT- $\mu_{CDCL}'$ -bound; simp)
    apply (rule rtrancpl- $\mu_{CDCL}'$ -bound-decreasing; simp)
    apply auto[]
    apply auto[]
    using cdclNOT-inv cdclNOT-no-dup apply blast
    using inv-restart apply auto[]
done

```

lemma *cdcl_{NOT}-with-restart- μ_{CDCL}' -le- μ_{CDCL}' -bound:*

assumes

cdcl_{NOT}: *cdcl_{NOT}-restart* (*T*, *a*) (*V*, *b*) **and**

cdcl_{NOT}-inv:

inv T

no-dup (*trail T*) **and**

bound-inv:

atms-of-mm (*clauses_{NOT} T*) \subseteq *atms-of-ms A*

atm-of ' *lits-of-l* (*trail T*) \subseteq *atms-of-ms A*

finite A

shows $\mu_{CDCL}' A V \leq \mu_{CDCL}'\text{-bound } A T$

using *cdcl_{NOT}-inv bound-inv*

proof (*induction rule: cdcl_{NOT}-with-restart-induct[OF cdcl_{NOT}]*)

case (*1 m S T n U*) **note** *U = this(3)*

show ?*case*

apply (*rule rtrancpl-cdcl_{NOT}- μ_{CDCL}' -bound-reduce-trail-to_{NOT}[of S T]*)

using $\langle (cdcl_{NOT} \rightsquigarrow m) S T \rangle$ **apply** (*fastforce dest!: relpowp-imp-rtrancpl*)

using 1 **by** *auto*

next

case (*2 S T n*) **note** *full = this(2)*

show ?*case*

apply (*rule rtrancpl-cdcl_{NOT}- μ_{CDCL}' -bound*)

using *full 2 unfolding full1-def* **by** *force+*

qed

lemma *cdcl_{NOT}-with-restart- μ_{CDCL}' -bound-le- μ_{CDCL}' -bound:*

assumes

cdcl_{NOT}: *cdcl_{NOT}-restart* (*T*, *a*) (*V*, *b*) **and**

cdcl_{NOT}-inv:

inv T

no-dup (*trail T*) **and**

bound-inv:

atms-of-mm (*clauses_{NOT} T*) \subseteq *atms-of-ms A*

atm-of ' *lits-of-l* (*trail T*) \subseteq *atms-of-ms A*

finite A

shows $\mu_{CDCL}'\text{-bound } A V \leq \mu_{CDCL}'\text{-bound } A T$

using *cdcl_{NOT}-inv bound-inv*

proof (*induction rule: cdcl_{NOT}-with-restart-induct[OF cdcl_{NOT}]*)

case (*1 m S T n U*) **note** *U = this(3)*

have $\mu_{CDCL}'\text{-bound } A T \leq \mu_{CDCL}'\text{-bound } A S$

apply (*rule rtrancpl- μ_{CDCL}' -bound-decreasing*)

using $\langle (cdcl_{NOT} \rightsquigarrow m) S T \rangle$ **apply** (*fastforce dest!: relpowp-imp-rtrancpl*)

using 1 **by** *auto*

then show ?*case* **using** *U unfolding μ_{CDCL}' -bound-def* **by** *auto*

```

next
  case (2 S T n) note full = this(2)
  show ?case
    apply (rule rtrancpl- $\mu_{CDCL}'$ -bound-decreasing)
    using full 2 unfolding full1-def by force+
qed

sublocale cdclNOT-increasing-restarts - - - - -
  f
   $\lambda S T. T \sim \text{reduce-trail-to}_{NOT} ([::'a \text{ list}) S$ 
   $\lambda A S. \text{atms-of-mm} (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms} A$ 
   $\wedge \text{atm-of} ' \text{lits-of-l} (\text{trail} S) \subseteq \text{atms-of-ms} A \wedge \text{finite} A$ 
   $\mu_{CDCL}' \text{ cdcl}_{NOT}$ 
   $\lambda S. \text{inv} S \wedge \text{no-dup} (\text{trail} S)$ 
   $\mu_{CDCL}'\text{-bound}$ 
  apply unfold-locales
  using cdclNOT-with-restart- $\mu_{CDCL}'$ -le- $\mu_{CDCL}'$ -bound apply simp
  using cdclNOT-with-restart- $\mu_{CDCL}'$ -bound-le- $\mu_{CDCL}'$ -bound apply simp
  done

lemma cdclNOT-restart-all-decomposition-implies:
  assumes cdclNOT-restart S T and
    inv (fst S) and
    no-dup (trail (fst S))
    all-decomposition-implies-m (clausesNOT (fst S)) (get-all-marked-decomposition (trail (fst S)))
  shows
    all-decomposition-implies-m (clausesNOT (fst T)) (get-all-marked-decomposition (trail (fst T)))
  using assms apply (induction)
  using rtrancpl-cdclNOT-all-decomposition-implies by (auto dest!: trancpl-into-rtrancpl
    simp: full1-def)

lemma rtrancpl-cdclNOT-restart-all-decomposition-implies:
  assumes cdclNOT-restart** S T and
    inv: inv (fst S) and
    n-d: no-dup (trail (fst S)) and
    decomp:
      all-decomposition-implies-m (clausesNOT (fst S)) (get-all-marked-decomposition (trail (fst S)))
  shows
    all-decomposition-implies-m (clausesNOT (fst T)) (get-all-marked-decomposition (trail (fst T)))
  using assms(1)
proof (induction rule: rtrancpl-induct)
  case base
  then show ?case using decomp by simp
next
  case (step T u) note st = this(1) and r = this(2) and IH = this(3)
  have inv (fst T)
    using rtrancpl-cdclNOT-with-restart-cdclNOT-inv[OF st] inv n-d by blast
  moreover have no-dup (trail (fst T))
    using rtrancpl-cdclNOT-with-restart-cdclNOT-inv[OF st] inv n-d by blast
  ultimately show ?case
    using cdclNOT-restart-all-decomposition-implies r IH n-d by fast
qed

lemma cdclNOT-restart-sat-ext-iff:
  assumes

```

```

  st: cdclNOT-restart  $S$   $T$  and
  n-d: no-dup (trail (fst  $S$ )) and
  inv: inv (fst  $S$ )
shows  $I \models_{\text{sextm}} \text{clauses}_{\text{NOT}} (\text{fst } S) \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{\text{NOT}} (\text{fst } T)$ 
using assms
proof (induction)
  case (restart-step  $m$   $S$   $T$   $n$   $U$ )
  then show ?case
    using rtrancpl-cdclNOT-bj-sat-ext-iff n-d by (fastforce dest!: relpowp-imp-rtrancpl)
next
  case restart-full
  then show ?case using rtrancpl-cdclNOT-bj-sat-ext-iff unfolding full1-def
  by (fastforce dest!: trancpl-into-rtrancpl)
qed

lemma rtrancpl-cdclNOT-restart-sat-ext-iff:
  assumes
    st: cdclNOT-restart**  $S$   $T$  and
    n-d: no-dup (trail (fst  $S$ )) and
    inv: inv (fst  $S$ )
  shows  $I \models_{\text{sextm}} \text{clauses}_{\text{NOT}} (\text{fst } S) \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{\text{NOT}} (\text{fst } T)$ 
  using st
proof (induction)
  case base
  then show ?case by simp
next
  case (step  $T$   $U$ ) note st = this(1) and r = this(2) and IH = this(3)
  have inv (fst  $T$ )
    using rtrancpl-cdclNOT-with-restart-cdclNOT-inv[OF st] inv n-d by blast+
  moreover have no-dup (trail (fst  $T$ ))
    using rtrancpl-cdclNOT-with-restart-cdclNOT-inv rtrancpl-cdclNOT-no-dup st inv n-d by blast
  ultimately show ?case
    using cdclNOT-restart-sat-ext-iff[OF r] IH by blast
qed

theorem full-cdclNOT-restart-backjump-final-state:
  fixes  $A :: 'v$  literal multiset set and  $S$   $T :: 'st$ 
  assumes
    full: full cdclNOT-restart ( $S$ ,  $n$ ) ( $T$ ,  $m$ ) and
    atms-S: atms-of-mm (clausesNOT  $S$ )  $\subseteq$  atms-of-ms  $A$  and
    atms-trail: atm-of ' lits-of-l (trail  $S$ )  $\subseteq$  atms-of-ms  $A$  and
    n-d: no-dup (trail  $S$ ) and
    fin-A[simp]: finite  $A$  and
    inv: inv  $S$  and
    decomp: all-decomposition-implies-m (clausesNOT  $S$ ) (get-all-marked-decomposition (trail  $S$ ))
  shows unsatisfiable (set-mset (clausesNOT  $S$ ))
     $\vee$  (lits-of-l (trail  $T$ )  $\models_{\text{sextm}} \text{clauses}_{\text{NOT}} S \wedge$  satisfiable (set-mset (clausesNOT  $S$ )))
proof -
  have st: cdclNOT-restart** ( $S$ ,  $n$ ) ( $T$ ,  $m$ ) and
    n-s: no-step cdclNOT-restart ( $T$ ,  $m$ )
    using full unfolding full-def by fast+
  have binv-T: atms-of-mm (clausesNOT  $T$ )  $\subseteq$  atms-of-ms  $A$ 
    atm-of ' lits-of-l (trail  $T$ )  $\subseteq$  atms-of-ms  $A$ 
    using rtrancpl-cdclNOT-with-restart-bound-inv[OF st, of  $A$ ] inv n-d atms-S atms-trail
    by auto

```



```

moreover have inv-T: no-dup (trail T) inv T
  using rtrancp-cdclNOT-with-restart-cdclNOT-inv[OF st] inv n-d by auto
moreover have all-decomposition-implies-m (clausesNOT T) (get-all-marked-decomposition (trail T))
  using rtrancp-cdclNOT-restart-all-decomposition-implies[OF st] inv n-d
  decomp by auto
ultimately have T: unsatisfiable (set-mset (clausesNOT T))
   $\vee$  (trail T  $\models_{asm}$  clausesNOT T  $\wedge$  satisfiable (set-mset (clausesNOT T)))
  using no-step-cdclNOT-restart-no-step-cdclNOT[of (T, m) A] n-s
  cdclNOT-final-state[of T A] unfolding cdclNOT-NOT-all-inv-def by auto
have eq-sat-S-T:  $\bigwedge I. I \models_{sextm} \text{clauses}_{NOT} S \longleftrightarrow I \models_{sextm} \text{clauses}_{NOT} T$ 
  using rtrancp-cdclNOT-restart-sat-ext-iff[OF st] inv n-d atms-S
  atms-trail by auto
have cons-T: consistent-interp (lits-of-l (trail T))
  using inv-T(1) distinct-consistent-interp by blast
consider
  (unsat) unsatisfiable (set-mset (clausesNOT T))
  | (sat) trail T  $\models_{asm}$  clausesNOT T and satisfiable (set-mset (clausesNOT T))
  using T by blast
then show ?thesis
proof cases
  case unsat
  then have unsatisfiable (set-mset (clausesNOT S))
    using eq-sat-S-T consistent-true-clss-ext-satisfiable true-clss-imp-true-clss-ext
    unfolding satisfiable-def by blast
  then show ?thesis by fast
next
  case sat
  then have lits-of-l (trail T)  $\models_{sextm}$  clausesNOT S
    using rtrancp-cdclNOT-restart-sat-ext-iff[OF st] inv n-d atms-S
    atms-trail by (auto simp: true-clss-imp-true-clss-ext true-annots-true-clss)
  moreover then have satisfiable (set-mset (clausesNOT S))
    using cons-T consistent-true-clss-ext-satisfiable by blast
  ultimately show ?thesis by blast
qed
qed
end — end of cdclNOT-with-backtrack-and-restarts locale

```

The restart does only reset the trail, contrary to Weidenbach's version. But there is a forget rule.

```

locale cdclNOT-merge-bj-learn-with-backtrack-restarts =
  cdclNOT-merge-bj-learn mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  trail raw-clauses prepend-trail tl-trail add-clssNOT remove-clssNOT
   $\lambda C C' L' S T. \text{distinct-mset } (C' + \{\#L'\# \}) \wedge \text{backjump-l-cond } C C' L' S T$ 
  propagate-conds forget-conds inv
for
  mset-clss:: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  trail :: 'st  $\Rightarrow$  ('v, unit, unit) marked-lits and

```

```

raw-clauses :: 'st  $\Rightarrow$  'cls and
prepend-trail :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
tl-trail :: 'st  $\Rightarrow$  'st and
add-clNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
remove-clNOT :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
propagate-conds :: ('v, unit, unit) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  bool and
inv :: 'st  $\Rightarrow$  bool and
forget-conds :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  bool and
backjump-l-cond :: 'v clause  $\Rightarrow$  'v clause  $\Rightarrow$  'v literal  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  bool
+
fixes f :: nat  $\Rightarrow$  nat
assumes
  unbounded: unbounded f and f-ge-1:  $\bigwedge n. n \geq 1 \Rightarrow f\ n \geq 1$  and
  inv-restart:  $\bigwedge S\ T. inv\ S \Rightarrow T \sim reduce\_trail\_to_{NOT} \ \Box\ S \Rightarrow inv\ T$ 
begin

definition not-simplified-cls A = {#C  $\in$  # A. tautology C  $\vee$   $\neg$ distinct-mset C#}

lemma simple-clss-or-not-simplified-cls:
  assumes atms-of-mm (clausesNOT S)  $\subseteq$  atms-of-ms A and
    x  $\in$  # clausesNOT S and finite A
  shows x  $\in$  simple-clss (atms-of-ms A)  $\vee$  x  $\in$  # not-simplified-cls (clausesNOT S)
proof -
  consider
    (simpl)  $\neg$ tautology x and distinct-mset x
  | (n-simp) tautology x  $\vee$   $\neg$ distinct-mset x
  by auto
  then show ?thesis
  proof cases
    case simpl
    then have x  $\in$  simple-clss (atms-of-ms A)
    by (meson assms atms-of-atms-of-ms-mono atms-of-ms-finite simple-clss-mono
      distinct-mset-not-tautology-implies-in-simple-clss finite-subset
      subsetCE)
    then show ?thesis by blast
  next
    case n-simp
    then have x  $\in$  # not-simplified-cls (clausesNOT S)
    using (x  $\in$  # clausesNOT S) unfolding not-simplified-cls-def by auto
    then show ?thesis by blast
  qed
qed

lemma cdclNOT-merged-bj-learn-clauses-bound:
  assumes
    cdclNOT-merged-bj-learn S T and
    inv: inv S and
    atms-clss: atms-of-mm (clausesNOT S)  $\subseteq$  atms-of-ms A and
    atms-trail: atm-of ('(lits-of-l (trail S))  $\subseteq$  atms-of-ms A and
    n-d: no-dup (trail S) and
    fin-A[simp]: finite A
  shows set-mset (clausesNOT T)  $\subseteq$  set-mset (not-simplified-cls (clausesNOT S))
     $\cup$  simple-clss (atms-of-ms A)
  using assms
proof (induction rule: cdclNOT-merged-bj-learn.induct)

```

```

case  $cdcl_{NOT}$ -merged-bj-learn-decide $_{NOT}$ 
then show ?case using  $dpll$ -bj-clauses by (force dest!: simple-clss-or-not-simplified-clss)
next
case  $cdcl_{NOT}$ -merged-bj-learn-propagate $_{NOT}$ 
then show ?case using  $dpll$ -bj-clauses by (force dest!: simple-clss-or-not-simplified-clss)
next
case  $cdcl_{NOT}$ -merged-bj-learn-forget $_{NOT}$ 
then show ?case using clauses-remove-clss $_{NOT}$  unfolding state-eq $_{NOT}$ -def
  by (force elim!: forget $_{NOT}$ E dest: simple-clss-or-not-simplified-clss)
next
case ( $cdcl_{NOT}$ -merged-bj-learn-backjump-l T) note  $bj = this(1)$  and  $inv = this(2)$  and
   $atms-clss = this(3)$  and  $atms-trail = this(4)$  and  $n-d = this(5)$ 

have  $cdcl_{NOT}^{**} S T$ 
  apply (rule  $rtranclp$ - $cdcl_{NOT}$ -merged-bj-learn-is- $rtranclp$ - $cdcl_{NOT}$ )
  using  $\langle backjump-l S T \rangle$   $inv$   $cdcl_{NOT}$ -merged-bj-learn.simps  $n-d$  by blast+
have  $atm-of \text{ '}(lits-of-l (trail T)) \subseteq atms-of-ms A$ 
  using  $rtranclp$ - $cdcl_{NOT}$ -trail-clauses-bound[OF  $\langle cdcl_{NOT}^{**} S T \rangle$ ]  $inv$   $atms-trail$   $atms-clss$ 
   $n-d$  by auto
have  $atms-of-mm (clauses_{NOT} T) \subseteq atms-of-ms A$ 
  using  $rtranclp$ - $cdcl_{NOT}$ -trail-clauses-bound[OF  $\langle cdcl_{NOT}^{**} S T \rangle$ ]  $inv$   $n-d$   $atms-clss$   $atms-trail$ 
  by fast
moreover have no-dup (trail T)
  using  $rtranclp$ - $cdcl_{NOT}$ -no-dup[OF  $\langle cdcl_{NOT}^{**} S T \rangle$ ]  $inv$   $n-d$  by fast

obtain  $F' K F L l C' C D$  where
   $tr-S: trail S = F' @ Marked K () \# F$  and
   $T: T \sim prepend-trail (Propagated L l) (reduce-trail-to_{NOT} F (add-clss_{NOT} D S))$  and
   $C \in \# clauses_{NOT} S$  and
   $trail S \models_{as} CNot C$  and
   $undef: undefined-lit F L$  and
   $clauses_{NOT} S \models_{pm} C' + \{\#L\# \}$  and
   $F \models_{as} CNot C'$  and
   $D: mset-clss D = C' + \{\#L\# \}$  and
   $dist: distinct-mset (C' + \{\#L\# \})$  and
   $tauto: \neg tautology (C' + \{\#L\# \})$  and
   $backjump-l-cond C C' L S T$ 
  using  $\langle backjump-l S T \rangle$  apply (elim backjump-lE) by auto

have  $atms-of C' \subseteq atm-of \text{ '}(lits-of-l F)$ 
  using  $\langle F \models_{as} CNot C' \rangle$  by (simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
     $atms-of-def$  image-subset-iff in-CNot-implies-uminus(2))
then have  $atms-of (C' + \{\#L\# \}) \subseteq atms-of-ms A$ 
  using  $T \langle atm-of \text{ '}(lits-of-l (trail T)) \subseteq atms-of-ms A \rangle$   $tr-S$   $undef$   $n-d$  by auto
then have simple-clss ( $atms-of (C' + \{\#L\# \})$ )  $\subseteq$  simple-clss ( $atms-of-ms A$ )
  apply – by (rule simple-clss-mono) (simp-all)
then have  $C' + \{\#L\# \} \in simple-clss (atms-of-ms A)$ 
  using distinct-mset-not-tautology-implies-in-simple-clss[OF dist tauto]
  by auto
then show ?case
  using  $T$   $inv$   $atms-clss$   $undef$   $tr-S$   $n-d$   $D$  by (force dest!: simple-clss-or-not-simplified-clss)
qed

lemma  $cdcl_{NOT}$ -merged-bj-learn-not-simplified-decreasing:
  assumes  $cdcl_{NOT}$ -merged-bj-learn S T

```

shows $(\text{not-simplified-cls } (\text{clauses}_{NOT} T)) \subseteq \# (\text{not-simplified-cls } (\text{clauses}_{NOT} S))$
using *assms apply induction*
prefer 4
unfolding *not-simplified-cls-def* **apply** $(\text{auto elim!}:\text{ backjump-LE forget}_{NOT} E)[3]$
by $(\text{elim backjump-LE}) \text{ auto}$

lemma *rtrancpl-cdcl_{NOT}-merged-bj-learn-not-simplified-decreasing*:
assumes *cdcl_{NOT}-merged-bj-learn** S T*
shows $(\text{not-simplified-cls } (\text{clauses}_{NOT} T)) \subseteq \# (\text{not-simplified-cls } (\text{clauses}_{NOT} S))$
using *assms apply induction*
apply *simp*
by $(\text{drule cdcl}_{NOT}\text{-merged-bj-learn-not-simplified-decreasing}) \text{ auto}$

lemma *rtrancpl-cdcl_{NOT}-merged-bj-learn-clauses-bound*:

assumes
*cdcl_{NOT}-merged-bj-learn** S T and*
inv S and
atms-of-mm (clauses_{NOT} S) \subseteq atms-of-ms A and
atm-of ' (lits-of-l (trail S)) \subseteq atms-of-ms A and
n-d: no-dup (trail S) and
finite[simp]: finite A
shows $\text{set-mset } (\text{clauses}_{NOT} T) \subseteq \text{set-mset } (\text{not-simplified-cls } (\text{clauses}_{NOT} S))$
 $\cup \text{simple-clss } (\text{atms-of-ms } A)$
using *assms(1-5)*

proof *induction*

case *base*

then show ?case **by** $(\text{auto dest!}:\text{ simple-clss-or-not-simplified-cls})$

next

case $(\text{step } T U)$ **note** *st = this(1) and cdcl_{NOT} = this(2) and IH = this(3)[OF this(4-7)] and*
inv = this(4) and atms-clss-S = this(5) and atms-trail-S = this(6) and finite-clss-S = this(7)

have *st': cdcl_{NOT}** S T*

using *inv rtrancpl-cdcl_{NOT}-merged-bj-learn-is-rtrancpl-cdcl_{NOT}-and-inv st n-d* **by** *blast*

have *inv T*

using *inv rtrancpl-cdcl_{NOT}-merged-bj-learn-inv st n-d* **by** *blast*

moreover

have *atms-of-mm (clauses_{NOT} T) \subseteq atms-of-ms A and*

atm-of ' lits-of-l (trail T) \subseteq atms-of-ms A

using *rtrancpl-cdcl_{NOT}-trail-clauses-bound[OF st'] inv atms-clss-S atms-trail-S n-d*
by *blast+*

moreover moreover have *no-dup (trail T)*

using *rtrancpl-cdcl_{NOT}-no-dup[OF 'cdcl_{NOT}** S T' inv n-d]* **by** *fast*

ultimately have *set-mset (clauses_{NOT} U)*

$\subseteq \text{set-mset } (\text{not-simplified-cls } (\text{clauses}_{NOT} T)) \cup \text{simple-clss } (\text{atms-of-ms } A)$

using *cdcl_{NOT} finite cdcl_{NOT}-merged-bj-learn-clauses-bound*

by $(\text{auto intro!}:\text{ cdcl}_{NOT}\text{-merged-bj-learn-clauses-bound})$

moreover have *set-mset (not-simplified-cls (clauses_{NOT} T))*

$\subseteq \text{set-mset } (\text{not-simplified-cls } (\text{clauses}_{NOT} S))$

using *rtrancpl-cdcl_{NOT}-merged-bj-learn-not-simplified-decreasing[OF st]* **by** *auto*

ultimately show ?case **using** *IH inv atms-clss-S*

by $(\text{auto dest!}:\text{ simple-clss-or-not-simplified-cls})$

qed

abbreviation $\mu_{CDCL}'\text{-bound}$ **where**

$\mu_{CDCL}'\text{-bound } A T == ((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))) * 2$
 $+ \text{card } (\text{set-mset } (\text{not-simplified-cls}(\text{clauses}_{NOT} T)))$

+ 3 \wedge card (atms-of-ms A)

lemma *rtrancpl-cdcl_{NOT}-merged-bj-learn-clauses-bound-card*:

assumes

*cdcl_{NOT}-merged-bj-learn*** S T **and**

inv S **and**

atms-of-mm (clauses_{NOT} S) \subseteq *atms-of-ms* A **and**

atm-of ' (*lits-of-l* (trail S)) \subseteq *atms-of-ms* A **and**

n-d: *no-dup* (trail S) **and**

finite: *finite* A

shows μ_{CDCL}' -merged A T \leq μ_{CDCL}' -bound A S

proof –

have *set-mset* (clauses_{NOT} T) \subseteq *set-mset* (*not-simplified-cls*(clauses_{NOT} S))

\cup *simple-clss* (atms-of-ms A)

using *rtrancpl-cdcl_{NOT}-merged-bj-learn-clauses-bound*[OF *assms*] .

moreover have card (*set-mset* (*not-simplified-cls*(clauses_{NOT} S))

\cup *simple-clss* (atms-of-ms A))

\leq card (*set-mset* (*not-simplified-cls*(clauses_{NOT} S))) + 3 \wedge card (atms-of-ms A)

by (*meson* *Nat.le-trans* *atms-of-ms-finite* *simple-clss-card* *card-Un-le* *finite*
nat-add-left-cancel-le)

ultimately have card (*set-mset* (clauses_{NOT} T))

\leq card (*set-mset* (*not-simplified-cls*(clauses_{NOT} S))) + 3 \wedge card (atms-of-ms A)

by (*meson* *Nat.le-trans* *atms-of-ms-finite* *simple-clss-finite* *card-mono*
finite-UnI *finite-set-mset* *local.finite*)

moreover have ((2 + card (atms-of-ms A)) \wedge (1 + card (atms-of-ms A)) – μ_C' A T) * 2

\leq (2 + card (atms-of-ms A)) \wedge (1 + card (atms-of-ms A)) * 2

by *auto*

ultimately show *?thesis unfolding* μ_{CDCL}' -merged-def **by** *auto*

qed

sublocale *cdcl_{NOT}-increasing-restarts-ops* λ S T. T \sim *reduce-trail-to_{NOT}* ($\llbracket :: 'a$ list) S

cdcl_{NOT}-merged-bj-learn f

λ A S. *atms-of-mm* (clauses_{NOT} S) \subseteq *atms-of-ms* A

\wedge *atm-of* ' *lits-of-l* (trail S) \subseteq *atms-of-ms* A \wedge *finite* A

μ_{CDCL}' -merged

λ S. *inv* S \wedge *no-dup* (trail S)

μ_{CDCL}' -bound

apply *unfold-locales*

using *unbounded apply simp*

using *f-ge-1 apply force*

apply (*blast dest!*: *cdcl_{NOT}-merged-bj-learn-is-trancpl-cdcl_{NOT}* *trancpl-into-rtrancpl*
rtrancpl-cdcl_{NOT}-trail-clauses-bound)

apply (*simp add*: *cdcl_{NOT}-decreasing-measure*)

using *rtrancpl-cdcl_{NOT}-merged-bj-learn-clauses-bound-card* **apply** *blast*

apply (*drule* *rtrancpl-cdcl_{NOT}-merged-bj-learn-not-simplified-decreasing*)

apply (*auto dest!*: *simp*: *card-mono set-mset-mono*) \llbracket

apply *simp*

apply *auto* \llbracket

using *cdcl_{NOT}-merged-bj-learn-no-dup-inv* *cdcl-merged-inv* **apply** *blast*

apply (*auto simp*: *inv-restart*) \llbracket

done

lemma *cdcl_{NOT}-restart- μ_{CDCL}' -merged-le- μ_{CDCL}' -bound*:

assumes

cdcl_{NOT}-restart T V

```

  inv (fst T) and
  no-dup (trail (fst T)) and
  atms-of-mm (clausesNOT (fst T))  $\subseteq$  atms-of-ms A and
  atm-of ' lits-of-l (trail (fst T))  $\subseteq$  atms-of-ms A and
  finite A
shows  $\mu_{CDCL}'$ -merged A (fst V)  $\leq$   $\mu_{CDCL}'$ -bound A (fst T)
using assms
proof induction
  case (restart-full S T n)
  show ?case
    unfolding fst-conv
    apply (rule rtrncpl-cdclNOT-merged-bj-learn-clauses-bound-card)
    using restart-full unfolding full1-def by (force dest!: trncpl-into-rtrncpl)+
next
  case (restart-step m S T n U) note st = this(1) and U = this(3) and inv = this(4) and
    n-d = this(5) and atms-clss = this(6) and atms-trail = this(7) and finite = this(8)
  then have st': cdclNOT-merged-bj-learn** S T
    by (blast dest: relpowp-imp-rtrncpl)
  then have st'': cdclNOT** S T
    using inv n-d apply - by (rule rtrncpl-cdclNOT-merged-bj-learn-is-rtrncpl-cdclNOT) auto
  have inv T
    apply (rule rtrncpl-cdclNOT-merged-bj-learn-inv)
    using inv st' n-d by auto
  then have inv U
    using U by (auto simp: inv-restart)
  have atms-of-mm (clausesNOT T)  $\subseteq$  atms-of-ms A
    using rtrncpl-cdclNOT-trail-clauses-bound[OF st''] inv atms-clss atms-trail n-d
    by simp
  then have atms-of-mm (clausesNOT U)  $\subseteq$  atms-of-ms A
    using U by simp
  have not-simplified-cls (clausesNOT U)  $\subseteq$  # not-simplified-cls (clausesNOT T)
    using  $\langle U \sim \text{reduce-trail-to}_{NOT} [] T \rangle$  by auto
  moreover have not-simplified-cls (clausesNOT T)  $\subseteq$  # not-simplified-cls (clausesNOT S)
    apply (rule rtrncpl-cdclNOT-merged-bj-learn-not-simplified-decreasing)
    using  $\langle \text{cdcl}_{NOT}\text{-merged-bj-learn} \widetilde{\sim} m \rangle S T$  by (auto dest!: relpowp-imp-rtrncpl)
  ultimately have U-S: not-simplified-cls (clausesNOT U)  $\subseteq$  # not-simplified-cls (clausesNOT S)
    by auto

  have (set-mset (clausesNOT U))
     $\subseteq$  set-mset (not-simplified-cls (clausesNOT U))  $\cup$  simple-clss (atms-of-ms A)
    apply (rule rtrncpl-cdclNOT-merged-bj-learn-clauses-bound)
    apply simp
    using  $\langle \text{inv } U \rangle$  apply simp
    using  $\langle \text{atms-of-mm (clauses}_{NOT} U) \subseteq \text{atms-of-ms A} \rangle$  apply simp
    using U apply simp
    using U apply simp
    using finite apply simp
  done
  then have f1: card (set-mset (clausesNOT U))  $\leq$  card (set-mset (not-simplified-cls (clausesNOT U))
     $\cup$  simple-clss (atms-of-ms A))
    by (simp add: simple-clss-finite card-mono local.finite)

  moreover have set-mset (not-simplified-cls (clausesNOT U))  $\cup$  simple-clss (atms-of-ms A)
     $\subseteq$  set-mset (not-simplified-cls (clausesNOT S))  $\cup$  simple-clss (atms-of-ms A)
    using U-S by auto

```

then have $f2$:

$\text{card } (\text{set-mset } (\text{not-simplified-cls } (\text{clauses}_{NOT} U)) \cup \text{simple-clss } (\text{atms-of-ms } A))$
 $\leq \text{card } (\text{set-mset } (\text{not-simplified-cls } (\text{clauses}_{NOT} S)) \cup \text{simple-clss } (\text{atms-of-ms } A))$
 by (*simp add: simple-clss-finite card-mono local.finite*)

moreover have $\text{card } (\text{set-mset } (\text{not-simplified-cls } (\text{clauses}_{NOT} S)) \cup \text{simple-clss } (\text{atms-of-ms } A))$
 $\leq \text{card } (\text{set-mset } (\text{not-simplified-cls } (\text{clauses}_{NOT} S))) + \text{card } (\text{simple-clss } (\text{atms-of-ms } A))$
 using *card-Un-le* by *blast*

moreover have $\text{card } (\text{simple-clss } (\text{atms-of-ms } A)) \leq 3 \wedge \text{card } (\text{atms-of-ms } A)$
 using *atms-of-ms-finite simple-clss-card local.finite* by *blast*

ultimately have $\text{card } (\text{set-mset } (\text{clauses}_{NOT} U))$
 $\leq \text{card } (\text{set-mset } (\text{not-simplified-cls } (\text{clauses}_{NOT} S))) + 3 \wedge \text{card } (\text{atms-of-ms } A)$
 by *linarith*

then show ?case **unfolding** μ_{CDCL}' -merged-def by *auto*

qed

lemma cdcl_{NOT} -restart- μ_{CDCL}' -bound-le- μ_{CDCL}' -bound:

assumes

cdcl_{NOT} -restart $T V$ **and**

no-dup (*trail* (*fst* T)) **and**

inv (*fst* T) **and**

fin: *finite* A

shows μ_{CDCL}' -bound A (*fst* V) $\leq \mu_{CDCL}'$ -bound A (*fst* T)

using *assms(1-3)*

proof *induction*

case (*restart-full* $S T n$)

have $\text{not-simplified-cls } (\text{clauses}_{NOT} T) \subseteq \# \text{not-simplified-cls } (\text{clauses}_{NOT} S)$

apply (*rule rtranclp-cdcl_{NOT}-merged-bj-learn-not-simplified-decreasing*)

using $\langle \text{full1 } \text{cdcl}_{NOT}\text{-merged-bj-learn } S T \rangle$ **unfolding** *full1-def*

by (*auto dest: tranclp-into-rtranclp*)

then show ?case **by** (*auto simp: card-mono set-mset-mono*)

next

case (*restart-step* $m S T n U$) **note** $st = \text{this}(1)$ **and** $U = \text{this}(3)$ **and** $n-d = \text{this}(4)$ **and** $inv = \text{this}(5)$

then have $st': \text{cdcl}_{NOT}\text{-merged-bj-learn}^{**} S T$

by (*blast dest: relpowp-imp-rtranclp*)

then have $st'': \text{cdcl}_{NOT}^{**} S T$

using *inv n-d* **apply** – **by** (*rule rtranclp-cdcl_{NOT}-merged-bj-learn-is-rtranclp-cdcl_{NOT}*) *auto*

have *inv* T

apply (*rule rtranclp-cdcl_{NOT}-merged-bj-learn-inv*)

using *inv st' n-d* **by** *auto*

then have *inv* U

using U **by** (*auto simp: inv-restart*)

have $\text{not-simplified-cls } (\text{clauses}_{NOT} U) \subseteq \# \text{not-simplified-cls } (\text{clauses}_{NOT} T)$

using $\langle U \sim \text{reduce-trail-to}_{NOT} [] T \rangle$ **by** *auto*

moreover have $\text{not-simplified-cls } (\text{clauses}_{NOT} T) \subseteq \# \text{not-simplified-cls } (\text{clauses}_{NOT} S)$

apply (*rule rtranclp-cdcl_{NOT}-merged-bj-learn-not-simplified-decreasing*)

using $\langle (\text{cdcl}_{NOT}\text{-merged-bj-learn} \sim m) S T \rangle$ **by** (*auto dest!: relpowp-imp-rtranclp*)

ultimately have $U-S: \text{not-simplified-cls } (\text{clauses}_{NOT} U) \subseteq \# \text{not-simplified-cls } (\text{clauses}_{NOT} S)$

by *auto*

then show ?case **by** (*auto simp: card-mono set-mset-mono*)

qed

sublocale *cdcl_{NOT}-increasing-restarts* - - - - - *f*
 $\lambda S T. T \sim \text{reduce-trail-to}_{NOT} ([::'a \text{ list}] S$
 $\lambda A S. \text{atms-of-mm} (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A$
 $\wedge \text{atm-of ' lits-of-l} (\text{trail } S) \subseteq \text{atms-of-ms } A \wedge \text{finite } A$
 $\mu_{CDCL}'\text{-merged } \text{cdcl}_{NOT}\text{-merged-bj-learn}$
 $\lambda S. \text{inv } S \wedge \text{no-dup} (\text{trail } S)$
 $\lambda A T. ((2 + \text{card} (\text{atms-of-ms } A)) \wedge (1 + \text{card} (\text{atms-of-ms } A))) * 2$
 $+ \text{card} (\text{set-mset} (\text{not-simplified-cl}(\text{clauses}_{NOT} T)))$
 $+ 3 \wedge \text{card} (\text{atms-of-ms } A)$
apply *unfold-locales*
using *cdcl_{NOT}-restart- μ_{CDCL}' -merged-le- μ_{CDCL}' -bound* **apply** *force*
using *cdcl_{NOT}-restart- μ_{CDCL}' -bound-le- μ_{CDCL}' -bound* **by** *fastforce*

lemma *cdcl_{NOT}-restart-eq-sat-iff*:

assumes
 $\text{cdcl}_{NOT}\text{-restart } S T$ **and**
 $\text{no-dup} (\text{trail} (\text{fst } S))$
 $\text{inv} (\text{fst } S)$
shows $I \models_{\text{sextm}} \text{clauses}_{NOT} (\text{fst } S) \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{NOT} (\text{fst } T)$
using *assms*

proof (*induction rule: cdcl_{NOT}-restart.induct*)

case (*restart-full* $S T n$)
then have $\text{cdcl}_{NOT}\text{-merged-bj-learn}^{**} S T$
by (*simp add: tranclp-into-rtranclp full1-def*)
then show *?case*
using *rtranclp-cdcl_{NOT}-bj-sat-ext-iff restart-full.prem1,2*
rtranclp-cdcl_{NOT}-merged-bj-learn-is-rtranclp-cdcl_{NOT} **by** *auto*

next

case (*restart-step* $m S T n U$)
then have $\text{cdcl}_{NOT}\text{-merged-bj-learn}^{**} S T$
by (*auto simp: tranclp-into-rtranclp full1-def dest!: relpowp-imp-rtranclp*)
then have $I \models_{\text{sextm}} \text{clauses}_{NOT} S \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{NOT} T$
using *rtranclp-cdcl_{NOT}-bj-sat-ext-iff restart-step.prem1,2*
rtranclp-cdcl_{NOT}-merged-bj-learn-is-rtranclp-cdcl_{NOT} **by** *auto*
moreover have $I \models_{\text{sextm}} \text{clauses}_{NOT} T \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{NOT} U$
using *restart-step.hyps(3)* **by** *auto*
ultimately show *?case* **by** *auto*

qed

lemma *rtranclp-cdcl_{NOT}-restart-eq-sat-iff*:

assumes
 $\text{cdcl}_{NOT}\text{-restart}^{**} S T$ **and**
 $\text{inv: inv} (\text{fst } S)$ **and** $n\text{-d: no-dup}(\text{trail} (\text{fst } S))$
shows $I \models_{\text{sextm}} \text{clauses}_{NOT} (\text{fst } S) \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{NOT} (\text{fst } T)$
using *assms(1)*

proof (*induction rule: rtranclp-induct*)

case *base*
then show *?case* **by** *simp*

next

case (*step* $T U$) **note** $st = \text{this}(1)$ **and** $cdcl = \text{this}(2)$ **and** $IH = \text{this}(3)$
have $\text{inv} (\text{fst } T)$ **and** $\text{no-dup} (\text{trail} (\text{fst } T))$
using *rtranclp-cdcl_{NOT}-with-restart-cdcl_{NOT}-inv* **using** $st \text{ inv } n\text{-d}$ **by** *blast+*
then have $I \models_{\text{sextm}} \text{clauses}_{NOT} (\text{fst } T) \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{NOT} (\text{fst } U)$
using *cdcl_{NOT}-restart-eq-sat-iff cdcl* **by** *blast*
then show *?case* **using** IH **by** *blast*

qed

lemma *cdcl_{NOT}-restart-all-decomposition-implies-m*:

assumes

cdcl_{NOT}-restart *S T* **and**

inv: *inv* (*fst S*) **and** *n-d*: *no-dup*(*trail* (*fst S*)) **and**

all-decomposition-implies-m (*clauses_{NOT}* (*fst S*))

(*get-all-marked-decomposition* (*trail* (*fst S*)))

shows *all-decomposition-implies-m* (*clauses_{NOT}* (*fst T*))

(*get-all-marked-decomposition* (*trail* (*fst T*)))

using *assms*

proof (*induction*)

case (*restart-full* *S T n*) **note** *full* = *this*(1) **and** *inv* = *this*(2) **and** *n-d* = *this*(3) **and**

decomp = *this*(4)

have *st*: *cdcl_{NOT}-merged-bj-learn*** *S T* **and**

n-s: *no-step cdcl_{NOT}-merged-bj-learn* *T*

using *full* **unfolding** *full1-def* **by** (*fast dest*: *trancpl-into-rtrancpl*)+

have *st'*: *cdcl_{NOT}*** *S T*

using *inv rtrancpl-cdcl_{NOT}-merged-bj-learn-is-rtrancpl-cdcl_{NOT}-and-inv st n-d* **by** *auto*

have *inv T*

using *rtrancpl-cdcl_{NOT}-cdcl_{NOT}-inv[OF st]* *inv n-d* **by** *auto*

then show ?*case*

using *rtrancpl-cdcl_{NOT}-all-decomposition-implies[OF - - n-d decomp]* *st' inv* **by** *auto*

next

case (*restart-step* *m S T n U*) **note** *st* = *this*(1) **and** *U* = *this*(3) **and** *inv* = *this*(4) **and**

n-d = *this*(5) **and** *decomp* = *this*(6)

show ?*case* **using** *U* **by** *auto*

qed

lemma *rtrancpl-cdcl_{NOT}-restart-all-decomposition-implies-m*:

assumes

*cdcl_{NOT}-restart*** *S T* **and**

inv: *inv* (*fst S*) **and** *n-d*: *no-dup*(*trail* (*fst S*)) **and**

decomp: *all-decomposition-implies-m* (*clauses_{NOT}* (*fst S*))

(*get-all-marked-decomposition* (*trail* (*fst S*)))

shows *all-decomposition-implies-m* (*clauses_{NOT}* (*fst T*))

(*get-all-marked-decomposition* (*trail* (*fst T*)))

using *assms*

proof (*induction*)

case *base*

then show ?*case* **using** *decomp* **by** *simp*

next

case (*step* *T U*) **note** *st* = *this*(1) **and** *cdcl* = *this*(2) **and** *IH* = *this*(3)[*OF this*(4-)] **and**

inv = *this*(4) **and** *n-d* = *this*(5) **and** *decomp* = *this*(6)

have *inv* (*fst T*) **and** *no-dup* (*trail* (*fst T*))

using *rtrancpl-cdcl_{NOT}-with-restart-cdcl_{NOT}-inv* **using** *st inv n-d* **by** *blast+*

then show ?*case*

using *cdcl_{NOT}-restart-all-decomposition-implies-m[OF cdcl]* *IH* **by** *auto*

qed

lemma *full-cdcl_{NOT}-restart-normal-form*:

assumes

full: *full cdcl_{NOT}-restart* *S T* **and**

inv: *inv* (*fst S*) **and** *n-d*: *no-dup*(*trail* (*fst S*)) **and**

decomp: *all-decomposition-implies-m* (*clauses_{NOT}* (*fst S*))

(get-all-marked-decomposition (trail (fst S))) and
 atms-cls: atms-of-mm (clauses_{NOT} (fst S)) \subseteq atms-of-ms A and
 atms-trail: atm-of ' lits-of-l (trail (fst S)) \subseteq atms-of-ms A and
 fin: finite A
shows unsatisfiable (set-mset (clauses_{NOT} (fst S)))
 \vee lits-of-l (trail (fst T)) \models_{sextm} clauses_{NOT} (fst S) \wedge satisfiable (set-mset (clauses_{NOT} (fst S)))
proof –
have inv-T: inv (fst T) and n-d-T: no-dup (trail (fst T))
using rtrancpl-cdcl_{NOT}-with-restart-cdcl_{NOT}-inv **using** full inv n-d **unfolding** full-def **by** blast+
moreover have
 atms-cls-T: atms-of-mm (clauses_{NOT} (fst T)) \subseteq atms-of-ms A and
 atms-trail-T: atm-of ' lits-of-l (trail (fst T)) \subseteq atms-of-ms A
using rtrancpl-cdcl_{NOT}-with-restart-bound-inv[of S T A] full atms-cls atms-trail fin inv n-d
unfolding full-def **by** blast+
ultimately have no-step cdcl_{NOT}-merged-bj-learn (fst T)
apply –
apply (rule no-step-cdcl_{NOT}-restart-no-step-cdcl_{NOT}[of - A])
using full **unfolding** full-def **apply** simp
apply simp
using fin **apply** simp
done
moreover have all-decomposition-implies-m (clauses_{NOT} (fst T))
 (get-all-marked-decomposition (trail (fst T)))
using rtrancpl-cdcl_{NOT}-restart-all-decomposition-implies-m[of S T] inv n-d decomp
 full **unfolding** full-def **by** auto
ultimately have unsatisfiable (set-mset (clauses_{NOT} (fst T)))
 \vee trail (fst T) \models_{asm} clauses_{NOT} (fst T) \wedge satisfiable (set-mset (clauses_{NOT} (fst T)))
apply –
apply (rule cdcl_{NOT}-merged-bj-learn-final-state)
using atms-cls-T atms-trail-T fin n-d-T fin inv-T **by** blast+
then consider
 (unsat) unsatisfiable (set-mset (clauses_{NOT} (fst T)))
 | (sat) trail (fst T) \models_{asm} clauses_{NOT} (fst T) and satisfiable (set-mset (clauses_{NOT} (fst T)))
by auto
then show unsatisfiable (set-mset (clauses_{NOT} (fst S)))
 \vee lits-of-l (trail (fst T)) \models_{sextm} clauses_{NOT} (fst S) \wedge satisfiable (set-mset (clauses_{NOT} (fst S)))
proof cases
case unsat
then have unsatisfiable (set-mset (clauses_{NOT} (fst S)))
unfolding satisfiable-def **apply** auto
using rtrancpl-cdcl_{NOT}-restart-eq-sat-iff[of S T] full inv n-d
 consistent-true-clss-ext-satisfiable true-clss-imp-true-clss-ext
unfolding satisfiable-def full-def **by** blast
then show ?thesis **by** blast
next
case sat
then have lits-of-l (trail (fst T)) \models_{sextm} clauses_{NOT} (fst T)
using true-clss-imp-true-clss-ext **by** (auto simp: true-annots-true-clss)
then have lits-of-l (trail (fst T)) \models_{sextm} clauses_{NOT} (fst S)
using rtrancpl-cdcl_{NOT}-restart-eq-sat-iff[of S T] full inv n-d **unfolding** full-def **by** blast
moreover then have satisfiable (set-mset (clauses_{NOT} (fst S)))
using consistent-true-clss-ext-satisfiable distinct-consistent-interp n-d-T **by** fast
ultimately show ?thesis **by** fast
qed
qed

corollary *full-cdcl_{NOT}-restart-normal-form-init-state*:
assumes
init-state: $\text{trail } S = [] \text{ clauses}_{NOT} S = N$ **and**
full: *full cdcl_{NOT}-restart* ($S, 0$) T **and**
inv: *inv* S
shows *unsatisfiable* (*set-mset* N)
 \vee *lits-of-l* (*trail* (*fst* T)) $\models_{sextm} N \wedge \text{satisfiable } (\text{set-mset } N)$
using *full-cdcl_{NOT}-restart-normal-form*[*of* ($S, 0$) T] *assms* **by** *auto*

end

end
theory *DPLL-NOT*
imports *CDCL-NOT*
begin

16 DPLL as an instance of NOT

16.1 DPLL with simple backtrack

We are using a concrete couple instead of an abstract state.

locale *dpll-with-backtrack*
begin

inductive *backtrack* :: ($'v, \text{unit}, \text{unit}$) *marked-lit list* \times $'v$ *clauses*
 \Rightarrow ($'v, \text{unit}, \text{unit}$) *marked-lit list* \times $'v$ *clauses* \Rightarrow *bool* **where**
backtrack-split (*fst* S) = ($M', L \# M$) \Longrightarrow *is-marked* $L \Longrightarrow D \in \# \text{ snd } S$
 $\Longrightarrow \text{fst } S \models_{as} CNot D \Longrightarrow \text{backtrack } S (\text{Propagated } (- (\text{lit-of } L)) () \# M, \text{snd } S)$

inductive-cases *backtrackE*[*elim*]: *backtrack* (M, N) (M', N')

lemma *backtrack-is-backjump*:

fixes $M M' :: (\text{'v}, \text{unit}, \text{unit})$ *marked-lit list*

assumes

backtrack: *backtrack* (M, N) (M', N') **and**

no-dup: (*no-dup* \circ *fst*) (M, N) **and**

decomp: *all-decomposition-implies-m* N (*get-all-marked-decomposition* M)

shows

$\exists C F' K F L l C'.$

$M = F' @ \text{Marked } K () \# F \wedge$

$M' = \text{Propagated } L l \# F \wedge N = N' \wedge C \in \# N \wedge F' @ \text{Marked } K d \# F \models_{as} CNot C \wedge$

undefined-lit $F L \wedge \text{atm-of } L \in \text{atms-of-mm } N \cup \text{atm-of } ' \text{ lits-of-l } (F' @ \text{Marked } K d \# F) \wedge$

$N \models_{pm} C' + \{\#L\# \} \wedge F \models_{as} CNot C'$

proof –

let $?S = (M, N)$

let $?T = (M', N')$

obtain $F F' P L D$ **where**

b-sp: *backtrack-split* $M = (F', L \# F)$ **and**

is-marked L **and**

$D \in \# \text{ snd } ?S$ **and**

$M \models_{as} CNot D$ **and**

bt: *backtrack* $?S (\text{Propagated } (- (\text{lit-of } L)) P \# F, N)$ **and**

M' : $M' = \text{Propagated } (- (\text{lit-of } L)) P \# F$ **and**

[*simp*]: $N' = N$

using *backtrackE*[*OF backtrack*] **by** (*metis backtrack fstI sndI*)

let $?K = \text{lit-of } L$

```

let ?C = image-mset lit-of {#K ∈ #mset M. is-marked K ∧ K ≠ L#} :: 'v literal multiset
let ?C' = set-mset (image-mset single (?C + {#?K#}))
obtain K where L: L = Marked K () using ⟨is-marked L⟩ by (cases L) auto

have M: M = F' @ Marked K () # F
  using b-sp by (metis L backtrack-split-list-eq fst-conv snd-conv)
moreover have F' @ Marked K () # F ⊨as CNot D
  using ⟨M ⊨as CNot D⟩ unfolding M .
moreover have undefined-lit F (−?K)
  using no-dup unfolding M L by (simp add: defined-lit-map)
moreover have atm-of (−K) ∈ atms-of-mm N ∪ atm-of ' lits-of-l (F' @ Marked K d # F)
  by auto
moreover
have set-mset N ∪ ?C' ⊨ps {{#}}
proof −
  have A: set-mset N ∪ ?C' ∪ unmark-l M =
    set-mset N ∪ unmark-l M
  unfolding M L by auto
  have set-mset N ∪ {{#lit-of L#} | L. is-marked L ∧ L ∈ set M}
    ⊨ps unmark-l M
  using all-decomposition-implies-propagated-lits-are-implied[OF decomp] .
  moreover have C': ?C' = {{#lit-of L#} | L. is-marked L ∧ L ∈ set M}
  unfolding M L apply standard
  apply force
  using IntI by auto
ultimately have N-C-M: set-mset N ∪ ?C' ⊨ps unmark-l M
  by auto
have set-mset N ∪ (λL. {#lit-of L#}) ' (set M) ⊨ps {{#}}
  unfolding true-clss-clss-def
proof (intro allI impI, goal-cases)
  case (1 I) note tot = this(1) and cons = this(2) and I-N-M = this(3)
  have I ⊨ D
    using I-N-M ⟨D ∈ # snd ?S⟩ unfolding true-clss-def by auto
  moreover have I ⊨s CNot D
    using ⟨M ⊨as CNot D⟩ unfolding M by (metis 1(3) ⟨M ⊨as CNot D⟩
      true-annots-true-clss true-clss-mono-set-mset-l true-clss-def
      true-clss-singleton-lit-of-implies-incl true-clss-union)
  ultimately show ?case using cons consistent-CNot-not by blast
qed
then show ?thesis
  using true-clss-clss-left-right[OF N-C-M, of {{#}}] unfolding A by auto
qed
have N ⊨pm image-mset uminus ?C + {#−?K#}
  unfolding true-clss-clss-def true-clss-clss-def total-over-m-def
proof (intro allI impI)
  fix I
  assume
    tot: total-over-set I (atms-of-ms (set-mset N ∪ {image-mset uminus ?C + {#−?K#}})) and
    cons: consistent-interp I and
    I ⊨sm N
  have (K ∈ I ∧ −K ∉ I) ∨ (−K ∈ I ∧ K ∉ I)
    using cons tot unfolding consistent-interp-def L by (cases K) auto
  have {a ∈ set M. is-marked a ∧ a ≠ Marked K ()} =
    set M ∩ {L. is-marked L ∧ L ≠ Marked K ()}
  by auto

```

then have
tI: *total-over-set* *I* (*atm-of* ‘*lit-of* ‘ (*set* *M* \cap {*L*. *is-marked* *L* \wedge *L* \neq *Marked* *K* *d*}))
using *tot by* (*auto simp add*: *L* *atms-of-uminus-lit-atm-of-lit-of*)

then have *H*: $\bigwedge x$.
lit-of *x* \notin *I* \implies *x* \in *set* *M* \implies *is-marked* *x*
 \implies *x* \neq *Marked* *K* *d* \implies \neg *lit-of* *x* \in *I*

proof –
fix *x* :: (*v*, *unit*, *unit*) *marked-lit*
assume *a1*: *x* \neq *Marked* *K* *d*
assume *a2*: *is-marked* *x*
assume *a3*: *x* \in *set* *M*
assume *a4*: *lit-of* *x* \notin *I*
have *atm-of* (*lit-of* *x*) \in *atm-of* ‘*lit-of* ‘
(*set* *M* \cap {*m*. *is-marked* *m* \wedge *m* \neq *Marked* *K* *d*}))
using *a3 a2 a1 by blast*
then have *Pos* (*atm-of* (*lit-of* *x*)) \in *I* \vee *Neg* (*atm-of* (*lit-of* *x*)) \in *I*
using *tI unfolding total-over-set-def by blast*
then show \neg *lit-of* *x* \in *I*
using *a4 by* (*metis* (*no-types*) *atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set*
literal.sel(1,2))

qed
have $\neg I \models_s ?C'$
using (*set-mset* *N* \cup *?C'* \models_{ps} {*{#}*} *tot cons* (*I* \models_{sm} *N*))
unfolding *true-clss-clss-def total-over-m-def*
by (*simp add*: *atms-of-uminus-lit-atm-of-lit-of atms-of-ms-single-image-atm-of-lit-of*)
then show *I* \models *image-mset* *uminus* *?C* + {*{#}* – *lit-of* *L* *#*}
unfolding *true-clss-def true-cls-def*
using (*K* \in *I* \wedge $\neg K \notin I$) \vee ($\neg K \in I \wedge K \notin I$)
unfolding *L* **by** (*auto dest!*: *H*)

qed
moreover
have *set* *F'* \cap {*K*. *is-marked* *K* \wedge *K* \neq *L*} = {}
using *backtrack-split-fst-not-marked*[*of* - *M*] *b-sp* **by** *auto*
then have *F* \models_{as} *CNot* (*image-mset* *uminus* *?C*)
unfolding *M CNot-def true-annots-def* **by** (*auto simp add*: *L* *lits-of-def*)
ultimately show *?thesis*
using *M'* (*D* \in *#* *snd* *?S*) *L* **by** *force*

qed

lemma *backtrack-is-backjump'*:
fixes *M M'* :: (*v*, *unit*, *unit*) *marked-lit* *list*
assumes
backtrack: *backtrack* *S* *T* **and**
no-dup: (*no-dup* \circ *fst*) *S* **and**
decomp: *all-decomposition-implies-m* (*snd* *S*) (*get-all-marked-decomposition* (*fst* *S*))
shows
 $\exists C F' K F L l C'$.
fst *S* = *F'* @ *Marked* *K* () $\#$ *F* \wedge
T = (*Propagated* *L* l $\#$ *F*, *snd* *S*) \wedge *C* \in *#* *snd* *S* \wedge *fst* *S* \models_{as} *CNot* *C*
 \wedge *undefined-lit* *F* *L* \wedge *atm-of* *L* \in *atms-of-mm* (*snd* *S*) \cup *atm-of* ‘*lits-of-l* (*fst* *S*) \wedge
snd *S* \models_{pm} *C'* + {*{#L#}*} \wedge *F* \models_{as} *CNot* *C'*

apply (*cases* *S*, *cases* *T*)
using *backtrack-is-backjump*[*of* *fst* *S* *snd* *S* *fst* *T* *snd* *T*] *assms* **by** *fastforce*

sublocale *dpll-state*

id $\lambda L C. C + \{\#L\# \}$ *remove1-mset*
id $op + op \in \# \lambda L C. C + \{\#L\# \}$ *remove1-mset*
fst snd $\lambda L (M, N). (L \# M, N) \lambda (M, N). (tl M, N)$
 $\lambda C (M, N). (M, \{\#C\# \} + N) \lambda C (M, N). (M, removeAll-mset C N)$
by *unfold-locales* (*auto simp: ac-simps*)

sublocale *backjumping-ops*

id $\lambda L C. C + \{\#L\# \}$ *remove1-mset*
id $op + op \in \# \lambda L C. C + \{\#L\# \}$ *remove1-mset*
fst snd $\lambda L (M, N). (L \# M, N) \lambda (M, N). (tl M, N)$
 $\lambda C (M, N). (M, \{\#C\# \} + N) \lambda C (M, N). (M, removeAll-mset C N) \lambda - - S T. backtrack S T$
by *unfold-locales*

lemma *reduce-trail-to_{NOT}-snd*:

snd (*reduce-trail-to_{NOT}* *F S*) = *snd S*
apply (*induction F S rule: reduce-trail-to_{NOT}.induct*)
by (*cases S, rename-tac F Sa, case-tac Sa*)
(simp add: less-imp-diff-less reduce-trail-to_{NOT}.simps)

lemma *reduce-trail-to_{NOT}*:

reduce-trail-to_{NOT} *F S* =
(if length (fst S) \geq length F
then drop (length (fst S) - length F) (fst S)
else [],
snd S) (**is** *?R = ?C*)

proof –

have *?R = (fst ?R, snd ?R)*
by *auto*
also have *(fst ?R, snd ?R) = ?C*
by (*auto simp: trail-reduce-trail-to_{NOT}-drop reduce-trail-to_{NOT}-snd*)
finally show *?thesis* .

qed

lemma *backtrack-is-backjump''*:

fixes *M M' :: ('v, unit, unit) marked-lit list*
assumes
backtrack: backtrack S T **and**
no-dup: (no-dup \circ fst) S **and**
decomp: all-decomposition-implies-m (snd S) (get-all-marked-decomposition (fst S))
shows *backjump S T*

proof –

obtain *C F' K F L l C'* **where**
1: *fst S = F' @ Marked K () # F* **and**
2: *T = (Propagated L l # F, snd S)* **and**
3: *C $\in \#$ snd S* **and**
4: *fst S \models_{as} CNot C* **and**
5: *undefined-lit F L* **and**
6: *atm-of L \in atms-of-mm (snd S) \cup atm-of ' lits-of-l (fst S)* **and**
7: *snd S \models_{pm} C' + {\#L\#}* **and**
8: *F \models_{as} CNot C'*
using *backtrack-is-backjump'[OF assms]* **by** *force*
show *?thesis*
apply (*cases S*)
using *backjump.intros[OF 1 - - 4 5 - - 8, of T] 2 backtrack 1 5 3 6 7*

```

    by (auto simp: state-eqNOT-def trail-reduce-trail-toNOT-drop
        reduce-trail-toNOT simp del: state-simpNOT)
qed

lemma can-do-bt-step:
  assumes
    M: fst S = F' @ Marked K d # F and
    C ∈# snd S and
    C: fst S ⊨as CNot C
  shows ¬ no-step backtrack S
proof -
  obtain L G' G where
    backtrack-split (fst S) = (G', L # G)
  unfolding M by (induction F' rule: marked-lit-list-induct) auto
  moreover then have is-marked L
    by (metis backtrack-split-snd-hd-marked list.distinct(1) list.sel(1) snd-conv)
  ultimately show ?thesis
    using backtrack.intros[of S G' L G C] ⟨C ∈# snd S⟩ C unfolding M by auto
qed

end

sublocale dpll-with-backtrack ⊆ dpll-with-backjumping-ops
  id λL C. C + {#L#} remove1-mset
  id op + op ∈# λL C. C + {#L#} remove1-mset
  fst snd λL (M, N). (L # M, N)
  λ(M, N). (tl M, N) λC (M, N). (M, {#C#} + N) λC (M, N). (M, removeAll-mset C N)
  λ(M, N). no-dup M ∧ all-decomposition-implies-m N (get-all-marked-decomposition M)
  λ- - S T. backtrack S T
  λ- -. True
  apply unfold-locales
  by (metis (mono-tags, lifting) case-prod-beta comp-def dpll-with-backtrack.backtrack-is-backjump''
      dpll-with-backtrack.can-do-bt-step id-apply)

sublocale dpll-with-backtrack ⊆ dpll-with-backjumping
  id λL C. C + {#L#} remove1-mset
  id op + op ∈# λL C. C + {#L#} remove1-mset
  fst snd λL (M, N). (L # M, N)
  λ(M, N). (tl M, N) λC (M, N). (M, {#C#} + N) λC (M, N). (M, removeAll-mset C N)
  λ(M, N). no-dup M ∧ all-decomposition-implies-m N (get-all-marked-decomposition M)
  λ- - S T. backtrack S T
  λ- -. True
  apply unfold-locales
  using dpll-bj-no-dup dpll-bj-all-decomposition-implies-inv apply fastforce
done

context dpll-with-backtrack
begin
term learn
end

context dpll-with-backtrack
begin
lemma wf-tranclp-dpll-inital-state:

```

assumes *fin*: *finite A*
shows *wf* $\{((M'::('v, unit, unit) \text{ marked-lits}, N'::'v \text{ clauses}), ([], N)) \mid M' N' N.$
 $dpll\text{-}bj^{++} ([], N) (M', N') \wedge \text{atms-of-mm } N \subseteq \text{atms-of-ms } A\}$
using *wf-tranclp-dpll-bj*[*OF assms(1)*] **by** (*rule wf-subset*) *auto*

corollary *full-dpll-final-state-conclusive*:

fixes *M M' :: ('v, unit, unit) marked-lit list*
assumes
full: *full dpll-bj* ($[], N$) (*M', N'*)
shows *unsatisfiable* (*set-mset N*) $\vee (M' \models_{asm} N \wedge \text{satisfiable } (\text{set-mset } N))$
using *assms full-dpll-backjump-final-state*[*of* ($[], N$) (*M', N'*) *set-mset N*] **by** *auto*

corollary *full-dpll-normal-form-from-init-state*:

fixes *M M' :: ('v, unit, unit) marked-lit list*
assumes
full: *full dpll-bj* ($[], N$) (*M', N'*)
shows $M' \models_{asm} N \longleftrightarrow \text{satisfiable } (\text{set-mset } N)$

proof –

have *no-dup M'*
using *rtranclp-dpll-bj-no-dup*[*of* ($[], N$) (*M', N'*)]
full unfolding full-def **by** *auto*
then have $M' \models_{asm} N \implies \text{satisfiable } (\text{set-mset } N)$
using *distinct-consistent-interp satisfiable-carac'* *true-annots-true-cls* **by** *blast*
then show *?thesis*
using *full-dpll-final-state-conclusive*[*OF full*] **by** *auto*

qed

interpretation *conflict-driven-clause-learning-ops*

id $\lambda L C. C + \{\#L\# \} \text{ remove1-mset}$
id $op + op \in \# \lambda L C. C + \{\#L\# \} \text{ remove1-mset}$
fst snd $\lambda L (M, N). (L \# M, N)$
 $\lambda (M, N). (tl M, N) \lambda C (M, N). (M, \{\#C\# \} + N) \lambda C (M, N). (M, \text{removeAll-mset } C N)$
 $\lambda (M, N). \text{no-dup } M \wedge \text{all-decomposition-implies-m } N (\text{get-all-marked-decomposition } M)$
 $\lambda - - S T. \text{backtrack } S T$
 $\lambda - -. \text{True } \lambda - -. \text{False } \lambda - -. \text{False}$
by *unfold-locales*

interpretation *conflict-driven-clause-learning*

id $\lambda L C. C + \{\#L\# \} \text{ remove1-mset}$
id $op + op \in \# \lambda L C. C + \{\#L\# \} \text{ remove1-mset}$
fst snd $\lambda L (M, N). (L \# M, N)$
 $\lambda (M, N). (tl M, N) \lambda C (M, N). (M, \{\#C\# \} + N) \lambda C (M, N). (M, \text{removeAll-mset } C N)$
 $\lambda (M, N). \text{no-dup } M \wedge \text{all-decomposition-implies-m } N (\text{get-all-marked-decomposition } M)$
 $\lambda - - S T. \text{backtrack } S T$
 $\lambda - -. \text{True } \lambda - -. \text{False } \lambda - -. \text{False}$
apply *unfold-locales*
using *cdcl_{NOT}-all-decomposition-implies cdcl_{NOT}-no-dup* **by** *fastforce*

lemma *cdcl_{NOT}-is-dpll*:

$cdcl_{NOT} S T \longleftrightarrow dpll\text{-}bj S T$
by (*auto simp: cdcl_{NOT}.simps learn.simps forget_{NOT}.simps*)

Another proof of termination:

lemma *wf* $\{(T, S). dpll\text{-}bj S T \wedge cdcl_{NOT}\text{-}NOT\text{-all-inv } A S\}$
unfolding *cdcl_{NOT}-is-dpll*[*symmetric*]


```

by (rule wf-cdclNOT-no-learn-and-forget-infinite-chain)
(auto simp: learn.simps forgetNOT.simps)
end

```

16.2 Adding restarts

This was mainly a test whether it was possible to instantiate the assumption of the locale.

```

locale dpll-withbacktrack-and-restarts =
  dpll-with-backtrack +
  fixes f :: nat  $\Rightarrow$  nat
  assumes unbounded: unbounded f and f-ge-1:  $\bigwedge n. n \geq 1 \implies f\ n \geq 1$ 
begin
  sublocale cdclNOT-increasing-restarts
    id  $\lambda L\ C. C + \{\#L\# \}$  remove1-mset
    id op + op  $\in \# \lambda L\ C. C + \{\#L\# \}$  remove1-mset
  fst snd  $\lambda L\ (M, N). (L \# M, N) \lambda (M, N). (tl\ M, N)$ 
     $\lambda C\ (M, N). (M, \{\#C\# \} + N) \lambda C\ (M, N). (M, removeAll-mset\ C\ N) f \lambda (-, N)\ S. S = ([], N)$ 
   $\lambda A\ (M, N). atms-of-mm\ N \subseteq atms-of-ms\ A \wedge atm-of\ ' lits-of-l\ M \subseteq atms-of-ms\ A \wedge finite\ A$ 
     $\wedge all-decomposition-implies-m\ N\ (get-all-marked-decomposition\ M)$ 
   $\lambda A\ T. (2 + card\ (atms-of-ms\ A)) \wedge (1 + card\ (atms-of-ms\ A))$ 
     $- \mu_C\ (1 + card\ (atms-of-ms\ A))\ (2 + card\ (atms-of-ms\ A))\ (trail-weight\ T)\ dpll-bj$ 
   $\lambda (M, N). no-dup\ M \wedge all-decomposition-implies-m\ N\ (get-all-marked-decomposition\ M)$ 
   $\lambda A\ -. (2 + card\ (atms-of-ms\ A)) \wedge (1 + card\ (atms-of-ms\ A))$ 
  apply unfold-locales
    apply (rule unbounded)
    using f-ge-1 apply fastforce
    apply (smt dpll-bj-all-decomposition-implies-inv dpll-bj-atms-in-trail-in-set
      dpll-bj-clauses id-apply prod.case-eq-if)
    apply (rule dpll-bj-trail-mes-decreasing-prop; auto)
    apply (rename-tac A T U, case-tac T, simp)
    apply (rename-tac A T U, case-tac U, simp)
    using dpll-bj-clauses dpll-bj-all-decomposition-implies-inv dpll-bj-no-dup by fastforce+
end

end
theory DPLL-W
imports Main Partial-Clausal-Logic Partial-Annotated-Clausal-Logic List-More Wellfounded-More
  DPLL-NOT
begin

```

17 DPLL

17.1 Rules

```

type-synonym 'a dpllW-marked-lit = ('a, unit, unit) marked-lit
type-synonym 'a dpllW-marked-lits = ('a, unit, unit) marked-lits
type-synonym 'v dpllW-state = 'v dpllW-marked-lits  $\times$  'v clauses

```

```

abbreviation trail :: 'v dpllW-state  $\Rightarrow$  'v dpllW-marked-lits where
  trail  $\equiv$  fst
abbreviation clauses :: 'v dpllW-state  $\Rightarrow$  'v clauses where
  clauses  $\equiv$  snd

```

The definition of DPLL is given in figure 2.13 page 70 of CW.

```

inductive dpllW :: 'v dpllW-state  $\Rightarrow$  'v dpllW-state  $\Rightarrow$  bool where

```

propagate: $C + \{\#L\# \} \in \# \text{ clauses } S \implies \text{trail } S \models_{as} C \text{Not } C \implies \text{undefined-lit } (\text{trail } S) L$
 $\implies \text{dpll}_W S (\text{Propagated } L () \# \text{trail } S, \text{clauses } S) \mid$
decided: $\text{undefined-lit } (\text{trail } S) L \implies \text{atm-of } L \in \text{atms-of-mm } (\text{clauses } S)$
 $\implies \text{dpll}_W S (\text{Marked } L () \# \text{trail } S, \text{clauses } S) \mid$
backtrack: $\text{backtrack-split } (\text{trail } S) = (M', L \# M) \implies \text{is-marked } L \implies D \in \# \text{ clauses } S$
 $\implies \text{trail } S \models_{as} C \text{Not } D \implies \text{dpll}_W S (\text{Propagated } (- (\text{lit-of } L)) () \# M, \text{clauses } S)$

17.2 Invariants

lemma *dpll_W-distinct-inv*:

assumes *dpll_W S S'*
and *no-dup (trail S)*
shows *no-dup (trail S')*
using *assms*

proof (*induct rule: dpll_W.induct*)

case (*decided L S*)

then show *?case using defined-lit-map by force*

next

case (*propagate C L S*)

then show *?case using defined-lit-map by force*

next

case (*backtrack S M' L M D*) **note** *extracted = this(1) and no-dup = this(5)*

show *?case*

using *no-dup backtrack-split-list-eq[of trail S, symmetric]* **unfolding** *extracted by auto*

qed

lemma *dpll_W-consistent-interp-inv*:

assumes *dpll_W S S'*
and *consistent-interp (lits-of-l (trail S))*
and *no-dup (trail S)*
shows *consistent-interp (lits-of-l (trail S'))*
using *assms*

proof (*induct rule: dpll_W.induct*)

case (*backtrack S M' L M D*) **note** *extracted = this(1) and marked = this(2) and D = this(4) and cons = this(5) and no-dup = this(6)*

have *no-dup': no-dup M*

by (*metis (no-types) backtrack-split-list-eq distinct.simps(2) distinct-append extracted list.simps(9) map-append no-dup snd-conv*)

then have *insert (lit-of L) (lits-of-l M) \subseteq lits-of-l (trail S)*

using *backtrack-split-list-eq[of trail S, symmetric]* **unfolding** *extracted by auto*

then have *cons: consistent-interp (insert (lit-of L) (lits-of-l M))*

using *consistent-interp-subset cons by blast*

moreover

have *lit-of L \notin lits-of-l M*

using *no-dup backtrack-split-list-eq[of trail S, symmetric]* *extracted*

unfolding *lits-of-def by force*

moreover

have *atm-of (-lit-of L) \notin ($\lambda m. \text{atm-of } (\text{lit-of } m)$) ' set M*

using *no-dup backtrack-split-list-eq[of trail S, symmetric]* **unfolding** *extracted by force*

then have *-lit-of L \notin lits-of-l M*

unfolding *lits-of-def by force*

ultimately show *?case by simp*

qed (*auto intro: consistent-add-undefined-lit-consistent*)

lemma *dpll_W-vars-in-snd-inv*:

assumes *dpll_W S S'*

and $\text{atm-of } \text{' (lits-of-l (trail S))} \subseteq \text{atms-of-mm (clauses S)}$
shows $\text{atm-of } \text{' (lits-of-l (trail S'))} \subseteq \text{atms-of-mm (clauses S')}$
using *assms*
proof (*induct rule: dpll_W.induct*)
case (*backtrack S M' L M D*)
then have $\text{atm-of (lit-of L)} \in \text{atms-of-mm (clauses S)}$
using *backtrack-split-list-eq*[*of trail S, symmetric*] **by** *auto*
moreover
have $\text{atm-of } \text{' lits-of-l (trail S)} \subseteq \text{atms-of-mm (clauses S)}$
using *backtrack*(5) **by** *simp*
then have $\bigwedge x. x \in \text{set } M \implies \text{atm-of (lit-of } x) \in \text{atms-of-mm (clauses S)}$
using *backtrack-split-list-eq*[*symmetric, of trail S*] *backtrack.hyps*(1)
unfolding *lits-of-def* **by** *auto*
ultimately show ?*case* **by** (*auto simp : lits-of-def*)
qed (*auto simp: in-plus-implies-atm-of-on-atms-of-ms*)

lemma *atms-of-ms-lit-of-atms-of*: $\text{atms-of-ms } ((\lambda a. \{\# \text{lit-of } a \# \}) \text{' } c) = \text{atm-of } \text{' lit-of } \text{' } c$
unfolding *atms-of-ms-def* **using** *image-iff* **by** *force*

Lemma theorem 2.8.2 page 71 of CW

lemma *dpll_W-propagate-is-conclusion*:
assumes *dpll_W S'*
and *all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S))*
and $\text{atm-of } \text{' lits-of-l (trail S)} \subseteq \text{atms-of-mm (clauses S)}$
shows *all-decomposition-implies-m (clauses S') (get-all-marked-decomposition (trail S'))*
using *assms*
proof (*induct rule: dpll_W.induct*)
case (*decided L S*)
then show ?*case* **unfolding** *all-decomposition-implies-def* **by** *simp*
next
case (*propagate C L S*) **note** *inS = this(1)* **and** *cnot = this(2)* **and** *IH = this(4)* **and** *undef = this(3)* **and** *atms-incl = this(5)*
let ?*I* = $\text{set (map } (\lambda a. \{\# \text{lit-of } a \# \}) (\text{trail S})) \cup \text{set-mset (clauses S)}$
have ?*I* $\models_p C + \{\#L\# \}$ **by** (*auto simp add: inS*)
moreover have ?*I* $\models_{ps} C \text{Not } C$ **using** *true-annots-true-clss-cls cnot* **by** *fastforce*
ultimately have ?*I* $\models_p \{\#L\# \}$ **using** *true-clss-cls-plus-CNot*[*of ?I C L*] *inS* **by** *blast*
{
assume *get-all-marked-decomposition (trail S) = []*
then have ?*case* **by** *blast*
}
moreover {
assume *n: get-all-marked-decomposition (trail S) $\neq []$*
have 1: $\bigwedge a b. (a, b) \in \text{set (tl (get-all-marked-decomposition (trail S)))}$
 $\implies (\text{unmark-l } a \cup \text{set-mset (clauses S)}) \models_{ps} \text{unmark-l } b$
using *IH* **unfolding** *all-decomposition-implies-def* **by** (*fastforce simp add: list.set-sel(2) n*)
moreover have 2: $\bigwedge a c. \text{hd (get-all-marked-decomposition (trail S))} = (a, c)$
 $\implies (\text{unmark-l } a \cup \text{set-mset (clauses S)}) \models_{ps} (\text{unmark-l } c)$
by (*metis IH all-decomposition-implies-cons-pair all-decomposition-implies-single list.collapse n*)
moreover have 3: $\bigwedge a c. \text{hd (get-all-marked-decomposition (trail S))} = (a, c)$
 $\implies (\text{unmark-l } a \cup \text{set-mset (clauses S)}) \models_p \{\#L\# \}$
proof –
fix *a c*
assume *h: hd (get-all-marked-decomposition (trail S)) = (a, c)*
have *h': trail S = c @ a* **using** *get-all-marked-decomposition-decomp h* **by** *blast*

```

have  $I$ :  $\text{set } (\text{map } (\lambda a. \{\# \text{lit-of } a \# \}) \ a) \cup \text{set-mset } (\text{clauses } S)$ 
   $\cup \text{unmark-l } c \models_{ps} \text{CNot } C$ 
  using  $\langle ?I \models_{ps} \text{CNot } C \rangle$  unfolding  $h'$  by  $(\text{simp add: Un-commute Un-left-commute})$ 
have
   $\text{atms-of-ms } (\text{CNot } C) \subseteq \text{atms-of-ms } (\text{set } (\text{map } (\lambda a. \{\# \text{lit-of } a \# \}) \ a) \cup \text{set-mset } (\text{clauses } S))$ 
  and
   $\text{atms-of-ms } (\text{unmark-l } c) \subseteq \text{atms-of-ms } (\text{set } (\text{map } (\lambda a. \{\# \text{lit-of } a \# \}) \ a)$ 
     $\cup \text{set-mset } (\text{clauses } S))$ 
  apply  $(\text{metis CNot-plus Un-subset-iff atms-of-atms-of-ms-mono atms-of-ms-CNot-atms-of}$ 
     $\text{atms-of-ms-union inS sup.coboundedI2})$ 
  using  $\text{inS atms-of-atms-of-ms-mono atms-incl}$  by  $(\text{fastforce simp: } h')$ 

  then have  $\text{unmark-l } a \cup \text{set-mset } (\text{clauses } S) \models_{ps} \text{CNot } C$ 
  using  $\text{true-clss-clss-left-right}[OF - I]$   $h$  2 by  $\text{auto}$ 
  then show  $\text{unmark-l } a \cup \text{set-mset } (\text{clauses } S) \models_p \{\# L \#\}$ 
  by  $(\text{metis (no-types) Un-insert-right inS insertI1 mk-disjoint-insert inS}$ 
     $\text{true-clss-clss-in true-clss-clss-plus-CNot})$ 
qed
ultimately have  $?case$ 
  by  $(\text{cases hd } (\text{get-all-marked-decomposition } (\text{trail } S)))$ 
   $(\text{auto simp: all-decomposition-implies-def})$ 
}
ultimately show  $?case$  by  $\text{auto}$ 
next
case  $(\text{backtrack } S \ M' \ L \ M \ D)$  note  $\text{extracted} = \text{this}(1)$  and  $\text{marked} = \text{this}(2)$  and  $D = \text{this}(3)$  and
   $\text{cnot} = \text{this}(4)$  and  $\text{cons} = \text{this}(4)$  and  $IH = \text{this}(5)$  and  $\text{atms-incl} = \text{this}(6)$ 
have  $S$ :  $\text{trail } S = M' \ @ \ L \ \# \ M$ 
  using  $\text{backtrack-split-list-eq}[of \ \text{trail } S]$  unfolding  $\text{extracted}$  by  $\text{auto}$ 
have  $M'$ :  $\forall l \in \text{set } M'. \neg \text{is-marked } l$ 
  using  $\text{extracted backtrack-split-fst-not-marked}[of - \ \text{trail } S]$  by  $\text{simp}$ 
have  $n$ :  $\text{get-all-marked-decomposition } (\text{trail } S) \neq []$  by  $\text{auto}$ 
then have  $\text{all-decomposition-implies-m } (\text{clauses } S) ((L \ \# \ M, M')$ 
   $\# \ \text{tl } (\text{get-all-marked-decomposition } (\text{trail } S)))$ 
  by  $(\text{metis (no-types) IH extracted get-all-marked-decomposition-backtrack-split list.exhaust-sel})$ 
then have  $1$ :  $\text{unmark-l } (L \ \# \ M) \cup \text{set-mset } (\text{clauses } S) \models_{ps} (\lambda a. \{\# \text{lit-of } a \# \}) \ ' \ \text{set } M'$ 
  by  $\text{simp}$ 
moreover
have  $\text{unmark-l } (L \ \# \ M) \cup \text{unmark-l } M' \models_{ps} \text{CNot } D$ 
  by  $(\text{metis (mono-tags, lifting) } S \ \text{Un-commute cons image-Un set-append}$ 
     $\text{true-annots-true-clss-clss})$ 
then have  $2$ :  $\text{unmark-l } (L \ \# \ M) \cup \text{set-mset } (\text{clauses } S) \cup \text{unmark-l } M'$ 
   $\models_{ps} \text{CNot } D$ 
  by  $(\text{metis (no-types, lifting) Un-assoc Un-left-commute true-clss-clss-union-l-r})$ 
ultimately
have  $\text{set } (\text{map } (\lambda a. \{\# \text{lit-of } a \# \}) \ (L \ \# \ M)) \cup \text{set-mset } (\text{clauses } S) \models_{ps} \text{CNot } D$ 
  using  $\text{true-clss-clss-left-right}$  by  $\text{fastforce}$ 
then have  $\text{set } (\text{map } (\lambda a. \{\# \text{lit-of } a \# \}) \ (L \ \# \ M)) \cup \text{set-mset } (\text{clauses } S) \models_p \{\# \}$ 
  by  $(\text{metis (mono-tags, lifting) } D \ \text{Un-def mem-Collect-eq}$ 
     $\text{true-clss-clss-contradiction-true-clss-clss-false})$ 
then have  $IL$ :  $\text{unmark-l } M \cup \text{set-mset } (\text{clauses } S) \models_p \{\# - \text{lit-of } L \#\}$ 
  using  $\text{true-clss-clss-false-left-right}$  by  $\text{auto}$ 
show  $?case$  unfolding  $S$   $\text{all-decomposition-implies-def}$ 
proof
  fix  $x \ P \ \text{level}$ 
  assume  $x$ :  $x \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$ 

```

```

    (fst (Propagated (– lit-of L) P # M, clauses S)))
let ?M' = Propagated (– lit-of L) P # M
let ?hd = hd (get-all-marked-decomposition ?M')
let ?tl = tl (get-all-marked-decomposition ?M')
have x = ?hd ∨ x ∈ set ?tl
  using x
  by (cases get-all-marked-decomposition ?M')
    auto
moreover {
  assume x': x ∈ set ?tl
  have L': Marked (lit-of L) () = L using marked by (cases L, auto)
  have x ∈ set (get-all-marked-decomposition (M' @ L # M))
    using x' get-all-marked-decomposition-except-last-choice-equal[of M' lit-of L P M]
    L' by (metis (no-types) M' list.set-sel(2) tl-Nil)
  then have case x of (Ls, seen) ⇒ unmark-l Ls ∪ set-mset (clauses S)
    by ps unmark-l seen
    using marked IH by (cases L) (auto simp add: S all-decomposition-implies-def)
}
moreover {
  assume x': x = ?hd
  have tl: tl (get-all-marked-decomposition (M' @ L # M)) ≠ []
  proof –
    have f1: ∧ms. length (get-all-marked-decomposition (M' @ ms))
      = length (get-all-marked-decomposition ms)
      by (simp add: M' get-all-marked-decomposition-remove-unmark-ssed-length)
    have Suc (length (get-all-marked-decomposition M)) ≠ Suc 0
      by blast
    then show ?thesis
      using f1 marked by (metis (no-types) get-all-marked-decomposition.simps(1) length-tl
        list.sel(3) list.size(3) marked-lit.collapse(1))
  qed
  obtain M0' M0 where
    L0: hd (tl (get-all-marked-decomposition (M' @ L # M))) = (M0, M0')
    by (cases hd (tl (get-all-marked-decomposition (M' @ L # M))))
  have x'': x = (M0, Propagated (– lit-of L) P # M0')
    unfolding x' using get-all-marked-decomposition-last-choice tl M' L0
    by (metis marked marked-lit.collapse(1))
  obtain l-get-all-marked-decomposition where
    get-all-marked-decomposition (trail S) = (L # M, M') # (M0, M0') #
    l-get-all-marked-decomposition
    using get-all-marked-decomposition-backtrack-split extracted by (metis (no-types) L0 S
      hd-Cons-tl n tl)
  then have M = M0' @ M0 using get-all-marked-decomposition-hd-hd by fastforce
  then have IL': unmark-l M0 ∪ set-mset (clauses S)
    ∪ unmark-l M0' by ps {#– lit-of L#}
    using IL by (simp add: Un-commute Un-left-commute image-Un)
  moreover have H: unmark-l M0 ∪ set-mset (clauses S)
    by ps unmark-l M0'
    using IH x'' unfolding all-decomposition-implies-def by (metis (no-types, lifting) L0 S
      list.set-sel(1) list.set-sel(2) old.prod.case tl tl-Nil)
  ultimately have case x of (Ls, seen) ⇒ unmark-l Ls ∪ set-mset (clauses S)
    by ps unmark-l seen
    using true-clss-clss-left-right unfolding x'' by auto
}
ultimately show case x of (Ls, seen) ⇒

```

$unmark-l\ Ls \cup set-mset\ (snd\ (?M',\ clauses\ S))$
 $\models_{ps}\ unmark-l\ seen$
unfolding *snd-conv* **by** *blast*
qed
qed

Lemma theorem 2.8.3 page 72 of CW

theorem *dpll_W-propagate-is-conclusion-of-decided*:
assumes *dpll_W S S'*
and *all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S))*
and *atm-of ' lits-of-l (trail S) \subseteq atms-of-mm (clauses S)*
shows *set-mset (clauses S') \cup { {#lit-of L#} | L. is-marked L \wedge L \in set (trail S') }*
 $\models_{ps}\ (\lambda a. \{ \#lit-of\ a\# \})\ ' \bigcup (set\ ' \textit{snd}\ ' set\ (get-all-marked-decomposition\ (trail\ S')))$
using *all-decomposition-implies-trail-is-implied*[OF *dpll_W-propagate-is-conclusion*[OF *assms*]] .

Lemma theorem 2.8.4 page 72 of CW

lemma *only-propagated-vars-unsat*:
assumes *marked: $\forall x \in set\ M. \neg is-marked\ x$*
and *DN: $D \in N$ and $D: M \models_{as}\ CNot\ D$*
and *inv: all-decomposition-implies N (get-all-marked-decomposition M)*
and *atm-incl: atm-of ' lits-of-l M \subseteq atms-of-ms N*
shows *unsatisfiable N*
proof (rule *ccontr*)
assume $\neg unsatisfiable\ N$
then obtain *I* **where**
 $I: I \models_s N$ **and**
cons: consistent-interp I **and**
tot: total-over-m I N
unfolding *satisfiable-def* **by** *auto*
then have *I-D: $I \models D$*
using *DN* **unfolding** *true-clss-def* **by** *auto*

have *l0: { {#lit-of L#} | L. is-marked L \wedge L \in set M } = { }* **using** *marked* **by** *auto*
have *atms-of-ms (N \cup unmark-l M) = atms-of-ms N*
using *atm-incl* **unfolding** *atms-of-ms-def lits-of-def* **by** *auto*

then have *total-over-m I (N \cup ($\lambda a. \{ \#lit-of\ a\# \})\ ' (set\ M))$*
using *tot* **unfolding** *total-over-m-def* **by** *auto*
then have $I \models_s (\lambda a. \{ \#lit-of\ a\# \})\ ' (set\ M)$
using *all-decomposition-implies-propagated-lits-are-implied*[OF *inv*] *cons I*
unfolding *true-clss-clss-def l0* **by** *auto*
then have *IM: $I \models_s unmark-l\ M$* **by** *auto*
{
fix *K*
assume $K \in\# D$
then have $-K \in lits-of-l\ M$
by (*auto split: if-split-asm*
intro: allE[OF *D*[*unfolded true-annots-def Ball-def*], of { $\#-K\#$ }])
then have $-K \in I$ **using** *IM true-clss-singleton-lit-of-implies-incl* **by** *fastforce*
}
then have $\neg I \models D$ **using** *cons* **unfolding** *true-clss-def consistent-interp-def* **by** *auto*
then show *False* **using** *I-D* **by** *blast*
qed

lemma *dpll_W-same-clauses*:

```

assumes  $dp\ll_W S S'$ 
shows  $clauses S = clauses S'$ 
using assms by (induct rule:  $dp\ll_W.induct$ , auto)

lemma rtrancp-dp\ll_W-inv:
  assumes rtrancp dp\ll_W S S'
  and inv: all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S))
  and atm-incl: atm-of ' lits-of-l (trail S)  $\subseteq$  atms-of-mm (clauses S)
  and consistent-interp (lits-of-l (trail S))
  and no-dup (trail S)
  shows all-decomposition-implies-m (clauses S') (get-all-marked-decomposition (trail S'))
  and atm-of ' lits-of-l (trail S')  $\subseteq$  atms-of-mm (clauses S')
  and  $clauses S = clauses S'$ 
  and consistent-interp (lits-of-l (trail S'))
  and no-dup (trail S')
  using assms
proof (induct rule: rtrancp-induct)
  case base
  show
    all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S)) and
    atm-of ' lits-of-l (trail S)  $\subseteq$  atms-of-mm (clauses S) and
     $clauses S = clauses S$  and
    consistent-interp (lits-of-l (trail S)) and
    no-dup (trail S) using assms by auto
  next
  case (step S' S'') note  $dp\ll_W Star = this(1)$  and  $IH = this(3,4,5,6,7)$  and
     $dp\ll_W = this(2)$ 
  moreover
    assume
      inv: all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S)) and
      atm-incl: atm-of ' lits-of-l (trail S)  $\subseteq$  atms-of-mm (clauses S) and
      cons: consistent-interp (lits-of-l (trail S)) and
      no-dup (trail S)
    ultimately have decomp: all-decomposition-implies-m (clauses S')
      (get-all-marked-decomposition (trail S')) and
      atm-incl': atm-of ' lits-of-l (trail S')  $\subseteq$  atms-of-mm (clauses S') and
      snd:  $clauses S = clauses S'$  and
      cons': consistent-interp (lits-of-l (trail S')) and
      no-dup': no-dup (trail S') by blast+
    show  $clauses S = clauses S''$  using  $dp\ll_W\text{-same-clauses}[OF\ dp\ll_W]$  snd by metis

  show all-decomposition-implies-m (clauses S'') (get-all-marked-decomposition (trail S''))
    using  $dp\ll_W\text{-propagate-is-conclusion}[OF\ dp\ll_W]$  decomp atm-incl' by auto
  show atm-of ' lits-of-l (trail S'')  $\subseteq$  atms-of-mm (clauses S'')
    using  $dp\ll_W\text{-vars-in-snd-inv}[OF\ dp\ll_W]$  atm-incl atm-incl' by auto
  show no-dup (trail S'') using  $dp\ll_W\text{-distinct-inv}[OF\ dp\ll_W]$  no-dup' dp\ll_W by auto
  show consistent-interp (lits-of-l (trail S''))
    using cons' no-dup'  $dp\ll_W\text{-consistent-interp-inv}[OF\ dp\ll_W]$  by auto
qed

definition  $dp\ll_W\text{-all-inv } S \equiv$ 
  (all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S)))
   $\wedge$  atm-of ' lits-of-l (trail S)  $\subseteq$  atms-of-mm (clauses S)
   $\wedge$  consistent-interp (lits-of-l (trail S))
   $\wedge$  no-dup (trail S)

```

```

lemma dpllW-all-inv-dest[dest]:
  assumes dpllW-all-inv S
  shows all-decomposition-implies-m (clauses S) (get-all-marked-decomposition (trail S))
  and atm-of ‘ lits-of-l (trail S)  $\subseteq$  atms-of-mm (clauses S)
  and consistent-interp (lits-of-l (trail S))  $\wedge$  no-dup (trail S)
  using assms unfolding dpllW-all-inv-def lits-of-def by auto

lemma rtrancpl-dpllW-all-inv:
  assumes rtrancpl dpllW S S'
  and dpllW-all-inv S
  shows dpllW-all-inv S'
  using assms rtrancpl-dpllW-inv[OF assms(1)] unfolding dpllW-all-inv-def lits-of-def by blast

lemma dpllW-all-inv:
  assumes dpllW S S'
  and dpllW-all-inv S
  shows dpllW-all-inv S'
  using assms rtrancpl-dpllW-all-inv by blast

lemma rtrancpl-dpllW-inv-starting-from-0:
  assumes rtrancpl dpllW S S'
  and inv: trail S = []
  shows dpllW-all-inv S'
proof –
  have dpllW-all-inv S
    using assms unfolding all-decomposition-implies-def dpllW-all-inv-def by auto
  then show ?thesis using rtrancpl-dpllW-all-inv[OF assms(1)] by blast
qed

lemma dpllW-can-do-step:
  assumes consistent-interp (set M)
  and distinct M
  and atm-of ‘ (set M)  $\subseteq$  atms-of-mm N
  shows rtrancpl dpllW ([], N) (map ( $\lambda M$ . Marked M ()) M, N)
  using assms
proof (induct M)
  case Nil
  then show ?case by auto
next
  case (Cons L M)
  then have undefined-lit (map ( $\lambda M$ . Marked M ()) M) L
    unfolding defined-lit-def consistent-interp-def by auto
  moreover have atm-of L  $\in$  atms-of-mm N using Cons.prem(3) by auto
  ultimately have dpllW (map ( $\lambda M$ . Marked M ()) M, N) (map ( $\lambda M$ . Marked M ()) (L # M), N)
    using dpllW.decided by auto
  moreover have consistent-interp (set M) and distinct M and atm-of ‘ set M  $\subseteq$  atms-of-mm N
    using Cons.prem unfolding consistent-interp-def by auto
  ultimately show ?case using Cons.hyps by auto
qed

definition conclusive-dpllW-state (S:: ‘v dpllW-state)  $\longleftrightarrow$ 
  (trail S  $\models$  asm clauses S  $\vee$  ( $\forall L \in$  set (trail S).  $\neg$  is-marked L)
   $\wedge$  ( $\exists C \in$  # clauses S. trail S  $\models$  as CNot C)))

```


lemma *dp_{ll}_W-strong-completeness*:

assumes *set* $M \models_{sm} N$
and *consistent-interp* (*set* M)
and *distinct* M
and *atm-of* ' (*set* M) \subseteq *atms-of-mm* N
shows $dp_{llW}^{**} ([], N) (map (\lambda M. \text{Marked } M ()) M, N)$
and *conclusive-dp_{ll}_W-state* ($map (\lambda M. \text{Marked } M ()) M, N$)

proof –

show *rtranc_lp* $dp_{llW} ([], N) (map (\lambda M. \text{Marked } M ()) M, N)$ **using** *dp_{ll}_W-can-do-step* *assms* **by** *auto*
have $map (\lambda M. \text{Marked } M ()) M \models_{asm} N$ **using** *assms*(1) *true-annots-marked-true-cls* **by** *auto*
then show *conclusive-dp_{ll}_W-state* ($map (\lambda M. \text{Marked } M ()) M, N$)
unfolding *conclusive-dp_{ll}_W-state-def* **by** *auto*

qed

lemma *dp_{ll}_W-sound*:

assumes
rtranc_lp $dp_{llW} ([], N) (M, N)$ **and**
 $\forall S. \neg dp_{llW} (M, N) S$
shows $M \models_{asm} N \longleftrightarrow \text{satisfiable } (set-mset N) (is ?A \longleftrightarrow ?B)$

proof

let $?M' = \text{lits-of-}l M$
assume $?A$
then have $?M' \models_{sm} N$ **by** (*simp add: true-annots-true-cls*)
moreover have *consistent-interp* $?M'$
using *rtranc_lp-dp_{ll}_W-inv-starting-from-0*[*OF assms*(1)] **by** *auto*
ultimately show $?B$ **by** *auto*

next

assume $?B$
show $?A$
proof (*rule ccontr*)
assume $n: \neg ?A$
have $(\exists L. \text{undefined-lit } M L \wedge \text{atm-of } L \in \text{atms-of-mm } N) \vee (\exists D \in \#N. M \models_{as} CNot D)$
proof –
obtain $D :: 'a \text{ clause}$ **where** $D: D \in \# N$ **and** $\neg M \models_a D$
using n **unfolding** *true-annots-def Ball-def* **by** *auto*
then have $(\exists L. \text{undefined-lit } M L \wedge \text{atm-of } L \in \text{atms-of } D) \vee M \models_{as} CNot D$
unfolding *true-annots-def Ball-def CNot-def true-annot-def*
using *atm-of-lit-in-atms-of true-annot-iff-marked-or-true-lit true-cls-def* **by** *blast*
then show *?thesis*
by (*metis Bex-def D atms-of-atms-of-ms-mono rev-subsetD*)

qed

moreover {

assume $\exists L. \text{undefined-lit } M L \wedge \text{atm-of } L \in \text{atms-of-mm } N$
then have *False* **using** *assms*(2) **decided by** *fastforce*

}

moreover {

assume $\exists D \in \#N. M \models_{as} CNot D$
then obtain D **where** $DN: D \in \# N$ **and** $MD: M \models_{as} CNot D$ **by** *auto*
{
assume $\forall l \in set M. \neg is-marked l$
moreover have *dp_{ll}_W-all-inv* ($[], N$)
using *assms* **unfolding** *all-decomposition-implies-def dp_{ll}_W-all-inv-def* **by** *auto*
ultimately have *unsatisfiable* (*set-mset* N)

```

    using only-propagated-vars-unsat[of M D set-mset N] DN MD
    rtranchp-dpllW-all-inv[OF assms(1)] by force
  then have False using ⟨?B⟩ by blast
}
moreover {
  assume l: ∃ l ∈ set M. is-marked l
  then have False
    using backtrack[of (M, N) - - D ] DN MD assms(2)
    backtrack-split-some-is-marked-then-snd-has-hd[OF l]
    by (metis backtrack-split-snd-hd-marked fst-conv list.distinct(1) list.sel(1) snd-conv)
}
ultimately have False by blast
}
ultimately show False by blast
qed
qed

```

17.3 Termination

definition $dpll_W\text{-mes } M n =$
 $\text{map } (\lambda l. \text{if is-marked } l \text{ then } 2 \text{ else } (1::\text{nat})) (\text{rev } M) @ \text{replicate } (n - \text{length } M) \ 3$

lemma $\text{length-dpll}_W\text{-mes}$:
assumes $\text{length } M \leq n$
shows $\text{length } (dpll_W\text{-mes } M n) = n$
using *assms* **unfolding** $dpll_W\text{-mes-def}$ **by** *auto*

lemma $\text{distinctcard-atm-of-lits-of-eq-length}$:
assumes *no-dup S*
shows $\text{card } (\text{atm-of } \text{'lits-of-l } S) = \text{length } S$
using *assms* **by** (*induct S*) (*auto simp add: image-image lits-of-def*)

lemma $dpll_W\text{-card-decrease}$:
assumes $dpll: dpll_W \ S \ S'$ **and** $\text{length } (\text{trail } S') \leq \text{card vars}$
and $\text{length } (\text{trail } S) \leq \text{card vars}$
shows $(dpll_W\text{-mes } (\text{trail } S') (\text{card vars}), dpll_W\text{-mes } (\text{trail } S) (\text{card vars}))$
 $\in \text{lexn } \{(a, b). a < b\} (\text{card vars})$
using *assms*

proof (*induct rule: dpll_W.induct*)
case (*propagate C L S*)
have $m: \text{map } (\lambda l. \text{if is-marked } l \text{ then } 2 \text{ else } 1) (\text{rev } (\text{trail } S))$
 $@ \text{replicate } (\text{card vars} - \text{length } (\text{trail } S)) \ 3$
 $= \text{map } (\lambda l. \text{if is-marked } l \text{ then } 2 \text{ else } 1) (\text{rev } (\text{trail } S)) @ 3$
 $\# \text{replicate } (\text{card vars} - \text{Suc } (\text{length } (\text{trail } S))) \ 3$
using *propagate.prem[simplified]* **using** *Suc-diff-le* **by** *fastforce*
then show *?case*
using *propagate.prem[s(1)]* **unfolding** $dpll_W\text{-mes-def}$ **by** (*fastforce simp add: lexn-conv assms(2)*)

next

case (*decided S L*)
have $m: \text{map } (\lambda l. \text{if is-marked } l \text{ then } 2 \text{ else } 1) (\text{rev } (\text{trail } S))$
 $@ \text{replicate } (\text{card vars} - \text{length } (\text{trail } S)) \ 3$
 $= \text{map } (\lambda l. \text{if is-marked } l \text{ then } 2 \text{ else } 1) (\text{rev } (\text{trail } S)) @ 3$
 $\# \text{replicate } (\text{card vars} - \text{Suc } (\text{length } (\text{trail } S))) \ 3$
using *decided.prem[simplified]* **using** *Suc-diff-le* **by** *fastforce*
then show *?case*
using *decided.prem* **unfolding** $dpll_W\text{-mes-def}$ **by** (*force simp add: lexn-conv assms(2)*)

next
 case (*backtrack* $S M' L M D$)
 have L : *is-marked* L **using** *backtrack.hyps*(2) **by** *auto*
 have S : *trail* $S = M' @ L \# M$
 using *backtrack.hyps*(1) *backtrack-split-list-eq*[*of trail S*] **by** *auto*
 show ?*case*
 using *backtrack.prem*s L **unfolding** *dpll_W-mes-def* S **by** (*fastforce simp add: lexn-conv assms*(2))
qed

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lemma *dpll_W-card-decrease'*:
 assumes *dpll*: *dpll_W* $S S'$
 and *atm-incl*: *atm-of* ' *lits-of-l* (*trail S*) \subseteq *atms-of-mm* (*clauses S*)

and *no-dup*: *no-dup* (*trail S*)

shows (*dpll_W-mes* (*trail S'*) (*card* (*atms-of-mm* (*clauses S'*)))),
dpll_W-mes (*trail S*) (*card* (*atms-of-mm* (*clauses S*)))) \in *lex* {(a, b). $a < b$ }

proof –

have *finite* (*atms-of-mm* (*clauses S*)) **unfolding** *atms-of-ms-def* **by** *auto*
 then have 1: *length* (*trail S*) \leq *card* (*atms-of-mm* (*clauses S*))
 using *distinctcard-atm-of-lit-of-eq-length*[*OF no-dup*] *atm-incl card-mono* **by** *metis*

moreover

have *no-dup'*: *no-dup* (*trail S'*) **using** *dpll dpll_W-distinct-inv no-dup* **by** *blast*
 have *SS'*: *clauses S' = clauses S* **using** *dpll* **by** (*auto dest!*: *dpll_W-same-clauses*)
 have *atm-incl'*: *atm-of* ' *lits-of-l* (*trail S'*) \subseteq *atms-of-mm* (*clauses S'*)
 using *atm-incl dpll dpll_W-vars-in-snd-inv*[*OF dpll*] **by** *force*
 have *finite* (*atms-of-mm* (*clauses S'*))
 unfolding *atms-of-ms-def* **by** *auto*
 then have 2: *length* (*trail S'*) \leq *card* (*atms-of-mm* (*clauses S'*))
 using *distinctcard-atm-of-lit-of-eq-length*[*OF no-dup*] *atm-incl' card-mono SS'* **by** *metis*

ultimately have (*dpll_W-mes* (*trail S'*) (*card* (*atms-of-mm* (*clauses S*)))),
dpll_W-mes (*trail S*) (*card* (*atms-of-mm* (*clauses S*))))
 \in *lexn* {(a, b). $a < b$ } (*card* (*atms-of-mm* (*clauses S*))))
 using *dpll_W-card-decrease*[*OF assms*(1), *of atms-of-mm* (*clauses S*)] **by** *blast*
then have (*dpll_W-mes* (*trail S'*) (*card* (*atms-of-mm* (*clauses S*)))),
dpll_W-mes (*trail S*) (*card* (*atms-of-mm* (*clauses S*)))) \in *lex* {(a, b). $a < b$ }
 unfolding *lex-def* **by** *auto*
then show (*dpll_W-mes* (*trail S'*) (*card* (*atms-of-mm* (*clauses S'*)))),
dpll_W-mes (*trail S*) (*card* (*atms-of-mm* (*clauses S*)))) \in *lex* {(a, b). $a < b$ }
 using *dpll_W-same-clauses*[*OF assms*(1)] **by** *auto*
qed

lemma *wf-lexn*: *wf* (*lexn* {(a, b). ($a::nat$) $< b$ } (*card* (*atms-of-mm* (*clauses S*))))

proof –

have m : {(a, b). $a < b$ } = *measure id* **by** *auto*
 show ?*thesis* **apply** (*rule wf-lexn*) **unfolding** m **by** *auto*
qed

lemma *dpll_W-wf*:

wf {(S', S). *dpll_W-all-inv* $S \wedge$ *dpll_W* $S S'$ }
apply (*rule wf-wf-if-measure'*[*OF wf-lex-less*, *of* - -
 λS . *dpll_W-mes* (*trail S*) (*card* (*atms-of-mm* (*clauses S*))))])
using *dpll_W-card-decrease'* **by** *fast*

lemma *dpll_W-trancp-star-commute*:
 $\{(S', S). \text{dpll}_W\text{-all-inv } S \wedge \text{dpll}_W S S'\}^+ = \{(S', S). \text{dpll}_W\text{-all-inv } S \wedge \text{trancp dpll}_W S S'\}$
 (is ?A = ?B)

proof

```

{ fix S S'
  assume (S, S') ∈ ?A
  then have (S, S') ∈ ?B
    by (induct rule: trancl.induct, auto)
}
then show ?A ⊆ ?B by blast
{ fix S S'
  assume (S, S') ∈ ?B
  then have dpllW++ S' S and dpllW-all-inv S' by auto
  then have (S, S') ∈ ?A
    proof (induct rule: trancp.induct)
      case r-into-trancl
      then show ?case by (simp-all add: r-into-trancl')
    next
      case (trancl-into-trancl S S' S'')
      then have (S', S) ∈ {a. case a of (S', S) ⇒ dpllW-all-inv S ∧ dpllW S S'}+ by blast
      moreover have dpllW-all-inv S'
        using rtrancp-dpllW-all-inv[OF tranclp-into-rtrancp[OF trancl-into-trancl.hyps(1)]]
        trancl-into-trancl.prem by auto
      ultimately have (S'', S') ∈ {(pa, p). dpllW-all-inv p ∧ dpllW p pa}+
        using ⟨dpllW-all-inv S'⟩ trancl-into-trancl.hyps(3) by blast
      then show ?case
        using ⟨(S', S) ∈ {a. case a of (S', S) ⇒ dpllW-all-inv S ∧ dpllW S S'}+⟩ by auto
    qed
  }
}
then show ?B ⊆ ?A by blast
qed

```

lemma *dpll_W-wf-trancp*: wf {(S', S). dpll_W-all-inv S ∧ dpll_W⁺⁺ S S'}
 unfolding dpll_W-trancp-star-commute[symmetric] by (simp add: dpll_W-wf wf-trancl)

lemma *dpll_W-wf-plus*:
 shows wf {(S', ([], N)) | S'. dpll_W⁺⁺ ([], N) S'} (is wf ?P)
 apply (rule wf-subset[OF dpll_W-wf-trancp, of ?P])
 using assms unfolding dpll_W-all-inv-def by auto

17.4 Final States

lemma *dpll_W-no-more-step-is-a-conclusive-state*:

assumes $\forall S'. \neg \text{dpll}_W S S'$
 shows conclusive-dpll_W-state S

proof —

```

have vars: ∀ s ∈ atms-of-mm (clauses S). s ∈ atm-of ' lits-of-l (trail S)
proof (rule ccontr)
  assume ¬ (∀ s ∈ atms-of-mm (clauses S). s ∈ atm-of ' lits-of-l (trail S))
  then obtain L where
    L-in-atms: L ∈ atms-of-mm (clauses S) and
    L-notin-trail: L ∉ atm-of ' lits-of-l (trail S) by metis
  obtain L' where L': atm-of L' = L by (meson literal.sel(2))
  then have undefined-lit (trail S) L'
    unfolding Marked-Propagated-in-iff-in-lits-of-l by (metis L-notin-trail atm-of-uminus imageI)

```

```

    then show False using dpllW.decided assms(1) L-in-atms L' by blast
  qed
show ?thesis
proof (rule ccontr)
  assume not-final:  $\neg$  ?thesis
  then have
     $\neg$  trail S  $\models_{asm}$  clauses S and
     $(\exists L \in set \ (trail \ S). \ is\_marked \ L) \vee (\forall C \in \# \ clauses \ S. \ \neg trail \ S \models_{as} \ CNot \ C)$ 
    unfolding conclusive-dpllW-state-def by auto
  moreover {
    assume  $\exists L \in set \ (trail \ S). \ is\_marked \ L$ 
    then obtain L M' M where L: backtrack-split (trail S) = (M', L # M)
      using backtrack-split-some-is-marked-then-snd-has-hd by blast
    obtain D where  $D \in \# \ clauses \ S$  and  $\neg trail \ S \models_a \ D$ 
      using  $\langle \neg trail \ S \models_{asm} \ clauses \ S \rangle$  unfolding true-annots-def by auto
    then have  $\forall s \in atms\_of\_ms \ \{D\}. \ s \in atm\_of \ ' \ lits\_of\_l \ (trail \ S)$ 
      using vars unfolding atms-of-ms-def by auto
    then have trail S  $\models_{as} \ CNot \ D$ 
      using all-variables-defined-not-imply-cnot[of D]  $\langle \neg trail \ S \models_a \ D \rangle$  by auto
    moreover have is-marked L
      using L by (metis backtrack-split-snd-hd-marked list.distinct(1) list.sel(1) snd-conv)
    ultimately have False
      using assms(1) dpllW.backtrack L  $\langle D \in \# \ clauses \ S \rangle \langle trail \ S \models_{as} \ CNot \ D \rangle$  by blast
  }
  moreover {
    assume tr:  $\forall C \in \# \ clauses \ S. \ \neg trail \ S \models_{as} \ CNot \ C$ 
    obtain C where C-in-cl:  $C \in \# \ clauses \ S$  and trC:  $\neg trail \ S \models_a \ C$ 
      using  $\langle \neg trail \ S \models_{asm} \ clauses \ S \rangle$  unfolding true-annots-def by auto
    have  $\forall s \in atms\_of\_ms \ \{C\}. \ s \in atm\_of \ ' \ lits\_of\_l \ (trail \ S)$ 
      using vars  $\langle C \in \# \ clauses \ S \rangle$  unfolding atms-of-ms-def by auto
    then have trail S  $\models_{as} \ CNot \ C$ 
      by (meson C-in-cl tr trC all-variables-defined-not-imply-cnot)
    then have False using tr C-in-cl by auto
  }
  ultimately show False by blast
qed
qed

```

lemma *dpll_W-conclusive-state-correct*:
 assumes *dpll_W*** ($\llbracket \cdot \rrbracket, N$) (*M*, *N*) and *conclusive-dpll_W-state* (*M*, *N*)
 shows $M \models_{asm} N \longleftrightarrow \text{satisfiable} \ (set_mset \ N)$ (*is* ?*A* \longleftrightarrow ?*B*)

```

proof
  let ?M' = lits-of-l M
  assume ?A
  then have ?M'  $\models_{sm} \ N$  by (simp add: true-annots-true-cls)
  moreover have consistent-interp ?M'
    using rtranclp-dpllW-inv-starting-from-0[OF assms(1)] by auto
  ultimately show ?B by auto

```

```

next
  assume ?B
  show ?A
  proof (rule ccontr)
    assume n:  $\neg$  ?A
    have no-mark:  $\forall L \in set \ M. \ \neg is\_marked \ L \ \exists C \in \# \ N. \ M \models_{as} \ CNot \ C$ 
      using n assms(2) unfolding conclusive-dpllW-state-def by auto

```

moreover obtain D where $DN: D \in \# N$ and $MD: M \models_{as} CNot D$ using *no-mark* by *auto*
 ultimately have *unsatisfiable (set-mset N)*
 using *only-propagated-vars-unsat rtrancpl-dpll_W-all-inv[OF assms(1)]*
 unfolding *dpll_W-all-inv-def* by *force*
 then show *False* using $\langle ?B \rangle$ by *blast*
 qed
 qed

17.5 Link with NOT's DPLL

interpretation *dpll_{W-NOT}*: *dpll-with-backtrack* .

declare *dpll_{W-NOT}.state-simp_{NOT}[simp del]*
 lemma *state-eq_{NOT}-iff-eq[iff, simp]: dpll_{W-NOT}.state-eq_{NOT} S T \longleftrightarrow S = T*
 unfolding *dpll_{W-NOT}.state-eq_{NOT}-def* by (cases *S*, cases *T*) *auto*
 lemma *dpll_W-dpll_W-bj*:
 assumes *inv: dpll_W-all-inv S* and *dpll: dpll_W S T*
 shows *dpll_{W-NOT}.dpll-bj S T*
 using *dpll inv*
 apply (induction rule: *dpll_W.induct*)
 apply (rule *dpll_{W-NOT}.bj-propagate_{NOT}*)
 apply (rule *dpll_{W-NOT}.propagate_{NOT}.propagate_{NOT}; simp?*)
 apply *fastforce*
 apply (rule *dpll_{W-NOT}.bj-decide_{NOT}*)
 apply (rule *dpll_{W-NOT}.decide_{NOT}.decide_{NOT}; simp?*)
 apply *fastforce*
 apply (frule *dpll_{W-NOT}.backtrack.intros[of - - - -], simp-all*)
 apply (rule *dpll_{W-NOT}.dpll-bj.bj-backjump*)
 apply (rule *dpll_{W-NOT}.backtrack-is-backjump'', simp-all add: dpll_W-all-inv-def*)
 done

 lemma *dpll_W-bj-dpll*:
 assumes *inv: dpll_W-all-inv S* and *dpll: dpll_{W-NOT}.dpll-bj S T*
 shows *dpll_W S T*
 using *dpll*
 apply (induction rule: *dpll_{W-NOT}.dpll-bj.induct*)
 apply (elim *dpll_{W-NOT}.decide_{NOT}E*, cases *S*)
 apply (frule *decided; simp*)

 apply (elim *dpll_{W-NOT}.propagate_{NOT}E*, cases *S*)
 apply (auto intro!: *propagate[of - - (-, -), simplified]*)[]
 apply (elim *dpll_{W-NOT}.backjumpE*, cases *S*)
 by (simp add: *dpll_W.simps dpll-with-backtrack.backtrack.simps*)

 lemma *rtrancpl-dpll_W-rtrancpl-dpll_{W-NOT}*:
 assumes *dpll_W** S T* and *dpll_W-all-inv S*
 shows *dpll_{W-NOT}.dpll-bj** S T*
 using *assms* apply (induction)
 apply *simp*
 by (auto intro: *rtrancpl-dpll_W-all-inv dpll_W-dpll_W-bj rtrancpl.rtrancpl-into-rtrancpl*)

 lemma *rtrancpl-dpll-rtrancpl-dpll_W*:
 assumes *dpll_{W-NOT}.dpll-bj** S T* and *dpll_W-all-inv S*
 shows *dpll_W** S T*
 using *assms* apply (induction)

```

  apply simp
by (auto intro: dpllW-bj-dpll rtrancpl.rtrancpl-into-rtrancpl rtrancpl-dpllW-all-inv)

lemma dpll-conclusive-state-correctness:
  assumes dpllW-NOT.dpll-bj** ([], N) (M, N) and conclusive-dpllW-state (M, N)
  shows M  $\models_{asm}$  N  $\longleftrightarrow$  satisfiable (set-mset N)
proof -
  have dpllW-all-inv ([], N)
  unfolding dpllW-all-inv-def by auto
  show ?thesis
  apply (rule dpllW-conclusive-state-correct)
  apply (simp add:  $\langle$ dpllW-all-inv ([], N) $\rangle$  assms(1) rtrancpl-dpll-rtrancpl-dpllW)
  using assms(2) by simp
qed

end
theory CDCL-W-Level
imports Partial-Annotated-Clausal-Logic
begin

```

17.5.1 Level of literals and clauses

Getting the level of a variable, implies that the list has to be reversed. Here is the funtion after reversing.

```

fun get-rev-level :: ('v, nat, 'a) marked-lits  $\Rightarrow$  nat  $\Rightarrow$  'v literal  $\Rightarrow$  nat where
get-rev-level [] - = 0 |
get-rev-level (Marked l level # Ls) n L =
  (if atm-of l = atm-of L then level else get-rev-level Ls level L) |
get-rev-level (Propagated l - # Ls) n L =
  (if atm-of l = atm-of L then n else get-rev-level Ls n L)

```

abbreviation get-level M L \equiv get-rev-level (rev M) 0 L

lemma get-rev-level-uminus[simp]: get-rev-level M n(-L) = get-rev-level M n L
 by (induct arbitrary: n rule: get-rev-level.induct) auto

lemma atm-of-notin-get-rev-level-eq-0:
 assumes atm-of L \notin atm-of ' lits-of-l M
 shows get-rev-level M n L = 0
 using assms by (induct M arbitrary: n rule: marked-lit-list-induct) auto

lemma get-rev-level-ge-0-atm-of-in:
 assumes get-rev-level M n L > n
 shows atm-of L \in atm-of ' lits-of-l M
 using assms by (induct M arbitrary: n rule: marked-lit-list-induct)
 (fastforce simp: atm-of-notin-get-rev-level-eq-0)+

In *get-rev-level* (resp. *get-level*), the beginning (resp. the end) can be skipped if the literal is not in the beginning (resp. the end).

lemma get-rev-level-skip[simp]:
 assumes atm-of L \notin atm-of ' lits-of-l M
 shows get-rev-level (M @ Marked K i # M') n L = get-rev-level (Marked K i # M') i L
 using assms by (induct M arbitrary: n i rule: marked-lit-list-induct) auto

lemma get-rev-level-notin-end[simp]:

assumes $\text{atm-of } L \notin \text{atm-of ' lits-of-l } M'$
shows $\text{get-rev-level } (M @ M') \ n \ L = \text{get-rev-level } M \ n \ L$
using *assms by (induct M arbitrary: n rule: marked-lit-list-induct)*
(auto simp: atm-of-notin-get-rev-level-eq-0)

If the literal is at the beginning, then the end can be skipped

lemma *get-rev-level-skip-end[simp]:*
assumes $\text{atm-of } L \in \text{atm-of ' lits-of-l } M$
shows $\text{get-rev-level } (M @ M') \ n \ L = \text{get-rev-level } M \ n \ L$
using *assms by (induct arbitrary: n rule: marked-lit-list-induct) auto*

lemma *get-level-skip-beginning:*
assumes $\text{atm-of } L' \neq \text{atm-of (lit-of } K)$
shows $\text{get-level } (K \# M) \ L' = \text{get-level } M \ L'$
using *assms by auto*

lemma *get-level-skip-beginning-not-marked-rev:*
assumes $\text{atm-of } L \notin \text{atm-of ' lit-of ' (set } S)$
and $\forall s \in \text{set } S. \neg \text{is-marked } s$
shows $\text{get-level } (M @ \text{rev } S) \ L = \text{get-level } M \ L$
using *assms by (induction S rule: marked-lit-list-induct) auto*

lemma *get-level-skip-beginning-not-marked[simp]:*
assumes $\text{atm-of } L \notin \text{atm-of ' lit-of ' (set } S)$
and $\forall s \in \text{set } S. \neg \text{is-marked } s$
shows $\text{get-level } (M @ S) \ L = \text{get-level } M \ L$
using *get-level-skip-beginning-not-marked-rev[of L rev S M] assms by auto*

lemma *get-rev-level-skip-beginning-not-marked[simp]:*
assumes $\text{atm-of } L \notin \text{atm-of ' lit-of ' (set } S)$
and $\forall s \in \text{set } S. \neg \text{is-marked } s$
shows $\text{get-rev-level } (\text{rev } S @ \text{rev } M) \ 0 \ L = \text{get-level } M \ L$
using *get-level-skip-beginning-not-marked-rev[of L rev S M] assms by auto*

lemma *get-level-skip-in-all-not-marked:*
fixes $M :: ('a, \text{nat}, 'b) \text{ marked-lit list}$ **and** $L :: 'a \text{ literal}$
assumes $\forall m \in \text{set } M. \neg \text{is-marked } m$
and $\text{atm-of } L \in \text{atm-of ' lit-of ' (set } M)$
shows $\text{get-rev-level } M \ n \ L = n$
using *assms by (induction M rule: marked-lit-list-induct) auto*

lemma *get-level-skip-all-not-marked[simp]:*
fixes M
defines $M' \equiv \text{rev } M$
assumes $\forall m \in \text{set } M. \neg \text{is-marked } m$
shows $\text{get-level } M \ L = 0$
proof –
have $M: M = \text{rev } M'$
unfolding $M'\text{-def}$ **by** *auto*
show *?thesis*
using *assms unfolding M by (induction M' rule: marked-lit-list-induct) auto*
qed

abbreviation $M\text{Max } M \equiv \text{Max } (\text{set-mset } M)$

the $\{\#0::'a\# \}$ is there to ensures that the set is not empty.

definition *get-maximum-level* :: ('a, nat, 'b) marked-lit list \Rightarrow 'a literal multiset \Rightarrow nat
where
get-maximum-level M D = MMax ({#0#} + image-mset (get-level M) D)

lemma *get-maximum-level-ge-get-level*:
 $L \in \# D \Rightarrow \text{get-maximum-level } M D \geq \text{get-level } M L$
unfolding *get-maximum-level-def* **by** *auto*

lemma *get-maximum-level-empty[simp]*:
 $\text{get-maximum-level } M \{ \# \} = 0$
unfolding *get-maximum-level-def* **by** *auto*

lemma *get-maximum-level-exists-lit-of-max-level*:
 $D \neq \{ \# \} \Rightarrow \exists L \in \# D. \text{get-level } M L = \text{get-maximum-level } M D$
unfolding *get-maximum-level-def*
apply (induct D)
apply *simp*
by (rename-tac D x, case-tac D = {#}) (auto simp add: max-def)

lemma *get-maximum-level-empty-list[simp]*:
 $\text{get-maximum-level } [] D = 0$
unfolding *get-maximum-level-def* **by** (simp add: image-constant-conv)

lemma *get-maximum-level-single[simp]*:
 $\text{get-maximum-level } M \{ \# L \# \} = \text{get-level } M L$
unfolding *get-maximum-level-def* **by** *simp*

lemma *get-maximum-level-plus*:
 $\text{get-maximum-level } M (D + D') = \max (\text{get-maximum-level } M D) (\text{get-maximum-level } M D')$
by (induct D) (auto simp add: get-maximum-level-def)

lemma *get-maximum-level-exists-lit*:
assumes n: $n > 0$
and max: $\text{get-maximum-level } M D = n$
shows $\exists L \in \# D. \text{get-level } M L = n$
proof –
have f: finite (insert 0 (($\lambda L. \text{get-level } M L$) ' set-mset D)) **by** *auto*
then have $n \in ((\lambda L. \text{get-level } M L) ' \text{ set-mset } D)$
using n max Max-in[OF f] **unfolding** *get-maximum-level-def* **by** *simp*
then show $\exists L \in \# D. \text{get-level } M L = n$ **by** *auto*
qed

lemma *get-maximum-level-skip-first[simp]*:
assumes atm-of L \notin atms-of D
shows $\text{get-maximum-level } (\text{Propagated } L C \# M) D = \text{get-maximum-level } M D$
using asms **unfolding** *get-maximum-level-def* *atms-of-def*
atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
by (smt atm-of-in-atm-of-set-in-uminus get-level-skip-beginning image-iff marked-lit.sel(2)
multiset.map-cong0)

lemma *get-maximum-level-skip-beginning*:
assumes DH: atms-of D \subseteq atm-of 'lits-of-l H
shows $\text{get-maximum-level } (c @ \text{Marked } Kh \ i \ \# \ H) D = \text{get-maximum-level } H D$
proof –

have (get-rev-level (rev H @ Marked Kh i # rev c) 0) ‘ set-mset D
 = (get-rev-level (rev H) 0) ‘ set-mset D
using DH **unfolding** atms-of-def
by (metis (no-types, lifting) get-rev-level-skip-end image-cong image-subset-iff set-rev)
then show ?thesis **using** DH **unfolding** get-maximum-level-def **by** auto
qed

lemma get-maximum-level-D-single-propagated:

get-maximum-level [Propagated x21 x22] D = 0

proof –

have A: insert 0 ((λL. 0) ‘ (set-mset D ∩ {L. atm-of x21 = atm-of L}))
 ∪ (λL. 0) ‘ (set-mset D ∩ {L. atm-of x21 ≠ atm-of L})) = {0}
by auto

show ?thesis **unfolding** get-maximum-level-def **by** (simp add: A)

qed

lemma get-maximum-level-skip-notin:

assumes D: ∀ L ∈ #D. atm-of L ∈ atm-of ‘ lits-of-l M

shows get-maximum-level M D = get-maximum-level (Propagated x21 x22 # M) D

proof –

have A: (get-rev-level (rev M @ [Propagated x21 x22]) 0) ‘ set-mset D
 = (get-rev-level (rev M) 0) ‘ set-mset D

using D **by** (auto intro!: image-cong simp add: lits-of-def)

show ?thesis **unfolding** get-maximum-level-def **by** (auto simp: A)

qed

lemma get-maximum-level-skip-un-marked-not-present:

assumes ∀ L ∈ #D. atm-of L ∈ atm-of ‘ lits-of-l aa **and**

∀ m ∈ set M. ¬ is-marked m

shows get-maximum-level aa D = get-maximum-level (M @ aa) D

using assms **by** (induction M rule: marked-lit-list-induct)

(auto intro!: get-maximum-level-skip-notin[of D - @ aa] simp add: image-Un)

lemma get-maximum-level-union-mset:

get-maximum-level M (A # ∪ B) = get-maximum-level M (A + B)

unfolding get-maximum-level-def **by** (auto simp: image-Un)

fun get-maximum-possible-level:: ('b, nat, 'c) marked-lit list ⇒ nat **where**

get-maximum-possible-level [] = 0 |

get-maximum-possible-level (Marked K i # l) = max i (get-maximum-possible-level l) |

get-maximum-possible-level (Propagated - - # l) = get-maximum-possible-level l

lemma get-maximum-possible-level-append[simp]:

get-maximum-possible-level (M @ M')

= max (get-maximum-possible-level M) (get-maximum-possible-level M')

by (induct M rule: marked-lit-list-induct) auto

lemma get-maximum-possible-level-rev[simp]:

get-maximum-possible-level (rev M) = get-maximum-possible-level M

by (induct M rule: marked-lit-list-induct) auto

lemma get-maximum-possible-level-ge-get-rev-level:

max (get-maximum-possible-level M) i ≥ get-rev-level M i L

by (induct M arbitrary: i rule: marked-lit-list-induct) (auto simp add: le-max-iff-disj)

lemma *get-maximum-possible-level-ge-get-level*[simp]:
get-maximum-possible-level $M \geq$ *get-level* M L
using *get-maximum-possible-level-ge-get-rev-level*[of *rev* - 0] **by** *auto*

lemma *get-maximum-possible-level-ge-get-maximum-level*[simp]:
get-maximum-possible-level $M \geq$ *get-maximum-level* M D
using *get-maximum-level-exists-lit-of-max-level* **unfolding** *Bex-def*
by (*metis* *get-maximum-level-empty* *get-maximum-possible-level-ge-get-level* *le0*)

fun *get-all-mark-of-propagated* **where**
get-all-mark-of-propagated [] = [] |
get-all-mark-of-propagated (*Marked* - - # L) = *get-all-mark-of-propagated* L |
get-all-mark-of-propagated (*Propagated* - mark # L) = mark # *get-all-mark-of-propagated* L

lemma *get-all-mark-of-propagated-append*[simp]:
get-all-mark-of-propagated ($A @ B$) = *get-all-mark-of-propagated* $A @$ *get-all-mark-of-propagated* B
by (*induct* A *rule: marked-lit-list-induct*) *auto*

17.5.2 Properties about the levels

fun *get-all-levels-of-marked* :: ('b, 'a, 'c) *marked-lit list* \Rightarrow 'a *list* **where**
get-all-levels-of-marked [] = [] |
get-all-levels-of-marked (*Marked* l level # Ls) = level # *get-all-levels-of-marked* Ls |
get-all-levels-of-marked (*Propagated* - - # Ls) = *get-all-levels-of-marked* Ls

lemma *get-all-levels-of-marked-nil-iff-not-is-marked*:
get-all-levels-of-marked $xs = [] \longleftrightarrow (\forall x \in \text{set } xs. \neg \text{is-marked } x)$
using *assms* **by** (*induction* xs *rule: marked-lit-list-induct*) *auto*

lemma *get-all-levels-of-marked-cons*:
get-all-levels-of-marked ($a \# b$) =
 (*if* *is-marked* a *then* [*level-of* a] *else* []) @ *get-all-levels-of-marked* b
by (*cases* a) *simp-all*

lemma *get-all-levels-of-marked-append*[simp]:
get-all-levels-of-marked ($a @ b$) = *get-all-levels-of-marked* $a @$ *get-all-levels-of-marked* b
by (*induct* a) (*simp-all* *add: get-all-levels-of-marked-cons*)

lemma *in-get-all-levels-of-marked-iff-decomp*:
 $i \in \text{set } (\text{get-all-levels-of-marked } M) \longleftrightarrow (\exists c K c'. M = c @ \text{Marked } K i \# c') \text{ (is } ?A \longleftrightarrow ?B)$

proof

assume $?B$

then show $?A$ **by** *auto*

next

assume $?A$

then show $?B$

apply (*induction* M *rule: marked-lit-list-induct*)

apply *auto*[]

apply (*metis* *append-Cons* *append-Nil* *get-all-levels-of-marked.simps(2)* *set-ConsD*)

by (*metis* *append-Cons* *get-all-levels-of-marked.simps(3)*)

qed

lemma *get-rev-level-less-max-get-all-levels-of-marked*:
get-rev-level M n $L \leq \text{Max } (\text{set } (n \# \text{get-all-levels-of-marked } M))$
by (*induct* M *arbitrary: n rule: get-all-levels-of-marked.induct*)
 (*simp-all* *add: max.coboundedI2*)

lemma *get-rev-level-ge-min-get-all-levels-of-marked*:
assumes *atm-of* $L \in \text{atm-of } \text{' } \text{ lits-of-l } M$
shows $\text{get-rev-level } M \ n \ L \geq \text{Min } (\text{set } (n \# \text{get-all-levels-of-marked } M))$
using *assms* **by** (*induct* M *arbitrary*: n *rule*: *get-all-levels-of-marked.induct*)
(auto simp add: min-le-iff-disj)

lemma *get-all-levels-of-marked-rev-eq-rev-get-all-levels-of-marked[simp]*:
 $\text{get-all-levels-of-marked } (\text{rev } M) = \text{rev } (\text{get-all-levels-of-marked } M)$
by (*induct* M *rule*: *get-all-levels-of-marked.induct*)
(simp-all add: max.coboundedI2)

lemma *get-maximum-possible-level-max-get-all-levels-of-marked*:
 $\text{get-maximum-possible-level } M = \text{Max } (\text{insert } 0 \ (\text{set } (\text{get-all-levels-of-marked } M)))$
by (*induct* M *rule*: *marked-lit-list-induct*) *(auto simp: insert-commute)*

lemma *get-rev-level-in-levels-of-marked*:
 $\text{get-rev-level } M \ n \ L \in \{0, n\} \cup \text{set } (\text{get-all-levels-of-marked } M)$
by (*induction* M *arbitrary*: n *rule*: *marked-lit-list-induct*) *(force simp add: atm-of-eq-atm-of)+*

lemma *get-rev-level-in-atms-in-levels-of-marked*:
 $\text{atm-of } L \in \text{atm-of } \text{' } (\text{lits-of-l } M) \implies$
 $\text{get-rev-level } M \ n \ L \in \{n\} \cup \text{set } (\text{get-all-levels-of-marked } M)$
by (*induction* M *arbitrary*: n *rule*: *marked-lit-list-induct*) *(auto simp add: atm-of-eq-atm-of)*

lemma *get-all-levels-of-marked-no-marked*:
 $(\forall l \in \text{set } Ls. \neg \text{is-marked } l) \longleftrightarrow \text{get-all-levels-of-marked } Ls = []$
by (*induction* Ls) *(auto simp add: get-all-levels-of-marked-cons)*

lemma *get-level-in-levels-of-marked*:
 $\text{get-level } M \ L \in \{0\} \cup \text{set } (\text{get-all-levels-of-marked } M)$
using *get-rev-level-in-levels-of-marked[of rev M 0 L]* **by** *auto*

The zero is here to avoid empty-list issues with *last*:

lemma *get-level-get-rev-level-get-all-levels-of-marked*:
assumes $\text{atm-of } L \notin \text{atm-of } \text{' } (\text{lits-of-l } M)$
shows
 $\text{get-level } (K @ M) \ L = \text{get-rev-level } (\text{rev } K) \ (\text{last } (0 \# \text{get-all-levels-of-marked } (\text{rev } M))) \ L$
using *assms*
proof (*induct* M *arbitrary*: K)
case *Nil*
then show *?case* **by** *auto*
next
case (*Cons* $a \ M$)
then have $H: \bigwedge K. \text{get-level } (K @ M) \ L$
 $= \text{get-rev-level } (\text{rev } K) \ (\text{last } (0 \# \text{get-all-levels-of-marked } (\text{rev } M))) \ L$
by *auto*
have $\text{get-level } ((K @ [a]) @ M) \ L$
 $= \text{get-rev-level } (a \# \text{rev } K) \ (\text{last } (0 \# \text{get-all-levels-of-marked } (\text{rev } M))) \ L$
using $H[\text{of } K @ [a]]$ **by** *simp*
then show *?case* **using** *Cons(2)* **by** (*cases* a) *auto*
qed

lemma *get-rev-level-can-skip-correctly-ordered*:
assumes

```

    no-dup M and
    atm-of L  $\notin$  atm-of ' (lits-of-l M) and
    get-all-levels-of-marked M = rev [Suc 0.. $\leq$  Suc (length (get-all-levels-of-marked M))]
  shows get-rev-level (rev M @ K) 0 L = get-rev-level K (length (get-all-levels-of-marked M)) L
  using assms
proof (induct M arbitrary: K rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
  case (marked L' i M K)
  then have
    i: i = Suc (length (get-all-levels-of-marked M)) and
    get-all-levels-of-marked M = rev [Suc 0.. $\leq$  Suc (length (get-all-levels-of-marked M))]
  by auto
  then have get-rev-level (rev M @ (Marked L' i # K)) 0 L
    = get-rev-level (Marked L' i # K) (length (get-all-levels-of-marked M)) L
  using marked by auto
  then show ?case using marked unfolding i by auto
next
  case (proped L' D M K)
  then have get-all-levels-of-marked M = rev [Suc 0.. $\leq$  Suc (length (get-all-levels-of-marked M))]
  by auto
  then have get-rev-level (rev M @ (Propagated L' D # K)) 0 L
    = get-rev-level (Propagated L' D # K) (length (get-all-levels-of-marked M)) L
  using proped by auto
  then show ?case using proped by auto
qed

lemma get-level-skip-beginning-hd-get-all-levels-of-marked:
  assumes atm-of L  $\notin$  atm-of ' lits-of-l S and get-all-levels-of-marked S  $\neq$  []
  shows get-level (M @ S) L = get-rev-level (rev M) (hd (get-all-levels-of-marked S)) L
  using assms
proof (induction S arbitrary: M rule: marked-lit-list-induct)
  case nil
  then show ?case by (auto simp add: lits-of-def)
next
  case (marked K m)
  note notin = this(2)
  then show ?case by (auto simp add: lits-of-def)
next
  case (proped L l)
  note IH = this(1) and L = this(2) and neq = this(3)
  show ?case using IH[of M@[Propagated L l]] L neq by (auto simp add: atm-of-eq-atm-of)
qed

end
theory CDCL-W
imports CDCL-Abstract-Clause-Representation List-More CDCL-W-Level Wellfounded-More

begin

```

18 Weidenbach's CDCL

```

declare upt.simps(2)[simp del]

```

18.1 The State

We will abstract the representation of clause and clauses via two locales. We expect our representation to behave like multiset, but the internal representation can be done using list or whatever other representation.

```

locale statew-ops =
  raw-clss mset-cl insert-cl remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
  +
  raw-ccls-union mset-ccls union-ccls insert-ccls remove-clit
for
  — Clause
  mset-cls:: 'cls ⇒ 'v clause and
  insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
  remove-lits :: 'v literal ⇒ 'cls ⇒ 'cls and

  — Multiset of Clauses
  mset-clss:: 'clss ⇒ 'v clauses and
  union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
  in-clss :: 'cls ⇒ 'clss ⇒ bool and
  insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
  remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and

  mset-ccls:: 'ccls ⇒ 'v clause and
  union-ccls :: 'ccls ⇒ 'ccls ⇒ 'ccls and
  insert-ccls :: 'v literal ⇒ 'ccls ⇒ 'ccls and
  remove-clits :: 'v literal ⇒ 'ccls ⇒ 'ccls
  +
fixes
  ccls-of-cls :: 'cls ⇒ 'ccls and
  cls-of-ccls :: 'ccls ⇒ 'cls and

  trails :: 'st ⇒ ('v, nat, 'v clause) marked-lits and
  hd-raw-trails :: 'st ⇒ ('v, nat, 'cls) marked-lit and
  raw-init-clss :: 'st ⇒ 'clss and
  raw-learned-clss :: 'st ⇒ 'clss and
  backtrack-lvls :: 'st ⇒ nat and
  raw-conflictings :: 'st ⇒ 'ccls option and

  cons-trails :: ('v, nat, 'cls) marked-lit ⇒ 'st ⇒ 'st and
  tl-trails :: 'st ⇒ 'st and
  add-init-cls :: 'cls ⇒ 'st ⇒ 'st and
  add-learned-cls :: 'cls ⇒ 'st ⇒ 'st and
  remove-cls :: 'cls ⇒ 'st ⇒ 'st and
  update-backtrack-lvls :: nat ⇒ 'st ⇒ 'st and
  update-conflictings :: 'ccls option ⇒ 'st ⇒ 'st and

  init-states :: 'clss ⇒ 'st and
  restart-states :: 'st ⇒ 'st
assumes
  mset-ccls-ccls-of-cls[simp]:
    mset-ccls (ccls-of-cls C) = mset-cls C and
  mset-cls-cls-of-ccls[simp]:
    mset-cls (cls-of-ccls D) = mset-ccls D and
  ex-mset-cls: ∃ a. mset-cls a = E

```

```

begin
fun mmset-of-mlit :: ('a, 'b, 'cls) marked-lit  $\Rightarrow$  ('a, 'b, 'v clause) marked-lit
  where
mmset-of-mlit (Propagated L C) = Propagated L (mset-clss C) |
mmset-of-mlit (Marked L i) = Marked L i

lemma lit-of-mmset-of-mlit[simp]:
  lit-of (mmset-of-mlit a) = lit-of a
  by (cases a) auto

lemma lit-of-mmset-of-mlit-set-lit-of-l[simp]:
  lit-of ' mmset-of-mlit ' set M' = lits-of-l M'
  by (induction M') auto

lemma map-mmset-of-mlit-true-annots-true-clss[simp]:
  map mmset-of-mlit M'  $\models_{as}$  C  $\longleftrightarrow$  M'  $\models_{as}$  C
  by (simp add: true-annots-true-clss lits-of-def)

abbreviation init-clss  $\equiv$   $\lambda S. \text{mset-clss } (\text{raw-init-clss } S)$ 
abbreviation learned-clss  $\equiv$   $\lambda S. \text{mset-clss } (\text{raw-learned-clss } S)$ 
abbreviation conflicting  $\equiv$   $\lambda S. \text{map-option mset-ccls } (\text{raw-conflicting } S)$ 

notation insert-clss (infix !++ 50)

notation in-clss (infix ! $\in$ ! 50)
notation union-clss (infix  $\oplus$  50)
notation insert-clss (infix !++! 50)

notation union-ccls (infix ! $\cup$  50)

definition raw-clauses :: 'st  $\Rightarrow$  'clss where
raw-clauses S = union-clss (raw-init-clss S) (raw-learned-clss S)

abbreviation clauses :: 'st  $\Rightarrow$  'v clauses where
clauses S  $\equiv$  mset-clss (raw-clauses S)

end

locale stateW =
  stateW-ops
  — functions for clauses:
  mset-clss insert-clss remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss

  — functions for the conflicting clause:
  mset-ccls union-ccls insert-ccls remove-clit

  — Conversion between conflicting and non-conflicting
  ccls-of-clss clss-of-ccls

  — functions for the state:
  — access functions:
  trail hd-raw-trail raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting
  — changing state:
  cons-trail tl-trail add-init-clss add-learned-clss remove-clss update-backtrack-lvl

```

update-conflicting

— get state:

init-state

restart-state

for

mset-cls :: *'cls* \Rightarrow *'v clause* **and**

insert-cls :: *'v literal* \Rightarrow *'cls* \Rightarrow *'cls* **and**

remove-lit :: *'v literal* \Rightarrow *'cls* \Rightarrow *'cls* **and**

mset-clss :: *'clss* \Rightarrow *'v clauses* **and**

union-clss :: *'clss* \Rightarrow *'clss* \Rightarrow *'clss* **and**

in-clss :: *'cls* \Rightarrow *'clss* \Rightarrow *bool* **and**

insert-clss :: *'cls* \Rightarrow *'clss* \Rightarrow *'clss* **and**

remove-from-clss :: *'cls* \Rightarrow *'clss* \Rightarrow *'clss* **and**

mset-ccls :: *'ccls* \Rightarrow *'v clause* **and**

union-ccls :: *'ccls* \Rightarrow *'ccls* \Rightarrow *'ccls* **and**

insert-ccls :: *'v literal* \Rightarrow *'ccls* \Rightarrow *'ccls* **and**

remove-clit :: *'v literal* \Rightarrow *'ccls* \Rightarrow *'ccls* **and**

ccls-of-cls :: *'cls* \Rightarrow *'ccls* **and**

cls-of-ccls :: *'ccls* \Rightarrow *'cls* **and**

trail :: *'st* \Rightarrow (*'v*, *nat*, *'v clause*) *marked-lits* **and**

hd-raw-trail :: *'st* \Rightarrow (*'v*, *nat*, *'cls*) *marked-lit* **and**

raw-init-clss :: *'st* \Rightarrow *'clss* **and**

raw-learned-clss :: *'st* \Rightarrow *'clss* **and**

backtrack-lvl :: *'st* \Rightarrow *nat* **and**

raw-conflicting :: *'st* \Rightarrow *'ccls option* **and**

cons-trail :: (*'v*, *nat*, *'cls*) *marked-lit* \Rightarrow *'st* \Rightarrow *'st* **and**

tl-trail :: *'st* \Rightarrow *'st* **and**

add-init-cls :: *'cls* \Rightarrow *'st* \Rightarrow *'st* **and**

add-learned-cls :: *'cls* \Rightarrow *'st* \Rightarrow *'st* **and**

remove-cls :: *'cls* \Rightarrow *'st* \Rightarrow *'st* **and**

update-backtrack-lvl :: *nat* \Rightarrow *'st* \Rightarrow *'st* **and**

update-conflicting :: *'ccls option* \Rightarrow *'st* \Rightarrow *'st* **and**

init-state :: *'clss* \Rightarrow *'st* **and**

restart-state :: *'st* \Rightarrow *'st* +

assumes

hd-raw-trail: *trail S* $\neq [] \implies \text{mmset-of-mlit } (\text{hd-raw-trail } S) = \text{hd } (\text{trail } S)$ **and**

trail-cons-trail[simp]:

$\bigwedge L \text{ st. } \text{undefined-lit } (\text{trail st}) (\text{lit-of } L) \implies$

$\text{trail } (\text{cons-trail } L \text{ st}) = \text{mmset-of-mlit } L \# \text{trail st}$ **and**

trail-tl-trail[simp]: $\bigwedge \text{st. } \text{trail } (\text{tl-trail st}) = \text{tl } (\text{trail st})$ **and**

trail-add-init-cls[simp]:

$\bigwedge \text{st } C. \text{no-dup } (\text{trail st}) \implies \text{trail } (\text{add-init-cls } C \text{ st}) = \text{trail st}$ **and**

trail-add-learned-cls[simp]:

$\bigwedge C \text{ st. no-dup } (\text{trail st}) \implies \text{trail } (\text{add-learned-cls } C \text{ st}) = \text{trail st}$ **and**

trail-remove-cls[simp]:

$\bigwedge C \text{ st. } \text{trail } (\text{remove-cls } C \text{ st}) = \text{trail st}$ **and**

trail-update-backtrack-lvl[simp]: $\bigwedge \text{st } C. \text{trail } (\text{update-backtrack-lvl } C \text{ st}) = \text{trail st}$ **and**

trail-update-conflicting[simp]: $\bigwedge C \text{ st. } \text{trail } (\text{update-conflicting } C \text{ st}) = \text{trail st}$ **and**

init-clss-cons-trail[simp]:
 $\bigwedge M \text{ st. } \text{undefined-lit } (\text{trail } st) (\text{lit-of } M) \implies$
 $\text{init-clss } (\text{cons-trail } M \text{ st}) = \text{init-clss } st$
and
init-clss-tl-trail[simp]:
 $\bigwedge st. \text{init-clss } (\text{tl-trail } st) = \text{init-clss } st$ **and**
init-clss-add-init-cls[simp]:
 $\bigwedge st \ C. \text{no-dup } (\text{trail } st) \implies \text{init-clss } (\text{add-init-cls } C \text{ st}) = \{\#mset-cls \ C\# \} + \text{init-clss } st$
and
init-clss-add-learned-cls[simp]:
 $\bigwedge C \text{ st. no-dup } (\text{trail } st) \implies \text{init-clss } (\text{add-learned-cls } C \text{ st}) = \text{init-clss } st$ **and**
init-clss-remove-cls[simp]:
 $\bigwedge C \text{ st. init-clss } (\text{remove-cls } C \text{ st}) = \text{removeAll-mset } (mset-cls \ C) (\text{init-clss } st)$ **and**
init-clss-update-backtrack-lvl[simp]:
 $\bigwedge st \ C. \text{init-clss } (\text{update-backtrack-lvl } C \text{ st}) = \text{init-clss } st$ **and**
init-clss-update-conflicting[simp]:
 $\bigwedge C \text{ st. init-clss } (\text{update-conflicting } C \text{ st}) = \text{init-clss } st$ **and**

learned-clss-cons-trail[simp]:
 $\bigwedge M \text{ st. } \text{undefined-lit } (\text{trail } st) (\text{lit-of } M) \implies$
 $\text{learned-clss } (\text{cons-trail } M \text{ st}) = \text{learned-clss } st$ **and**
learned-clss-tl-trail[simp]:
 $\bigwedge st. \text{learned-clss } (\text{tl-trail } st) = \text{learned-clss } st$ **and**
learned-clss-add-init-cls[simp]:
 $\bigwedge st \ C. \text{no-dup } (\text{trail } st) \implies \text{learned-clss } (\text{add-init-cls } C \text{ st}) = \text{learned-clss } st$ **and**
learned-clss-add-learned-cls[simp]:
 $\bigwedge C \text{ st. no-dup } (\text{trail } st) \implies$
 $\text{learned-clss } (\text{add-learned-cls } C \text{ st}) = \{\#mset-cls \ C\# \} + \text{learned-clss } st$ **and**
learned-clss-remove-cls[simp]:
 $\bigwedge C \text{ st. learned-clss } (\text{remove-cls } C \text{ st}) = \text{removeAll-mset } (mset-cls \ C) (\text{learned-clss } st)$ **and**
learned-clss-update-backtrack-lvl[simp]:
 $\bigwedge st \ C. \text{learned-clss } (\text{update-backtrack-lvl } C \text{ st}) = \text{learned-clss } st$ **and**
learned-clss-update-conflicting[simp]:
 $\bigwedge C \text{ st. learned-clss } (\text{update-conflicting } C \text{ st}) = \text{learned-clss } st$ **and**

backtrack-lvl-cons-trail[simp]:
 $\bigwedge M \text{ st. } \text{undefined-lit } (\text{trail } st) (\text{lit-of } M) \implies$
 $\text{backtrack-lvl } (\text{cons-trail } M \text{ st}) = \text{backtrack-lvl } st$ **and**
backtrack-lvl-tl-trail[simp]:
 $\bigwedge st. \text{backtrack-lvl } (\text{tl-trail } st) = \text{backtrack-lvl } st$ **and**
backtrack-lvl-add-init-cls[simp]:
 $\bigwedge st \ C. \text{no-dup } (\text{trail } st) \implies \text{backtrack-lvl } (\text{add-init-cls } C \text{ st}) = \text{backtrack-lvl } st$ **and**
backtrack-lvl-add-learned-cls[simp]:
 $\bigwedge C \text{ st. no-dup } (\text{trail } st) \implies \text{backtrack-lvl } (\text{add-learned-cls } C \text{ st}) = \text{backtrack-lvl } st$ **and**
backtrack-lvl-remove-cls[simp]:
 $\bigwedge C \text{ st. backtrack-lvl } (\text{remove-cls } C \text{ st}) = \text{backtrack-lvl } st$ **and**
backtrack-lvl-update-backtrack-lvl[simp]:
 $\bigwedge st \ k. \text{backtrack-lvl } (\text{update-backtrack-lvl } k \text{ st}) = k$ **and**
backtrack-lvl-update-conflicting[simp]:
 $\bigwedge C \text{ st. backtrack-lvl } (\text{update-conflicting } C \text{ st}) = \text{backtrack-lvl } st$ **and**

conflicting-cons-trail[simp]:
 $\bigwedge M \text{ st. } \text{undefined-lit } (\text{trail } st) (\text{lit-of } M) \implies$
 $\text{conflicting } (\text{cons-trail } M \text{ st}) = \text{conflicting } st$ **and**

conflicting-tl-trail[simp]:
 $\bigwedge st. \text{ conflicting } (tl\text{-trail } st) = \text{ conflicting } st \text{ and}$
conflicting-add-init-cls[simp]:
 $\bigwedge st \ C. \text{ no-dup } (trail \ st) \implies \text{ conflicting } (add\text{-init-cls } C \ st) = \text{ conflicting } st \text{ and}$
conflicting-add-learned-cls[simp]:
 $\bigwedge C \ st. \text{ no-dup } (trail \ st) \implies \text{ conflicting } (add\text{-learned-cls } C \ st) = \text{ conflicting } st$
and
conflicting-remove-cls[simp]:
 $\bigwedge C \ st. \text{ conflicting } (remove\text{-cls } C \ st) = \text{ conflicting } st \text{ and}$
conflicting-update-backtrack-lvl[simp]:
 $\bigwedge st \ C. \text{ conflicting } (update\text{-backtrack-lvl } C \ st) = \text{ conflicting } st \text{ and}$
conflicting-update-conflicting[simp]:
 $\bigwedge C \ st. \text{ raw-conflicting } (update\text{-conflicting } C \ st) = C \text{ and}$

init-state-trail[simp]: $\bigwedge N. \text{ trail } (init\text{-state } N) = [] \text{ and}$
init-state-clss[simp]: $\bigwedge N. (init\text{-clss } (init\text{-state } N)) = mset\text{-clss } N \text{ and}$
init-state-learned-clss[simp]: $\bigwedge N. \text{ learned-clss } (init\text{-state } N) = \{\#\} \text{ and}$
init-state-backtrack-lvl[simp]: $\bigwedge N. \text{ backtrack-lvl } (init\text{-state } N) = 0 \text{ and}$
init-state-conflicting[simp]: $\bigwedge N. \text{ conflicting } (init\text{-state } N) = \text{None} \text{ and}$

trail-restart-state[simp]: $\text{ trail } (restart\text{-state } S) = [] \text{ and}$
init-clss-restart-state[simp]: $\text{ init-clss } (restart\text{-state } S) = \text{ init-clss } S \text{ and}$
learned-clss-restart-state[intro]:
 $\text{ learned-clss } (restart\text{-state } S) \subseteq \# \text{ learned-clss } S \text{ and}$
backtrack-lvl-restart-state[simp]: $\text{ backtrack-lvl } (restart\text{-state } S) = 0 \text{ and}$
conflicting-restart-state[simp]: $\text{ conflicting } (restart\text{-state } S) = \text{None}$

begin

lemma
shows

clauses-cons-trail[simp]:
 $\text{ undefined-lit } (trail \ S) \ (\text{lit-of } M) \implies \text{ clauses } (cons\text{-trail } M \ S) = \text{ clauses } S \text{ and}$
clss-tl-trail[simp]: $\text{ clauses } (tl\text{-trail } S) = \text{ clauses } S \text{ and}$
clauses-add-learned-cls-unfolded:
 $\text{ no-dup } (trail \ S) \implies \text{ clauses } (add\text{-learned-cls } U \ S) =$
 $\{\#mset\text{-cls } U\# \} + \text{ learned-clss } S + \text{ init-clss } S$
and
clauses-add-init-cls[simp]:
 $\text{ no-dup } (trail \ S) \implies$
 $\text{ clauses } (add\text{-init-cls } N \ S) = \{\#mset\text{-cls } N\# \} + \text{ init-clss } S + \text{ learned-clss } S \text{ and}$
clauses-update-backtrack-lvl[simp]: $\text{ clauses } (update\text{-backtrack-lvl } k \ S) = \text{ clauses } S \text{ and}$
clauses-update-conflicting[simp]: $\text{ clauses } (update\text{-conflicting } D \ S) = \text{ clauses } S \text{ and}$
clauses-remove-cls[simp]:
 $\text{ clauses } (remove\text{-cls } C \ S) = \text{ removeAll-mset } (mset\text{-cls } C) \ (\text{clauses } S) \text{ and}$
clauses-add-learned-cls[simp]:
 $\text{ no-dup } (trail \ S) \implies \text{ clauses } (add\text{-learned-cls } C \ S) = \{\#mset\text{-cls } C\# \} + \text{ clauses } S \text{ and}$
clauses-restart[simp]: $\text{ clauses } (restart\text{-state } S) \subseteq \# \text{ clauses } S \text{ and}$
clauses-init-state[simp]: $\bigwedge N. \text{ clauses } (init\text{-state } N) = mset\text{-clss } N$
prefer 9 using raw-clauses-def learned-clss-restart-state apply fastforce
by (auto simp: ac-simps replicate-mset-plus raw-clauses-def intro: multiset-eqI)

abbreviation *state* :: $'st \Rightarrow ('v, \text{ nat}, 'v \text{ clause}) \text{ marked-lit list} \times 'v \text{ clauses} \times 'v \text{ clauses}$
 $\times \text{ nat} \times 'v \text{ clause option}$ **where**
 $\text{ state } S \equiv (\text{ trail } S, \text{ init-clss } S, \text{ learned-clss } S, \text{ backtrack-lvl } S, \text{ conflicting } S)$

abbreviation *incr-lvl* :: 'st \Rightarrow 'st **where**
incr-lvl *S* \equiv *update-backtrack-lvl* (*backtrack-lvl* *S* + 1) *S*

definition *state-eq* :: 'st \Rightarrow 'st \Rightarrow bool (**infix** \sim 50) **where**
S \sim *T* \longleftrightarrow *state* *S* = *state* *T*

lemma *state-eq-ref*[*simp*, *intro*]:
S \sim *S*
unfolding *state-eq-def* **by** *auto*

lemma *state-eq-sym*:
S \sim *T* \longleftrightarrow *T* \sim *S*
unfolding *state-eq-def* **by** *auto*

lemma *state-eq-trans*:
S \sim *T* \Longrightarrow *T* \sim *U* \Longrightarrow *S* \sim *U*
unfolding *state-eq-def* **by** *auto*

lemma
shows
state-eq-trail: *S* \sim *T* \Longrightarrow *trail* *S* = *trail* *T* **and**
state-eq-init-clss: *S* \sim *T* \Longrightarrow *init-clss* *S* = *init-clss* *T* **and**
state-eq-learned-clss: *S* \sim *T* \Longrightarrow *learned-clss* *S* = *learned-clss* *T* **and**
state-eq-backtrack-lvl: *S* \sim *T* \Longrightarrow *backtrack-lvl* *S* = *backtrack-lvl* *T* **and**
state-eq-conflicting: *S* \sim *T* \Longrightarrow *conflicting* *S* = *conflicting* *T* **and**
state-eq-clauses: *S* \sim *T* \Longrightarrow *clauses* *S* = *clauses* *T* **and**
state-eq-undefined-lit: *S* \sim *T* \Longrightarrow *undefined-lit* (*trail* *S*) *L* = *undefined-lit* (*trail* *T*) *L*
unfolding *state-eq-def* *raw-clauses-def* **by** *auto*

lemma *state-eq-raw-conflicting-None*:
S \sim *T* \Longrightarrow *conflicting* *T* = *None* \Longrightarrow *raw-conflicting* *S* = *None*
unfolding *state-eq-def* *raw-clauses-def* **by** *auto*

lemmas *state-simp*[*simp*] = *state-eq-trail* *state-eq-init-clss* *state-eq-learned-clss*
state-eq-backtrack-lvl *state-eq-conflicting* *state-eq-clauses* *state-eq-undefined-lit*
state-eq-raw-conflicting-None

lemma *atms-of-ms-learned-clss-restart-state-in-atms-of-ms-learned-clssI*[*intro*]:
x \in *atms-of-mm* (*learned-clss* (*restart-state* *S*)) \Longrightarrow *x* \in *atms-of-mm* (*learned-clss* *S*)
by (*meson* *atms-of-ms-mono* *learned-clss-restart-state* *set-mset-mono* *subsetCE*)

function *reduce-trail-to* :: 'a list \Rightarrow 'st \Rightarrow 'st **where**
reduce-trail-to *F* *S* =
 (if *length* (*trail* *S*) = *length* *F* \vee *trail* *S* = [] then *S* else *reduce-trail-to* *F* (*tl-trail* *S*))
by *fast+*
termination
by (*relation* *measure* ($\lambda(-, S). \text{length } (\text{trail } S)$)) *simp-all*

declare *reduce-trail-to.simps*[*simp* *del*]

lemma
shows
reduce-trail-to-nil[*simp*]: *trail* *S* = [] \Longrightarrow *reduce-trail-to* *F* *S* = *S* **and**
reduce-trail-to-eq-length[*simp*]: *length* (*trail* *S*) = *length* *F* \Longrightarrow *reduce-trail-to* *F* *S* = *S*

by (auto simp: reduce-trail-to.simps)

lemma *reduce-trail-to-length-ne*:

$\text{length } (\text{trail } S) \neq \text{length } F \implies \text{trail } S \neq [] \implies$
 $\text{reduce-trail-to } F S = \text{reduce-trail-to } F (\text{tl-trail } S)$
 by (auto simp: reduce-trail-to.simps)

lemma *trail-reduce-trail-to-length-le*:

assumes $\text{length } F > \text{length } (\text{trail } S)$
shows $\text{trail } (\text{reduce-trail-to } F S) = []$
using *assms* **apply** (*induction* $F S$ *rule*: *reduce-trail-to.induct*)
by (*metis* (*no-types*, *hide-lams*) *length-tl less-imp-diff-less less-irrefl trail-tl-trail*
reduce-trail-to.simps)

lemma *trail-reduce-trail-to-nil[simp]*:

$\text{trail } (\text{reduce-trail-to } [] S) = []$
apply (*induction* $[]::'v, \text{nat}, 'v$ *clause*) *marked-lits S rule: reduce-trail-to.induct*)
by (*metis* *length-0-conv reduce-trail-to-length-ne reduce-trail-to-nil*)

lemma *clauses-reduce-trail-to-nil*:

$\text{clauses } (\text{reduce-trail-to } [] S) = \text{clauses } S$

proof (*induction* $[] S$ *rule*: *reduce-trail-to.induct*)

case ($1 Sa$)

then have $\text{clauses } (\text{reduce-trail-to } ([]::'a \text{ list}) (\text{tl-trail } Sa)) = \text{clauses } (\text{tl-trail } Sa)$
 $\vee \text{trail } Sa = []$

by *fastforce*

then show $\text{clauses } (\text{reduce-trail-to } ([]::'a \text{ list}) Sa) = \text{clauses } Sa$

by (*metis* (*no-types*) *length-0-conv reduce-trail-to-eq-length clss-tl-trail*
reduce-trail-to-length-ne)

qed

lemma *reduce-trail-to-skip-beginning*:

assumes $\text{trail } S = F' @ F$

shows $\text{trail } (\text{reduce-trail-to } F S) = F$

using *assms* **by** (*induction* F' *arbitrary: S*) (*auto simp: reduce-trail-to-length-ne*)

lemma *clauses-reduce-trail-to[simp]*:

$\text{clauses } (\text{reduce-trail-to } F S) = \text{clauses } S$

apply (*induction* $F S$ *rule*: *reduce-trail-to.induct*)

by (*metis* *clss-tl-trail reduce-trail-to.simps*)

lemma *conflicting-update-trail[simp]*:

$\text{conflicting } (\text{reduce-trail-to } F S) = \text{conflicting } S$

apply (*induction* $F S$ *rule*: *reduce-trail-to.induct*)

by (*metis* *conflicting-tl-trail reduce-trail-to.simps*)

lemma *backtrack-lvl-update-trail[simp]*:

$\text{backtrack-lvl } (\text{reduce-trail-to } F S) = \text{backtrack-lvl } S$

apply (*induction* $F S$ *rule*: *reduce-trail-to.induct*)

by (*metis* *backtrack-lvl-tl-trail reduce-trail-to.simps*)

lemma *init-clss-update-trail[simp]*:

$\text{init-clss } (\text{reduce-trail-to } F S) = \text{init-clss } S$

apply (*induction* $F S$ *rule*: *reduce-trail-to.induct*)

by (*metis* *init-clss-tl-trail reduce-trail-to.simps*)

lemma *learned-clss-update-trail[simp]*:
learned-clss (*reduce-trail-to* *F S*) = *learned-clss S*
apply (*induction F S rule: reduce-trail-to.induct*)
by (*metis learned-clss-tl-trail reduce-trail-to.simps*)

lemma *raw-conflicting-reduce-trail-to[simp]*:
raw-conflicting (*reduce-trail-to F S*) = *None* \longleftrightarrow *raw-conflicting S* = *None*
apply (*induction F S rule: reduce-trail-to.induct*)
by (*metis conflicting-update-trail map-option-is-None*)

lemma *trail-eq-reduce-trail-to-eq*:
trail S = *trail T* \implies *trail* (*reduce-trail-to F S*) = *trail* (*reduce-trail-to F T*)
apply (*induction F S arbitrary: T rule: reduce-trail-to.induct*)
by (*metis trail-tl-trail reduce-trail-to.simps*)

lemma *reduce-trail-to-state-eq_{NOT}-compatible*:
assumes *ST*: *S* \sim *T*
shows *reduce-trail-to F S* \sim *reduce-trail-to F T*
proof –
have *trail* (*reduce-trail-to F S*) = *trail* (*reduce-trail-to F T*)
using *trail-eq-reduce-trail-to-eq*[*of S T F*] *ST* **by** *auto*
then show ?thesis **using** *ST* **by** (*auto simp del: state-simp simp: state-eq-def*)
qed

lemma *reduce-trail-to-trail-tl-trail-decomp[simp]*:
trail S = *F' @ Marked K d # F* \implies (*trail* (*reduce-trail-to F S*)) = *F*
apply (*rule reduce-trail-to-skip-beginning*[*of - F' @ Marked K d # []*])
by (*cases F'*) (*auto simp add:tl-append reduce-trail-to-skip-beginning*)

lemma *reduce-trail-to-add-learned-cls[simp]*:
no-dup (*trail S*) \implies
trail (*reduce-trail-to F* (*add-learned-cls C S*)) = *trail* (*reduce-trail-to F S*)
by (*rule trail-eq-reduce-trail-to-eq*) *auto*

lemma *reduce-trail-to-add-init-cls[simp]*:
no-dup (*trail S*) \implies
trail (*reduce-trail-to F* (*add-init-cls C S*)) = *trail* (*reduce-trail-to F S*)
by (*rule trail-eq-reduce-trail-to-eq*) *auto*

lemma *reduce-trail-to-remove-learned-cls[simp]*:
trail (*reduce-trail-to F* (*remove-cls C S*)) = *trail* (*reduce-trail-to F S*)
by (*rule trail-eq-reduce-trail-to-eq*) *auto*

lemma *reduce-trail-to-update-conflicting[simp]*:
trail (*reduce-trail-to F* (*update-conflicting C S*)) = *trail* (*reduce-trail-to F S*)
by (*rule trail-eq-reduce-trail-to-eq*) *auto*

lemma *reduce-trail-to-update-backtrack-lvl[simp]*:
trail (*reduce-trail-to F* (*update-backtrack-lvl C S*)) = *trail* (*reduce-trail-to F S*)
by (*rule trail-eq-reduce-trail-to-eq*) *auto*

lemma *in-get-all-marked-decomposition-marked-or-empty*:
assumes (*a, b*) \in *set* (*get-all-marked-decomposition M*)

```

shows a = [] ∨ (is-marked (hd a))
using assms
proof (induct M arbitrary: a b)
  case Nil then show ?case by simp
next
  case (Cons m M)
  show ?case
    proof (cases m)
      case (Marked l mark)
      then show ?thesis using Cons by auto
    next
      case (Propagated l mark)
      then show ?thesis using Cons by (cases get-all-marked-decomposition M) force+
    qed
  qed
qed

```

lemma *reduce-trail-to-length*:

```

length M = length M'  $\implies$  reduce-trail-to M S = reduce-trail-to M' S
apply (induction M S arbitrary: rule: reduce-trail-to.induct)
by (simp add: reduce-trail-to.simps)

```

lemma *trail-reduce-trail-to-drop*:

```

trail (reduce-trail-to F S) =
  (if length (trail S)  $\geq$  length F
   then drop (length (trail S) - length F) (trail S)
   else [])
apply (induction F S rule: reduce-trail-to.induct)
apply (rename-tac F S, case-tac trail S)
apply auto[]
apply (rename-tac list, case-tac Suc (length list) > length F)
prefer 2 apply (metis diff-is-0-eq drop-Cons' length-Cons nat-le-linear nat-less-le
  reduce-trail-to-eq-length trail-reduce-trail-to-length-le)
apply (subgoal-tac Suc (length list) - length F = Suc (length list - length F))
by (auto simp add: reduce-trail-to-length-ne)

```

lemma *in-get-all-marked-decomposition-trail-update-trail[simp]*:

```

assumes H: (L # M1, M2)  $\in$  set (get-all-marked-decomposition (trail S))
shows trail (reduce-trail-to M1 S) = M1
proof -
  obtain K mark where
    L: L = Marked K mark
  using H by (cases L) (auto dest!: in-get-all-marked-decomposition-marked-or-empty)
  obtain c where
    tr-S: trail S = c @ M2 @ L # M1
  using H by auto
  show ?thesis
    by (rule reduce-trail-to-trail-tl-trail-decomp[of - c @ M2 K mark])
    (auto simp: tr-S L)

```

qed

lemma *raw-conflicting-cons-trail[simp]*:

```

assumes undefined-lit (trail S) (lit-of L)
shows
  raw-conflicting (cons-trail L S) = None  $\longleftrightarrow$  raw-conflicting S = None
using assms conflicting-cons-trail[of S L] map-option-is-None by fastforce+

```

```

lemma raw-conflicting-add-init-cls[simp]:
  no-dup (trail S)  $\implies$ 
    raw-conflicting (add-init-cls C S) = None  $\longleftrightarrow$  raw-conflicting S = None
  using map-option-is-None conflicting-add-init-cls[of S C] by fastforce+

lemma raw-conflicting-add-learned-cls[simp]:
  no-dup (trail S)  $\implies$ 
    raw-conflicting (add-learned-cls C S) = None  $\longleftrightarrow$  raw-conflicting S = None
  using map-option-is-None conflicting-add-learned-cls[of S C] by fastforce+

lemma raw-conflicting-update-backtrack-lvl[simp]:
  raw-conflicting (update-backtrack-lvl k S) = None  $\longleftrightarrow$  raw-conflicting S = None
  using map-option-is-None conflicting-update-backtrack-lvl[of k S] by fastforce+

end — end of stateW locale

```

18.2 CDCL Rules

Because of the strategy we will later use, we distinguish propagate, conflict from the other rules

```

locale conflict-driven-clause-learningW =
  stateW
  — functions for clauses:
  mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss

  — functions for the conflicting clause:
  mset-ccls union-ccls insert-ccls remove-clit

  — conversion
  ccls-of-cls cls-of-ccls

  — functions for the state:
  — access functions:
  trail hd-trail raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting
  — changing state:
  cons-trail tl-trail add-init-cls add-learned-cls remove-cls update-backtrack-lvl
  update-conflicting

  — get state:
  init-state
  restart-state
for
  mset-cls:: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and

  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and

  mset-ccls:: 'ccls  $\Rightarrow$  'v clause and
  union-ccls :: 'ccls  $\Rightarrow$  'ccls  $\Rightarrow$  'ccls and

```

insert-ccls :: 'v literal \Rightarrow 'ccls \Rightarrow 'ccls **and**
remove-clit :: 'v literal \Rightarrow 'ccls \Rightarrow 'ccls **and**

ccls-of-cls :: 'cls \Rightarrow 'ccls **and**
cls-of-ccls :: 'ccls \Rightarrow 'cls **and**

trail :: 'st \Rightarrow ('v, nat, 'v clause) marked-lits **and**
hd-raw-trail :: 'st \Rightarrow ('v, nat, 'cls) marked-lit **and**
raw-init-clss :: 'st \Rightarrow 'clss **and**
raw-learned-clss :: 'st \Rightarrow 'clss **and**
backtrack-lvl :: 'st \Rightarrow nat **and**
raw-conflicting :: 'st \Rightarrow 'ccls option **and**

cons-trail :: ('v, nat, 'cls) marked-lit \Rightarrow 'st \Rightarrow 'st **and**
tl-trail :: 'st \Rightarrow 'st **and**
add-init-cls :: 'cls \Rightarrow 'st \Rightarrow 'st **and**
add-learned-cls :: 'cls \Rightarrow 'st \Rightarrow 'st **and**
remove-cls :: 'cls \Rightarrow 'st \Rightarrow 'st **and**
update-backtrack-lvl :: nat \Rightarrow 'st \Rightarrow 'st **and**
update-conflicting :: 'ccls option \Rightarrow 'st \Rightarrow 'st **and**

init-state :: 'clss \Rightarrow 'st **and**
restart-state :: 'st \Rightarrow 'st

begin

inductive *propagate* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

propagate-rule: *conflicting S* = None \Rightarrow

E ! \in ! *raw-clauses S* \Rightarrow

L \in # *mset-cls E* \Rightarrow

trail S \models_{as} CNot (*mset-cls* (*remove-lit L E*)) \Rightarrow

undefined-lit (*trail S*) *L* \Rightarrow

T \sim *cons-trail* (*Propagated L E*) *S* \Rightarrow

propagate S T

inductive-cases *propagateE*: *propagate S T*

inductive *conflict* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

conflict-rule:

conflicting S = None \Rightarrow

D ! \in ! *raw-clauses S* \Rightarrow

trail S \models_{as} CNot (*mset-cls D*) \Rightarrow

T \sim *update-conflicting* (*Some* (*ccls-of-cls D*)) *S* \Rightarrow

conflict S T

inductive-cases *conflictE*: *conflict S T*

inductive *backtrack* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

backtrack-rule:

raw-conflicting S = *Some D* \Rightarrow

L \in # *mset-ccls D* \Rightarrow

(*Marked K* (*i*+1) # *M1, M2*) \in *set* (*get-all-marked-decomposition* (*trail S*)) \Rightarrow

get-level (*trail S*) *L* = *backtrack-lvl S* \Rightarrow

get-level (*trail S*) *L* = *get-maximum-level* (*trail S*) (*mset-ccls D*) \Rightarrow

get-maximum-level (*trail S*) (*mset-ccls* (*remove-clit L D*)) \equiv *i* \Rightarrow

T \sim *cons-trail* (*Propagated L* (*cls-of-ccls D*))

$(\text{reduce-trail-to } M1$
 $(\text{add-learned-cls } (\text{cls-of-ccls } D)$
 $(\text{update-backtrack-lvl } i$
 $(\text{update-conflicting } \text{None } S)))) \implies$
 $\text{backtrack } S \ T$

inductive-cases *backtrackE*: $\text{backtrack } S \ T$
thm *backtrackE*

inductive *decide* :: $'st \Rightarrow 'st \Rightarrow \text{bool}$ **for** $S :: 'st$ **where**
decide-rule:

$\text{conflicting } S = \text{None} \implies$
 $\text{undefined-lit } (\text{trail } S) \ L \implies$
 $\text{atm-of } L \in \text{atms-of-mm } (\text{init-cls } S) \implies$
 $T \sim \text{cons-trail } (\text{Marked } L \ (\text{backtrack-lvl } S + 1)) \ (\text{incr-lvl } S) \implies$
 $\text{decide } S \ T$

inductive-cases *decideE*: $\text{decide } S \ T$

inductive *skip* :: $'st \Rightarrow 'st \Rightarrow \text{bool}$ **for** $S :: 'st$ **where**
skip-rule:

$\text{trail } S = \text{Propagated } L \ C' \ \# \ M \implies$
 $\text{raw-conflicting } S = \text{Some } E \implies$
 $-L \notin \# \ \text{mset-ccls } E \implies$
 $\text{mset-ccls } E \neq \{\#\} \implies$
 $T \sim \text{tl-trail } S \implies$
 $\text{skip } S \ T$

inductive-cases *skipE*: $\text{skip } S \ T$

$\text{get-maximum-level } (\text{Propagated } L \ (C + \{\#L\}) \ \# \ M) \ D = k \vee k = 0$ (that was in a previous version of the book) is equivalent to $\text{get-maximum-level } (\text{Propagated } L \ (C + \{\#L\}) \ \# \ M) \ D = k$, when the structural invariants holds.

inductive *resolve* :: $'st \Rightarrow 'st \Rightarrow \text{bool}$ **for** $S :: 'st$ **where**

resolve-rule: $\text{trail } S \neq [] \implies$
 $\text{hd-raw-trail } S = \text{Propagated } L \ E \implies$
 $L \in \# \ \text{mset-cls } E \implies$
 $\text{raw-conflicting } S = \text{Some } D' \implies$
 $-L \in \# \ \text{mset-ccls } D' \implies$
 $\text{get-maximum-level } (\text{trail } S) \ (\text{mset-ccls } (\text{remove-clit } (-L) \ D')) = \text{backtrack-lvl } S \implies$
 $T \sim \text{update-conflicting } (\text{Some } (\text{union-ccls } (\text{remove-clit } (-L) \ D') \ (\text{ccls-of-cls } (\text{remove-lit } L \ E))))$
 $(\text{tl-trail } S) \implies$
 $\text{resolve } S \ T$

inductive-cases *resolveE*: $\text{resolve } S \ T$

inductive *restart* :: $'st \Rightarrow 'st \Rightarrow \text{bool}$ **for** $S :: 'st$ **where**

restart: $\text{state } S = (M, N, U, k, \text{None}) \implies \neg M \models_{\text{asm}} \text{clauses } S$
 $\implies T \sim \text{restart-state } S$
 $\implies \text{restart } S \ T$

inductive-cases *restartE*: $\text{restart } S \ T$

We add the condition $C \notin \# \ \text{init-cls } S$, to maintain consistency even without the strategy.

inductive *forget*:: 'st \Rightarrow 'st \Rightarrow bool **where**

forget-rule:

conflicting $S = \text{None} \Rightarrow$
 $C \notin \text{raw-learned-clss } S \Rightarrow$
 $\neg(\text{trail } S) \models_{\text{asm}} \text{clauses } S \Rightarrow$
 $\text{mset-clss } C \notin \text{set } (\text{get-all-mark-of-propagated } (\text{trail } S)) \Rightarrow$
 $\text{mset-clss } C \notin \# \text{init-clss } S \Rightarrow$
 $T \sim \text{remove-clss } C \ S \Rightarrow$
forget $S \ T$

inductive-cases *forgetE*: *forget* $S \ T$

inductive *cdcl_W-rf* :: 'st \Rightarrow 'st \Rightarrow bool **for** $S :: \text{'st}$ **where**

restart: *restart* $S \ T \Rightarrow \text{cdcl}_W\text{-rf } S \ T \mid$

forget: *forget* $S \ T \Rightarrow \text{cdcl}_W\text{-rf } S \ T$

inductive *cdcl_W-bj* :: 'st \Rightarrow 'st \Rightarrow bool **where**

skip: *skip* $S \ S' \Rightarrow \text{cdcl}_W\text{-bj } S \ S' \mid$

resolve: *resolve* $S \ S' \Rightarrow \text{cdcl}_W\text{-bj } S \ S' \mid$

backtrack: *backtrack* $S \ S' \Rightarrow \text{cdcl}_W\text{-bj } S \ S'$

inductive-cases *cdcl_W-bjE*: *cdcl_W-bj* $S \ T$

inductive *cdcl_W-o*:: 'st \Rightarrow 'st \Rightarrow bool **for** $S :: \text{'st}$ **where**

decide: *decide* $S \ S' \Rightarrow \text{cdcl}_W\text{-o } S \ S' \mid$

bj: *cdcl_W-bj* $S \ S' \Rightarrow \text{cdcl}_W\text{-o } S \ S'$

inductive *cdcl_W* :: 'st \Rightarrow 'st \Rightarrow bool **for** $S :: \text{'st}$ **where**

propagate: *propagate* $S \ S' \Rightarrow \text{cdcl}_W \ S \ S' \mid$

conflict: *conflict* $S \ S' \Rightarrow \text{cdcl}_W \ S \ S' \mid$

other: *cdcl_W-o* $S \ S' \Rightarrow \text{cdcl}_W \ S \ S' \mid$

rf: *cdcl_W-rf* $S \ S' \Rightarrow \text{cdcl}_W \ S \ S'$

lemma *rtranclp-propagate-is-rtranclp-cdcl_W*:

*propagate*** $S \ S' \Rightarrow \text{cdcl}_W^{**} \ S \ S'$

apply (*induction rule*: *rtranclp-induct*)

apply *simp*

apply (*frule* *propagate*)

using *rtranclp-trans*[*of cdcl_W*] **by** *blast*

lemma *cdcl_W-all-rules-induct*[*consumes 1*, *case-names propagate conflict forget restart decide skip resolve backtrack*]:

fixes $S :: \text{'st}$

assumes

cdcl_W: *cdcl_W* $S \ S'$ **and**

propagate: $\bigwedge T. \text{propagate } S \ T \Rightarrow P \ S \ T$ **and**

conflict: $\bigwedge T. \text{conflict } S \ T \Rightarrow P \ S \ T$ **and**

forget: $\bigwedge T. \text{forget } S \ T \Rightarrow P \ S \ T$ **and**

restart: $\bigwedge T. \text{restart } S \ T \Rightarrow P \ S \ T$ **and**

decide: $\bigwedge T. \text{decide } S \ T \Rightarrow P \ S \ T$ **and**

skip: $\bigwedge T. \text{skip } S \ T \Rightarrow P \ S \ T$ **and**

resolve: $\bigwedge T. \text{resolve } S \ T \Rightarrow P \ S \ T$ **and**

backtrack: $\bigwedge T. \text{backtrack } S \ T \Rightarrow P \ S \ T$

shows $P \ S \ S'$

using *assms*(1)

```

proof (induct S' rule: cdclW.induct)
  case (propagate S') note propagate = this(1)
  then show ?case using assms(2) by auto
next
  case (conflict S')
  then show ?case using assms(3) by auto
next
  case (other S')
  then show ?case
    proof (induct rule: cdclW-o.induct)
      case (decide U)
      then show ?case using assms(6) by auto
    next
      case (bj S')
      then show ?case using assms(7-9) by (induction rule: cdclW-bj.induct) auto
    qed
  next
  case (rf S')
  then show ?case
    by (induct rule: cdclW-rf.induct) (fast dest: forget restart)+
  qed

lemma cdclW-all-induct[consumes 1, case-names propagate conflict forget restart decide skip
  resolve backtrack]:
fixes S :: 'st
assumes
  cdclW: cdclW S S' and
  propagateH:  $\bigwedge C L T. \text{conflicting } S = \text{None} \implies$ 
    C ! $\in$ ! raw-clauses S  $\implies$ 
    L  $\in$  # mset-cls C  $\implies$ 
    trail S  $\models_{\text{as}}$  CNot (remove1-mset L (mset-cls C))  $\implies$ 
    undefined-lit (trail S) L  $\implies$ 
    T  $\sim$  cons-trail (Propagated L C) S  $\implies$ 
    P S T and
  conflictH:  $\bigwedge D T. \text{conflicting } S = \text{None} \implies$ 
    D ! $\in$ ! raw-clauses S  $\implies$ 
    trail S  $\models_{\text{as}}$  CNot (mset-cls D)  $\implies$ 
    T  $\sim$  update-conflicting (Some (ccls-of-cls D)) S  $\implies$ 
    P S T and
  forgetH:  $\bigwedge C U T. \text{conflicting } S = \text{None} \implies$ 
    C ! $\in$ ! raw-learned-clss S  $\implies$ 
     $\neg$ (trail S)  $\models_{\text{asm}}$  clauses S  $\implies$ 
    mset-cls C  $\notin$  set (get-all-mark-of-propagated (trail S))  $\implies$ 
    mset-cls C  $\notin$  # init-clss S  $\implies$ 
    T  $\sim$  remove-cls C S  $\implies$ 
    P S T and
  restartH:  $\bigwedge T. \neg \text{trail } S \models_{\text{asm}}$  clauses S  $\implies$ 
    conflicting S = None  $\implies$ 
    T  $\sim$  restart-state S  $\implies$ 
    P S T and
  decideH:  $\bigwedge L T. \text{conflicting } S = \text{None} \implies$ 
    undefined-lit (trail S) L  $\implies$ 
    atm-of L  $\in$  atms-of-mm (init-clss S)  $\implies$ 
    T  $\sim$  cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S)  $\implies$ 
    P S T and

```

$skipH: \bigwedge L C' M E T.$
 $trail S = \text{Propagated } L C' \# M \implies$
 $raw\text{-conflicting } S = \text{Some } E \implies$
 $-L \notin \# \text{ mset-ccls } E \implies \text{ mset-ccls } E \neq \{\#\} \implies$
 $T \sim tl\text{-trail } S \implies$
 $P S T \text{ and}$
 $resolveH: \bigwedge L E M D T.$
 $trail S = \text{Propagated } L (\text{mset-cls } E) \# M \implies$
 $L \in \# \text{ mset-cls } E \implies$
 $hd\text{-raw-trail } S = \text{Propagated } L E \implies$
 $raw\text{-conflicting } S = \text{Some } D \implies$
 $-L \in \# \text{ mset-ccls } D \implies$
 $get\text{-maximum-level } (trail S) (\text{mset-ccls } (\text{remove-clit } (-L) D)) = \text{backtrack-lvl } S \implies$
 $T \sim \text{update-conflicting}$
 $(\text{Some } (\text{union-ccls } (\text{remove-clit } (-L) D) (\text{ccls-of-cls } (\text{remove-lit } L E)))) (tl\text{-trail } S) \implies$
 $P S T \text{ and}$
 $backtrackH: \bigwedge L D K i M1 M2 T.$
 $raw\text{-conflicting } S = \text{Some } D \implies$
 $L \in \# \text{ mset-ccls } D \implies$
 $(\text{Marked } K (i+1) \# M1, M2) \in \text{set } (get\text{-all-marked-decomposition } (trail S)) \implies$
 $get\text{-level } (trail S) L = \text{backtrack-lvl } S \implies$
 $get\text{-level } (trail S) L = get\text{-maximum-level } (trail S) (\text{mset-ccls } D) \implies$
 $get\text{-maximum-level } (trail S) (\text{remove1-mset } L (\text{mset-ccls } D)) \equiv i \implies$
 $T \sim \text{cons-trail } (\text{Propagated } L (\text{cls-of-ccls } D))$
 $(\text{reduce-trail-to } M1$
 $(\text{add-learned-cls } (\text{cls-of-ccls } D)$
 $(\text{update-backtrack-lvl } i$
 $(\text{update-conflicting } \text{None } S)))) \implies$
 $P S T$
shows $P S S'$
using $cdcl_W$
proof ($\text{induct } S S' \text{ rule: } cdcl_W\text{-all-rules-induct}$)
case ($\text{propagate } S'$)
then show $?case$
by ($\text{auto elim!}:$ $\text{propagateE intro!}:$ propagateH)
next
case ($\text{conflict } S'$)
then show $?case$
by ($\text{auto elim!}:$ $\text{conflictE intro!}:$ conflictH)
next
case ($\text{restart } S'$)
then show $?case$
by ($\text{auto elim!}:$ $\text{restartE intro!}:$ restartH)
next
case ($\text{decide } T$)
then show $?case$
by ($\text{auto elim!}:$ $\text{decideE intro!}:$ decideH)
next
case ($\text{backtrack } S'$)
then show $?case$ **by** ($\text{auto elim!}:$ $\text{backtrackE intro!}:$ backtrackH
 $\text{simp del: state-simp simp add: state-eq-def}$)
next
case ($\text{forget } S'$)
then show $?case$ **by** ($\text{auto elim!}:$ $\text{forgetE intro!}:$ forgetH)
next

```

case (skip S')
then show ?case by (auto elim!: skipE intro!: skipH)
next
case (resolve S')
then show ?case
  using hd-raw-trail[of S] by (cases trail S) (auto elim!: resolveE intro!: resolveH)
qed

```

lemma *cdcl_W-o-induct*[consumes 1, case-names decide skip resolve backtrack]:
fixes *S* :: 'st
assumes *cdcl_W*: *cdcl_W-o S T* **and**
decideH: $\bigwedge L T. \text{conflicting } S = \text{None} \implies \text{undefined-lit } (\text{trail } S) L$
 $\implies \text{atm-of } L \in \text{atms-of-mm } (\text{init-clss } S)$
 $\implies T \sim \text{cons-trail } (\text{Marked } L (\text{backtrack-lvl } S + 1)) (\text{incr-lvl } S)$
 $\implies P S T$ **and**
skipH: $\bigwedge L C' M E T.$
 $\text{trail } S = \text{Propagated } L C' \# M \implies$
 $\text{raw-conflicting } S = \text{Some } E \implies$
 $-L \notin \# \text{mset-ccls } E \implies \text{mset-ccls } E \neq \{\#\} \implies$
 $T \sim \text{tl-trail } S \implies$
 $P S T$ **and**
resolveH: $\bigwedge L E M D T.$
 $\text{trail } S = \text{Propagated } L (\text{mset-clc } E) \# M \implies$
 $L \in \# \text{mset-clc } E \implies$
 $\text{hd-raw-trail } S = \text{Propagated } L E \implies$
 $\text{raw-conflicting } S = \text{Some } D \implies$
 $-L \in \# \text{mset-ccls } D \implies$
 $\text{get-maximum-level } (\text{trail } S) (\text{mset-ccls } (\text{remove-clit } (-L) D)) = \text{backtrack-lvl } S \implies$
 $T \sim \text{update-conflicting}$
 $(\text{Some } (\text{union-ccls } (\text{remove-clit } (-L) D) (\text{ccls-of-clc } (\text{remove-lit } L E)))) (\text{tl-trail } S) \implies$
 $P S T$ **and**
backtrackH: $\bigwedge L D K i M1 M2 T.$
 $\text{raw-conflicting } S = \text{Some } D \implies$
 $L \in \# \text{mset-ccls } D \implies$
 $(\text{Marked } K (i+1) \# M1, M2) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S)) \implies$
 $\text{get-level } (\text{trail } S) L = \text{backtrack-lvl } S \implies$
 $\text{get-level } (\text{trail } S) L = \text{get-maximum-level } (\text{trail } S) (\text{mset-ccls } D) \implies$
 $\text{get-maximum-level } (\text{trail } S) (\text{remove1-mset } L (\text{mset-ccls } D)) \equiv i \implies$
 $T \sim \text{cons-trail } (\text{Propagated } L (\text{cls-of-ccls } D))$
 $(\text{reduce-trail-to } M1$
 $(\text{add-learned-clc } (\text{cls-of-ccls } D)$
 $(\text{update-backtrack-lvl } i$
 $(\text{update-conflicting } \text{None } S)))) \implies$
 $P S T$
shows $P S T$
using *cdcl_W* **apply** (*induct T rule: cdcl_W-o.induct*)
using *assms*(2) **apply** (*auto elim: decideE*)[1]
apply (*elim cdcl_W-bjE skipE resolveE backtrackE*)
apply (*frule skipH; simp*)
using *hd-raw-trail*[of *S*] **apply** (*cases trail S; auto elim!: resolveE intro!: resolveH*)
apply (*frule backtrackH; simp-all del: state-simp add: state-eq-def*)
done

thm *cdcl_W-o.induct*

lemma *cdcl_W-o-all-rules-induct*[consumes 1, case-names decide backtrack skip resolve]:

```

fixes  $S\ T :: 'st$ 
assumes
   $cdcl_W-o\ S\ T$  and
   $\bigwedge T. decide\ S\ T \implies P\ S\ T$  and
   $\bigwedge T. backtrack\ S\ T \implies P\ S\ T$  and
   $\bigwedge T. skip\ S\ T \implies P\ S\ T$  and
   $\bigwedge T. resolve\ S\ T \implies P\ S\ T$ 
shows  $P\ S\ T$ 
using assms by (induct  $T$  rule:  $cdcl_W-o.induct$ ) (auto simp:  $cdcl_W-bj.simps$ )

```

lemma $cdcl_W-o-rule-cases[consumes\ 1, case-names\ decide\ backtrack\ skip\ resolve]$:

```

fixes  $S\ T :: 'st$ 
assumes
   $cdcl_W-o\ S\ T$  and
   $decide\ S\ T \implies P$  and
   $backtrack\ S\ T \implies P$  and
   $skip\ S\ T \implies P$  and
   $resolve\ S\ T \implies P$ 
shows  $P$ 
using assms by (auto simp:  $cdcl_W-o.simps\ cdcl_W-bj.simps$ )

```

18.3 Invariants

18.3.1 Properties of the trail

We here establish that: * the marks are exactly 1..k where k is the level * the consistency of the trail * the fact that there is no duplicate in the trail.

lemma *backtrack-lit-skipped*:

```

assumes
   $L: get-level\ (trail\ S)\ L = backtrack-lvl\ S$  and
   $M1: (Marked\ K\ (i + 1) \# M1, M2) \in set\ (get-all-marked-decomposition\ (trail\ S))$  and
   $no-dup: no-dup\ (trail\ S)$  and
   $bt-l: backtrack-lvl\ S = length\ (get-all-levels-of-marked\ (trail\ S))$  and
   $order: get-all-levels-of-marked\ (trail\ S)$ 
     $= rev\ [1..<(1+length\ (get-all-levels-of-marked\ (trail\ S)))]$ 
shows  $atm-of\ L \notin atm-of\ 'lits-of-l\ M1$ 
proof (rule ccontr)
  let  $?M = trail\ S$ 
  assume  $L-in-M1: \neg atm-of\ L \notin atm-of\ 'lits-of-l\ M1$ 
  obtain  $c$  where
     $Mc: trail\ S = c @ M2 @ Marked\ K\ (i + 1) \# M1$ 
  using  $M1$  by blast
  have  $atm-of\ L \notin atm-of\ 'lits-of-l\ c$ 
  using  $L-in-M1\ no-dup$  unfolding  $Mc\ lits-of-def$  by force
  have  $g-M-eq-g-M1: get-level\ ?M\ L = get-level\ M1\ L$ 
  using  $L-in-M1$  unfolding  $Mc$  by auto
  have  $g: get-all-levels-of-marked\ M1 = rev\ [1..<Suc\ i]$ 
  using order unfolding  $Mc$  by (auto simp del:  $upt-simps\ simp: rev-swap[symmetric]$ 
    dest:  $append-cons-eq-upt-length-i$ )
  then have  $Max\ (set\ (0 \# get-all-levels-of-marked\ (rev\ M1))) < Suc\ i$  by auto
  then have  $get-level\ M1\ L < Suc\ i$ 
  using  $get-rev-level-less-max-get-all-levels-of-marked[of\ rev\ M1\ 0\ L]$  by linarith
  moreover have  $Suc\ i \leq backtrack-lvl\ S$  using  $bt-l$  by (simp add:  $Mc\ g$ )
  ultimately show  $False$  using  $L\ g-M-eq-g-M1$  by auto
qed

```

lemma *cdcl_W-distinctinv-1*:

assumes

cdcl_W S S' **and**

no-dup (trail S) **and**

backtrack-lvl S = length (get-all-levels-of-marked (trail S)) **and**

get-all-levels-of-marked (trail S) = rev [1.. $1 + \text{length (get-all-levels-of-marked (trail S))}$]

shows *no-dup (trail S')*

using *assms*

proof (*induct rule: cdcl_W-all-induct*)

case (*backtrack L D K i M1 M2 T*) **note** *decomp = this(3)* **and** *L = this(4)* **and** *T = this(7)* **and** *n-d = this(8)*

obtain *c* **where** *Mc: trail S = c @ M2 @ Marked K (i + 1) # M1*

using *decomp* **by** *auto*

have *no-dup (M2 @ Marked K (i + 1) # M1)*

using *Mc n-d* **by** *fastforce*

moreover **have** *atm-of L \notin ($\lambda l. \text{atm-of (lit-of l)}$)* ‘*set M1*

using *backtrack-lit-skipped[of S L K i M1 M2] L decomp backtrack.prem*

by (*fastforce simp: lits-of-def*)

moreover **then** **have** *undefined-lit M1 L*

by (*simp add: defined-lit-map*)

ultimately **show** *?case* **using** *decomp T n-d* **by** *simp*

qed (*auto simp: defined-lit-map*)

lemma *cdcl_W-consistent-inv-2*:

assumes

cdcl_W S S' **and**

no-dup (trail S) **and**

backtrack-lvl S = length (get-all-levels-of-marked (trail S)) **and**

get-all-levels-of-marked (trail S) = rev [1.. $1 + \text{length (get-all-levels-of-marked (trail S))}$]

shows *consistent-interp (lits-of-l (trail S'))*

using *cdcl_W-distinctinv-1[OF assms] distinct-consistent-interp* **by** *fast*

lemma *cdcl_W-o-bt*:

assumes

cdcl_W-o S S' **and**

backtrack-lvl S = length (get-all-levels-of-marked (trail S)) **and**

get-all-levels-of-marked (trail S) =

rev ([1.. $1 + \text{length (get-all-levels-of-marked (trail S))}$]) **and**

n-d[simp]: no-dup (trail S)

shows *backtrack-lvl S' = length (get-all-levels-of-marked (trail S'))*

using *assms*

proof (*induct rule: cdcl_W-o-induct*)

case (*backtrack L D K i M1 M2 T*) **note** *decomp = this(3)* **and** *T = this(7)* **and** *level = this(9)*

have [*simp*]: *trail (reduce-trail-to M1 S) = M1*

using *decomp* **by** *auto*

obtain *c* **where** *M: trail S = c @ M2 @ Marked K (i + 1) # M1* **using** *decomp* **by** *auto*

have *rev (get-all-levels-of-marked (trail S))*

= [1.. $1 + (\text{length (get-all-levels-of-marked (trail S))})$]

using *level* **by** (*auto simp: rev-swap[symmetric]*)

moreover **have** *atm-of L \notin ($\lambda l. \text{atm-of (lit-of l)}$)* ‘*set M1*

using *backtrack-lit-skipped[of S L K i M1 M2] backtrack(4,8,9) decomp*

by (*fastforce simp add: lits-of-def*)

moreover **then** **have** *undefined-lit M1 L*

by (*simp add: defined-lit-map*)

moreover then have *no-dup* (trail *T*)
using *T decomp n-d* **by** (auto simp: defined-lit-map *M*)
ultimately show ?case
using *T n-d unfolding M* **by** (auto dest!: append-cons-eq-upt-length simp del: upt-simps)
qed auto

lemma *cdcl_W-rf-bt*:
assumes
cdcl_W-rf S S' **and**
backtrack-lvl S = length (get-all-levels-of-marked (trail S)) **and**
get-all-levels-of-marked (trail S) = rev [1.. $(1 + \text{length (get-all-levels-of-marked (trail S))})$]
shows *backtrack-lvl S' = length (get-all-levels-of-marked (trail S'))*
using *assms* **by** (induct rule: *cdcl_W-rf.induct*) (auto elim: restartE forgetE)

lemma *cdcl_W-bt*:
assumes
cdcl_W S S' **and**
backtrack-lvl S = length (get-all-levels-of-marked (trail S)) **and**
get-all-levels-of-marked (trail S)
 $= \text{rev } ([1.. $(1 + \text{length (get-all-levels-of-marked (trail S))})$])$ **and**
no-dup (trail S)
shows *backtrack-lvl S' = length (get-all-levels-of-marked (trail S'))*
using *assms* **by** (induct rule: *cdcl_W.induct*) (auto simp add: *cdcl_W-o-bt cdcl_W-rf-bt*
elim: conflictE propagateE)

lemma *cdcl_W-bt-level'*:
assumes
cdcl_W S S' **and**
backtrack-lvl S = length (get-all-levels-of-marked (trail S)) **and**
get-all-levels-of-marked (trail S)
 $= \text{rev } ([1.. $(1 + \text{length (get-all-levels-of-marked (trail S))})$])$ **and**
n-d: no-dup (trail S)
shows *get-all-levels-of-marked (trail S')*
 $= \text{rev } [1.. $(1 + \text{length (get-all-levels-of-marked (trail S'))})$]$
using *assms*
proof (induct rule: *cdcl_W-all-induct*)
case (decide *L T*) **note** *undef = this(2)* **and** *T = this(4)*
let ?k = *backtrack-lvl S*
let ?M = *trail S*
let ?M' = *Marked L (?k + 1) # trail S*
have *H: get-all-levels-of-marked ?M = rev [Suc 0.. $(1 + \text{length (get-all-levels-of-marked ?M)})$]*
using *decide.prem*s **by** simp
have *k: ?k = length (get-all-levels-of-marked ?M)*
using *decide.prem*s **by** auto
have *get-all-levels-of-marked ?M' = Suc ?k # get-all-levels-of-marked ?M* **by** simp
then have *get-all-levels-of-marked ?M' = Suc ?k #*
 $\text{rev } [\text{Suc } 0.. $(1 + \text{length (get-all-levels-of-marked ?M)})$]$
using *H* **by** auto
moreover have $\dots = \text{rev } [\text{Suc } 0.. $(1 + \text{length (get-all-levels-of-marked ?M)})$]$
unfolding *k* **by** simp
finally show ?case **using** *T undef* **by** (auto simp add: defined-lit-map)
next
case (*backtrack L D K i M1 M2 T*) **note** *decomp = this(3)* **and** *confli = this(1)* **and** *T = this(7)*
and
all-marked = this(9) **and** *bt-lvl = this(8)*


```

have atm-of L  $\notin$  atm-of ' lits-of-l M1
  using backtrack-lit-skipped[of S L K i M1 M2] backtrack(4,8-10) decomp
  by (fastforce simp add: lits-of-def)
moreover then have undefined-lit M1 L
  by (auto simp: defined-lit-map lits-of-def)
then have [simp]: trail T = Propagated L (mset-ccls D) # M1
  using T decomp n-d by auto
obtain c where M: trail S = c @ M2 @ Marked K (i + 1) # M1 using decomp by auto
have get-all-levels-of-marked (rev (trail S))
  = [Suc 0.. $2 + \text{length (get-all-levels-of-marked c)} + (\text{length (get-all-levels-of-marked M2)} + \text{length (get-all-levels-of-marked M1)})]$ 
  using all-marked bt-lvl unfolding M by (auto simp: rev-swap[symmetric] simp del: upt-simps)
then show ?case
  using T by (auto simp: rev-swap M simp del: upt-simps dest!: append-cons-eq-upt(1))
qed auto

```

We write $1 + \text{length (get-all-levels-of-marked (trail S))}$ instead of $\text{backtrack-lvl } S$ to avoid non termination of rewriting.

definition $\text{cdcl}_W\text{-}M\text{-level-inv} :: 'st \Rightarrow \text{bool}$ **where**
 $\text{cdcl}_W\text{-}M\text{-level-inv } S \longleftrightarrow$
 $\text{consistent-interp (lits-of-l (trail S))}$
 $\wedge \text{no-dup (trail S)}$
 $\wedge \text{backtrack-lvl } S = \text{length (get-all-levels-of-marked (trail S))}$
 $\wedge \text{get-all-levels-of-marked (trail S)}$
 $= \text{rev [1.. $1 + \text{length (get-all-levels-of-marked (trail S))}$]}$

lemma $\text{cdcl}_W\text{-}M\text{-level-inv-decomp}$:
assumes $\text{cdcl}_W\text{-}M\text{-level-inv } S$
shows
 $\text{consistent-interp (lits-of-l (trail S))}$ **and**
 no-dup (trail S)
using *assms* **unfolding** $\text{cdcl}_W\text{-}M\text{-level-inv-def}$ **by** *fastforce+*

lemma $\text{cdcl}_W\text{-consistent-inv}$:
fixes $S S' :: 'st$
assumes
 $\text{cdcl}_W S S'$ **and**
 $\text{cdcl}_W\text{-}M\text{-level-inv } S$
shows $\text{cdcl}_W\text{-}M\text{-level-inv } S'$
using *assms* $\text{cdcl}_W\text{-consistent-inv-2}$ $\text{cdcl}_W\text{-distinctinv-1}$ $\text{cdcl}_W\text{-bt}$ $\text{cdcl}_W\text{-bt-level'}$
unfolding $\text{cdcl}_W\text{-}M\text{-level-inv-def}$ **by** *meson+*

lemma $\text{rtrancpl-cdcl}_W\text{-consistent-inv}$:
assumes
 $\text{cdcl}_W^{**} S S'$ **and**
 $\text{cdcl}_W\text{-}M\text{-level-inv } S$
shows $\text{cdcl}_W\text{-}M\text{-level-inv } S'$
using *assms* **by** (*induct* rule: rtrancpl-induct) (*auto* *intro*: $\text{cdcl}_W\text{-consistent-inv}$)

lemma $\text{trancpl-cdcl}_W\text{-consistent-inv}$:
assumes
 $\text{cdcl}_W^{++} S S'$ **and**
 $\text{cdcl}_W\text{-}M\text{-level-inv } S$
shows $\text{cdcl}_W\text{-}M\text{-level-inv } S'$
using *assms* **by** (*induct* rule: trancpl-induct)

(auto intro: cdcl_W-consistent-inv)

lemma *cdcl_W-M-level-inv-S0-cdcl_W[simp]:*
cdcl_W-M-level-inv (init-state N)
unfolding *cdcl_W-M-level-inv-def* **by** *auto*

lemma *cdcl_W-M-level-inv-get-level-le-backtrack-lvl:*
assumes *inv: cdcl_W-M-level-inv S*
shows *get-level (trail S) L ≤ backtrack-lvl S*

proof –

have *get-all-levels-of-marked (trail S) = rev [1.. $1 + \text{backtrack-lvl } S$]*
using *inv* **unfolding** *cdcl_W-M-level-inv-def* **by** *auto*
then show *?thesis*
using *get-rev-level-less-max-get-all-levels-of-marked[of rev (trail S) 0 L]*
by (*auto simp: Max-n-upt*)

qed

lemma *backtrack-ex-decomp:*

assumes

M-l: cdcl_W-M-level-inv S **and**

i-S: i < backtrack-lvl S

shows $\exists K M1 M2. (\text{Marked } K (i + 1) \# M1, M2) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$

proof –

let *?M = trail S*

have

g: get-all-levels-of-marked (trail S) = rev [Suc 0.. $\text{Suc } (\text{backtrack-lvl } S)$]

using *M-l* **unfolding** *cdcl_W-M-level-inv-def* **by** *simp-all*

then have $i+1 \in \text{set } (\text{get-all-levels-of-marked } (\text{trail } S))$

using *i-S* **by** *auto*

then obtain *c K c'* **where** *tr-S: trail S = c @ Marked K (i + 1) # c'*

using *in-get-all-levels-of-marked-iff-decomp[of i+1 trail S]* **by** *auto*

obtain *M1 M2* **where** $(\text{Marked } K (i + 1) \# M1, M2) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$

using *Marked-cons-in-get-all-marked-decomposition-append-Marked-cons* **unfolding** *tr-S* **by** *fast*

then show *?thesis* **by** *blast*

qed

18.3.2 Better-Suited Induction Principle

We generalise the induction principle defined previously: the induction case for *backtrack* now includes the assumption that *undefined-lit M1 L*. This helps the simplifier and thus the automation.

lemma *backtrack-induction-lev[consumes 1, case-names M-devel-inv backtrack]:*

assumes

bt: backtrack S T **and**

inv: cdcl_W-M-level-inv S **and**

backtrackH: $\bigwedge K i M1 M2 L D T.$

raw-conflicting S = Some D \implies

L $\in \# \text{mset-ccls } D \implies$

(Marked K (Suc i) # M1, M2) $\in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S)) \implies$

get-level (trail S) L = backtrack-lvl S \implies

get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) \implies

get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) $\equiv i \implies$

undefined-lit M1 L \implies

$T \sim \text{cons-trail } (\text{Propagated } L \text{ (cls-of-ccls } D))$
 $(\text{reduce-trail-to } M1$
 $(\text{add-learned-cls } (\text{cls-of-ccls } D)$
 $(\text{update-backtrack-lvl } i$
 $(\text{update-conflicting } \text{None } S)))) \implies$
 $P \ S \ T$
shows $P \ S \ T$
proof –
obtain $K \ i \ M1 \ M2 \ L \ D$ **where**
 $\text{decomp: } (\text{Marked } K \ (\text{Suc } i) \ \# \ M1, \ M2) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$ **and**
 $L: \text{get-level } (\text{trail } S) \ L = \text{backtrack-lvl } S$ **and**
 $\text{confl: raw-conflicting } S = \text{Some } D$ **and**
 $LD: L \in \# \text{ mset-ccls } D$ **and**
 $\text{lev-L: get-level } (\text{trail } S) \ L = \text{get-maximum-level } (\text{trail } S) \ (\text{mset-ccls } D)$ **and**
 $\text{lev-D: get-maximum-level } (\text{trail } S) \ (\text{remove1-mset } L \ (\text{mset-ccls } D)) \equiv i$ **and**
 $T: T \sim \text{cons-trail } (\text{Propagated } L \text{ (cls-of-ccls } D))$
 $(\text{reduce-trail-to } M1$
 $(\text{add-learned-cls } (\text{cls-of-ccls } D)$
 $(\text{update-backtrack-lvl } i$
 $(\text{update-conflicting } \text{None } S))))$
using bt **by** (elim backtrackE) metis

have $\text{atm-of } L \notin \text{atm-of ' lits-of-l } M1$
using $\text{backtrack-lit-skipped[of } S \ L \ K \ i \ M1 \ M2] \ L \ \text{decomp } bt \ \text{confl } \text{lev-L } \text{lev-D } \text{inv}$
unfolding $\text{cdcl}_W\text{-M-level-inv-def}$ **by** force
then have $\text{undefined-lit } M1 \ L$
by $(\text{auto simp: defined-lit-map lits-of-def})$
then show $?thesis$
using $\text{backtrackH}[OF \ \text{confl } LD \ \text{decomp } L \ \text{lev-L } \text{lev-D} - T]$ **by** simp
qed

lemmas $\text{backtrack-induction-lev2} = \text{backtrack-induction-lev}[\text{consumes } 2, \text{case-names backtrack}]$

lemma $\text{cdcl}_W\text{-all-induct-lev-full:}$
fixes $S :: 'st$
assumes
 $\text{cdcl}_W: \text{cdcl}_W \ S \ S'$ **and**
 $\text{inv[simp]: cdcl}_W\text{-M-level-inv } S$ **and**
 $\text{propagateH: } \bigwedge C \ L \ T. \ \text{conflicting } S = \text{None} \implies$
 $C \ !\in! \text{ raw-clauses } S \implies$
 $L \in \# \text{ mset-cls } C \implies$
 $\text{trail } S \models_{\text{as}} C \text{Not } (\text{remove1-mset } L \ (\text{mset-cls } C)) \implies$
 $\text{undefined-lit } (\text{trail } S) \ L \implies$
 $T \sim \text{cons-trail } (\text{Propagated } L \ C) \ S \implies$
 $P \ S \ T$ **and**
 $\text{conflictH: } \bigwedge D \ T. \ \text{conflicting } S = \text{None} \implies$
 $D \ !\in! \text{ raw-clauses } S \implies$
 $\text{trail } S \models_{\text{as}} C \text{Not } (\text{mset-cls } D) \implies$
 $T \sim \text{update-conflicting } (\text{Some } (\text{ccls-of-cls } D)) \ S \implies$
 $P \ S \ T$ **and**
 $\text{forgetH: } \bigwedge C \ T. \ \text{conflicting } S = \text{None} \implies$
 $C \ !\in! \text{ raw-learned-clss } S \implies$
 $\neg(\text{trail } S) \models_{\text{asm}} \text{clauses } S \implies$
 $\text{mset-cls } C \notin \text{set } (\text{get-all-mark-of-propagated } (\text{trail } S)) \implies$
 $\text{mset-cls } C \notin \# \text{ init-clss } S \implies$

$T \sim \text{remove-cls } C \ S \implies$
 $P \ S \ T \text{ and}$
 $\text{restartH: } \bigwedge T. \neg \text{trail } S \models \text{asm clauses } S \implies$
 $\text{conflicting } S = \text{None} \implies$
 $T \sim \text{restart-state } S \implies$
 $P \ S \ T \text{ and}$
 $\text{decideH: } \bigwedge L \ T. \text{ conflicting } S = \text{None} \implies$
 $\text{undefined-lit } (\text{trail } S) \ L \implies$
 $\text{atm-of } L \in \text{atms-of-mm } (\text{init-clss } S) \implies$
 $T \sim \text{cons-trail } (\text{Marked } L \ (\text{backtrack-lvl } S + 1)) \ (\text{incr-lvl } S) \implies$
 $P \ S \ T \text{ and}$
 $\text{skipH: } \bigwedge L \ C' \ M \ E \ T.$
 $\text{trail } S = \text{Propagated } L \ C' \ \# \ M \implies$
 $\text{raw-conflicting } S = \text{Some } E \implies$
 $-L \notin \# \text{ mset-ccls } E \implies \text{ mset-ccls } E \neq \{\#\} \implies$
 $T \sim \text{tl-trail } S \implies$
 $P \ S \ T \text{ and}$
 $\text{resolveH: } \bigwedge L \ E \ M \ D \ T.$
 $\text{trail } S = \text{Propagated } L \ (\text{mset-cls } E) \ \# \ M \implies$
 $L \in \# \text{ mset-cls } E \implies$
 $\text{hd-raw-trail } S = \text{Propagated } L \ E \implies$
 $\text{raw-conflicting } S = \text{Some } D \implies$
 $-L \in \# \text{ mset-ccls } D \implies$
 $\text{get-maximum-level } (\text{trail } S) \ (\text{mset-ccls } (\text{remove-clit } (-L) \ D)) = \text{backtrack-lvl } S \implies$
 $T \sim \text{update-conflicting}$
 $(\text{Some } (\text{union-ccls } (\text{remove-clit } (-L) \ D) \ (\text{ccls-of-cls } (\text{remove-lit } L \ E)))) \ (\text{tl-trail } S) \implies$
 $P \ S \ T \text{ and}$
 $\text{backtrackH: } \bigwedge K \ i \ M1 \ M2 \ L \ D \ T.$
 $\text{raw-conflicting } S = \text{Some } D \implies$
 $L \in \# \text{ mset-ccls } D \implies$
 $(\text{Marked } K \ (\text{Suc } i) \ \# \ M1, M2) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S)) \implies$
 $\text{get-level } (\text{trail } S) \ L = \text{backtrack-lvl } S \implies$
 $\text{get-level } (\text{trail } S) \ L = \text{get-maximum-level } (\text{trail } S) \ (\text{mset-ccls } D) \implies$
 $\text{get-maximum-level } (\text{trail } S) \ (\text{remove1-mset } L \ (\text{mset-ccls } D)) \equiv i \implies$
 $\text{undefined-lit } M1 \ L \implies$
 $T \sim \text{cons-trail } (\text{Propagated } L \ (\text{cls-of-ccls } D))$
 $(\text{reduce-trail-to } M1$
 $(\text{add-learned-cls } (\text{cls-of-ccls } D)$
 $(\text{update-backtrack-lvl } i$
 $(\text{update-conflicting } \text{None } S)))) \implies$
 $P \ S \ T$
shows $P \ S \ S'$
using cdcl_W
proof ($\text{induct } S' \text{ rule: } \text{cdcl}_W\text{-all-rules-induct}$)
case ($\text{propagate } S'$)
then show $?case$
by ($\text{auto elim!} \text{: } \text{propagateE intro!} \text{: } \text{propagateH}$)
next
case ($\text{conflict } S'$)
then show $?case$
by ($\text{auto elim!} \text{: } \text{conflictE intro!} \text{: } \text{conflictH}$)
next
case ($\text{restart } S'$)
then show $?case$
by ($\text{auto elim!} \text{: } \text{restartE intro!} \text{: } \text{restartH}$)

```

next
  case (decide T)
  then show ?case
    by (auto elim!: decideE intro!: decideH)
next
  case (backtrack S')
  then show ?case
    apply (induction rule: backtrack-induction-lev)
    apply (rule inv)
    by (rule backtrackH;
        fastforce simp del: state-simp simp add: state-eq-def dest!: HOL.meta-eq-to-obj-eq)
next
  case (forget S')
  then show ?case by (auto elim!: forgetE intro!: forgetH)
next
  case (skip S')
  then show ?case by (auto elim!: skipE intro!: skipH)
next
  case (resolve S')
  then show ?case
    using hd-raw-trail[of S] by (cases trail S) (auto elim!: resolveE intro!: resolveH)
qed

lemmas cdclW-all-induct-lev2 = cdclW-all-induct-lev-full[consumes 2, case-names propagate conflict
forget restart decide skip resolve backtrack]

lemmas cdclW-all-induct-lev = cdclW-all-induct-lev-full[consumes 1, case-names lev-inv propagate
conflict forget restart decide skip resolve backtrack]

thm cdclW-o-induct
lemma cdclW-o-induct-lev[consumes 1, case-names M-lev decide skip resolve backtrack]:
  fixes S :: 'st
  assumes
    cdclW: cdclW-o S T and
    inv[simp]: cdclW-M-level-inv S and
    decideH:  $\bigwedge L T. \text{conflicting } S = \text{None} \implies$ 
      undefined-lit (trail S) L  $\implies$ 
      atm-of L  $\in$  atms-of-mm (init-clss S)  $\implies$ 
      T  $\sim$  cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S)  $\implies$ 
      P S T and
    skipH:  $\bigwedge L C' M E T. \text{trail } S = \text{Propagated } L C' \# M \implies$ 
      raw-conflicting S = Some E  $\implies$ 
       $\neg L \in \# \text{mset-ccls } E \implies \text{mset-ccls } E \neq \{\#\} \implies$ 
      T  $\sim$  tl-trail S  $\implies$ 
      P S T and
    resolveH:  $\bigwedge L E M D T. \text{trail } S = \text{Propagated } L (\text{mset-clss } E) \# M \implies$ 
      L  $\in \# \text{mset-clss } E \implies$ 
      hd-raw-trail S = Propagated L E  $\implies$ 
      raw-conflicting S = Some D  $\implies$ 
       $\neg L \in \# \text{mset-ccls } D \implies$ 
      get-maximum-level (trail S) (mset-ccls (remove-clit ( $\neg$ L) D)) = backtrack-lvl S  $\implies$ 
      T  $\sim$  update-conflicting
      (Some (union-ccls (remove-clit ( $\neg$ L) D) (ccls-of-clss (remove-lit L E)))) (tl-trail S)  $\implies$ 

```

```

  P S T and
  backtrackH:  $\bigwedge K i M1 M2 L D T.$ 
  raw-conflicting S = Some D  $\implies$ 
  L  $\in \#$  mset-ccls D  $\implies$ 
  (Marked K (Suc i) # M1, M2)  $\in$  set (get-all-marked-decomposition (trail S))  $\implies$ 
  get-level (trail S) L = backtrack-lvl S  $\implies$ 
  get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D)  $\implies$ 
  get-maximum-level (trail S) (remove1-mset L (mset-ccls D))  $\equiv$  i  $\implies$ 
  undefined-lit M1 L  $\implies$ 
  T  $\sim$  cons-trail (Propagated L (cls-of-ccls D))
    (reduce-trail-to M1
      (add-learned-cls (cls-of-ccls D)
        (update-backtrack-lvl i
          (update-conflicting None S))))  $\implies$ 

  P S T
  shows P S T
  using cdclW
proof (induct S T rule: cdclW-o-all-rules-induct)
  case (decide T)
  then show ?case
    by (auto elim!: decideE intro!: decideH)
next
  case (backtrack S')
  then show ?case
    apply (induction rule: backtrack-induction-lev)
    apply (rule inv)
    by (rule backtrackH;
      fastforce simp del: state-simp simp add: state-eq-def dest!: HOL.meta-eq-to-obj-eq)
next
  case (skip S')
  then show ?case by (auto elim!: skipE intro!: skipH)
next
  case (resolve S')
  then show ?case
    using hd-raw-trail[of S] by (cases trail S) (auto elim!: resolveE intro!: resolveH)
qed

lemmas cdclW-o-induct-lev2 = cdclW-o-induct-lev[consumes 2, case-names decide skip resolve
  backtrack]

```

18.3.3 Compatibility with $op \sim$

```

lemma propagate-state-eq-compatible:
  assumes
    propa: propagate S T and
    SS': S  $\sim$  S' and
    TT': T  $\sim$  T'
  shows propagate S' T'
proof -
  obtain C L where
    conf: conflicting S = None and
    C: C ! $\in$ ! raw-clauses S and
    L: L  $\in \#$  mset-cls C and
    tr: trail S  $\models_{as}$  CNot (remove1-mset L (mset-cls C)) and
    undef: undefined-lit (trail S) L and
    T: T  $\sim$  cons-trail (Propagated L C) S

```

```

using propa by (elim propagateE) auto

obtain C' where
  CC': mset-cls C' = mset-cls C and
  C': C' !∈! raw-clauses S'
using SS' C
  in-mset-clss-exists-preimage[of mset-cls C raw-learned-clss S']
  in-mset-clss-exists-preimage[of mset-cls C raw-init-clss S']
apply –
apply (frule in-clss-mset-clss)
by (auto simp: state-eq-def raw-clauses-def simp del: state-simp dest: in-clss-mset-clss)

show ?thesis
apply (rule propagate-rule[of - C'])
using state-eq-sym[of S S'] SS' conf C' CC' L tr undef TT' T
by (auto simp: state-eq-def simp del: state-simp)
qed

lemma conflict-state-eq-compatible:
assumes
  confl: conflict S T and
  TT': T ~ T' and
  SS': S ~ S'
shows conflict S' T'
proof –
obtain D where
  conf: conflicting S = None and
  D: D !∈! raw-clauses S and
  tr: trail S ⊨as CNot (mset-cls D) and
  T: T ~ update-conflicting (Some (ccls-of-cls D)) S
using confl by (elim conflictE) auto

obtain D' where
  DD': mset-cls D' = mset-cls D and
  D': D' !∈! raw-clauses S'
using D SS' in-mset-clss-exists-preimage by fastforce

show ?thesis
apply (rule conflict-rule[of - D'])
using state-eq-sym[of S S'] SS' conf D' DD' tr TT' T
by (auto simp: state-eq-def simp del: state-simp)
qed

lemma backtrack-levE[consumes 2]:
  backtrack S S' ⇒ cdclW-M-level-inv S ⇒
  (⋀K i M1 M2 L D.
    raw-conflicting S = Some D ⇒
    L ∈# mset-ccls D ⇒
    (Marked K (Suc i) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) ⇒
    get-level (trail S) L = backtrack-lvl S ⇒
    get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) ⇒
    get-maximum-level (trail S) (remove1-mset L (mset-ccls D)) ≡ i ⇒
    undefined-lit M1 L ⇒
    S' ~ cons-trail (Propagated L (cls-of-ccls D))
    (reduce-trail-to M1

```

```

      (add-learned-cls (cls-of-ccls D)
        (update-backtrack-lvl i
          (update-conflicting None S))))  $\implies$  P)  $\implies$ 
P
using assms by (induction rule: backtrack-induction-lev2) metis
thm allI

lemma backtrack-state-eq-compatible:
assumes
  bt: backtrack S T and
  SS':  $S \sim S'$  and
  TT':  $T \sim T'$  and
  inv: cdclW-M-level-inv S
shows backtrack S' T'
proof -
obtain D L K i M1 M2 where
  conf: raw-conflicting S = Some D and
  L:  $L \in \#$  mset-ccls D and
  decomp: (Marked K (Suc i)  $\#$  M1, M2)  $\in$  set (get-all-marked-decomposition (trail S)) and
  lev: get-level (trail S) L = backtrack-lvl S and
  max: get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) and
  max-D: get-maximum-level (trail S) (remove1-mset L (mset-ccls D))  $\equiv$  i and
  undef: undefined-lit M1 L and
  T:  $T \sim$  cons-trail (Propagated L (cls-of-ccls D))
    (reduce-trail-to M1
      (add-learned-cls (cls-of-ccls D)
        (update-backtrack-lvl i
          (update-conflicting None S))))
using bt inv by (elim backtrack-levE) metis
obtain D' where
  D': raw-conflicting S' = Some D'
  using SS' conf by (cases raw-conflicting S') auto
have [simp]: mset-ccls D = mset-ccls D'
  using SS' D' conf by (auto simp: state-eq-def simp del: state-simp)[]

have T':  $T' \sim$  cons-trail (Propagated L (cls-of-ccls D'))
  (reduce-trail-to M1 (add-learned-cls (cls-of-ccls D')
    (update-backtrack-lvl i (update-conflicting None S'))))
using TT' unfolding state-eq-def
using decomp undef inv SS' T by (auto simp add: cdclW-M-level-inv-def)

show ?thesis
apply (rule backtrack-rule[of - D'])
apply (rule D')
using state-eq-sym[of S S'] TT' SS' D' conf L decomp lev max max-D undef T
apply (auto simp: state-eq-def simp del: state-simp)[]
using decomp SS' lev SS' max-D max T' by (auto simp: state-eq-def simp del: state-simp)
qed

```

```

lemma decide-state-eq-compatible:
assumes
  decide S T and
   $S \sim S'$  and
   $T \sim T'$ 

```


shows decide $S' T'$
 using *assms* **apply** (*elim decideE*)
 by (rule *decide-rule*) (auto simp: *state-eq-def raw-clauses-def simp del: state-simp*)

lemma *skip-state-eq-compatible*:

assumes

skip: *skip S T* and

SS': $S \sim S'$ and

TT': $T \sim T'$

shows *skip S' T'*

proof –

obtain $L C' M E$ where

tr: *trail S = Propagated L C' # M* and

raw: *raw-conflicting S = Some E* and

L: $-L \notin \# \text{ mset-ccls } E$ and

E: *mset-ccls E* $\neq \{\#\}$ and

T: $T \sim \text{tl-trail } S$

using *skip* **by** (*elim skipE*) *simp*

obtain E' where *E'*: *raw-conflicting S' = Some E'*

using *SS'* *raw* **by** (*cases raw-conflicting S'*) (auto simp: *state-eq-def simp del: state-simp*)

show ?thesis

apply (rule *skip-rule*)

using *tr raw L E T SS'* **apply** (auto simp: *simp del:*) []

using *E'* **apply** *simp*

using *E' SS' L raw E* **apply** (auto simp: *state-eq-def simp del: state-simp*)[2]

using *T TT' SS'* **by** (auto simp: *state-eq-def simp del: state-simp*)

qed

lemma *resolve-state-eq-compatible*:

assumes

res: *resolve S T* and

TT': $T \sim T'$ and

SS': $S \sim S'$

shows *resolve S' T'*

proof –

obtain $E D L$ where

tr: *trail S* $\neq []$ and

hd: *hd-raw-trail S = Propagated L E* and

L: $L \in \# \text{ mset-cls } E$ and

raw: *raw-conflicting S = Some D* and

LD: $-L \in \# \text{ mset-ccls } D$ and

i: *get-maximum-level (trail S) (mset-ccls (remove-clit (-L) D)) = backtrack-lvl S* and

T: $T \sim \text{update-conflicting (Some (union-ccls (remove-clit (-L) D) (ccls-of-cls (remove-lit L E)))) (tl-trail S)}$

using *assms* **by** (*elim resolveE*) *simp*

obtain E' where

E': *hd-raw-trail S' = Propagated L E'*

using *SS' hd* **by** (*metis (trail S* $\neq []$) *hd-raw-trail is-proped-def marked-lit.disc(3)*
marked-lit.inject(2) *mmset-of-mlit.elims state-eq-trail*)

have [*simp*]: *mset-cls E = mset-cls E'*

using *hd-raw-trail[of S] tr hd-raw-trail[of S'] tr SS' hd E'*

by (*metis marked-lit.inject(2)* *mmset-of-mlit.simps(1)* *state-eq-trail*)

obtain D' where

D': *raw-conflicting S' = Some D'*

```

    using SS' raw by fastforce
have [simp]: mset-ccls D = mset-ccls D'
    using D' SS' raw state-simp(5) by fastforce
have T'T: T' ~ T
    using TT' state-eq-sym by auto
show ?thesis
    apply (rule resolve-rule)
      using tr SS' apply simp
      using E' apply simp
      using L apply simp
      using D' apply simp
      using D' SS' raw LD apply (auto simp add: state-eq-def simp del: state-simp)[]
      using D' SS' raw LD apply (auto simp add: state-eq-def simp del: state-simp)[]
      using raw SS' i apply (auto simp add: state-eq-def simp del: state-simp)[]
      using T T'T SS' by (auto simp: state-eq-def simp del: state-simp )
qed

```

lemma *forget-state-eq-compatible*:

```

assumes
  forget: forget S T and
  SS': S ~ S' and
  TT': T ~ T'
shows forget S' T'
proof -
  obtain C where
    conf: conflicting S = None and
    C !∈! raw-learned-clss S and
    tr: ¬(trail S) ⊨asm clauses S and
    C1: mset-cls C ∉ set (get-all-mark-of-propagated (trail S)) and
    C2: mset-cls C ∉# init-clss S and
    T: T ~ remove-cls C S
    using forget by (elim forgetE) simp

```

obtain C' where

```

  C': C' !∈! raw-learned-clss S' and
  [simp]: mset-cls C' = mset-cls C
  using ⟨C !∈! raw-learned-clss S⟩ SS' in-mset-clss-exists-preimage by fastforce
show ?thesis
  apply (rule forget-rule)
    using SS' conf apply simp
    using C' apply simp
    using SS' tr apply simp
    using SS' C1 apply simp
    using SS' C2 apply simp
  using T TT' SS' by (auto simp: state-eq-def simp del: state-simp)
qed

```

lemma *cdcl_W-state-eq-compatible*:

```

assumes
  cdclW S T and ¬restart S T and
  S ~ S'
  T ~ T' and
  cdclW-M-level-inv S
shows cdclW S' T'
using assms by (meson backtrack backtrack-state-eq-compatible bj cdclW.sims cdclW-o-rule-cases

```

*cdcl_W-rf.cases conflict-state-eq-compatible decide decide-state-eq-compatible forget
forget-state-eq-compatible propagate-state-eq-compatible resolve resolve-state-eq-compatible
skip skip-state-eq-compatible state-eq-ref)*

lemma *cdcl_W-bj-state-eq-compatible:*

assumes

cdcl_W-bj S T and cdcl_W-M-level-inv S

T ~ T'

shows *cdcl_W-bj S T'*

using *assms by (meson backtrack backtrack-state-eq-compatible cdcl_W-bjE resolve
resolve-state-eq-compatible skip skip-state-eq-compatible state-eq-ref)*

lemma *trancpl-cdcl_W-bj-state-eq-compatible:*

assumes

cdcl_W-bj⁺⁺ S T and inv: cdcl_W-M-level-inv S and

S ~ S' and

T ~ T'

shows *cdcl_W-bj⁺⁺ S' T'*

using *assms*

proof (*induction arbitrary: S' T'*)

case *base*

then show *?case*

unfolding *trancpl-unfold-end by (meson backtrack-state-eq-compatible cdcl_W-bj.simps
resolve-state-eq-compatible rtrancpl-unfold skip-state-eq-compatible)*

next

case (*step T U*) **note** *IH = this(3)[OF this(4-5)]*

have *cdcl_W⁺⁺ S T*

using *trancpl-mono[of cdcl_W-bj cdcl_W] step.hyps(1) cdcl_W.other cdcl_W-o.bj by blast*

then have *cdcl_W-M-level-inv T*

using *inv trancpl-cdcl_W-consistent-inv by blast*

then have *cdcl_W-bj⁺⁺ T T'*

using *⟨U ~ T'⟩ cdcl_W-bj-state-eq-compatible[of T U] ⟨cdcl_W-bj T U⟩ by auto*

then show *?case*

using *IH[of T] by auto*

qed

18.3.4 Conservation of some Properties

lemma *cdcl_W-o-no-more-init-clss:*

assumes

cdcl_W-o S S' and

inv: cdcl_W-M-level-inv S

shows *init-clss S = init-clss S'*

using *assms by (induct rule: cdcl_W-o-induct-lev2) (auto simp: inv cdcl_W-M-level-inv-decomp)*

lemma *trancpl-cdcl_W-o-no-more-init-clss:*

assumes

cdcl_W-o⁺⁺ S S' and

inv: cdcl_W-M-level-inv S

shows *init-clss S = init-clss S'*

using *assms apply (induct rule: trancpl.induct)*

by (*auto dest: cdcl_W-o-no-more-init-clss*

dest!: trancpl-cdcl_W-consistent-inv dest: trancpl-mono-explicit[of cdcl_W-o - - cdcl_W]

simp: other)

lemma *rtrancpl-cdcl_W-o-no-more-init-clss:*

assumes
 $cdcl_W\text{-}o^{**} S S'$ **and**
 $inv: cdcl_W\text{-}M\text{-level-inv } S$
shows $init\text{-}clss S = init\text{-}clss S'$
using *assms* **unfolding** *rtranclp-unfold* **by** (*auto intro: tranclp-cdcl_W-o-no-more-init-clss*)

lemma *cdcl_W-init-clss*:

assumes
 $cdcl_W S T$ **and**
 $inv: cdcl_W\text{-}M\text{-level-inv } S$
shows $init\text{-}clss S = init\text{-}clss T$
using *assms* **by** (*induct rule: cdcl_W-all-induct-lev2*)
(*auto simp: inv cdcl_W-M-level-inv-decomp not-in-iff*)

lemma *rtranclp-cdcl_W-init-clss*:

$cdcl_W^{**} S T \implies cdcl_W\text{-}M\text{-level-inv } S \implies init\text{-}clss S = init\text{-}clss T$
by (*induct rule: rtranclp-induct*) (*auto dest: cdcl_W-init-clss rtranclp-cdcl_W-consistent-inv*)

lemma *tranclp-cdcl_W-init-clss*:

$cdcl_W^{++} S T \implies cdcl_W\text{-}M\text{-level-inv } S \implies init\text{-}clss S = init\text{-}clss T$
using *rtranclp-cdcl_W-init-clss*[*of S T*] **unfolding** *rtranclp-unfold* **by** *auto*

18.3.5 Learned Clause

This invariant shows that:

- the learned clauses are entailed by the initial set of clauses.
- the conflicting clause is entailed by the initial set of clauses.
- the marks are entailed by the clauses. A more precise version would be to show that either these marked are learned or are in the set of clauses

definition *cdcl_W-learned-clause* ($S:: 'st$) \longleftrightarrow

($init\text{-}clss S \models_{psm} learned\text{-}clss S$
 $\wedge (\forall T. conflicting S = Some T \longrightarrow init\text{-}clss S \models_{pm} T)$
 $\wedge set (get\text{-}all\text{-}mark\text{-}of\text{-}propagated (trail S)) \subseteq set\text{-}mset (clauses S))$

lemma *cdcl_W-learned-clause-S0-cdcl_W[simp]*:

$cdcl_W\text{-}learned\text{-}clause (init\text{-}state N)$
unfolding *cdcl_W-learned-clause-def* **by** *auto*

lemma *cdcl_W-learned-clss*:

assumes
 $cdcl_W S S'$ **and**
 $learned: cdcl_W\text{-}learned\text{-}clause S$ **and**
 $lev\text{-}inv: cdcl_W\text{-}M\text{-level-inv } S$
shows $cdcl_W\text{-}learned\text{-}clause S'$
using *assms*(1) *lev-inv learned*

proof (*induct rule: cdcl_W-all-induct-lev2*)

case (*backtrack K i M1 M2 L D T*) **note** $decomp = this(3)$ **and** $confl = this(1)$ **and** $undef = this(7)$
and $T = this(8)$

show *?case*

using *decomp confl learned undef T* **unfolding** *cdcl_W-learned-clause-def*

```

    by (auto dest!: get-all-marked-decomposition-exists-prepend
        simp: raw-clauses-def lev-inv cdclW-M-level-inv-decomp dest: true-clss-clss-left-right)
next
case (resolve L C M D) note trail = this(1) and CL = this(2) and confl = this(4) and DL = this(5)
    and lw = this(6)
    and T = this(7)
moreover
    have init-clss S  $\models_{psm}$  learned-clss S
        using learned trail unfolding cdclW-learned-clause-def raw-clauses-def by auto
    then have init-clss S  $\models_{pm}$  mset-cl C + {#L#}
        using trail learned unfolding cdclW-learned-clause-def raw-clauses-def
        by (auto dest: true-clss-clss-in-imp-true-clss-clss)
    moreover have remove1-mset (- L) (mset-ccls D) + {#- L#} = mset-ccls D
        using DL by (auto simp: multiset-eq-iff)
    moreover have remove1-mset L (mset-cl C) + {#L#} = mset-cl C
        using CL by (auto simp: multiset-eq-iff)
    ultimately show ?case
        using learned T
        by (auto dest: mk-disjoint-insert
            simp add: cdclW-learned-clause-def raw-clauses-def
            intro!: true-clss-clss-union-mset-true-clss-clss-or-not-true-clss-clss-or[of - - L])
next
case (restart T)
then show ?case
    using learned learned-clss-restart-state[of T]
    by (auto
        simp: raw-clauses-def state-eq-def cdclW-learned-clause-def
        simp del: state-simp
        dest: true-clss-clssm-subsetE)
next
case propagate
then show ?case using learned by (auto simp: cdclW-learned-clause-def)
next
case conflict
then show ?case using learned
    by (fastforce simp: cdclW-learned-clause-def raw-clauses-def
        true-clss-clss-in-imp-true-clss-clss)
next
case (forget U)
then show ?case using learned
    by (auto simp: cdclW-learned-clause-def raw-clauses-def split: if-split-asm)
qed (auto simp: cdclW-learned-clause-def raw-clauses-def)

lemma rtranclp-cdclW-learned-clss:
  assumes
    cdclW** S S' and
    cdclW-M-level-inv S
    cdclW-learned-clause S
  shows cdclW-learned-clause S'
  using assms by induction (auto dest: cdclW-learned-clss intro: rtranclp-cdclW-consistent-inv)

```

18.3.6 No alien atom in the state

This invariant means that all the literals are in the set of clauses.

definition *no-strange-atm* $S' \longleftrightarrow ($

$(\forall T. \text{conflicting } S' = \text{Some } T \longrightarrow \text{atms-of } T \subseteq \text{atms-of-mm } (\text{init-clss } S'))$
 $\wedge (\forall L \text{ mark. Propagated } L \text{ mark} \in \text{set } (\text{trail } S') \longrightarrow \text{atms-of } (\text{mark}) \subseteq \text{atms-of-mm } (\text{init-clss } S'))$
 $\wedge \text{atms-of-mm } (\text{learned-clss } S') \subseteq \text{atms-of-mm } (\text{init-clss } S')$
 $\wedge \text{atm-of } ' (\text{lits-of-l } (\text{trail } S')) \subseteq \text{atms-of-mm } (\text{init-clss } S')$

lemma *no-strange-atm-decomp*:

assumes *no-strange-atm* S
shows *conflicting* $S = \text{Some } T \implies \text{atms-of } T \subseteq \text{atms-of-mm } (\text{init-clss } S)$
and $(\forall L \text{ mark. Propagated } L \text{ mark} \in \text{set } (\text{trail } S) \longrightarrow \text{atms-of } (\text{mark}) \subseteq \text{atms-of-mm } (\text{init-clss } S))$
and $\text{atms-of-mm } (\text{learned-clss } S) \subseteq \text{atms-of-mm } (\text{init-clss } S)$
and $\text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-mm } (\text{init-clss } S)$
using *assms* **unfolding** *no-strange-atm-def* **by** *blast*

lemma *no-strange-atm-S0* [*simp*]: *no-strange-atm* (*init-state* N)
unfolding *no-strange-atm-def* **by** *auto*

lemma *in-atms-of-implies-atm-of-on-atms-of-ms*:

$C + \{\#L\# \} \in \# A \implies x \in \text{atms-of } C \implies x \in \text{atms-of-mm } A$
using *multi-member-split* **by** *fastforce*

lemma *propagate-no-strange-atm-inv*:

assumes
propagate S T **and**
alien: *no-strange-atm* S
shows *no-strange-atm* T
using *assms*(1)

proof (*induction*)

case (*propagate-rule* C L T) **note** *confl* = *this*(1) **and** $C = \text{this}(2)$ **and** $C-L = \text{this}(3)$ **and**
 $tr = \text{this}(4)$ **and** *undef* = *this*(5) **and** $T = \text{this}(6)$
have *atm-CL*: $\text{atms-of } (\text{mset-clss } C) \subseteq \text{atms-of-mm } (\text{init-clss } S)$
using C *alien* **unfolding** *no-strange-atm-def*
by (*auto simp: raw-clauses-def atms-of-ms-def dest!: in-clss-mset-clss*)

show ?*case*

unfolding *no-strange-atm-def*

proof (*intro conjI allI impI, goal-cases*)

case 1

then show ?*case*

using *confl* T *undef* **by** *auto*

next

case (2 L' *mark'*)

then show ?*case*

using $C-L$ T *alien* *undef* *atm-CL*

unfolding *no-strange-atm-def* *raw-clauses-def* **apply** *auto* **by** *blast*

next

case (3)

show ?*case* **using** T *alien* *undef* **unfolding** *no-strange-atm-def* **by** *auto*

next

case (4)

show ?*case*

using T *alien* *undef* $C-L$ *atm-CL* **unfolding** *no-strange-atm-def* **by** (*auto simp: atms-of-def*)

qed

qed

lemma *in-atms-of-remove1-mset-in-atms-of*:

$x \in \text{atms-of } (\text{remove1-mset } L \ C) \implies x \in \text{atms-of } C$

using *in-diffD* **unfolding** *atms-of-def* **by** *fastforce*

lemma *cdcl_W-no-strange-atm-explicit*:

assumes

cdcl_W S S' **and**

lev: cdcl_W-M-level-inv S **and**

conf: $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{atms-of } T \subseteq \text{atms-of-mm } (\text{init-clss } S)$ **and**

marked: $\forall L \text{ mark. Propagated } L \text{ mark} \in \text{set } (\text{trail } S)$

$\longrightarrow \text{atms-of mark} \subseteq \text{atms-of-mm } (\text{init-clss } S)$ **and**

learned: $\text{atms-of-mm } (\text{learned-clss } S) \subseteq \text{atms-of-mm } (\text{init-clss } S)$ **and**

trail: $\text{atm-of } '(\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-mm } (\text{init-clss } S)$

shows

$(\forall T. \text{conflicting } S' = \text{Some } T \longrightarrow \text{atms-of } T \subseteq \text{atms-of-mm } (\text{init-clss } S')) \wedge$

$(\forall L \text{ mark. Propagated } L \text{ mark} \in \text{set } (\text{trail } S'))$

$\longrightarrow \text{atms-of } (\text{mark}) \subseteq \text{atms-of-mm } (\text{init-clss } S') \wedge$

$\text{atms-of-mm } (\text{learned-clss } S') \subseteq \text{atms-of-mm } (\text{init-clss } S') \wedge$

$\text{atm-of } '(\text{lits-of-l } (\text{trail } S')) \subseteq \text{atms-of-mm } (\text{init-clss } S')$

(is $?C \ S' \wedge ?M \ S' \wedge ?U \ S' \wedge ?V \ S')$

using *assms(1,2)*

proof (*induct rule: cdcl_W-all-induct-lev2*)

case (*propagate C L T*) **note** *confl = this(1)* **and** *C-L = this(2)* **and** *tr = this(3)* **and** *undef = this(4)*

and *T = this(5)*

show *?case*

using *propagate-rule[OF propagate.hyps(1-3) - propagate.hyps(5,6), simplified]*

propagate.hyps(4) propagate-no-strange-atm-inv[of S T]

conf marked learned trail **unfolding** *no-strange-atm-def* **by** *presburger*

next

case (*decide L*)

then show *?case* **using** *learned marked conf trail* **unfolding** *raw-clauses-def* **by** *auto*

next

case (*skip L C M D*)

then show *?case* **using** *learned marked conf trail* **by** *auto*

next

case (*conflict D T*) **note** *D-S = this(2)* **and** *T = this(4)*

have *D: atm-of 'set-mset (mset-cls D) $\subseteq \bigcup (\text{atms-of } '(\text{set-mset } (\text{clauses } S)))$*

using *D-S* **by** (*auto simp add: atms-of-def atms-of-ms-def*)

moreover {

fix *xa :: 'v literal*

assume *a1: atm-of 'set-mset (mset-cls D) $\subseteq (\bigcup_{x \in \text{set-mset } (\text{init-clss } S). \text{atms-of } x} \cup (\bigcup_{x \in \text{set-mset } (\text{learned-clss } S). \text{atms-of } x})$*

assume *a2:*

$(\bigcup_{x \in \text{set-mset } (\text{learned-clss } S). \text{atms-of } x} \subseteq (\bigcup_{x \in \text{set-mset } (\text{init-clss } S). \text{atms-of } x})$

assume *xa $\in \#$ mset-cls D*

then have *atm-of xa $\in \text{UNION } (\text{set-mset } (\text{init-clss } S)) \text{ atms-of}$*

using *a2 a1* **by** (*metis (no-types) Un-iff atm-of-lit-in-atms-of atms-of-def subset-Un-eq*)

then have $\exists m \in \text{set-mset } (\text{init-clss } S). \text{atm-of } xa \in \text{atms-of } m$

by *blast*

} note *H = this*

ultimately show *?case* **using** *conflict.premis T learned marked conf trail*

unfolding *atms-of-def atms-of-ms-def raw-clauses-def*

by (*auto simp add: H*)

```

next
  case (restart T)
  then show ?case using learned marked conf trail by auto
next
case (forget C T) note confl = this(1) and C = this(4) and C-le = this(5) and
  T = this(6)
have H:  $\bigwedge L$  mark. Propagated L mark  $\in$  set (trail S)  $\implies$  atms-of mark  $\subseteq$  atms-of-mm (init-clss S)
  using marked by simp
show ?case unfolding raw-clauses-def apply (intro conjI)
  using conf confl T trail C unfolding raw-clauses-def apply (auto dest!: H)[]
  using T trail C C-le apply (auto dest!: H)[]
  using T learned C-le atms-of-ms-remove-subset[of set-mset (learned-clss S)] apply auto[]
  using T trail C-le apply (auto simp: raw-clauses-def lits-of-def)[]
done
next
case (backtrack K i M1 M2 L D T) note confl = this(1) and LD = this(2) and decomp = this(3)
and
  undef = this(7)
  and T = this(8)
have ?C T
  using conf T decomp undef lev by (auto simp: cdclW-M-level-inv-decomp)
moreover have set M1  $\subseteq$  set (trail S)
  using decomp by auto
then have M: ?M T
  using marked conf undef confl T decomp lev
  by (auto simp: image-subset-iff raw-clauses-def cdclW-M-level-inv-decomp)
moreover have ?U T
  using learned decomp conf confl T undef lev unfolding raw-clauses-def
  by (auto simp: cdclW-M-level-inv-decomp)
moreover have ?V T
  using M conf confl trail T undef decomp lev LD
  by (auto simp: cdclW-M-level-inv-decomp atms-of-def
    dest!: get-all-marked-decomposition-exists-prepend)
ultimately show ?case by blast
next
case (resolve L C M D T) note trail-S = this(1) and confl = this(4) and T = this(7)
let ?T = update-conflicting (Some ((remove-clit (-L) D) ! $\cup$  ccls-of-cls ((remove-lit L C))))
  (tl-trail S)
have ?C ?T
  using confl trail-S conf marked by (auto dest!: in-atms-of-remove1-mset-in-atms-of)
moreover have ?M ?T
  using confl trail-S conf marked by auto
moreover have ?U ?T
  using trail learned by auto
moreover have ?V ?T
  using confl trail-S trail by auto
ultimately show ?case using T by simp
qed

lemma cdclW-no-strange-atm-inv:
  assumes cdclW S S' and no-strange-atm S and cdclW-M-level-inv S
  shows no-strange-atm S'
  using cdclW-no-strange-atm-explicit[OF assms(1)] assms(2,3) unfolding no-strange-atm-def by fast

lemma rtrancpl-cdclW-no-strange-atm-inv:

```


assumes $cdcl_W^{**} S S'$ **and** $no\text{-}strange\text{-}atm S$ **and** $cdcl_W\text{-}M\text{-}level\text{-}inv S$
shows $no\text{-}strange\text{-}atm S'$
using *assms* **by** *induction* (*auto intro: cdcl_W-no-strange-atm-inv rtrancplp-cdcl_W-consistent-inv*)

18.3.7 No duplicates all around

This invariant shows that there is no duplicate (no literal appearing twice in the formula). The last part could be proven using the previous invariant moreover.

definition $distinct\text{-}cdcl_W\text{-}state (S::'st)$
 $\longleftrightarrow ((\forall T. \text{conflicting } S = \text{Some } T \longrightarrow distinct\text{-}mset T)$
 $\wedge distinct\text{-}mset\text{-}mset (\text{learned-clss } S)$
 $\wedge distinct\text{-}mset\text{-}mset (\text{init-clss } S)$
 $\wedge (\forall L \text{ mark. } (\text{Propagated } L \text{ mark} \in \text{set } (\text{trail } S) \longrightarrow distinct\text{-}mset (\text{mark}))))$

lemma $distinct\text{-}cdcl_W\text{-}state\text{-}decomp$:
assumes $distinct\text{-}cdcl_W\text{-}state (S::'st)$
shows $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow distinct\text{-}mset T$
and $distinct\text{-}mset\text{-}mset (\text{learned-clss } S)$
and $distinct\text{-}mset\text{-}mset (\text{init-clss } S)$
and $\forall L \text{ mark. } (\text{Propagated } L \text{ mark} \in \text{set } (\text{trail } S) \longrightarrow distinct\text{-}mset (\text{mark}))$
using *assms* **unfolding** $distinct\text{-}cdcl_W\text{-}state\text{-}def$ **by** *blast+*

lemma $distinct\text{-}cdcl_W\text{-}state\text{-}decomp\text{-}2$:
assumes $distinct\text{-}cdcl_W\text{-}state (S::'st)$
shows $\text{conflicting } S = \text{Some } T \implies distinct\text{-}mset T$
using *assms* **unfolding** $distinct\text{-}cdcl_W\text{-}state\text{-}def$ **by** *auto*

lemma $distinct\text{-}cdcl_W\text{-}state\text{-}S0\text{-}cdcl_W[simp]$:
 $distinct\text{-}mset\text{-}mset (\text{mset-clss } N) \implies distinct\text{-}cdcl_W\text{-}state (\text{init-state } N)$
unfolding $distinct\text{-}cdcl_W\text{-}state\text{-}def$ **by** *auto*

lemma $distinct\text{-}cdcl_W\text{-}state\text{-}inv$:
assumes
 $cdcl_W S S'$ **and**
 $lev\text{-}inv: cdcl_W\text{-}M\text{-}level\text{-}inv S$ **and**
 $distinct\text{-}cdcl_W\text{-}state S$
shows $distinct\text{-}cdcl_W\text{-}state S'$
using *assms*(1,2,2,3)
proof (*induct rule: cdcl_W-all-induct-lev2*)
case (*backtrack L D K i M1 M2*)
then show *?case*
using $lev\text{-}inv$ **unfolding** $distinct\text{-}cdcl_W\text{-}state\text{-}def$
by (*auto dest: get-all-marked-decomposition-incl simp: cdcl_W-M-level-inv-decomp*)
next
case *restart*
then show *?case*
unfolding $distinct\text{-}cdcl_W\text{-}state\text{-}def$ $distinct\text{-}mset\text{-}set\text{-}def$ *raw-clauses-def*
using $learned\text{-}clss\text{-}restart\text{-}state[of S]$ **by** *auto*
next
case *resolve*
then show *?case*
by (*auto simp add: distinct-cdcl_W-state-def distinct-mset-set-def raw-clauses-def*
 $distinct\text{-}mset\text{-}single\text{-}add$
 $intro!: distinct\text{-}mset\text{-}union\text{-}mset$)
qed (*auto simp: distinct-cdcl_W-state-def distinct-mset-set-def raw-clauses-def*)

dest!: *in-clss-mset-clss in-diffD*)

lemma *rtanclp-distinct-cdcl_W-state-inv*:

assumes

*cdcl_W** S S' and*

cdcl_W-M-level-inv S and

distinct-cdcl_W-state S

shows *distinct-cdcl_W-state S'*

using *assms apply* (*induct rule: rtanclp-induct*)

using *distinct-cdcl_W-state-inv rtanclp-cdcl_W-consistent-inv by blast+*

18.3.8 Conflicts and co

This invariant shows that each mark contains a contradiction only related to the previously defined variable.

abbreviation *every-mark-is-a-conflict* :: *'st* \Rightarrow *bool* **where**

every-mark-is-a-conflict S \equiv

$\forall L \text{ mark } a \ b. \ a \ @ \ \text{Propagated } L \text{ mark} \ \# \ b = (\text{trail } S)$

$\longrightarrow (b \models_{\text{as}} \text{CNot } (\text{mark} - \{\#L\}) \wedge L \in \# \text{ mark})$

definition *cdcl_W-conflicting S* \equiv

$(\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{\text{as}} \text{CNot } T)$

$\wedge \text{every-mark-is-a-conflict } S$

lemma *backtrack-atms-of-D-in-M1*:

fixes *M1* :: (*'v*, *nat*, *'v clause*) *marked-lits*

assumes

inv: cdcl_W-M-level-inv S and

undef: undefined-lit M1 L and

i: get-maximum-level (trail S) (mset-ccls (remove-clit L D)) \equiv i and

decomp: (Marked K (Suc i) $\#$ M1, M2)

\in set (get-all-marked-decomposition (trail S)) and

S-lvl: backtrack-lvl S = get-maximum-level (trail S) (mset-ccls D) and

S-conf: raw-conflicting S = Some D and

undef: undefined-lit M1 L and

T: T \sim cons-trail (Propagated L (cls-of-ccls D))

(reduce-trail-to M1

(add-learned-cls (cls-of-ccls D)

(update-backtrack-lvl i

(update-conflicting None S)))) and

conf: $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{\text{as}} \text{CNot } T$

shows *atms-of (mset-ccls (remove-clit L D)) \subseteq atm-of ' lits-of-l (tl (trail T))*

proof (*rule ccontr*)

let *?k* = *get-maximum-level (trail S) (mset-ccls D)*

let *?D* = *mset-ccls D*

let *?D'* = *mset-ccls (remove-clit L D)*

have *trail S $\models_{\text{as}} \text{CNot } ?D$ using conf S-conf by auto*

then have *vars-of-D: atms-of ?D \subseteq atm-of ' lits-of-l (trail S) unfolding atms-of-def*

by (meson image-subsetI true-annots-CNot-all-atms-defined)

obtain *M0* **where** *M: trail S = M0 @ M2 @ Marked K (Suc i) $\#$ M1*

using decomp by auto

have *max: ?k = length (get-all-levels-of-marked (M0 @ M2 @ Marked K (Suc i) $\#$ M1))*

using inv unfolding cdcl_W-M-level-inv-def S-lvl M by simp

```

assume  $a: \neg \text{?thesis}$ 
then obtain  $L'$  where
   $L': L' \in \text{atms-of } ?D'$  and
   $L'\text{-notin-}M1: L' \notin \text{atm-of } \text{' lits-of-}l\ M1$ 
  using  $T \text{ undef decomp inv}$  by ( $\text{auto simp: cdcl}_W\text{-}M\text{-level-inv-decomp}$ )
then have  $L'\text{-in}: L' \in \text{atm-of } \text{' lits-of-}l\ (M0 @ M2 @ \text{Marked } K\ (i + 1) \# [])$ 
  using  $\text{vars-of-}D$  unfolding  $M$  by ( $\text{auto dest: in-atms-of-remove1-mset-in-atms-of}$ )
then obtain  $L''$  where
   $L'' \in \# ?D'$  and
   $L'': L' = \text{atm-of } L''$ 
  using  $L' L'\text{-notin-}M1$  unfolding  $\text{atms-of-def}$  by  $\text{auto}$ 
have  $\text{lev-}L''$ :
   $\text{get-level } (\text{trail } S)\ L'' = \text{get-rev-level } (\text{Marked } K\ (\text{Suc } i) \# \text{rev } M2 @ \text{rev } M0)\ (\text{Suc } i)\ L''$ 
  using  $L'\text{-notin-}M1\ L''\ M$  by ( $\text{auto simp del: get-rev-level.simps}$ )
have  $\text{get-all-levels-of-marked } (\text{trail } S) = \text{rev } [1..<1+?k]$ 
  using  $\text{inv } S\text{-lvl}$  unfolding  $\text{cdcl}_W\text{-}M\text{-level-inv-def}$  by  $\text{auto}$ 
then have  $\text{get-all-levels-of-marked } (M0 @ M2) = \text{rev } [\text{Suc } (\text{Suc } i)..<\text{Suc } ?k]$ 
  unfolding  $M$  by ( $\text{auto simp: rev-swap[symmetric] dest!: append-cons-eq-upt-length-i-end}$ )

then have  $M: \text{get-all-levels-of-marked } M0 @ \text{get-all-levels-of-marked } M2$ 
   $= \text{rev } [\text{Suc } (\text{Suc } i)..<\text{Suc } (\text{length } (\text{get-all-levels-of-marked } (M0 @ M2 @ \text{Marked } K\ (\text{Suc } i) \# M1)))]$ 
  unfolding  $\text{max}$  unfolding  $M$  by  $\text{simp}$ 

have  $\text{get-rev-level } (\text{Marked } K\ (\text{Suc } i) \# \text{rev } (M0 @ M2))\ (\text{Suc } i)\ L''$ 
   $\geq \text{Min } (\text{set } ((\text{Suc } i) \# \text{get-all-levels-of-marked } (\text{Marked } K\ (\text{Suc } i) \# \text{rev } (M0 @ M2))))$ 
  using  $\text{get-rev-level-ge-min-get-all-levels-of-marked[of } L''$ 
   $\text{rev } (M0 @ M2 @ [\text{Marked } K\ (\text{Suc } i)])\ \text{Suc } i]\ L'\text{-in}$ 
  unfolding  $L''$  by ( $\text{fastforce simp add: lits-of-def}$ )
also have  $\text{Min } (\text{set } ((\text{Suc } i) \# \text{get-all-levels-of-marked } (\text{Marked } K\ (\text{Suc } i) \# \text{rev } (M0 @ M2))))$ 
   $= \text{Min } (\text{set } ((\text{Suc } i) \# \text{get-all-levels-of-marked } (\text{rev } (M0 @ M2))))$  by  $\text{auto}$ 
also have  $\dots = \text{Min } (\text{set } ((\text{Suc } i) \# \text{get-all-levels-of-marked } M0 @ \text{get-all-levels-of-marked } M2))$ 
  by ( $\text{simp add: Un-commute}$ )
also have  $\dots = \text{Min } (\text{set } ((\text{Suc } i) \# [\text{Suc } (\text{Suc } i)..<2 + \text{length } (\text{get-all-levels-of-marked } M0)$ 
   $+ (\text{length } (\text{get-all-levels-of-marked } M2) + \text{length } (\text{get-all-levels-of-marked } M1))]))$ 
  unfolding  $M$  by ( $\text{auto simp add: Un-commute}$ )
also have  $\dots = \text{Suc } i$  by ( $\text{auto intro: Min-eqI}$ )
finally have  $\text{get-rev-level } (\text{Marked } K\ (\text{Suc } i) \# \text{rev } (M0 @ M2))\ (\text{Suc } i)\ L'' \geq \text{Suc } i$  .
then have  $\text{get-level } (\text{trail } S)\ L'' \geq i + 1$ 
  using  $\text{lev-}L''$  by  $\text{simp}$ 
then have  $\text{get-maximum-level } (\text{trail } S)\ ?D' \geq i + 1$ 
  using  $\text{get-maximum-level-ge-get-level[OF } \langle L'' \in \# ?D' \rangle, \text{ of trail } S]$  by  $\text{auto}$ 
then show  $\text{False}$  using  $i$  by  $\text{auto}$ 
qed

```

lemma *distinct-atms-of-incl-not-in-other:*

```

assumes
   $a1: \text{no-dup } (M @ M')$  and
   $a2: \text{atms-of } D \subseteq \text{atm-of } \text{' lits-of-}l\ M'$  and
   $a3: x \in \text{atms-of } D$ 
shows  $x \notin \text{atm-of } \text{' lits-of-}l\ M$ 
proof –
have  $\text{ff1}: \bigwedge l\ ms. \text{undefined-lit } ms\ l \vee \text{atm-of } l$ 
   $\in \text{set } (\text{map } (\lambda m. \text{atm-of } (\text{lit-of } (m::('a, 'b, 'c) \text{marked-lit})))\ ms)$ 
  by ( $\text{simp add: defined-lit-map}$ )
have  $\text{ff2}: \bigwedge a. a \notin \text{atms-of } D \vee a \in \text{atm-of } \text{' lits-of-}l\ M'$ 

```

```

    using a2 by (meson subsetCE)
  have ff3:  $\bigwedge a. a \notin \text{set } (\text{map } (\lambda m. \text{atm-of } (\text{lit-of } m)) M')$ 
     $\vee a \notin \text{set } (\text{map } (\lambda m. \text{atm-of } (\text{lit-of } m)) M)$ 
    using a1 by (metis (lifting) IntI distinct-append empty-iff map-append)
  have  $\forall L a f. \exists l. ((a::'a) \notin f \wedge L \vee (l::'a \text{ literal}) \in L) \wedge (a \notin f \wedge L \vee f l = a)$ 
    by blast
  then show  $x \notin \text{atm-of } ' \text{ lits-of-l } M$ 
    using ff3 ff2 ff1 a3 by (metis (no-types) Marked-Propagated-in-iff-in-lits-of-l)
qed

```

lemma *true-annot-CNot-remove1-mset-remove1-mset:*
 $I \models_{\text{as}} \text{CNot } C \implies I \models_{\text{as}} \text{CNot } (\text{remove1-mset } L C)$
 by (auto simp: true-annots-true-cls-def-iff-negation-in-model dest: in-diffD)

lemma *cdcl_W-propagate-is-conclusion:*

```

  assumes
    cdclW S S' and
    inv: cdclW-M-level-inv S and
    decomp: all-decomposition-implies-m (init-cls S) (get-all-marked-decomposition (trail S)) and
    learned: cdclW-learned-clause S and
    confl:  $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{\text{as}} \text{CNot } T$  and
    alien: no-strange-atm S
  shows all-decomposition-implies-m (init-cls S') (get-all-marked-decomposition (trail S'))
  using assms(1,2)
proof (induct rule: cdclW-all-induct-lev2)
  case restart
  then show ?case by auto
next
  case forget
  then show ?case using decomp by auto
next
  case conflict
  then show ?case using decomp by auto
next
  case (resolve L C M D)
  note tr = this(1) and T = this(7)
  let ?decomp = get-all-marked-decomposition M
  have M: set ?decomp = insert (hd ?decomp) (set (tl ?decomp))
    by (cases ?decomp) auto
  show ?case
    using decomp tr T unfolding all-decomposition-implies-def
    by (cases hd (get-all-marked-decomposition M))
      (auto simp: M)
next
  case (skip L C' M D)
  note tr = this(1) and T = this(5)
  have M: set (get-all-marked-decomposition M)
    = insert (hd (get-all-marked-decomposition M)) (set (tl (get-all-marked-decomposition M)))
    by (cases get-all-marked-decomposition M) auto
  show ?case
    using decomp tr T unfolding all-decomposition-implies-def
    by (cases hd (get-all-marked-decomposition M))
      (auto simp add: M)
next
  case decide
  note S = this(1) and undef = this(2) and T = this(4)
  show ?case using decomp T undef unfolding S all-decomposition-implies-def by auto

```

next

case (propagate $C\ L\ T$) **note** $propa = \text{this}(2)$ **and** $L = \text{this}(3)$ **and** $undef = \text{this}(5)$ **and** $T = \text{this}(6)$
obtain $a\ y$ **where** $ay: \text{hd}(\text{get-all-marked-decomposition}(\text{trail } S)) = (a, y)$
by (cases $\text{hd}(\text{get-all-marked-decomposition}(\text{trail } S))$)
then have $M: \text{trail } S = y @ a$ **using** $\text{get-all-marked-decomposition-decomp}$ **by** blast
have $M': \text{set}(\text{get-all-marked-decomposition}(\text{trail } S))$
 $= \text{insert}(a, y)(\text{set}(\text{tl}(\text{get-all-marked-decomposition}(\text{trail } S))))$
using ay **by** (cases $\text{get-all-marked-decomposition}(\text{trail } S)$) auto
have $\text{unmark-l } a \cup \text{set-mset}(\text{init-clss } S) \models_{ps} \text{unmark-l } y$
using $\text{decomp } ay$ **unfolding** $\text{all-decomposition-implies-def}$
by (cases $\text{get-all-marked-decomposition}(\text{trail } S)$) fastforce+
then have $a\text{-Un-N-M}: \text{unmark-l } a \cup \text{set-mset}(\text{init-clss } S)$
 $\models_{ps} \text{unmark-l}(\text{trail } S)$
unfolding M **by** ($\text{auto simp add: all-in-true-clss-clss image-Un}$)

have $\text{unmark-l } a \cup \text{set-mset}(\text{init-clss } S) \models_p \{\#L\# \}$ (**is** $?I \models_p -$)

proof ($\text{rule true-clss-clss-plus-CNot}$)

show $?I \models_p \text{remove1-mset } L(\text{mset-clss } C) + \{\#L\# \}$

apply ($\text{rule true-clss-clss-in-imp-true-clss-clss[of -}$
 $\text{set-mset}(\text{init-clss } S) \cup \text{set-mset}(\text{learned-clss } S)]$)

using $\text{learned propa } L$ **by** ($\text{auto simp: raw-clauses-def cdcl}_W\text{-learned-clause-def}$
 $\text{true-annot-CNot-remove1-mset-remove1-mset}$)

next

have $(\lambda m. \{\#lit\text{-of } m\# \}) ' \text{set}(\text{trail } S) \models_{ps} \text{CNot}(\text{remove1-mset } L(\text{mset-clss } C))$

using $\langle (\text{trail } S) \models_{as} \text{CNot}(\text{remove1-mset } L(\text{mset-clss } C)) \rangle \text{true-annots-true-clss-clss}$

by blast

then show $?I \models_{ps} \text{CNot}(\text{remove1-mset } L(\text{mset-clss } C))$

using $a\text{-Un-N-M true-clss-clss-left-right true-clss-clss-union-l-r}$ **by** blast

qed

moreover have $\bigwedge aa\ b.$

$\forall (Ls, \text{seen}) \in \text{set}(\text{get-all-marked-decomposition}(y @ a)).$

$\text{unmark-l } Ls \cup \text{set-mset}(\text{init-clss } S) \models_{ps} \text{unmark-l } \text{seen}$

$\implies (aa, b) \in \text{set}(\text{tl}(\text{get-all-marked-decomposition}(y @ a)))$

$\implies \text{unmark-l } aa \cup \text{set-mset}(\text{init-clss } S) \models_{ps} \text{unmark-l } b$

by ($\text{metis (no-types, lifting) case-prod-conv get-all-marked-decomposition-never-empty-sym}$
 $\text{list.collapse list.set-intros}(2))$

ultimately show $?case$

using $\text{decomp } T\ undef$ **unfolding** ay $\text{all-decomposition-implies-def}$

using $M(\text{unmark-l } a \cup \text{set-mset}(\text{init-clss } S) \models_{ps} \text{unmark-l } y)$

ay **by** auto

next

case ($\text{backtrack } K\ i\ M1\ M2\ L\ D\ T$) **note** $\text{conf} = \text{this}(1)$ **and** $LD = \text{this}(2)$ **and** $\text{decomp}' = \text{this}(3)$

and

$\text{lev-L} = \text{this}(4)$ **and** $\text{undef} = \text{this}(7)$ **and** $T = \text{this}(8)$

let $?D = \text{mset-ccls } D$

let $?D' = \text{mset-ccls}(\text{remove-clit } L\ D)$

have $\forall l \in \text{set } M2. \neg \text{is-marked } l$

using $\text{get-all-marked-decomposition-snd-not-marked decomp}'$ **by** blast

obtain $M0$ **where** $M: \text{trail } S = M0 @ M2 @ \text{Marked } K(i + 1) \# M1$

using decomp' **by** auto

show $?case$ **unfolding** $\text{all-decomposition-implies-def}$

proof

fix x

assume $x \in \text{set}(\text{get-all-marked-decomposition}(\text{trail } T))$

```

then have  $x$ :  $x \in \text{set } (\text{get-all-marked-decomposition } (\text{Propagated } L \text{ ?}D \# M1))$ 
  using  $T \text{ decomp}' \text{ undef inv}$  by ( $\text{simp add: cdcl}_W\text{-}M\text{-level-inv-decomp}$ )
let  $?m = \text{get-all-marked-decomposition } (\text{Propagated } L \text{ ?}D \# M1)$ 
let  $?hd = hd \text{ ?}m$ 
let  $?tl = tl \text{ ?}m$ 
consider
  ( $hd$ )  $x = ?hd$ 
  | ( $tl$ )  $x \in \text{set } ?tl$ 
  using  $x$  by ( $\text{cases ?}m$ ) auto
then show  $\text{case } x \text{ of } (Ls, \text{seen}) \Rightarrow \text{unmark-l } Ls \cup \text{set-mset } (\text{init-clss } T)$ 
   $\models_{ps} \text{unmark-l seen}$ 
proof cases
  case  $tl$ 
  then have  $x \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$ 
  using  $tl\text{-get-all-marked-decomposition-skip-some}[of \text{ } x]$  by ( $\text{simp add: list.set-sel}(2) \text{ } M$ )
  then show  $?thesis$ 
  using  $\text{decomp learned decomp confl alien inv } T \text{ undef } M$ 
  unfolding  $\text{all-decomposition-implies-def cdcl}_W\text{-}M\text{-level-inv-def}$ 
  by auto
next
  case  $hd$ 
  obtain  $M1' \text{ } M1''$  where  $M1: hd (\text{get-all-marked-decomposition } M1) = (M1', M1'')$ 
  by ( $\text{cases } hd (\text{get-all-marked-decomposition } M1)$ )
  then have  $x': x = (M1', \text{Propagated } L \text{ ?}D \# M1'')$ 
  using  $\langle x = ?hd \rangle$  by auto
  have  $(M1', M1'') \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$ 
  using  $M1[\text{symmetric}] \text{ } hd\text{-get-all-marked-decomposition-skip-some}[OF \text{ } M1[\text{symmetric}],$ 
     $of \text{ } M0 @ M2 - i + 1]$  unfolding  $M$  by fastforce
  then have  $1: \text{unmark-l } M1' \cup \text{set-mset } (\text{init-clss } S) \models_{ps} \text{unmark-l } M1''$ 
  using  $\text{decomp unfolding all-decomposition-implies-def}$  by auto

moreover
  have  $\text{vars-of-}D: \text{atms-of } ?D' \subseteq \text{atm-of ' lits-of-l } M1$ 
  using  $\text{backtrack-atms-of-}D\text{-in-}M1[of \text{ } S \text{ } M1 \text{ } L \text{ } D \text{ } i \text{ } K \text{ } M2 \text{ } T]$   $\text{backtrack.hyps inv conf confl}$ 
  by ( $\text{auto simp: cdcl}_W\text{-}M\text{-level-inv-decomp}$ )
  have  $\text{no-dup } (\text{trail } S)$  using  $\text{inv}$  by ( $\text{auto simp: cdcl}_W\text{-}M\text{-level-inv-decomp}$ )
  then have  $\text{vars-in-}M1:$ 
     $\forall x \in \text{atms-of } ?D'. x \notin \text{atm-of ' lits-of-l } (M0 @ M2 @ \text{Marked } K (i + 1) \# [])$ 
  using  $\text{vars-of-}D \text{ distinct-atms-of-incl-not-in-other}[of$ 
     $M0 @ M2 @ \text{Marked } K (i + 1) \# [] \text{ } M1]$  unfolding  $M$  by auto
  have  $\text{trail } S \models_{as} CNot (\text{remove1-mset } L (\text{mset-ccls } D))$ 
  using  $\text{conf confl LD unfolding } M \text{ true-annots-true-clss-def-iff-negation-in-model}$ 
  by ( $\text{auto dest!: Multiset.in-diffD}$ )
  then have  $M1 \models_{as} CNot ?D'$ 
  using  $\text{vars-in-}M1 \text{ true-annots-remove-if-not-in-vars}[of \text{ } M0 @ M2 @ \text{Marked } K (i + 1) \# []$ 
     $M1 \text{ } CNot ?D']$   $\text{conf confl}$  unfolding  $M \text{ lits-of-def}$  by simp
  have  $M1 = M1'' @ M1'$  by ( $\text{simp add: } M1 \text{ get-all-marked-decomposition-decomp}$ )
  have  $TT: \text{unmark-l } M1' \cup \text{set-mset } (\text{init-clss } S) \models_{ps} CNot ?D'$ 
  using  $\text{true-annots-true-clss-clss}[OF \text{ } \langle M1 \models_{as} CNot ?D' \rangle \text{ true-clss-clss-left-right}[OF \text{ } 1]$ 
  unfolding  $\langle M1 = M1'' @ M1' \rangle$  by ( $\text{auto simp add: inf-sup-aci}(5, 7)$ )
  have  $\text{init-clss } S \models_{pm} ?D' + \{\#L\# \}$ 
  using  $\text{conf learned confl LD unfolding cdcl}_W\text{-learned-clause-def}$  by auto
  then have  $T': \text{unmark-l } M1' \cup \text{set-mset } (\text{init-clss } S) \models_p ?D' + \{\#L\# \}$  by auto
  have  $\text{atms-of } (?D' + \{\#L\# \}) \subseteq \text{atms-of-mm } (\text{clauses } S)$ 
  using  $\text{alien conf LD unfolding no-strange-atm-def raw-clauses-def}$  by auto

```

```

    then have unmark-l  $M1' \cup \text{set-mset}(\text{init-clss } S) \models_p \{\#L\# \}$ 
      using true-clss-clss-plus-CNot[OF T' TT] by auto

    ultimately show ?thesis
      using T' T decomp' undef inv unfolding x' by (simp add: cdclW-M-level-inv-decomp)
  qed
qed
qed

lemma cdclW-propagate-is-false:
  assumes
    cdclW S S' and
    lev: cdclW-M-level-inv S and
    learned: cdclW-learned-clause S and
    decomp: all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S)) and
    confl:  $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{as} \text{CNot } T$  and
    alien: no-strange-atm S and
    mark-confl: every-mark-is-a-conflict S
  shows every-mark-is-a-conflict S'
  using assms(1,2)
proof (induct rule: cdclW-all-induct-lev2)
  case (propagate C L T) note LC = this(3) and confl = this(4) and undef = this(5) and T = this(6)
  show ?case
  proof (intro allI impI)
    fix L' mark a b
    assume a @ Propagated L' mark # b = trail T
    then consider
      (hd) a = [] and L = L' and mark = mset-clss C and b = trail S
      | (tl) tl a @ Propagated L' mark # b = trail S
    using T undef by (cases a) fastforce+
    then show b  $\models_{as} \text{CNot} (\text{mark} - \{\#L'\#\}) \wedge L' \in \# \text{ mark}$ 
      using mark-confl confl LC by cases auto
  qed
next
  case (decide L) note undef[simp] = this(2) and T = this(4)
  have  $\bigwedge a \text{ La mark } b. a @ \text{Propagated La mark \# b} = \text{Marked L (backtrack-lvl S+1) \# trail S}$ 
     $\implies \text{tl } a @ \text{Propagated La mark \# b} = \text{trail S}$  by (case-tac a) auto
  then show ?case using mark-confl T unfolding decide.hyps(1) by fastforce
next
  case (skip L C' M D T) note tr = this(1) and T = this(5)
  show ?case
  proof (intro allI impI)
    fix L' mark a b
    assume a @ Propagated L' mark # b = trail T
    then have a @ Propagated L' mark # b = M using tr T by simp
    then have (Propagated L C' # a) @ Propagated L' mark # b = Propagated L C' # M by auto
    moreover have  $\forall \text{La mark } a b. a @ \text{Propagated La mark \# b} = \text{Propagated L C' \# M}$ 
       $\longrightarrow b \models_{as} \text{CNot} (\text{mark} - \{\#La\#\}) \wedge La \in \# \text{ mark}$ 
      using mark-confl unfolding skip.hyps(1) by simp
    ultimately show b  $\models_{as} \text{CNot} (\text{mark} - \{\#L'\#\}) \wedge L' \in \# \text{ mark}$  by blast
  qed
next
  case (conflict D)
  then show ?case using mark-confl by simp
next

```

```

case (resolve L C M D T) note tr-S = this(1) and T = this(7)
show ?case unfolding resolve.hyps(1)
proof (intro allI impI)
  fix L' mark a b
  assume a @ Propagated L' mark # b = trail T
  then have (Propagated L (mset-cls (L !++ C)) # a) @ Propagated L' mark # b
    = Propagated L (mset-cls (L !++ C)) # M
  using T tr-S by auto
  then show b  $\models_{as}$  CNot (mark - {#L'#})  $\wedge$  L'  $\in$  # mark
    using mark-confl unfolding tr-S by (metis Cons-eq-appendI list.sel(3))
qed
next
  case restart
  then show ?case by auto
next
  case forget
  then show ?case using mark-confl by auto
next
  case (backtrack K i M1 M2 L D T) note conf = this(1) and LD = this(2) and decomp = this(3)
and
  undef = this(7) and T = this(8)
  have  $\forall l \in \text{set } M2. \neg \text{is-marked } l$ 
    using get-all-marked-decomposition-snd-not-marked decomp by blast
  obtain M0 where M: trail S = M0 @ M2 @ Marked K (i + 1) # M1
  using decomp by auto
  have [simp]: trail (reduce-trail-to M1 (add-learned-cls (cls-of-ccls (insert-ccls L D))
    (update-backtrack-lvl i (update-conflicting None S)))) = M1
  using decomp lev by (auto simp: cdclW-M-level-inv-decomp)
  let ?D = mset-ccls D
  let ?D' = mset-ccls (remove-clit L D)
  show ?case
  proof (intro allI impI)
    fix La :: 'v literal and mark :: 'v literal multiset and
      a b :: ('v, nat, 'v literal multiset) marked-lit list
    assume a @ Propagated La mark # b = trail T
    then consider
      (hd-tr) a = [] and
      (Propagated La mark :: ('v, nat, 'v literal multiset) marked-lit)
        = Propagated L ?D and
      b = M1
    | (tl-tr) tl a @ Propagated La mark # b = M1
    using M T decomp undef lev by (cases a) (auto simp: cdclW-M-level-inv-def)
  then show b  $\models_{as}$  CNot (mark - {#La#})  $\wedge$  La  $\in$  # mark
    proof cases
      case hd-tr note A = this(1) and P = this(2) and b = this(3)
      have trail S  $\models_{as}$  CNot ?D using conf confl by auto
      then have vars-of-D: atms-of ?D  $\subseteq$  atm-of ' lits-of-l (trail S)
        unfolding atms-of-def
        by (meson image-subsetI true-annots-CNot-all-atms-defined)
      have vars-of-D: atms-of ?D'  $\subseteq$  atm-of ' lits-of-l M1
        using backtrack-atms-of-D-in-M1[of S M1 L D i K M2 T] T backtrack lev confl
        by (auto simp: cdclW-M-level-inv-decomp)
      have no-dup (trail S) using lev by (auto simp: cdclW-M-level-inv-decomp)
      then have  $\forall x \in \text{atms-of } ?D'. x \notin \text{atm-of ' lits-of-l } (M0 @ M2 @ \text{Marked } K (i + 1) \# [])$ 
        using vars-of-D distinct-atms-of-incl-not-in-other[of

```



```

    M0 @ M2 @ Marked K (i + 1) # [] M1] unfolding M by auto
then have M1  $\models_{as}$  CNot ?D'
    using true-annots-remove-if-notin-vars[of M0 @ M2 @ Marked K (i + 1) # []
    M1 CNot ?D']  $\langle$ trail S  $\models_{as}$  CNot ?D $\rangle$  unfolding M lits-of-def
    by (simp add: true-annot-CNot-remove1-mset-remove1-mset)
then show b  $\models_{as}$  CNot (mark - {#La#})  $\wedge$  La  $\in$  # mark
    using P LD b by auto
next
  case tl-tr
    then obtain c' where c' @ Propagated La mark # b = trail S
    unfolding M by auto
    then show b  $\models_{as}$  CNot (mark - {#La#})  $\wedge$  La  $\in$  # mark
    using mark-conf1 by auto
  qed
qed
qed

lemma cdclW-conflicting-is-false:
assumes
  cdclW S S' and
  M-lev: cdclW-M-level-inv S and
  confl-inv:  $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{as} \text{CNot } T$  and
  marked-conf1:  $\forall L \text{ mark } a \ b. \ a \ @ \ \text{Propagated } L \ \text{mark} \ \# \ b = (\text{trail } S)$ 
 $\longrightarrow (b \models_{as} \text{CNot } (\text{mark} - \{\#L\}) \wedge L \in \# \text{ mark})$  and
  dist: distinct-cdclW-state S
shows  $\forall T. \text{conflicting } S' = \text{Some } T \longrightarrow \text{trail } S' \models_{as} \text{CNot } T$ 
using assms(1,2)
proof (induct rule: cdclW-all-induct-lev2)
case (skip L C' M D T) note tr-S = this(1) and confl = this(2) and L-D = this(3) and T = this(5)
let ?D = mset-ccls D
have D: Propagated L C' # M  $\models_{as}$  CNot (mset-ccls D) using assms skip by auto
moreover
  have L  $\notin$  # ?D
  proof (rule ccontr)
    assume  $\neg$  ?thesis
    then have - L  $\in$  lits-of-l M
    using in-CNot-implies-uminus(2)[of L ?D Propagated L C' # M]
     $\langle$ Propagated L C' # M  $\models_{as}$  CNot ?D $\rangle$  by simp
    then show False
    by (metis (no-types, hide-lams) M-lev cdclW-M-level-inv-decomp(1) consistent-interp-def
    image-insert insert-iff list.set(2) lits-of-def marked-lit.sel(2) tr-S)
  qed
ultimately show ?case
  using tr-S confl L-D T unfolding cdclW-M-level-inv-def
  by (auto intro: true-annots-CNot-lit-of-notin-skip)
next
  case (resolve L C M D T) note tr = this(1) and LC = this(2) and confl = this(4) and LD =
  this(5)
  and T = this(7)
  let ?C = remove1-mset L (mset-ccls C)
  let ?D = remove1-mset (-L) (mset-ccls D)
  show ?case
  proof (intro allI impI)
    fix T'
    have tl (trail S)  $\models_{as}$  CNot ?C using tr marked-conf1 by fastforce

```

```

moreover
  have distinct-mset ( $?D + \{\# - L\# \}$ ) using confl dist LD
    unfolding distinct-cdclW-state-def by auto
  then have  $-L \notin \# ?D$  unfolding distinct-mset-def
    by (meson  $\langle \text{distinct-mset } (?D + \{\# - L\# \}) \rangle$  distinct-mset-single-add)
  have  $M \models_{as} CNot ?D$ 
  proof -
    have Propagated L ( $?C + \{\# L\# \}$ )  $\# M \models_{as} CNot ?D \cup CNot \{\# - L\# \}$ 
      using confl tr confl-inv LC by (metis CNot-plus LD insert-DiffM2 option.simps(9))
    then show ?thesis
      using M-lev  $\langle - L \notin \# ?D \rangle$  tr true-annots-lit-of-notin-skip
      unfolding cdclW-M-level-inv-def by force
    qed
  moreover assume conflicting T = Some T'
  ultimately
    show trail T  $\models_{as} CNot T'$ 
    using tr T by auto
  qed
qed (auto simp: M-lev cdclW-M-level-inv-decomp)

```

lemma *cdcl_W-conflicting-decomp*:

```

assumes cdclW-conflicting S
shows  $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{as} CNot T$ 
and  $\forall L \text{ mark } a \ b. a @ \text{Propagated } L \text{ mark } \# \ b = (\text{trail } S)$ 
   $\longrightarrow (b \models_{as} CNot (\text{mark} - \{\# L\# \}) \wedge L \in \# \text{ mark})$ 
using assms unfolding cdclW-conflicting-def by blast+

```

lemma *cdcl_W-conflicting-decomp2*:

```

assumes cdclW-conflicting S and conflicting S = Some T
shows trail S  $\models_{as} CNot T$ 
using assms unfolding cdclW-conflicting-def by blast+

```

lemma *cdcl_W-conflicting-S0-cdcl_W[simp]*:

```

cdclW-conflicting (init-state N)
unfolding cdclW-conflicting-def by auto

```

18.3.9 Putting all the invariants together

lemma *cdcl_W-all-inv*:

```

assumes
  cdclW: cdclW S S' and
  1: all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S)) and
  2: cdclW-learned-clause S and
  4: cdclW-M-level-inv S and
  5: no-strange-atm S and
  7: distinct-cdclW-state S and
  8: cdclW-conflicting S
shows
  all-decomposition-implies-m (init-clss S') (get-all-marked-decomposition (trail S')) and
  cdclW-learned-clause S' and
  cdclW-M-level-inv S' and
  no-strange-atm S' and
  distinct-cdclW-state S' and
  cdclW-conflicting S'
proof -
  show S1: all-decomposition-implies-m (init-clss S') (get-all-marked-decomposition (trail S'))

```

```

    using cdclW-propagate-is-conclusion[OF cdclW 4 1 2 - 5] 8 unfolding cdclW-conflicting-def
    by blast
show S2: cdclW-learned-clause S' using cdclW-learned-clss[OF cdclW 2 4] .
show S4: cdclW-M-level-inv S' using cdclW-consistent-inv[OF cdclW 4] .
show S5: no-strange-atm S' using cdclW-no-strange-atm-inv[OF cdclW 5 4] .
show S7: distinct-cdclW-state S' using distinct-cdclW-state-inv[OF cdclW 4 7] .
show S8: cdclW-conflicting S'
  using cdclW-conflicting-is-false[OF cdclW 4 - - 7] 8 cdclW-propagate-is-false[OF cdclW 4 2 1 -
    5]
  unfolding cdclW-conflicting-def by fast
qed

```

lemma *rtrancpl-cdcl_W-all-inv*:

```

assumes
  cdclW: rtrancpl cdclW S S' and
  1: all-decomposition-implies-m (init-clss S) (get-all-marked-decomposition (trail S)) and
  2: cdclW-learned-clause S and
  4: cdclW-M-level-inv S and
  5: no-strange-atm S and
  7: distinct-cdclW-state S and
  8: cdclW-conflicting S
shows
  all-decomposition-implies-m (init-clss S') (get-all-marked-decomposition (trail S')) and
  cdclW-learned-clause S' and
  cdclW-M-level-inv S' and
  no-strange-atm S' and
  distinct-cdclW-state S' and
  cdclW-conflicting S'
using assms
proof (induct rule: rtrancpl-induct)
  case base
  case 1 then show ?case by blast
  case 2 then show ?case by blast
  case 3 then show ?case by blast
  case 4 then show ?case by blast
  case 5 then show ?case by blast
  case 6 then show ?case by blast
next
  case (step S' S'') note H = this
  case 1 with H(3-7)[OF this(1-6)] show ?case using cdclW-all-inv[OF H(2)]
    H by presburger
  case 2 with H(3-7)[OF this(1-6)] show ?case using cdclW-all-inv[OF H(2)]
    H by presburger
  case 3 with H(3-7)[OF this(1-6)] show ?case using cdclW-all-inv[OF H(2)]
    H by presburger
  case 4 with H(3-7)[OF this(1-6)] show ?case using cdclW-all-inv[OF H(2)]
    H by presburger
  case 5 with H(3-7)[OF this(1-6)] show ?case using cdclW-all-inv[OF H(2)]
    H by presburger
  case 6 with H(3-7)[OF this(1-6)] show ?case using cdclW-all-inv[OF H(2)]
    H by presburger
qed

```

lemma *all-invariant-S0-cdcl_W*:

assumes *distinct-mset-mset* (mset-clss N)

```

all-decomposition-implies-m (init-clss (init-state N))
  (get-all-marked-decomposition (trail (init-state N))) and
cdclW-learned-clause (init-state N) and
 $\forall T. \text{conflicting (init-state } N) = \text{Some } T \longrightarrow (\text{trail (init-state } N)) \models_{as} CNot\ T$  and
no-strange-atm (init-state N) and
consistent-interp (lits-of-l (trail (init-state N))) and
 $\forall L \text{ mark } a \ b. \ a \ @ \ \text{Propagated } L \text{ mark } \# \ b = \text{trail (init-state } N) \longrightarrow$ 
 $(b \models_{as} CNot \ (\text{mark} - \{\#L\}) \wedge L \in \# \ \text{mark})$  and
distinct-cdclW-state (init-state N)
using assms by auto

```

assumes
marked: $\forall x \in \text{set } M. \neg \text{is-marked } x$ **and**
DN: $D \in \# \text{ clauses } S$ **and**
D: $M \models_{as} CNot\ D$ **and**
inv: *all-decomposition-implies-m* N (*get-all-marked-decomposition* M) **and**
state: *state* $S = (M, N, U, k, C)$ **and**
learned-cl: *cdcl_W-learned-clause* S **and**
atm-incl: *no-strange-atm* S
shows *unsatisfiable* (*set-mset* N)

then have $-K \in \text{ lits-of-}l\ M$

```

    using D unfolding true-annots-def Ball-def CNot-def true-annot-def true-clss-def true-lit-def
    Bex-def by force
    then have  $-K \in I$  using IM true-clss-singleton-lit-of-implies-incl lits-of-def by fastforce }
    then have  $\neg I \models D$  using cons unfolding true-clss-def true-lit-def consistent-interp-def by auto
    then show False using I-D by blast
qed

```

We have actually a much stronger theorem, namely *all-decomposition-implies ?N* (*get-all-marked-decomposition ?M*) $\implies ?N \cup \{unmark\ L \mid L. is_marked\ L \wedge L \in set\ ?M\} \models_{ps} unmark-l\ ?M$, that show that the only choices we made are marked in the formula

```

lemma
  assumes all-decomposition-implies-m N (get-all-marked-decomposition M)
  and  $\forall m \in set\ M. \neg is\_marked\ m$ 
  shows set-mset N  $\models_{ps} unmark-l\ M$ 
proof -
  have T:  $\{unmark\ L \mid L. is\_marked\ L \wedge L \in set\ M\} = \{\}$  using assms(2) by auto
  then show ?thesis
    using all-decomposition-implies-propagated-lits-are-implied[OF assms(1)] unfolding T by simp
qed

```

lemma *conflict-with-false-implies-unsat:*

```

  assumes
    cdclW: cdclW S S' and
    lev: cdclW-M-level-inv S and
    [simp]: conflicting S' = Some {#} and
    learned: cdclW-learned-clause S
  shows unsatisfiable (set-mset (init-clss S))
  using assms
proof -
  have cdclW-learned-clause S' using cdclW-learned-clss cdclW learned lev by auto
  then have init-clss S'  $\models_{pm} \{\#\}$  using assms(3) unfolding cdclW-learned-clause-def by auto
  then have init-clss S  $\models_{pm} \{\#\}$ 
    using cdclW-init-clss[OF assms(1) lev] by auto
  then show ?thesis unfolding satisfiable-def true-clss-clss-def by auto
qed

```

lemma *conflict-with-false-implies-terminated:*

```

  assumes cdclW S S'
  and conflicting S = Some {#}
  shows False
  using assms by (induct rule: cdclW-all-induct) auto

```

18.3.10 No tautology is learned

This is a simple consequence of all we have shown previously. It is not strictly necessary, but helps finding a better bound on the number of learned clauses.

lemma *learned-clss-are-not-tautologies:*

```

  assumes
    cdclW S S' and
    lev: cdclW-M-level-inv S and
    conflicting: cdclW-conflicting S and
    no-tauto:  $\forall s \in \# learned-clss\ S. \neg tautology\ s$ 
  shows  $\forall s \in \# learned-clss\ S'. \neg tautology\ s$ 

```

```

using assms
proof (induct rule: cdclW-all-induct-lev2)
  case (backtrack K i M1 M2 L D T) note confl = this(1)
  have consistent-interp (lits-of-l (trail S)) using lev by (auto simp: cdclW-M-level-inv-decomp)
  moreover
    have trail S ⊨as CNot (mset-ccls D)
      using conflicting confl unfolding cdclW-conflicting-def by auto
    then have lits-of-l (trail S) ⊨s CNot (mset-ccls D)
      using true-annots-true-clb by blast
    ultimately have  $\neg$ tautology (mset-ccls D) using consistent-CNot-not-tautology by blast
    then show ?case using backtrack no-tauto lev
      by (auto simp: cdclW-M-level-inv-decomp split: if-split-asm)
  next
    case restart
    then show ?case using learned-clss-restart-state state-eq-learned-clss no-tauto
      by (metis (no-types, lifting) set-mset-mono subsetCE)
qed (auto dest!: in-diffD)

```

definition *final-cdcl_W-state* (*S:: 'st*)
 \longleftrightarrow (*trail S ⊨_{asm} init-clss S*
 $\vee ((\forall L \in \text{set } (\text{trail } S). \neg \text{is-marked } L) \wedge$
 $(\exists C \in \# \text{ init-clss } S. \text{trail } S \models_{\text{as}} \text{CNot } C)))$

definition *termination-cdcl_W-state* (*S:: 'st*)
 \longleftrightarrow (*trail S ⊨_{asm} init-clss S*
 $\vee ((\forall L \in \text{atms-of-mm } (\text{init-clss } S). L \in \text{atm-of ' lits-of-l } (\text{trail } S))$
 $\wedge (\exists C \in \# \text{ init-clss } S. \text{trail } S \models_{\text{as}} \text{CNot } C)))$

18.4 CDCL Strong Completeness

fun *mapi* :: (*'a* \Rightarrow *nat* \Rightarrow *'b*) \Rightarrow *nat* \Rightarrow *'a list* \Rightarrow *'b list* **where**
mapi - - [] = [] |
mapi *f* *n* (*x* # *xs*) = *f* *x* *n* # *mapi* *f* (*n* - 1) *xs*

lemma *mark-not-in-set-mapi[simp]*: $L \notin \text{set } M \implies \text{Marked } L \ k \notin \text{set } (\text{mapi } \text{Marked } i \ M)$
by (*induct M arbitrary: i*) *auto*

lemma *propagated-not-in-set-mapi[simp]*: $L \notin \text{set } M \implies \text{Propagated } L \ k \notin \text{set } (\text{mapi } \text{Marked } i \ M)$
by (*induct M arbitrary: i*) *auto*

lemma *image-set-mapi*:
 $f \text{ ' set } (\text{mapi } g \ i \ M) = \text{set } (\text{mapi } (\lambda x \ i. f \ (g \ x \ i)) \ i \ M)$
by (*induction M arbitrary: i*) *auto*

lemma *mapi-map-convert*:
 $\forall x \ i \ j. f \ x \ i = f \ x \ j \implies \text{mapi } f \ i \ M = \text{map } (\lambda x. f \ x \ 0) \ M$
by (*induction M arbitrary: i*) *auto*

lemma *defined-lit-mapi*: $\text{defined-lit } (\text{mapi } \text{Marked } i \ M) \ L \longleftrightarrow \text{atm-of } L \in \text{atm-of ' set } M$
by (*induction M*) (*auto simp: defined-lit-map image-set-mapi mapi-map-convert*)

lemma *cdcl_W-can-do-step*:
assumes
consistent-interp (set M) **and**
distinct M **and**
 $\text{atm-of ' (set } M) \subseteq \text{atms-of-mm (mset-clss } N)$

```

shows  $\exists S. \text{rtrancp\_cdcl}_W (\text{init-state } N) S$ 
   $\wedge \text{state } S = (\text{mapi Marked } (\text{length } M) M, \text{mset-clss } N, \{\#\}, \text{length } M, \text{None})$ 
using assms
proof (induct  $M$ )
  case Nil
  then show ?case apply – by (rule exI[of - init-state  $N$ ]) auto
next
  case (Cons  $L M$ ) note  $IH = \text{this}(1)$ 
  have consistent-interp (set  $M$ ) and distinct  $M$  and atm-of ‘set  $M \subseteq \text{atms-of-mm } (\text{mset-clss } N)$ ’
    using Cons.prems(1–3) unfolding consistent-interp-def by auto
  then obtain  $S$  where
     $st: \text{cdcl}_W^{**} (\text{init-state } N) S$  and
     $S: \text{state } S = (\text{mapi Marked } (\text{length } M) M, \text{mset-clss } N, \{\#\}, \text{length } M, \text{None})$ 
    using  $IH$  by blast
  let  $?S_0 = \text{incr-lvl } (\text{cons-trail } (\text{Marked } L (\text{length } M + 1)) S)$ 
  have undefined-lit (mapi Marked (length  $M$ )  $M$ )  $L$ 
    using Cons.prems(1,2) unfolding defined-lit-def consistent-interp-def by fastforce
  moreover have init-clss  $S = \text{mset-clss } N$ 
    using  $S$  by blast
  moreover have atm-of  $L \in \text{atms-of-mm } (\text{mset-clss } N)$  using Cons.prems(3) by auto
  moreover have undef: undefined-lit (trail  $S$ )  $L$ 
    using  $S \langle \text{distinct } (L \# M) \rangle$  calculation(1) by (auto simp: defined-lit-mapi defined-lit-map)
  ultimately have  $\text{cdcl}_W S ?S_0$ 
    using  $\text{cdcl}_W.\text{other}[OF \text{cdcl}_W.\text{o.decide}[OF \text{decide-rule}[of  $S L ?S_0$ ]]]  $S$ 
    by (auto simp: state-eq-def simp del: state-simp)
  then have  $\text{cdcl}_W^{**} (\text{init-state } N) ?S_0$ 
    using  $st$  by auto
  then show ?case
    using  $S$  undef by (auto intro!: exI[of -  $?S_0$ ] del: simp del: )
qed$ 
```

lemma *cdcl_W-strong-completeness*:

```

assumes
   $MN: \text{set } M \models_{sm} \text{mset-clss } N$  and
   $cons: \text{consistent-interp } (\text{set } M)$  and
   $dist: \text{distinct } M$  and
   $atm: \text{atm-of } (\text{set } M) \subseteq \text{atms-of-mm } (\text{mset-clss } N)$ 
obtains  $S$  where
   $\text{state } S = (\text{mapi Marked } (\text{length } M) M, \text{mset-clss } N, \{\#\}, \text{length } M, \text{None})$  and
   $\text{rtrancp\_cdcl}_W (\text{init-state } N) S$  and
  final-cdclW-state  $S$ 
proof –
  obtain  $S$  where
     $st: \text{rtrancp\_cdcl}_W (\text{init-state } N) S$  and
     $S: \text{state } S = (\text{mapi Marked } (\text{length } M) M, \text{mset-clss } N, \{\#\}, \text{length } M, \text{None})$ 
    using  $\text{cdcl}_W.\text{can-do-step}[OF \text{cons dist atm}]$  by auto
  have lits-of-l (mapi Marked (length  $M$ )  $M$ ) = set  $M$ 
    by (induct  $M$ , auto)
  then have  $\text{mapi Marked } (\text{length } M) M \models_{asm} \text{mset-clss } N$  using  $MN$  true-annots-true-cls by metis
  then have final-cdclW-state  $S$ 
    using  $S$  unfolding final-cdclW-state-def by auto
  then show ?thesis using that  $st S$  by blast
qed

```

18.5 Higher level strategy

The rules described previously do not lead to a conclusive state. We have to add a strategy.

18.5.1 Definition

```

lemma tranclp-conflict:
  tranclp conflict S S'  $\implies$  conflict S S'
  apply (induct rule: tranclp.induct)
  apply simp
  by (metis conflictE conflicting-update-conflicting option.distinct(1) option.simps(8,9)
    state-eq-conflicting)

lemma tranclp-conflict-iff[iff]:
  full1 conflict S S'  $\longleftrightarrow$  conflict S S'
proof –
  have tranclp conflict S S'  $\implies$  conflict S S' by (meson tranclp-conflict rtranclpD)
  then show ?thesis unfolding full1-def
  by (metis conflict.simps conflicting-update-conflicting option.distinct(1) option.simps(9)
    state-eq-conflicting tranclp.intros(1))
qed

inductive cdclW-cp :: 'st  $\Rightarrow$  'st  $\Rightarrow$  bool where
  conflict'[intro]: conflict S S'  $\implies$  cdclW-cp S S' |
  propagate': propagate S S'  $\implies$  cdclW-cp S S'

lemma rtranclp-cdclW-cp-rtranclp-cdclW:
  cdclW-cp** S T  $\implies$  cdclW** S T
  by (induction rule: rtranclp-induct) (auto simp: cdclW-cp.simps dest: cdclW.intros)

lemma cdclW-cp-state-eq-compatible:
  assumes
    cdclW-cp S T and
    S  $\sim$  S' and
    T  $\sim$  T'
  shows cdclW-cp S' T'
  using assms
  apply (induction)
  using conflict-state-eq-compatible apply auto[1]
  using propagate' propagate-state-eq-compatible by auto

lemma tranclp-cdclW-cp-state-eq-compatible:
  assumes
    cdclW-cp** S T and
    S  $\sim$  S' and
    T  $\sim$  T'
  shows cdclW-cp** S' T'
  using assms
proof induction
  case base
  then show ?case
    using cdclW-cp-state-eq-compatible by blast
next
  case (step U V)
  obtain ss :: 'st where

```



```

  cdclW-cp S ss ∧ cdclW-cp** ss U
  by (metis (no-types) step(1) tranclpD)
then show ?case
  by (meson cdclW-cp-state-eq-compatible rtranclp.rtrancl-into-rtrancl rtranclp-into-tranclp2
      state-eq-ref step(2) step(4) step(5))
qed

```

```

lemma option-full-cdclW-cp:
  conflicting S ≠ None ⇒ full cdclW-cp S S
  unfolding full-def rtranclp-unfold tranclp-unfold
  by (auto simp add: cdclW-cp.simps elim: conflictE propagateE)

```

```

lemma skip-unique:
  skip S T ⇒ skip S T' ⇒ T ∼ T'
  by (fastforce simp: state-eq-def simp del: state-simp elim: skipE)

```

```

lemma resolve-unique:
  resolve S T ⇒ resolve S T' ⇒ T ∼ T'
  by (fastforce simp: state-eq-def simp del: state-simp elim: resolveE)

```

```

lemma cdclW-cp-no-more-clauses:
  assumes cdclW-cp S S'
  shows clauses S = clauses S'
  using assms by (induct rule: cdclW-cp.induct) (auto elim!: conflictE propagateE)

```

```

lemma tranclp-cdclW-cp-no-more-clauses:
  assumes cdclW-cp++ S S'
  shows clauses S = clauses S'
  using assms by (induct rule: tranclp.induct) (auto dest: cdclW-cp-no-more-clauses)

```

```

lemma rtranclp-cdclW-cp-no-more-clauses:
  assumes cdclW-cp** S S'
  shows clauses S = clauses S'
  using assms by (induct rule: rtranclp.induct) (fastforce dest: cdclW-cp-no-more-clauses)+

```

```

lemma no-conflict-after-conflict:
  conflict S T ⇒ ¬conflict T U
  by (metis None-eq-map-option-iff conflictE conflicting-update-conflicting option.distinct(1)
      state-simp(5))

```

```

lemma no-propagate-after-conflict:
  conflict S T ⇒ ¬propagate T U
  by (metis conflictE conflicting-update-conflicting map-option-is-None option.distinct(1)
      propagate.cases state-eq-conflicting)

```

```

lemma tranclp-cdclW-cp-propagate-with-conflict-or-not:
  assumes cdclW-cp++ S U
  shows (propagate++ S U ∧ conflicting U = None)
    ∨ (∃ T D. propagate** S T ∧ conflict T U ∧ conflicting U = Some D)
proof -
  have propagate++ S U ∨ (∃ T. propagate** S T ∧ conflict T U)
  using assms by induction
  (force simp: cdclW-cp.simps tranclp-into-rtranclp dest: no-conflict-after-conflict
      no-propagate-after-conflict)+
moreover

```

have $\text{propagate}^{++} S U \implies \text{conflicting } U = \text{None}$
unfolding $\text{trancpl-unfold-end}$ **by** $(\text{auto elim!}: \text{propagateE})$
moreover
have $\bigwedge T. \text{conflict } T U \implies \exists D. \text{conflicting } U = \text{Some } D$
by $(\text{auto elim!}: \text{conflictE simp: state-eq-def simp del: state-simp})$
ultimately show $?thesis$ **by** meson
qed

lemma $\text{cdcl}_W\text{-cp-conflicting-not-empty}[simp]: \text{conflicting } S = \text{Some } D \implies \neg \text{cdcl}_W\text{-cp } S S'$
proof
assume $\text{cdcl}_W\text{-cp } S S'$ **and** $\text{conflicting } S = \text{Some } D$
then show False **by** $(\text{induct rule: cdcl}_W\text{-cp.induct})$
 $(\text{auto elim: conflictE propagateE simp: state-eq-def simp del: state-simp})$
qed

lemma $\text{no-step-cdcl}_W\text{-cp-no-conflict-no-propagate}$:
assumes $\text{no-step cdcl}_W\text{-cp } S$
shows $\text{no-step conflict } S$ **and** $\text{no-step propagate } S$
using $\text{assms conflict' apply blast}$
by $(\text{meson assms conflict' propagate'})$

CDCL with the reasonable strategy: we fully propagate the conflict and propagate, then we apply any other possible rule $\text{cdcl}_W\text{-o } S S'$ and re-apply conflict and propagate $\text{full cdcl}_W\text{-cp } S' S''$

inductive $\text{cdcl}_W\text{-stgy} :: 'st \Rightarrow 'st \Rightarrow \text{bool}$ **for** $S :: 'st$ **where**
 $\text{conflict'}: \text{full1 cdcl}_W\text{-cp } S S' \implies \text{cdcl}_W\text{-stgy } S S' \mid$
 $\text{other'}: \text{cdcl}_W\text{-o } S S' \implies \text{no-step cdcl}_W\text{-cp } S \implies \text{full cdcl}_W\text{-cp } S' S'' \implies \text{cdcl}_W\text{-stgy } S S''$

18.5.2 Invariants

These are the same invariants as before, but lifted

lemma $\text{cdcl}_W\text{-cp-learned-clause-inv}$:
assumes $\text{cdcl}_W\text{-cp } S S'$
shows $\text{learned-clss } S = \text{learned-clss } S'$
using $\text{assms by (induct rule: cdcl}_W\text{-cp.induct) (fastforce elim: conflictE propagateE)+}$

lemma $\text{rtrancpl-cdcl}_W\text{-cp-learned-clause-inv}$:
assumes $\text{cdcl}_W\text{-cp}^{**} S S'$
shows $\text{learned-clss } S = \text{learned-clss } S'$
using $\text{assms by (induct rule: rtrancpl-induct) (fastforce dest: cdcl}_W\text{-cp-learned-clause-inv)+}$

lemma $\text{trancpl-cdcl}_W\text{-cp-learned-clause-inv}$:
assumes $\text{cdcl}_W\text{-cp}^{++} S S'$
shows $\text{learned-clss } S = \text{learned-clss } S'$
using $\text{assms by (simp add: rtrancpl-cdcl}_W\text{-cp-learned-clause-inv trancpl-into-rtrancpl)}$

lemma $\text{cdcl}_W\text{-cp-backtrack-lvl}$:
assumes $\text{cdcl}_W\text{-cp } S S'$
shows $\text{backtrack-lvl } S = \text{backtrack-lvl } S'$
using $\text{assms by (induct rule: cdcl}_W\text{-cp.induct) (fastforce elim: conflictE propagateE)+}$

lemma $\text{rtrancpl-cdcl}_W\text{-cp-backtrack-lvl}$:
assumes $\text{cdcl}_W\text{-cp}^{**} S S'$
shows $\text{backtrack-lvl } S = \text{backtrack-lvl } S'$
using $\text{assms by (induct rule: rtrancpl-induct) (fastforce dest: cdcl}_W\text{-cp-backtrack-lvl)+}$

```

lemma cdclW-cp-consistent-inv:
  assumes cdclW-cp S S'
  and cdclW-M-level-inv S
  shows cdclW-M-level-inv S'
  using assms
proof (induct rule: cdclW-cp.induct)
  case (conflict')
  then show ?case using cdclW-consistent-inv cdclW.conflict by blast
next
  case (propagate' S S')
  have cdclW S S'
    using propagate'.hyps(1) propagate by blast
  then show cdclW-M-level-inv S'
    using propagate'.prems(1) cdclW-consistent-inv propagate by blast
qed

lemma full1-cdclW-cp-consistent-inv:
  assumes full1 cdclW-cp S S'
  and cdclW-M-level-inv S
  shows cdclW-M-level-inv S'
  using assms unfolding full1-def
  by (metis rtrancpl-cdclW-cp-rtrancpl-cdclW rtrancpl-unfold trancpl-cdclW-consistent-inv)

lemma rtrancpl-cdclW-cp-consistent-inv:
  assumes rtrancpl cdclW-cp S S'
  and cdclW-M-level-inv S
  shows cdclW-M-level-inv S'
  using assms unfolding full1-def
  by (induction rule: rtrancpl-induct) (blast intro: cdclW-cp-consistent-inv)+

lemma cdclW-stgy-consistent-inv:
  assumes cdclW-stgy S S'
  and cdclW-M-level-inv S
  shows cdclW-M-level-inv S'
  using assms apply (induct rule: cdclW-stgy.induct)
  unfolding full-unfold by (blast intro: cdclW-consistent-inv full1-cdclW-cp-consistent-inv
    cdclW.other)+

lemma rtrancpl-cdclW-stgy-consistent-inv:
  assumes cdclW-stgy** S S'
  and cdclW-M-level-inv S
  shows cdclW-M-level-inv S'
  using assms by induction (auto dest!: cdclW-stgy-consistent-inv)

lemma cdclW-cp-no-more-init-clss:
  assumes cdclW-cp S S'
  shows init-clss S = init-clss S'
  using assms by (induct rule: cdclW-cp.induct) (auto elim: conflictE propagateE)

lemma trancpl-cdclW-cp-no-more-init-clss:
  assumes cdclW-cp++ S S'
  shows init-clss S = init-clss S'
  using assms by (induct rule: trancpl.induct) (auto dest: cdclW-cp-no-more-init-clss)

```

lemma *cdcl_W-stgy-no-more-init-clss*:
assumes *cdcl_W-stgy S S'* **and** *cdcl_W-M-level-inv S*
shows *init-clss S = init-clss S'*
using *assms*
apply (*induct rule: cdcl_W-stgy.induct*)
unfolding *full1-def full-def* **apply** (*blast dest: tranclp-cdcl_W-cp-no-more-init-clss*
tranclp-cdcl_W-o-no-more-init-clss)
by (*metis cdcl_W-o-no-more-init-clss rtranclp-unfold tranclp-cdcl_W-cp-no-more-init-clss*)

lemma *rtranclp-cdcl_W-stgy-no-more-init-clss*:
assumes *cdcl_W-stgy** S S'* **and** *cdcl_W-M-level-inv S*
shows *init-clss S = init-clss S'*
using *assms*
apply (*induct rule: rtranclp-induct, simp*)
using *cdcl_W-stgy-no-more-init-clss* **by** (*simp add: rtranclp-cdcl_W-stgy-consistent-inv*)

lemma *cdcl_W-cp-dropWhile-trail'*:
assumes *cdcl_W-cp S S'*
obtains *M* **where** *trail S' = M @ trail S* **and** $(\forall l \in \text{set } M. \neg \text{is-marked } l)$
using *assms* **by** *induction (fastforce elim: conflictE propagateE)+*

lemma *rtranclp-cdcl_W-cp-dropWhile-trail'*:
assumes *cdcl_W-cp** S S'*
obtains *M :: ('v, nat, 'v clause) marked-lit list* **where**
trail S' = M @ trail S **and** $\forall l \in \text{set } M. \neg \text{is-marked } l$
using *assms* **by** *induction (fastforce dest!: cdcl_W-cp-dropWhile-trail')+*

lemma *cdcl_W-cp-dropWhile-trail*:
assumes *cdcl_W-cp S S'*
shows $\exists M. \text{trail } S' = M @ \text{trail } S \wedge (\forall l \in \text{set } M. \neg \text{is-marked } l)$
using *assms* **by** *induction (fastforce elim: conflictE propagateE)+*

lemma *rtranclp-cdcl_W-cp-dropWhile-trail*:
assumes *cdcl_W-cp** S S'*
shows $\exists M. \text{trail } S' = M @ \text{trail } S \wedge (\forall l \in \text{set } M. \neg \text{is-marked } l)$
using *assms* **by** *induction (fastforce dest: cdcl_W-cp-dropWhile-trail)+*

This theorem can be seen a a termination theorem for *cdcl_W-cp*.

lemma *length-model-le-vars*:
assumes
no-strange-atm S **and**
no-d: no-dup (trail S) **and**
finite (atms-of-mm (init-clss S))
shows $\text{length } (\text{trail } S) \leq \text{card } (\text{atms-of-mm } (\text{init-clss } S))$

proof –

obtain *M N U k D* **where** *S: state S = (M, N, U, k, D)* **by** (*cases state S, auto*)
have *finite (atm-of ' lits-of-l (trail S))*
using *assms(1,3)* **unfolding** *S* **by** (*auto simp add: finite-subset*)
have $\text{length } (\text{trail } S) = \text{card } (\text{atm-of ' lits-of-l } (\text{trail } S))$
using *no-dup-length-eq-card-atm-of-lits-of-l no-d* **by** *blast*
then show *?thesis* **using** *assms(1)* **unfolding** *no-strange-atm-def*
by (*auto simp add: assms(3) card-mono*)

qed

lemma *cdcl_W-cp-decreasing-measure*:

```

assumes
  cdclW: cdclW-cp S T and
  M-lev: cdclW-M-level-inv S and
  alien: no-strange-atm S
shows ( $\lambda S. \text{card}(\text{atms-of-mm}(\text{init-clss } S)) - \text{length}(\text{trail } S)$ 
  + (if conflicting S = None then 1 else 0)) S
  > ( $\lambda S. \text{card}(\text{atms-of-mm}(\text{init-clss } S)) - \text{length}(\text{trail } S)$ 
  + (if conflicting S = None then 1 else 0)) T
using assms
proof -
  have  $\text{length}(\text{trail } T) \leq \text{card}(\text{atms-of-mm}(\text{init-clss } T))$ 
  apply (rule length-model-le-vars)
    using cdclW-no-strange-atm-inv alien M-lev apply (meson cdclW cdclW.simps cdclW-cp.cases)
    using M-lev cdclW cdclW-cp-consistent-inv cdclW-M-level-inv-def apply blast
    using cdclW by (auto simp: cdclW-cp.simps)
  with assms
  show ?thesis by induction (auto elim!: conflictE propagateE
    simp del: state-simp simp: state-eq-def)+
qed

lemma cdclW-cp-wf: wf {(b,a). (cdclW-M-level-inv a ∧ no-strange-atm a)
  ∧ cdclW-cp a b}
apply (rule wf-wf-if-measure'[of less-than - -
  ( $\lambda S. \text{card}(\text{atms-of-mm}(\text{init-clss } S)) - \text{length}(\text{trail } S)$ 
  + (if conflicting S = None then 1 else 0)))]
  apply simp
using cdclW-cp-decreasing-measure unfolding less-than-iff by blast

lemma rtranclp-cdclW-all-struct-inv-cdclW-cp-iff-rtranclp-cdclW-cp:
assumes
  lev: cdclW-M-level-inv S and
  alien: no-strange-atm S
shows ( $\lambda a b. (\text{cdcl}_W\text{-M-level-inv } a \wedge \text{no-strange-atm } a) \wedge \text{cdcl}_W\text{-cp } a b)^{**} S T$ 
   $\longleftrightarrow \text{cdcl}_W\text{-cp}^{**} S T$ 
  (is ?I S T  $\longleftrightarrow$  ?C S T)
proof
  assume
    ?I S T
  then show ?C S T by induction auto
next
  assume
    ?C S T
  then show ?I S T
    proof induction
      case base
      then show ?case by simp
    next
      case (step T U) note st = this(1) and cp = this(2) and IH = this(3)
      have  $\text{cdcl}_W^{**} S T$ 
      by (metis rtranclp-unfold cdclW-cp-conflicting-not-empty cp st
        rtranclp-propagate-is-rtranclp-cdclW tranclp-cdclW-cp-propagate-with-conflict-or-not)
      then have
        cdclW-M-level-inv T and
        no-strange-atm T
      using  $\langle \text{cdcl}_W^{**} S T \rangle$  apply (simp add: assms(1) rtranclp-cdclW-consistent-inv)

```

```

    using ⟨cdclW** S T⟩ alien rtrancp-cdclW-no-strange-atm-inv lev by blast
  then have (λa b. (cdclW-M-level-inv a ∧ no-strange-atm a)
    ∧ cdclW-cp a b)** T U
    using cp by auto
  then show ?case using IH by auto
qed
qed

lemma cdclW-cp-normalized-element:
  assumes
    lev: cdclW-M-level-inv S and
    no-strange-atm S
  obtains T where full cdclW-cp S T
proof -
  let ?inv = λa. (cdclW-M-level-inv a ∧ no-strange-atm a)
  obtain T where T: full (λa b. ?inv a ∧ cdclW-cp a b) S T
  using cdclW-cp-wf wf-exists-normal-form[of λa b. ?inv a ∧ cdclW-cp a b]
  unfolding full-def by blast
  then have cdclW-cp** S T
    using rtrancp-cdclW-all-struct-inv-cdclW-cp-iff-rtrancp-cdclW-cp assms unfolding full-def
    by blast
  moreover
  then have cdclW** S T
    using rtrancp-cdclW-cp-rtrancp-cdclW by blast
  then have
    cdclW-M-level-inv T and
    no-strange-atm T
    using ⟨cdclW** S T⟩ apply (simp add: assms(1) rtrancp-cdclW-consistent-inv)
    using ⟨cdclW** S T⟩ assms(2) rtrancp-cdclW-no-strange-atm-inv lev by blast
  then have no-step cdclW-cp T
    using T unfolding full-def by auto
  ultimately show thesis using that unfolding full-def by blast
qed

lemma always-exists-full-cdclW-cp-step:
  assumes no-strange-atm S
  shows ∃ S''. full cdclW-cp S S''
  using assms
proof (induct card (atms-of-mm (init-clss S) - atm-of 'lits-of-l (trail S)) arbitrary: S)
  case 0 note card = this(1) and alien = this(2)
  then have atm: atms-of-mm (init-clss S) = atm-of 'lits-of-l (trail S)
    unfolding no-strange-atm-def by auto
  { assume a: ∃ S'. conflict S S'
    then obtain S' where S': conflict S S' by metis
    then have ∀ S''. ¬cdclW-cp S' S''
      by (auto simp: cdclW-cp.simps elim!: conflictE propagateE
        simp del: state-simp simp: state-eq-def)
    then have ?case using a S' cdclW-cp.conflict' unfolding full-def by blast
  }
  moreover {
    assume a: ∃ S'. propagate S S'
    then obtain S' where propagate S S' by blast
    then obtain E L where
      S: conflicting S = None and
      E: E !∈ raw-clauses S and

```

```

  LE:  $L \in \# \text{ mset-cls } E$  and
  tr:  $\text{trail } S \models_{\text{as}} \text{CNot } (\text{mset-cls } (\text{remove-lit } L \ E))$  and
  undef:  $\text{undefined-lit } (\text{trail } S) \ L$  and
  S':  $S' \sim \text{cons-trail } (\text{Propagated } L \ E) \ S$ 
  by (elim propagateE) simp
have  $\text{atms-of-mm } (\text{learned-clss } S) \subseteq \text{atms-of-mm } (\text{init-clss } S)$ 
  using alien S unfolding no-strange-atm-def by auto
then have  $\text{atm-of } L \in \text{atms-of-mm } (\text{init-clss } S)$ 
  using E LE S undef unfolding raw-clauses-def by (force simp: in-implies-atm-of-on-atms-of-ms)
then have False using undef S unfolding atm unfolding lits-of-def
  by (auto simp add: defined-lit-map)
}
ultimately show ?case unfolding full-def by (metis cdclW-cp.cases rtranclp.rtrancl-refl)
next
case (Suc n) note IH = this(1) and card = this(2) and alien = this(3)
{ assume a:  $\exists S'. \text{conflict } S \ S'$ 
  then obtain S' where S':  $\text{conflict } S \ S'$  by metis
  then have  $\forall S''. \neg \text{cdcl}_W\text{-cp } S' \ S''$ 
    by (auto simp: cdclW-cp.simps elim!: conflictE propagateE
      simp del: state-simp simp: state-eq-def)
  then have ?case unfolding full-def Ex-def using S' cdclW-cp.conflict' by blast
}
moreover {
  assume a:  $\exists S'. \text{propagate } S \ S'$ 
  then obtain S' where propagate:  $\text{propagate } S \ S'$  by blast
  then obtain E L where
    S:  $\text{conflicting } S = \text{None}$  and
    E:  $E \notin \text{raw-clauses } S$  and
    LE:  $L \in \# \text{ mset-cls } E$  and
    tr:  $\text{trail } S \models_{\text{as}} \text{CNot } (\text{mset-cls } (\text{remove-lit } L \ E))$  and
    undef:  $\text{undefined-lit } (\text{trail } S) \ L$  and
    S':  $S' \sim \text{cons-trail } (\text{Propagated } L \ E) \ S$ 
    by (elim propagateE) simp
  then have  $\text{atm-of } L \notin \text{atm-of ' lits-of-l } (\text{trail } S)$ 
    unfolding lits-of-def by (auto simp add: defined-lit-map)
  moreover
    have no-strange-atm S' using alien propagate propagate-no-strange-atm-inv by blast
    then have  $\text{atm-of } L \in \text{atms-of-mm } (\text{init-clss } S)$ 
      using S' LE E undef unfolding no-strange-atm-def
      by (auto simp: raw-clauses-def in-implies-atm-of-on-atms-of-ms)
    then have  $\bigwedge A. \{\text{atm-of } L\} \subseteq \text{atms-of-mm } (\text{init-clss } S) - A \vee \text{atm-of } L \in A$  by force
  moreover have  $\text{Suc } n - \text{card } \{\text{atm-of } L\} = n$  by simp
  moreover have  $\text{card } (\text{atms-of-mm } (\text{init-clss } S) - \text{atm-of ' lits-of-l } (\text{trail } S)) = \text{Suc } n$ 
    using card S S' by simp
  ultimately
    have  $\text{card } (\text{atms-of-mm } (\text{init-clss } S) - \text{atm-of ' insert } L \ (\text{lits-of-l } (\text{trail } S))) = n$ 
      by (metis (no-types) Diff-insert card-Diff-subset finite.emptyI finite.insertI image-insert)
    then have  $n = \text{card } (\text{atms-of-mm } (\text{init-clss } S') - \text{atm-of ' lits-of-l } (\text{trail } S'))$ 
      using card S S' undef by simp
  then have a1:  $\text{Ex } (\text{full cdcl}_W\text{-cp } S')$  using IH <no-strange-atm S'> by blast
  have ?case
    proof -
      obtain S'' :: 'st where
        ff1:  $\text{cdcl}_W\text{-cp}^{**} \ S' \ S'' \wedge \text{no-step cdcl}_W\text{-cp } S''$ 
        using a1 unfolding full-def by blast
    }
}

```

```

    have  $cdcl_W\text{-}cp^{**} S S''$ 
      using  $ff1\ cdcl_W\text{-}cp.intros(2)[OF\ propagate]$ 
      by (metis (no-types) converse-rtrancl-into-rtranclp)
    then have  $\exists S''.\ cdcl_W\text{-}cp^{**} S S'' \wedge (\forall S'''. \neg cdcl_W\text{-}cp S'' S''')$ 
      using  $ff1$  by blast
    then show ?thesis unfolding full-def
      by meson
  qed
}
ultimately show ?case unfolding full-def by (metis  $cdcl_W\text{-}cp.cases\ rtranclp.rtrancl\text{-}refl$ )
qed

```

18.5.3 Literal of highest level in conflicting clauses

One important property of the $cdcl_W$ with strategy is that, whenever a conflict takes place, there is at least a literal of level k involved (except if we have derived the false clause). The reason is that we apply conflicts before a decision is taken.

abbreviation $no\text{-}clause\text{-}is\text{-}false :: 'st \Rightarrow bool$ **where**

$no\text{-}clause\text{-}is\text{-}false \equiv$

$\lambda S. (conflicting\ S = None \longrightarrow (\forall D \in \# \text{ clauses } S. \neg trail\ S \models_{as} CNot\ D))$

abbreviation $conflict\text{-}is\text{-}false\text{-}with\text{-}level :: 'st \Rightarrow bool$ **where**

$conflict\text{-}is\text{-}false\text{-}with\text{-}level\ S \equiv \forall D. conflicting\ S = Some\ D \longrightarrow D \neq \{\#\}$

$\longrightarrow (\exists L \in \# D. get\text{-}level\ (trail\ S)\ L = backtrack\text{-}lvl\ S)$

lemma $not\text{-}conflict\text{-}not\text{-}any\text{-}negated\text{-}init\text{-}clss$:

assumes $\forall S'. \neg conflict\ S S'$

shows $no\text{-}clause\text{-}is\text{-}false\ S$

proof (clarify)

fix D

assume $D \in \# local.clauses\ S$ **and** $raw\text{-}conflicting\ S = None$ **and** $trail\ S \models_{as} CNot\ D$

moreover then obtain D' **where**

$mset\text{-}cls\ D' = D$ **and**

$D' \notin \# raw\text{-}clauses\ S$

using $in\text{-}mset\text{-}clss\text{-}exists\text{-}preimage$ **unfolding** $raw\text{-}clauses\text{-}def$ **by** $blast$

ultimately show $False$

using $conflict\text{-}rule[of\ S\ D'\ update\text{-}conflicting\ (Some\ (ccls\text{-}of\text{-}cls\ D'))\ S]$ $assms$

by $auto$

qed

lemma $full\text{-}cdcl_W\text{-}cp\text{-}not\text{-}any\text{-}negated\text{-}init\text{-}clss$:

assumes $full\ cdcl_W\text{-}cp\ S S'$

shows $no\text{-}clause\text{-}is\text{-}false\ S'$

using $assms\ not\text{-}conflict\text{-}not\text{-}any\text{-}negated\text{-}init\text{-}clss$ **unfolding** $full\text{-}def$ **by** $auto$

lemma $full1\text{-}cdcl_W\text{-}cp\text{-}not\text{-}any\text{-}negated\text{-}init\text{-}clss$:

assumes $full1\ cdcl_W\text{-}cp\ S S'$

shows $no\text{-}clause\text{-}is\text{-}false\ S'$

using $assms\ not\text{-}conflict\text{-}not\text{-}any\text{-}negated\text{-}init\text{-}clss$ **unfolding** $full1\text{-}def$ **by** $auto$

lemma $cdcl_W\text{-}stgy\text{-}not\text{-}non\text{-}negated\text{-}init\text{-}clss$:

assumes $cdcl_W\text{-}stgy\ S S'$

shows $no\text{-}clause\text{-}is\text{-}false\ S'$

using $assms$ **apply** (induct rule: $cdcl_W\text{-}stgy.induct$)

using $full1\text{-}cdcl_W\text{-}cp\text{-}not\text{-}any\text{-}negated\text{-}init\text{-}clss\ full\text{-}cdcl_W\text{-}cp\text{-}not\text{-}any\text{-}negated\text{-}init\text{-}clss$ **by** $metis+$

lemma *rtrancpl-cdcl_W-stgy-not-non-negated-init-clss*:
assumes *cdcl_W-stgy^{**} S S'* **and** *no-clause-is-false S*
shows *no-clause-is-false S'*
using *assms* **by** (*induct rule: rtrancpl-induct*) (*auto simp: cdcl_W-stgy-not-non-negated-init-clss*)

lemma *cdcl_W-stgy-conflict-ex-lit-of-max-level*:

assumes *cdcl_W-cp S S'*
and *no-clause-is-false S*
and *cdcl_W-M-level-inv S*
shows *conflict-is-false-with-level S'*
using *assms*

proof (*induct rule: cdcl_W-cp.induct*)

case *conflict'*
then show *?case* **by** (*auto elim: conflictE*)

next

case *propagate'*
then show *?case* **by** (*auto elim: propagateE*)

qed

lemma *no-chained-conflict*:

assumes *conflict S S'*
and *conflict S' S''*
shows *False*
using *assms* **unfolding** *conflict.simps*
by (*metis conflicting-update-conflicting option.distinct(1) option.simps(9) state-eq-conflicting*)

lemma *rtrancpl-cdcl_W-cp-propa-or-propa-conf*:

assumes *cdcl_W-cp^{**} S U*
shows *propagate^{**} S U* \vee ($\exists T. \text{propagate^{**} } S \ T \ \wedge \ \text{conflict } T \ U$)
using *assms*

proof *induction*

case *base*
then show *?case* **by** *auto*

next

case (*step U V*) **note** *SU = this(1)* **and** *UV = this(2)* **and** *IH = this(3)*
consider (*confl*) *T* **where** *propagate^{**} S T* **and** *conflict T U*
| (*propa*) *propagate^{**} S U* **using** *IH* **by** *auto*

then show *?case*

proof *cases*

case *confl*
then have *False* **using** *UV* **by** (*auto elim: conflictE*)
then show *?thesis* **by** *fast*

next

case *propa*
also have *conflict U V* \vee *propagate U V* **using** *UV* **by** (*auto simp add: cdcl_W-cp.simps*)
ultimately show *?thesis* **by** *force*

qed

qed

lemma *rtrancpl-cdcl_W-co-conflict-ex-lit-of-max-level*:

assumes *full: full cdcl_W-cp S U*
and *cls-f: no-clause-is-false S*
and *conflict-is-false-with-level S*
and *lev: cdcl_W-M-level-inv S*

```

shows conflict-is-false-with-level U
proof (intro allI impI)
  fix D
  assume
    confl: conflicting U = Some D and
    D: D ≠ {#}
  consider (CT) conflicting S = None | (SD) D' where conflicting S = Some D'
  by (cases conflicting S) auto
  then show ∃ L ∈ #D. get-level (trail U) L = backtrack-lvl U
  proof cases
    case SD
    then have S = U
      by (metis (no-types) assms(1) cdclW-cp-conflicting-not-empty full-def rtrancpD trancpD)
    then show ?thesis using assms(3) confl D by blast-
  next
  case CT
  have init-clss U = init-clss S and learned-clss U = learned-clss S
    using full unfolding full-def
    apply (metis (no-types) rtrancpD trancp-cdclW-cp-no-more-init-clss)
    by (metis (mono-tags, lifting) full full-def rtrancp-cdclW-cp-learned-clause-inv)
  obtain T where propagate** S T and TU: conflict T U
  proof -
    have f5: U ≠ S
      using confl CT by force
    then have cdclW-cp++ S U
      by (metis full full-def rtrancpD)
    have ∧ p pa. ¬ propagate p pa ∨ conflicting pa =
      (None::'v clause option)
      by (auto elim: propagateE)
    then show ?thesis
      using f5 that trancp-cdclW-cp-propagate-with-conflict-or-not[OF ⟨cdclW-cp++ S U⟩]
      full confl CT unfolding full-def by auto
  qed
  obtain D' where
    raw-conflicting T = None and
    D': D' !∈ ! raw-clauses T and
    tr: trail T ⊨as CNot (mset-cls D') and
    U: U ∼ update-conflicting (Some (ccls-of-cls D')) T
    using TU by (auto elim!: conflictE)
  have init-clss T = init-clss S and learned-clss T = learned-clss S
    using U ⟨init-clss U = init-clss S⟩ ⟨learned-clss U = learned-clss S⟩ by auto
  then have D ∈ # clauses S
    using confl U D' by (auto simp: raw-clauses-def)
  then have ¬ trail S ⊨as CNot D
    using cls-f CT by simp

  moreover
    obtain M where tr-U: trail U = M @ trail S and nm: ∀ m ∈ set M. ¬ is-marked m
      by (metis (mono-tags, lifting) assms(1) full-def rtrancp-cdclW-cp-dropWhile-trail)
    have trail U ⊨as CNot D
      using tr confl U by (auto elim!: conflictE)
  ultimately obtain L where L ∈ # D and -L ∈ lits-of-l M
    unfolding tr-U CNot-def true-annots-def Ball-def true-annot-def true-cls-def by force

  moreover have inv-U: cdclW-M-level-inv U

```

```

    by (metis cdclW-stgy.conflict' cdclW-stgy-consistent-inv full full-unfold lev)
  moreover
    have backtrack-lvl U = backtrack-lvl S
      using full unfolding full-def by (auto dest: rtranclp-cdclW-cp-backtrack-lvl)

  moreover
    have no-dup (trail U)
      using inv-U unfolding cdclW-M-level-inv-def by auto
    { fix x :: ('v, nat, 'v clause) marked-lit and
      xb :: ('v, nat, 'v clause) marked-lit
      assume a1: atm-of L = atm-of (lit-of xb)
      moreover assume a2: - L = lit-of x
      moreover assume a3: (λl. atm-of (lit-of l)) ' set M
        ∩ (λl. atm-of (lit-of l)) ' set (trail S) = {}
      moreover assume a4: x ∈ set M
      moreover assume a5: xb ∈ set (trail S)
      moreover have atm-of (- L) = atm-of L
        by auto
      ultimately have False
        by auto
    }
  then have LS: atm-of L ∉ atm-of ' lits-of-l (trail S)
    using ⟨-L ∈ lits-of-l M⟩ ⟨no-dup (trail U)⟩ unfolding tr-U lits-of-def by auto
  ultimately have get-level (trail U) L = backtrack-lvl U
  proof (cases get-all-levels-of-marked (trail S) ≠ [], goal-cases)
    case 2 note LD = this(1) and LM = this(2) and inv-U = this(3) and US = this(4) and
      LS = this(5) and ne = this(6)
    have backtrack-lvl S = 0
      using lev ne unfolding cdclW-M-level-inv-def by auto
    moreover have get-rev-level (rev M) 0 L = 0
      using nm by auto
    ultimately show ?thesis using LS ne US unfolding tr-U
      by (simp add: get-all-levels-of-marked-nil-iff-not-is-marked lits-of-def)
  next
    case 1 note LD = this(1) and LM = this(2) and inv-U = this(3) and US = this(4) and
      LS = this(5) and ne = this(6)

    have hd (get-all-levels-of-marked (trail S)) = backtrack-lvl S
      using ne lev unfolding cdclW-M-level-inv-def
      by (cases get-all-levels-of-marked (trail S)) auto
    moreover have atm-of L ∈ atm-of ' lits-of-l M
      using ⟨-L ∈ lits-of-l M⟩ by (simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
        lits-of-def)
    ultimately show ?thesis
      using nm ne get-level-skip-beginning-hd-get-all-levels-of-marked[OF LS, of M]
        get-level-skip-in-all-not-marked[of rev M L backtrack-lvl S]
        unfolding lits-of-def US tr-U
        by auto
    qed
  then show ∃ L ∈ #D. get-level (trail U) L = backtrack-lvl U
    using ⟨L ∈ #D⟩ by blast
  qed
qed

```

18.5.4 Literal of highest level in marked literals

definition *mark-is-false-with-level* :: 'st \Rightarrow bool **where**

mark-is-false-with-level $S' \equiv$

$\forall D \ M1 \ M2 \ L. \ M1 \ @ \ \text{Propagated } L \ D \ \# \ M2 = \text{trail } S' \longrightarrow D - \{\#L\# \} \neq \{\#\}$
 $\longrightarrow (\exists L. L \in \# \ D \wedge \text{get-level } (\text{trail } S') \ L = \text{get-maximum-possible-level } M1)$

definition *no-more-propagation-to-do*:: 'st \Rightarrow bool **where**

no-more-propagation-to-do $S \equiv$

$\forall D \ M \ M' \ L. \ D + \{\#L\#\} \in \# \ \text{clauses } S \longrightarrow \text{trail } S = M' \ @ \ M \longrightarrow M \models_{as} CNot \ D$
 $\longrightarrow \text{undefined-lit } M \ L \longrightarrow \text{get-maximum-possible-level } M < \text{backtrack-lvl } S$
 $\longrightarrow (\exists L. L \in \# \ D \wedge \text{get-level } (\text{trail } S) \ L = \text{get-maximum-possible-level } M)$

lemma *propagate-no-more-propagation-to-do*:

assumes *propagate*: *propagate* $S \ S'$

and H : *no-more-propagation-to-do* S

and *lev-inv*: *cdcl_W-M-level-inv* S

shows *no-more-propagation-to-do* S'

using *assms*

proof –

obtain $E \ L$ **where**

S : *conflicting* $S = None$ **and**

E : $E \ !\in \text{raw-clauses } S$ **and**

LE : $L \in \# \ \text{mset-cls } E$ **and**

tr : $\text{trail } S \models_{as} CNot \ (\text{mset-cls } (\text{remove-lit } L \ E))$ **and**

$undefL$: *undefined-lit* $(\text{trail } S) \ L$ **and**

S' : $S' \sim \text{cons-trail } (\text{Propagated } L \ E) \ S$

using *propagate* **by** $(\text{elim } \text{propagate} E) \ \text{simp}$

let $?M' = \text{Propagated } L \ (\text{mset-cls } E) \ \# \ \text{trail } S$

show *?thesis unfolding no-more-propagation-to-do-def*

proof (intro allI impI)

fix $D \ M1 \ M2 \ L'$

assume

$D-L$: $D + \{\#L'\#\} \in \# \ \text{clauses } S'$ **and**

$\text{trail } S' = M2 \ @ \ M1$ **and**

get-max : *get-maximum-possible-level* $M1 < \text{backtrack-lvl } S'$ **and**

$M1 \models_{as} CNot \ D$ **and**

$undef$: *undefined-lit* $M1 \ L'$

have $tl \ M2 \ @ \ M1 = \text{trail } S \vee (M2 = [] \wedge M1 = \text{Propagated } L \ (\text{mset-cls } E) \ \# \ \text{trail } S)$

using $(\text{trail } S' = M2 \ @ \ M1) \ S' \ S \ \text{undefL} \ \text{lev-inv}$

by $(\text{cases } M2) \ (\text{auto simp:cdcl}_W\text{-M-level-inv-decomp})$

moreover {

assume $tl \ M2 \ @ \ M1 = \text{trail } S$

moreover **have** $D + \{\#L'\#\} \in \# \ \text{clauses } S$

using $D-L \ S \ S' \ \text{undefL} \ \text{unfolding raw-clauses-def}$ **by** *auto*

moreover **have** *get-maximum-possible-level* $M1 < \text{backtrack-lvl } S$

using *get-max* $S \ S' \ \text{undefL}$ **by** *auto*

ultimately obtain L' **where** $L' \in \# \ D$ **and**

get-level $(\text{trail } S) \ L' = \text{get-maximum-possible-level } M1$

using $H \ (M1 \models_{as} CNot \ D) \ \text{undef} \ \text{unfolding no-more-propagation-to-do-def}$ **by** *metis*

moreover

{ **have** *cdcl_W-M-level-inv* S'

using *cdcl_W-consistent-inv lev-inv cdcl_W.propagate[OF propagate]* **by** *blast*

then have *no-dup* $?M'$ **using** $S' \ \text{undefL} \ \text{unfolding cdcl}_W\text{-M-level-inv-def}$ **by** *auto*

moreover

have *atm-of* $L' \in \text{atm-of } (\text{ lits-of-l } M1)$

```

    using  $\langle L' \in \# D \rangle \langle M1 \models_{as} CNot D \rangle$  by (metis atm-of-uminus image-eqI
      in-CNot-implies-uminus(2))
    then have atm-of  $L' \in atm-of \text{ ' (lits-of-l (trail S))}$ 
      using  $\langle tl M2 @ M1 = trail S \rangle [symmetric] S undefL$  by auto
    ultimately have atm-of  $L \neq atm-of L'$  unfolding lits-of-def by auto
  }
  ultimately have  $\exists L' \in \# D. get-level (trail S') L' = get-maximum-possible-level M1$ 
    using  $S S' undefL$  by auto
}
moreover {
  assume  $M2 = []$  and  $M1: M1 = Propagated L (mset-cls E) \# trail S$ 
  have  $cdcl_W\text{-}M\text{-level-inv } S'$ 
    using  $cdcl_W\text{-consistent-inv}[OF - lev-inv] cdcl_W.propagate[OF propagate]$  by blast
  then have  $get-all-levels-of-marked (trail S') = rev [Suc 0..<(Suc 0+backtrack-lvl S)]$ 
    using  $S' undefL$  unfolding  $cdcl_W\text{-}M\text{-level-inv-def}$  by auto
  then have  $get-maximum-possible-level M1 = backtrack-lvl S'$ 
    using  $get-maximum-possible-level-max-get-all-levels-of-marked[of M1] S' M1 undefL$ 
    by (auto intro: Max-eqI)
  then have False using get-max by auto
}
ultimately show  $\exists L. L \in \# D \wedge get-level (trail S') L = get-maximum-possible-level M1$ 
  by fast
qed
qed

```

lemma *conflict-no-more-propagation-to-do:*

```

assumes
  conflict: conflict  $S S'$  and
  H: no-more-propagation-to-do  $S$  and
  M:  $cdcl_W\text{-}M\text{-level-inv } S$ 
shows no-more-propagation-to-do  $S'$ 
using assms unfolding no-more-propagation-to-do-def by (force elim!: conflictE)

```

lemma *cdcl_W-cp-no-more-propagation-to-do:*

```

assumes
  conflict:  $cdcl_W\text{-cp } S S'$  and
  H: no-more-propagation-to-do  $S$  and
  M:  $cdcl_W\text{-}M\text{-level-inv } S$ 
shows no-more-propagation-to-do  $S'$ 
using assms
proof (induct rule:  $cdcl_W\text{-cp.induct}$ )
case (conflict'  $S S'$ )
then show ?case using conflict-no-more-propagation-to-do[of  $S S'$ ] by blast
next
case (propagate'  $S S'$ ) note  $S = this$ 
show 1: no-more-propagation-to-do  $S'$ 
  using propagate-no-more-propagation-to-do[of  $S S'$ ]  $S$  by blast
qed

```

lemma *cdcl_W-then-exists-cdcl_W-stgy-step:*

```

assumes
  o:  $cdcl_W\text{-o } S S'$  and
  alien: no-strange-atm  $S$  and
  lev:  $cdcl_W\text{-}M\text{-level-inv } S$ 
shows  $\exists S'. cdcl_W\text{-stgy } S S'$ 

```

proof –
obtain S'' **where** $\text{full_cdcl}_W\text{-cp } S' S''$
using $\text{always-exists-full-cdcl}_W\text{-cp-step}$ alien $\text{cdcl}_W\text{-no-strange-atm-inv}$ $\text{cdcl}_W\text{-o-no-more-init-clss}$
 o other lev **by** $(\text{meson } \text{cdcl}_W\text{-consistent-inv})$
then show $?thesis$
using assms **by** $(\text{metis } \text{always-exists-full-cdcl}_W\text{-cp-step } \text{cdcl}_W\text{-stgy.conflict' full-unfold other'})$
qed

lemma $\text{backtrack-no-decomp}$:

assumes
 S : $\text{raw-conflicting } S = \text{Some } E$ **and**
 LE : $L \in \# \text{ mset-ccls } E$ **and**
 L : $\text{get-level } (\text{trail } S) L = \text{backtrack-lvl } S$ **and**
 D : $\text{get-maximum-level } (\text{trail } S) (\text{remove1-mset } L (\text{mset-ccls } E)) < \text{backtrack-lvl } S$ **and**
 bt : $\text{backtrack-lvl } S = \text{get-maximum-level } (\text{trail } S) (\text{mset-ccls } E)$ **and**
 $M-L$: $\text{cdcl}_W\text{-M-level-inv } S$
shows $\exists S'. \text{cdcl}_W\text{-o } S S'$

proof –

have $L-D$: $\text{get-level } (\text{trail } S) L = \text{get-maximum-level } (\text{trail } S) (\text{mset-ccls } E)$
using $L D bt$ **by** $(\text{simp add: get-maximum-level-plus})$
let $?i = \text{get-maximum-level } (\text{trail } S) (\text{remove1-mset } L (\text{mset-ccls } E))$
obtain $K M1 M2$ **where**
 K : $(\text{Marked } K (?i + 1) \# M1, M2) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$
using $\text{backtrack-ex-decomp}[OF M-L, of ?i] D S$ **by** auto
show $?thesis$ **using** $\text{backtrack-rule}[OF S LE K L] bt L bj \text{cdcl}_W\text{-bj.simps}$ **by** auto
qed

lemma $\text{cdcl}_W\text{-stgy-final-state-conclusive}$:

assumes
 termi : $\forall S'. \neg \text{cdcl}_W\text{-stgy } S S'$ **and**
 decomp : $\text{all-decomposition-implies-m } (\text{init-clss } S) (\text{get-all-marked-decomposition } (\text{trail } S))$ **and**
 learned : $\text{cdcl}_W\text{-learned-clause } S$ **and**
 level-inv : $\text{cdcl}_W\text{-M-level-inv } S$ **and**
 alien : $\text{no-strange-atm } S$ **and**
 no-dup : $\text{distinct-cdcl}_W\text{-state } S$ **and**
 confl : $\text{cdcl}_W\text{-conflicting } S$ **and**
 confl-k : $\text{conflict-is-false-with-level } S$
shows $(\text{conflicting } S = \text{Some } \{\#\} \wedge \text{unsatisfiable } (\text{set-mset } (\text{init-clss } S)))$
 $\vee (\text{conflicting } S = \text{None} \wedge \text{trail } S \models_{\text{as}} \text{set-mset } (\text{init-clss } S))$

proof –

let $?M = \text{trail } S$
let $?N = \text{init-clss } S$
let $?k = \text{backtrack-lvl } S$
let $?U = \text{learned-clss } S$
consider
 $(\text{None}) \text{raw-conflicting } S = \text{None}$
 $| (\text{Some-Empty}) E$ **where** $\text{raw-conflicting } S = \text{Some } E$ **and** $\text{mset-ccls } E = \{\#\}$
 $| (\text{Some}) E'$ **where** $\text{raw-conflicting } S = \text{Some } E'$ **and**
 $\text{conflicting } S = \text{Some } (\text{mset-ccls } E')$ **and** $\text{mset-ccls } E' \neq \{\#\}$
by $(\text{cases } \text{conflicting } S, \text{simp}) \text{ auto}$
then show $?thesis$
proof cases
case $(\text{Some-Empty } E)$
then have $\text{conflicting } S = \text{Some } \{\#\}$ **by** auto
then have $\text{unsatisfiable } (\text{set-mset } (\text{init-clss } S))$

```

using assms(3) unfolding cdclW-learned-clause-def true-clss-clb-def
by (metis (no-types, lifting) Un-insert-right atms-of-empty satisfiable-def
sup-bot.right-neutral total-over-m-insert total-over-set-empty true-clb-empty)
then show ?thesis using Some-Empty by auto
next
case None
{ assume  $\neg ?M \models_{asm} ?N$ 
  have atm-of ' (lits-of-l ?M) = atms-of-mm ?N (is ?A = ?B)
  proof
    show ?A  $\subseteq$  ?B using alien unfolding no-strange-atm-def by auto
    show ?B  $\subseteq$  ?A
    proof (rule ccontr)
      assume  $\neg ?B \subseteq ?A$ 
      then obtain l where  $l \in ?B$  and  $l \notin ?A$  by auto
      then have undefined-lit ?M (Pos l)
        using  $\langle l \notin ?A \rangle$  unfolding lits-of-def by (auto simp add: defined-lit-map)
      moreover have conflicting S = None
        using None by auto
      ultimately have  $\exists S'. \text{cdcl}_W\text{-o } S \ S'$ 
        using cdclW-o.decide decide-rule  $\langle l \in ?B \rangle$  no-strange-atm-def
        by (metis literal.sel(1) state-eq-def)
      then show False
        using termi cdclW-then-exists-cdclW-stgy-step[OF - alien] level-inv by blast
    qed
  qed
obtain D where  $\neg ?M \models_a D$  and  $D \in \# ?N$ 
  using  $\langle \neg ?M \models_{asm} ?N \rangle$  unfolding lits-of-def true-annots-def Ball-def by auto
have atms-of D  $\subseteq$  atm-of ' (lits-of-l ?M)
  using  $\langle D \in \# ?N \rangle$  unfolding  $\langle \text{atm-of ' (lits-of-l ?M) = atms-of-mm ?N} \rangle$  atms-of-ms-def
  by (auto simp add: atms-of-def)
then have a1: atm-of ' set-mset D  $\subseteq$  atm-of ' lits-of-l (trail S)
  by (auto simp add: atms-of-def lits-of-def)
have total-over-m (lits-of-l ?M) {D}
  using  $\langle \text{atms-of } D \subseteq \text{atm-of ' (lits-of-l ?M)} \rangle$ 
atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set by (fastforce simp: total-over-set-def)
then have ?M  $\models_{as} CNot \ D$ 
  using total-not-true-clb-true-clss-CNot  $\langle \neg \text{trail } S \models_a D \rangle$  true-annot-def
true-annots-true-clb by fastforce
then have False
  proof -
    obtain S' where
      f2: full cdclW-cp S S'
      by (meson alien always-exists-full-cdclW-cp-step level-inv)
    then have S' = S
      using cdclW-stgy.conflict'[of S] by (metis (no-types) full-unfold termi)
    then show ?thesis
      using f2  $\langle D \in \# \text{init-clss } S \rangle$  None  $\langle \text{trail } S \models_{as} CNot \ D \rangle$ 
raw-clauses-def full-cdclW-cp-not-any-negated-init-clss by auto
  qed
}
then have ?M  $\models_{asm} ?N$  by blast
then show ?thesis
  using None by auto
next
case (Some E') note raw-conf = this(1) and LD = this(2) and nempty = this(3)

```

then obtain $L \ D$ where
 $E'[simp]: mset-ccls \ E' = D + \{\#L\# \}$ and
 $lev-L: get-level \ ?M \ L = ?k$
 by (metis (mono-tags) confl-k insert-DiffM2)
 let $?D = D + \{\#L\# \}$
 have $?D \neq \{\# \}$ by auto
 have $?M \models_{as} CNot \ ?D$ using confl LD unfolding cdcl_W-conflicting-def by auto
 then have $?M \neq []$ unfolding true-annots-def Ball-def true-annot-def true-clc-def by force
 have $M: ?M = hd \ ?M \ \# \ tl \ ?M$ using $\langle ?M \neq [] \rangle$ list.collapse by fastforce
 have $g-a-l: get-all-levels-of-marked \ ?M = rev \ [1..<1 + ?k]$
 using level-inv lev-L M unfolding cdcl_W-M-level-inv-def by auto
 have $g-k: get-maximum-level \ (trail \ S) \ D \leq ?k$
 using get-maximum-possible-level-ge-get-maximum-level[of ?M]
 get-maximum-possible-level-max-get-all-levels-of-marked[of ?M]
 by (auto simp add: Max-n-upt g-a-l)
 {
 assume marked: is-marked (hd ?M)
 then obtain k' where $k': k' + 1 = ?k$
 using level-inv M unfolding cdcl_W-M-level-inv-def
 by (cases hd (trail S); cases trail S) auto
 obtain $L' \ l'$ where $L': hd \ ?M = Marked \ L' \ l'$ using marked by (cases hd ?M) auto
 have marked-hd-tl: $get-all-levels-of-marked \ (hd \ (trail \ S) \ \# \ tl \ (trail \ S))$
 $= rev \ [1..<1 + length \ (get-all-levels-of-marked \ ?M)]$
 using level-inv lev-L M unfolding cdcl_W-M-level-inv-def M[symmetric]
 by blast
 then have $l'-tl: l' \ \# \ get-all-levels-of-marked \ (tl \ ?M)$
 $= rev \ [1..<1 + length \ (get-all-levels-of-marked \ ?M)]$ unfolding L' by simp
 moreover have $\dots = length \ (get-all-levels-of-marked \ ?M)$
 $\# \ rev \ [1..<length \ (get-all-levels-of-marked \ ?M)]$
 using M Suc-le-mono calculation by (fastforce simp add: upt.simps(2))
 finally have
 $l'-cons: l' \ \# \ get-all-levels-of-marked \ (tl \ (trail \ S)) =$
 $length \ (get-all-levels-of-marked \ (trail \ S))$
 $\# \ rev \ [1..<length \ (get-all-levels-of-marked \ (trail \ S))]$ and
 $l' = ?k$ and
 $g-r: get-all-levels-of-marked \ (tl \ (trail \ S))$
 $= rev \ [1..<length \ (get-all-levels-of-marked \ (trail \ S))]$
 using level-inv lev-L M unfolding cdcl_W-M-level-inv-def by auto

 have *: $\bigwedge list. no-dup \ list \implies$
 $- L \in lits-of-l \ list \implies atm-of \ L \in atm-of \ 'lits-of-l \ list$
 by (metis atm-of-uminus imageI)
 have $L'-L: L' = -L$
 proof (rule ccontr)
 assume $\neg ?thesis$
 moreover have $-L \in lits-of-l \ ?M$ using confl LD unfolding cdcl_W-conflicting-def by auto
 ultimately have $get-level \ (hd \ (trail \ S) \ \# \ tl \ (trail \ S)) \ L = get-level \ (tl \ ?M) \ L$
 using cdcl_W-M-level-inv-decomp(1)[OF level-inv] L' M atm-of-eq-atm-of
 unfolding lits-of-def consistent-interp-def
 by (metis (mono-tags, hide-lams) marked-lit.sel(1) get-level-skip-beginning image-eqI
 list.set-intros(1))
 moreover
 have $length \ (get-all-levels-of-marked \ (trail \ S)) = ?k$
 using level-inv unfolding cdcl_W-M-level-inv-def by auto
 then have $Max \ (set \ (0 \ \# \ get-all-levels-of-marked \ (tl \ (trail \ S)))) = ?k - 1$


```

    unfolding g-r by (auto simp add: Max-n-upt)
  then have get-level (tl ?M) L < ?k
    using get-maximum-possible-level-ge-get-level[of tl ?M L]
    by (metis One-nat-def add.right-neutral add-Suc-right diff-add-inverse2
        get-maximum-possible-level-max-get-all-levels-of-marked k' le-imp-less-Suc
        list.simps(15))
  finally show False using lev-L M by auto
qed
have L: hd ?M = Marked (-L) ?k using ⟨l' = ?k⟩ L'-L L' by auto

have get-maximum-level (trail S) D < ?k
proof (rule ccontr)
  assume ¬ ?thesis
  then have get-maximum-level (trail S) D = ?k using M g-k unfolding L by auto
  then obtain L'' where L'' ∈# D and L-k: get-level ?M L'' = ?k
    using get-maximum-level-exists-lit[of ?k ?M D] unfolding k'[symmetric] by auto
  have L ≠ L'' using no-dup ⟨L'' ∈# D⟩
    unfolding distinct-cdclW-state-def LD
    by (metis E' add.right-neutral add-diff-cancel-right'
        distinct-mem-diff-mset union-commute union-single-eq-member)
  have L'' = -L
  proof (rule ccontr)
    assume ¬ ?thesis
    then have get-level ?M L'' = get-level (tl ?M) L''
      using M ⟨L ≠ L''⟩ get-level-skip-beginning[of L'' hd ?M tl ?M] unfolding L
      by (auto simp: atm-of-eq-atm-of)
    then show False
      by (metis L-k Max-n-upt One-nat-def Suc-n-not-le-n ⟨l' = backtrack-lvl S⟩
          add-Suc-right add-implies-diff g-r
          get-all-levels-of-marked-rev-eq-rev-get-all-levels-of-marked list.set(2)
          get-rev-level-less-max-get-all-levels-of-marked k' l'-cons list.sel(1)
          rev-rev-ident semiring-normalization-rules(6) set-upt)
  qed
  then have taut: tautology (D + {#L#})
    using ⟨L'' ∈# D⟩ by (metis add.commute mset-leD mset-le-add-left multi-member-this
        tautology-minus)
  have consistent-interp (lits-of-l ?M)
    using level-inv unfolding cdclW-M-level-inv-def by auto
  then have ¬?M ⊨as CNot ?D
    using taut by (metis ⟨L'' = -L⟩ ⟨L'' ∈# D⟩ add.commute consistent-interp-def
        diff-union-cancelR in-CNot-implies-uminus(2) in-diffD multi-member-this)
  moreover have ?M ⊨as CNot ?D
    using confl no-dup LD unfolding cdclW-conflicting-def by auto
  ultimately show False by blast
qed note H = this
have get-maximum-level (trail S) D < get-maximum-level (trail S) (D + {#L#})
  using H by (auto simp: get-maximum-level-plus lev-L max-def)
moreover have backtrack-lvl S = get-maximum-level (trail S) (D + {#L#})
  using H by (auto simp: get-maximum-level-plus lev-L max-def)
ultimately have False
  using backtrack-no-decomp[OF raw-conf - lev-L] level-inv termi
  cdclW-then-exists-cdclW-stgy-step[of S] alien unfolding E'
  by (auto simp add: lev-L max-def)
} note not-is-marked = this

```

```

moreover {
  let ?D = D + {#L#}
  have ?D ≠ {#} by auto
  have ?M ⊨as CNot ?D using confl LD unfolding cdclW-conflicting-def by auto
  then have ?M ≠ [] unfolding true-annots-def Ball-def true-annot-def true-cls-def by force
  assume nm: ¬is-marked (hd ?M)
  then obtain L' C where L'C: hd-raw-trail S = Propagated L' C
    by (metis (trail S ≠ []) hd-raw-trail is-marked-def mset-of-mlit.elims)
  then have hd ?M = Propagated L' (mset-cls C)
    using (trail S ≠ []) hd-raw-trail mset-of-mlit.simps(1) by fastforce
  then have M: ?M = Propagated L' (mset-cls C) # tl ?M
    using (?M ≠ []) list.collapse by fastforce
  then obtain C' where C': mset-cls C = C' + {#L'#}
    using confl unfolding cdclW-conflicting-def by (metis append-Nil diff-single-eq-union)
  { assume -L' ∈# ?D
    then have Ex (skip S)
      using skip-rule[OF M raw-conf] unfolding E' by auto
    then have False
      using cdclW-then-exists-cdclW-stgy-step[of S] alien level-inv termi
      by (auto dest: cdclW-o.intros cdclW-bj.intros)
  }
moreover {
  assume L'D: -L' ∈# ?D
  then obtain D' where D': ?D = D' + {#-L'#} by (metis insert-DiffM2)
  have g-r: get-all-levels-of-marked (Propagated L' (mset-cls C) # tl (trail S))
    = rev [Suc 0..<Suc (length (get-all-levels-of-marked (trail S)))]
    using level-inv M unfolding cdclW-M-level-inv-def by auto
  have Max (insert 0
    (set (get-all-levels-of-marked (Propagated L' (mset-cls C) # tl (trail S))))) = ?k
    using level-inv M unfolding g-r cdclW-M-level-inv-def set-rev
    by (auto simp add:Max-n-upt)
  then have get-maximum-level (trail S) D' ≤ ?k
    using get-maximum-possible-level-ge-get-maximum-level[of
      Propagated L' (mset-cls C) # tl ?M] M
    unfolding get-maximum-possible-level-max-get-all-levels-of-marked by auto
  then have get-maximum-level (trail S) D' = ?k
    ∨ get-maximum-level (trail S) D' < ?k
    using le-neq-implies-less by blast
  moreover {
    assume g-D'-k: get-maximum-level (trail S) D' = ?k
    then have f1: get-maximum-level (trail S) D' = backtrack-lvl S
      using M by auto
    then have Ex (cdclW-o S)
      using f1 resolve-rule[of S L' C , OF (trail S ≠ []) - - raw-conf] raw-conf g-D'-k
      L'C L'D unfolding C' D' E'
      by (fastforce simp add: D' intro: cdclW-o.intros cdclW-bj.intros)
    then have False
      by (meson alien cdclW-then-exists-cdclW-stgy-step termi level-inv)
  }
  moreover {
    assume a1: get-maximum-level (trail S) D' < ?k
    then have f3: get-maximum-level (trail S) D' < get-level (trail S) (-L')
      using a1 lev-L by (metis D' get-maximum-level-ge-get-level insert-noteq-member
        not-less)
    moreover have backtrack-lvl S = get-level (trail S) L'

```

```

    apply (subst M)
    unfolding rev.simps
    apply (subst get-rev-level-can-skip-correctly-ordered)
    using level-inv unfolding cdclW-M-level-inv-def
    apply (subst (asm) (2) M) apply (simp add: cdclW-M-level-inv-decomp)
    using level-inv unfolding cdclW-M-level-inv-def
    apply (subst (asm) (2) M) apply (auto simp: cdclW-M-level-inv-decomp lits-of-def)[]
    using level-inv unfolding cdclW-M-level-inv-def
    apply (subst (asm) (4) M) apply (auto simp add: cdclW-M-level-inv-decomp)[]
    using level-inv unfolding cdclW-M-level-inv-def
    apply (subst (asm) (4) M) by (auto simp add: cdclW-M-level-inv-decomp)[]
  moreover
    then have get-level (trail S) L' = get-maximum-level (trail S) (D' + {#- L'#})
      using a1 by (auto simp add: get-maximum-level-plus max-def)
    ultimately have False
      using M backtrack-no-decomp[of S - -L', OF raw-conf]
      cdclW-then-exists-cdclW-stgy-step L'D level-inv termi alien
      unfolding D' E' by auto
  }
  ultimately have False by blast
}
ultimately have False by blast
}
ultimately show ?thesis by blast
qed
qed

```

```

lemma cdclW-cp-tranclp-cdclW:
  cdclW-cp S S'  $\implies$  cdclW++ S S'
  apply (induct rule: cdclW-cp.induct)
  by (meson cdclW.conflict cdclW.propagate tranclp.r-into-trancl tranclp.trancl-into-trancl)+

```

```

lemma tranclp-cdclW-cp-tranclp-cdclW:
  cdclW-cp++ S S'  $\implies$  cdclW++ S S'
  apply (induct rule: tranclp.induct)
  apply (simp add: cdclW-cp-tranclp-cdclW)
  by (meson cdclW-cp-tranclp-cdclW tranclp-trans)

```

```

lemma cdclW-stgy-tranclp-cdclW:
  cdclW-stgy S S'  $\implies$  cdclW++ S S'
proof (induct rule: cdclW-stgy.induct)
  case conflict'
  then show ?case
    unfolding full1-def by (simp add: tranclp-cdclW-cp-tranclp-cdclW)
next
  case (other' S' S'')
  then have S' = S''  $\vee$  cdclW-cp++ S' S''
    by (simp add: rtranclp-unfold full-def)
  then show ?case
    using other' by (meson cdclW.other tranclp.r-into-trancl
      tranclp-cdclW-cp-tranclp-cdclW tranclp-trans)
qed

```

```

lemma tranclp-cdclW-stgy-tranclp-cdclW:
  cdclW-stgy++ S S'  $\implies$  cdclW++ S S'

```

apply (*induct rule: tranclp.induct*)
using $cdcl_W\text{-stgy-tranclp-cdcl}_W$ **apply** *blast*
by (*meson cdcl_W-stgy-tranclp-cdcl_W tranclp-trans*)

lemma $rtranclp\text{-}cdcl_W\text{-stgy-rtranclp-cdcl}_W$:
 $cdcl_W\text{-stgy}^{**} S S' \implies cdcl_W^{**} S S'$
using $rtranclp\text{-}unfold[of\ cdcl_W\text{-stgy}\ S\ S']\ tranclp\text{-}cdcl_W\text{-stgy-tranclp-cdcl}_W[of\ S\ S']$ **by** *auto*

lemma *not-empty-get-maximum-level-exists-lit*:
assumes $n: D \neq \{\#\}$
and $max: get\text{-}maximum\text{-}level\ M\ D = n$
shows $\exists L \in \#D. get\text{-}level\ M\ L = n$

proof –
have $f: finite\ (insert\ 0\ ((\lambda L. get\text{-}level\ M\ L)\ 'set\text{-}mset\ D))$ **by** *auto*
then have $n \in ((\lambda L. get\text{-}level\ M\ L)\ 'set\text{-}mset\ D)$
using $n\ max\ get\text{-}maximum\text{-}level\text{-}exists\text{-}lit\text{-}of\text{-}max\text{-}level\ image\text{-}iff$
unfolding $get\text{-}maximum\text{-}level\text{-}def$ **by** *force*
then show $\exists L \in \#D. get\text{-}level\ M\ L = n$ **by** *auto*
qed

lemma $cdcl_W\text{-}o\text{-}conflict\text{-}is\text{-}false\text{-}with\text{-}level\text{-}inv$:

assumes
 $cdcl_W\text{-}o\ S\ S'$ **and**
 $lev: cdcl_W\text{-}M\text{-}level\text{-}inv\ S$ **and**
 $confl\text{-}inv: conflict\text{-}is\text{-}false\text{-}with\text{-}level\ S$ **and**
 $n\text{-}d: distinct\text{-}cdcl_W\text{-}state\ S$ **and**
 $conflicting: cdcl_W\text{-}conflicting\ S$
shows $conflict\text{-}is\text{-}false\text{-}with\text{-}level\ S'$
using $assms(1,2)$

proof (*induct rule: cdcl_W-o-induct-lev2*)
case ($resolve\ L\ C\ M\ D\ T$) **note** $tr\text{-}S = this(1)$ **and** $confl = this(4)$ **and** $LD = this(5)$ **and** $T = this(7)$

have $uL\text{-}not\text{-}D: -L \notin \# remove1\text{-}mset\ (-L)\ (mset\text{-}ccls\ D)$
using $n\text{-}d\ confl$ **unfolding** $distinct\text{-}cdcl_W\text{-}state\text{-}def\ distinct\text{-}mset\text{-}def$
by ($metis\ distinct\text{-}cdcl_W\text{-}state\text{-}def\ distinct\text{-}mem\text{-}diff\text{-}mset\ multi\text{-}member\text{-}last\ n\text{-}d\ option.simps(9)$)

moreover have $L\text{-}not\text{-}D: L \notin \# remove1\text{-}mset\ (-L)\ (mset\text{-}ccls\ D)$

proof (*rule ccontr*)

assume $\neg ?thesis$

then have $L \in \# mset\text{-}ccls\ D$

by ($auto\ simp: in\text{-}remove1\text{-}mset\text{-}neg$)

moreover have $Propagated\ L\ (mset\text{-}cls\ C) \# M \models_{as} CNot\ (mset\text{-}ccls\ D)$

using $conflicting\ confl\ tr\text{-}S$ **unfolding** $cdcl_W\text{-}conflicting\text{-}def$ **by** *auto*

ultimately have $-L \in lits\text{-}of\text{-}l\ (Propagated\ L\ (mset\text{-}cls\ C) \# M)$

using $in\text{-}CNot\text{-}implies\text{-}uminus(2)$ **by** *blast*

moreover have $no\text{-}dup\ (Propagated\ L\ (mset\text{-}cls\ C) \# M)$

using $lev\ tr\text{-}S$ **unfolding** $cdcl_W\text{-}M\text{-}level\text{-}inv\text{-}def$ **by** *auto*

ultimately show *False* **unfolding** $lits\text{-}of\text{-}def$ **by** ($metis\ consistent\text{-}interp\text{-}def\ image\text{-}eqI$

$list.set\text{-}intros(1)\ lits\text{-}of\text{-}def\ marked\text{-}lit.sel(2)\ distinct\text{-}consistent\text{-}interp$)

qed

ultimately

have $g\text{-}D: get\text{-}maximum\text{-}level\ (Propagated\ L\ (mset\text{-}cls\ C) \# M)\ (remove1\text{-}mset\ (-L)\ (mset\text{-}ccls\ D))$
 $= get\text{-}maximum\text{-}level\ M\ (remove1\text{-}mset\ (-L)\ (mset\text{-}ccls\ D))$

proof –

have $\forall a\ f\ L. ((a::'v) \in f\ 'L) = (\exists l. (l::'v\ literal) \in L \wedge a = f\ l)$

```

    by blast
  then show ?thesis
    using get-maximum-level-skip-first[of L remove1-mset (-L) (mset-ccls D) mset-cls C M]
    unfolding atms-of-def
    by (metis (no-types) uL-not-D L-not-D atm-of-eq-atm-of)
  qed
have lev-L[simp]: get-level M L = 0
  apply (rule atm-of-notin-get-rev-level-eq-0)
  using lev unfolding cdclW-M-level-inv-def tr-S by (auto simp: lits-of-def)

have D: get-maximum-level M (remove1-mset (-L) (mset-ccls D)) = backtrack-lvl S
  using resolve.hyps(6) LD unfolding tr-S by (auto simp: get-maximum-level-plus max-def g-D)
have get-all-levels-of-marked M = rev [Suc 0.. $\text{Suc } (\text{backtrack-lvl } S)$ ]
  using lev unfolding tr-S cdclW-M-level-inv-def by auto
then have get-maximum-level M (remove1-mset L (mset-cls C))  $\leq$  backtrack-lvl S
  using get-maximum-possible-level-ge-get-maximum-level[of M]
  get-maximum-possible-level-max-get-all-levels-of-marked[of M] by (auto simp: Max-n-upt)
then have
  get-maximum-level M (remove1-mset (- L) (mset-ccls D)  $\# \cup$  remove1-mset L (mset-cls C)) =
    backtrack-lvl S
  by (auto simp: get-maximum-level-union-mset get-maximum-level-plus max-def D)
then show ?case
  using tr-S not-empty-get-maximum-level-exists-lit[of
    remove1-mset (- L) (mset-ccls D)  $\# \cup$  remove1-mset L (mset-cls C) M] T
  by auto
next
case (skip L C' M D T) note tr-S = this(1) and D = this(2) and T = this(5)
then obtain La where
  La  $\in \#$  mset-ccls D and
  get-level (Propagated L C'  $\#$  M) La = backtrack-lvl S
  using skip confl-inv by auto
moreover
  have atm-of La  $\neq$  atm-of L
  proof (rule ccontr)
    assume  $\neg$  ?thesis
    then have La: La = L using  $\langle \text{La} \in \# \text{ mset-ccls } D \rangle \langle - L \notin \# \text{ mset-ccls } D \rangle$ 
      by (auto simp add: atm-of-eq-atm-of)
    have Propagated L C'  $\#$  M  $\models_{\text{as}}$  CNot (mset-ccls D)
      using conflicting tr-S D unfolding cdclW-conflicting-def by auto
    then have  $-L \in \text{lits-of-l } M$ 
      using  $\langle \text{La} \in \# \text{ mset-ccls } D \rangle$  in-CNot-implies-uminus(2)[of L mset-ccls D
        Propagated L C'  $\#$  M] unfolding La
      by auto
    then show False using lev tr-S unfolding cdclW-M-level-inv-def consistent-interp-def by auto
  qed
  then have get-level (Propagated L C'  $\#$  M) La = get-level M La by auto
ultimately show ?case using D tr-S T by auto
next
case backtrack
then show ?case
  by (auto split: if-split-asm simp: cdclW-M-level-inv-decomp lev)
qed auto

```

18.5.5 Strong completeness

lemma $\text{cdcl}_W\text{-cp-propagate-confl}$:

assumes $cdcl_W\text{-}cp\ S\ T$
shows $propagate^{**}\ S\ T \vee (\exists S'.\ propagate^{**}\ S\ S' \wedge conflict\ S'\ T)$
using *assms* **by** *induction blast+*

lemma *rtranclp-cdcl_W-cp-propagate-conf*:
assumes $cdcl_W\text{-}cp^{**}\ S\ T$
shows $propagate^{**}\ S\ T \vee (\exists S'.\ propagate^{**}\ S\ S' \wedge conflict\ S'\ T)$
by (*simp add: assms rtranclp-cdcl_W-cp-propa-or-propa-conf*)

lemma *propagate-high-levelE*:
assumes $propagate\ S\ T$
obtains $M'\ N'\ U\ k\ L\ C$ **where**
 $state\ S = (M',\ N',\ U,\ k,\ None)$ **and**
 $state\ T = (Propagated\ L\ (C + \{\#L\#\})\ \# \ M',\ N',\ U,\ k,\ None)$ **and**
 $C + \{\#L\#\} \in \# local.clauses\ S$ **and**
 $M' \models_{as} CNot\ C$ **and**
 $undefined\text{-}lit\ (trail\ S)\ L$

proof –

obtain $E\ L$ **where**
 $conf: conflicting\ S = None$ **and**
 $E: E \notin raw.clauses\ S$ **and**
 $LE: L \in \# mset\text{-}cls\ E$ **and**
 $tr: trail\ S \models_{as} CNot\ (mset\text{-}cls\ (remove\text{-}lit\ L\ E))$ **and**
 $undef: undefined\text{-}lit\ (trail\ S)\ L$ **and**
 $T: T \sim cons\text{-}trail\ (Propagated\ L\ E)\ S$
using *assms* **by** (*elim propagateE*) *simp*
obtain $M\ N\ U\ k$ **where**
 $S: state\ S = (M,\ N,\ U,\ k,\ None)$
using *conf* **by** *auto*
show *thesis*
using *that*[*of* $M\ N\ U\ k\ L\ remove1\text{-}mset\ L\ (mset\text{-}cls\ E)]\ S\ T\ LE\ E\ tr\ undef$
by *auto*

qed

lemma *cdcl_W-cp-propagate-completeness*:
assumes $MN: set\ M \models_s set\text{-}mset\ N$ **and**
 $cons: consistent\text{-}interp\ (set\ M)$ **and**
 $tot: total\text{-}over\text{-}m\ (set\ M)\ (set\text{-}mset\ N)$ **and**
 $lits\text{-}of\text{-}l\ (trail\ S) \subseteq set\ M$ **and**
 $init\text{-}clss\ S = N$ **and**
 $propagate^{**}\ S\ S'$ **and**
 $learned\text{-}clss\ S = \{\#\}$
shows $length\ (trail\ S) \leq length\ (trail\ S') \wedge lits\text{-}of\text{-}l\ (trail\ S') \subseteq set\ M$
using *assms*(6,4,5,7)

proof (*induction rule: rtranclp-induct*)

case *base*
then show *?case* **by** *auto*

next

case (*step Y Z*)
note $st = this(1)$ **and** $propa = this(2)$ **and** $IH = this(3)$ **and** $lits' = this(4)$ **and** $NS = this(5)$ **and**
 $learned = this(6)$
then have $len: length\ (trail\ S) \leq length\ (trail\ Y)$ **and** $LM: lits\text{-}of\text{-}l\ (trail\ Y) \subseteq set\ M$
by *blast+*

obtain $M'\ N'\ U\ k\ C\ L$ **where**

Y : state $Y = (M', N', U, k, \text{None})$ **and**
 Z : state $Z = (\text{Propagated } L (C + \{\#L\}) \# M', N', U, k, \text{None})$ **and**
 C : $C + \{\#L\} \in \# \text{ clauses } Y$ **and**
 $M'-C$: $M' \models_{as} C \text{Not } C$ **and**
 $\text{undefined-lit } (\text{trail } Y) L$
using *propa* **by** (*auto elim: propagate-high-levelE*)
have *init-clss* $S = \text{init-clss } Y$
using *st* **by** *induction* (*auto elim: propagateE*)
then have [*simp*]: $N' = N$ **using** *NS* $Y Z$ **by** *simp*
have *learned-clss* $Y = \{\#\}$
using *st* **learned** **by** *induction* (*auto elim: propagateE*)
then have [*simp*]: $U = \{\#\}$ **using** Y **by** *auto*
have *set* $M \models_s C \text{Not } C$
using $M'-C$ *LM* Y **unfolding** *true-annots-def* *Ball-def* *true-annot-def* *true-clss-def* *true-clss-def*
by *force*
moreover
have *set* $M \models C + \{\#L\}$
using *MN* C *learned* Y *NS* $\langle \text{init-clss } S = \text{init-clss } Y \rangle \langle \text{learned-clss } Y = \{\#\} \rangle$
unfolding *true-clss-def* *raw-clauses-def* **by** *fastforce*
ultimately have $L \in \text{set } M$ **by** (*simp add: cons consistent-CNot-not*)
then show ?*case* **using** *LM* *len* $Y Z$ **by** *auto*
qed

lemma

assumes *propagate*** $S X$
shows
 $\text{rtrancp-propagate-init-clss: init-clss } X = \text{init-clss } S$ **and**
 $\text{rtrancp-propagate-learned-clss: learned-clss } X = \text{learned-clss } S$
using *assms* **by** (*induction rule: rtrancp-induct*) (*auto elim: propagateE*)

lemma *completeness-is-a-full1-propagation:*

fixes $S :: 'st$ **and** $M :: 'v$ *literal list*
assumes *MN*: $\text{set } M \models_s \text{set-mset } N$
and *cons*: *consistent-interp* ($\text{set } M$)
and *tot*: *total-over-m* ($\text{set } M$) ($\text{set-mset } N$)
and *alien*: *no-strange-atm* S
and *learned*: *learned-clss* $S = \{\#\}$
and *clsS*[*simp*]: $\text{init-clss } S = N$
and *lits*: *lits-of-l* ($\text{trail } S$) $\subseteq \text{set } M$
shows $\exists S'. \text{propagate** } S S' \wedge \text{full } \text{cdcl}_W\text{-cp } S S'$

proof –

obtain S' **where** *full*: $\text{full } \text{cdcl}_W\text{-cp } S S'$
using *always-exists-full-cdcl_W-cp-step* *alien* **by** *blast*
then consider (*propa*) $\text{propagate** } S S'$
 $|$ (*confl*) $\exists X. \text{propagate** } S X \wedge \text{conflict } X S'$
using *rtrancp-cdcl_W-cp-propagate-confl* **unfolding** *full-def* **by** *blast*
then show ?*thesis*
proof *cases*
case *propa* **then show** ?*thesis* **using** *full* **by** *blast*
next
case *confl*
then obtain X **where**
 X : $\text{propagate** } S X$ **and**
 $X\text{conf}$: $\text{conflict } X S'$
by *blast*

```

have clsX: init-clss X = init-clss S
  using X by (blast dest: rtrancpl-propagate-init-clss)
have learnedX: learned-clss X = {#}
  using X learned by (auto dest: rtrancpl-propagate-learned-clss)
obtain E where
  E: E ∈ # init-clss X + learned-clss X and
  Not-E: trail X ⊨as CNot E
  using Xconf by (auto simp add: raw-clauses-def elim!: conflictE)
have lits-of-l (trail X) ⊆ set M
  using cdclW-cp-propagate-completeness[OF assms(1-3) lits - X learned] learned by auto
then have MNE: set M ⊨s CNot E
  using Not-E
  by (fastforce simp add: true-annot-def true-annot-def true-clss-def true-cl-def)
have ¬ set M ⊨s set-mset N
  using E consistent-CNot-not[OF cons MNE]
  unfolding learnedX true-clss-def unfolding clsX clsS by auto
then show ?thesis using MN by blast
qed
qed

```

See also *cdcl_W-cp** ?S ?S' ⇒ ∃ M. trail ?S' = M @ trail ?S ∧ (∀ l ∈ set M. ¬ is-marked l)*

lemma *rtrancpl-propagate-is-trail-append*:
*propagate** S T ⇒ ∃ c. trail T = c @ trail S*
by (*induction rule: rtrancpl-induct*) (*auto elim: propagateE*)

lemma *rtrancpl-propagate-is-update-trail*:
*propagate** S T ⇒ cdcl_W-M-level-inv S ⇒*
init-clss S = init-clss T ∧ learned-clss S = learned-clss T ∧ backtrack-lvl S = backtrack-lvl T
∧ conflicting S = conflicting T

proof (*induction rule: rtrancpl-induct*)
case *base*
then **show** ?*case* **unfolding** *state-eq-def* **by** (*auto simp: cdcl_W-M-level-inv-decomp*)
next
case (*step T U*) **note** *IH=this(3)[OF this(4)]*
moreover **have** *cdcl_W-M-level-inv U*
using *rtrancpl-cdcl_W-consistent-inv* ⟨*propagate** S T*⟩ ⟨*propagate T U*⟩
rtrancpl-mono[*of propagate cdcl_W*] *cdcl_W-cp-consistent-inv propagate'*
rtrancpl-propagate-is-rtrancpl-cdcl_W step.prem **by** *blast*
then **have** *no-dup* (*trail U*) **unfolding** *cdcl_W-M-level-inv-def* **by** *auto*
ultimately **show** ?*case* **using** ⟨*propagate T U*⟩ **unfolding** *state-eq-def*
by (*fastforce simp: elim: propagateE*)
qed

lemma *cdcl_W-stgy-strong-completeness-n*:

assumes
MN: *set* *M* ⊨_s *set-mset* (*mset-clss* *N*) **and**
cons: *consistent-interp* (*set* *M*) **and**
tot: *total-over-m* (*set* *M*) (*set-mset* (*mset-clss* *N*)) **and**
atm-incl: *atm-of* ' (*set* *M*) ⊆ *atms-of-mm* (*mset-clss* *N*) **and**
distM: *distinct* *M* **and**
length: *n* ≤ *length* *M*
shows
 ∃ *M'* *k S. length* *M'* ≥ *n* ∧
lits-of-l *M'* ⊆ *set* *M* ∧
no-dup *M'* ∧


```

    state S = (M', mset-cls N, {#}, k, None) ∧
    cdclW-stgy** (init-state N) S
  using length
proof (induction n)
  case 0
  have state (init-state N) = ([], mset-cls N, {#}, 0, None)
    by (auto simp: state-eq-def simp del: state-simp)
  moreover have
    0 ≤ length [] and
    lits-of-l [] ⊆ set M and
    cdclW-stgy** (init-state N) (init-state N)
    and no-dup []
    by (auto simp: state-eq-def simp del: state-simp)
  ultimately show ?case using state-eq-sym by blast
next
case (Suc n) note IH = this(1) and n = this(2)
then obtain M' k S where
  l-M': length M' ≥ n and
  M': lits-of-l M' ⊆ set M and
  n-d[simp]: no-dup M' and
  S: state S = (M', mset-cls N, {#}, k, None) and
  st: cdclW-stgy** (init-state N) S
  by auto
have
  M: cdclW-M-level-inv S and
  alien: no-strange-atm S
  using cdclW-M-level-inv-S0-cdclW rtrancp-cdclW-stgy-consistent-inv st apply blast
  using cdclW-M-level-inv-S0-cdclW no-strange-atm-S0 rtrancp-cdclW-no-strange-atm-inv
  rtrancp-cdclW-stgy-rtrancp-cdclW st by blast

{ assume no-step: ¬no-step propagate S
  obtain S' where S': propagate** S S' and full: full cdclW-cp S S'
    using completeness-is-a-full1-propagation[OF assms(1-3), of S] alien M' S
    by (auto simp: comp-def)
  have lev: cdclW-M-level-inv S'
    using M S' rtrancp-cdclW-consistent-inv rtrancp-propagate-is-rtrancp-cdclW by blast
  then have n-d'[simp]: no-dup (trail S')
    unfolding cdclW-M-level-inv-def by auto
  have length (trail S) ≤ length (trail S') ∧ lits-of-l (trail S') ⊆ set M
    using S' full cdclW-cp-propagate-completeness[OF assms(1-3), of S] M' S
    by (auto simp: comp-def)
  moreover
    have full: full1 cdclW-cp S S'
      using full no-step no-step-cdclW-cp-no-conflict-no-propagate(2) unfolding full1-def full-def
      rtrancp-unfold by blast
    then have cdclW-stgy S S' by (simp add: cdclW-stgy.conflict')
  moreover
    have propa: propagate++ S S' using S' full unfolding full1-def by (metis rtrancpD trancpD)
    have trail S = M'
      using S by (auto simp: comp-def rev-map)
    with propa have length (trail S') > n
      using l-M' propa by (induction rule: trancp.induct) (auto elim: propagateE)
  moreover
    have stS': cdclW-stgy** (init-state N) S'
      using st cdclW-stgy.conflict'[OF full] by auto

```

```

then have init-clss  $S' = \text{mset-clss } N$ 
  using stS' rtranclp-cdclW-stgy-no-more-init-clss by fastforce
moreover
  have
    [simp]:learned-clss  $S' = \{\#\}$  and
    [simp]: init-clss  $S' = \text{init-clss } S$  and
    [simp]: conflicting  $S' = \text{None}$ 
    using trancplp-into-rtranclp[OF  $\langle \text{propagate}^{++} S S' \rangle$ ]  $S$ 
    rtranclp-propagate-is-update-trail[of  $S S'$ ]  $S M$  unfolding state-eq-def
    by (auto simp: comp-def)
  have  $S\text{-}S'$ : state  $S' = (\text{trail } S', \text{mset-clss } N, \{\#\}, \text{backtrack-lvl } S', \text{None})$ 
    using  $S$  by auto
  have cdclW-stgy** (init-state  $N$ )  $S'$ 
    apply (rule rtranclp.rtrancl-into-rtrancl)
    using st apply simp
    using  $\langle \text{cdcl}_W\text{-stgy } S S' \rangle$  by simp
ultimately have ?case
  apply –
  apply (rule exI[of - trail  $S'$ ], rule exI[of - backtrack-lvl  $S'$ ], rule exI[of -  $S'$ ])
  using  $S\text{-}S'$  by (auto simp: state-eq-def simp del: state-simp)
}
moreover {
  assume no-step: no-step propagate S
  have ?case
    proof (cases length  $M' \geq \text{Suc } n$ )
      case True
        then show ?thesis using l-M' M' st M alien S n-d by blast
      next
        case False
          then have  $n'$ : length  $M' = n$  using l-M' by auto
          have no-conflict: no-step conflict S
            proof –
              { fix  $D$ 
                assume  $D \in \# \text{mset-clss } N$  and  $M' \models_{as} \text{CNot } D$ 
                then have set  $M \models D$  using MN unfolding true-clss-def by auto
                moreover have set  $M \models_s \text{CNot } D$ 
                  using  $\langle M' \models_{as} \text{CNot } D \rangle M'$ 
                  by (metis le-iff-sup true-annots-true-clss true-clss-union-increase)
                ultimately have False using cons consistent-CNot-not by blast
              }
            then show ?thesis
              using  $S$  by (auto simp: true-clss-def comp-def rev-map
                raw-clauses-def dest!: in-clss-mset-clss elim!: conflictE)
          qed
          have lenM: length  $M = \text{card } (\text{set } M)$  using distM by (induction M) auto
          have no-dup  $M'$  using  $S M$  unfolding cdclW-M-level-inv-def by auto
          then have card (lits-of-l  $M'$ ) = length  $M'$ 
            by (induction M') (auto simp add: lits-of-def card-insert-if)
          then have lits-of-l  $M' \subset \text{set } M$ 
            using  $n M' n' \text{lenM}$  by auto
          then obtain  $m$  where  $m: m \in \text{set } M$  and undef-m:  $m \notin \text{lits-of-l } M'$  by auto
          moreover have undef: undefined-lit  $M' m$ 
            using  $M'$  Marked-Propagated-in-iff-in-lits-of-l calculation(1,2) cons
            consistent-interp-def by (metis (no-types, lifting) subset-eq)
          moreover have atm-of  $m \in \text{atms-of-mm } (\text{init-clss } S)$ 

```

```

    using atm-incl calculation S by auto
ultimately
  have dec: decide S (cons-trail (Marked m (k+1)) (incr-lvl S))
    using decide-rule[of S -
      cons-trail (Marked m (k + 1)) (incr-lvl S)] S
    by auto
  let ?S' = cons-trail (Marked m (k+1)) (incr-lvl S)
  have lits-of-l (trail ?S')  $\subseteq$  set M using m M' S undef by auto
  moreover have no-strange-atm ?S'
    using alien dec M by (meson cdclW-no-strange-atm-inv decide other)
  ultimately obtain S'' where S'': propagate** ?S' S'' and full: full cdclW-cp ?S' S''
    using completeness-is-a-full1-propagation[OF assms(1-3), of ?S'] S undef
    by auto
  have cdclW-M-level-inv ?S'
    using M dec rtranclp-mono[of decide cdclW] by (meson cdclW-consistent-inv decide other)
  then have lev'': cdclW-M-level-inv S''
    using S'' rtranclp-cdclW-consistent-inv rtranclp-propagate-is-rtranclp-cdclW by blast
  then have n-d'': no-dup (trail S'')
    unfolding cdclW-M-level-inv-def by auto
  have length (trail ?S')  $\leq$  length (trail S'')  $\wedge$  lits-of-l (trail S'')  $\subseteq$  set M
    using S'' full cdclW-cp-propagate-completeness[OF assms(1-3), of ?S' S''] m M' S undef
    by simp
  then have Suc n  $\leq$  length (trail S'')  $\wedge$  lits-of-l (trail S'')  $\subseteq$  set M
    using l-M' S undef by auto
  moreover
    have cdclW-M-level-inv (cons-trail (Marked m (Suc (backtrack-lvl S))))
      (update-backtrack-lvl (Suc (backtrack-lvl S)) S))
      using S (cdclW-M-level-inv (cons-trail (Marked m (k + 1)) (incr-lvl S))) by auto
    then have S'':
      state S'' = (trail S'', mset-cls N, {#}, backtrack-lvl S'', None)
      using rtranclp-propagate-is-update-trail[OF S''] S undef n-d'' lev''
      by auto
    then have cdclW-stgy** (init-state N) S''
      using cdclW-stgy.intros(2)[OF decide[OF dec] - full] no-step no-confl st
      by (auto simp: cdclW-cp.simps)
    ultimately show ?thesis using S'' n-d'' by blast
  qed
}
ultimately show ?case by blast
qed

```

lemma *cdcl_W-stgy-strong-completeness:*

assumes

MN: set M \models_s set-mset (mset-cls N) **and**

cons: consistent-interp (set M) **and**

tot: total-over-m (set M) (set-mset (mset-cls N)) **and**

atm-incl: atm-of ' (set M) \subseteq atms-of-mm (mset-cls N) **and**

distM: distinct M

shows

$\exists M' k S.$

lits-of-l M' = set M \wedge

state S = (M', mset-cls N, {#}, k, None) \wedge

cdcl_W-stgy** (init-state N) S \wedge

final-cdcl_W-state S

proof —

```

from  $cdcl_W$ -stgy-strong-completeness- $n$ [ $OF$   $assms$ , of length  $M$ ]
obtain  $M' k T$  where
   $l$ : length  $M \leq$  length  $M'$  and
   $M'-M$ : lits-of- $l$   $M' \subseteq$  set  $M$  and
   $no-dup$ : no-dup  $M'$  and
   $T$ : state  $T = (M', mset-clss\ N, \{\#\}, k, None)$  and
   $st$ :  $cdcl_W$ -stgy** (init-state  $N$ )  $T$ 
  by auto
have card (set  $M$ ) = length  $M$  using  $distM$  by (simp add: distinct-card)
moreover
  have  $cdcl_W$ - $M$ -level-inv  $T$ 
    using  $rtrancp$ - $cdcl_W$ -stgy-consistent-inv[ $OF$   $st$ ]  $T$  by auto
  then have card (set ((map ( $\lambda l$ . atm-of (lit-of  $l$ ))  $M'$ ))) = length  $M'$ 
    using distinct-card no-dup by fastforce
moreover have card (lits-of- $l$   $M'$ ) = card (set ((map ( $\lambda l$ . atm-of (lit-of  $l$ ))  $M'$ )))
  using no-dup unfolding lits-of-def apply (induction  $M'$ ) by (auto simp add: card-insert-if)
ultimately have card (set  $M$ )  $\leq$  card (lits-of- $l$   $M'$ ) using  $l$  unfolding lits-of-def by auto
then have set  $M =$  lits-of- $l$   $M'$ 
  using  $M'-M$  card-seteq by blast
moreover
  then have  $M' \models_{asm} mset-clss\ N$ 
    using  $MN$  unfolding true-annots-def Ball-def true-annot-def true-clss-def by auto
  then have final- $cdcl_W$ -state  $T$ 
    using  $T$  no-dup unfolding final- $cdcl_W$ -state-def by auto
ultimately show ?thesis using  $st\ T$  by blast
qed

```

18.5.6 No conflict with only variables of level less than backtrack level

This invariant is stronger than the previous argument in the sense that it is a property about all possible conflicts.

definition no-smaller-conf (S::'st) \equiv
 $(\forall M\ K\ i\ M'\ D. M' @ \text{Marked } K\ i \ \# \ M = \text{trail } S \longrightarrow D \in \# \text{ clauses } S$
 $\longrightarrow \neg M \models_{as} CNot\ D)$

lemma no-smaller-conf-init-sate[*simp*]:
 no-smaller-conf (init-state N) **unfolding** no-smaller-conf-def **by** *auto*

lemma $cdcl_W$ -o-no-smaller-conf-inv:

```

fixes  $S\ S' :: 'st$ 
assumes
   $cdcl_W$ -o  $S\ S'$  and
  lev:  $cdcl_W$ - $M$ -level-inv  $S$  and
  max-lev: conflict-is-false-with-level  $S$  and
  smaller: no-smaller-conf  $S$  and
  no-f: no-clause-is-false  $S$ 
shows no-smaller-conf  $S'$ 
using  $assms(1,2)$  unfolding no-smaller-conf-def
proof (induct rule:  $cdcl_W$ -o-induct-lev2)
case (decide  $L\ T$ ) note conf = this(1) and undef = this(2) and  $T =$  this(4)
have [simp]: clauses  $T =$  clauses  $S$ 
  using  $T$  undef by auto
show ?case
  proof (intro allI impI)
    fix  $M''\ K\ i\ M'\ Da$ 

```

```

assume  $M'' @ \text{Marked } K \ i \ \# \ M' = \text{trail } T$ 
and  $D: Da \in \# \text{ local.clauses } T$ 
then have  $tl \ M'' @ \text{Marked } K \ i \ \# \ M' = \text{trail } S$ 
   $\vee (M'' = [] \wedge \text{Marked } K \ i \ \# \ M' = \text{Marked } L \ (\text{backtrack-lvl } S + 1) \ \# \ \text{trail } S)$ 
  using  $T \text{ undef by (cases } M'') \text{ auto}$ 
moreover {
  assume  $tl \ M'' @ \text{Marked } K \ i \ \# \ M' = \text{trail } S$ 
  then have  $\neg M' \models_{as} CNot \ Da$ 
    using  $D \ T \text{ undef no-f confl smaller unfolding no-smaller-confl-def smaller by fastforce}$ 
  }
moreover {
  assume  $\text{Marked } K \ i \ \# \ M' = \text{Marked } L \ (\text{backtrack-lvl } S + 1) \ \# \ \text{trail } S$ 
  then have  $\neg M' \models_{as} CNot \ Da$  using  $\text{no-f } D \text{ confl } T \text{ by auto}$ 
  }
ultimately show  $\neg M' \models_{as} CNot \ Da$  by fast
qed
next
  case resolve
    then show  $?case$  using  $\text{smaller no-f max-lev unfolding no-smaller-confl-def by auto}$ 
  next
    case skip
      then show  $?case$  using  $\text{smaller no-f max-lev unfolding no-smaller-confl-def by auto}$ 
  next
    case  $(\text{backtrack } K \ i \ M1 \ M2 \ L \ D \ T)$  note  $\text{confl} = \text{this}(1)$  and  $LD = \text{this}(2)$  and  $\text{decomp} = \text{this}(3)$ 
and
   $\text{undef} = \text{this}(7)$  and  $T = \text{this}(8)$ 
obtain  $c$  where  $M: \text{trail } S = c @ M2 @ \text{Marked } K \ (i+1) \ \# \ M1$ 
  using  $\text{decomp by auto}$ 

show  $?case$ 
proof  $(\text{intro allI impI})$ 
  fix  $M \text{ ia } K' \ M' \ Da$ 

  assume  $M' @ \text{Marked } K' \ \text{ia} \ \# \ M = \text{trail } T$ 
  then have  $tl \ M' @ \text{Marked } K' \ \text{ia} \ \# \ M = M1$ 
    using  $T \text{ decomp undef lev by (cases } M') \text{ (auto simp: cdcl}_W\text{-M-level-inv-decomp)}$ 
  let  $?S' = (\text{cons-trail } (\text{Propagated } L \ (\text{cls-of-ccls } D))$ 
     $(\text{reduce-trail-to } M1 \ (\text{add-learned-cls } (\text{cls-of-ccls } D))$ 
     $(\text{update-backtrack-lvl } i \ (\text{update-conflicting } \text{None } S))))$ 
  assume  $D: Da \in \# \text{ clauses } T$ 
  moreover{
    assume  $Da \in \# \text{ clauses } S$ 
    then have  $\neg M \models_{as} CNot \ Da$  using  $\langle tl \ M' @ \text{Marked } K' \ \text{ia} \ \# \ M = M1 \rangle M \text{ confl undef smaller}$ 
      unfolding no-smaller-confl-def by auto
  }
  moreover {
    assume  $Da: Da = \text{mset-ccls } D$ 
    have  $\neg M \models_{as} CNot \ Da$ 
    proof  $(\text{rule ccontr})$ 
      assume  $\neg ?thesis$ 
      then have  $-L \in \text{lits-of-l } M$ 
        using  $LD$  unfolding  $Da$  by  $(\text{simp add: in-CNot-implies-uminus}(2))$ 
      then have  $-L \in \text{lits-of-l } (\text{Propagated } L \ (\text{mset-ccls } D) \ \# \ M1)$ 
        using  $UnI2 \ \langle tl \ M' @ \text{Marked } K' \ \text{ia} \ \# \ M = M1 \rangle$ 
        by auto
  }

```

```

moreover
  have backtrack  $S \ ?S'$ 
    using backtrack-rule[of  $S$ ] backtrack.hyps
    by (force simp: state-eq-def simp del: state-simp)
  then have cdclW-M-level-inv  $?S'$ 
    using cdclW-consistent-inv[OF - lev] other[OF bj] by (auto intro: cdclW-bj.intros)
  then have no-dup (Propagated  $L$  (mset-ccls  $D$ )  $\#$   $M1$ )
    using decomp undef lev unfolding cdclW-M-level-inv-def by auto
  ultimately show False
    using undef by (auto simp: Marked-Propagated-in-iff-in-lits-of-l)
qed
}
ultimately show  $\neg M \models_{as} CNot\ Da$ 
  using T undef decomp lev unfolding cdclW-M-level-inv-def by fastforce
qed
qed

```

lemma *conflict-no-smaller-conflict-inv*:

```

assumes conflict  $S\ S'$ 
and no-smaller-conflict  $S$ 
shows no-smaller-conflict  $S'$ 
using assms unfolding no-smaller-conflict-def by (fastforce elim: conflictE)

```

lemma *propagate-no-smaller-conflict-inv*:

```

assumes propagate: propagate  $S\ S'$ 
and n-l: no-smaller-conflict  $S$ 
shows no-smaller-conflict  $S'$ 
unfolding no-smaller-conflict-def
proof (intro allI impI)
  fix  $M' K i M'' D$ 
  assume  $M': M'' @ Marked\ K\ i \# M' = trail\ S'$ 
  and  $D \in \# clauses\ S'$ 
  obtain  $M\ N\ U\ k\ C\ L$  where
     $S$ : state  $S = (M, N, U, k, None)$  and
     $S'$ : state  $S' = (Propagated\ L\ (C + \{\#L\# \}) \# M, N, U, k, None)$  and
     $C + \{\#L\# \} \in \# clauses\ S$  and
     $M \models_{as} CNot\ C$  and
    undefined-lit  $M\ L$ 
  using propagate by (auto elim: propagate-high-levelE)
  have  $tl\ M'' @ Marked\ K\ i \# M' = trail\ S$  using  $M'\ S\ S'$ 
    by (metis Pair-inject list.inject list.sel(3) marked-lit.distinct(1) self-append-conv2
      tl-append2)
  then have  $\neg M' \models_{as} CNot\ D$ 
    using  $\langle D \in \# clauses\ S' \rangle$  n-l  $S\ S'$  raw-clauses-def unfolding no-smaller-conflict-def by auto
  then show  $\neg M' \models_{as} CNot\ D$  by auto
qed

```

lemma *cdcl_W-cp-no-smaller-conflict-inv*:

```

assumes propagate: cdclW-cp  $S\ S'$ 
and n-l: no-smaller-conflict  $S$ 
shows no-smaller-conflict  $S'$ 
using assms
proof (induct rule: cdclW-cp.induct)
  case (conflict'  $S\ S'$ )
  then show  $?case$  using conflict-no-smaller-conflict-inv[of  $S\ S'$ ] by blast

```

```

next
  case (propagate' S S')
  then show ?case using propagate-no-smaller-conflict-inv[of S S'] by fastforce
qed

lemma rtrancp-cdclW-cp-no-smaller-conflict-inv:
  assumes propagate: cdclW-cp** S S'
  and n-l: no-smaller-conflict S
  shows no-smaller-conflict S'
  using assms
proof (induct rule: rtrancp-induct)
  case base
  then show ?case by simp
next
  case (step S' S'')
  then show ?case using cdclW-cp-no-smaller-conflict-inv[of S' S''] by fast
qed

lemma trancp-cdclW-cp-no-smaller-conflict-inv:
  assumes propagate: cdclW-cp++ S S'
  and n-l: no-smaller-conflict S
  shows no-smaller-conflict S'
  using assms
proof (induct rule: trancp.induct)
  case (r-into-tranc S S')
  then show ?case using cdclW-cp-no-smaller-conflict-inv[of S S'] by blast
next
  case (tranc-into-tranc S S' S'')
  then show ?case using cdclW-cp-no-smaller-conflict-inv[of S' S''] by fast
qed

lemma full-cdclW-cp-no-smaller-conflict-inv:
  assumes full cdclW-cp S S'
  and n-l: no-smaller-conflict S
  shows no-smaller-conflict S'
  using assms unfolding full-def
  using rtrancp-cdclW-cp-no-smaller-conflict-inv[of S S'] by blast

lemma full1-cdclW-cp-no-smaller-conflict-inv:
  assumes full1 cdclW-cp S S'
  and n-l: no-smaller-conflict S
  shows no-smaller-conflict S'
  using assms unfolding full1-def
  using trancp-cdclW-cp-no-smaller-conflict-inv[of S S'] by blast

lemma cdclW-stgy-no-smaller-conflict-inv:
  assumes cdclW-stgy S S'
  and n-l: no-smaller-conflict S
  and conflict-is-false-with-level S
  and cdclW-M-level-inv S
  shows no-smaller-conflict S'
  using assms
proof (induct rule: cdclW-stgy.induct)
  case (conflict' S')
  then show ?case using full1-cdclW-cp-no-smaller-conflict-inv[of S S'] by blast

```

```

next
case (other' S' S'')
have no-smaller-confl S'
  using cdclW-o-no-smaller-confl-inv[OF other'.hyps(1) other'.prems(3,2,1)]
  not-conflict-not-any-negated-init-clss other'.hyps(2) cdclW-cp.simps by auto
then show ?case using full-cdclW-cp-no-smaller-confl-inv[of S' S''] other'.hyps by blast
qed

lemma is-conflicting-exists-conflict:
  assumes  $\neg(\forall D \in \# \text{init-clss } S' + \text{learned-clss } S'. \neg \text{trail } S' \models_{\text{as}} \text{CNot } D)$ 
  and conflicting S' = None
  shows  $\exists S''. \text{conflict } S' S''$ 
  using assms raw-clauses-def not-conflict-not-any-negated-init-clss by fastforce

lemma cdclW-o-conflict-is-no-clause-is-false:
  fixes S S' :: 'st
  assumes
    cdclW-o S S' and
    lev: cdclW-M-level-inv S and
    max-lev: conflict-is-false-with-level S and
    no-f: no-clause-is-false S and
    no-l: no-smaller-confl S
  shows no-clause-is-false S'
     $\vee (\text{conflicting } S' = \text{None}$ 
       $\longrightarrow (\forall D \in \# \text{clauses } S'. \text{trail } S' \models_{\text{as}} \text{CNot } D$ 
         $\longrightarrow (\exists L. L \in \# D \wedge \text{get-level } (\text{trail } S') L = \text{backtrack-lvl } S')))$ 
  using assms(1,2)
proof (induct rule: cdclW-o-induct-lev2)
case (decide L T) note S = this(1) and undef = this(2) and T = this(4)
show ?case
proof (rule HOL.disjI2, clarify)
fix D
assume D:  $D \in \# \text{clauses } T$  and M-D:  $\text{trail } T \models_{\text{as}} \text{CNot } D$ 
let ?M = trail S
let ?M' = trail T
let ?k = backtrack-lvl S
have  $\neg ?M \models_{\text{as}} \text{CNot } D$ 
  using no-f D S T undef by auto
have  $-L \in \# D$ 
proof (rule ccontr)
assume  $\neg ?thesis$ 
have  $?M \models_{\text{as}} \text{CNot } D$ 
  unfolding true-annots-def Ball-def true-annot-def CNot-def true-cls-def
proof (intro allI impI)
fix x
assume x:  $x \in \{\{\# - L\# \} \mid L. L \in \# D\}$ 

  then obtain L' where  $L': x = \{\# - L'\# \} \mid L' \in \# D$  by auto
  obtain L'' where  $L'' \in \# x$  and lits-of-l (Marked L (?k + 1) # ?M)  $\models_l L''$ 
    using M-D x T undef unfolding true-annots-def Ball-def true-annot-def CNot-def
    true-cls-def Bex-def by auto
  show  $\exists L \in \# x. \text{lits-of-l } ?M \models_l L$  unfolding Bex-def
    using L'(1) L'(2)  $\langle - L \notin \# D \rangle \langle L'' \in \# x \rangle$ 
     $\langle \text{lits-of-l } (\text{Marked } L (\text{backtrack-lvl } S + 1) \# \text{trail } S) \models_l L'' \rangle$  by auto
qed

```



```

    then show False using  $\langle \neg ?M \models_{as} CNot\ D \rangle$  by auto
  qed
  have atm-of  $L \notin \text{atm-of } \langle \text{lits-of-l } ?M \rangle$ 
    using undef defined-lit-map unfolding lits-of-def by fastforce
  then have get-level  $(\text{Marked } L\ (?k + 1)\ \# ?M)\ (-L) = ?k + 1$  by simp
  then show  $\exists La. La \in \# D \wedge \text{get-level } ?M'\ La = \text{backtrack-lvl } T$ 
    using  $\langle -L \in \# D \rangle\ T$  undef by auto
  qed
next
  case resolve
  then show ?case by auto
next
  case skip
  then show ?case by auto
next
  case  $(\text{backtrack } K\ i\ M1\ M2\ L\ D\ T)$  note decomp = this(3) and undef = this(7) and  $T = \text{this}(8)$ 
  show ?case
  proof (rule HOL.disjI2, clarify)
    fix Da
    assume Da:  $Da \in \# \text{clauses } T$ 
    and M-D:  $\text{trail } T \models_{as} CNot\ Da$ 
    obtain c where  $M: \text{trail } S = c @ M2 @ \text{Marked } K\ (i + 1)\ \# M1$ 
      using decomp by auto
    have tr-T:  $\text{trail } T = \text{Propagated } L\ (\text{mset-ccls } D)\ \# M1$ 
      using T decomp undef lev by (auto simp: cdclW-M-level-inv-decomp)
    have backtrack S T
      using backtrack-rule[of S] backtrack.hyps T
      by (force simp del: state-simp simp: state-eq-def)
    then have lev':  $\text{cdcl}_W\text{-M-level-inv } T$ 
      using cdclW-consistent-inv lev other cdclW-bj.backtrack cdclW-o.bj by blast
    then have  $-L \notin \text{lits-of-l } M1$ 
      using lev cdclW-M-level-inv-def Marked-Propagated-in-iff-in-lits-of-l undef by blast
    { assume  $Da \in \# \text{clauses } S$ 
      then have  $\neg M1 \models_{as} CNot\ Da$  using no-l M unfolding no-smaller-confl-def by auto
    }
    moreover {
      assume Da:  $Da = \text{mset-ccls } D$ 
      have  $\neg M1 \models_{as} CNot\ Da$  using  $\langle -L \notin \text{lits-of-l } M1 \rangle$  unfolding Da
        using backtrack.hyps(2) in-CNot-implies-uminus(2) by auto
    }
    ultimately have  $\neg M1 \models_{as} CNot\ Da$ 
      using Da T undef decomp lev by (fastforce simp: cdclW-M-level-inv-decomp)
    then have  $-L \in \# Da$ 
      using M-D  $\langle -L \notin \text{lits-of-l } M1 \rangle\ T$  unfolding tr-T true-annots-true-clss true-clss-def
      by (auto simp: uminus-lit-swap)
    have g-M1:  $\text{get-all-levels-of-marked } M1 = \text{rev } [1..<i+1]$ 
      using lev lev' T decomp undef unfolding cdclW-M-level-inv-def by auto
    have no-dup  $(\text{Propagated } L\ (\text{mset-ccls } D)\ \# M1)$ 
      using lev lev' T decomp undef unfolding cdclW-M-level-inv-def by auto
    then have L:  $\text{atm-of } L \notin \text{atm-of } \langle \text{lits-of-l } M1 \rangle$  unfolding lits-of-def by auto
    have get-level  $(\text{Propagated } L\ (\text{mset-ccls } D)\ \# M1)\ (-L) = i$ 
      using get-level-get-rev-level-get-all-levels-of-marked[OF L,
        of [Propagated L (mset-ccls D)]]
      by (simp add: g-M1 split: if-splits)
    then show  $\exists La. La \in \# Da \wedge \text{get-level } (\text{trail } T)\ La = \text{backtrack-lvl } T$ 

```

```

    using  $\neg L \in \# \text{ Da}$   $T$  decomp undef lev by (auto simp: cdclW-M-level-inv-def)
qed
qed

lemma full1-cdclW-cp-exists-conflict-decompose:
  assumes
    confl:  $\exists D \in \# \text{ clauses } S. \text{ trail } S \models_{as} CNot \ D$  and
    full: full cdclW-cp  $S \ U$  and
    no-confl: conflicting  $S = None$  and
    lev: cdclW-M-level-inv  $S$ 
  shows  $\exists T. \text{ propagate}^{**} \ S \ T \wedge \text{ conflict } T \ U$ 
proof -
  consider (propa)  $\text{ propagate}^{**} \ S \ U$ 
    | (confl)  $T$  where  $\text{ propagate}^{**} \ S \ T$  and  $\text{ conflict } T \ U$ 
  using full unfolding full-def by (blast dest: rtranclp-cdclW-cp-propa-or-propa-confl)
  then show ?thesis
  proof cases
    case confl
    then show ?thesis by blast
  next
    case propa
    then have conflicting  $U = None$  and
      [simp]: learned-clss  $U = \text{ learned-clss } S$  and
      [simp]: init-clss  $U = \text{ init-clss } S$ 
    using no-confl rtranclp-propagate-is-update-trail lev by auto
    moreover
      obtain  $D$  where  $D: D \in \# \text{ clauses } U$  and
        trS:  $\text{ trail } S \models_{as} CNot \ D$ 
      using confl raw-clauses-def by auto
      obtain  $M$  where  $M: \text{ trail } U = M @ \text{ trail } S$ 
      using full rtranclp-cdclW-cp-dropWhile-trail unfolding full-def by meson
      have tr-U:  $\text{ trail } U \models_{as} CNot \ D$ 
      apply (rule true-annots-mono)
      using trS unfolding  $M$  by simp-all
    have  $\exists V. \text{ conflict } U \ V$ 
      using  $\langle \text{ conflicting } U = None \rangle \ D$  raw-clauses-def not-conflict-not-any-negated-init-clss tr-U
      by meson
    then have False using full cdclW-cp.conflict' unfolding full-def by blast
    then show ?thesis by fast
  qed
qed

```

```

lemma full1-cdclW-cp-exists-conflict-full1-decompose:
  assumes
    confl:  $\exists D \in \# \text{ clauses } S. \text{ trail } S \models_{as} CNot \ D$  and
    full: full cdclW-cp  $S \ U$  and
    no-confl: conflicting  $S = None$  and
    lev: cdclW-M-level-inv  $S$ 
  shows  $\exists T \ D. \text{ propagate}^{**} \ S \ T \wedge \text{ conflict } T \ U$ 
     $\wedge \text{ trail } T \models_{as} CNot \ D \wedge \text{ conflicting } U = \text{ Some } D \wedge D \in \# \text{ clauses } S$ 
proof -
  obtain  $T$  where propa:  $\text{ propagate}^{**} \ S \ T$  and confl:  $\text{ conflict } T \ U$ 
  using full1-cdclW-cp-exists-conflict-decompose[OF assms] by blast
  have p: learned-clss  $T = \text{ learned-clss } S$  init-clss  $T = \text{ init-clss } S$ 
  using propa lev rtranclp-propagate-is-update-trail by auto

```

```

have c: learned-clss U = learned-clss T init-clss U = init-clss T
  using conf by (auto elim: conflictE)
obtain D where trail T  $\models_{as}$  CNot D  $\wedge$  conflicting U = Some D  $\wedge$  D  $\in \#$  clauses S
  using conf p c by (fastforce simp: raw-clauses-def elim!: conflictE)
then show ?thesis
  using propa conf by blast
qed

```

lemma *cdcl_W-stgy-no-smaller-confl*:

```

assumes
  cdclW-stgy S S' and
  n-l: no-smaller-confl S and
  conflict-is-false-with-level S and
  cdclW-M-level-inv S and
  no-clause-is-false S and
  distinct-cdclW-state S and
  cdclW-conflicting S
shows no-smaller-confl S'
  using assms
proof (induct rule: cdclW-stgy.induct)
  case (conflict' S')
  show no-smaller-confl S'
    using conflict'.hyps conflict'.prems(1) full1-cdclW-cp-no-smaller-confl-inv by blast
next
  case (other' S' S'')
  have lev': cdclW-M-level-inv S'
    using cdclW-consistent-inv other other'.hyps(1) other'.prems(3) by blast
  show no-smaller-confl S''
    using cdclW-stgy-no-smaller-confl-inv[OF cdclW-stgy.other'[OF other'.hyps(1-3)]]
      other'.prems(1-3) by blast
qed

```

lemma *cdcl_W-stgy-ex-lit-of-max-level*:

```

assumes
  cdclW-stgy S S' and
  n-l: no-smaller-confl S and
  conflict-is-false-with-level S and
  cdclW-M-level-inv S and
  no-clause-is-false S and
  distinct-cdclW-state S and
  cdclW-conflicting S
shows conflict-is-false-with-level S'
  using assms
proof (induct rule: cdclW-stgy.induct)
  case (conflict' S')
  have no-smaller-confl S'
    using conflict'.hyps conflict'.prems(1) full1-cdclW-cp-no-smaller-confl-inv by blast
  moreover have conflict-is-false-with-level S'
    using conflict'.hyps conflict'.prems(2-4)
      rtranclp-cdclW-co-conflict-ex-lit-of-max-level[of S S']
    unfolding full-def full1-def rtranclp-unfold by presburger
  then show ?case by blast
next
  case (other' S' S'')
  have lev': cdclW-M-level-inv S'

```

```

using cdclW-consistent-inv other other'.hyps(1) other'.prems(3) by blast
moreover
  have no-clause-is-false S'
     $\vee$  (conflicting S' = None  $\longrightarrow$  ( $\forall D \in \# \text{clauses } S'. \text{trail } S' \models_{as} CNot \ D$ 
       $\longrightarrow$  ( $\exists L. L \in \# \ D \wedge \text{get-level } (\text{trail } S') \ L = \text{backtrack-lvl } S')$ ))
    using cdclW-o-conflict-is-no-clause-is-false[of S S'] other'.hyps(1) other'.prems(1-4) by fast
moreover {
  assume no-clause-is-false S'
  {
    assume conflicting S' = None
    then have conflict-is-false-with-level S' by auto
    moreover have full cdclW-cp S' S''
      by (metis (no-types) other'.hyps(3))
    ultimately have conflict-is-false-with-level S''
      using rtranclp-cdclW-co-conflict-ex-lit-of-max-level[of S' S''] lev' <no-clause-is-false S'
      by blast
  }
moreover
  {
    assume c: conflicting S'  $\neq$  None
    have conflicting S  $\neq$  None using other'.hyps(1) c
      by (induct rule: cdclW-o-induct) auto
    then have conflict-is-false-with-level S'
      using cdclW-o-conflict-is-false-with-level-inv[OF other'.hyps(1)]
      other'.prems(3,5,6,2) by blast
    moreover have cdclW-cp** S' S'' using other'.hyps(3) unfolding full-def by auto
    then have S' = S'' using c
      by (induct rule: rtranclp-induct)
      (fastforce intro: option.exhaust) +
    ultimately have conflict-is-false-with-level S'' by auto
  }
  ultimately have conflict-is-false-with-level S'' by blast
}
moreover {
  assume
    confl: conflicting S' = None and
    D-L:  $\forall D \in \# \text{clauses } S'. \text{trail } S' \models_{as} CNot \ D$ 
       $\longrightarrow$  ( $\exists L. L \in \# \ D \wedge \text{get-level } (\text{trail } S') \ L = \text{backtrack-lvl } S')$ 
  { assume  $\forall D \in \# \text{clauses } S'. \neg \text{trail } S' \models_{as} CNot \ D$ 
    then have no-clause-is-false S' using confl by simp
    then have conflict-is-false-with-level S'' using calculation(3) by presburger
  }
moreover {
  assume  $\neg(\forall D \in \# \text{clauses } S'. \neg \text{trail } S' \models_{as} CNot \ D)$ 
  then obtain T D where
    propagate** S' T and
    conflict T S'' and
    D: D  $\in \# \text{clauses } S'$  and
    trail S''  $\models_{as} CNot \ D$  and
    conflicting S'' = Some D
    using full1-cdclW-cp-exists-conflict-full1-decompose[OF - - confl]
    other'(3) lev' by (metis (mono-tags, lifting) conflictE state-eq-trail
      trail-update-conflicting)
  obtain M where M: trail S'' = M @ trail S' and nm:  $\forall m \in \text{set } M. \neg \text{is-marked } m$ 
    using rtranclp-cdclW-cp-dropWhile-trail other'(3) unfolding full-def by meson
  }
}

```

```

have btS: backtrack-lvl S'' = backtrack-lvl S'
  using other'.hyps(3) unfolding full-def by (metis rtrancpl-cdclW-cp-backtrack-lvl)
have inv: cdclW-M-level-inv S''
  by (metis (no-types) cdclW-stgy.conflict' cdclW-stgy-consistent-inv full-unfold lev'
      other'.hyps(3))
then have nd: no-dup (trail S'')
  by (metis (no-types) cdclW-M-level-inv-decomp(2))
have conflict-is-false-with-level S''
proof cases
  assume trail S' ⊨as CNot D
  moreover then obtain L where
    L ∈ # D and
    lev-L: get-level (trail S') L = backtrack-lvl S'
    using D-L D by blast
  moreover
    have LS': -L ∈ lits-of-l (trail S')
      using ⟨trail S' ⊨as CNot D⟩ ⟨L ∈ # D⟩ in-CNot-implies-uminus(2) by blast
    { fix x :: ('v, nat, 'v clause) marked-lit and
      xb :: ('v, nat, 'v clause) marked-lit
      assume a1: x ∈ set (trail S') and
        a2: xb ∈ set M and
        a3: (λl. atm-of (lit-of l)) ' set M ∩ (λl. atm-of (lit-of l)) ' set (trail S')
          = {} and
        a4: - L = lit-of x and
        a5: atm-of L = atm-of (lit-of xb)
      moreover have atm-of (lit-of x) = atm-of L
        using a4 by (metis (no-types) atm-of-uminus)
      ultimately have False
        using a5 a3 a2 a1 by auto
    }
  then have atm-of L ∉ atm-of ' lits-of-l M
    using nd LS' unfolding M by (auto simp add: lits-of-def)
  then have get-level (trail S'') L = get-level (trail S') L
    unfolding M by (simp add: lits-of-def)
  ultimately show ?thesis using btS ⟨conflicting S'' = Some D⟩ by auto
next
  assume ¬trail S' ⊨as CNot D
  then obtain L where L ∈ # D and LM: -L ∈ lits-of-l M
    using ⟨trail S'' ⊨as CNot D⟩ unfolding M
    by (auto simp add: true-cls-def M true-annots-def true-annot-def
        split: if-split-asm)
  { fix x :: ('v, nat, 'v clause) marked-lit and
    xb :: ('v, nat, 'v clause) marked-lit
    assume a1: xb ∈ set (trail S') and
      a2: x ∈ set M and
      a3: atm-of L = atm-of (lit-of xb) and
      a4: - L = lit-of x and
      a5: (λl. atm-of (lit-of l)) ' set M ∩ (λl. atm-of (lit-of l)) ' set (trail S')
        = {}
    moreover have atm-of (lit-of xb) = atm-of (- L)
      using a3 by simp
    ultimately have False
      by auto
  }
  then have LS': atm-of L ∉ atm-of ' lits-of-l (trail S')
    using nd ⟨L ∈ # D⟩ LM unfolding M by (auto simp add: lits-of-def)

```

```

show ?thesis
proof cases
  assume ne: get-all-levels-of-marked (trail S') = []
  have backtrack-lvl S'' = 0
    using inv ne nm unfolding cdclW-M-level-inv-def M
    by (simp add: get-all-levels-of-marked-nil-iff-not-is-marked)
  moreover
    have a1: get-level M L = 0
      using nm by auto
    then have get-level (M @ trail S') L = 0
      by (metis LS' get-all-levels-of-marked-nil-iff-not-is-marked
        get-level-skip-beginning-not-marked lits-of-def ne)
    ultimately show ?thesis using ⟨conflicting S'' = Some D⟩ ⟨L ∈ # D⟩ unfolding M
      by auto
  next
    assume ne: get-all-levels-of-marked (trail S') ≠ []
    have hd (get-all-levels-of-marked (trail S')) = backtrack-lvl S'
      using ne lev' M nm unfolding cdclW-M-level-inv-def
      by (cases get-all-levels-of-marked (trail S'))
        (simp-all add: get-all-levels-of-marked-nil-iff-not-is-marked[symmetric])
    moreover have atm-of L ∈ atm-of ' lits-of-l M
      using ⟨¬L ∈ lits-of-l M⟩
      by (simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set lits-of-def)
    ultimately show ?thesis
      using nm ne ⟨L ∈ # D⟩ ⟨conflicting S'' = Some D⟩
        get-level-skip-beginning-hd-get-all-levels-of-marked[OF LS', of M]
        get-level-skip-in-all-not-marked[of rev M L backtrack-lvl S']
      unfolding lits-of-def btS M
      by auto
    qed
  qed
}
ultimately have conflict-is-false-with-level S'' by blast
}
moreover
{
  assume conflicting S' ≠ None
  have no-clause-is-false S' using ⟨conflicting S' ≠ None⟩ by auto
  then have conflict-is-false-with-level S'' using calculation(3) by presburger
}
ultimately show ?case by fast
qed

```

lemma rtrancp-cdcl_W-stgy-no-smaller-confl-inv:

assumes

cdcl_W-stgy** S S' **and**

n-l: no-smaller-confl S **and**

cls-false: conflict-is-false-with-level S **and**

lev: cdcl_W-M-level-inv S **and**

no-f: no-clause-is-false S **and**

dist: distinct-cdcl_W-state S **and**

conflicting: cdcl_W-conflicting S **and**

decomp: all-decomposition-implies-m (init-cls S) (get-all-marked-decomposition (trail S)) **and**

learned: cdcl_W-learned-clause S **and**

alien: no-strange-atm S

shows *no-smaller-confl* $S' \wedge$ *conflict-is-false-with-level* S'
using *assms*(1)
proof (*induct rule: rtranclp-induct*)
case *base*
then show ?*case* **using** *n-l cls-false* **by** *auto*
next
case (*step* $S' S''$) **note** $st = \text{this}(1)$ **and** $cdcl = \text{this}(2)$ **and** $IH = \text{this}(3)$
have *no-smaller-confl* S' **and** *conflict-is-false-with-level* S'
using IH **by** *blast+*
moreover have *cdcl_W-M-level-inv* S'
using st *lev rtranclp-cdcl_W-stgy-rtranclp-cdcl_W*
by (*blast intro: rtranclp-cdcl_W-consistent-inv*)+
moreover have *no-clause-is-false* S'
using st *no-f rtranclp-cdcl_W-stgy-not-non-negated-init-clss* **by** *presburger*
moreover have *distinct-cdcl_W-state* S'
using *rtanclp-distinct-cdcl_W-state-inv*[*of* $S S'$] *lev rtranclp-cdcl_W-stgy-rtranclp-cdcl_W*[*OF* st]
dist **by** *auto*
moreover have *cdcl_W-conflicting* S'
using *rtranclp-cdcl_W-all-inv*(6)[*of* $S S'$] st *alien conflicting decomp dist learned lev*
rtranclp-cdcl_W-stgy-rtranclp-cdcl_W **by** *blast*
ultimately show ?*case*
using *cdcl_W-stgy-no-smaller-confl*[*OF* $cdcl$] *cdcl_W-stgy-ex-lit-of-max-level*[*OF* $cdcl$] **by** *fast*
qed

18.5.7 Final States are Conclusive

lemma *full-cdcl_W-stgy-final-state-conclusive-non-false*:
fixes $S' :: 'st$
assumes *full: full cdcl_W-stgy (init-state N) S'*
and *no-d: distinct-mset-mset (mset-clss N)*
and *no-empty: $\forall D \in \#mset-clss N. D \neq \{\#\}$*
shows (*conflicting* $S' = \text{Some } \{\#\} \wedge$ *unsatisfiable (set-mset (init-clss S'))*)
 \vee (*conflicting* $S' = \text{None} \wedge$ *trail S' \models_{asm} init-clss S'*)
proof –
let ? $S = \text{init-state } N$
have
termi: $\forall S''. \neg \text{cdcl}_W\text{-stgy } S' S''$ and
*step: $\text{cdcl}_W\text{-stgy}^{**} ?S S'$ using full unfolding full-def by auto*
moreover have
learned: cdcl_W-learned-clause S' and
level-inv: cdcl_W-M-level-inv S' and
alien: no-strange-atm S' and
no-dup: distinct-cdcl_W-state S' and
confl: cdcl_W-conflicting S' and
decomp: all-decomposition-implies-m (init-clss S') (get-all-marked-decomposition (trail S'))
using *no-d tranclp-cdcl_W-stgy-tranclp-cdcl_W*[*of* ? $S S'$] *step rtranclp-cdcl_W-all-inv*(1–6)[*of* ? $S S'$]
unfolding *rtranclp-unfold* **by** *auto*
moreover
have $\forall D \in \#mset-clss N. \neg [] \models_{as} C \text{Not } D$ **using** *no-empty* **by** *auto*
then have *confl-k: conflict-is-false-with-level S'*
using *rtranclp-cdcl_W-stgy-no-smaller-confl-inv*[*OF* *step*] *no-d* **by** *auto*
show ?*thesis*
using *cdcl_W-stgy-final-state-conclusive*[*OF* *termi decomp learned level-inv alien no-dup confl*
confl-k] .
qed

lemma *conflict-is-full1-cdcl_W-cp*:
 assumes *cp*: *conflict S S'*
 shows *full1 cdcl_W-cp S S'*
proof –
 have *cdcl_W-cp S S'* and *conflicting S' ≠ None*
 using *cp cdcl_W-cp.intros* by (auto elim!: *conflictE simp: state-eq-def simp del: state-simp*)
 then have *cdcl_W-cp⁺⁺ S S'* by *blast*
 moreover have *no-step cdcl_W-cp S'*
 using *⟨conflicting S' ≠ None⟩* by (metis *cdcl_W-cp-conflicting-not-empty option.exhaust*)
 ultimately show *full1 cdcl_W-cp S S'* unfolding *full1-def* by *blast+*
qed

lemma *cdcl_W-cp-fst-empty-conflicting-false*:
 assumes
 cdcl_W-cp S S' and
 trail S = [] and
 conflicting S ≠ None
 shows *False*
 using *assms* by (induct rule: *cdcl_W-cp.induct*) (auto elim: *propagateE conflictE*)

lemma *cdcl_W-o-fst-empty-conflicting-false*:
 assumes *cdcl_W-o S S'*
 and *trail S = []*
 and *conflicting S ≠ None*
 shows *False*
 using *assms* by (induct rule: *cdcl_W-o.induct*) auto

lemma *cdcl_W-stgy-fst-empty-conflicting-false*:
 assumes *cdcl_W-stgy S S'*
 and *trail S = []*
 and *conflicting S ≠ None*
 shows *False*
 using *assms* apply (induct rule: *cdcl_W-stgy.induct*)
 using *trancpD cdcl_W-cp-fst-empty-conflicting-false* unfolding *full1-def* apply *metis*
 using *cdcl_W-o-fst-empty-conflicting-false* by *blast*
thm *cdcl_W-cp.induct[split-format(complete)]*

lemma *cdcl_W-cp-conflicting-is-false*:
cdcl_W-cp S S' ⟹ conflicting S = Some {#} ⟹ False
 by (induction rule: *cdcl_W-cp.induct*) (auto elim: *propagateE conflictE*)

lemma *rtrancp-cdcl_W-cp-conflicting-is-false*:
cdcl_W-cp⁺⁺ S S' ⟹ conflicting S = Some {#} ⟹ False
 apply (induction rule: *trancp.induct*)
 by (auto dest: *cdcl_W-cp-conflicting-is-false*)

lemma *cdcl_W-o-conflicting-is-false*:
cdcl_W-o S S' ⟹ conflicting S = Some {#} ⟹ False
 by (induction rule: *cdcl_W-o.induct*) auto

lemma *cdcl_W-stgy-conflicting-is-false*:
cdcl_W-stgy S S' ⟹ conflicting S = Some {#} ⟹ False
 apply (induction rule: *cdcl_W-stgy.induct*)

unfolding *full1-def* **apply** (*metis* (*no-types*) *cdcl_W-cp-conflicting-not-empty* *trancplD*)
unfolding *full-def* **by** (*metis* *conflict-with-false-implies-terminated* *other*)

lemma *rtrancpl-cdcl_W-stgy-conflicting-is-false*:

*cdcl_W-stgy^{**} S S' \implies conflicting S = Some {#} \implies S' = S*

apply (*induction rule*: *rtrancpl-induct*)

apply *simp*

using *cdcl_W-stgy-conflicting-is-false* **by** *blast*

lemma *full-cdcl_W-init-clss-with-false-normal-form*:

assumes

$\forall m \in \text{set } M. \neg \text{is-marked } m$ **and**

$E = \text{Some } D$ **and**

$\text{state } S = (M, N, U, 0, E)$

full cdcl_W-stgy S S' **and**

all-decomposition-implies-m (*init-clss S*) (*get-all-marked-decomposition* (*trail S*))

cdcl_W-learned-clause S

cdcl_W-M-level-inv S

no-strange-atm S

distinct-cdcl_W-state S

cdcl_W-conflicting S

shows $\exists M''. \text{state } S' = (M'', N, U, 0, \text{Some } \{ \# \})$

using *assms*(10,9,8,7,6,5,4,3,2,1)

proof (*induction M arbitrary: E D S*)

case *Nil*

then show *?case*

using *rtrancpl-cdcl_W-stgy-conflicting-is-false* **unfolding** *full-def cdcl_W-conflicting-def*

by *fastforce*

next

case (*Cons L M*) **note** *IH = this(1)* **and** *full = this(8)* **and** *E = this(10)* **and** *inv = this(2-7)* **and**
 $S = \text{this}(9)$ **and** $nm = \text{this}(11)$

obtain *K p* **where** $K: L = \text{Propagated } K p$

using *nm* **by** (*cases L*) *auto*

have *every-mark-is-a-conflict S* **using** *inv* **unfolding** *cdcl_W-conflicting-def* **by** *auto*

then have *MpK: M \models_{as} CNot (p - {#K#})* **and** *Kp: K $\in \#$ p*

using *S* **unfolding** *K* **by** *fastforce+*

then have *p: p = (p - {#K#}) + {#K#}*

by (*auto simp add: multiset-eq-iff*)

then have *K': L = Propagated K ((p - {#K#}) + {#K#})*

using *K* **by** *auto*

obtain *p'* **where**

p': hd-raw-trail S = Propagated K p' **and**

pp': mset-cls p' = p

using *hd-raw-trail[of S] S K* **by** (*cases hd-raw-trail S*) *auto*

obtain *raw-D* **where**

raw-D: raw-conflicting S = Some raw-D

using *S E* **by** (*cases raw-conflicting S*) *auto*

then have *raw-DD: mset-ccls raw-D = D*

using *S E* **by** *auto*

consider (*D*) $D = \{ \# \} \mid (D') D \neq \{ \# \}$ **by** *blast*

then show *?case*

proof *cases*

case *D*

then show *?thesis*

using *full* *rtrancpl-cdcl_W-stgy-conflicting-is-false S* **unfolding** *full-def E D* **by** *auto*

```

next
case  $D'$ 
then have no-p: no-step propagate S and no-c: no-step conflict S
  using  $S E$  by (auto elim: propagateE conflictE)
then have no-step cdclW-cp S by (auto simp: cdclW-cp.simps)
have res-skip:  $\exists T. (resolve\ S\ T \wedge no\text{-}step\ skip\ S \wedge full\ cdcl_W\text{-}cp\ T\ T)$ 
   $\vee (skip\ S\ T \wedge no\text{-}step\ resolve\ S \wedge full\ cdcl_W\text{-}cp\ T\ T)$ 
proof cases
  assume  $\neg lit\text{-}of\ L \notin \# D$ 
  then obtain  $T$  where sk: skip S T
    using  $S D' K$  skip-rule unfolding  $E$  by fastforce
  then have res: no-step resolve S
    using  $\langle \neg lit\text{-}of\ L \notin \# D \rangle S D' K$  hd-raw-trail[of  $S$ ] unfolding  $E$ 
    by (auto elim!: skipE resolveE)
  have full cdclW-cp T T
    using sk by (auto intro!: option-full-cdclW-cp elim: skipE)
  then show ?thesis
    using sk res by blast
next
assume  $LD: \neg \neg lit\text{-}of\ L \notin \# D$ 
then have  $D: Some\ D = Some\ ((D - \{\#lit\text{-}of\ L\}) + \{\#lit\text{-}of\ L\})$ 
  by (auto simp add: multiset-eq-iff)

have  $\bigwedge L. get\text{-}level\ M\ L = 0$ 
  by (simp add: nm)
then have get-maximum-level (Propagated K (p - {#K#} + {#K#}) # M) (D - {#-
K#}) = 0
  using  $LD$  get-maximum-level-exists-lit-of-max-level
proof -
  obtain  $L'$  where get-level (L#M) L' = get-maximum-level (L#M) D
    using  $LD$  get-maximum-level-exists-lit-of-max-level[of  $D\ L\#M$ ] by fastforce
  then show ?thesis by (metis (mono-tags)  $K'$  get-level-skip-all-not-marked
    get-maximum-level-exists-lit nm not-gr0)
qed
then obtain  $T$  where sk: resolve S T
  using resolve-rule[of  $S\ K\ p'$  raw- $D$ ]  $S\ p' \langle K \in \# p \rangle$  raw- $D\ LD$ 
  unfolding  $K' D E\ pp'$  raw- $DD$  by auto
then have res: no-step skip S
  using  $LD\ S D' K$  hd-raw-trail[of  $S$ ] unfolding  $E$ 
  by (auto elim!: skipE resolveE)
have full cdclW-cp T T
  using sk by (auto simp: option-full-cdclW-cp elim: resolveE)
then show ?thesis
  using sk res by blast
qed
then have step-s:  $\exists T. cdcl_W\text{-}stgy\ S\ T$ 
  using (no-step cdclW-cp  $S$ ) other' by (meson bj resolve skip)
have get-all-marked-decomposition (L # M) = [([], L#M)]
  using nm unfolding  $K$  apply (induction  $M$  rule: marked-lit-list-induct, simp)
  by (rename-tac  $L\ l\ xs$ , case-tac hd (get-all-marked-decomposition  $xs$ ), auto)+
then have no-b: no-step backtrack S
  using nm  $S$  by (auto elim: backtrackE)
have no-d: no-step decide S
  using  $S E$  by (auto elim: decideE)

```

```

have full-S-S: full cdclW-cp S S
  using S E by (auto simp add: option-full-cdclW-cp)
then have no-f: no-step (full1 cdclW-cp) S
  unfolding full-def full1-def rtrancpl-unfold by (meson trancplD)
obtain T where
  s: cdclW-stgy S T and st: cdclW-stgy** T S'
  using full step-s full unfolding full-def by (metis rtrancpl-unfold trancplD)
have resolve S T ∨ skip S T
  using s no-b no-d res-skip full-S-S cdclW-cp-state-eq-compatible resolve-unique
  skip-unique unfolding cdclW-stgy.simps cdclW-o.simps full-unfold
  full1-def by (blast dest!: trancplD elim!: cdclW-bj.cases)+
then obtain D' where T: state T = (M, N, U, 0, Some D')
  using S E by (auto elim!: skipE resolveE simp: state-eq-def simp del: state-simp)

have st-c: cdclW** S T
  using E T rtrancpl-cdclW-stgy-rtrancpl-cdclW s by blast
have cdclW-conflicting T
  using rtrancpl-cdclW-all-inv(6)[OF st-c inv(6,5,4,3,2,1)] .
show ?thesis
  apply (rule IH[of T])
    using rtrancpl-cdclW-all-inv(6)[OF st-c inv(6,5,4,3,2,1)] apply blast
    using rtrancpl-cdclW-all-inv(5)[OF st-c inv(6,5,4,3,2,1)] apply blast
    using rtrancpl-cdclW-all-inv(4)[OF st-c inv(6,5,4,3,2,1)] apply blast
    using rtrancpl-cdclW-all-inv(3)[OF st-c inv(6,5,4,3,2,1)] apply blast
    using rtrancpl-cdclW-all-inv(2)[OF st-c inv(6,5,4,3,2,1)] apply blast
    using rtrancpl-cdclW-all-inv(1)[OF st-c inv(6,5,4,3,2,1)] apply blast
  apply (metis full-def st full)
  using T E apply blast
  apply auto[]
  using nm by simp
qed
qed

lemma full-cdclW-stgy-final-state-conclusive-is-one-false:
  fixes S' :: 'st
  assumes full: full cdclW-stgy (init-state N) S'
  and no-d: distinct-mset-mset (mset-clss N)
  and empty: {#} ∈# (mset-clss N)
  shows conflicting S' = Some {#} ∧ unsatisfiable (set-mset (init-clss S'))
proof -
  let ?S = init-state N
  have cdclW-stgy** ?S S' and no-step cdclW-stgy S' using full unfolding full-def by auto
  then have plus-or-eq: cdclW-stgy++ ?S S' ∨ S' = ?S unfolding rtrancpl-unfold by auto
  have ∃ S''. conflict ?S S''
    using empty not-conflict-not-any-negated-init-clss[of init-state N] by auto

  then have cdclW-stgy: ∃ S'. cdclW-stgy ?S S'
    using cdclW-cp.conflict'[of ?S] conflict-is-full1-cdclW-cp cdclW-stgy.intros(1) by metis
  have S' ≠ ?S using (no-step cdclW-stgy S') cdclW-stgy by blast

  then obtain St:: 'st where St: cdclW-stgy ?S St and cdclW-stgy** St S'
    using plus-or-eq by (metis (no-types) ⟨cdclW-stgy** ?S S'⟩ converse-rtrancplE)
  have st: cdclW** ?S St
    by (simp add: rtrancpl-unfold ⟨cdclW-stgy ?S St⟩ cdclW-stgy-trancpl-cdclW)

```

```

have  $\exists T. \text{conflict } ?S \ T$ 
  using empty not-conflict-not-any-negated-init-clss[of ?S] by force
then have fullSt: full1 cdclW-cp ?S St
  using St unfolding cdclW-stgy.simps by blast
then have bt: backtrack-lvl St = (0::nat)
  using rtranclp-cdclW-cp-backtrack-lvl unfolding full1-def
  by (fastforce dest!: tranclp-into-rtranclp)
have cls-St: init-clss St = mset-clss N
  using fullSt cdclW-stgy-no-more-init-clss[OF St] by auto
have conflicting St  $\neq$  None
proof (rule ccontr)
  assume conf:  $\neg ?thesis$ 
  obtain E where
    ES: E ! $\in$ ! raw-init-clss St and
    E: mset-clss E = {#}
    using empty cls-St by (metis in-mset-clss-exists-preimage)
  then have  $\exists T. \text{conflict } St \ T$ 
    using empty cls-St conflict-rule[of St E] ES conf unfolding E
    by (auto simp: raw-clauses-def dest: in-mset-clss-exists-preimage)
  then show False using fullSt unfolding full1-def by blast
qed

have 1:  $\forall m \in \text{set } (\text{trail } St). \neg \text{is-marked } m$ 
  using fullSt unfolding full1-def by (auto dest!: tranclp-into-rtranclp
    rtranclp-cdclW-cp-dropWhile-trail)
have 2: full cdclW-stgy St S'
  using cdclW-stgy** St S' no-step cdclW-stgy S' bt unfolding full-def by auto
have 3: all-decomposition-implies-m
  (init-clss St)
  (get-all-marked-decomposition
    (trail St))
  using rtranclp-cdclW-all-inv(1)[OF st] no-d bt by simp
have 4: cdclW-learned-clause St
  using rtranclp-cdclW-all-inv(2)[OF st] no-d bt by simp
have 5: cdclW-M-level-inv St
  using rtranclp-cdclW-all-inv(3)[OF st] no-d bt by simp
have 6: no-strange-atm St
  using rtranclp-cdclW-all-inv(4)[OF st] no-d bt by simp
have 7: distinct-cdclW-state St
  using rtranclp-cdclW-all-inv(5)[OF st] no-d bt by simp
have 8: cdclW-conflicting St
  using rtranclp-cdclW-all-inv(6)[OF st] no-d bt by simp
have init-clss S' = init-clss St and conflicting S' = Some {#}
  using conflicting St  $\neq$  None full-cdclW-init-clss-with-false-normal-form[OF 1, of - - St]
  2 3 4 5 6 7 8 St apply (metis cdclW-stgy** St S' rtranclp-cdclW-stgy-no-more-init-clss)
  using conflicting St  $\neq$  None full-cdclW-init-clss-with-false-normal-form[OF 1, of - - St - -
    S] 2 3 4 5 6 7 8 by (metis bt option.exhaust prod.inject)

moreover have init-clss S' = mset-clss N
  using cdclW-stgy** (init-state N) S' rtranclp-cdclW-stgy-no-more-init-clss by fastforce
moreover have unsatisfiable (set-mset (mset-clss N))
  by (meson empty satisfiable-def true-clss-empty true-clss-def)
ultimately show ?thesis by auto
qed

```

```

lemma full-cdclW-stgy-final-state-conclusive:
  fixes  $S' :: 'st$ 
  assumes full: full cdclW-stgy (init-state N) S' and no-d: distinct-mset-mset (mset-clss N)
  shows (conflicting S' = Some {#}  $\wedge$  unsatisfiable (set-mset (init-clss S')))
     $\vee$  (conflicting S' = None  $\wedge$  trail S'  $\models_{asm}$  init-clss S')
  using assms full-cdclW-stgy-final-state-conclusive-is-one-false
full-cdclW-stgy-final-state-conclusive-non-false by blast

lemma full-cdclW-stgy-final-state-conclusive-from-init-state:
  fixes  $S' :: 'st$ 
  assumes full: full cdclW-stgy (init-state N) S'
  and no-d: distinct-mset-mset (mset-clss N)
  shows (conflicting S' = Some {#}  $\wedge$  unsatisfiable (set-mset (mset-clss N)))
     $\vee$  (conflicting S' = None  $\wedge$  trail S'  $\models_{asm}$  (mset-clss N)  $\wedge$  satisfiable (set-mset (mset-clss N)))
proof –
  have  $N$ : init-clss S' = (mset-clss N)
    using full unfolding full-def by (auto dest: rtrancp-cdclW-stgy-no-more-init-clss)
  consider
    (confl) conflicting S' = Some {#} and unsatisfiable (set-mset (init-clss S'))
  | (sat) conflicting S' = None and trail S'  $\models_{asm}$  init-clss S'
    using full-cdclW-stgy-final-state-conclusive[OF assms] by auto
  then show ?thesis
    proof cases
      case confl
        then show ?thesis by (auto simp: N)
      next
        case sat
          have cdclW-M-level-inv (init-state N) by auto
          then have cdclW-M-level-inv S'
            using full rtrancp-cdclW-stgy-consistent-inv unfolding full-def by blast
          then have consistent-interp (lits-of-l (trail S')) unfolding cdclW-M-level-inv-def by blast
          moreover have lits-of-l (trail S')  $\models_s$  set-mset (init-clss S')
            using sat(2) by (auto simp add: true-annot-def true-annot-def true-clss-def)
          ultimately have satisfiable (set-mset (init-clss S')) by simp
          then show ?thesis using sat unfolding N by blast
        qed
      qed
    end
  end
theory CDCL-W-Termination
imports CDCL-W
begin

context conflict-driven-clause-learningW
begin

```

18.6 Termination

The condition that no learned clause is a tautology is overkill (in the sense that the no-duplicate condition is enough), but we can reuse *simple-clss*.

The invariant contains all the structural invariants that holds,

definition *cdcl_W-all-struct-inv* **where**

cdcl_W-all-struct-inv S \longleftrightarrow

$no\text{-}strange\text{-}atm\ S \ \wedge$
 $cdcl_W\text{-}M\text{-}level\text{-}inv\ S \ \wedge$
 $(\forall s \in \# \text{ learned-clss } S. \neg \text{tautology } s) \ \wedge$
 $distinct\text{-}cdcl_W\text{-}state\ S \ \wedge$
 $cdcl_W\text{-}conflicting\ S \ \wedge$
 $all\text{-}decomposition\text{-}implies\text{-}m\ (init\text{-}clss\ S)\ (get\text{-}all\text{-}marked\text{-}decomposition\ (trail\ S)) \ \wedge$
 $cdcl_W\text{-}learned\text{-}clause\ S$

lemma $cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}inv$:

assumes $cdcl_W\ S\ S'$ **and** $cdcl_W\text{-}all\text{-}struct\text{-}inv\ S$

shows $cdcl_W\text{-}all\text{-}struct\text{-}inv\ S'$

unfolding $cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}def$

proof (*intro HOL.conjI*)

show $no\text{-}strange\text{-}atm\ S'$

using $cdcl_W\text{-}all\text{-}inv[OF\ assms(1)]\ assms(2)$ **unfolding** $cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}def$ **by** *auto*

show $cdcl_W\text{-}M\text{-}level\text{-}inv\ S'$

using $cdcl_W\text{-}all\text{-}inv[OF\ assms(1)]\ assms(2)$ **unfolding** $cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}def$ **by** *fast*

show $distinct\text{-}cdcl_W\text{-}state\ S'$

using $cdcl_W\text{-}all\text{-}inv[OF\ assms(1)]\ assms(2)$ **unfolding** $cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}def$ **by** *fast*

show $cdcl_W\text{-}conflicting\ S'$

using $cdcl_W\text{-}all\text{-}inv[OF\ assms(1)]\ assms(2)$ **unfolding** $cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}def$ **by** *fast*

show $all\text{-}decomposition\text{-}implies\text{-}m\ (init\text{-}clss\ S')\ (get\text{-}all\text{-}marked\text{-}decomposition\ (trail\ S'))$

using $cdcl_W\text{-}all\text{-}inv[OF\ assms(1)]\ assms(2)$ **unfolding** $cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}def$ **by** *fast*

show $cdcl_W\text{-}learned\text{-}clause\ S'$

using $cdcl_W\text{-}all\text{-}inv[OF\ assms(1)]\ assms(2)$ **unfolding** $cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}def$ **by** *fast*

show $\forall s \in \# \text{ learned-clss } S'. \neg \text{tautology } s$

using $assms(1)[THEN\ learned\text{-}clss\text{-}are\text{-}not\text{-}tautologies]\ assms(2)$

unfolding $cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}def$ **by** *fast*

qed

lemma $rtranclp\text{-}cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}inv$:

assumes $cdcl_W^{**}\ S\ S'$ **and** $cdcl_W\text{-}all\text{-}struct\text{-}inv\ S$

shows $cdcl_W\text{-}all\text{-}struct\text{-}inv\ S'$

using $assms$ **by** *induction* (*auto intro: cdcl_W-all-struct-inv-inv*)

lemma $cdcl_W\text{-}stgy\text{-}cdcl_W\text{-}all\text{-}struct\text{-}inv$:

$cdcl_W\text{-}stgy\ S\ T \implies cdcl_W\text{-}all\text{-}struct\text{-}inv\ S \implies cdcl_W\text{-}all\text{-}struct\text{-}inv\ T$

by (*meson cdcl_W-stgy-tranclp-cdcl_W rtranclp-cdcl_W-all-struct-inv-inv rtranclp-unfold*)

lemma $rtranclp\text{-}cdcl_W\text{-}stgy\text{-}cdcl_W\text{-}all\text{-}struct\text{-}inv$:

$cdcl_W\text{-}stgy^{**}\ S\ T \implies cdcl_W\text{-}all\text{-}struct\text{-}inv\ S \implies cdcl_W\text{-}all\text{-}struct\text{-}inv\ T$

by (*induction rule: rtranclp-induct*) (*auto intro: cdcl_W-stgy-cdcl_W-all-struct-inv*)

18.7 No Relearning of a clause

lemma $cdcl_W\text{-}o\text{-}new\text{-}clause\text{-}learned\text{-}is\text{-}backtrack\text{-}step$:

assumes $learned: D \in \# \text{ learned-clss } T$ **and**

$new: D \notin \# \text{ learned-clss } S$ **and**

$cdcl_W: cdcl_W\text{-}o\ S\ T$ **and**

$lev: cdcl_W\text{-}M\text{-}level\text{-}inv\ S$

shows $backtrack\ S\ T \wedge conflicting\ S = Some\ D$

using $cdcl_W\ lev\ learned\ new$

proof (*induction rule: cdcl_W-o-induct-lev2*)

case ($backtrack\ K\ i\ M1\ M2\ L\ C\ T$) **note** $decomp = this(3)$ **and** $undef = this(6)$ **and** $andef = this(7)$

and

$T = \text{this}(8)$ and $D-T = \text{this}(9)$ and $D-S = \text{this}(10)$
then have $D = \text{mset-clss } C$
using *not-gr0 lev by (auto simp: cdcl_W-M-level-inv-decomp)*
then show ?case
using $T \text{ backtrack.hyps}(1-5) \text{ backtrack.intros}[OF \text{ backtrack.hyps}(1,2)] \text{ backtrack.hyps}(3-6)$
by auto
qed auto

lemma *cdcl_W-cp-new-clause-learned-has-backtrack-step:*

assumes *learned: $D \in \# \text{ learned-clss } T$ and*
new: $D \notin \# \text{ learned-clss } S$ and
cdcl_W: cdcl_W-stgy $S \ T$ and
lev: cdcl_W-M-level-inv S
shows $\exists S'. \text{backtrack } S \ S' \wedge \text{cdcl}_W\text{-stgy}^{**} S' \ T \wedge \text{conflicting } S = \text{Some } D$
using *cdcl_W learned new*
proof (*induction rule: cdcl_W-stgy.induct*)
case (*conflict' S'*)
then show ?case
unfolding *full1-def by (metis (mono-tags, lifting) rtranclp-cdcl_W-cp-learned-clause-inv*
trancplp-into-rtranclp)
next
case (*other' $S' \ S''$*)
then have $D \in \# \text{ learned-clss } S'$
unfolding *full-def by (auto dest: rtranclp-cdcl_W-cp-learned-clause-inv)*
then show ?case
using *cdcl_W-o-new-clause-learned-is-backtrack-step[OF - $\langle D \notin \# \text{ learned-clss } S \rangle \langle \text{cdcl}_W\text{-o } S \ S' \rangle]$*
 $\langle \text{full cdcl}_W\text{-cp } S' \ S'' \rangle \text{ lev by (metis cdcl}_W\text{-stgy.conflict' full-unfold r-into-rtranclp}$
 $\text{rtranclp.rtrancl-refl})$
qed

lemma *rtranclp-cdcl_W-cp-new-clause-learned-has-backtrack-step:*

assumes *learned: $D \in \# \text{ learned-clss } T$ and*
new: $D \notin \# \text{ learned-clss } S$ and
*cdcl_W: cdcl_W-stgy^{**} $S \ T$ and*
lev: cdcl_W-M-level-inv S
shows $\exists S' \ S''. \text{cdcl}_W\text{-stgy}^{**} S \ S' \wedge \text{backtrack } S' \ S'' \wedge \text{conflicting } S' = \text{Some } D \wedge$
 $\text{cdcl}_W\text{-stgy}^{**} S'' \ T$
using *cdcl_W learned new*
proof (*induction rule: rtranclp-induct*)
case *base*
then show ?case **by blast**
next
case (*step $T \ U$*) **note** $st = \text{this}(1)$ and $o = \text{this}(2)$ and $IH = \text{this}(3)$ and
 $D-U = \text{this}(4)$ and $D-S = \text{this}(5)$
show ?case
proof (*cases $D \in \# \text{ learned-clss } T$*)
case *True*
then obtain $S' \ S''$ **where**
 $st': \text{cdcl}_W\text{-stgy}^{**} S \ S'$ **and**
 $bt: \text{backtrack } S' \ S''$ **and**
 $confl: \text{conflicting } S' = \text{Some } D$ **and**
 $st'': \text{cdcl}_W\text{-stgy}^{**} S'' \ T$
using $IH \ D-S$ **by metis**
have $\text{cdcl}_W\text{-stgy}^{++} S'' \ U$
using $st'' \ o$ **by force**

```

then show ?thesis
  by (meson bt confl rtrancpl-unfold st')
next
  case False
  have cdclW-M-level-inv T
    using lev rtrancpl-cdclW-stgy-consistent-inv st by blast
  then obtain S' where
    bt: backtrack T S' and
    st': cdclW-stgy** S' U and
    confl: conflicting T = Some D
    using cdclW-cp-new-clause-learned-has-backtrack-step[OF D-U False o]
    by metis
  then have cdclW-stgy** S T and
    backtrack T S' and
    conflicting T = Some D and
    cdclW-stgy** S' U
    using o st by auto
  then show ?thesis by blast
qed
qed

```

```

lemma propagate-no-more-Marked-lit:
  assumes propagate S S'
  shows Marked K i ∈ set (trail S)  $\longleftrightarrow$  Marked K i ∈ set (trail S')
  using assms by (auto elim: propagateE)

```

```

lemma conflict-no-more-Marked-lit:
  assumes conflict S S'
  shows Marked K i ∈ set (trail S)  $\longleftrightarrow$  Marked K i ∈ set (trail S')
  using assms by (auto elim: conflictE)

```

```

lemma cdclW-cp-no-more-Marked-lit:
  assumes cdclW-cp S S'
  shows Marked K i ∈ set (trail S)  $\longleftrightarrow$  Marked K i ∈ set (trail S')
  using assms apply (induct rule: cdclW-cp.induct)
  using conflict-no-more-Marked-lit propagate-no-more-Marked-lit by auto

```

```

lemma rtrancpl-cdclW-cp-no-more-Marked-lit:
  assumes cdclW-cp** S S'
  shows Marked K i ∈ set (trail S)  $\longleftrightarrow$  Marked K i ∈ set (trail S')
  using assms apply (induct rule: rtrancpl-induct)
  using cdclW-cp-no-more-Marked-lit by blast+

```

```

lemma cdclW-o-no-more-Marked-lit:
  assumes cdclW-o S S' and lev: cdclW-M-level-inv S and  $\neg$ decide S S'
  shows Marked K i ∈ set (trail S')  $\longrightarrow$  Marked K i ∈ set (trail S)
  using assms
proof (induct rule: cdclW-o-induct-lev2)
  case backtrack note decomp = this(3) and undef = this(7) and T = this(8)
  then show ?case using lev by (auto simp: cdclW-M-level-inv-decomp)
next
  case (decide L T)
  then show ?case using decide-rule[OF decide.hyps] by blast
qed auto

```


lemma *cdcl_W-new-marked-at-beginning-is-decide:*

assumes *cdcl_W-stgy* $S\ S'$ **and**
lev: cdcl_W-M-level-inv S **and**
trail $S' = M' @ \text{Marked } L\ i \# M$ **and**
trail $S = M$
shows $\exists T. \text{decide } S\ T \wedge \text{no-step } \text{cdcl}_W\text{-cp } S$
using *assms*

proof (*induct rule: cdcl_W-stgy.induct*)

case (*conflict'* S') **note** $st = \text{this}(1)$ **and** $\text{no-dup} = \text{this}(2)$ **and** $S' = \text{this}(3)$ **and** $S = \text{this}(4)$
have *cdcl_W-M-level-inv* S'
using *full1-cdcl_W-cp-consistent-inv no-dup st* **by** *blast*
then have *Marked* $L\ i \in \text{set } (\text{trail } S')$ **and** *Marked* $L\ i \notin \text{set } (\text{trail } S)$
using *no-dup unfolding* $S\ S'$ *cdcl_W-M-level-inv-def* **by** (*auto simp add: rev-image-eqI*)
then have *False*
using *st rtranclp-cdcl_W-cp-no-more-Marked-lit[of S S']*
unfolding *full1-def rtranclp-unfold* **by** *blast*
then show *?case* **by** *fast*

next

case (*other'* $T\ U$) **note** $o = \text{this}(1)$ **and** $ns = \text{this}(2)$ **and** $st = \text{this}(3)$ **and** $\text{no-dup} = \text{this}(4)$ **and**
 $S' = \text{this}(5)$ **and** $S = \text{this}(6)$
have *cdcl_W-M-level-inv* U
by (*metis (full-types) lev cdcl_W.simps cdcl_W-consistent-inv full-def o*
other'.hyps(3) rtranclp-cdcl_W-cp-consistent-inv)
then have *Marked* $L\ i \in \text{set } (\text{trail } U)$ **and** *Marked* $L\ i \notin \text{set } (\text{trail } S)$
using *no-dup unfolding* $S\ S'$ *cdcl_W-M-level-inv-def* **by** (*auto simp add: rev-image-eqI*)
then have *Marked* $L\ i \in \text{set } (\text{trail } T)$
using *st rtranclp-cdcl_W-cp-no-more-Marked-lit* **unfolding** *full-def* **by** *blast*
then show *?case*
using *cdcl_W-o-no-more-Marked-lit[OF o] (Marked L i (trail S) ns lev* **by** *meson*
qed

lemma *cdcl_W-o-is-decide:*

assumes *cdcl_W-o* $S\ T$ **and** *lev: cdcl_W-M-level-inv* S
trail $T = \text{drop } (\text{length } M_0)\ M' @ \text{Marked } L\ i \# H @ M$ **and**
 $\neg (\exists M'. \text{trail } S = M' @ \text{Marked } L\ i \# H @ M)$
shows *decide* $S\ T$
using *assms*

proof (*induction rule: cdcl_W-o-induct-lev2*)

case (*backtrack* $K\ i\ M1\ M2\ L\ D\ T$)
then obtain c **where** $\text{trail } S = c @ M2 @ \text{Marked } K\ (Suc\ i) \# M1$
by *auto*
show *?case*
using *backtrack lev*
apply (*cases drop (length M₀) M'*)
apply (*auto simp: cdcl_W-M-level-inv-decomp*)
using $\langle \text{trail } S = c @ M2 @ \text{Marked } K\ (Suc\ i) \# M1 \rangle$
by (*auto simp: cdcl_W-M-level-inv-decomp*)

next

case *decide*
show *?case* **using** *decide-rule[of S] decide(1-4)* **by** *auto*
qed *auto*

lemma *rtranclp-cdcl_W-new-marked-at-beginning-is-decide:*

assumes *cdcl_W-stgy*** $R\ U$ **and**
trail $U = M' @ \text{Marked } L\ i \# H @ M$ **and**

$trail\ R = M$ **and**
 $cdcl_W\text{-}M\text{-level-inv}\ R$
shows
 $\exists S\ T\ T'.\ cdcl_W\text{-}stgy^{**}\ R\ S \wedge decide\ S\ T \wedge cdcl_W\text{-}stgy^{**}\ T\ U \wedge cdcl_W\text{-}stgy^{**}\ S\ U \wedge$
 $no\text{-}step\ cdcl_W\text{-}cp\ S \wedge trail\ T = Marked\ L\ i\ \# H\ @\ M \wedge trail\ S = H\ @\ M \wedge cdcl_W\text{-}stgy\ S\ T' \wedge$
 $cdcl_W\text{-}stgy^{**}\ T'\ U$
using *assms*
proof (*induct arbitrary: M H M' i rule: rtranclp-induct*)
case *base*
then show *?case* **by** *auto*
next
case (*step T U*) **note** $st = this(1)$ **and** $IH = this(3)$ **and** $s = this(2)$ **and**
 $U = this(4)$ **and** $S = this(5)$ **and** $lev = this(6)$
show *?case*
proof (*cases* $\exists M'.\ trail\ T = M' @ Marked\ L\ i\ \# H @ M$)
case *False*
with s **show** *?thesis* **using** $U\ s\ st\ S$
proof *induction*
case (*conflict' W*) **note** $cp = this(1)$ **and** $nd = this(2)$ **and** $W = this(3)$
then obtain M_0 **where** $trail\ W = M_0 @ trail\ T$ **and** $nmarked: \forall l \in set\ M_0.\ \neg is\text{-}marked\ l$
using *rtranclp-cdcl_W-cp-dropWhile-trail unfolding full1-def rtranclp-unfold* **by** *meson*
then have $MV: M' @ Marked\ L\ i\ \# H @ M = M_0 @ trail\ T$ **unfolding** W **by** *simp*
then have $V: trail\ T = drop\ (length\ M_0)\ (M' @ Marked\ L\ i\ \# H @ M)$
by *auto*
have $takeWhile\ (Not\ o\ is\text{-}marked)\ M' = M_0 @ takeWhile\ (Not\ o\ is\text{-}marked)\ (trail\ T)$
using *arg-cong[OF MV, of takeWhile (Not o is-marked)] nmarked*
by (*simp add: takeWhile-tail*)
from *arg-cong[OF this, of length]* **have** $length\ M_0 \leq length\ M'$
unfolding *length-append* **by** (*metis (no-types, lifting) Nat.le-trans le-add1 length-takeWhile-le*)
then have *False* **using** $nd\ V$ **by** *auto*
then show *?case* **by** *fast*
next
case (*other' T' U*) **note** $o = this(1)$ **and** $ns = this(2)$ **and** $cp = this(3)$ **and** $nd = this(4)$
and $U = this(5)$ **and** $st = this(6)$
obtain M_0 **where** $trail\ U = M_0 @ trail\ T'$ **and** $nmarked: \forall l \in set\ M_0.\ \neg is\text{-}marked\ l$
using *rtranclp-cdcl_W-cp-dropWhile-trail cp unfolding full-def* **by** *meson*
then have $MV: M' @ Marked\ L\ i\ \# H @ M = M_0 @ trail\ T'$ **unfolding** U **by** *simp*
then have $V: trail\ T' = drop\ (length\ M_0)\ (M' @ Marked\ L\ i\ \# H @ M)$
by *auto*
have $takeWhile\ (Not\ o\ is\text{-}marked)\ M' = M_0 @ takeWhile\ (Not\ o\ is\text{-}marked)\ (trail\ T')$
using *arg-cong[OF MV, of takeWhile (Not o is-marked)] nmarked*
by (*simp add: takeWhile-tail*)
from *arg-cong[OF this, of length]* **have** $length\ M_0 \leq length\ M'$
unfolding *length-append* **by** (*metis (no-types, lifting) Nat.le-trans le-add1 length-takeWhile-le*)
then have $tr\text{-}T': trail\ T' = drop\ (length\ M_0)\ M' @ Marked\ L\ i\ \# H @ M$ **using** V **by** *auto*
then have $LT': Marked\ L\ i \in set\ (trail\ T')$ **by** *auto*
moreover
have $cdcl_W\text{-}M\text{-level-inv}\ T$
using $lev\ rtranclp\text{-}cdcl_W\text{-}stgy\text{-}consistent\text{-}inv\ step.hyps(1)$ **by** *blast*
then have $decide\ T\ T'$ **using** $o\ nd\ tr\text{-}T'\ cdcl_W\text{-}o\text{-is-decide}$ **by** *metis*
ultimately have $decide\ T\ T'$ **using** $cdcl_W\text{-}o\text{-no-more-Marked-lit[OF o]$ **by** *blast*
then have $1: cdcl_W\text{-}stgy^{**}\ R\ T$ **and** $2: decide\ T\ T'$ **and** $3: cdcl_W\text{-}stgy^{**}\ T'\ U$
using $st\ other'.prems(4)$

```

    by (metis cdclW-stgy.conflict' cp full-unfold r-into-rtrancp rtrancp.rtrancp-refl)+
  have [simp]: drop (length M0) M' = []
    using ⟨decide T T'⟩ ⟨Marked L i ∈ set (trail T')⟩ nd tr-T'
    by (auto simp add: Cons-eq-append-conv elim: decideE)
  have T': drop (length M0) M' @ Marked L i # H @ M = Marked L i # trail T
    using ⟨decide T T'⟩ ⟨Marked L i ∈ set (trail T')⟩ nd tr-T'
    by (auto elim: decideE)
  have trail T' = Marked L i # trail T
    using ⟨decide T T'⟩ ⟨Marked L i ∈ set (trail T')⟩ tr-T'
    by (auto elim: decideE)
  then have 5: trail T' = Marked L i # H @ M
    using append.simps(1) list.sel(3) local.other'(5) tl-append2 by (simp add: tr-T')
  have 6: trail T = H @ M
    by (metis (no-types) ⟨trail T' = Marked L i # trail T⟩
      ⟨trail T' = drop (length M0) M' @ Marked L i # H @ M⟩ append-Nil list.sel(3) nd
      tl-append2)
  have 7: cdclW-stgy** T U using other'.prems(4) st by auto
  have 8: cdclW-stgy T U cdclW-stgy** U U
    using cdclW-stgy.other'[OF other'(1-3)] by simp-all
  show ?case apply (rule exI[of - T], rule exI[of - T], rule exI[of - U])
    using ns 1 2 3 5 6 7 8 by fast
qed
next
case True
then obtain M' where T: trail T = M' @ Marked L i # H @ M by metis
from IH[OF this S lev] obtain S' S'' S''' where
  1: cdclW-stgy** R S' and
  2: decide S' S'' and
  3: cdclW-stgy** S'' T and
  4: no-step cdclW-cp S' and
  6: trail S'' = Marked L i # H @ M and
  7: trail S' = H @ M and
  8: cdclW-stgy** S' T and
  9: cdclW-stgy S' S''' and
  10: cdclW-stgy** S''' T
    by blast
  have cdclW-stgy** S'' U using s ⟨cdclW-stgy** S'' T⟩ by auto
  moreover have cdclW-stgy** S' U using 8 s by auto
  moreover have cdclW-stgy** S''' U using 10 s by auto
  ultimately show ?thesis apply - apply (rule exI[of - S], rule exI[of - S'])
    using 1 2 4 6 7 8 9 by blast
qed
qed

lemma rtrancp-cdclW-new-marked-at-beginning-is-decide':
  assumes cdclW-stgy** R U and
  trail U = M' @ Marked L i # H @ M and
  trail R = M and
  cdclW-M-level-inv R
  shows ∃ y y'. cdclW-stgy** R y ∧ cdclW-stgy y y' ∧ ¬ (∃ c. trail y = c @ Marked L i # H @ M)
    ∧ (λ a b. cdclW-stgy a b ∧ (∃ c. trail a = c @ Marked L i # H @ M))** y' U
proof -
  fix T'
  obtain S' T T' where
    st: cdclW-stgy** R S' and

```

decide $S' T$ **and**
 TU : $cdcl_W\text{-stgy}^{**} T U$ **and**
no-step $cdcl_W\text{-cp}$ S' **and**
 trT : $trail T = \text{Marked } L i \# H @ M$ **and**
 trS' : $trail S' = H @ M$ **and**
 $S'U$: $cdcl_W\text{-stgy}^{**} S' U$ **and**
 $S'T'$: $cdcl_W\text{-stgy} S' T'$ **and**
 $T'U$: $cdcl_W\text{-stgy}^{**} T' U$
using $rtranclp\text{-}cdcl_W\text{-new-marked-at-beginning-is-decide}[OF\ assms]$ **by** *blast*
have n : $\neg (\exists c. trail S' = c @ \text{Marked } L i \# H @ M)$ **using** trS' **by** *auto*
show *?thesis*
using $rtranclp\text{-trans}[OF\ st]$ $rtranclp\text{-exists-last-with-prop}[of\ cdcl_W\text{-stgy}\ S'\ T' -$
 $\lambda a\ -. \neg (\exists c. trail a = c @ \text{Marked } L i \# H @ M), OF\ S'T'\ T'U\ n]$
by *meson*
qed

lemma *beginning-not-marked-invert*:

assumes $A: M @ A = M' @ \text{Marked } K i \# H$ **and**
 $nm: \forall m \in set\ M. \neg is\text{-marked } m$
shows $\exists M. A = M @ \text{Marked } K i \# H$
proof $-$
have $A = drop\ (length\ M)\ (M' @ \text{Marked } K i \# H)$
using $arg\text{-cong}[OF\ A, of\ drop\ (length\ M)]$ **by** *auto*
moreover **have** $drop\ (length\ M)\ (M' @ \text{Marked } K i \# H) = drop\ (length\ M)\ M' @ \text{Marked } K i \# H$
using nm **by** $(metis\ (no\text{-types},\ lifting)\ A\ drop\text{-Cons}'\ drop\text{-append}\ marked\text{-lit.disc}(1)\ not\text{-gr0}\ nth\text{-append}\ nth\text{-append-length}\ nth\text{-mem}\ zero\text{-less-diff})$
finally **show** *?thesis* **by** *fast*
qed

lemma $cdcl_W\text{-stgy-trail-has-new-marked-is-decide-step}$:

assumes $cdcl_W\text{-stgy}\ S\ T$
 $\neg (\exists c. trail\ S = c @ \text{Marked } L i \# H @ M)$ **and**
 $(\lambda a\ b. cdcl_W\text{-stgy}\ a\ b \wedge (\exists c. trail\ a = c @ \text{Marked } L i \# H @ M))^{**} T\ U$ **and**
 $\exists M'. trail\ U = M' @ \text{Marked } L i \# H @ M$ **and**
 $lev: cdcl_W\text{-M-level-inv}\ S$
shows $\exists S'. decide\ S\ S' \wedge full\ cdcl_W\text{-cp}\ S'\ T \wedge no\text{-step}\ cdcl_W\text{-cp}\ S$
using $assms(3,1,2,4,5)$
proof *induction*
case $(step\ T\ U)$
then **show** *?case* **by** *fastforce*
next
case *base*
then **show** *?case*
proof $(induction\ rule: cdcl_W\text{-stgy.induct})$
case $(conflict'\ T)$ **note** $cp = this(1)$ **and** $nd = this(2)$ **and** $M' = this(3)$ **and** $no\text{-dup} = this(3)$
then **obtain** M' **where** $M': trail\ T = M' @ \text{Marked } L i \# H @ M$ **by** *metis*
obtain M'' **where** $M'': trail\ T = M'' @ trail\ S$ **and** $nm: \forall m \in set\ M''. \neg is\text{-marked } m$
using cp **unfolding** *full1-def*
by $(metis\ rtranclp\text{-}cdcl_W\text{-cp-dropWhile-trail'\ tranclp\text{-into-rtranclp})$
have *False*
using *beginning-not-marked-invert* $[of\ M''\ trail\ S\ M'\ L\ i\ H @ M]\ M'\ nm\ nd$ **unfolding** M''
by *fast*
then **show** *?case* **by** *fast*
next
case $(other'\ T\ U)$ **note** $o = this(1)$ **and** $ns = this(2)$ **and** $cp = this(3)$ **and** $nd = this(4)$

```

    and trU' = this(5)
  have cdclW-cp** T U' using cp unfolding full-def by blast
  from rtrancp-cdclW-cp-dropWhile-trail[OF this]
  have  $\exists M'. \text{trail } T = M' @ \text{Marked } L \ i \ \# \ H @ M$ 
    using trU' beginning-not-marked-invert[of - trail T - L i H @ M] by metis
  then obtain M' where M': trail T = M' @ Marked L i # H @ M
    by auto
  with o lev nd cp ns
  show ?case
  proof (induction rule: cdclW-o-induct-lev2)
    case (decide L) note dec = this(1) and cp = this(5) and ns = this(4)
    then have decide S (cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S))
      using decide.hyps decide.intros[of S] by force
    then show ?case using cp decide.premis by (meson decide-state-eq-compatible ns state-eq-ref
      state-eq-sym)
  next
    case (backtrack K j M1 M2 L' D T) note decomp = this(3) and undef = this(7) and
      T = this(8) and trT = this(12)
    obtain MS3 where MS3: trail S = MS3 @ M2 @ Marked K (Suc j) # M1
      using get-all-marked-decomposition-exists-prepend[OF decomp] by metis
    have tl (M' @ Marked L i # H @ M) = tl M' @ Marked L i # H @ M
      using lev trT T lev undef decomp by (cases M') (auto simp: cdclW-M-level-inv-decomp)
    then have M'': M1 = tl M' @ Marked L i # H @ M
      using arg-cong[OF trT[simplified], of tl] T decomp undef lev
      by (simp add: cdclW-M-level-inv-decomp)
    have False using nd MS3 T undef decomp unfolding M'' by auto
    then show ?case by fast
  qed auto
qed
qed
qed

```

lemma rtrancp-cdcl_W-stgy-with-trail-end-has-trail-end:

```

  assumes (λa b. cdclW-stgy a b ∧ (∃ c. trail a = c @ Marked L i # H @ M))** T U and
  ∃ M'. trail U = M' @ Marked L i # H @ M
  shows ∃ M'. trail T = M' @ Marked L i # H @ M
  using assms by (induction rule: rtrancp-induct) auto

```

lemma remove1-mset-eq-remove1-mset-same:

```

  remove1-mset L D = remove1-mset L' D  $\implies$  L ∈ # D  $\implies$  L = L'
  by (metis diff-single-trivial insert-DiffM multi-drop-mem-not-eq single-eq-single
    union-right-cancel)

```

lemma cdcl_W-o-cannot-learn:

```

  assumes
    cdclW-o y z and
    lev: cdclW-M-level-inv y and
    trM: trail y = c @ Marked Kh i # H and
    DL: D ∉ # learned-clss y and
    LD: L ∈ # D and
    DH: atm-of (remove1-mset L D) ⊆ atm-of 'lits-of-l H and
    LH: atm-of L ∉ atm-of 'lits-of-l H and
    learned: ∀ T. conflicting y = Some T  $\longrightarrow$  trail y ⊨as CNot T and
    z: trail z = c' @ Marked Kh i # H
  shows D ∉ # learned-clss z
  using assms(1-2) trM DL DH LH learned z

```

proof (induction rule: $cdcl_W$ -o-induct-lev2)

case ($backtrack\ K\ j\ M1\ M2\ L'\ D'\ T$) **note** $confl = this(1)$ **and** $LD' = this(2)$ **and** $decomp = this(3)$
and $levL = this(4)$ **and** $levD = this(5)$ **and** $j = this(6)$ **and** $undef = this(7)$ **and** $T = this(8)$ **and**
 $z = this(14)$

obtain $M3$ **where** $M3$: $trail\ y = M3\ @\ M2\ @\ Marked\ K\ (Suc\ j)\ \# \ M1$
using $decomp\ get\ all\ marked\ decomposition\ exists\ prepend$ **by** $metis$
have M : $trail\ y = c\ @\ Marked\ Kh\ i\ \# \ H$ **using** trM **by** $simp$
have H : $get\ all\ levels\ of\ marked\ (trail\ y) = rev\ [1..<1 + backtrack\ lvl\ y]$
using $lev\ unfolding\ cdcl_W\ M\ level\ inv\ def$ **by** $auto$
have $c' @ Marked\ Kh\ i\ \# \ H = Propagated\ L'\ (mset\ ccls\ D')\ \# \ trail\ (reduce\ trail\ to\ M1\ y)$
using $z\ decomp\ undef\ T\ lev$ **by** ($force\ simp$: $cdcl_W\ M\ level\ inv\ def$)
then obtain d **where** d : $M1 = d @ Marked\ Kh\ i\ \# \ H$
by ($metis\ (no\ types)\ decomp\ in\ get\ all\ marked\ decomposition\ trail\ update\ trail\ list.\ inject$
 $list.sel(3)\ marked\ lit.\ distinct(1)\ self\ append\ conv2\ tl\ append2$)
have $i \in set\ (get\ all\ levels\ of\ marked\ (M3\ @\ M2\ @\ Marked\ K\ (Suc\ j)\ \# \ d @ Marked\ Kh\ i\ \# \ H))$
by $auto$
then have $i > 0$ **unfolding** $H[unfolding\ M3\ d]$ **by** $auto$
show $?case$

proof

assume $D \in \# \ learned\ cls\ T$
then have DLD' : $D = mset\ ccls\ D'$
using $DL\ T\ neq0\ conv\ undef\ decomp\ lev$ **by** ($fastforce\ simp$: $cdcl_W\ M\ level\ inv\ def$)
have $L\ cKh$: $atm\ of\ L \in atm\ of\ ' \ lits\ of\ l\ (c @ [Marked\ Kh\ i])$
using $LH\ learned\ M\ DLD'[symmetric]\ confl\ LD'\ LD$

apply ($auto\ simp\ add$: $image\ iff\ dest!$: $in\ CNot\ implies\ uminus$)

apply ($metis\ atm\ of\ uminus$) **done**

have $get\ all\ levels\ of\ marked\ (M3\ @\ M2\ @\ Marked\ K\ (j + 1)\ \# \ M1)$
 $= rev\ [1..<1 + backtrack\ lvl\ y]$

using $lev\ unfolding\ cdcl_W\ M\ level\ inv\ def\ M3$ **by** $auto$

from $arg\ cong[OF\ this,\ of\ \lambda a.\ (Suc\ j) \in set\ a]$ **have** $backtrack\ lvl\ y \geq j$ **by** $auto$

have $DD'[simp]$: $remove1\ mset\ L\ D = mset\ ccls\ D' - \{\#L'\#\}$

proof (rule $ccontr$)

assume DD' : $\neg ?thesis$

then have $L' \in \# \ remove1\ mset\ L\ D$ **using** $DLD'\ LD$ **by** ($metis\ LD'\ in\ remove1\ mset\ neq$)

then have $get\ level\ (trail\ y)\ L' \leq get\ maximum\ level\ (trail\ y)\ (remove1\ mset\ L\ D)$

using $get\ maximum\ level\ ge\ get\ level$ **by** $blast$

moreover {

have $get\ maximum\ level\ (trail\ y)\ (remove1\ mset\ L\ D) =$

$get\ maximum\ level\ H\ (remove1\ mset\ L\ D)$

using $DH\ unfolding\ M$ **by** ($simp\ add$: $get\ maximum\ level\ skip\ beginning$)

moreover

have $get\ all\ levels\ of\ marked\ (trail\ y) = rev\ [1..<1 + backtrack\ lvl\ y]$

using $lev\ unfolding\ cdcl_W\ M\ level\ inv\ def$ **by** $auto$

then have $get\ all\ levels\ of\ marked\ H = rev\ [1..< i]$

unfolding M **by** ($auto\ dest$: $append\ cons\ eq\ upt\ length\ i$
 $simp\ add$: $rev\ swap[symmetric]$)

then have $get\ maximum\ possible\ level\ H < i$

using $get\ maximum\ possible\ level\ max\ get\ all\ levels\ of\ marked[of\ H]\ \langle i > 0 \rangle$ **by** $auto$

ultimately have $get\ maximum\ level\ (trail\ y)\ (remove1\ mset\ L\ D) < i$

by ($metis\ (full\ types)\ dual\ order.\ strict\ trans\ nat\ neq\ iff\ not\ le$
 $get\ maximum\ possible\ level\ ge\ get\ maximum\ level$) }

moreover

```

have L ∈# remove1-mset L' (mset-ccls D')
  using DLD'[symmetric] DD' LD by (metis in-remove1-mset-neg)
then have get-maximum-level (trail y) (remove1-mset L' (mset-ccls D')) ≥
  get-level (trail y) L
  using get-maximum-level-ge-get-level by blast
moreover {
  have get-all-levels-of-marked (c @ [Marked Kh i]) = rev [i..< backtrack-lvl y+1]
    using append-cons-eq-upt-length-i-end[of rev (get-all-levels-of-marked H) i
      rev (get-all-levels-of-marked c) Suc 0 Suc (backtrack-lvl y)] H
  unfolding M apply (auto simp add: rev-swap[symmetric])
    by (metis (no-types, hide-lams) Nil-is-append-conv Suc-le-eq less-Suc-eq list.sel(1)
      rev.simps(2) rev-rev-ident upt-Suc upt-rec)
  have get-level (trail y) L = get-level (c @ [Marked Kh i]) L
    using L-cKh LH unfolding M by simp
  have get-level (c @ [Marked Kh i]) L ≥ i
    using L-cKh levL
      ⟨get-all-levels-of-marked (c @ [Marked Kh i]) = rev [i..<backtrack-lvl y + 1]⟩
      calculation(1,2) by auto
  then have get-level (trail y) L ≥ i
    using M ⟨get-level (trail y) L = get-level (c @ [Marked Kh i]) L⟩ by auto }
moreover
  have get-maximum-level (trail y) (remove1-mset L' (mset-ccls D'))
    < get-level (trail y) L
    using ⟨j ≤ backtrack-lvl y⟩ levL j calculation(1-4) by linarith
  ultimately show False using backtrack.hyps(4) by linarith
qed
then have LL': L = L'
  using LD LD' remove1-mset-eq-remove1-mset-same unfolding DLD'[symmetric] by fast
have nd: no-dup (trail y) using lev unfolding cdclW-M-level-inv-def by auto

{ assume D: remove1-mset L (mset-ccls D') = {#}
  then have j: j = 0 using levD j by (simp add: LL')
  have ∀m ∈ set M1. ¬is-marked m
    using H unfolding M3 j
    by (auto simp add: rev-swap[symmetric] get-all-levels-of-marked-no-marked
      dest!: append-cons-eq-upt-length-i)
  then have False using d by auto
}
moreover {
  assume D[simp]: remove1-mset L (mset-ccls D') ≠ {#}
  have i ≤ j
    using H unfolding M3 d by (auto simp add: rev-swap[symmetric]
      dest: upt-decomp-lt)
  have j > 0 apply (rule ccontr)
    using H ⟨i > 0⟩ unfolding M3 d
    by (auto simp add: rev-swap[symmetric] dest!: upt-decomp-lt)
  obtain L'' where
    L'' ∈# remove1-mset L (mset-ccls D') and
    L''D': get-level (trail y) L'' = get-maximum-level (trail y)
      (remove1-mset L (mset-ccls D'))
    using get-maximum-level-exists-lit-of-max-level[OF D, of trail y] by auto
  have L''M: atm-of L'' ∈ atm-of ' lits-of-l (trail y)
    using get-rev-level-ge-0-atm-of-in[of 0 rev (trail y) L''] ⟨j>0 levD L''D'
      ⟨j ≤ backtrack-lvl y⟩ levL by (simp add: LL' j)
  then have L'' ∈ lits-of-l (Marked Kh i # d)

```

```

proof -
{
  assume  $L''H: \text{atm-of } L'' \in \text{atm-of ' lits-of-l } H$ 
  have  $\text{get-all-levels-of-marked } H = \text{rev } [1..<i]$ 
    using  $H$  unfolding  $M$ 
    by (auto simp add: rev-swap[symmetric] dest!: append-cons-eq-upt-length-i)
  moreover have  $\text{get-level } (\text{trail } y) L'' = \text{get-level } H L''$ 
    using  $L''H$  unfolding  $M$  by simp
  ultimately have  $\text{False}$ 
    using  $\text{levD } \langle j > 0 \rangle \text{ get-rev-level-in-levels-of-marked } [\text{of rev } H \ 0 \ L'] \langle i \leq j \rangle$ 
    unfolding  $L''D$  [symmetric] nd
    by (metis  $L''D' LL' \text{Max-n-upt Nat.le-trans One-nat-def Suc-pred } \langle 0 < i \rangle$ 
       $\text{get-all-levels-of-marked-rev-eq-rev-get-all-levels-of-marked}$ 
       $\text{get-rev-level-less-max-get-all-levels-of-marked } j \text{ lessI list.simps(15)}$ 
       $\text{not-less rev-rev-ident set-upt}$ )
}
moreover
  have  $\text{atm-of } L'' \in \text{atm-of ' lits-of-l } H$ 
    using  $DD' DH \langle L'' \in \# \text{remove1-mset } L (\text{mset-ccls } D') \rangle \text{atm-of-lit-in-atms-of } LL' LD$ 
     $LD'$  by fastforce
  ultimately show ?thesis
    using  $DD' DH \langle L'' \in \# \text{remove1-mset } L (\text{mset-ccls } D') \rangle \text{atm-of-lit-in-atms-of}$ 
    by auto
qed
moreover
  have  $\text{atm-of } L'' \in \text{atms-of } (\text{remove1-mset } L (\text{mset-ccls } D'))$ 
    using  $\langle L'' \in \# \text{remove1-mset } L (\text{mset-ccls } D') \rangle$  by (auto simp: atms-of-def)

  then have  $\text{atm-of } L'' \in \text{atm-of ' lits-of-l } H$ 
    using  $DH$  unfolding  $DD'$  unfolding  $LL'$  by blast
  ultimately have  $\text{False}$ 
    using nd unfolding  $M3$  d  $LL'$  by (auto simp: lits-of-def)
}
ultimately show  $\text{False}$  by blast
qed
qed auto

```

lemma $\text{cdcl}_W\text{-stgy-with-trail-end-has-not-been-learned}$:

```

assumes
   $\text{cdcl}_W\text{-stgy } y \ z$  and
   $\text{cdcl}_W\text{-M-level-inv } y$  and
   $\text{trail } y = c @ \text{Marked } Kh \ i \ \# \ H$  and
   $D \notin \# \text{learned-clss } y$  and
   $LD: L \in \# \ D$  and
   $DH: \text{atms-of } (\text{remove1-mset } L \ D) \subseteq \text{atm-of ' lits-of-l } H$  and
   $LH: \text{atm-of } L \notin \text{atm-of ' lits-of-l } H$  and
   $\forall T. \text{conflicting } y = \text{Some } T \longrightarrow \text{trail } y \models_{\text{as}} C\text{Not } T$  and
   $\text{trail } z = c' @ \text{Marked } Kh \ i \ \# \ H$ 
shows  $D \notin \# \text{learned-clss } z$ 
using assms
proof induction
  case conflict'
  then show ?case
    unfolding full1-def using tranclp-cdclW-cp-learned-clause-inv by auto
next

```


case (*other'* $T\ U$) **note** $o = \text{this}(1)$ **and** $cp = \text{this}(3)$ **and** $lev = \text{this}(4)$ **and** $trY = \text{this}(5)$ **and**
 $notin = \text{this}(6)$ **and** $LD = \text{this}(7)$ **and** $DH = \text{this}(8)$ **and** $LH = \text{this}(9)$ **and** $confl = \text{this}(10)$ **and**
 $trU = \text{this}(11)$
obtain c' **where** c' : $\text{trail } T = c' @ \text{Marked } Kh\ i \# H$
using cp *beginning-not-marked-invert*[*of* - $\text{trail } T\ c' Kh\ i\ H$]
 $\text{rtrancplp-cdcl}_W\text{-cp-dropWhile-trail}$ [*of* $T\ U$] **unfolding** trU *full-def* **by** *fastforce*
show *?case*
using $\text{cdcl}_W\text{-o-cannot-learn}$ [*OF* $o\ lev\ trY\ notin\ LD\ DH\ LH\ confl\ c'$]
 $\text{rtrancplp-cdcl}_W\text{-cp-learned-clause-inv } cp$ **unfolding** *full-def* **by** *auto*
qed

lemma $\text{rtrancplp-cdcl}_W\text{-stgy-with-trail-end-has-not-been-learned}$:

assumes
 $(\lambda a\ b. \text{cdcl}_W\text{-stgy } a\ b \wedge (\exists c. \text{trail } a = c @ \text{Marked } K\ i \# H @ []))^{**} S\ z$ **and**
 $\text{cdcl}_W\text{-all-struct-inv } S$ **and**
 $\text{trail } S = c @ \text{Marked } K\ i \# H$ **and**
 $D \notin \# \text{learned-clss } S$ **and**
 $LD: L \in \# D$ **and**
 $DH: \text{atms-of } (\text{remove1-mset } L\ D) \subseteq \text{atm-of } ' \text{ lits-of-l } H$ **and**
 $LH: \text{atm-of } L \notin \text{atm-of } ' \text{ lits-of-l } H$ **and**
 $\exists c'. \text{trail } z = c' @ \text{Marked } K\ i \# H$
shows $D \notin \# \text{learned-clss } z$
using *assms*(1-4,8)
proof (*induction rule*: rtrancplp-induct)
case *base*
then show *?case* **by** *auto*[1]
next
case (*step* $T\ U$) **note** $st = \text{this}(1)$ **and** $s = \text{this}(2)$ **and** $IH = \text{this}(3)$ [*OF* $\text{this}(4-6)$]
and $lev = \text{this}(4)$ **and** $trS = \text{this}(5)$ **and** $DL-S = \text{this}(6)$ **and** $trU = \text{this}(7)$
obtain c **where** c : $\text{trail } T = c @ \text{Marked } K\ i \# H$ **using** s **by** *auto*
obtain c' **where** c' : $\text{trail } U = c' @ \text{Marked } K\ i \# H$ **using** trU **by** *blast*
have $\text{cdcl}_W^{**} S\ T$
proof –
have $\forall p\ pa. \exists s\ sa. \forall sb\ sc\ sd\ se. (\neg p^{**} (sb::'st)\ sc \vee p\ s\ sa \vee pa^{**} sb\ sc)$
 $\wedge (\neg pa\ s\ sa \vee \neg p^{**} sd\ se \vee pa^{**} sd\ se)$
by (*metis* (*no-types*) *mono-rtrancplp*)
then have $\text{cdcl}_W\text{-stgy}^{**} S\ T$
using st **by** *blast*
then show *?thesis*
using $\text{rtrancplp-cdcl}_W\text{-stgy-rtrancplp-cdcl}_W$ **by** *blast*
qed
then have lev' : $\text{cdcl}_W\text{-all-struct-inv } T$
using $\text{rtrancplp-cdcl}_W\text{-all-struct-inv-inv}$ [*of* $S\ T$] lev **by** *auto*
then have $confl'$: $\forall Ta. \text{conflicting } T = \text{Some } Ta \longrightarrow \text{trail } T \models_{as} CNot\ Ta$
unfolding $\text{cdcl}_W\text{-all-struct-inv-def}$ $\text{cdcl}_W\text{-conflicting-def}$ **by** *blast*
show *?case*
apply (*rule* $\text{cdcl}_W\text{-stgy-with-trail-end-has-not-been-learned}$ [*OF* - - $c - LD\ DH\ LH\ confl'\ c'$])
using $s\ lev'\ IH\ c$ **unfolding** $\text{cdcl}_W\text{-all-struct-inv-def}$ **by** *blast* +
qed

lemma $\text{cdcl}_W\text{-stgy-new-learned-clause}$:

assumes $\text{cdcl}_W\text{-stgy } S\ T$ **and**
 lev : $\text{cdcl}_W\text{-M-level-inv } S$ **and**
 $E \notin \# \text{learned-clss } S$ **and**
 $E \in \# \text{learned-clss } T$

shows $\exists S'. \text{ backtrack } S S' \wedge \text{ conflicting } S = \text{Some } E \wedge \text{ full cdcl}_W\text{-cp } S' T$
using *assms*
proof *induction*
case *conflict'*
then show ?case **unfolding** *full1-def* **by** (*auto dest: tranclp-cdcl_W-cp-learned-clause-inv*)
next
case (*other' T U*) **note** *o = this(1)* **and** *cp = this(3)* **and** *not-yet = this(5)* **and** *learned = this(6)*
have *E ∈# learned-clss T*
using *learned cp rtranclp-cdcl_W-cp-learned-clause-inv* **unfolding** *full-def* **by** *auto*
then have *backtrack S T* **and** *conflicting S = Some E*
using *cdcl_W-o-new-clause-learned-is-backtrack-step[OF - not-yet o]* *lev* **by** *blast+*
then show ?case **using** *cp* **by** *blast*
qed

lemma *cdcl_W-stgy-no-relearned-clause:*

assumes
invR: cdcl_W-all-struct-inv R **and**
*st': cdcl_W-stgy** R S* **and**
bt: backtrack S T **and**
confl: raw-conflicting S = Some E **and**
already-learned: mset-ccls E ∈# clauses S **and**
R: trail R = []
shows *False*
proof –
have *M-lev: cdcl_W-M-level-inv R*
using *invR* **unfolding** *cdcl_W-all-struct-inv-def* **by** *auto*
have *cdcl_W-M-level-inv S*
using *M-lev assms(2) rtranclp-cdcl_W-stgy-consistent-inv* **by** *blast*
with *bt* **obtain** *L M1 M2-loc K i* **where**
T: T ~ cons-trail (Propagated L (cls-of-ccls E))
(reduce-trail-to M1 (add-learned-cls (cls-of-ccls E)
(update-backtrack-lvl i (update-conflicting None S))))
and
decomp: (Marked K (Suc i) # M1, M2-loc) ∈
set (get-all-marked-decomposition (trail S)) **and**
LD: L ∈# mset-ccls E **and**
k: get-level (trail S) L = backtrack-lvl S **and**
level: get-level (trail S) L = get-maximum-level (trail S) (mset-ccls E) **and**
confl-S: raw-conflicting S = Some E **and**
i: i = get-maximum-level (trail S) (remove1-mset L (mset-ccls E)) **and**
undef: undefined-lit M1 L
using *confl* **by** (*induction rule: backtrack-induction-lev2*) *fastforce*
obtain *M2* **where**
M: trail S = M2 @ Marked K (Suc i) # M1
using *get-all-marked-decomposition-exists-prepend[OF decomp]* **unfolding** *i* **by** (*metis append-assoc*)
let ?E = *mset-ccls E*
let ?E' = *remove1-mset L ?E*
have *invS: cdcl_W-all-struct-inv S*
using *invR rtranclp-cdcl_W-all-struct-inv-inv rtranclp-cdcl_W-stgy-rtranclp-cdcl_W st'* **by** *blast*
then have *confl: cdcl_W-conflicting S* **unfolding** *cdcl_W-all-struct-inv-def* **by** *blast*
then have *trail S ⊨_{as} CNot ?E* **unfolding** *cdcl_W-conflicting-def confl-S* **by** *auto*
then have *MD: trail S ⊨_{as} CNot ?E* **by** *auto*
then have *MD': trail S ⊨_{as} CNot ?E'* **using** *true-annot-CNot-remove1-mset-remove1-mset* **by** *blast*
have *lev': cdcl_W-M-level-inv S* **using** *invS* **unfolding** *cdcl_W-all-struct-inv-def* **by** *blast*

have *get-lvls-M*: *get-all-levels-of-marked* (*trail S*) = *rev* [*1..<Suc* (*backtrack-lvl S*)]
using *lev'* **unfolding** *cdcl_W-M-level-inv-def* **by** *auto*

have *lev*: *cdcl_W-M-level-inv* *R* **using** *invR* **unfolding** *cdcl_W-all-struct-inv-def* **by** *blast*
then have *vars-of-D*: *atms-of* ?*E'* \subseteq *atm-of* ' *lits-of-l* *M1*
using *backtrack-atms-of-D-in-M1* [*OF* *lev'* *undef* - *decomp* - - *T*] *conft-S* *conf T* *decomp k* *level*
lev' i *undef* **unfolding** *cdcl_W-conflicting-def* **by** (*auto simp*: *cdcl_W-M-level-inv-decomp*)
have *no-dup* (*trail S*) **using** *lev'* **by** (*auto simp*: *cdcl_W-M-level-inv-decomp*)
have *vars-in-M1*:
 $\forall x \in \text{atms-of } ?E'. x \notin \text{atm-of ' lits-of-l } (M2 \text{ @ } [\text{Marked } K \text{ (} i + 1)])$
unfolding *Set.Ball-def* **apply** (*intro impI allI*)
apply (*rule vars-of-D distinct-atms-of-incl-not-in-other*[*of*
 $M2 \text{ @ } \text{Marked } K \text{ (} i + 1) \# [] M1 ?E'$])
using $\langle \text{no-dup } (\text{trail } S) \rangle M \text{ vars-of-D}$ **by** *simp-all*
have *M1-D*: *M1* \models_{as} *CNot* ?*E'*
using *vars-in-M1 true-annots-remove-if-notin-vars*[*of* $M2 \text{ @ } \text{Marked } K \text{ (} i + 1) \# [] M1 \text{ CNot } ?E'$]
MD' M **by** *simp*

have *get-lvls-M*: *get-all-levels-of-marked* (*trail S*) = *rev* [*1..<Suc* (*backtrack-lvl S*)]
using *lev'* **unfolding** *cdcl_W-M-level-inv-def* **by** *auto*
then have *backtrack-lvl S* > 0 **unfolding** *M* **by** (*auto split*: *if-split-asm simp add*: *upt.simps*(2))

obtain *M1' K' Ls* **where**
M': *trail S* = *Ls* @ *Marked K'* (*backtrack-lvl S*) # *M1'* **and**
Ls: $\forall l \in \text{set } Ls. \neg \text{is-marked } l$ **and**
 $\text{set } M1 \subseteq \text{set } M1'$
proof –
let ?*Ls* = *takeWhile* (*Not o is-marked*) (*trail S*)
have *MLs*: *trail S* = ?*Ls* @ *dropWhile* (*Not o is-marked*) (*trail S*)
by *auto*
have *dropWhile* (*Not o is-marked*) (*trail S*) $\neq []$ **unfolding** *M* **by** *auto*
moreover
from *hd-dropWhile*[*OF this*] **have** *is-marked*(*hd* (*dropWhile* (*Not o is-marked*) (*trail S*)))
by *simp*
ultimately
obtain *K' K'k* **where**
K'k: *dropWhile* (*Not o is-marked*) (*trail S*)
= *Marked K' K'k* # *tl* (*dropWhile* (*Not o is-marked*) (*trail S*))
by (*cases* *dropWhile* (*Not o is-marked*) (*trail S*);
cases *hd* (*dropWhile* (*Not o is-marked*) (*trail S*)))
simp-all
moreover have $\forall l \in \text{set } ?Ls. \neg \text{is-marked } l$ **using** *set-takeWhileD* **by** *force*
moreover
have *get-all-levels-of-marked* (*trail S*)
= *K'k* # *get-all-levels-of-marked*(*tl* (*dropWhile* (*Not o is-marked*) (*trail S*)))
apply (*subst MLs, subst K'k*)
using *calculation*(2) **by** (*auto simp add*: *get-all-levels-of-marked-no-marked*)
then have *K'k* = *backtrack-lvl S*
using *calculation*(2) **by** (*auto split*: *if-split-asm simp add*: *get-lvls-M upt.simps*(2))
moreover have $\text{set } M1 \subseteq \text{set } (\text{tl } (\text{dropWhile } (\text{Not } o \text{ is-marked}) (\text{trail } S)))$
unfolding *M* **by** (*induction M2*) *auto*
ultimately show ?*thesis* **using** *that MLs* **by** *metis*
qed

have *get-lvls-M*: *get-all-levels-of-marked* (*trail S*) = *rev* [*1..<Suc* (*backtrack-lvl S*)]

```

  using lev' unfolding cdclW-M-level-inv-def by auto
then have backtrack-lvl S > 0 unfolding M by (auto split: if-split-asm simp add: upt.simps(2) i)

have M1'-D: M1' ⊨as CNot ?E' using M1-D ⟨set M1 ⊆ set M1'⟩ by (auto intro: true-annots-mono)
have -L ∈ lits-of-l (trail S) using conf confl-S LD unfolding cdclW-conflicting-def
  by (auto simp: in-CNot-implies-uminus)
have lvs-M1': get-all-levels-of-marked M1' = rev [1..backtrack-lvl S]
  using get-lvs-M Ls by (auto simp add: get-all-levels-of-marked-no-marked M' upt.simps(2)
    split: if-split-asm)
have L-notin: atm-of L ∈ atm-of ' lits-of-l Ls ∨ atm-of L = atm-of K'
  proof (rule ccontr)
    assume ¬ ?thesis
    then have atm-of L ∉ atm-of ' lits-of-l (Marked K' (backtrack-lvl S) # rev Ls) by simp
    then have get-level (trail S) L = get-level M1' L
      unfolding M' by auto
    then show False using get-level-in-levels-of-marked[of M1' L] ⟨backtrack-lvl S > 0⟩
      unfolding k lvs-M1' by auto
  qed
obtain Y Z where
  RY: cdclW-stgy** R Y and
  YZ: cdclW-stgy Y Z and
  nt: ¬ (∃ c. trail Y = c @ Marked K' (backtrack-lvl S) # M1' @ []) and
  Z: (λa b. cdclW-stgy a b ∧ (∃ c. trail a = c @ Marked K' (backtrack-lvl S) # M1' @ []))** Z S
  using rtranclp-cdclW-new-marked-at-beginning-is-decide'[OF st' - lev, of Ls K'
    backtrack-lvl S M1' []] unfolding R M' by auto
have [simp]: cdclW-M-level-inv Y
  using RY lev rtranclp-cdclW-stgy-consistent-inv by blast
obtain M' where trZ: trail Z = M' @ Marked K' (backtrack-lvl S) # M1'
  using rtranclp-cdclW-stgy-with-trail-end-has-trail-end[OF Z] M' by auto
have no-dup (trail Y)
  using RY lev rtranclp-cdclW-stgy-consistent-inv unfolding cdclW-M-level-inv-def by blast
then obtain Y' where
  dec: decide Y Y' and
  Y'Z: full cdclW-cp Y' Z and
  no-step cdclW-cp Y
  using cdclW-stgy-trail-has-new-marked-is-decide-step[OF YZ nt Z] M' by auto
have trY: trail Y = M1'
  proof -
    obtain M' where M: trail Z = M' @ Marked K' (backtrack-lvl S) # M1'
      using rtranclp-cdclW-stgy-with-trail-end-has-trail-end[OF Z] M' by auto
    obtain M'' where M'': trail Z = M'' @ trail Y' and ∀ m ∈ set M''. ¬ is-marked m
      using Y'Z rtranclp-cdclW-cp-dropWhile-trail' unfolding full-def by blast
    obtain M''' where trail Y' = M''' @ Marked K' (backtrack-lvl S) # M1'
      using M'' unfolding M
      by (metis (no-types, lifting) ⟨∀ m ∈ set M''. ¬ is-marked m⟩ beginning-not-marked-invert)
    then show ?thesis using dec nt by (induction M''') (auto elim: decideE)
  qed
have Y-CT: conflicting Y = None using ⟨decide Y Y'⟩ by (auto elim: decideE)
have cdclW** R Y by (simp add: RY rtranclp-cdclW-stgy-rtranclp-cdclW)
then have init-clss Y = init-clss R using rtranclp-cdclW-init-clss[of R Y] M-lev by auto
{ assume DL: mset-ccls E ∈ # clauses Y
  have atm-of L ∉ atm-of ' lits-of-l M1
    apply (rule backtrack-lit-skipped[of S])
    using decomp i k lev' unfolding cdclW-M-level-inv-def by auto
  then have LM1: undefined-lit M1 L

```

```

    by (metis Marked-Propagated-in-iff-in-lits-of-l atm-of-uminus image-eqI)
have L-trY: undefined-lit (trail Y) L
  using L-notin ⟨no-dup (trail S)⟩ unfolding defined-lit-map trY M'
  by (auto simp add: image-iff lits-of-def)
obtain E' where
  E': E' !∈! raw-clauses Y and
  EE': mset-cls E' = mset-ccls E
  using DL in-mset-clss-exists-preimage by blast
have Ex (propagate Y)
  using propagate-rule[of Y E' L] DL M1'-D L-trY Y-CT trY LD E'
  by (auto simp: EE')
then have False using ⟨no-step cdclW-cp Y⟩ propagate' by blast
}
moreover {
  assume DL: mset-ccls E ∉# clauses Y
  have lY-lZ: learned-clss Y = learned-clss Z
    using dec Y'Z rtranclp-cdclW-cp-learned-clause-inv[of Y' Z] unfolding full-def
    by (auto elim: decideE)
  have invZ: cdclW-all-struct-inv Z
    by (meson RY YZ invR r-into-rtranclp rtranclp-cdclW-all-struct-inv-inv
        rtranclp-cdclW-stgy-rtranclp-cdclW)
  have n: mset-ccls E ∉# learned-clss Z
    using DL lY-lZ YZ unfolding raw-clauses-def by auto
  have ?E ∉# learned-clss S
    apply (rule rtranclp-cdclW-stgy-with-trail-end-has-not-been-learned[OF Z invZ trZ])
    apply (simp add: n)
    using LD apply simp
    apply (metis (no-types, lifting) ⟨set M1 ⊆ set M1'⟩ image-mono order-trans
        vars-of-D lits-of-def)
    using L-notin ⟨no-dup (trail S)⟩ unfolding M' by (auto simp add: image-iff lits-of-def)
  then have False
    using already-learned DL confl st' M-lev rtranclp-cdclW-stgy-no-more-init-clss[of R S]
    unfolding M'
    by (simp add: ⟨init-clss Y = init-clss R⟩ raw-clauses-def confl-S
        rtranclp-cdclW-stgy-no-more-init-clss)
}
ultimately show False by blast
qed

```

lemma *rtranclp-cdcl_W-stgy-distinct-mset-clauses:*

```

assumes
  invR: cdclW-all-struct-inv R and
  st: cdclW-stgy** R S and
  dist: distinct-mset (clauses R) and
  R: trail R = []
shows distinct-mset (clauses S)
using st
proof (induction)
  case base
  then show ?case using dist by simp
next
  case (step S T) note st = this(1) and s = this(2) and IH = this(3)
  from s show ?case
  proof (cases rule: cdclW-stgy.cases)
    case conflict'

```

```

then show ?thesis
  using IH unfolding full1-def by (auto dest: trancpl-cdclW-cp-no-more-clauses)
next
case (other' S') note o = this(1) and full = this(3)
have [simp]: clauses T = clauses S'
  using full unfolding full-def by (auto dest: rtrancpl-cdclW-cp-no-more-clauses)
show ?thesis
  using o IH
  proof (cases rule: cdclW-o-rule-cases)
    case backtrack
    moreover
    have cdclW-all-struct-inv S
      using invR rtrancpl-cdclW-stgy-cdclW-all-struct-inv st by blast
    then have cdclW-M-level-inv S
      unfolding cdclW-all-struct-inv-def by auto
    ultimately obtain E where
      conflicting S = Some E and
      cls-S': clauses S' = {#E#} + clauses S
    using ⟨cdclW-M-level-inv S⟩
    by (induction rule: backtrack-induction-lev2) (auto simp: cdclW-M-level-inv-decomp)
    then have E ∉ # clauses S
      using cdclW-stgy-no-relearned-clause R invR local.backtrack st by blast
    then show ?thesis using IH by (simp add: distinct-mset-add-single cls-S')
  qed (auto elim: decideE skipE resolveE)
qed
qed

```

```

lemma cdclW-stgy-distinct-mset-clauses:
  assumes
    st: cdclW-stgy** (init-state N) S and
    no-duplicate-clause: distinct-mset (mset-clss N) and
    no-duplicate-in-clause: distinct-mset-mset (mset-clss N)
  shows distinct-mset (clauses S)
  using rtrancpl-cdclW-stgy-distinct-mset-clauses[OF - st] assms
  by (auto simp: cdclW-all-struct-inv-def distinct-cdclW-state-def)

```

18.8 Decrease of a measure

```

fun cdclW-measure where
  cdclW-measure S =
    [(?::nat) ^ (card (atms-of-mm (init-clss S))) - card (set-mset (learned-clss S)),
     if conflicting S = None then 1 else 0,
     if conflicting S = None then card (atms-of-mm (init-clss S)) - length (trail S)
     else length (trail S)
    ]

```

```

lemma length-model-le-vars-all-inv:
  assumes cdclW-all-struct-inv S
  shows length (trail S) ≤ card (atms-of-mm (init-clss S))
  using assms length-model-le-vars[of S] unfolding cdclW-all-struct-inv-def
  by (auto simp: cdclW-M-level-inv-decomp)
end

```

```

context conflict-driven-clause-learningW
begin

```

```

lemma learned-clss-less-upper-bound:
  fixes  $S :: 'st$ 
  assumes
    distinct-cdclW-state S and
     $\forall s \in \# \text{ learned-clss } S. \neg \text{tautology } s$ 
  shows  $\text{card}(\text{set-mset } (\text{learned-clss } S)) \leq 3 \wedge \text{card } (\text{atms-of-mm } (\text{learned-clss } S))$ 
proof –
  have  $\text{set-mset } (\text{learned-clss } S) \subseteq \text{simple-clss } (\text{atms-of-mm } (\text{learned-clss } S))$ 
  apply (rule simplified-in-simple-clss)
  using assms unfolding distinct-cdclW-state-def by auto
  then have  $\text{card}(\text{set-mset } (\text{learned-clss } S))$ 
     $\leq \text{card } (\text{simple-clss } (\text{atms-of-mm } (\text{learned-clss } S)))$ 
  by (simp add: simple-clss-finite card-mono)
  then show ?thesis
  by (meson atms-of-ms-finite simple-clss-card finite-set-mset order-trans)
qed

lemma lexn3[intro!, simp]:
   $a < a' \vee (a = a' \wedge b < b') \vee (a = a' \wedge b = b' \wedge c < c')$ 
   $\implies ([a::nat, b, c], [a', b', c']) \in \text{lexn } \{(x, y). x < y\} \ 3$ 
  apply auto
  unfolding lexn-conv apply fastforce
  unfolding lexn-conv apply auto
  apply (metis append.simps(1) append.simps(2)) +
  done

lemma cdclW-measure-decreasing:
  fixes  $S :: 'st$ 
  assumes
    cdclW S S' and
    no-restart:
       $\neg(\text{learned-clss } S \subseteq \# \text{ learned-clss } S' \wedge [] = \text{trail } S' \wedge \text{conflicting } S' = \text{None})$ 
    and
    no-forget:  $\text{learned-clss } S \subseteq \# \text{ learned-clss } S'$  and
    no-relearn:  $\bigwedge S'. \text{backtrack } S S' \implies \forall T. \text{conflicting } S = \text{Some } T \longrightarrow T \notin \# \text{ learned-clss } S$ 
    and
    alien: no-strange-atm S and
    M-level: cdclW-M-level-inv S and
    no-taut:  $\forall s \in \# \text{ learned-clss } S. \neg \text{tautology } s$  and
    no-dup: distinct-cdclW-state S and
    confl: cdclW-conflicting S
  shows  $(\text{cdcl}_W\text{-measure } S', \text{cdcl}_W\text{-measure } S) \in \text{lexn } \{(a, b). a < b\} \ 3$ 
  using assms(1) M-level assms(2,3)
proof (induct rule: cdclW-all-induct-lev2)
  case (propagate C L) note conf = this(1) and undef = this(5) and T = this(6)
  have propa: propagate S (cons-trail (Propagated L C) S)
  using propagate-rule[OF propagate.hyps(1,2)] propagate.hyps by auto
  then have no-dup': no-dup (Propagated L (mset-cls C) # trail S)
  using M-level cdclW-M-level-inv-decomp(2) undef defined-lit-map by auto

  let ?N = init-clss S
  have no-strange-atm (cons-trail (Propagated L C) S)
  using alien cdclW.propagate cdclW-no-strange-atm-inv propa M-level by blast
  then have atm-of ' lits-of-l (Propagated L (mset-cls C) # trail S)
     $\subseteq \text{atms-of-mm } (\text{init-clss } S)$ 

```

```

    using undef unfolding no-strange-atm-def by auto
  then have card (atm-of ' lits-of-l (Propagated L (mset-cls C) # trail S))
    ≤ card (atms-of-mm (init-clss S))
    by (meson atms-of-ms-finite card-mono finite-set-mset)
  then have length (Propagated L (mset-cls C) # trail S) ≤ card (atms-of-mm ?N)
    using no-dup-length-eq-card-atm-of-lits-of-l no-dup' by fastforce
  then have H: card (atms-of-mm (init-clss S)) - length (trail S)
    = Suc (card (atms-of-mm (init-clss S)) - Suc (length (trail S)))
    by simp
  show ?case using conf T undef by (auto simp: H)
next
case (decide L) note conf = this(1) and undef = this(2) and T = this(4)
moreover
  have dec: decide S (cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S))
    using decide-rule decide.hyps by force
  then have cdclW:cdclW S (cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S))
    using cdclW.simps cdclW-o.intros by blast
moreover
  have lev: cdclW-M-level-inv (cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S))
    using cdclW M-level cdclW-consistent-inv[OF cdclW] by auto
  then have no-dup: no-dup (Marked L (backtrack-lvl S + 1) # trail S)
    using undef unfolding cdclW-M-level-inv-def by auto
  have no-strange-atm (cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S))
    using M-level alien calculation(4) cdclW-no-strange-atm-inv by blast
  then have length (Marked L ((backtrack-lvl S) + 1) # (trail S))
    ≤ card (atms-of-mm (init-clss S))
    using no-dup undef
    length-model-le-vars[of cons-trail (Marked L (backtrack-lvl S + 1)) (incr-lvl S)]
    by fastforce
  ultimately show ?case using conf by auto
next
case (skip L C' M D) note tr = this(1) and conf = this(2) and T = this(5)
show ?case using conf T by (simp add: tr)
next
case conflict
then show ?case by simp
next
case resolve
then show ?case using finite by simp
next
case (backtrack K i M1 M2 L D T) note conf = this(1) and decomp = this(3) and undef = this(7)
and
  T = this(8) and lev = this(9)
let ?S' = T
have bt: backtrack S ?S'
  using backtrack.hyps backtrack.intros[of S D L K i] by auto
have mset-ccls D ∉ # learned-clss S
  using no-relearn conf bt by auto
then have card-T:
  card (set-mset ({#mset-ccls D#} + learned-clss S)) = Suc (card (set-mset (learned-clss S)))
  by simp
have distinct-cdclW-state ?S'
  using bt M-level distinct-cdclW-state-inv no-dup other cdclW-o.intros cdclW-bj.intros by blast
moreover have ∀ s ∈ #learned-clss ?S'. ¬ tautology s
  using learned-clss-are-not-tautologies[OF cdclW.other[OF cdclW-o.bj[OF

```



```

    cdclW-bj.backtrack[OF bt]]] M-level no-taut confl by auto
ultimately have card (set-mset (learned-clss T)) ≤ 3 ^ card (atms-of-mm (learned-clss T))
  by (auto simp: learned-clss-less-upper-bound)
then have H: card (set-mset ({#mset-ccls D#} + learned-clss S))
  ≤ 3 ^ card (atms-of-mm ({#mset-ccls D#} + learned-clss S))
  using T undef decomp M-level by (simp add: cdclW-M-level-inv-decomp)
moreover
  have atms-of-mm ({#mset-ccls D#} + learned-clss S) ⊆ atms-of-mm (init-clss S)
    using alien conf unfolding no-strange-atm-def by auto
  then have card-f: card (atms-of-mm ({#mset-ccls D#} + learned-clss S))
    ≤ card (atms-of-mm (init-clss S))
    by (meson atms-of-ms-finite card-mono finite-set-mset)
  then have (3::nat) ^ card (atms-of-mm ({#mset-ccls D#} + learned-clss S))
    ≤ 3 ^ card (atms-of-mm (init-clss S)) by simp
ultimately have (3::nat) ^ card (atms-of-mm (init-clss S))
  ≥ card (set-mset ({#mset-ccls D#} + learned-clss S))
  using le-trans by blast
then show ?case using decomp undef diff-less-mono2 card-T T M-level
  by (auto simp: cdclW-M-level-inv-decomp)
next
  case restart
  then show ?case using alien by (auto simp: state-eq-def simp del: state-simp)
next
  case (forget C T) note no-forget = this(8)
  then have mset-cls C ∈# learned-clss S and mset-cls C ∉# learned-clss T
    using forget.hyps by auto
  then show ?case using no-forget by (auto simp add: mset-leD)
qed

```

```

lemma propagate-measure-decreasing:
  fixes S :: 'st
  assumes propagate S S' and cdclW-all-struct-inv S
  shows (cdclW-measure S', cdclW-measure S) ∈ lexn {(a, b). a < b} 3
  apply (rule cdclW-measure-decreasing)
  using assms(1) propagate apply blast
    using assms(1) apply (auto simp add: propagate.simps)[3]
    using assms(2) apply (auto simp add: cdclW-all-struct-inv-def)
  done

```

```

lemma conflict-measure-decreasing:
  fixes S :: 'st
  assumes conflict S S' and cdclW-all-struct-inv S
  shows (cdclW-measure S', cdclW-measure S) ∈ lexn {(a, b). a < b} 3
  apply (rule cdclW-measure-decreasing)
  using assms(1) conflict apply blast
    using assms(1) apply (auto simp: state-eq-def simp del: state-simp elim!: conflictE)[3]
    using assms(2) apply (auto simp add: cdclW-all-struct-inv-def elim: conflictE)
  done

```

```

lemma decide-measure-decreasing:
  fixes S :: 'st
  assumes decide S S' and cdclW-all-struct-inv S
  shows (cdclW-measure S', cdclW-measure S) ∈ lexn {(a, b). a < b} 3
  apply (rule cdclW-measure-decreasing)
  using assms(1) decide other apply blast

```

```

    using assms(1) apply (auto simp: state-eq-def simp del: state-simp elim!: decideE)[3]
    using assms(2) apply (auto simp add: cdclW-all-struct-inv-def elim: decideE)
done

lemma trans-le:
  trans {(a, (b::nat)). a < b}
  unfolding trans-def by auto

lemma cdclW-cp-measure-decreasing:
  fixes S :: 'st
  assumes cdclW-cp S S' and cdclW-all-struct-inv S
  shows (cdclW-measure S', cdclW-measure S) ∈ lexn {(a, b). a < b} 3
  using assms
proof induction
  case conflict'
  then show ?case using conflict-measure-decreasing by blast
next
  case propagate'
  then show ?case using propagate-measure-decreasing by blast
qed

lemma tranclp-cdclW-cp-measure-decreasing:
  fixes S :: 'st
  assumes cdclW-cp++ S S' and cdclW-all-struct-inv S
  shows (cdclW-measure S', cdclW-measure S) ∈ lexn {(a, b). a < b} 3
  using assms
proof induction
  case base
  then show ?case using cdclW-cp-measure-decreasing by blast
next
  case (step T U) note st = this(1) and step = this(2) and IH = this(3) and inv = this(4)
  then have (cdclW-measure T, cdclW-measure S) ∈ lexn {a. case a of (a, b) ⇒ a < b} 3 by blast

  moreover have (cdclW-measure U, cdclW-measure T) ∈ lexn {a. case a of (a, b) ⇒ a < b} 3
  using cdclW-cp-measure-decreasing[OF step] rtranclp-cdclW-all-struct-inv-inv inv
  tranclp-cdclW-cp-tranclp-cdclW[OF st]
  unfolding trans-def rtranclp-unfold
  by blast
  ultimately show ?case using lexn-transI[OF trans-le] unfolding trans-def by blast
qed

lemma cdclW-stgy-step-decreasing:
  fixes R S T :: 'st
  assumes cdclW-stgy S T and
  cdclW-stgy** R S
  trail R = [] and
  cdclW-all-struct-inv R
  shows (cdclW-measure T, cdclW-measure S) ∈ lexn {(a, b). a < b} 3
proof -
  have cdclW-all-struct-inv S
  using assms
  by (metis rtranclp-unfold rtranclp-cdclW-all-struct-inv-inv tranclp-cdclW-stgy-tranclp-cdclW)
  with assms show ?thesis
  proof induction
    case (conflict' V) note cp = this(1) and inv = this(5)

```

```

show ?case
  using tranclp-cdclW-cp-measure-decreasing[OF HOL.conjunct1[OF cp[unfolded full1-def]] inv]
.
next
case (other' T U) note st = this(1) and H = this(4,5,6,7) and cp = this(3)
have cdclW-all-struct-inv T
  using cdclW-all-struct-inv-inv other other'.hyps(1) other'.prems(4) by blast
from tranclp-cdclW-cp-measure-decreasing[OF - this]
have le-or-eq: (cdclW-measure U, cdclW-measure T) ∈ lexn {a. case a of (a, b) ⇒ a < b} 3 ∨
  cdclW-measure U = cdclW-measure T
  using cp unfolding full-def rtranclp-unfold by blast
moreover
have cdclW-M-level-inv S
  using cdclW-all-struct-inv-def other'.prems(4) by blast
with st have (cdclW-measure T, cdclW-measure S) ∈ lexn {a. case a of (a, b) ⇒ a < b} 3
proof (induction rule:cdclW-o-induct-lev2)
  case (decide T)
  then show ?case using decide-measure-decreasing H decide.intros[OF decide.hyps] by blast
next
case (backtrack K i M1 M2 L D T) note conf = this(1) and decomp = this(3) and
  undef = this(7) and T = this(8)
have bt: backtrack S T
  apply (rule backtrack-rule)
  using backtrack.hyps by auto
then have no-relearn: ∀ T. conflicting S = Some T ⟶ T ∉ learned-clss S
  using cdclW-stgy-no-relearned-clause[of R S T] H conf
  unfolding cdclW-all-struct-inv-def raw-clauses-def by auto
have inv: cdclW-all-struct-inv S
  using ⟨cdclW-all-struct-inv S⟩ by blast
show ?case
  apply (rule cdclW-measure-decreasing)
  using bt cdclW-bj.backtrack cdclW-o.bj other apply simp
  using bt T undef decomp inv unfolding cdclW-all-struct-inv-def
    cdclW-M-level-inv-def apply auto[]
  using bt T undef decomp inv unfolding cdclW-all-struct-inv-def
    cdclW-M-level-inv-def apply auto[]
  using bt no-relearn apply auto[]
  using inv unfolding cdclW-all-struct-inv-def apply simp
  using inv unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def apply simp
  using inv unfolding cdclW-all-struct-inv-def apply simp
  using inv unfolding cdclW-all-struct-inv-def apply simp
  using inv unfolding cdclW-all-struct-inv-def by simp
next
case skip
then show ?case by force
next
case resolve
then show ?case by force
qed
ultimately show ?case
  by (metis lexn-transI transD trans-le)
qed
qed

```

lemma tranclp-cdcl_W-stgy-decreasing:

```

fixes  $R\ S\ T :: 'st$ 
assumes  $cdcl_W\text{-}stgy^{++}\ R\ S$ 
 $trail\ R = []$  and
 $cdcl_W\text{-}all\text{-}struct\text{-}inv\ R$ 
shows  $(cdcl_W\text{-}measure\ S,\ cdcl_W\text{-}measure\ R) \in le_{rn}\ \{(a,\ b).\ a < b\}\ \exists$ 
using  $assms$ 
apply  $induction$ 
  using  $cdcl_W\text{-}stgy\text{-}step\text{-}decreasing[of\ R - R]$  apply  $blast$ 
using  $cdcl_W\text{-}stgy\text{-}step\text{-}decreasing[of\ - - R]$   $trancpl\text{-}into\text{-}rtrancpl[of\ cdcl_W\text{-}stgy\ R]$ 
 $le_{rn}\text{-}transI[OF\ trans\text{-}le,\ of\ \exists]$  unfolding  $trans\text{-}def$  by  $blast$ 

lemma  $trancpl\text{-}cdcl_W\text{-}stgy\text{-}S0\text{-}decreasing$ :
fixes  $R\ S\ T :: 'st$ 
assumes
   $pl: cdcl_W\text{-}stgy^{++}\ (init\text{-}state\ N)\ S$  and
   $no\text{-}dup: distinct\text{-}mset\text{-}mset\ (mset\text{-}class\ N)$ 
shows  $(cdcl_W\text{-}measure\ S,\ cdcl_W\text{-}measure\ (init\text{-}state\ N)) \in le_{rn}\ \{(a,\ b).\ a < b\}\ \exists$ 
proof -
  have  $cdcl_W\text{-}all\text{-}struct\text{-}inv\ (init\text{-}state\ N)$ 
    using  $no\text{-}dup$  unfolding  $cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}def$  by  $auto$ 
  then show  $?thesis$  using  $pl\ trancpl\text{-}cdcl_W\text{-}stgy\text{-}decreasing\ init\text{-}state\text{-}trail$  by  $blast$ 
qed

lemma  $wf\text{-}trancpl\text{-}cdcl_W\text{-}stgy$ :
 $wf\ \{(S::'st,\ init\text{-}state\ N) |$ 
   $S\ N.\ distinct\text{-}mset\text{-}mset\ (mset\text{-}class\ N) \wedge cdcl_W\text{-}stgy^{++}\ (init\text{-}state\ N)\ S\}$ 
apply  $(rule\ wf\text{-}wf\text{-}if\text{-}measure'\text{-}notation2[of\ le_{rn}\ \{(a,\ b).\ a < b\}\ \exists - - cdcl_W\text{-}measure])$ 
apply  $(simp\ add: wf\ wf\text{-}le_{rn})$ 
using  $trancpl\text{-}cdcl_W\text{-}stgy\text{-}S0\text{-}decreasing$  by  $blast$ 

lemma  $cdcl_W\text{-}cp\text{-}wf\text{-}all\text{-}inv$ :
 $wf\ \{(S',\ S).\ cdcl_W\text{-}all\text{-}struct\text{-}inv\ S \wedge cdcl_W\text{-}cp\ S\ S'\}$ 
 $(is\ wf\ ?R)$ 
proof  $(rule\ wf\text{-}bounded\text{-}measure[of\ -$ 
   $\lambda S.\ card\ (atms\text{-}of\text{-}mm\ (init\text{-}class\ S)) + 1$ 
   $\lambda S.\ length\ (trail\ S) + (if\ conflicting\ S = None\ then\ 0\ else\ 1)]$ ,  $goal\text{-}cases)$ 
case  $(1\ S\ S')$ 
then have  $cdcl_W\text{-}all\text{-}struct\text{-}inv\ S$  and  $cdcl_W\text{-}cp\ S\ S'$  by  $auto$ 
moreover then have  $cdcl_W\text{-}all\text{-}struct\text{-}inv\ S'$ 
  using  $cdcl_W\text{-}cp.simps\ cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}inv\ conflict\ cdcl_W.intros\ cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}inv$ 
by  $blast+$ 
ultimately show  $?case$ 
  by  $(auto\ simp: cdcl_W\text{-}cp.simps\ state\text{-}eq\text{-}def\ simp\ del: state\text{-}simp\ elim!: conflictE\ propagateE$ 
     $dest: length\text{-}model\text{-}le\text{-}vars\text{-}all\text{-}inv)$ 
qed

end

end

theory  $DPLL\text{-}CDCL\text{-}W\text{-}Implementation$ 
imports  $Partial\text{-}Annotated\text{-}Clausal\text{-}Logic$ 
begin

```

19 Simple Implementation of the DPLL and CDCL

19.1 Common Rules

19.1.1 Propagation

The following theorem holds:

lemma *lits-of-l-unfold*[iff]:

$(\forall c \in \text{set } C. -c \in \text{lits-of-l } Ms) \longleftrightarrow Ms \models_{as} CNot \ (mset \ C)$
unfolding *true-annot-def Ball-def true-annot-def CNot-def* **by** *auto*

The right-hand version is written at a high-level, but only the left-hand side is executable.

definition *is-unit-clause* :: 'a literal list \Rightarrow ('a, 'b, 'c) marked-lit list \Rightarrow 'a literal option

where

is-unit-clause *l* *M* =

(case *List.filter* ($\lambda a. \text{atm-of } a \notin \text{atm-of ' lits-of-l } M$) *l* of
 $a \# [] \Rightarrow \text{if } M \models_{as} CNot \ (mset \ l - \{\#a\# \}) \text{ then } Some \ a \text{ else } None$
 $| - \Rightarrow None$)

definition *is-unit-clause-code* :: 'a literal list \Rightarrow ('a, 'b, 'c) marked-lit list

\Rightarrow 'a literal option **where**

is-unit-clause-code *l* *M* =

(case *List.filter* ($\lambda a. \text{atm-of } a \notin \text{atm-of ' lits-of-l } M$) *l* of
 $a \# [] \Rightarrow \text{if } (\forall c \in \text{set } (remove1 \ a \ l). -c \in \text{lits-of-l } M) \text{ then } Some \ a \text{ else } None$
 $| - \Rightarrow None$)

lemma *is-unit-clause-is-unit-clause-code*[code]:

is-unit-clause *l* *M* = *is-unit-clause-code* *l* *M*

proof –

have $1: \bigwedge a. (\forall c \in \text{set } (remove1 \ a \ l). -c \in \text{lits-of-l } M) \longleftrightarrow M \models_{as} CNot \ (mset \ l - \{\#a\# \})$

using *lits-of-l-unfold*[of *remove1* - *l*, of - *M*] **by** *simp*

thus *?thesis*

unfolding *is-unit-clause-code-def is-unit-clause-def 1* **by** *blast*

qed

lemma *is-unit-clause-some-undef*:

assumes *is-unit-clause* *l* *M* = *Some a*

shows *undefined-lit* *M* *a*

proof –

have (case [*a* ← *l* . *atm-of* *a* \notin *atm-of ' lits-of-l* *M*] of [] $\Rightarrow None$

| [*a*] $\Rightarrow \text{if } M \models_{as} CNot \ (mset \ l - \{\#a\# \}) \text{ then } Some \ a \text{ else } None$

| *a* # *ab* # *xa* $\Rightarrow Map.empty \ xa$) = *Some a*

using *assms* **unfolding** *is-unit-clause-def* .

hence *a* $\in \text{set } [a \leftarrow l . \text{atm-of } a \notin \text{atm-of ' lits-of-l } M]$

apply (cases [*a* ← *l* . *atm-of* *a* \notin *atm-of ' lits-of-l* *M*])

apply *simp*

apply (*rename-tac aa list*; *case-tac list*) **by** (*auto split: if-split-asm*)

hence *atm-of* *a* \notin *atm-of ' lits-of-l* *M* **by** *auto*

thus *?thesis*

by (*simp add: Marked-Propagated-in-iff-in-lits-of-l*
atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)

qed

lemma *is-unit-clause-some-CNot*: *is-unit-clause* *l* *M* = *Some a* $\implies M \models_{as} CNot \ (mset \ l - \{\#a\# \})$

unfolding *is-unit-clause-def*

proof –

```

assume (case [a←l . atm-of a ∉ atm-of ‘ lits-of-l M] of [] ⇒ None
  | [a] ⇒ if M ⊨as CNot (mset l - {#a#}) then Some a else None
  | a # ab # xa ⇒ Map.empty xa) = Some a
thus ?thesis
apply (cases [a←l . atm-of a ∉ atm-of ‘ lits-of-l M], simp)
apply simp
apply (rename-tac aa list, case-tac list) by (auto split: if-split-asm)
qed

```

lemma *is-unit-clause-some-in*: *is-unit-clause l M = Some a ⇒ a ∈ set l*
unfolding *is-unit-clause-def*

proof –

```

assume (case [a←l . atm-of a ∉ atm-of ‘ lits-of-l M] of [] ⇒ None
  | [a] ⇒ if M ⊨as CNot (mset l - {#a#}) then Some a else None
  | a # ab # xa ⇒ Map.empty xa) = Some a
thus a ∈ set l
by (cases [a←l . atm-of a ∉ atm-of ‘ lits-of-l M])
  (fastforce dest: filter-eq-ConsD split: if-split-asm split: list.splits)+
qed

```

lemma *is-unit-clause-nil[simp]*: *is-unit-clause [] M = None*
unfolding *is-unit-clause-def* **by** auto

19.1.2 Unit propagation for all clauses

Finding the first clause to propagate

```

fun find-first-unit-clause :: 'a literal list list ⇒ ('a, 'b, 'c) marked-lit list
  ⇒ ('a literal × 'a literal list) option where
find-first-unit-clause (a # l) M =
  (case is-unit-clause a M of
    None ⇒ find-first-unit-clause l M
  | Some L ⇒ Some (L, a)) |
find-first-unit-clause [] - = None

```

lemma *find-first-unit-clause-some*:

```

find-first-unit-clause l M = Some (a, c)
⇒ c ∈ set l ∧ M ⊨as CNot (mset c - {#a#}) ∧ undefined-lit M a ∧ a ∈ set c
apply (induction l)
apply simp
by (auto split: option.splits dest: is-unit-clause-some-in is-unit-clause-some-CNot
  is-unit-clause-some-undef)

```

lemma *propagate-is-unit-clause-not-None*:

```

assumes dist: distinct c and
M: M ⊨as CNot (mset c - {#a#}) and
undef: undefined-lit M a and
ac: a ∈ set c
shows is-unit-clause c M ≠ None

```

proof –

```

have [a←c . atm-of a ∉ atm-of ‘ lits-of-l M] = [a]
using assms
proof (induction c)
case Nil thus ?case by simp
next
case (Cons ac c)

```

```

show ?case
  proof (cases a = ac)
    case True
      thus ?thesis using Cons
        by (auto simp del: lits-of-l-unfold
          simp add: lits-of-l-unfold[symmetric] Marked-Propagated-in-iff-in-lits-of-l
            atm-of-eq-atm-of atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)
    next
      case False
        hence T: mset c + {#ac#} - {#a#} = mset c - {#a#} + {#ac#}
          by (auto simp add: multiset-eq-iff)
        show ?thesis using False Cons
          by (auto simp add: T atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)
      qed
    qed
thus ?thesis
using M unfolding is-unit-clause-def by auto
qed

```

lemma find-first-unit-clause-none:
 $distinct\ c \implies c \in set\ l \implies M \models_{as} CNot\ (mset\ c - \{ \#a\# \}) \implies undefined\text{-}lit\ M\ a \implies a \in set\ c$
 $\implies find\text{-}first\text{-}unit\text{-}clause\ l\ M \neq None$
by (induction l)
 (auto split: option.split simp add: propagate-is-unit-clause-not-None)

19.1.3 Decide

fun find-first-unused-var :: 'a literal list list \Rightarrow 'a literal set \Rightarrow 'a literal option **where**
 find-first-unused-var (a # l) M =
 (case List.find ($\lambda lit. lit \notin M \wedge \neg lit \notin M$) a of
 None \Rightarrow find-first-unused-var l M
 | Some a \Rightarrow Some a) |
 find-first-unused-var [] - = None

lemma find-none[iff]:
 $List.find\ (\lambda lit. lit \notin M \wedge \neg lit \notin M)\ a = None \iff atm\text{-}of\ 'set\ a \subseteq atm\text{-}of\ 'M$
apply (induct a)
using atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
by (force simp add: atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)+

lemma find-some: $List.find\ (\lambda lit. lit \notin M \wedge \neg lit \notin M)\ a = Some\ b \implies b \in set\ a \wedge b \notin M \wedge \neg b \notin M$
unfolding find-Some-iff **by** (metis nth-mem)

lemma find-first-unused-var-None[iff]:
 $find\text{-}first\text{-}unused\text{-}var\ l\ M = None \iff (\forall a \in set\ l. atm\text{-}of\ 'set\ a \subseteq atm\text{-}of\ 'M)$
by (induct l)
 (auto split: option.splits dest!: find-some
 simp add: image-subset-iff atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set)

lemma find-first-unused-var-Some-not-all-incl:
assumes find-first-unused-var l M = Some c
shows $\neg(\forall a \in set\ l. atm\text{-}of\ 'set\ a \subseteq atm\text{-}of\ 'M)$
proof -
have find-first-unused-var l M $\neq None$
using assms **by** (cases find-first-unused-var l M) auto
thus $\neg(\forall a \in set\ l. atm\text{-}of\ 'set\ a \subseteq atm\text{-}of\ 'M)$ **by** auto

qed

lemma *find-first-unused-var-Some*:

find-first-unused-var l $M = \text{Some } a \implies (\exists m \in \text{set } l. a \in \text{set } m \wedge a \notin M \wedge -a \notin M)$
by (induct l) (auto split: option.splits dest: find-some)

lemma *find-first-unused-var-undefined*:

find-first-unused-var l (*lits-of-l* Ms) = *Some* $a \implies \text{undefined-lit } Ms$ a
using *find-first-unused-var-Some*[of l *lits-of-l* Ms a] *Marked-Propagated-in-iff-in-lits-of-l*
by blast

end

theory *DPLL-W-Implementation*

imports *DPLL-CDCL-W-Implementation* *DPLL-W* $\sim\sim$ /src/HOL/Library/Code-Target-Numeral

begin

19.2 Simple Implementation of DPLL

19.2.1 Combining the propagate and decide: a DPLL step

definition *DPLL-step* :: $\text{int } dpll_W\text{-marked-lits} \times \text{int literal list list}$

$\Rightarrow \text{int } dpll_W\text{-marked-lits} \times \text{int literal list list}$ **where**

DPLL-step = $(\lambda(Ms, N).$

(*case find-first-unit-clause* N Ms of

Some ($L, -$) \Rightarrow (*Propagated* L ()) $\#$ Ms, N)

| $- \Rightarrow$

if $\exists C \in \text{set } N. (\forall c \in \text{set } C. -c \in \text{lits-of-l } Ms)$

then

(*case backtrack-split* Ms of

($-, L \# M$) \Rightarrow (*Propagated* ($-(\text{lit-of } L)$) ()) $\#$ M, N)

| ($-, -$) \Rightarrow (Ms, N)

)

else

(*case find-first-unused-var* N (*lits-of-l* Ms) of

Some $a \Rightarrow$ (*Marked* a ()) $\#$ Ms, N)

| *None* \Rightarrow (Ms, N))))

Example of propagation:

value *DPLL-step* ($[\text{Marked } (\text{Neg } 1) ()], [[\text{Pos } (1::\text{int}), \text{Neg } 2]]$)

We define the conversion function between the states as defined in *Prop-DPLL* (with multisets) and here (with lists).

abbreviation *toS* $\equiv \lambda(Ms::(\text{int}, \text{unit}, \text{unit}) \text{ marked-lit list})$

$(N:: \text{int literal list list}). (Ms, \text{mset } (\text{map mset } N))$

abbreviation *toS'* $\equiv \lambda(Ms::(\text{int}, \text{unit}, \text{unit}) \text{ marked-lit list},$

$N:: \text{int literal list list}). (Ms, \text{mset } (\text{map mset } N))$

Proof of correctness of *DPLL-step*

lemma *DPLL-step-is-a-dpll_W-step*:

assumes *step*: $(Ms', N') = \text{DPLL-step } (Ms, N)$

and *neg*: $(Ms, N) \neq (Ms', N')$

shows *dpll_W* (*toS* Ms N) (*toS* Ms' N')

proof –

let $?S = (Ms, \text{mset } (\text{map mset } N))$

{ fix L E


```

assume unit: find-first-unit-clause N Ms = Some (L, E)
hence Ms'N: (Ms', N') = (Propagated L () # Ms, N)
  using step unfolding DPLL-step-def by auto
obtain C where
  C: C ∈ set N and
  Ms: Ms ⊨as CNot (mset C − {#L#}) and
  undef: undefined-lit Ms L and
  L ∈ set C using find-first-unit-clause-some[OF unit] by metis
have dpllW (Ms, mset (map mset N))
  (Propagated L () # fst (Ms, mset (map mset N)), snd (Ms, mset (map mset N)))
  apply (rule dpllW.propagate)
  using Ms undef C (L ∈ set C) by (auto simp add: C)
hence ?thesis using Ms'N by auto
}
moreover
{ assume unit: find-first-unit-clause N Ms = None
  assume exC: ∃ C ∈ set N. Ms ⊨as CNot (mset C)
  then obtain C where C: C ∈ set N and Ms: Ms ⊨as CNot (mset C) by auto
  then obtain L M M' where bt: backtrack-split Ms = (M', L # M)
    using step exC neq unfolding DPLL-step-def prod.case unit
    by (cases backtrack-split Ms, rename-tac b, case-tac b) auto
  hence is-marked L using backtrack-split-snd-hd-marked[of Ms] by auto
  have 1: dpllW (Ms, mset (map mset N))
    (Propagated (− lit-of L) () # M, snd (Ms, mset (map mset N)))
    apply (rule dpllW.backtrack[OF - is-marked L, of ])
    using C Ms bt by auto
  moreover have (Ms', N') = (Propagated (− (lit-of L)) () # M, N)
    using step exC unfolding DPLL-step-def bt prod.case unit by auto
  ultimately have ?thesis by auto
}
moreover
{ assume unit: find-first-unit-clause N Ms = None
  assume exC: ¬ (∃ C ∈ set N. Ms ⊨as CNot (mset C))
  obtain L where unused: find-first-unused-var N (lits-of-l Ms) = Some L
    using step exC neq unfolding DPLL-step-def prod.case unit
    by (cases find-first-unused-var N (lits-of-l Ms)) auto
  have dpllW (Ms, mset (map mset N))
    (Marked L () # fst (Ms, mset (map mset N)), snd (Ms, mset (map mset N)))
    apply (rule dpllW.decided[of ?S L])
    using find-first-unused-var-Some[OF unused]
    by (auto simp add: Marked-Propagated-in-iff-in-lits-of-l atms-of-ms-def)
  moreover have (Ms', N') = (Marked L () # Ms, N)
    using step exC unfolding DPLL-step-def unused prod.case unit by auto
  ultimately have ?thesis by auto
}
ultimately show ?thesis by (cases find-first-unit-clause N Ms) auto
qed

```

lemma *DPLL-step-stuck-final-state*:

assumes *step*: (*Ms*, *N*) = *DPLL-step* (*Ms*, *N*)
shows *conclusive-dpll_W-state* (*toS Ms N*)

proof −

have *unit*: *find-first-unit-clause* *N Ms* = *None*
using *step unfolding DPLL-step-def* **by** (*auto split:option.splits*)

```

{ assume n:  $\exists C \in \text{set } N. Ms \models_{as} CNot (mset C)$ 
  hence Ms:  $(Ms, N) = (\text{case backtrack-split } Ms \text{ of } (x, []) \Rightarrow (Ms, N) \mid (x, L \# M) \Rightarrow (Propagated (- \text{lit-of } L) () \# M, N))$ 
    using step unfolding DPLL-step-def by (simp add:unit)

have snd (backtrack-split Ms) = []
proof (cases backtrack-split Ms, cases snd (backtrack-split Ms))
  fix a b
  assume backtrack-split Ms = (a, b) and snd (backtrack-split Ms) = []
  thus snd (backtrack-split Ms) = [] by blast
next
fix a b aa list
assume
  bt: backtrack-split Ms = (a, b) and
  bt': snd (backtrack-split Ms) = aa # list
hence Ms: Ms = Propagated (- lit-of aa) () # list using Ms by auto
have is-marked aa using backtrack-split-snd-hd-marked[of Ms] bt bt' by auto
moreover have fst (backtrack-split Ms) @ aa # list = Ms
  using backtrack-split-list-eq[of Ms] bt' by auto
ultimately have False unfolding Ms by auto
thus snd (backtrack-split Ms) = [] by blast
qed

hence ?thesis
  using n backtrack-snd-empty-not-marked[of Ms] unfolding conclusive-dpllW-state-def
  by (cases backtrack-split Ms) auto
}
moreover {
  assume n:  $\neg (\exists C \in \text{set } N. Ms \models_{as} CNot (mset C))$ 
  hence find-first-unused-var N (lits-of-l Ms) = None
    using step unfolding DPLL-step-def by (simp add: unit split: option.splits)
  hence a:  $\forall a \in \text{set } N. \text{atm-of 'set } a \subseteq \text{atm-of ' (lits-of-l Ms)}$  by auto
  have fst (toS Ms N)  $\models_{asm} \text{snd (toS Ms N)}$  unfolding true-annots-def CNot-def Ball-def
  proof clarify
    fix x
    assume x:  $x \in \text{set-mset (clauses (toS Ms N))}$ 
    hence  $\neg Ms \models_{as} CNot x$  using n unfolding true-annots-def CNot-def Ball-def by auto
    moreover have total-over-m (lits-of-l Ms) {x}
      using a x image-iff in-mono atms-of-s-def
      unfolding total-over-m-def total-over-set-def lits-of-def by fastforce
    ultimately show fst (toS Ms N)  $\models_a x$ 
      using total-not-CNot[of lits-of-l Ms x] by (simp add: true-annot-def true-annots-true-cl)
    qed
  hence ?thesis unfolding conclusive-dpllW-state-def by blast
}
ultimately show ?thesis by blast
qed

```

19.2.2 Adding invariants

Invariant tested in the function `function DPLL-ci :: int dpllW-marked-lits \Rightarrow int literal list list`
 `\Rightarrow int dpllW-marked-lits \times int literal list list` **where**
`DPLL-ci Ms N =`
`(if $\neg \text{dpll}_W\text{-all-inv (Ms, mset (map mset N))}$`
`then (Ms, N)`

```

else
  let (Ms', N') = DPLL-step (Ms, N) in
  if (Ms', N') = (Ms, N) then (Ms, N) else DPLL-ci Ms' N)
by fast+
termination
proof (relation {(S', S). (toS' S', toS' S) ∈ {(S', S). dpllW-all-inv S ∧ dpllW S S'}})
  show wf {(S', S). (toS' S', toS' S) ∈ {(S', S). dpllW-all-inv S ∧ dpllW S S'}}
    using wf-if-measure-f[OF dpllW-wf, of toS'] by auto
next
fix Ms :: int dpllW-marked-lits and N x xa y
assume ¬ ¬ dpllW-all-inv (toS Ms N)
and step: x = DPLL-step (Ms, N)
and x: (xa, y) = x
and (xa, y) ≠ (Ms, N)
thus ((xa, N), Ms, N) ∈ {(S', S). (toS' S', toS' S) ∈ {(S', S). dpllW-all-inv S ∧ dpllW S S'}}
  using DPLL-step-is-a-dpllW-step dpllW-same-clauses split-conv by fastforce
qed

No invariant tested  function (domintros) DPLL-part:: int dpllW-marked-lits ⇒ int literal list list
⇒
  int dpllW-marked-lits × int literal list list where
DPLL-part Ms N =
  (let (Ms', N') = DPLL-step (Ms, N) in
  if (Ms', N') = (Ms, N) then (Ms, N) else DPLL-part Ms' N)
by fast+

lemma snd-DPLL-step[simp]:
  snd (DPLL-step (Ms, N)) = N
  unfolding DPLL-step-def by (auto split: if-split option.splits prod.splits list.splits)

lemma dpllW-all-inv-implicS-2-eq3-and-dom:
  assumes dpllW-all-inv (Ms, mset (map mset N))
  shows DPLL-ci Ms N = DPLL-part Ms N ∧ DPLL-part-dom (Ms, N)
  using assms
proof (induct rule: DPLL-ci.induct)
  case (1 Ms N)
  have snd (DPLL-step (Ms, N)) = N by auto
  then obtain Ms' where Ms': DPLL-step (Ms, N) = (Ms', N) by (cases DPLL-step (Ms, N)) auto
  have inv': dpllW-all-inv (toS Ms' N) by (metis (mono-tags) 1.prem DPLL-step-is-a-dpllW-step
    Ms' dpllW-all-inv old.prod.inject)
  { assume (Ms', N) ≠ (Ms, N)
    hence DPLL-ci Ms' N = DPLL-part Ms' N ∧ DPLL-part-dom (Ms', N) using 1(1)[of - Ms' N]
  }
  Ms'
  1(2) inv' by auto
  hence DPLL-part-dom (Ms, N) using DPLL-part.domintros Ms' by fastforce
  moreover have DPLL-ci Ms N = DPLL-part Ms N using 1.prem DPLL-part.psims Ms'
    ⟨DPLL-ci Ms' N = DPLL-part Ms' N ∧ DPLL-part-dom (Ms', N)⟩ ⟨DPLL-part-dom (Ms, N)⟩ by
  auto
  ultimately have ?case by blast
}
moreover {
  assume (Ms', N) = (Ms, N)
  hence ?case using DPLL-part.domintros DPLL-part.psims Ms' by fastforce
}
ultimately show ?case by blast

```

qed

lemma *DPLL-ci-dpll_W-rtrancp*:

assumes *DPLL-ci* $Ms\ N = (Ms', N')$
shows $dpll_W^{**} (toS\ Ms\ N) (toS\ Ms'\ N)$
using *assms*

proof (*induct* $Ms\ N$ *arbitrary*: $Ms'\ N'$ *rule*: *DPLL-ci.induct*)

case ($1\ Ms\ N\ Ms'\ N'$) **note** $IH = this(1)$ **and** $step = this(2)$

obtain $S_1\ S_2$ **where** $S: (S_1, S_2) = DPLL-step\ (Ms, N)$ **by** (*cases* *DPLL-step* (Ms, N)) *auto*

{ **assume** $\neg dpll_W\text{-all-inv}\ (toS\ Ms\ N)$
hence $(Ms, N) = (Ms', N)$ **using** *step* **by** *auto*
hence *?case* **by** *auto*

}

moreover

{ **assume** $dpll_W\text{-all-inv}\ (toS\ Ms\ N)$
and $(S_1, S_2) = (Ms, N)$
hence *?case* **using** *S step* **by** *auto*

}

moreover

{ **assume** $dpll_W\text{-all-inv}\ (toS\ Ms\ N)$
and $(S_1, S_2) \neq (Ms, N)$

moreover obtain $S_1'\ S_2'$ **where** $DPLL-ci\ S_1\ N = (S_1', S_2')$ **by** (*cases* *DPLL-ci* $S_1\ N$) *auto*

moreover have $DPLL-ci\ Ms\ N = DPLL-ci\ S_1\ N$ **using** *DPLL-ci.simps*[*of* $Ms\ N$] *calculation*

proof –

have (*case* (S_1, S_2) *of* $(ms, lss) \Rightarrow$

if $(ms, lss) = (Ms, N)$ *then* (Ms, N) *else* $DPLL-ci\ ms\ N = DPLL-ci\ Ms\ N$

using *S DPLL-ci.simps*[*of* $Ms\ N$] *calculation* **by** *presburger*

hence (*if* $(S_1, S_2) = (Ms, N)$ *then* (Ms, N) *else* $DPLL-ci\ S_1\ N = DPLL-ci\ Ms\ N$

by *fastforce*

thus *?thesis*

using *calculation*(2) **by** *presburger*

qed

ultimately have $dpll_W^{**} (toS\ S_1'\ N) (toS\ Ms'\ N)$ **using** $IH[of\ (S_1, S_2)\ S_1\ S_2]$ *S step* **by** *simp*

moreover have $dpll_W\ (toS\ Ms\ N) (toS\ S_1\ N)$

by (*metis* *DPLL-step-is-a-dpll_W-step* $S\ \langle (S_1, S_2) \neq (Ms, N) \rangle$ *prod.sel*(2) *snd-DPLL-step*)

ultimately have *?case* **by** (*metis* (*mono-tags*, *hide-lams*) $IH\ S\ \langle (S_1, S_2) \neq (Ms, N) \rangle$

$\langle DPLL-ci\ Ms\ N = DPLL-ci\ S_1\ N \rangle$ $\langle dpll_W\text{-all-inv}\ (toS\ Ms\ N) \rangle$ *converse-rtrancp-into-rtrancp*
local.step)

}

ultimately show *?case* **by** *blast*

qed

lemma *dpll_W-all-inv-dpll_W-trancp-irrefl*:

assumes $dpll_W\text{-all-inv}\ (Ms, N)$

and $dpll_W^{++} (Ms, N) (Ms, N)$

shows *False*

proof –

have $1: wf\ \{(S', S). dpll_W\text{-all-inv}\ S \wedge dpll_W^{++}\ S\ S'\}$ **using** *dpll_W-wf-trancp* **by** *auto*

have $((Ms, N), (Ms, N)) \in \{(S', S). dpll_W\text{-all-inv}\ S \wedge dpll_W^{++}\ S\ S'\}$ **using** *assms* **by** *auto*

thus *False* **using** *wf-not-refl*[*OF* 1] **by** *blast*

qed

lemma *DPLL-ci-final-state*:

assumes *step*: $DPLL\text{-}ci\ Ms\ N = (Ms, N)$
and *inv*: $dpll_W\text{-}all\text{-}inv\ (toS\ Ms\ N)$
shows *conclusive-dpll_W-state* $(toS\ Ms\ N)$
proof –
have *st*: $dpll_W^{**}\ (toS\ Ms\ N)\ (toS\ Ms\ N)$ **using** $DPLL\text{-}ci\text{-}dpll_W\text{-}rtrancpl[OF\ step]$.
have $DPLL\text{-}step\ (Ms, N) = (Ms, N)$
proof (*rule ccontr*)
obtain $Ms'\ N'$ **where** $Ms'N: (Ms', N') = DPLL\text{-}step\ (Ms, N)$
by (*cases* $DPLL\text{-}step\ (Ms, N)$) *auto*
assume $\neg\ ?thesis$
hence $DPLL\text{-}ci\ Ms'\ N = (Ms, N)$ **using** *step inv st Ms'N[symmetric]* **by** *fastforce*
hence $dpll_W^{++}\ (toS\ Ms\ N)\ (toS\ Ms\ N)$
by (*metis* $DPLL\text{-}ci\text{-}dpll_W\text{-}rtrancpl\ DPLL\text{-}step\text{-}is\text{-}a\text{-}dpll_W\text{-}step\ Ms'\ N\ \langle DPLL\text{-}step\ (Ms, N) \neq (Ms,$
 $N) \rangle$
 $prod.sel(2)\ rtrancpl\text{-}into\text{-}trancpl2\ snd\text{-}DPLL\text{-}step \rangle$)
thus *False* **using** $dpll_W\text{-}all\text{-}inv\text{-}dpll_W\text{-}trancpl\text{-}irrefl\ inv$ **by** *auto*
qed
thus *?thesis* **using** $DPLL\text{-}step\text{-}stuck\text{-}final\text{-}state[of\ Ms\ N]$ **by** *simp*
qed

lemma *DPLL-step-obtains*:

obtains Ms' **where** $(Ms', N) = DPLL\text{-}step\ (Ms, N)$
unfolding $DPLL\text{-}step\text{-}def$ **by** (*metis* (*no-types, lifting*) $DPLL\text{-}step\text{-}def\ prod.collapse\ snd\text{-}DPLL\text{-}step$)

lemma *DPLL-ci-obtains*:

obtains Ms' **where** $(Ms', N) = DPLL\text{-}ci\ Ms\ N$

proof (*induct rule: DPLL-ci.induct*)

case $(1\ Ms\ N)$ **note** $IH = this(1)$ **and** $that = this(2)$

obtain S **where** $SN: (S, N) = DPLL\text{-}step\ (Ms, N)$ **using** $DPLL\text{-}step\text{-}obtains$ **by** *metis*

{ **assume** $\neg\ dpll_W\text{-}all\text{-}inv\ (toS\ Ms\ N)$

hence *?case* **using** *that* **by** *auto*

}

moreover **{**

assume $n: (S, N) \neq (Ms, N)$

and *inv*: $dpll_W\text{-}all\text{-}inv\ (toS\ Ms\ N)$

have $\exists\ ms. DPLL\text{-}step\ (Ms, N) = (ms, N)$

by (*metis* $\langle \bigwedge thesisa. (\bigwedge S. (S, N) = DPLL\text{-}step\ (Ms, N) \implies thesisa) \implies thesisa \rangle$)

hence *?thesis*

using *IH that* **by** *fastforce*

}

moreover **{**

assume $n: (S, N) = (Ms, N)$

hence *?case* **using** *SN that* **by** *fastforce*

}

ultimately show *?case* **by** *blast*

qed

lemma *DPLL-ci-no-more-step*:

assumes *step*: $DPLL\text{-}ci\ Ms\ N = (Ms', N')$

shows $DPLL\text{-}ci\ Ms'\ N' = (Ms', N')$

using *assms*

proof (*induct arbitrary: Ms' N' rule: DPLL-ci.induct*)

case $(1\ Ms\ N\ Ms'\ N')$ **note** $IH = this(1)$ **and** $step = this(2)$

obtain S_1 **where** $S: (S_1, N) = DPLL\text{-}step\ (Ms, N)$ **using** $DPLL\text{-}step\text{-}obtains$ **by** *auto*

```

{ assume  $\neg dpll_W\text{-all-inv}$  (toS Ms N)
  hence ?case using step by auto
}
moreover {
  assume  $dpll_W\text{-all-inv}$  (toS Ms N)
  and  $(S_1, N) = (Ms, N)$ 
  hence ?case using S step by auto
}
moreover
{ assume inv:  $dpll_W\text{-all-inv}$  (toS Ms N)
  assume n:  $(S_1, N) \neq (Ms, N)$ 
  obtain  $S_1'$  where SS:  $(S_1', N) = DPLL\text{-ci } S_1 N$  using DPLL-ci-obtains by blast
  moreover have  $DPLL\text{-ci } Ms N = DPLL\text{-ci } S_1 N$ 
  proof -
    have (case  $(S_1, N)$  of (ms, lss)  $\Rightarrow$  if (ms, lss) = (Ms, N) then (Ms, N) else  $DPLL\text{-ci } ms N$ )
      =  $DPLL\text{-ci } Ms N$ 
    using S DPLL-ci.simps[of Ms N] calculation inv by presburger
    hence (if  $(S_1, N) = (Ms, N)$  then (Ms, N) else  $DPLL\text{-ci } S_1 N$ ) =  $DPLL\text{-ci } Ms N$ 
    by fastforce
    thus ?thesis
    using calculation n by presburger
  qed
  moreover
    have  $DPLL\text{-ci } S_1' N = (S_1', N)$  using step IH[OF - - S n SS[symmetric]] inv by blast
    ultimately have ?case using step by fastforce
  }
ultimately show ?case by blast
qed

```

lemma *DPLL-part-dpll_W-all-inv-final*:

```

fixes M Ms': (int, unit, unit) marked-lit list and
  N :: int literal list list
assumes inv:  $dpll_W\text{-all-inv}$  (Ms, mset (map mset N))
and MsN:  $DPLL\text{-part } Ms N = (Ms', N)$ 
shows conclusive-dpllW-state (toS Ms' N)  $\wedge$   $dpll_W^{**}$  (toS Ms N) (toS Ms' N)
proof -
  have 2:  $DPLL\text{-ci } Ms N = DPLL\text{-part } Ms N$  using inv  $dpll_W\text{-all-inv-implieS-2-eq3-and-dom}$  by blast
  hence star:  $dpll_W^{**}$  (toS Ms N) (toS Ms' N) unfolding MsN using DPLL-ci-dpllW-rtranclp by
blast
  hence inv':  $dpll_W\text{-all-inv}$  (toS Ms' N) using inv rtranclp-dpllW-all-inv by blast
  show ?thesis using star DPLL-ci-final-state[OF DPLL-ci-no-more-step inv] 2 unfolding MsN by
blast
qed

```

Embedding the invariant into the type

Defining the type `typedef dpllW-state =`

```

{(M::(int, unit, unit) marked-lit list, N::int literal list list).
   $dpll_W\text{-all-inv}$  (toS M N)}
```

`morphisms rough-state-of state-of`

proof

```

show  $([], []) \in \{(M, N). dpll_W\text{-all-inv}$  (toS M N) $\}$  by (auto simp add:  $dpll_W\text{-all-inv-def}$ )
```

qed

lemma

DPLL-part-dom (\square , N)

using *assms* *dpll_W-all-inv-implicS-2-eq3-and-dom*[*of* \square N] **by** (*simp add: dpll_W-all-inv-def*)

Some type classes instantiation *dpll_W-state* :: *equal*

begin

definition *equal-dpll_W-state* :: *dpll_W-state* \Rightarrow *dpll_W-state* \Rightarrow *bool* **where**

equal-dpll_W-state $S S' = (\text{rough-state-of } S = \text{rough-state-of } S')$

instance

by *standard* (*simp add: rough-state-of-inject equal-dpll_W-state-def*)

end

DPLL definition *DPLL-step'* :: *dpll_W-state* \Rightarrow *dpll_W-state* **where**

DPLL-step' $S = \text{state-of } (DPLL\text{-step } (\text{rough-state-of } S))$

declare *rough-state-of-inverse*[*simp*]

lemma *DPLL-step-dpll_W-conc-inv*:

DPLL-step (*rough-state-of* S) $\in \{(M, N). \text{dpll}_W\text{-all-inv } (toS\ M\ N)\}$

by (*smt DPLL-ci.simps DPLL-ci-dpll_W-rtrancpl case-prodE case-prodI2 rough-state-of mem-Collect-eq old.prod.case prod.sel(2) rtrancpl-dpll_W-all-inv snd-DPLL-step*)

lemma *rough-state-of-DPLL-step'-DPLL-step*[*simp*]:

rough-state-of (*DPLL-step'* S) = *DPLL-step* (*rough-state-of* S)

using *DPLL-step-dpll_W-conc-inv DPLL-step'-def state-of-inverse* **by** *auto*

function *DPLL-tot*:: *dpll_W-state* \Rightarrow *dpll_W-state* **where**

DPLL-tot $S =$

(*let* $S' = DPLL\text{-step}'\ S$ *in*

if $S' = S$ *then* S *else* *DPLL-tot* $S')$

by *fast+*

termination

proof (*relation* $\{(T', T)\}$.

(*rough-state-of* T' , *rough-state-of* T)

$\in \{(S', S). (toS'\ S', toS'\ S)$

$\in \{(S', S). \text{dpll}_W\text{-all-inv } S \wedge \text{dpll}_W\ S\ S')\}\}$)

show *wf* $\{(b, a).$

(*rough-state-of* b , *rough-state-of* a)

$\in \{(b, a). (toS'\ b, toS'\ a)$

$\in \{(b, a). \text{dpll}_W\text{-all-inv } a \wedge \text{dpll}_W\ a\ b)\}\}$

using *wf-if-measure-f*[*OF wf-if-measure-f*[*OF dpll_W-wf*, *of toS'*], *of rough-state-of*] .

next

fix $S\ x$

assume $x: x = DPLL\text{-step}'\ S$

and $x \neq S$

have *dpll_W-all-inv* (*case rough-state-of* S *of* (Ms, N) \Rightarrow ($Ms, \text{mset } (\text{map } \text{mset } N)$))

by (*metis* (*no-types*, *lifting*) *case-prodE mem-Collect-eq old.prod.case rough-state-of*)

moreover have *dpll_W* (*case rough-state-of* S *of* (Ms, N) \Rightarrow ($Ms, \text{mset } (\text{map } \text{mset } N)$))

(*case rough-state-of* (*DPLL-step'* S) *of* (Ms, N) \Rightarrow ($Ms, \text{mset } (\text{map } \text{mset } N)$))

proof –

obtain $Ms\ N$ **where** $Ms: (Ms, N) = \text{rough-state-of } S$ **by** (*cases rough-state-of* S) *auto*

have *dpll_W-all-inv* (*toS'* (Ms, N)) **using** *calculation unfolding Ms* **by** *blast*

moreover obtain $Ms'\ N'$ **where** $Ms': (Ms', N') = \text{rough-state-of } (DPLL\text{-step}'\ S)$

by (*cases rough-state-of* (*DPLL-step'* S)) *auto*

ultimately have *dpll_W-all-inv* (*toS'* (Ms', N')) **unfolding** Ms'

```

    by (metis (no-types, lifting) case-prod-unfold mem-Collect-eq rough-state-of)

  have dpllW (toS Ms N) (toS Ms' N')
    apply (rule DPLL-step-is-a-dpllW-step[of Ms' N' Ms N])
    unfolding Ms Ms' using ⟨x ≠ S⟩ rough-state-of-inject x by fastforce+
    thus ?thesis unfolding Ms[symmetric] Ms'[symmetric] by auto
  qed
ultimately show (x, S) ∈ {(T', T). (rough-state-of T', rough-state-of T)
  ∈ {(S', S). (toS' S', toS' S) ∈ {(S', S). dpllW-all-inv S ∧ dpllW S S'}}}
  by (auto simp add: x)
qed

lemma [code]:
DPLL-tot S =
  (let S' = DPLL-step' S in
   if S' = S then S else DPLL-tot S') by auto

lemma DPLL-tot-DPLL-step-DPLL-tot[simp]: DPLL-tot (DPLL-step' S) = DPLL-tot S
  apply (cases DPLL-step' S = S)
  apply simp
  unfolding DPLL-tot.simps[of S] by (simp del: DPLL-tot.simps)

lemma DOPLL-step'-DPLL-tot[simp]:
  DPLL-step' (DPLL-tot S) = DPLL-tot S
  by (rule DPLL-tot.induct[of λS. DPLL-step' (DPLL-tot S) = DPLL-tot S S])
  (metis (full-types) DPLL-tot.simps)

lemma DPLL-tot-final-state:
  assumes DPLL-tot S = S
  shows conclusive-dpllW-state (toS' (rough-state-of S))
proof -
  have DPLL-step' S = S using assms[symmetric] DOPLL-step'-DPLL-tot by metis
  hence DPLL-step (rough-state-of S) = (rough-state-of S)
    unfolding DPLL-step'-def using DPLL-step-dpllW-conc-inv rough-state-of-inverse
    by (metis rough-state-of-DPLL-step'-DPLL-step)
  thus ?thesis
    by (metis (mono-tags, lifting) DPLL-step-stuck-final-state old.prod.exhaust split-conv)
qed

lemma DPLL-tot-star:
  assumes rough-state-of (DPLL-tot S) = S'
  shows dpllW** (toS' (rough-state-of S)) (toS' S')
  using assms
proof (induction arbitrary: S' rule: DPLL-tot.induct)
  case (1 S S')
  let ?x = DPLL-step' S
  { assume ?x = S
    then have ?case using 1(2) by simp
  }
  moreover {
    assume S: ?x ≠ S
    have ?case
      apply (cases DPLL-step' S = S)

```



```

    using S apply blast
  by (smt 1.IH 1.prem1 DPLL-step-is-a-dpllW-step DPLL-tot.simps case-prodE2
      rough-state-of-DPLL-step'-DPLL-step rtranclp.rtrancl-into-rtrancl rtranclp.rtrancl-refl
      rtranclp-idemp split-conv)
}
ultimately show ?case by auto
qed

```

```

lemma rough-state-of-rough-state-of-nil[simp]:
  rough-state-of (state-of ([], N)) = ([], N)
  apply (rule DPLL-W-Implementation.dpllW-state.state-of-inverse)
  unfolding dpllW-all-inv-def by auto

```

Theorem of correctness

```

lemma DPLL-tot-correct:
  assumes rough-state-of (DPLL-tot (state-of ([], N))) = (M, N')
  and (M', N'') = toS' (M, N')
  shows M' ⊨asm N'' ⟷ satisfiable (set-mset N'')
proof -
  have dpllW** (toS' ([], N)) (toS' (M, N')) using DPLL-tot-star[OF assms(1)] by auto
  moreover have conclusive-dpllW-state (toS' (M, N'))
    using DPLL-tot-final-state by (metis (mono-tags, lifting) DOPLL-step'-DPLL-tot DPLL-tot.simps
    assms(1))
  ultimately show ?thesis using dpllW-conclusive-state-correct by (smt DPLL-ci.simps
    DPLL-ci-dpllW-rtranclp assms(2) dpllW-all-inv-def prod.case prod.sel(1) prod.sel(2)
    rtranclp-dpllW-inv(3) rtranclp-dpllW-inv-starting-from-0)
qed

```

19.2.3 Code export

A conversion to *DPLL-W-Implementation.dpll_W-state* **definition** *Con* :: (int, unit, unit) marked-lit list × int literal list list

⇒ dpll_W-state **where**

Con *xs* = state-of (if dpll_W-all-inv (toS (fst *xs*) (snd *xs*)) then *xs* else ([], []))

lemma [code abstype]:

Con (rough-state-of *S*) = *S*

using rough-state-of[of *S*] unfolding *Con*-def by auto

declare rough-state-of-DPLL-step'-DPLL-step[code abstract]

lemma *Con-DPLL-step-rough-state-of-state-of*[simp]:

Con (DPLL-step (rough-state-of *s*)) = state-of (DPLL-step (rough-state-of *s*))

unfolding *Con*-def by (metis (mono-tags, lifting) DPLL-step-dpll_W-conc-inv mem-Collect-eq prod.case-eq-if)

A slightly different version of *DPLL-tot* where the returned boolean indicates the result.

definition *DPLL-tot-rep* **where**

DPLL-tot-rep *S* =

(let (M, N) = (rough-state-of (DPLL-tot *S*)) in (∀ A ∈ set N. (∃ a ∈ set A. a ∈ lits-of-l (M)), M))

One version of the generated SML code is here, but not included in the generated document. The only differences are:

- export '*a* literal from the SML Module *Clausal-Logic*;
- export the constructor *Con* from *DPLL-W-Implementation*;

- export the *int* constructor from *Arith*.

All these allows to test on the code on some examples.

```

end
theory CDCL-W-Implementation
imports DPLL-CDCL-W-Implementation CDCL-W-Termination
begin

notation image-mset (infixr '# 90)

type-synonym 'a cdclW-mark = 'a literal list
type-synonym cdclW-marked-level = nat

type-synonym 'v cdclW-marked-lit = ('v, cdclW-marked-level, 'v cdclW-mark) marked-lit
type-synonym 'v cdclW-marked-lits = ('v, cdclW-marked-level, 'v cdclW-mark) marked-lits
type-synonym 'v cdclW-state =
  'v cdclW-marked-lits × 'v literal list list × 'v literal list list × nat ×
  'v literal list option

abbreviation raw-trail :: 'a × 'b × 'c × 'd × 'e ⇒ 'a where
raw-trail ≡ (λ(M, -). M)

abbreviation raw-cons-trail :: 'a ⇒ 'a list × 'b × 'c × 'd × 'e ⇒ 'a list × 'b × 'c × 'd × 'e
  where
raw-cons-trail ≡ (λL (M, S). (L#M, S))

abbreviation raw-tl-trail :: 'a list × 'b × 'c × 'd × 'e ⇒ 'a list × 'b × 'c × 'd × 'e where
raw-tl-trail ≡ (λ(M, S). (tl M, S))

abbreviation raw-init-clss :: 'a × 'b × 'c × 'd × 'e ⇒ 'b where
raw-init-clss ≡ λ(M, N, -). N

abbreviation raw-learned-clss :: 'a × 'b × 'c × 'd × 'e ⇒ 'c where
raw-learned-clss ≡ λ(M, N, U, -). U

abbreviation raw-backtrack-lvl :: 'a × 'b × 'c × 'd × 'e ⇒ 'd where
raw-backtrack-lvl ≡ λ(M, N, U, k, -). k

abbreviation raw-update-backtrack-lvl :: 'd ⇒ 'a × 'b × 'c × 'd × 'e ⇒ 'a × 'b × 'c × 'd × 'e
  where
raw-update-backtrack-lvl ≡ λk (M, N, U, -, S). (M, N, U, k, S)

abbreviation raw-conflicting :: 'a × 'b × 'c × 'd × 'e ⇒ 'e where
raw-conflicting ≡ λ(M, N, U, k, D). D

abbreviation raw-update-conflicting :: 'e ⇒ 'a × 'b × 'c × 'd × 'e ⇒ 'a × 'b × 'c × 'd × 'e
  where
raw-update-conflicting ≡ λS (M, N, U, k, -). (M, N, U, k, S)

abbreviation raw-add-learned-cls where
raw-add-learned-cls ≡ λC (M, N, U, S). (M, N, {#C#} + U, S)

abbreviation raw-remove-cls where
raw-remove-cls ≡ λC (M, N, U, S). (M, removeAll-mset C N, removeAll-mset C U, S)

```

type-synonym $'v \text{ cdcl}_W\text{-state-inv-st} = ('v, \text{nat}, 'v \text{ literal list}) \text{ marked-lit list} \times 'v \text{ literal list list} \times 'v \text{ literal list list} \times \text{nat} \times 'v \text{ literal list option}$

abbreviation $\text{raw-}S0\text{-cdcl}_W \ N \equiv (([], N, [], 0, \text{None}) :: 'v \text{ cdcl}_W\text{-state-inv-st})$

fun $\text{mmset-of-mlit}' :: ('v, \text{nat}, 'v \text{ literal list}) \text{ marked-lit} \Rightarrow ('v, \text{nat}, 'v \text{ clause}) \text{ marked-lit}$
where

$\text{mmset-of-mlit}' (\text{Propagated } L \ C) = \text{Propagated } L \ (\text{mset } C) \mid$
 $\text{mmset-of-mlit}' (\text{Marked } L \ i) = \text{Marked } L \ i$

lemma $\text{lit-of-mmset-of-mlit}'[\text{simp}]$:
 $\text{lit-of } (\text{mmset-of-mlit}' \ x a) = \text{lit-of } x a$
by $(\text{induction } x a) \text{ auto}$

abbreviation trail **where**
 $\text{trail } S \equiv \text{map } \text{mmset-of-mlit}' (\text{raw-trail } S)$

abbreviation clauses-of-l **where**
 $\text{clauses-of-l} \equiv \lambda L. \text{mset } (\text{map } \text{mset } L)$

global-interpretation $\text{state}_W\text{-ops}$

$\text{mset} :: 'v \text{ literal list} \Rightarrow 'v \text{ clause}$
 $\text{op} \# \text{remove1}$

$\text{clauses-of-l } \text{op} \ @ \ \lambda L \ C. L \in \text{set } C \ \text{op} \# \ \lambda C. \text{remove1-cond } (\lambda L. \text{mset } L = \text{mset } C)$

$\text{mset } \lambda x s \ y s. \text{case-prod } \text{append } (\text{fold } (\lambda x \ (y s, z s). (\text{remove1 } x \ y s, x \# z s)) \ x s \ (y s, []))$
 $\text{op} \# \text{remove1}$

id id

$\lambda(M, -). \text{map } \text{mmset-of-mlit}' \ M \ \lambda(M, -). \text{hd } M$
 $\lambda(-, N, -). N$
 $\lambda(-, -, U, -). U$
 $\lambda(-, -, -, k, -). k$
 $\lambda(-, -, -, -, C). C$

$\lambda L \ (M, S). (L \# M, S)$
 $\lambda(M, S). (\text{tl } M, S)$
 $\lambda C \ (M, N, S). (M, C \# N, S)$
 $\lambda C \ (M, N, U, S). (M, N, C \# U, S)$
 $\lambda C \ (M, N, U, S). (M, \text{filter } (\lambda L. \text{mset } L \neq \text{mset } C) \ N, \text{filter } (\lambda L. \text{mset } L \neq \text{mset } C) \ U, S)$
 $\lambda(k :: \text{nat}) \ (M, N, U, -, D). (M, N, U, k, D)$
 $\lambda D \ (M, N, U, k, -). (M, N, U, k, D)$
 $\lambda N. ([], N, [], 0, \text{None})$
 $\lambda(-, N, U, -). ([], N, U, 0, \text{None})$
apply unfold-locales **by** $(\text{auto simp: hd-map comp-def map-tl ac-simps union-mset-list mset-map-mset-remove1-cond ex-mset})$

lemma $\text{mmset-of-mlit}'\text{-mmset-of-mlit}$: $\text{mmset-of-mlit}' \ l = \text{mmset-of-mlit } l$
apply $(\text{induct } l)$
apply auto
done

lemma $\text{clauses-of-l-filter-removeAll}$:

clauses-of-l [$L \leftarrow a . \text{mset } L \neq \text{mset } C$] = *mset* (*removeAll* (*mset* *C*) (*map mset a*))
by (*induct a*) *auto*

interpretation *state_w*

mset::'v literal list \Rightarrow *'v clause*

op # *remove1*

clauses-of-l op @ $\lambda L . C . L \in \text{set } C \text{ op } \# \lambda C . \text{remove1-cond } (\lambda L . \text{mset } L = \text{mset } C)$

mset $\lambda xs \text{ ys} . \text{case-prod append } (\text{fold } (\lambda x \text{ (ys, zs)} . (\text{remove1 } x \text{ ys}, x \# \text{zs})) \text{ xs } (\text{ys}, []))$
op # *remove1*

id id

$\lambda(M, -) . \text{map mmset-of-mlit}' M \lambda(M, -) . \text{hd } M$

$\lambda(-, N, -) . N$

$\lambda(-, -, U, -) . U$

$\lambda(-, -, -, k, -) . k$

$\lambda(-, -, -, -, C) . C$

$\lambda L (M, S) . (L \# M, S)$

$\lambda(M, S) . (\text{tl } M, S)$

$\lambda C (M, N, S) . (M, C \# N, S)$

$\lambda C (M, N, U, S) . (M, N, C \# U, S)$

$\lambda C (M, N, U, S) . (M, \text{filter } (\lambda L . \text{mset } L \neq \text{mset } C) N, \text{filter } (\lambda L . \text{mset } L \neq \text{mset } C) U, S)$

$\lambda(k::\text{nat}) (M, N, U, -, D) . (M, N, U, k, D)$

$\lambda D (M, N, U, k, -) . (M, N, U, k, D)$

$\lambda N . ([], N, [], 0, \text{None})$

$\lambda(-, N, U, -) . ([], N, U, 0, \text{None})$

apply *unfold-locales*

apply (*rename-tac S, case-tac S*)

by (*auto simp: hd-map comp-def map-tl ac-simps clauses-of-l-filter-removeAll*
mmset-of-mlit'-mmset-of-mlit)

global-interpretation *conflict-driven-clause-learning_w*

mset::'v literal list \Rightarrow *'v clause*

op # *remove1*

clauses-of-l op @ $\lambda L . C . L \in \text{set } C \text{ op } \# \lambda C . \text{remove1-cond } (\lambda L . \text{mset } L = \text{mset } C)$

mset $\lambda xs \text{ ys} . \text{case-prod append } (\text{fold } (\lambda x \text{ (ys, zs)} . (\text{remove1 } x \text{ ys}, x \# \text{zs})) \text{ xs } (\text{ys}, []))$
op # *remove1*

id id

$\lambda(M, -) . \text{map mmset-of-mlit}' M \lambda(M, -) . \text{hd } M$

$\lambda(-, N, -) . N$

$\lambda(-, -, U, -) . U$

$\lambda(-, -, -, k, -) . k$

$\lambda(-, -, -, -, C) . C$

$\lambda L (M, S) . (L \# M, S)$

$\lambda(M, S) . (\text{tl } M, S)$

$\lambda C (M, N, S) . (M, C \# N, S)$

$\lambda C (M, N, U, S) . (M, N, C \# U, S)$

$\lambda C (M, N, U, S). (M, \text{filter } (\lambda L. \text{mset } L \neq \text{mset } C) N, \text{filter } (\lambda L. \text{mset } L \neq \text{mset } C) U, S)$
 $\lambda(k::\text{nat}) (M, N, U, -, D). (M, N, U, k, D)$
 $\lambda D (M, N, U, k, -). (M, N, U, k, D)$
 $\lambda N. ([], N, [], 0, \text{None})$
 $\lambda(-, N, U, -). ([], N, U, 0, \text{None})$
by *intro-locales*

declare *state-simp*[*simp del*] *raw-clauses-def*[*simp*] *state-eq-def*[*simp*]
notation *state-eq* (**infix** \sim 50)
term *reduce-trail-to*

lemma *reduce-trail-to-map*[*simp*]:
reduce-trail-to (map *f* *M1*) = *reduce-trail-to* *M1*
by (*rule ext*) (*auto intro: reduce-trail-to-length*)

19.3 CDCL Implementation

19.3.1 Definition of the rules

Types **lemma** *true-clss-remdups*[*simp*]:
 $I \models_s (\text{mset} \circ \text{remdups}) 'N \longleftrightarrow I \models_s \text{mset} 'N$
by (*simp add: true-clss-def*)

lemma *satisfiable-mset-remdups*[*simp*]:
satisfiable ((*mset* \circ *remdups*) 'N) \longleftrightarrow *satisfiable* (*mset* 'N)
unfolding *satisfiable-carac*[*symmetric*] **by** *simp*

We need some functions to convert between our abstract state *nat cdcl_W-state* and the concrete state *'v cdcl_W-state-inv-st*.

abbreviation *convertC* :: 'a list option \Rightarrow 'a multiset option **where**
convertC \equiv *map-option mset*

lemma *convert-Propagated*[*elim!*]:
 $\text{mmset-of-mlit}' z = \text{Propagated } L C \implies (\exists C'. z = \text{Propagated } L C' \wedge C = \text{mset } C')$
by (*cases z*) *auto*

lemma *get-rev-level-map-convert*:
 $\text{get-rev-level} (\text{map } \text{mmset-of-mlit}' M) n x = \text{get-rev-level } M n x$
by (*induction M arbitrary: n rule: marked-lit-list-induct*) *auto*

lemma *get-level-map-convert*[*simp*]:
 $\text{get-level} (\text{map } \text{mmset-of-mlit}' M) = \text{get-level } M$
using *get-rev-level-map-convert*[*of rev M*] **by** (*simp add: rev-map*)

lemma *get-rev-level-map-mmsetof-mlit*[*simp*]:
 $\text{get-rev-level} (\text{map } \text{mmset-of-mlit } M) = \text{get-rev-level } M$
by (*induction M rule: marked-lit-list-induct*) (*auto intro!: ext*)

lemma *get-level-map-mmsetof-mlit*[*simp*]:
 $\text{get-level} (\text{map } \text{mmset-of-mlit } M) = \text{get-level } M$
using *get-rev-level-map-mmsetof-mlit*[*of rev M*] **unfolding** *rev-map* **by** *simp*

lemma *get-maximum-level-map-convert*[*simp*]:
 $\text{get-maximum-level} (\text{map } \text{mmset-of-mlit}' M) D = \text{get-maximum-level } M D$
by (*induction D*) (*auto simp add: get-maximum-level-plus*)

lemma *get-all-levels-of-marked-map-convert*[simp]:
get-all-levels-of-marked (map mmset-of-mlit' M) = (*get-all-levels-of-marked* M)
by (induction M rule: marked-lit-list-induct) auto

lemma *reduce-trail-to-empty-trail*[simp]:
reduce-trail-to F ([], aa, ab, ac, b) = ([], aa, ab, ac, b)
using *reduce-trail-to.simps* **by** auto

lemma *raw-trail-reduce-trail-to-length-le*:
assumes *length* F > *length* (raw-trail S)
shows raw-trail (*reduce-trail-to* F S) = []
using *assms* *trail-reduce-trail-to-length-le*[of S F]
by (cases S, cases *reduce-trail-to* F S) auto

lemma *reduce-trail-to*:
reduce-trail-to F S =
 ((if *length* (raw-trail S) ≥ *length* F
 then drop (*length* (raw-trail S) − *length* F) (raw-trail S)
 else []), raw-init-clss S, raw-learned-clss S, raw-backtrack-lvl S, raw-conflicting S)
 (is ?S = -)

proof (induction F S rule: *reduce-trail-to.induct*)
case (1 F S) **note** IH = this
show ?case
proof (cases raw-trail S)
case Nil
then show ?thesis **using** IH **by** (cases S) auto
next
case (Cons L M)
then show ?thesis
apply (cases Suc (*length* M) > *length* F)
prefer 2 **using** IH *reduce-trail-to-length-ne*[of S F] **apply** (cases S) **apply** auto[]
apply (subgoal-tac Suc (*length* M) − *length* F = Suc (*length* M − *length* F))
using *reduce-trail-to-length-ne*[of S F] IH **by** (cases S) (auto simp add:)
qed
qed

Definition an abstract type

typedef 'v cdcl_W-state-inv = {S::'v cdcl_W-state-inv-st. cdcl_W-all-struct-inv S}
morphisms *rough-state-of* state-of

proof
show ([], [], [], 0, None) ∈ {S. cdcl_W-all-struct-inv S}
by (auto simp add: cdcl_W-all-struct-inv-def)

qed

instantiation cdcl_W-state-inv :: (type) equal

begin

definition *equal-cdcl_W-state-inv* :: 'v cdcl_W-state-inv ⇒ 'v cdcl_W-state-inv ⇒ bool **where**
equal-cdcl_W-state-inv S S' = (*rough-state-of* S = *rough-state-of* S')

instance

by standard (simp add: *rough-state-of-inject* *equal-cdcl_W-state-inv-def*)
end

lemma *lits-of-map-convert*[simp]: *lits-of-l* (map mmset-of-mlit' M) = *lits-of-l* M
by (induction M rule: marked-lit-list-induct) simp-all

lemma *undefined-lit-map-convert*[*iff*]:
undefined-lit (map *mmset-of-mlit'* *M*) *L* \longleftrightarrow *undefined-lit* *M* *L*
by (auto *simp* add: *defined-lit-map image-image mmset-of-mlit'-mmset-of-mlit*)

lemma *true-annot-map-convert*[*simp*]: map *mmset-of-mlit'* *M* $\models_a N \longleftrightarrow M \models_a N$
by (*induction* *M* rule: *marked-lit-list-induct*) (*simp-all* add: *true-annot-def mmset-of-mlit'-mmset-of-mlit lits-of-def*)

lemma *true-annots-map-convert*[*simp*]: map *mmset-of-mlit'* *M* $\models_{as} N \longleftrightarrow M \models_{as} N$
unfolding *true-annots-def* **by** *auto*

lemmas *propagateE*

lemma *find-first-unit-clause-some-is-propagate*:
assumes *H*: *find-first-unit-clause* (*N* @ *U*) *M* = *Some* (*L*, *C*)
shows *propagate* (*M*, *N*, *U*, *k*, *None*) (*Propagated* *L* *C* # *M*, *N*, *U*, *k*, *None*)
using *assms*
by (auto *dest!*: *find-first-unit-clause-some intro!*: *propagate-rule*)

19.3.2 The Transitions

Propagate **definition** *do-propagate-step* **where**

do-propagate-step *S* =
(case *S* of
(*M*, *N*, *U*, *k*, *None*) \Rightarrow
(case *find-first-unit-clause* (*N* @ *U*) *M* of
Some (*L*, *C*) \Rightarrow (*Propagated* *L* *C* # *M*, *N*, *U*, *k*, *None*)
| *None* \Rightarrow (*M*, *N*, *U*, *k*, *None*))
| *S* \Rightarrow *S*)

lemma *do-propagate-step*:
do-propagate-step *S* $\neq S \implies$ *propagate* *S* (*do-propagate-step* *S*)
apply (*cases* *S*, *cases conflicting* *S*)
using *find-first-unit-clause-some-is-propagate*[*of raw-init-clss S raw-learned-clss S*]
by (auto *simp* add: *do-propagate-step-def split: option.splits*)

lemma *do-propagate-step-option*[*simp*]:
conflicting *S* $\neq \text{None} \implies$ *do-propagate-step* *S* = *S*
unfolding *do-propagate-step-def* **by** (*cases* *S*, *cases conflicting* *S*) *auto*
thm *prod-cases*

lemma *do-propagate-step-no-step*:
assumes *dist*: $\forall c \in \text{set } (\text{raw-clauses } S). \text{distinct } c$ **and**
prop-step: *do-propagate-step* *S* = *S*
shows *no-step propagate* *S*
proof (*standard*, *standard*)
fix *T*
assume *propagate* *S* *T*
then obtain *C* *L* **where**
toSS: *conflicting* *S* = *None* **and**
C: *C* $\in \text{set } (\text{raw-clauses } S)$ **and**
L: *L* $\in \text{set } C$ **and**
MC: *raw-trail* *S* $\models_{as} C \text{Not } (\text{mset } (\text{remove1 } L \ C))$ **and**
T: *T* $\sim \text{raw-cons-trail } (\text{Propagated } L \ C) \ S$ **and**
undef: *undefined-lit* (*raw-trail* *S*) *L*
apply (*cases* *S* rule: *prod-cases5*)
by (*elim propagateE*) *simp*

```

let ?M = raw-trail S
let ?N = raw-init-clss S
let ?U = raw-learned-clss S
let ?k = raw-backtrack-lvl S
let ?D = None
have S: S = (?M, ?N, ?U, ?k, ?D)
  using toSS by (cases S, cases conflicting S) simp-all

have find-first-unit-clause (?N @ ?U) ?M ≠ None
  apply (rule dist find-first-unit-clause-none[of C ?N @ ?U ?M L, OF -])
    using C dist apply auto[]
    using C apply auto[1]
    using MC apply auto[1]
    using undef apply auto[1]
    using L by auto
then show False using prop-step S unfolding do-propagate-step-def by (cases S) auto
qed

```

Conflict fun find-conflict where

```

find-conflict M [] = None |
find-conflict M (N # Ns) = (if (∀ c ∈ set N. ¬c ∈ lits-of-l M) then Some N else find-conflict M Ns)

```

lemma find-conflict-Some:

```

find-conflict M Ns = Some N ⟹ N ∈ set Ns ∧ M ⊨as CNot (mset N)
by (induction Ns rule: find-conflict.induct)
  (auto split: if-split-asm)

```

lemma find-conflict-None:

```

find-conflict M Ns = None ⟷ (∀ N ∈ set Ns. ¬M ⊨as CNot (mset N))
by (induction Ns) auto

```

lemma find-conflict-None-no-conf:

```

find-conflict M (N@U) = None ⟷ no-step conflict (M, N, U, k, None)
by (auto simp add: find-conflict-None conflict.simps)

```

definition do-conflict-step where

```

do-conflict-step S =
  (case S of
    (M, N, U, k, None) ⇒
      (case find-conflict M (N @ U) of
        Some a ⇒ (M, N, U, k, Some a)
      | None ⇒ (M, N, U, k, None))
  | S ⇒ S)

```

lemma do-conflict-step:

```

do-conflict-step S ≠ S ⟹ conflict S (do-conflict-step S)
apply (cases S, cases conflicting S)
unfolding conflict.simps do-conflict-step-def
by (auto dest!: find-conflict-Some split: option.splits simp: state-eq-def)

```

lemma do-conflict-step-no-step:

```

do-conflict-step S = S ⟹ no-step conflict S
apply (cases S, cases conflicting S)
unfolding do-conflict-step-def
using find-conflict-None-no-conf[of raw-trail S raw-init-clss S raw-learned-clss S]

```



```

    raw-backtrack-lvl S]
  by (auto split: option.split elim: conflictE)

lemma do-conflict-step-option[simp]:
  conflicting S ≠ None ⇒ do-conflict-step S = S
  unfolding do-conflict-step-def by (cases S, cases conflicting S) auto

lemma do-conflict-step-conflicting[dest]:
  do-conflict-step S ≠ S ⇒ conflicting (do-conflict-step S) ≠ None
  unfolding do-conflict-step-def by (cases S, cases conflicting S) (auto split: option.splits)

definition do-cp-step where
  do-cp-step S =
    (do-propagate-step o do-conflict-step) S

lemma cp-step-is-cdclW-cp:
  assumes H: do-cp-step S ≠ S
  shows cdclW-cp S (do-cp-step S)
proof -
  show ?thesis
proof (cases do-conflict-step S ≠ S)
  case True
  then have do-propagate-step (do-conflict-step S) = do-conflict-step S
    by auto
  then show ?thesis
    by (auto simp add: do-conflict-step do-conflict-step-conflicting do-cp-step-def True)
next
  case False
  then have confl[simp]: do-conflict-step S = S by simp
  show ?thesis
  proof (cases do-propagate-step S = S)
    case True
    then show ?thesis
      using H by (simp add: do-cp-step-def)
  next
    case False
    let ?S = S
    let ?T = (do-propagate-step S)
    let ?U = (do-conflict-step (do-propagate-step S))
    have propa: propagate S ?T using False do-propagate-step by blast
    moreover have ns: no-step conflict S using confl do-conflict-step-no-step by blast
    ultimately show ?thesis
      using cdclW-cp.intros(2)[of ?S ?T] confl unfolding do-cp-step-def by auto
  qed
qed
qed

lemma do-cp-step-eq-no-prop-no-confl:
  do-cp-step S = S ⇒ do-conflict-step S = S ∧ do-propagate-step S = S
  by (cases S, cases raw-conflicting S)
  (auto simp add: do-conflict-step-def do-propagate-step-def do-cp-step-def split: option.splits)

lemma no-cdclW-cp-iff-no-propagate-no-conflict:
  no-step cdclW-cp S ⇔ no-step propagate S ∧ no-step conflict S
  by (auto simp: cdclW-cp.simps)

```

lemma *do-cp-step-eq-no-step*:

assumes

H: *do-cp-step S = S* **and**

$\forall c \in \text{set } (\text{raw-init-clss } S @ \text{raw-learned-clss } S). \text{ distinct } c$

shows *no-step cdcl_W-cp S*

unfolding *no-cdcl_W-cp-iff-no-propagate-no-conflict*

using *assms* **apply** (*cases S, cases conflicting S*)

using *do-propagate-step-no-step[of S]*

by (*auto dest!: do-cp-step-eq-no-prop-no-confl[simplified] do-conflict-step-no-step split: option.splits*)

lemma *cdcl_W-cp-cdcl_W-st*: *cdcl_W-cp S S' \implies cdcl_W** S S'*

by (*simp add: cdcl_W-cp-tranclp-cdcl_W tranclp-into-rtranclp*)

lemma *cdcl_W-all-struct-inv-rough-state[simp]*: *cdcl_W-all-struct-inv (rough-state-of S)*

using *rough-state-of* **by** *auto*

lemma [*simp*]: *cdcl_W-all-struct-inv S \implies rough-state-of (state-of S) = S*

by (*simp add: state-of-inverse*)

lemma *rough-state-of-state-of-do-cp-step[simp]*:

rough-state-of (state-of (do-cp-step (rough-state-of S))) = do-cp-step (rough-state-of S)

proof –

have *cdcl_W-all-struct-inv (do-cp-step (rough-state-of S))*

apply (*cases do-cp-step (rough-state-of S) = (rough-state-of S)*)

apply *simp*

using *cp-step-is-cdcl_W-cp[of rough-state-of S] cdcl_W-all-struct-inv-rough-state[of S]*

cdcl_W-cp-cdcl_W-st rtranclp-cdcl_W-all-struct-inv-inv **by** *blast*

then show *?thesis* **by** *auto*

qed

Skip fun *do-skip-step* :: '*v* *cdcl_W-state-inv-st \Rightarrow 'v cdcl_W-state-inv-st* **where**

do-skip-step (Propagated L C # Ls, N, U, k, Some D) =

(if $-L \notin \text{set } D \wedge D \neq []$

then (Ls, N, U, k, Some D)

else (Propagated L C # Ls, N, U, k, Some D)) |

do-skip-step S = S

lemma *do-skip-step*:

do-skip-step S \neq S \implies skip S (do-skip-step S)

apply (*induction S rule: do-skip-step.induct*)

by (*auto simp add: skip.simps*)

lemma *do-skip-step-no*:

do-skip-step S = S \implies no-step skip S

by (*induction S rule: do-skip-step.induct*)

(auto simp add: other split: if-split-asm elim!: skipE)

lemma *do-skip-step-trail-is-None[iff]*:

do-skip-step S = (a, b, c, d, None) \longleftrightarrow S = (a, b, c, d, None)

by (*cases S rule: do-skip-step.cases*) *auto*

Resolve fun *maximum-level-code*:: '*a* *literal list \Rightarrow ('a, nat, 'a literal list) marked-lit list \Rightarrow nat*

where

$\text{maximum-level-code } [] = 0$ |
 $\text{maximum-level-code } (L \# Ls) M = \max (\text{get-level } M L) (\text{maximum-level-code } Ls M)$

lemma *maximum-level-code-eq-get-maximum-level*[code, simp]:
 $\text{maximum-level-code } D M = \text{get-maximum-level } M (\text{mset } D)$
by (induction D) (auto simp add: get-maximum-level-plus)

fun *do-resolve-step* :: 'v *cdcl_W-state-inv-st* \Rightarrow 'v *cdcl_W-state-inv-st* **where**
do-resolve-step (Propagated L C # Ls, N, U, k, Some D) =
 (if $-L \in \text{set } D \wedge \text{maximum-level-code } (\text{remove1 } (-L) D) (\text{Propagated } L C \# Ls) = k$
 then (Ls, N, U, k, Some (remdups (remove1 L C @ remove1 (-L) D)))
 else (Propagated L C # Ls, N, U, k, Some D)) |
do-resolve-step S = S

lemma *do-resolve-step*:
 $\text{cdcl}_W\text{-all-struct-inv } S \implies \text{do-resolve-step } S \neq S$
 $\implies \text{resolve } S (\text{do-resolve-step } S)$

proof (induction S rule: *do-resolve-step.induct*)

case (1 L C M N U k D)

then have

LD: $-L \in \text{set } D$ **and**

M: $\text{maximum-level-code } (\text{remove1 } (-L) D) (\text{Propagated } L C \# M) = k$

by (cases mset D - {#- L#} = {#},

auto dest!: *get-maximum-level-exists-lit-of-max-level*[of - Propagated L C # M]

split: *if-split-asm*) +

have *every-mark-is-a-conflict* (Propagated L C # M, N, U, k, Some D)

using 1(1) **unfolding** *cdcl_W-all-struct-inv-def* *cdcl_W-conflicting-def* **by** *fast*

then have LC: $L \in \text{set } C$ **by** *fastforce*

then obtain C' **where** C: $\text{mset } C = C' + \{\#L\# \}$

by (*metis* add.commute in-multiset-in-set insert-DiffM)

obtain D' **where** D: $\text{mset } D = D' + \{\#-L\# \}$

using $\langle -L \in \text{set } D \rangle$ **by** (*metis* add.commute in-multiset-in-set insert-DiffM)

have D'L: $D' + \{\#-L\# \} - \{\#-L\# \} = D'$ **by** (auto simp add: *multiset-eq-iff*)

have CL: $\text{mset } C - \{\#L\# \} + \{\#L\# \} = \text{mset } C$ **using** $\langle L \in \text{set } C \rangle$ **by** (auto simp add: *multiset-eq-iff*)

have max: $\text{get-maximum-level } (\text{Propagated } L (C' + \{\#L\# \}) \# \text{map mmset-of-mlit } M) D' = k$

using M[simplified] **unfolding** *maximum-level-code-eq-get-maximum-level* C[symmetric] CL

by (*metis* D D'L *get-maximum-level-map-convert list.simps(9) mmset-of-mlit'.simps(1)*)

have *distinct-mset* (mset C) **and** *distinct-mset* (mset D)

using $\langle \text{cdcl}_W\text{-all-struct-inv } (\text{Propagated } L C \# M, N, U, k, \text{Some } D) \rangle$

unfolding *cdcl_W-all-struct-inv-def* *distinct-cdcl_W-state-def*

by *auto*

then have conf: $(\text{mset } C - \{\#L\# \}) \# \cup (\text{mset } D - \{\#-L\# \}) =$

$\text{remdups-mset } (\text{mset } C - \{\#L\# \} + (\text{mset } D - \{\#-L\# \}))$

by (auto simp: *distinct-mset-remdups-union-mset*)

show ?case

apply (rule *resolve-rule*)

using LC LD max M conf C D **by** (auto simp: *subset-mset.sup.commute*)

qed *auto*

lemma *do-resolve-step-no*:

$\text{do-resolve-step } S = S \implies \text{no-step resolve } S$

apply (cases S; cases (raw-trail S); cases *raw-conflicting* S)

by (auto

elim!: *resolveE* split: *if-split-asm*

dest!: union-single-eq-member
simp del: in-multiset-in-set get-maximum-level-map-convert
simp: get-maximum-level-map-convert[symmetric] do-resolve-step

lemma *rough-state-of-state-of-resolve*[simp]:
 $cdcl_W\text{-all-struct-inv } S \implies \text{rough-state-of } (\text{state-of } (\text{do-resolve-step } S)) = \text{do-resolve-step } S$
apply (rule *state-of-inverse*)
apply (cases *do-resolve-step* $S = S$)
apply *simp*
by (blast *dest*: other resolve *bj* *do-resolve-step* *cdcl_W*-all-struct-inv-inv)

lemma *do-resolve-step-trail-is-None*[iff]:
 $\text{do-resolve-step } S = (a, b, c, d, \text{None}) \longleftrightarrow S = (a, b, c, d, \text{None})$
by (cases S rule: *do-resolve-step.cases*) auto

Backjumping **fun** *find-level-decomp* **where**

find-level-decomp $M \ [] \ D \ k = \text{None} \mid$
find-level-decomp $M \ (L \ \# \ Ls) \ D \ k =$
 (case (*get-level* $M \ L$, *maximum-level-code* ($D \ @ \ Ls$) M) of
 (i, j) \Rightarrow if $i = k \wedge j < i$ then *Some* (L, j) else *find-level-decomp* $M \ Ls \ (L \ \# \ D) \ k$
)

lemma *find-level-decomp-some*:
assumes *find-level-decomp* $M \ Ls \ D \ k = \text{Some } (L, j)$
shows $L \in \text{set } Ls \wedge \text{get-maximum-level } M \ (\text{mset } (\text{remove1 } L \ (Ls \ @ \ D))) = j \wedge \text{get-level } M \ L = k$
using *assms*

proof (induction Ls arbitrary: D)

case *Nil*
then show ?case **by** *simp*

next

case (*Cons* $L' \ Ls$) **note** $IH = \text{this}(1)$ **and** $H = \text{this}(2)$

def *find* \equiv (if *get-level* $M \ L' \neq k \vee \neg \text{get-maximum-level } M \ (\text{mset } D + \text{mset } Ls) < \text{get-level } M \ L'$
 then *find-level-decomp* $M \ Ls \ (L' \ \# \ D) \ k$
 else *Some* ($L', \text{get-maximum-level } M \ (\text{mset } D + \text{mset } Ls)$))

have $a1: \bigwedge D. \text{find-level-decomp } M \ Ls \ D \ k = \text{Some } (L, j) \implies$
 $L \in \text{set } Ls \wedge \text{get-maximum-level } M \ (\text{mset } Ls + \text{mset } D - \{\#L\# \}) = j \wedge \text{get-level } M \ L = k$
using IH **by** *simp*

have $a2: \text{find} = \text{Some } (L, j)$
using H **unfolding** *find-def* **by** (auto *split*: *if-split-asm*)

{ assume *Some* ($L', \text{get-maximum-level } M \ (\text{mset } D + \text{mset } Ls)$) $\neq \text{find}$
then have $f3: L \in \text{set } Ls$ **and** $\text{get-maximum-level } M \ (\text{mset } Ls + \text{mset } (L' \ \# \ D) - \{\#L\# \}) = j$
using $a1 \ IH \ a2$ **unfolding** *find-def* **by** *meson*+

moreover then have $\text{mset } Ls + \text{mset } D - \{\#L\# \} + \{\#L'\# \} = \{\#L'\# \} + \text{mset } D + (\text{mset } Ls - \{\#L\# \})$
by (auto *simp*: *ac-simps* *multiset-eq-iff* *Suc-leI*)

ultimately have $f4: \text{get-maximum-level } M \ (\text{mset } Ls + \text{mset } D - \{\#L\# \} + \{\#L'\# \}) = j$
by (*metis* *add.commute* *diff-union-single-conv* *in-multiset-in-set* *mset.simps*(2))

} note $f4 = \text{this}$

have $\{\#L'\# \} + (\text{mset } Ls + \text{mset } D) = \text{mset } Ls + (\text{mset } D + \{\#L'\# \})$
by (auto *simp*: *ac-simps*)

then have

$(L = L' \longrightarrow \text{get-maximum-level } M \ (\text{mset } Ls + \text{mset } D) = j \wedge \text{get-level } M \ L' = k)$ **and**
 $(L \neq L' \longrightarrow L \in \text{set } Ls \wedge \text{get-maximum-level } M \ (\text{mset } Ls + \text{mset } D - \{\#L\# \} + \{\#L'\# \}) = j \wedge \text{get-level } M \ L = k)$

```

using f4 a2 a1[of L' # D] unfolding find-def by (metis (no-types) add-diff-cancel-left'
  mset.simps(2) option.inject prod.inject union-commute) +
then show ?case by simp
qed

lemma find-level-decomp-none:
  assumes find-level-decomp M Ls E k = None and mset (L#D) = mset (Ls @ E)
  shows  $\neg(L \in \text{set } Ls \wedge \text{get-maximum-level } M (\text{mset } D) < k \wedge k = \text{get-level } M L)$ 
  using assms
proof (induction Ls arbitrary: E L D)
  case Nil
  then show ?case by simp
next
  case (Cons L' Ls) note IH = this(1) and find-none = this(2) and LD = this(3)
  have mset D + {#L'#} = mset E + (mset Ls + {#L'#})  $\implies$  mset D = mset E + mset Ls
    by (metis add-right-imp-eq union-assoc)
  then show ?case
    using find-none IH[of L' # E L D] LD by (auto simp add: ac-simps split: if-split-asm)
qed

```

```

fun bt-cut where
  bt-cut i (Propagated - - # Ls) = bt-cut i Ls |
  bt-cut i (Marked K k # Ls) = (if k = Suc i then Some (Marked K k # Ls) else bt-cut i Ls) |
  bt-cut i [] = None

```

```

lemma bt-cut-some-decomp:
  bt-cut i M = Some M'  $\implies \exists K M2 M1. M = M2 @ M' \wedge M' = \text{Marked } K (i+1) \# M1$ 
  by (induction i M rule: bt-cut.induct) (auto split: if-split-asm)

```

```

lemma bt-cut-not-none: M = M2 @ Marked K (Suc i) # M'  $\implies$  bt-cut i M  $\neq$  None
  by (induction M2 arbitrary: M rule: marked-lit-list-induct) auto

```

```

lemma get-all-marked-decomposition-ex:
   $\exists N. (\text{Marked } K (\text{Suc } i) \# M', N) \in \text{set } (\text{get-all-marked-decomposition } (M2 @ \text{Marked } K (\text{Suc } i) \# M'))$ 
  apply (induction M2 rule: marked-lit-list-induct)
  apply auto[2]
  by (rename-tac L m xs, case-tac get-all-marked-decomposition (xs @ Marked K (Suc i) # M'))
  auto

```

```

lemma bt-cut-in-get-all-marked-decomposition:
  bt-cut i M = Some M'  $\implies \exists M2. (M', M2) \in \text{set } (\text{get-all-marked-decomposition } M)$ 
  by (auto dest!: bt-cut-some-decomp simp add: get-all-marked-decomposition-ex)

```

```

fun do-backtrack-step where
  do-backtrack-step (M, N, U, k, Some D) =
    (case find-level-decomp M D [] k of
      None  $\Rightarrow$  (M, N, U, k, Some D)
    | Some (L, j)  $\Rightarrow$ 
      (case bt-cut j M of
        Some (Marked - - # Ls)  $\Rightarrow$  (Propagated L D # Ls, N, D # U, j, None)
      | -  $\Rightarrow$  (M, N, U, k, Some D))
    ) |
  do-backtrack-step S = S

```

```

lemma get-all-marked-decomposition-map-convert:
  (get-all-marked-decomposition (map mmset-of-mlit' M)) =
    map (λ(a, b). (map mmset-of-mlit' a, map mmset-of-mlit' b)) (get-all-marked-decomposition M)
apply (induction M rule: marked-lit-list-induct)
apply simp
by (rename-tac L l xs, case-tac get-all-marked-decomposition xs; auto)+

lemma do-backtrack-step:
assumes
  db: do-backtrack-step S ≠ S and
  inv: cdclW-all-struct-inv S
shows backtrack S (do-backtrack-step S)
proof (cases S, cases raw-conflicting S, goal-cases)
  case (1 M N U k E)
  then show ?case using db by auto
next
  case (2 M N U k E C) note S = this(1) and confl = this(2)
  have E: E = Some C using S confl by auto

  obtain L j where fd: find-level-decomp M C [] k = Some (L, j)
    using db unfolding S E by (cases C) (auto split: if-split-asm option.splits)
  have
    L ∈ set C and
    j: get-maximum-level M (mset (remove1 L C)) = j and
    levL: get-level M L = k
    using find-level-decomp-some[OF fd] by auto
  obtain C' where C: mset C = mset C' + {#L#}
    using ⟨L ∈ set C⟩ by (metis add.commute ex-mset in-multiset-in-set insert-DiffM)
  obtain M2 where M2: bt-cut j M = Some M2
    using db fd unfolding S E by (auto split: option.splits)
  obtain M1 K where M1: M2 = Marked K (Suc j) # M1
    using bt-cut-some-decomp[OF M2] by (cases M2) auto
  obtain c where c: M = c @ Marked K (Suc j) # M1
    using bt-cut-in-get-all-marked-decomposition[OF M2]
    unfolding M1 by fastforce
  have get-all-levels-of-marked (map mmset-of-mlit' M) = rev [1..Suc k]
    using inv unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def S by auto
  from arg-cong[OF this, of λa. Suc j ∈ set a] have j ≤ k unfolding c by auto
  have max-l-j: maximum-level-code C' M = j
    using db fd M2 C unfolding S E by (auto
      split: option.splits list.splits marked-lit.splits
      dest!: find-level-decomp-some)[1]
  have get-maximum-level M (mset C) ≥ k
    using ⟨L ∈ set C⟩ levL get-maximum-level-ge-get-level by (metis set-mset-mset)
  moreover have get-maximum-level M (mset C) ≤ k
    using get-maximum-level-exists-lit-of-max-level[of mset C M] inv
    cdclW-M-level-inv-get-level-le-backtrack-lvl[of S]
    unfolding C cdclW-all-struct-inv-def S by (auto dest: sym[of get-level - -])
  ultimately have get-maximum-level M (mset C) = k by auto

  obtain M2 where M2: (M2, M2) ∈ set (get-all-marked-decomposition M)
    using bt-cut-in-get-all-marked-decomposition[OF M2] by metis
  have decomp:
    (Marked K (Suc (get-maximum-level M (remove1-mset L (mset C)))) # (map mmset-of-mlit' M1),
    (map mmset-of-mlit' M2)) ∈

```

```

    set (get-all-marked-decomposition (map mmset-of-mlit' M))
    using imageI[of - - λ(a, b). (map mmset-of-mlit' a, map mmset-of-mlit' b), OF M2] j
    unfolding S E M1 by (auto simp add: get-all-marked-decomposition-map-convert)
have red: (reduce-trail-to (map mmset-of-mlit' M1)
  (M, N, C # U, get-maximum-level M (remove1-mset L (mset C)), None))
  = (M1, N, C # U, get-maximum-level M (remove1-mset L (mset C)), None)
    using M2 M1 by (auto simp: reduce-trail-to)
show ?case
  apply (rule backtrack-rule)
    using M2 fd confl ⟨L ∈ set C⟩ j decomp levL ⟨get-maximum-level M (mset C) = k⟩
    unfolding S E M1 apply (auto simp: mset-map)[6]
    unfolding CDCL-W-Implementation.state-eq-def
    using M2 fd confl ⟨L ∈ set C⟩ j decomp levL ⟨get-maximum-level M (mset C) = k⟩ red
    unfolding S E M1
    by auto
qed

lemma map-eq-list-length:
  map f L = L' ⟹ length L = length L'
  by auto

lemma map-mmset-of-mlit-eq-cons:
  assumes map mmset-of-mlit' M = a @ c
  obtains a' c' where
    M = a' @ c' and
    a = map mmset-of-mlit' a' and
    c = map mmset-of-mlit' c'
  using that[of take (length a) M drop (length a) M]
  assms by (metis append-eq-conv-conj append-take-drop-id drop-map take-map)

lemma do-backtrack-step-no:
  assumes
    db: do-backtrack-step S = S and
    inv: cdclW-all-struct-inv S
  shows no-step backtrack S
proof (rule ccontr, cases S, cases conflicting S, goal-cases)
  case 1
  then show ?case using db by (auto split: option.splits elim: backtrackE)
next
  case (2 M N U k E C) note bt = this(1) and S = this(2) and confl = this(3)
  obtain K j M1 M2 L D where
    CE: raw-conflicting S = Some D and
    LD: L ∈# mset D and
    decomp: (Marked K (Suc j) # M1, M2) ∈ set (get-all-marked-decomposition (trail S)) and
    levL: get-level (raw-trail S) L = raw-backtrack-lvl S and
    k: get-level (raw-trail S) L = get-maximum-level (raw-trail S) (mset D) and
    j: get-maximum-level (raw-trail S) (remove1-mset L (mset D)) ≡ j and
    undef: undefined-lit M1 L
  using bt apply clarsimp
  apply (elim backtrack-levE)
    using inv unfolding cdclW-all-struct-inv-def apply fast
  apply (cases S)
  by (auto simp add: get-all-marked-decomposition-map-convert)

  obtain c where c: trail S = c @ M2 @ Marked K (Suc j) # M1

```

```

    using decomp by blast
have get-all-levels-of-marked (trail S) = rev [1..<Suc k]
    using inv unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def S by auto
from arg-cong[OF this, of  $\lambda a. \text{Suc } j \in \text{set } a$ ] have k > j
    unfolding c by (auto simp: get-all-marked-decomposition-map-convert)
have [simp]: L ∈ set D
    using LD by auto
have CD: C = mset D
    using CE confl by auto
obtain D' where
  E: E = Some D and
  DD': mset D = {#L#} + mset D'
    using that[of remove1 L D]
    using S CE confl LD by (auto simp add: insert-DiffM)
have find-level-decomp M D [] k ≠ None
    apply rule
    apply (drule find-level-decomp-none[of - - - L D'])
    using DD' <k > j mset-eq-setD S levL unfolding k[symmetric] j[symmetric]
    by (auto simp: ac-simps)
then obtain L' j' where fd-some: find-level-decomp M D [] k = Some (L', j')
    by (cases find-level-decomp M D [] k) auto
have L': L' = L
    proof (rule ccontr)
      assume  $\neg$  ?thesis
      then have L' ∈ # mset (remove1 L D)
        by (metis fd-some find-level-decomp-some in-set-remove1 set-mset-mset)
      then have get-level M L' ≤ get-maximum-level M (mset (remove1 L D))
        using get-maximum-level-ge-get-level by blast
      then show False using <k > j j find-level-decomp-some[OF fd-some] S DD' by auto
    qed
then have j': j' = j using find-level-decomp-some[OF fd-some] j S DD' by auto

obtain c' M1' where cM: M = c' @ Marked K (Suc j) # M1'
    apply (rule map-mmset-of-mlit-eq-cons[of M c @ M2 Marked K (Suc j) # M1])
    using c S apply simp
    apply (rule map-mmset-of-mlit-eq-cons[of - [Marked K (Suc j)] M1])
    apply auto[]
    apply (rename-tac a b' aa b, case-tac aa)
    apply auto[]
    apply (rename-tac a b' aa b, case-tac aa)
    by auto
have btc-none: bt-cut j M ≠ None
    apply (rule bt-cut-not-none[of M ])
    using cM by simp
show ?case using db unfolding S E
    by (auto split: option.splits list.splits marked-lit.splits
      simp add: fd-some L' j' btc-none
      dest: bt-cut-some-decomp)
qed

lemma rough-state-of-state-of-backtrack[simp]:
  assumes inv: cdclW-all-struct-inv S
  shows rough-state-of (state-of (do-backtrack-step S)) = do-backtrack-step S
proof (rule state-of-inverse)
  have f2: backtrack S (do-backtrack-step S) ∨ do-backtrack-step S = S

```



```

    using do-backtrack-step inv by blast
have  $\bigwedge p. \neg \text{cdcl}_W\text{-o } S \ p \vee \text{cdcl}_W\text{-all-struct-inv } p$ 
    using inv cdclW-all-struct-inv-inv other by blast
then have do-backtrack-step  $S = S \vee \text{cdcl}_W\text{-all-struct-inv } (\text{do-backtrack-step } S)$ 
    using f2 inv cdclW-o.intros cdclW-bj.intros by blast
then show do-backtrack-step  $S \in \{S. \text{cdcl}_W\text{-all-struct-inv } S\}$ 
    using inv by fastforce
qed

```

Decide fun do-decide-step where
do-decide-step (M, N, U, k, None) =
(case find-first-unused-var N (lits-of-l M) of
None $\Rightarrow (M, N, U, k, \text{None})$
| Some $L \Rightarrow (\text{Marked } L (\text{Suc } k) \# M, N, U, k+1, \text{None}))$ |
do-decide-step $S = S$

lemma do-decide-step:
fixes $S :: 'v \text{ cdcl}_W\text{-state-inv-st}$
assumes do-decide-step $S \neq S$
shows decide S (do-decide-step S)
using assms
apply (cases S , cases conflicting S)
defer
apply (auto split: option.splits simp add: decide.simps Marked-Propagated-in-iff-in-lits-of-l
dest: find-first-unused-var-undefined find-first-unused-var-Some
intro:)[1]

proof –
fix $a :: ('v, \text{nat}, 'v \text{ literal list}) \text{ marked-lit list and}$
 $b :: 'v \text{ literal list list and } c :: 'v \text{ literal list list and}$
 $d :: \text{nat and } e :: 'v \text{ literal list option}$
{
fix $a :: ('v, \text{nat}, 'v \text{ literal list}) \text{ marked-lit list and}$
 $b :: 'v \text{ literal list list and } c :: 'v \text{ literal list list and}$
 $d :: \text{nat and } x2 :: 'v \text{ literal and } m :: 'v \text{ literal list}$
assume $a1: m \in \text{set } b$
assume $x2 \in \text{set } m$
then have $f2: \text{atm-of } x2 \in \text{atms-of } (\text{mset } m)$
by simp
have $\bigwedge f. (f \ m :: 'v \text{ clause}) \in f \text{ ' set } b$
using $a1$ by blast
then have $\bigwedge f. (\text{atms-of } (f \ m) :: 'v \text{ set}) \subseteq \text{atms-of-ms } (f \text{ ' set } b)$
by simp
then have $\bigwedge n f. (n :: 'v) \in \text{atms-of-ms } (f \text{ ' set } b) \vee n \notin \text{atms-of } (f \ m)$
by (meson contra-subsetD)
then have $\text{atm-of } x2 \in \text{atms-of-ms } (\text{mset ' set } b)$
using $f2$ by blast
} note $H = \text{this}$
{
fix $m :: 'v \text{ literal list and } x2$
have $m \in \text{set } b \implies x2 \in \text{set } m \implies x2 \notin \text{lits-of-l } a \implies \neg x2 \notin \text{lits-of-l } a \implies$
 $\exists aa \in \text{set } b. \neg \text{atm-of ' set } aa \subseteq \text{atm-of ' lits-of-l } a$
by (meson atm-of-in-atm-of-set-in-uminus contra-subsetD rev-image-eqI)
} note $H' = \text{this}$

assume do-decide-step $S \neq S$ **and**

```

    S = (a, b, c, d, e) and
    conflicting S = None
  then show decide S (do-decide-step S)
  using H H' by (auto split: option.splits simp: lits-of-def decide.simps
    Marked-Propagated-in-iff-in-lits-of-l
    dest!: find-first-unused-var-Some)
qed

lemma mmset-of-mlit'-eq-Marked[iff]: mmset-of-mlit' z = Marked x k  $\longleftrightarrow$  z = Marked x k
  by (cases z) auto

lemma do-decide-step-no:
  do-decide-step S = S  $\implies$  no-step decide S
  apply (cases S, cases conflicting S)

  apply (auto simp: atms-of-ms-mset-unfold Marked-Propagated-in-iff-in-lits-of-l lits-of-def
    dest!: atm-of-in-atm-of-set-in-uminus
    elim!: decideE
    split: option.splits)+
  using atm-of-eq-atm-of by blast

lemma rough-state-of-state-of-do-decide-step[simp]:
  cdclW-all-struct-inv S  $\implies$  rough-state-of (state-of (do-decide-step S)) = do-decide-step S
proof (subst state-of-inverse, goal-cases)
  case 1
  then show ?case
  by (cases do-decide-step S = S)
    (auto dest: do-decide-step decide other intro: cdclW-all-struct-inv-inv)
qed simp

lemma rough-state-of-state-of-do-skip-step[simp]:
  cdclW-all-struct-inv S  $\implies$  rough-state-of (state-of (do-skip-step S)) = do-skip-step S
  apply (subst state-of-inverse, cases do-skip-step S = S)
  apply simp
  by (blast dest: other skip bj do-skip-step cdclW-all-struct-inv-inv)+

```

19.3.3 Code generation

Type definition There are two invariants: one while applying conflict and propagate and one for the other rules

```

declare rough-state-of-inverse[simp add]
definition Con where
  Con xs = state-of (if cdclW-all-struct-inv xs then xs else ([], [], [], 0, None))

```

```

lemma [code abstype]:
  Con (rough-state-of S) = S
  using rough-state-of[of S] unfolding Con-def by simp

```

```

definition do-cp-step' where
  do-cp-step' S = state-of (do-cp-step (rough-state-of S))

```

```

typedef 'v cdclW-state-inv-from-init-state = {S::'v cdclW-state-inv-st. cdclW-all-struct-inv S
   $\wedge$  cdclW-stgy** (raw-S0-cdclW (raw-init-clss S)) S}
morphisms rough-state-from-init-state-of state-from-init-state-of
proof

```

show ($\square, \square, \square, 0, None$) $\in \{S. \text{cdcl}_W\text{-all-struct-inv } S$
 $\wedge \text{cdcl}_W\text{-stgy}^{**} (\text{raw-}S0\text{-cdcl}_W (\text{raw-init-clss } S)) S\}$
by (*auto simp add: cdcl_W-all-struct-inv-def*)
qed

instantiation $\text{cdcl}_W\text{-state-inv-from-init-state} :: (\text{type}) \text{ equal}$
begin

definition $\text{equal-cdcl}_W\text{-state-inv-from-init-state} :: 'v \text{ cdcl}_W\text{-state-inv-from-init-state} \Rightarrow$
 $'v \text{ cdcl}_W\text{-state-inv-from-init-state} \Rightarrow \text{bool}$ **where**
 $\text{equal-cdcl}_W\text{-state-inv-from-init-state } S S' \longleftrightarrow$
 $(\text{rough-state-from-init-state-of } S = \text{rough-state-from-init-state-of } S')$

instance

by *standard (simp add: rough-state-from-init-state-of-inject*
 $\text{equal-cdcl}_W\text{-state-inv-from-init-state-def})$
end

definition ConI **where**

$\text{ConI } S = \text{state-from-init-state-of (if } \text{cdcl}_W\text{-all-struct-inv } S$
 $\wedge \text{cdcl}_W\text{-stgy}^{**} (\text{raw-}S0\text{-cdcl}_W (\text{raw-init-clss } S)) S \text{ then } S \text{ else } (\square, \square, \square, 0, None))$

lemma [*code abstype*]:

$\text{ConI } (\text{rough-state-from-init-state-of } S) = S$
using $\text{rough-state-from-init-state-of[of } S]$ **unfolding** ConI-def
by (*simp add: rough-state-from-init-state-of-inverse*)

definition $\text{id-of-I-to} :: 'v \text{ cdcl}_W\text{-state-inv-from-init-state} \Rightarrow 'v \text{ cdcl}_W\text{-state-inv}$ **where**
 $\text{id-of-I-to } S = \text{state-of } (\text{rough-state-from-init-state-of } S)$

lemma [*code abstract*]:

$\text{rough-state-of } (\text{id-of-I-to } S) = \text{rough-state-from-init-state-of } S$
unfolding id-of-I-to-def **using** $\text{rough-state-from-init-state-of[of } S]$ **by** *auto*

Conflict and Propagate function $\text{do-full1-cp-step} :: 'v \text{ cdcl}_W\text{-state-inv} \Rightarrow 'v \text{ cdcl}_W\text{-state-inv}$
where

$\text{do-full1-cp-step } S =$
 $(\text{let } S' = \text{do-cp-step}' S \text{ in}$
 $\text{if } S = S' \text{ then } S \text{ else } \text{do-full1-cp-step } S')$

by *auto*

termination

proof (*relation* $\{(T', T). (\text{rough-state-of } T', \text{rough-state-of } T) \in \{(S', S).$
 $(S', S) \in \{(S', S). \text{cdcl}_W\text{-all-struct-inv } S \wedge \text{cdcl}_W\text{-cp } S S'\}\}, \text{goal-cases})$

case 1

show *?case*

using $\text{wf-if-measure-f[OF wf-if-measure-f[OF cdcl}_W\text{-cp-wf-all-inv, of }], \text{ of rough-state-of}]$.

next

case (2 $S' S$)

then show *?case*

unfolding $\text{do-cp-step}'\text{-def}$

apply *simp*

by (*metis cp-step-is-cdcl_W-cp rough-state-of-inverse*)

qed

lemma $\text{do-full1-cp-step-fix-point-of-do-full1-cp-step}$:

$\text{do-cp-step}(\text{rough-state-of } (\text{do-full1-cp-step } S)) = \text{rough-state-of } (\text{do-full1-cp-step } S)$
by (*rule do-full1-cp-step.induct[of $\lambda S. \text{do-cp-step}(\text{rough-state-of } (\text{do-full1-cp-step } S))$*)

$= \text{rough-state-of } (\text{do-full1-cp-step } S))$
 $(\text{metis } (\text{full-types}) \text{ do-full1-cp-step.elims } \text{rough-state-of-state-of-do-cp-step } \text{do-cp-step'-def})$

lemma *in-clauses-rough-state-of-is-distinct*:

$c \in \text{set } (\text{raw-init-clss } (\text{rough-state-of } S) \text{ @ } \text{raw-learned-clss } (\text{rough-state-of } S)) \implies \text{distinct } c$
apply $(\text{cases } \text{rough-state-of } S)$
using $\text{rough-state-of}[of\ S]$ **by** $(\text{auto simp add: distinct-mset-set-distinct } \text{cdcl}_W\text{-all-struct-inv-def } \text{distinct-cdcl}_W\text{-state-def})$

lemma *do-full1-cp-step-full*:

$\text{full } \text{cdcl}_W\text{-cp } (\text{rough-state-of } S)$
 $(\text{rough-state-of } (\text{do-full1-cp-step } S))$
unfolding full-def

proof $(\text{rule } \text{conjI}, \text{induction } S \text{ rule: } \text{do-full1-cp-step.induct})$

case $(1\ S)$

then have $f1$:

$\text{cdcl}_W\text{-cp}^{**} ((\text{do-cp-step } (\text{rough-state-of } S))) ($
 $\text{rough-state-of } (\text{do-full1-cp-step } (\text{state-of } (\text{do-cp-step } (\text{rough-state-of } S))))))$
 $\vee \text{state-of } (\text{do-cp-step } (\text{rough-state-of } S)) = S$

using $\text{rough-state-of-state-of-do-cp-step}[of\ S]$ **unfolding** do-cp-step'-def **by** fastforce

have $f2$: $\bigwedge c. (\text{if } c = \text{state-of } (\text{do-cp-step } (\text{rough-state-of } c))$
 $\text{then } c \text{ else } \text{do-full1-cp-step } (\text{state-of } (\text{do-cp-step } (\text{rough-state-of } c))))$
 $= \text{do-full1-cp-step } c$

by $(\text{metis } (\text{full-types}) \text{ do-cp-step'-def } \text{do-full1-cp-step.simps})$

have $f3$: $\neg \text{cdcl}_W\text{-cp } (\text{rough-state-of } S) (\text{do-cp-step } (\text{rough-state-of } S))$

$\vee \text{state-of } (\text{do-cp-step } (\text{rough-state-of } S)) = S$

$\vee \text{cdcl}_W\text{-cp}^{++} (\text{rough-state-of } S)$

$(\text{rough-state-of } (\text{do-full1-cp-step } (\text{state-of } (\text{do-cp-step } (\text{rough-state-of } S))))))$

using $f1$ **by** $(\text{meson } \text{rtranclp-into-tranclp2})$

{ assume $\text{do-full1-cp-step } S \neq S$

then have $\text{do-cp-step } (\text{rough-state-of } S) = \text{rough-state-of } S$

$\longrightarrow \text{cdcl}_W\text{-cp}^{**} (\text{rough-state-of } S) (\text{rough-state-of } (\text{do-full1-cp-step } S))$

$\vee \text{do-cp-step } (\text{rough-state-of } S) \neq \text{rough-state-of } S$

$\wedge \text{state-of } (\text{do-cp-step } (\text{rough-state-of } S)) \neq S$

using $f2\ f1$ **by** $(\text{metis } (\text{no-types}))$

then have $\text{do-cp-step } (\text{rough-state-of } S) \neq \text{rough-state-of } S$

$\wedge \text{state-of } (\text{do-cp-step } (\text{rough-state-of } S)) \neq S$

$\vee \text{cdcl}_W\text{-cp}^{**} (\text{rough-state-of } S) (\text{rough-state-of } (\text{do-full1-cp-step } S))$

by $(\text{metis } \text{rough-state-of-state-of-do-cp-step})$

then have $\text{cdcl}_W\text{-cp}^{**} (\text{rough-state-of } S) (\text{rough-state-of } (\text{do-full1-cp-step } S))$

using $f3\ f2$ **by** $(\text{metis } (\text{no-types}) \text{ cp-step-is-cdcl}_W\text{-cp } \text{tranclp-into-rtranclp})$ **}**

then show $?case$

by fastforce

next

show $\text{no-step } \text{cdcl}_W\text{-cp } (\text{rough-state-of } (\text{do-full1-cp-step } S))$

apply $(\text{rule } \text{do-cp-step-eq-no-step}[OF\ \text{do-full1-cp-step-fix-point-of-do-full1-cp-step}[of\ S]])$

using $\text{in-clauses-rough-state-of-is-distinct}$ **unfolding** do-cp-step'-def **by** blast

qed

lemma *[code abstract]*:

$\text{rough-state-of } (\text{do-cp-step}'\ S) = \text{do-cp-step } (\text{rough-state-of } S)$

unfolding do-cp-step'-def **by** auto

The other rules **fun** do-other-step **where**

$\text{do-other-step } S =$

```

(let  $T = \text{do-skip-step } S$  in
  if  $T \neq S$ 
  then  $T$ 
  else
    (let  $U = \text{do-resolve-step } T$  in
      if  $U \neq T$ 
      then  $U$  else
        (let  $V = \text{do-backtrack-step } U$  in
          if  $V \neq U$  then  $V$  else  $\text{do-decide-step } V$ )))

```

lemma *do-other-step*:
assumes *inv*: $\text{cdcl}_W\text{-all-struct-inv } S$ **and**
st: $\text{do-other-step } S \neq S$
shows $\text{cdcl}_W\text{-o } S$ ($\text{do-other-step } S$)
using *st inv* **by** (*auto split: if-split-asm*
simp add: Let-def
intro: do-skip-step do-resolve-step do-backtrack-step do-decide-step
 $\text{cdcl}_W\text{-o.intros cdcl}_W\text{-bj.intros}$)

lemma *do-other-step-no*:
assumes *inv*: $\text{cdcl}_W\text{-all-struct-inv } S$ **and**
st: $\text{do-other-step } S = S$
shows $\text{no-step cdcl}_W\text{-o } S$
using *st inv* **by** (*auto split: if-split-asm elim: cdcl}_W\text{-bjE}*
simp add: Let-def cdcl}_W\text{-bj.simps elim!: cdcl}_W\text{-o.cases}
dest!: do-skip-step-no do-resolve-step-no do-backtrack-step-no do-decide-step-no)

lemma *rough-state-of-state-of-do-other-step[simp]*:
 $\text{rough-state-of } (\text{state-of } (\text{do-other-step } (\text{rough-state-of } S))) = \text{do-other-step } (\text{rough-state-of } S)$
proof (*cases do-other-step (rough-state-of S) = rough-state-of S*)
case True
then show *?thesis* **by** *simp*
next
case False
have $\text{cdcl}_W\text{-o } (\text{rough-state-of } S)$ ($\text{do-other-step } (\text{rough-state-of } S)$)
by (*metis False cdcl}_W\text{-all-struct-inv-rough-state do-other-step[of rough-state-of S]*)
then have $\text{cdcl}_W\text{-all-struct-inv } (\text{do-other-step } (\text{rough-state-of } S))$
using $\text{cdcl}_W\text{-all-struct-inv-inv cdcl}_W\text{-all-struct-inv-rough-state other}$ **by** *blast*
then show *?thesis*
by (*simp add: CollectI state-of-inverse*)
qed

definition *do-other-step'* **where**
 $\text{do-other-step}' S =$
 $\text{state-of } (\text{do-other-step } (\text{rough-state-of } S))$

lemma *rough-state-of-do-other-step'[code abstract]*:
 $\text{rough-state-of } (\text{do-other-step}' S) = \text{do-other-step } (\text{rough-state-of } S)$
apply (*cases do-other-step (rough-state-of S) = rough-state-of S*)
unfolding *do-other-step'-def* **apply** *simp*
using $\text{do-other-step[of rough-state-of S]}$ **by** (*auto intro: cdcl}_W\text{-all-struct-inv-inv*
 $\text{cdcl}_W\text{-all-struct-inv-rough-state other state-of-inverse}$)

definition *do-cdcl}_W\text{-stgy-step}* **where**
 $\text{do-cdcl}_W\text{-stgy-step } S =$

```

(let T = do-full1-cp-step S in
  if T ≠ S
  then T
  else
    (let U = (do-other-step' T) in
      (do-full1-cp-step U)))

```

definition *do-cdcl_W-stgy-step'* **where**

do-cdcl_W-stgy-step' S = state-from-init-state-of (rough-state-of (do-cdcl_W-stgy-step (id-of-I-to S)))

lemma *toS-do-full1-cp-step-not-eq*: *do-full1-cp-step S ≠ S* ⇒
 rough-state-of S ≠ rough-state-of (do-full1-cp-step S)

proof –

```

assume a1: do-full1-cp-step S ≠ S
then have S ≠ do-cp-step' S
  by fastforce
then show ?thesis
  by (metis (no-types) do-cp-step'-def do-full1-cp-step-fix-point-of-do-full1-cp-step
    rough-state-of-inverse)

```

qed

do-full1-cp-step should not be unfolded anymore:

declare *do-full1-cp-step.simps*[*simp del*]

Correction of the transformation **lemma** *do-cdcl_W-stgy-step*:

```

assumes do-cdclW-stgy-step S ≠ S
shows cdclW-stgy (rough-state-of S) (rough-state-of (do-cdclW-stgy-step S))

```

proof (*cases do-full1-cp-step S = S*)

case *False*

then show ?thesis

```

using assms do-full1-cp-step-full[of S] unfolding full-unfold do-cdclW-stgy-step-def
by (auto intro!: cdclW-stgy.intros dest: toS-do-full1-cp-step-not-eq)

```

next

case *True*

have *cdcl_W-o* (rough-state-of S) (rough-state-of (do-other-step' S))

```

by (smt True assms cdclW-all-struct-inv-rough-state do-cdclW-stgy-step-def do-other-step
  rough-state-of-do-other-step' rough-state-of-inverse)

```

moreover

have

np: no-step propagate (rough-state-of S) **and**

nc: no-step conflict (rough-state-of S)

apply (metis True *cdcl_W-cp.simps* *do-cp-step-eq-no-step*

do-full1-cp-step-fix-point-of-do-full1-cp-step in-clauses-rough-state-of-is-distinct)

by (metis True *do-conflict-step-no-step do-cp-step-eq-no-prop-no-conf*

do-full1-cp-step-fix-point-of-do-full1-cp-step)

then have *no-step cdcl_W-cp* (rough-state-of S)

by (*simp add: cdcl_W-cp.simps*)

moreover have *full cdcl_W-cp* (rough-state-of (do-other-step' S))

(rough-state-of (do-full1-cp-step (do-other-step' S)))

using *do-full1-cp-step-full* **by** *auto*

ultimately show ?thesis

using *assms True* **unfolding** *do-cdcl_W-stgy-step-def*

by (*auto intro!: cdcl_W-stgy.other' dest: toS-do-full1-cp-step-not-eq*)

qed

```

lemma do-skip-step-trail-changed-or-conflict:
  assumes d: do-other-step S  $\neq$  S
  and inv: cdclW-all-struct-inv S
  shows trail S  $\neq$  trail (do-other-step S)
proof –
  have M:  $\bigwedge M\ K\ M1\ c.\ M = c @ K \# M1 \implies \text{Suc}(\text{length } M1) \leq \text{length } M$ 
    by auto
  have cdclW-M-level-inv S
    using inv unfolding cdclW-all-struct-inv-def by auto
  have cdclW-o S (do-other-step S) using do-other-step[OF inv d] .
  then show ?thesis
    using  $\langle \text{cdcl}_W\text{-M-level-inv } S \rangle$ 
  proof (induction do-other-step S rule: cdclW-o-induct-lev2)
    case decide
    then show ?thesis
      apply (cases S)
      apply (auto dest!: find-first-unused-var-Some
        simp: split: option.splits)
      by (meson atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set contra-subsetD)
    next
    case (skip)
    then show ?case
      by (cases S; cases do-other-step S) force
    next
    case (resolve)
    then show ?case
      by (cases S, cases do-other-step S) force
    next
    case (backtrack K i M1 M2 L D) note decomp = this(1) and confl-S = this(3) and undef =
this(6)
      and U = this(7)
    then show ?case
      apply (cases do-other-step S)
      apply (auto split: if-split-asm simp: Let-def)
      apply (cases S rule: do-skip-step.cases; auto split: if-split-asm)
      apply (cases S rule: do-skip-step.cases; auto split: if-split-asm)

      apply (cases S rule: do-backtrack-step.cases;
        auto split: if-split-asm option.splits list.splits marked-lit.splits
        dest!: bt-cut-some-decomp simp: Let-def)
      using d apply (cases S rule: do-decide-step.cases; auto split: option.splits)[]
    done
  qed
qed

```

```

lemma do-full1-cp-step-induct:
   $(\bigwedge S. (S \neq \text{do-cp-step}' S \implies P(\text{do-cp-step}' S)) \implies P S) \implies P a0$ 
  using do-full1-cp-step.induct by metis

```

```

lemma do-cp-step-neq-trail-increase:
   $\exists c. \text{raw-trail}(\text{do-cp-step } S) = c @ \text{raw-trail } S \wedge (\forall m \in \text{set } c. \neg \text{is-marked } m)$ 
  by (cases S, cases raw-conflicting S)
    (auto simp add: do-cp-step-def do-conflict-step-def do-propagate-step-def split: option.splits)

```

```

lemma do-full1-cp-step-neq-trail-increase:

```

$\exists c. \text{raw-trail } (\text{rough-state-of } (\text{do-full1-cp-step } S)) = c @ \text{raw-trail } (\text{rough-state-of } S)$
 $\wedge (\forall m \in \text{set } c. \neg \text{is-marked } m)$
apply (induction rule: do-full1-cp-step-induct)
apply (rename-tac S , case-tac $\text{do-cp-step}' S = S$)
apply (simp add: do-full1-cp-step.simps)
by (smt Un-iff append-assoc do-cp-step'-def do-cp-step-neq-trail-increase do-full1-cp-step.simps
rough-state-of-state-of-do-cp-step set-append)

lemma do-cp-step-conflicting:

conflicting (rough-state-of S) \neq None \implies do-cp-step' $S = S$
unfolding do-cp-step'-def do-cp-step-def **by** simp

lemma do-full1-cp-step-conflicting:

conflicting (rough-state-of S) \neq None \implies do-full1-cp-step $S = S$
unfolding do-cp-step'-def do-cp-step-def
apply (induction rule: do-full1-cp-step-induct)
by (rename-tac S , case-tac $S \neq \text{do-cp-step}' S$)
(auto simp add: do-full1-cp-step.simps do-cp-step-conflicting)

lemma do-decide-step-not-conflicting-one-more-decide:

assumes
conflicting $S = \text{None}$ **and**
do-decide-step $S \neq S$
shows Suc (length (filter is-marked (raw-trail S)))
= length (filter is-marked (raw-trail (do-decide-step S)))
using assms **unfolding** do-other-step'-def
by (cases S) (force simp: Let-def split: if-split-asm option.splits
dest!: find-first-unused-var-Some-not-all-incl)

lemma do-decide-step-not-conflicting-one-more-decide-bt:

assumes conflicting $S \neq \text{None}$ **and**
do-decide-step $S \neq S$
shows length (filter is-marked (raw-trail S)) <
length (filter is-marked (raw-trail (do-decide-step S)))
using assms **unfolding** do-other-step'-def **by** (cases S , cases conflicting S)
(auto simp add: Let-def split: if-split-asm option.splits)

lemma do-other-step-not-conflicting-one-more-decide-bt:

assumes
conflicting (rough-state-of S) \neq None **and**
conflicting (rough-state-of (do-other-step' S)) = None **and**
do-other-step' $S \neq S$
shows length (filter is-marked (raw-trail (rough-state-of S)))
> length (filter is-marked (raw-trail (rough-state-of (do-other-step' S))))

proof (cases S , goal-cases)

case (1 y) **note** $S = \text{this}(1)$ **and** $\text{inv} = \text{this}(2)$
obtain $M N U k E$ **where** $y: y = (M, N, U, k, \text{Some } E)$
using assms(1) S **inv** **by** (cases y , cases conflicting y) **auto**
have M : rough-state-of (state-of ($M, N, U, k, \text{Some } E$)) = ($M, N, U, k, \text{Some } E$)
using $\text{inv } y$ **by** (auto simp add: state-of-inverse)
have bt : do-other-step' $S = \text{state-of } (\text{do-backtrack-step } (\text{rough-state-of } S))$
proof (cases rough-state-of S rule: do-decide-step.cases)
case 1
then show ?thesis
using assms(1,2) **by** auto[]


```

next
case (2 v vb vd vf vh)
have f3:  $\bigwedge c. (if\ do\_skip\_step\ (rough\_state\_of\ c) \neq rough\_state\_of\ c$ 
  then  $do\_skip\_step\ (rough\_state\_of\ c)$ 
  else if  $do\_resolve\_step\ (do\_skip\_step\ (rough\_state\_of\ c)) \neq do\_skip\_step\ (rough\_state\_of\ c)$ 
    then  $do\_resolve\_step\ (do\_skip\_step\ (rough\_state\_of\ c))$ 
    else if  $do\_backtrack\_step\ (do\_resolve\_step\ (do\_skip\_step\ (rough\_state\_of\ c)))$ 
       $\neq do\_resolve\_step\ (do\_skip\_step\ (rough\_state\_of\ c))$ 
      then  $do\_backtrack\_step\ (do\_resolve\_step\ (do\_skip\_step\ (rough\_state\_of\ c)))$ 
      else  $do\_decide\_step\ (do\_backtrack\_step\ (do\_resolve\_step\ (do\_skip\_step\ (rough\_state\_of\ c))))$ 
    =  $rough\_state\_of\ (do\_other\_step'\ c)$ 
  by (simp add: rough-state-of-do-other-step')
have
  (raw-trail (rough-state-of (do-other-step' S)),
   raw-init-clss (rough-state-of (do-other-step' S)),
   raw-learned-clss (rough-state-of (do-other-step' S)),
   raw-backtrack-lvl (rough-state-of (do-other-step' S)), None)
  = rough-state-of (do-other-step' S)
using assms(2) by (cases do-other-step' S) auto
then show ?thesis
  using f3 2 by (metis (no-types) do-decide-step.simps(2) do-resolve-step-trail-is-None
    do-skip-step-trail-is-None rough-state-of-inverse)
qed
show ?case
using assms(2) S unfolding bt y inv
apply simp
by (auto simp add: M bt-cut-not-none
  split: option.splits
  dest!: bt-cut-some-decomp)
qed

lemma do-other-step-not-conflicting-one-more-decide:
  assumes conflicting (rough-state-of S) = None and
    do-other-step' S  $\neq$  S
  shows 1 + length (filter is-marked (raw-trail (rough-state-of S)))
    = length (filter is-marked (raw-trail (rough-state-of (do-other-step' S))))
proof (cases S, goal-cases)
case (1 y) note S = this(1) and inv = this(2)
obtain M N U k where y: y = (M, N, U, k, None) using assms(1) S inv by (cases y) auto
have M: rough-state-of (state-of (M, N, U, k, None)) = (M, N, U, k, None)
  using inv y by (auto simp add: state-of-inverse)
have state-of (do-decide-step (M, N, U, k, None))  $\neq$  state-of (M, N, U, k, None)
  using assms(2) unfolding do-other-step'-def y inv S by (auto simp add: M)
then have f4: do-skip-step (rough-state-of S) = rough-state-of S
  unfolding S M y by (metis (full-types) do-skip-step.simps(4))
have f5: do-resolve-step (rough-state-of S) = rough-state-of S
  unfolding S M y by (metis (no-types) do-resolve-step.simps(4))
have f6: do-backtrack-step (rough-state-of S) = rough-state-of S
  unfolding S M y by (metis (no-types) do-backtrack-step.simps(2))
have do-other-step (rough-state-of S)  $\neq$  rough-state-of S
  using assms(2) unfolding S M y do-other-step'-def by (metis (no-types))
then show ?case
  using f6 f5 f4 by (simp add: assms(1) do-decide-step-not-conflicting-one-more-decide
    do-other-step'-def)

```

qed

lemma *rough-state-of-state-of-do-skip-step-rough-state-of*[simp]:
 rough-state-of (*state-of* (*do-skip-step* (*rough-state-of* *S*))) = *do-skip-step* (*rough-state-of* *S*)
 by (smt *do-other-step.simps rough-state-of-inverse rough-state-of-state-of-do-other-step*)

lemma *conflicting-do-resolve-step-iff*[iff]:
 conflicting (*do-resolve-step* *S*) = None \longleftrightarrow *conflicting* *S* = None
 by (cases *S* rule: *do-resolve-step.cases*)
 (auto simp add: *Let-def split: option.splits*)

lemma *conflicting-do-skip-step-iff*[iff]:
 conflicting (*do-skip-step* *S*) = None \longleftrightarrow *conflicting* *S* = None
 by (cases *S* rule: *do-skip-step.cases*)
 (auto simp add: *Let-def split: option.splits*)

lemma *conflicting-do-decide-step-iff*[iff]:
 conflicting (*do-decide-step* *S*) = None \longleftrightarrow *conflicting* *S* = None
 by (cases *S* rule: *do-decide-step.cases*)
 (auto simp add: *Let-def split: option.splits*)

lemma *conflicting-do-backtrack-step-imp*[simp]:
 do-backtrack-step *S* \neq *S* \implies *conflicting* (*do-backtrack-step* *S*) = None
 by (cases *S* rule: *do-backtrack-step.cases*)
 (auto simp add: *Let-def split: list.splits option.splits marked-lit.splits*)

lemma *do-skip-step-eq-iff-trail-eq*:
 do-skip-step *S* = *S* \longleftrightarrow *trail* (*do-skip-step* *S*) = *trail* *S*
 by (cases *S* rule: *do-skip-step.cases*) auto

lemma *do-decide-step-eq-iff-trail-eq*:
 do-decide-step *S* = *S* \longleftrightarrow *trail* (*do-decide-step* *S*) = *trail* *S*
 by (cases *S* rule: *do-decide-step.cases*) (auto split: *option.split*)

lemma *do-backtrack-step-eq-iff-trail-eq*:
 do-backtrack-step *S* = *S* \longleftrightarrow *raw-trail* (*do-backtrack-step* *S*) = *raw-trail* *S*
 by (cases *S* rule: *do-backtrack-step.cases*)
 (auto split: *option.split list.splits marked-lit.splits*
 dest!: *bt-cut-in-get-all-marked-decomposition*)

lemma *do-resolve-step-eq-iff-trail-eq*:
 do-resolve-step *S* = *S* \longleftrightarrow *trail* (*do-resolve-step* *S*) = *trail* *S*
 by (cases *S* rule: *do-resolve-step.cases*) auto

lemma *do-other-step-eq-iff-trail-eq*:
 do-other-step *S* = *S* \longleftrightarrow *raw-trail* (*do-other-step* *S*) = *raw-trail* *S*

apply

(auto simp add: *Let-def do-skip-step-eq-iff-trail-eq*
 do-decide-step-eq-iff-trail-eq do-backtrack-step-eq-iff-trail-eq
 do-resolve-step-eq-iff-trail-eq
)

apply (*simp* add: *do-resolve-step-eq-iff-trail-eq[symmetric]*
 do-skip-step-eq-iff-trail-eq[symmetric])

apply (*simp* add: *do-skip-step-eq-iff-trail-eq[symmetric]*)

```

do-decide-step-eq-iff-trail-eq do-backtrack-step-eq-iff-trail-eq[symmetric]
do-resolve-step-eq-iff-trail-eq[symmetric]
)
done

lemma do-full1-cp-step-do-other-step'-normal-form[dest!]:
  assumes H: do-full1-cp-step (do-other-step' S) = S
  shows do-other-step' S = S ∧ do-full1-cp-step S = S
proof -
  let ?T = do-other-step' S
  { assume confl: conflicting (rough-state-of ?T) ≠ None
    then have tr: trail (rough-state-of (do-full1-cp-step ?T)) = trail (rough-state-of ?T)
      using do-full1-cp-step-conflicting by fastforce
    have raw-trail (rough-state-of (do-full1-cp-step (do-other-step' S))) =
      raw-trail (rough-state-of S)
      using arg-cong[OF H, of λS. raw-trail (rough-state-of S)] .
    then have raw-trail (rough-state-of (do-other-step' S)) = raw-trail (rough-state-of S)
      using confl by (auto simp add: do-full1-cp-step-conflicting)
    then have do-other-step' S = S
      by (simp add: do-other-step-eq-iff-trail-eq[symmetric] do-other-step'-def
        del: do-other-step.simps)
  }
  moreover {
    assume eq[simp]: do-other-step' S = S
    obtain c where c: raw-trail (rough-state-of (do-full1-cp-step S)) =
      c @ raw-trail (rough-state-of S)
      using do-full1-cp-step-neq-trail-increase by auto

    moreover have raw-trail (rough-state-of (do-full1-cp-step S)) = raw-trail (rough-state-of S)
      using arg-cong[OF H, of λS. raw-trail (rough-state-of S)] by simp
    finally have c = [] by blast
    then have do-full1-cp-step S = S using assms by auto
  }
  moreover {
    assume confl: conflicting (rough-state-of ?T) = None and neq: do-other-step' S ≠ S
    obtain c where
      c: raw-trail (rough-state-of (do-full1-cp-step ?T)) = c @ raw-trail (rough-state-of ?T) and
      nm: ∀ m ∈ set c. ¬ is-marked m
      using do-full1-cp-step-neq-trail-increase by auto
    have length (filter is-marked (raw-trail (rough-state-of (do-full1-cp-step ?T))))
      = length (filter is-marked (raw-trail (rough-state-of ?T)))
      using nm unfolding c by force
    moreover have length (filter is-marked (raw-trail (rough-state-of S)))
      ≠ length (filter is-marked (raw-trail (rough-state-of ?T)))
      using do-other-step-not-conflicting-one-more-decide[OF - neq]
        do-other-step-not-conflicting-one-more-decide-bt[of S, OF - confl neq]
        by linarith
    finally have False unfolding H by blast
  }
  ultimately show ?thesis by blast
qed

```

```

lemma do-cdclW-stgy-step-no:
  assumes S: do-cdclW-stgy-step S = S
  shows no-step cdclW-stgy (rough-state-of S)

```

```

proof -
{
  fix S'
  assume full1 cdclW-cp (rough-state-of S) S'
  then have False
    using do-full1-cp-step-full[of S] unfolding full-def S rtrancpl-unfold full1-def
    by (smt assms do-cdclW-stgy-step-def trancplD)
}
moreover {
  fix S' S''
  assume cdclW-o (rough-state-of S) S' and
    no-step propagate (rough-state-of S) and
    no-step conflict (rough-state-of S) and
    full cdclW-cp S' S''
  then have False
    using assms unfolding do-cdclW-stgy-step-def
    by (smt cdclW-all-struct-inv-rough-state do-full1-cp-step-do-other-step'-normal-form
      do-other-step-no rough-state-of-do-other-step')
}
ultimately show ?thesis using assms by (force simp: cdclW-cp.simps cdclW-stgy.simps)
qed

lemma toS-rough-state-of-state-of-rough-state-from-init-state-of[simp]:
  rough-state-of (state-of (rough-state-from-init-state-of S))
  = rough-state-from-init-state-of S
  using rough-state-from-init-state-of[of S] by (auto simp add: state-of-inverse)

lemma cdclW-cp-is-rtrancpl-cdclW: cdclW-cp S T  $\implies$  cdclW** S T
  apply (induction rule: cdclW-cp.induct)
  using conflict apply blast
  using propagate by blast

lemma rtrancpl-cdclW-cp-is-rtrancpl-cdclW: cdclW-cp** S T  $\implies$  cdclW** S T
  apply (induction rule: rtrancpl-induct)
  apply simp
  by (fastforce dest!: cdclW-cp-is-rtrancpl-cdclW)

lemma cdclW-stgy-is-rtrancpl-cdclW:
  cdclW-stgy S T  $\implies$  cdclW** S T
  apply (induction rule: cdclW-stgy.induct)
  using cdclW-stgy.conflict' rtrancpl-cdclW-stgy-rtrancpl-cdclW apply blast
  unfolding full-def by (fastforce dest!: other rtrancpl-cdclW-cp-is-rtrancpl-cdclW)

lemma cdclW-stgy-init-clss: cdclW-stgy S T  $\implies$  cdclW-M-level-inv S  $\implies$  init-clss S = init-clss T
  using rtrancpl-cdclW-init-clss cdclW-stgy-is-rtrancpl-cdclW by fast

lemma clauses-toS-rough-state-of-do-cdclW-stgy-step[simp]:
  init-clss (rough-state-of (do-cdclW-stgy-step (state-of (rough-state-from-init-state-of S))))
  = init-clss (rough-state-from-init-state-of S) (is - = init-clss ?S)
proof (cases do-cdclW-stgy-step (state-of ?S) = state-of ?S)
  case True
  then show ?thesis by simp
next
  case False
  have  $\bigwedge c$ . cdclW-M-level-inv (rough-state-of c)

```

using *cdcl_W-all-struct-inv-def cdcl_W-all-struct-inv-rough-state* **by** *blast*
then have $\bigwedge c. \text{init-clss} (\text{rough-state-of } c) = \text{init-clss} (\text{rough-state-of } (\text{do-cdcl}_W\text{-stgy-step } c))$
 $\vee \text{do-cdcl}_W\text{-stgy-step } c = c$
using *cdcl_W-stgy-no-more-init-clss do-cdcl_W-stgy-step* **by** *blast*
then show *?thesis*
using *False* **by** *force*
qed

lemma *raw-init-clss-do-cp-step[simp]:*
 $\text{raw-init-clss} (\text{do-cp-step } S) = \text{raw-init-clss } S$
by (*cases S*) (*auto simp: do-cp-step-def do-propagate-step-def do-conflict-step-def*
split: option.splits)
lemma *raw-init-clss-do-cp-step'[simp]:*
 $\text{raw-init-clss} (\text{rough-state-of } (\text{do-cp-step}' S)) = \text{raw-init-clss} (\text{rough-state-of } S)$
by (*simp add: do-cp-step'-def*)

lemma *raw-init-clss-rough-state-of-do-full1-cp-step[simp]:*
 $\text{raw-init-clss} (\text{rough-state-of } (\text{do-full1-cp-step } S))$
 $= \text{raw-init-clss} (\text{rough-state-of } S)$
apply (*rule do-full1-cp-step.induct[of $\lambda S.$*
 $\text{raw-init-clss} (\text{rough-state-of } (\text{do-full1-cp-step } S))$
 $= \text{raw-init-clss} (\text{rough-state-of } S)]$)
by (*metis (mono-tags, lifting) do-full1-cp-step.simps raw-init-clss-do-cp-step'*)

lemma *raw-init-clss-do-skip-def[simp]:*
 $\text{raw-init-clss} (\text{do-skip-step } S) = \text{raw-init-clss } S$
by (*cases S rule: do-skip-step.cases*) (*auto simp: do-other-step'-def Let-def*
split: option.splits)

lemma *raw-init-clss-do-resolve-def[simp]:*
 $\text{raw-init-clss} (\text{do-resolve-step } S) = \text{raw-init-clss } S$
by (*cases S rule: do-resolve-step.cases*) (*auto simp: do-other-step'-def Let-def*
split: option.splits)

lemma *raw-init-clss-do-backtrack-def[simp]:*
 $\text{raw-init-clss} (\text{do-backtrack-step } S) = \text{raw-init-clss } S$
by (*cases S rule: do-backtrack-step.cases*) (*auto simp: do-other-step'-def Let-def*
split: option.splits list.splits marked-lit.splits)

lemma *raw-init-clss-do-decide-def[simp]:*
 $\text{raw-init-clss} (\text{do-decide-step } S) = \text{raw-init-clss } S$
by (*cases S rule: do-decide-step.cases*) (*auto simp: do-other-step'-def Let-def*
split: option.splits)

lemma *raw-init-clss-rough-state-of-do-other-step'[simp]:*
 $\text{raw-init-clss} (\text{rough-state-of } (\text{do-other-step}' S))$
 $= \text{raw-init-clss} (\text{rough-state-of } S)$
by (*cases S*) (*auto simp: do-other-step'-def Let-def do-skip-step.cases*
split: option.splits)

lemma [*simp*]:
 $\text{raw-init-clss} (\text{rough-state-of } (\text{do-cdcl}_W\text{-stgy-step } (\text{state-of } (\text{rough-state-from-init-state-of } S))))$
 $=$
 $\text{raw-init-clss} (\text{rough-state-from-init-state-of } S)$

unfolding *do-cdcl_W-stgy-step-def* **by** (*auto simp: Let-def*)

lemma *rough-state-from-init-state-of-do-cdcl_W-stgy-step'*[*code abstract*]:

rough-state-from-init-state-of (*do-cdcl_W-stgy-step'* *S*) =
rough-state-of (*do-cdcl_W-stgy-step* (*id-of-I-to* *S*))

proof –

let *?S* = (*rough-state-from-init-state-of* *S*)

have *cdcl_W-stgy*** (*raw-S0-cdcl_W* (*raw-init-clss* (*rough-state-from-init-state-of* *S*)))
(*rough-state-from-init-state-of* *S*)

using *rough-state-from-init-state-of*[*of S*] **by** *auto*

moreover have *cdcl_W-stgy***

(*rough-state-from-init-state-of* *S*)

(*rough-state-of* (*do-cdcl_W-stgy-step*

(*state-of* (*rough-state-from-init-state-of* *S*))))

using *do-cdcl_W-stgy-step*[*of state-of ?S*]

by (*cases do-cdcl_W-stgy-step* (*state-of ?S*) = *state-of ?S*) *auto*

ultimately show *?thesis*

unfolding *do-cdcl_W-stgy-step'-def id-of-I-to-def*

by (*auto intro: state-from-init-state-of-inverse*)

qed

All rules together **function** *do-all-cdcl_W-stgy* **where**

do-all-cdcl_W-stgy *S* =

(*let* *T* = *do-cdcl_W-stgy-step'* *S* *in*

if *T* = *S* *then S* *else do-all-cdcl_W-stgy* *T*)

by *fast+*

termination

proof (*relation* {(*T*, *S*).

(*cdcl_W-measure* (*rough-state-from-init-state-of* *T*),

cdcl_W-measure (*rough-state-from-init-state-of* *S*))

∈ *lexn* {(*a*, *b*). *a* < *b*} *3*}, *goal-cases*)

case 1

show *?case* **by** (*rule wf-if-measure-f*) (*auto intro!: wf-lexn wf-less*)

next

case (2 *S T*) **note** *T* = *this*(1) **and** *ST* = *this*(2)

let *?S* = *rough-state-from-init-state-of* *S*

have *S*: *cdcl_W-stgy*** (*raw-S0-cdcl_W* (*raw-init-clss* *?S*)) *?S*

using *rough-state-from-init-state-of*[*of S*] **by** *auto*

moreover have *cdcl_W-stgy* (*rough-state-from-init-state-of* *S*)

(*rough-state-from-init-state-of* *T*)

proof –

have $\bigwedge c.$ *rough-state-of* (*state-of* (*rough-state-from-init-state-of* *c*)) =
rough-state-from-init-state-of *c*

using *rough-state-from-init-state-of* **by** *force*

then have *do-cdcl_W-stgy-step* (*state-of* (*rough-state-from-init-state-of* *S*))

≠ *state-of* (*rough-state-from-init-state-of* *S*)

using *ST T rough-state-from-init-state-of-inverse*

unfolding *id-of-I-to-def do-cdcl_W-stgy-step'-def*

by *fastforce*

from *do-cdcl_W-stgy-step*[*OF this*] **show** *?thesis*

by (*simp add: T id-of-I-to-def rough-state-from-init-state-of-do-cdcl_W-stgy-step'*)

qed

moreover

have *cdcl_W-all-struct-inv* (*rough-state-from-init-state-of* *S*)

```

    using rough-state-from-init-state-of[of S] by auto
  then have cdclW-all-struct-inv (raw-S0-cdclW (raw-init-clss (rough-state-from-init-state-of S)))
    by (cases rough-state-from-init-state-of S)
      (auto simp add: cdclW-all-struct-inv-def distinct-cdclW-state-def)
  ultimately show ?case
    by (auto intro!: cdclW-stgy-step-decreasing[of - - raw-S0-cdclW (raw-init-clss ?S)]
      simp del: cdclW-measure.simps)
qed

```

thm *do-all-cdcl_W-stgy.induct*

lemma *do-all-cdcl_W-stgy.induct:*

```

  (∧ S. (do-cdclW-stgy-step' S ≠ S ⇒ P (do-cdclW-stgy-step' S)) ⇒ P S) ⇒ P a0
  using do-all-cdclW-stgy.induct by metis

```

lemma [*simp*]: *raw-init-clss (rough-state-from-init-state-of (do-all-cdcl_W-stgy S)) =*
raw-init-clss (rough-state-from-init-state-of S)

apply (*induction rule: do-all-cdcl_W-stgy.induct*)

```

  by (smt do-all-cdclW-stgy.simps do-cdclW-stgy-step-def id-of-I-to-def
    raw-init-clss-rough-state-of-do-full1-cp-step raw-init-clss-rough-state-of-do-other-step'
    rough-state-from-init-state-of-do-cdclW-stgy-step'
    toS-rough-state-of-state-of-rough-state-from-init-state-of)

```

lemma *no-step-cdcl_W-stgy-cdcl_W-all:*

fixes *S :: 'a cdcl_W-state-inv-from-init-state*

shows *no-step cdcl_W-stgy (rough-state-from-init-state-of (do-all-cdcl_W-stgy S))*

apply (*induction S rule: do-all-cdcl_W-stgy.induct*)

apply (*rename-tac S, case-tac do-cdcl_W-stgy-step' S ≠ S*)

proof –

fix *Sa :: 'a cdcl_W-state-inv-from-init-state*

assume *a1: ¬ do-cdcl_W-stgy-step' Sa ≠ Sa*

{ fix *pp*

have (*if True then Sa else do-all-cdcl_W-stgy Sa*) = *do-all-cdcl_W-stgy Sa*

using *a1* **by** *auto*

then have *¬ cdcl_W-stgy (rough-state-from-init-state-of (do-all-cdcl_W-stgy Sa)) pp*

using *a1* **by** (*smt do-cdcl_W-stgy-step-no id-of-I-to-def*
rough-state-from-init-state-of-do-cdcl_W-stgy-step' rough-state-of-inverse) }

then show *no-step cdcl_W-stgy (rough-state-from-init-state-of (do-all-cdcl_W-stgy Sa))*

by *fastforce*

next

fix *Sa :: 'a cdcl_W-state-inv-from-init-state*

assume *a1: do-cdcl_W-stgy-step' Sa ≠ Sa*

⇒ *no-step cdcl_W-stgy (rough-state-from-init-state-of*
(do-all-cdcl_W-stgy (do-cdcl_W-stgy-step' Sa)))

assume *a2: do-cdcl_W-stgy-step' Sa ≠ Sa*

have *do-all-cdcl_W-stgy Sa = do-all-cdcl_W-stgy (do-cdcl_W-stgy-step' Sa)*

by (*metis (full-types) do-all-cdcl_W-stgy.simps*)

then show *no-step cdcl_W-stgy (rough-state-from-init-state-of (do-all-cdcl_W-stgy Sa))*

using *a2 a1* **by** *presburger*

qed

lemma *do-all-cdcl_W-stgy-is-rtrancpl-cdcl_W-stgy:*

*cdcl_W-stgy** (rough-state-from-init-state-of S)*

(rough-state-from-init-state-of (do-all-cdcl_W-stgy S))

proof (*induction S rule: do-all-cdcl_W-stgy.induct*)

case (*1 S*) **note** *IH = this(1)*

```

show ?case
proof (cases do-cdclW-stgy-step' S = S)
  case True
  then show ?thesis by simp
next
  case False
  have f2: do-cdclW-stgy-step (id-of-I-to S) = id-of-I-to S  $\longrightarrow$ 
    rough-state-from-init-state-of (do-cdclW-stgy-step' S)
    = rough-state-of (state-of (rough-state-from-init-state-of S))
  unfolding rough-state-from-init-state-of-do-cdclW-stgy-step'
    id-of-I-to-def by presburger
  have f3: do-all-cdclW-stgy S = do-all-cdclW-stgy (do-cdclW-stgy-step' S)
    by (metis (full-types) do-all-cdclW-stgy.simps)
  have cdclW-stgy (rough-state-from-init-state-of S)
    (rough-state-from-init-state-of (do-cdclW-stgy-step' S))
    = cdclW-stgy (rough-state-of (id-of-I-to S))
    (rough-state-of (do-cdclW-stgy-step (id-of-I-to S)))
  unfolding id-of-I-to-def rough-state-from-init-state-of-do-cdclW-stgy-step'
    toS-rough-state-of-state-of-rough-state-from-init-state-of by presburger
  then show ?thesis
  using f3 f2 IH do-cdclW-stgy-step
  by (smt False toS-rough-state-of-state-of-rough-state-from-init-state-of tranclp.intros(1)
    tranclp-into-rtranclp transitive-closurep-trans'(2))
qed
qed

```

Final theorem:

lemma *consistent-interp-mmset-of-mlit[simp]*:
 consistent-interp (lit-of 'mmset-of-mlit' 'set M') \longleftrightarrow
 consistent-interp (lit-of 'set M')
 by (auto simp: image-image)

lemma *DPLL-tot-correct*:

assumes

r: rough-state-from-init-state-of (do-all-cdcl_W-stgy (state-from-init-state-of
 ([], map remdups N, [], 0, None))) = S **and**

S: (M', N', U', k, E) = S

shows (E \neq Some [] \wedge satisfiable (set (map mset N)))
 \vee (E = Some [] \wedge unsatisfiable (set (map mset N)))

proof –

let ?N = map remdups N

have *inv*: cdcl_W-all-struct-inv ([], map remdups N, [], 0, None)

unfolding cdcl_W-all-struct-inv-def distinct-cdcl_W-state-def distinct-mset-set-def **by** auto

then have S0: rough-state-of (state-of ([], map remdups N, [], 0, None))

= ([], map remdups N, [], 0, None) **by** simp

have 1: full cdcl_W-stgy ([], ?N, [], 0, None) S

unfolding full-def **apply** rule

using do-all-cdcl_W-stgy-is-rtranclp-cdcl_W-stgy[of

state-from-init-state-of ([], map remdups N, [], 0, None)] *inv*

by (auto simp del: do-all-cdcl_W-stgy.simps simp: state-from-init-state-of-inverse
 r[symmetric] no-step-cdcl_W-stgy-cdcl_W-all)+

moreover have 2: finite (set (map mset ?N)) **by** auto

moreover have 3: distinct-mset-set (set (map mset ?N))

unfolding distinct-mset-set-def **by** auto

moreover


```

have cdclW-all-struct-inv S
  by (metis (no-types) cdclW-all-struct-inv-rough-state r
    toS-rough-state-of-state-of-rough-state-from-init-state-of)
then have cons: consistent-interp (lits-of-l M')
  unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def S[symmetric]
  by (auto simp: lits-of-def)
moreover
have [simp]:
  rough-state-from-init-state-of (state-from-init-state-of (raw-S0-cdclW (map remdups N)))
  = raw-S0-cdclW (map remdups N)
apply (rule cdclW-state-inv-from-init-state.state-from-init-state-of-inverse)
using 3 by (auto simp: cdclW-all-struct-inv-def distinct-cdclW-state-def
  image-image comp-def)
have raw-init-clss ([], ?N, [], 0, None) = raw-init-clss S
  using arg-cong[OF r, of raw-init-clss] unfolding S[symmetric]
  by (simp del: do-all-cdclW-stgy.simps)
then have N': N' = map remdups N
  using S[symmetric] by auto
have conflicting S = Some {#} ∧ unsatisfiable (set-mset (init-clss S)) ∨
  conflicting S = None ∧ (case S of (M, uu-) ⇒ map mmset-of-mlit' M) ⊨asm init-clss S
apply (rule full-cdclW-stgy-final-state-conclusive)
  using 1 apply simp
  using 2 apply simp
  using 3 by simp
then have (E ≠ Some [] ∧ satisfiable (set (map mset ?N)))
  ∨ (E = Some [] ∧ unsatisfiable (set (map mset ?N)))
  using cons unfolding S[symmetric] N' apply (auto simp: comp-def)
  by (simp add: true-annots-true-clss)
then show ?thesis by auto
qed

```

The Code The SML code is skipped in the documentation, but stays to ensure that some version of the exported code is working. The only difference between the generated code and the one used here is the export of the constructor `ConI`.

```

end
theory CDCL-W-Merge
imports CDCL-W-Termination
begin

```

20 Link between Weidenbach's and NOT's CDCL

20.1 Inclusion of the states

```

context conflict-driven-clause-learningW
begin
declare cdclW.intros[intro] cdclW-bj.intros[intro] cdclW-o.intros[intro]

lemma backtrack-no-cdclW-bj:
  assumes cdcl: cdclW-bj T U and inv: cdclW-M-level-inv V
  shows ¬backtrack V T
  using cdcl inv
  apply (induction rule: cdclW-bj.induct)
  apply (elim skipE, force elim!: backtrack-levE[OF - inv] simp: cdclW-M-level-inv-def)

```

```

  apply (elim resolveE, force elim!: backtrack-levE[OF - inv] simp: cdclW-M-level-inv-def)
  apply standard
  apply (elim backtrack-levE[OF - inv], elim backtrackE)
  apply (force simp del: state-simp simp add: state-eq-def cdclW-M-level-inv-decomp)
done

```

inductive *skip-or-resolve* :: '*st* ⇒ '*st* ⇒ bool **where**
s-or-r-skip[intro]: *skip S T* ⇒ *skip-or-resolve S T* |
s-or-r-resolve[intro]: *resolve S T* ⇒ *skip-or-resolve S T*

lemma *rtrancpl-cdcl_W-bj-skip-or-resolve-backtrack*:
assumes *cdcl_W-bj^{**} S U* **and** *inv: cdcl_W-M-level-inv S*
shows *skip-or-resolve^{**} S U* ∨ (∃ *T*. *skip-or-resolve^{**} S T* ∧ *backtrack T U*)
using *assms*
proof (*induction*)
case *base*
then show ?*case* **by** *simp*
next
case (*step U V*) **note** *st = this(1)* **and** *bj = this(2)* **and** *IH = this(3)[OF this(4)]*
consider
 (*SU*) *S = U*
 | (*SUp*) *cdcl_W-bj⁺⁺ S U*
using *st* **unfolding** *rtrancpl-unfold* **by** *blast*
then show ?*case*
proof *cases*
case *SUp*
have $\bigwedge T. \text{skip-or-resolve}^{**} S T \Rightarrow \text{cdcl}_W^{**} S T$
using *mono-rtrancpl[of skip-or-resolve cdcl_W]*
by (*blast intro: skip-or-resolve.cases*)
then have *skip-or-resolve^{**} S U*
using *bj IH inv backtrack-no-cdcl_W-bj rtrancpl-cdcl_W-consistent-inv[OF - inv]* **by** *meson*
then show ?*thesis*
using *bj* **by** (*auto simp: cdcl_W-bj.simps dest!: skip-or-resolve.intros*)
next
case *SU*
then show ?*thesis*
using *bj* **by** (*auto simp: cdcl_W-bj.simps dest!: skip-or-resolve.intros*)
qed
qed

lemma *rtrancpl-skip-or-resolve-rtrancpl-cdcl_W*:
*skip-or-resolve^{**} S T* ⇒ *cdcl_W^{**} S T*
by (*induction rule: rtrancpl-induct*)
(auto dest!: cdcl_W-bj.intros cdcl_W.intros cdcl_W-o.intros simp: skip-or-resolve.simps)

definition *backjump-l-cond* :: '*v* clause ⇒ '*v* clause ⇒ '*v* literal ⇒ '*st* ⇒ '*st* ⇒ bool **where**
backjump-l-cond ≡ λ*C C' L' S T*. *True*

definition *inv_{NOT}* :: '*st* ⇒ bool **where**
inv_{NOT} ≡ λ*S*. *no-dup (trail S)*

declare *inv_{NOT}-def[simp]*
end

context *conflict-driven-clause-learning_W*
begin

20.2 More lemmas conflict-propagate and backjumping

20.2.1 Termination

lemma *cdcl_W-cp-normalized-element-all-inv*:
assumes *inv*: *cdcl_W-all-struct-inv S*
obtains *T* **where** *full cdcl_W-cp S T*
using *assms cdcl_W-cp-normalized-element* **unfolding** *cdcl_W-all-struct-inv-def* **by** *blast*
thm *backtrackE*

lemma *cdcl_W-bj-measure*:
assumes *cdcl_W-bj S T* **and** *cdcl_W-M-level-inv S*
shows *length (trail S) + (if conflicting S = None then 0 else 1)*
> length (trail T) + (if conflicting T = None then 0 else 1)
using *assms* **by** (*induction rule: cdcl_W-bj.induct*)
(force dest:arg-cong[of - - length]
intro: get-all-marked-decomposition-exists-prepend
elim!: backtrack-levE skipE resolveE
simp: cdcl_W-M-level-inv-def)+

lemma *wf-cdcl_W-bj*:
wf {(b,a). cdcl_W-bj a b ∧ cdcl_W-M-level-inv a}
apply (*rule wfP-if-measure[of λ-. True*
- λT. length (trail T) + (if conflicting T = None then 0 else 1), simplified])
using *cdcl_W-bj-measure* **by** *simp*

lemma *cdcl_W-bj-exists-normal-form*:
assumes *lev: cdcl_W-M-level-inv S*
shows $\exists T. \text{full } cdcl_W\text{-bj } S \ T$
proof –
obtain *T* **where** *T: full (λa b. cdcl_W-bj a b ∧ cdcl_W-M-level-inv a) S T*
using *wf-exists-normal-form-full[OF wf-cdcl_W-bj]* **by** *auto*
then have *cdcl_W-bj** S T*
by (*auto dest: rtrancp-and-rtrancp-left simp: full-def*)
moreover
then have *cdcl_W** S T*
using *mono-rtrancp[of cdcl_W-bj cdcl_W]* **by** *blast*
then have *cdcl_W-M-level-inv T*
using *rtrancp-cdcl_W-consistent-inv lev* **by** *auto*
ultimately show *?thesis* **using** *T* **unfolding** *full-def* **by** *auto*
qed

lemma *rtrancp-skip-state-decomp*:
assumes *skip** S T* **and** *no-dup (trail S)*
shows
 $\exists M. \text{trail } S = M @ \text{trail } T \wedge (\forall m \in \text{set } M. \neg \text{is-marked } m)$
init-clss S = init-clss T
learned-clss S = learned-clss T
backtrack-lvl S = backtrack-lvl T
conflicting S = conflicting T
using *assms* **by** (*induction rule: rtrancp-induct*)
(auto simp del: state-simp simp: state-eq-def elim!: skipE)

20.2.2 More backjumping

Backjumping after skipping or jump directly lemma *rtrancpl-skip-backtrack-backtrack*:

```

assumes
  skip** S T and
  backtrack T W and
  cdclW-all-struct-inv S
shows backtrack S W
using assms
proof induction
  case base
  then show ?case by simp
next
  case (step T V) note st = this(1) and skip = this(2) and IH = this(3) and bt = this(4) and
    inv = this(5)
  have skip** S V
    using st skip by auto
  then have cdclW-all-struct-inv V
    using rtrancpl-mono[of skip cdclW] assms(3) rtrancpl-cdclW-all-struct-inv-inv mono-rtrancpl
    by (auto dest!: bj other cdclW-bj.skip)
  then have cdclW-M-level-inv V
    unfolding cdclW-all-struct-inv-def by auto
  then obtain K i M1 M2 L D where
    conf: raw-conflicting V = Some D and
    LD: L ∈ # mset-ccls D and
    decomp: (Marked K (Suc i) # M1, M2) ∈ set (get-all-marked-decomposition (trail V)) and
    lev-L: get-level (trail V) L = backtrack-lvl V and
    max: get-level (trail V) L = get-maximum-level (trail V) (mset-ccls D) and
    max-D: get-maximum-level (trail V) (remove1-mset L (mset-ccls D)) ≡ i and
    undef: undefined-lit M1 L and
    W: W ∼ cons-trail (Propagated L (cls-of-ccls D))
      (reduce-trail-to M1
        (add-learned-cls (cls-of-ccls D)
          (update-backtrack-lvl i
            (update-conflicting None V))))
  using bt inv by (elim backtrack-levE) metis+
  obtain L' C' M E where
    tr: trail T = Propagated L' C' # M and
    raw: raw-conflicting T = Some E and
    LE: -L' ∉ # mset-ccls E and
    E: mset-ccls E ≠ {#} and
    V: V ∼ tl-trail T
    using skip by (elim skipE) metis
  let ?M = Propagated L' C' # trail V
  have tr-M: trail T = ?M
    using tr V by auto
  have MT: M = tl (trail T) and MV: M = trail V
    using tr V by auto
  have DE[simp]: mset-ccls D = mset-ccls E
    using V conf raw by (auto simp add: state-eq-def simp del: state-simp)
  have cdclW** S T using bj cdclW-bj.skip mono-rtrancpl[of skip cdclW S T] other st by meson
  then have inv': cdclW-all-struct-inv T
    using rtrancpl-cdclW-all-struct-inv-inv inv by blast
  have M-lev: cdclW-M-level-inv T using inv' unfolding cdclW-all-struct-inv-def by auto
  then have n-d': no-dup ?M
    using tr-M unfolding cdclW-M-level-inv-def by auto

```

```

let ?k = backtrack-lvl T
have [simp]:
  backtrack-lvl V = ?k
  using V by simp
have ?k > 0
  using decomp M-lev V tr unfolding cdclW-M-level-inv-def by auto
then have atm-of L ∈ atm-of ' lits-of-l (trail V)
  using lev-L get-rev-level-ge-0-atm-of-in[of 0 rev (trail V) L] by auto
then have L-L': atm-of L ≠ atm-of L'
  using n-d' unfolding lits-of-def by auto
have L'-M: atm-of L' ∉ atm-of ' lits-of-l (trail V)
  using n-d' unfolding lits-of-def by auto
have ?M ⊨as CNot (mset-ccls D)
  using inv' raw unfolding cdclW-conflicting-def cdclW-all-struct-inv-def tr-M by auto
then have L' ∉ # mset-ccls (remove-clit L D)
  using L-L' L'-M ⟨Propagated L' C' # trail V ⊨as CNot (mset-ccls D)⟩
  unfolding true-annots-true-cls true-clss-def
  by (auto simp: uminus-lit-swap atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set dest!: in-diffD)
have [simp]: trail (reduce-trail-to M1 T) = M1
  using decomp undef tr W V by auto
have skip** S V
  using st skip by auto
have no-dup (trail S)
  using inv unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def by auto
then have [simp]: init-cls S = init-cls V and [simp]: learned-cls S = learned-cls V
  using rtrancp-skip-state-decomp[OF ⟨skip** S V⟩] V
  by (auto simp del: state-simp simp: state-eq-def)
then have
  W-S: W ∼ cons-trail (Propagated L (cls-of-ccls E)) (reduce-trail-to M1
    (add-learned-cls (cls-of-ccls E) (update-backtrack-lvl i (update-conflicting None T))))
  using W V undef M-lev decomp tr
  by (auto simp del: state-simp simp: state-eq-def cdclW-M-level-inv-def)

obtain M2' where
  decomp': (Marked K (i+1) # M1, M2') ∈ set (get-all-marked-decomposition (trail T))
  using decomp V unfolding tr-M by (cases hd (get-all-marked-decomposition (trail V)),
    cases get-all-marked-decomposition (trail V)) auto
moreover
  from L-L' have get-level ?M L = ?k
    using lev-L V by (auto split: if-split-asm)
moreover
  have atm-of L' ∉ atms-of (mset-ccls D)
    by (metis DE LE L-L' ⟨L' ∉ # mset-ccls (remove-clit L D)⟩ in-remove1-mset-neq remove-clit
      atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set atms-of-def)
  then have get-level ?M L = get-maximum-level ?M (mset-ccls D)
    using calculation(2) lev-L max by auto
moreover
  have atm-of L' ∉ atms-of (mset-ccls (remove-clit L D))
    by (metis DE LE L-L' ⟨L' ∉ # mset-ccls (remove-clit L D)⟩
      atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set atms-of-def in-remove1-mset-neq remove-clit
      in-atms-of-remove1-mset-in-atms-of)
  have i = get-maximum-level ?M (mset-ccls (remove-clit L D))
    using max-D ⟨atm-of L' ∉ atms-of (mset-ccls (remove-clit L D))⟩ by auto

ultimately have backtrack T W

```

```

apply –
apply (rule backtrack-rule[of T - L K i M1 M2' W, OF raw])
unfolding tr-M[symmetric]
  using LD apply simp
  apply simp
  apply simp
  apply simp
  apply auto[]
  using W-S by auto
then show ?thesis using IH inv by blast
qed

```

```

lemma fst-get-all-marked-decomposition-prepend-not-marked:
assumes  $\forall m \in \text{set } MS. \neg \text{is-marked } m$ 
shows set (map fst (get-all-marked-decomposition M))
  = set (map fst (get-all-marked-decomposition (MS @ M)))
using assms apply (induction MS rule: marked-lit-list-induct)
apply auto[2]
by (rename-tac L m xs; case-tac get-all-marked-decomposition (xs @ M) simp-all)

```

See also $\llbracket \text{skip}^{**} ?S ?T; \text{backtrack} ?T ?W; \text{cdcl}_W\text{-all-struct-inv} ?S \rrbracket \implies \text{backtrack} ?S ?W$

```

lemma rtrancpl-skip-backtrack-backtrack-end:

```

```

assumes
  skip: skip** S T and
  bt: backtrack S W and
  inv: cdclW-all-struct-inv S

```

```

shows backtrack T W

```

```

using assms

```

```

proof –

```

```

  have M-lev: cdclW-M-level-inv S

```

```

    using bt inv unfolding cdclW-all-struct-inv-def by (auto elim!: backtrack-levE)

```

```

then obtain K i M1 M2 L D where

```

```

  raw-S: raw-conflicting S = Some D and

```

```

  LD:  $L \in \# \text{mset-ccls } D$  and

```

```

  decomp:  $(\text{Marked } K (\text{Suc } i) \# M1, M2) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$  and

```

```

  lev-l: get-level (trail S) L = backtrack-lvl S and

```

```

  lev-l-D: get-level (trail S) L = get-maximum-level (trail S) (mset-ccls D) and

```

```

  i: get-maximum-level (trail S) (remove1-mset L (mset-ccls D))  $\equiv i$  and

```

```

  undef: undefined-lit M1 L and

```

```

  W:  $W \sim \text{cons-trail } (\text{Propagated } L (\text{cls-of-ccls } D))$ 

```

```

    (reduce-trail-to M1

```

```

      (add-learned-cls (cls-of-ccls D)

```

```

        (update-backtrack-lvl i

```

```

          (update-conflicting None S))))

```

```

  using bt by (elim backtrack-levE)

```

```

  (simp-all add: cdclW-M-level-inv-decomp state-eq-def del: state-simp)

```

```

let ?D = remove1-mset L (mset-ccls D)

```

```

have [simp]: no-dup (trail S)

```

```

  using M-lev by (auto simp: cdclW-M-level-inv-decomp)

```

```

have cdclW-all-struct-inv T

```

```

  using mono-rtrancpl[of skip cdclW] by (smt bj cdclW-bj.skip inv local.skip other

```

```

    rtrancpl-cdclW-all-struct-inv-inv)

```

```

then have [simp]: no-dup (trail T)

```

```

unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def by auto

obtain MS MT where M: trail S = MS @ MT and MT: MT = trail T and nm:  $\forall m \in \text{set } MS. \neg \text{is-marked } m$ 
using rtrancpl-skip-state-decomp(1)[OF skip] raw-S M-lev by auto
have T: state T = (MT, init-clss S, learned-clss S, backtrack-lvl S, Some (mset-ccls D))
using MT rtrancpl-skip-state-decomp[of S T] skip raw-S
by (auto simp del: state-simp simp: state-eq-def)

have cdclW-all-struct-inv T
apply (rule rtrancpl-cdclW-all-struct-inv-inv[OF - inv])
using bj cdclW-bj.skip local.skip other rtrancpl-mono[of skip cdclW] by blast
then have MT  $\models_{as}$  CNot (mset-ccls D)
unfolding cdclW-all-struct-inv-def cdclW-conflicting-def using T by blast
then have  $\forall L \in \# \text{mset-ccls } D. \text{atm-of } L \in \text{atm-of 'lits-of-l } M_T$ 
by (meson atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set
true-annots-true-cls-def-iff-negation-in-model)
moreover have no-dup (trail S)
using inv unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def by auto
ultimately have  $\forall L \in \# \text{mset-ccls } D. \text{atm-of } L \notin \text{atm-of 'lits-of-l } MS$ 
unfolding M unfolding lits-of-def by auto
then have H:  $\bigwedge L. L \in \# \text{mset-ccls } D \implies \text{get-level (trail S) } L = \text{get-level } M_T L$ 
unfolding M by (fastforce simp: lits-of-def)
have [simp]: get-maximum-level (trail S) (mset-ccls D) = get-maximum-level MT (mset-ccls D)
by (metis  $\langle M_T \models_{as} CNot (mset-ccls D) \rangle$  M nm true-annots-CNot-all-atms-defined
get-maximum-level-skip-un-marked-not-present)

have lev-l': get-level MT L = backtrack-lvl S
using lev-l LD by (auto simp: H)
have [simp]: trail (reduce-trail-to M1 T) = M1
using T decomp M nm by (smt MT append-assoc beginning-not-marked-invert
get-all-marked-decomposition-exists-prepend reduce-trail-to-trail-tl-trail-decomp)
have W: W  $\sim$  cons-trail (Propagated L (cls-of-ccls D)) (reduce-trail-to M1
(add-learned-cls (cls-of-ccls D) (update-backtrack-lvl i (update-conflicting None T)))))
using W T i decomp undef by (auto simp del: state-simp simp: state-eq-def)
have lev-l-D': get-level MT L = get-maximum-level MT (mset-ccls D)
using lev-l-D LD by (auto simp: H)
have [simp]: get-maximum-level (trail S) ?D = get-maximum-level MT ?D
by (smt H get-maximum-level-exists-lit get-maximum-level-ge-get-level in-diffD le-antisym
not-gr0 not-less)
then have i': i = get-maximum-level MT ?D
using i by auto
have Marked K (i + 1) # M1  $\in$  set (map fst (get-all-marked-decomposition (trail S)))
using Set.imageI[OF decomp, of fst] by auto
then have Marked K (i + 1) # M1  $\in$  set (map fst (get-all-marked-decomposition MT))
using fst-get-all-marked-decomposition-prepend-not-marked[OF nm] unfolding M by auto
then obtain M2' where decomp': (Marked K (i+1) # M1, M2')  $\in$  set (get-all-marked-decomposition
MT)
by auto
then show backtrack T W
using T decomp' lev-l' lev-l-D' i' W LD undef
by (force intro!: backtrack.intros simp del: state-simp simp: state-eq-def)
qed

```

lemma *cdcl_W-bj-decomp-resolve-skip-and-bj:*

```

assumes  $cdcl_W$ -bj**  $S$   $T$  and  $inv$ :  $cdcl_W$ -M-level- $inv$   $S$ 
shows ( $skip$ -or- $resolve$ **  $S$   $T$ 
   $\vee (\exists U. skip$ -or- $resolve$ **  $S$   $U \wedge backtrack$   $U$   $T$ ))
using  $assms$ 
proof  $induction$ 
  case  $base$ 
  then show ? $case$  by  $simp$ 
next
  case ( $step$   $T$   $U$ ) note  $st = this(1)$  and  $bj = this(2)$  and  $IH = this(3)$ 
  have  $IH$ :  $skip$ -or- $resolve$ **  $S$   $T$ 
  proof –
    { assume ( $\exists U. skip$ -or- $resolve$ **  $S$   $U \wedge backtrack$   $U$   $T$ )
      then obtain  $V$  where
         $bt$ :  $backtrack$   $V$   $T$  and
         $skip$ -or- $resolve$ **  $S$   $V$ 
        by  $blast$ 
        have  $cdcl_W$ **  $S$   $V$ 
        using ( $skip$ -or- $resolve$ **  $S$   $V$ )  $rtranclp$ - $skip$ -or- $resolve$ - $rtranclp$ - $cdcl_W$  by  $blast$ 
        then have  $cdcl_W$ -M-level- $inv$   $V$  and  $cdcl_W$ -M-level- $inv$   $S$ 
        using  $rtranclp$ - $cdcl_W$ -consistent- $inv$   $inv$  by  $blast$ +
        with  $bj$   $bt$  have  $False$  using  $backtrack$ -no- $cdcl_W$ - $bj$  by  $simp$ 
      }
    then show ? $thesis$  using  $IH$   $inv$  by  $blast$ 
  qed
show ? $case$ 
  using  $bj$ 
  proof ( $cases$   $rule$ :  $cdcl_W$ - $bj$ . $cases$ )
    case  $backtrack$ 
    then show ? $thesis$  using  $IH$  by  $blast$ 
  qed ( $metis$  ( $no$ -types,  $lifting$ )  $IH$   $rtranclp$ . $simps$   $skip$ -or- $resolve$ . $simps$ ) +
qed

```

lemma $resolve$ - $skip$ -deterministic:

$resolve$ S $T \implies skip$ S $U \implies False$

by ($auto$ $elim$!: $skipE$ $resolveE$ $dest$: hd -raw-trail)

lemma $backtrack$ -unique:

```

assumes
   $bt$ - $T$ :  $backtrack$   $S$   $T$  and
   $bt$ - $U$ :  $backtrack$   $S$   $U$  and
   $inv$ :  $cdcl_W$ -all-struct- $inv$   $S$ 
shows  $T \sim U$ 
proof –
  have  $lev$ :  $cdcl_W$ -M-level- $inv$   $S$ 
  using  $inv$  unfolding  $cdcl_W$ -all-struct- $inv$ -def by  $auto$ 
  then obtain  $K$   $i$   $M1$   $M2$   $L$   $D$  where
    raw- $S$ : raw-conflicting  $S = Some$   $D$  and
     $LD$ :  $L \in \#$   $mset$ -ccls  $D$  and
     $decomp$ : ( $Marked$   $K$  ( $Suc$   $i$ )  $\#$   $M1, M2$ )  $\in set$  ( $get$ -all-marked-decomposition ( $trail$   $S$ )) and
     $lev$ - $l$ :  $get$ -level ( $trail$   $S$ )  $L = backtrack$ -lvl  $S$  and
     $lev$ - $l$ - $D$ :  $get$ -level ( $trail$   $S$ )  $L = get$ -maximum-level ( $trail$   $S$ ) ( $mset$ -ccls  $D$ ) and
     $i$ :  $get$ -maximum-level ( $trail$   $S$ ) ( $remove1$ - $mset$   $L$  ( $mset$ -ccls  $D$ ))  $\equiv i$  and
     $undef$ :  $undefined$ -lit  $M1$   $L$  and
     $T$ :  $T \sim cons$ -trail ( $Propagated$   $L$  ( $cls$ -of-ccls  $D$ ))
    ( $reduce$ -trail-to  $M1$ 

```



```

      (add-learned-cls (cls-of-ccls D)
        (update-backtrack-lvl i
          (update-conflicting None S))))
    using bt-T by (elim backtrack-levE) (force simp: cdclW-M-level-inv-def)+

obtain  $K' i' M1' M2' L' D'$  where
  raw-S': raw-conflicting  $S = \text{Some } D'$  and
  LD':  $L' \in \# \text{ mset-ccls } D'$  and
  decomp':  $(\text{Marked } K' (\text{Suc } i') \# M1', M2') \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$  and
  lev-l:  $\text{get-level } (\text{trail } S) L' = \text{backtrack-lvl } S$  and
  lev-l-D:  $\text{get-level } (\text{trail } S) L' = \text{get-maximum-level } (\text{trail } S) (\text{mset-ccls } D')$  and
  i':  $\text{get-maximum-level } (\text{trail } S) (\text{remove1-mset } L' (\text{mset-ccls } D')) \equiv i'$  and
  undef': undefined-lit  $M1' L'$  and
  U:  $U \sim \text{cons-trail } (\text{Propagated } L' (\text{cls-of-ccls } D'))$ 
    (reduce-trail-to  $M1'$ 
      (add-learned-cls (cls-of-ccls  $D'$ )
        (update-backtrack-lvl  $i'$ 
          (update-conflicting None S))))
    using bt-U lev by (elim backtrack-levE) (force simp: cdclW-M-level-inv-def)+
obtain  $c$  where  $M: \text{trail } S = c @ M2 @ \text{Marked } K (i + 1) \# M1$ 
  using decomp by auto
obtain  $c'$  where  $M': \text{trail } S = c' @ M2' @ \text{Marked } K' (i' + 1) \# M1'$ 
  using decomp' by auto
have marked:  $\text{get-all-levels-of-marked } (\text{trail } S) = \text{rev } [1..<1+\text{backtrack-lvl } S]$ 
  using inv unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def by auto
then have  $i < \text{backtrack-lvl } S$ 
  unfolding  $M$  by (force simp add: rev-swap[symmetric] dest!: arg-cong[of - - set])

have [simp]:  $L' = L$ 
proof (rule ccontr)
  assume  $\neg ?thesis$ 
  then have  $L' \in \# \text{ remove1-mset } L (\text{mset-ccls } D)$ 
    using raw-S raw-S' LD LD' by (simp add: in-remove1-mset-neq)
  then have  $\text{get-maximum-level } (\text{trail } S) (\text{remove1-mset } L (\text{mset-ccls } D)) \geq \text{backtrack-lvl } S$ 
    using  $\langle \text{get-level } (\text{trail } S) L' = \text{backtrack-lvl } S \rangle$  get-maximum-level-ge-get-level
    by metis
  then show  $\text{False}$  using  $i' i \langle i < \text{backtrack-lvl } S \rangle$  by auto
qed
then have [simp]:  $\text{mset-ccls } D' = \text{mset-ccls } D$ 
  using raw-S raw-S' by auto
have [simp]:  $i' = i$ 
  using  $i i'$  by auto

```

Automation in a step later...

```

have  $H: \bigwedge a A B. \text{insert } a A = B \implies a : B$ 
  by blast
have  $\text{get-all-levels-of-marked } (c @ M2) = \text{rev } [i+2..<1+\text{backtrack-lvl } S]$  and
   $\text{get-all-levels-of-marked } (c' @ M2') = \text{rev } [i+2..<1+\text{backtrack-lvl } S]$ 
  using marked unfolding  $M$ 
  using marked unfolding  $M'$ 
  unfolding rev-swap[symmetric] by (auto dest: append-cons-eq-upt-length-i-end)
from arg-cong[OF this(1), of set] arg-cong[OF this(2), of set]
have
  dropWhile  $(\lambda L. \neg \text{is-marked } L \vee \text{level-of } L \neq \text{Suc } i) (c @ M2) = []$  and

```

$\text{dropWhile } (\lambda L. \neg \text{is-marked } L \vee \text{level-of } L \neq \text{Suc } i) (c' @ M2') = []$
unfolding $\text{dropWhile-eq-Nil-conv Ball-def}$
by $(\text{intro allI}; \text{rename-tac } x; \text{case-tac } x; \text{auto dest!}; H \text{ simp add: in-set-conv-decomp})+$

then have $[\text{simp}]: M1' = M1$
using $\text{arg-cong}[OF M, \text{of dropWhile } (\lambda L. \neg \text{is-marked } L \vee \text{level-of } L \neq \text{Suc } i)]$
unfolding M' **by** auto
show $?thesis$ **using** $T U \text{ undef inv decomp by } (\text{auto simp del: state-simp simp: state-eq-def}$
 $\text{cdcl}_W\text{-all-struct-inv-def cdcl}_W\text{-M-level-inv-decomp})$
qed

lemma *if-can-apply-backtrack-no-more-resolve:*
assumes
 $\text{skip: skip}^{**} S U$ **and**
 $\text{bt: backtrack } S T$ **and**
 $\text{inv: cdcl}_W\text{-all-struct-inv } S$
shows $\neg \text{resolve } U V$

proof (rule ccontr)
assume $\text{resolve: } \neg \neg \text{resolve } U V$

obtain $L E D$ **where**
 $U: \text{trail } U \neq []$ **and**
 $\text{tr-}U: \text{hd-raw-trail } U = \text{Propagated } L E$ **and**
 $LE: L \in \# \text{ mset-cls } E$ **and**
 $\text{raw-}U: \text{raw-conflicting } U = \text{Some } D$ **and**
 $LD: -L \in \# \text{ mset-ccls } D$ **and**
 $\text{get-maximum-level } (\text{trail } U) (\text{mset-ccls } (\text{remove-clit } (-L) D)) = \text{backtrack-lvl } U$ **and**
 $V: V \sim \text{update-conflicting } (\text{Some } (\text{union-ccls } (\text{remove-clit } (-L) D) (\text{ccls-of-cls } (\text{remove-lit } L E))))$
 $(\text{tl-trail } U)$
using $\text{resolve by } (\text{auto elim!}; \text{resolveE})$

have $\text{cdcl}_W\text{-all-struct-inv } U$
using $\text{mono-rtrancpl}[of \text{skip cdcl}_W]$ **by** $(\text{meson } bj \text{ cdcl}_W\text{-bj.skip inv local.skip other}$
 $\text{rtrancpl-cdcl}_W\text{-all-struct-inv-inv})$

then have $[\text{iff}]: \text{no-dup } (\text{trail } S) \text{ cdcl}_W\text{-M-level-inv } S$ **and** $[\text{iff}]: \text{no-dup } (\text{trail } U)$
using $\text{inv unfolding cdcl}_W\text{-all-struct-inv-def cdcl}_W\text{-M-level-inv-def by blast+}$

then have
 $S: \text{init-clss } U = \text{init-clss } S$
 $\text{learned-clss } U = \text{learned-clss } S$
 $\text{backtrack-lvl } U = \text{backtrack-lvl } S$
 $\text{conflicting } S = \text{Some } (\text{mset-ccls } D)$
using $\text{rtrancpl-skip-state-decomp}[OF \text{skip}] U \text{ raw-}U$
by $(\text{auto simp del: state-simp simp: state-eq-def})$

obtain M_0 **where**
 $\text{tr-}S: \text{trail } S = M_0 @ \text{trail } U$ **and**
 $\text{nm: } \forall m \in \text{set } M_0. \neg \text{is-marked } m$
using $\text{rtrancpl-skip-state-decomp}[OF \text{skip}]$ **by** blast

obtain $K' i' M1' M2' L' D'$ **where**
 $\text{raw-}S': \text{raw-conflicting } S = \text{Some } D'$ **and**
 $LD': L' \in \# \text{ mset-ccls } D'$ **and**
 $\text{decomp}': (\text{Marked } K' (\text{Suc } i') \# M1', M2') \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } S))$ **and**
 $\text{lev-l: get-level } (\text{trail } S) L' = \text{backtrack-lvl } S$ **and**
 $\text{lev-l-D: get-level } (\text{trail } S) L' = \text{get-maximum-level } (\text{trail } S) (\text{mset-ccls } D')$ **and**
 $i': \text{get-maximum-level } (\text{trail } S) (\text{remove1-mset } L' (\text{mset-ccls } D')) \equiv i'$ **and**

$undef'$: undefined-lit $M1' L'$ **and**
 $R: T \sim cons-trail (Propagated L' (cls-of-ccls D'))$
 $(reduce-trail-to M1'$
 $(add-learned-ccls (cls-of-ccls D'))$
 $(update-backtrack-lvl i'$
 $(update-conflicting None S))))$
using bt **by** $(elim backtrack-levE) (fastforce simp: S state-eq-def simp del:state-simp)+$
obtain c **where** $M: trail S = c @ M2' @ Marked K' (i' + 1) \# M1'$
using $get-all-marked-decomposition-exists-prepend[OF decomp]$ **by** $auto$
have $marked: get-all-levels-of-marked (trail S) = rev [1..<1+backtrack-lvl S]$
using $inv unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def$ **by** $auto$
then have $i' < backtrack-lvl S$
unfolding M **by** $(force simp add: rev-swap[symmetric] dest!: arg-cong[of - - set])$

have $U: trail U = Propagated L (mset-ccls E) \# trail V$
using $tr-S hd-raw-trail[OF U] U S V tr-U$ **by** $(auto simp: lits-of-def)$
have $DD'[simp]: mset-ccls D' = mset-ccls D$
using $raw-U raw-S' S$ **by** $auto$
have $[simp]: L' = -L$
proof $(rule ccontr)$
assume $\neg ?thesis$
then have $-L \in \# remove1-mset L' (mset-ccls D')$
using $DD' LD' LD$ **by** $(simp add: in-remove1-mset-neq)$
moreover
have $M': trail S = M_0 @ Propagated L (mset-ccls E) \# trail V$
using $tr-S unfolding U$ **by** $auto$
have $no-dup (trail S)$
using $inv U unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def$ **by** $auto$
then have $atm-L-notin-M: atm-of L \notin atm-of ' (lits-of-l (trail V))$
using $M' U S$ **by** $(auto simp: lits-of-def)$
have $get-all-levels-of-marked (trail S) = rev [1..<1+backtrack-lvl S]$
using $inv U unfolding cdcl_W-all-struct-inv-def cdcl_W-M-level-inv-def$ **by** $auto$
then have $get-all-levels-of-marked (trail U) = rev [1..<1+backtrack-lvl S]$
using $nm M' U$ **by** $(simp add: get-all-levels-of-marked-no-marked)$
then have $get-lev-L:$
 $get-level(Propagated L (mset-ccls E) \# trail V) L = backtrack-lvl S$
using $get-level-get-rev-level-get-all-levels-of-marked[OF atm-L-notin-M,$
 $of [Propagated L (mset-ccls E)]] U$ **by** $auto$
have $atm-of L \notin atm-of ' (lits-of-l (rev M_0))$
using $\langle no-dup (trail S) \rangle M'$ **by** $(auto simp: lits-of-def)$
then have $get-level (trail S) L = backtrack-lvl S$
by $(metis M' get-lev-L get-rev-level-notin-end rev-append)$
ultimately
have $get-maximum-level (trail S) (remove1-mset L' (mset-ccls D')) \geq backtrack-lvl S$
by $(metis get-maximum-level-ge-get-level get-rev-level-uminus)$
then show $False$
using $\langle i' < backtrack-lvl S \rangle i'$ **by** $auto$
qed

have $cdcl_W^{**} S U$
using $bj cdcl_W-bj.skip local.skip mono-rtrancp[of skip cdcl_W S U]$ **other by** $meson$
then have $cdcl_W-all-struct-inv U$
using $inv rtrancp-cdcl_W-all-struct-inv-inv$ **by** $blast$
then have $Propagated L (mset-ccls E) \# trail V \models_{as} CNot (mset-ccls D')$
using $cdcl_W-all-struct-inv-def cdcl_W-conflicting-def raw-U U$ **by** $auto$
then have $\forall L' \in \# (remove1-mset L' (mset-ccls D')) . atm-of L' \in atm-of ' lits-of-l (Propagated L$

```

(mset-cls E) # trail U)
  using U atm-of-in-atm-of-set-iff-in-set-or-uminus-in-set in-CNot-implies-uminus(2)
  by (fastforce dest: in-diffD)
then have get-maximum-level (trail S) (remove1-mset L' (mset-ccls D')) = backtrack-lvl S
  using get-maximum-level-skip-un-marked-not-present[of remove1-mset L' (mset-ccls D')
    trail U M0] tr-S nm U
  ⟨get-maximum-level (trail U) (mset-ccls (remove-clit (- L) D)) = backtrack-lvl U⟩
  by (auto simp: S)
then show False
  using i' ⟨i' < backtrack-lvl S⟩ by auto
qed

```

lemma *if-can-apply-resolve-no-more-backtrack:*

```

assumes
  skip: skip** S U and
  resolve: resolve S T and
  inv: cdclW-all-struct-inv S
shows ¬backtrack U V
using assms
by (meson if-can-apply-backtrack-no-more-resolve rtranclp.rtrancl-refl
  rtranclp-skip-backtrack-backtrack)

```

lemma *if-can-apply-backtrack-skip-or-resolve-is-skip:*

```

assumes
  bt: backtrack S T and
  skip: skip-or-resolve** S U and
  inv: cdclW-all-struct-inv S
shows skip** S U
using assms(2,3,1)
by induction (simp-all add: if-can-apply-backtrack-no-more-resolve skip-or-resolve.simps)

```

lemma *cdcl_W-bj-bj-decomp:*

```

assumes cdclW-bj** S W and cdclW-all-struct-inv S
shows
  (∃ T U V. (λS T. skip-or-resolve S T ∧ no-step backtrack S)** S T
    ∧ (λT U. resolve T U ∧ no-step backtrack T) T U
    ∧ skip** U V ∧ backtrack V W)
  ∨ (∃ T U. (λS T. skip-or-resolve S T ∧ no-step backtrack S)** S T
    ∧ (λT U. resolve T U ∧ no-step backtrack T) T U ∧ skip** U W)
  ∨ (∃ T. skip** S T ∧ backtrack T W)
  ∨ skip** S W (is ?RB S W ∨ ?R S W ∨ ?SB S W ∨ ?S S W)

```

using assms

proof *induction*

case *base*

then show ?case **by** *simp*

next

case (step W X) **note** st = this(1) **and** bj = this(2) **and** IH = this(3)[OF this(4)] **and** inv = this(4)

have ¬?RB S W **and** ¬?SB S W

proof (*clarify, goal-cases*)

case (1 T U V)

have skip-or-resolve** S T

using 1(1) **by** (auto dest!: rtranclp-and-rtranclp-left)

then show False

by (metis (no-types, lifting) 1(2) 1(4) 1(5) backtrack-no-cdcl_W-bj

```

    cdclW-all-struct-inv-def cdclW-all-struct-inv-inv cdclW-o.bj local.bj other
    resolve rtrancpl-cdclW-all-struct-inv-inv rtrancpl-skip-backtrack-backtrack
    rtrancpl-skip-or-resolve-rtrancpl-cdclW step.premis)
next
  case 2
  then show ?case by (meson assms(2) cdclW-all-struct-inv-def backtrack-no-cdclW-bj
    local.bj rtrancpl-skip-backtrack-backtrack)
qed
then have IH: ?R S W  $\vee$  ?S S W using IH by blast

have cdclW** S W using mono-rtrancpl[of cdclW-bj cdclW] st by blast
then have inv-W: cdclW-all-struct-inv W by (simp add: rtrancpl-cdclW-all-struct-inv-inv
  step.premis)
consider
  (BT) X' where backtrack W X'
| (skip) no-step backtrack W and skip W X
| (resolve) no-step backtrack W and resolve W X
using bj cdclW-bj.cases by meson
then show ?case
proof cases
  case (BT X')
  then consider
    (bt) backtrack W X
  | (sk) skip W X
  using bj if-can-apply-backtrack-no-more-resolve[of W W X' X] inv-W cdclW-bj.cases by fast
then show ?thesis
proof cases
  case bt
  then show ?thesis using IH by auto
next
  case sk
  then show ?thesis using IH by (meson rtrancpl-trans r-into-rtrancpl)
qed
next
  case skip
  then show ?thesis using IH by (meson rtrancpl.rtrancpl-into-rtrancpl)
next
  case resolve note no-bt = this(1) and res = this(2)
  consider
    (RS) T U where
      ( $\lambda S T$ . skip-or-resolve S T  $\wedge$  no-step backtrack S)** S T and
      resolve T U and
      no-step backtrack T and
      skip** U W
  | (S) skip** S W
  using IH by auto
then show ?thesis
proof cases
  case (RS T U)
  have cdclW** S T
  using RS(1) cdclW-bj.resolve cdclW-o.bj other skip
  mono-rtrancpl[of ( $\lambda S T$ . skip-or-resolve S T  $\wedge$  no-step backtrack S) cdclW S T]
  by (meson skip-or-resolve.cases)
  then have cdclW-all-struct-inv U
  by (meson RS(2) cdclW-all-struct-inv-inv cdclW-bj.resolve cdclW-o.bj other

```

```

    rtrancpl-cdclW-all-struct-inv-inv step.prem)
{ fix U'
  assume skip** U U' and skip** U' W
  have cdclW-all-struct-inv U'
    using ⟨cdclW-all-struct-inv U⟩ ⟨skip** U U'⟩ rtrancpl-cdclW-all-struct-inv-inv
    cdclW-o.bj rtrancpl-mono[of skip cdclW] other skip by blast
  then have no-step backtrack U'
    using if-can-apply-backtrack-no-more-resolve[OF ⟨skip** U' W⟩] res by blast
}
with ⟨skip** U W⟩
have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** U W
  proof induction
    case base
    then show ?case by simp
  next
  case (step V W) note st = this(1) and skip = this(2) and IH = this(3) and H = this(4)
  have ∧U'. skip** U' V ⇒ skip** U' W
    using skip by auto
  then have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** U V
    using IH H by blast
  moreover have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** V W
    by (simp add: local.skip r-into-rtrancpl st step.prem skip-or-resolve.intros)
  ultimately show ?case by simp
qed
then show ?thesis
  proof -
    have f1: ∀ p pa pb pc. ¬ p (pa) pb ∨ ¬ p** pb pc ∨ p** pa pc
      by (meson converse-rtrancpl-into-rtrancpl)
    have skip-or-resolve T U ∧ no-step backtrack T
      using RS(2) RS(3) by force
    then have (λp pa. skip-or-resolve p pa ∧ no-step backtrack p)** T W
      proof -
        have (∃ vr19 vr16 vr17 vr18. vr19 (vr16::'st) vr17 ∧ vr19** vr17 vr18
          ∧ ¬ vr19** vr16 vr18)
          ∨ ¬ (skip-or-resolve T U ∧ no-step backtrack T)
          ∨ ¬ (λuu uua. skip-or-resolve uu uua ∧ no-step backtrack uu)** U W
          ∨ (λuu uua. skip-or-resolve uu uua ∧ no-step backtrack uu)** T W
          by force
        then show ?thesis
          by (metis (no-types) ⟨(λS T. skip-or-resolve S T ∧ no-step backtrack S)** U W⟩
            ⟨skip-or-resolve T U ∧ no-step backtrack T⟩ f1)
      qed
    then have (λp pa. skip-or-resolve p pa ∧ no-step backtrack p)** S W
      using RS(1) by force
    then show ?thesis
      using no-bt res by blast
  qed
next
case S
{ fix U'
  assume skip** S U' and skip** U' W
  then have cdclW** S U'
    using mono-rtrancpl[of skip cdclW S U'] by (simp add: cdclW-o.bj other skip)
  then have cdclW-all-struct-inv U'

```

```

    by (metis (no-types, hide-lams) ⟨cdclW-all-struct-inv S⟩
      rtrancpl-cdclW-all-struct-inv-inv)
  then have no-step backtrack U'
    using if-can-apply-backtrack-no-more-resolve[OF ⟨skip** U' W⟩] res by blast
}
with S
have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** S W
  proof induction
    case base
    then show ?case by simp
  next
  case (step V W) note st = this(1) and skip = this(2) and IH = this(3) and H = this(4)
  have ∧ U'. skip** U' V ⇒ skip** U' W
    using skip by auto
  then have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** S V
    using IH H by blast
  moreover have (λS T. skip-or-resolve S T ∧ no-step backtrack S)** V W

    by (simp add: local.skip r-into-rtrancpl st step.prem skip-or-resolve.intros)
  ultimately show ?case by simp
qed
then show ?thesis using res no-bt by blast
qed
qed
qed

```

The case distinction is needed, since $T \sim V$ does not imply that $R^{**} T V$.

lemma *cdcl_W-bj-strongly-confluent*:

```

  assumes
    cdclW-bj** S V and
    cdclW-bj** S T and
    n-s: no-step cdclW-bj V and
    inv: cdclW-all-struct-inv S
  shows T ~ V ∨ cdclW-bj** T V
  using assms(2)
proof induction
  case base
  then show ?case by (simp add: assms(1))
next
  case (step T U) note st = this(1) and s-o-r = this(2) and IH = this(3)
  have cdclW** S T
    using st mono-rtrancpl[of cdclW-bj cdclW] other by blast
  then have lev-T: cdclW-M-level-inv T
    using inv rtrancpl-cdclW-consistent-inv[of S T]
    unfolding cdclW-all-struct-inv-def by auto

  consider
    (TV) T ~ V
    | (bj-TV) cdclW-bj** T V
  using IH by blast
then show ?case
proof cases
  case TV
  have no-step cdclW-bj T
    using ⟨cdclW-M-level-inv T⟩ n-s cdclW-bj-state-eq-compatible[of T - V] TV

```

```

    by (meson backtrack-state-eq-compatible cdclW-bj.simps resolve-state-eq-compatible
        skip-state-eq-compatible state-eq-ref)
  then show ?thesis
    using s-o-r by auto
next
case bj-TV
then obtain U' where
  T-U': cdclW-bj T U' and
  cdclW-bj** U' V
  using IH n-s s-o-r by (metis rtrancpl-unfold trancplD)
have cdclW** S T
  by (metis (no-types, hide-lams) bj mono-rtrancpl[of cdclW-bj cdclW] other st)
then have inv-T: cdclW-all-struct-inv T
  by (metis (no-types, hide-lams) inv rtrancpl-cdclW-all-struct-inv-inv)

have lev-U: cdclW-M-level-inv U
  using s-o-r cdclW-consistent-inv lev-T other by blast
show ?thesis
  using s-o-r
  proof cases
    case backtrack
    then obtain V0 where skip** T V0 and backtrack V0 V
      using IH if-can-apply-backtrack-skip-or-resolve-is-skip[OF backtrack - inv-T]
        cdclW-bj-decomp-resolve-skip-and-bj
      by (meson bj-TV cdclW-bj.backtrack inv-T lev-T n-s
          rtrancpl-skip-backtrack-backtrack-end)
    then have cdclW-bj** T V0 and cdclW-bj V0 V
      using rtrancpl-mono[of skip cdclW-bj] by blast+
    then show ?thesis
      using ⟨backtrack V0 V⟩ ⟨skip** T V0⟩ backtrack-unique inv-T local.backtrack
        rtrancpl-skip-backtrack-backtrack by auto
  next
    case resolve
    then have U ~ U'
      by (meson T-U' cdclW-bj.simps if-can-apply-backtrack-no-more-resolve inv-T
          resolve-skip-deterministic resolve-unique rtrancpl.rtrancpl-refl)
    then show ?thesis
      using ⟨cdclW-bj** U' V⟩ unfolding rtrancpl-unfold
      by (meson T-U' bj cdclW-consistent-inv lev-T other state-eq-ref state-eq-sym
          trancpl-cdclW-bj-state-eq-compatible)
  next
    case skip
    consider
      (sk) skip T U'
    | (bt) backtrack T U'
    using T-U' by (meson cdclW-bj.cases local.skip resolve-skip-deterministic)
    then show ?thesis
      proof cases
        case sk
        then show ?thesis
          using ⟨cdclW-bj** U' V⟩ unfolding rtrancpl-unfold
          by (meson T-U' bj cdclW-all-inv(3) cdclW-all-struct-inv-def inv-T local.skip other
              trancpl-cdclW-bj-state-eq-compatible skip-unique state-eq-ref)
      next
        case bt

```



```

    have skip++ T U
      using local.skip by blast
    have cdclW-bj U U'
      by (meson ⟨skip++ T U⟩ backtrack bt inv-T rtrancpl-skip-backtrack-backtrack-end
            trancpl-into-rtrancpl)
    then have cdclW-bj++ U V
      using ⟨cdclW-bj** U' V⟩ by auto
    then show ?thesis
      by (meson trancpl-into-rtrancpl)
  qed
qed
qed
qed

```

lemma *cdcl_W-bj-unique-normal-form*:

```

assumes
  ST: cdclW-bj** S T and SU: cdclW-bj** S U and
  n-s-U: no-step cdclW-bj U and
  n-s-T: no-step cdclW-bj T and
  inv: cdclW-all-struct-inv S
shows T ~ U

```

proof –

```

  have T ~ U ∨ cdclW-bj** T U
    using ST SU cdclW-bj-strongly-confluent inv n-s-U by blast
  then show ?thesis
    by (metis (no-types) n-s-T rtrancpl-unfold state-eq-ref trancpl-unfold-begin)

```

qed

lemma *full-cdcl_W-bj-unique-normal-form*:

```

assumes full cdclW-bj S T and full cdclW-bj S U and
  inv: cdclW-all-struct-inv S
shows T ~ U
  using cdclW-bj-unique-normal-form assms unfolding full-def by blast

```

20.3 CDCL FW

inductive *cdcl_W-merge-restart* :: 'st ⇒ 'st ⇒ bool **where**

fw-r-propagate: *propagate S S' ⇒ cdcl_W-merge-restart S S' |*

fw-r-conflict: *conflict S T ⇒ full cdcl_W-bj T U ⇒ cdcl_W-merge-restart S U |*

fw-r-decide: *decide S S' ⇒ cdcl_W-merge-restart S S' |*

fw-r-rf: *cdcl_W-rf S S' ⇒ cdcl_W-merge-restart S S'*

lemma *rtrancpl-cdcl_W-bj-rtrancpl-cdcl_W*:

```

  cdclW-bj** S T ⇒ cdclW** S T
  using mono-rtrancpl[of cdclW-bj cdclW] by blast

```

lemma *cdcl_W-merge-restart-cdcl_W*:

```

assumes cdclW-merge-restart S T
shows cdclW** S T
  using assms

```

proof *induction*

```

  case (fw-r-conflict S T U) note confl = this(1) and bj = this(2)

```

```

  have cdclW S T using confl by (simp add: cdclW.intros r-into-rtrancpl)

```

moreover

```

  have cdclW-bj** T U using bj unfolding full-def by auto

```

then have $cdcl_W^{**} T U$ **using** $rtrancpl-cdcl_W-bj-rtrancpl-cdcl_W$ **by** *blast*
ultimately show *?case* **by** *auto*
qed (*simp-all add: cdcl_W-o.intros cdcl_W.intros r-into-rtrancpl*)

lemma $cdcl_W$ -merge-restart-conflicting-true-or-no-step:

assumes $cdcl_W$ -merge-restart $S T$
shows $conflicting T = None \vee no_step cdcl_W T$
using *assms*

proof *induction*

case (*fw-r-conflict* $S T U$) **note** $confl = this(1)$ **and** $n-s = this(2)$
{ **fix** $D V$
assume $cdcl_W U V$ **and** $conflicting U = Some D$
then have *False*
using $n-s$ **unfolding** *full-def*
by (*induction rule: cdcl_W-all-rules-induct*)
(*auto dest!: cdcl_W-bj.intros elim: decideE propagateE conflictE forgetE restartE*)
}
then show *?case* **by** (*cases conflicting U*) *fastforce+*
qed (*auto simp add: cdcl_W-rf.simps elim: propagateE decideE restartE forgetE*)

inductive $cdcl_W$ -merge :: $'st \Rightarrow 'st \Rightarrow bool$ **where**
fw-propagate: $propagate S S' \Longrightarrow cdcl_W$ -merge $S S' \mid$
fw-conflict: $conflict S T \Longrightarrow full\ cdcl_W$ -bj $T U \Longrightarrow cdcl_W$ -merge $S U \mid$
fw-decide: $decide S S' \Longrightarrow cdcl_W$ -merge $S S' \mid$
fw-forget: $forget S S' \Longrightarrow cdcl_W$ -merge $S S'$

lemma $cdcl_W$ -merge- $cdcl_W$ -merge-restart:

$cdcl_W$ -merge $S T \Longrightarrow cdcl_W$ -merge-restart $S T$
by (*meson cdcl_W-merge.cases cdcl_W-merge-restart.simps forget*)

lemma $rtrancpl-cdcl_W$ -merge- $trancpl-cdcl_W$ -merge-restart:

$cdcl_W$ -merge $^{**} S T \Longrightarrow cdcl_W$ -merge-restart $^{**} S T$
using $rtrancpl$ -mono[*of cdcl_W-merge cdcl_W-merge-restart*] $cdcl_W$ -merge- $cdcl_W$ -merge-restart **by** *blast*

lemma $cdcl_W$ -merge- $rtrancpl-cdcl_W$:

$cdcl_W$ -merge $S T \Longrightarrow cdcl_W^{**} S T$
using $cdcl_W$ -merge- $cdcl_W$ -merge-restart $cdcl_W$ -merge-restart- $cdcl_W$ **by** *blast*

lemma $rtrancpl-cdcl_W$ -merge- $rtrancpl-cdcl_W$:

$cdcl_W$ -merge $^{**} S T \Longrightarrow cdcl_W^{**} S T$
using $rtrancpl$ -mono[*of cdcl_W-merge cdcl_W^{**}*] $cdcl_W$ -merge- $rtrancpl-cdcl_W$ **by** *auto*

lemmas *rulesE* =

skipE resolveE backtrackE propagateE conflictE decideE restartE forgetE

lemma $cdcl_W$ -all-struct-inv- $trancpl-cdcl_W$ -merge- $trancpl-cdcl_W$ -merge- $cdcl_W$ -all-struct-inv:

assumes
inv: $cdcl_W$ -all-struct-inv b
 $cdcl_W$ -merge $^{++} b a$
shows $(\lambda S T. cdcl_W$ -all-struct-inv $S \wedge cdcl_W$ -merge $S T)^{++} b a$
using *assms(2)*

proof *induction*

case *base*
then show *?case* **using** *inv* **by** *auto*

next

```

case (step c d) note st = this(1) and fw = this(2) and IH = this(3)
have cdclW-all-struct-inv c
  using tranclp-into-rtranclp[OF st] cdclW-merge-rtranclp-cdclW
  assms(1) rtranclp-cdclW-all-struct-inv-inv rtranclp-mono[of cdclW-merge cdclW**] by fastforce
then have (λS T. cdclW-all-struct-inv S ∧ cdclW-merge S T)++ c d
  using fw by auto
then show ?case using IH by auto
qed

lemma backtrack-is-full1-cdclW-bj:
  assumes bt: backtrack S T and inv: cdclW-M-level-inv S
  shows full1 cdclW-bj S T
  using bt inv backtrack-no-cdclW-bj unfolding full1-def by blast

lemma rtrancl-cdclW-conflicting-true-cdclW-merge-restart:
  assumes cdclW** S V and inv: cdclW-M-level-inv S and conflicting S = None
  shows (cdclW-merge-restart** S V ∧ conflicting V = None)
    ∨ (∃ T U. cdclW-merge-restart** S T ∧ conflicting V ≠ None ∧ conflict T U ∧ cdclW-bj** U V)
  using assms
proof induction
  case base
  then show ?case by simp
next
  case (step U V) note st = this(1) and cdclW = this(2) and IH = this(3)[OF this(4-)] and
    confl[simp] = this(5) and inv = this(4)
  from cdclW
  show ?case
  proof (cases)
    case propagate
    moreover then have conflicting U = None and conflicting V = None
      by (auto elim: propagateE)
    ultimately show ?thesis using IH cdclW-merge-restart.fw-r-propagate[of U V] by auto
  next
    case conflict
    moreover then have conflicting U = None and conflicting V ≠ None
      by (auto elim!: conflictE simp del: state-simp simp: state-eq-def)
    ultimately show ?thesis using IH by auto
  next
    case other
    then show ?thesis
    proof cases
      case decide
      then show ?thesis using IH cdclW-merge-restart.fw-r-decide[of U V] by (auto elim: decideE)
    next
      case bj
      moreover {
        assume skip-or-resolve U V
        have f1: cdclW-bj++ U V
          by (simp add: local.bj tranclp.r-into-trancl)
        obtain T T' :: 'st where
          f2: cdclW-merge-restart** S U
            ∨ cdclW-merge-restart** S T ∧ conflicting U ≠ None
            ∧ conflict T T' ∧ cdclW-bj** T' U
          using IH confl by blast
        have conflicting V ≠ None ∧ conflicting U ≠ None

```

```

    using ⟨skip-or-resolve U V⟩
    by (auto simp: skip-or-resolve.simps state-eq-def elim!: skipE resolveE
        simp del: state-simp)
  then have ?thesis
    by (metis (full-types) IH f1 rtranclp-trans tranclp-into-rtranclp)
}
moreover {
  assume backtrack U V
  then have conflicting U ≠ None by (auto elim: backtrackE)
  then obtain T T' where
    cdclW-merge-restart** S T and
    conflicting U ≠ None and
    conflict T T' and
    cdclW-bj** T' U
    using IH confl by meson
  have invU: cdclW-M-level-inv U
    using inv rtranclp-cdclW-consistent-inv step.hyps(1) by blast
  then have conflicting V = None
    using ⟨backtrack U V⟩ inv by (auto elim: backtrack-levE
        simp: cdclW-M-level-inv-decomp)
  have full cdclW-bj T' V
    apply (rule rtranclp-fullI[of cdclW-bj T' U V])
    using ⟨cdclW-bj** T' U⟩ apply fast
    using ⟨backtrack U V⟩ backtrack-is-full1-cdclW-bj invU unfolding full1-def full-def
    by blast
  then have ?thesis
    using cdclW-merge-restart.fw-r-conflict[of T T' V] ⟨conflict T T'⟩
    ⟨cdclW-merge-restart** S T⟩ ⟨conflicting V = None⟩ by auto
}
ultimately show ?thesis by (auto simp: cdclW-bj.simps)
qed
next
case rf
moreover then have conflicting U = None and conflicting V = None
  by (auto simp: cdclW-rf.simps elim: restartE forgetE)
ultimately show ?thesis using IH cdclW-merge-restart.fw-r-rf[of U V] by auto
qed
qed

lemma no-step-cdclW-no-step-cdclW-merge-restart: no-step cdclW S ⇒ no-step cdclW-merge-restart S
  by (auto simp: cdclW.simps cdclW-merge-restart.simps cdclW-o.simps cdclW-bj.simps)

lemma no-step-cdclW-merge-restart-no-step-cdclW:
  assumes
    conflicting S = None and
    cdclW-M-level-inv S and
    no-step cdclW-merge-restart S
  shows no-step cdclW S
proof -
  { fix S'
    assume conflict S S'
    then have cdclW S S' using cdclW.conflict by auto
    then have cdclW-M-level-inv S'
      using assms(2) cdclW-consistent-inv by blast
  }

```

```

    then obtain  $S''$  where  $\text{full } \text{cdcl}_W\text{-bj } S' S''$ 
    using  $\text{cdcl}_W\text{-bj-exists-normal-form[of } S']$  by auto
    then have  $\text{False}$ 
    using  $\langle \text{conflict } S S' \rangle \text{ assms}(3) \text{ fw-r-conflict}$  by blast
  }
  then show ?thesis
  using assms unfolding  $\text{cdcl}_W.\text{simps}$   $\text{cdcl}_W\text{-merge-restart.simps}$   $\text{cdcl}_W\text{-o.simps}$   $\text{cdcl}_W\text{-bj.simps}$ 
  by (auto elim: skipE resolveE backtrackE conflictE decideE restartE)
qed

```

```

lemma  $\text{cdcl}_W\text{-merge-restart-no-step-cdcl}_W\text{-bj}$ :
  assumes
     $\text{cdcl}_W\text{-merge-restart } S T$ 
  shows  $\text{no-step } \text{cdcl}_W\text{-bj } T$ 
  using assms
  by (induction rule:  $\text{cdcl}_W\text{-merge-restart.induct}$ )
  (force simp:  $\text{cdcl}_W\text{-bj.simps}$   $\text{cdcl}_W\text{-rf.simps}$   $\text{cdcl}_W\text{-merge-restart.simps}$   $\text{full-def}$ 
    elim!: rulesE)+

```

```

lemma  $\text{rtrancpl-cdcl}_W\text{-merge-restart-no-step-cdcl}_W\text{-bj}$ :
  assumes
     $\text{cdcl}_W\text{-merge-restart}^{**} S T$  and
     $\text{conflicting } S = \text{None}$ 
  shows  $\text{no-step } \text{cdcl}_W\text{-bj } T$ 
  using assms unfolding  $\text{rtrancpl-unfold}$ 
  apply (elim disjE)
  apply (force simp:  $\text{cdcl}_W\text{-bj.simps}$   $\text{cdcl}_W\text{-rf.simps}$  elim!: rulesE)
  by (auto simp:  $\text{trancpl-unfold-end}$  simp:  $\text{cdcl}_W\text{-merge-restart-no-step-cdcl}_W\text{-bj}$ )

```

If $\text{conflicting } S \neq \text{None}$, we cannot say anything.

Remark that this theorem does not say anything about well-foundedness: even if you know that one relation is well-founded, it only states that the normal forms are shared.

```

lemma  $\text{conflicting-true-full-cdcl}_W\text{-iff-full-cdcl}_W\text{-merge}$ :
  assumes  $\text{confl: conflicting } S = \text{None}$  and  $\text{lev: cdcl}_W\text{-M-level-inv } S$ 
  shows  $\text{full } \text{cdcl}_W S V \iff \text{full } \text{cdcl}_W\text{-merge-restart } S V$ 
proof
  assume  $\text{full: full } \text{cdcl}_W\text{-merge-restart } S V$ 
  then have  $\text{st: cdcl}_W^{**} S V$ 
    using  $\text{rtrancpl-mono[of cdcl}_W\text{-merge-restart cdcl}_W^{**}]$   $\text{cdcl}_W\text{-merge-restart-cdcl}_W$ 
    unfolding  $\text{full-def}$  by auto

```

```

  have  $n\text{-s: no-step } \text{cdcl}_W\text{-merge-restart } V$ 
    using  $\text{full}$  unfolding  $\text{full-def}$  by auto
  have  $n\text{-s-bj: no-step } \text{cdcl}_W\text{-bj } V$ 
    using  $\text{rtrancpl-cdcl}_W\text{-merge-restart-no-step-cdcl}_W\text{-bj confl full}$  unfolding  $\text{full-def}$  by auto
  have  $\bigwedge S'. \text{conflict } V S' \implies \text{cdcl}_W\text{-M-level-inv } S'$ 
    using  $\text{cdcl}_W.\text{conflict}$   $\text{cdcl}_W\text{-consistent-inv lev rtrancpl-cdcl}_W\text{-consistent-inv st}$  by blast
  then have  $\bigwedge S'. \text{conflict } V S' \implies \text{False}$ 
    using  $n\text{-s } n\text{-s-bj}$   $\text{cdcl}_W\text{-bj-exists-normal-form}$   $\text{cdcl}_W\text{-merge-restart.simps}$  by meson
  then have  $n\text{-s-cdcl}_W\text{: no-step } \text{cdcl}_W V$ 
    using  $n\text{-s } n\text{-s-bj}$  by (auto simp:  $\text{cdcl}_W.\text{simps}$   $\text{cdcl}_W\text{-o.simps}$   $\text{cdcl}_W\text{-merge-restart.simps}$ )
  then show  $\text{full } \text{cdcl}_W S V$  using  $\text{st}$  unfolding  $\text{full-def}$  by auto
next
  assume  $\text{full: full } \text{cdcl}_W S V$ 
  have  $\text{no-step } \text{cdcl}_W\text{-merge-restart } V$ 

```

```

using full no-step-cdclW-no-step-cdclW-merge-restart unfolding full-def by blast
moreover
consider
  (fw) cdclW-merge-restart** S V and conflicting V = None
| (bj) T U where
  cdclW-merge-restart** S T and
  conflicting V ≠ None and
  conflict T U and
  cdclW-bj** U V
using full rtrancp-cdclW-conflicting-true-cdclW-merge-restart confl lev unfolding full-def
by meson
then have cdclW-merge-restart** S V
proof cases
  case fw
  then show ?thesis by fast
next
  case (bj T U)
  have no-step cdclW-bj V
  using full unfolding full-def by (meson cdclW-o.bj other)
  then have full cdclW-bj U V
  using ⟨ cdclW-bj** U V ⟩ unfolding full-def by auto
  then have cdclW-merge-restart T V
  using ⟨ conflict T U ⟩ cdclW-merge-restart.fw-r-conflict by blast
  then show ?thesis using ⟨ cdclW-merge-restart** S T ⟩ by auto
qed
ultimately show full cdclW-merge-restart S V unfolding full-def by fast
qed

```

lemma *init-state-true-full-cdcl_W-iff-full-cdcl_W-merge:*
shows full cdcl_W (init-state N) V \longleftrightarrow full cdcl_W-merge-restart (init-state N) V
by (rule conflicting-true-full-cdcl_W-iff-full-cdcl_W-merge) auto

20.4 FW with strategy

20.4.1 The intermediate step

inductive cdcl_W-s' :: 'st \Rightarrow 'st \Rightarrow bool **where**
conflict': full1 cdcl_W-cp S S' \Longrightarrow cdcl_W-s' S S' |
decide': decide S S' \Longrightarrow no-step cdcl_W-cp S \Longrightarrow full cdcl_W-cp S' S'' \Longrightarrow cdcl_W-s' S S'' |
bj': full1 cdcl_W-bj S S' \Longrightarrow no-step cdcl_W-cp S \Longrightarrow full cdcl_W-cp S' S'' \Longrightarrow cdcl_W-s' S S''

inductive-cases cdcl_W-s'E: cdcl_W-s' S T

lemma rtrancp-cdcl_W-bj-full1-cdclp-cdcl_W-stgy:
 cdcl_W-bj** S S' \Longrightarrow full cdcl_W-cp S' S'' \Longrightarrow cdcl_W-stgy** S S''

proof (induction rule: converse-rtrancp-induct)

case base

then show ?case **by** (metis cdcl_W-stgy.conflict' full-unfold rtrancp.simps)

next

case (step T U) **note** st = this(2) **and** bj = this(1) **and** IH = this(3)[OF this(4)]

have no-step cdcl_W-cp T

using bj **by** (auto simp add: cdcl_W-bj.simps cdcl_W-cp.simps elim!: rulesE)

consider

(U) U = S'

| (U') U' **where** cdcl_W-bj U U' **and** cdcl_W-bj** U' S'

using st **by** (metis converse-rtrancpE)

```

then show ?case
proof cases
  case U
  then show ?thesis
    using ⟨no-step cdclW-cp T⟩ cdclW-o.bj local.bj other' step.prem by (meson r-into-rtrancp)
next
  case U' note U' = this(1)
  have no-step cdclW-cp U
    using U' by (fastforce simp: cdclW-cp.simps cdclW-bj.simps elim: rulesE)
  then have full cdclW-cp U U
    by (simp add: full-unfold)
  then have cdclW-stgy T U
    using ⟨no-step cdclW-cp T⟩ cdclW-stgy.simps local.bj cdclW-o.bj by meson
  then show ?thesis using IH by auto
qed
qed

```

```

lemma cdclW-s'-is-rtrancp-cdclW-stgy:
  cdclW-s' S T  $\implies$  cdclW-stgy** S T
  apply (induction rule: cdclW-s'.induct)
  apply (auto intro: cdclW-stgy.intros)[]
  apply (meson decide other' r-into-rtrancp)
  by (metis full1-def rtrancp-cdclW-bj-full1-cdclp-cdclW-stgy trancp-into-rtrancp)

```

lemma cdcl_W-cp-cdcl_W-bj-bissimulation:

```

assumes
  full cdclW-cp T U and
  cdclW-bj** T T' and
  cdclW-all-struct-inv T and
  no-step cdclW-bj T'
shows full cdclW-cp T' U
   $\vee (\exists U' U''. \text{full cdcl}_W\text{-cp } T' U'' \wedge \text{full1 cdcl}_W\text{-bj } U U' \wedge \text{full cdcl}_W\text{-cp } U' U''$ 
     $\wedge \text{cdcl}_W\text{-s}^{**} U U'')$ 
  using assms(2,1,3,4)
proof (induction rule: rtrancp-induct)
  case base
  then show ?case by blast
next
  case (step T' T'') note st = this(1) and bj = this(2) and IH = this(3)[OF this(4,5)] and
    full = this(4) and inv = this(5)
  have cdclW-bj** T T''
    using local.bj st by auto
  then have cdclW** T T''
    using rtrancp-cdclW-bj-rtrancp-cdclW by blast
  then have inv-T'': cdclW-all-struct-inv T''
    using inv rtrancp-cdclW-all-struct-inv-inv by blast
  have cdclW-bj++ T T''
    using local.bj st by auto
  have full1 cdclW-bj T T''
    by (metis ⟨cdclW-bj++ T T'⟩ full1-def step.prem(3))
  then have T = U
  proof -
    obtain Z where cdclW-bj T Z
      using ⟨cdclW-bj++ T T'⟩ by (blast dest: trancpD)
    { assume cdclW-cp++ T U

```

```

    then obtain  $Z'$  where  $cdcl_W\text{-}cp\ T\ Z'$ 
      by (meson tranclpD)
    then have False
      using  $\langle cdcl_W\text{-}bj\ T\ Z \rangle$  by (fastforce simp:  $cdcl_W\text{-}bj.simps\ cdcl_W\text{-}cp.simps$ 
        elim: rulesE)
  }
  then show ?thesis
    using full unfolding full-def rtranclp-unfold by blast
qed
obtain  $U''$  where full  $cdcl_W\text{-}cp\ T''\ U''$ 
  using  $cdcl_W\text{-}cp\text{-}normalized\text{-}element\text{-}all\text{-}inv\ inv\text{-}T''$  by blast
moreover then have  $cdcl_W\text{-}stgy^{**}\ U\ U''$ 
  by (metis  $\langle T = U \rangle \langle cdcl_W\text{-}bj^{++}\ T\ T'' \rangle rtranclp\text{-}cdcl_W\text{-}bj\text{-}full1\text{-}cdclp\text{-}cdcl_W\text{-}stgy\ rtranclp\text{-}unfold$ )
moreover have  $cdcl_W\text{-}s^{**}\ U\ U''$ 
proof -
  obtain  $ss :: 'st \Rightarrow 'st$  where
     $f1: \forall x2. (\exists v3. cdcl_W\text{-}cp\ x2\ v3) = cdcl_W\text{-}cp\ x2\ (ss\ x2)$ 
    by moura
  have  $\neg cdcl_W\text{-}cp\ U\ (ss\ U)$ 
    by (meson full full-def)
  then show ?thesis
    using  $f1$  by (metis (no-types)  $\langle T = U \rangle \langle full1\ cdcl_W\text{-}bj\ T\ T'' \rangle bj'\ calculation(1)$ 
      r-into-rtranclp)
qed
ultimately show ?case
  using  $\langle full1\ cdcl_W\text{-}bj\ T\ T'' \rangle \langle full\ cdcl_W\text{-}cp\ T''\ U'' \rangle$  unfolding  $\langle T = U \rangle$  by blast
qed

```

lemma $cdcl_W\text{-}cp\text{-}cdcl_W\text{-}bj\text{-}bissimulation'$:

```

assumes
  full  $cdcl_W\text{-}cp\ T\ U$  and
   $cdcl_W\text{-}bj^{**}\ T\ T'$  and
   $cdcl_W\text{-}all\text{-}struct\text{-}inv\ T$  and
  no-step  $cdcl_W\text{-}bj\ T'$ 
shows full  $cdcl_W\text{-}cp\ T'\ U$ 
   $\vee (\exists U'. full1\ cdcl_W\text{-}bj\ U\ U' \wedge (\forall U''. full\ cdcl_W\text{-}cp\ U'\ U'' \longrightarrow full\ cdcl_W\text{-}cp\ T'\ U''$ 
     $\wedge cdcl_W\text{-}s^{**}\ U\ U''))$ 
  using assms(2,1,3,4)
proof (induction rule: rtranclp-induct)
  case base
  then show ?case by blast
next
  case (step  $T'\ T''$ ) note  $st = this(1)$  and  $bj = this(2)$  and  $IH = this(3)[OF\ this(4,5)]$  and
     $full = this(4)$  and  $inv = this(5)$ 
  have  $cdcl_W^{**}\ T\ T''$ 
    by (metis local.bj rtranclp.simps rtranclp-cdcl_W-bj-rtranclp-cdcl_W st)
  then have  $inv\text{-}T''$ :  $cdcl_W\text{-}all\text{-}struct\text{-}inv\ T''$ 
    using  $inv\ rtranclp\text{-}cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}inv$  by blast
  have  $cdcl_W\text{-}bj^{++}\ T\ T''$ 
    using local.bj st by auto
  have  $full1\ cdcl_W\text{-}bj\ T\ T''$ 
    by (metis  $\langle cdcl_W\text{-}bj^{++}\ T\ T'' \rangle full1\text{-}def\ step.premis(3)$ )
  then have  $T = U$ 
  proof -
    obtain  $Z$  where  $cdcl_W\text{-}bj\ T\ Z$ 

```



```

    using  $\langle cdcl_W\text{-}bj^{++} \ T \ T' \rangle$  by (blast dest: tranclpD)
  { assume  $cdcl_W\text{-}cp^{++} \ T \ U$ 
    then obtain  $Z'$  where  $cdcl_W\text{-}cp \ T \ Z'$ 
      by (meson tranclpD)
    then have False
      using  $\langle cdcl_W\text{-}bj \ T \ Z \rangle$  by (fastforce simp:  $cdcl_W\text{-}bj.simps \ cdcl_W\text{-}cp.simps \ elim: rulesE$ )
    }
  then show ?thesis
    using full unfolding full-def rtranclp-unfold by blast
qed
{ fix  $U''$ 
  assume full  $cdcl_W\text{-}cp \ T'' \ U''$ 
  moreover then have  $cdcl_W\text{-}stgy^{**} \ U \ U''$ 
    by (metis  $\langle T = U \rangle \langle cdcl_W\text{-}bj^{++} \ T \ T' \rangle \ rtranclp\text{-}cdcl_W\text{-}bj\text{-}full1\text{-}cdclp\text{-}cdcl_W\text{-}stgy \ rtranclp\text{-}unfold$ )
  moreover have  $cdcl_W\text{-}s^{**} \ U \ U''$ 
  proof -
    obtain  $ss :: 'st \Rightarrow 'st$  where
       $f1: \forall x2. (\exists v3. \ cdcl_W\text{-}cp \ x2 \ v3) = cdcl_W\text{-}cp \ x2 \ (ss \ x2)$ 
      by moura
    have  $\neg \ cdcl_W\text{-}cp \ U \ (ss \ U)$ 
      by (meson assms(1) full-def)
    then show ?thesis
      using f1 by (metis (no-types)  $\langle T = U \rangle \langle full1 \ cdcl_W\text{-}bj \ T \ T' \rangle \ bj' \ calculation(1) \ r\text{-}into\text{-}rtranclp$ )
    qed
  ultimately have  $full1 \ cdcl_W\text{-}bj \ U \ T''$  and  $cdcl_W\text{-}s'^{**} \ T'' \ U''$ 
    using  $\langle full1 \ cdcl_W\text{-}bj \ T \ T' \rangle \langle full \ cdcl_W\text{-}cp \ T'' \ U'' \rangle$  unfolding  $\langle T = U \rangle$ 
    apply blast
    by (metis  $\langle full \ cdcl_W\text{-}cp \ T'' \ U'' \rangle \ cdcl_W\text{-}s'.simps \ full\text{-}unfold \ rtranclp.simps$ )
  }
then show ?case
  using  $\langle full1 \ cdcl_W\text{-}bj \ T \ T' \rangle \ full \ bj' \ unfolding \langle T = U \rangle \ full\text{-}def$  by (metis  $r\text{-}into\text{-}rtranclp$ )
qed

```

lemma $cdcl_W\text{-}stgy\text{-}cdcl_W\text{-}s'\text{-}connected$:

```

  assumes  $cdcl_W\text{-}stgy \ S \ U$  and  $cdcl_W\text{-}all\text{-}struct\text{-}inv \ S$ 
  shows  $cdcl_W\text{-}s' \ S \ U$ 
     $\vee (\exists U'. \ full1 \ cdcl_W\text{-}bj \ U \ U' \wedge (\forall U''. \ full \ cdcl_W\text{-}cp \ U' \ U'' \longrightarrow cdcl_W\text{-}s' \ S \ U''))$ 
  using assms
proof (induction rule:  $cdcl_W\text{-}stgy.induct$ )
  case (conflict' T)
  then have  $cdcl_W\text{-}s' \ S \ T$ 
    using  $cdcl_W\text{-}s'.conflict'$  by blast
  then show ?case
    by blast
next
  case (other' T U) note  $o = this(1)$  and  $n\text{-}s = this(2)$  and  $full = this(3)$  and  $inv = this(4)$ 
  show ?case
    using o
  proof cases
    case decide
    then show ?thesis using  $cdcl_W\text{-}s'.simps \ full \ n\text{-}s$  by blast
  next
    case bj
    have  $inv\text{-}T: \ cdcl_W\text{-}all\text{-}struct\text{-}inv \ T$ 

```

```

    using cdclW-all-struct-inv-inv o other other'.prems by blast
consider
  (cp) full cdclW-cp T U and no-step cdclW-bj T
| (fbj) T' where full1 cdclW-bj T T'
apply (cases no-step cdclW-bj T)
  using full apply blast
  using cdclW-bj-exists-normal-form[of T] inv-T unfolding cdclW-all-struct-inv-def
  by (metis full-unfold)
then show ?thesis
proof cases
  case cp
  then show ?thesis
  proof -
    obtain ss :: 'st ⇒ 'st where
      f1: ∀ s sa sb. (¬ full1 cdclW-bj s sa ∨ cdclW-cp s (ss s) ∨ ¬ full cdclW-cp sa sb)
      ∨ cdclW-s' s sb
    using bj' by moura
    have full1 cdclW-bj S T
    by (simp add: cp(2) full1-def local.bj tranclp.r-into-trancl)
    then show ?thesis
    using f1 full n-s by blast
  qed
next
  case (fbj U')
  then have full1 cdclW-bj S U'
    using bj unfolding full1-def by auto
  moreover have no-step cdclW-cp S
    using n-s by blast
  moreover have T = U
    using full fbj unfolding full1-def full-def rtranclp-unfold
    by (force dest!: tranclpD simp:cdclW-bj.simps elim: rulesE)
  ultimately show ?thesis using cdclW-s'.bj'[of S U'] using fbj by blast
qed
qed
qed

lemma cdclW-stgy-cdclW-s'-connected':
  assumes cdclW-stgy S U and cdclW-all-struct-inv S
  shows cdclW-s' S U
    ∨ (∃ U' U''. cdclW-s' S U'' ∧ full1 cdclW-bj U U' ∧ full cdclW-cp U' U'')
  using assms
proof (induction rule: cdclW-stgy.induct)
  case (conflict' T)
  then have cdclW-s' S T
    using cdclW-s'.conflict' by blast
  then show ?case
    by blast
next
  case (other' T U) note o = this(1) and n-s = this(2) and full = this(3) and inv = this(4)
  show ?case
    using o
  proof cases
    case decide
    then show ?thesis using cdclW-s'.simps full n-s by blast
  next

```

```

case bj
have cdclW-all-struct-inv T
  using cdclW-all-struct-inv-inv o other other'.prems by blast
then obtain T' where T': full cdclW-bj T T'
  using cdclW-bj-exists-normal-form unfolding full-def cdclW-all-struct-inv-def by metis
then have full cdclW-bj S T'
  proof –
    have f1: cdclW-bj** T T' ∧ no-step cdclW-bj T'
      by (metis (no-types) T' full-def)
    then have cdclW-bj** S T'
      by (meson converse-rtranclp-into-rtranclp local.bj)
    then show ?thesis
      using f1 by (simp add: full-def)
  qed
have cdclW-bj** T T'
  using T' unfolding full-def by simp
have cdclW-all-struct-inv T
  using cdclW-all-struct-inv-inv o other other'.prems by blast
then consider
  (T'U) full cdclW-cp T' U
  | (U) U' U'' where
    full cdclW-cp T' U'' and
    full1 cdclW-bj U U' and
    full cdclW-cp U' U'' and
    cdclW-s*** U U''
  using cdclW-cp-cdclW-bj-bissimulation[OF full ⟨cdclW-bj** T T'⟩ T' unfolding full-def
  by blast
then show ?thesis by (metis T' cdclW-s'.simps full-full1 local.bj n-s)
qed
qed

```

lemma *cdcl_W-stgy-cdcl_W-s'-no-step:*

```

assumes cdclW-stgy S U and cdclW-all-struct-inv S and no-step cdclW-bj U
shows cdclW-s' S U
using cdclW-stgy-cdclW-s'-connected[OF assms(1,2)] assms(3)
by (metis (no-types, lifting) full1-def tranclpD)

```

lemma *rtranclp-cdcl_W-stgy-connected-to-rtranclp-cdcl_W-s':*

```

assumes cdclW-stgy** S U and inv: cdclW-M-level-inv S
shows cdclW-s*** S U ∨ (∃ T. cdclW-s*** S T ∧ cdclW-bj++ T U ∧ conflicting U ≠ None)
using assms(1)

```

proof *induction*

case *base*

then show *?case* **by** *simp*

next

case (*step T V*) **note** *st = this(1)* **and** *o = this(2)* **and** *IH = this(3)*

from *o* **show** *?case*

proof *cases*

case *conflict'*

then have *f2: cdcl_W-s' T V*

using *cdcl_W-s'.conflict'* **by** *blast*

obtain *ss :: 'st* **where**

*f3: S = T ∨ cdcl_W-stgy** S ss ∧ cdcl_W-stgy ss T*

by (*metis (full-types) rtranclp.simps st*)

obtain *ssa :: 'st* **where**

```

    ssa: cdclW-cp T ssa
    using conflict' by (metis (no-types) full1-def tranclpD)
have  $\forall s. \neg \text{full } \text{cdcl}_W\text{-cp } s \text{ } T$ 
  by (meson ssa full-def)
then have  $S = T$ 
  by (metis (full-types) f3 ssa cdclW-stgy.cases full1-def)
then show ?thesis
  using f2 by blast
next
case (other' U) note  $o = \text{this}(1)$  and  $n\text{-}s = \text{this}(2)$  and  $\text{full} = \text{this}(3)$ 
then show ?thesis
  using o
  proof (cases rule: cdclW-o-rule-cases)
    case decide
    then have cdclW-s'** S T
      using IH by (auto elim: rulesE)
    then show ?thesis
      by (meson decide decide' full n-s rtranclp.rtrancl-into-rtrancl)
  next
  case backtrack
  consider
    (s') cdclW-s'** S T
  | (bj) S' where cdclW-s'** S S' and cdclW-bj++ S' T and conflicting T ≠ None
  using IH by blast
then show ?thesis
  proof cases
    case s'
    moreover
      have cdclW-M-level-inv T
        using inv local.step(1) rtranclp-cdclW-stgy-consistent-inv by auto
      then have full1 cdclW-bj T U
        using backtrack-is-full1-cdclW-bj backtrack by blast
      then have cdclW-s' T V
        using full bj' n-s by blast
      ultimately show ?thesis by auto
    next
    case (bj S') note  $S\text{-}S' = \text{this}(1)$  and  $\text{bj}\text{-}T = \text{this}(2)$ 
    have no-step cdclW-cp S'
      using bj-T by (fastforce simp: cdclW-cp.simps cdclW-bj.simps dest!: tranclpD
        elim: rulesE)
    moreover
      have cdclW-M-level-inv T
        using inv local.step(1) rtranclp-cdclW-stgy-consistent-inv by auto
      then have full1 cdclW-bj T U
        using backtrack-is-full1-cdclW-bj backtrack by blast
      then have full1 cdclW-bj S' U
        using bj-T unfolding full1-def by fastforce
      ultimately have cdclW-s' S' V using full by (simp add: bj')
      then show ?thesis using  $S\text{-}S'$  by auto
  qed
next
case skip
then have [simp]:  $U = V$ 
  using full converse-rtranclpE unfolding full-def by (fastforce elim: rulesE)
then have confl-V: conflicting V ≠ None

```

```

    using skip by (auto elim!: rulesE simp del: state-simp simp: state-eq-def)
  consider
    (s') cdclW-s'** S T
  | (bj) S' where cdclW-s'** S S' and cdclW-bj++ S' T and conflicting T ≠ None
    using IH by blast
  then show ?thesis
  proof cases
    case s'
    show ?thesis using s' confl-V skip by force
  next
    case (bj S') note S-S' = this(1) and bj-T = this(2)
    have cdclW-bj++ S' V
      using skip bj-T by (metis (U = V) cdclW-bj.skip tranclp.simps)
    then show ?thesis using S-S' confl-V by auto
  qed
next
case resolve
then have [simp]: U = V
  using full unfolding full-def rtranclp-unfold
  by (auto elim!: rulesE dest!: tranclpD
    simp del: state-simp simp: state-eq-def cdclW-cp.simps)
have confl-V: conflicting V ≠ None
  using resolve by (auto elim!: rulesE simp del: state-simp simp: state-eq-def)

consider
  (s') cdclW-s'** S T
| (bj) S' where cdclW-s'** S S' and cdclW-bj++ S' T and conflicting T ≠ None
  using IH by blast
then show ?thesis
proof cases
  case s'
  have cdclW-bj++ T V
    using resolve by force
  then show ?thesis using s' confl-V by auto
next
  case (bj S') note S-S' = this(1) and bj-T = this(2)
  have cdclW-bj++ S' V
    using resolve bj-T by (metis (U = V) cdclW-bj.resolve tranclp.simps)
  then show ?thesis using confl-V S-S' by auto
qed
qed
qed
qed

lemma n-step-cdclW-stgy-iff-no-step-cdclW-cl-cdclW-o:
  assumes inv: cdclW-all-struct-inv S
  shows no-step cdclW-s' S ⟷ no-step cdclW-cp S ∧ no-step cdclW-o S (is ?S' S ⟷ ?C S ∧ ?O S)
proof
  assume ?C S ∧ ?O S
  then show ?S' S
    by (auto simp: cdclW-s'.simps full1-def tranclp-unfold-begin)
next
  assume n-s: ?S' S
  have ?C S
    proof (rule ccontr)

```

```

    assume  $\neg ?thesis$ 
    then obtain  $S'$  where  $cdcl_W\text{-}cp\ S\ S'$ 
      by auto
    then obtain  $T$  where  $full1\ cdcl_W\text{-}cp\ S\ T$ 
      using  $cdcl_W\text{-}cp\text{-}normalized\text{-}element\text{-}all\text{-}inv\ inv$  by (metis (no-types, lifting) full-unfold)
    then show  $False$  using  $n\text{-}s\ cdcl_W\text{-}s'\text{-}conflict'$  by blast
  qed
moreover have  $?O\ S$ 
proof (rule ccontr)
  assume  $\neg ?thesis$ 
  then obtain  $S'$  where  $cdcl_W\text{-}o\ S\ S'$ 
    by auto
  then obtain  $T$  where  $full1\ cdcl_W\text{-}cp\ S'\ T$ 
    using  $cdcl_W\text{-}cp\text{-}normalized\text{-}element\text{-}all\text{-}inv\ inv$ 
    by (meson  $cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}def\ n\text{-}s$ 
       $cdcl_W\text{-}stgy\text{-}cdcl_W\text{-}s'\text{-}connected'\ cdcl_W\text{-}then\text{-}exists\text{-}cdcl_W\text{-}stgy\text{-}step$ )
  then show  $False$  using  $n\text{-}s$  by (meson  $\langle cdcl_W\text{-}o\ S\ S' \rangle\ cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}def$ 
     $cdcl_W\text{-}stgy\text{-}cdcl_W\text{-}s'\text{-}connected'\ cdcl_W\text{-}then\text{-}exists\text{-}cdcl_W\text{-}stgy\text{-}step\ inv$ )
  qed
ultimately show  $?C\ S \wedge ?O\ S$  by auto
qed

lemma  $cdcl_W\text{-}s'\text{-}trancpl\text{-}cdcl_W$ :
   $cdcl_W\text{-}s'\ S\ S' \implies cdcl_W^{++}\ S\ S'$ 
proof (induct rule:  $cdcl_W\text{-}s'\text{-}induct$ )
  case  $conflict'$ 
  then show  $?case$ 
    by (simp add:  $full1\text{-}def\ trancpl\text{-}cdcl_W\text{-}cp\text{-}trancpl\text{-}cdcl_W$ )
next
  case  $decide'$ 
  then show  $?case$ 
    using  $cdcl_W\text{-}stgy.simps\ cdcl_W\text{-}stgy\text{-}trancpl\text{-}cdcl_W$  by (meson  $cdcl_W\text{-}o.simps$ )
next
  case ( $bj'\ Sa\ S'a\ S''$ ) note  $a2 = this(1)$  and  $a1 = this(2)$  and  $n\text{-}s = this(3)$ 
  obtain  $ss :: 'st \Rightarrow 'st \Rightarrow ('st \Rightarrow 'st \Rightarrow bool) \Rightarrow 'st$  where
     $\forall x0\ x1\ x2. (\exists v3. x2\ x1\ v3 \wedge x2^{**}\ v3\ x0) = (x2\ x1\ (ss\ x0\ x1\ x2) \wedge x2^{**}\ (ss\ x0\ x1\ x2)\ x0)$ 
    by moura
  then have  $f3: \forall p\ s\ sa. \neg p^{++}\ s\ sa \vee p\ s\ (ss\ sa\ s\ p) \wedge p^{**}\ (ss\ sa\ s\ p)\ sa$ 
    by (metis (full-types)  $trancplD$ )
  have  $cdcl_W\text{-}bj^{++}\ Sa\ S'a \wedge no\text{-}step\ cdcl_W\text{-}bj\ S'a$ 
    using  $a2$  by (simp add:  $full1\text{-}def$ )
  then have  $cdcl_W\text{-}bj\ Sa\ (ss\ S'a\ Sa\ cdcl_W\text{-}bj) \wedge cdcl_W\text{-}bj^{**}\ (ss\ S'a\ Sa\ cdcl_W\text{-}bj)\ S'a$ 
    using  $f3$  by auto
  then show  $cdcl_W^{++}\ Sa\ S''$ 
    using  $a1\ n\text{-}s$  by (meson  $bj\ other\ rtrancpl\text{-}cdcl_W\text{-}bj\text{-}full1\text{-}cdclp\text{-}cdcl_W\text{-}stgy$ 
       $rtrancpl\text{-}cdcl_W\text{-}stgy\text{-}rtrancpl\text{-}cdcl_W\ rtrancpl\text{-}into\text{-}trancpl2$ )
  qed

lemma  $trancpl\text{-}cdcl_W\text{-}s'\text{-}trancpl\text{-}cdcl_W$ :
   $cdcl_W\text{-}s'^{++}\ S\ S' \implies cdcl_W^{++}\ S\ S'$ 
  apply (induct rule:  $trancpl.induct$ )
  using  $cdcl_W\text{-}s'\text{-}trancpl\text{-}cdcl_W$  apply blast
  by (meson  $cdcl_W\text{-}s'\text{-}trancpl\text{-}cdcl_W\ trancpl\text{-}trans$ )

lemma  $rtrancpl\text{-}cdcl_W\text{-}s'\text{-}rtrancpl\text{-}cdcl_W$ :

```

$cdcl_W-s'^{**} S S' \implies cdcl_W^{**} S S'$
using *rtrancpl-unfold*[*of cdcl_W-s' S S'*] *trancpl-cdcl_W-s'-trancpl-cdcl_W*[*of S S'*] **by** *auto*

lemma *full-cdcl_W-stgy-iff-full-cdcl_W-s'*:
assumes *inv*: *cdcl_W-all-struct-inv S*
shows *full cdcl_W-stgy S T \longleftrightarrow full cdcl_W-s' S T* (**is** $?S \longleftrightarrow ?S'$)

proof
assume $?S'$
then have *cdcl_W^{**} S T*
using *rtrancpl-cdcl_W-s'-rtrancpl-cdcl_W*[*of S T*] **unfolding** *full-def* **by** *blast*
then have *inv'*: *cdcl_W-all-struct-inv T*
using *rtrancpl-cdcl_W-all-struct-inv-inv inv* **by** *blast*
have *cdcl_W-stgy^{**} S T*
using $\langle ?S' \rangle$ **unfolding** *full-def*
using *cdcl_W-s'-is-rtrancpl-cdcl_W-stgy rtrancpl-mono*[*of cdcl_W-s' cdcl_W-stgy^{**}*] **by** *auto*
then show $?S$
using $\langle ?S' \rangle$ *inv' cdcl_W-stgy-cdcl_W-s'-connected'* **unfolding** *full-def* **by** *blast*

next
assume $?S$
then have *inv-T:cdcl_W-all-struct-inv T*
by (*metis assms full-def rtrancpl-cdcl_W-all-struct-inv-inv rtrancpl-cdcl_W-stgy-rtrancpl-cdcl_W*)

consider
 $(s') \text{ } cdcl_W-s'^{**} S T$
 $| (st) S' \text{ where } cdcl_W-s'^{**} S S' \text{ and } cdcl_W-bj^{++} S' T \text{ and conflicting } T \neq None$
using *rtrancpl-cdcl_W-stgy-connected-to-rtrancpl-cdcl_W-s'*[*of S T*] *inv* $\langle ?S \rangle$
unfolding *full-def cdcl_W-all-struct-inv-def*
by *blast*

then show $?S'$
proof cases
case s'
have *no-step cdcl_W-s' T*
using $\langle full \text{ } cdcl_W-stgy \text{ } S \text{ } T \rangle$ **unfolding** *full-def*
by (*meson cdcl_W-all-struct-inv-def cdcl_W-s'E cdcl_W-stgy.conflict'*
cdcl_W-then-exists-cdcl_W-stgy-step inv-T n-step-cdcl_W-stgy-iff-no-step-cdcl_W-cl-cdcl_W-o)
then show *?thesis*
using s' **unfolding** *full-def* **by** *blast*

next
case $(st \text{ } S')$
have *full cdcl_W-cp T T*
using *option-full-cdcl_W-cp st(3)* **by** *blast*
moreover
have *n-s: no-step cdcl_W-bj T*
by (*metis* $\langle full \text{ } cdcl_W-stgy \text{ } S \text{ } T \rangle$ *bj inv-T cdcl_W-all-struct-inv-def*
cdcl_W-then-exists-cdcl_W-stgy-step full-def)
then have *full1 cdcl_W-bj S' T*
using *st(2)* **unfolding** *full1-def* **by** *blast*
moreover have *no-step cdcl_W-cp S'*
using *st(2)* **by** (*fastforce dest!*: *trancplD simp: cdcl_W-cp.simps cdcl_W-bj.simps*
elim: rulesE)
ultimately have *cdcl_W-s' S' T*
using *cdcl_W-s'.bj'*[*of S' T T*] **by** *blast*
then have *cdcl_W-s'^{**} S T*
using *st(1)* **by** *auto*
moreover have *no-step cdcl_W-s' T*

```

    using inv-T ⟨full cdclW-cp T T⟩ ⟨full cdclW-stgy S T⟩ unfolding full-def
    by (metis cdclW-all-struct-inv-def cdclW-then-exists-cdclW-stgy-step
        n-step-cdclW-stgy-iff-no-step-cdclW-cl-cdclW-o)
    ultimately show ?thesis
    unfolding full-def by blast
qed
qed

lemma conflict-step-cdclW-stgy-step:
  assumes
    conflict S T
    cdclW-all-struct-inv S
  shows ∃ T. cdclW-stgy S T
proof -
  obtain U where full cdclW-cp S U
  using cdclW-cp-normalized-element-all-inv assms by blast
  then have full1 cdclW-cp S U
  by (metis cdclW-cp.conflict' assms(1) full-unfold)
  then show ?thesis using cdclW-stgy.conflict' by blast
qed

lemma decide-step-cdclW-stgy-step:
  assumes
    decide S T
    cdclW-all-struct-inv S
  shows ∃ T. cdclW-stgy S T
proof -
  obtain U where full cdclW-cp T U
  using cdclW-cp-normalized-element-all-inv by (meson assms(1) assms(2) cdclW-all-struct-inv-inv
    cdclW-cp-normalized-element-all-inv decide other)
  then show ?thesis
  by (metis assms cdclW-cp-normalized-element-all-inv cdclW-stgy.conflict' decide full-unfold
    other')
qed

lemma rtranclp-cdclW-cp-conflicting-Some:
  cdclW-cp** S T ⟹ conflicting S = Some D ⟹ S = T
  using rtranclpD tranclpD by fastforce

inductive cdclW-merge-cp :: 'st ⇒ 'st ⇒ bool where
  conflict'[intro]: conflict S T ⟹ full cdclW-bj T U ⟹ cdclW-merge-cp S U |
  propagate'[intro]: propagate++ S S' ⟹ cdclW-merge-cp S S'

lemma cdclW-merge-restart-cases[consumes 1, case-names conflict propagate]:
  assumes
    cdclW-merge-cp S U and
    ∧ T. conflict S T ⟹ full cdclW-bj T U ⟹ P and
    propagate++ S U ⟹ P
  shows P
  using assms unfolding cdclW-merge-cp.simps by auto

lemma cdclW-merge-cp-tranclp-cdclW-merge:
  cdclW-merge-cp S T ⟹ cdclW-merge++ S T
  apply (induction rule: cdclW-merge-cp.induct)
  using cdclW-merge.simps apply auto[1]

```


using *trancpl-mono*[of *propagate cdcl_W-merge*] *fw-propagate* **by** *blast*

lemma *rtrancpl-cdcl_W-merge-cp-rtrancpl-cdcl_W*:
*cdcl_W-merge-cp*** *S T* \implies *cdcl_W*** *S T*
apply (*induction rule: rtrancpl-induct*)
apply *simp*
unfolding *cdcl_W-merge-cp.simps* **by** (*meson cdcl_W-merge-restart-cdcl_W fw-r-conflict*
rtrancpl-propagate-is-rtrancpl-cdcl_W rtrancpl-trans trancpl-into-rtrancpl)

lemma *full1-cdcl_W-bj-no-step-cdcl_W-bj*:
full1 cdcl_W-bj S T \implies *no-step cdcl_W-cp S*
unfolding *full1-def* **by** (*metis rtrancpl-unfold cdcl_W-cp-conflicting-not-empty option.exhaust*
rtrancpl-cdcl_W-merge-restart-no-step-cdcl_W-bj trancplD)

inductive *cdcl_W-s'-without-decide* **where**
conflict'-without-decide[*intro*]: *full1 cdcl_W-cp S S'* \implies *cdcl_W-s'-without-decide S S'* |
bj'-without-decide[*intro*]: *full1 cdcl_W-bj S S'* \implies *no-step cdcl_W-cp S* \implies *full cdcl_W-cp S' S''*
 \implies *cdcl_W-s'-without-decide S S''*

lemma *rtrancpl-cdcl_W-s'-without-decide-rtrancpl-cdcl_W*:
*cdcl_W-s'-without-decide*** *S T* \implies *cdcl_W*** *S T*
apply (*induction rule: rtrancpl-induct*)
apply *simp*
by (*meson cdcl_W-s'.simps cdcl_W-s'-trancpl-cdcl_W cdcl_W-s'-without-decide.simps*
rtrancpl-trancpl-trancpl trancpl-into-rtrancpl)

lemma *rtrancpl-cdcl_W-s'-without-decide-rtrancpl-cdcl_W-s'*:
*cdcl_W-s'-without-decide*** *S T* \implies *cdcl_W-s'*** *S T*
proof (*induction rule: rtrancpl-induct*)
case *base*
then show ?*case* **by** *simp*
next
case (*step y z*) **note** *a2 = this(2)* **and** *a1 = this(3)*
have *cdcl_W-s' y z*
using *a2* **by** (*metis (no-types) bj' cdcl_W-s'.conflict' cdcl_W-s'-without-decide.cases*)
then show *cdcl_W-s'*** *S z*
using *a1* **by** (*meson r-into-rtrancpl rtrancpl-trans*)
qed

lemma *rtrancpl-cdcl_W-merge-cp-is-rtrancpl-cdcl_W-s'-without-decide*:
assumes
*cdcl_W-merge-cp*** *S V*
conflicting S = None
shows
(*cdcl_W-s'-without-decide*** *S V*)
 \vee ($\exists T. \text{cdcl}_{W-s'-without-decide}^{**} S T \wedge \text{propagate}^{++} T V$)
 \vee ($\exists T U. \text{cdcl}_{W-s'-without-decide}^{**} S T \wedge \text{full1 cdcl}_{W-bj} T U \wedge \text{propagate}^{**} U V$)
using *assms*
proof (*induction rule: rtrancpl-induct*)
case *base*
then show ?*case* **by** *simp*
next
case (*step U V*) **note** *st = this(1)* **and** *cp = this(2)* **and** *IH = this(3)[OF this(4)]*
from *cp* **show** ?*case*
proof (*cases rule: cdcl_W-merge-restart-cases*)

```

case propagate
then show ?thesis using IH by (meson rtrancpl-trancpl-trancpl trancpl-into-rtrancpl)
next
case (conflict U') note confl = this(1) and bj = this(2)
have full1-U-U': full1 cdclW-cp U U'
  by (simp add: conflict-is-full1-cdclW-cp local.conflict(1))
consider
  (s') cdclW-s'-without-decide** S U
| (propa) T' where cdclW-s'-without-decide** S T' and propagate++ T' U
| (bj-prop) T' T'' where
  cdclW-s'-without-decide** S T' and
  full1 cdclW-bj T' T'' and
  propagate** T'' U
using IH by blast
then show ?thesis
proof cases
case s'
have cdclW-s'-without-decide U U'
  using full1-U-U' conflict'-without-decide by blast
then have cdclW-s'-without-decide** S U'
  using ⟨cdclW-s'-without-decide** S U⟩ by auto
moreover have U' = V ∨ full1 cdclW-bj U' V
  using bj by (meson full-unfold)
ultimately show ?thesis by blast
next
case propa note s' = this(1) and T'-U = this(2)
have full1 cdclW-cp T' U'
  using rtrancpl-mono[of propagate cdclW-cp] T'-U cdclW-cp.propagate' full1-U-U'
  rtrancpl-full1I[of cdclW-cp T'] by (metis (full-types) predicate2D predicate2I
    trancpl-into-rtrancpl)
have cdclW-s'-without-decide** S U'
  using ⟨full1 cdclW-cp T' U'⟩ conflict'-without-decide s' by force
have full1 cdclW-bj U' V ∨ V = U'
  by (metis (lifting) full-unfold local.bj)
then show ?thesis
  using ⟨cdclW-s'-without-decide** S U'⟩ by blast
next
case bj-prop note s' = this(1) and bj-T' = this(2) and T''-U = this(3)
have no-step cdclW-cp T'
  using bj-T' full1-cdclW-bj-no-step-cdclW-bj by blast
moreover have full1 cdclW-cp T'' U'
  using rtrancpl-mono[of propagate cdclW-cp] T''-U cdclW-cp.propagate' full1-U-U'
  rtrancpl-full1I[of cdclW-cp T''] by blast
ultimately have cdclW-s'-without-decide T' U'
  using bj'-without-decide[of T' T'' U] bj-T' by (simp add: full-unfold)
then have cdclW-s'-without-decide** S U'
  using s' rtrancpl.intros(2)[of - S T' U] by blast
then show ?thesis
  by (metis full-unfold local.bj rtrancpl.rtrancpl-refl)
qed
qed
qed

```

lemma *rtrancpl-cdcl_W-s'-without-decide-is-rtrancpl-cdcl_W-merge-cp:*

```

assumes
  cdclW-s'-without-decide** S V and
  confl: conflicting S = None
shows
  (cdclW-merge-cp** S V  $\wedge$  conflicting V = None)
 $\vee$  (cdclW-merge-cp** S V  $\wedge$  conflicting V  $\neq$  None  $\wedge$  no-step cdclW-cp V  $\wedge$  no-step cdclW-bj V)
 $\vee$  ( $\exists T. \textit{cdcl}_W\text{-merge-cp}^{**} S T \wedge \textit{conflict} T V$ )
using assms(1)
proof (induction)
  case base
  then show ?case using confl by auto
next
  case (step U V) note st = this(1) and s = this(2) and IH = this(3)
  from s show ?case
  proof (cases rule: cdclW-s'-without-decide.cases)
    case conflict'-without-decide
    then have rt: cdclW-cp++ U V unfolding full1-def by fast
    then have conflicting U = None
    using trancp-cdclW-cp-propagate-with-conflict-or-not[of U V]
    conflict by (auto dest!: trancpD simp: rtrancp-unfold elim: rulesE)
    then have cdclW-merge-cp** S U using IH by (auto elim: rulesE
      simp del: state-simp simp: state-eq-def)
    consider
      (propa) propagate++ U V
      | (confl') conflict U V
      | (propa-confl') U' where propagate++ U U' conflict U' V
    using trancp-cdclW-cp-propagate-with-conflict-or-not[OF rt] unfolding rtrancp-unfold
    by fastforce
    then show ?thesis
    proof cases
      case propa
      then have cdclW-merge-cp U V
      by auto
      moreover have conflicting V = None
      using propa unfolding trancp-unfold-end by (auto elim: rulesE)
      ultimately show ?thesis using (cdclW-merge-cp** S U) by (auto elim!: rulesE
        simp del: state-simp simp: state-eq-def)
    next
    case confl'
    then show ?thesis using (cdclW-merge-cp** S U) by auto
    next
    case propa-confl' note propa = this(1) and confl' = this(2)
    then have cdclW-merge-cp U U' by auto
    then have cdclW-merge-cp** S U' using (cdclW-merge-cp** S U) by auto
    then show ?thesis using (cdclW-merge-cp** S U) confl' by auto
    qed
  next
  case (bj'-without-decide U') note full-bj = this(1) and cp = this(3)
  then have conflicting U  $\neq$  None
  using full-bj unfolding full1-def by (fastforce dest!: trancpD simp: cdclW-bj.simps
    elim: rulesE)
  with IH obtain T where
    S-T: cdclW-merge-cp** S T and T-U: conflict T U
    using full-bj unfolding full1-def by (blast dest: trancpD)
  then have cdclW-merge-cp T U'

```

```

    using cdclW-merge-cp.conflict'[of T U U'] full-bj by (simp add: full-unfold)
  then have S-U': cdclW-merge-cp** S U' using S-T by auto
  consider
    (n-s) U' = V
    | (propa) propagate++ U' V
    | (confl') conflict U' V
    | (propa-confl') U'' where propagate++ U' U'' conflict U'' V
  using tranclp-cdclW-cp-propagate-with-conflict-or-not cp
  unfolding rtranclp-unfold full-def by metis
  then show ?thesis
  proof cases
    case propa
    then have cdclW-merge-cp U' V by auto
    moreover have conflicting V = None
    using propa unfolding tranclp-unfold-end by (auto elim: rulesE)
    ultimately show ?thesis using S-U' by (auto elim: rulesE
      simp del: state-simp simp: state-eq-def)
  next
    case confl'
    then show ?thesis using S-U' by auto
  next
    case propa-confl' note propa = this(1) and confl = this(2)
    have cdclW-merge-cp U' U'' using propa by auto
    then show ?thesis using S-U' confl by (meson rtranclp.rtrancl-into-rtrancl)
  next
    case n-s
    then show ?thesis
    using S-U' apply (cases conflicting V = None)
    using full-bj apply simp
    by (metis cp full-def full-unfold full-bj)
  qed
qed
qed

lemma no-step-cdclW-s'-no-ste-cdclW-merge-cp:
  assumes
    cdclW-all-struct-inv S
    conflicting S = None
    no-step cdclW-s' S
  shows no-step cdclW-merge-cp S
  using assms apply (auto simp: cdclW-s'.simps cdclW-merge-cp.simps)
  using conflict-is-full1-cdclW-cp apply blast
  using cdclW-cp-normalized-element-all-inv cdclW-cp.propagate' by (metis cdclW-cp.propagate'
    full-unfold tranclpD)

```

The *no-step decide S* is needed, since *cdcl_W-merge-cp* is *cdcl_W-s'* without *decide*.

lemma *conflicting-true-no-step-cdcl_W-merge-cp-no-step-s'-without-decide*:

```

  assumes
    confl: conflicting S = None and
    inv: cdclW-M-level-inv S and
    n-s: no-step cdclW-merge-cp S
  shows no-step cdclW-s'-without-decide S
  proof (rule ccontr)
    assume ¬ no-step cdclW-s'-without-decide S
    then obtain T where

```

```

    cdclW: cdclW-s'-without-decide S T
  by auto
then have inv-T: cdclW-M-level-inv T
  using rtrancp-cdclW-s'-without-decide-rtrancp-cdclW[of S T]
  rtrancp-cdclW-consistent-inv inv by blast
from cdclW show False
proof cases
  case conflict'-without-decide
  have no-step propagate S
    using n-s by blast
  then have conflict S T
    using local.conflict' trancp-cdclW-cp-propagate-with-conflict-or-not[of S T]
    unfolding full1-def by (metis full1-def local.conflict'-without-decide rtrancp-unfold
      trancp-unfold-begin)
  moreover
    then obtain T' where full cdclW-bj T T'
      using cdclW-bj-exists-normal-form inv-T by blast
    ultimately show False using cdclW-merge-cp.conflict' n-s by meson
next
  case (bj'-without-decide S')
  then show ?thesis
    using confl unfolding full1-def by (fastforce simp: cdclW-bj.simps dest: trancpD
      elim: rulesE)
qed
qed

```

lemma *conflicting-true-no-step-s'-without-decide-no-step-cdcl_W-merge-cp:*

```

  assumes
    inv: cdclW-all-struct-inv S and
    n-s: no-step cdclW-s'-without-decide S
  shows no-step cdclW-merge-cp S
proof (rule ccontr)
  assume ¬ ?thesis
  then obtain T where cdclW-merge-cp S T
    by auto
  then show False
  proof cases
    case (conflict' S')
    then show False using n-s conflict'-without-decide conflict-is-full1-cdclW-cp by blast
  next
    case propagate'
    moreover
      have cdclW-all-struct-inv T
        using inv by (meson local.propagate' rtrancp-cdclW-all-struct-inv-inv
          rtrancp-propagate-is-rtrancp-cdclW trancp-into-rtrancp)
      then obtain U where full cdclW-cp T U
        using cdclW-cp-normalized-element-all-inv by auto
      ultimately have full1 cdclW-cp S U
        using trancp-full-full1I[of cdclW-cp S T U] cdclW-cp.propagate'
        trancp-mono[of propagate cdclW-cp] by blast
      then show False using conflict'-without-decide n-s by blast
    qed
  qed
qed

```

lemma *no-step-cdcl_W-merge-cp-no-step-cdcl_W-cp:*

$no\text{-}step\ cdcl_W\text{-}merge\text{-}cp\ S \implies cdcl_W\text{-}M\text{-}level\text{-}inv\ S \implies no\text{-}step\ cdcl_W\text{-}cp\ S$
using $cdcl_W\text{-}bj\text{-}exists\text{-}normal\text{-}form\ cdcl_W\text{-}consistent\text{-}inv[OF\ cdcl_W.conflict,\ of\ S]$
by $(metis\ cdcl_W\text{-}cp.cases\ cdcl_W\text{-}merge\text{-}cp.simps\ tranclp.intros(1))$

lemma *conflicting-not-true-rtranclp-cdcl_W-merge-cp-no-step-cdcl_W-bj:*

assumes
 $conflicting\ S = None$ **and**
 $cdcl_W\text{-}merge\text{-}cp^{**}\ S\ T$
shows $no\text{-}step\ cdcl_W\text{-}bj\ T$
using $assms(2,1)$ **by** $(induction)$
 $(fastforce\ simp:\ cdcl_W\text{-}merge\text{-}cp.simps\ full\text{-}def\ tranclp\text{-}unfold\text{-}end\ cdcl_W\text{-}bj.simps$
 $elim:\ rulesE)+$

lemma *conflicting-true-full-cdcl_W-merge-cp-iff-full-cdcl_W-s'-without-decode:*

assumes
 $confl:\ conflicting\ S = None$ **and**
 $inv:\ cdcl_W\text{-}all\text{-}struct\text{-}inv\ S$
shows
 $full\ cdcl_W\text{-}merge\text{-}cp\ S\ V \longleftrightarrow full\ cdcl_W\text{-}s'\text{-}without\text{-}decide\ S\ V\ (is\ ?fw \longleftrightarrow ?s')$

proof

assume $?fw$
then have $st:\ cdcl_W\text{-}merge\text{-}cp^{**}\ S\ V$ **and** $n\text{-}s:\ no\text{-}step\ cdcl_W\text{-}merge\text{-}cp\ V$
unfolding $full\text{-}def$ **by** $blast+$
have $inv\text{-}V:\ cdcl_W\text{-}all\text{-}struct\text{-}inv\ V$
using $rtranclp\text{-}cdcl_W\text{-}merge\text{-}cp\text{-}rtranclp\text{-}cdcl_W[of\ S\ V]\ \langle ?fw \rangle$ **unfolding** $full\text{-}def$
by $(simp\ add:\ inv\ rtranclp\text{-}cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}inv)$
consider
 $(s')\ cdcl_W\text{-}s'\text{-}without\text{-}decide^{**}\ S\ V$
 $| (propa)\ T\ \mathbf{where}\ cdcl_W\text{-}s'\text{-}without\text{-}decide^{**}\ S\ T\ \mathbf{and}\ propagate^{++}\ T\ V$
 $| (bj)\ T\ U\ \mathbf{where}\ cdcl_W\text{-}s'\text{-}without\text{-}decide^{**}\ S\ T\ \mathbf{and}\ full1\ cdcl_W\text{-}bj\ T\ U\ \mathbf{and}\ propagate^{**}\ U\ V$
using $rtranclp\text{-}cdcl_W\text{-}merge\text{-}cp\text{-}is\text{-}rtranclp\text{-}cdcl_W\text{-}s'\text{-}without\text{-}decide\ confl\ st\ n\text{-}s$ **by** $metis$
then have $cdcl_W\text{-}s'\text{-}without\text{-}decide^{**}\ S\ V$

proof cases

case s'
then show $?thesis$.

next

case $propa$ **note** $s' = this(1)$ **and** $propa = this(2)$
have $no\text{-}step\ cdcl_W\text{-}cp\ V$
using $no\text{-}step\text{-}cdcl_W\text{-}merge\text{-}cp\text{-}no\text{-}step\text{-}cdcl_W\text{-}cp\ n\text{-}s\ inv\text{-}V$
unfolding $cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}def$ **by** $blast$
then have $full1\ cdcl_W\text{-}cp\ T\ V$
using $propa\ tranclp\text{-}mono[of\ propagate\ cdcl_W\text{-}cp]\ cdcl_W\text{-}cp.propagate'$ **unfolding** $full1\text{-}def$
by $blast$
then have $cdcl_W\text{-}s'\text{-}without\text{-}decide\ T\ V$
using $conflict'\text{-}without\text{-}decide$ **by** $blast$
then show $?thesis$ **using** s' **by** $auto$

next

case bj **note** $s' = this(1)$ **and** $bj = this(2)$ **and** $propa = this(3)$
have $no\text{-}step\ cdcl_W\text{-}cp\ V$
using $no\text{-}step\text{-}cdcl_W\text{-}merge\text{-}cp\text{-}no\text{-}step\text{-}cdcl_W\text{-}cp\ n\text{-}s\ inv\text{-}V$
unfolding $cdcl_W\text{-}all\text{-}struct\text{-}inv\text{-}def$ **by** $blast$
then have $full\ cdcl_W\text{-}cp\ U\ V$
using $propa\ rtranclp\text{-}mono[of\ propagate\ cdcl_W\text{-}cp]\ cdcl_W\text{-}cp.propagate'$ **unfolding** $full\text{-}def$
by $blast$
moreover have $no\text{-}step\ cdcl_W\text{-}cp\ T$

```

    using bj unfolding full1-def by (fastforce dest!: tranclpD simp:cdclW-bj.simps elim: rulesE)
ultimately have cdclW-s'-without-decide T V
  using bj'-without-decide[of T U V] bj by blast
then show ?thesis using s' by auto
qed
moreover have no-step cdclW-s'-without-decide V
proof (cases conflicting V = None)
  case False
  { fix ss :: 'st
    have ff1:  $\forall s sa. \neg cdclW-s' s sa \vee full1\ cdclW-cp s sa$ 
       $\vee (\exists sb. decide\ s\ sb \wedge no-step\ cdclW-cp\ s \wedge full\ cdclW-cp\ sb\ sa)$ 
       $\vee (\exists sb. full1\ cdclW-bj\ s\ sb \wedge no-step\ cdclW-cp\ s \wedge full\ cdclW-cp\ sb\ sa)$ 
      by (metis cdclW-s'.cases)
    have ff2:  $(\forall p s sa. \neg full1\ p\ (s::'st)\ sa \vee p^{++}\ s\ sa \wedge no-step\ p\ sa)$ 
       $\wedge (\forall p s sa. (\neg p^{++}\ (s::'st)\ sa \vee (\exists s. p\ sa\ s)) \vee full1\ p\ s\ sa)$ 
      by (meson full1-def)
    obtain ssa :: ('st  $\Rightarrow$  'st  $\Rightarrow$  bool)  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  'st where
      ff3:  $\forall p s sa. \neg p^{++}\ s\ sa \vee p\ s\ (ssa\ p\ s\ sa) \wedge p^{**}\ (ssa\ p\ s\ sa)\ sa$ 
      by (metis (no-types) tranclpD)
    then have a3:  $\neg cdclW-cp^{++}\ V\ ss$ 
      using False by (metis option-full-cdclW-cp full-def)
    have  $\bigwedge s. \neg cdclW-bj^{++}\ V\ s$ 
      using ff3 False by (metis confl st
        conflicting-not-true-rtranclp-cdclW-merge-cp-no-step-cdclW-bj)
    then have  $\neg cdclW-s'-without-decide\ V\ ss$ 
      using ff1 a3 ff2 by (metis cdclW-s'-without-decide.cases)
  }
then show ?thesis
  by fastforce
next
  case True
  then show ?thesis
    using conflicting-true-no-step-cdclW-merge-cp-no-step-s'-without-decide n-s inv-V
    unfolding cdclW-all-struct-inv-def by simp
qed
ultimately show ?s' unfolding full-def by blast
next
assume s': ?s'
then have st:  $cdclW-s'-without-decide^{**}\ S\ V$  and n-s:  $no-step\ cdclW-s'-without-decide\ V$ 
  unfolding full-def by auto
then have  $cdclW^{**}\ S\ V$ 
  using rtranclp-cdclW-s'-without-decide-rtranclp-cdclW st by blast
then have inv-V:  $cdclW-all-struct-inv\ V$  using inv rtranclp-cdclW-all-struct-inv-inv by blast
then have n-s-cp-V:  $no-step\ cdclW-cp\ V$ 
  using  $cdclW-cp-normalized-element-all-inv$ [of V] full-fullI[of cdclW-cp V] n-s
  conflict'-without-decide conflicting-true-no-step-s'-without-decide-no-step-cdclW-merge-cp
  no-step-cdclW-merge-cp-no-step-cdclW-cp
  unfolding cdclW-all-struct-inv-def by presburger
have n-s-bj:  $no-step\ cdclW-bj\ V$ 
proof (rule ccontr)
  assume  $\neg ?thesis$ 
  then obtain W where  $W: cdclW-bj\ V\ W$  by blast
  have  $cdclW-all-struct-inv\ W$ 
    using W cdclW.simps cdclW-all-struct-inv-inv inv-V by blast
  then obtain W' where  $full1\ cdclW-bj\ V\ W'$ 

```

```

    using cdclW-bj-exists-normal-form[of W] full-fullI[of cdclW-bj V W] W
    unfolding cdclW-all-struct-inv-def
    by blast
  moreover
    then have cdclW++ V W'
      using tranclp-mono[of cdclW-bj cdclW] cdclW.other cdclW-o.bj unfolding full1-def by blast
    then have cdclW-all-struct-inv W'
      by (meson inv-V rtranclp-cdclW-all-struct-inv-inv tranclp-into-rtranclp)
    then obtain X where full cdclW-cp W' X
      using cdclW-cp-normalized-element-all-inv by blast
    ultimately show False
      using bj'-without-decide n-s-cp-V n-s by blast
  qed
from s' consider
  (cp-true) cdclW-merge-cp** S V and conflicting V = None
| (cp-false) cdclW-merge-cp** S V and conflicting V ≠ None and no-step cdclW-cp V and
  no-step cdclW-bj V
| (cp-conf) T where cdclW-merge-cp** S T conflict T V
using rtranclp-cdclW-s'-without-decide-is-rtranclp-cdclW-merge-cp[of S V] confl
unfolding full-def by meson
then have cdclW-merge-cp** S V
proof cases
  case cp-conf note S-T = this(1) and conf-V = this(2)
  have full cdclW-bj V V
    using conf-V n-s-bj unfolding full-def by fast
  then have cdclW-merge-cp T V
    using cdclW-merge-cp.conflict' conf-V by auto
  then show ?thesis using S-T by auto
qed fast+
moreover
  then have cdclW** S V using rtranclp-cdclW-merge-cp-rtranclp-cdclW by blast
  then have cdclW-all-struct-inv V
    using inv rtranclp-cdclW-all-struct-inv-inv by blast
  then have no-step cdclW-merge-cp V
    using conflicting-true-no-step-s'-without-decide-no-step-cdclW-merge-cp s'
    unfolding full-def by blast
  ultimately show ?fw unfolding full-def by auto
qed

lemma conflicting-true-full1-cdclW-merge-cp-iff-full1-cdclW-s'-without-decode:
  assumes
    confl: conflicting S = None and
    inv: cdclW-all-struct-inv S
  shows
    full1 cdclW-merge-cp S V ⟷ full1 cdclW-s'-without-decode S V
proof -
  have full cdclW-merge-cp S V = full cdclW-s'-without-decode S V
    using confl conflicting-true-full-cdclW-merge-cp-iff-full-cdclW-s'-without-decode inv
    by simp
  then show ?thesis unfolding full-unfold full1-def tranclp-unfold-begin by blast
qed

lemma conflicting-true-full1-cdclW-merge-cp-imp-full1-cdclW-s'-without-decode:
  assumes
    fw: full1 cdclW-merge-cp S V and

```


$inv: cdcl_W\text{-all-struct-inv } S$
shows
 $full1\ cdcl_W\text{-s'-without-decide } S\ V$
proof –
have $conflicting\ S = None$
using $fw\ unfolding\ full1\text{-def}$ **by** $(auto\ dest!: tranclpD\ simp: cdcl_W\text{-merge-cp.simps}\ elim: rulesE)$
then show $?thesis$
using $conflicting\text{-true-full1-cdcl}_W\text{-merge-cp-iff-full1-cdcl}_W\text{-s'-without-decide } fw\ inv$ **by** $simp$
qed

inductive $cdcl_W\text{-merge-stgy}$ **where**
 $fw\text{-s-cp[intro]: full1\ cdcl}_W\text{-merge-cp } S\ T \implies cdcl_W\text{-merge-stgy } S\ T \mid$
 $fw\text{-s-decide[intro]: decide } S\ T \implies no\text{-step } cdcl_W\text{-merge-cp } S \implies full\ cdcl}_W\text{-merge-cp } T\ U$
 $\implies cdcl_W\text{-merge-stgy } S\ U$

lemma $cdcl_W\text{-merge-stgy-tranclp-cdcl}_W\text{-merge}$:
assumes $fw: cdcl_W\text{-merge-stgy } S\ T$
shows $cdcl_W\text{-merge}^{++}\ S\ T$
proof –
{ fix $S\ T$
assume $full1\ cdcl_W\text{-merge-cp } S\ T$
then have $cdcl_W\text{-merge}^{++}\ S\ T$
using $tranclp\text{-mono}[of\ cdcl_W\text{-merge-cp } cdcl_W\text{-merge}^{++}]$ $cdcl_W\text{-merge-cp-tranclp-cdcl}_W\text{-merge}$
 $unfolding\ full1\text{-def}$
by $auto$
} note $full1\text{-cdcl}_W\text{-merge-cp-cdcl}_W\text{-merge} = this$
show $?thesis$
using fw
apply $(induction\ rule: cdcl_W\text{-merge-stgy.induct})$
using $full1\text{-cdcl}_W\text{-merge-cp-cdcl}_W\text{-merge}$ **apply** $simp$
unfolding $full\text{-unfold}$ **by** $(auto\ dest!: full1\text{-cdcl}_W\text{-merge-cp-cdcl}_W\text{-merge } fw\text{-decide})$
qed

lemma $rtranclp\text{-cdcl}_W\text{-merge-stgy-rtranclp-cdcl}_W\text{-merge}$:
assumes $fw: cdcl_W\text{-merge-stgy}^{**}\ S\ T$
shows $cdcl_W\text{-merge}^{**}\ S\ T$
using $fw\ cdcl_W\text{-merge-stgy-tranclp-cdcl}_W\text{-merge}$ $rtranclp\text{-mono}[of\ cdcl_W\text{-merge-stgy } cdcl_W\text{-merge}^{++}]$
unfolding $tranclp\text{-rtranclp-rtranclp}$ **by** $blast$

lemma $cdcl_W\text{-merge-stgy-rtranclp-cdcl}_W$:
 $cdcl_W\text{-merge-stgy } S\ T \implies cdcl_W^{**}\ S\ T$
apply $(induction\ rule: cdcl_W\text{-merge-stgy.induct})$
using $rtranclp\text{-cdcl}_W\text{-merge-cp-rtranclp-cdcl}_W$ **unfolding** $full1\text{-def}$
apply $(simp\ add: tranclp\text{-into-rtranclp})$
using $rtranclp\text{-cdcl}_W\text{-merge-cp-rtranclp-cdcl}_W\ cdcl_W\text{-o.decide } cdcl_W\text{-other}$ **unfolding** $full\text{-def}$
by $(meson\ r\text{-into-rtranclp } rtranclp\text{-trans})$

lemma $rtranclp\text{-cdcl}_W\text{-merge-stgy-rtranclp-cdcl}_W$:
 $cdcl_W\text{-merge-stgy}^{**}\ S\ T \implies cdcl_W^{**}\ S\ T$
using $rtranclp\text{-mono}[of\ cdcl_W\text{-merge-stgy } cdcl_W^{**}]$ $cdcl_W\text{-merge-stgy-rtranclp-cdcl}_W$ **by** $auto$

lemma $cdcl_W\text{-merge-stgy-cases}[consumes\ 1,\ case\text{-names } fw\text{-s-cp } fw\text{-s-decide}]$:
assumes
 $cdcl_W\text{-merge-stgy } S\ U$
 $full1\ cdcl_W\text{-merge-cp } S\ U \implies P$

$\bigwedge T. \text{decide } S \ T \implies \text{no-step } \text{cdcl}_W\text{-merge-cp } S \implies \text{full } \text{cdcl}_W\text{-merge-cp } T \ U \implies P$
shows P
using *assms* **by** (*auto simp: cdcl_W-merge-stgy.simps*)

inductive $\text{cdcl}_W\text{-s'-w} :: 'st \Rightarrow 'st \Rightarrow \text{bool}$ **where**
conflict': $\text{full1 } \text{cdcl}_W\text{-s'-without-decide } S \ S' \implies \text{cdcl}_W\text{-s'-w } S \ S' \mid$
decide': $\text{decide } S \ S' \implies \text{no-step } \text{cdcl}_W\text{-s'-without-decide } S \implies \text{full } \text{cdcl}_W\text{-s'-without-decide } S' \ S''$
 $\implies \text{cdcl}_W\text{-s'-w } S \ S''$

lemma $\text{cdcl}_W\text{-s'-w-rtrancpl-cdcl}_W$:
 $\text{cdcl}_W\text{-s'-w } S \ T \implies \text{cdcl}_W^{**} \ S \ T$
apply (*induction rule: cdcl_W-s'-w.induct*)
using *rtrancpl-cdcl_W-s'-without-decide-rtrancpl-cdcl_W* **unfolding** *full1-def*
apply (*simp add: trancpl-into-rtrancpl*)
using *rtrancpl-cdcl_W-s'-without-decide-rtrancpl-cdcl_W* **unfolding** *full-def*
by (*meson decide other rtrancpl-into-trancpl2 trancpl-into-rtrancpl*)

lemma $\text{rtrancpl-cdcl}_W\text{-s'-w-rtrancpl-cdcl}_W$:
 $\text{cdcl}_W\text{-s'-w}^{**} \ S \ T \implies \text{cdcl}_W^{**} \ S \ T$
using *rtrancpl-mono[of cdcl_W-s'-w cdcl_W**]* $\text{cdcl}_W\text{-s'-w-rtrancpl-cdcl}_W$ **by** *auto*

lemma *no-step-cdcl_W-cp-no-step-cdcl_W-s'-without-decide*:
assumes *no-step cdcl_W-cp S and conflicting S = None and inv: cdcl_W-M-level-inv S*
shows *no-step cdcl_W-s'-without-decide S*
by (*metis assms cdcl_W-cp.conflict' cdcl_W-cp.propagate' cdcl_W-merge-restart-cases trancplD*
conflicting-true-no-step-cdcl_W-merge-cp-no-step-s'-without-decide)

lemma *no-step-cdcl_W-cp-no-step-cdcl_W-merge-restart*:
assumes *no-step cdcl_W-cp S and conflicting S = None*
shows *no-step cdcl_W-merge-cp S*
by (*metis assms(1) cdcl_W-cp.conflict' cdcl_W-cp.propagate' cdcl_W-merge-restart-cases trancplD*)

lemma *after-cdcl_W-s'-without-decide-no-step-cdcl_W-cp*:
assumes *cdcl_W-s'-without-decide S T*
shows *no-step cdcl_W-cp T*
using *assms* **by** (*induction rule: cdcl_W-s'-without-decide.induct*) (*auto simp: full1-def full-def*)

lemma *no-step-cdcl_W-s'-without-decide-no-step-cdcl_W-cp*:
 $\text{cdcl}_W\text{-all-struct-inv } S \implies \text{no-step } \text{cdcl}_W\text{-s'-without-decide } S \implies \text{no-step } \text{cdcl}_W\text{-cp } S$
by (*simp add: conflicting-true-no-step-s'-without-decide-no-step-cdcl_W-merge-cp*
no-step-cdcl_W-merge-cp-no-step-cdcl_W-cp cdcl_W-all-struct-inv-def)

lemma *after-cdcl_W-s'-w-no-step-cdcl_W-cp*:
assumes $\text{cdcl}_W\text{-s'-w } S \ T$ **and** *cdcl_W-all-struct-inv S*
shows *no-step cdcl_W-cp T*
using *assms*
proof (*induction rule: cdcl_W-s'-w.induct*)
case *conflict'*
then show *?case*
by (*auto simp: full1-def trancpl-unfold-end after-cdcl_W-s'-without-decide-no-step-cdcl_W-cp*)
next
case (*decide' S T U*)
moreover
then have $\text{cdcl}_W^{**} \ S \ U$
using *rtrancpl-cdcl_W-s'-without-decide-rtrancpl-cdcl_W* [*of T U*] *cdcl_W.other* [*of S T*]
cdcl_W-o.decide **unfolding** *full-def* **by** *auto*

then have $cdcl_W\text{-all-struct-inv } U$
 using $decide'.prems \text{ rtranclp-cdcl}_W\text{-all-struct-inv-inv}$ by $blast$
 ultimately show $?case$
 using $no\text{-step-cdcl}_W\text{-s'without-decide-no-step-cdcl}_W\text{-cp}$ unfolding $full\text{-def}$ by $blast$
 qed

lemma $rtranclp\text{-cdcl}_W\text{-s'-w-no-step-cdcl}_W\text{-cp-or-eq}$:
 assumes $cdcl_W\text{-s'-w}^{**} S T$ and $cdcl_W\text{-all-struct-inv } S$
 shows $S = T \vee no\text{-step } cdcl_W\text{-cp } T$
 using $assms$
proof (induction rule: $rtranclp\text{-induct}$)
 case $base$
 then show $?case$ by $simp$
next
 case (step $T U$)
 moreover have $cdcl_W\text{-all-struct-inv } T$
 using $rtranclp\text{-cdcl}_W\text{-s'-w-rtranclp-cdcl}_W[of S U] \text{ assms}(2) \text{ rtranclp-cdcl}_W\text{-all-struct-inv-inv}$
 $rtranclp\text{-cdcl}_W\text{-s'-w-rtranclp-cdcl}_W \text{ step.hyps}(1)$ by $blast$
 ultimately show $?case$ using $after\text{-cdcl}_W\text{-s'-w-no-step-cdcl}_W\text{-cp}$ by $fast$
 qed

lemma $rtranclp\text{-cdcl}_W\text{-merge-stgy'-no-step-cdcl}_W\text{-cp-or-eq}$:
 assumes $cdcl_W\text{-merge-stgy}^{**} S T$ and $inv: cdcl_W\text{-all-struct-inv } S$
 shows $S = T \vee no\text{-step } cdcl_W\text{-cp } T$
 using $assms$
proof (induction rule: $rtranclp\text{-induct}$)
 case $base$
 then show $?case$ by $simp$
next
 case (step $T U$)
 moreover have $cdcl_W\text{-all-struct-inv } T$
 using $rtranclp\text{-cdcl}_W\text{-merge-stgy-rtranclp-cdcl}_W[of S U] \text{ assms}(2) \text{ rtranclp-cdcl}_W\text{-all-struct-inv-inv}$
 $rtranclp\text{-cdcl}_W\text{-s'-w-rtranclp-cdcl}_W \text{ step.hyps}(1)$
 by (meson $rtranclp\text{-cdcl}_W\text{-merge-stgy-rtranclp-cdcl}_W$)
 ultimately show $?case$
 using $after\text{-cdcl}_W\text{-s'-w-no-step-cdcl}_W\text{-cp } inv$ unfolding $cdcl_W\text{-all-struct-inv-def}$
 by (metis $cdcl_W\text{-all-struct-inv-def } cdcl_W\text{-merge-stgy.simps full1-def full-def}$
 $no\text{-step-cdcl}_W\text{-merge-cp-no-step-cdcl}_W\text{-cp } rtranclp\text{-cdcl}_W\text{-all-struct-inv-inv}$
 $rtranclp\text{-cdcl}_W\text{-merge-stgy-rtranclp-cdcl}_W \text{ tranclp.intros}(1) \text{ tranclp-into-rtranclp}$)
 qed

lemma $no\text{-step-cdcl}_W\text{-s'without-decide-no-step-cdcl}_W\text{-bj}$:
 assumes $no\text{-step } cdcl_W\text{-s'without-decide } S$ and $inv: cdcl_W\text{-all-struct-inv } S$
 shows $no\text{-step } cdcl_W\text{-bj } S$
proof (rule $ccontr$)
 assume $\neg ?thesis$
 then obtain T where $S\text{-}T: cdcl_W\text{-bj } S T$
 by $auto$
 have $cdcl_W\text{-all-struct-inv } T$
 using $S\text{-}T \text{ cdcl}_W\text{-all-struct-inv-inv } inv$ other by $blast$
 then obtain T' where $full1 \text{ cdcl}_W\text{-bj } S T'$
 using $cdcl_W\text{-bj-exists-normal-form}[of T] \text{ full-fullI } S\text{-}T$ unfolding $cdcl_W\text{-all-struct-inv-def}$
 by $metis$
 moreover
 then have $cdcl_W^{**} S T'$

using *rtrancp-mono*[*of cdcl_W-bj cdcl_W*] *cdcl_W.other cdcl_W-o.bj trancp-into-rtrancp*[*of cdcl_W-bj*]
unfolding *full1-def* **by** *blast*
then have *cdcl_W-all-struct-inv T'*
using *inv rtrancp-cdcl_W-all-struct-inv-inv* **by** *blast*
then obtain *U* **where** *full cdcl_W-cp T' U*
using *cdcl_W-cp-normalized-element-all-inv* **by** *blast*
moreover have *no-step cdcl_W-cp S*
using *S-T* **by** (*auto simp: cdcl_W-bj.simps elim: rulesE*)
ultimately show *False*
using *assms cdcl_W-s'-without-decide.intros(2)*[*of S T' U*] **by** *fast*
qed

lemma *cdcl_W-s'-w-no-step-cdcl_W-bj*:
assumes *cdcl_W-s'-w S T* **and** *cdcl_W-all-struct-inv S*
shows *no-step cdcl_W-bj T*
using *assms* **apply** *induction*
using *rtrancp-cdcl_W-s'-without-decide-rtrancp-cdcl_W rtrancp-cdcl_W-all-struct-inv-inv*
no-step-cdcl_W-s'-without-decide-no-step-cdcl_W-bj **unfolding** *full1-def*
apply (*meson trancp-into-rtrancp*)
using *rtrancp-cdcl_W-s'-without-decide-rtrancp-cdcl_W rtrancp-cdcl_W-all-struct-inv-inv*
no-step-cdcl_W-s'-without-decide-no-step-cdcl_W-bj **unfolding** *full-def*
by (*meson cdcl_W-merge-restart-cdcl_W fw-r-decide*)

lemma *rtrancp-cdcl_W-s'-w-no-step-cdcl_W-bj-or-eq*:
assumes *cdcl_W-s'-w** S T* **and** *cdcl_W-all-struct-inv S*
shows *S = T ∨ no-step cdcl_W-bj T*
using *assms* **apply** *induction*
apply *simp*
using *rtrancp-cdcl_W-s'-w-rtrancp-cdcl_W rtrancp-cdcl_W-all-struct-inv-inv*
cdcl_W-s'-w-no-step-cdcl_W-bj **by** *meson*

lemma *rtrancp-cdcl_W-s'-no-step-cdcl_W-s'-without-decide-decomp-into-cdcl_W-merge*:

assumes
cdcl_W-s'^{/} R V* **and**
conflicting R = None **and**
inv: cdcl_W-all-struct-inv R
shows (*cdcl_W-merge-stgy** R V ∧ conflicting V = None*)
 \vee (*cdcl_W-merge-stgy** R V ∧ conflicting V ≠ None ∧ no-step cdcl_W-bj V*)
 \vee ($\exists S T U. \text{cdcl}_W\text{-merge-stgy}^{**} R S \wedge \text{no-step cdcl}_W\text{-merge-cp } S \wedge \text{decide } S T$
 $\wedge \text{cdcl}_W\text{-merge-cp}^{**} T U \wedge \text{conflict } U V$)
 \vee ($\exists S T. \text{cdcl}_W\text{-merge-stgy}^{**} R S \wedge \text{no-step cdcl}_W\text{-merge-cp } S \wedge \text{decide } S T$
 $\wedge \text{cdcl}_W\text{-merge-cp}^{**} T V$
 $\wedge \text{conflicting } V = \text{None}$)
 \vee (*cdcl_W-merge-cp** R V ∧ conflicting V = None*)
 \vee ($\exists U. \text{cdcl}_W\text{-merge-cp}^{**} R U \wedge \text{conflict } U V$)
using *assms(1,2)*

proof *induction*

case *base*

then show *?case* **by** *simp*

next

case (*step V W*) **note** *st = this(1)* **and** *s' = this(2)* **and** *IH = this(3)[OF this(4)]* **and**
n-s-R = this(4)

from *s'*

show *?case*

proof *cases*

```

case conflict'
consider
  (s') cdclW-merge-stgy** R V
  | (dec-conf) S T U where cdclW-merge-stgy** R S and no-step cdclW-merge-cp S and
    decide S T and cdclW-merge-cp** T U and conflict U V
  | (dec) S T where cdclW-merge-stgy** R S and no-step cdclW-merge-cp S and decide S T
    and cdclW-merge-cp** T V and conflicting V = None
  | (cp) cdclW-merge-cp** R V
  | (cp-conf) U where cdclW-merge-cp** R U and conflict U V
using IH by meson
then show ?thesis
proof cases
next
  case s'
  then have R = V
    by (metis full1-def inv local.conflict' tranclp-unfold-begin
      rtranclp-cdclW-merge-stgy'-no-step-cdclW-cp-or-eq)
  consider
    (V-W) V = W
    | (propa) propagate** V W and conflicting W = None
    | (propa-conf) V' where propagate** V V' and conflict V' W
    using tranclp-cdclW-cp-propagate-with-conflict-or-not[of V W] conflict'
    unfolding full-unfold full1-def by meson
  then show ?thesis
  proof cases
    case V-W
    then show ?thesis using  $\langle R = V \rangle$  n-s-R by simp
  next
    case propa
    then show ?thesis using  $\langle R = V \rangle$  by auto
  next
    case propa-conf
    moreover
      then have cdclW-merge-cp** V V'
      by (metis rtranclp-unfold cdclW-merge-cp.propagate' r-into-rtranclp)
      ultimately show ?thesis using  $s' \langle R = V \rangle$  by blast
    qed
  next
    case dec-conf note - = this(5)
    then have False using conflict' unfolding full1-def by (auto dest!: tranclpD elim: rulesE)
    then show ?thesis by fast
  next
    case dec note T-V = this(4)
    consider
      (propa) propagate** V W and conflicting W = None
      | (propa-conf) V' where propagate** V V' and conflict V' W
      using tranclp-cdclW-cp-propagate-with-conflict-or-not[of V W] conflict'
      unfolding full1-def by meson
    then show ?thesis
    proof cases
      case propa
      then show ?thesis
      by (meson T-V cdclW-merge-cp.propagate' dec rtranclp.rtrancl-into-rtrancl)
    next
      case propa-conf

```

```

    then have  $cdcl_W\text{-merge-cp}^{**} T V'$ 
      using  $T\text{-}V$  by (metis  $rtrancl\text{-}unfold$   $cdcl_W\text{-merge-cp.propagate'}$   $rtrancl\text{-}simps$ )
    then show ?thesis using dec  $propa\text{-}confl(2)$  by metis
  qed
next
case cp
consider
  (propa)  $propagate^{++} V W$  and  $conflicting W = None$ 
  | (propa-confl)  $V'$  where  $propagate^{**} V V'$  and  $conflict V' W$ 
  using  $trancpl\text{-}cdcl_W\text{-cp-propagate-with-conflict-or-not[of } V W] conflict'$ 
  unfolding  $full1\text{-}def$  by meson
then show ?thesis
proof cases
case propa
then show ?thesis by (meson  $cdcl_W\text{-merge-cp.propagate'}$  cp
   $rtrancl\text{-}rtrancl\text{-}into\text{-}rtrancl$ )
next
case propa-confl
then show ?thesis
  using  $propa\text{-}confl(2)$  cp
  by (metis (full-types)  $cdcl_W\text{-merge-cp.propagate'}$   $rtrancl\text{-}rtrancl\text{-}into\text{-}rtrancl$ 
     $rtrancl\text{-}unfold$ )
qed
next
case cp-confl
then show ?thesis using  $conflict'$  unfolding  $full1\text{-}def$  by (fastforce dest!:  $trancplD$ 
  elim!: rulesE)
qed
next
case (decide'  $V'$ )
then have  $conf\text{-}V$ :  $conflicting V = None$ 
  by (auto elim: rulesE)
consider
  ( $s'$ )  $cdcl_W\text{-merge-stgy}^{**} R V$ 
  | (dec-confl)  $S T U$  where  $cdcl_W\text{-merge-stgy}^{**} R S$  and no-step  $cdcl_W\text{-merge-cp } S$  and
    decide  $S T$  and  $cdcl_W\text{-merge-cp}^{**} T U$  and  $conflict U V$ 
  | (dec)  $S T$  where  $cdcl_W\text{-merge-stgy}^{**} R S$  and no-step  $cdcl_W\text{-merge-cp } S$  and decide  $S T$ 
    and  $cdcl_W\text{-merge-cp}^{**} T V$  and  $conflicting V = None$ 
  | (cp)  $cdcl_W\text{-merge-cp}^{**} R V$ 
  | (cp-confl)  $U$  where  $cdcl_W\text{-merge-cp}^{**} R U$  and  $conflict U V$ 
  using IH by meson
then show ?thesis
proof cases
case  $s'$ 
have  $conf\text{-}V'$ :  $conflicting V' = None$  using  $decide'(1)$  by (auto elim: rulesE)
have full:  $full1\ cdcl_W\text{-cp } V' W \vee (V' = W \wedge no\text{-}step\ cdcl_W\text{-cp } W)$ 
  using  $decide'(3)$  unfolding  $full\text{-}unfold$  by blast
consider
  ( $V'\text{-}W$ )  $V' = W$ 
  | (propa)  $propagate^{++} V' W$  and  $conflicting W = None$ 
  | (propa-confl)  $V''$  where  $propagate^{**} V' V''$  and  $conflict V'' W$ 
  using  $trancpl\text{-}cdcl_W\text{-cp-propagate-with-conflict-or-not[of } V W] decide'$ 
   $\langle full1\ cdcl_W\text{-cp } V' W \vee V' = W \wedge no\text{-}step\ cdcl_W\text{-cp } W \rangle$  unfolding  $full1\text{-}def$ 
  by (metis  $trancpl\text{-}cdcl_W\text{-cp-propagate-with-conflict-or-not}$ )
then show ?thesis

```

```

proof cases
  case  $V'-W$ 
  then show ?thesis
    using confl-V' local.decide'(1,2) s' conf-V
    no-step-cdclW-cp-no-step-cdclW-merge-restart[of V] by (auto elim: rulesE)
  next
    case propa
    then show ?thesis using local.decide'(1,2) s' by (metis cdclW-merge-cp.simps conf-V
      no-step-cdclW-cp-no-step-cdclW-merge-restart r-into-rtranclp)
    next
      case propa-confl
      then have cdclW-merge-cp** V' V''
        by (metis rtranclp-unfold cdclW-merge-cp.propagate' r-into-rtranclp)
      then show ?thesis
        using local.decide'(1,2) propa-confl(2) s' conf-V
        no-step-cdclW-cp-no-step-cdclW-merge-restart
        by metis
      qed
    next
      case (dec) note  $s' = \text{this}(1)$  and  $\text{dec} = \text{this}(2)$  and  $\text{cp} = \text{this}(3)$  and  $\text{ns-cp-T} = \text{this}(4)$ 
      have full cdclW-merge-cp T V
        unfolding full-def by (simp add: conf-V local.decide'(2)
          no-step-cdclW-cp-no-step-cdclW-merge-restart ns-cp-T)
      moreover have no-step cdclW-merge-cp V
        by (simp add: conf-V local.decide'(2) no-step-cdclW-cp-no-step-cdclW-merge-restart)
      moreover have no-step cdclW-merge-cp S
        by (metis dec)
      ultimately have cdclW-merge-stgy S V
        using cp by blast
      then have cdclW-merge-stgy** R V using s' by auto
      consider
        ( $V'-W$ )  $V' = W$ 
        | (propa) propagate++ V' W and conflicting W = None
        | (propa-confl)  $V''$  where propagate** V' V'' and conflict V'' W
        using trancpl-cdclW-cp-propagate-with-conflict-or-not[of V' W] decide'
        unfolding full-unfold full1-def by meson
      then show ?thesis
      proof cases
        case  $V'-W$ 
        moreover have conflicting V' = None
          using decide'(1) by (auto elim: rulesE)
        ultimately show ?thesis
          using  $\langle \text{cdcl}_W\text{-merge-stgy}^{**} R V \rangle \text{decide}' \langle \text{no-step cdcl}_W\text{-merge-cp } V \rangle$  by blast
        next
          case propa
          moreover then have cdclW-merge-cp V' W
            by auto
          ultimately show ?thesis
            using  $\langle \text{cdcl}_W\text{-merge-stgy}^{**} R V \rangle \text{decide}' \langle \text{no-step cdcl}_W\text{-merge-cp } V \rangle$ 
            by (meson r-into-rtranclp)
          next
            case propa-confl
            moreover then have cdclW-merge-cp** V' V''
              by (metis cdclW-merge-cp.propagate' rtranclp-unfold trancpl-unfold-end)
            ultimately show ?thesis using  $\langle \text{cdcl}_W\text{-merge-stgy}^{**} R V \rangle \text{decide}'$ 

```

```

      ⟨no-step cdclW-merge-cp V⟩ by (meson r-into-rtrancp)
    qed
  next
    case cp
    have no-step cdclW-merge-cp V
      using conf-V local.decide'(2) no-step-cdclW-cp-no-step-cdclW-merge-restart by auto
    then have full cdclW-merge-cp R V
      unfolding full-def using cp by fast
    then have cdclW-merge-stgy** R V
      unfolding full-unfold by auto
    have full1 cdclW-cp V' W ∨ (V' = W ∧ no-step cdclW-cp W)
      using decide'(3) unfolding full-unfold by blast

  consider
    (V'-W) V' = W
  | (propa) propagate++ V' W and conflicting W = None
  | (propa-confl) V'' where propagate** V' V'' and conflict V'' W
    using trancp-cdclW-cp-propagate-with-conflict-or-not[of V' W] decide'
    unfolding full-unfold full1-def by meson
  then show ?thesis

  proof cases
    case V'-W
    moreover have conflicting V' = None
      using decide'(1) by (auto elim: rulesE)
    ultimately show ?thesis
      using ⟨cdclW-merge-stgy** R V⟩ decide' ⟨no-step cdclW-merge-cp V⟩ by blast
  next
    case propa
    moreover then have cdclW-merge-cp V' W
      by auto
    ultimately show ?thesis using ⟨cdclW-merge-stgy** R V⟩ decide'
      ⟨no-step cdclW-merge-cp V⟩ by (meson r-into-rtrancp)
  next
    case propa-confl
    moreover then have cdclW-merge-cp** V' V''
      by (metis cdclW-merge-cp.propagate' rtrancp-unfold trancp-unfold-end)
    ultimately show ?thesis using ⟨cdclW-merge-stgy** R V⟩ decide'
      ⟨no-step cdclW-merge-cp V⟩ by (meson r-into-rtrancp)
  qed
next
  case (dec-confl)
  show ?thesis using conf-V dec-confl(5) by (auto elim!: rulesE
    simp del: state-simp simp: state-eq-def)
next
  case cp-confl
  then show ?thesis using decide' apply - by (intro HOL.disjI2) (fastforce elim: rulesE
    simp del: state-simp simp: state-eq-def)
qed
next
  case (bj' V')
  then have ¬no-step cdclW-bj V
    by (auto dest: trancpD simp: full1-def)
  then consider
    (s') cdclW-merge-stgy** R V and conflicting V = None

```



```

| (dec-confl) S T U where cdclW-merge-stgy** R S and no-step cdclW-merge-cp S and
  decide S T and cdclW-merge-cp** T U and conflict U V
| (dec) S T where cdclW-merge-stgy** R S and no-step cdclW-merge-cp S and decide S T
  and cdclW-merge-cp** T V and conflicting V = None
| (cp) cdclW-merge-cp** R V and conflicting V = None
| (cp-confl) U where cdclW-merge-cp** R U and conflict U V
using IH by meson
then show ?thesis
proof cases
  case s' note - = this(2)
  then have False
    using bj'(1) unfolding full1-def by (force dest!: tranclpD simp: cdclW-bj.simps
      elim: rulesE)
  then show ?thesis by fast
next
  case dec note - = this(5)
  then have False
    using bj'(1) unfolding full1-def by (force dest!: tranclpD simp: cdclW-bj.simps
      elim: rulesE)
  then show ?thesis by fast
next
  case dec-confl
  then have cdclW-merge-cp U V'
    using bj' cdclW-merge-cp.intros(1)[of U V V'] by (simp add: full-unfold)
  then have cdclW-merge-cp** T V'
    using dec-confl(4) by simp
  consider
    (V'-W) V' = W
  | (propa) propagate++ V' W and conflicting W = None
  | (propa-confl) V'' where propagate** V' V'' and conflict V'' W
  using tranclp-cdclW-cp-propagate-with-conflict-or-not[of V' W] bj'(3)
  unfolding full-unfold full1-def by meson
then show ?thesis
proof cases
  case V'-W
  then have no-step cdclW-cp V'
    using bj'(3) unfolding full-def by auto
  then have no-step cdclW-merge-cp V'
    by (metis cdclW-cp.propagate' cdclW-merge-cp.cases tranclpD
      no-step-cdclW-cp-no-conflict-no-propagate(1) )
  then have full1 cdclW-merge-cp T V'
    unfolding full1-def using ⟨cdclW-merge-cp U V'⟩ dec-confl(4) by auto
  then have full cdclW-merge-cp T V'
    by (simp add: full-unfold)
  then have cdclW-merge-stgy S V'
    using dec-confl(3) cdclW-merge-stgy.fw-s-decide ⟨no-step cdclW-merge-cp S⟩ by blast
  then have cdclW-merge-stgy** R V'
    using ⟨cdclW-merge-stgy** R S⟩ by auto
  show ?thesis
  proof cases
    assume conflicting W = None
    then show ?thesis using ⟨cdclW-merge-stgy** R V'⟩ ⟨V' = W⟩ by auto
  next
    assume conflicting W ≠ None
    then show ?thesis

```

```

    using ⟨cdclW-merge-stgy** R V'⟩ ⟨V' = W⟩ by (metis ⟨cdclW-merge-cp U V'⟩
      conflictE conflicting-not-true-rtrancp-cdclW-merge-cp-no-step-cdclW-bj
      dec-confl(5) map-option-is-None r-into-rtrancp)
  qed
next
  case propa
  moreover then have cdclW-merge-cp V' W
    by auto
  ultimately show ?thesis using decide' by (meson ⟨cdclW-merge-cp** T V'⟩ dec-confl(1-3)
    rtrancp.rtrancp-into-rtrancp)
next
  case propa-confl
  moreover then have cdclW-merge-cp** V' V''
    by (metis cdclW-merge-cp.propagate' rtrancp-unfold trancp-unfold-end)
  ultimately show ?thesis by (meson ⟨cdclW-merge-cp** T V'⟩ dec-confl(1-3) rtrancp-trans)
  qed
next
  case cp note - = this(2)
  then show ?thesis using bj'(1) ⟨¬ no-step cdclW-bj V'⟩
    conflicting-not-true-rtrancp-cdclW-merge-cp-no-step-cdclW-bj by auto
next
  case cp-confl
  then have cdclW-merge-cp U V' by (simp add: cdclW-merge-cp.conflict' full-unfold
    local.bj'(1))
  consider
    (V'-W) V' = W
  | (propa) propagate++ V' W and conflicting W = None
  | (propa-confl) V'' where propagate** V' V'' and conflict V'' W
  using trancp-cdclW-cp-propagate-with-conflict-or-not[of V' W] bj'
  unfolding full-unfold full1-def by meson
  then show ?thesis

proof cases
  case V'-W
  show ?thesis
  proof cases
    assume conflicting V' = None
    then show ?thesis
      using V'-W ⟨cdclW-merge-cp U V'⟩ cp-confl(1) by force
  next
    assume confl: conflicting V' ≠ None
    then have no-step cdclW-merge-stgy V'
      by (fastforce simp: cdclW-merge-stgy.simps full1-def full-def
        cdclW-merge-cp.simps dest!: trancpD elim: rulesE)
    have no-step cdclW-merge-cp V'
      using confl by (auto simp: full1-def full-def cdclW-merge-cp.simps
        dest!: trancpD elim: rulesE)
    moreover have cdclW-merge-cp U W
      using V'-W ⟨cdclW-merge-cp U V'⟩ by blast
    ultimately have full1 cdclW-merge-cp R V'
      using cp-confl(1) V'-W unfolding full1-def by auto
    then have cdclW-merge-stgy R V'
      by auto
    moreover have no-step cdclW-merge-stgy V'
      using confl ⟨no-step cdclW-merge-cp V'⟩ by (auto simp: cdclW-merge-stgy.simps

```

```

      full1-def dest!: tranclpD elim: rulesE)
ultimately have cdclW-merge-stgy** R V' by auto
{ fix ss :: 'st
  have cdclW-merge-cp U W
    using V'-W ⟨cdclW-merge-cp U V'⟩ by blast
  then have ¬ cdclW-bj W ss
    by (meson conflicting-not-true-rtranclp-cdclW-merge-cp-no-step-cdclW-bj
        cp-confl(1) rtranclp.rtrancl-into-rtrancl step.prem)
  then have cdclW-merge-stgy** R W ∧ conflicting W = None ∨
    cdclW-merge-stgy** R W ∧ ¬ cdclW-bj W ss
    using V'-W ⟨cdclW-merge-stgy** R V'⟩ by presburger }
  then show ?thesis
    by presburger
qed
next
case propa
moreover then have cdclW-merge-cp V' W
  by auto
ultimately show ?thesis using ⟨cdclW-merge-cp U V'⟩ cp-confl(1) by force
next
case propa-confl
moreover then have cdclW-merge-cp** V' V''
  by (metis cdclW-merge-cp.propagate' rtranclp-unfold tranclp-unfold-end)
ultimately show ?thesis
  using ⟨cdclW-merge-cp U V'⟩ cp-confl(1) by (metis rtranclp.rtrancl-into-rtrancl
    rtranclp-trans)
qed
qed
qed
qed

```

lemma *decide-rtranclp-cdcl_W-s'-rtranclp-cdcl_W-s'*:

```

assumes
  dec: decide S T and
  cdclW-s'*** T U and
  n-s-S: no-step cdclW-cp S and
  no-step cdclW-cp U
shows cdclW-s'*** S U
using assms(2,4)
proof induction
case (step U V) note st = this(1) and s' = this(2) and IH = this(3) and n-s = this(4)
consider
  (TU) T = U
  | (s'-st) T' where cdclW-s' T T' and cdclW-s'*** T' U
  using st[unfolded rtranclp-unfold] by (auto dest!: tranclpD)
then show ?case
proof cases
case TU
then show ?thesis
proof -
  assume a1: T = U
  then have f2: cdclW-s' T V
    using s' by force
  obtain ss :: 'st where
    ss: cdclW-s'*** S T ∨ cdclW-cp T ss

```

```

    using a1 step.IH by blast-
  obtain ssa :: 'st  $\Rightarrow$  'st where
    f3:  $\forall s\ sa\ sb. (\neg decide\ s\ sa \vee cdcl_W\text{-}cp\ s\ (ssa\ s) \vee \neg full\ cdcl_W\text{-}cp\ sa\ sb) \vee cdcl_W\text{-}s'\ s\ sb$ 
    using cdcl_W-s'.decide' by moura
  have  $\forall s\ sa. \neg cdcl_W\text{-}s'\ s\ sa \vee full1\ cdcl_W\text{-}cp\ s\ sa \vee (\exists sb. decide\ s\ sb \wedge no\text{-}step\ cdcl_W\text{-}cp\ s \wedge full\ cdcl_W\text{-}cp\ sb\ sa) \vee (\exists sb. full1\ cdcl_W\text{-}bj\ s\ sb \wedge no\text{-}step\ cdcl_W\text{-}cp\ s \wedge full\ cdcl_W\text{-}cp\ sb\ sa)$ 
    by (metis cdcl_W-s'E)
  then have  $\exists s. cdcl_W\text{-}s'^{**}\ S\ s \wedge cdcl_W\text{-}s'\ s\ V$ 
    using f3 ss f2 by (metis dec full1-is-full n-s-S rtrancp-unfold)
  then show ?thesis
    by force
qed
next
case (s'-st T') note s'-T' = this(1) and st = this(2)
have cdcl_W-s'^{**}\ S\ T'
  using s'-T'
proof cases
  case conflict'
  then have cdcl_W-s'\ S\ T'
    using dec cdcl_W-s'.decide' n-s-S by (simp add: full-unfold)
  then show ?thesis
    using st by auto
next
case (decide' T'')
  then have cdcl_W-s'\ S\ T
    using dec cdcl_W-s'.decide' n-s-S by (simp add: full-unfold)
  then show ?thesis using decide' s'-T' by auto
next
case bj'
  then have False
    using dec unfolding full1-def by (fastforce dest!: trancpD simp: cdcl_W-bj.simps elim: rulesE)
  then show ?thesis by fast
qed
then show ?thesis using s' st by auto
qed
next
case base
  then have full cdcl_W-cp\ T\ T
    by (simp add: full-unfold)
  then show ?case
    using cdcl_W-s'.simps dec n-s-S by auto
qed

lemma rtrancp-cdcl_W-merge-stgy-rtrancp-cdcl_W-s':
  assumes
    cdcl_W-merge-stgy^{**}\ R\ V and
    inv: cdcl_W-all-struct-inv R
  shows cdcl_W-s'^{**}\ R\ V
  using assms(1)
proof induction
  case base
  then show ?case by simp

```

next

```

case (step S T) note st = this(1) and fw = this(2) and IH = this(3)
have cdclW-all-struct-inv S
  using inv rtrancpl-cdclW-all-struct-inv-inv rtrancpl-cdclW-merge-stgy-rtrancpl-cdclW st by blast
from fw show ?case
proof (cases rule: cdclW-merge-stgy-cases)
  case fw-s-cp
  then show ?thesis
  proof -
    assume a1: full1 cdclW-merge-cp S T
    obtain ss :: ('st ⇒ 'st ⇒ bool) ⇒ 'st ⇒ 'st where
      f2: ∧ p s sa pa sb sc sd pb se sf. (¬ full1 p (s::'st) sa ∨ p++ s sa)
        ∧ (¬ pa (sb::'st) sc ∨ ¬ full1 pa sd sb) ∧ (¬ pb++ se sf ∨ pb sf (ss pb sf)
        ∨ full1 pb se sf)
    by (metis (no-types) full1-def)
    then have f3: cdclW-merge-cp++ S T
    using a1 by auto
    obtain ssa :: ('st ⇒ 'st ⇒ bool) ⇒ 'st ⇒ 'st ⇒ 'st where
      f4: ∧ p s sa. ¬ p++ s sa ∨ p s (ssa p s sa)
    by (meson trancpl-unfold-begin)
    then have f5: ∧ s. ¬ full1 cdclW-merge-cp s S
    using f3 f2 by (metis (full-types))
    have ∧ s. ¬ full cdclW-merge-cp s S
    using f4 f3 by (meson full-def)
    then have S = R
    using f5 by (metis cdclW-cp.conflict' cdclW-cp.propagate' cdclW-merge-cp.cases f3 f4 inv
      rtrancpl-cdclW-merge-stgy'-no-step-cdclW-cp-or-eq st)
    then show ?thesis
    using f2 a1 by (metis (no-types) ⟨cdclW-all-struct-inv S⟩
      conflicting-true-full1-cdclW-merge-cp-imp-full1-cdclW-s'-without-decode
      rtrancpl-cdclW-s'-without-decide-rtrancpl-cdclW-s' rtrancpl-unfold)
  qed
qed
next
case (fw-s-decide S') note dec = this(1) and n-S = this(2) and full = this(3)
moreover then have conflicting S' = None
  by (auto elim: rulesE)
ultimately have full cdclW-s'-without-decide S' T
  by (meson ⟨cdclW-all-struct-inv S⟩ cdclW-merge-restart-cdclW fw-r-decide
    rtrancpl-cdclW-all-struct-inv-inv
    conflicting-true-full-cdclW-merge-cp-iff-full-cdclW-s'-without-decode)
then have a1: cdclW-s*** S' T
  unfolding full-def by (metis (full-types) rtrancpl-cdclW-s'-without-decide-rtrancpl-cdclW-s')
have cdclW-merge-stgy** S T
  using fw by blast
then have cdclW-s*** S T
  using decide-rtrancpl-cdclW-s'-rtrancpl-cdclW-s' a1 by (metis ⟨cdclW-all-struct-inv S⟩ dec
    n-S no-step-cdclW-merge-cp-no-step-cdclW-cp cdclW-all-struct-inv-def
    rtrancpl-cdclW-merge-stgy'-no-step-cdclW-cp-or-eq)
then show ?thesis using IH by auto
qed
qed

```

lemma rtrancpl-cdcl_W-merge-stgy-distinct-mset-clauses:

assumes invR: cdcl_W-all-struct-inv R and
 st: cdcl_W-merge-stgy^{**} R S and

```

dist: distinct-mset (clauses R) and
R: trail R = []
shows distinct-mset (clauses S)
using rtrancpl-cdclW-stgy-distinct-mset-clauses[OF invR - dist R]
invR st rtrancpl-mono[of cdclW-s' cdclW-stgy**] cdclW-s'-is-rtrancpl-cdclW-stgy
by (auto dest!: cdclW-s'-is-rtrancpl-cdclW-stgy rtrancpl-cdclW-merge-stgy-rtrancpl-cdclW-s')

lemma no-step-cdclW-s'-no-step-cdclW-merge-stgy:
assumes
  inv: cdclW-all-struct-inv R and s': no-step cdclW-s' R
shows no-step cdclW-merge-stgy R
proof –
{ fix ss :: 'st
  obtain ssa :: 'st ⇒ 'st ⇒ 'st where
    ff1:  $\bigwedge s \text{ sa. } \neg \text{cdcl}_W\text{-merge-stgy } s \text{ sa} \vee \text{full1 } \text{cdcl}_W\text{-merge-cp } s \text{ sa} \vee \text{decide } s \text{ (ssa } s \text{ sa)}$ 
    using cdclW-merge-stgy.cases by moura
  obtain ssb :: ('st ⇒ 'st ⇒ bool) ⇒ 'st ⇒ 'st ⇒ 'st where
    ff2:  $\bigwedge p \text{ s sa. } \neg p^{++} \text{ s sa} \vee p \text{ s (ssb } p \text{ s sa)}$ 
    by (meson trancpl-unfold-begin)
  obtain ssc :: 'st ⇒ 'st where
    ff3:  $\bigwedge s \text{ sa sb. } (\neg \text{cdcl}_W\text{-all-struct-inv } s \vee \neg \text{cdcl}_W\text{-cp } s \text{ sa} \vee \text{cdcl}_W\text{-s' } s \text{ (ssc } s))$ 
     $\wedge (\neg \text{cdcl}_W\text{-all-struct-inv } s \vee \neg \text{cdcl}_W\text{-o } s \text{ sb} \vee \text{cdcl}_W\text{-s' } s \text{ (ssc } s))$ 
    using n-step-cdclW-stgy-iff-no-step-cdclW-cl-cdclW-o by moura
  then have ff4:  $\bigwedge s. \neg \text{cdcl}_W\text{-o } R \text{ s}$ 
    using s' inv by blast
  have ff5:  $\bigwedge s. \neg \text{cdcl}_W\text{-cp}^{++} R \text{ s}$ 
    using ff3 ff2 s' by (metis inv)
  have  $\bigwedge s. \neg \text{cdcl}_W\text{-bj}^{++} R \text{ s}$ 
    using ff4 ff2 by (metis bj)
  then have  $\bigwedge s. \neg \text{cdcl}_W\text{-s'-without-decide } R \text{ s}$ 
    using ff5 by (simp add: cdclW-s'-without-decide.simps full1-def)
  then have  $\neg \text{cdcl}_W\text{-s'-without-decide}^{++} R \text{ ss}$ 
    using ff2 by blast
  then have  $\neg \text{full1 } \text{cdcl}_W\text{-s'-without-decide } R \text{ ss}$ 
    by (simp add: full1-def)
  then have  $\neg \text{cdcl}_W\text{-merge-stgy } R \text{ ss}$ 
    using ff4 ff1 conflicting-true-full1-cdclW-merge-cp-imp-full1-cdclW-s'-without-decode inv
    by blast }
then show ?thesis
  by fastforce
qed
end

```

We will discharge the assumption later.

```

locale conflict-driven-clause-learningW-termination =
  conflict-driven-clause-learningW +
  assumes wf-cdclW-merge-inv: wf {(T, S). cdclW-all-struct-inv S ∧ cdclW-merge S T}
begin

```

```

lemma wf-trancpl-cdclW-merge: wf {(T, S). cdclW-all-struct-inv S ∧ cdclW-merge++ S T}
using wf-trancpl[OF wf-cdclW-merge-inv]
apply (rule wf-subset)
by (auto simp: trancpl-set-trancpl
  cdclW-all-struct-inv-trancpl-cdclW-merge-trancpl-cdclW-merge-cdclW-all-struct-inv)

```

lemma *wf-cdcl_W-merge-cp*:
wf{(*T*, *S*). *cdcl_W-all-struct-inv S* \wedge *cdcl_W-merge-cp S T*}
using *wf-tranclp-cdcl_W-merge* **by** (*rule wf-subset*) (*auto simp: cdcl_W-merge-cp-tranclp-cdcl_W-merge*)

lemma *wf-cdcl_W-merge-stgy*:
wf{(*T*, *S*). *cdcl_W-all-struct-inv S* \wedge *cdcl_W-merge-stgy S T*}
using *wf-tranclp-cdcl_W-merge* **by** (*rule wf-subset*)
(*auto simp add: cdcl_W-merge-stgy-tranclp-cdcl_W-merge*)

lemma *cdcl_W-merge-cp-obtain-normal-form*:

assumes *inv: cdcl_W-all-struct-inv R*
obtains *S* **where** *full cdcl_W-merge-cp R S*

proof –

obtain *S* **where** *full* ($\lambda S T. \text{cdcl}_W\text{-all-struct-inv } S \wedge \text{cdcl}_W\text{-merge-cp } S T$) *R S*
using *wf-exists-normal-form-full*[*OF wf-cdcl_W-merge-cp*] **by** *blast*

then have

st: ($\lambda S T. \text{cdcl}_W\text{-all-struct-inv } S \wedge \text{cdcl}_W\text{-merge-cp } S T$)** *R S* **and**
n-s: no-step ($\lambda S T. \text{cdcl}_W\text{-all-struct-inv } S \wedge \text{cdcl}_W\text{-merge-cp } S T$) *S*

unfolding *full-def* **by** *blast+*

have *cdcl_W-merge-cp** R S*

using *st* **by** *induction auto*

moreover

have *cdcl_W-all-struct-inv S*

using *st inv*

apply (*induction rule: rtranclp-induct*)

apply *simp*

by (*meson r-into-rtranclp rtranclp-cdcl_W-all-struct-inv-inv*
rtranclp-cdcl_W-merge-cp-rtranclp-cdcl_W)

then have *no-step cdcl_W-merge-cp S*

using *n-s* **by** *auto*

ultimately show *?thesis*

using *that unfolding full-def* **by** *blast*

qed

lemma *no-step-cdcl_W-merge-stgy-no-step-cdcl_W-s'*:

assumes

inv: cdcl_W-all-struct-inv R **and**

confl: conflicting R = None **and**

n-s: no-step cdcl_W-merge-stgy R

shows *no-step cdcl_W-s' R*

proof (*rule ccontr*)

assume $\neg ?thesis$

then obtain *S* **where** *cdcl_W-s' R S* **by** *auto*

then show *False*

proof *cases*

case *conflict'*

then obtain *S'* **where** *full1 cdcl_W-merge-cp R S'*

proof –

obtain *R' :: 'e* **where**

cdcl_W-merge-cp R R'

using *inv* **unfolding** *cdcl_W-all-struct-inv-def* **by** (*meson confl*

cdcl_W-s'-without-decide.simps conflict'

conflicting-true-no-step-cdcl_W-merge-cp-no-step-s'-without-decide)

then show *?thesis*

using *that* **by** (*metis cdcl_W-merge-cp-obtain-normal-form full-unfold inv*)

```

    qed
  then show False using n-s by blast
next
case (decide' R')
then have cdclW-all-struct-inv R'
  using inv cdclW-all-struct-inv-inv cdclW.other cdclW-o.decide by meson
then obtain R'' where full cdclW-merge-cp R' R''
  using cdclW-merge-cp-obtain-normal-form by blast
moreover have no-step cdclW-merge-cp R
  by (simp add: confl local.decide'(2) no-step-cdclW-cp-no-step-cdclW-merge-restart)
ultimately show False using n-s cdclW-merge-stgy.intros local.decide'(1) by blast
next
case (bj' R')
then show False
  using confl no-step-cdclW-cp-no-step-cdclW-s'-without-decide inv
  unfolding cdclW-all-struct-inv-def by auto
qed
qed

lemma rtranclp-cdclW-merge-cp-no-step-cdclW-bj:
  assumes conflicting R = None and cdclW-merge-cp** R S
  shows no-step cdclW-bj S
  using assms conflicting-not-true-rtranclp-cdclW-merge-cp-no-step-cdclW-bj by auto

lemma rtranclp-cdclW-merge-stgy-no-step-cdclW-bj:
  assumes confl: conflicting R = None and cdclW-merge-stgy** R S
  shows no-step cdclW-bj S
  using assms(2)
proof induction
  case base
  then show ?case
    using confl by (auto simp: cdclW-bj.simps elim: rulesE)
next
case (step S T) note st = this(1) and fw = this(2) and IH = this(3)
have confl-S: conflicting S = None
  using fw apply cases
  by (auto simp: full1-def cdclW-merge-cp.simps dest!: tranclpD elim: rulesE)
from fw show ?case
  proof cases
    case fw-s-cp
    then show ?thesis
      using rtranclp-cdclW-merge-cp-no-step-cdclW-bj confl-S
      by (simp add: full1-def tranclp-into-rtranclp)
  next
  case (fw-s-decide S')
  moreover then have conflicting S' = None by (auto elim: rulesE)
  ultimately show ?thesis
    using conflicting-not-true-rtranclp-cdclW-merge-cp-no-step-cdclW-bj
    unfolding full-def by meson
  qed
qed
qed

end

end

```



```

theory CDCL-W-Restart
imports CDCL-W-Merge
begin

```

20.5 Adding Restarts

```

locale cdclW-restart =
  conflict-driven-clause-learningW
  — functions for clauses:
  mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss

  — functions for the conflicting clause:
  mset-ccls union-ccls insert-ccls remove-clit

  — conversion
  ccls-of-cls cls-of-ccls

  — functions for the state:
  — access functions:
  trail hd-trail raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting
  — changing state:
  cons-trail tl-trail add-init-cls add-learned-cls remove-cls update-backtrack-lvl
  update-conflicting

  — get state:
  init-state
  restart-state
for
  mset-cls:: 'cls  $\Rightarrow$  'v clause and
  insert-cls :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and
  remove-lit :: 'v literal  $\Rightarrow$  'cls  $\Rightarrow$  'cls and

  mset-clss:: 'clss  $\Rightarrow$  'v clauses and
  union-clss :: 'clss  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  in-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  bool and
  insert-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and
  remove-from-clss :: 'cls  $\Rightarrow$  'clss  $\Rightarrow$  'clss and

  mset-ccls:: 'ccls  $\Rightarrow$  'v clause and
  union-ccls :: 'ccls  $\Rightarrow$  'ccls  $\Rightarrow$  'ccls and
  insert-ccls :: 'v literal  $\Rightarrow$  'ccls  $\Rightarrow$  'ccls and
  remove-clit :: 'v literal  $\Rightarrow$  'ccls  $\Rightarrow$  'ccls and

  ccls-of-cls :: 'cls  $\Rightarrow$  'ccls and
  cls-of-ccls :: 'ccls  $\Rightarrow$  'cls and

  trail :: 'st  $\Rightarrow$  ('v, nat, 'v clause) marked-lits and
  hd-trail :: 'st  $\Rightarrow$  ('v, nat, 'cls) marked-lit and
  raw-init-clss :: 'st  $\Rightarrow$  'clss and
  raw-learned-clss :: 'st  $\Rightarrow$  'clss and
  backtrack-lvl :: 'st  $\Rightarrow$  nat and
  raw-conflicting :: 'st  $\Rightarrow$  'ccls option and

  cons-trail :: ('v, nat, 'cls) marked-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
  tl-trail :: 'st  $\Rightarrow$  'st and

```

```

add-init-cls :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
add-learned-cls :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
remove-cls :: 'cls  $\Rightarrow$  'st  $\Rightarrow$  'st and
update-backtrack-lvl :: nat  $\Rightarrow$  'st  $\Rightarrow$  'st and
update-conflicting :: 'ccls option  $\Rightarrow$  'st  $\Rightarrow$  'st and

init-state :: 'clss  $\Rightarrow$  'st and
restart-state :: 'st  $\Rightarrow$  'st +
fixes f :: nat  $\Rightarrow$  nat
assumes f: unbounded f
begin

```

The condition of the differences of cardinality has to be strict. Otherwise, you could be in a strange state, where nothing remains to do, but a restart is done. See the proof of well-foundedness.

inductive *cdcl_W-merge-with-restart* **where**

restart-step:

```

(cdclW-merge-stgy  $\sim$  (card (set-mset (learned-clss T)) - card (set-mset (learned-clss S)))) S T
 $\implies$  card (set-mset (learned-clss T)) - card (set-mset (learned-clss S)) > f n
 $\implies$  restart T U  $\implies$  cdclW-merge-with-restart (S, n) (U, Suc n) |

```

restart-full: full1 cdcl_W-merge-stgy S T \implies cdcl_W-merge-with-restart (S, n) (T, Suc n)

lemma *cdcl_W-merge-with-restart* S T \implies cdcl_W-merge-restart** (fst S) (fst T)

by (induction rule: *cdcl_W-merge-with-restart.induct*)

```

(auto dest!: relpowp-imp-rtranclp cdclW-merge-stgy-tranclp-cdclW-merge tranclp-into-rtranclp
  rtranclp-cdclW-merge-stgy-rtranclp-cdclW-merge rtranclp-cdclW-merge-tranclp-cdclW-merge-restart
  fw-r-rf cdclW-rf.restart
  simp: full1-def)

```

lemma *cdcl_W-merge-with-restart-rtranclp-cdcl_W*:

cdcl_W-merge-with-restart S T \implies cdcl_W** (fst S) (fst T)

by (induction rule: *cdcl_W-merge-with-restart.induct*)

```

(auto dest!: relpowp-imp-rtranclp rtranclp-cdclW-merge-stgy-rtranclp-cdclW cdclW.rf
  cdclW-rf.restart tranclp-into-rtranclp simp: full1-def)

```

lemma *cdcl_W-merge-with-restart-increasing-number*:

cdcl_W-merge-with-restart S T \implies snd T = 1 + snd S

by (induction rule: *cdcl_W-merge-with-restart.induct*) auto

lemma full1 cdcl_W-merge-stgy S T \implies cdcl_W-merge-with-restart (S, n) (T, Suc n)

using *restart-full* **by** blast

lemma *cdcl_W-all-struct-inv-learned-clss-bound*:

assumes *inv*: cdcl_W-all-struct-inv S

shows set-mset (learned-clss S) \subseteq simple-clss (atms-of-mm (init-clss S))

proof

fix C

assume C: C \in set-mset (learned-clss S)

have *distinct-mset* C

using C *inv* **unfolding** *cdcl_W-all-struct-inv-def* *distinct-cdcl_W-state-def* *distinct-mset-set-def*

by auto

moreover **have** \neg tautology C

using C *inv* **unfolding** *cdcl_W-all-struct-inv-def* *cdcl_W-learned-clause-def* **by** auto

moreover

have atms-of C \subseteq atms-of-mm (learned-clss S)

```

    using C by auto
    then have  $\text{atms-of } C \subseteq \text{atms-of-mm } (\text{init-clss } S)$ 
    using inv unfolding  $\text{cdcl}_W\text{-all-struct-inv-def no-strange-atm-def}$  by force
    moreover have  $\text{finite } (\text{atms-of-mm } (\text{init-clss } S))$ 
    using inv unfolding  $\text{cdcl}_W\text{-all-struct-inv-def}$  by auto
    ultimately show  $C \in \text{simple-clss } (\text{atms-of-mm } (\text{init-clss } S))$ 
    using  $\text{distinct-mset-not-tautology-implies-in-simple-clss simple-clss-mono}$ 
    by blast
qed

```

lemma $\text{cdcl}_W\text{-merge-with-restart-init-clss}$:

```

 $\text{cdcl}_W\text{-merge-with-restart } S \ T \implies \text{cdcl}_W\text{-M-level-inv } (\text{fst } S) \implies$ 
 $\text{init-clss } (\text{fst } S) = \text{init-clss } (\text{fst } T)$ 
using  $\text{cdcl}_W\text{-merge-with-restart-rtrancpl-cdcl}_W \text{ rtrancpl-cdcl}_W\text{-init-clss}$  by blast

```

lemma

```

wf {(T, S).  $\text{cdcl}_W\text{-all-struct-inv } (\text{fst } S) \wedge \text{cdcl}_W\text{-merge-with-restart } S \ T$ }
proof (rule ccontr)
  assume  $\neg ?thesis$ 
  then obtain g where
    g:  $\bigwedge i. \text{cdcl}_W\text{-merge-with-restart } (g \ i) \ (g \ (\text{Suc } i))$  and
    inv:  $\bigwedge i. \text{cdcl}_W\text{-all-struct-inv } (\text{fst } (g \ i))$ 
  unfolding wf-iff-no-infinite-down-chain by fast
  { fix i
    have  $\text{init-clss } (\text{fst } (g \ i)) = \text{init-clss } (\text{fst } (g \ 0))$ 
    apply (induction i)
    apply simp
    using g inv unfolding  $\text{cdcl}_W\text{-all-struct-inv-def}$  by (metis  $\text{cdcl}_W\text{-merge-with-restart-init-clss}$ )
  } note init-g = this
  let ?S = g 0
  have  $\text{finite } (\text{atms-of-mm } (\text{init-clss } (\text{fst } ?S)))$ 
  using inv unfolding  $\text{cdcl}_W\text{-all-struct-inv-def}$  by auto
  have  $\text{snd-g}: \bigwedge i. \text{snd } (g \ i) = i + \text{snd } (g \ 0)$ 
  apply (induct-tac i)
  apply simp
  by (metis  $\text{Suc-eq-plus1-left add-Suc cdcl}_W\text{-merge-with-restart-increasing-number } g$ )
  then have  $\text{snd-g-0}: \bigwedge i. i > 0 \implies \text{snd } (g \ i) = i + \text{snd } (g \ 0)$ 
  by blast
  have  $\text{unbounded-f-g}: \text{unbounded } (\lambda i. f \ (\text{snd } (g \ i)))$ 
  using f unfolding bounded-def by (metis  $\text{add.commute } f \text{ less-or-eq-imp-le snd-g}$ 
    not-bounded-nat-exists-larger not-le le-iff-add)

```

obtain k where

```

f-g-k:  $f \ (\text{snd } (g \ k)) > \text{card } (\text{simple-clss } (\text{atms-of-mm } (\text{init-clss } (\text{fst } ?S))))$  and
 $k > \text{card } (\text{simple-clss } (\text{atms-of-mm } (\text{init-clss } (\text{fst } ?S))))$ 
using not-bounded-nat-exists-larger[OF unbounded-f-g] by blast

```

The following does not hold anymore with the non-strict version of cardinality in the definition.

```

{ fix i
  assume no-step  $\text{cdcl}_W\text{-merge-stgy } (\text{fst } (g \ i))$ 
  with g[of i]
  have False
  proof (induction rule:  $\text{cdcl}_W\text{-merge-with-restart.induct}$ )
    case (restart-step T S n) note H = this(1) and c = this(2) and n-s = this(4)
    obtain S' where  $\text{cdcl}_W\text{-merge-stgy } S \ S'$ 

```

```

    using  $H$   $c$  by (metis gr-implies-not0 relpowp-E2)
  then show False using  $n$ -s by auto
next
  case (restart-full  $S$   $T$ )
  then show False unfolding full1-def by (auto dest: tranclpD)
qed
} note  $H = \text{this}$ 
obtain  $m$   $T$  where
   $m$ :  $m = \text{card} (\text{set-mset} (\text{learned-clss } T)) - \text{card} (\text{set-mset} (\text{learned-clss} (\text{fst } (g \ k))))$  and
   $m > f (\text{snd } (g \ k))$  and
  restart  $T$  (fst ( $g$  ( $k+1$ ))) and
   $\text{cdcl}_W\text{-merge-stgy}$ : ( $\text{cdcl}_W\text{-merge-stgy} \rightsquigarrow m$ ) (fst ( $g \ k$ ))  $T$ 
  using  $g[\text{of } k]$   $H[\text{of } \text{Suc } k]$  by (force simp:  $\text{cdcl}_W\text{-merge-with-restart.simps full1-def}$ )
have  $\text{cdcl}_W\text{-merge-stgy}^{**}$  (fst ( $g \ k$ ))  $T$ 
  using  $\text{cdcl}_W\text{-merge-stgy relpowp-imp-rtranclp}$  by metis
then have  $\text{cdcl}_W\text{-all-struct-inv } T$ 
  using  $\text{inv}[\text{of } k]$   $\text{rtranclp-cdcl}_W\text{-all-struct-inv-inv rtranclp-cdcl}_W\text{-merge-stgy-rtranclp-cdcl}_W$ 
  by blast
moreover have  $\text{card} (\text{set-mset} (\text{learned-clss } T)) - \text{card} (\text{set-mset} (\text{learned-clss} (\text{fst } (g \ k))))$ 
  >  $\text{card} (\text{simple-clss} (\text{atms-of-mm} (\text{init-clss} (\text{fst } ?S))))$ 
  unfolding  $m[\text{symmetric}]$  using  $\langle m > f (\text{snd } (g \ k)) \rangle f\text{-}g\text{-}k$  by linarith
then have  $\text{card} (\text{set-mset} (\text{learned-clss } T))$ 
  >  $\text{card} (\text{simple-clss} (\text{atms-of-mm} (\text{init-clss} (\text{fst } ?S))))$ 
  by linarith
moreover
  have  $\text{init-clss} (\text{fst } (g \ k)) = \text{init-clss } T$ 
  using  $\langle \text{cdcl}_W\text{-merge-stgy}^{**} (\text{fst } (g \ k)) \ T \rangle \text{rtranclp-cdcl}_W\text{-merge-stgy-rtranclp-cdcl}_W$ 
   $\text{rtranclp-cdcl}_W\text{-init-clss inv}$  unfolding  $\text{cdcl}_W\text{-all-struct-inv-def}$  by blast
  then have  $\text{init-clss} (\text{fst } ?S) = \text{init-clss } T$ 
  using  $\text{init-g}[\text{of } k]$  by auto
ultimately show False
  using  $\text{cdcl}_W\text{-all-struct-inv-learned-clss-bound}$ 
  by (simp add:  $\langle \text{finite} (\text{atms-of-mm} (\text{init-clss} (\text{fst } (g \ 0)))) \rangle \text{simple-clss-finite}$ 
     $\text{card-mono leD}$ )
qed

lemma  $\text{cdcl}_W\text{-merge-with-restart-distinct-mset-clauses}$ :
  assumes  $\text{invR}$ :  $\text{cdcl}_W\text{-all-struct-inv} (\text{fst } R)$  and
   $st$ :  $\text{cdcl}_W\text{-merge-with-restart } R \ S$  and
   $dist$ :  $\text{distinct-mset} (\text{clauses} (\text{fst } R))$  and
   $R$ :  $\text{trail} (\text{fst } R) = []$ 
  shows  $\text{distinct-mset} (\text{clauses} (\text{fst } S))$ 
  using  $\text{assms}(2,1,3,4)$ 
proof (induction)
  case (restart-full  $S$   $T$ )
  then show ?case using  $\text{rtranclp-cdcl}_W\text{-merge-stgy-distinct-mset-clauses}[\text{of } S \ T]$  unfolding full1-def
    by (auto dest: tranclp-into-rtranclp)
next
  case (restart-step  $T \ S \ n \ U$ )
  then have  $\text{distinct-mset} (\text{clauses } T)$ 
    using  $\text{rtranclp-cdcl}_W\text{-merge-stgy-distinct-mset-clauses}[\text{of } S \ T]$  unfolding full1-def
    by (auto dest: relpowp-imp-rtranclp)
  then show ?case using  $\langle \text{restart } T \ U \rangle$  by (metis clauses-restart distinct-mset-union fstI
     $\text{mset-le-exists-conv restart.cases state-eq-clauses}$ )
qed

```

inductive *cdcl_W-with-restart* **where**

restart-step:

$(\text{cdcl}_W\text{-stgy} \sim (\text{card } (\text{set-mset } (\text{learned-clss } T)) - \text{card } (\text{set-mset } (\text{learned-clss } S)))) S T \implies$
 $\text{card } (\text{set-mset } (\text{learned-clss } T)) - \text{card } (\text{set-mset } (\text{learned-clss } S)) > f n \implies$
 $\text{restart } T U \implies$

$\text{cdcl}_W\text{-with-restart } (S, n) (U, \text{Suc } n) \mid$

restart-full: $\text{full1 } \text{cdcl}_W\text{-stgy } S T \implies \text{cdcl}_W\text{-with-restart } (S, n) (T, \text{Suc } n)$

lemma *cdcl_W-with-restart-rtranclp-cdcl_W*:

$\text{cdcl}_W\text{-with-restart } S T \implies \text{cdcl}_W^{**} (\text{fst } S) (\text{fst } T)$

apply (*induction rule*: *cdcl_W-with-restart.induct*)

by (*auto dest!*: *relopwp-imp-rtranclp tranclp-into-rtranclp fw-r-rf*

cdcl_W-rf.restart rtranclp-cdcl_W-stgy-rtranclp-cdcl_W cdcl_W-merge-restart-cdcl_W

simp: *full1-def*)

lemma *cdcl_W-with-restart-increasing-number*:

$\text{cdcl}_W\text{-with-restart } S T \implies \text{snd } T = 1 + \text{snd } S$

by (*induction rule*: *cdcl_W-with-restart.induct*) *auto*

lemma *full1 cdcl_W-stgy* $S T \implies \text{cdcl}_W\text{-with-restart } (S, n) (T, \text{Suc } n)$

using *restart-full* **by** *blast*

lemma *cdcl_W-with-restart-init-clss*:

$\text{cdcl}_W\text{-with-restart } S T \implies \text{cdcl}_W\text{-M-level-inv } (\text{fst } S) \implies \text{init-clss } (\text{fst } S) = \text{init-clss } (\text{fst } T)$

using *cdcl_W-with-restart-rtranclp-cdcl_W rtranclp-cdcl_W-init-clss* **by** *blast*

lemma

wf $\{(T, S). \text{cdcl}_W\text{-all-struct-inv } (\text{fst } S) \wedge \text{cdcl}_W\text{-with-restart } S T\}$

proof (*rule ccontr*)

assume $\neg ?thesis$

then obtain *g* **where**

g: $\bigwedge i. \text{cdcl}_W\text{-with-restart } (g \ i) (g (\text{Suc } i))$ **and**

inv: $\bigwedge i. \text{cdcl}_W\text{-all-struct-inv } (\text{fst } (g \ i))$

unfolding *wf-iff-no-infinite-down-chain* **by** *fast*

{ fix *i*

have $\text{init-clss } (\text{fst } (g \ i)) = \text{init-clss } (\text{fst } (g \ 0))$

apply (*induction i*)

apply *simp*

using *g inv unfolding cdcl_W-all-struct-inv-def* **by** (*metis cdcl_W-with-restart-init-clss*)

} note *init-g = this*

let $?S = g \ 0$

have *finite* (*atms-of-mm* ($\text{init-clss } (\text{fst } ?S)$))

using *inv unfolding cdcl_W-all-struct-inv-def* **by** *auto*

have *snd-g*: $\bigwedge i. \text{snd } (g \ i) = i + \text{snd } (g \ 0)$

apply (*induct-tac i*)

apply *simp*

by (*metis Suc-eq-plus1-left add-Suc cdcl_W-with-restart-increasing-number g*)

then have *snd-g-0*: $\bigwedge i. i > 0 \implies \text{snd } (g \ i) = i + \text{snd } (g \ 0)$

by *blast*

have *unbounded-f-g*: *unbounded* $(\lambda i. f (\text{snd } (g \ i)))$

using *f unfolding bounded-def* **by** (*metis add.commute f less-or-eq-imp-le snd-g*

not-bounded-nat-exists-larger not-le le-iff-add)

obtain *k* **where**

f-g-k: $f \text{ (snd } (g \ k)) > \text{card (simple-clss (atms-of-mm (init-clss (fst ?S))))}$ **and**
 $k > \text{card (simple-clss (atms-of-mm (init-clss (fst ?S))))}$
using *not-bounded-nat-exists-larger*[*OF unbounded-f-g*] **by** *blast*

The following does not hold anymore with the non-strict version of cardinality in the definition.

```
{ fix i
  assume no-step cdclW-stgy (fst (g i))
  with g[of i]
  have False
  proof (induction rule: cdclW-with-restart.induct)
    case (restart-step T S n) note H = this(1) and c = this(2) and n-s = this(4)
    obtain S' where cdclW-stgy S S'
    using H c by (metis gr-implies-not0 relpowp-E2)
    then show False using n-s by auto
  next
    case (restart-full S T)
    then show False unfolding full1-def by (auto dest: tranclpD)
  qed
} note H = this
obtain m T where
  m: m = card (set-mset (learned-clss T)) - card (set-mset (learned-clss (fst (g k)))) and
  m > f (snd (g k)) and
  restart T (fst (g (k+1))) and
  cdclW-merge-stgy: (cdclW-stgy  $\sim$  m) (fst (g k)) T
using g[of k] H[of Suc k] by (force simp: cdclW-with-restart.simps full1-def)
have cdclW-stgy** (fst (g k)) T
  using cdclW-merge-stgy relpowp-imp-rtranclp by metis
then have cdclW-all-struct-inv T
  using inv[of k] rtranclp-cdclW-all-struct-inv-inv rtranclp-cdclW-stgy-rtranclp-cdclW by blast
moreover have card (set-mset (learned-clss T)) - card (set-mset (learned-clss (fst (g k))))
  > card (simple-clss (atms-of-mm (init-clss (fst ?S))))
  unfolding m[symmetric] using (m > f (snd (g k))) f-g-k by linarith
then have card (set-mset (learned-clss T))
  > card (simple-clss (atms-of-mm (init-clss (fst ?S))))
  by linarith
moreover
  have init-clss (fst (g k)) = init-clss T
    using (cdclW-stgy** (fst (g k)) T) rtranclp-cdclW-stgy-rtranclp-cdclW rtranclp-cdclW-init-clss
    inv unfolding cdclW-all-struct-inv-def
    by blast
  then have init-clss (fst ?S) = init-clss T
    using init-g[of k] by auto
ultimately show False
  using cdclW-all-struct-inv-learned-clss-bound
  by (simp add: (finite (atms-of-mm (init-clss (fst (g 0))))) simple-clss-finite
    card-mono leD)
qed
```

lemma *cdcl_W-with-restart-distinct-mset-clauses*:
assumes *invR*: *cdcl_W-all-struct-inv* (fst *R*) **and**
st: *cdcl_W-with-restart* *R S* **and**
dist: *distinct-mset* (*clauses* (fst *R*)) **and**
R: *trail* (fst *R*) = []
shows *distinct-mset* (*clauses* (fst *S*))
using *assms*(2,1,3,4)

```

proof (induction)
  case (restart-full S T)
  then show ?case using rtrancpl-cdclW-stgy-distinct-mset-clauses[of S T] unfolding full1-def
    by (auto dest: trancpl-into-rtrancpl)
next
  case (restart-step T S n U)
  then have distinct-mset (clauses T) using rtrancpl-cdclW-stgy-distinct-mset-clauses[of S T]
    unfolding full1-def by (auto dest: relpowp-imp-rtrancpl)
  then show ?case using (restart T U) by (metis clauses-restart distinct-mset-union fstI
    mset-le-exists-conv restart.cases state-eq-clauses)
qed
end

locale luby-sequence =
  fixes ur :: nat
  assumes ur > 0
begin

lemma exists-luby-decomp:
  fixes i :: nat
  shows  $\exists k::nat. (2^{k-1} \leq i \wedge i < 2^k - 1) \vee i = 2^k - 1$ 
proof (induction i)
  case 0
  then show ?case
    by (rule exI[of - 0], simp)
next
  case (Suc n)
  then obtain k where  $2^{k-1} \leq n \wedge n < 2^k - 1 \vee n = 2^k - 1$ 
    by blast
  then consider
    (st-interv)  $2^{k-1} \leq n$  and  $n \leq 2^k - 2$ 
  | (end-interv)  $2^{k-1} \leq n$  and  $n = 2^k - 2$ 
  | (pow2)  $n = 2^k - 1$ 
    by linarith
  then show ?case
  proof cases
  case st-interv
  then show ?thesis apply - apply (rule exI[of - k])
    by (metis (no-types, lifting) One-nat-def Suc-diff-Suc Suc-lessI
      (2^{k-1} \leq n \wedge n < 2^k - 1 \vee n = 2^k - 1) diff-self-eq-0
      dual-order.trans le-SucI le-imp-less-Suc numeral-2-eq-2 one-le-numeral
      one-le-power zero-less-numeral zero-less-power)
  next
  case end-interv
  then show ?thesis apply - apply (rule exI[of - k]) by auto
  next
  case pow2
  then show ?thesis apply - apply (rule exI[of - k+1]) by auto
qed
qed

```

Luby sequences are defined by:

- $2^k - 1$, if $i = (2::'a)^k - (1::'a)$
- $\text{luby-sequence-core } (i - 2^k - 1 + 1)$, if $(2::'a)^{k-1} \leq i$ and $i \leq (2::'a)^k - (1::'a)$

Then the sequence is then scaled by a constant unit run (called *ur* here), strictly positive.

```

function luby-sequence-core :: nat ⇒ nat where
  luby-sequence-core i =
    (if ∃ k. i = 2k - 1
     then 2((SOME k. i = 2k - 1) - 1)
     else luby-sequence-core (i - 2((SOME k. 2(k-1) ≤ i ∧ i < 2k - 1) - 1) + 1))
by auto
termination
proof (relation less-than, goal-cases)
  case 1
  then show ?case by auto
next
  case (2 i)
  let ?k = (SOME k. 2(k - 1) ≤ i ∧ i < 2k - 1)
  have 2(?k - 1) ≤ i ∧ i < 2?k - 1
  apply (rule someI-ex)
  using 2 exists-luby-decomp by blast
  then show ?case

proof -
  have ∀ n na. ¬ (1::nat) ≤ n ∨ 1 ≤ n^ na
  by (meson one-le-power)
  then have f1: (1::nat) ≤ 2(?k - 1)
  using one-le-numeral by blast
  have f2: i - 2(?k - 1) + 2(?k - 1) = i
  using (2(?k - 1) ≤ i ∧ i < 2?k - 1) le-add-diff-inverse2 by blast
  have f3: 2?k - 1 ≠ Suc 0
  using f1 (2(?k - 1) ≤ i ∧ i < 2?k - 1) by linarith
  have 2?k - (1::nat) ≠ 0
  using (2(?k - 1) ≤ i ∧ i < 2?k - 1) gr-implies-not0 by blast
  then have f4: 2?k ≠ (1::nat)
  by linarith
  have f5: ∀ n na. if na = 0 then (n::nat)^ na = 1 else n^ na = n * n^ (na - 1)
  by (simp add: power-eq-if)
  then have ?k ≠ 0
  using f4 by meson
  then have 2(?k - 1) ≠ Suc 0
  using f5 f3 by presburger
  then have Suc 0 < 2(?k - 1)
  using f1 by linarith
  then show ?thesis
  using f2 less-than-iff by presburger
qed
qed

```

```

function natlog2 :: nat ⇒ nat where
  natlog2 n = (if n = 0 then 0 else 1 + natlog2 (n div 2))
  using not0-implies-Suc by auto
termination by (relation measure (λn. n)) auto

```

```

declare natlog2.simps[simp del]

```

```

declare luby-sequence-core.simps[simp del]

```

```

lemma two-pover-n-eq-two-power-n'-eq:

```


assumes $H: (2::nat) \wedge (k::nat) - 1 = 2 \wedge k' - 1$
shows $k' = k$
proof –
have $(2::nat) \wedge (k::nat) = 2 \wedge k'$
using H **by** (*metis One-nat-def Suc-pred zero-less-numeral zero-less-power*)
then show *?thesis* **by** *simp*
qed

lemma *luby-sequence-core-two-power-minus-one:*
 $luby-sequence-core (2^k - 1) = 2^{(k-1)}$ (**is** $?L = ?K$)
proof –
have *decomp*: $\exists ka. 2^k - 1 = 2^{ka} - 1$
by *auto*
have $?L = 2^{((SOME k'. (2::nat)^k - 1 = 2^{k'} - 1) - 1)}$
apply (*subst luby-sequence-core.simps, subst decomp*)
by *simp*
moreover have $(SOME k'. (2::nat)^k - 1 = 2^{k'} - 1) = k$
apply (*rule some-equality*)
apply *simp*
using *two-pover-n-eq-two-power-n'-eq* **by** *blast*
ultimately show *?thesis* **by** *presburger*
qed

lemma *different-luby-decomposition-false:*
assumes
 $H: 2^k - (k - Suc\ 0) \leq i$ **and**
 $k': i < 2^{k'} - Suc\ 0$ **and**
 $k-k': k > k'$
shows *False*
proof –
have $2^{k'} - Suc\ 0 < 2^k - (k - Suc\ 0)$
using $k-k'$ *less-eq-Suc-le* **by** *auto*
then show *?thesis*
using $H\ k'$ **by** *linarith*
qed

lemma *luby-sequence-core-not-two-power-minus-one:*
assumes
 $k-i: 2^k - 1 \leq i$ **and**
 $i-k: i < 2^k - 1$
shows $luby-sequence-core\ i = luby-sequence-core\ (i - 2^k + 1)$
proof –
have $H: \neg (\exists ka. i = 2^{ka} - 1)$
proof (*rule ccontr*)
assume $\neg ?thesis$
then obtain $k': nat$ **where** $k': i = 2^{k'} - 1$ **by** *blast*
have $(2::nat)^{k'} - 1 < 2^k - 1$
using $i-k$ **unfolding** k' .
then have $(2::nat)^{k'} < 2^k$
by *linarith*
then have $k' < k$
by *simp*
have $2^k - 1 \leq 2^{k'} - 1$
using $k-i$ **unfolding** k' .
then have $(2::nat)^k < (2::nat)^{k'}$

```

    by (metis Suc-diff-1 not-le not-less-eq zero-less-numeral zero-less-power)
  then have  $k-1 < k'$ 
    by simp

  show False using  $\langle k' < k \rangle \langle k-1 < k' \rangle$  by linarith
qed
have  $\bigwedge k k'. 2^{\wedge (k - \text{Suc } 0)} \leq i \implies i < 2^{\wedge k - \text{Suc } 0} \implies 2^{\wedge (k' - \text{Suc } 0)} \leq i \implies$ 
 $i < 2^{\wedge k' - \text{Suc } 0} \implies k = k'$ 
  by (meson different-luby-decomposition-false linorder-neqE-nat)
then have k: (SOME k.  $2^{\wedge (k - \text{Suc } 0)} \leq i \wedge i < 2^{\wedge k - \text{Suc } 0}$ ) = k
  using k-i i-k by auto
show ?thesis
  apply (subst luby-sequence-core.simps[of i], subst H)
  by (simp add: k)
qed

lemma unbounded-luby-sequence-core: unbounded luby-sequence-core
  unfolding bounded-def
proof
  assume  $\exists b. \forall n. \text{luby-sequence-core } n \leq b$ 
  then obtain b where  $b: \bigwedge n. \text{luby-sequence-core } n \leq b$ 
    by metis
  have luby-sequence-core ( $2^{\wedge (b+1)} - 1$ ) =  $2^{\wedge b}$ 
    using luby-sequence-core-two-power-minus-one[of b+1] by simp
  moreover have ( $2::\text{nat}$ ) $^{\wedge b} > b$ 
    by (induction b) auto
  ultimately show False using b[of  $2^{\wedge (b+1)} - 1$ ] by linarith
qed

abbreviation luby-sequence :: nat  $\Rightarrow$  nat where
  luby-sequence n  $\equiv$  ur * luby-sequence-core n

lemma bounded-luby-sequence: unbounded luby-sequence
  using bounded-const-product[of ur] luby-sequence-axioms
  luby-sequence-def unbounded-luby-sequence-core by blast

lemma luby-sequence-core-0: luby-sequence-core 0 = 1
proof -
  have 0: ( $0::\text{nat}$ ) =  $2^{\wedge 0} - 1$ 
    by auto
  show ?thesis
    by (subst 0, subst luby-sequence-core-two-power-minus-one) simp
qed

lemma luby-sequence-core n  $\geq 1$ 
proof (induction n rule: nat-less-induct-case)
  case 0
  then show ?case by (simp add: luby-sequence-core-0)
next
  case (Suc n) note IH = this

  consider
    (interv) k where  $2^{\wedge (k - 1)} \leq \text{Suc } n$  and  $\text{Suc } n < 2^{\wedge k} - 1$ 
  | (pow2) k where  $\text{Suc } n = 2^{\wedge k} - \text{Suc } 0$ 
  using exists-luby-decomp[of Suc n] by auto

```

```

then show ?case
proof cases
case pow2
show ?thesis
using luby-sequence-core-two-power-minus-one pow2 by auto
next
case interv
have n: Suc n - 2 ^ (k - 1) + 1 < Suc n
by (metis Suc-1 Suc-eq-plus1 add.commute add-diff-cancel-left' add-less-mono1 gr0I
interv(1) interv(2) le-add-diff-inverse2 less-Suc-eq not-le power-0 power-one-right
power-strict-increasing-iff)
show ?thesis
apply (subst luby-sequence-core-not-two-power-minus-one[OF interv])
using IH n by auto
qed
qed
end

locale luby-sequence-restart =
luby-sequence ur +
conflict-driven-clause-learningW — functions for clauses:
mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss

— functions for the conflicting clause:
mset-ccls union-ccls insert-ccls remove-clit

— conversion
ccls-of-cls cls-of-ccls

— functions for the state:
— access functions:
trail hd-raw-trail raw-init-clss raw-learned-clss backtrack-lvl raw-conflicting
— changing state:
cons-trail tl-trail add-init-cls add-learned-cls remove-cls update-backtrack-lvl
update-conflicting

— get state:
init-state
restart-state
for
ur :: nat and
mset-cls:: 'cls ⇒ 'v clause and
insert-cls :: 'v literal ⇒ 'cls ⇒ 'cls and
remove-lit :: 'v literal ⇒ 'cls ⇒ 'cls and

mset-clss:: 'clss ⇒ 'v clauses and
union-clss :: 'clss ⇒ 'clss ⇒ 'clss and
in-clss :: 'cls ⇒ 'clss ⇒ bool and
insert-clss :: 'cls ⇒ 'clss ⇒ 'clss and
remove-from-clss :: 'cls ⇒ 'clss ⇒ 'clss and

mset-ccls:: 'ccls ⇒ 'v clause and
union-ccls :: 'ccls ⇒ 'ccls ⇒ 'ccls and

```

```

insert-ccls :: 'v literal ⇒ 'ccls ⇒ 'ccls and
remove-clit :: 'v literal ⇒ 'ccls ⇒ 'ccls and

ccls-of-cls :: 'cls ⇒ 'ccls and
cls-of-ccls :: 'ccls ⇒ 'cls and

trail :: 'st ⇒ ('v, nat, 'v clause) marked-lits and
hd-raw-trail :: 'st ⇒ ('v, nat, 'cls) marked-lit and
raw-init-clss :: 'st ⇒ 'clss and
raw-learned-clss :: 'st ⇒ 'clss and
backtrack-lvl :: 'st ⇒ nat and
raw-conflicting :: 'st ⇒ 'ccls option and

cons-trail :: ('v, nat, 'cls) marked-lit ⇒ 'st ⇒ 'st and
tl-trail :: 'st ⇒ 'st and
add-init-cls :: 'cls ⇒ 'st ⇒ 'st and
add-learned-cls :: 'cls ⇒ 'st ⇒ 'st and
remove-cls :: 'cls ⇒ 'st ⇒ 'st and
update-backtrack-lvl :: nat ⇒ 'st ⇒ 'st and
update-conflicting :: 'ccls option ⇒ 'st ⇒ 'st and

init-state :: 'clss ⇒ 'st and
restart-state :: 'st ⇒ 'st
begin

sublocale cdclW-restart - - - - - luby-sequence
apply unfold-locales
using bounded-luby-sequence by blast

end
end
theory CDCL-WNOT
imports CDCL-NOT CDCL-W-Termination CDCL-W-Merge
begin

```

21 Link between Weidenbach's and NOT's CDCL

21.1 Inclusion of the states

```

declare upt.simps(2)[simp del]

fun convert-marked-lit-from-W where
convert-marked-lit-from-W (Propagated L -) = Propagated L () |
convert-marked-lit-from-W (Marked L -) = Marked L ()

abbreviation convert-trail-from-W ::
('v, 'lvl, 'a) marked-lit list
⇒ ('v, unit, unit) marked-lit list where
convert-trail-from-W ≡ map convert-marked-lit-from-W

lemma lits-of-l-convert-trail-from-W[simp]:
lits-of-l (convert-trail-from-W M) = lits-of-l M
by (induction rule: marked-lit-list-induct) simp-all

lemma lit-of-convert-trail-from-W[simp]:

```

lit-of (*convert-marked-lit-from-W* *L*) = *lit-of* *L*
by (*cases* *L*) *auto*

lemma *no-dup-convert-from-W*[*simp*]:
no-dup (*convert-trail-from-W* *M*) \longleftrightarrow *no-dup* *M*
by (*auto simp: comp-def*)

lemma *convert-trail-from-W-true-annots*[*simp*]:
convert-trail-from-W *M* \models_{as} *C* \longleftrightarrow *M* \models_{as} *C*
by (*auto simp: true-annots-true-cls image-image lits-of-def*)

lemma *defined-lit-convert-trail-from-W*[*simp*]:
defined-lit (*convert-trail-from-W* *S*) *L* \longleftrightarrow *defined-lit* *S* *L*
by (*auto simp: defined-lit-map image-comp*)

The values *0* and $\{\#\}$ are dummy values.

consts *dummy-cls* :: '*cls*
fun *convert-marked-lit-from-NOT*
 :: ('*a*, '*e*, '*b*) *marked-lit* \Rightarrow ('*a*, *nat*, '*cls*) *marked-lit* **where**
convert-marked-lit-from-NOT (*Propagated* *L* -) = *Propagated* *L* *dummy-cls* |
convert-marked-lit-from-NOT (*Marked* *L* -) = *Marked* *L* *0*

abbreviation *convert-trail-from-NOT* **where**
convert-trail-from-NOT \equiv *map convert-marked-lit-from-NOT*

lemma *undefined-lit-convert-trail-from-NOT*[*simp*]:
undefined-lit (*convert-trail-from-NOT* *F*) *L* \longleftrightarrow *undefined-lit* *F* *L*
by (*induction F rule: marked-lit-list-induct*) (*auto simp: defined-lit-map*)

lemma *lits-of-l-convert-trail-from-NOT*:
lits-of-l (*convert-trail-from-NOT* *F*) = *lits-of-l* *F*
by (*induction F rule: marked-lit-list-induct*) *auto*

lemma *convert-trail-from-W-from-NOT*[*simp*]:
convert-trail-from-W (*convert-trail-from-NOT* *M*) = *M*
by (*induction rule: marked-lit-list-induct*) *auto*

lemma *convert-trail-from-W-convert-lit-from-NOT*[*simp*]:
convert-marked-lit-from-W (*convert-marked-lit-from-NOT* *L*) = *L*
by (*cases* *L*) *auto*

abbreviation *trail_{NOT}* **where**
trail_{NOT} *S* \equiv *convert-trail-from-W* (*fst* *S*)

lemma *undefined-lit-convert-trail-from-W*[*iff*]:
undefined-lit (*convert-trail-from-W* *M*) *L* \longleftrightarrow *undefined-lit* *M* *L*
by (*auto simp: defined-lit-map image-comp*)

lemma *lit-of-convert-marked-lit-from-NOT*[*iff*]:
lit-of (*convert-marked-lit-from-NOT* *L*) = *lit-of* *L*
by (*cases* *L*) *auto*

sublocale *state_W* \subseteq *dpll-state-ops*
mset-cls insert-cls remove-lit
mset-clss union-clss in-clss insert-clss remove-from-clss

```

λS. convert-trail-from-W (trail S)
raw-clauses
λL S. cons-trail (convert-marked-lit-from-NOT L) S
λS. tl-trail S
λC S. add-learned-cls C S
λC S. remove-cls C S
by unfold-locales

context state_W
begin
lemma convert-marked-lit-from-W-convert-marked-lit-from-NOT[simp]:
  convert-marked-lit-from-W (mset-of-lit (convert-marked-lit-from-NOT L)) = L
  by (cases L) auto
end

sublocale state_W ⊆ dpll-state
  mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
λS. convert-trail-from-W (trail S)
raw-clauses
λL S. cons-trail (convert-marked-lit-from-NOT L) S
λS. tl-trail S
λC S. add-learned-cls C S
λC S. remove-cls C S
by unfold-locales (auto simp: map-tl o-def)

context state_W
begin
declare state-simp_NOT[simp del]
end

sublocale conflict-driven-clause-learning_W ⊆ cdcl_NOT-merge-bj-learn-ops
  mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
λS. convert-trail-from-W (trail S)
raw-clauses
λL S. cons-trail (convert-marked-lit-from-NOT L) S
λS. tl-trail S
λC S. add-learned-cls C S
λC S. remove-cls C S
λ- -. True
λ- S. raw-conflicting S = None
λC C' L' S T. backjump-l-cond C C' L' S T
  ∧ distinct-mset (C' + {#L'#}) ∧ ¬tautology (C' + {#L'#})
by unfold-locales

thm cdcl_NOT-merge-bj-learn-proxy.axioms
sublocale conflict-driven-clause-learning_W ⊆ cdcl_NOT-merge-bj-learn-proxy
  mset-cls insert-cls remove-lit
  mset-clss union-clss in-clss insert-clss remove-from-clss
λS. convert-trail-from-W (trail S)
raw-clauses

```

```

λL S. cons-trail (convert-marked-lit-from-NOT L) S
λS. tl-trail S
λC S. add-learned-cls C S
λC S. remove-cls C S

λ- -. True
λ- S. raw-conflicting S = None
backjump-l-cond
invNOT
proof (unfold-locales, goal-cases)
  case 2
  then show ?case using cdclNOT-merged-bj-learn-no-dup-inv by (auto simp: comp-def)
next
  case (1 C' S C F' K F L)
  moreover
    let ?C' = remdups-mset C'
    have L ∉ # C'
      using ⟨F ⊨as CNot C'⟩ ⟨undefined-lit F L⟩ Marked-Propagated-in-iff-in-lits-of-l
      in-CNot-implies-uminus(2) by fast
    then have distinct-mset (?C' + {#L#})
      by (simp add: distinct-mset-single-add)
  moreover
    have no-dup F
      using ⟨invNOT S⟩ ⟨convert-trail-from-W (trail S) = F' @ Marked K () # F⟩
      unfolding invNOT-def
      by (smt comp-apply distinct.simps(2) distinct-append list.simps(9) map-append
        no-dup-convert-from-W)
    then have consistent-interp (lits-of-l F)
      using distinct-consistent-interp by blast
    then have ¬ tautology (C')
      using ⟨F ⊨as CNot C'⟩ consistent-CNot-not-tautology true-annots-true-cls by blast
    then have ¬ tautology (?C' + {#L#})
      using ⟨F ⊨as CNot C'⟩ ⟨undefined-lit F L⟩ by (metis CNot-remdups-mset
        Marked-Propagated-in-iff-in-lits-of-l add.commute in-CNot-uminus tautology-add-single
        tautology-remdups-mset true-annot-singleton true-annots-def)
  show ?case
  proof -
    have f2: no-dup (convert-trail-from-W (trail S))
      using ⟨invNOT S⟩ unfolding invNOT-def by (simp add: o-def)
    have f3: atm-of L ∈ atms-of-mm (clauses S)
      ∪ atm-of ' lits-of-l (convert-trail-from-W (trail S))
      using ⟨convert-trail-from-W (trail S) = F' @ Marked K () # F⟩
      ⟨atm-of L ∈ atms-of-mm (clauses S) ∪ atm-of ' lits-of-l (F' @ Marked K () # F)⟩ by auto
    have f4: clauses S ⊨pm remdups-mset C' + {#L#}
      by (metis (no-types) ⟨L ∉ # C'⟩ ⟨clauses S ⊨pm C' + {#L#}⟩ remdups-mset-singleton-sum(2)
        true-clss-cls-remdups-mset union-commute)
    have F ⊨as CNot (remdups-mset C')
      by (simp add: ⟨F ⊨as CNot C'⟩)
    obtain D where D: mset-cls D = remdups-mset C' + {#L#}
      using ex-mset-cls by blast
    have Ex (backjump-l S)
      apply standard
      apply (rule backjump-l.intros[OF - f2, of - -])
      using f4 f3 f2 ⟨¬ tautology (remdups-mset C' + {#L#})⟩
      calculation(2-5,9) ⟨F ⊨as CNot (remdups-mset C')⟩

```

```

      state-eqNOT-ref D unfolding backjump-l-cond-def by blast+
    then show ?thesis
      by blast
    qed
qed

sublocale conflict-driven-clause-learningW  $\subseteq$  cdclNOT-merge-bj-learn-proxy2 - - - - -
   $\lambda S.$  convert-trail-from-W (trail S)
  raw-clauses
   $\lambda L S.$  cons-trail (convert-marked-lit-from-NOT L) S
   $\lambda S.$  tl-trail S
   $\lambda C S.$  add-learned-cls C S
   $\lambda C S.$  remove-cls C S
   $\lambda -.$  True
   $\lambda - S.$  raw-conflicting S = None backjump-l-cond invNOT
  by unfold-locales

sublocale conflict-driven-clause-learningW  $\subseteq$  cdclNOT-merge-bj-learn - - - - -
   $\lambda S.$  convert-trail-from-W (trail S)
  raw-clauses
   $\lambda L S.$  cons-trail (convert-marked-lit-from-NOT L) S
   $\lambda S.$  tl-trail S
   $\lambda C S.$  add-learned-cls C S
   $\lambda C S.$  remove-cls C S
  backjump-l-cond
   $\lambda -.$  True
   $\lambda - S.$  raw-conflicting S = None invNOT
  apply unfold-locales
  using dpll-bj-no-dup apply (simp add: comp-def)
  using cdclNOT.simps cdclNOT-no-dup no-dup-convert-from-W unfolding invNOT-def by blast

context conflict-driven-clause-learningW
begin

Notations are lost while proving locale inclusion:
notation state-eqNOT (infix  $\sim_{NOT}$  50)

```

21.2 Additional Lemmas between NOT and W states

lemma *trail*_W-eq-reduce-trail-to_{NOT}-eq:

trail S = *trail T* \implies *trail* (*reduce-trail-to*_{NOT} *F S*) = *trail* (*reduce-trail-to*_{NOT} *F T*)

proof (*induction F S arbitrary: T rule: reduce-trail-to*_{NOT}.*induct*)

case (1 *F S T*) **note** *IH* = *this*(1) **and** *tr* = *this*(2)

then have \square = *convert-trail-from-W* (*trail S*)

\vee *length F* = *length* (*convert-trail-from-W* (*trail S*))

\vee *trail* (*reduce-trail-to*_{NOT} *F* (*tl-trail S*)) = *trail* (*reduce-trail-to*_{NOT} *F* (*tl-trail T*))

using *IH* **by** (*metis* (*no-types*) *trail-tl-trail*)

then show *trail* (*reduce-trail-to*_{NOT} *F S*) = *trail* (*reduce-trail-to*_{NOT} *F T*)

using *tr* **by** (*metis* (*no-types*) *reduce-trail-to*_{NOT}.*elims*)

qed

lemma *trail-reduce-trail-to*_{NOT}-add-learned-cls:

no-dup (*trail S*) \implies

trail (*reduce-trail-to*_{NOT} *M* (*add-learned-cls D S*)) = *trail* (*reduce-trail-to*_{NOT} *M S*)

by (*rule* *trail*_W-eq-reduce-trail-to_{NOT}-eq) *simp*


```

lemma reduce-trail-toNOT-reduce-trail-convert:
  reduce-trail-toNOT C S = reduce-trail-to (convert-trail-from-NOT C) S
apply (induction C S rule: reduce-trail-toNOT.induct)
apply (subst reduce-trail-toNOT.simps, subst reduce-trail-to.simps)
by auto

lemma reduce-trail-to-map[simp]:
  reduce-trail-to (map f M) S = reduce-trail-to M S
by (rule reduce-trail-to-length) simp

lemma reduce-trail-toNOT-map[simp]:
  reduce-trail-toNOT (map f M) S = reduce-trail-toNOT M S
by (rule reduce-trail-toNOT-length) simp

lemma skip-or-resolve-state-change:
  assumes skip-or-resolve** S T
  shows
     $\exists M. \text{trail } S = M @ \text{trail } T \wedge (\forall m \in \text{set } M. \neg \text{is-marked } m)$ 
    clauses S = clauses T
    backtrack-lvl S = backtrack-lvl T
  using assms
proof (induction rule: rtranclp-induct)
  case base
  case 1 show ?case by simp
  case 2 show ?case by simp
  case 3 show ?case by simp
next
  case (step T U) note st = this(1) and s-o-r = this(2) and IH = this(3) and IH' = this(3-5)

  case 2 show ?case using IH' s-o-r by (auto elim!: rulesE simp: skip-or-resolve.simps)
  case 3 show ?case using IH' s-o-r by (auto elim!: rulesE simp: skip-or-resolve.simps)
  case 1 show ?case
    using s-o-r
    proof cases
      case s-or-r-skip
      then show ?thesis using IH by (auto elim!: rulesE simp: skip-or-resolve.simps)
    next
      case s-or-r-resolve
      then show ?thesis
        using IH by (cases trail T) (auto elim!: rulesE simp: skip-or-resolve.simps dest!: hd-raw-trail)
    qed
  qed

```

21.3 More lemmas conflict-propagate and backjumping

21.4 CDCL FW

```

lemma cdclW-merge-is-cdclNOT-merged-bj-learn:
  assumes
    inv: cdclW-all-struct-inv S and
    cdclW:cdclW-merge S T
  shows cdclNOT-merged-bj-learn S T
     $\vee (\text{no-step } \text{cdcl}_W\text{-merge } T \wedge \text{conflicting } T \neq \text{None})$ 
  using cdclW inv
proof induction

```

```

case (fw-propagate S T) note propa = this(1)
then obtain M N U k L C where
  H: state S = (M, N, U, k, None) and
  CL: C + {#L#} ∈ # clauses S and
  M-C: M ⊨as CNot C and
  undef: undefined-lit (trail S) L and
  T: state T = (Propagated L (C + {#L#}) # M, N, U, k, None)
  by (auto elim: propagate-high-levelE)
have propagateNOT S T
  using H CL T undef M-C by (auto simp: state-eqNOT-def state-eq-def raw-clauses-def
    simp del: state-simp)
then show ?case
  using cdclNOT-merged-bj-learn.intros(2) by blast
next
case (fw-decide S T) note dec = this(1) and inv = this(2)
then obtain L where
  undef-L: undefined-lit (trail S) L and
  atm-L: atm-of L ∈ atms-of-mm (init-clss S) and
  T: T ∼ cons-trail (Marked L (Suc (backtrack-lvl S)))
    (update-backtrack-lvl (Suc (backtrack-lvl S)) S)
  by (auto elim: decideE)
have decideNOT S T
  apply (rule decideNOT.decideNOT)
  using undef-L apply simp
  using atm-L inv unfolding cdclW-all-struct-inv-def no-strange-atm-def raw-clauses-def
  apply auto[]
  using T undef-L unfolding state-eq-def state-eqNOT-def by (auto simp: raw-clauses-def)
then show ?case using cdclNOT-merged-bj-learn-decideNOT by blast
next
case (fw-forget S T) note rf = this(1) and inv = this(2)
then obtain C where
  S: conflicting S = None and
  C-le: C !∈ raw-learned-clss S and
  ¬(trail S) ⊨asm clauses S and
  mset-cls C ∉ set (get-all-mark-of-propagated (trail S)) and
  C-init: mset-cls C ∉ # init-clss S and
  T: T ∼ remove-cls C S
  by (auto elim: forgetE)
have init-clss S ⊨pm mset-cls C
  using inv C-le unfolding cdclW-all-struct-inv-def cdclW-learned-clause-def raw-clauses-def
  by (meson in-clss-mset-clss true-clss-clss-in-imp-true-clss-clss)
then have S-C: removeAll-mset (mset-cls C) (clauses S) ⊨pm mset-cls C
  using C-init C-le unfolding raw-clauses-def by (auto simp add: Un-Diff ac-simps)
have forgetNOT S T
  apply (rule forgetNOT.forgetNOT)
  using S-C apply blast
  using S apply simp
  using C-init C-le apply (simp add: raw-clauses-def)
  using T C-le C-init by (auto
    simp: state-eq-def Un-Diff state-eqNOT-def raw-clauses-def ac-simps
    simp del: state-simp)
then show ?case using cdclNOT-merged-bj-learn-forgetNOT by blast
next
case (fw-conflict S T U) note confl = this(1) and bj = this(2) and inv = this(3)
obtain CS CT where

```

```

confl-T: raw-conflicting T = Some CT and
CT: mset-ccls CT = mset-cls CS and
CS: CS !∈! raw-clauses S and
tr-S-CS: trail S ⊨as CNot (mset-cls CS)
using confl by (elim conflictE) (auto simp del: state-simp simp: state-eq-def)
have cdclW-all-struct-inv T
  using cdclW.simps cdclW-all-struct-inv-inv confl inv by blast
then have cdclW-M-level-inv T
  unfolding cdclW-all-struct-inv-def by auto
then consider
  (no-bt) skip-or-resolve** T U
| (bt) T' where skip-or-resolve** T T' and backtrack T' U
  using bj rtrancp-cdclW-bj-skip-or-resolve-backtrack unfolding full-def by meson
then show ?case
proof cases
case no-bt
then have conflicting U ≠ None
  using confl by (induction rule: rtrancp-induct)
  (auto simp del: state-simp simp: skip-or-resolve.simps state-eq-def elim!: rulesE)
moreover then have no-step cdclW-merge U
  by (auto simp: cdclW-merge.simps elim: rulesE)
ultimately show ?thesis by blast
next
case bt note s-or-r = this(1) and bt = this(2)
have cdclW** T T'
  using s-or-r mono-rtrancp[of skip-or-resolve cdclW] rtrancp-skip-or-resolve-rtrancp-cdclW
  by blast
then have cdclW-M-level-inv T'
  using rtrancp-cdclW-consistent-inv (cdclW-M-level-inv T) by blast
then obtain M1 M2 i D L K where
  confl-T': raw-conflicting T' = Some D and
  LD: L ∈# mset-ccls D and
  M1-M2: (Marked K (i+1) # M1, M2) ∈ set (get-all-marked-decomposition (trail T')) and
  get-level (trail T') L = backtrack-lvl T' and
  get-level (trail T') L = get-maximum-level (trail T') (mset-ccls D) and
  get-maximum-level (trail T') (mset-ccls (remove-clit L D)) = i and
  undef-L: undefined-lit M1 L and
  U: U ∼ cons-trail (Propagated L (cls-of-ccls D))
  (reduce-trail-to M1
    (add-learned-cls (cls-of-ccls D)
      (update-backtrack-lvl i
        (update-conflicting None T')))))
  using bt by (auto elim: backtrack-levE)
have [simp]: clauses S = clauses T
  using confl by (auto elim: rulesE)
have [simp]: clauses T = clauses T'
  using s-or-r
proof (induction)
case base
then show ?case by simp
next
case (step U V) note st = this(1) and s-o-r = this(2) and IH = this(3)
have clauses U = clauses V
  using s-o-r by (auto simp: skip-or-resolve.simps elim: rulesE)
then show ?case using IH by auto

```

```

qed
have inv-T: cdclW-all-struct-inv T
  by (meson cdclW-cp.simps confl inv r-into-rtrancp rtrancp-cdclW-all-struct-inv-inv
      rtrancp-cdclW-cp-rtrancp-cdclW)
have cdclW** T T'
  using rtrancp-skip-or-resolve-rtrancp-cdclW s-or-r by blast
have inv-T': cdclW-all-struct-inv T'
  using <cdclW** T T'> inv-T rtrancp-cdclW-all-struct-inv-inv by blast
have inv-U: cdclW-all-struct-inv U
  using cdclW-merge-restart-cdclW confl fw-r-conflict inv local.bj
  rtrancp-cdclW-all-struct-inv-inv by blast

have [simp]: init-clss S = init-clss T'
  using <cdclW** T T'> cdclW-init-clss confl cdclW-all-struct-inv-def conflict inv
  by (metis <cdclW-M-level-inv T'> rtrancp-cdclW-init-clss)
then have atm-L: atm-of L ∈ atms-of-mm (clauses S)
  using inv-T' confl-T' LD unfolding cdclW-all-struct-inv-def no-strange-atm-def
  raw-clauses-def
  by (simp add: atms-of-def image-subset-iff)
obtain M where tr-T: trail T = M @ trail T'
  using s-or-r skip-or-resolve-state-change by meson
obtain M' where
  tr-T': trail T' = M' @ Marked K (i+1) # tl (trail U) and
  tr-U: trail U = Propagated L (mset-ccls D) # tl (trail U)
  using U M1-M2 undef-L inv-T' unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def
  by fastforce
def M'' ≡ M @ M'
have tr-T: trail S = M'' @ Marked K (i+1) # tl (trail U)
  using tr-T tr-T' confl unfolding M''-def by (auto elim: rulesE)
have init-clss T' + learned-clss S ⊢pm mset-ccls D
  using inv-T' confl-T' unfolding cdclW-all-struct-inv-def cdclW-learned-clause-def
  raw-clauses-def by simp
have reduce-trail-to (convert-trail-from-NOT (convert-trail-from-W M1)) S =
  reduce-trail-to M1 S
  by (rule reduce-trail-to-length) simp
moreover have trail (reduce-trail-to M1 S) = M1
  apply (rule reduce-trail-to-skip-beginning[of - M @ - @ M2 @ [Marked K (Suc i)]])
  using confl M1-M2 <trail T = M @ trail T'>
  apply (auto dest!: get-all-marked-decomposition-exists-prepend
      elim!: conflictE)
  by (rule sym) auto
ultimately have [simp]: trail (reduce-trail-toNOT M1 S) = M1
  using M1-M2 confl by (subst reduce-trail-toNOT-reduce-trail-convert)
  (auto simp: comp-def elim: rulesE)
have every-mark-is-a-conflict U
  using inv-U unfolding cdclW-all-struct-inv-def cdclW-conflicting-def by simp
then have U-D: tl (trail U) ⊢as CNot (remove1-mset L (mset-ccls D))
  by (metis append-self-conv2 tr-U)
thm backjump-l[of - - - - L cls-of-ccls D - remove1-mset L (mset-ccls D)]
have backjump-l S U
  apply (rule backjump-l[of - - - - L cls-of-ccls D - remove1-mset L (mset-ccls D)])
  using tr-T apply simp
  using inv unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def
  apply (simp add: comp-def)
  using U M1-M2 confl undef-L M1-M2 inv-T' inv undef-L unfolding cdclW-all-struct-inv-def

```

```

    cdclW-M-level-inv-def apply (auto simp: state-eqNOT-def
      trail-reduce-trail-toNOT-add-learned-cls)[]
  using CS apply auto[]
using tr-S-CS apply simp

  using U undef-L M1-M2 inv-T' inv unfolding cdclW-all-struct-inv-def
  cdclW-M-level-inv-def apply auto[]
  using undef-L atm-L apply (simp add: trail-reduce-trail-toNOT-add-learned-cls)
using (init-clss T' + learned-clss S  $\models_{pm}$  mset-ccls D) LD unfolding raw-clauses-def
apply simp
using LD apply simp
apply (metis U-D convert-trail-from-W-true-annots)
using inv-T' inv-U U confl-T' undef-L M1-M2 LD unfolding cdclW-all-struct-inv-def
  distinct-cdclW-state-def by (simp add: cdclW-M-level-inv-decomp backjump-l-cond-def)
then show ?thesis using cdclNOT-merged-bj-learn-backjump-l by fast
qed
qed

abbreviation cdclNOT-restart where
  cdclNOT-restart  $\equiv$  restart-ops.cdclNOT-raw-restart cdclNOT restart

lemma cdclW-merge-restart-is-cdclNOT-merged-bj-learn-restart-no-step:
  assumes
    inv: cdclW-all-struct-inv S and
    cdclW:cdclW-merge-restart S T
  shows cdclNOT-restart** S T  $\vee$  (no-step cdclW-merge T  $\wedge$  conflicting T  $\neq$  None)
proof –
consider
  (fw) cdclW-merge S T
| (fw-r) restart S T
using cdclW by (meson cdclW-merge-restart.simps cdclW-rf.cases fw-conflict fw-decide fw-forget
  fw-propagate)
then show ?thesis
proof cases
  case fw
  then have IH: cdclNOT-merged-bj-learn S T  $\vee$  (no-step cdclW-merge T  $\wedge$  conflicting T  $\neq$  None)
    using inv cdclW-merge-is-cdclNOT-merged-bj-learn by blast
  have invS: invNOT S
    using inv unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def by auto
  have ff2: cdclNOT++ S T  $\longrightarrow$  cdclNOT** S T
    by (meson tranclp-into-rtranclp)
  have ff3: no-dup (convert-trail-from-W (trail S))
    using invS by (simp add: comp-def)
  have cdclNOT  $\leq$  cdclNOT-restart
    by (auto simp: restart-ops.cdclNOT-raw-restart.simps)
  then show ?thesis
    using ff3 ff2 IH cdclNOT-merged-bj-learn-is-tranclp-cdclNOT
    rtranclp-mono[of cdclNOT cdclNOT-restart] invS predicate2D by blast
  next
  case fw-r
  then show ?thesis by (blast intro: restart-ops.cdclNOT-raw-restart.intros)
qed
qed

```

abbreviation $\mu_{FW} :: 'st \Rightarrow nat$ **where**

$\mu_{FW} S \equiv (\text{if no-step } cdcl_W\text{-merge } S \text{ then } 0 \text{ else } 1 + \mu_{CDCL}'\text{-merged } (\text{set-mset } (\text{init-clss } S)) S)$

lemma *cdcl_W-merge- μ_{FW} -decreasing*:

assumes

inv: *cdcl_W-all-struct-inv* *S* **and**

fw: *cdcl_W-merge* *S* *T*

shows $\mu_{FW} T < \mu_{FW} S$

proof –

let *?A* = *init-clss* *S*

have *atm-clauses*: *atms-of-mm* (*clauses* *S*) \subseteq *atms-of-mm* *?A*

using *inv* **unfolding** *cdcl_W-all-struct-inv-def* *no-strange-atm-def* *raw-clauses-def* **by** *auto*

have *atm-trail*: *atm-of* ‘*lits-of-l* (*trail* *S*) \subseteq *atms-of-mm* *?A*

using *inv* **unfolding** *cdcl_W-all-struct-inv-def* *no-strange-atm-def* *raw-clauses-def* **by** *auto*

have *n-d*: *no-dup* (*trail* *S*)

using *inv* **unfolding** *cdcl_W-all-struct-inv-def* **by** (*auto simp*: *cdcl_W-M-level-inv-decomp*)

have [*simp*]: \neg *no-step* *cdcl_W-merge* *S*

using *fw* **by** *auto*

have [*simp*]: *init-clss* *S* = *init-clss* *T*

using *cdcl_W-merge-restart-cdcl_W*[*of* *S* *T*] *inv* *rtranclp-cdcl_W-init-clss*

unfolding *cdcl_W-all-struct-inv-def*

by (*meson* *cdcl_W-merge.simps* *cdcl_W-merge-restart.simps* *cdcl_W-rf.simps* *fw*)

consider

(*merged*) *cdcl_{NOT}-merged-bj-learn* *S* *T*

| (*n-s*) *no-step* *cdcl_W-merge* *T*

using *cdcl_W-merge-is-cdcl_{NOT}-merged-bj-learn* *inv* *fw* **by** *blast*

then show *?thesis*

proof *cases*

case *merged*

then show *?thesis*

using *cdcl_{NOT}-decreasing-measure*‘[*OF* - - *atm-clauses*, *of* *T*] *atm-trail* *n-d*

by (*auto split*: *if-split simp*: *comp-def image-image lits-of-def*)

next

case *n-s*

then show *?thesis* **by** *simp*

qed

qed

lemma *wf-cdcl_W-merge*: *wf* $\{(T, S). \text{cdcl}_W\text{-all-struct-inv } S \wedge \text{cdcl}_W\text{-merge } S T\}$

apply (*rule* *wfP-if-measure*[*of* - - μ_{FW}])

using *cdcl_W-merge- μ_{FW} -decreasing* **by** *blast*

sublocale *conflict-driven-clause-learning_W-termination*

by *unfold-locales* (*simp add*: *wf-cdcl_W-merge*)

lemma *full-cdcl_W-s'-full-cdcl_W-merge-restart*:

assumes

conflicting *R* = *None* **and**

inv: *cdcl_W-all-struct-inv* *R*

shows *full cdcl_W-s'* *R* *V* \longleftrightarrow *full cdcl_W-merge-stgy* *R* *V* (**is** *?s'* \longleftrightarrow *?fw*)

proof

assume *?s'*

then have *cdcl_W-s^{***}* *R* *V* **unfolding** *full-def* **by** *blast*

have *cdcl_W-all-struct-inv* *V*

using $\langle \text{cdcl}_W\text{-s^{***} } R \text{ } V \rangle$ *inv* *rtranclp-cdcl_W-all-struct-inv-inv* *rtranclp-cdcl_W-s'-rtranclp-cdcl_W*

by *blast*

```

then have n-s: no-step cdclW-merge-stgy V
  using no-step-cdclW-s'-no-step-cdclW-merge-stgy by (meson ⟨full cdclW-s' R V⟩ full-def)
have n-s-bj: no-step cdclW-bj V
  by (metis ⟨cdclW-all-struct-inv V⟩ ⟨full cdclW-s' R V⟩ bj full-def
      n-step-cdclW-stgy-iff-no-step-cdclW-cl-cdclW-o)
have n-s-cp: no-step cdclW-merge-cp V
proof -
  { fix ss :: 'st
    obtain ssa :: 'st ⇒ 'st where
      ff1: ∀ s. ¬ cdclW-all-struct-inv s ∨ cdclW-s'-without-decide s (ssa s)
        ∨ no-step cdclW-merge-cp s
      using conflicting-true-no-step-s'-without-decide-no-step-cdclW-merge-cp by moura
    have (∀ p s sa. ¬ full p (s::'st) sa ∨ p** s sa ∧ no-step p sa) and
      (∀ p s sa. (¬ p** (s::'st) sa ∨ (∃ s. p sa s)) ∨ full p s sa)
      by (meson full-def)+
    then have ¬ cdclW-merge-cp V ss
      using ff1 by (metis (no-types) ⟨cdclW-all-struct-inv V⟩ ⟨full cdclW-s' R V⟩ cdclW-s'.simps
          cdclW-s'-without-decide.cases) }
    then show ?thesis
      by blast
  qed
consider
  (fw-no-confl) cdclW-merge-stgy** R V and conflicting V = None
| (fw-confl) cdclW-merge-stgy** R V and conflicting V ≠ None and no-step cdclW-bj V
| (fw-dec-confl) S T U where cdclW-merge-stgy** R S and no-step cdclW-merge-cp S and
  decide S T and cdclW-merge-cp** T U and conflict U V
| (fw-dec-no-confl) S T where cdclW-merge-stgy** R S and no-step cdclW-merge-cp S and
  decide S T and cdclW-merge-cp** T V and conflicting V = None
| (cp-no-confl) cdclW-merge-cp** R V and conflicting V = None
| (cp-confl) U where cdclW-merge-cp** R U and conflict U V
using rtranclp-cdclW-s'-no-step-cdclW-s'-without-decide-decomp-into-cdclW-merge[OF
  ⟨cdclW-s'* R V⟩ assms] by auto
then show ?fw
proof cases
  case fw-no-confl
  then show ?thesis using n-s unfolding full-def by blast
next
  case fw-confl
  then show ?thesis using n-s unfolding full-def by blast
next
  case fw-dec-confl
  have cdclW-merge-cp U V
  using n-s-bj by (metis cdclW-merge-cp.simps full-unfold fw-dec-confl(5))
  then have full1 cdclW-merge-cp T V
  unfolding full1-def by (metis fw-dec-confl(4) n-s-cp tranclp-unfold-end)
  then have cdclW-merge-stgy S V using ⟨decide S T⟩ ⟨no-step cdclW-merge-cp S⟩ by auto
  then show ?thesis using n-s ⟨cdclW-merge-stgy** R S⟩ unfolding full-def by auto
next
  case fw-dec-no-confl
  then have full cdclW-merge-cp T V
  using n-s-cp unfolding full-def by blast
  then have cdclW-merge-stgy S V using ⟨decide S T⟩ ⟨no-step cdclW-merge-cp S⟩ by auto
  then show ?thesis using n-s ⟨cdclW-merge-stgy** R S⟩ unfolding full-def by auto
next
  case cp-no-confl

```

```

    then have full cdclW-merge-cp R V
      by (simp add: full-def n-s-cp)
    then have R = V ∨ cdclW-merge-stgy++ R V
      using fw-s-cp unfolding full-unfold fw-s-cp
      by (metis (no-types) rtrancpl-unfold trancpl-unfold-end)
    then show ?thesis
      by (simp add: full-def n-s rtrancpl-unfold)
  next
  case cp-conf
  have full cdclW-bj V V
    using n-s-bj unfolding full-def by blast
  then have full1 cdclW-merge-cp R V
    unfolding full1-def by (meson cdclW-merge-cp.conflict' cp-conf(1,2) n-s-cp
      rtrancpl-into-trancpl1)
  then show ?thesis using n-s unfolding full-def by auto
qed
next
assume ?fw
then have cdclW** R V using rtrancpl-mono[of cdclW-merge-stgy cdclW**]
  cdclW-merge-stgy-rtrancpl-cdclW unfolding full-def by auto
then have inv': cdclW-all-struct-inv V using inv rtrancpl-cdclW-all-struct-inv-inv by blast
have cdclW-s'** R V
  using ⟨?fw⟩ by (simp add: full-def inv rtrancpl-cdclW-merge-stgy-rtrancpl-cdclW-s')
moreover have no-step cdclW-s' V
proof cases
  assume conflicting V = None
  then show ?thesis
    by (metis inv' ⟨full cdclW-merge-stgy R V⟩ full-def
      no-step-cdclW-merge-stgy-no-step-cdclW-s')
  next
  assume confl-V: conflicting V ≠ None
  then have no-step cdclW-bj V
    using rtrancpl-cdclW-merge-stgy-no-step-cdclW-bj by (meson ⟨full cdclW-merge-stgy R V⟩
      assms(1) full-def)
  then show ?thesis using confl-V by (fastforce simp: cdclW-s'.simps full1-def cdclW-cp.simps
    dest!: trancplD elim: rulesE)
qed
ultimately show ?s' unfolding full-def by blast
qed

```

lemma full-cdcl_W-stgy-full-cdcl_W-merge:

```

assumes
  conflicting R = None and
  inv: cdclW-all-struct-inv R
shows full cdclW-stgy R V ⟷ full cdclW-merge-stgy R V
by (simp add: assms(1) full-cdclW-s'-full-cdclW-merge-restart full-cdclW-stgy-iff-full-cdclW-s'
  inv)

```

lemma full-cdcl_W-merge-stgy-final-state-conclusive':

```

fixes S' :: 'st
assumes full: full cdclW-merge-stgy (init-state N) S'
and no-d: distinct-mset-mset (mset-cls N)
shows (conflicting S' = Some {#} ∧ unsatisfiable (set-mset (mset-cls N)))
  ∨ (conflicting S' = None ∧ trail S' ⊨asm mset-cls N ∧ satisfiable (set-mset (mset-cls N)))

```

proof —


```

have cdclW-all-struct-inv (init-state N)
  using no-d unfolding cdclW-all-struct-inv-def by auto
moreover have conflicting (init-state N) = None
  by auto
ultimately show ?thesis
  using full full-cdclW-stgy-final-state-conclusive-from-init-state
  full-cdclW-stgy-full-cdclW-merge no-d by presburger
qed
end

end
theory CDCL-W-Incremental
imports CDCL-W-Termination
begin

```

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```

context conflict-driven-clause-learningW
begin

```

This invariant holds all the invariant related to the strategy. See the structural invariant in *cdcl_W-all-struct-inv*

```

definition cdclW-stgy-invariant where
cdclW-stgy-invariant  $S \longleftrightarrow$ 
  conflict-is-false-with-level  $S$ 
   $\wedge$  no-clause-is-false  $S$ 
   $\wedge$  no-smaller-confl  $S$ 
   $\wedge$  no-clause-is-false  $S$ 

```

lemma *cdcl_W-stgy-cdcl_W-stgy-invariant*:

```

assumes
  cdclW: cdclW-stgy  $S$   $T$  and
  inv-s: cdclW-stgy-invariant  $S$  and
  inv: cdclW-all-struct-inv  $S$ 
shows
  cdclW-stgy-invariant  $T$ 
unfolding cdclW-stgy-invariant-def cdclW-all-struct-inv-def apply (intro conjI)
apply (rule cdclW-stgy-ex-lit-of-max-level[of S])
using assms unfolding cdclW-stgy-invariant-def cdclW-all-struct-inv-def apply auto[7]
using cdclW cdclW-stgy-not-non-negated-init-clss apply simp
apply (rule cdclW-stgy-no-smaller-confl-inv)
using assms unfolding cdclW-stgy-invariant-def cdclW-all-struct-inv-def apply auto[4]
using cdclW cdclW-stgy-not-non-negated-init-clss by auto

```

lemma *rtrancp-cdcl_W-stgy-cdcl_W-stgy-invariant*:

```

assumes
  cdclW: cdclW-stgy**  $S$   $T$  and
  inv-s: cdclW-stgy-invariant  $S$  and
  inv: cdclW-all-struct-inv  $S$ 
shows
  cdclW-stgy-invariant  $T$ 
using assms apply (induction)
apply simp
using cdclW-stgy-cdclW-stgy-invariant rtrancp-cdclW-all-struct-inv-inv
rtrancp-cdclW-stgy-rtrancp-cdclW by blast

```

abbreviation *decr-bt-lvl* **where**

decr-bt-lvl $S \equiv \text{update-backtrack-lvl } (\text{backtrack-lvl } S - 1) S$

When we add a new clause, we reduce the trail until we get to the first literal included in C . Then we can mark the conflict.

fun *cut-trail-wrt-clause* **where**

cut-trail-wrt-clause $C \ [] S = S \mid$
cut-trail-wrt-clause $C \ (\text{Marked } L - \# M) S =$
 (if $-L \in \# C$ then S
 else *cut-trail-wrt-clause* $C \ M \ (\text{decr-bt-lvl } (\text{tl-trail } S))) \mid$
cut-trail-wrt-clause $C \ (\text{Propagated } L - \# M) S =$
 (if $-L \in \# C$ then S
 else *cut-trail-wrt-clause* $C \ M \ (\text{tl-trail } S))$

definition *add-new-clause-and-update* :: $'\text{ccls} \Rightarrow '\text{st} \Rightarrow '\text{st}$ **where**

add-new-clause-and-update $C \ S =$
 (if $\text{trail } S \models_{\text{as}} C \text{Not } (\text{mset-ccls } C)$
 then *update-conflicting* $(\text{Some } C) \ (\text{add-init-cls } (\text{cls-of-ccls } C)$
 (*cut-trail-wrt-clause* $(\text{mset-ccls } C) \ (\text{trail } S) \ S))$
 else *add-init-cls* $(\text{cls-of-ccls } C) \ S)$

thm *cut-trail-wrt-clause.induct*

lemma *init-clss-cut-trail-wrt-clause[simp]*:
init-clss $(\text{cut-trail-wrt-clause } C \ M \ S) = \text{init-clss } S$
by (*induction rule: cut-trail-wrt-clause.induct*) *auto*

lemma *learned-clss-cut-trail-wrt-clause[simp]*:
learned-clss $(\text{cut-trail-wrt-clause } C \ M \ S) = \text{learned-clss } S$
by (*induction rule: cut-trail-wrt-clause.induct*) *auto*

lemma *conflicting-clss-cut-trail-wrt-clause[simp]*:
conflicting $(\text{cut-trail-wrt-clause } C \ M \ S) = \text{conflicting } S$
by (*induction rule: cut-trail-wrt-clause.induct*) *auto*

lemma *trail-cut-trail-wrt-clause*:

$\exists M. \text{trail } S = M @ \text{trail } (\text{cut-trail-wrt-clause } C \ (\text{trail } S) \ S)$

proof (*induction trail S arbitrary: S rule: marked-lit-list-induct*)

case *nil*

then show ?*case* **by** *simp*

next

case $(\text{marked } L \ l \ M)$ **note** $IH = \text{this}(1)[\text{of } \text{decr-bt-lvl } (\text{tl-trail } S)]$ **and** $M = \text{this}(2)[\text{symmetric}]$

then show ?*case* **using** *Cons-eq-appendI* **by** *fastforce+*

next

case $(\text{proped } L \ l \ M)$ **note** $IH = \text{this}(1)[\text{of } \text{tl-trail } S]$ **and** $M = \text{this}(2)[\text{symmetric}]$

then show ?*case* **using** *Cons-eq-appendI* **by** *fastforce+*

qed

lemma *n-dup-no-dup-trail-cut-trail-wrt-clause[simp]*:

assumes *n-d: no-dup* $(\text{trail } T)$

shows *no-dup* $(\text{trail } (\text{cut-trail-wrt-clause } C \ (\text{trail } T) \ T))$

proof –

obtain M **where**

$M: \text{trail } T = M @ \text{trail } (\text{cut-trail-wrt-clause } C \ (\text{trail } T) \ T)$

using *trail-cut-trail-wrt-clause[of T C]* **by** *auto*

```

show ?thesis
  using n-d unfolding arg-cong[OF M, of no-dup] by auto
qed

lemma cut-trail-wrt-clause-backtrack-lvl-length-marked:
  assumes
    backtrack-lvl T = length (get-all-levels-of-marked (trail T))
  shows
    backtrack-lvl (cut-trail-wrt-clause C (trail T) T) =
      length (get-all-levels-of-marked (trail (cut-trail-wrt-clause C (trail T) T)))
  using assms
proof (induction trail T arbitrary:T rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
  case (marked L l M) note IH = this(1)[of decr-bt-lvl (tl-trail T)] and M = this(2)[symmetric]
    and bt = this(3)
  then show ?case by auto
next
  case (proped L l M) note IH = this(1)[of tl-trail T] and M = this(2)[symmetric] and bt = this(3)
  then show ?case by auto
qed

lemma cut-trail-wrt-clause-get-all-levels-of-marked:
  assumes get-all-levels-of-marked (trail T) = rev [Suc 0..  

    Suc (length (get-all-levels-of-marked (trail T)))]
  shows
    get-all-levels-of-marked (trail ((cut-trail-wrt-clause C (trail T) T))) = rev [Suc 0..  

    Suc (length (get-all-levels-of-marked (trail ((cut-trail-wrt-clause C (trail T) T)))]
  using assms
proof (induction trail T arbitrary:T rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
  case (marked L l M) note IH = this(1)[of decr-bt-lvl (tl-trail T)] and M = this(2)[symmetric]
    and bt = this(3)
  then show ?case by (cases count C L = 0) auto
next
  case (proped L l M) note IH = this(1)[of tl-trail T] and M = this(2)[symmetric] and bt = this(3)
  then show ?case by (cases count C L = 0) auto
qed

lemma cut-trail-wrt-clause-CNot-trail:
  assumes trail T  $\models_{as}$  CNot C
  shows
    (trail ((cut-trail-wrt-clause C (trail T) T)))  $\models_{as}$  CNot C
  using assms
proof (induction trail T arbitrary:T rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
  case (marked L l M) note IH = this(1)[of decr-bt-lvl (tl-trail T)] and M = this(2)[symmetric]
    and bt = this(3)
  show ?case
    proof (cases count C (-L) = 0)

```

```

    case False
    then show ?thesis
      using IH M bt by (auto simp: true-annots-true-cls)
  next
    case True
    obtain mma :: 'v literal multiset where
      f6: (mma ∈ {{#- l#} | l. l ∈# C} → M ⊨a mma) → M ⊨as {{#- l#} | l. l ∈# C}
      using true-annots-def by blast
    have mma ∈ {{#- l#} | l. l ∈# C} → trail T ⊨a mma
      using CNot-def M bt by (metis (no-types) true-annots-def)
    then have M ⊨as {{#- l#} | l. l ∈# C}
      using f6 True M bt by (force simp: count-eq-zero-iff)
    then show ?thesis
      using IH true-annots-true-cls M by (auto simp: CNot-def)
  qed
next
case (proped L l M) note IH = this(1)[of tl-trail T] and M = this(2)[symmetric] and bt = this(3)
show ?case
proof (cases count C (-L) = 0)
  case False
  then show ?thesis
    using IH M bt by (auto simp: true-annots-true-cls)
  next
    case True
    obtain mma :: 'v literal multiset where
      f6: (mma ∈ {{#- l#} | l. l ∈# C} → M ⊨a mma) → M ⊨as {{#- l#} | l. l ∈# C}
      using true-annots-def by blast
    have mma ∈ {{#- l#} | l. l ∈# C} → trail T ⊨a mma
      using CNot-def M bt by (metis (no-types) true-annots-def)
    then have M ⊨as {{#- l#} | l. l ∈# C}
      using f6 True M bt by (force simp: count-eq-zero-iff)
    then show ?thesis
      using IH true-annots-true-cls M by (auto simp: CNot-def)
  qed
qed

lemma cut-trail-wrt-clause-hd-trail-in-or-empty-trail:
  ((∀ L ∈# C. -L ∉ lits-of-l (trail T)) ∧ trail (cut-trail-wrt-clause C (trail T) T) = [])
  ∨ (-lit-of (hd (trail (cut-trail-wrt-clause C (trail T) T))) ∈# C
    ∧ length (trail (cut-trail-wrt-clause C (trail T) T)) ≥ 1)
  using assms
proof (induction trail T arbitrary: T rule: marked-lit-list-induct)
  case nil
  then show ?case by simp
next
  case (marked L l M) note IH = this(1)[of decr-bt-lvl (tl-trail T)] and M = this(2)[symmetric]
  then show ?case by simp force
next
  case (proped L l M) note IH = this(1)[of tl-trail T] and M = this(2)[symmetric]
  then show ?case by simp force
qed

```

We can fully run $cdcl_W$ -s or add a clause. Remark that we use $cdcl_W$ -s to avoid an explicit *skip*, *resolve*, and *backtrack* normalisation to get rid of the conflict C if possible.

inductive $incremental-cdcl_W :: 'st \Rightarrow 'st \Rightarrow bool$ **for** S **where**

add-conflict:

$\text{trail } S \models_{\text{asm}} \text{init-clss } S \implies \text{distinct-mset } (\text{mset-ccls } C) \implies \text{conflicting } S = \text{None} \implies$
 $\text{trail } S \models_{\text{as}} \text{CNot } (\text{mset-ccls } C) \implies$
 $\text{full cdcl}_W\text{-stgy}$
 $(\text{update-conflicting } (\text{Some } C)$
 $(\text{add-init-cls } (\text{cls-of-ccls } C) (\text{cut-trail-wrt-clause } (\text{mset-ccls } C) (\text{trail } S) S))) T \implies$
 $\text{incremental-cdcl}_W S T \mid$

add-no-conflict:

$\text{trail } S \models_{\text{asm}} \text{init-clss } S \implies \text{distinct-mset } (\text{mset-ccls } C) \implies \text{conflicting } S = \text{None} \implies$
 $\neg \text{trail } S \models_{\text{as}} \text{CNot } (\text{mset-ccls } C) \implies$
 $\text{full cdcl}_W\text{-stgy } (\text{add-init-cls } (\text{cls-of-ccls } C) S) T \implies$
 $\text{incremental-cdcl}_W S T$

lemma *cdcl_W-all-struct-inv-add-new-clause-and-update-cdcl_W-all-struct-inv*:

assumes

$\text{inv-}T$: $\text{cdcl}_W\text{-all-struct-inv } T$ **and**
 $\text{tr-}T\text{-}N[\text{simp}]$: $\text{trail } T \models_{\text{asm}} N$ **and**
 $\text{tr-}C[\text{simp}]$: $\text{trail } T \models_{\text{as}} \text{CNot } (\text{mset-ccls } C)$ **and**
 $[\text{simp}]$: $\text{distinct-mset } (\text{mset-ccls } C)$

shows $\text{cdcl}_W\text{-all-struct-inv } (\text{add-new-clause-and-update } C T)$ (**is** $\text{cdcl}_W\text{-all-struct-inv } ?T'$)

proof –

let $?T = \text{update-conflicting } (\text{Some } C)$
 $(\text{add-init-cls } (\text{cls-of-ccls } C) (\text{cut-trail-wrt-clause } (\text{mset-ccls } C) (\text{trail } T) T))$

obtain M **where**

M : $\text{trail } T = M @ \text{trail } (\text{cut-trail-wrt-clause } (\text{mset-ccls } C) (\text{trail } T) T)$

using $\text{trail-cut-trail-wrt-clause}[of T \text{ mset-ccls } C]$ **by** *blast*

have $H[\text{dest}]$: $\bigwedge x. x \in \text{lits-of-l } (\text{trail } (\text{cut-trail-wrt-clause } (\text{mset-ccls } C) (\text{trail } T) T)) \implies$
 $x \in \text{lits-of-l } (\text{trail } T)$

using $\text{inv-}T \text{ arg-cong}[OF M, \text{ of lits-of-l}]$ **by** *auto*

have $H'[\text{dest}]$: $\bigwedge x. x \in \text{set } (\text{trail } (\text{cut-trail-wrt-clause } (\text{mset-ccls } C) (\text{trail } T) T)) \implies$
 $x \in \text{set } (\text{trail } T)$

using $\text{inv-}T \text{ arg-cong}[OF M, \text{ of set}]$ **by** *auto*

have $H\text{-proped}$: $\bigwedge x. x \in \text{set } (\text{get-all-mark-of-propagated } (\text{trail } (\text{cut-trail-wrt-clause } (\text{mset-ccls } C) (\text{trail } T) T))) \implies$
 $x \in \text{set } (\text{get-all-mark-of-propagated } (\text{trail } T))$

using $\text{inv-}T \text{ arg-cong}[OF M, \text{ of get-all-mark-of-propagated}]$ **by** *auto*

have $[\text{simp}]$: $\text{no-strange-atm } ?T$

using $\text{inv-}T$ **unfolding** $\text{cdcl}_W\text{-all-struct-inv-def no-strange-atm-def add-new-clause-and-update-def}$
 $\text{cdcl}_W\text{-}M\text{-level-inv-def}$ **by** $(\text{auto } 20 \ 1)$

have $M\text{-lev}$: $\text{cdcl}_W\text{-}M\text{-level-inv } T$

using $\text{inv-}T$ **unfolding** $\text{cdcl}_W\text{-all-struct-inv-def}$ **by** *blast*

then have $\text{no-dup } (M @ \text{trail } (\text{cut-trail-wrt-clause } (\text{mset-ccls } C) (\text{trail } T) T))$

unfolding $\text{cdcl}_W\text{-}M\text{-level-inv-def}$ **unfolding** $M[\text{symmetric}]$ **by** *auto*

then have $[\text{simp}]$: $\text{no-dup } (\text{trail } (\text{cut-trail-wrt-clause } (\text{mset-ccls } C) (\text{trail } T) T))$
by *auto*

have $\text{consistent-interp } (\text{lits-of-l } (M @ \text{trail } (\text{cut-trail-wrt-clause } (\text{mset-ccls } C) (\text{trail } T) T)))$

using $M\text{-lev}$ **unfolding** $\text{cdcl}_W\text{-}M\text{-level-inv-def}$ **unfolding** $M[\text{symmetric}]$ **by** *auto*

then have $[\text{simp}]$: $\text{consistent-interp } (\text{lits-of-l } (\text{trail } (\text{cut-trail-wrt-clause } (\text{mset-ccls } C) (\text{trail } T) T)))$

unfolding $\text{consistent-interp-def}$ **by** *auto*

have $[\text{simp}]$: $\text{cdcl}_W\text{-}M\text{-level-inv } ?T$

using $M\text{-lev cut-trail-wrt-clause-get-all-levels-of-marked}[of T \text{ mset-ccls } C]$

```

unfolding cdclW-M-level-inv-def by (auto dest: H H'
  simp: M-lev cdclW-M-level-inv-def cut-trail-wrt-clause-backtrack-lvl-length-marked)

have [simp]:  $\bigwedge s. s \in \# \text{ learned-clss } T \implies \neg \text{tautology } s$ 
  using inv-T unfolding cdclW-all-struct-inv-def by auto

have distinct-cdclW-state T
  using inv-T unfolding cdclW-all-struct-inv-def by auto
then have [simp]: distinct-cdclW-state ?T
  unfolding distinct-cdclW-state-def by auto

have cdclW-conflicting T
  using inv-T unfolding cdclW-all-struct-inv-def by auto
have trail ?T  $\models_{\text{as}}$  CNot (mset-ccls C)
  by (simp add: cut-trail-wrt-clause-CNot-trail)
then have [simp]: cdclW-conflicting ?T
  unfolding cdclW-conflicting-def apply simp
  by (metis M  $\langle$ cdclW-conflicting T $\rangle$  append-assoc cdclW-conflicting-decomp(2))

have
  decomp-T: all-decomposition-implies-m (init-clss T) (get-all-marked-decomposition (trail T))
  using inv-T unfolding cdclW-all-struct-inv-def by auto
have all-decomposition-implies-m (init-clss ?T)
  (get-all-marked-decomposition (trail ?T))
  unfolding all-decomposition-implies-def
  proof clarify
    fix a b
    assume  $(a, b) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } ?T))$ 
    from in-get-all-marked-decomposition-in-get-all-marked-decomposition-prepend[OF this, of M]
    obtain b' where
       $(a, b' @ b) \in \text{set } (\text{get-all-marked-decomposition } (\text{trail } T))$ 
      using M by auto
    then have unmark-l a  $\cup$  set-mset (init-clss T)  $\models_{\text{ps}}$  unmark-l (b' @ b)
      using decomp-T unfolding all-decomposition-implies-def by fastforce
    then have unmark-l a  $\cup$  set-mset (init-clss ?T)  $\models_{\text{ps}}$  unmark-l (b @ b')
      by (simp add: Un-commute)
    then show unmark-l a  $\cup$  set-mset (init-clss ?T)  $\models_{\text{ps}}$  unmark-l b
      by (auto simp: image-Un)
  qed

have [simp]: cdclW-learned-clause ?T
  using inv-T unfolding cdclW-all-struct-inv-def cdclW-learned-clause-def
  by (auto dest!: H-proped simp: raw-clauses-def)
show ?thesis
  using  $\langle \text{all-decomposition-implies-m } (\text{init-clss } ?T) \text{ (get-all-marked-decomposition (trail ?T))} \rangle$ 
  unfolding cdclW-all-struct-inv-def by (auto simp: add-new-clause-and-update-def)
qed

lemma cdclW-all-struct-inv-add-new-clause-and-update-cdclW-stgy-inv:
assumes
  inv-s: cdclW-stgy-invariant T and
  inv: cdclW-all-struct-inv T and
  tr-T-N[simp]: trail T  $\models_{\text{asm}}$  N and
  tr-C[simp]: trail T  $\models_{\text{as}}$  CNot (mset-ccls C) and

```

```

[simp]: distinct-mset (mset-ccls C)
shows cdclW-stgy-invariant (add-new-clause-and-update C T)
  (is cdclW-stgy-invariant ?T')
proof -
  have cdclW-all-struct-inv ?T'
    using cdclW-all-struct-inv-add-new-clause-and-update-cdclW-all-struct-inv assms by blast
  then have
    no-dup-cut-T[simp]: no-dup (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)) and
    n-d[simp]: no-dup (trail T)
    using cdclW-M-level-inv-decomp(2) cdclW-all-struct-inv-def inv
    n-dup-no-dup-trail-cut-trail-wrt-clause by blast+
  then have trail (add-new-clause-and-update C T)  $\models$ as CNot (mset-ccls C)
    by (simp add: add-new-clause-and-update-def cut-trail-wrt-clause-CNot-trail
      cdclW-M-level-inv-def cdclW-all-struct-inv-def)
  obtain MT where
    MT: trail T = MT @ trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)
    using trail-cut-trail-wrt-clause by blast
  consider
    (false)  $\forall L \in \#mset\text{-}ccls\ C. - L \notin \text{ lits-of-}l\ (\text{trail } T)$  and
    trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T) = []
  | (not-false)
    - lit-of (hd (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))  $\in \#(mset\text{-}ccls\ C)$  and
    1  $\leq$  length (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))
    using cut-trail-wrt-clause-hd-trail-in-or-empty-trail[of mset-ccls C T] by auto
  then show ?thesis
  proof cases
    case false note C = this(1) and empty-tr = this(2)
    then have [simp]: mset-ccls C = {#}
      by (simp add: in-CNot-implies-uminus(2) multiset-eqI)
    show ?thesis
      using empty-tr unfolding cdclW-stgy-invariant-def no-smaller-conflict-def
      cdclW-all-struct-inv-def by (auto simp: add-new-clause-and-update-def)
  next
    case not-false note C = this(1) and l = this(2)
    let ?L = - lit-of (hd (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
    have get-all-levels-of-marked (trail (add-new-clause-and-update C T)) =
      rev [1.. $1 + \text{length}(\text{get-all-levels-of-marked}(\text{trail}(\text{add-new-clause-and-update } C\ T)))$ ]
      using  $\langle \text{cdcl}_W\text{-all-struct-inv } ?T' \rangle$  unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def
      by blast
    moreover
      have backtrack-lvl (cut-trail-wrt-clause (mset-ccls C) (trail T) T) =
        length (get-all-levels-of-marked (trail (add-new-clause-and-update C T)))
        using  $\langle \text{cdcl}_W\text{-all-struct-inv } ?T' \rangle$  unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def
        by (auto simp: add-new-clause-and-update-def)
    moreover
      have no-dup (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))
        using  $\langle \text{cdcl}_W\text{-all-struct-inv } ?T' \rangle$  unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def
        by (auto simp: add-new-clause-and-update-def)
      then have atm-of ?L  $\notin$  atm-of ' lits-of-l
        (tl (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
        by (cases trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))
        (auto simp: lits-of-def)
  ultimately have L: get-level (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)) (-?L)
    = length (get-all-levels-of-marked (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))

```

```

using get-level-get-rev-level-get-all-levels-of-marked[OF
  ‹atm-of ?L ∉ atm-of ‘ lits-of-l (tl (trail (cut-trail-wrt-clause (mset-ccls C)
    (trail T) T)))›,
  of [hd (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))]]

apply (cases trail (add-init-cls (cls-of-ccls C)
  (cut-trail-wrt-clause (mset-ccls C) (trail T) T));
  cases hd (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
using l by (auto split: if-split-asm
  simp: rev-swap[symmetric] add-new-clause-and-update-def)

have L': length (get-all-levels-of-marked (trail (cut-trail-wrt-clause (mset-ccls C)
  (trail T) T)))
  = backtrack-lvl (cut-trail-wrt-clause (mset-ccls C) (trail T) T)
using ‹cdclW-all-struct-inv ?T'› unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def
by (auto simp: add-new-clause-and-update-def)

have [simp]: no-smaller-confl (update-conflicting (Some C)
  (add-init-cls (cls-of-ccls C) (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
unfolding no-smaller-confl-def
proof (clarify, goal-cases)
  case (1 M K i M' D)
  then consider
    (DC) D = mset-ccls C
    | (D-T) D ∈ # clauses T
  by (auto simp: raw-clauses-def split: if-split-asm)
then show False
proof cases
  case D-T
  have no-smaller-confl T
    using inv-s unfolding cdclW-stgy-invariant-def by auto
  have (MT @ M') @ Marked K i # M = trail T
    using MT 1(1) by auto
  thus False using D-T ‹no-smaller-confl T› 1(3) unfolding no-smaller-confl-def by blast
next
  case DC note -[simp] = this
  then have atm-of (−?L) ∈ atm-of ‘ (lits-of-l M)
    using 1(3) C in-CNot-implies-uminus(2) by blast
  moreover
    have lit-of (hd (M' @ Marked K i # [])) = −?L
      using l 1(1)[symmetric] inv
    by (cases trail (add-init-cls (cls-of-ccls C)
      (cut-trail-wrt-clause (mset-ccls C) (trail T) T)))
      (auto dest!: arg-cong[of - # - hd] simp: hd-append cdclW-all-struct-inv-def
        cdclW-M-level-inv-def)
    from arg-cong[OF this, of atm-of]
    have atm-of (−?L) ∈ atm-of ‘ (lits-of-l (M' @ Marked K i # []))
      by (cases (M' @ Marked K i # [])) auto
  moreover have no-dup (trail (cut-trail-wrt-clause (mset-ccls C) (trail T) T))
    using ‹cdclW-all-struct-inv ?T'› unfolding cdclW-all-struct-inv-def
    cdclW-M-level-inv-def by (auto simp: add-new-clause-and-update-def)
  ultimately show False
    unfolding 1(1)[symmetric, simplified] by (auto simp: lits-of-def)
qed
qed

```



```

show ?thesis using L L' C
  unfolding cdclW-stgy-invariant-def
  unfolding cdclW-all-struct-inv-def by (auto simp: add-new-clause-and-update-def)
qed
qed

lemma full-cdclW-stgy-inv-normal-form:
assumes
  full: full cdclW-stgy S T and
  inv-s: cdclW-stgy-invariant S and
  inv: cdclW-all-struct-inv S
shows conflicting T = Some {#}  $\wedge$  unsatisfiable (set-mset (init-clss S))
   $\vee$  conflicting T = None  $\wedge$  trail T  $\models_{asm}$  init-clss S  $\wedge$  satisfiable (set-mset (init-clss S))
proof -
have no-step cdclW-stgy T
  using full unfolding full-def by blast
moreover have cdclW-all-struct-inv T and inv-s: cdclW-stgy-invariant T
apply (metis rtrancpl-cdclW-stgy-rtrancpl-cdclW full full-def inv
  rtrancpl-cdclW-all-struct-inv-inv)
by (metis full full-def inv inv-s rtrancpl-cdclW-stgy-cdclW-stgy-invariant)
ultimately have conflicting T = Some {#}  $\wedge$  unsatisfiable (set-mset (init-clss T))
   $\vee$  conflicting T = None  $\wedge$  trail T  $\models_{asm}$  init-clss T
using cdclW-stgy-final-state-conclusive[of T] full
unfolding cdclW-all-struct-inv-def cdclW-stgy-invariant-def full-def by fast
moreover have consistent-interp (lits-of-l (trail T))
using  $\langle$ cdclW-all-struct-inv T $\rangle$  unfolding cdclW-all-struct-inv-def cdclW-M-level-inv-def
by auto
moreover have init-clss S = init-clss T
using inv unfolding cdclW-all-struct-inv-def
by (metis rtrancpl-cdclW-stgy-no-more-init-clss full full-def)
ultimately show ?thesis
by (metis satisfiable-carac' true-annot-def true-annots-def true-clss-def)
qed

```

```

lemma incremental-cdclW-inv:
assumes
  inc: incremental-cdclW S T and
  inv: cdclW-all-struct-inv S and
  s-inv: cdclW-stgy-invariant S
shows
  cdclW-all-struct-inv T and
  cdclW-stgy-invariant T
using inc
proof (induction)
case (add-confl C T)
let ?T = (update-conflicting (Some C) (add-init-cls (cls-of-ccls C)
  (cut-trail-wrt-clause (mset-ccls C) (trail S) S)))
have cdclW-all-struct-inv ?T and inv-s-T: cdclW-stgy-invariant ?T
using add-confl.hyps(1,2,4) add-new-clause-and-update-def
  cdclW-all-struct-inv-add-new-clause-and-update-cdclW-all-struct-inv inv apply auto[1]
using add-confl.hyps(1,2,4) add-new-clause-and-update-def
  cdclW-all-struct-inv-add-new-clause-and-update-cdclW-stgy-inv inv s-inv by auto
case 1 show ?case
by (metis add-confl.hyps(1,2,4,5) add-new-clause-and-update-def
  cdclW-all-struct-inv-add-new-clause-and-update-cdclW-all-struct-inv)

```

```

    rtrancpl-cdclW-all-struct-inv-inv rtrancpl-cdclW-stgy-rtrancpl-cdclW full-def inv)

case 2 show ?case
  by (metis inv-s-T add-confl.hyps(1,2,4,5) add-new-clause-and-update-def
    cdclW-all-struct-inv-add-new-clause-and-update-cdclW-all-struct-inv full-def inv
    rtrancpl-cdclW-stgy-cdclW-stgy-invariant)
next
case (add-no-confl C T)
case 1
have cdclW-all-struct-inv (add-init-cls (cls-of-ccls C) S)
  using inv <distinct-mset (mset-ccls C)> unfolding cdclW-all-struct-inv-def no-strange-atm-def
  cdclW-M-level-inv-def distinct-cdclW-state-def cdclW-conflicting-def cdclW-learned-clause-def
  by (auto 9 1 simp: all-decomposition-implies-insert-single raw-clauses-def)

then show ?case
  using add-no-confl(5) unfolding full-def by (auto intro: rtrancpl-cdclW-stgy-cdclW-all-struct-inv)
case 2
have nc:  $\forall M. (\exists K i M'. \text{trail } S = M' @ \text{Marked } K i \# M) \longrightarrow \neg M \models_{\text{as}} \text{CNot } (\text{mset-ccls } C)$ 
  using < $\neg \text{trail } S \models_{\text{as}} \text{CNot } (\text{mset-ccls } C)$ >
  by (auto simp: true-annots-true-cls-def-iff-negation-in-model)

have cdclW-stgy-invariant (add-init-cls (cls-of-ccls C) S)
  using s-inv < $\neg \text{trail } S \models_{\text{as}} \text{CNot } (\text{mset-ccls } C)$ > inv unfolding cdclW-stgy-invariant-def
  no-smaller-confl-def eq-commute[of - trail -] cdclW-M-level-inv-def cdclW-all-struct-inv-def
  by (auto simp: raw-clauses-def nc)
then show ?case
  by (metis <cdclW-all-struct-inv (add-init-cls (cls-of-ccls C) S)> add-no-confl.hyps(5) full-def
    rtrancpl-cdclW-stgy-cdclW-stgy-invariant)
qed

lemma rtrancpl-incremental-cdclW-inv:
assumes
  inc: incremental-cdclW** S T and
  inv: cdclW-all-struct-inv S and
  s-inv: cdclW-stgy-invariant S
shows
  cdclW-all-struct-inv T and
  cdclW-stgy-invariant T
  using inc apply induction
  using inv apply simp
  using s-inv apply simp
using incremental-cdclW-inv by blast+

lemma incremental-conclusive-state:
assumes
  inc: incremental-cdclW S T and
  inv: cdclW-all-struct-inv S and
  s-inv: cdclW-stgy-invariant S
shows conflicting T = Some {#}  $\wedge$  unsatisfiable (set-mset (init-cls T))
   $\vee$  conflicting T = None  $\wedge$  trail T  $\models_{\text{asm}}$  init-cls T  $\wedge$  satisfiable (set-mset (init-cls T))
using inc
proof induction
print-cases
case (add-confl C T) note tr = this(1) and dist = this(2) and conf = this(3) and C = this(4) and
  full = this(5)

```

```

have full cdclW-stgy T T
  using full unfolding full-def by auto
then show ?case
  using full C conf dist tr
  by (metis full-cdclW-stgy-inv-normal-form incremental-cdclW.simps incremental-cdclW-inv(1)
      incremental-cdclW-inv(2) inv s-inv)
next
case (add-no-conf C T) note tr = this(1) and dist = this(2) and conf = this(3) and C = this(4)
  and full = this(5)

have full cdclW-stgy T T
  using full unfolding full-def by auto
then show ?case
  by (meson C conf dist full full-cdclW-stgy-inv-normal-form incremental-cdclW.add-no-conf
      incremental-cdclW-inv(1) incremental-cdclW-inv(2) inv s-inv tr)
qed

lemma tranclp-incremental-correct:
  assumes
    inc: incremental-cdclW++ S T and
    inv: cdclW-all-struct-inv S and
    s-inv: cdclW-stgy-invariant S
  shows conflicting T = Some {#} ∧ unsatisfiable (set-mset (init-clss T))
    ∨ conflicting T = None ∧ trail T ⊨asm init-clss T ∧ satisfiable (set-mset (init-clss T))
  using inc apply induction
  using assms incremental-conclusive-state apply blast
  by (meson incremental-conclusive-state inv rtranclp-incremental-cdclW-inv s-inv
      tranclp-into-rtranclp)

end

end

```

23 2-Watched-Literal

```

theory CDCL-Two-Watched-Literals
imports CDCL-WNOT
begin

```

We will directly on the two-watched literals datastructure with lists: it could be also seen as a state over some abstract clause representation we would later refine as lists. However, as we need a way to select element from a clause, working on lists is better.

23.1 Datastructure and Access Functions

Only the 2-watched literals have to be verified here: the backtrack level and the trail that appear in the state are not related to the 2-watched algorithm.

```

datatype 'v twl-clause =
  TWL-Clause (watched: 'v literal list) (unwatched: 'v literal list)

datatype 'v twl-state =
  TWL-State (raw-trail: ('v, nat, 'v twl-clause) marked-lit list)
    (raw-init-clss: 'v twl-clause list)

```

(*raw-learned-clss*: 'v twl-clause list) (*backtrack-lvl*: nat)
 (*raw-conflicting*: 'v literal list option)

fun *mmset-of-mlit'* :: ('v, nat, 'v twl-clause) marked-lit \Rightarrow ('v, nat, 'v clause) marked-lit
where
mmset-of-mlit' (*Propagated* L C) = *Propagated* L (mset (watched C @ unwatched C)) |
mmset-of-mlit' (*Marked* L i) = *Marked* L i

lemma *lit-of-mmset-of-mlit'[simp]*: *lit-of* (*mmset-of-mlit'* x) = *lit-of* x
by (*cases* x) *auto*

lemma *lits-of-mmset-of-mlit'[simp]*: *lits-of* (*mmset-of-mlit'* ' S) = *lits-of* S
by (*auto simp: lits-of-def image-image*)

abbreviation *trail* **where**
trail S \equiv map *mmset-of-mlit'* (*raw-trail* S)

abbreviation *clauses-of-l* **where**
clauses-of-l \equiv $\lambda L. \text{mset } (\text{map } \text{mset } L)$

definition *raw-clause* :: 'v twl-clause \Rightarrow 'v literal list **where**
raw-clause C \equiv watched C @ unwatched C

abbreviation *raw-clss* :: 'v twl-state \Rightarrow 'v clauses **where**
raw-clss S \equiv *clauses-of-l* (map *raw-clause* (*raw-init-clss* S @ *raw-learned-clss* S))

interpretation *raw-cl*
 $\lambda C. \text{mset } (\text{raw-clause } C)$
 $\lambda L \ C. \text{TWL-Clause } (\text{watched } C) (L \# \text{unwatched } C)$
 $\lambda L \ C. \text{TWL-Clause } [] (\text{remove1 } L (\text{raw-clause } C))$
apply (*unfold-locales*)
by (*auto simp:hd-map comp-def map-tl ac-simps*
mset-map-mset-remove1-cond ex-mset raw-clause-def
simp del:)

lemma XXX:
 $\text{mset } (\text{map } (\lambda x. \text{mset } (\text{unwatched } x) + \text{mset } (\text{watched } x))$
 $(\text{remove1-cond } (\lambda D. \text{mset } (\text{raw-clause } D) = \text{mset } (\text{raw-clause } a)) \text{ Cs})) =$
 $\text{remove1-mset } (\text{mset } (\text{raw-clause } a)) (\text{mset } (\text{map } (\lambda x. \text{mset } (\text{raw-clause } x)) \text{ Cs}))$
apply (*induction* Cs)
apply *simp*
by (*auto simp: ac-simps remove1-mset-single-add raw-clause-def*)

interpretation *raw-clss*
 $\lambda C. \text{mset } (\text{raw-clause } C)$
 $\lambda L \ C. \text{TWL-Clause } (\text{watched } C) (L \# \text{unwatched } C)$
 $\lambda L \ C. \text{TWL-Clause } [] (\text{remove1 } L (\text{raw-clause } C))$
 $\lambda C. \text{clauses-of-l } (\text{map } \text{raw-clause } C) \text{ op } @$
 $\lambda L \ C. L \in \text{set } C \text{ op } \# \lambda C. \text{remove1-cond } (\lambda D. \text{mset } (\text{raw-clause } D) = \text{mset } (\text{raw-clause } C))$
apply (*unfold-locales*)
using XXX **by** (*auto simp:hd-map comp-def map-tl ac-simps raw-clause-def*
union-mset-list mset-map-mset-remove1-cond ex-mset
simp del:)

lemma *ex-mset-unwatched-watched*:
 $\exists a. \text{mset } (\text{unwatched } a) + \text{mset } (\text{watched } a) = E$

proof –
obtain *e* **where** $\text{mset } e = E$
using *ex-mset* **by** *blast*
then have $\text{mset } (\text{unwatched } (\text{TWL-Clause } [] e)) + \text{mset } (\text{watched } (\text{TWL-Clause } [] e)) = E$
by *auto*
then show *?thesis* **by** *fast*
qed

thm *CDCL-Two-Watched-Literals.raw-cls-axioms*

interpretation *twl: state_W-ops*
 $\lambda C. \text{mset } (\text{raw-clause } C)$

$\lambda L C. \text{TWL-Clause } (\text{watched } C) (L \# \text{unwatched } C)$
 $\lambda L C. \text{TWL-Clause } [] (\text{remove1 } L (\text{raw-clause } C))$
 $\lambda C. \text{clauses-of-l } (\text{map raw-clause } C) \text{ op } @$
 $\lambda L C. L \in \text{set } C \text{ op } \# \lambda C. \text{remove1-cond } (\lambda D. \text{mset } (\text{raw-clause } D) = \text{mset } (\text{raw-clause } C))$

$\text{mset } \lambda xs \ ys. \text{case-prod append } (\text{fold } (\lambda x \ (ys, zs). (\text{remove1 } x \ ys, x \# \ zs)) \ xs \ (ys, []))$
 $\text{op } \# \text{remove1}$

raw-clause $\lambda C. \text{TWL-Clause } [] C$
trail $\lambda S. \text{hd } (\text{raw-trail } S)$
raw-init-clss *raw-learned-clss* *backtrack-lvl* *raw-conflicting*

apply *unfold-locales* **apply** (*auto simp: hd-map comp-def map-tl ac-simps raw-clause-def*
union-mset-list mset-map-mset-remove1-cond ex-mset-unwatched-watched)
done

declare *CDCL-Two-Watched-Literals.twl.mset-ccls-ccls-of-cls[simp del]*

lemma *mmset-of-mlit'-mmset-of-mlit[simp]*:
 $\text{twl.mmset-of-mlit } L = \text{mmset-of-mlit}' L$
by (*metis mmset-of-mlit'.simps(1) mmset-of-mlit'.simps(2) twl.mmset-of-mlit.elims raw-clause-def*)

definition
candidates-propagate $:: 'v \text{ twl-state} \Rightarrow ('v \text{ literal} \times 'v \text{ twl-clause}) \text{ set}$
where
candidates-propagate $S =$
 $\{(L, C) \mid L C. \text{C} \in \text{set } (\text{twl.raw-clauses } S) \wedge$
 $\text{set } (\text{watched } C) - (\text{uminus } ' \text{ lits-of-l } (\text{trail } S)) = \{L\} \wedge$
 $\text{undefined-lit } (\text{trail } S) L\}$

definition *candidates-conflict* $:: 'v \text{ twl-state} \Rightarrow 'v \text{ twl-clause set}$ **where**
candidates-conflict $S =$
 $\{C. C \in \text{set } (\text{twl.raw-clauses } S) \wedge$
 $\text{set } (\text{watched } C) \subseteq \text{uminus } ' \text{ lits-of-l } (\text{trail } S)\}$

primrec (*nonexhaustive*) *index* $:: 'a \text{ list} \Rightarrow 'a \Rightarrow \text{nat}$ **where**
 $\text{index } (a \# l) \ c = (\text{if } a = c \text{ then } 0 \text{ else } 1 + \text{index } l \ c)$

lemma *index-nth*:
 $a \in \text{set } l \Longrightarrow l ! (\text{index } l \ a) = a$
by (*induction l*) *auto*

23.2 Invariants

We need the following property about updates: if there is a literal L with $-L$ in the trail, and L is not watched, then it stays unwatched; i.e., while updating with *rewatch* it does not get swap with a watched literal L' such that $-L'$ is in the trail.

```

primrec watched-decided-most-recently :: ('v, 'wl, 'mark) marked-lit list  $\Rightarrow$ 
  'v twl-clause  $\Rightarrow$  bool
where
  watched-decided-most-recently M (TWL-Clause W UW)  $\longleftrightarrow$ 
    ( $\forall L' \in \text{set } W. \forall L \in \text{set } UW.
      -L' \in \text{lits-of-l } M \longrightarrow -L \in \text{lits-of-l } M \longrightarrow L \notin \# \text{ mset } W \longrightarrow
      \text{index (map lit-of } M) (-L') \leq \text{index (map lit-of } M) (-L)$ )

```

Here are the invariant strictly related to the 2-WL data structure.

```

primrec wf-tw-cl-cls :: ('v, 'wl, 'mark) marked-lit list  $\Rightarrow$  'v twl-clause  $\Rightarrow$  bool where
  wf-tw-cl-cls M (TWL-Clause W UW)  $\longleftrightarrow$ 
    distinct W  $\wedge$  length W  $\leq 2 \wedge$  (length W  $< 2 \longrightarrow \text{set } UW \subseteq \text{set } W$ )  $\wedge$ 
    ( $\forall L \in \text{set } W. -L \in \text{lits-of-l } M \longrightarrow (\forall L' \in \text{set } UW. L' \notin \text{set } W \longrightarrow -L' \in \text{lits-of-l } M)$ )  $\wedge$ 
    watched-decided-most-recently M (TWL-Clause W UW)

```

```

lemma size-mset-2: size x1 = 2  $\longleftrightarrow$  ( $\exists a b. x1 = \{\#a, b\# \}$ )
apply (cases x1)
apply simp
by (metis (no-types, hide-lams) Suc-eq-plus1 one-add-one size-1-singleton-mset
  size-Diff-singleton size-Suc-Diff1 size-single union-single-eq-diff union-single-eq-member)

```

```

lemma distinct-mset-size-2: distinct-mset  $\{\#a, b\# \} \longleftrightarrow a \neq b$ 
unfolding distinct-mset-def by auto

```

```

lemma wf-tw-cl-cls-annotation-indepndant:
  assumes M: map lit-of M = map lit-of M'
  shows wf-tw-cl-cls M (TWL-Clause W UW)  $\longleftrightarrow$  wf-tw-cl-cls M' (TWL-Clause W UW)
proof -
  have lits-of-l M = lits-of-l M'
    using arg-cong[OF M, of set] by (simp add: lits-of-def)
  then show ?thesis
    by (simp add: lits-of-def M)
qed

```

```

lemma wf-tw-cl-cls-wf-tw-cl-cls-tl:
  assumes wf: wf-tw-cl-cls M C and n-d: no-dup M
  shows wf-tw-cl-cls (tl M) C
proof (cases M)
  case Nil
  then show ?thesis using wf
    by (cases C) (simp add: wf-tw-cl-cls.simps[of tl -])
next
  case (Cons l M') note M = this(1)
  obtain W UW where C: C = TWL-Clause W UW
    by (cases C)
  { fix L L'
    assume
      LW: L  $\in$  set W and
      LM:  $-L \in$  lits-of-l M' and
      L'UW: L'  $\in$  set UW and

```

```

  L' ∉ set W
then have
  L'M: - L' ∈ lits-of-l M
  using wf by (auto simp: C M)
have watched-decided-most-recently M C
  using wf by (auto simp: C)
then have
  index (map lit-of M) (-L) ≤ index (map lit-of M) (-L')
  using LM L'M L'UW LW ⟨L' ∉ set W⟩ C M unfolding lits-of-def
  by (fastforce simp: lits-of-def)
then have - L' ∈ lits-of-l M'
  using ⟨L' ∉ set W⟩ LW L'M by (auto simp: C M split: if-split-asm)
}
moreover
{
  fix L' L
  assume
    L' ∈ set W and
    L ∈ set UW and
    L'M: - L' ∈ lits-of-l M' and
    - L ∈ lits-of-l M' and
    L ∉ set W
  moreover
    have lit-of l ≠ - L'
    using n-d unfolding M
      by (metis (no-types) L'M M Marked-Propagated-in-iff-in-lits-of-l defined-lit-map
        distinct.simps(2) list.simps(9) set-map)
  moreover have watched-decided-most-recently M C
    using wf by (auto simp: C)
  ultimately have index (map lit-of M') (- L') ≤ index (map lit-of M') (- L)
    by (fastforce simp: M C split: if-split-asm)
}
moreover have distinct W and length W ≤ 2 and (length W < 2 → set UW ⊆ set W)
  using wf by (auto simp: C M)
ultimately show ?thesis by (auto simp add: M C)
qed

```

definition *wf-twl-state* :: 'v twl-state ⇒ bool **where**
wf-twl-state S ⇔ (∀ C ∈ set (twl.raw-clauses S). *wf-twl-cl* (trail S) C) ∧ no-dup (trail S)

lemma *length-list-Suc-0*:
 length W = Suc 0 ⇔ (∃ L. W = [L])
apply (cases W)
apply simp
apply (rename-tac a W', case-tac W')
apply auto
done

lemma *wf-candidates-propagate-sound*:
assumes wf: *wf-twl-state* S **and**
 cand: (L, C) ∈ candidates-propagate S
shows trail S ⊨_{as} CNot (mset (removeAll L (raw-clause C))) ∧ undefined-lit (trail S) L
 (is ?Not ∧ ?undef)
proof
 def M ≡ trail S

```

def  $N \equiv \text{raw-init-clss } S$ 
def  $U \equiv \text{raw-learned-clss } S$ 

note  $MNU\text{-defs } [simp] = M\text{-def } N\text{-def } U\text{-def}$ 

have  $cw$ :
   $C \in \text{set } (N @ U)$ 
   $\text{set } (\text{watched } C) - \text{uminus ' lits-of-l } M = \{L\}$ 
   $\text{undefined-lit } M L$ 
  using cand unfolding candidates-propagate-def MNU-defs twl.raw-clauses-def by auto

obtain  $W UW$  where  $cw\text{-eq}: C = \text{TWL-Clause } W UW$ 
  by (cases C)

have  $l\text{-}w: L \in \text{set } W$ 
  using  $cw(2)$   $cw\text{-eq}$  by auto

have  $wf\text{-}c: wf\text{-}twl\text{-cls } M C$ 
  using  $wf\ cw(1)$  unfolding  $wf\text{-}twl\text{-state-def}$  by (simp add: twl.raw-clauses-def)

have  $w\text{-}nw$ :
   $\text{distinct } W$ 
   $\text{length } W < 2 \implies \text{set } UW \subseteq \text{set } W$ 
   $\bigwedge L L'. L \in \text{set } W \implies -L \in \text{lits-of-l } M \implies L' \in \text{set } UW \implies L' \notin \text{set } W \implies -L' \in \text{lits-of-l } M$ 
  using  $wf\text{-}c$  unfolding  $cw\text{-eq}$  by (auto simp: image-image)

have  $\forall L' \in \text{set } (\text{raw-clause } C) - \{L\}. -L' \in \text{lits-of-l } M$ 
proof (cases length W < 2)
  case True
    moreover have  $\text{size } W \neq 0$ 
      using  $cw(2)$   $cw\text{-eq}$  by auto
    ultimately have  $\text{size } W = 1$ 
      by linarith
    then have  $w: W = [L]$ 
      using  $l\text{-}w$  by (auto simp: length-list-Suc-0)
    from True have  $\text{set } UW \subseteq \text{set } W$ 
      using  $w\text{-}nw(2)$  by blast
    then show ?thesis
      using  $w\ cw(1)$   $cw\text{-eq}$  by (auto simp: raw-clause-def)
  next
    case  $sz2: \text{False}$ 
    show ?thesis
  proof
    fix  $L'$ 
    assume  $l': L' \in \text{set } (\text{raw-clause } C) - \{L\}$ 
    have  $ex\text{-}la: \exists La. La \neq L \wedge La \in \text{set } W$ 
    proof (cases W)
      case  $w: \text{Nil}$ 
      thus ?thesis
      using  $l\text{-}w$  by auto
    next
      case  $lb: (\text{Cons } Lb\ W')$ 
      show ?thesis
      proof (cases W')
        case Nil

```



```

    thus ?thesis
      using lb sz2 by simp
  next
    case lc: (Cons Lc W'')
    thus ?thesis
      by (metis distinct-length-2-or-more lb list.set-intros(1) list.set-intros(2) w-nw(1))
  qed
qed
then obtain La where la: La ≠ L La ∈ set W
  by blast
then have La ∈ uminus ' lits-of-l M
  using cw(2)[unfolded cw-eq, simplified, folded M-def] ⟨La ∈ set W⟩ ⟨La ≠ L⟩ by auto
then have nla: ¬La ∈ lits-of-l M
  by (auto simp: image-image)
then show ¬L' ∈ lits-of-l M

proof -
  have f1: L' ∈ set (raw-clause C)
    using l' by blast
  have f2: L' ∉ {L}
    using l' by fastforce
  have ∧l L. ¬ (l::'a literal) ∈ L ∨ l ∉ uminus ' L
    by force
  then show ?thesis
    using cw(1) cw-eq w-nw(3) raw-clause-def by (metis DiffI Un-iff cw(2) f1 f2 la(2) nla
      set-append twl-clause.sel(1) twl-clause.sel(2))
  qed
qed
qed
then show ?Not
  unfolding true-annots-def by (auto simp: image-image Ball-def CNot-def)

show ?undef
  using cw(3) M-def by blast
qed

lemma wf-candidates-propagate-complete:
  assumes wf: wf-twl-state S and
    c-mem: C ∈ set (twl.raw-clauses S) and
    l-mem: L ∈ set (raw-clause C) and
    unsat: trail S ⊨as CNot (mset-set (set (raw-clause C) - {L})) and
    undef: undefined-lit (trail S) L
  shows (L, C) ∈ candidates-propagate S
proof -
  def M ≡ trail S
  def N ≡ raw-init-clss S
  def U ≡ raw-learned-clss S

  note MNU-defs [simp] = M-def N-def U-def

  obtain W UW where cw-eq: C = TWL-Clause W UW
    by (cases C, blast)

  have wf-c: wf-twl-clss M C
    using wf c-mem unfolding wf-twl-state-def by simp

```

```

have w-nw:
  distinct W
  length W < 2  $\implies$  set UW  $\subseteq$  set W
   $\bigwedge L L'. L \in \text{set } W \implies -L \in \text{lits-of-l } M \implies L' \in \text{set } UW \implies L' \notin \text{set } W \implies -L' \in \text{lits-of-l } M$ 
  using wf-c unfolding cw-eq by (auto simp: image-image)

have unit-set: set W  $-(\text{uminus } \text{'lits-of-l } M) = \{L\}$  (is ?W = ?L)
proof
  show ?W  $\subseteq \{L\}$ 
  proof
    fix L'
    assume l': L'  $\in$  ?W
    hence l'-mem-w: L'  $\in$  set W
      by (simp add: in-diffD)
    have L'  $\notin$  uminus 'lits-of-l M
      using l' by blast
    then have  $\neg M \models_a \{\#-L'\#$ 
      by (auto simp: lits-of-def uminus-lit-swap image-image)
    moreover have L'  $\in$  set (raw-clause C)
      using c-mem cw-eq l'-mem-w by (auto simp: raw-clause-def)
    ultimately have L' = L
      using unsat[unfolded CNot-def true-annots-def, simplified]
      unfolding M-def by fastforce
    then show L'  $\in \{L\}$ 
      by simp
  qed
next
show  $\{L\} \subseteq ?W$ 
proof clarify
  have L  $\in$  set W
  proof (cases W)
    case Nil
    thus ?thesis
      using w-nw(2) cw-eq l-mem by (auto simp: raw-clause-def)
  next
    case (Cons La W')
    thus ?thesis
      proof (cases La = L)
        case True
        thus ?thesis
          using Cons by simp
      next
        case False
        have  $-La \in \text{lits-of-l } M$ 
          using False Cons cw-eq unsat[unfolded CNot-def true-annots-def, simplified]
          by (fastforce simp: raw-clause-def)
        then show ?thesis
          using Cons cw-eq l-mem undef w-nw(3)
          by (auto simp: Marked-Propagated-in-iff-in-lits-of-l raw-clause-def)
      qed
  qed
  moreover have L  $\notin \#$  mset-set (uminus 'lits-of-l M)
    using undef by (auto simp: Marked-Propagated-in-iff-in-lits-of-l image-image)
  ultimately show L  $\in$  ?W

```

```

    by simp
  qed
qed

show ?thesis
  unfolding candidates-propagate-def using unit-set undef c-mem unfolding cw-eq M-def
  by (auto simp: image-image cw-eq intro!: exI[of - C])
qed

lemma wf-candidates-conflict-sound:
  assumes wf: wf-twl-state S and
    cand: C ∈ candidates-conflict S
  shows trail S ⊨as CNot (mset (raw-clause C)) ∧ C ∈ set (twl.raw-clauses S)
proof
  def M ≡ trail S
  def N ≡ raw-init-clss S
  def U ≡ raw-learned-clss S

  note MNU-defs [simp] = M-def N-def U-def

  have cw:
    C ∈ set (N @ U)
    set (watched C) ⊆ uminus ' lits-of-l (trail S)
    using cand[unfolded candidates-conflict-def, simplified] unfolding twl.raw-clauses-def by auto

  obtain W UW where cw-eq: C = TWL-Clause W UW
    by (cases C, blast)

  have wf-c: wf-twl-clss M C
    using wf cw(1) unfolding wf-twl-state-def by (simp add: comp-def twl.raw-clauses-def)

  have w-nw:
    distinct W
    length W < 2 ⟹ set UW ⊆ set W
    ∧ L L'. L ∈ set W ⟹ ¬L ∈ lits-of-l M ⟹ L' ∈ set UW ⟹ L' ∉ set W ⟹ ¬L' ∈ lits-of-l M
    using wf-c unfolding cw-eq by (auto simp: image-image)

  have ∀ L ∈ set (raw-clause C). ¬L ∈ lits-of-l M
  proof (cases W)
    case Nil
    then have raw-clause C = []
      using cw(1) cw-eq w-nw(2) by (auto simp: raw-clause-def)
    then show ?thesis
      by simp
  next
    case (Cons La W') note W' = this(1)
    show ?thesis
    proof
      fix L
      assume l: L ∈ set (raw-clause C)
      show ¬L ∈ lits-of-l M
      proof (cases L ∈ set W)
        case True
        thus ?thesis
          using cw(2) cw-eq by fastforce
      end
    end
  end

```

```

next
  case False
  thus ?thesis
    by (metis (no-types, hide-lams) M-def UnE W' contra-subsetD cw(2) cw-eq imageE
        insertI1 l list.set(2) set-append twl-clause.sel(1) twl-clause.sel(2)
        uminus-of-uminus-id w-nw(3) raw-clause-def)
qed
qed
qed
then show trail S  $\models_{as}$  CNot (mset (raw-clause C))
  unfolding CNot-def true-annots-def by auto

show C  $\in$  set (twl.raw-clauses S)
  using cw unfolding twl.raw-clauses-def by auto
qed

lemma wf-candidates-conflict-complete:
  assumes wf: wf-twl-state S and
    c-mem: C  $\in$  set (twl.raw-clauses S) and
    unsat: trail S  $\models_{as}$  CNot (mset (raw-clause C))
  shows C  $\in$  candidates-conflict S
proof -
  def M  $\equiv$  trail S
  def N  $\equiv$  twl.init-clss S
  def U  $\equiv$  twl.learned-clss S

  note MNU-defs [simp] = M-def N-def U-def

  obtain W UW where cw-eq: C = TWL-Clause W UW
    by (cases C, blast)

  have wf-c: wf-twl-clss M C
    using wf c-mem unfolding wf-twl-state-def by simp

  have w-nw:
    distinct W
    length W < 2  $\implies$  set UW  $\subseteq$  set W
     $\bigwedge L L'. L \in$  set W  $\implies -L \in$  lits-of-l M  $\implies L' \in$  set UW  $\implies L' \notin$  set W  $\implies -L' \in$  lits-of-l M
    using wf-c unfolding cw-eq by (auto simp: image-image)

  have  $\bigwedge L. L \in$  set (raw-clause C)  $\implies -L \in$  lits-of-l M
    unfolding M-def using unsat[unfolded CNot-def true-annots-def, simplified] by auto
  then have set (raw-clause C)  $\subseteq$  uminus ' lits-of-l M
    by (metis imageI subsetI uminus-of-uminus-id)
  then have set W  $\subseteq$  uminus ' lits-of-l M
    using cw-eq by (auto simp: raw-clause-def)
  then have subset: set W  $\subseteq$  uminus ' lits-of-l M
    by (simp add: w-nw(1))

  have W = watched C
    using cw-eq twl-clause.sel(1) by simp
  then show ?thesis
    using MNU-defs c-mem subset candidates-conflict-def by blast
qed

```

typedef 'v wf-twl = {S::'v twl-state. wf-twl-state S}
morphisms rough-state-of-twl twl-of-rough-state
proof –
 have TWL-State ([::('v, nat, 'v twl-clause) marked-lits)
 [] [] 0 None ∈ {S:: 'v twl-state. wf-twl-state S}
 by (auto simp: wf-twl-state-def twl.raw-clauses-def)
 then show ?thesis **by** auto
qed

lemma [code abstype]:
 twl-of-rough-state (rough-state-of-twl S) = S
 by (fact CDCL-Two-Watched-Literals.wf-twl.rough-state-of-twl-inverse)

lemma wf-twl-state-rough-state-of-twl[simp]: wf-twl-state (rough-state-of-twl S)
 using rough-state-of-twl **by** auto

abbreviation candidates-conflict-twl :: 'v wf-twl ⇒ 'v twl-clause set **where**
 candidates-conflict-twl S ≡ candidates-conflict (rough-state-of-twl S)

abbreviation candidates-propagate-twl :: 'v wf-twl ⇒ ('v literal × 'v twl-clause) set **where**
 candidates-propagate-twl S ≡ candidates-propagate (rough-state-of-twl S)

abbreviation raw-trail-twl :: 'a wf-twl ⇒ ('a, nat, 'a twl-clause) marked-lit list **where**
 raw-trail-twl S ≡ raw-trail (rough-state-of-twl S)

abbreviation trail-twl :: 'a wf-twl ⇒ ('a, nat, 'a literal multiset) marked-lit list **where**
 trail-twl S ≡ trail (rough-state-of-twl S)

abbreviation raw-clauses-twl :: 'a wf-twl ⇒ 'a twl-clause list **where**
 raw-clauses-twl S ≡ twl.raw-clauses (rough-state-of-twl S)

abbreviation raw-init-clss-twl :: 'a wf-twl ⇒ 'a twl-clause list **where**
 raw-init-clss-twl S ≡ raw-init-clss (rough-state-of-twl S)

abbreviation raw-learned-clss-twl :: 'a wf-twl ⇒ 'a twl-clause list **where**
 raw-learned-clss-twl S ≡ raw-learned-clss (rough-state-of-twl S)

abbreviation backtrack-lvl-twl **where**
 backtrack-lvl-twl S ≡ backtrack-lvl (rough-state-of-twl S)

abbreviation raw-conflicting-twl **where**
 raw-conflicting-twl S ≡ raw-conflicting (rough-state-of-twl S)

lemma wf-candidates-twl-conflict-complete:
 assumes
 c-mem: C ∈ set (raw-clauses-twl S) **and**
 unsat: trail-twl S ⊨_{as} CNot (mset (raw-clause C))
 shows C ∈ candidates-conflict-twl S
 using c-mem unsat wf-candidates-conflict-complete wf-twl-state-rough-state-of-twl **by** blast

abbreviation update-backtrack-lvl **where**
 update-backtrack-lvl k S ≡
 TWL-State (raw-trail S) (raw-init-clss S) (raw-learned-clss S) k (raw-conflicting S)

abbreviation update-conflicting **where**

update-conflicting $C\ S \equiv$
 $TWL\text{-}State\ (raw\text{-}trail\ S)\ (raw\text{-}init\text{-}clss\ S)\ (raw\text{-}learned\text{-}clss\ S)\ (backtrack\text{-}lvl\ S)\ C$

23.3 Abstract 2-WL

definition *tl-trail* **where**

tl-trail $S =$
 $TWL\text{-}State\ (tl\ (raw\text{-}trail\ S))\ (raw\text{-}init\text{-}clss\ S)\ (raw\text{-}learned\text{-}clss\ S)\ (backtrack\text{-}lvl\ S)$
 $(raw\text{-}conflicting\ S)$

locale *abstract-twl* =

fixes

watch :: $'v\ twl\text{-}state \Rightarrow 'v\ literal\ list \Rightarrow 'v\ twl\text{-}clause$ **and**

rewatch :: $'v\ literal \Rightarrow 'v\ twl\text{-}state \Rightarrow$

$'v\ twl\text{-}clause \Rightarrow 'v\ twl\text{-}clause$ **and**

restart-learned :: $'v\ twl\text{-}state \Rightarrow 'v\ twl\text{-}clause\ list$

assumes

clause-watch: $no\text{-}dup\ (trail\ S) \implies mset\ (raw\text{-}clause\ (watch\ S\ C)) = mset\ C$ **and**

wf-watch: $no\text{-}dup\ (trail\ S) \implies wf\text{-}twl\text{-}cls\ (trail\ S)\ (watch\ S\ C)$ **and**

clause-rewatch: $mset\ (raw\text{-}clause\ (rewatch\ L'\ S\ C')) = mset\ (raw\text{-}clause\ C')$ **and**

wf-rewatch:

$no\text{-}dup\ (trail\ S) \implies undefined\text{-}lit\ (trail\ S)\ (lit\text{-}of\ L) \implies wf\text{-}twl\text{-}cls\ (trail\ S)\ C' \implies$

$wf\text{-}twl\text{-}cls\ (L\ \# \ trail\ S)\ (rewatch\ (lit\text{-}of\ L)\ S\ C')$

and

restart-learned: $mset\ (restart\text{-}learned\ S) \subseteq\# \ mset\ (raw\text{-}learned\text{-}clss\ S)$ — We need *mset* and not *set*

to take care of duplicates.

begin

definition

cons-trail :: $('v, nat, 'v\ twl\text{-}clause)\ marked\text{-}lit \Rightarrow 'v\ twl\text{-}state \Rightarrow 'v\ twl\text{-}state$

where

cons-trail $L\ S =$

$TWL\text{-}State\ (L\ \# \ raw\text{-}trail\ S)\ (map\ (rewatch\ (lit\text{-}of\ L)\ S)\ (raw\text{-}init\text{-}clss\ S))$

$(map\ (rewatch\ (lit\text{-}of\ L)\ S)\ (raw\text{-}learned\text{-}clss\ S))\ (backtrack\text{-}lvl\ S)\ (raw\text{-}conflicting\ S)$

definition

add-init-cls :: $'v\ literal\ list \Rightarrow 'v\ twl\text{-}state \Rightarrow 'v\ twl\text{-}state$

where

add-init-cls $C\ S =$

$TWL\text{-}State\ (raw\text{-}trail\ S)\ (watch\ S\ C\ \# \ raw\text{-}init\text{-}clss\ S)\ (raw\text{-}learned\text{-}clss\ S)\ (backtrack\text{-}lvl\ S)$

$(raw\text{-}conflicting\ S)$

definition

add-learned-cls :: $'v\ literal\ list \Rightarrow 'v\ twl\text{-}state \Rightarrow 'v\ twl\text{-}state$

where

add-learned-cls $C\ S =$

$TWL\text{-}State\ (raw\text{-}trail\ S)\ (raw\text{-}init\text{-}clss\ S)\ (watch\ S\ C\ \# \ raw\text{-}learned\text{-}clss\ S)\ (backtrack\text{-}lvl\ S)$

$(raw\text{-}conflicting\ S)$

definition

remove-cls :: $'v\ literal\ list \Rightarrow 'v\ twl\text{-}state \Rightarrow 'v\ twl\text{-}state$

where

remove-cls $C\ S =$

$TWL\text{-}State\ (raw\text{-}trail\ S)$

$(removeAll\text{-}cond\ (\lambda D. mset\ (raw\text{-}clause\ D) = mset\ C)\ (raw\text{-}init\text{-}clss\ S))$

$(removeAll\text{-}cond\ (\lambda D. mset\ (raw\text{-}clause\ D) = mset\ C)\ (raw\text{-}learned\text{-}clss\ S))$

(backtrack-lvl S)
 (raw-conflicting S)

definition $\text{init-state} :: 'v \text{ literal list list} \Rightarrow 'v \text{ twl-state}$ **where**
 $\text{init-state } N = \text{fold add-init-cls } N \text{ (TWL-State } [] [] 0 \text{ None)}$

lemma $\text{unchanged-fold-add-init-cls}$:

$\text{raw-trail (fold add-init-cls } Cs \text{ (TWL-State } M \ N \ U \ k \ C)) = M$
 $\text{raw-learned-clss (fold add-init-cls } Cs \text{ (TWL-State } M \ N \ U \ k \ C)) = U$
 $\text{backtrack-lvl (fold add-init-cls } Cs \text{ (TWL-State } M \ N \ U \ k \ C)) = k$
 $\text{raw-conflicting (fold add-init-cls } Cs \text{ (TWL-State } M \ N \ U \ k \ C)) = C$
by (induct Cs arbitrary: N) (auto simp: add-init-cls-def)

lemma $\text{unchanged-init-state[simp]}$:

$\text{raw-trail (init-state } N) = []$
 $\text{raw-learned-clss (init-state } N) = []$
 $\text{backtrack-lvl (init-state } N) = 0$
 $\text{raw-conflicting (init-state } N) = \text{None}$
unfolding init-state-def **by** (rule unchanged-fold-add-init-cls)+

lemma $\text{clauses-init-fold-add-init}$:

$\text{no-dup } M \Longrightarrow$
 $\text{twl.init-clss (fold add-init-cls } Cs \text{ (TWL-State } M \ N \ U \ k \ C)) =$
 $\text{clauses-of-l } Cs + \text{clauses-of-l (map raw-clause } N)$
by (induct Cs arbitrary: N) (auto simp: add-init-cls-def clause-watch comp-def ac-simps)

lemma $\text{init-clss-init-state[simp]}$: $\text{twl.init-clss (init-state } N) = \text{clauses-of-l } N$

unfolding init-state-def **by** (subst clauses-init-fold-add-init) simp-all

definition restart' **where**

$\text{restart' } S = \text{TWL-State } [] \text{ (raw-init-clss } S) \text{ (restart-learned } S) 0 \text{ None}$

end

23.4 Instantiation of the previous locale

definition $\text{watch-nat} :: 'v \text{ twl-state} \Rightarrow 'v \text{ literal list} \Rightarrow 'v \text{ twl-clause}$ **where**

$\text{watch-nat } S \ C =$
 (let
 $C' = \text{remdups } C;$
 $\text{neg-not-assigned} = \text{filter } (\lambda L. -L \notin \text{lits-of-l (raw-trail } S)) \ C';$
 $\text{neg-assigned-sorted-by-trail} = \text{filter } (\lambda L. L \in \text{set } C) \ (\text{map } (\lambda L. -\text{lit-of } L) \ (\text{raw-trail } S));$
 $W = \text{take } 2 \ (\text{neg-not-assigned} @ \text{neg-assigned-sorted-by-trail});$
 $UW = \text{foldr remove1 } W \ C$
 in $\text{TWL-Clause } W \ UW)$

lemma list-cases2 :

fixes $l :: 'a \text{ list}$

assumes

$l = [] \Longrightarrow P$ **and**

$\bigwedge x. l = [x] \Longrightarrow P$ **and**

$\bigwedge x \ y \ xs. l = x \# y \# xs \Longrightarrow P$

shows P

by (metis assms list.collapse)

lemma $\text{filter-in-list-prop-verifiedD}$:

assumes $[L \leftarrow P \ . \ Q \ L] = l$
shows $\forall x \in \text{set } l. x \in \text{set } P \wedge Q \ x$
using *assms* **by** *auto*

lemma *no-dup-filter-diff*:

assumes *n-d*: *no-dup* *M* **and** *H*: $[L \leftarrow \text{map } (\lambda L. - \text{lit-of } L) \ M. L \in \text{set } C] = l$
shows *distinct* *l*
unfolding *H*[*symmetric*]
apply (*rule distinct-filter*)
using *n-d* **by** (*induction* *M*) *auto*

lemma *watch-nat-lists-disjointD*:

assumes
 $l: [L \leftarrow \text{remdups } C. - L \notin \text{lits-of-l } (\text{raw-trail } S)] = l$ **and**
 $l': [L \leftarrow \text{map } (\lambda L. - \text{lit-of } L) (\text{raw-trail } S) . L \in \text{set } C] = l'$
shows $\forall x \in \text{set } l. \forall y \in \text{set } l'. x \neq y$
by (*auto simp: l[symmetric] l'[symmetric] lits-of-def image-image*)

lemma *watch-nat-list-cases-witness*[*consumes* 2, *case-names* *nil-nil nil-single nil-other single-nil single-other other*]:

fixes

C :: '*v* literal list **and**
S :: '*v* twl-state

defines

xs $\equiv [L \leftarrow \text{remdups } C. - L \notin \text{lits-of-l } (\text{raw-trail } S)]$ **and**
ys $\equiv [L \leftarrow \text{map } (\lambda L. - \text{lit-of } L) (\text{raw-trail } S) . L \in \text{set } C]$

assumes

n-d: *no-dup* (*raw-trail* *S*) **and**
nil-nil: *xs* = [] \implies *ys* = [] \implies *P* **and**
nil-single:
 $\bigwedge a. xs = [] \implies ys = [a] \implies a \in \text{set } C \implies P$ **and**
nil-other: $\bigwedge a \ b \ ys'. xs = [] \implies ys = a \# b \# ys' \implies a \neq b \implies P$ **and**
single-nil: $\bigwedge a. xs = [a] \implies ys = [] \implies P$ **and**
single-other: $\bigwedge a \ b \ ys'. xs = [a] \implies ys = b \# ys' \implies a \neq b \implies P$ **and**
other: $\bigwedge a \ b \ xs'. xs = a \# b \# xs' \implies a \neq b \implies P$

shows *P*

proof –

note *xs-def*[*simp*] **and** *ys-def*[*simp*]
have *dist*: $\bigwedge P. \text{distinct } [L \leftarrow \text{remdups } C . P \ L]$
by *auto*
then have *H*: $\bigwedge a \ b \ P \ xs. [L \leftarrow \text{remdups } C . P \ L] = a \# b \# xs \implies a \neq b$
by (*metis distinct-length-2-or-more*)
show ?thesis
apply (*cases* $[L \leftarrow \text{remdups } C. - L \notin \text{lits-of-l } (\text{raw-trail } S)]$
 $\text{rule: list-cases2};$
 $\text{cases } [L \leftarrow \text{map } (\lambda L. - \text{lit-of } L) (\text{raw-trail } S) . L \in \text{set } C]$ rule: list-cases2)
using *nil-nil* **apply** *simp*
using *nil-single* **apply** (*force dest: filter-in-list-prop-verifiedD*)
using *nil-other no-dup-filter-diff*[*OF* *n-d*, *of* *C*]
apply *fastforce*
using *single-nil* **apply** *simp*
using *single-other xs-def ys-def* **apply** (*metis list.set-intros(1) watch-nat-lists-disjointD*)
using *single-other* **unfolding** *xs-def ys-def* **apply** (*metis list.set-intros(1) watch-nat-lists-disjointD*)
using *other xs-def ys-def* **by** (*metis H*)**+**

qed

lemma *watch-nat-list-cases* [consumes 1, case-names *nil-nil nil-single nil-other single-nil single-other other*]:

fixes

$C :: 'v$ literal list **and**

$S :: 'v$ twl-state

defines

$xs \equiv [L \leftarrow \text{remdups } C . - L \notin \text{lits-of-l (raw-trail } S)]$ **and**

$ys \equiv [L \leftarrow \text{map } (\lambda L. - \text{lit-of } L) (\text{raw-trail } S) . L \in \text{set } C]$

assumes

n-d: no-dup (raw-trail S) **and**

nil-nil: $xs = [] \implies ys = [] \implies P$ **and**

nil-single:

$\bigwedge a. xs = [] \implies ys = [a] \implies a \in \text{set } C \implies P$ **and**

nil-other: $\bigwedge a b ys'. xs = [] \implies ys = a \# b \# ys' \implies a \neq b \implies P$ **and**

single-nil: $\bigwedge a. xs = [a] \implies ys = [] \implies P$ **and**

single-other: $\bigwedge a b ys'. xs = [a] \implies ys = b \# ys' \implies a \neq b \implies P$ **and**

other: $\bigwedge a b xs'. xs = a \# b \# xs' \implies a \neq b \implies P$

shows P

using *watch-nat-list-cases-witness*[OF *n-d*, of C P]

nil-nil nil-single nil-other single-nil single-other other

unfolding *xs-def*[symmetric] *ys-def*[symmetric] **by** *auto*

lemma *watch-nat-lists-set-union-witness*:

fixes

$C :: 'v$ literal list **and**

$S :: 'v$ twl-state

defines

$xs \equiv [L \leftarrow \text{remdups } C . - L \notin \text{lits-of-l (raw-trail } S)]$ **and**

$ys \equiv [L \leftarrow \text{map } (\lambda L. - \text{lit-of } L) (\text{raw-trail } S) . L \in \text{set } C]$

assumes *n-d*: no-dup (raw-trail S)

shows $\text{set } C = \text{set } xs \cup \text{set } ys$

using *n-d* **unfolding** *xs-def* *ys-def* **by** (*auto simp: lits-of-def comp-def uminus-lit-swap*)

lemma *mset-intersection-inclusion*: $A + (B - A) = B \longleftrightarrow A \subseteq \# B$

apply (*rule iffI*)

apply (*metis mset-le-add-left*)

by (*auto simp: ac-simps multiset-eq-iff subseteq-mset-def*)

lemma *clause-watch-nat*:

assumes *no-dup* (raw-trail S)

shows $\text{mset (raw-clause (watch-nat } S \ C))} = \text{mset } C$

using *assms*

apply (*cases rule: watch-nat-list-cases*[OF *assms*(1), of C])

by (*auto dest: filter-in-list-prop-verifiedD simp: watch-nat-def multiset-eq-iff raw-clause-def*)

lemma *index-uminus-index-map-uminus*:

$-a \in \text{set } L \implies \text{index } L \ (-a) = \text{index (map uminus } L) \ (a::'a \text{ literal})$

by (*induction L*) *auto*

lemma *index-filter*:

$a \in \text{set } L \implies b \in \text{set } L \implies P \ a \implies P \ b \implies$

$\text{index } L \ a \leq \text{index } L \ b \longleftrightarrow \text{index (filter } P \ L) \ a \leq \text{index (filter } P \ L) \ b$

by (*induction L*) *auto*

```

lemma foldr-remove1-W-Nil[simp]: foldr remove1 W [] = []
  by (induct W) auto

lemma image-lit-of-mmset-of-mlit'[simp]:
  lit-of ' mmset-of-mlit' ' A = lit-of ' A
  by (auto simp: image-image comp-def)

lemma distinct-filter-eq:
  assumes distinct xs
  shows [L ← xs. L = a] = (if a ∈ set xs then [a] else [])
  using assms by (induction xs) auto

lemma no-dup-distinct-map-uminus-lit-of:
  no-dup xs ⇒ distinct (map (λL. - lit-of L) xs)
  by (induction xs) auto

lemma wf-watch-witness:
  fixes C :: 'v literal list and
    S :: 'v twl-state
  defines
    ass: neg-not-assigned ≡ filter (λL. -L ∉ lits-of-l (raw-trail S)) (remdups C) and
    tr: neg-assigned-sorted-by-trail ≡ filter (λL. L ∈ set C) (map (λL. -lit-of L) (raw-trail S))
  defines
    W: W ≡ take 2 (neg-not-assigned @ neg-assigned-sorted-by-trail)
  assumes
    n-d[simp]: no-dup (raw-trail S)
  shows wf-twlc (trail S) (TWL-Clause W (foldr remove1 W C))
  unfolding wf-twlc.simps
proof (intro conjI, goal-cases)
  case 1
  then show ?case using n-d W unfolding ass tr
    apply (cases rule: watch-nat-list-cases-witness[of S C, OF n-d])
    by (auto simp: distinct-mset-add-single)
next
  case 2
  then show ?case unfolding W by simp
next
  case 3
  show ?case using n-d
  proof (cases rule: watch-nat-list-cases-witness[of S C])
  case nil-nil
  then have set C = set [] ∪ set []
    using watch-nat-lists-set-union-witness n-d by metis
  then show ?thesis
    by simp
  next
  case (nil-single a)
  moreover have ∧x. set C = {a} ⇒ - a ∈ lits-of-l (trail S) ⇒ x ∈ set (remove1 a C) ⇒
    x = a
    using notin-set-remove1 by auto
  ultimately show ?thesis
    using watch-nat-lists-set-union-witness[of S C] 3 by (auto simp: W ass tr comp-def)
next
  case nil-other

```

```

    then show ?thesis
    using 3 by (auto simp: W ass tr)
next
case (single-nil a)
show ?thesis
  using watch-nat-lists-set-union-witness[of S C] 3
  by (fastforce simp add: W ass tr single-nil comp-def distinct-filter-eq
      no-dup-distinct-map-uminus-lit-of min-def)
next
case single-other
then show ?thesis
  using 3 by (auto simp: W ass tr)
next
case other
then show ?thesis
  using 3 by (auto simp: W ass tr)
qed
next
case 4 note -[simp] = this
show ?case
  using n-d apply (cases rule: watch-nat-list-cases-witness[of S C])
  apply (auto dest: filter-in-list-prop-verifiedD
      simp: W ass tr lits-of-def filter-empty-conv)[4]
  using watch-nat-lists-set-union-witness[of S C]
  by (force dest: filter-in-list-prop-verifiedD simp: W ass tr lits-of-def)+
next
case 5
from n-d show ?case
proof (cases rule: watch-nat-list-cases-witness[of S C])
  case nil-nil
  then show ?thesis by (auto simp: W ass tr)
next
case nil-single
then show ?thesis
  using watch-nat-lists-set-union-witness[of S C] tr by (fastforce simp: W ass)
next
case nil-other
then show ?thesis
  unfolding watched-decided-most-recently.simps Ball-def
  apply (intro allI impI)
  apply (subst index-uminus-index-map-uminus,
      simp add: index-uminus-index-map-uminus lits-of-def o-def)
  apply (subst index-uminus-index-map-uminus,
      simp add: index-uminus-index-map-uminus lits-of-def o-def)

  apply (subst index-filter[of - - λL. L ∈ set C])
  by (auto dest: filter-in-list-prop-verifiedD
      simp: uminus-lit-swap lits-of-def o-def W ass tr dest: in-diffD)
next
case single-nil
then show ?thesis
  using watch-nat-lists-set-union-witness[of S C] tr by (fastforce simp: W ass)
next
case single-other
then show ?thesis

```

```

unfolding watched-decided-most-recently.simps Ball-def
apply (clarify)
apply (subst index-uminus-index-map-uminus,
  simp add: index-uminus-index-map-uminus lits-of-def image-image o-def)
apply (subst index-uminus-index-map-uminus,
  simp add: index-uminus-index-map-uminus lits-of-def o-def)

apply (subst index-filter[of - -  $\lambda L. L \in \text{set } C$ ])
by (auto dest: filter-in-list-prop-verifiedD
  simp: W ass tr uminus-lit-swap lits-of-def o-def dest: in-diffD)
next
case other
then show ?thesis
  unfolding watched-decided-most-recently.simps
  apply clarify
  apply (subst index-uminus-index-map-uminus,
    simp add: index-uminus-index-map-uminus lits-of-def o-def)[1]
  apply (subst index-uminus-index-map-uminus,
    simp add: index-uminus-index-map-uminus lits-of-def o-def)[1]

  apply (subst index-filter[of - -  $\lambda L. L \in \text{set } C$ ])
  by (auto dest: filter-in-list-prop-verifiedD
    simp: index-uminus-index-map-uminus lits-of-def o-def uminus-lit-swap
    W ass tr)
qed
qed

lemma wf-watch-nat: no-dup (raw-trail S)  $\implies$  wf-twl-cls (trail S) (watch-nat S C)
using wf-watch-witness[of S C] watch-nat-def by metis

definition
  rewatch-nat ::
    'v literal  $\Rightarrow$  'v twl-state  $\Rightarrow$  'v twl-clause  $\Rightarrow$  'v twl-clause
where
  rewatch-nat L S C =
    (if  $\neg L \in \text{set } (\text{watched } C)$  then
      case filter ( $\lambda L'. L' \notin \text{set } (\text{watched } C) \wedge \neg L' \notin \text{insert } L (\text{lits-of-l } (\text{trail } S))$ )
        (unwatched C) of
        []  $\Rightarrow$  C
      | L' # -  $\Rightarrow$ 
        TWL-Clause (L' # remove1 ( $\neg L$ ) (watched C)) ( $\neg L$  # remove1 L' (unwatched C))
    else
      C)

lemma clause-rewatch-nat:
fixes UW :: 'v literal list and
  S :: 'v twl-state and
  L :: 'v literal and C :: 'v twl-clause
shows mset (raw-clause (rewatch-nat L S C)) = mset (raw-clause C)
using List.set-remove1-subset[of  $\neg L$  watched C]
apply (cases C)
by (auto simp: raw-clause-def rewatch-nat-def ac-simps multiset-eq-iff
  split: list.split
  dest: filter-in-list-prop-verifiedD)

```

lemma *filter-sorted-list-of-multiset-Nil*:

$[x \leftarrow \text{sorted-list-of-multiset } M. p \ x] = [] \longleftrightarrow (\forall x \in\# M. \neg p \ x)$
by *auto (metis empty-iff filter-set list.set(1) member-filter set-sorted-list-of-multiset)*

lemma *filter-sorted-list-of-multiset-ConsD*:

$[x \leftarrow \text{sorted-list-of-multiset } M. p \ x] = x \# xs \implies p \ x$
by *(metis filter-set insert-iff list.set(2) member-filter)*

lemma *mset-minus-single-eq-empty*:

$a - \{\#b\} = \{\#\} \longleftrightarrow a = \{\#b\} \vee a = \{\#\}$
by *(metis Multiset.diff-cancel add.right-neutral diff-single-eq-union diff-single-trivial zero-diff)*

lemma *size-mset-le-2-cases*:

assumes $size \ W \leq 2$
shows $W = \{\#\} \vee (\exists a. W = \{\#a\}) \vee (\exists a \ b. W = \{\#a, b\})$
by *(metis One-nat-def Suc-1 Suc-eq-plus1-left assms linorder-not-less nat-less-le not-less-eq-eq le-iff-add size-1-singleton-mset size-eq-0-iff-empty size-mset-2)*

lemma *filter-sorted-list-of-multiset-eqD*:

assumes $[x \leftarrow \text{sorted-list-of-multiset } A. p \ x] = x \# xs$ (**is** $?comp = -$)
shows $x \in\# A$

proof $-$

have $x \in \text{set } ?comp$
using *assms* **by** *simp*
then have $x \in \text{set } (\text{sorted-list-of-multiset } A)$
by *simp*
then show $x \in\# A$
by *simp*

qed

lemma *clause-rewatch-witness'*:

assumes
 $wf: wf\text{-}twl\text{-}cls \ (\text{trail } S) \ C$ **and**
 $n\text{-}d: no\text{-}dup \ (\text{raw-trail } S)$ **and**
 $undef: undefined\text{-}lit \ (\text{trail } S) \ (\text{lit-of } L)$
shows $wf\text{-}twl\text{-}cls \ (L \# \text{trail } S) \ (\text{rewatch-nat } (\text{lit-of } L) \ S \ C)$

proof *(cases - lit-of L ∈ set (watched C))*

case *False*

then show *?thesis*

apply *(cases C)*
using $wf \ n\text{-}d \ undef$ **unfolding** *rewatch-nat-def*
by *(auto simp: uminus-lit-swap Marked-Propagated-in-iff-in-lits-of-l comp-def)*

next

case *falsified: True*

let *?unwatched-nonfalsified* =

$[L' \leftarrow \text{unwatched } C. L' \notin \text{set } (\text{watched } C) \wedge - L' \notin \text{insert } (\text{lit-of } L) \ (\text{lits-of-l } (\text{trail } S))]$

obtain $W \ UW$ **where** $C: C = \text{TWL-Clause } W \ UW$

by *(cases C)*

show *?thesis*

proof *(cases ?unwatched-nonfalsified)*

case *Nil*

```

show ?thesis
  using falsified Nil
  apply (simp only: wf-twl-cls.simps if-True list.cases C rewatch-nat-def)
  apply (intro conjI)
  proof goal-cases
    case 1
    then show ?case using wf C by simp
  next
    case 2
    then show ?case using wf C by simp
  next
    case 3
    then show ?case using wf C by simp
  next
    case 4
    have  $\bigwedge p l. \text{filter } p (\text{unwatched } C) \neq [] \vee l \notin \text{set } UW \vee \neg p l$ 
      unfolding C by (metis (no-types) filter-empty-conv twl-clause.sel(2))
    then show ?case
      using 4(2) C by auto
  next
    case 5
    then show ?case
      using wf by (fastforce simp add: C comp-def uminus-lit-swap)
  qed
next
case (Cons L' Ls)
show ?thesis
  unfolding rewatch-nat-def
  using falsified Cons
  apply (simp only: wf-twl-cls.simps if-True list.cases C)
  apply (intro conjI)
  proof goal-cases
    case 1
    have distinct (watched (TWL-Clause W UW))
      using wf unfolding C by auto
    moreover have  $L' \notin \text{set } (\text{remove1 } (-\text{lit-of } L) (\text{watched } (TWL-Clause W UW)))$ 
      using 1(2) not-gr0 by (fastforce dest: filter-in-list-prop-verifiedD in-diffD)
    ultimately show ?case
      by (auto simp: distinct-mset-single-add)
  next
    case 2
    have f2:  $[l \leftarrow \text{unwatched } (TWL-Clause W UW) . l \notin \text{set } (\text{watched } (TWL-Clause W UW)) \wedge -l \notin \text{insert } (\text{lit-of } L) (\text{lits-of-l } (\text{trail } S))] \neq []$ 
      using 2(2) by simp
    then have  $\neg \text{set } UW \subseteq \text{set } W$ 
      using 2 by (auto simp add: filter-empty-conv)
    then show ?case
      using wf C 2(1) by (auto simp: length-remove1)
  next
    case 3
    have  $W: \text{length } W \leq \text{Suc } 0 \longleftrightarrow \text{length } W = 0 \vee \text{length } W = \text{Suc } 0$ 
      by linarith
    show ?case
      using wf C 3 by (auto simp: length-remove1 W length-list-Suc-0 dest!: subset-singletonD)
  next

```

```

case 4
have H:  $\forall L \in \text{set } W. - L \in \text{lits-of-l } (\text{trail } S) \longrightarrow$ 
  ( $\forall L' \in \text{set } UW. L' \notin \text{set } W \longrightarrow - L' \in \text{lits-of-l } (\text{trail } S)$ )
  using wf by (auto simp: C)
have W:  $\text{length } W \leq 2$  and W-UW:  $\text{length } W < 2 \longrightarrow \text{set } UW \subseteq \text{set } W$ 
  using wf by (auto simp: C)
have distinct: distinct W
  using wf by (auto simp: C)
show ?case
  using 4
  unfolding C watched-decided-most-recently.simps Ball-def twl-clause.sel
  apply (intro allI impI)
  apply (rename-tac xW xUW)
  apply (case-tac - lit-of L = xW; case-tac xW = xUW; case-tac L' = xW)
    apply (auto simp: uminus-lit-swap)[2]
    apply (force dest: filter-in-list-prop-verifiedD)
    using H distinct apply (fastforce split: if-split-asm)
    using distinct apply (fastforce split: if-split-asm)
    using distinct apply (fastforce split: if-split-asm)
    apply (force dest: filter-in-list-prop-verifiedD)
    using H by (auto simp: uminus-lit-swap)
next
case 5
have H:  $\forall x. x \in \text{set } W \longrightarrow - x \in \text{lits-of-l } (\text{trail } S) \longrightarrow (\forall x. x \in \text{set } UW \longrightarrow x \notin \text{set } W$ 
   $\longrightarrow - x \in \text{lits-of-l } (\text{trail } S))$ 
  using wf by (auto simp: C)
show ?case
  unfolding C watched-decided-most-recently.simps Ball-def
  proof (intro allI impI conjI, goal-cases)
    case (1 xW x)
    show ?case
      proof (cases - lit-of L = xW)
        case True
        then show ?thesis
          by (cases xW = x) (auto simp: uminus-lit-swap)
      next
        case False note LxW = this
        have f9:  $L' \in \text{set } [l \leftarrow \text{unwatched } C. l \notin \text{set } (\text{watched } (TWL\text{-Clause } W \ UW))$ 
           $\wedge - l \notin \text{lits-of-l } (L \# \text{trail } S)]$ 
          using 1(2) 5 C by auto
        moreover then have f11:  $- xW \in \text{lits-of-l } (\text{trail } S)$ 
          using 1(3) LxW by (auto simp: uminus-lit-swap)
        moreover then have xW  $\notin \text{set } W$ 
          using f9 1(2) H by (auto simp: C)
        ultimately have False
          using 1 by auto
        then show ?thesis
          by fast
      qed
    qed
  qed
qed

```

```

interpretation twl: abstract-twl watch-nat rewatch-nat raw-learned-clss
  apply unfold-locales
  apply (rule clause-watch-nat; simp add: image-image comp-def)
  apply (rule wf-watch-nat; simp add: image-image comp-def)
  apply (rule clause-rewatch-nat)
  apply (rule clause-rewatch-witness'; simp add: image-image comp-def)
  apply (simp)
done

```

```

interpretation twl2: abstract-twl watch-nat rewatch-nat  $\lambda$ -. []
  apply unfold-locales
  apply (rule clause-watch-nat; simp add: image-image comp-def)
  apply (rule wf-watch-nat; simp add: image-image comp-def)
  apply (rule clause-rewatch-nat)
  apply (rule clause-rewatch-witness'; simp add: image-image comp-def)
  apply (simp)
done

```

end

24 Implementation for 2 Watched-Literals

```

theory CDCL-Two-Watched-Literals-Invariant
imports CDCL-Two-Watched-Literals DPLL-CDCL-W-Implementation
begin

```

24.1 Interpretation for *conflict-driven-clause-learning_W.cdcl_W*

We define here the 2-WL with the invariant and show the role of the candidates.

```

context abstract-twl
begin

```

24.1.1 Direct Interpretation

```

lemma mset-map-removeAll-cond:
  mset (map ( $\lambda$ x. mset (raw-clause x))
    (removeAll-cond ( $\lambda$ D. mset (raw-clause D) = mset (raw-clause C)) N))
  = mset (removeAll (mset (raw-clause C)) (map ( $\lambda$ x. mset (raw-clause x)) N))
  by (induction N) auto

```

```

lemma mset-raw-init-clss-init-state:
  mset (map ( $\lambda$ x. mset (raw-clause x)) (raw-init-clss (init-state (map raw-clause N))))
  = mset (map ( $\lambda$ x. mset (raw-clause x)) N)
  by (metis (no-types, lifting) init-clss-init-state map-eq-conv map-map o-def)

```

```

interpretation rough-cdcl: stateW
   $\lambda$ C. mset (raw-clause C)

```

```

 $\lambda$ L C. TWL-Clause (watched C) (L # unwatched C)
 $\lambda$ L C. TWL-Clause [] (remove1 L (raw-clause C))
 $\lambda$ C. clauses-of-l (map raw-clause C) op @
 $\lambda$ L C. L  $\in$  set C op #  $\lambda$ C. remove1-cond ( $\lambda$ D. mset (raw-clause D) = mset (raw-clause C))

mset  $\lambda$ xs ys. case-prod append (fold ( $\lambda$ x (ys, zs). (remove1 x ys, x # zs)) xs (ys, []))

```


op # remove1

raw-clause $\lambda C. \text{TWL-Clause } [] \ C$
trail $\lambda S. \text{hd } (\text{raw-trail } S)$
raw-init-clss *raw-learned-clss* *backtrack-lvl* *raw-conflicting*
cons-trail *tl-trail* $\lambda C. \text{add-init-cls } (\text{raw-clause } C) \ \lambda C. \text{add-learned-cls } (\text{raw-clause } C)$
 $\lambda C. \text{remove-cls } (\text{raw-clause } C)$
update-backtrack-lvl
update-conflicting $\lambda N. \text{init-state } (\text{map raw-clause } N) \ \text{restart}'$
apply *unfold-locales*
apply *(case-tac raw-trail S)*
apply *(simp-all add: add-init-cls-def add-learned-cls-def clause-rewatch clause-watch*
cons-trail-def remove-cls-def restart'-def tl-trail-def map-tl comp-def
ac-simps mset-map-removeAll-cond mset-raw-init-clss-init-state)
apply *(auto simp: mset-map image-mset-subseteq-mono[OF restart-learned])*
done

interpretation *rough-cdcl: conflict-driven-clause-learning_w*
 $\lambda C. \text{mset } (\text{raw-clause } C)$

$\lambda L \ C. \text{TWL-Clause } (\text{watched } C) \ (L \# \text{unwatched } C)$
 $\lambda L \ C. \text{TWL-Clause } [] \ (\text{remove1 } L \ (\text{raw-clause } C))$
 $\lambda C. \text{clauses-of-l } (\text{map raw-clause } C) \ \text{op } @$
 $\lambda L \ C. L \in \text{set } C \ \text{op } \# \ \lambda C. \text{remove1-cond } (\lambda D. \text{mset } (\text{raw-clause } D) = \text{mset } (\text{raw-clause } C))$

mset $\lambda xs \ ys. \text{case-prod append } (\text{fold } (\lambda x \ (ys, zs). (\text{remove1 } x \ ys, x \# \ zs)) \ xs \ (ys, []))$
op # remove1

raw-clause $\lambda C. \text{TWL-Clause } [] \ C$
trail $\lambda S. \text{hd } (\text{raw-trail } S)$
raw-init-clss *raw-learned-clss* *backtrack-lvl* *raw-conflicting*
cons-trail *tl-trail* $\lambda C. \text{add-init-cls } (\text{raw-clause } C) \ \lambda C. \text{add-learned-cls } (\text{raw-clause } C)$
 $\lambda C. \text{remove-cls } (\text{raw-clause } C)$
update-backtrack-lvl
update-conflicting $\lambda N. \text{init-state } (\text{map raw-clause } N) \ \text{restart}'$
by *unfold-locales*

declare *local.rough-cdcl.mset-ccls-ccls-of-cls[simp del]*

24.1.2 Opaque Type with Invariant

declare *rough-cdcl.state-simp[simp del]*

definition *cons-trail-twl* $:: ('v, \text{nat}, 'v \text{ twl-clause}) \text{marked-lit} \Rightarrow 'v \text{ wf-twl} \Rightarrow 'v \text{ wf-twl}$
where
cons-trail-twl $L \ S \equiv \text{twl-of-rough-state } (\text{cons-trail } L \ (\text{rough-state-of-twl } S))$

lemma *wf-twl-state-cons-trail:*

assumes

undef: undefined-lit (trail S) (lit-of L) and
wf: wf-twl-state S

shows *wf-twl-state (cons-trail L S)*

using *undef wf wf-rewatch[of S mset-of-mlit' L] unfolding wf-twl-state-def Ball-def*
by *(auto simp: cons-trail-def defined-lit-map comp-def image-def twl.raw-clauses-def)*

lemma *rough-state-of-twl-cons-trail*:

undefined-lit (trail-twl S) (lit-of L) \implies
rough-state-of-twl (cons-trail-twl L S) = cons-trail L (rough-state-of-twl S)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-cons-trail*
unfolding *cons-trail-twl-def* **by** *blast*

abbreviation *add-init-cls-twl* **where**

add-init-cls-twl C S \equiv twl-of-rough-state (add-init-cls C (rough-state-of-twl S))

lemma *wf-twl-add-init-cls*: *wf-twl-state S \implies wf-twl-state (add-init-cls L S)*

unfolding *wf-twl-state-def* **by** (*auto simp: wf-watch add-init-cls-def comp-def twl.raw-clauses-def*
split: if-split-asm)

lemma *rough-state-of-twl-add-init-cls*:

rough-state-of-twl (add-init-cls-twl L S) = add-init-cls L (rough-state-of-twl S)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-add-init-cls* **by** *blast*

abbreviation *add-learned-cls-twl* **where**

add-learned-cls-twl C S \equiv twl-of-rough-state (add-learned-cls C (rough-state-of-twl S))

lemma *wf-twl-add-learned-cls*: *wf-twl-state S \implies wf-twl-state (add-learned-cls L S)*

unfolding *wf-twl-state-def* **by** (*auto simp: wf-watch add-learned-cls-def twl.raw-clauses-def*
split: if-split-asm)

lemma *rough-state-of-twl-add-learned-cls*:

rough-state-of-twl (add-learned-cls-twl L S) = add-learned-cls L (rough-state-of-twl S)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-add-learned-cls* **by** *blast*

abbreviation *remove-cls-twl* **where**

remove-cls-twl C S \equiv twl-of-rough-state (remove-cls C (rough-state-of-twl S))

lemma *set-removeAll-condD*: *x \in set (removeAll-cond f xs) \implies x \in set xs*

by (*induction xs*) (*auto split: if-split-asm*)

lemma *wf-twl-remove-cls*: *wf-twl-state S \implies wf-twl-state (remove-cls L S)*

unfolding *wf-twl-state-def* **by** (*auto simp: wf-watch remove-cls-def twl.raw-clauses-def comp-def*
split: if-split-asm dest: set-removeAll-condD)

lemma *rough-state-of-twl-remove-cls*:

rough-state-of-twl (remove-cls-twl L S) = remove-cls L (rough-state-of-twl S)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-remove-cls* **by** *blast*

abbreviation *init-state-twl* **where**

init-state-twl N \equiv twl-of-rough-state (init-state N)

lemma *wf-twl-state-wf-twl-state-fold-add-init-cls*:

assumes *wf-twl-state S*
shows *wf-twl-state (fold add-init-cls N S)*
using *assms* **apply** (*induction N arbitrary: S*)
apply (*auto simp: wf-twl-state-def*)
by (*simp add: wf-twl-add-init-cls*)

lemma *wf-twl-state-epsilon-state[simp]*:

wf-twl-state (TWL-State [] [] 0 None)
by (*auto simp: wf-twl-state-def twl.raw-clauses-def*)

lemma *wf-twl-init-state*: *wf-twl-state* (*init-state* *N*)
unfolding *init-state-def* **by** (*auto intro!*: *wf-twl-state-wf-twl-state-fold-add-init-cls*)

lemma *rough-state-of-twl-init-state*:
rough-state-of-twl (*init-state-twl* *N*) = *init-state* *N*
by (*simp add*: *twl-of-rough-state-inverse wf-twl-init-state*)

abbreviation *tl-trail-twl* **where**
tl-trail-twl *S* \equiv *twl-of-rough-state* (*tl-trail* (*rough-state-of-twl* *S*))

lemma *wf-twl-state-tl-trail*: *wf-twl-state* *S* \implies *wf-twl-state* (*tl-trail* *S*)
by (*auto simp add*: *twl-of-rough-state-inverse wf-twl-init-state wf-twl-cls-wf-twl-cls-tl*
tl-trail-def wf-twl-state-def distinct-tl map-tl comp-def twl.raw-clauses-def)

lemma *rough-state-of-twl-tl-trail*:
rough-state-of-twl (*tl-trail-twl* *S*) = *tl-trail* (*rough-state-of-twl* *S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-tl-trail* **by** *blast*

abbreviation *update-backtrack-lvl-twl* **where**
update-backtrack-lvl-twl *k* *S* \equiv *twl-of-rough-state* (*update-backtrack-lvl* *k* (*rough-state-of-twl* *S*))

lemma *wf-twl-state-update-backtrack-lvl*:
wf-twl-state *S* \implies *wf-twl-state* (*update-backtrack-lvl* *k* *S*)
unfolding *wf-twl-state-def* **by** (*auto simp*: *comp-def twl.raw-clauses-def*)

lemma *rough-state-of-twl-update-backtrack-lvl*:
rough-state-of-twl (*update-backtrack-lvl-twl* *k* *S*) = *update-backtrack-lvl* *k*
(*rough-state-of-twl* *S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-update-backtrack-lvl* **by** *fast*

abbreviation *update-conflicting-twl* **where**
update-conflicting-twl *k* *S* \equiv *twl-of-rough-state* (*update-conflicting* *k* (*rough-state-of-twl* *S*))

lemma *wf-twl-state-update-conflicting*:
wf-twl-state *S* \implies *wf-twl-state* (*update-conflicting* *k* *S*)
unfolding *wf-twl-state-def* **by** (*auto simp*: *twl.raw-clauses-def comp-def*)

lemma *rough-state-of-twl-update-conflicting*:
rough-state-of-twl (*update-conflicting-twl* *k* *S*) = *update-conflicting* *k*
(*rough-state-of-twl* *S*)
using *rough-state-of-twl twl-of-rough-state-inverse wf-twl-state-update-conflicting* **by** *fast*

abbreviation *raw-clauses-twl* **where**
raw-clauses-twl *S* \equiv *twl.raw-clauses* (*rough-state-of-twl* *S*)

abbreviation *restart-twl* **where**
restart-twl *S* \equiv *twl-of-rough-state* (*restart'* (*rough-state-of-twl* *S*))

lemma *mset-union-mset-setD*:
mset *A* $\subseteq\#$ *mset* *B* \implies *set* *A* \subseteq *set* *B*
by *auto*

lemma *wf-wf-restart'*: *wf-twl-state* *S* \implies *wf-twl-state* (*restart'* *S*)
unfolding *restart'-def wf-twl-state-def* **apply** *standard*
apply *clarify*

```

apply (rename-tac x)
apply (subgoal-tac wf-twl-cls (trail S) x)
apply (case-tac x)
using restart-learned by (auto simp: twl.raw-clauses-def comp-def dest: mset-union-mset-setD)

```

lemma *rough-state-of-twl-restart-twl*:
rough-state-of-twl (restart-twl S) = restart' (rough-state-of-twl S)
by (simp add: twl-of-rough-state-inverse wf-wf-restart')

sublocale *conflict-driven-clause-learning_w*
 $\lambda C. \text{mset (raw-clause } C)$

```

 $\lambda L C. \text{TWL-Clause (watched } C) (L \# \text{unwatched } C)$ 
 $\lambda L C. \text{TWL-Clause } [] (\text{remove1 } L (\text{raw-clause } C))$ 
 $\lambda C. \text{clauses-of-l (map raw-clause } C) \text{ op } @$ 
 $\lambda L C. L \in \text{set } C \text{ op } \# \lambda C. \text{remove1-cond } (\lambda D. \text{mset (raw-clause } D) = \text{mset (raw-clause } C))$ 

```

```

mset  $\lambda xs \ ys. \text{case-prod append (fold } (\lambda x (ys, zs). (\text{remove1 } x \ ys, x \# \ zs)) \ xs \ (ys, []))$ 
op  $\# \text{remove1}$ 

```

```

 $\lambda C. \text{raw-clause } C \lambda C. \text{TWL-Clause } [] C$ 
trail-twl  $\lambda S. \text{hd (raw-trail-twl } S)$ 
raw-init-clss-twl
raw-learned-clss-twl
backtrack-lvl-twl
raw-conflicting-twl
cons-trail-twl
tl-trail-twl
 $\lambda C. \text{add-init-clss-twl (raw-clause } C)$ 
 $\lambda C. \text{add-learned-clss-twl (raw-clause } C)$ 
 $\lambda C. \text{remove-clss-twl (raw-clause } C)$ 
update-backtrack-lvl-twl
update-conflicting-twl
 $\lambda N. \text{init-state-twl (map raw-clause } N)$ 
restart-twl

```

```

apply unfold-locales
using rough-cdcl.hd-raw-trail apply blast
apply (simp-all add: rough-state-of-twl-cons-trail rough-state-of-twl-tl-trail
  rough-state-of-twl-add-init-clss rough-state-of-twl-add-learned-clss
  rough-state-of-twl-remove-clss rough-state-of-twl-update-backtrack-lvl
  rough-state-of-twl-update-conflicting)[7]
using rough-cdcl.init-clss-cons-trail rough-cdcl.init-clss-tl-trail
  rough-cdcl.init-clss-add-init-clss rough-cdcl.init-clss-remove-clss
  rough-cdcl.init-clss-add-learned-clss
  rough-cdcl.init-clss-update-backtrack-lvl
  rough-cdcl.init-clss-update-conflicting
apply (auto simp add: rough-state-of-twl-cons-trail rough-state-of-twl-tl-trail
  rough-state-of-twl-add-init-clss rough-state-of-twl-add-learned-clss
  rough-state-of-twl-remove-clss rough-state-of-twl-update-backtrack-lvl
  rough-state-of-twl-update-conflicting comp-def)[7]
using rough-cdcl.learned-clss-cons-trail rough-cdcl.learned-clss-tl-trail
  rough-cdcl.learned-clss-add-init-clss rough-cdcl.learned-clss-remove-clss
  rough-cdcl.learned-clss-add-learned-clss
  rough-cdcl.learned-clss-update-backtrack-lvl
  rough-cdcl.learned-clss-update-conflicting

```

```

apply (auto simp add: rough-state-of-twl-cons-trail rough-state-of-twl-tl-trail
  rough-state-of-twl-add-init-cls rough-state-of-twl-add-learned-cls
  rough-state-of-twl-remove-cls rough-state-of-twl-update-backtrack-lvl
  rough-state-of-twl-update-conflicting comp-def)[7]
apply (auto simp add: rough-state-of-twl-cons-trail rough-state-of-twl-tl-trail
  rough-state-of-twl-add-init-cls rough-state-of-twl-add-learned-cls
  rough-state-of-twl-remove-cls rough-state-of-twl-update-backtrack-lvl
  rough-state-of-twl-update-conflicting comp-def)[14]
using init-clss-init-state apply (auto simp: rough-state-of-twl-init-state)[5]
using rough-cdcl.init-clss-restart-state rough-cdcl.learned-clss-restart-state
apply (auto simp: rough-state-of-twl-restart-twl)[5]
done

```

```

declare local.rough-cdcl.mset-ccls-ccls-of-clcs[simp del]
abbreviation state-eq-twl (infix  $\sim$  TWL 51) where
  state-eq-twl  $S S' \equiv$  rough-cdcl.state-eq (rough-state-of-twl  $S$ ) (rough-state-of-twl  $S'$ )
notation state-eq (infix  $\sim$  51)
declare state-simp[simp del]

```

To avoid ambiguities:

```

no-notation state-eq-twl (infix  $\sim$  51)

```

```

inductive propagate-twl :: 'v wf-twl  $\Rightarrow$  'v wf-twl  $\Rightarrow$  bool where
  propagate-twl-rule:  $(L, C) \in$  candidates-propagate-twl  $S \Rightarrow$ 
     $S' \sim$  cons-trail-twl (Propagated  $L C$ )  $S \Rightarrow$ 
    raw-conflicting-twl  $S =$  None  $\Rightarrow$ 
    propagate-twl  $S S'$ 

```

```

inductive-cases propagate-twlE: propagate-twl  $S T$ 
thm propagateE

```

```

lemma distinct-filter-eq-if:
  distinct  $C \Rightarrow$  length (filter (op =  $L$ )  $C$ ) = (if  $L \in$  set  $C$  then 1 else 0)
by (induction  $C$ ) auto

```

```

lemma distinct-mset-remove1-All:
  distinct-mset  $C \Rightarrow$  remove1-mset  $L C =$  removeAll-mset  $L C$ 
by (auto simp: multiset-eq-iff distinct-mset-count-less-1)

```

```

lemma propagate-twl-iff-propagate:
  assumes inv: cdclW-all-struct-inv  $S$ 
  shows propagate  $S T \longleftrightarrow$  propagate-twl  $S T$  (is ? $P \longleftrightarrow$  ? $T$ )

```

proof

```

  assume ? $P$ 

```

```

  then obtain  $L E$  where

```

```

    raw-conflicting-twl  $S =$  None and

```

```

    CL-Clauses:  $E \in$  set (twl.raw-clauses  $S$ ) and

```

```

    LE:  $L \in \#$  mset (raw-clause  $E$ ) and

```

```

    tr-CNot: trail-twl  $S \models$  CNot (remove1-mset  $L$  (mset (raw-clause  $E$ ))) and

```

```

    undef-lot[simp]: undefined-lit (trail-twl  $S$ )  $L$  and

```

```

     $T \sim$  cons-trail-twl (Propagated  $L E$ )  $S$ 

```

```

    by (blast elim: propagateE)

```

```

have distinct (raw-clause  $E$ )

```

```

  using inv CL-Clauses unfolding cdclW-all-struct-inv-def distinct-mset-set-def
  distinct-cdclW-state-def raw-clauses-def by auto

```

```

then have  $X$ :  $\text{remove1-mset } L \ (\text{mset } (\text{raw-clause } E)) = \text{mset-set } (\text{set } (\text{raw-clause } E) - \{L\})$ 
  by (auto simp: multiset-eq-iff raw-clause-def count-mset distinct-filter-eq-if)
have  $(L, E) \in \text{candidates-propagate-twl } S$ 
apply (rule wf-candidates-propagate-complete)
  using rough-state-of-twl apply auto[]
  using CL-Clauses unfolding raw-clauses-def twl.raw-clauses-def
  apply auto[]
  using LE apply simp
  using tr-CNot X apply simp
  using undef-lot apply blast
done
show  $?T$ 
apply (rule propagate-twl-rule)
  apply (rule  $\langle (L, E) \in \text{candidates-propagate-twl } S \rangle$ )
  using  $\langle T \sim \text{cons-trail-twl } (\text{Propagated } L \ E) \ S \rangle$ 
  apply (auto simp: raw-conflicting-twl S = None twl.state-eq-def)
done
next
assume  $?T$ 
then obtain  $L \ C$  where
   $LC$ :  $(L, C) \in \text{candidates-propagate-twl } S$  and
   $T$ :  $T \sim \text{cons-trail-twl } (\text{Propagated } L \ C) \ S$  and
   $\text{confl}$ :  $\text{raw-conflicting-twl } S = \text{None}$ 
  by (auto elim: propagate-twlE)
have
   $C'S$ :  $C \in \text{set } (\text{raw-clauses-twl } S)$  and
   $L$ :  $\text{set } (\text{watched } C) - \text{uminus ' lits-of-l } (\text{trail-twl } S) = \{L\}$  and
   $\text{undef}$ :  $\text{undefined-lit } (\text{trail-twl } S) \ L$ 
  using  $LC$  unfolding candidates-propagate-def raw-clauses-def by auto
have  $\text{dist}$ :  $\text{distinct } (\text{raw-clause } C)$ 
  using inv C'S unfolding cdclW-all-struct-inv-def distinct-cdclW-state-def
  distinct-mset-set-def twl.raw-clauses-def by fastforce
then have  $C-L-L$ :  $\text{mset-set } (\text{set } (\text{raw-clause } C) - \{L\}) = \text{mset } (\text{raw-clause } C) - \{\#L\# \}$ 
  by (metis distinct-mset-distinct distinct-mset-minus distinct-mset-set-mset-ident mset-remove1
  set-mset-mset set-remove1-eq)

show  $?P$ 
apply (rule propagate-rule[of S C L])
  using  $\text{confl}$  apply auto[]
  using  $C'S$  unfolding twl.raw-clauses-def apply (simp add: raw-clauses-def)
  using  $L$  unfolding candidates-propagate-def apply (auto simp: raw-clause-def)[]
  using wf-candidates-propagate-sound[OF - LC] rough-state-of-twl dist
  apply (simp add: distinct-mset-remove1-All)
  using  $\text{undef}$  apply simp
  using  $T$   $\text{undef}$  by (smt backtrack-lvl-cons-trail confl init-clss-cons-trail
  learned-clss-cons-trail marked-lit.sel(2) raw-conflicting-cons-trail state-eq-def
  trail-cons-trail twl2.mmset-of-mlit.simps(1) twl2.mset-cls-cls-of-ccls)
qed

```

no-notation twl.state-eq-twl (**infix** $\sim \text{TWL}$ 51)

inductive *conflict-twl* **where**

conflict-twl-rule:

$C \in \text{candidates-conflict-twl } S \implies$

$S' \sim \text{update-conflicting-twl } (\text{Some } (\text{raw-clause } C)) \ S \implies$

raw-conflicting-twl $S = \text{None} \implies$
conflict-twl $S S'$

inductive-cases *conflict-twlE*: *conflict-twl* $S T$

lemma *conflict-twl-iff-conflict*:

shows *conflict* $S T \longleftrightarrow \text{conflict-twl } S T$ (**is** $?C \longleftrightarrow ?T$)

proof

assume $?C$

then obtain D **where**

S : *raw-conflicting-twl* $S = \text{None}$ **and**

D : $D \in \text{set } (\text{raw-clauses } S)$ **and**

MD : *trail-twl* $S \models_{\text{as}} \text{CNot } (\text{mset } (\text{raw-clause } D))$ **and**

T : $T \sim \text{update-conflicting-twl } (\text{Some } (\text{raw-clause } D)) S$

by (*elim conflictE*)

have $D \in \text{candidates-conflict-twl } S$

apply (*rule wf-candidates-conflict-complete*)

apply *simp*

using D **apply** (*auto simp: raw-clauses-def twl.raw-clauses-def*)[]

using $MD S$ **by** *auto*

moreover have $T \sim \text{twl-of-rough-state } (\text{update-conflicting } (\text{Some } (\text{raw-clause } D)))$
(rough-state-of-twl S)

using T **unfolding** *rough-cdcl.state-eq-def state-eq-def* **by** *auto*

ultimately show $?T$

using S **by** (*auto intro: conflict-twl-rule*)

next

assume $?T$

then obtain C **where**

C : $C \in \text{candidates-conflict-twl } S$ **and**

T : $T \sim \text{update-conflicting-twl } (\text{Some } (\text{raw-clause } C)) S$ **and**

confl: *raw-conflicting-twl* $S = \text{None}$

by (*auto elim: conflict-twlE*)

have

$C \in \text{set } (\text{raw-clauses } S)$

using C **unfolding** *candidates-conflict-def raw-clauses-def twl.raw-clauses-def* **by** *auto*

moreover have *trail-twl* $S \models_{\text{as}} \text{CNot } (\text{mset } (\text{raw-clause } C))$

using *wf-candidates-conflict-sound*[*OF - C*] **by** *auto*

ultimately show $?C$ **apply** –

apply (*rule conflict.conflict-rule*[*of - C*])

using *confl T* **unfolding** *rough-cdcl.state-eq-def* **by** (*auto simp del: map-map*)

qed

inductive *cdcl_W-twl* :: ' v *wf-twl* \Rightarrow ' v *wf-twl* \Rightarrow bool **for** S :: ' v *wf-twl* **where**

propagate: *propagate-twl* $S S' \implies \text{cdcl}_W\text{-twl } S S' \mid$

conflict: *conflict-twl* $S S' \implies \text{cdcl}_W\text{-twl } S S' \mid$

other: *cdcl_W-o* $S S' \implies \text{cdcl}_W\text{-twl } S S' \mid$

rf: *cdcl_W-rf* $S S' \implies \text{cdcl}_W\text{-twl } S S'$

lemma *cdcl_W-twl-iff-cdcl_W*:

assumes *cdcl_W-all-struct-inv* S

shows *cdcl_W-twl* $S T \longleftrightarrow \text{cdcl}_W S T$

by (*simp add: assms cdcl_W.simps cdcl_W-twl.simps conflict-twl-iff-conflict*
propagate-twl-iff-propagate del: map-map)

```

lemma rtrancpl-cdclW-twl-all-struct-inv-inv:
  assumes cdclW-twl** S T and cdclW-all-struct-inv S
  shows cdclW-all-struct-inv T
  using assms by (induction rule: rtrancpl-induct)
  (simp-all add: cdclW-twl-iff-cdclW cdclW-all-struct-inv-inv del: map-map)

lemma rtrancpl-cdclW-twl-iff-rtrancpl-cdclW:
  assumes cdclW-all-struct-inv S
  shows cdclW-twl** S T  $\longleftrightarrow$  cdclW** S T (is ?T  $\longleftrightarrow$  ?W)
proof
  assume ?W
  then show ?T
    proof (induction rule: rtrancpl-induct)
      case base
        then show ?case by simp
      next
        case (step T U) note st = this(1) and cdcl = this(2) and IH = this(3)
        have cdclW-twl T U
          using assms st cdcl rtrancpl-cdclW-all-struct-inv-inv cdclW-twl-iff-cdclW
          by blast
        then show ?case using IH by auto
      qed
    next
      assume ?T
      then show ?W
        proof (induction rule: rtrancpl-induct)
          case base
            then show ?case by simp
          next
            case (step T U) note st = this(1) and cdcl = this(2) and IH = this(3)
            have cdclW T U
              using assms st cdcl rtrancpl-cdclW-twl-all-struct-inv-inv cdclW-twl-iff-cdclW
              by blast
            then show ?case using IH by auto
          qed
        qed
      qed
    end
  end
theory Prop-Superposition
imports Partial-Clausal-Logic ../lib/Herbrand-Interpretation
begin
sledgehammer-params[verbose]
no-notation Herbrand-Interpretation.true-cls (infix  $\models$  50)
notation Herbrand-Interpretation.true-cls (infix  $\models_h$  50)

no-notation Herbrand-Interpretation.true-clss (infix  $\models_s$  50)
notation Herbrand-Interpretation.true-clss (infix  $\models_{hs}$  50)

lemma herbrand-interp-iff-partial-interp-cls:
  S  $\models_h C \longleftrightarrow \{Pos P | P. P \in S\} \cup \{Neg P | P. P \notin S\} \models C$ 
  unfolding Herbrand-Interpretation.true-cls-def Partial-Clausal-Logic.true-cls-def
  by auto

```


lemma *herbrand-consistent-interp:*

consistent-interp ($\{Pos\ P|P. P \in S\} \cup \{Neg\ P|P. P \notin S\}$)

unfolding *consistent-interp-def* **by** *auto*

lemma *herbrand-total-over-set:*

total-over-set ($\{Pos\ P|P. P \in S\} \cup \{Neg\ P|P. P \notin S\}$) *T*

unfolding *total-over-set-def* **by** *auto*

lemma *herbrand-total-over-m:*

total-over-m ($\{Pos\ P|P. P \in S\} \cup \{Neg\ P|P. P \notin S\}$) *T*

unfolding *total-over-m-def* **by** (*auto simp add: herbrand-total-over-set*)

lemma *herbrand-interp-iff-partial-interp-clss:*

$S \models_{hs} C \iff \{Pos\ P|P. P \in S\} \cup \{Neg\ P|P. P \notin S\} \models_s C$

unfolding *true-clss-def Ball-def herbrand-interp-iff-partial-interp-clss*

Partial-Clausal-Logic.true-clss-def **by** *auto*

definition *clss-lt* :: *'a::wellorder clauses* \Rightarrow *'a clause* \Rightarrow *'a clauses* **where**

clss-lt *N C* = $\{D \in N. D \# \subset \# C\}$

notation (*latex output*)

clss-lt ($-\hat{<}^{bsup} -\hat{<}^{esup}$)

locale *selection* =

fixes *S* :: *'a clause* \Rightarrow *'a clause*

assumes

S-selects-subseteq: $\bigwedge C. S\ C \leq \# C$ **and**

S-selects-neg-lits: $\bigwedge C\ L. L \in \# S\ C \implies is_neg\ L$

locale *ground-resolution-with-selection* =

selection S **for** *S* :: (*'a* :: *wellorder*) *clause* \Rightarrow *'a clause*

begin

context

fixes *N* :: *'a clause set*

begin

We do not create an equivalent of δ , but we directly defined N_C by inlining the definition.

function

production :: *'a clause* \Rightarrow *'a interp*

where

production C =

$\{A. C \in N \wedge C \neq \{\#\} \wedge \text{Max}(\text{set-mset } C) = \text{Pos } A \wedge \text{count } C (\text{Pos } A) \leq 1$
 $\wedge \neg (\bigcup D \in \{D. D \# \subset \# C\}. \text{production } D) \models_h C \wedge S\ C = \{\#\}\}$

by *auto*

termination by (*relation* $\{(D, C). D \# \subset \# C\}$) (*auto simp: wf-less-multiset*)

declare *production.simps*[*simp del*]

definition *interp* :: *'a clause* \Rightarrow *'a interp* **where**

interp C = $(\bigcup D \in \{D. D \# \subset \# C\}. \text{production } D)$

lemma *production-unfold:*

production C = $\{A. C \in N \wedge C \neq \{\#\} \wedge \text{Max}(\text{set-mset } C) = \text{Pos } A \wedge \text{count } C (\text{Pos } A) \leq 1 \wedge \neg$

interp C $\models_h C \wedge S\ C = \{\#\}\}$

unfolding *interp-def* **by** (*rule production.simps*)

abbreviation *productive* $A \equiv (\text{production } A \neq \{\})$

abbreviation *produces* $:: 'a \text{ clause} \Rightarrow 'a \Rightarrow \text{bool}$ **where**
produces $C \ A \equiv \text{production } C = \{A\}$

lemma *producesD*:
produces $C \ A \Longrightarrow C \in N \wedge C \neq \{\#\} \wedge \text{Pos } A = \text{Max } (\text{set-mset } C) \wedge \text{count } C (\text{Pos } A) \leq 1 \wedge \neg$
interp $C \models_h C \wedge S \ C = \{\#\}$
unfolding *production-unfold* **by** *auto*

lemma *produces* $C \ A \Longrightarrow \text{Pos } A \in \# \ C$
by (*simp add: Max-in-lits producesD*)

lemma *interp'-def-in-set*:
interp $C = (\bigcup D \in \{D \in N. D \# \subseteq \# \ C\}. \text{production } D)$
unfolding *interp-def* **apply** *auto*
unfolding *production-unfold* **apply** *auto*
done

lemma *production-iff-produces*:
produces $D \ A \longleftrightarrow A \in \text{production } D$
unfolding *production-unfold* **by** *auto*

definition *Interp* $:: 'a \text{ clause} \Rightarrow 'a \text{ interp}$ **where**
Interp $C = \text{interp } C \cup \text{production } C$

lemma
assumes *produces* $C \ P$
shows *Interp* $C \models_h C$
unfolding *Interp-def* *assms* **using** *producesD*[*OF assms*]
by (*metis Max-in-lits Un-insert-right insertI1 pos-literal-in-imp-true-cls*)

definition *INTERP* $:: 'a \text{ interp}$ **where**
INTERP $= (\bigcup D \in N. \text{production } D)$

lemma *interp-subseteq-Interp*[*simp*]: *interp* $C \subseteq \text{Interp } C$
unfolding *Interp-def* **by** *simp*

lemma *Interp-as-UNION*: *Interp* $C = (\bigcup D \in \{D. D \# \subseteq \# \ C\}. \text{production } D)$
unfolding *Interp-def* *interp-def* *le-multiset-def* **by** *fast*

lemma *productive-not-empty*: *productive* $C \Longrightarrow C \neq \{\#\}$
unfolding *production-unfold* **by** *auto*

lemma *productive-imp-produces-Max-literal*: *productive* $C \Longrightarrow \text{produces } C (\text{atm-of } (\text{Max } (\text{set-mset } C)))$
unfolding *production-unfold* **by** (*auto simp del: atm-of-Max-lit*)

lemma *productive-imp-produces-Max-atom*: *productive* $C \Longrightarrow \text{produces } C (\text{Max } (\text{atms-of } C))$
unfolding *atms-of-def* *Max-atm-of-set-mset-commute*[*OF productive-not-empty*]
by (*rule productive-imp-produces-Max-literal*)

lemma *produces-imp-Max-literal*: *produces* $C \ A \Longrightarrow A = \text{atm-of } (\text{Max } (\text{set-mset } C))$

by (metis Max-singleton insert-not-empty productive-imp-produces-Max-literal)

lemma produces-imp-Max-atom: produces C $A \implies A = \text{Max} (\text{atms-of } C)$
 by (metis Max-singleton insert-not-empty productive-imp-produces-Max-atom)

lemma produces-imp-Pos-in-lits: produces C $A \implies \text{Pos } A \in\# C$
 by (auto intro: Max-in-lits dest!: producesD)

lemma productive-in-N: productive $C \implies C \in N$
 unfolding production-unfold by auto

lemma produces-imp-atms-leq: produces C $A \implies B \in \text{atms-of } C \implies B \leq A$
 by (metis Max-ge finite-atms-of insert-not-empty productive-imp-produces-Max-atom singleton-inject)

lemma produces-imp-neg-notin-lits: produces C $A \implies \neg \text{Neg } A \in\# C$
 by (rule pos-Max-imp-neg-notin) (auto dest: producesD)

lemma less-eq-imp-interp-subseteq-interp: $C \# \subseteq\# D \implies \text{interp } C \subseteq \text{interp } D$
 unfolding interp-def by auto (metis multiset-order.order.strict-trans2)

lemma less-eq-imp-interp-subseteq-Interp: $C \# \subseteq\# D \implies \text{interp } C \subseteq \text{Interp } D$
 unfolding Interp-def using less-eq-imp-interp-subseteq-interp by blast

lemma less-imp-production-subseteq-interp: $C \# \subset\# D \implies \text{production } C \subseteq \text{interp } D$
 unfolding interp-def by fast

lemma less-eq-imp-production-subseteq-Interp: $C \# \subseteq\# D \implies \text{production } C \subseteq \text{Interp } D$
 unfolding Interp-def using less-imp-production-subseteq-interp
 by (metis multiset-order.le-imp-less-or-eq le-supI1 sup-ge2)

lemma less-imp-Interp-subseteq-interp: $C \# \subset\# D \implies \text{Interp } C \subseteq \text{interp } D$
 unfolding Interp-def
 by (auto simp: less-eq-imp-interp-subseteq-interp less-imp-production-subseteq-interp)

lemma less-eq-imp-Interp-subseteq-Interp: $C \# \subseteq\# D \implies \text{Interp } C \subseteq \text{Interp } D$
 using less-imp-Interp-subseteq-interp
 unfolding Interp-def by (metis multiset-order.le-imp-less-or-eq le-supI2 subset-refl sup-commute)

lemma false-Interp-to-true-interp-imp-less-multiset: $A \notin \text{Interp } C \implies A \in \text{interp } D \implies C \# \subset\# D$
 using less-eq-imp-interp-subseteq-Interp multiset-linorder.not-less by blast

lemma false-interp-to-true-interp-imp-less-multiset: $A \notin \text{interp } C \implies A \in \text{interp } D \implies C \# \subset\# D$
 using less-eq-imp-interp-subseteq-interp multiset-linorder.not-less by blast

lemma false-Interp-to-true-Interp-imp-less-multiset: $A \notin \text{Interp } C \implies A \in \text{Interp } D \implies C \# \subset\# D$
 using less-eq-imp-Interp-subseteq-Interp multiset-linorder.not-less by blast

lemma false-interp-to-true-Interp-imp-le-multiset: $A \notin \text{interp } C \implies A \in \text{Interp } D \implies C \# \subseteq\# D$
 using less-imp-Interp-subseteq-interp multiset-linorder.not-less by blast

lemma interp-subseteq-INTERP: $\text{interp } C \subseteq \text{INTERP}$
 unfolding interp-def INTERP-def by (auto simp: production-unfold)

lemma production-subseteq-INTERP: $\text{production } C \subseteq \text{INTERP}$

unfolding *INTERP-def* **using** *production-unfold* **by** *blast*

lemma *Interp-subseteq-INTERP*: *Interp C* \subseteq *INTERP*

unfolding *Interp-def* **by** (*auto intro!*: *interp-subseteq-INTERP* *production-subseteq-INTERP*)

This lemma corresponds to theorem 2.7.6 page 66 of CW.

lemma *produces-imp-in-interp*:

assumes *a-in-c*: *Neg A* $\in\#$ *C* **and** *d*: *produces D A*

shows *A* \in *interp C*

proof –

from *d* **have** *Max (set-mset D) = Pos A*

using *production-unfold* **by** *blast*

hence *D* $\# \subset \#$ $\{\#Neg A\# \}$

by (*auto intro*: *Max-pos-neg-less-multiset*)

moreover have $\{\#Neg A\# \}$ $\# \subseteq \#$ *C*

by (*rule less-eq-imp-le-multiset*) (*rule mset-le-single*[*OF a-in-c*])

ultimately show *?thesis*

using *d* **by** (*blast dest*: *less-eq-imp-interp-subseteq-interp* *less-imp-production-subseteq-interp*)

qed

lemma *neg-notin-Interp-not-produce*: *Neg A* $\in\#$ *C* $\implies A \notin$ *Interp D* $\implies C \# \subseteq \# D \implies \neg$ *produces D'' A*

by (*auto dest*: *produces-imp-in-interp* *less-eq-imp-interp-subseteq-Interp*)

lemma *in-production-imp-produces*: *A* \in *production C* \implies *produces C A*

by (*metis insert-absorb productive-imp-produces-Max-atom singleton-insert-inj-eq'*)

lemma *not-produces-imp-notin-production*: \neg *produces C A* $\implies A \notin$ *production C*

by (*metis in-production-imp-produces*)

lemma *not-produces-imp-notin-interp*: $(\bigwedge D. \neg$ *produces D A*) $\implies A \notin$ *interp C*

unfolding *interp-def* **by** (*fast intro!*: *in-production-imp-produces*)

The results below corresponds to Lemma 3.4.

Nitpicking: If $D = D'$ and D is productive, $I^D \subseteq I_{D'}$ does not hold.

lemma *true-Interp-imp-general*:

assumes

c-le-d: *C* $\# \subseteq \#$ *D* **and**

d-lt-d': *D* $\# \subset \#$ *D'* **and**

c-at-d: *Interp D* $\models_h C$ **and**

subs: *interp D'* $\subseteq (\bigcup C \in CC. \text{production } C)$

shows $(\bigcup C \in CC. \text{production } C) \models_h C$

proof (*cases* $\exists A. Pos A \in\# C \wedge A \in Interp D$)

case *True*

then obtain *A* **where** *a-in-c*: *Pos A* $\in\#$ *C* **and** *a-at-d*: *A* \in *Interp D*

by *blast*

from *a-at-d* **have** *A* \in *interp D'*

using *d-lt-d'* *less-imp-Interp-subseteq-interp* **by** *blast*

thus *?thesis*

using *subs a-in-c* **by** (*blast dest*: *contra-subsetD*)

next

case *False*

then obtain *A* **where** *a-in-c*: *Neg A* $\in\#$ *C* **and** *A* \notin *Interp D*

using *c-at-d* **unfolding** *true-cls-def* **by** *blast*

hence $\bigwedge D''. \neg$ *produces D'' A*

```

    using c-le-d neg-notin-Interp-not-produce by simp
  thus ?thesis
    using a-in-c subs not-produces-imp-notin-production by auto
qed

lemma true-Interp-imp-imp:  $C \# \subseteq \# D \implies D \# \subset \# D' \implies \text{Interp } D \models_h C \implies \text{interp } D' \models_h C$ 
  using interp-def true-Interp-imp-general by simp

lemma true-Interp-imp-Interp:  $C \# \subseteq \# D \implies D \# \subset \# D' \implies \text{Interp } D \models_h C \implies \text{Interp } D' \models_h C$ 
  using Interp-as-UNION interp-subseteq-Interp true-Interp-imp-general by simp

lemma true-Interp-imp-INTERP:  $C \# \subseteq \# D \implies \text{Interp } D \models_h C \implies \text{INTERP} \models_h C$ 
  using INTERP-def interp-subseteq-INTERP
    true-Interp-imp-general[OF - less-multiset-right-total]
  by simp

lemma true-interp-imp-general:
  assumes
    c-le-d:  $C \# \subseteq \# D$  and
    d-lt-d':  $D \# \subset \# D'$  and
    c-at-d:  $\text{interp } D \models_h C$  and
    subs:  $\text{interp } D' \subseteq (\bigcup C \in CC. \text{production } C)$ 
  shows  $(\bigcup C \in CC. \text{production } C) \models_h C$ 
proof (cases  $\exists A. \text{Pos } A \in \# C \wedge A \in \text{interp } D$ )
case True
  then obtain A where a-in-c:  $\text{Pos } A \in \# C$  and a-at-d:  $A \in \text{interp } D$ 
  by blast
  from a-at-d have  $A \in \text{interp } D'$ 
  using d-lt-d' less-eq-imp-interp-subseteq-interp[OF multiset-order.less-imp-le] by blast
  thus ?thesis
    using subs a-in-c by (blast dest: contra-subsetD)
next
case False
  then obtain A where a-in-c:  $\text{Neg } A \in \# C$  and  $A \notin \text{interp } D$ 
  using c-at-d unfolding true-cls-def by blast
  hence  $\bigwedge D''. \neg \text{produces } D'' A$ 
  using c-le-d by (auto dest: produces-imp-in-interp less-eq-imp-interp-subseteq-interp)
  thus ?thesis
    using a-in-c subs not-produces-imp-notin-production by auto
qed

This lemma corresponds to theorem 2.7.6 page 66 of CW. Here the strict maximality is important

lemma true-interp-imp-imp:  $C \# \subseteq \# D \implies D \# \subset \# D' \implies \text{interp } D \models_h C \implies \text{interp } D' \models_h C$ 
  using interp-def true-interp-imp-general by simp

lemma true-interp-imp-Interp:  $C \# \subseteq \# D \implies D \# \subset \# D' \implies \text{interp } D \models_h C \implies \text{Interp } D' \models_h C$ 
  using Interp-as-UNION interp-subseteq-Interp[of D'] true-interp-imp-general by simp

lemma true-interp-imp-INTERP:  $C \# \subseteq \# D \implies \text{interp } D \models_h C \implies \text{INTERP} \models_h C$ 
  using INTERP-def interp-subseteq-INTERP
    true-interp-imp-general[OF - less-multiset-right-total]
  by simp

lemma productive-imp-false-interp:  $\text{productive } C \implies \neg \text{interp } C \models_h C$ 
  unfolding production-unfold by auto

```

This lemma corresponds to theorem 2.7.6 page 66 of CW. Here the strict maximality is important

```

lemma cls-gt-double-pos-no-production:
  assumes  $D: \{\#Pos\ P, Pos\ P\#\} \# \subset \# C$ 
  shows  $\neg produces\ C\ P$ 
proof -
  let  $?D = \{\#Pos\ P, Pos\ P\#\}$ 
  note  $D' = D[unfolded\ less-multiset_{HO}]$ 
  consider
     $(P)\ count\ C\ (Pos\ P) \geq 2$ 
  |  $(Q)\ Q\ \textbf{where}\ Q > Pos\ P\ \textbf{and}\ Q \in \# C$ 
    using  $HOL.spec[OF\ HOL.conjunct2[OF\ D'],\ of\ Pos\ P]$  by (auto split: if-split-asm)
  thus ?thesis
  proof cases
    case  $Q$ 
    have  $Q \in set-mset\ C$ 
    using  $Q(2)$  by (auto split: if-split-asm)
    then have  $Max\ (set-mset\ C) > Pos\ P$ 
    using  $Q(1)\ Max-gr-iff$  by blast
    thus ?thesis
    unfolding production-unfold by auto
  next
    case  $P$ 
    thus ?thesis
    unfolding production-unfold by auto
  qed
qed

```

This lemma corresponds to theorem 2.7.6 page 66 of CW.

```

lemma
  assumes  $D: C + \{\#Neg\ P\#\} \# \subset \# D$ 
  shows  $production\ D \neq \{P\}$ 
proof -
  note  $D' = D[unfolded\ less-multiset_{HO}]$ 
  consider
     $(P)\ Neg\ P \in \# D$ 
  |  $(Q)\ Q\ \textbf{where}\ Q > Neg\ P\ \textbf{and}\ count\ D\ Q > count\ (C + \{\#Neg\ P\#\})\ Q$ 
    using  $HOL.spec[OF\ HOL.conjunct2[OF\ D'],\ of\ Neg\ P]\ count-greater-zero-iff$  by fastforce
  thus ?thesis
  proof cases
    case  $Q$ 
    have  $Q \in set-mset\ D$ 
    using  $Q(2)\ gr-implies-not0$  by fastforce
    then have  $Max\ (set-mset\ D) > Neg\ P$ 
    using  $Q(1)\ Max-gr-iff$  by blast
    hence  $Max\ (set-mset\ D) > Pos\ P$ 
    using less-trans[of Pos P Neg P Max (set-mset D)] by auto
    thus ?thesis
    unfolding production-unfold by auto
  next
    case  $P$ 
    hence  $Max\ (set-mset\ D) > Pos\ P$ 
    by (meson Max-ge finite-set-mset le-less-trans linorder-not-le pos-less-neg)
    thus ?thesis
    unfolding production-unfold by auto
  qed

```

qed

lemma *in-interp-is-produced*:

assumes $P \in \text{INTERP}$

shows $\exists D. D + \{\#Pos\ P\# \} \in N \wedge \text{produces } (D + \{\#Pos\ P\# \})\ P$

using *assms unfolding INTERP-def UN-iff production-iff-produces Ball-def*

by (*metis ground-resolution-with-selection.produces-imp-Pos-in-lits insert-DiffM2*
ground-resolution-with-selection-axioms not-produces-imp-notin-production)

end

end

abbreviation $MMax\ M \equiv Max\ (set-mset\ M)$

24.2 We can now define the rules of the calculus

inductive *superposition-rules* :: '*a clause* \Rightarrow '*a clause* \Rightarrow '*a clause* \Rightarrow *bool* **where**

factoring: *superposition-rules* $(C + \{\#Pos\ P\# \} + \{\#Pos\ P\# \})\ B\ (C + \{\#Pos\ P\# \})\ |$

superposition-l: *superposition-rules* $(C_1 + \{\#Pos\ P\# \})\ (C_2 + \{\#Neg\ P\# \})\ (C_1 + C_2)$

inductive *superposition* :: '*a clauses* \Rightarrow '*a clauses* \Rightarrow *bool* **where**

superposition: $A \in N \Longrightarrow B \in N \Longrightarrow \text{superposition-rules } A\ B\ C$

$\Longrightarrow \text{superposition } N\ (N \cup \{C\})$

definition *abstract-red* :: '*a::wellorder clause* \Rightarrow '*a clauses* \Rightarrow *bool* **where**

abstract-red $C\ N = (clss-lt\ N\ C \models_p C)$

lemma *less-multiset[iff]*: $M < N \longleftrightarrow M \# \subset \# N$

unfolding *less-multiset-def* **by** *auto*

lemma *less-eq-multiset[iff]*: $M \leq N \longleftrightarrow M \# \subseteq \# N$

unfolding *less-eq-multiset-def* **by** *auto*

lemma *herbrand-true-clss-true-clss-clss-herbrand-true-clss*:

assumes

$AB: A \models_{hs} B$ **and**

$BC: B \models_p C$

shows $A \models_h C$

proof –

let $?I = \{Pos\ P\ | P. P \in A\} \cup \{Neg\ P\ | P. P \notin A\}$

have $B: ?I \models_s B$ **using** AB

by (*auto simp add: herbrand-interp-iff-partial-interp-clss*)

have $IH: \bigwedge I. \text{total-over-set } I\ (\text{atms-of } C) \Longrightarrow \text{total-over-m } I\ B \Longrightarrow \text{consistent-interp } I$

$\Longrightarrow I \models_s B \Longrightarrow I \models C$ **using** BC

by (*auto simp add: true-clss-clss-def*)

show *?thesis*

unfolding *herbrand-interp-iff-partial-interp-clss*

by (*auto intro: IH[of ?I] simp add: herbrand-total-over-set herbrand-total-over-m*
herbrand-consistent-interp B)

qed

lemma *abstract-red-subset-mset-abstract-red*:

assumes

abstr: *abstract-red* $C\ N$ **and**

$c\text{-lt-d}: C \subseteq\# D$
shows *abstract-red* $D N$
proof –
have $\{D \in N. D \# \subset\# C\} \subseteq \{D' \in N. D' \# \subset\# D\}$
using *c-lt-d less-eq-imp-le-multiset* **by** *fastforce*
thus *?thesis*
using *abstr unfolding abstract-red-def class-lt-def*
by (*metis (no-types, lifting) c-lt-d subset-mset.diff-add true-class-cls-mono-r'*
true-class-cls-subset)
qed

lemma *true-class-cls-extended*:

assumes
 $A \models_p B$ **and**
tot: *total-over-m* $I (A)$ **and**
cons: *consistent-interp* I **and**
 $I \models_s A$
shows $I \models B$
proof –
let $?I = I \cup \{Pos P \mid P. P \in \text{atms-of } B \wedge P \notin \text{atms-of-s } I\}$
have *consistent-interp* $?I$
using *cons unfolding consistent-interp-def atms-of-s-def atms-of-def*
apply (*auto 1 5 simp add: image-iff*)
by (*metis atm-of-uminus literal.sel(1)*)
moreover have *total-over-m* $?I (A \cup \{B\})$
proof –
obtain $aa :: 'a \text{ set} \Rightarrow 'a \text{ literal set} \Rightarrow 'a$ **where**
 $f2: \forall x0 x1. (\exists v2. v2 \in x0 \wedge Pos v2 \notin x1 \wedge Neg v2 \notin x1)$
 $\longleftrightarrow (aa x0 x1 \in x0 \wedge Pos (aa x0 x1) \notin x1 \wedge Neg (aa x0 x1) \notin x1)$
by *moura*
have $\forall a. a \notin \text{atms-of-ms } A \vee Pos a \in I \vee Neg a \in I$
using *tot by (simp add: total-over-m-def total-over-set-def)*
hence $aa (\text{atms-of-ms } A \cup \text{atms-of-ms } \{B\}) (I \cup \{Pos a \mid a. a \in \text{atms-of } B \wedge a \notin \text{atms-of-s } I\})$
 $\notin \text{atms-of-ms } A \cup \text{atms-of-ms } \{B\} \vee Pos (aa (\text{atms-of-ms } A \cup \text{atms-of-ms } \{B\})$
 $(I \cup \{Pos a \mid a. a \in \text{atms-of } B \wedge a \notin \text{atms-of-s } I\})) \in I$
 $\cup \{Pos a \mid a. a \in \text{atms-of } B \wedge a \notin \text{atms-of-s } I\}$
 $\vee Neg (aa (\text{atms-of-ms } A \cup \text{atms-of-ms } \{B\})$
 $(I \cup \{Pos a \mid a. a \in \text{atms-of } B \wedge a \notin \text{atms-of-s } I\})) \in I$
 $\cup \{Pos a \mid a. a \in \text{atms-of } B \wedge a \notin \text{atms-of-s } I\}$
by *auto*
hence *total-over-set* $(I \cup \{Pos a \mid a. a \in \text{atms-of } B \wedge a \notin \text{atms-of-s } I\}) (\text{atms-of-ms } A \cup \text{atms-of-ms } \{B\})$
using $f2$ **by** (*meson total-over-set-def*)
thus *?thesis*
by (*simp add: total-over-m-def*)
qed
moreover have $?I \models_s A$
using *I-A* **by** *auto*
ultimately have $?I \models B$
using $\langle A \models_p B \rangle$ **unfolding** *true-class-cls-def* **by** *auto*
thus *?thesis*
oops
lemma
assumes

$CP: \neg \text{clss-lt } N (\{ \#C\# \} + \{ \#E\# \}) \models_p \{ \#C\# \} + \{ \#Neg P\# \}$ **and**
 $\text{clss-lt } N (\{ \#C\# \} + \{ \#E\# \}) \models_p \{ \#E\# \} + \{ \#Pos P\# \} \vee \text{clss-lt } N (\{ \#C\# \} + \{ \#E\# \}) \models_p$
 $\{ \#C\# \} + \{ \#Neg P\# \}$
shows $\text{clss-lt } N (\{ \#C\# \} + \{ \#E\# \}) \models_p \{ \#E\# \} + \{ \#Pos P\# \}$

oops

locale *ground-ordered-resolution-with-redundancy* =

ground-resolution-with-selection +

fixes *redundant* :: 'a::wellorder clause \Rightarrow 'a clauses \Rightarrow bool

assumes

redundant-iff-abstract: $\text{redundant } A \ N \longleftrightarrow \text{abstract-red } A \ N$

begin

definition *saturated* :: 'a clauses \Rightarrow bool **where**

$\text{saturated } N \longleftrightarrow (\forall A \ B \ C. A \in N \longrightarrow B \in N \longrightarrow \neg \text{redundant } A \ N \longrightarrow \neg \text{redundant } B \ N$
 $\longrightarrow \text{superposition-rules } A \ B \ C \longrightarrow \text{redundant } C \ N \vee C \in N)$

lemma

assumes

saturated: $\text{saturated } N$ **and**

finite: $\text{finite } N$ **and**

empty: $\{ \# \} \notin N$

shows $\text{INTERP } N \models_{hs} N$

proof (*rule ccontr*)

let $?N_{\mathcal{I}} = \text{INTERP } N$

assume $\neg ?thesis$

hence *not-empty*: $\{ E \in N. \neg ?N_{\mathcal{I}} \models_h E \} \neq \{ \}$

unfolding *true-clss-def Ball-def* **by** *auto*

def $D \equiv \text{Min } \{ E \in N. \neg ?N_{\mathcal{I}} \models_h E \}$

have [*simp*]: $D \in N$

unfolding *D-def*

by (*metis (mono-tags, lifting) Min-in not-empty finite mem-Collect-eq rev-finite-subset subsetI*)

have *not-d-interp*: $\neg ?N_{\mathcal{I}} \models_h D$

unfolding *D-def*

by (*metis (mono-tags, lifting) Min-in finite mem-Collect-eq not-empty rev-finite-subset subsetI*)

have *cls-not-D*: $\bigwedge E. E \in N \implies E \neq D \implies \neg ?N_{\mathcal{I}} \models_h E \implies D \leq E$

using *finite D-def* **by** (*auto simp del: less-eq-multiset*)

obtain $C \ L$ **where** $D: D = C + \{ \#L\# \}$ **and** $LSD: L \in \# \ S \ D \vee (S \ D = \{ \# \} \wedge \text{Max } (\text{set-mset } D) = L)$

proof (*cases* $S \ D = \{ \# \}$)

case *False*

then obtain L **where** $L \in \# \ S \ D$

using *Max-in-lits* **by** *blast*

moreover

hence $L \in \# \ D$

using *S-selects-subseteq[of D]* **by** *auto*

hence $D = (D - \{ \#L\# \}) + \{ \#L\# \}$

by *auto*

ultimately show *?thesis* **using** *that* **by** *blast*

next

let $?L = \text{MMax } D$

case *True*

moreover

have $?L \in \# \ D$

by (*metis (no-types, lifting) Max-in-lits (D ∈ N) empty*)

```

    hence  $D = (D - \{\#?L\#\}) + \{\#?L\#\}$ 
    by auto
    ultimately show ?thesis using that by blast
qed
have red:  $\neg \text{redundant } D \ N$ 
proof (rule ccontr)
  assume red[simplified]:  $\sim \sim \text{redundant } D \ N$ 
  have  $\forall E < D. E \in N \longrightarrow ?N_{\mathcal{I}} \models_h E$ 
    using cls-not-D not-le by fastforce
  hence  $?N_{\mathcal{I}} \models_{hs} \text{clss-lt } N \ D$ 
    unfolding clss-lt-def true-clss-def Ball-def by blast
  thus False
    using red not-d-interp unfolding abstract-red-def redundant-iff-abstract
    using herbrand-true-clss-true-clss-clss-herbrand-true-clss by fast
qed

consider
  (L) P where  $L = \text{Pos } P$  and  $S \ D = \{\#\}$  and  $\text{Max } (\text{set-mset } D) = \text{Pos } P$ 
| (Lneg) P where  $L = \text{Neg } P$ 
  using LSD S-selects-neg-lits[of L D] by (cases L) auto
thus False
proof cases
  case L note  $P = \text{this}(1)$  and  $S = \text{this}(2)$  and  $\text{max} = \text{this}(3)$ 
  have count  $D \ L > 1$ 
  proof (rule ccontr)
    assume  $\sim ?thesis$ 
    hence count:  $\text{count } D \ L = 1$ 
    unfolding D by (auto simp: not-in-iff)
    have  $\neg ?N_{\mathcal{I}} \models_h D$ 
    using not-d-interp true-interp-imp-INTERP ground-resolution-with-selection-axioms
    by blast
    hence produces  $N \ D \ P$ 
    using not-empty empty finite  $\langle D \in N \rangle$  count L
    true-interp-imp-INTERP unfolding production-iff-produces unfolding production-unfold
    by (auto simp add: max not-empty)
    hence  $\text{INTERP } N \models_h D$ 
    unfolding D
    by (metis pos-literal-in-imp-true-clss produces-imp-Pos-in-lits
    production-subseteq-INTERP singletonI subsetCE)
    thus False
    using not-d-interp by blast
  qed
  then have  $\text{Pos } P \in \# \ C$ 
    by (simp add: P D)
  then obtain C' where  $C':D = C' + \{\#\text{Pos } P\#\} + \{\#\text{Pos } P\#\}$ 
    unfolding D by (metis (full-types) P insert-DiffM2)
  have sup: superposition-rules  $D \ D \ (D - \{\#L\#\})$ 
    unfolding C' L by (auto simp add: superposition-rules.simps)
  have  $C' + \{\#\text{Pos } P\#\} \ \# \subset \# \ C' + \{\#\text{Pos } P\#\} + \{\#\text{Pos } P\#\}$ 
    by auto
  moreover have  $\neg ?N_{\mathcal{I}} \models_h (D - \{\#L\#\})$ 
    using not-d-interp unfolding C' L by auto
  ultimately have  $C' + \{\#\text{Pos } P\#\} \notin N$ 
    by (metis (no-types, lifting) C' P add-diff-cancel-right' cls-not-D less-multiset
    multi-self-add-other-not-self not-le)

```

```

have  $D - \{\#L\} \# \subset \# D$ 
  unfolding  $C' L$  by auto
have  $c'-p-p: C' + \{\#Pos P\} + \{\#Pos P\} - \{\#Pos P\} = C' + \{\#Pos P\}$ 
  by auto
have redundant  $(C' + \{\#Pos P\}) N$ 
  using saturated red sup  $\langle D \in N \rangle \langle C' + \{\#Pos P\} \notin N \rangle$  unfolding saturated-def  $C' L c'-p-p$ 
  by blast
moreover have  $C' + \{\#Pos P\} \subseteq \# C' + \{\#Pos P\} + \{\#Pos P\}$ 
  by auto
ultimately show False
  using red unfolding  $C'$  redundant-iff-abstract by (blast dest:
    abstract-red-subset-mset-abstract-red)
next
case Lneg note  $L = this(1)$ 
have  $P \in ?N_{\mathcal{I}}$ 
  using not-d-interp unfolding  $D$  true-cls-def  $L$  by (auto split: if-split-asm)
then obtain  $E$  where
  DPN:  $E + \{\#Pos P\} \in N$  and
  prod: production  $N (E + \{\#Pos P\}) = \{P\}$ 
  using in-interp-is-produced by blast
have sup-EC: superposition-rules  $(E + \{\#Pos P\}) (C + \{\#Neg P\}) (E + C)$ 
  using superposition-l by fast
hence superposition  $N (N \cup \{E+C\})$ 
  using DPN  $\langle D \in N \rangle$  unfolding  $D L$  by (auto simp add: superposition.simps)
have
  PMax:  $Pos P = MMax (E + \{\#Pos P\})$  and
  count  $(E + \{\#Pos P\}) (Pos P) \leq 1$  and
   $S (E + \{\#Pos P\}) = \{\#\}$  and
   $\neg \text{interp } N (E + \{\#Pos P\}) \models_h E + \{\#Pos P\}$ 
  using prod unfolding production-unfold by auto
have  $Neg P \notin \# E$ 
  using prod produces-imp-neg-notin-lits by force
hence  $\bigwedge y. y \in \# (E + \{\#Pos P\})$ 
 $\implies \text{count } (E + \{\#Pos P\}) (Neg P) < \text{count } (C + \{\#Neg P\}) (Neg P)$ 
  using count-greater-zero-iff by fastforce
moreover have  $\bigwedge y. y \in \# (E + \{\#Pos P\}) \implies y < Neg P$ 
  using PMax by (metis DPN Max-less-iff empty finite-set-mset pos-less-neg
    set-mset-eq-empty-iff)
moreover have  $E + \{\#Pos P\} \neq C + \{\#Neg P\}$ 
  using prod produces-imp-neg-notin-lits by force
ultimately have  $E + \{\#Pos P\} \# \subset \# C + \{\#Neg P\}$ 
  unfolding less-multisetHO by (metis count-greater-zero-iff less-iff-Suc-add zero-less-Suc)
have ce-lt-d:  $C + E \# \subset \# D$ 
  unfolding  $D L$  by (simp add:  $\langle \bigwedge y. y \in \# E + \{\#Pos P\} \implies y < Neg P \rangle$  ex-gt-imp-less-multiset)
have  $?N_{\mathcal{I}} \models_h E + \{\#Pos P\}$ 
  using  $\langle P \in ?N_{\mathcal{I}} \rangle$  by blast
have  $?N_{\mathcal{I}} \models_h C+E \vee C+E \notin N$ 
  using ce-lt-d cls-not-D unfolding  $D$ -def by fastforce
have  $Pos P \notin \# C+E$ 
  using  $D \langle P \in \text{ground-resolution-with-selection.INTERP } S N \rangle$ 
 $\langle \text{count } (E + \{\#Pos P\}) (Pos P) \leq 1 \rangle$  multi-member-skip not-d-interp
  by (auto simp: not-in-iff)
hence  $\bigwedge y. y \in \# C+E$ 
 $\implies \text{count } (C+E) (Pos P) < \text{count } (E + \{\#Pos P\}) (Pos P)$ 
  using set-mset-def by fastforce

```

```

have  $\neg$ redundant  $(C + E) N$ 
proof (rule ccontr)
  assume  $red'[simplified]: \neg ?thesis$ 
  have  $abs: clss\text{-}lt\ N\ (C + E) \models_p C + E$ 
  using redundant-iff-abstract  $red'$  unfolding abstract-red-def by auto
  have  $clss\text{-}lt\ N\ (C + E) \models_p E + \{\#Pos\ P\#\} \vee clss\text{-}lt\ N\ (C + E) \models_p C + \{\#Neg\ P\#\}$ 
  proof clarify
    assume  $CP: \neg clss\text{-}lt\ N\ (C + E) \models_p C + \{\#Neg\ P\#\}$ 
    { fix  $I$ 
      assume
         $total\text{-}over\text{-}m\ I\ (clss\text{-}lt\ N\ (C + E) \cup \{E + \{\#Pos\ P\#\}\})$  and
         $consistent\text{-}interp\ I$  and
         $I \models_s clss\text{-}lt\ N\ (C + E)$ 
      hence  $I \models C + E$ 
      using abs sorry
      moreover have  $\neg I \models C + \{\#Neg\ P\#\}$ 
      using  $CP$  unfolding true-clss-clss-def
      sorry
      ultimately have  $I \models E + \{\#Pos\ P\#\}$  by auto
    }
    then show  $clss\text{-}lt\ N\ (C + E) \models_p E + \{\#Pos\ P\#\}$ 
      unfolding true-clss-clss-def by auto
  qed
  moreover have  $clss\text{-}lt\ N\ (C + E) \subseteq clss\text{-}lt\ N\ (C + \{\#Neg\ P\#\})$ 
  using ce-lt-d mult-less-trans unfolding clss-lt-def D L by force
  ultimately have  $redundant\ (C + \{\#Neg\ P\#\})\ N \vee clss\text{-}lt\ N\ (C + E) \models_p E + \{\#Pos\ P\#\}$ 
  unfolding redundant-iff-abstract abstract-red-def using true-clss-clss-subset by blast
  show False sorry
qed
moreover have  $\neg redundant\ (E + \{\#Pos\ P\#\})\ N$ 
sorry
ultimately have  $CEN: C + E \in N$ 
  using  $\langle D \in N \rangle \langle E + \{\#Pos\ P\#\} \in N \rangle$  saturated sup-EC red unfolding saturated-def D L
  by (metis union-commute)
have  $CED: C + E \neq D$ 
  using  $D$  ce-lt-d by auto
have  $interp: \neg INTERP\ N \models_h C + E$ 
sorry
show False
  using cls-not-D[OF CEN CED interp] ce-lt-d unfolding INTERP-def less-eq-multiset-def by
auto
qed
qed
end

lemma tautology-is-redundant:
  assumes tautology C
  shows abstract-red C N
  using assms unfolding abstract-red-def true-clss-clss-def tautology-def by auto

lemma subsumed-is-redundant:
  assumes  $AB: A \subset\# B$ 
  and  $AN: A \in N$ 

```

```

    shows abstract-red B N
proof -
  have  $A \in \text{class-lt } N \ B$  using  $AN \ AB$  unfolding class-lt-def
    by (auto dest: less-eq-imp-le-multiset simp add: multiset-order.dual-order.order-iff-strict)
  thus ?thesis
    using  $AB$  unfolding abstract-red-def true-class-cls-def Partial-Clausal-Logic.true-class-def
    by blast
qed

inductive redundant :: 'a clause  $\Rightarrow$  'a clauses  $\Rightarrow$  bool where
  subsumption:  $A \in N \Longrightarrow A \subset\# B \Longrightarrow \text{redundant } B \ N$ 

lemma redundant-is-redundancy-criterion:
  fixes  $A :: 'a :: \text{wellorder clause}$  and  $N :: 'a :: \text{wellorder clauses}$ 
  assumes redundant A N
  shows abstract-red A N
  using assms
proof (induction rule: redundant.induct)
  case (subsumption A B N)
  thus ?case
    using subsumed-is-redundant[of A N B] unfolding abstract-red-def class-lt-def by auto
qed

lemma redundant-mono:
   $\text{redundant } A \ N \Longrightarrow A \subseteq\# B \Longrightarrow \text{redundant } B \ N$ 
  apply (induction rule: redundant.induct)
  by (meson subset-mset.less-le-trans subsumption)

locale truc=
  selection S for S :: nat clause  $\Rightarrow$  nat clause
begin

end

end
theory Weidenbach-Book
imports
  Prop-Normalisation

  Prop-Resolution

  Prop-Superposition

  CDCL-NOT DPLL-NOT DPLL-W-Implementation CDCL-W-Implementation CDCL-W-Incremental
  CDCL-WNOT

begin

end

```