Linear Algebra

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Preface

This book mainly follows the structure of [1].

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Chapter 1 Vector Spaces

1.1 Vector Spaces

Through this book, we use the symbol $\mathbb{F}$ (F for field) to denote $\mathbb{R}$ or $\mathbb{C}$. Many mathematical definitions arises from simple concrete objects with certain properties. The abstraction of **vector spaces** comes from the nice properties possessed by $\mathbb{F}^n$.

Definition 1.1.1

Let V be a set. We say V is a vector space over field F (or we simply say V is a vector space if $\mathbb{F}$ is clear from the context) if the following properties hold:

1. ***Commutativity*** $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ *for all* $\mathbf{u},\mathbf{v}\in V$.
2. ***Associativity*** $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ *for all* $\mathbf{u},\mathbf{v},\mathbf{w}\in V$.
3. ***Additive Identity*** *There exists an element* $\mathbf{0}\in V$ *such that* $\mathbf{v}+\mathbf{0}=\mathbf{v}$ *for all* $\mathbf{v}\in V$. *(We only assume the existence of element* $\mathbf{0}$*, nothing is said about its uniqueness. Although it is indeed unique, we still need to prove this.)*
4. ***Additive Inverse*** *For every* $\mathbf{v}\in V$*, there exists an element* $\mathbf{w}\in V$ *such that* $\mathbf{v}+\mathbf{w}=\mathbf{0}$ *where* $\mathbf{0}$ *is the same as the one given above.*
5. ***Multiplicative Identity*** $1\mathbf{v}=\mathbf{v}$ *for all* $\mathbf{v}\in V$ *where* 1 *is simply the real number* 1.
6. ***Distributive Property of Scalar Multiplication over Vector Addition*** $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$ *for all* $a\in\mathbb{F}$ *and all* $\mathbf{u},\mathbf{v}\in V$.
7. ***Distributive Property of Scaler Multiplication over Scalar Addition*** $(a+b)\mathbf{v}=a\mathbf{v}+b\mathbf{v}$ *for all* $a,b\in F$ *and all* $\mathbf{v}\in V$.

♣

Note *In fact, if a set satisfies property 1 - 4, we say that it is an* ***Abelian group***

As noted, we need to prove that such element $\mathbf{0}$ is unique.

Proposition 1.1.1

Let V be an Abelian group. Suppose both elements $\mathbf{0}$ and $\mathbf{0}'$ satisfy properties 3 in Definition 1.1.1. Then $\mathbf{0}=\mathbf{0}'$.

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Proof Regarding $\mathbf{0}$ as *an* additive identity, by property 3, we have

$$\mathbf{0}'+\mathbf{0}=\mathbf{0}' \tag{1.1}$$

On the other hand, regarding $\mathbf{0}'$ as *an* additive identity, we also have

$$\mathbf{0}+\mathbf{0}'=\mathbf{0} \tag{1.2}$$

Because the vector addition is commutative (property 1), the left-hand sides of (1.1) and (1.2) are equal and hence $\mathbf{0}=\mathbf{0}'$. ■

Now, we can safely say $\mathbf{0}$ is *the* additive identity. We also say $\mathbf{0}$ is the zero vector.

The next proposition is known as the **cancellation property**.

Proposition 1.1.2 (Cancellation Property)

Let V be an Abelian group. Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, we have

$$\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w} \implies \mathbf{u} = \mathbf{v}$$

Proof By property 4 in Definition 1.1.1, there exists $\mathbf{w}'$ such that $\mathbf{w} + \mathbf{w}' = \mathbf{0}$. It then follows that

$$\begin{aligned} & \mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w} \\ \implies & \mathbf{u} + (\mathbf{w} + \mathbf{w}') = \mathbf{v} + (\mathbf{w} + \mathbf{w}') \\ \implies & \mathbf{u} + \mathbf{0} = \mathbf{v} + \mathbf{0} \\ \implies & \mathbf{u} = \mathbf{v} \end{aligned}$$

■

The number zero 0 times any vector is the zero vector $\mathbf{0}$.

Proposition 1.1.3

Let V be a vector space. Then for any $\mathbf{v} \in V$, we have $0 \cdot \mathbf{v} = \mathbf{0}$.

The proof is simple. But we do not skip any intermediate steps to demonstrate how each property is applied.

Proof Let $\mathbf{v} \in V$ be arbitrary. We have

$$\mathbf{v} + 0 \cdot \mathbf{v} = 1 \cdot \mathbf{v} + 0 \cdot \mathbf{v} = (1 + 0) \cdot \mathbf{v} = 1 \cdot \mathbf{v} = \mathbf{v} = \mathbf{v} + \mathbf{0}$$

Then applying the cancellation property yields $0 \cdot \mathbf{v} = \mathbf{0}$. ■

Any scalar in $\mathbb{F}$ times the zero vector $\mathbf{0}$ is the zero vector itself.

Proposition 1.1.4

Let V be a vector space. For any $a \in \mathbb{F}$, we have $a\mathbf{0} = \mathbf{0}$.

Proof We have

$$a\mathbf{0} + \mathbf{0} = a\mathbf{0} = a(\mathbf{0} + \mathbf{0}) = a\mathbf{0} + a\mathbf{0}$$

Then by the cancellation property, we conclude that $a\mathbf{0} = \mathbf{0}$. ■

The element $\mathbf{w}$ in property 4 is also unique as we shall prove in the following proposition.

Proposition 1.1.5

Let V be an Abelian group. Pick an element $\mathbf{v} \in V$. Suppose both elements $\mathbf{w}$ and $\mathbf{w}'$ satisfy properties 4 in Definition 1.1.1. Then $\mathbf{w} = \mathbf{w}'$.

Proof We have

$$\mathbf{w}' + \mathbf{0} = \mathbf{w}'$$

Replacing $\mathbf{0}$ with $\mathbf{v}+\mathbf{w}$ yields

$$\begin{aligned} &\mathbf{w}'+(\mathbf{v}+\mathbf{w})=\mathbf{w} \\ \Longrightarrow\ &(\mathbf{w}'+\mathbf{v})+\mathbf{w}=\mathbf{w}' \\ \Longrightarrow\ &\mathbf{0}+\mathbf{w}=\mathbf{w}' \\ \Longrightarrow\ &\mathbf{w}=\mathbf{w}' \end{aligned}$$

where the last equality follows from Proposition 1.1.3. ■

Therefore, for each $\mathbf{v}$, the choice of its additive inverse $\mathbf{w}$ is unique. We may then say that such $\mathbf{w}$ is *the* additive inverse of $\mathbf{v}$. And to make the notation more intuitive, we shall denote the additive inverse of $\mathbf{v}$ by $-\mathbf{v}$.

The additive inverse $-\mathbf{v}$ of $\mathbf{v}$ can be computed by the scalar multiplication $-1\cdot\mathbf{v}$.

Proposition 1.1.6

Let V be a vector space and $\mathbf{v}\in V$. We have $-\mathbf{v}=-1\cdot\mathbf{v}$.

Proof We have

$$\mathbf{v}+(-1\cdot\mathbf{v})=1\cdot\mathbf{v}+(-1)\cdot\mathbf{v}=(1+(-1))\mathbf{v}=0\cdot\mathbf{v}=\mathbf{0}$$

This shows $-1\cdot\mathbf{v}$ is indeed the additive inverse of $\mathbf{v}$. ■

The subtraction operator $-: V\times V\to V$ is defined as $\mathbf{u}-\mathbf{v}:=\mathbf{u}+(-\mathbf{v})$.

1.2 Subspaces

Let U be a subset of the vector space V. We say that U is a **vector subspace** of V (or simply subspace of V) if U is also a vector space with the same addition and scalar multiplication defined on V.

To check whether a given subset $U\subseteq V$, we may simply check that if U contains the zero vector and if it is closed under the addition and scalar multiplication.

Proposition 1.2.1

Let U be a subset of V. Then U is a vector subspace of V if and only if

1. $\mathbf{0}\in U$,
2. $\mathbf{u}+\mathbf{v}\in U$ *for all* $\mathbf{u},\mathbf{v}\in U$, *and*
3. $a\mathbf{u}\in U$ *for all* $a\in\mathbb{F}$ *and* $\mathbf{u}\in U$.

By simple observations, one may notice that the additive identity in the subspace U is exactly the one in the superspace V and $\mathbf{w}$ is the additive inverse of $\mathbf{u}$ in U if and only if it is also the additive inverse of u in V.

Example 1.1 $\{\mathbf{0}\}$ and V are subspaces of V, which are the simplest examples of vector spaces.

Example 1.2 $\{(x,0,0)\mid x\in\mathbb{F}\}$ and $\{(x,y,0)\mid x,y\in\mathbb{F}\}$ are subspaces of $\mathbb{F}^3$. Specially, when $\mathbb{F}=\mathbb{R}$, this means that the 1D x-axis line and the 2D x-y plane are subspaces of the 3D space.

In fact, all 1D lines and 2D planes that pass through the origin are subspaces of the 3D space.

1.3 Sums and Direct Sums

References

[1] Sheldon Jay Axler. *Linear Algebra Done Right*. 2nd ed. Undergraduate Texts in Mathematics. New York: Springer, 1997. ISBN: 978-0-387-98259-5 978-0-387-98258-8.

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