Probability and Statistics

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Part I

Measure Theory

Chapter 1 Measures

1.1 Semi-Algebras, Algebras and Sigma-Algebras

Definition 1.1.1 (Semi-Algebras)

A family of subsets $\mathcal{S}$ of Ω is a semi-algebra if it

1. *contains the empty set, i.e., $\emptyset \in \mathcal{S}$,*
2. *closed under finite Intersections, i.e., $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$, and*
3. *the complement of each set in $\mathcal{S}$ can be written as a finite disjoint union of other sets in $\mathcal{S}$, i.e., $A \in \mathcal{S} \implies \exists E_1, \ldots, E_n \in \mathcal{S},\ A = \biguplus_{i=1}^{n} E_i$.*

♣

Example 1.1 Consider the following semi-algebra on $\mathbb{R}$:

$$\mathcal{S} = \emptyset \cup \{(a, b] \mid a, b \in \mathbb{R}, a < b\} \cup \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b] \mid b \in \mathbb{R}\} \cup \mathbb{R}$$

It is easy to verify that it is indeed a semi-algebra. In fact, the definition of semi-algebra arises from the study of this very example. We will construct the famous Lebesgue measure from this semi-algebra, which defines the length of a set.

Definition 1.1.2 (σ-Algebras)

A family of subsets $\mathcal{F}$ of Ω is a σ-algebra if it satisfies the following:

1. *$\emptyset \in \mathcal{F}$*
2. *$\mathcal{F}$ is closed under complements, i.e., $A \in \mathcal{F} \implies A^{\complement} \in \mathcal{F}$*
3. *$\mathcal{F}$ is closed under countable unions, i.e., $A_i \in \mathcal{F}\ \forall i \in \mathbb{N}^* \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$*

♣

Definition 1.1.3

The Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ on $\mathbb{R}^d$ is defined as the σ-algebra generated by the collection of all open sets in $\mathbb{R}^d$. Mathematically,

$$\mathcal{B}(\mathbb{R}^d) := \sigma(\tau)$$

where τ is the Euclidean topology on $\mathbb{R}^d$.

♣

Lemma 1.1.1

Suppose $\mathcal{C}_1, \mathcal{C}_2 \subset \Omega$. If for any $E \in \mathcal{C}_1$, either one of the following holds:

1. *$E = F^{\complement}$ for some $F \in \mathcal{C}_2$*
2. *$E = \bigcup_{i=1}^{\infty} F_i$ where $F_i \in \mathcal{C}_2\ \forall i \in \mathbb{N}^*$*
3. *$E = \bigcap_{i=1}^{\infty} F_i$ where $F_i \in \mathcal{C}_2\ \forall i \in \mathbb{N}^*$*

then we have

$$\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{C}_2)$$

♡

1.2 Measures

A **set function** is a function that maps from collections of subsets of Ω to the extended real numbers $\mathbb{R} \cup \{\pm\infty\}$.

We say a set function μ is **finitely additive** if

$$\mu(A \uplus B) = \mu(A) + \mu(B)$$

The above equation also holds for finitely many disjoint unions of sets, that is,

$$\mu\left(\biguplus_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i) \tag{1.1}$$

which can be proved by the mathematical induction.

If (1.1) holds for countably infinite disjoint unions of sets, i.e.,

$$\mu\left(\biguplus_{i=1}^{n} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Then we say μ is **σ-additive** .

Definition 1.2.1

*Let $(\Omega, \mathcal{F})$ be a measurable space. A set function $\mu : \mathcal{F} \to [0, \infty]$ is called a **measure** if*

1. $\mu(\emptyset) = 0$, *and*
2. μ *is σ-additive.*

*The triplet $(\Omega, \mathcal{F}, \mu)$ is then called a **measure space** .*

♣

If μ is only finitely additive, we say that μ is a **finitely additive measure** .

Remark If we assume $\mu : \mathcal{F} \to [0, \infty]$ is σ-additive, and there exists some set $A \in \mathcal{F}$ such that $\mu(A) < \infty$, then $\mu(\emptyset) = 0$ holds naturally and hence condition 1 is redundant. To see this, we note that $\mu(A) = \mu(A \uplus \emptyset) = \mu(A) + \mu(\emptyset)$, which implies $\mu(\emptyset) = 0$ provided that $\mu(A)$ is finite.

The following proposition shows the monotonicity of a measure.

Proposition 1.2.1

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For $A, B \in \mathcal{F}$, we have

$$A \subset B \implies \mu(A) \le \mu(B) \tag{1.2}$$

Moreover, if $\mu(A) < \infty$ in (1.2), we have

$$\mu(B \setminus A) = \mu(B) - \mu(A)$$

♠

Proof Suppose $A \subset B$. We have

$$B = A \uplus (B \setminus A)$$

By the σ-additivity (or weaker, the finite additivity), it follows that

$$\mu(B) = \mu(A) + \mu(B \setminus A)$$

(1.2) follows since $\mu(B \setminus A) \ge 0$. If $\mu(A) < \infty$, by subtracting $\mu(A)$ from both sides of the above equation, we obtain

$$\mu(B) - \mu(A) = \mu(B \setminus A)$$

■

Given a sequence of sets $\{A_k\}$, we need this sequence of sets to be mutually disjoint in order to apply the σ-additivity of a measure. However, it is not the case in general. But we can easily construct another sequence of mutually disjoint sets from $\{A_k\}$ while keeping the union of first n sets unchanged. The procedure is illustrated in the following proposition.

Proposition 1.2.2 (Construction of Mutually Disjoint Sets)

Let $\{A_k\}_{k\in\mathbb{N}^*}$ be sequence of subsets of Ω. Let B_k be given by

$$B_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i$$

where $A_0 := \emptyset$. Then $\{B_k\}$ is a family of mutually disjoint sets, and

$$\biguplus_{k=1}^{n} B_k = \bigcup_{k=1}^{n} A_k \quad \forall n \in \mathbb{N}^*$$

Specially,

$$\biguplus_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k$$

♠

Remark This technique will be frequently used in the proofs of upcoming propositions and theorem.

Proof ■

Proposition 1.2.3

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $\{A_k\}$ a sequence of sets in $\mathcal{F}$. We have

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k) \tag{1.3}$$

♠

Remark If a set function satisfies (1.3), we say that it is **σ-subadditive** .

Proof Let B_k be given by

$$B_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i$$

as in Proposition 1.2.2. Then we have

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\biguplus_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$$

The last inequality follows from Proposition 1.2.1. ■

Proposition 1.2.4

♠

1.3 Extension of Set Functions on Semi-Algebras

Theorem 1.3.1

Let $\mathcal{S}$ be a semi-algebra on Ω, and $\mu : \mathcal{S} \to [0, \infty]$ a nonnegative additive (resp. σ-additive) set function. Then μ can be extended uniquely to an additive (resp. σ-additive) function ν on $\mathcal{A}(\mathcal{S})$. That is, $\exists! \nu : \mathcal{A}(\mathcal{S}) \to [0, \infty]$ such that

1. *ν is additive (resp. σ-additive), and*
2. *$\nu|_{\mathcal{S}} = \mu$.*

To be specific, this extension ν is given by

$$\nu(A) = \sum_{i=1}^{n} \mu(E_i)$$

where $\{E_1, \ldots, E_n\}$ is a family of mutually disjoint sets in $\mathcal{S}$ satisfying $A = \biguplus_{i=1}^{n} E_i$. ♡

1.4 Carathéodory's Extension Theorem

Definition 1.4.1

*An **outer measure** on Ω is a set function $\mu^* : \mathcal{P}(\Omega) \to [0, \infty]$ such that it*

1. *assumes zero at empty set, i.e., $\mu^*(\emptyset) = 0$, and*
2. *is σ-subadditive, i.e., $E \subset \bigcup_{i=1}^{\infty} E_i, \; \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$* ♣

Definition 1.4.2

*Suppose that μ^* is an outer measure on Ω. The collection of **measurable sets** with respect to μ^* is defined by*

$$\mathcal{M} = \left\{ A \subset \Omega \;\middle|\; \forall E \subset \Omega, \; \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^{\complement}) \right\}$$

Sometimes, we also say the sets in $\mathcal{M}$ are μ^*-measurable. ♣

Theorem 1.4.1

Let μ^* be an outer measure on Ω and $\mathcal{M}$ the collection of μ^*-measurable sets. We claim that

1. *$\mathcal{M}$ is a σ-algebra, and*
2. *$\mu^*|_{\mathcal{M}}$ is σ-additive.* ♡

Consider the equality

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^{\complement})$$

in the definition of $\mathcal{M}$. Note that it always holds that

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^{\complement})$$

because $E \subset (E \cap A) \cup (E \cap A^{\complement})$ and μ^* is an outer measure and hence σ-subadditive. Therefore, in the following proofs, we only need to show

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^\complement)$$

in order to prove the equality.

Before proving that $\mathcal{M}$ is a σ-algebra, we first show that it is an algebra.

Proof We shall check each condition in the definition of an algebra.

(Containment of the Empty Set) Note that

$$\mu^*(E \cap \Omega) + \mu^*(E \cap \Omega^\complement) = \mu^*(E) + 0 = \mu^*(E)$$

Therefore, clearly $\Omega \in \mathcal{M}$.

(Closure under Complements) Suppose $A \in \mathcal{M}$, then $\forall E \subset \Omega$, we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^\complement) = \mu^*(E \cap (A^\complement)^\complement) + \mu^*(E \cap A^\complement)$$

The last equality above implies that $A^\complement \in \mathcal{M}$.

(Closure under Finite Intersections) Suppose $A_1, A_2 \in \mathcal{M}$. Let $E \subset \Omega$ be arbitrary, then

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A_1) + \mu^*(E \cap {A_1}^\complement) && \text{since } A_1 \in \mathcal{M} \\ &= \mu^*(E \cap A_1 \cap A_2) + \mu^*(E \cap A_1 \cap {A_2}^\complement) + \mu^*(E \cap {A_1}^\complement) && \text{since } A_2 \in \mathcal{M} \end{aligned} \tag{1.4}$$

On the other hand, consider $\mu^*(E \cap (A_1 \cap A_2)^\complement)$. We have

$$\begin{aligned} \mu^*(E \cap (A_1 \cap A_2)^\complement) &= \mu^*((E \cap {A_1}^\complement) \cup (E \cap {A_2}^\complement)) \\ &= \mu^*(E \cap A_1 \cap {A_2}^\complement) + \mu^*(E \cap {A_1}^\complement) \quad \text{since } A_1 \in \mathcal{M} \end{aligned} \tag{1.5}$$

Combining equations (1.4) and (1.5), we obtain

$$\mu^*(E) = \mu^*(E \cap A_1 \cap A_2) + \mu^*(E \cap (A_1 \cap A_2)^\complement)$$

Therefore, $A_1 \cap A_2 \in \mathcal{M}$. ■

In the following, we show that $\mu^*|_{\mathcal{M}}$ is σ-additive.

Proof We prove this by first showing μ^* is additive on $\mathcal{M}$ and then applying Proposition 1.2.4 to conclude that μ^* is actually σ-additive on $\mathcal{M}$.

(Additivity) First, clearly $\mu^*(\emptyset) = 0$ since μ^* is an outer measure. Suppose $A_1, A_2 \in \mathcal{M}$ and $A = A_1 \uplus A_2$. Note that $A \in \mathcal{M}$ since we have shown that $\mathcal{M}$ is an algebra. It follows that

$$\mu^*(A) = \mu^*(A \cap A_1) + \mu^*(A \cap {A_1}^\complement)$$

Note that $A \cap A_1 = A_1$ and $A \cap {A_1}^\complement = A_2$. Therefore,

$$\mu^*(A) = \mu^*(A_1) + \mu^*(A_2)$$

Then by induction, we can show that for $A_i \in \mathcal{M}$,

$$\mu^*\left(\biguplus_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu^*(A_i)$$

(σ-Additivity) Note that μ^* is σ-subadditive on $\mathcal{P}(\Omega)$ (and of course it is also σ-subadditive on $\mathcal{M}$) because μ^* is an outer measure. Moreover, we have already shown that μ^* is also additive on $\mathcal{M}$. Then Proposition 1.2.4 immediately implies that μ^* is σ-additive. ■

Finally, we show that $\mathcal{M}$ is actually a σ-algebra.

Proof Recall that we have already shown $\mathcal{M}$ is an algebra. Hence, we only need to show that $\mathcal{M}$ is closed under countable unions. Suppose that $A_i \in \mathcal{M}$ and $A = \bigcup_{i=1}^{\infty} A_i$. Fix a set $E \subset \Omega$. Note that $\bigcap_{i=1}^{n} A_i \in \mathcal{M}$ since $\mathcal{M}$ is an algebra. It then follows that

$$\begin{aligned}\mu^*(E) &= \mu^*\left(E \cap \bigcap_{i=1}^{n} A_i\right) + \mu^*\left(E \cap \left(\bigcap_{i=1}^{n} A_i\right)^{\complement}\right) \\ &= \mu^*\left(E \cap \bigcap_{i=1}^{n} A_i\right) + \mu^*\left(E \cap \bigcup_{i=1}^{n} A_i{}^{\complement}\right)\end{aligned}$$

And since $E \cap \bigcap_{i=1}^{n} A_i \supset E \cap \bigcap_{i=1}^{\infty} A_i = E \cap A$, we have

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*\left(E \cap \bigcup_{i=1}^{n} A_i{}^{\complement}\right) \tag{1.6}$$

For the convenience of the notations, we denote

$$F_i = E \cap \bigcup_{j=1}^{i} A_j{}^{\complement}$$

Define sets G_i as follows.

$$G_1 = F_1 \qquad\qquad G_i = F_i \setminus \bigcup_{j=1}^{i-1} F_j \quad i \geq 2$$

One can show that all G_i's are mutually disjoint and $\bigcup_{i=1}^{n} G_i = \bigcup_{i=1}^{n} F_i$. It is also true that $\bigcup_{i=1}^{\infty} G_i = \bigcup_{i=1}^{\infty} F_i$. In summary,

$$\biguplus_{i=1}^{n} G_i = \bigcup_{i=1}^{n} F_i \qquad\qquad \biguplus_{i=1}^{\infty} G_i = \bigcup_{i=1}^{\infty} F_i = E \cap A^{\complement}$$

Applying the additivity of μ^* (this is valid because $G_i \in \mathcal{M}$) to inequality (1.6), we obtain

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*\left(\biguplus_{i=1}^{n} G_i\right) = \mu^*(E \cap A) + \sum_{i=1}^{n} \mu^*(G_i)$$

Letting $n \to \infty$,

$$\mu^*(E) \geq \mu^*(E \cap A) + \sum_{i=1}^{\infty} \mu^*(G_i)$$

Then we apply the σ-subadditivity of μ^*,

$$\mu^*(E) \geq \mu^*(E \cap A) + \sum_{i=1}^{\infty} \mu^*(G_i) \geq \mu^*(E \cap A) + \mu^*\left(\biguplus_{i=1}^{\infty} G_i\right) = \mu^*(E \cap A) + \mu^*(E \cap A^{\complement})$$

Therefore, we have shown that

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^{\complement})$$

In fact, the inequality above can be replaced by equality as we have explained before. Therefore, $A \in \mathcal{M}$ and hence $\mathcal{M}$ is indeed a σ-algebra. ■

The following theorem extends a pre-measure, i.e., a σ-additive nonnegative set function on an algebra $\mathcal{A}$ to a measure on the σ-algebra generated by $\mathcal{A}$.

Theorem 1.4.2 (Carathéodory's Extension Theorem)

Let $\mathcal{A}$ be an algebra on Ω, and μ_0 a pre-measure on $\mathcal{A}$. Then μ_0 can be extended to a measure μ on $\mathcal{F} = \sigma(\mathcal{A})$, i.e., there exists a measure $\mu : \mathcal{F} \to [0, \infty]$ such that $\mu|_{\mathcal{A}} = \mu_0$. Furthermore, if μ_0 is σ-fintie, then the extension is unique. ♡

Proof ■

Theorem 1.4.3 (Extension of Set Funcitons on Semi-Algebras)

Let $\mathcal{S}$ be a semi-algebra on Ω, and $\nu : \mathcal{S} \to [0, \infty]$ a σ-additive set function. Then, ν can be first uniquely extended to a pre-measure μ_0 on $\mathcal{A}(\mathcal{S})$. After that, if μ_0 is σ-fintie, it can be extended uniquely to a measure μ on $\sigma(\mathcal{S})$. (Note that $\sigma(\mathcal{A}(\mathcal{S})) = \sigma(\mathcal{S})$). ♡

1.5 Lebesgue Measure

Part II

Probability Theory

Chapter 2 Random Variables, Expectations, and Independence

2.1 Random Variables

We define random variables formally.

Part III

Mathematical Statistics

Chapter 3 Fundamentals of Statistics

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We shall introduce basic concepts.

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