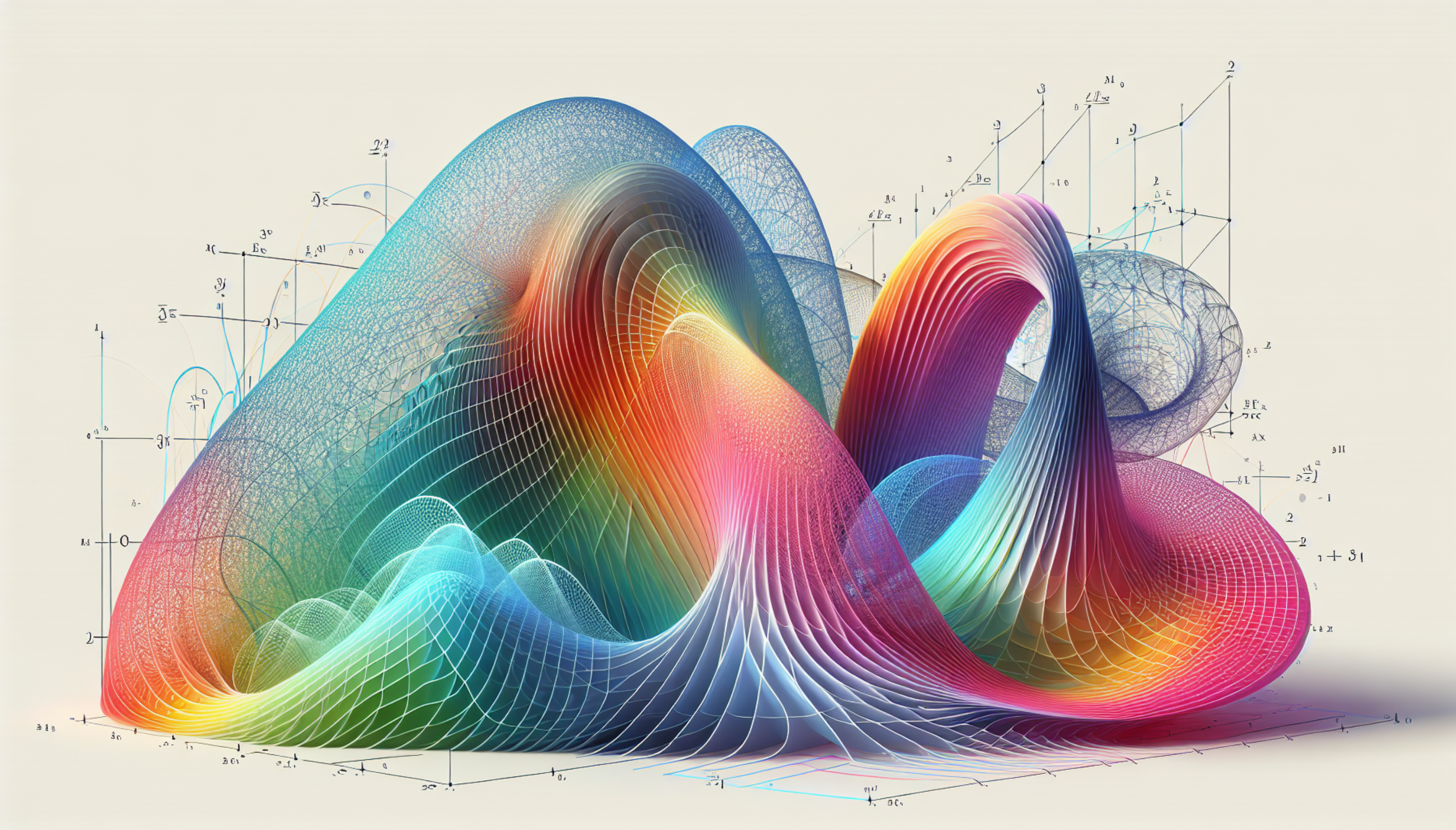

Mathematical Analysis

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Preface

This book mainly follows the structure of [1].

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Chapter 1 Point-Set Topology

Chapter 2 Functions of Bounded Variation and Rectifiable Curves

2.1 Functions of Bounded Variation

Definition 2.1.1

Let $[a, b]$ be an interval. A set of points

$$P = \{x_0, x_1, \dots, x_n\}$$

satisfying

$$a = x_0 < x_1 < \cdots < x_n = b$$

*is called a **partition** of $[a, b]$.*
The interval $[x_{k-1}, x_k]$ is called the k-th subinterval of P, and we often write $\Delta x_k = x_k - x_{k-1}$.
The collection of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$. ♣

Note *In mathematics texts, we have another definition of partitions, which states that a partition of a set S is a collection of subsets of S such that they are disjoint and their union equals S. We should not confuse these two definitions.*

Definition 2.1.2

Let f be a real-valued function on $[a, b]$. If $P = \{x_0, \dots x_n\}$ is a partition of $[a, b]$, write $\Delta f_k = f(x_k) - f(x_{k-1})$. If there exists a positive number M such that

$$\sum_{k=1}^{n} |\Delta f_k| \leq M \tag{2.1}$$

*for all partitions P of $[a, b]$, then we say that f is **of bounded variation** on $[a, b]$.* ♣

Note *A geometric interpretation of the sum $\sum_{k=1}^{n} |\Delta f_k|$ is the total vertical length of several pieces of the function. Imagine a point moving along the curve of the function from the left to the right. If the partition gets finer and finer, then $\sum_{k=1}^{n} |\Delta f_k|$ will become the length of its journey projected on the y-axis. In fact, it is defined as the total variation as we shall introduce later.*

Sometimes, it is convenient to denote the sum $\sum_{k=1}^{n} |\Delta f_k|$ by the symbol

$$v(P, f) := \sum_{k=1}^{n} |\Delta f_k|$$

We do not use the capital letter V here for it is reserved for the total variation.

A simple observation is that a function of bounded variation is also bounded.

Proposition 2.1.1

Let f be a function of bounded variation on $[a, b]$. Then f is bounded on $[a, b]$. ♠

Proof By definition, there exists $M > 0$ such that (2.1) holds for any partitions of $[a, b]$. For any $x \in (a, b)$, consider the partition $P = \{a, x, b\}$. We have

$$|f(x) - f(a)| + |f(b) - f(x)| \leq M$$

This implies that $|f(x) - f(a)| \leq M$, which further implies $|f(x)| \leq |f(a)| + M$. Note that x is arbitrarily chosen from (a, b). Therefore, f is indeed bounded on $[a, b]$. ■

But a bounded function is not necessarily of bounded variation.

Example 2.1 Consider the function

$$f(x) = \begin{cases} \cos \frac{1}{x}, & x \in (0, 1] \\ 0, & x = 0 \end{cases}$$

Its graph is shown in Figure 2.1.

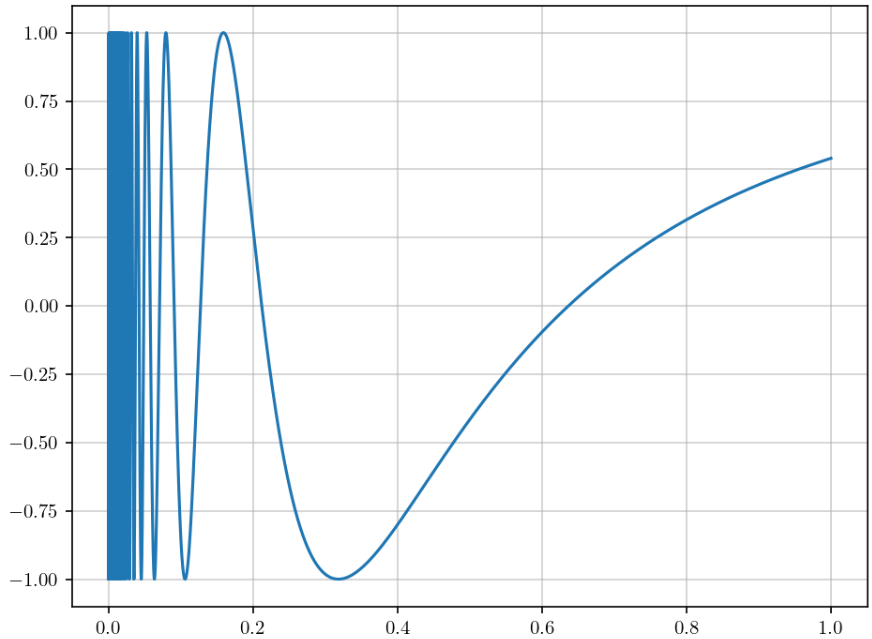

Figure 2.1: Graph of the function $f(x) = \cos \frac{1}{x}$ for $x \in (0, 1]$ and $f(0) = 0$. It is bounded on $[0, 1]$ but not of bounded variation for it varies rapidly near $x = 0$.

Clearly, this function is bounded by 1. But intuitively, it is not of bounded variation since it varies rapidly near $x = 0$. Let P be a partition of $[0, 1]$ where each x_k is given by

$$x_k = \begin{cases} \frac{1}{(n-k)\pi}, & 1 \leq k \leq n-1 \\ 0, & k = 0 \\ 1, & k = n \end{cases}$$

For $k = 1, \ldots, n-1$, we have

$$f(x_k) = \cos((n-k)\pi) \in \{-1, 1\}$$

The function value is either 1 or -1 and the sign alternates between each two consecutive points.

Hence,

$$\sum_{k=1}^{n} |\Delta f_k| \geq \sum_{k=2}^{n-1} |\Delta f_k| = 2(n-2)$$

As we increase the number of points in the partition, $\sum |\Delta f_k|$ will exceeds any given number. Therefore, f is not of bounded variation on $[0, 1]$.

Proposition 2.1.2

If f is monotonic on $[a, b]$, then f is of bounded variation on $[a, b]$.

Proof Assume f is increasing. For any partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$, we have

$$\sum_{k=1}^{n} |\Delta f_k| = \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) = f(b) - f(a)$$

Therefore, f is of bounded variation on $[a, b]$.

If f is decreasing, then $-f$ is increasing. Applying what we have proved, we may conclude that $-f$ is of bounded variation. Hence, f is also of bounded variation since $\sum |\Delta(-f)_k| = \sum |\Delta f_k|$. ■

Proposition 2.1.3

Suppose that f is continuous on $[a, b]$ and the derivative f' exists in (a, b). If $|f'(x)| \leq A$ for all $x \in (a, b)$, then f is of bounded variation on $[a, b]$.

Note *The assumption that f being continuous on $[a, b]$, and f' exists in (a, b) coincides with the mean value theorem. And indeed, it is the key of this proof.*

Proof Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. By the mean value theorem, there exists $t_k \in (x_{k-1}, x_k)$ for all $k = 1, \ldots, n$ such that

$$f(x_k) - f(x_{k-1}) = f'(t_k)(x_k - x_{k-1})$$

It then follows that

$$\begin{aligned}
\sum_{k=1}^{n} |\Delta f_k| &= \sum_{k=1}^{n} |f'(t_k)| \, (x_k - x_{k-1}) \\
&\leq \sum_{k=1}^{n} A(x_k - x_{k-1}) \\
&= A(f(b) - f(a))
\end{aligned}$$

Therefore, f is of bounded variation on $[a, b]$. ■

The following is a well crafted example of showing that a continuous and differentiable function is not necessarily of bounded variation if we do not impose that its derivative is bounded in the interior.

Example 2.2 Consider function defined on $[0, 1]$ given by

$$f(x) = \begin{cases} x \cos \frac{\pi}{2x}, & x \in (0, 1] \\ 0, & x = 0 \end{cases}$$

Its graph is shown in Figure 2.2.

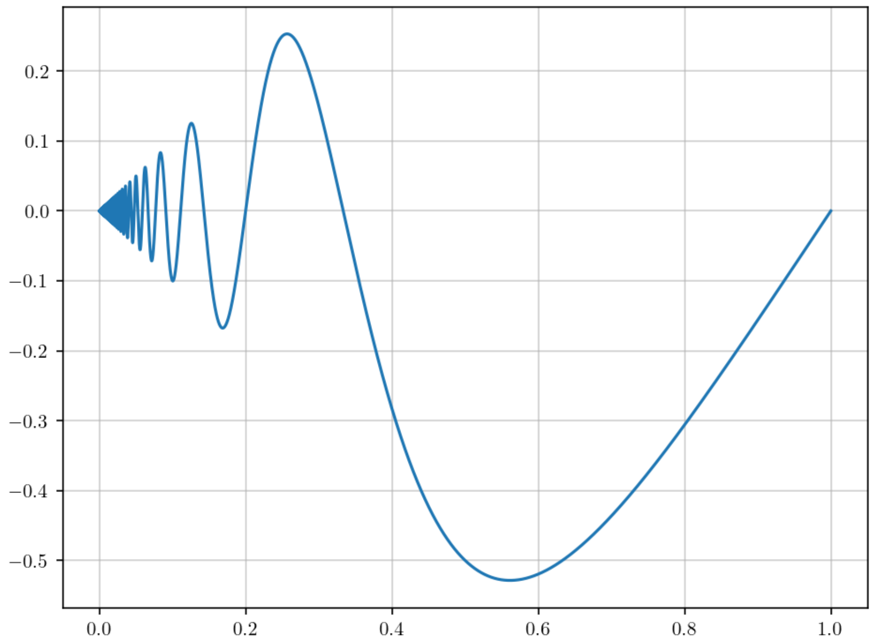

Figure 2.2: Graph of the function $f(x) = x\cos\frac{\pi}{2x}$ for $x \in (0,1]$ and $f(0) = 0$. This function is continuous and its derivative exists in $(0,1)$. But the derivative is unbounded.

The fact that this function is not of bounded variation may be less intuitive than the one given in Example 2.1. The function still varies rapidly near $x = 0$. However, it does not range from -1 and 1. Instead, it damps out at $x = 0$ and becomes 0. But we will show in the following that we can find a partition so fine that by collecting each small function variation, the overall sum may still increase to infinity.

Consider the partition

$$P = \{0, \frac{1}{2n}, \frac{1}{2n-1}, \ldots, \frac{1}{3}, \frac{1}{2}, 1\}$$

We have

$$\begin{aligned}\sum_k |\Delta f_k| &= \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n-1} + \frac{1}{2n-1} + \cdots + \frac{1}{2} + \frac{1}{2} \\ &= 1 + \cdots + \frac{1}{n}\end{aligned}$$

As n gets larger and larger, the sum on the right hand-side will increase infinitely for we know that the harmonic series $\sum \frac{1}{n}$ diverges. Therefore, this function is not of bounded variation.

Of course, the condition of the derivative being bounded is not necessary for a function to be of bounded variation.

Example 2.3 The derivative of the square root function $f(x) = \sqrt{x}$ in $(0,1)$ is $f'(x) = \frac{1}{2\sqrt{x}}$, which tends to infinity as $x \to 0$. But f is clearly of bounded variation on $[0,1]$ by Proposition 2.1.2 for it is increasing.

Let P be a partition of $[a,b]$. If we make it finer by adding some intermediate points, then the sum of variations will increase. This result may be helpful in some proofs.

Proposition 2.1.4

Let f be defined on $[a, b]$, and P a partition of $[a, b]$. If P' is finer than P, i.e., $P' \supset P$, then

$$v(P', f) \geq v(P, f)$$

Note *Compare this to the upper and lower Darboux sums when we introduce them in a later section.*

Proof It suffices to that prove for the case where P' is one point finer than P. Suppose $P = \{x_0, \ldots, x_n\}$ and $P' = P \sqcup \{c\}$. We have

$$\begin{aligned} v(P', f) &= |f(x_1) - f(x_0)| + \cdots + |f(c) - f(x_{j-1})| + |f(x_j) - f(c)| + \cdots + |f(x_n) - f(x_{n-1})| \\ &\geq |f(x_1) - f(x_0)| + \cdots + |f(x_j) - f(x_{j-1})| + \cdots + |f(x_n) - f(x_{n-1})| \\ &= v(P, f) \end{aligned}$$

Note *Note that j may equal to 1 or n in the above notations. We write down the summation in the expanded form to make the proof easier to read.*

This completes the proof. ∎

2.2 Total Variation

Recall that f is said to be of bounded of variation on $[a, b]$ if, equivalently to what we stated, the set

$$\left\{ \sum_{k=1}^{n} |\Delta f_k| \;\middle|\; P \in \mathcal{P}[a, b] \right\} \tag{2.2}$$

or with our shortened notation

$$\{v(P, f) \mid P \in \mathcal{P}[a, b]\}$$

is bounded above. This set is of course nonempty for $\{a, b\}$ is clearly a partition. By the least upper bound property, the set in (2.2) has a supremum, which is referred to as **total variation** of f on $[a, b]$.

Definition 2.2.1

Let f be of bounded variation on $[a, b]$. The total variation, denoted by $V_a^b(f)$, of f on $[a, b]$ is defined as

$$V_a^b(f) := \sup_{P \in \mathcal{P}[a,b]} v(P, f) = \sup \left\{ \sum_{k=1}^{n} |\Delta f_k| \;\middle|\; P \in \mathcal{P}[a, b] \right\}$$

Note *We adopt the notation $V_a^b(f)$, which is inspired by the notion of a definite integral $\int_a^b f(x)\,\mathrm{d}x$. And as we shall see, these two concepts indeed share some similar properties, namely, the linear properties.*

Notations are very important for they provide intuitive expressions of the intrinsic mathematical concepts.

From this definition, we have some simple observations. First, the value of $V_a^b(f)$ is nonnegative. And it is easy to prove that $V_a^b(f) = 0$ if and only if f is constant on $[a, b]$.

The simplest function of bounded variation (well, apart from a constant function) is monotonic function. It is natural to ask what is its total variation. With a little thought, one can imagine that it should be the absolute value of the difference at the endpoints.

Proposition 2.2.1

If f is a monotonic function on $[a,b]$, then its total variation is the absolute value of the difference of the function values at the endpoints, i.e.,

$$V_a^b(f) = |f(a) - f(b)|$$

Proof We only prove the case that f is increasing. For any partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$, we have

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})] = f(b) - f(a)$$

Note that the sum is independent of the partition. Hence, the set in (2.2) is just a constant. Therefore, the total variation $V_a^b(f) = f(b) - f(a)$. ■

When studying functions of bounded variation, in most cases, we are often interested in monotonic functions or continuous and differentiable functions. (Proposition 2.1.2 and 2.1.3.)

Note *On one hand, we will see in Theorem 2.2.5, a function is of bounded variation if and only if it can be expressed as a difference of two increasing functions, the need of studying monotonic functions arises naturally.*

On the other hand, as we shall see in the chapter on Riemann-Stieltjes integrals, we assume the integrator α is of bounded variation. Since integrator α will be put after the differential operator, $\mathrm{d}\alpha$, and we often hope to express it as $\alpha'(t)\,\mathrm{d}t$ to reduce the integral to Riemann integral and compute its value, we would like α to be differentiable.

But if we are curious about whether some piecewise functions are of bounded variation, then Proposition 2.1.2 and 2.1.3 will not be enough.

Example 2.4 For example, consider the following function defined on $[0,3]$:

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ -(x-1)(x-3), & 1 < x \le 3 \end{cases}$$

Figure 2.3 depicts its graph.

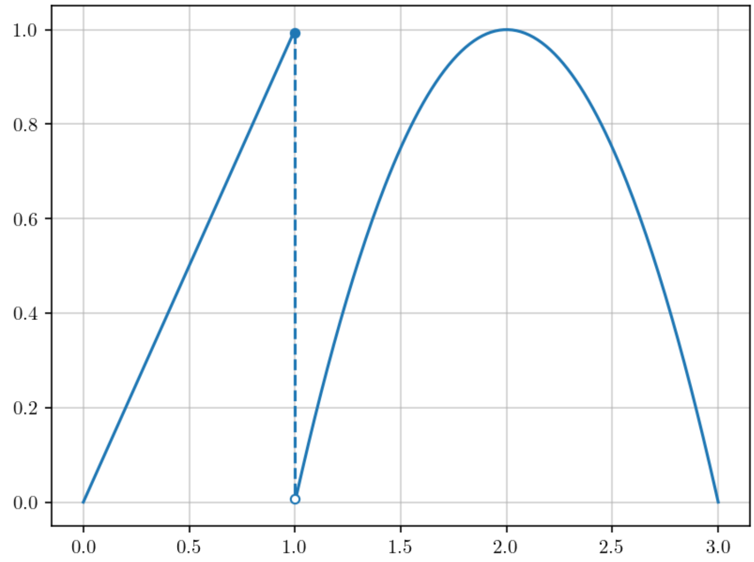

Figure 2.3: Is this function of bounded variation on $[0,3]$?

Intuitively, the function in Figure 2.3 should be of bounded variation. But we must be careful about the jump point, which we have not covered in the previous discussion.

Proposition 2.2.2

Suppose f is of bounded variation on $[a,b]$, and is continuous at $x=a$. If function g is defined by revising the value at $x=a$, i.e.,

$$g(x)=\begin{cases} f(x), & x\in(a,b] \\ y, & x=a \end{cases}$$

then g is still of bounded variation on $[a,b]$. And its total variation is given by

$$V_a^b(g)=V_a^b(f)+|y-f(a)|$$

Proof Let P be a partition of $[a,b]$. We have

$$\begin{aligned} v(P,g) &= |g(x_1)-g(a)|+\cdots|g(x_n)-g(x_{n-1})| \\ &= |f(x_1)-y|+\cdots|f(x_n)-f(x_{n-1})| \\ &\le [|y-f(a)|+|f(x_1)-f(a)|]+\cdots|f(x_n)-f(x_{n-1})| \\ &= |y-f(a)|+v(P,f) \\ &\le |y-f(a)|+V_a^b(f) \end{aligned} \tag{2.3}$$

This shows that g is of bounded variation on $[a,b]$.

Now, we compute its total variation. Let $\varepsilon>0$ be arbitrary. Because f is continuous at $x=a$, there exists $\delta>0$ such that

$$|x-a|<\delta \implies |f(x)-f(a)|<\varepsilon/4$$

By the definition of total variation and Proposition 2.1.4, there exists a fine enough partition P such that the minimum length of the subinterval is less than δ, and

$$v(P, f) > V_a^b(f) - \varepsilon/2$$

On the subinterval $[a = x_0, x_1]$, we have

$$\begin{aligned}
|\Delta g_1| &= |g(x_1) - g(x_0)| \\
&= |f(x_1) - y| \\
&\geq |f(a) - y| - |f(x_1) - f(a)| \\
&= |f(a) - y| + |f(x_1) - f(a)| - 2\,|f(x_1) - f(a)| \\
&\text{Note that } |x_1 - x_0| < \delta \text{, hence we may estimate the last term as follows} \\
&> |f(a) - y| + |f(x_1) - f(a)| - 2 \cdot \varepsilon/4 \\
&> |f(a) - y| + |f(x_1) - f(a)| - \varepsilon/2
\end{aligned}$$

Note *When reaching*

$$|\Delta g_1| \geq |f(a) - y| - |f(x_1) - f(a)|$$

in the above derivation, one may be worried that it is not proceeding towards the goal since we have a minus sign before $|f(x_1) - f(a)|$. But since this term $|f(x_1) - f(a)|$ can be made arbitrarily small, we can always add it (to construct the sum $v(P, f)$) and then subtract it, and make the trailing negative term $-2\,|f(x_1) - f(a)|$ negligible, as what we did above.

It then follows that

$$v(P, g) > |f(a) - y| + v(P, f) - \varepsilon/2 > |f(a) - y| + V_a^b(f) - \varepsilon$$

Therefore, we have

$$V_a^b(g) \geq v(P, g) \geq |f(a) - y| + V_a^b(f)$$

Compare this to (2.3), we may conclude

$$V_a^b(g) = V_a^b(f) + |f(a) - y|$$

■

The function f in Example 2.4 can be regarded as a sum of two functions on $[0, 3]$, $f(x) = g(x) + h(x)$ where

$$g(x) = \begin{cases} x, & x \in [0, 1] \\ 0, & x \in (1, 3] \end{cases} = (x \mapsto x)\mathbb{1}_{[0,1]}$$

$$h(x) = \begin{cases} 0, & x \in [0, 1) \\ \tilde{h}(x), & x \in [1, 3] \end{cases} = \tilde{h}\mathbb{1}_{[1,3]}$$

where

$$\tilde{h}(x) = \begin{cases} -(x-1)(x-3), & x \in (1, 3] \\ 0, & x = 1 \end{cases}$$

We have already seen that functions like $\tilde{h}$ are of bounded variation in Proposition 2.2.2. If we know the sum of two functions of bounded variation (on the same interval) is also of bounded variation

(Theorem 2.2.1), we may then conclude that piecewise functions like f in Example 2.4 are indeed of bounded variation.

Hence, the next step to do is studying whether functions like g and h are of bounded variation. Describing in words, such functions are constructed by extending a function of bounded variation to a larger interval by defining function values of everywhere else in the larger interval to be zeros.

Theorem 2.2.1

Let f and g be of bounded variation on $[a, b]$, then so are their sum, difference and product. Moreover, we have the following inequalities:

$$V_a^b(f \pm g) \leq V_a^b(f) + V_a^b(g) \tag{2.4}$$

and

$$V_a^b(fg) \leq \sup_{x\in[a,b]} |g(x)|\, V_a^b(f) + \sup_{x\in[a,b]} |f(x)|\, V_a^b(g) \tag{2.5}$$

♡

Note *Note that the supremums in* (2.5) *indeed exist since the functions f and g are bounded due to Proposition 2.1.1.*

Proof We first show that the sum and the difference of two functions are of bounded variation, and satisfy (2.4). Let P be an arbitrary partition of $[a, b]$. On each subinterval, we have

$$\begin{aligned} |\Delta(f \pm g)_k| &= |f(x_k) \pm g(x_k) - [f(x_{k-1}) \pm g(x_{k-1})]| \\ &= |\Delta f_k \pm \Delta g_k| \\ &\leq |\Delta f_k| + |\Delta g_k| \end{aligned}$$

Taking the sum over k, we have

$$\sum_k |\Delta(f \pm g)_k| \leq \sum_k |\Delta f_k| + \sum_k |\Delta g_k| \leq V_a^b(f) + V_a^b(g)$$

The above inequality shows that $f \pm g$ is of bounded variation on $[a, b]$, and (2.4) is satisfied.

In the following, we show that the product of two Functions are of bounded variation and satisfies (2.5). Let P be an arbitrary partition of $[a, b]$. On each subinterval, we have

$$\begin{aligned} |\Delta(fg)_k| &= |f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1})| \\ &\quad \text{Add and subtract the term } f(x_{k-1})g(x_k) \\ &= |g(x_k)[f(x_k) - f(x_{k-1})] + f(x_{k-1})[g(x_k) - g(x_{k-1})]| \\ &\leq |g(x_k)|\,|\Delta f_k| + |f(x_{k-1})|\,|\Delta g_k| \\ &\leq \sup_{x\in[a,b]} |g(x)|\,|\Delta f_k| + \sup_{x\in[a,b]} |f(x)|\,|\Delta g_k| \end{aligned}$$

Summing over k, we have

$$\begin{aligned} \sum_k |\Delta(fg)_k| &\leq \sup_{x\in[a,b]} |g(x)| \sum_k |\Delta f_k| + \sup_{x\in[a,b]} |f(x)| \sum_k |\Delta g_k| \\ &\leq \sup_{x\in[a,b]} |g(x)|\, V_a^b(f) + \sup_{x\in[a,b]} |f(x)|\, V_a^b(g) \end{aligned}$$

This shows the product fg is in fact of bounded variation on $[a, b]$, and (2.5) is satisfied. ■

We must exclude the quotients from the above theorem since the reciprocal $\frac{1}{f}$ of f may not be of bounded variation even though f is.

Example 2.5 Consider function

$$f(x) = \begin{cases} 1-x, & 0 \le x < 1 \\ -x, & 1 \le x \le 2 \end{cases}$$

Function f is of bounded variation on $[0,2]$ since it is decreasing. Its reciprocal is

$$\frac{1}{f(x)} = \begin{cases} \frac{1}{1-x}, & 0 \le x < 1 \\ -\frac{1}{x}, & 1 \le x \le 2 \end{cases}$$

Figure 2.4 depicts the graphs of f and $\frac{1}{f}$.

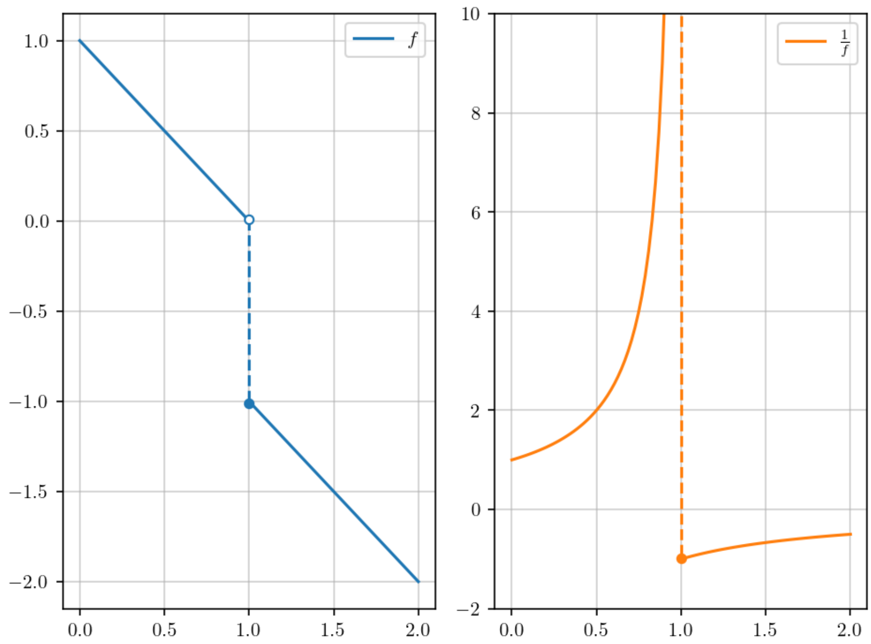

Figure 2.4: Left: f is of bounded variation for it is decreasing. Right: $\frac{1}{f}$ is not of bounded variation for it is unbounded.

Note that $\frac{1}{f}$ goes to positive infinity when $x \to 1^-$. Therefore, by Proposition 2.1.1, $\frac{1}{f}$ is not of bounded variation on $[0,2]$ since it is not bounded.

To extend Theorem 2.2.1 to quotients, we need to required that f is bounded away from zero in the interval.

Theorem 2.2.2

Let f be of bounded variation on $[a,b]$. And there exists $m > 0$ such that $f(x) \ge m$ for all $x \in [a,b]$. Then the reciprocal of f is of bounded variation on $[a,b]$, and

$$V_a^b(\frac{1}{f}) \le \frac{1}{m^2} V_a^b(f)$$

♡

Proof Let P be any partition of $[a,b]$. On each subinterval $[x_{k-1}, x_k]$, we have

$$\begin{aligned}\left|\Delta(\frac{1}{f})_k\right| &= \left|\frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})}\right| \\ &= \left|\frac{\Delta f_k}{f(x_{k-1})f(x_k)}\right| \\ &\leq \frac{|\Delta f_k|}{m^2}\end{aligned}$$

Summing over k, we have

$$\sum_k \left|\Delta(\frac{1}{f})_k\right| \leq \frac{1}{m^2}\sum_k |\Delta f_k| \leq \frac{1}{m^2}V_a^b(f)$$

Therefore, $\frac{1}{f}$ is of bounded variation on $[a,b]$. ■

2.2.1 Additive Property of Total Variation

Theorem 2.2.3

Let f be of bounded variation on $[a,b]$, and $c \in (a,b)$. Then f is of bounded variation on the subintervals $[c,b]$ and $[a,c]$. Moreover, we have

$$V_a^b(f) = V_a^c(f) + V_c^b(f) \tag{2.6}$$

♡

Proof We will first show that f is of bounded variation on each subinterval, and

$$V_a^c(f) + V_c^b(f) \leq V_a^b(f) \tag{2.7}$$

Let P' and P'' be partitions of $[a,c]$ and $[c,b]$, respectively, and let $P = P' \cup P''$. Note that P is a partition of $[a,b]$, and by reviewing the notation of $v(P,f)$ one may easily conclude that $v(P',f) + v(P'',f) = v(P,f)$. Since f is of bounded of variation on $[a,b]$, we have

$$v(P',f) + v(P'',f) = v(P,f) \leq V_a^b(f) \tag{2.8}$$

The above inequality holds for any partition p' of $[a,c]$ and any partition P'' of $[c,b]$. Therefore, by definition, f is of bounded variation on $[a,c]$ and $[c,b]$. Moreover, taking the supremum over P' and then over P'' on both sides of (2.8), we will obtain exactly (2.7).

To show the equality (2.6), we also need to show

$$V_a^c(f) + V_c^b(f) \geq V_a^b(f) \tag{2.9}$$

Let $\varepsilon > 0$ be arbitrary. There exists a partition P of $[a,b]$ such that $v(P,f) > V_a^b(f) - \varepsilon$. Let

$$P' = (P \cap [a,c]) \cup \{c\} \quad \text{and} \quad P'' = (P \cap [c,b]) \cup \{c\}$$

It is clear that P' and P'' are partitions of $[a,c]$ and $[c,b]$, respectively. By Proposition 2.1.4, we have

$$V_a^c(f) + V_c^b(f) \geq v(P',f) + v(P'',f) \geq v(P,f) > V_a^b(f) - \varepsilon \tag{2.10}$$

Because (2.10) holds for every $\varepsilon > 0$, (2.9) is proved. ■

Applying the above theorem, we can immediately conclude that f is also of bounded variation on any interval contained in $[a,b]$.

Corollary 2.2.1

If f is of bounded variation on $[a, b]$, and $[c, d] \subseteq [a, b]$, then f is also of bounded variation on $[c, d]$.

Proof With the given condition, we have $a \leq c < d \leq b$. If $c = a$ or $d = b$, then the assumption of this corollary reduces to the one in Theorem 2.2.3.

Now, we assume that $a < c < d < b$. Regarding c as an intermediate point in $[a, b]$, Theorem 2.2.3 shows that f is of bounded variation on $[c, b]$. Next, because $d \in (c, b)$, applying Theorem 2.2.3 again, we conclude that f is of bounded variation on $[c, d]$. ■

2.2.2 Total Variation as a Function of the Right Endpoint

Suppose f is of bounded variation on $[a, b]$. For any $x \in (a, b)$. Theorem 2.2.3 tells us that f is of bounded variation on $[a, x]$. Therefore, we can regard $V_a^x(f)$ as a function of x.

Note *This is very similar to considering $\int_a^x f(t)\,\mathrm{d}t$ as a function of the upper limit of the integral, which again shows that our notation of the total variation rather helpful.*

When $x = b$, it is just the total variation of f on the entire interval. We don't have definition for $x = a$ yet. But we can easily fix this by naturally defining $V_a^a(f) := 0$. Now, function $V_a^x(f)$ is defined on the entire interval $[a, b]$.

In the next chapter, we will study the Riemann-Stieltjes integral, which is more generalized definition of the Riemann integral. In the texts of the Riemann-Stieltjes integral $\int_a^b f\,\mathrm{d}\alpha$, we often assume that the integrator α is increasing (or slightly more generalized, monotonic)[2]. But we can extend the results easily to a even more general assumption that the integrator α is of bounded variation on $[a, b]$.

The key of achieving this is that a function of bounded variation can be written as a difference of two increasing functions, and conversely, the difference of two increasing functions is of bounded variation (Theorem 2.2.5). And the following theorem tells us exactly how to find such increasing functions.

Theorem 2.2.4

Let f be of bounded variation on $[a, b]$. Then

1. *$V_a^x(f)$ is increasing on $[a, b]$, and*
2. *$V_a^x(f) - f(x)$ is also increasing.*

Proof Let $h > 0$ (and $x + h \leq b$), by Theorem 2.2.3, we have

$$V_a^x(f) + V_x^{x+h}(f) = V_a^{x+h}(f)$$

Note *We have seen in Corollary 2.2.1 that $V_x^{x+h}(f)$ indeed exists.*

It then follows that

$$V_a^{x+h}(f) - V_a^x(f) = V_x^{x+h}(f) \geq 0$$

This shows $V_a^x(f)$ is increasing.

Next, we will prove $V_a^x(f) - f(x)$ is creasing. To ease the notation, let $g(x) = V_a^x(f) - f(x)$. Similarly, suppose $h > 0$ and $x + h \leq b$, consider the difference

$$g(x+h) - g(x) = V_x^{x+h}(f) + [f(x+h) - f(x)] \tag{2.11}$$

Note *Seeing the term $f(x+h) - f(x)$ in the context of total variation, we immediately think of the partition $P = \{x, x+h\}$ of $[x, x+h]$.*

We have

$$|f(x+h) - f(x)| \leq V_x^{x+h}(f)$$

It then follows that

$$V_x^{x+h}(f) \geq |f(x+h) - f(x)| \geq -[f(x+h) - f(x)]$$

which further implies

$$V_x^{x+h}(f) + [f(x+h) - f(x)] \geq 0 \tag{2.12}$$

Comparing (2.11) and (2.12), we conclude that $g(x)$ is indeed increasing. ■

2.2.3 Characterization of Functions of Bounded Variation

With the help of Theorem 2.2.4, we can easily prove the following elegant theorem, which characterizes functions of bounded variation. It states that a function on $[a, b]$ is of bounded variation if and only if it can be written as a difference of two increasing functions. The difficult part of find such increasing functions is already handled by Theorem 2.2.4.

Theorem 2.2.5

Let f be defined on $[a, b]$, then f is of bounded variation if and only if it can be expressed as a difference of two increasing functions. ♡

Proof We first suppose that f is of bounded variation. Then Theorem 2.2.4 shows that $V_a^x(f)$ and $V_a^x(f) - f(x)$ are both increasing on $[a, b]$. Since we can write

$$f(x) = V_a^x(f) - [V_a^x(f) - f(x)]$$

It is proved.

Reversely, suppose that f can be expressed as a difference of two increasing functions g and h on $[a, b]$, $f = g - h$. Proposition 2.1.2 tells us that g and h are of bounded variation since they are increasing functions. Then by Theorem 2.2.1, we know that $g - h$ is also of bounded variation. This completes the proof. ■

Note *We can also make these two increasing functions strict. Suppose $f = g - h$. We can easily achieve this by defining $\tilde{g}(x) = g(x) + x$ and $\tilde{h}(x) = h(x) + x$.*

2.3 Continuous Functions of Bounded Variation

Previously, we have shown that a function f of bounded variation can be written as the difference of two increasing functions, $f = g - h$. Now, suppose f is continuous. We will show in this section that the two increasing functions g and h can also be made continuous as well.

Theorem 2.3.1

Suppose f is of bounded variation on $[a, b]$. Then f is continuous at $x_0 \in [a, b]$ if and only if $V_a^x(f)$ is continuous at x_0. In other words, every point of continuity of f is also a point of continuity of $V_a^x(f)$ and vice versa.

Proof We first suppose that $V_a^x(f)$ is continuous at x_0 and show that f is also continuous at x_0, which is the easier direction to prove.

We will only show that f is continuous from the right at x_0 ($x_0 \neq b$), and the continuity from the left is similarly proved (including $x_0 = b$). Let $\varepsilon > 0$ be arbitrary. Because $V_a^x(f)$ is continuous at x_0, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |V_a^x(f) - V_a^x(f)| < \varepsilon$$

For all h satisfying $0 < h < \delta$, we have

$$\begin{aligned}
|f(x_0 + h) - f(x_0)| &= v(P, f) \quad \text{where } P = \{x_0, x_0 + h\} \text{ is a partition of } [x_0, x_0 + h] \\
&\leq V_{x_0}^{x_0+h}(f) \\
&= V_a^{x_0+h}(f) - V_a^{x_0}(f) \\
&< \varepsilon
\end{aligned}$$

This shows that f is continuous at x_0 from the right. Applying a similar argument, one may show that it is also continuous at x_0 from the left by considering the interval $[x_0 - h, x_0]$.

We now prove the reverse direction. Suppose f is continuous at x_0. Again, we will only prove that $V_a^x(f)$ is continuous at x_0 ($x_0 \neq b$) from the right. Let $\varepsilon > 0$ be arbitrary. Since f is continuous at x_0, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon/2$$

Consider the total variation $V_{x_0}^b(f)$. For any h satisfying $0 < h < \delta$, There exists a partition P such that

$$x_1 - x_0 \leq \delta$$

where $x_1 = x_0 + h$ and

$$v(P, f) > V_{x_0}^b(f) - \varepsilon/2 \tag{2.13}$$

Note *If one is confusing about how finding such P is possible, we can think of finding it with the following process. First, find a partition P of $[x_0, b]$ such that*

$$v(P, f) > V_{x_0}^b(f) - \varepsilon/2$$

and then refine P to P' by adding a point c in between x_0 and x_1 such that $c - x_0 < \delta$. Note that $v(P', f) \geq v(P, f)$ (Proposition 2.1.4). Therefore,

$$v(P', f) > V_{x_0}^b(f) - \varepsilon/2$$

is satisfied. Finally, rename P' to P.

We can express $v(P, f)$ as

$$
\begin{aligned}
v(P, f) &= |\Delta f_1| + \underbrace{|\Delta f_2| + \cdots + |\Delta f_n|}_{= v(P', f) \text{ where } P' \text{ is a partition of } [x_1, b]} \\
&= |f(x_1) - f(x_0)| + v(P', f) \\
&\leq |f(x_1) - f(x_0)| + V_{x_1}^b(f) \\
&\text{Recall that } x_1 - x_0 < \delta \text{ and } f \text{ is continuous at } x_0 \\
&< \varepsilon/2 + V_{x_1}^b(f) \qquad (2.14)
\end{aligned}
$$

Combining (2.13) and (2.14), we obtain

$$\varepsilon/2 + V_{x_1}^b(f) > v(P, f) > V_{x_0}^b(f) - \varepsilon/2$$

Rearranging the terms yields

$$\left|V_a^{x_0+h}(f) - V_a^{x_0}(f)\right| = V_{x_0}^{x_0+h}(f) = V_{x_0}^{x_1}(f) = V_{x_0}^b(f) - V_{x_1}^b(f) < \varepsilon$$

(Specially, it also holds for $x_0 = 1$.) This shows $V_a^x(f)$ is continuous at x_0 from the right. And considering $V_a^{x_0}(f)$ and $V_a^{x_0-h}(f)$ and applying a similar argument, one can also show that $V_a^x(f)$ is continuous at x_0 from the left. ■

Chapter 3 The Riemann-Stieltjes Integral

3.1 The Definition of the Riemann-Stieltjes Integral

Definition 3.1.1

Let f and α be real-valued functions on $[a, b]$. Assume α is bounded. We say f is ***Riemann-Stieltjes integrable*** *with respect to α on $[a, b]$ if there exists a number A such that for any choice of $\varepsilon > 0$, we can always find a partition P_ε of $[a, b]$ such that for any partition P finer than P_ε, $P \supseteq P_\varepsilon$, and for any list of representatives T of P, the* ***Riemann-Stieltjes sum*** *satisfies*

$$|S(P, T, f, \alpha) - A| < \varepsilon$$

The number A is denoted by $\int_a^b f \,\mathrm{d}\alpha$ or more verbose, $\int_a^b f(x) \,\mathrm{d}\alpha(x)$, and is referred to as the (value of) ***Riemann-Stieltjes integral*** *(of f w.r.t. α on $[a, b]$).* ♣

In Apostol's definition [1], the function f is assumed to be bounded. This assumption is made because if f is unbounded, the integral is bound not to exist. Consequently, Apostol chose not to explore integrals of unbounded functions, excluding them from his definition.

However, for educational purposes, we aim to demonstrate explicitly that the integral does not exist when f is unbounded. Therefore, we modify the definition to allow f to be unbounded and subsequently prove the non-existence of the integral.

Proposition 3.1.1

If f is unbounded, then $f \notin \mathfrak{R}(\alpha)$ on $[a, b]$. ♠

Proof We shall prove by contradiction. Assume, on the contrary, $f \in \mathfrak{R}(\alpha)$ on $[a, b]$ and $\int_a^b f \,\mathrm{d}\alpha = A$. Then there exists a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that for any list of representatives T of P,

$$|S(P, T, f, \alpha) - A| < \frac{1}{2} \tag{3.1}$$

Let T_0 be a particular list of representatives. Because f is unbounded on $[a, b]$, there exists $j \in \{1, \ldots, n\}$ such that f is unbounded on $[x_{j-1}, x_j]$. It then follows that we may choose a point $t'_j \in [x_{j-1}, x_j]$ such that

$$\left|f(t'_j) - f(t_j)\right| |\Delta\alpha_j| > 1$$

Let T' be constructed by replacing the j-th point t_j with t'_j in T_0. We have

$$|S(P, T', f, \alpha) - S(P, T_0, f, \alpha)| = \left|[f(t'_j) - f(t_j)]\Delta\alpha_j\right| > 1$$

It then follows that

$$\begin{aligned} |S(P, T', f, \alpha) - A| &> |S(P, T', f, \alpha) - S(P, T_0, f, \alpha)| - |S(P, T_0, f, \alpha) - A| \\ &> 1 - \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

This results in a contradiction with (3.1). ■

3.2 Linear Properties

The following theorem shows the linearity of integrals in the fashion of the integrands.

Theorem 3.2.1

If $f, g \in \mathfrak{R}(\alpha)$ on $[a, b]$, then $c_1 f + c_2 g \in \mathfrak{R}(\alpha)$ on $[a, b]$. And

$$\int_a^b c_1 f + c_2 g \,\mathrm{d}\alpha = c_1 \int_a^b f \,\mathrm{d}\alpha + c_2 \int_a^b g \,\mathrm{d}\alpha \tag{3.2}$$

Proof Let $\varepsilon > 0$ be arbitrary. Because f and g are both Riemann integrable on $[a, b]$, there exists a partition P_ε of $[a, b]$ such that for any $P \supseteq P_\varepsilon$ and set of representatives T of P satisfying

$$\left| S(P, T, f, \alpha) - \int_a^b f \,\mathrm{d}\alpha \right| < \frac{\varepsilon}{|c_1| + |c_2| + 1} \quad \text{and} \quad \left| S(P, T, g, \alpha) - \int_a^b g \,\mathrm{d}\alpha \right| < \frac{\varepsilon}{|c_1| + |c_2| + 1} \tag{3.3}$$

Note *The reason of the choice of the small number $\frac{\varepsilon}{|c_1|+|c_2|+1}$ will be clear later. And the $+1$ in the denominator is designed for the case that both c_1 and c_2 are zeros.*

Consider the Riemann-Stieltjes sum $S(P, T, c_1 f + c_2 g, \alpha)$. We have

$$\begin{aligned}
&\left| S(P, T, c_1 f + c_2 g, \alpha) - c_1 \int_a^b f \,\mathrm{d}\alpha - c_2 \int_a^b g \,\mathrm{d}\alpha \right| \\
&= \left| \sum_k (c_1 \Delta f_k + c_2 \Delta g_k) - c_1 \int_a^b f \,\mathrm{d}\alpha - c_2 \int_a^b g \,\mathrm{d}\alpha \right| \\
&= \left| c_1 \sum_k \Delta f_k + c_2 \sum_k \Delta g_k - c_1 \int_a^b f \,\mathrm{d}\alpha - c_2 \int_a^b g \,\mathrm{d}\alpha \right| \\
&= \left| c_1 S(P, T, f, \alpha) + c_2 S(P, T, g, \alpha) - c_1 \int_a^b f \,\mathrm{d}\alpha - c_2 \int_a^b g \,\mathrm{d}\alpha \right| \\
&\leq |c_1| \left| S(P, T, f, \alpha) - \int_a^b f \,\mathrm{d}\alpha \right| + |c_2| \left| S(P, T, g, \alpha) - \int_a^b g \,\mathrm{d}\alpha \right|
\end{aligned}$$

Applying (3.3), we obtain

$$\begin{aligned}
\left| S(P, T, c_1 f + c_2 g, \alpha) - c_1 \int_a^b f \,\mathrm{d}\alpha - c_2 \int_a^b g \,\mathrm{d}\alpha \right| &< |c_1| \frac{\varepsilon}{|c_1| + |c_2| + 1} + |c_2| \frac{\varepsilon}{|c_1| + |c_2| + 1} \\
&= \frac{\varepsilon(|c_1| + |c_2|)}{|c_1| + |c_2| + 1} \\
&< \varepsilon
\end{aligned}$$

This shows that $c_1 f + c_2 g$ is also Riemann integrable on $[a, b]$, and (3.2) is satisfied. ■

Analogously, we can prove that the integral is linear in the integrators.

Theorem 3.2.2

If $f \in \mathfrak{R}(\alpha)$ *and* $f \in \mathfrak{R}(\beta)$ *on* $[a, b]$, *then* $f \in \mathfrak{R}(c_1\alpha + c_2\beta)$ *on* $[a, b]$. *And*

$$\int_a^b f \, \mathrm{d}(c_1\alpha + c_2\beta) = c_1 \int_a^b f \, \mathrm{d}\alpha + c_2 \int_a^b f \, \mathrm{d}\beta$$

Theorem 3.2.3

Assume $c \in (a, b)$. *If two of the three integrals in* (3.4) *exist, then the other one also exists, and* (3.4) *holds.*

$$\int_a^c f \, \mathrm{d}\alpha + \int_c^b f \, \mathrm{d}\alpha = \int_a^b f \, \mathrm{d}\alpha \tag{3.4}$$

3.3 Integration by Parts

Theorem 3.3.1

If $f \in \mathfrak{R}(\alpha)$ *on* $[a, b]$, *then* $\alpha \in \mathfrak{R}(f)$ *on* $[a, b]$, *and*

$$\int_a^b f \, \mathrm{d}\alpha + \int_a^b \alpha \, \mathrm{d}f = f(b)\alpha(b) - f(a)\alpha(a) \tag{3.5}$$

Note *Take a second and appreciate the beauty of symmetry of the equation* (3.5). *This can be regarded as a reciprocal rule for Riemann-Stieltjes integrals. Indeed, it tells us the value of the integral when the integrand and the integrator are swapped.*

Proof Let $\varepsilon > 0$ be arbitrary. Because $f \in \mathfrak{R}(\alpha)$ on $[a, b]$, by definition, there exists P_ε such that for any refinement $P \supseteq P_\varepsilon$ and any set of representatives T of P, the Riemann-Stieltjes sum $S(P, T, f, \alpha)$ satisfies that

$$\left| S(P, T, f, \alpha) - \int_a^b f \, \mathrm{d}\alpha \right| < \varepsilon$$

Consider an arbitrary refinement $P' \supseteq P_\varepsilon$. And let T' be a list of representatives of P'. We want to show that $S(P', T', \alpha, f)$ is near the desired value. Write $P' = \{x_0, \ldots, x_n\}$. The Riemann-Stieltjes sum $S(P', T', \alpha, f)$ can be then written as

$$S(P', T', \alpha, f) = \sum_{k=1}^{n} \alpha(t_k)[f(x_k) - f(x_{k-1}] = \sum_{k=1}^{n} \alpha(t_k) f(x_k) - \sum_{k=1}^{n} \alpha(t_k) f(x_{k-1}) \tag{3.6}$$

Meanwhile, the difference $A = f(b)\alpha(b) - f(a)\alpha(a)$ on the right-hand side of (3.5) can be written as

$$\begin{aligned}
A =& f(b)\alpha(b) - f(a)\alpha(a) \\
=& f(x_n)\alpha(x_n) - f(x_{n-1})\alpha(x_{n-1}) \\
&+ f(x_{n-1})\alpha(x_{n-1}) - \cdots - f(x_1)\alpha(x_1) \\
&+ f(x_1)\alpha(x_1) - f(x_0)\alpha(x_0) \\
=& \sum_{k=1}^{n} f(x_k)\alpha(x_k) - \sum_{k=1}^{n} f(x_{k-1})\alpha(x_{k-1})
\end{aligned} \tag{3.7}$$

Subtracting (3.6) from (3.7), we obtain

$$A - S(P', T', \alpha, f) = \sum_{k=1}^{n} f(x_k)[\alpha(x_k) - \alpha(t_k)] + \sum_{k=1}^{n} f(x_{k-1})[\alpha(t_k) - \alpha(x_{k-1})] \tag{3.8}$$

Taking a close look at the right-hand side of (3.8), one may realize that it is also a Riemann-Stieltjes sum. To see this, let $P'' = P' \cup T'$, and let T'' be the list of representatives constructed as follows. Choose x_k in $[t_k, x_k]$ (if $t_k < x_k$) and chose x_{k-1} in $[x_{k-1}, t_k]$ (if $x_{k-1} < t_k$).

Note *There are chances that $t_k = x_k$ or $t_k = x_{k-1}$. In that case, the term $f(x_k)[\alpha(x_k) - \alpha(t_k)]$ or $f(x_{k-1})[\alpha(t_k) - \alpha(x_{k-1})]$ would be zero.*

Consider the diagram shown in Figure 3.1.

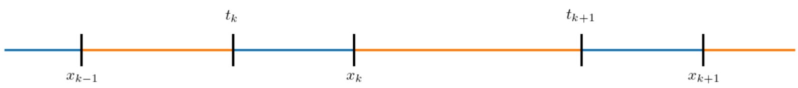

Figure 3.1: The blue part is associated with $f(x_k)[\alpha(x_k) - \alpha(t_k)]$ while the orange part is associated with $f(x_{k-1})[\alpha(t_k) - \alpha(x_{k-1})]$. We see that summing them up yields $S(P'', T'', f, \alpha)$.

The right-hand side of (3.8) is just $S(P'', T'', f, \alpha)$. Since $P'' \supseteq P_\varepsilon$, we have

$$\left| S(P'', T'', f, \alpha) - \int_a^b f \,\mathrm{d}\alpha \right| < \varepsilon \implies \left| A - S(P', T', \alpha, f) - \int_a^b f \,\mathrm{d}\alpha \right| < \varepsilon$$

This shows that $\alpha \in \mathfrak{R}(f)$ on $[a, b]$, and (3.5) is proved. ■

3.4 Change of Variables in Riemann-Stieltjes Integrals

Theorem 3.4.1

Suppose $f \in \mathfrak{R}(f)$ on $[a, b]$. Let g be a strictly monotonic and surjective function defined on an interval J having endpoints c and d. Suppose also that $g(c) = a$ and $g(d) = b$. Let

$$h(x) = f[g(x)] \quad \text{and} \quad \beta(x) = \alpha[g(x)] \quad x \in J$$

Then $h \in \mathfrak{R}(\beta)$ on J, and $\int_a^b f \,\mathrm{d}\alpha = \int_c^d h \,\mathrm{d}\beta$. That is,

$$\int_{a=g(c)}^{b=g(d)} f(t) \,\mathrm{d}t = \int_c^d f[g(x)] \,\mathrm{d}\alpha[g(x)]$$

♡

Note *Originally in [1], the condition of function g is that it is strictly monotonic and continuous.*

As a matter of fact, these two conditions are equivalent. At the end of the day, we impose these to conditions of g so that it has an inverse g^{-1} defined on $[a, b]$. (g^{-1} is also continuous of course.)

The key of this proof is constructing a one-to-one relation using function g between a partition P of $[a, b]$ and a partition P' of J. In the proof below, we write $P = g(P')$ and $P' = g^{-1}(P)$.

Proof Without loss of generality, we may assume that g is strictly increasing. If g is decreasing, we may easily obtain the same result by applying the linearity of the integrals.

Let $\varepsilon > 0$ be arbitrary. There exists a partition P_ε of $[a, b]$ satisfying the property described in Definition 3.1.1. As noted previously, we may construct a partition of $[c, d]$, $P'_\varepsilon = g^{-1}(P_\varepsilon)$. Let $P' \supseteq P'_\varepsilon$ be any refinement. Similarly, we may construct a partition of $[a, b]$ associated with P', $P = g(P')$.

Note *To exploit the property of P_ε, we would want to have $P \supseteq P_\varepsilon$. Luckily, this is indeed true.*

Now, we show that $P \supseteq P_\varepsilon$. For any point $x \in P_\varepsilon$, we have $g^{-1}(x) \in P'_\varepsilon \subseteq P'$. Since $x = g[g^{-1}(x)]$, it then follows that $x \in g(P') = P$. This shows $P \supseteq P_\varepsilon$.

Write $P' = \{y_0, \ldots, y_n\}$. Suppose T' is a list of representatives of P'. Let $T = g(T')$. Then, of course, T is a list of representatives of P. This implies that

$$
\begin{aligned}
S(P', T', h, \beta) &= \sum_{k=1}^{n} f[g(s_k)]\,\{\alpha[g(y_k)] - \alpha[y_{k-1}]\} \\
&= \sum_{k=1}^{n} f(t_k)[\alpha(x_k) - \alpha(x_{k-1})] \\
&= S(P, T, f, \alpha)
\end{aligned}
$$

The remainder of the proof is straightforward and is therefore omitted. ■

3.5 Reduction to Riemann Integrals

The next theorem tells us that we may replace the symbol $\mathrm{d}\alpha$ with $\alpha'(x)\,\mathrm{d}x$ under some conditions.

Theorem 3.5.1

Suppose $f \in \mathfrak{R}(\alpha)$ on $[a, b]$, and α has a continuous derivative on $[a, b]$. Then $f\alpha' \in \mathfrak{R}$ on $[a, b]$, and

$$\int_a^b f(x)\,\mathrm{d}\alpha(x) = \int_a^b f(x)\alpha'(x)\,\mathrm{d}x$$

Proof First, suppose f is bounded by $M > 0$, i.e.,

$$|f(x)| \leq M \quad \forall x \in [a, b] \tag{3.9}$$

Let $\varepsilon > 0$ be arbitrary.

Because α' is continuous on $[a, b]$, it is continuous uniformly there. There exists $\delta > 0$ such that

$$|s - t| \implies |\alpha'(s) - \alpha'(t)| < \frac{\varepsilon}{2M(b-a)} \tag{3.10}$$

Since f is integrable w.r.t. α on $[a, b]$, there exists a partition P_ε of $[a, b]$ such that for any refinement P of P_ε, and any list of representatives T of P, we have

$$\left|S(P,T,f,\alpha) - \int_a^b f\,\mathrm{d}\alpha\right| < \varepsilon/2 \tag{3.11}$$

Then, we can find a finer partition $P'_\varepsilon \supseteq P_\varepsilon$ such that $\|P'_\varepsilon\| < \delta$.

Let $P \supseteq P'_\varepsilon$ be a refinement such that and T be a list of representatives of P. Note that P is of course also a refinement of P_ε. Applying the mean value theorem, we have

$$\begin{aligned} S(P,T,f,\alpha) &= \sum_{k=1}^{n} f(t_k)[\alpha(t_k) - \alpha(t_{k-1})] \\ &= \sum_{k=1}^{n} f(t_k)\alpha'(s_k)\Delta x_k \end{aligned}$$

where each $s_k \in (x_{k-1}, x_k)$.

Taking the difference of $S(P,T,f\alpha',x)$ and $S(P,T,f,\alpha)$, we have

$$\begin{aligned} |S(P,T,f\alpha',x) - S(P,T,f,\alpha)| &= \left|\sum_{k=1}^{n} f(t_k)[\alpha'(t_k) - \alpha'(s_k)]\Delta x_k\right| \\ &\le \sum_{k=1}^{n} |f(t_k)[\alpha'(t_k) - \alpha'(s_k)]\Delta x_k| \\ &= \sum_{k=1}^{n} |f(t_k)|\,|\alpha'(t_k) - \alpha'(s_k)|\,\Delta x_k \end{aligned}$$

Then applying (3.9) and (3.10), the above difference is further bounded by

$$|S(P,T,f\alpha',x) - S(P,T,f,\alpha)| < M\frac{\varepsilon}{2M(b-a)}\sum_{k=1}^{n}\Delta x_k = \varepsilon/2 \tag{3.12}$$

Recall $P \supseteq P_\varepsilon$. Then we may conclude this proof by comparing (3.11) and (3.12). ■

3.6 Step Functions as Integrators

Proposition 3.6.1

Suppose α is constant on $[a,b]$ except possibly at point $x = a$, that is, $\alpha(x) = \alpha(b)$ for all $a < x \le b$. If f is continuous from the right at a, then $f \in \mathfrak{R}(\alpha)$ on $[a,b]$, and

$$\int_a^b f\,\mathrm{d}\alpha = f(a)[\alpha(b) - \alpha(a)]$$

Note *Note that we assume α possibly has a different value at a. If α is constant on the entire interval $[a,b]$, the integral clearly exists and is zero.*

An analogous result holds when we assume α is constant on $[a,b]$ except possibly at the right endpoint $x = b$.

Proof If $\alpha(a) = \alpha(b)$, the conclusion is trivial. In the following proof, we assume $\alpha(a) \neq \alpha(b)$.

Let $\varepsilon > 0$ be arbitrary. Because f is continuous at $x = a$, there exists $\delta > 0$ such that

$$x - a < \delta \implies |f(x) - f(a)| < \frac{\varepsilon}{|\alpha(b) - \alpha(a)|}$$

Let $P_\varepsilon = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$ such that $x_1 < x_0 + \delta$. For any refinement $P \supseteq P_\varepsilon$, we have

$$S(P, T, f, \alpha) = \sum_{k=1}^{n} f(x_k)\Delta\alpha_k = f(t_1)[\alpha(x_1) - \alpha(x_0)] = f(t_1)[\alpha(b) - \alpha(a)]$$

It then follows that

$$\begin{aligned} |S(P, T, f, \alpha) - f(a)[\alpha(b) - \alpha(a)]| &= |f(t_1) - f(a)|\, |\alpha(b) - \alpha(a)| \\ &< \frac{\varepsilon}{|\alpha(b) - \alpha(a)|} |\alpha(b) - \alpha(a)| \\ &= \varepsilon \end{aligned}$$

This completes the proof. ■

Theorem 3.6.1

Given $a < c < b$. Let α be constant on $[a, b]$ except at point $x = c$. That is, let $\alpha(a)$, $\alpha(c)$ and $\alpha(b)$ be arbitrary. Define

$$\alpha(x) := \begin{cases} \alpha(a), & a \le x < c \\ \alpha(c), & x = c \\ \alpha(b), & c < x \le b \end{cases}$$

If function f is defined in such a way that

1. *At least one of f and α is continuous from the left at c, and*
2. *at least one of f and α is continuous from the right at c,*

then $f \in \mathfrak{R}(\alpha)$ on $[a, b]$, and

$$\int_a^b f \,\mathrm{d}\alpha = f(c)[\alpha(c+) - \alpha(c-)]$$

♡

Proof It follows from Proposition 3.6.1 and its analogous result that $f \in \mathfrak{R}(\alpha)$ on $[a, c]$ and $f \in \mathfrak{R}$ on $[c, b]$. The values of the integrals are

$$\int_a^c f \,\mathrm{d}\alpha = f(c)[\alpha(c) - \alpha(a)] \quad \text{and} \quad \int_c^b f \,\mathrm{d}\alpha = f(c)[\alpha(b) - \alpha(c)]$$

Then Theorem 3.2.3 implies that $f \in \mathfrak{R}(\alpha)$ on $[a, b]$, and

$$\int_a^b f \,\mathrm{d}\alpha = \int_a^c f \,\mathrm{d}\alpha + \int_c^b f \,\mathrm{d}\alpha = f(c)[\alpha(b) - \alpha(a)] = f(c)[\alpha(c+) - \alpha(c-)]$$

■

3.7 Reduction of Riemann-Stieltjes Integrals to Finite Sums

Function α in Theorem 3.6.1 is a special case of step functions.

Definition 3.7.1 (Step Functions)

A function defined on $[a, b]$ is called a ***step function*** *if there is a partition*

$$a = x_1 < x_2 < \cdots < x_n = b$$

such that α is constant on (x_{k-1}, x_k) for $k = 2, \ldots, n$.
The number $\alpha(x_k+) - \alpha(x_k-)$ is defined as the ***jump*** *at point x_k for $k = 2, \ldots, n-1$.*
At the left endpoint $x = a$, the jump is defined as $\alpha(a+) - \alpha(a)$. Similarly, at the right endpoint $x = b$, the jump is defined as $\alpha(b) - \alpha(b-)$. ♣

Note *It is possible that $\alpha(x_k+) = \alpha(x_k-)$. In this case, the jump at x_k is zero. But this does not mean that α is constant on (x_{k-1}, x_{k+1}) because we might have $x_k \neq x_k+$.*

Step functions provide the link between the Riemann-Stieltjes integrals and the finite sums of functions.

Theorem 3.7.1

Let α be a step function on $[a, b]$. Let $x_1, \ldots, x_n$ be the same as in Definition 3.7.1 and α_k be the jump at x_k.
Let f be a function defined on $[a, b]$ such that

1. *at least one of f and α is continuous from the left at x_k, and*
2. *at least one of f and α is continuous from the right at x_k*

for $k = 2, \ldots, n-1$. And for $k = 1$ and $k = n$, at least one of f and α is continuous from one side at the endpoint.
Then, $f \in \mathfrak{R}(\alpha)$ on $[a, b]$, and we have

$$\int_a^b f \, \mathrm{d}\alpha = \sum_{k=1}^{n} f(x_k)\alpha_k \tag{3.13}$$

♡

Proof Consider a partition $P = \{s_0, s_1, \ldots, s_n\}$ on $[a, b]$ where s_k satisfies that

$$x_k < s_k < s_{k+1} \quad \forall k = 2, \ldots, n-1$$

Then, we have $x_k \in (s_{k-1}, s_k)$ $\forall k = 2, \ldots, n-1$. Note that the condition of Theorem 3.6.1 is satisfied on each subinterval $[s_{k-1}, s_k]$, $k = 2, \ldots, n-1$. Therefore, $f \in \mathfrak{R}(\alpha)$ on $[s_{k-1}, s_k]$, and

$$\int_{s_{k-1}}^{s_k} f \, \mathrm{d}\alpha = f(x_k)[\alpha(x_k+) - \alpha(x_k-)] = f(x_k)\alpha_k \quad \forall k = 2, \ldots, n-1 \tag{3.14}$$

By Proposition 3.6.1, we know that (3.14) also holds for $k = 1$ and $k = n$. Then by Theorem 3.2.3, f is integrable on the entire interval $[a, b]$. We may then conclude this proof by summing up (3.14) over all k. ■

Substitute $\alpha(x) = \lfloor x \rfloor$ in (3.13), we will obtain a formula for representing any finite sum using an integral.

Theorem 3.7.2

Given a finite sum $\sum_{k=1}^{n} a_k$. We have

$$\sum_{k=1}^{n} a_k = \int_0^n a_{\lceil x \rceil} \,\mathrm{d}\lfloor x \rfloor \tag{3.15}$$

where a_0 is an arbitrary constant. ♡

Proof Let $\alpha(x) = \lfloor x \rfloor$ on $[a, b]$. Define a function f on $[0, n]$ by

$$f(x) = a_{\lceil x \rceil}, \quad x \in [0, n]$$

Note that α is continuous from the right at $x = 0, 1, \ldots, n-1$, and f is continuous from the left at $x = 1, 2, \ldots, n$. Therefore, (3.13) is applicable. It yields that

$$\int_0^n f \,\mathrm{d}\alpha = \sum_{k=1}^{n} f(k)\alpha_k = \sum_{k=1}^{n} a_k \cdot 1$$

This completes the proof. ■

Of course, the construction of f is not unique. The construction is valid as long as $f(k) = a_k$ and is continuous from the left at $x = 1, \ldots, n$. One may define f by applying the linear interpolation (or polynomial interpolation or spline interpolation, etc.) on the data $(1, a_1), \ldots, (n, a_n)$, in which case f is continuous on the entire interval $[0, n]$. But I prefer the one given in the proof since this makes both floor and ceiling functions appear in (3.15), which makes the formula prettier.

3.8 Euler's Summation Formula

Euler's summation formula compares a sum $\sum f(n)$ with its associated integral $\int f(x)\,\mathrm{d}x$. See Figure 3.2 for an illustration.

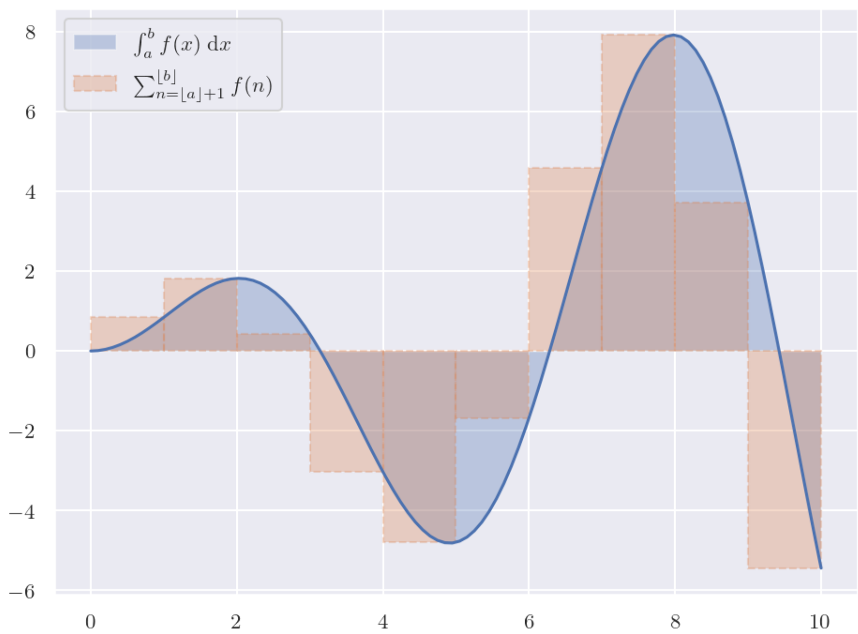

Figure 3.2: Euler's Summation Formula.

Theorem 3.8.1 (Euler's Summation Formula)

If f has a continuous derivative f' on $[a,b]$, then we have

$$\sum_{n=\lfloor a\rfloor+1}^{\lfloor b\rfloor} f(n) = \int_a^b f(x)\,\mathrm{d}x + \int_a^b f'(x)\{x\}\,\mathrm{d}x + f(a)\{a\} - f(b)\{b\} \tag{3.16}$$

In particular, if $a, b \in \mathbb{Z}$ we have

$$\sum_{n=a+1}^{b} f(n) = \int_a^b f(x)\,\mathrm{d}x + \int_a^b f'(x)\{x\}\,\mathrm{d}x \tag{3.17}$$

By adding the term $f(a)$ on both sides and applying the fundamental theorem of calculus, one may obtain a more symmetric formula:

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)\,\mathrm{d}x + \int_a^b f'(x)\left(\{x\} - \frac{1}{2}\right)\mathrm{d}x + \frac{f(a)+f(b)}{2} \tag{3.18}$$

Note (3.17) *is easier to use in practice while* (3.18) *is more elegant in the sense of symmetry. To derive* (3.18)*, we need to use the fundamental theorem of calculus, which will be introduced later.*

Proof Consider a partition of $[\lfloor a\rfloor + 1, \lfloor b\rfloor]$:

$$P = \{\lfloor a\rfloor + 1, \lfloor a\rfloor + 2, \ldots, \lfloor b\rfloor\}$$

Applying Theorem 3.7.1, we have

$$\int_{\lfloor a\rfloor+1}^{\lfloor b\rfloor} f(x)\,\mathrm{d}\lfloor x\rfloor = \sum_{n=\lfloor a\rfloor+2}^{b} f(n) \tag{3.19}$$

Note *Note that n starts at $\lfloor a\rfloor + 2$ since the jump of the floor function $\lfloor x\rfloor$ at $x = \lfloor a\rfloor + 1$ is* 0.

Next, we examine the integrals $\int_a^{\lfloor a\rfloor+1} f(x)\,\mathrm{d}\lfloor x\rfloor$ and $\int_{\lfloor b\rfloor}^{b} f(x)\,\mathrm{d}\lfloor x\rfloor$. It is clear that

$$\int_{\lfloor b\rfloor}^{b} f(x)\,\mathrm{d}\lfloor x\rfloor = 0 \tag{3.20}$$

because it is zero be definition if $b \in \mathbb{Z}$ and when $b \neq \mathbb{Z}$, $\lfloor x\rfloor = \lfloor b\rfloor$ is constant on $[\lfloor b\rfloor, b]$. Now, we consider $\lfloor x\rfloor$ on $[\lfloor a\rfloor, \lfloor a\rfloor + 1]$. We have

$$\lfloor x\rfloor = \begin{cases} \lfloor a\rfloor, & \lfloor a\rfloor \le x < \lfloor a\rfloor + 1 \\ \lfloor a\rfloor + 1, & x = \lfloor a\rfloor + 1 \end{cases}$$

Therefore, $\lfloor x\rfloor$ has a jump 1 at $x = \lfloor a\rfloor + 1$. It then follows that

$$\int_a^{\lfloor a\rfloor+1} f(x)\,\mathrm{d}\lfloor x\rfloor = f(\lfloor a\rfloor + 1) \tag{3.21}$$

Combining (3.19), (3.20) and (3.21), we obtain

$$\int_a^b f(x)\,\mathrm{d}\lfloor x\rfloor = \sum_{n=\lfloor a\rfloor+1}^{b} f(n) \tag{3.22}$$

This expresses the summation in (3.16) as an integral.

The rest of the proof leverages the theorems of integration by parts and reduction of Riemann integrals. Applying integration by parts, we have

$$\int_a^b f(x)\,\mathrm{d}x + \int_a^b x\,\mathrm{d}f(x) = f(b)b - f(a)a \tag{3.23}$$

and

$$\int_a^b f(x)\,\mathrm{d}\lfloor x\rfloor + \int_a^b \lfloor x\rfloor\,\mathrm{d}f(x) = f(b)\lfloor b\rfloor - f(a)\lfloor a\rfloor \tag{3.24}$$

Subtracting (3.24) from (3.23) yields

$$\int_a^b f(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}\lfloor x\rfloor + \int_a^b \{x\}\,\mathrm{d}f(x) = f(b)\{b\} - f(a)\{a\}$$

Since f has a continuous derivative, by Theorem 3.5.1, we can replace $\mathrm{d}f(x)$ with $f'(x)\,\mathrm{d}x$. Then rearranging the terms, we obtain

$$\int_a^b f(x)\,\mathrm{d}\lfloor x\rfloor = \int_a^b f(x)\,\mathrm{d}x + \int_a^b f'(x)\{x\}\,\mathrm{d}x + f(a)\{a\} - f(b)\{b\} \tag{3.25}$$

(3.16) is proved by comparing (3.22) and (3.25). ■

Example 3.1 Using the Euler's summation formula (3.17), we can derive the following identities related to summing up terms of the form $\frac{1}{k^s}$:

1. If $s \neq 1$
$$\sum_{k=1}^{n} \frac{1}{k^s} = \frac{1}{n^{s-1}} + s\int_1^n \frac{\lfloor x\rfloor}{x^{s+1}}\,\mathrm{d}x$$
2. If $s = 1$
$$\sum_{k=1}^{n} \frac{1}{k} = \ln n - \int_1^n \frac{\{x\}}{x^2}\,\mathrm{d}x + 1 \tag{3.26}$$

(3.26) provides another way of proving the sequence $\left\{\sum_1^n \frac{1}{k} - \ln n\right\}$ converges (to Euler's constant γ). And hence, we obtain an integral form of the Euler's constant:

$$\gamma = 1 - \int_1^\infty \frac{\{x\}}{x^2}\,\mathrm{d}x$$

3.9 Darboux Integration – Defining Integrals with Upper and Lower Integrals

From now on, the theory of Riemann-Stieltjes integration will be developed for increasing integrators.

One may have the feeling that it is troublesome to prove the existence of Integrals because not only are we required to find a partition but also consider all possible choices T of points in subintervals.

The definition of upper and lower sums will get rid of T in $S(P, T, f, \alpha)$.

Definition 3.9.1

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. For $k = 1, \dots, n$, define

$$M_k(f) := \sup_{x \in [x_{k-1}, x_k]} f(x)$$

$$m_k(f) := \inf_{x \in [x_{k-1}, x_k]} f(x)$$

The sums

$$U(P, f, \alpha) := \sum_{k=1}^{n} M_k(f) \Delta\alpha_k$$

$$L(P, f, \alpha) := \sum_{k=1}^{n} m_k(f) \Delta\alpha_k$$

*are called **upper and lower Darboux sums** respectively.* ♣

Let $t_k \in [x_{k-1}, x_k]$, clearly, we have

$$m_k(f) \leq f(t_k) \leq M_k$$

If the integrator α is increasing, then

$$m_k(f)\Delta\alpha_k \leq f(t_k)\Delta\alpha_k \leq M_k\Delta\alpha_k$$

Summing over k yields

$$L(P, f, \alpha) \leq S(P, T, f, \alpha) \leq U(P, f, \alpha)$$

Theorem 3.9.1

Assume α is increasing on $[a, b]$. Then

1. *if $P' \supseteq P$, we have*

$$U(P', f, \alpha) \leq U(P, f, \alpha) \quad \text{and} \quad L(P', f, \alpha) \geq L(P, f, \alpha)$$

   In other words, as the partition gets finer the upper Darboux sum decreases while the lower Darboux sum increases.

2. *For any two partitions P_1 and P_2, we have*

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

   That is, any lower Darboux sum is no greater than any upper Darboux sum. ♡

Think about how to prove $U(P', f, \alpha) \leq U(P, f, \alpha)$ in 1. One way to do this is by designing notations to explicitly write down the expressions for P' and $U(P', f, \alpha)$. My way is as follows. Let

$P = \{x_0, \dots, x_n\}$. Since $P \supseteq P$, we can express P' as

$$
\begin{aligned}
P' =\{&y_0,\\
&y_1, \dots y_{m_1},\\
&\dots,\\
&y_{m_{k-1}+1}, \dots y_{m_k},\\
&\dots,\\
&y_{m_{n-1}+1}, \dots y_{m_n}\}
\end{aligned}
$$

where $y_{m_k} = x_k$ ($m_0 = 0$) for $k = 0, 1, \dots, n$. And we have

$$U(P', f, \alpha) = \sum_{k=1}^{n} \sum_{j=m_{k-1}+1}^{m_k} \sup_{x\in[y_{j-1},y_j]} f(x)[\alpha(y_j) - \alpha(y_{j-1})]$$

And the rest of the proof can be done easily.

However, we can be a little smarter about this proof. Note that the major difficulty is that the form of the refinement P' is undetermined. It may contain many extra points scattered in different locations. But we can start by consider the simplest case where P' has only one point more than P. And then we can extend the conclusion to any larger refinements by applying the mathematical induction.

Proof **Proof of 1:** We only prove $U(P', f, \alpha) \le U(P, f, \alpha)$. First, consider the case where P' has only one point y more than $P = \{x_0, \dots, x_n\}$. Suppose $y \in (x_{j-1}, x_j)$. On the subinterval $[x_{j-1}, x_j]$, we have

$$
\begin{aligned}
&\sup_{x\in[x_{j-1},y]} f(x)[\alpha(y) - \alpha(x_{j-1})] + \sup_{x\in[y,x_j]} f(x)[\alpha(x_j) - \alpha(y)]\\
&\qquad \le \sup_{x\in[x_{j-1},x_j]} f(x)[\alpha(x_j) - \alpha(x_{j-1})] = M_j
\end{aligned}
$$

Then it is clear that $U(P', f, \alpha) \le U(P, f, \alpha)$.

In general, suppose $|P'| = |P| + n$, one may then prove this easily by applying the mathematical induction.

Proof of 2: Let $P = P_1 \cup P_2$. Then P is a refinement of both P_1 and P_2. Applying the first part of this theorem and the inequality that

$$L(P, f, \alpha) \le U(P, f, \alpha)$$

we obtain

$$L(P_1, f, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P_2, f, \alpha)$$

■

Let $P_0 = \{a, b\}$ be the trivial partition on $[a, b]$. Then $U(P, f, \alpha) \ge L(P_0, f, \alpha)$ and $L(P, f, \alpha) \le U(P_0, f, \alpha)$ for every partition P, which means that the set of all upper Darboux sums is bounded below, and the set of all lower Darboux sums is bounded above. Then we may take the infimum and supremum, respectively, of the two sets to introduce the definitions of the upper and lower integrals.

Definition 3.9.2

*Assume α is increasing on $[a,b]$. The **upper Darboux integral** is defined by*

$$\overline{\int_a^b} f \, \mathrm{d}\alpha := \inf_{P \in \mathcal{P}[a,b]} U(P,f,\alpha)$$

*and the **lower Darboux integral** is defined by*

$$\underline{\int_a^b} f \, \mathrm{d}\alpha := \sup_{P \in \mathcal{P}[a,b]} L(P,f,\alpha)$$

Note *Upper and lower integrals always exist assuming f is bounded and α is increasing, of course.*

Intuitively, the lower integral should be no greater than the upper integral.

Theorem 3.9.2

Assume α is increasing on $[a,b]$, we have

$$\underline{\int_a^b} f \, \mathrm{d}\alpha \leq \overline{\int_a^b} f \, \mathrm{d}\alpha \tag{3.27}$$

Proof Let $\varepsilon > 0$ be arbitrary. By the property of infimums, there exists a partition P_1 of $[a,b]$ such that

$$\underline{\int_a^b} f \, \mathrm{d}\alpha < L(P_1,f,\alpha) + \varepsilon/2$$

Similarly, by the property of supremums, there exists a partition P_2 of $[a,b]$ such that

$$\overline{\int_a^b} f \, \mathrm{d}\alpha > U(P_2,f,\alpha) - \varepsilon/2$$

It then follows that

$$\begin{aligned}
\underline{\int_a^b} f \, \mathrm{d}\alpha &< L(P_1,f,\alpha) + \varepsilon/2 \\
&\leq U(P_2,f,\alpha) + \varepsilon/2 \\
&< \overline{\int_a^b} f \, \mathrm{d}\alpha + \varepsilon/2 + \varepsilon/2 \\
&= \overline{\int_a^b} f \, \mathrm{d}\alpha + \varepsilon
\end{aligned}$$

In summary, we have

$$\underline{\int_a^b} f \, \mathrm{d}\alpha < \overline{\int_a^b} f \, \mathrm{d}\alpha + \varepsilon \quad \forall \varepsilon > 0$$

This implies that $\underline{\int_a^b} f \, \mathrm{d}\alpha \leq \overline{\int_a^b} f \, \mathrm{d}\alpha$. ■

There are examples where the inequality in (3.27) is strict.

Example 3.2 Consider the Dirichlet function $\mathbb{1}_{\mathbb{Q}}(x)$ restricted on $[a,b]$. Let P an arbitrary partition of $[a,b]$. On any subinterval $[x_{k-1},x_k]$, we have

$$\sup_{[x_{k-1},x_k]} \mathbb{1}_{\mathbb{Q}}(x) = 1 \quad \text{and} \quad \inf_{[x_{k-1},x_k]} \mathbb{1}_{\mathbb{Q}}(x) = 0$$

It then follows that

$$U(P, \mathbb{1}_{\mathbb{Q}}, x) = b - a \quad \text{and} \quad L(P, \mathbb{1}_{\mathbb{Q}}, x) = 0$$

Since the above equations hold for all $P \in \mathcal{P}[a, b]$, the upper and lower integrals are

$$\overline{\int_a^b} \mathbb{1}_{\mathbb{Q}}(x)\,\mathrm{d}x = b - a \quad \text{and} \quad \underline{\int_a^b} \mathbb{1}_{\mathbb{Q}}(x)\,\mathrm{d}x = 0$$

3.10 Additive and Linearity Properties of Upper and Lower Integrals

Proposition 3.10.1

Let $c \in (a, b)$. We have the following identities for upper and lower integrals:

$$\overline{\int_a^b} f\,\mathrm{d}\alpha = \overline{\int_a^c} f\,\mathrm{d}\alpha + \overline{\int_c^b} f\,\mathrm{d}\alpha$$

and

$$\underline{\int_a^b} f\,\mathrm{d}\alpha = \underline{\int_a^c} f\,\mathrm{d}\alpha + \underline{\int_c^b} f\,\mathrm{d}\alpha$$

♠

Proof We only prove the identity for upper integrals. We will prove this identity by showing both LHS $\geq$ RHS and LHS $\leq$ RHS.

Proof of LHS $\geq$ RHS: Let P_1 and P_2 be partitions of $[a, c]$ and $[c, b]$, respectively. Let $P = P_1 \cup P_2$. We have

$$U(P, f, \alpha) = U(P_1, f, \alpha) + U(P_2, f, \alpha) \geq \overline{\int_a^c} f\,\mathrm{d}\alpha + \overline{\int_c^b} f\,\mathrm{d}\alpha$$

Taking the infimum over P yields

$$\overline{\int_a^b} f\,\mathrm{d}\alpha = \inf_{P \in \mathcal{P}[a,b]} U(P, f, \alpha) \geq \overline{\int_a^c} f\,\mathrm{d}\alpha + \overline{\int_c^b} f\,\mathrm{d}\alpha$$

Proof of LHS $\leq$ RHS: Let $\varepsilon > 0$ be arbitrary. There exist a partition P_1 of $[a, c]$ and a partition P_2 of $[c, b]$ such that

$$U(P_1, f, \alpha) - \varepsilon/2 < \overline{\int_a^c} f\,\mathrm{d}\alpha$$
$$U(P_2, f, \alpha) - \varepsilon/2 < \overline{\int_c^b} f\,\mathrm{d}\alpha$$

Let $P = P_1 \cup P_2$. Adding the above two inequalities yields

$$U(P, f, \alpha) - \varepsilon = U(P_1, f, \alpha) - \varepsilon/2 + U(P_2, f, \alpha) - \varepsilon/2 < \overline{\int_a^c} f\,\mathrm{d}\alpha + \overline{\int_c^b} f\,\mathrm{d}\alpha$$

Taking the infimum over P on both sides yields

$$\overline{\int_a^b} f\,\mathrm{d}\alpha - \varepsilon = \inf_{P \in \mathcal{P}[a,b]} (U(P, f, \alpha) - \varepsilon) \leq \overline{\int_a^c} f\,\mathrm{d}\alpha + \overline{\int_c^b} f\,\mathrm{d}\alpha$$

Since

$$\overline{\int_a^b} f \, \mathrm{d}\alpha - \varepsilon \leq \overline{\int_a^c} f \, \mathrm{d}\alpha + \overline{\int_c^b} f \, \mathrm{d}\alpha$$

holds for every $\varepsilon > 0$,

$$\overline{\int_a^b} f \, \mathrm{d}\alpha \leq \overline{\int_a^c} f \, \mathrm{d}\alpha + \overline{\int_c^b} f \, \mathrm{d}\alpha$$

■

Proposition 3.10.2

We have the following inequalities about upper and lower integrals of sums of two functions:

$$\overline{\int_a^b} f + g \, \mathrm{d}\alpha \leq \overline{\int_a^b} f \, \mathrm{d}\alpha + \overline{\int_a^b} g \, \mathrm{d}\alpha \tag{3.28}$$

and

$$\underline{\int_a^b} f + g \, \mathrm{d}\alpha \geq \underline{\int_a^b} f \, \mathrm{d}\alpha + \underline{\int_a^b} g \, \mathrm{d}\alpha$$

♠

Proof We only prove (3.28). Let P be any partition of $[a, b]$. On each interval $[x_{k-1}, x_k]$, we have

$$f(x) + g(x) \leq \sup_{x \in [x_{k-1}, x_k]} f(x) + \sup_{x \in [x_{k-1}, x_k]} g(x) \quad \forall x \in [x_{k-1}, x_k]$$

Taking the supremum on both sides over x yields

$$\sup_{x \in [x_{k-1}, x_k]} f(x) + g(x) \leq \sup_{x \in [x_{k-1}, x_k]} f(x) + \sup_{x \in [x_{k-1}, x_k]} g(x)$$

Then summing over k:

$$\begin{aligned} U(P, f+g, \alpha) &= \sum_k \sup_{x \in [x_{k-1}, x_k]} f(x) + g(x) \\ &\leq \sum_k \sup_{x \in [x_{k-1}, x_k]} f(x) + \sum_k \sup_{x \in [x_{k-1}, x_k]} g(x) \\ &= U(P, f, \alpha) + U(P, g, \alpha) \end{aligned}$$

Finally, the inequality is proved by taking the infimum on both sides over P. ■

Example 3.3 The inequality (3.28) may be strict. Consider $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ and $g(x) = -\mathbb{1}_{\mathbb{Q}}(x)$ restricted on $[a, b]$. We have

$$\overline{\int_a^b} f + g \, \mathrm{d}\alpha = \overline{\int_a^b} 0 \, \mathrm{d}\alpha = 0$$

and

$$\overline{\int_a^b} f \, \mathrm{d}\alpha = b - a \quad \text{and} \quad \overline{\int_a^b} g \, \mathrm{d}\alpha = 0$$

3.11 Riemann's Condition

When we are discussing about whether the limit $\lim$ exists or not, we check that if the limit inferior $\liminf$ and the limit superior $\limsup$ are equal.

Similarly, it is reasonable to guess that the existence of the integral $\int$ depends on the equality of the lower and upper integrals (whether $\underline{\int} = \overline{\int}$).

Definition 3.11.1

*We say that a function f satisfies the **Riemann's condition** with respect to α on $[a, b]$ if for any $\varepsilon > 0$, there exists a partition P_ε such that*

$$P \supseteq P_\varepsilon \implies 0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon \tag{3.29}$$

or equivalently, just saying that

$$0 \leq U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon \tag{3.30}$$

Note (3.29) *is the definition given in [1]. But clearly* (3.29) $\iff$ (3.30)*.* ((3.29) $\implies$ (3.30) *is obvious.) If* (3.30) *holds, then by Theorem 3.9.1 we know that the upper sum will decrease while the lower sum will increase as the partition gets finer, and the lower sum is always no greater than the upper sum. Therefore,* (3.29) *holds.*

Proposition 3.11.1

Let f be a bounded function on $[a, b]$. We have

$$\sup_{x,y\in[a,b]} [f(x) - f(y)] = \sup_{x\in[a,b]} f(x) - \inf_{x\in[a,b]} f(x)$$

Proof Let $M = \sup_{x\in[a,b]} f(x)$ and $m = \inf_{x\in[a,b]} f(x)$. Clearly

$$f(x) - f(y) \leq M - m \quad \forall x, y \in [a, b]$$

That is, $M - m$ is an upper bound of the set $\{f(x) - f(y) \mid x, y \in [a, b]\}$. We now show that $M - m$ is the least upper bound. Let $\varepsilon > 0$ be arbitrary. There exists $x_0 \in [a, b]$ such that $f(x_0) > M - \varepsilon/2$, and there exists $y_0 \in [a, b]$ such that $f(y_0) < m + \varepsilon/2$. It then follows that

$$f(x_0) - f(y_0) > M - m - \varepsilon$$

This completes the proof. ■

The following theorem ease the procedure of proving the existence of an integral.

Theorem 3.11.1

Assume α is increasing on $[a, b]$. Then the following statements are equivalent:

1. *$f \in \mathfrak{R}(\alpha)$ on $[a, b]$.*
2. *f satisfies the Riemann's condition with respect to α on $[a, b]$.*
3. *$\underline{\int_a^b} f \, d\alpha = \overline{\int_a^b} f \, d\alpha$.*

Proof **Proof of 1 $\implies$ 2:** Let $\varepsilon > 0$ be arbitrary. Because $f \in \mathfrak{R}(\alpha)$ on $[a, b]$, there exists a partition P_ε such that for all $P \supseteq P_\varepsilon$ and any list of representatives T of P, we have

$$\left| S(P, T, f, \alpha) - \int_a^b f \, d\alpha \right| < \varepsilon/4 \tag{3.31}$$

We now construct two particular lists of representatives T_1 and T_2 of P as follows. Write

$P = \{x_0, \ldots, x_n\}$. On each subinterval $[x_{k-1}, x_k]$, we can find a t_k such that

$$f(t_k)\Delta\alpha_k < m_k + \frac{\varepsilon}{4n}$$

and we can find a t_k' such that

$$f(t_k')\Delta\alpha_k > M_k - \frac{\varepsilon}{4n}$$

Let T_1 be the list of all such t_k's and T_2 be the list of all t_k''s. It then follows that

$$S(P, T_1, f, \alpha) < L(P, f, \alpha) + \varepsilon/4$$
$$S(P, T_2, f, \alpha) > U(P, f, \alpha) - \varepsilon/4$$

Taking the difference yields

$$U(P, f, \alpha) - L(P, f, \alpha) < S(P, T_1, f, \alpha) - S(P, T_2, f, \alpha) + \varepsilon/2 \tag{3.32}$$

But (3.31) implies that

$$|S(P, T_1, f, \alpha) - S(P, T_2, f, \alpha)| < \varepsilon/2$$

Combined with (3.32), we obtain

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

which holds for any refinement P of P_ε. Therefore, f satisfies the Riemann's condition with respect to α on $[a, b]$.

Proof of 2 $\implies$ 3: Let $\varepsilon > 0$ be arbitrary. Because f satisfies the Riemann's condition. There exists a partition P_ε such that

$$U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$$

But

$$0 \le \overline{\int_a^b} f \, \mathrm{d}\alpha - \underline{\int_a^b} f \, \mathrm{d}\alpha \le U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha)$$

Therefore,

$$0 \le \overline{\int_a^b} f \, \mathrm{d}\alpha - \underline{\int_a^b} f \, \mathrm{d}\alpha < \varepsilon$$

Since the above inequality holds for every ε, we have $\underline{\int_a^b} f \, \mathrm{d}\alpha = \overline{\int_a^b} f \, \mathrm{d}\alpha$.

Proof of 3 $\implies$ 1: Let $\varepsilon > 0$ be arbitrary. There exists a partition P_ε' such that

$$\overline{\int_a^b} f \, \mathrm{d}\alpha + \varepsilon > U(P_\varepsilon', f, \alpha)$$

and a partition P_ε'' such that

$$\underline{\int_a^b} f \, \mathrm{d}\alpha - \varepsilon < L(P_\varepsilon'', f, \alpha)$$

Let $P_\varepsilon = P_\varepsilon' \cup P_\varepsilon''$. For any refinement $P \supseteq P_\varepsilon$, and any list of representatives T of P, we have

$$\underline{\int_a^b} f \, \mathrm{d}\alpha - \varepsilon < L(P, f, \alpha) \le S(P, T, f, \alpha) \le U(P, f, \alpha) < \overline{\int_a^b} f \, \mathrm{d}\alpha + \varepsilon$$

Because the lower and upper integrals are equal, we can write $A = \underline{\int_a^b} f \, \mathrm{d}\alpha - \varepsilon = \overline{\int_a^b} f \, \mathrm{d}\alpha - \varepsilon$. It then follows that

$$|S(P, T, f, \alpha) - A| < \varepsilon$$

This implies that $f \in \mathfrak{R}(\alpha)$ on $[a, b]$, and the value of the integral equals to that of both lower and upper integrals. ■

3.12 Comparison Theorems

Theorem 3.12.1

Assume α is increasing and $f \geq 0$ on $[a, b]$. If $f \in \mathfrak{R}(\alpha)$, then

$$\int_a^b f \, \mathrm{d}\alpha \geq 0$$

♡

Proof Let P be any partition of $[a, b]$. On any subinterval $[x_{k-1}, x_k]$, we have $f(x) \geq 0$ and hence

$$\sup_{[x_{k-1}, x_k]} f(x)\Delta\alpha_k \geq 0$$

Summing up over k, we obtain

$$U(P, f, \alpha) \geq 0$$

Finally, taking the infimum over P yields

$$\overline{\int_a^b} f \, \mathrm{d}\alpha \geq 0$$

Because $f \in \mathfrak{R}(\alpha)$, its integral equals to the upper integral, hence greater than or equal to zero. ■

Corollary 3.12.1

Assume α is increasing and $f \leq g$ on $[a, b]$. If $f, g \in \mathfrak{R}(\alpha)$, then

$$\int_a^b f \, \mathrm{d}\alpha \leq \int_a^b g \, \mathrm{d}\alpha$$

♡

Theorem 3.12.2

Assume α is increasing on $[a, b]$. If $f \in \mathfrak{R}(\alpha)$ on $[a, b]$, then $|f| \in \mathfrak{R}(\alpha)$ on $[a, b]$, and

$$\left| \int_a^b f \, \mathrm{d}\alpha \right| \leq \int_a^b |f| \, \mathrm{d}\alpha \tag{3.33}$$

♡

Proof Let $\varepsilon > 0$ be arbitrary. Because $f \in \mathfrak{R}(\alpha)$ on $[a, b]$ equivalently, satisfying the Riemann's condition, there exists a partition $P_\varepsilon = \{x_0, \ldots, x_n\}$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$. On each subinterval $I_k = [x_{k-1}, x_k]$, we have

$$\omega_{|f|}(I_k) = \sup_{x,y \in I_k} (|f(x)| - |f(y)|) \leq \sup_{x,y \in I_k} |f(x) - f(y)| = \omega_f(I_k)$$

Multiplying by $\Delta\alpha_k$ on both sides and then summing over k, we obtain

$$\begin{aligned} U(P_\varepsilon, |f|, \alpha) - L(P_\varepsilon, |f|, \alpha) &= \sum_{k=1}^{n} \omega_{|f|}(I_k)\Delta\alpha_k \\ &\leq \sum_{k=1}^{n} \omega_f(I_k)\Delta\alpha_k \\ &= U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) \\ &< \varepsilon \end{aligned}$$

This implies that $|f|$ also satisfies the Riemann's condition and hence integrable.

Then, since $-|f| \leq f \leq |f|$, (3.33) is proved by applying Corollary 3.12.1. ■

Example 3.4 If $|f|$ is integrable then f does not need to be integrable. Consider a function f on $[a, b]$ defined by

$$f(x) = \mathbb{1}_{\mathbb{Q}}(x) - \mathbb{1}_{\mathbb{R}\setminus\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$$

We have $|f(x)| = 1$, which is a constant and hence integrable. But f is not.

Theorem 3.12.3

Assume α is increasing on $[a, b]$. If $f \in \mathfrak{R}(\alpha)$ on $[a, b]$ then $f^2 \in \mathfrak{R}(\alpha)$ on $[a, b]$. ♡

Proof Suppose f is bounded by $M > 0$ on $[a, b]$. Let $\varepsilon > 0$ be arbitrary. Because $f \in \mathfrak{R}(\alpha)$ on $[a, b]$, there exists a partition P_ε of $[a, b]$ such that

$$U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \frac{\varepsilon}{2M}$$

We are going to show that f^2 also satisfies the Riemann's condition.

On each subinterval I_k, we have

$$\begin{aligned} \left|f^2(x) - f^2(y)\right| &= |f(x) + f(y)| \, |f(x) - f(y)| \\ &\leq 2M \sup_{x,y \in I_k} |f(x) - f(y)| \\ &= 2M\omega_f(I_k) \end{aligned}$$

It then follows that

$$\omega_{f^2}(I_k) = \sup_{x,y \in I_k} \left|f^2(x) - f^2(y)\right| \leq 2M\omega_f(I_k)$$

Multiplying by $\Delta\alpha_k$ and then summing over k yields

$$\begin{aligned} U(P_\varepsilon, f^2, \alpha) - L(P_\varepsilon, f^2, \alpha) &= \sum_k \omega_{f^2}(I_k)\Delta\alpha_k \\ &\leq 2M \sum_k \omega_f(I_k)\Delta\alpha_k \\ &= 2M[U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha)] \\ &< 2M\frac{\varepsilon}{2M} \\ &= \varepsilon \end{aligned}$$

■

An immediate consequence of the previous theorem is that the product of integrable functions is also integrable.

Theorem 3.12.4

Assume α is increasing on $[a, b]$. If $f, g \in \mathfrak{R}(\alpha)$ on $[a, b]$ then the product is also integrable: $fg \in \mathfrak{R}(\alpha)$ on $[a, b]$.

Proof The product fg can be written as

$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$$

Because the sum and difference $f \pm g$ are integrable, so do their squares $(f \pm g)^2$ by the previous theorem. Therefore, $fg \in \mathfrak{R}(\alpha)$ on $[a, b]$. ■

3.13 Integrators of Bounded Variation

Recall in the previous section we always assume that the integrator α is increasing. Now, we want to extend the theorems about existence of integrals to the case when the integrator is of bounded variation.

For example, we want to generalize Theorem 3.12.3 to that assuming α is of bounded variation on $[a, b]$, if $f \in \mathfrak{R}(\alpha)$ on $[a, b]$, then $f^2 \in \mathfrak{R}(\alpha)$ on $[a, b]$.

The key of achieving this is that a function α of bounded variation can be written as a difference of two increasing functions α_1 and α_2, (Theorem 2.2.5).

$$\alpha = \alpha_1 - \alpha_2$$

Then, to exploit the condition that $f \in \mathfrak{R}(\alpha)$, we would want that $f \in \mathfrak{R}(\alpha_1)$ and $f \in \mathfrak{R}(\alpha_2)$. But this is not true in general due to the nonuniqueness of the decomposition of α into two increasing functions. For example, consider $\alpha(x) = 0$ on $[a, b]$. We can write α as a difference of two identity functions $\alpha_1(x) = \alpha(x) = x$. Then the Dirichlet function $\mathbb{1}_{\mathbb{Q}}$ is not integrable w.r.t. α_1 nor α_2. But it is integrable w.r.t. α for the integrator is constant.

However, if we decompose α (in the canonical way) as

$$\alpha(x) = V_a^x(\alpha) - [V_a^x(\alpha) - \alpha(x)]$$

then we will see in Theorem 3.13.1 that f is also integrable w.r.t. $V_a^x(\alpha)$ and $V_a^x(\alpha) - \alpha(x)$.

Lemma 3.13.1

Assume α is of bounded variation on $[a, b]$. If $f \in \mathfrak{R}(\alpha)$ on $[a, b]$, then for any given $\varepsilon > 0$ there exists a partition P_ε of $[a, b]$ such that for all refinement $P \supseteq P_\varepsilon$, $P = \{x_0, \ldots, x_n\}$, we have

$$\sum_{k=1}^{n} \omega_f(I_k)\, |\Delta\alpha_k| < \varepsilon \tag{3.34}$$

Note *If α is increasing, then what this lemma states is exactly that f satisfies the Riemann's condition.*

The idea of the proof is as follows. The appearance of the oscillation $\omega_f(I_k)$ in (3.34) suggests us to consider the difference $f(t_k) - f(t_k')$. We have for all $t_k, t_k' \in [x_{k-1}, x_k]$ that

$$\left| \sum_{k=1}^{n} [f(t_k) - f(t_k')] \Delta\alpha_k \right| < c\varepsilon$$

The difficulty is that $\Delta\alpha_k$ is not nonnegative anymore as in the previous section. To solve this, we simply consider the set A of indices of which $\Delta\alpha_k \geq 0$ and the set B of indices of which $\Delta\alpha_k < 0$ separately. Then we have

$$\sum_{k=1}^{n} [f(t_k) - f(t_k')] \Delta\alpha_k = \sum_{k\in A} [f(t_k) - f(t_k')] \left|\Delta\alpha_k\right| + \sum_{k\in B} [f(t_k') - f(t_k)] \left|\Delta\alpha_k\right|$$

The rest of the proof is done by choosing the t_k and t_k' and then comparing $|f(t_k) - f(t_k')|$ with $\omega_f(I_k)$.

Proof If $V_a^b(\alpha) = 0$, then α is constant and hence the conclusion is trivial. In the following context, we assume $V_a^b(\alpha) > 0$.

Let $\varepsilon > 0$ be arbitrary. Because $f \in \mathfrak{R}(\alpha)$ on $[a, b]$, there exists a partition P_ε of $[a, b]$ such that for any refinement $P = \{x_0, \ldots, x_n\}$ and any list of representatives T of P, we have

$$\left| S(P, T, f, \alpha) - \int_a^b f \, \mathrm{d}\alpha \right| < \varepsilon/4$$

It then follows that

$$\left| \sum_{k=1}^{n} [f(t_k) - f(t_k')] \Delta\alpha_k \right| < \varepsilon/2 \quad \forall t_k, t_k' \in [x_{k-1}, x_k]$$

Let subsets A and B of indices $\{1, \ldots, n\}$ be defined by

$$A = \{\Delta\alpha_k \geq 0 \mid k \in \{1, \ldots, n\}\}$$
$$B = \{\Delta\alpha_k < 0 \mid k \in \{1, \ldots, n\}\}$$

Clearly, A and B forms a partition of $\{1, \ldots, n\}$, i.e., $A \cup B = \{1, \ldots, n\}$ and $A \cap B = \emptyset$. It then follows that

$$\sum_{k=1}^{n} [f(t_k) - f(t_k')] \Delta\alpha_k = \sum_{k\in A} [f(t_k) - f(t_k')] \left|\Delta\alpha_k\right| + \sum_{k\in B} [f(t_k') - f(t_k)] \left|\Delta\alpha_k\right|$$

Therefore,

$$\left| \sum_{k\in A} [f(t_k) - f(t_k')] \left|\Delta\alpha_k\right| + \sum_{k\in B} [f(t_k') - f(t_k)] \left|\Delta\alpha_k\right| \right| < \varepsilon/2 \quad \forall t_k, t_k' \in [x_{k-1}, x_k] \tag{3.35}$$

Now, we will strategically select t_k and t_k' to accomplish our goal.

1. For $k \in A$, we may select t_k and t_k' such that $f(t_k) - f(t_k') > \max\left\{0, \omega_f(I_k) - \frac{\varepsilon}{2V_a^b(\alpha)}\right\}$, and
2. For $k \in B$, we may select t_k and t_k' such that $f(t_k') - f(t_k) > \max\left\{0, \omega_f(I_k) - \frac{\varepsilon}{2V_a^b(\alpha)}\right\}$.

Plugging the particular choices of t_k's and t_k''s into (3.35), we obtain

$$\begin{aligned}
\varepsilon/2 &> \left|\sum_{k\in A}[f(t_k)-f(t_k')]\,|\Delta\alpha_k| + \sum_{k\in B}[f(t_k')-f(t_k)]\,|\Delta\alpha_k|\right| \\
&= \sum_{k\in A}[f(t_k)-f(t_k')]\,|\Delta\alpha_k| + \sum_{k\in B}[f(t_k')-f(t_k)]\,|\Delta\alpha_k| \\
&> \sum_{k=1}^{n}\left(\omega_f(I_k) - \frac{\varepsilon}{2V_a^b(\alpha)}\right)|\Delta\alpha_k| \\
&= \sum_{k=1}^{n}\omega_f(I_k)\,|\Delta\alpha_k| - \frac{\varepsilon}{2V_a^b(\alpha)}\sum_{k=1}^{n}|\Delta\alpha_k| \\
&\geq \sum_{k=1}^{n}\omega_f(I_k)\,|\Delta\alpha_k| - \varepsilon/2
\end{aligned}$$

where the last inequality follows from the property of total variations $\sum_{k=1}^{n}|\Delta\alpha_k| \leq V_a^b(\alpha)$. Moving the term $-\varepsilon/2$ to the left in the above inequality yields

$$\sum_{k=1}^{n}\omega_f(I_k)\,|\Delta\alpha_k| < \varepsilon$$

■

Theorem 3.13.1

Assume α is of bounded variation on $[a,b]$. If $f \in \mathfrak{R}(\alpha)$ on $[a,b]$ then $f \in \mathfrak{R}(V_a^x(\alpha))$ on $[a,b]$. And of course, applying the linearity, we also have $f \in \mathfrak{R}(V_a^x(\alpha)-\alpha)$ on $[a,b]$. ♡

Proof Suppose f is bounded by $M > 0$.

Let $\varepsilon > 0$ be arbitrary. By Lemma 3.13.1, there exists a partition P_ε' of $[a,b]$ such that for all $P \supseteq P_\varepsilon'$, we have

$$\sum_k \omega_f(I_k)\,|\Delta\alpha_k| < \varepsilon/2 \tag{3.36}$$

By the definition of total variations and Proposition 2.1.4, there exists a partition P_ε'' such that for all its refinement we have

$$V_a^b(\alpha) < \sum_k |\Delta\alpha_k| + \frac{\varepsilon}{4M} \tag{3.37}$$

Let $P_\varepsilon = P_\varepsilon' \cup P_\varepsilon''$. For any its refinement P, both (3.36) and (3.37) hold. And we have

$$\begin{aligned}
\sum_k \omega_f(I_k)(V_{x_{k-1}}^{x_k}(\alpha) - |\Delta\alpha_k|) &\leq 2M\sum_k (V_{x_{k-1}}^{x_k}(\alpha) - |\Delta\alpha_k|) \\
&= 2M\left(V_a^b(\alpha) - \sum_k |\Delta\alpha_k|\right)
\end{aligned}$$

Then applying (3.37) yields

$$\sum_k \omega_f(I_k)(V_{x_{k-1}}^{x_k}(\alpha) - |\Delta\alpha_k|) < \varepsilon/2 \tag{3.38}$$

Adding (3.36) and (3.38), we obtain

$$\sum_k \omega_f(I_k) V_{x_{k-1}}^{x_k}(\alpha) < \varepsilon$$

This implies that f satisfies the Riemann's condition w.r.t. $V_a^x(\alpha)$ on $[a, b]$ and hence the proof is complete. ■

Theorem 3.13.2

Assume α is of bounded variation on $[a, b]$. If $f \in \mathfrak{R}(\alpha)$ on $[a, b]$ then f is also integrable on any subinterval. That is, if $[c, d] \subseteq [a, b]$, then $f \in \mathfrak{R}(\alpha)$ on $[a, d]$. ♡

Proof Thanks to Theorem 3.13.1 and Theorem 3.2.2, we only need to prove this theorem for increasing integrators. In what follows, we assume α is increasing.

We are going to show that f satisfies the Riemann's condition w.r.t. α on $[c, d]$. Let $\varepsilon > 0$ be arbitrary. Because $f \in \mathfrak{R}(\alpha)$ on $[a, b]$, there exists a partition P_ε of $[a, b]$ such that

$$U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$$

Let $P' = P_\varepsilon \cup \{c, d\}$. Since P' is a refinement of P_ε. It also holds that

$$U(P', f, \alpha) - L(P', f, \alpha) < \varepsilon$$

Let $P = P' \cap [c, d]$. We note that P is a partition of $[c, d]$. And if we write

$$U(P', f, \alpha) - L(P', f, \alpha) = \sum_{k \in J} \omega_f(I_k) \Delta\alpha_k$$

then we will find that $U(P, f, \alpha) - L(P, f, \alpha)$ is the sum of a subcollection of terms from the above sum. That is, we can write

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{k \in K} \omega_f(I_k) \Delta\alpha_k$$

where $K \subseteq J$. Since each term $\omega_f(I_k)\Delta\alpha_k$ is nonnegative, we have

$$U(P, f, \alpha) - L(P, f, \alpha) \le U(P', f, \alpha) - L(P', f, \alpha) < \varepsilon$$

This implies that f satisfies the Riemann's condition w.r.t. α on $[c, d]$. ■

Theorem 3.13.3

Assume $f, g \in \mathfrak{R}(\alpha)$ on $[a, b]$ where the integrator α is increasing. Define functions F and G on $[a, b]$ by

$$F(x) = \int_a^x f \, \mathrm{d}\alpha$$

and

$$G(x) = \int_a^x g \, \mathrm{d}\alpha$$

Then $f \in \mathfrak{R}(G)$, $g \in \mathfrak{R}(F)$ and $fg \in \mathfrak{R}(\alpha)$ on $[a, b]$. And we have

$$\int_a^b fg \, \mathrm{d}\alpha = \int_a^b f \, \mathrm{d}G = \int_a^b g \, \mathrm{d}F$$ ♡

Proof We first show that F and G are well defined. When $x = a$, the lower and upper limits are equal, hence the integrals are zeros by the definition. And when $a < x \le b$, by Theorem 3.13.2, f

and g are integrable w.r.t. α on $[a, x]$.

The conclusion that $fg \in \mathfrak{R}(\alpha)$ on $[a, b]$ is exactly Theorem 3.12.4.

In the following, we will only prove that $f \in \mathfrak{R}(G)$ and

$$\int_a^b fg \, \mathrm{d}\alpha = \int_a^b f \, \mathrm{d}G$$

Suppose f is bounded by $M > 0$. Let $\varepsilon > 0$ be arbitrary. Because $g \in \mathfrak{R}(\alpha)$ on $[a, b]$, equivalently, g satisfies the Riemann's condition. There exists a partition P'_ε such that

$$\sum_{k=1}^{n} \omega_g(I_k)\Delta\alpha_k < \frac{\varepsilon}{2M} \quad \forall P \supseteq P'_\varepsilon \tag{3.39}$$

Because the product fg is integrable w.r.t. α, there exists a partition P''_ε such that

$$\left| S(P, T, fg, \alpha) - \int_a^b fg \, \mathrm{d}\alpha \right| < \varepsilon/2 \quad \forall P \supseteq P''_\varepsilon \quad \forall T \text{ of } P \tag{3.40}$$

Let $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$. Let $P \supseteq P_\varepsilon$ and T be any list of representatives of P. We will compare $S(P, T, f, G)$ and $S(P, T, fg, \alpha)$. We have

$$\begin{aligned} |S(P, T, f, G) - S(P, T, fg, \alpha)| &= \left| \sum_{k=1}^{n} f(t_k)[\Delta G_k - g(t_k)\Delta\alpha_k] \right| \\ &\le \sum_{k=1}^{n} |f(t_k)| \, |\Delta G_k - g(t_k)\Delta\alpha_k| \\ &\le M \sum_{k=1}^{n} |\Delta G_k - g(t_k)\Delta\alpha_k| \end{aligned} \tag{3.41}$$

where

$$\Delta G_k = \int_{x_{k-1}}^{x_k} g \, \mathrm{d}\alpha$$

Note *The well-definedness of ΔG_k also follows from Theorem 3.13.2.*

Let $m_k = \inf_{x \in [x_{k-1}, x_k]} g(x)$ and $M_k = \sup_{x \in [x_{k-1}, x_k]} g(x)$. Because $m_k \le g(x) \le M_k \; \forall x \in [x_{k-1}, x_k]$, by the comparison theorem of integrals (Corollary 3.12.1), we have

$$m_k \Delta\alpha_k = \int_a^n m_k \, \mathrm{d}\alpha \le \int_{x_{k-1}}^{x_k} g \, \mathrm{d}\alpha \le \int_a^n M_k \, \mathrm{d}\alpha = M_k \Delta\alpha_k$$

That is,

$$m_k \Delta\alpha_k \le \Delta G_k \le M_k \Delta\alpha_k$$

It then follows that

$$|\Delta G_k - g(t_k)\Delta\alpha_k| \le \omega_g(I_k)\Delta\alpha_k \tag{3.42}$$

Combining (3.41), (3.42) and (3.39) yields

$$|S(P, T, f, G) - S(P, T, fg, \alpha)| \le M \sum_{k=1}^{n} \omega_g(I_k)\Delta\alpha_k < M \frac{\varepsilon}{2M} = \varepsilon/2 \tag{3.43}$$

Finally, compare (3.43) and (3.40) and we may conclude that

$$\left|S(P,T,f,G) - \int_a^b fg\,\mathrm{d}\alpha\right| \\ \le |S(P,T,f,G) - S(P,T,fg,\alpha)| + \left|S(P,T,fg,\alpha) - \int_a^b fg\,\mathrm{d}\alpha\right| < \varepsilon$$

This implies that $f \in \mathfrak{R}(G)$ on $[a,b]$ and $\int_a^b f\,\mathrm{d}G = \int_a^b fg\,\mathrm{d}\alpha$. ■

3.14 Sufficient Conditions for Existence of Riemann-Stieltjes Integrals

We begin to study when does the integral exist.

One may recall from the calculus that the continuous function is the most intuitive and straightforward type of function that possesses the Riemann integrals.

We will show in the following theorem that this is also true for Riemann-Stieltjes integrals with integrators of bounded variation.

Theorem 3.14.1

If f is continuous on $[a,b]$ and if α is of bounded variation on $[a,b]$, then $f \in \mathfrak{R}(\alpha)$ on $[a,b]$. ♡

Proof It suffices to prove this theorem for increasing α.

If $\alpha(a) = \alpha(b)$, then α is a constant function, in which case the conclusion is trivial. In what follows, we assume $\alpha(a) < \alpha(b)$.

Let $\varepsilon > 0$ be arbitrary. Because f is continuous on $[a,b]$, f is uniformly continuous there. Then there exists $\delta > 0$ such that

$$|x-y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon/2}{\alpha(b) - \alpha(a)}$$

Let P_ε be a partition such that $x_k - x_{k-1} < \delta$ for all k. It then follows that

$$\omega_f(I_k) = \sup_{x,y\in I_k} |f(x) - f(y)| \le \frac{\varepsilon/2}{\alpha(b)-\alpha(a)} < \frac{\varepsilon}{\alpha(b)-\alpha(a)}$$

Multiply by $\Delta\alpha_k$ and sum over k, and we will obtain

$$\sum_k \omega_f(I_k)\Delta\alpha_k < \frac{\varepsilon}{\alpha(b)-\alpha(a)} \sum_k \Delta\alpha_k = \varepsilon$$

This implies that f satisfies the Riemann's condition w.r.t. α on $[a,b]$. ■

Thanks to the theorem of integration by parts (Theorem 3.3.1), by swapping the assumptions for the integrand and the integrator, we can immediately obtain another sufficient condition for existence of Riemann-Stieltjes integrals.

Theorem 3.14.2

If f is of bounded variation on $[a,b]$ and α is continuous, then $f \in \mathfrak{R}(\alpha)$ on $[a,b]$. ♡

Put $\alpha(x) = x$ in Theorem 3.14.1 and Theorem 3.14.2, and we may conclude that continuous functions and functions of bounded variation are Riemann integrable.

Theorem 3.14.3

If on $[a,b]$,

1. *f is continuous, or*
2. *f is of bounded variation*

then $f \in \mathfrak{R}$ on $[a,b]$, i.e., $\int_a^b f(x)\,\mathrm{d}x$ exists.

Back to Theorem 3.14.1. In the following proposition, we study one of the corner cases that f is continuous on the interval except possibly at the endpoints.

Proposition 3.14.1

Let α be of bounded variation on $[a,b]$. Assume f is bounded on $[a,b]$ and continuous on $[a,b]$ except at endpoint a or b or both. If α is continuous at the points where f is not, then $f \in \mathfrak{R}(\alpha)$ on $[a,b]$.

Proof We only prove the case where f is discontinuous at a and α is continuous at a.

We want to prove this using Riemann's condition. Hence, the condition that the integral is increasing is needed. In the following, we will first show that if we were able to prove this proposition for increasing integrators, then we can extend to integrators being bounded of variation. Now, assume this proposition is proved for increasing integrators. Since α is of bounded of variation, we can decompose it into two increasing functions:

$$\alpha(x) = \underbrace{V_a^x(\alpha)}_{\alpha_1(x)} - \underbrace{[V_a^x(\alpha) - \alpha(x)]}_{\alpha_2(x)}$$

Recall Theorem 2.3.1, which states that every point of continuity of α is a point of continuity of $V_a^x(\alpha)$ and vice versa, it then follows that both α_1 and α_2 are continuous at a. Therefore, applying this proposition, we have $f \in \mathfrak{R}(\alpha_1)$ and $f \in \mathfrak{R}(\alpha_2)$ on $[a,b]$. Then by the linearity of the integrals, $f \in \mathfrak{R}(\alpha)$ on $[a,b]$, which proves this proposition for integrators of bounded variation.

In the following, we will prove this proposition assuming α is increasing.

In the following, we will prove this proposition assuming α is increasing. Suppose f is bounded by $M > 0$. Let $\varepsilon > 0$ be arbitrary. We will construct a partition P_ε of $[a,b]$ in two steps.

Because α is continuous at a, there exists $\delta > 0$ such that

$$x - a < \delta \implies \alpha(x) - \alpha(a) < \frac{\varepsilon}{2M}$$

Let $x_1 = a + \delta/2$. This point will set as the second point in P_ε.

Next, we note that f is continuous on the interval $[x_1, b]$, and hence it is continuous uniformly there. There exists $\delta' > 0$ such that

$$|s - t| < \delta' \implies |f(s) - f(t)| < \frac{\varepsilon}{2(b-a)}$$

Choose the rest of the points in P_ε in such a way that

$$\Delta x_k = x_k - x_{k-1} < \delta' \quad \forall 2 \le k \le n$$

It then follows that

$$\omega_f(I_k) \leq \frac{\varepsilon}{2(b-a)} \quad \forall 2 \leq k \leq n$$

With the above construction of P_ε, we have

$$\begin{aligned} U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) &= \sum_{k=1}^{n} \omega_f(I_k)\Delta x_k \\ &= \omega_f(I_1)(x_1 - a) + \sum_{k=2}^{n} \omega_f(I_k)\Delta x_k \\ &< M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2(b-a)} \sum_{k=2}^{n} \Delta x_k \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)} \cdot (b-a) \\ &= \varepsilon \end{aligned}$$

This proves $f \in \mathfrak{R}(\alpha)$ on $[a, b]$. ■

Example 3.5 Consider

$$f(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \in (0, 1] \\ 0, & x = 0 \end{cases}$$

Function f is bounded on $[0, 1]$ and it is continuous on $[0, 1]$ except at $x = 0$. The Riemann integral $\int_0^1 f(x)\,\mathrm{d}x$ exists.

3.15 Necessary Conditions for Existence of Riemann-Stieltjes Integrals

Theorem 3.15.1

Assume α is increasing on $[a, b]$. Let $a \leq c < b$. If f and α are discontinuous from the right at c, then $f \notin \mathfrak{R}(\alpha)$ on $[a, b]$.
Analogously, letting $a < c \leq b$, if f and α are discontinuous from the left at c, then $f \notin \mathfrak{R}(\alpha)$ on $[a, b]$.

Proof We only prove the case when f and α are discontinuous from the right at c.

Because f and α are discontinuous from the right at c, there exists $\varepsilon > 0$ such that for every $\delta > 0$, we can find $x, y \in (c, c + \delta)$ such that

$$|f(x) - f(c)| > \varepsilon \quad \text{and} \quad |\alpha(y) - \alpha(c)| = \alpha(y) - \alpha(c) > \varepsilon$$

We want to show that f does not satisfy the Riemann's condition. More precisely, choosing the number $\varepsilon_0 = \varepsilon^2$, we claim that for every partition P_{ε_0} of $[a, b]$, there exists a refinement P such that $U(P, f, \alpha) - L(P, f, \alpha) > \varepsilon_0$.

First, exploiting the fact that α is discontinuous from the right at c we can find a number $d > c$ such that $\alpha(d) - \alpha(c) > \varepsilon$. Let partition $P = P_{\varepsilon_0} \cup \{c, d\}$. In this way, we guarantee that the subinterval $[c, d]$ is contained in the partition. Then, since f is discontinuous from the right at c, we can find $t \in (c, d)$ such that $|f(t) - f(c)| > \varepsilon$. It then follows that

$$\omega_f([c, d]) = \sup_{x,y \in [c,d]} |f(x) - f(y)| \geq |f(t) - f(c)| > \varepsilon$$

Multiplying by $[\alpha(d) - \alpha(c)]$ yields

$$\omega_f([c, d])[\alpha(d) - \alpha(c)] > \varepsilon^2 = \varepsilon_0$$

Therefore,

$$U(P, f, \alpha) - L(P, f, \alpha) \geq \omega_f([c, d])[\alpha(d) - \alpha(c)] > \varepsilon_0$$

This completes the proof. ■

3.16 Mean Value Theorems for Riemann-Stieltjes Integrals

Theorem 3.16.1 (First Mean Value Theorem for Riemann-Stieltjes Integrals)

Assume α is increasing on $[a, b]$ and $f \in \mathfrak{R}(\alpha)$ on $[a, b]$. Let $m = \inf_{x \in [a,b]} f(x)$ and $M = \sup_{x \in [a,b]} f(x)$. Then there exists $c \in [m, M]$ such that

$$\int_a^b f \, \mathrm{d}\alpha = c \int_a^b \mathrm{d}\alpha = c[\alpha(b) - \alpha(a)]$$

In particular, if f is continuous on $[a, b]$ then f can attain the value c, i.e., exists $x_0 \in [a, b]$ such that $f(x_0) = c$. ♡

Proof If $\alpha(a) = \alpha(b)$, then α is constant and the integral evaluates to zero. In this case, simply choose $c = f(a)$. Now, assume $\alpha(a) < \alpha(b)$.

Because $m \leq f(x) \leq M$ (regard m and M as constant functions) on $[a, b]$, applying the comparison theorem (Corollary 3.12.1), we have

$$\int_a^b m \, \mathrm{d}\alpha \leq \int_a^b f \, \mathrm{d}\alpha \leq \int_a^b M \, \mathrm{d}\alpha$$

The number c defined by

$$c = \frac{1}{\alpha(b) - \alpha(a)} \int_a^b f \, \mathrm{d}\alpha$$

is as desired. ■

Example 3.6

Consider $f(x) = -x^2 + 2x + 3$ defined on $[0, 3]$. We have $m = 0$ and $M = 4$. The integral is $\int_0^3 f(x) \, \mathrm{d}x = 9$, which is represented by the blue area in Figure 3.3.

Theorem 3.16.1 says that we can find a line $y = c$ such that the area under $y = c$ equals the area under f where $0 = m \leq c \leq M = 4$. In this example, $c = 3$. And because f is continuous, it attains the value $c = 3$ on $[0, 3]$. Indeed, we have $f(0) = f(2) = 3$.

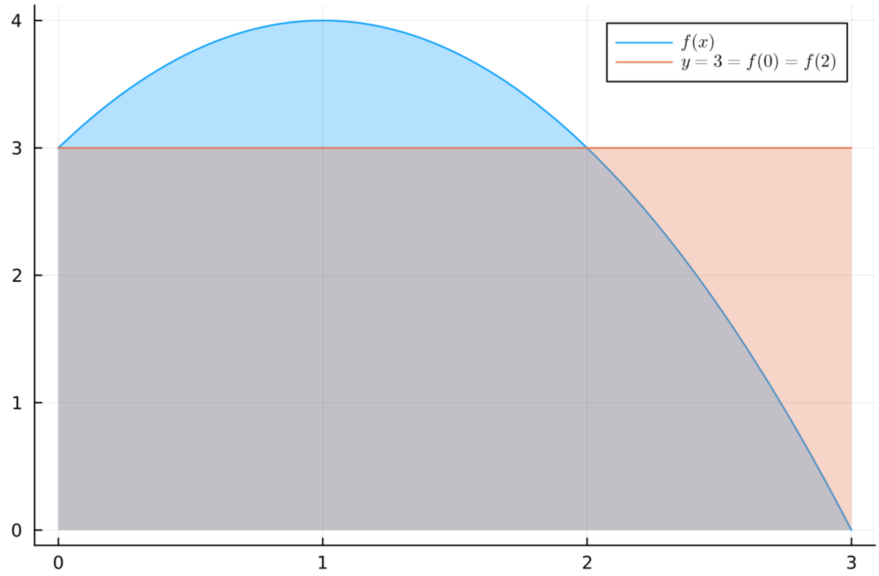

Figure 3.3: The area under f equals the area under $y = 3$.

Theorem 3.16.2 (Second Mean Value Theorem for Riemann-Stieltjes Integrals)

Assume α is continuous and f is increasing on $[a, b]$, then there exists a point $x_0 \in [a, b]$ such that

$$\int_a^b f \, \mathrm{d}\alpha = f(a) \int_a^{x_0} \mathrm{d}\alpha + f(b) \int_{x_0}^b \mathrm{d}\alpha$$

♡

The second mean value theorem can be proved easily using the first mean value theorem and integration by parts.

Proof

Because α is continuous and f is increasing, it is clear that $f \in \mathfrak{R}(\alpha)$ on $[a, b]$ and, of course, $\alpha \in \mathfrak{R}(f)$ on $[a, b]$.

Applying the first mean value theorem (Theorem 3.16.1), we have

$$\int_a^b \alpha \, \mathrm{d}f = \alpha(x_0)[f(b) - f(a)] \tag{3.44}$$

for some $x_0 \in [a, b]$. On the other hand, by the theorem of integration by parts, we have

$$\int_a^b f \, \mathrm{d}\alpha + \int_a^b \alpha \, \mathrm{d}f = f(b)\alpha(b) - f(a)\alpha(a) \tag{3.45}$$

Combining (3.44) and (3.45), we obtain that

$$\begin{aligned}
\int_a^b f \, \mathrm{d}\alpha &= f(b)\alpha(b) - f(a)\alpha(a) - f(b)\alpha(x_0) + f(a)\alpha(x_0) \\
&= f(a)[\alpha(x_0) - \alpha(a)] + f(b)[\alpha(b) - \alpha(x_0)] \\
&= f(a) \int_a^{x_0} \mathrm{d}\alpha + f(b) \int_{x_0}^b \mathrm{d}\alpha
\end{aligned}$$

which is as desired. ■

Example 3.7 For Riemann integrals, we can interpret Theorem 3.16.2 as follows. For example, consider $f(x) = 2 - \cos(x),\ x \in [0, \pi]$. By the second mean value theorem, we can find a point

$x_0 \in [0, \pi]$ (in this case, $x_0 = \pi/2$) such that the area under f is equal to the sum of areas of two rectangles. One rectangle has width $x_0 - a = \pi/2 - 0$ and height $f(a) = f(0)$. And the other one has width $b - x_0 = \pi - \pi/2$ and height $f(b) = f(\pi)$. See Figure 3.4 for an illustration.

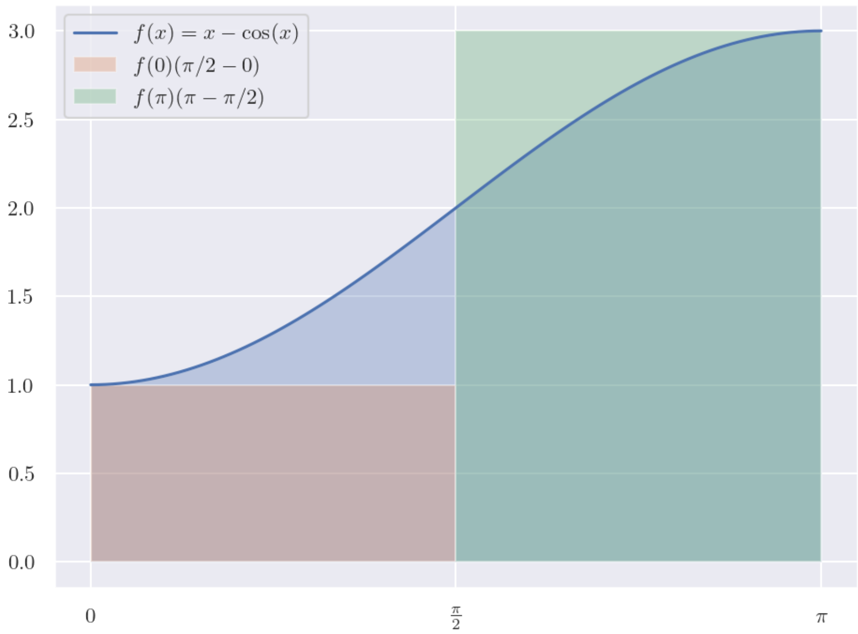

Figure 3.4: The area under f can be decomposed of two rectangles.

3.17 The Function of Variable Upper Limit Integrals

Theorem 3.17.1

Let α be of bounded variation on $[a, b]$ and $f \in \mathfrak{R}(\alpha)$ on $[a, b]$. Let function $F(x)$ be defined by

$$F(x) = \int_a^x f \,\mathrm{d}\alpha, \quad x \in [a, b]$$

(By Theorem 3.13.2 we see that F is well defined.) Then we have the following statements:

1. *F is of bounded variation on $[a, b]$.*
2. *Every point of continuity of α is also a point of continuity of F. In other words, if α is continuous at $x_0 \in [a, b]$, then F is also continuous at x_0.*
3. *Assume further that α is increasing on $[a, b]$. Then $F'(x)$ exists at $x \in (a, b)$ where $\alpha'(x)$ exists and where f is continuous. For such x, we have*

   $$F'(x) = f(x)\alpha'(x)$$

   And if the left side derivative $\alpha'_+(a)$ exists and f is continuous at the left endpoint, then the left derivate of F also exists. An analogous statement holds for the right endpoint.

   $$F'(a)_+ = f(a)\alpha'_+(a) \quad \text{or} \quad F'(b)_- = f(b)\alpha'_-(b)$$

♡

Proof **Proof of 1:** We will use the first mean value theorem and the condition that α is of bounded variation on $[a, b]$.

Because α is of bounded variation on $[a, b]$, by definition, there exists $M_1 > 0$ such that

$$\sum_k |\Delta\alpha_k| < M_1$$

for all partition P of $[a, b]$.

Suppose f is bounded by $M_2 > 0$. Let $M = M_1 M_2$. We will show that $\sum_k |\Delta F_k| < M$ for all partition P.

Let $P = \{x_0, \ldots, x_n\}$ be an approximate partition of $[a, b]$. On each subinterval $[x_{k-1}, x_k]$, applying the first mean value theorem, we obtain

$$|\Delta F_k| = \left| \int_{x_{k-1}}^{x_k} f \, \mathrm{d}\alpha \right| = |c_k| \, |\Delta\alpha_k| < M_2 \, |\Delta\alpha_k|$$

where c_k satisfies that $\inf_{x \in [x_{k-1}, x_k]} f(x) \le c_k \le \sup_{x \in [x_{k-1}, x_k]} f(x)$. Summing over k yields

$$\sum_{k=1}^{n} |\Delta F_k| < M_2 \sum_{k=1}^{n} |\Delta\alpha_k| < M_2 M_1 = M$$

This implies that F is indeed of bounded variation on $[a, b]$.

Proof of 2: Suppose f is bounded by $M > 0$. Let $\varepsilon > 0$ be arbitrary. Suppose α is continuous at $x_0 \in [a, b]$. Then there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |\alpha(x) - \alpha(x_0)| < \frac{\varepsilon}{2M}$$

Assume $0 < |x - x_0| < \delta$, we have

$$|F(x) - F(x_0)| = \left| \int_{x_0}^{x} f \, \mathrm{d}\alpha \right|$$

By the first mean value theorem, there exists a number c in between the infimum and supremum of f on the interval with endpoints x_0 and x such that

$$\int_{x_0}^{x} f \, \mathrm{d}\alpha = c[\alpha(x) - \alpha(x_0)]$$

It then follows that

$$|F(x) - F(x_0)| = |c| \, |\alpha(x) - \alpha(x_0)| \le M \frac{\varepsilon}{2M} = \varepsilon/2 < \varepsilon$$

This proves that F is continuous at x_0.

Proof of 3: For any $h \neq 0$ such that $x + h \in (a, b)$, we have

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_{x}^{x+h} f \, \mathrm{d}\alpha$$

By the first mean value theorem and the condition that f is continuous, there exists t in between x and $x + h$ (we have $|t - x| < |h|$) such that

$$\frac{1}{h} \int_{x}^{x+h} f \, \mathrm{d}\alpha = f(t) \frac{\alpha(x+h) - \alpha(x)}{h}$$

Because $\alpha'(x)$ exists and f is continuous at x, the limit of RHS of the above equation exists as $h \to 0$, we have

$$\lim_{h \to 0} f(t) \frac{\alpha(x+h) - \alpha(x)}{h} = f(x)\alpha'(x)$$

This completes the proof. ■

Specially, if we assume the integrator $\alpha(x) = x$, then we will obtain the **first fundamental**

theorem of calculus stated as follows.

Theorem 3.17.2 (First Fundamental Theorem of Calculus)

Let $F(x)$ be defined by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \quad x \in [a, b]$$

Then F is of bounded variation on $[a, b]$, and $F'(x)$ exists at $x \in (a, b)$ where f is continuous. For such x, we have

$$F'(x) = f(x)$$

If f is continuous at the endpoint a or b, then the one-sided derivate of F also exists there. In that case,

$$F'_+(a) = f(a) \quad \text{or} \quad F'_-(b) = f(b)$$

Example 3.8 Consider $f(x) = |x|\,,\ x \in [-1, 1]$ and $F(x) = \int_{-1}^x f(t)\,\mathrm{d}t, x \in [-1, 1]$. We know that $F^(x)$ exists in $(-1, 1)$ and its one-sided derivatives exist at endpoints. Consider the function $\tilde{F}$ defined by

$$\tilde{F}(x) = \begin{cases} -\frac{1}{2}x^2, & x \in [-1, 0) \\ \frac{1}{2}x^2, & x \in [0, 1] \end{cases}$$

By calculating the derivative of $\tilde{F}$, we find that

$$\tilde{F}'(x) = |x| = f(x) \quad \forall x \in (-1, 1)$$

And the one-sided derivatives $\tilde{F}'_+(-1) = 1 = f(-1)$ and $\tilde{F}'_-(1) = 1 = f(1)$. Therefore, by Rolle's theorem, F and $t\tilde{F}$ differs by a constant c, $F(x) = \tilde{F}(x) + c,\ x \in [-1, 1]$. Since $F(-1) = 0$, it follows that $c = \frac{1}{2}$. In conclusion, we have

$$F(x) = \frac{1}{2}\mathrm{sgn}(x)x^2 + \frac{1}{2}, \quad x \in [-1, 1]$$

The graphs of f and F are shown in Figure 3.5.

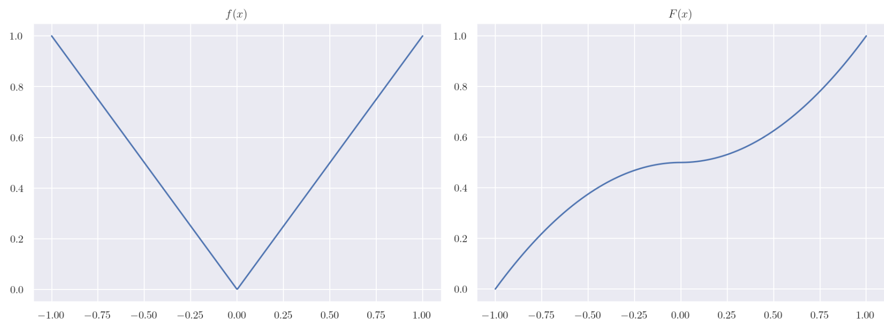

Figure 3.5: Left: function $f(x) = |x|$; Right: $F(x) = \int_{-1}^x f(t)\,\mathrm{d}t$.

Observe that the original function f has a sharp turn at $x = 0$. However, upon integration, the resulting function becomes smoother in the sense that it gains differentiability.

Combining Theorem 3.13.3 and Theorem 3.17.1, we obtain the following theorem which converts the Riemann integral of product fg to a Riemann-Stieltjes integral $\int_a^b f\,\mathrm{d}G$ or $\int_a^b g\,\mathrm{d}F$.

Theorem 3.17.3

Assume $f, g \in \mathfrak{R}$ on $[a, b]$. Let

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \quad G(x) = \int_a^x g(t)\,\mathrm{d}t, \quad x \in [a, b]$$

Then F and G are continuous functions of bounded variation on $[a, b]$. Moreover, $f \in \mathfrak{R}(G)$ and $g \in \mathfrak{R}(F)$ on $[a, b]$, and we have

$$\int_a^b f(x)g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}G(x) = \int_a^b g(x)\,\mathrm{d}F(x)$$

♡

3.18 Second Fundamental Theorem of Calculus

The first fundamental theorem of calculus tells us about nice properties of the function constructed from integrating a integrable function.

The following theorem, on the other hand, provides a method for evaluating the integral of a derivative.

Theorem 3.18.1 (Second Fundamental Theorem of Calculus)

Assume $f \in \mathfrak{R}$ on $[a, b]$. Let F be a continuous function defined on $[a, b]$ such that the derivative F' exists in (a, b) and

$$F'(x) = f(x) \quad \forall x \in (a, b)$$

Then we have

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

♡

Proof Let $\varepsilon > 0$ be arbitrary. Since $f \in \mathfrak{R}$ on $[a, b]$, there exists a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that

$$\left| S(P, T, f, x) - \int_a^b f(x)\,\mathrm{d}x \right| < \varepsilon$$

for any list of representatives. On each subinterval $[x_{k-1}, x_k]$, by the mean value theorem of differential calculus, we have

$$F(x_k) - F(x_{k-1}) = f(t_k)\Delta x_k$$

where $t_k \in (x_{k-1}, x_k)$. Summing over k yields

$$F(b) - F(a) = \sum_{k=1}^{n} f(t_k)\Delta x_k$$

where the RHS is a Riemann-Stieltjes sum. Therefore,

$$\left| (F(b) - F(a)) - \int_a^b f(x)\,\mathrm{d}x \right| < \varepsilon$$

Note that the above inequality holds for any $\varepsilon > 0$. This completes the proof. ■

✍ **Exercise 3.1** Assume further that f is continuous in Theorem 3.18.1. Construct a simpler proof using Theorem 3.5.1.

Example 3.9 Let

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \in (0,1] \\ 0, & x = 0 \end{cases}$$

Note that F is continuous on $[0,1]$, and in the interior $(0,1)$, we have

$$F'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

Define f on $[0,1]$ by

$$f(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \in (0,1] \\ 0, & x = 0 \end{cases}$$

Then $F'(x) = f(x)\ \forall x \in (0,1)$. In Example 3.5, we have seen that $f \in \mathfrak{R}$ on $[0,1]$. Using the second fundamental theorem of calculus, we can evaluate the integral $\int_0^1 f(x)\,\mathrm{d}x$:

$$\int_0^1 f(x)\,\mathrm{d}x = F(1) - F(0) = \sin(1)$$

Just for extra exercises, one may also compute that

$$\int_0^{1/\pi} f(x)\,\mathrm{d}x = F(1/\pi) - F(0) = 0 \quad \text{and} \quad \int_{1/\pi}^1 f(x)\,\mathrm{d}x = F(1) - F(1/\pi) = \sin(1)$$

A figure of f and F is shown in Figure 3.6.

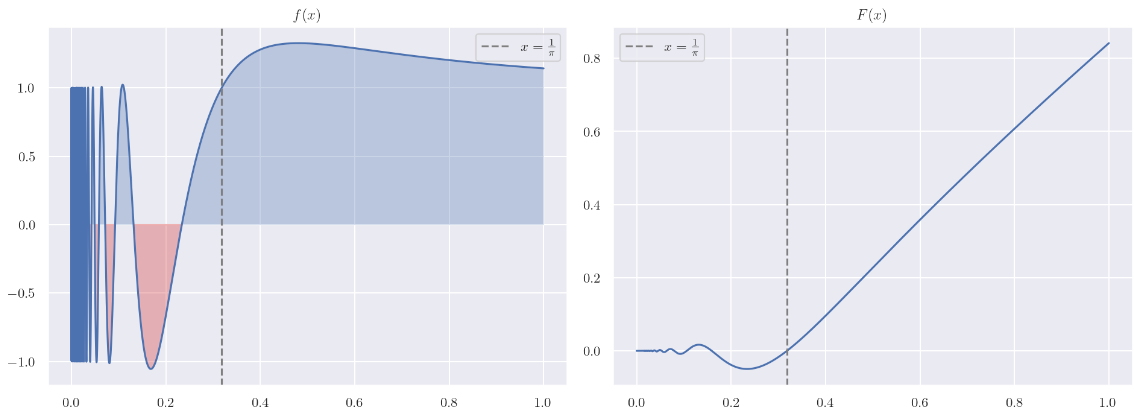

Figure 3.6: Left: $f(x)$, the signed area on the left of the line $x = \frac{1}{\pi}$ is 0, and the area on the right of the line $x = \frac{1}{\pi}$ equalling to the overall signed area, which is $\sin(1)$; Right: $F(x)$, note that $F(1/\pi) = 0$.

The second fundamental theorem of calculus combined with Theorem 3.18.1 can be used to strength Theorem 3.5.1.

Theorem 3.18.2

Assume $f \in \mathfrak{R}$ on $[a,b]$. If α is continuous on $[a,b]$ and its derivative $\alpha' \in \mathfrak{R}$ on $[a,b]$. Then the following integrals exist and are equal:

$$\int_a^b f(x)\,\mathrm{d}\alpha(x) = \int_a^b f(x)\alpha'(x)\,\mathrm{d}x$$

Note *This theorem strengthens Theorem 3.5.1 by relaxing the condition on α': instead of requiring*

α' to be continuous, it only requires α' to be integrable.

Proof Let

$$A(x) = \int_a^x \alpha'(t)\,\mathrm{d}t, \quad x \in [a, b]$$

By Theorem 3.17.3, we have $f \in \mathfrak{R}(A)$ on $[a, b]$, and

$$\int_a^b f(x)\alpha'(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}A(x)$$

Meanwhile, by the second fundamental theorem of calculus, we know

$$A(x) = \alpha(x) - \alpha(a)$$

Due to the linearity, $f \in \mathfrak{R}(\alpha)$ and

$$\begin{aligned}
\int_a^b f(x)\,\mathrm{d}\alpha(x) &= \int_a^b f(x)\,\mathrm{d}[A(x) + \alpha(a)] \\
&= \int_a^b f(x)\,\mathrm{d}A(x) + \int_a^b f(x)\,\mathrm{d}\alpha(a) \\
&= \int_a^b f(x)\,\mathrm{d}A(x) + 0 \\
&= \int_a^b f(x)\alpha'(x)\,\mathrm{d}x
\end{aligned}$$

■

3.19 Change of Variables in a Riemann Integral

Theorem 3.19.1

Assume that g has a continuous derivative g' on $[c, d]$. Let f be continuous on $g([c, d])$ and define F by

$$F(x) = \int_{g(c)}^x f(t)\,\mathrm{d}t, \quad x \in g([c, d])$$

Then

$$F[g(x)] = \int_{g(c)}^{g(x)} f(t)\,\mathrm{d}t = \int_c^x f[g(t)]g'(t)\,\mathrm{d}t \quad \forall x \in [c, d] \tag{3.46}$$

In particular, taking $x = d$ yields

$$\int_{g(c)}^{g(d)} f(t)\,\mathrm{d}t = \int_c^d f[g(t)]g'(t)\,\mathrm{d}t$$

♡

Note *Recall that a continuous function maps a connected sets to a connected sets, and maps compact sets to compact sets. Because g is continuous on $[c, d]$, the image $g([c, d])$ is actually a closed interval (when g is nonconstant). If g is constant, than $g([c, d])$ is just a single point.*

Proof When g is constant, the conclusion is trivial. In what follows, we assume g is nonconstant. As we have noted, $g([c, d])$ is a closed interval, say $[a, b]$. Since f is continuous on $[a, b]$, by the first fundamental theorem of calculus (Theorem 3.17.2), we have

$$F'(x) = f(x) \quad \forall x \in [a,b] = g([c,d])$$

For any $t \in [c,d]$, we have $g(t) \in [a,b]$. Substituting $x = g(t)$ and then multiplying by $g'(t)$ on both sides, we obtain

$$F'[g(t)]g'(t) = f[g(t)]g'(t) \quad \forall t \in [c,d]$$

Define function G on $[c,d]$ by

$$G(t) = F[g(t)], \quad t \in [c,d]$$

The chain rules implies

$$G'(t) = F'[g(t)]g'(t) = f[g(t)]g'(t) \quad \forall t \in [c,d]$$

Note that the derivative $G'(t)$ is continuous on $[c,d]$. Therefore, $G' \in \mathfrak{R}$ on $[c,d]$ (and hence on any subinterval, in particular, on $[c,x]$). Moreover, G is continuous on $[c,d]$. Hence, we are allowed use the second fundamental theorem of calculus (Theorem 3.18.1) to integrate the derivate $G'(t)$ on $[c,x]$. We have

$$\int_c^x G'(t)\,\mathrm{d}t = G(x) - G(c)$$

Plug in the formula for $H(t)$:

$$\int_c^d f'[g(t)]g'(t)\,\mathrm{d}t = F[g(x)] - F[g(c)] = \int_{g(c)}^{g(x)} f(t)\,\mathrm{d}t$$

Note that the above equality holds for every $x \in (c,d]$, and it also holds when $x = c$. ■

Example 3.10 Consider

$$I(x) = \int_0^x \sin^2 t \cos t\,\mathrm{d}t, \quad x \in \mathbb{R}$$

Let $f(t) = t^2$ and $g(t) = \sin t$. We have

$$I(x) = \int_0^x f[g(t)]g'(t)\,\mathrm{d}t$$

Apply (3.46) in reverse direction:

$$I(x) = \int_0^x f[g(t)]g'(t)\,\mathrm{d}t = \int_{g(0)}^{g(x)} f(t)\,\mathrm{d}t = \int_0^{\sin x} t^2\,\mathrm{d}t = \frac{1}{3}t^3\Big|_0^{\sin x} = \frac{1}{3}\sin^3 x$$

A graph of the function $\sin^2(x)\cos(x)$ is shown in Figure 3.7.

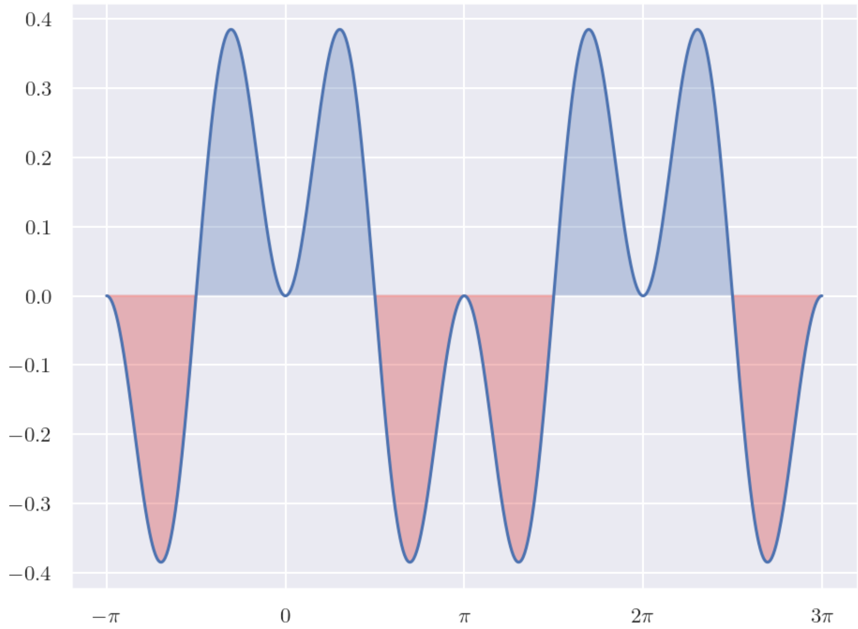

Figure 3.7: Graph of the function $\sin^2(x)\cos(x)$.

3.20 Second Mean-Value Theorem for Riemann Integrals

The second mean-value theorem provides a way of estimating the integral of a product $f(x)g(x)$ when one function is *monotonic*.

Theorem 3.20.1

Let g be continuous and f be increasing on $[a, b]$. Let A and B be two real numbers satisfying

$$A \leq f(a+) \quad \text{and} \quad B \geq f(b-)$$

Then there exists a point $x_0 \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = A\int_a^{x_0} g(x)\,\mathrm{d}x + B\int_{x_0}^b g(x)\,\mathrm{d}x \tag{3.47}$$

(Note that $f, g \in \mathfrak{R}$ on $[a, b]$, and so does their product (Theorem 3.12.4).)
In particular, if $f(x) \geq 0$ for all $x \in [a, b]$, we have

$$\int_a^b f(x)g(x)\,\mathrm{d}x = B\int_{x_0}^b g(x)\,\mathrm{d}x \tag{3.48}$$

♡

Proof **Proof of** (3.47)**:** We first define a function $\tilde{f}$ on $[a, b]$ as follows:

$$\tilde{f}(x) = \begin{cases} f(x), & x \in (a, b) \\ A, & x = a \\ B, & x = b \end{cases}$$

With the given properties of A and B, we note that $\tilde{f}$ is also increasing on $[a, b]$. And since $\tilde{f}g$ differs

fg by two points, the integrals are equal:

$$\int_a^b \tilde{f}(x)g(x)\,\mathrm{d}x = \int_a^b f(x)g(x)\,\mathrm{d}x \tag{3.49}$$

Let

$$G(x) = \int_a^x g(t)\,\mathrm{d}t, \quad x \in [a,b]$$

By Theorem 3.17.1, we have $G'(x) = g(x)\ \forall x \in [a,b]$. Then applying Theorem 3.18.2, we have

$$\int_a^b \tilde{f}(x)g(x)\,\mathrm{d}x = \int_a^b \tilde{f}(x)\,\mathrm{d}G(x) \tag{3.50}$$

Next, by the second mean value theorem for Riemann-Stieltjes integrals (Theorem 3.16.2), there exists $x_0 \in [a,b]$ such that

$$\begin{aligned}\int_a^b \tilde{f}(x)\,\mathrm{d}G(x) &= \tilde{f}(a)\int_a^{x_0} \mathrm{d}G(x) + \tilde{f}(b)\int_{x_0}^b \mathrm{d}G(x)\\ &= A\int_a^{x_0} \mathrm{d}G(x) + B\int_{x_0}^b \mathrm{d}G(x)\end{aligned}$$

Apply Theorem 3.18.2 again to replace $\mathrm{d}G(x)$ with $g(x)\,\mathrm{d}x$:

$$\int_a^b \tilde{f}(x)\,\mathrm{d}G(x) = A\int_a^{x_0} g(x)\,\mathrm{d}x + B\int_{x_0}^b g(x)\,\mathrm{d}x \tag{3.51}$$

Finally, combining (3.49), (3.50) and (3.51) together, we obtain

$$\int_a^b f(x)g(x)\,\mathrm{d}x = A\int_a^{x_0} g(x)\,\mathrm{d}x + B\int_{x_0}^b g(x)\,\mathrm{d}x$$

Proof of (3.48)**:** Let $A = 0$ and $B \geq f(b-)$. Since f is assumed to be nonnegative on $[a,b]$, it holds that $A \leq f(a+)$, and hence (3.47) is applicable. We have

$$\int_a^b f(x)g(x)\,\mathrm{d}x = A\int_a^{x_0} g(x)\,\mathrm{d}x + B\int_{x_0}^b g(x)\,\mathrm{d}x = B\int_{x_0}^b g(x)\,\mathrm{d}x$$

for some $x_0 \in [a,b]$.

■

An analogous result holds when f is decreasing on $[a,b]$. In that case, if $A \geq f(a+)$ and $B \leq f(b-)$, then there exists $x_0 \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = A\int_a^{x_0} g(x)\,\mathrm{d}x + B\int_{x_0}^b g(x)\,\mathrm{d}x$$

In particular, if f is nonpositive, then

$$\int_a^b f(x)g(x)\,\mathrm{d}x = A\int_a^{x_0} g(x)\,\mathrm{d}x$$

✍ **Exercise 3.2** Show that

$$\left|\int_a^b \frac{\sin x}{x}\,\mathrm{d}x\right| \leq \frac{2}{a}$$

where $0 < a < b$.

Proof Let $f(x) = \frac{1}{x}$ and $g(x) = \sin x$. Note that f is decreasing and nonnegative on $[a,b]$, and g is continuous on $[a, b]$. The second mean-value theorem implies that

$$\int_a^b \frac{\sin x}{x} \, \mathrm{d}x = \frac{1}{a} \int_a^{x_0} \sin x \, \mathrm{d}x$$

for some $x_0 \in [a, b]$. A few steps of calculation yield

$$\begin{aligned} \frac{1}{a} \int_a^{x_0} \sin x \, \mathrm{d}x &= \frac{1}{a} \left. (-\cos x) \right|_a^{x_0} \\ &= \frac{1}{a} (\cos a - \cos x_0) \end{aligned}$$

Finally, the proof is concluded by noting that $|\cos a - \cos x| \leq 2$. ■

References

[1] Tom M. Apostol. *Mathematical Analysis: A Modern Approach to Advanced Calculus*. 2. ed. Addison-Wesley Series in Mathematics. Reading, Mass.: Addison-Wesley, 1974. ISBN: 978-0-201-00288-1.

[2] Walter Rudin. *Principles of Mathematical Analysis*. 3d ed. International Series in Pure and Applied Mathematics. New York: McGraw-Hill, 1976. ISBN: 978-0-07-054235-8.

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