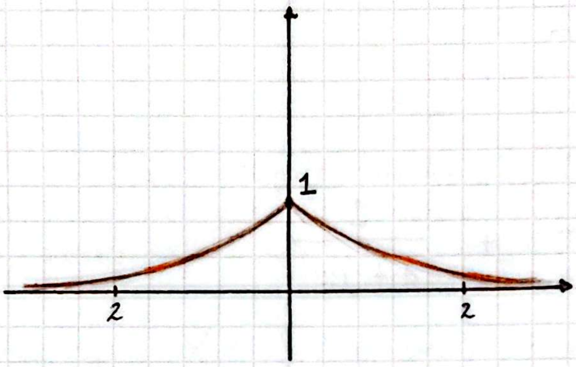
Taller 2

1.3. a) $x(t) = e^{-a|t|}$; $a \in \mathbb{R}^+$

∘ Para $a = 1$:



Como la señal es par, podemos decir:

$$x(t) = \begin{cases} e^{-at} & ; \quad t \geq 0 \\ e^{at} & ; \quad t < 0 \end{cases}$$

$$X(\omega) = \int_{-\infty}^{0} (e^{at} \cdot e^{-j\omega t})\, dt + \int_{0}^{\infty} (e^{-j\omega t} e^{-at})\, dt$$

∘ Resolviendo la primera integral:

$$\int_{-\infty}^{0} (e^{at} \cdot e^{-j\omega t})\, dt = \int_{-\infty}^{0} e^{at - j\omega t}\, dt = \left[\frac{e^{(a-j\omega)t}}{a - j\omega} \right]_{-\infty}^{0}$$

$$= \frac{1}{a - j\omega} - 0 = \frac{1}{a - j\omega}$$

Resolviendo la Segunda integral:

$$\int_0^{\infty} (e^{-(a+jw)t} dt = \left[\frac{e^{-(a+jw)t}}{-(a+jw)} \right]_0^{\infty} = \emptyset - \left[-\frac{1}{a+jw} \right]$$

$$= \frac{1}{a+jw}$$

$$X(w) = \frac{1}{a-jw} + \frac{1}{a+jw}$$

$$X(w) = \frac{2}{a^2 + w^2}$$

b) $\cos(w_c t)$; $w_c \in \mathbb{R}$

$$X(w) = \int_{-\infty}^{\infty} \cos(w_c t)\, e^{-jwt}\, dt$$

$$\cos(w_c t) = \frac{e^{jw_c t} + e^{-jw_c t}}{2}$$

Reemplazamos:

$$X(w) = \int_{-\infty}^{\infty} \frac{e^{jw_c t} + e^{-jw_c t}}{2} \left(e^{-jwt} \right) dt$$

$$X(\omega) = \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{j(\omega_c-\omega)t}\, dt + \int_{-\infty}^{\infty} e^{-j(\omega_c+\omega)t}\, dt\right]$$

Viendo que la integral es una exponencial compleja, es un delta de Dirac:

$$\int_{-\infty}^{\infty} e^{j(a)t}\, dt = 2\pi\,\delta(a)$$

Aplicandolo en la integral anterior ($X(\omega)$):

$$X(\omega) = \frac{1}{\cancel{2}}\left[\cancel{2}\pi\,\delta(\omega-\omega_c) + \cancel{2}\pi\,\delta(\omega+\omega_c)\right]$$

$$X(\omega) = \pi\left[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)\right]$$

C) $x(t) = \operatorname{Sen}(\omega_s t)$; $\omega_s \in \mathbb{R}$

Aplicamos la definición de FT:

$$X(\omega) = \int_{-\infty}^{\infty} \operatorname{Sin}(\omega_s t)\cdot e^{-j\omega t}\, dt$$

$$\operatorname{Sen}\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Reemplazamos:

$$X(w) = \int_{-\infty}^{\infty} \frac{e^{jw_s t} - e^{-jw_s t}}{2j} \left(e^{-jwt} \right) dt$$

$$= \frac{1}{2j} \int_{-\infty}^{\infty} \left(e^{j(w_s - w)t} - e^{-j(w_s + w)t} \right) dt$$

$$X(w) = \frac{1}{2j} \left[\int_{-\infty}^{\infty} e^{j(w_s - w)t} \, dt - \int_{-\infty}^{\infty} e^{-j(w_s + w)t} \, dt \right]$$

Resolviendo la Primera integral:

$$\int_{-\infty}^{\infty} e^{j(w_s - w)t} \, dt \quad ; \quad \text{Para} \quad w_s - w = \alpha$$

$$= \int_{-\infty}^{\infty} e^{j\alpha t} \, dt = 2\pi \delta(\alpha) = 2\pi \delta(w - w_s)$$

Para la segunda integral aplicamos el mismo concepto del delta de Dirac.

$$\int_{-\infty}^{\infty} e^{j(w_s + w)t} \, dt = \int_{-\infty}^{\infty} e^{j\alpha t} \, dt = 2\pi \delta(\alpha)$$

$$= 2\pi \delta(w + w_s)$$

Entonces: $$X(w) = -\frac{j}{2} \left[2\pi \delta(w - w_s) - 2\pi \delta(w + w_s) \right]$$

$$X(w) = j\pi \left(\delta(w + w_s) - \delta(w - w_s) \right)$$

d) $x(t) = f(t) \cdot \cos(w_c t)$; $w_c \in \mathbb{R}$, $f(t) \in \mathbb{R}, \mathbb{C}$

Aplicamos F.T.

$$X(w) = \int_{-\infty}^{\infty} f(t) \cos(w_c t) \cdot e^{-jw_c t} dt$$

- $\cos(w_c t) = \frac{1}{2}\left(e^{jw_c t} + e^{-jw_c t}\right)$

Reemplazamos:

$$X(w) = \int_{-\infty}^{\infty} f(t) \cdot \frac{1}{2}\left(e^{jw_c t} + e^{-jw_c t}\right) \cdot e^{-jw_c t} dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(t) \cdot \left(e^{j(w_c - w)t} + e^{-j(w_c + w)t}\right) dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(t) \cdot e^{j(w_c - w)t} dt + \int_{-\infty}^{\infty} f(t)\, e^{-j(w_c + w)t} dt\right]$$

Ya que cada integral corresponde a la F.T. de $f(t)$ evaluada en frecuencias desplazadas, por definición tenemos:

$$\cancel{\mathbb{R}}\ X(w) = \frac{1}{2}\left[F(w - w_c) + F(w + w_c)\right]$$

Donde $F(w) = f\{t\}$

e) $x(t) = e^{-a|t|^2}$, $a \in \mathbb{R}^+$ Señal Gaussiana

$$X(w) = \int_{-\infty}^{\infty} e^{-a|t|^2} dt$$, donde $|t|^2 = t^2$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at^2} \cdot e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{-at^2 - j\omega t}\, dt$$

Llevando $-at^2 - j\omega t$ a la forma $t^2 - bt$:

$$= -a\left(t^2 + \frac{j\omega}{a} \cdot t\right)$$

Si $(x-y)^2 \cdot (x+y)^2 = x^2 + 2xy + y^2$; $x = t$

$$= 2yt = \frac{j\omega}{a} t$$

$$y = \frac{j\omega}{2a}$$

Reemplazando:

$$t^2 + \frac{j\omega}{a} t + \left(\frac{j\omega}{2a}\right)^2 - \left(\frac{j\omega}{2a}\right)^2$$

Entonces:

$$-at^2 - j\omega t = -a\left[\left(t + \frac{j\omega}{2a}\right)^2 - \left(\frac{j\omega}{2a}\right)^2\right]$$

$$= -\cancel{a}\left[\left(t + \frac{j\omega}{2a}\right)^2 - \frac{\overset{-1}{\cancel{j^2}}\,\omega^2}{4a^{\cancel{2}}}\right]$$

$$= -a\left(t + \frac{j\omega}{2a}\right)^2 + \frac{\omega^2}{4a}$$

Sustituimos en la integral:

$$X(\omega) = \int_{-\infty}^{\infty} e^{\left[-a\left(t + j\omega/2a\right)^2 + \omega^2/4a\right]} dt$$

$$X(\omega) = e^{-\frac{\omega^2}{4a}} \int_{-\infty}^{\infty} e^{-a\left(t + \frac{j\omega}{2a}\right)^2} dt$$

$$u = t + \frac{j\omega}{2a} \quad ; \quad du = dt$$

$$t \to -\infty \quad , \quad u \to -\infty + \frac{j\omega}{2a} = -\infty$$

$$t \to +\infty \quad , \quad u \to +\infty + \frac{j\omega}{2a} = +\infty$$

$$X(\omega) = e^{-\frac{\omega^2}{4a}} \int_{-\infty}^{\infty} e^{-au} du$$

Resolviendo la integral (Gaussiana-estandar).

$$\int_{-\infty}^{\infty} e^{-au} du \quad , \text{ Podemos asimilarla } \int_{-\infty}^{\infty} e^{-au^2} du = \sqrt{\frac{\pi}{a}}$$

Entonces:

R/ $$X(\omega) = e^{-\frac{\omega^2}{4a}} \cdot \sqrt{\frac{\pi}{a}}$$

f) $x(t) = A \cdot rect_d(t)$, $A, d \in \mathbb{R}$

$$rect_d(t) = \begin{cases} 1 & ; \ |t| \leq \frac{d}{2} \\ 0 & ; \ c.c \end{cases} \quad , \quad x(t) = A \quad , \quad |t| \leq \frac{t}{2}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

$$X(\omega) = \int_{-d/2}^{d/2} A \cdot e^{-j\omega t}\, dt = A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-d/2}^{d/2}$$

$$= A\,\frac{1}{-j\omega}\left(e^{-j\omega(d/2)} - e^{-j\omega(-d/2)}\right)$$

Aplicando la identidad: $e^{-j\theta} - e^{j\theta} = -2j\,\text{Sen}\,\theta$

$$X(\omega) = \frac{A}{-j\omega}\left(-2j\,\text{Sen}(\omega d/2)\right)$$

$$= \frac{A\,2\,\text{Sen}(\omega d/2)}{\omega}$$

Si sabemos que: $\text{Sinc} = \frac{\text{Sen}(x)}{x}$

Podemos decir que:

$$\frac{\text{Sen}(\omega d/2)}{\omega d/2} = \text{Sinc}(\omega d/2)$$

Entonces:

$$\frac{d/2}{d/2} \cdot \frac{A2\,\text{Sen}(\omega d/2)}{\omega} = A2\frac{d}{2}\,\text{Sinc}(\omega d/2)$$

$$\therefore X(\omega) = Ad\,\text{Sinc}(\omega d/2)$$