Probability and Statistics

Author: Isaac FEI

Contents

Part I

Measure Theory

Chapter 1 Measures

1.1 Semi-Algebras, Algebras and Sigma-Algebras

Definition 1.1.1 (Semi-Algebras)

A family of subsets $\mathcal{S}$ of Ω is a semi-algebra if it

1. *contains the empty set, i.e., $\emptyset \in \mathcal{S}$,*
2. *closed under finite Intersections, i.e., $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$, and*
3. *the complement of each set in $\mathcal{S}$ can be written as a finite disjoint union of other sets in $\mathcal{S}$, i.e., $A \in \mathcal{S} \implies \exists E_1, \dots, E_n \in \mathcal{S},\ A = \biguplus_{i=1}^{n} E_i$.*

♣

Example 1.1 Consider the following semi-algebra on $\mathbb{R}$:

$$\mathcal{S} = \{\emptyset\} \cup \{(a, b] \mid a, b \in \mathbb{R}, a < b\} \cup \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b] \mid b \in \mathbb{R}\} \cup \{\mathbb{R}\}$$

It is easy to verify that it is indeed a semi-algebra. In fact, the definition of semi-algebra arises from the study of this very example. We will construct the famous Lebesgue measure from this semi-algebra, which defines the length of a set.

Definition 1.1.2 (σ-Algebras)

A family of subsets $\mathcal{F}$ of Ω is a σ-algebra if it satisfies the following:

1. *$\emptyset \in \mathcal{F}$*
2. *$\mathcal{F}$ is closed under complements, i.e., $A \in \mathcal{F} \implies A^{\complement} \in \mathcal{F}$*
3. *$\mathcal{F}$ is closed under countable unions, i.e., $A_i \in \mathcal{F}\ \forall i \in \mathbb{N}^* \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$*

♣

Proposition 1.1.1

Let $\mathcal{S}$ be a semi-algebra on Ω, and $\mathcal{A}(\mathcal{S})$ the algebra generated by $\mathcal{S}$. Then $\mathcal{A}(\mathcal{S})$ consists of all finite disjoint unions of sets in $\mathcal{S}$. Mathematically,

$$A \in \mathcal{A}(\mathcal{S}) \iff \exists \{S_i\}_{i=1}^{n} \subset \mathcal{S},\ A = \biguplus_{i=1}^{n} S_i$$

♠

Proof ■

Definition 1.1.3

The Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ on $\mathbb{R}^d$ is defined as the σ-algebra generated by the collection of all open sets in $\mathbb{R}^d$. Mathematically,

$$\mathcal{B}(\mathbb{R}^d) := \sigma(\tau)$$

where τ is the Euclidean topology on $\mathbb{R}^d$.

♣

Lemma 1.1.1

Suppose $\mathcal{C}_1, \mathcal{C}_2 \subset \Omega$. If for any $E \in \mathcal{C}_1$, either one of the following holds:

1. $E = F^{\complement}$ *for some* $F \in \mathcal{C}_2$
2. $E = \bigcup_{i=1}^{\infty} F_i$ *where* $F_i \in \mathcal{C}_2$ $\forall i \in \mathbb{N}^*$
3. $E = \bigcap_{i=1}^{\infty} F_i$ *where* $F_i \in \mathcal{C}_2$ $\forall i \in \mathbb{N}^*$

then we have

$$\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{C}_2)$$

♡

1.2 Measures

A **set function** is a function that maps from collections of subsets of Ω to the extended real numbers $\mathbb{R} \cup \{\pm\infty\}$.

We say a set function μ is **finitely additive** if

$$\mu(A \uplus B) = \mu(A) + \mu(B)$$

The above equation also holds for finitely many disjoint unions of sets, that is,

$$\mu\left(\biguplus_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i) \tag{1.1}$$

which can be proved by the mathematical induction.

If (1.1) holds for countably infinite disjoint unions of sets, i.e.,

$$\mu\left(\biguplus_{i=1}^{n} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Then we say μ is σ**-additive** .

Definition 1.2.1

Let $(\Omega, \mathcal{F})$ be a measurable space. A set function $\mu : \mathcal{F} \to [0, \infty]$ is called a ***measure*** *if*

1. $\mu(\emptyset) = 0$*, and*
2. μ *is σ-additive.*

The triplet $(\Omega, \mathcal{F}, \mu)$ is then called a ***measure space*** *.*

♣

If μ is only finitely additive, we say that μ is a **finitely additive measure** .

Remark If we assume $\mu : \mathcal{F} \to [0, \infty]$ is σ-additive, and there exists some set $A \in \mathcal{F}$ such that $\mu(A) < \infty$, then $\mu(\emptyset) = 0$ holds naturally and hence condition 1 is redundant. To see this, we note that $\mu(A) = \mu(A \uplus \emptyset) = \mu(A) + \mu(\emptyset)$, which implies $\mu(\emptyset) = 0$ provided that $\mu(A)$ is finite.

The following proposition shows the monotonicity of a measure.

Proposition 1.2.1

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For $A, B \in \mathcal{F}$, we have

$$A \subset B \implies \mu(A) \le \mu(B) \tag{1.2}$$

Moreover, if $\mu(A) < \infty$ *in* (1.2)*, we have*

$$\mu(B \setminus A) = \mu(B) - \mu(A)$$

Proof Suppose $A \subset B$. We have

$$B = A \uplus (B \setminus A)$$

By the σ-additivity (or weaker, the finite additivity), it follows that

$$\mu(B) = \mu(A) + \mu(B \setminus A)$$

(1.2) follows since $\mu(B \setminus A) \geq 0$. If $\mu(A) < \infty$, by subtracting $\mu(A)$ from both sides of the above equation, we obtain

$$\mu(B) - \mu(A) = \mu(B \setminus A)$$

■

Given a sequence of sets $\{A_k\}$, we need this sequence of sets to be mutually disjoint in order to apply the σ-additivity of a measure. However, it is not the case in general. But we can easily construct another sequence of mutually disjoint sets from $\{A_k\}$ while keeping the union of first n sets unchanged. The procedure is illustrated in the following proposition.

Proposition 1.2.2 (Construction of Mutually Disjoint Sets)

Let $\{A_k\}_{k \in \mathbb{N}^*}$ *be sequence of subsets of* Ω*. Let* B_k *be given by*

$$B_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i$$

where $A_0 := \emptyset$*. Then* $\{B_k\}$ *is a family of mutually disjoint sets, and*

$$\biguplus_{k=1}^{n} B_k = \bigcup_{k=1}^{n} A_k \quad \forall n \in \mathbb{N}^*$$

Specially,

$$\biguplus_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k$$

Remark This technique will be frequently used in the proofs of upcoming propositions and theorem.

Proof ■

Proposition 1.2.3

Let $(\Omega, \mathcal{F}, \mu)$ *be a measure space, and* $\{A_k\}$ *a sequence of sets in* $\mathcal{F}$*. We have*

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k) \tag{1.3}$$

Remark If a set function satisfies (1.3), we say that it is σ-**subadditive** .

Proof Let B_k be given by

$$B_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i$$

as in Proposition 1.2.2. Then we have

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\biguplus_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) \le \sum_{k=1}^{\infty} \mu(A_k)$$

The last inequality follows from Proposition 1.2.1. ■

Proposition 1.2.4

1.3 Extension of Set Functions on Semi-Algebras

Theorem 1.3.1

Let $\mathcal{S}$ be a semi-algebra on Ω, and $\mu : \mathcal{S} \to [0, \infty]$ a nonnegative additive (resp. σ-additive) set function. Then μ can be extended uniquely to an additive (resp. σ-additive) function ν on $\mathcal{A}(\mathcal{S})$. That is, $\exists! \nu : \mathcal{A}(\mathcal{S}) \to [0, \infty]$ such that

1. *ν is additive (resp. σ-additive), and*
2. *$\nu|_{\mathcal{S}} = \mu$.*

To be specific, this extension ν is given by

$$\nu(A) = \sum_{i=1}^{n} \mu(E_i)$$

where $\{E_1, \ldots, E_n\}$ is a family of mutually disjoint sets in $\mathcal{S}$ satisfying $A = \biguplus_{i=1}^{n} E_i$. ♡

1.4 Carathéodory's Extension Theorem

Definition 1.4.1

*An **outer measure** on Ω is a set function $\mu^* : \mathcal{P}(\Omega) \to [0, \infty]$ such that it*

1. *assumes zero at empty set, i.e., $\mu^*(\emptyset) = 0$, and*
2. *is σ-subadditive, i.e., $E \subset \bigcup_{i=1}^{\infty} E_i,\ \mu^*(E) \le \sum_{i=1}^{\infty} \mu^*(E_i)$* ♣

Definition 1.4.2

*Suppose that μ^* is an outer measure on Ω. The collection of **measurable sets** with respect to μ^* is defined by*

$$\mathcal{M} = \left\{ A \subset \Omega \,\middle|\, \forall E \subset \Omega,\ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^{\complement}) \right\}$$

Sometimes, we also say the sets in $\mathcal{M}$ are μ^*-measurable. ♣

Theorem 1.4.1

Let μ^* be an outer measure on Ω and $\mathcal{M}$ the collection of μ^*-measurable sets. We claim that

1. *$\mathcal{M}$ is a σ-algebra, and*
2. *$\mu^*|_{\mathcal{M}}$ is σ-additive.*

♡

Consider the equality

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^\complement)$$

in the definition of $\mathcal{M}$. Note that it always holds that

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^\complement)$$

because $E \subset (E \cap A) \cup (E \cap A^\complement)$ and μ^* is an outer measure and hence σ-subadditive. Therefore, in the following proofs, we only need to show

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^\complement)$$

in order to prove the equality.

Before proving that $\mathcal{M}$ is a σ-algebra, we first show that it is an algebra.

Proof We shall check each condition in the definition of an algebra.

(Containment of the Empty Set) Note that

$$\mu^*(E \cap \Omega) + \mu^*(E \cap \Omega^\complement) = \mu^*(E) + 0 = \mu^*(E)$$

Therefore, clearly $\Omega \in \mathcal{M}$.

(Closure under Complements) Suppose $A \in \mathcal{M}$, then $\forall E \subset \Omega$, we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^\complement) = \mu^*(E \cap (A^\complement)^\complement) + \mu^*(E \cap A^\complement)$$

The last equality above implies that $A^\complement \in \mathcal{M}$.

(Closure under Finite Intersections) Suppose $A_1, A_2 \in \mathcal{M}$. Let $E \subset \Omega$ be arbitrary, then

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A_1) + \mu^*(E \cap {A_1}^\complement) && \text{since } A_1 \in \mathcal{M} \\ &= \mu^*(E \cap A_1 \cap A_2) + \mu^*(E \cap A_1 \cap {A_2}^\complement) + \mu^*(E \cap {A_1}^\complement) && \text{since } A_2 \in \mathcal{M} \end{aligned} \tag{1.4}$$

On the other hand, consider $\mu^*(E \cap (A_1 \cap A_2)^\complement)$. We have

$$\begin{aligned} \mu^*(E \cap (A_1 \cap A_2)^\complement) &= \mu^*((E \cap {A_1}^\complement) \cup (E \cap {A_2}^\complement)) \\ &= \mu^*(E \cap A_1 \cap {A_2}^\complement) + \mu^*(E \cap {A_1}^\complement) && \text{since } A_1 \in \mathcal{M} \end{aligned} \tag{1.5}$$

Combining equations (1.4) and (1.5), we obtain

$$\mu^*(E) = \mu^*(E \cap A_1 \cap A_2) + \mu^*(E \cap (A_1 \cap A_2)^\complement)$$

Therefore, $A_1 \cap A_2 \in \mathcal{M}$.

■

In the following, we show that $\mu^*|_{\mathcal{M}}$ is σ-additive.

Proof We prove this by first showing μ^* is additive on $\mathcal{M}$ and then applying Proposition 1.2.4 to conclude that μ^* is actually σ-additive on $\mathcal{M}$.

(Additivity) First, clearly $\mu^*(\emptyset) = 0$ since μ^* is an outer measure. Suppose $A_1, A_2 \in \mathcal{M}$ and

$A = A_1 \uplus A_2$. Note that $A \in \mathcal{M}$ since we have shown that $\mathcal{M}$ is an algebra. It follows that

$$\mu^*(A) = \mu^*(A \cap A_1) + \mu^*(A \cap A_1{}^\complement)$$

Note that $A \cap A_1 = A_1$ and $A \cap A_1{}^\complement = A_2$. Therefore,

$$\mu^*(A) = \mu^*(A_1) + \mu^*(A_2)$$

Then by induction, we can show that for $A_i \in \mathcal{M}$,

$$\mu^*\left(\biguplus_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu^*(A_i)$$

(σ-Additivity) Note that μ^* is σ-subadditive on $\mathcal{P}(\Omega)$ (and of course it is also σ-subadditive on $\mathcal{M}$) because μ^* is an outer measure. Moreover, we have already shown that μ^* is also additive on $\mathcal{M}$. Then Proposition 1.2.4 immediately implies that μ^* is σ-additive. ■

Finally, we show that $\mathcal{M}$ is actually a σ-algebra.

Proof Recall that we have already shown $\mathcal{M}$ is an algebra. Hence, we only need to show that $\mathcal{M}$ is closed under countable unions. Suppose that $A_i \in \mathcal{M}$ and $A = \bigcup_{i=1}^{\infty} A_i$. Fix a set $E \subset \Omega$. Note that $\bigcap_{i=1}^{n} A_i \in \mathcal{M}$ since $\mathcal{M}$ is an algebra. It then follows that

$$\begin{aligned}
\mu^*(E) &= \mu^*\left(E \cap \bigcap_{i=1}^{n} A_i\right) + \mu^*\left(E \cap \left(\bigcap_{i=1}^{n} A_i\right)^\complement\right) \\
&= \mu^*\left(E \cap \bigcap_{i=1}^{n} A_i\right) + \mu^*\left(E \cap \bigcup_{i=1}^{n} A_i{}^\complement\right)
\end{aligned}$$

And since $E \cap \bigcap_{i=1}^{n} A_i \supset E \cap \bigcap_{i=1}^{\infty} A_i = E \cap A$, we have

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*\left(E \cap \bigcup_{i=1}^{n} A_i{}^\complement\right) \tag{1.6}$$

For the convenience of the notations, we denote

$$F_i = E \cap \bigcup_{j=1}^{i} A_j{}^\complement$$

Define sets G_i as follows.

$$G_1 = F_1 \qquad\qquad G_i = F_i \setminus \bigcup_{j=1}^{i-1} F_j \quad i \geq 2$$

One can show that all G_i's are mutually disjoint and $\bigcup_{i=1}^{n} G_i = \bigcup_{i=1}^{n} F_i$. It is also true that $\bigcup_{i=1}^{\infty} G_i = \bigcup_{i=1}^{\infty} F_i$. In summary,

$$\biguplus_{i=1}^{n} G_i = \bigcup_{i=1}^{n} F_i \qquad\qquad \biguplus_{i=1}^{\infty} G_i = \bigcup_{i=1}^{\infty} F_i = E \cap A^\complement$$

Applying the additivity of μ^* (this is valid because $G_i \in \mathcal{M}$) to inequality (1.6), we obtain

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*\left(\biguplus_{i=1}^{n} G_i\right) = \mu^*(E \cap A) + \sum_{i=1}^{n} \mu^*(G_i)$$

Letting $n \to \infty$,

$$\mu^*(E) \geq \mu^*(E \cap A) + \sum_{i=1}^{\infty} \mu^*(G_i)$$

Then we apply the σ-subadditivity of μ^*,

$$\mu^*(E) \geq \mu^*(E \cap A) + \sum_{i=1}^{\infty} \mu^*(G_i) \geq \mu^*(E \cap A) + \mu^* \left(\biguplus_{i=1}^{\infty} G_i \right) = \mu^*(E \cap A) + \mu^*(E \cap A^{\complement})$$

Therefore, we have shown that

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^{\complement})$$

In fact, the inequality above can be replaced by equality as we have explained before. Therefore, $A \in \mathcal{M}$ and hence $\mathcal{M}$ is indeed a σ-algebra. ■

The following theorem extends a pre-measure, i.e., a σ-additive nonnegative set function on an algebra $\mathcal{A}$ to a measure on the σ-algebra generated by $\mathcal{A}$.

Theorem 1.4.2 (Carathéodory's Extension Theorem)

Let $\mathcal{A}$ be an algebra on Ω, and μ_0 a pre-measure on $\mathcal{A}$. Then μ_0 can be extended to a measure μ on $\mathcal{F} = \sigma(\mathcal{A})$, i.e., there exists a measure $\mu : \mathcal{F} \to [0, \infty]$ such that $\mu|_{\mathcal{A}} = \mu_0$. Furthermore, if μ_0 is σ-fintie, then the extension is unique. ♡

Proof ■

Theorem 1.4.3 (Extension of Set Funcitons on Semi-Algebras)

Let $\mathcal{S}$ be a semi-algebra on Ω, and $\nu : \mathcal{S} \to [0, \infty]$ a σ-additive set function. Then, ν can be first uniquely extended to a pre-measure μ_0 on $\mathcal{A}(\mathcal{S})$. After that, if μ_0 is σ-fintie, it can be extended uniquely to a measure μ on $\sigma(\mathcal{S})$. (Note that $\sigma(\mathcal{A}(\mathcal{S})) = \sigma(\mathcal{S})$). ♡

1.5 Lebesgue Measure

In this section, we shall construct the Lebesgue measure on $\mathbb{R}$ mainly with the Carathéodory's Extension Theorem.

We start by defining an additive nonnegative set function ℓ on the semi-algebra $\mathcal{S}$ in Example 1.1. Recall $\mathcal{S}$ consists of the following five kinds of subsets in $\mathbb{R}$:

1. $\emptyset$
2. $(a, b]$
3. (a, ∞)
4. $(-\infty, b]$
5. $\mathbb{R}$

Define

1. $\ell(\emptyset) := 0$
2. $\ell(a, b] := b - a$
3. $\ell(a, \infty) := \infty$

4. $\ell(-\infty, b] := \infty$
5. $\ell(\mathbb{R}) := \infty$

As we can see, the function ℓ is simply the length of the intervals, and it is clearly *finitely* additive. We wish to extend this measurement of length to a larger collection of subsets of $\mathbb{R}$, which gives rise to the Lebesgue measure. As a custom, we use λ to denote the Lebesgue measure.

It is temping to apply Theorem 1.4.3 to extend ℓ. But then, as required in this theorem, we need to show that ℓ is σ-additive on $\mathcal{S}$, which is somehow difficult to prove *directly* even though it may seem to hold naturally.

Our strategy is to first apply Theorem 1.3.1 to extend ℓ to a *finitely* additive set function λ_0 on the algebra $\mathcal{A}(\mathcal{S})$ generated by $\mathcal{S}$. And then we prove λ_0 is σ-additive on $\mathcal{S}$. In other words, we prove that ℓ is σ-additive by proving the restricted function $\lambda_0|_{\mathcal{S}} = \ell$, which itself is defined on an algebra, is σ-additive. The reason why it is easier to prove λ is σ-additive on $\mathcal{S}$ is simply that $\mathcal{A}(\mathcal{S})$ is a larger collection of sets than $\mathcal{S}$.

After that, we are allowed to extend ℓ to the Lebesgue measure λ on $\sigma(\mathcal{S})$ as soon as we prove that λ_0 is σ-finite, which is easy to prove, to guarantee the uniqueness of the extension.

Therefore, our major goal is to prove

$$\lambda_0(S) = \sum_{k=1}^{\infty} \lambda(S_k) \tag{1.7}$$

where $S = \biguplus_{k=1}^{\infty} S_k$.

To prove the equation (1.7), we need to show that the left-hand side is less than or equal to the right-hand side as well as that the right-hand side is less than or equal to the left-hand side. We note that one of these two inequalities is easy to show, which is stated in the following lemma.

Lemma 1.5.1

If

$$A = \biguplus_{k=1}^{\infty} A_k$$

where $A, A_k \in \mathcal{A}(\mathcal{S})$, then we have the following inequality:

$$\lambda_0(A) \geq \sum_{k=1}^{\infty} \lambda_0(A_k) \tag{1.8}$$

♡

Proof By the finite additivity and monotonicity of λ_0, for each $n \in \mathbb{N}^*$, we have

$$\lambda_0(A) = \lambda_0\left(\biguplus_{k=1}^{\infty} A_k\right) \geq \lambda_0\left(\biguplus_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \lambda_0(A_k)$$

Hence, (1.8) follows by letting $n \to \infty$. ■

As we can see, there are several forms of set S, which makes the proof of (1.7) rather complicated. We shall first consider the finite intervals.

Lemma 1.5.2

Let $S = (a, b]$. *If*

$$S = \biguplus_{k=1}^{\infty} S_k$$

where $S_k \in \mathcal{S}$, *then*

$$\lambda_0(S) = \sum_{k=1}^{\infty} \lambda_0(S_k) \tag{1.9}$$

♡

Proof Without loss of generality, we may assume that $S_k \neq \emptyset\ \forall k \in \mathbb{N}^*$. We observe that each S_k must have the form $(a_k, b_k]$ since $S = (a, b]$. Otherwise, S cannot be a finite interval.

By Lemma 1.5.1, we have

$$\lambda_0(S) \geq \sum_{k=1}^{\infty} \lambda_0(S_k) \tag{1.10}$$

On the other hand, for an arbitrary $\varepsilon > 0$, we have

$$[a + \varepsilon, b] \subset (a, b] = S = \biguplus_{k=1}^{\infty} S_k = \biguplus_{k=1}^{\infty} (a_k, b_k] \subset \bigcup_{k=1}^{\infty} (a_k, b_k + \varepsilon/2^k)$$

Hence,

$$[a + \varepsilon, b] \subset \bigcup_{k=1}^{\infty} (a_k, b_k + \varepsilon/2^k)$$

Note that $[a + \varepsilon, b]$ is a compact set in $\mathbb{R}$, and $\left\{(a_k, b_k + \varepsilon/2^k)\right\}$ forms an open cover. Therefore, there exists $n \in \mathbb{N}^*$ such that

$$[a + \varepsilon, b] \subset \bigcup_{k=1}^{n} (a_k, b_k + \varepsilon/2^k)$$

Then by the monotonicity and finite subadditivity of λ_0, we have

$$\begin{aligned}
b - a - \varepsilon = \lambda_0(a + \varepsilon, b] &\leq \lambda_0[a + \varepsilon/2, b] \\
&\leq \lambda_0\left(\bigcup_{k=1}^{n} (a_k, b_k + \varepsilon/2^k)\right) \\
&\leq \sum_{k=1}^{n} \lambda_0(a_k, b_k + \varepsilon/2^k) \\
&\leq \sum_{k=1}^{n} \lambda_0(a_k, b_k + \varepsilon/2^k] \\
&= \sum_{k=1}^{n} (b_k - a_k) + \sum_{k=1}^{n} \varepsilon/2^k \\
&< \varepsilon + \sum_{k=1}^{\infty} (b_k - a_k)
\end{aligned}$$

In summary, we have obtained

$$\lambda_0(a,b] = b-a < 2\varepsilon + \sum_{k=1}^{\infty}(b_k - a_k) = 2\varepsilon + \sum_{k=1}^{\infty}\lambda_0(a_k, b_k] \quad \forall \varepsilon > 0$$

Therefore, we have

$$\lambda_0(S) \leq \sum_{k=1}^{\infty}\lambda_0(S_k) \tag{1.11}$$

by letting $\varepsilon \to 0$.

Finally, (1.10) follows from (1.8) and (1.11). ■

Before extending the Lemma 1.5.2 to general S, we need the following limit formula.

Lemma 1.5.3

Let $E_n = (-n, n]$ where $n \in \mathbb{N}^*$. Then

$$\lim_{n\to\infty}\lambda_0(S \cap E_n) = \lambda_0(S) \tag{1.12}$$

where $S \in \mathcal{S}$. ♡

Proof First, we observe that indeed $S \cap E_n \in \mathcal{S} \subset \mathcal{A}(\mathcal{S})\ \forall n \in \mathbb{N}^*$. We then prove (1.12) by considering each form of S.

($S = \emptyset$) (1.12) holds because

$$\lambda_0(S \cap E_n) = \lambda_0(\emptyset) = 0$$

($S = (a, b]$) There exists a large enough $N \in \mathbb{N}^*$ such that $N > \max\{-a, b\}$. It follows that

$$S \cap E_n = (a,b] \cap (-n,n] = (a,b] = S \quad \forall n \geq N$$

Thus,

$$\lambda_0(S \cap E_n) = \lambda_0(S) \quad \forall n \geq N$$

which implies (1.12).

($S = (a, \infty)$) There exists $N \in \mathbb{N}^*$ such that $N > -a$. Then, we have

$$S \cap E_n = (a, \infty) \cap (-n, n] = (a, n] \quad \forall n \geq N$$

It then follows that

$$\lambda_0(S \cap E_n) = \lambda_0(a, n] = n - a \quad \forall n \geq N$$

By letting $n \to \infty$, we obtain

$$\lim_{n\to\infty}\lambda_0(S \cap E_n) = \lim_{n\to\infty}(n-a) = \infty = \lambda_0(a, \infty) = \lambda_0(S)$$

which is exactly (1.12).

($S = (-\infty, b]$) Similar to the proof of the preceding case, we have

$$S \cap E_n = (-n, b] \quad \forall n \geq N$$

where N is a constant integer larger than b. It follows that

$$\lim_{n\to\infty}\lambda_0(S \cap E_n) = \lim_{n\to\infty}(n+b) = \infty = \lambda_0(-\infty, b] = \lambda_0(S)$$

($S = \mathbb{R}$) In this case, we have

$$S \cap E_n = \mathbb{R} \cap (-n, n] = (-n, n] \quad \forall n \in \mathbb{N}^*$$

It follows that

$$\lim_{n\to\infty} \lambda_0(S \cap E_n) = \lim_{n\to\infty} 2n = \infty = \lambda_0(\mathbb{R}) = \lambda_0(S)$$

■

We are now ready to prove (1.7) for all possible forms of S.

Lemma 1.5.4

Let $S \in \mathcal{S}$. If

$$S = \biguplus_{k=1}^{\infty} S_k$$

where $S_k \in \mathcal{S}$, then

$$\lambda_0(S) = \sum_{k=1}^{\infty} \lambda_0(S_k) \tag{1.13}$$

♡

Proof Firstly, by Lemma 1.5.1, we have

$$\lambda_0(S) \geq \sum_{k=1}^{\infty} \lambda_0(S_k) \tag{1.14}$$

What is left to prove is

$$\lambda_0(S) \leq \sum_{k=1}^{\infty} \lambda_0(S_k) \tag{1.15}$$

If $S = \emptyset$, then (1.15) holds trivially, since all S_k's must also be empty sets. In the rest of the proof, we assume that $S \neq \emptyset$. Let $E_n = (-n, n]$. It is clear that $S \cap E_n \in \mathcal{S}$ since $S, E_n \in \mathcal{S}$. Moreover, we observe that the set $S \cap E_n$ has the form $(a, b]$ for $n \geq N$ where $N \in \mathbb{N}^*$ is some constant large enough integer. It then follows from Lemma 1.5.2 that

$$\lambda_0(S \cap E_n) = \sum_{k=1}^{\infty} \lambda_0(S_k \cap E_n) \leq \sum_{k=1}^{\infty} \lambda_0 S_k \tag{1.16}$$

The last inequality follows from the monotonicity of λ_0. We then send $n \to \infty$ on both sides of (1.16). It follows from Lemma 1.5.3 that

$$\lambda_0(S) = \lim_{n\to\infty} \lambda_0(S \cap E_n) \leq \sum_{k=1}^{\infty} \lambda_0(S_k)$$

which is exactly (1.15). This completes the proof. ■

Part II

Probability Theory

Chapter 2 Random Variables, Expectations, and Independence

2.1 Random Variables

We define random variables formally.

Part III

Mathematical Statistics

Chapter 3 Fundamentals of Statistics

3.1 Populations, Samples and Models

We shall introduce basic concepts.

3.2 Statistical Decision Theory

We shall discuss decision theory.

Index