Lambda Calculus

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Contents

Functions

named function

- $f : X \to Y, x \mapsto f(x)$, ex. $s : \mathbb{N} \to \mathbb{N}, n \mapsto n \cup \{n\}$.
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Example in OCaml (Review)

```
# let rec len l = match l with
     [] -> 0
   | a::l1 ->  1 + (len l1);;
val len : 'a list -> int = <fun>
# len [1; 2; 3];;
- : int = 3
# let rec sum l = match l with
     [] -> 0
   | a::l1 ->  a + (sum l1);;
val sum : int list -> int = <fun>
# sum [1; 2; 3];;
- : int = 6
# let rec rev l = match l with
     [] -> []
   | a::l1 -> (rev l1) @ [a];;
val rev : 'a list -> 'a list = <fun>
# rev [1; 2; 3];;
- : int list = [3; 2; 1]
```

Evaluation Processus of len

```
let rec len l = match l with
| [] -> 0
| a::l1 -> 1 + len l1;;
```

```
  len [1; 2; 3]
= 1 + (len [2; 3])
= 1 + (1 + (len [3]))
= 1 + (1 + (1 + (len [])))
= 1 + (1 + (1 + (  0   )))
```

Abstraction `1 + (len l)` with function `f(a, len l)`, we have

```
  len [1; 2; 3]
= f(1, len [2; 3])
= f(1, f(2, len [3]))
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```

Evaluation Processus of rev

```
let rec rev l = match l with
| [] -> []
| a::l1 -> 1 + rev l1;;
```

```
  rev [1; 2; 3]
= (rev [2; 3]) @ [1]
= ((rev [3]) @ [2]) @ [1]
= (((rev []) @ [3])@ [2]) @ [1]
= (((  []  ) @ [3])@ [2]) @ [1]
```

Abstraction `(rev l) @ a` with function `f(a, rev l)`, we have

```
  rev [1; 2; 3]
= f(1, rev [2; 3])
= f(1, f(2, rev [3]))
= f(1, f(2, f(3, rev [])))
= f(1, f(2, f(3,   []   )))
```

Evaluation Processus of rev

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let rec rev l = match l with
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| a::l1 -> 1 + rev l1;;
```

```
  rev [1; 2; 3]
= (rev [2; 3]) @ [1]
= ((rev [3]) @ [2]) @ [1]
= (((rev []) @ [3])@ [2]) @ [1]
= (((  []  ) @ [3])@ [2]) @ [1]
```

Abstraction `(rev l) @ a` with function `f(a, rev l)`, we have

```
  rev [1; 2; 3]
= f(1, rev [2; 3])
= f(1, f(2, rev [3]))
= f(1, f(2, f(3, rev [])))
= f(1, f(2, f(3,   []   )))
```

Evaluation Processus of rev

```
let rec rev l = match l with
| [] -> []
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  rev [1; 2; 3]
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Evaluation Processus of rev

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let rec rev l = match l with
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  rev [1; 2; 3]
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```

Function as data

- The above 3 functions have the same behaviors: applying consecutively the every list element from right to left to a function f:

$$f(a_1, f(a_2, f(a_3, f(\cdots f(a_n, b)\cdots)))).$$

  where $(a_1, a_2, \ldots a_n)$ is the list, and $f : X \times Y \to Y$ is the abstract function which operates on a list element & the result of the application f to the rest of the list. The intial element b will correspond the result of the empty list.
- for `len`, f can be taked $(x, y) \mapsto 1 + y$, $b = 0$.
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Evaluation Processus of sum

define new function `fold_right`, take `sum` as an argument

```
  sum [1; 2; 3]
= 1 + (sum [2; 3])
= 1 + (2 + (sum [3]))
= 1 + (2 + (3 + (sum [])))
= 1 + (2 + (3 + (  0   )))

  fold_right f [1; 2; 3] b
= f(1, fold_right f [2; 3] b)
= f(1, f(2, fold_right f [3] b))
= f(1, f(2, f(3, fold_right f [] b)))
= f(1, f(2, f(3,           b        )))
```

b is the initial element.

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b is the initial element.

Example in OCaml

```
# let rec fold_right f l b = match l with
    [] -> b
  | a::l1 -> f a (fold_right f l1 b);;
val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b -> 'b = <fun>
# let len l = fold_right (fun x y -> 1 + y)  l 0;;
val len : 'a list -> int = <fun>
# len [1; 2; 3];;
- : int = 3
# let sum l = fold_right (+) l 0;;
val sum : int list -> int = <fun>
# sum [1; 2; 3];;
- : int = 6
# let rev l = fold_right (fun a l1 -> l1 @ [a]) l [];;
val rev : 'a list -> 'a list = <fun>
# reve [1; 2; 3];;
- : int list = [3; 2; 1]
```

Change to tail recursion

- `fold_right` is not tail recursive, so the execution is not efficient.
- Because compiler can transform the tail recursion to while-loop, the more efficient way is define the function as tail recursion.
- The tips is change the recursion result to recursion argument, so called "accumulator":

```
let rec sum l = match l with
  [] -> 0
| a::l1 -> a + (sum l1);;
```

  could transform to:

```
let rec sum a l = match l with
  [] -> a
| b::l1 -> sum (a + b) l1;;
```

- The same way, define `fold_left` as

$$f(f(\cdots f(f(f(a, b_1), b_2), b_3), \ldots, b_{n-1}), b_n).$$

  where $(b_1, b_2, \ldots b_n)$ is the list, and $f : X \times Y \to X$ is the abstract function which operates on an intial element a and list element, produces the element of same type of the initial element.

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Evaluation Processus of sum

define new function `fold_left`, take `sum` as an argument `f`

```
  sum  0 [1; 2; 3]
= sum (0 + 1) [2; 3]
= sum (0 + 1 + 2) [3]
= sum (0 + 1 + 2 + 3) []
=      0 + 1 + 2 + 3
```

```
  fold_left f a [1; 2; 3]
= fold_left (f(a, 1)) [2; 3]
= fold_left (f(f(a, 1), 2)) [3]
= fold_left (f(f(f(a, 1), 2), 3)) []
=           (f(f(f(a, 1), 2), 3))
```

a is the initial element.

Evaluation Processus of sum

define new function `fold_left`, take `sum` as an argument `f`

```
  sum  0 [1; 2; 3]
= sum (0 + 1) [2; 3]
= sum (0 + 1 + 2) [3]
= sum (0 + 1 + 2 + 3) []
=      0 + 1 + 2 + 3
```

```
let rec sum a l = match l with
| [] -> a
| b::l1 -> sum (a + b) l1;;
```

```
let rec fold_left f a l = match l with
| [] -> a
| b::l1 -> fold_left (f a b) l1;;
```

```
  fold_left f a [1; 2; 3]
= fold_left (f(a, 1)) [2; 3]
= fold_left (f(f(a, 1), 2)) [3]
= fold_left (f(f(f(a, 1), 2), 3)) []
=           (f(f(f(a, 1), 2), 3))
```

a is the initial element.

Evaluation Processus of sum

define new function fold_left, take sum as an argument f

```
  sum  0 [1; 2; 3]
= sum (0 + 1) [2; 3]
= sum (0 + 1 + 2) [3]
= sum (0 + 1 + 2 + 3) []
=      0 + 1 + 2 + 3
```

```
let rec sum a l = match l with
| [] -> a
| b::l1 -> sum (a + b) l1;;
```

```
let rec fold_left f a l = match l with
| [] -> a
| b::l1 -> fold_left (f a b) l1;;
```

```
  fold_left f a [1; 2; 3]
= fold_left (f(a, 1)) [2; 3]
= fold_left (f(f(a, 1), 2)) [3]
= fold_left (f(f(f(a, 1), 2), 3)) []
=           (f(f(f(a, 1), 2), 3))
```

a is the initial element.

Evaluation Processus of sum

define new function `fold_left`, take `sum` as an argument `f`

```
  sum  0 [1; 2; 3]
= sum (0 + 1) [2; 3]
= sum (0 + 1 + 2) [3]
= sum (0 + 1 + 2 + 3) []
=      0 + 1 + 2 + 3
```

```
let rec sum a l = match l with
| [] -> a
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a is the initial element.

Evaluation Processus of sum

define new function fold_left, take sum as an argument f

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Example in OCaml

```
# let rec fold_left f a l = match l with
    [] -> a
  | b::l1 -> fold_left f (f a b) l1;;
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a = <fun>
# let len l = fold_left (fun x y -> x + 1) 0 l;;
val len : 'a list -> int = <fun>
# len [1;2;3];;
- : int = 3
# let sum l = fold_left (+) 0 l;;
val sum : int list -> int = <fun>
# sum [1;2;3];;
- : int = 6
# let rev l = fold_left (fun l1 a -> a::l1) [] l;;
val rev : 'a list -> 'a list = <fun>
# reve [1;2;3];;
- : int list = [3; 2; 1]
```

fold_left is an iterator

```
# let rec aux l a = match l with
| [] -> [a]
| b :: l1 -> if a <= b then a::l else b::(aux l1 a);;
val insert_sort : 'a list -> 'a list = <fun>
# let insert_sort l = fold_left aux [] l;;
val insert_sort : 'a list -> 'a list = <fun>
# insert_sort [3; 1; 6; 2; 4; 5];;
- : int list = [1; 2; 3; 4; 5; 6]
# insert_sort [3; 1; 6; 2; 4; 5; 1; 2]
- : int list = [1; 1; 2; 2; 3; 4; 5; 6]
```

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λ-calculus

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Terms

Definition

The terms of the λ-calculus, known as, λ-terms, are constructed recursively from a given set of variables $x, y, z, \ldots$. They are inductively defined as: following forms:

- all variables are terms (called atoms);
- if M and N are any terms, then (MN) is a term (called an application);
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the set of all terms is denoted by Λ.

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- Syntactic identity of terms will be denoted by '$\equiv$' which means two term are the same alphabetic string (after add the omitted parentheses). so $MP \equiv NQ$ iff $M \equiv P$ and $N \equiv Q$. $(\lambda x.(MN)) \equiv \lambda x.MN$.
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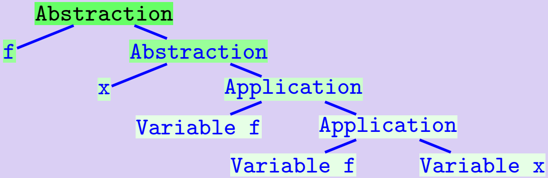
Data type of terms

```
type lambdaExpression =
    Variable of string
  | Abstraction of string * lambdaExpression
  | Apply of lambdaExpression * lambdaExpression;;

#lambda "@fx.f(fx)";;
- : lambdaExpression =
Abstraction ("f", Abstraction ("x", Apply
  (Variable "f", Apply (Variable "f", Variable "x"))))
```

Abstract Syntax Three



Length of the terms

```
let rec lgh = function
| (Variable var) ->  1
| (Abstraction (var, body)) -> 1 + lgh body
| (Apply (func, arg)) ->  lgh func + lgh arg;;

#lgh (lambda "(@x.(@f.(f (f (f (f (f (f (f x)))))))))");;
- : int = 10
```

the length is very useful for induction on the terms.

Free and bound variables

```
let bounds term = let rec bv = function
  | (Variable var) -> []
  | (Abstraction (var, body)) -> union [var] (bv body)
  | (Apply (func, arg)) -> union (bv func) (bv arg)
  in   bv (lambda term);;

let rec fv = function
  | (Variable var) -> [var]
  | (Abstraction (var, body)) -> exclude var (fv body)
  | (Apply (func, arg)) -> union (fv func) (fv arg)
and free term =  fv (lambda term) ;;

#bounds "(@y.yx(@x.y(@y.z)x))vw";;
- : string list = ["x"; "y"]
#free "(@y.yx(@x.y(@y.z)x))vw";;
- : string list = ["v"; "w"; "x"; "z"]
```

Remarks

- the notions of bound and free are the same of the first order formulas, or
- the integral $\int_y^z f(x)dx$ where x is bound, and y, z are free.
- x occurs both bound and free in $(\lambda y.yx(\lambda x.y(\lambda y.z)x))vw$, just like the global and argument with the same name in PL. It's better to avoid this name conflict in practice.
- A closed term is a term without any free variables. ex. $\lambda fx.f(f(fx))$. and we will concentrate only the close terms.

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Substitution

the substitution L for every every free occurrence y in the term M, denoted by $M[L/y]$ is inductively defined as

$$x[L/y] \equiv \begin{cases} L & \text{if } x \equiv y \\ x & \text{otherwise} \end{cases}$$

$$(\lambda x.M)[L/y] \equiv \begin{cases} \lambda x.M & \text{if } y \notin \mathrm{FV}(M) \\ \lambda x.(M[L/y]) & x \notin \mathrm{FV}(L) \wedge y \in \mathrm{FV}(M) \\ \lambda z.(M[z/x][L/y]) & x \in \mathrm{FV}(L) \wedge y \in \mathrm{FV}(M) \wedge \\ & z \text{ is new variable not in } \mathrm{FV}(LM) \end{cases}$$

$$(MN)[L/y] \equiv (M[L/y])(N[L/y])$$

Examples

the substitution is similar to the substution rule of first order logic, and the substituted term L in the result $M[L/y]$ must not introduce the new binding. so the free variables in L, should not be the bound variable in M.

- $(\lambda fx.f(fx))[\lambda fx.f(fx)/f] \equiv \lambda fx.f(fx)$ (no free occurrence of f in M).

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implementation of substitution

```
let var_counter = ref 0 ;;

let uniqueVar () = var_counter := !var_counter + 1 ;
  "v" ^ (string_of_int !var_counter);;

let rec substitution e x t = match e with
  | (Variable v) ->
      if v = x then t else e
  | (Abstraction (v, b)) ->
      if v = x then e     (* e has no free ocurrences of x *)
      else if not (belongs v (fv t)) then
        (* no free ocurrences of v in t, so no capture *)
        Abstraction (v, substitution b x t)
      else   (* there are free ocurrences of v in t and they
                are all captured -> use alpha equivalence *)
        let z = uniqueVar () in
        let newBody = substitution b v (Variable z) in
          Abstraction (z, substitution newBody x t)
  | (Apply (f,n)) ->
      Apply (substitution f x t, substitution n x t)
  and subst e x t =
    print (substitution (lambda e) x (lambda t));;

#subst "@fx.f(yx)" "y" "@f.f(fx)";;
(@f.(@v1.(f ((@f.(f (f x))) v1))))
```

α-conversions

Definition

Let a term P has an subterm $\lambda x.M$, and let $y \notin FV(M)$.
The act of replacing $\lambda x.M$ by $\lambda y.M[y/x]$ is called a change of bound variable or an α-conversion in P. If P can be changed to Q by a finite (perhaps empty) series of α-conversions, we shall say P α-converts to Q, and denoted by $P \equiv_\alpha Q$.

Example

$$\begin{aligned}\lambda xy.x(xy) &\equiv \lambda x.(\lambda y.x(xy))\\ &\equiv_\alpha \lambda x.(\lambda v.x(xv))\\ &\equiv_\alpha \lambda u.(\lambda v.u(uv))\\ &\equiv \lambda uv.u(uv).\end{aligned}$$

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Properties of α-conversions

Theorem

- The relation $\equiv_\alpha$ is reflexive, transitive and symmetric (equivalent). That is, for all P, Q, R, we have:
  - reflexivity: $P \equiv_\alpha P$,
  - transitivity: $P \equiv_\alpha Q \wedge Q \equiv_\alpha R \Rightarrow P \equiv_\alpha R$,
  - symmetry: $P \equiv_\alpha Q \Rightarrow Q \equiv_\alpha P$.
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- By the symmetry, α-conversion is reversible.
- the α-conversion is congruent relation under the substitution. the α-conversion guarantees that the substitution works correctly. e.g. $\lambda x.y[x/y] = \lambda z.y$, if z is new introduced variable; but $\lambda x.y[x/y] = \lambda w.y$, if w is the new one. and $\lambda z.y \equiv_\alpha \lambda w.y$. if no α-conversion, we get two different term.

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β-conversion

Definition

let P a term, any subterm of form

$$(\lambda x.M)N$$

is called a β-redex and the corresponding term

$$M[N/x]$$

is called its contractum. if P' is the result of replacing that occurrence by $M[N/x]$, we say we have contracted the redex-occurrence in P, and P β-converts (reduces) to P' and denoted by

$$P \rhd_{1\beta} P'.$$

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Examples

- $(\lambda x.x(xy))N \equiv \underline{(\lambda x.x(xy))N} \rhd_{1\beta} x(xy)[N/x] \equiv_\alpha N(Ny)$.
- $(\lambda x.x)N \equiv \underline{(\lambda x.x)N} \rhd_{1\beta} x[N/x] \equiv_\alpha N$ (identity $\mathbb{1}$).
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- $(\lambda x.(\lambda y.yx)z)v \equiv \underline{(\lambda x.(\lambda y.yx)z)v} \rhd_{1\beta} (\lambda y.yx)z[v/x] \equiv_\alpha (\lambda y.yv)z \equiv \underline{(\lambda y.yv)z} \rhd_{1\beta} yv[z/y] \equiv_\alpha zv$.
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Remark

- β-conversion is just like subroutine call in PL which replacing the formal parameter in function $\lambda x.M$ with the actual parameter N.
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- We simply denote $\equiv_\alpha$ by $\equiv$.

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β-normal form

Definition

A term Q which contains no β-redexes is called a β-normal form (or simply nf).
If $P \rhd_\beta Q$ and Q is nf, then Q is called nf of P.
If Q is nf, there is not Q' such that $Q \rhd_{1\beta} Q'$.

Examples

- $(\lambda x.(\lambda y.yx)z)v \rhd_\beta zv$ and zv has no redex, so is nf of $(\lambda x.(\lambda y.yx)z)v$.
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Strategy of reduction

- $\rhd_{1\beta}$ is multivalued relationship.
- if a term has more than one redexes, we must choose one to do the β-conversion.
- so the different strategy of reduction maybe results the different nf. this nondeterminism doesn't conformed with the notion of computability.
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Church Rosser theorem

If $P \rhd_\beta M$ and $P \rhd_\beta N$, then there exists a term T such that

$$M \rhd_\beta T \wedge N \rhd_\beta T.$$

The theorem garantees the uniqueness of nf, and the term can be reduced to two different terms then these two terms can be further reduced to one term, is called confluence.

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$$\begin{tikzcd}
 & P \arrow[dl] \arrow[dr] & \\
M \arrow[dr, dashed] & & N \arrow[dl, dashed] \\
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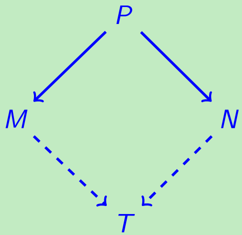

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Different cases of reductions

- disjoint: $\cdots(\lambda x.M)N\cdots(\lambda y.P)Q\cdots$

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 & a((\lambda x.x)b) \arrow[dr] & \\
((\lambda x.x)a)((\lambda x.x)b) \arrow[ur] \arrow[dr] & & ab \\
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reduction one of the redexes will not effect the another.

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where $\mathbf{I} = \lambda x.x$.

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- disjoint: $\cdots(\lambda x.M)N\cdots(\lambda y.P)Q\cdots$

$$\begin{tikzcd}[row sep=small] & a((\lambda x.x)b) \arrow[dr] & \\ ((\lambda x.x)a)((\lambda x.x)b) \arrow[ur] \arrow[dr] & & ab \\ & ((\lambda x.x)a)b \arrow[ur] & \end{tikzcd}$$

reduction one of the redexes will not effect the another.

- substitution: $\cdots(\lambda x.(\cdots(\lambda y.M)N\cdots))Q\cdots$

$$\begin{tikzcd}[row sep=small] & (\lambda y.yv)z \arrow[dr] & \\ (\lambda x.(\lambda y.yx)z)v \arrow[ur] \arrow[dr] & & zv \\ & (\lambda x.(zx)) \arrow[ur] & \end{tikzcd}$$

- duplication: $\cdots(\lambda x.M)(\cdots(\lambda y.N)P\cdots)\cdots$

$$\begin{tikzcd}[row sep=small] & (\lambda x.xx)a \arrow[drr] & & \\ (\lambda x.xx)(\mathbf{I}\,a) \arrow[ur] \arrow[dr] & & & aa \\ & (\mathbf{I}\,a)\mathbf{I}\,a \arrow[r] & a(\mathbf{I}\,a) \arrow[ur] & \end{tikzcd}$$

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Discussions

- we can transform the substitution case to duplication case by
  $$\cdots(\lambda z.(\lambda x.(\cdots z\cdots))((\lambda y.M)N))Q\cdots$$
  $$\rhd_{1\beta}\cdots(\lambda x.(\cdots(\lambda y.M)N\cdots))Q\cdots$$
- this case corresponds the local declarations in OCaml:
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  let x = e in f;;
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  the compiler will transform it to $(\lambda x.f)e$. e.g.
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  let s = fun x -> x + 1 in s 2;;
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- normally, reduce first the outside redex is more efficient than the inside. but it is not always true.
- the Church-Rosser theorem can be proved by using the strip lemma: if $M \rhd_{1\beta} P$ and $M \rhd_{\beta} Q$, then there is T such that $P \rhd_{\beta} T \wedge Q \rhd_{\beta} T$.

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β-equality

- β-reduction is not inversible, so $\rhd_\beta$ is not symmetric relation.
- the symmetric and transitive closure of $\rhd_\beta$ is equivalent relation, called β-equality, denoted by $=_\beta$.
- $P =_\beta Q$ iff Q can be obtained from P by a finite (perhaps empty) series of β-reduction, reversed β-reduction and α-conversion.

Example

$(\lambda xyz.xzy)(\lambda xy.x) =_\beta (\lambda xy.x)(\lambda x.x)$ En fact

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If $P =_\beta Q$, then there exists a term T such that $P \rhd_\beta T \wedge Q \rhd_\beta T$.

Illustration of proof by induction

if $P =_\beta Q$ by 0 step $\rhd_{1\beta}$ or the dual, it's $P \equiv Q$.

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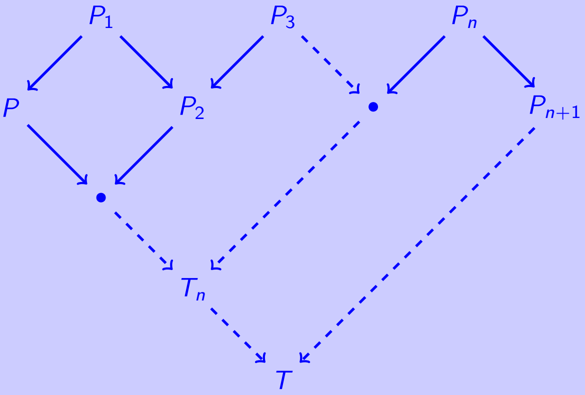

Undecidability of λ-calculus

- There is no algorithm that takes as input two λ terms and outputs TRUE or FALSE depending on whether or not the two term are β-equal.
- The problem can be reduce to determining whether a given term has a nf. it's the halting problem for λ-calculus. Church assumes that's decidable. then there is term e based on the Gödel numbering, and if e is applied to its own Gödel number, a contradiction results.
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Different reduction strategies

let $\Omega = (\lambda x.xx)(\lambda x.xx)$, then

$$(\lambda y.a)\Omega \longrightarrow a \qquad \text{call by name}$$

$$(\lambda y.a)\Omega \longrightarrow (\lambda y.a)\Omega \longrightarrow (\lambda y.a)\Omega \cdots\cdots\cdots \qquad \text{call by value}$$

- it can be obtained by call-by-name strategy of reduction: function argument (Ω) is not reduced but substituted 'as is' into the body of the abstraction (a). so the substitution erases the argument.
- if reduce the argument (Ω) first (call-by-value), then reductions are trapped into Ω without termination and never reach the nf.
- whether a term has nf or not, and how much work needs to be done in reaching it if there is, depends to a large extent on the reduction strategy used.
- the compiler of PL must choose the reduction strategies for it works as determinstic program.

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Applicative order

- The rightmost, innermost redex is always reduced first. Intuitively this means a function's arguments are always reduced before the function itself. Applicative order always attempts to apply functions to normal forms, even when this is not possible.
- most FP (including Lisp, ML) use this strategy, it also called "eager (strict) evaluation"
- because a redex is reduced only when its right hand side (function argument) has reduced to nf. It is also called call-by-value. most imperative languages like C and Java use this convention for function call. e.g.
  `(x -> x + x) (3 * 4) => (x -> x + x) 7 => 7 + 7`
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Examples

$(\lambda x.a((\lambda y.by)x))((\lambda z.(\lambda u.ud)z)c)$

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\triangleright_{1\beta} & (\lambda x.a((\lambda y.by)x))(cd) \\
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OCaml Example

OCaml use eager evaluation as default reduction strategy:

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cdab- : int = 11
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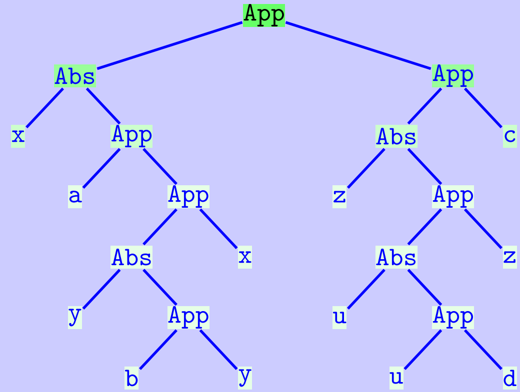

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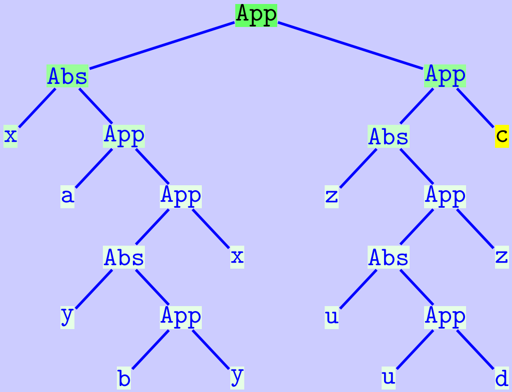

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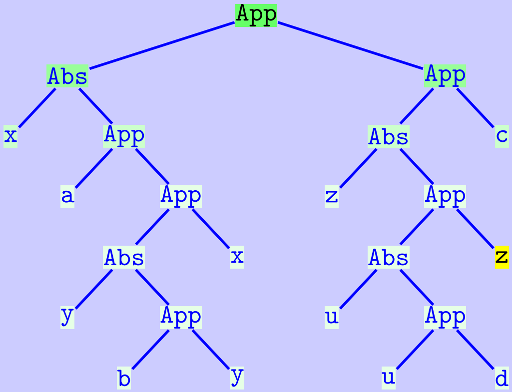

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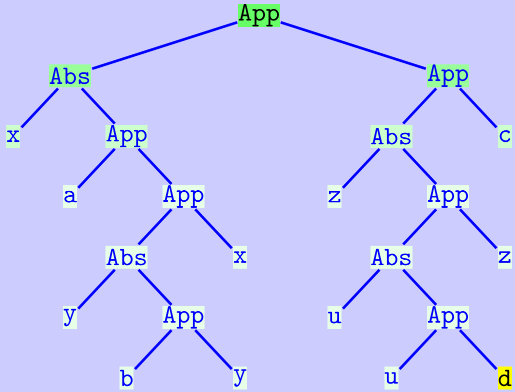

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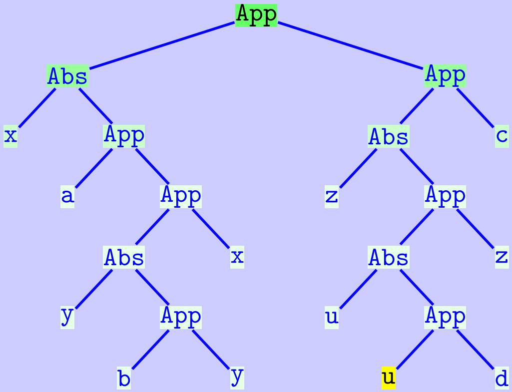

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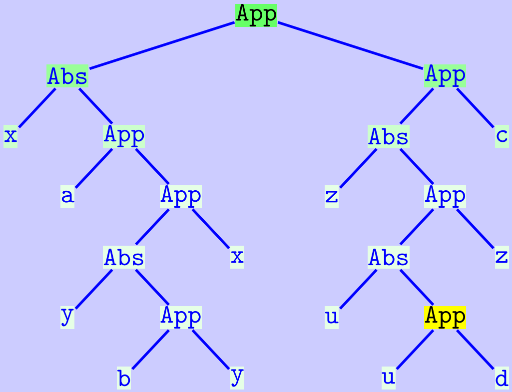

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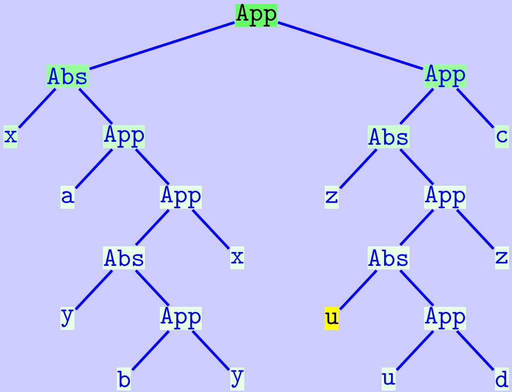

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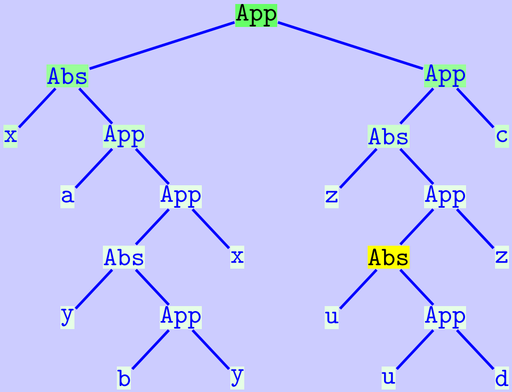

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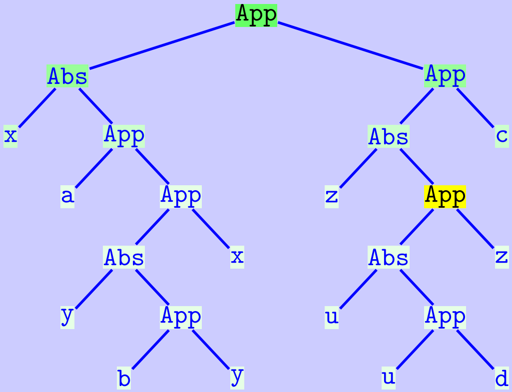

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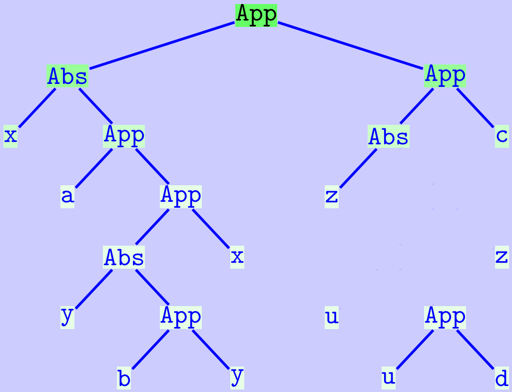

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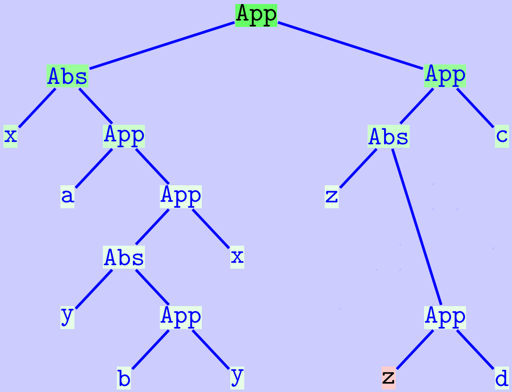

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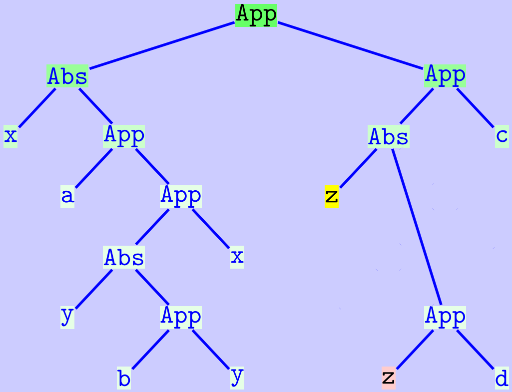

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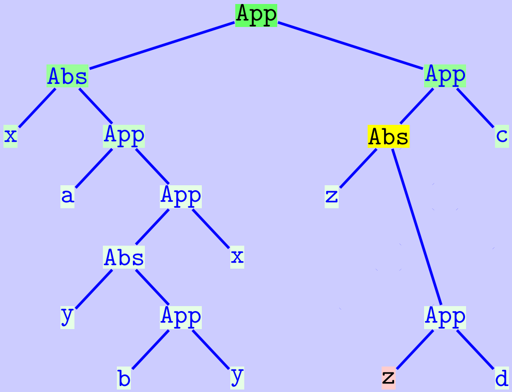

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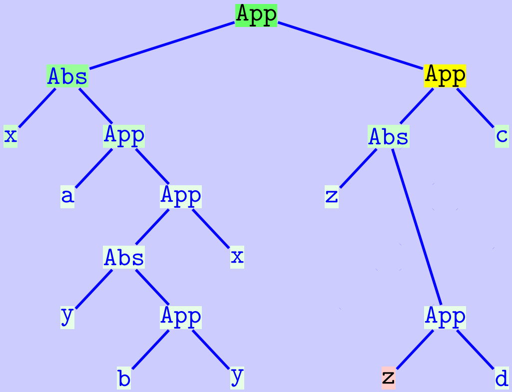

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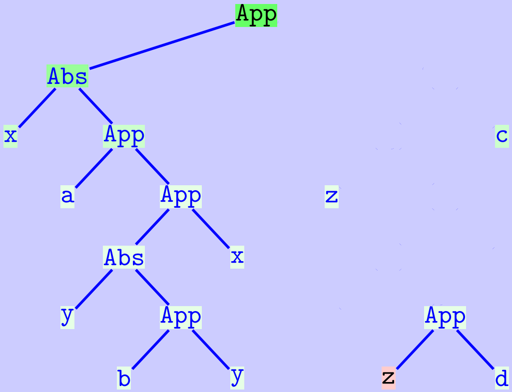

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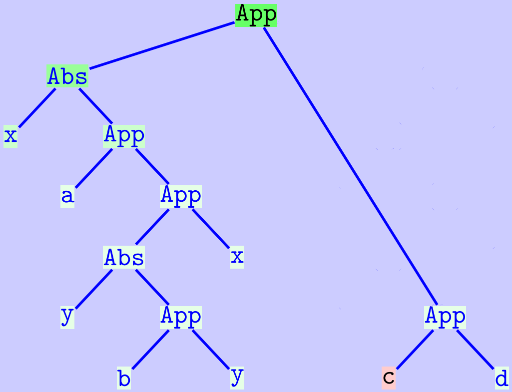

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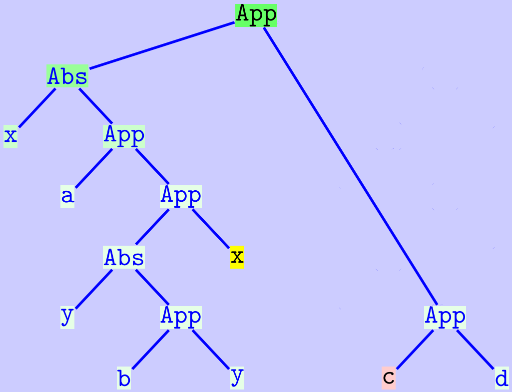

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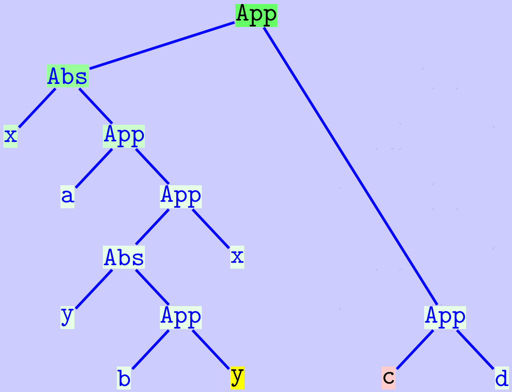

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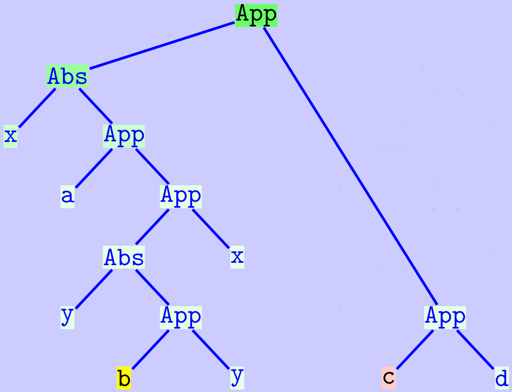

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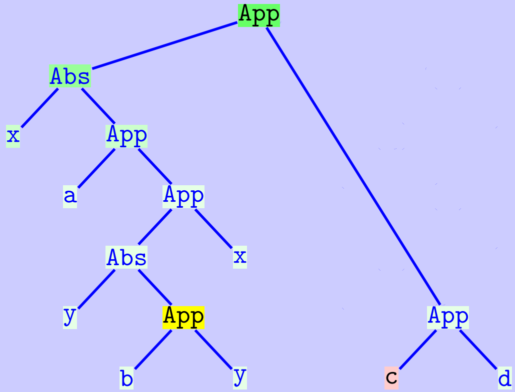

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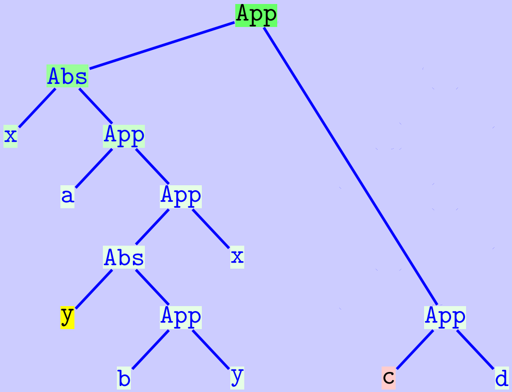

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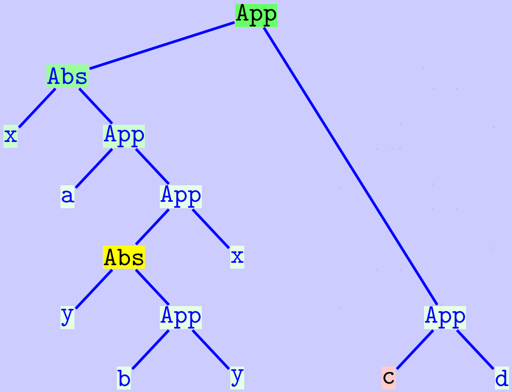

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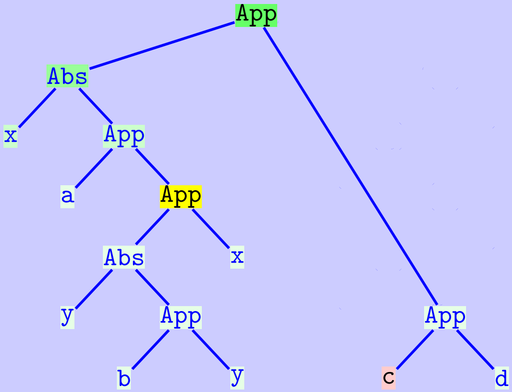

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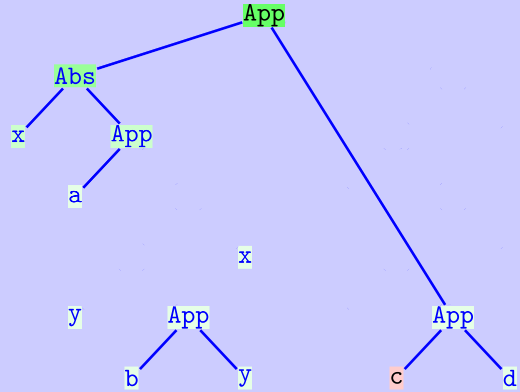

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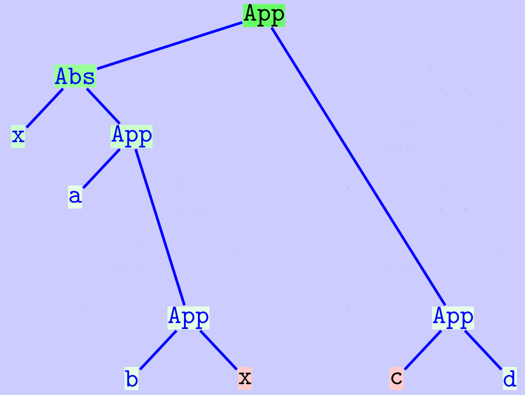

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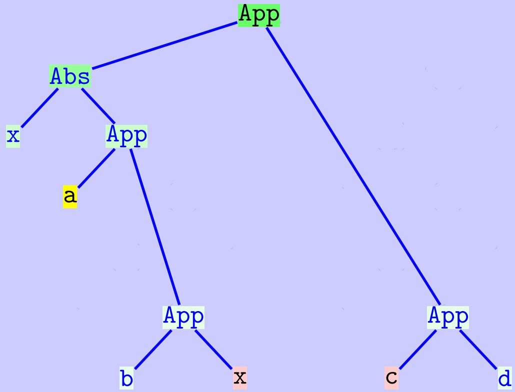

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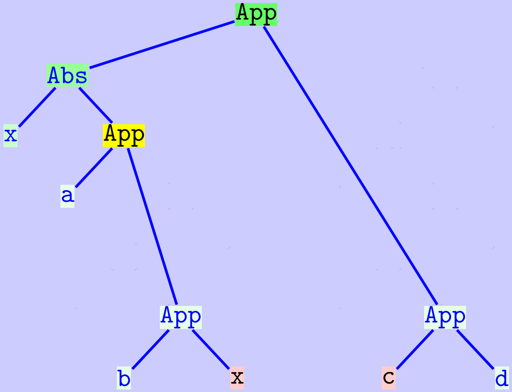

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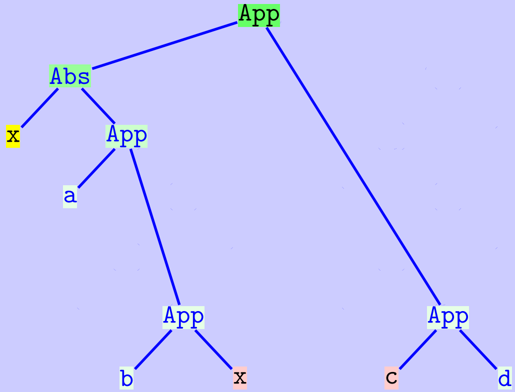

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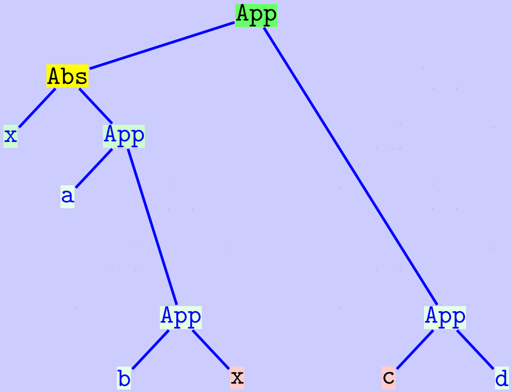

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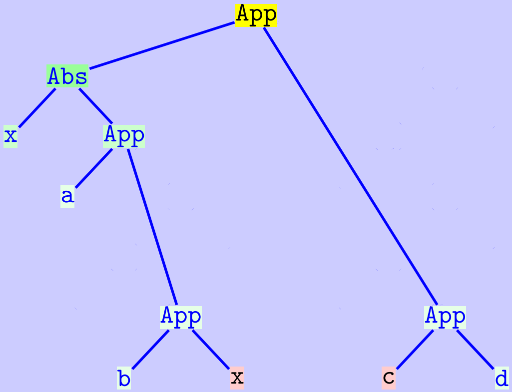

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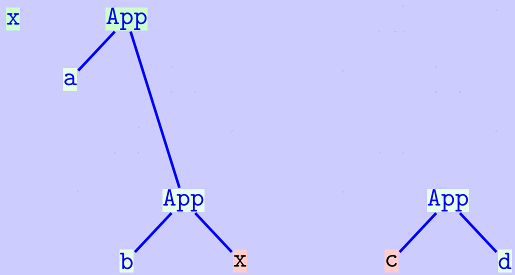

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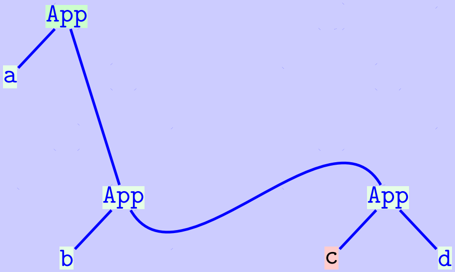

implementation of reduction of applicative order

```
let rec reductionStepInnerRightOrder = function
  | (Variable var) -> raise Lfail
  | (Abstraction (var, body)) ->
      Abstraction (var, reductionStepInnerRightOrder body)
  | (Apply (func, arg)) ->
       try Apply (func, reductionStepInnerRightOrder arg)
       with Lfail ->
         try Apply (reductionStepInnerRightOrder func, arg)
         with Lfail ->
           match func with
             | Abstraction (var, body) ->
                 (* beta reduction *)
                 substitution body var arg
             | _ -> raise Lfail
;;
```

Normal order

- the leftmost outermost redex is always reduced first, applying functions before evaluating function arguments.
- it correspond preorder traversal of abstract syntax tree.
- because the reduction of the right hand side of the redex (function argument) is delayed. It is also called call-by-name. ALGOL 60 uses this convention. e.g.
  `(x -> x + x) (3 * 4) => (3 * 4) + (3 * 4) => 7 + (3 * 4) => 7 + 7`
- it isn't efficient, but it's the normalising strategy (which always obtains the nf if there is).

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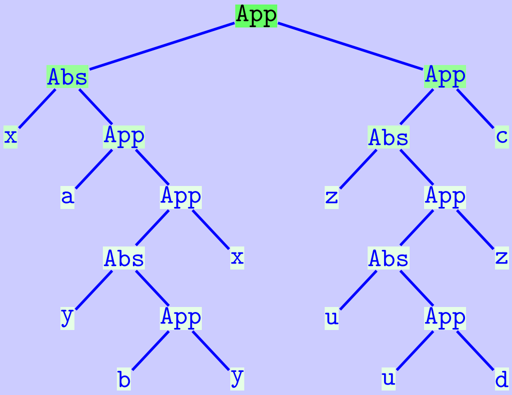
Examples

$(\lambda x.a((\lambda y.by)x))((\lambda z.(\lambda u.ud)z)c)$

$$
\begin{aligned}
&(\lambda x.a((\lambda y.by)x))((\lambda z.(\lambda u.ud)z)c)\\
\triangleright_{1\beta}\, & a((\lambda y.by)((\lambda z.(\lambda u.ud)z)c))\\
\triangleright_{1\beta}\, & a((b((\lambda z.(\lambda u.ud)z)c)))\\
\triangleright_{1\beta}\, & a(b((\lambda u.ud)c))\\
\triangleright_{1\beta}\, & a(b(cd))
\end{aligned}
$$

Example: normal order animation

$(\lambda x.a((\lambda y.by)x))((\lambda z.(\lambda u.ud)z)c)$



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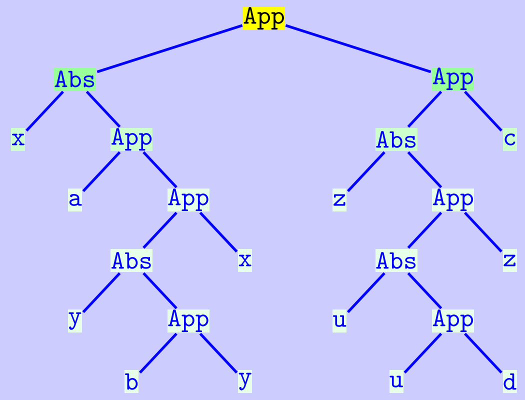

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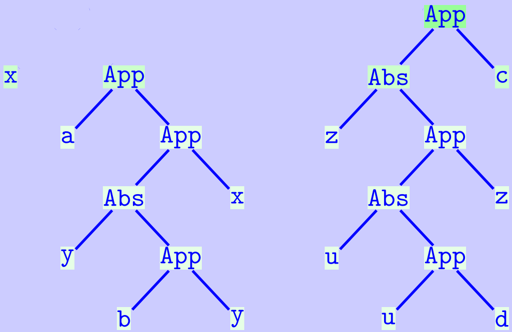

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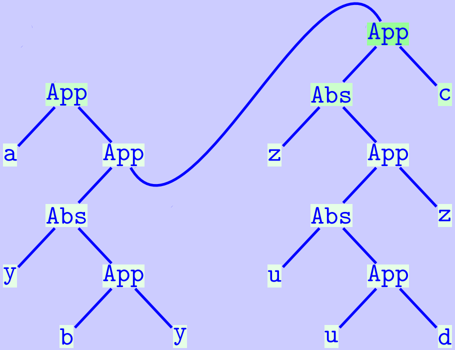

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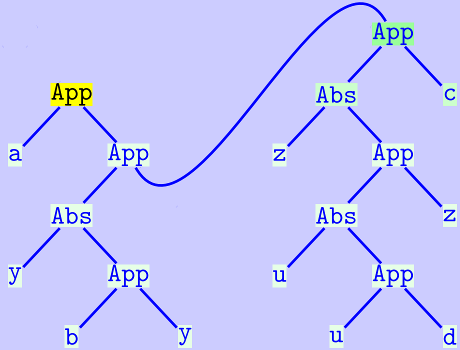

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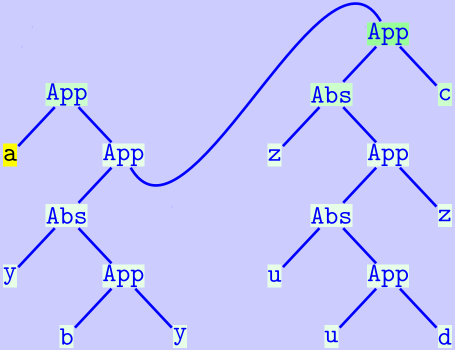

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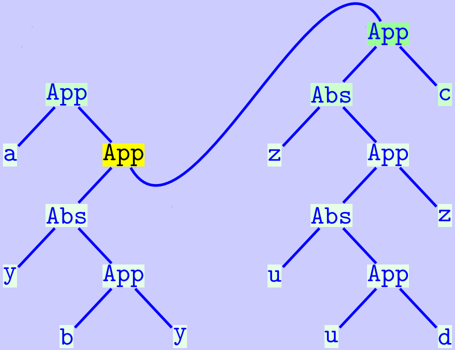

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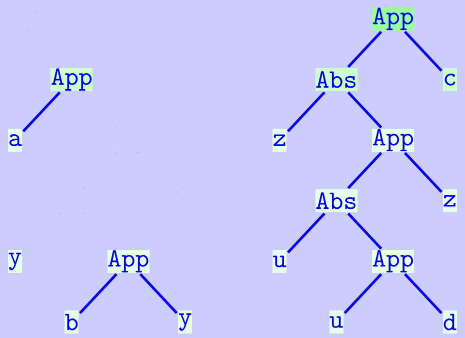

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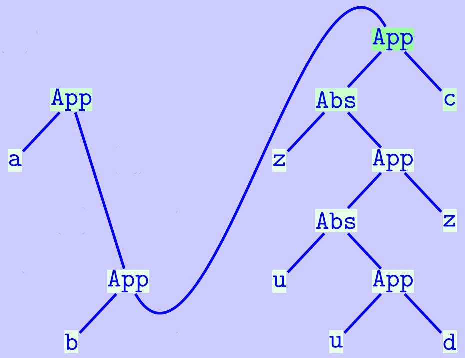

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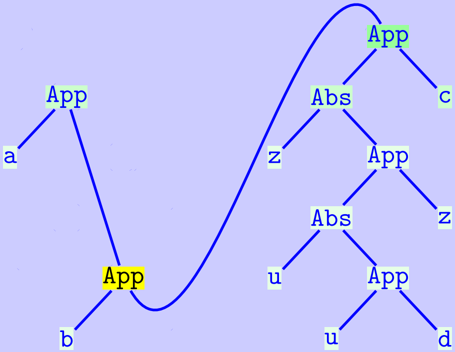

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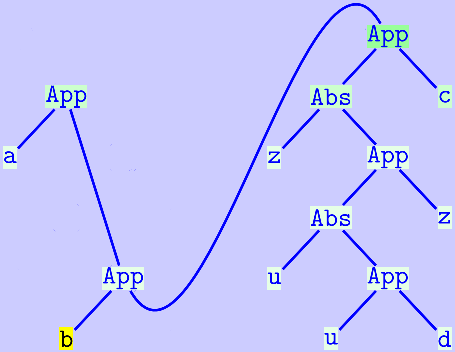

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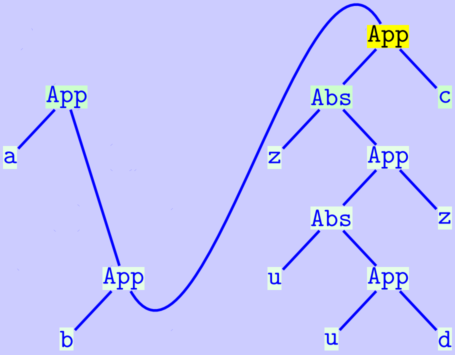

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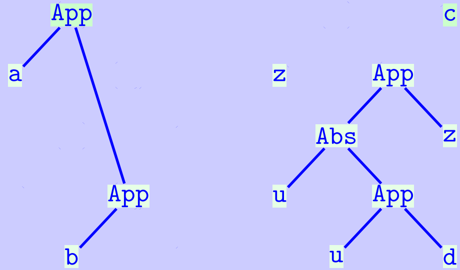

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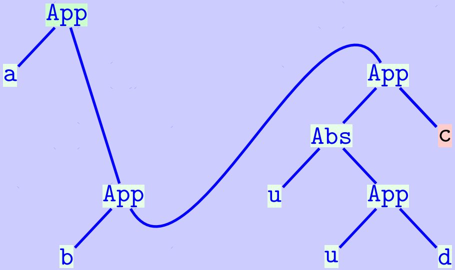

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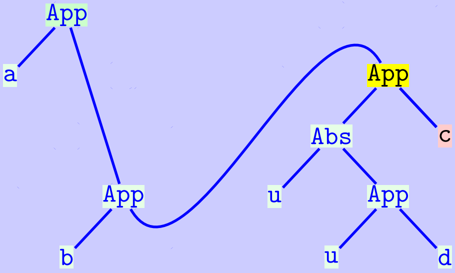

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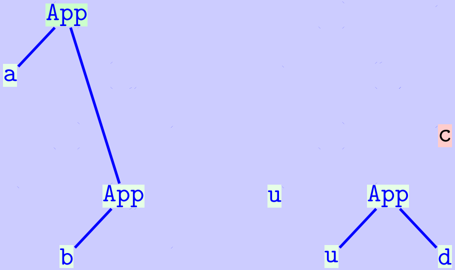

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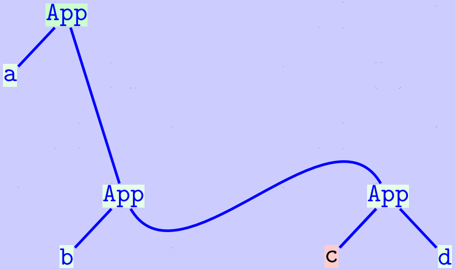

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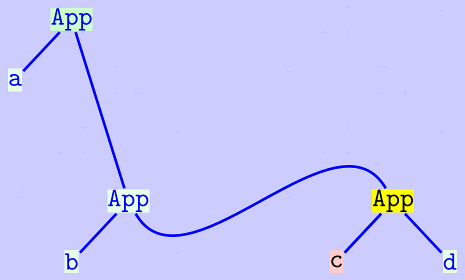

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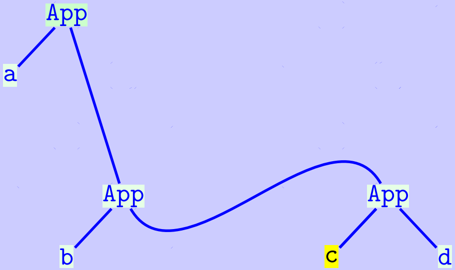

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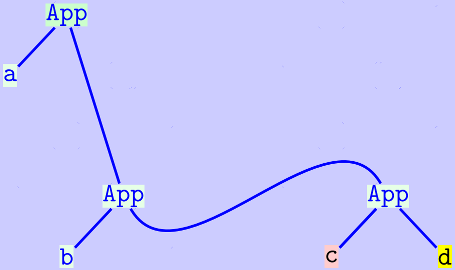

implementation of reduction of applicative order

```
let rec reductionStepOutLeftOrder = function
  | (Variable var) -> raise Lfail
  | (Abstraction (var, body)) ->
      Abstraction (var, reductionStepOutLeftOrder body)
  | (Apply (func, arg)) ->
      match func with
        | Abstraction (var, body) -> (* beta reduction *)
            substitution body var arg
        | _ ->
            try Apply (reductionStepOutLeftOrder func, arg)
            with Lfail ->
              Apply (func, reductionStepOutLeftOrder arg)
;;
```

Lazy evaluation

- if there are multiple occurrences of bound variable in function body, the normal reduction must evaluate the same function argument multiple times if the argument has redexes. So it's inefficient.
  $(\lambda x.xx)((\lambda x.x)y) \rhd_{1\beta} ((\lambda x.x)y)((\lambda x.x)y) \rhd_{1\beta} y((\lambda x.x)y) \rhd_{1\beta} yy$
- Lazy evaluation (or call-by-need) is an improved normal reduction. which never evaluates an argument more than once. it evaluate the argument until its value is actually required and the next occurrence of the argument will share the result of the first one. So it's optimal. e.g.
  $(\lambda x.xx)((\lambda x.x)y) \rhd_{1\beta} ((\lambda x.x)y)((\lambda x.x)y) \rhd_{\beta} yy$
- it also called non-strict evaluation.
- it can be implemented by representing the term by a graph rather than a tree.

Lazy evaluation

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Lazy evaluation (cont'd)

- most purely functional programming languages (Miranda, Haskell) use lazy evaluation as default reduction strategy.
- OCaml use `lazy` and `Lazy.force` to change the eager evaluation to the lazy. e.g.

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# let x = lazy (print_string "Hello"; 3*4);;
val x : int lazy_t = <lazy>
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Hello- : int = 12
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- .NET can simulate lazy evaluation using the type Lazy<T>.
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Encoding data in the λ-calculus

- In the imperative programming languages, data and controls are different objects. e.g. "Algorithms + Data Structures = Program" by N. Wirth.
- In the λ-calculus, data and controls are unified to the same objects — terms.
- The λ-calculus is expressive enough to encode boolean values, ordered pairs, natural numbers and lists as terms
- So the encoded data can carry their control with them. this mecanism let us realize high level abstract function. e.g. fold_left in OCaml.

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Booleans

- **if** can be seen as 3 argument function. **if true** M N will return M and **if false** M N return N. so **true** and **false** will be 2 argument functions.
- encoding **if**, **true** and **false** as:

$$\begin{aligned}\textbf{true} &\equiv \lambda xy.x\\ \textbf{false} &\equiv \lambda xy.y\\ \textbf{if} &\equiv \lambda pxy.pxy\end{aligned}$$

  so **if true** $M\,N =_\beta M$ and **if false** $M\,N =_\beta N$

- conjunction, disjunction and negation can be expressed as:

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Ordered Pairs

- the pair is the control which contains two element in order. **fst** and **snd** will return the first and second element
- encoding **pair**, **fst** and **snd** as:

$$\textbf{pair} \equiv \lambda xyf.fxy$$
$$\textbf{fst} \equiv \lambda p.p\,\textbf{true}$$
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  so for any terms M, N, **pair** $M\ N =_\beta \lambda f.f\ M\ N$, packaging M and N consecutively. f will be the place of control for output the first and second element.
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Natural numbers

- Church numerals are the representations of natural numbers under Church encoding. the "value" $\underline{n}$ is equivalent to the number of times the function encapsulates its argument:

$$f^n = f \circ f \circ \cdots \circ f$$

- so the Church numerals are defines as

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Arithmetic on Church numberals

- for function compositions, we have

$$f^m \circ f^n = f^{m+n}$$
$$(f^m)^n = f^{mn}$$

  and the monoid $\langle f \rangle$ is isomorphic to $\mathbb{N}$.

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Arithmetic on Church numberals (cont'd)

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$$\begin{aligned}\textbf{mult}\ \underline{m}\ \underline{n} &\rhd_\beta (\lambda mnfx.m(nf)x)\underline{m}\ \underline{n}\\ &\rhd_\beta \lambda fx.\underline{m}(\underline{n}f)x)\\ &\rhd_\beta \lambda fx.(\underline{n}f)^m x\\ &\rhd_\beta \lambda fx.(f^n)^m x)\\ &\rhd_\beta \lambda fx.f^{m\times n}x\end{aligned}$$

- and

$$\begin{aligned}\textbf{expt}\ \underline{m}\ \underline{n} &\rhd_\beta (\lambda mnfx.nmfx)\underline{m}\ \underline{n}\\ &\rhd_\beta \lambda fx.\underline{n}\ \underline{m}fx\\ &\rhd_\beta \lambda fx.\underline{m}^n fx\\ &\rhd_\beta \lambda fx.f^{m^n}x\end{aligned}$$

- because the Church numerals have an inbuilt source of repetition, we can encode arithmetic operation without the recursion.

Basic operations on Church numberals

- the successor and zero test can be encoded as

$$\begin{aligned}\textbf{succ} &\equiv \lambda nfx.f(nfx)\\ \textbf{iszero} &\equiv \lambda n.n(\lambda x.\textbf{false})\textbf{true}\end{aligned}$$

- so for any $\underline{n}$, we have:

$$\begin{aligned}\textbf{succ}\ \underline{n} &\rhd_\beta \underline{n+1}\\ \textbf{iszero}\ \underline{0} &\rhd_\beta (\lambda n.n(\lambda x.\textbf{false})\textbf{true})\underline{0}\\ &\rhd_\beta \underline{0}(\lambda x.\textbf{false})\textbf{true}\\ &\rhd_\beta (\lambda fx.x)(\lambda x.\textbf{false})\textbf{true}\\ &\rhd_\beta \textbf{true}\\ \textbf{iszero}\ \underline{n+1} &\rhd_\beta (\lambda n.n(\lambda x.\textbf{false})\textbf{true})\underline{n+1}\\ &\rhd_\beta \underline{n+1}(\lambda x.\textbf{false})\textbf{true}\\ &\rhd_\beta (\lambda x.\textbf{false})^{n+1}\ \textbf{true}\\ &\equiv\ (\lambda x.\textbf{false})^{n}((\lambda x.\textbf{false})\ \textbf{true})\\ &\rhd_\beta \textbf{false}\end{aligned}$$

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&\rhd_\beta (\lambda fx.x)(\lambda x.\textbf{false})\textbf{true}\\
&\rhd_\beta \textbf{true}\\
\textbf{iszero}\ \underline{n+1} &\rhd_\beta (\lambda n.n(\lambda x.\textbf{false})\textbf{true})\underline{n+1}\\
&\rhd_\beta \underline{n+1}(\lambda x.\textbf{false})\textbf{true}\\
&\rhd_\beta (\lambda x.\textbf{false})^{n+1}\ \textbf{true}\\
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Basic operations on Church numberals (cont'd)

- because the Church numeral is an iterator, we must use the $n+1$ iterator to generate the one of n. if choosing $\textbf{predfn}(f)\langle x, x\rangle = \langle f(x), x\rangle$ as first argument and $\langle x, x\rangle$ as second argument of $\underline{n+1}$. then

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\begin{aligned}
&\underline{n+1}(\textbf{predfn}\ f)\langle x, x\rangle \\
=&(\textbf{predfn}\ f)^n((\textbf{predfn}\ f)\langle x, x\rangle) \\
=&(\textbf{predfn}\ f)^n\langle f(x), x\rangle \\
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\begin{aligned}
\textbf{predfn} &\equiv \lambda fp.\textbf{pair}(f(\textbf{fst}\ p))(\textbf{fst}\ p) \\
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List

- in maths, a list $[x_1, x_2, \cdots, x_n]$ can be expressed as an n tuple $\langle x_1, x_2, \cdots, x_n \rangle \triangleq \langle x_1, \langle x_2, \langle \cdots, \langle x_n, [] \rangle \cdots \rangle \rangle \rangle$.
- so the list can be encoded as nested pairs :

$$\begin{aligned}
\textbf{cons} &\equiv \textbf{pair} \equiv \lambda xyf.fxy \\
\textbf{hd} &\equiv \textbf{fst} \equiv \lambda p.p\,\textbf{true} \\
\textbf{tl} &\equiv \textbf{snd} \equiv \lambda p.p\,\textbf{false} \\
\textbf{nil} &\equiv \lambda x.\textbf{true} \\
\textbf{null} &\equiv \lambda l.l\lambda xy.\textbf{false}
\end{aligned}$$

- then for any term M and N, we have

$$\begin{aligned}
\textbf{null}(\textbf{cons}\,M\,N) &\rhd_\beta (\textbf{cons}\,M\,N)\,\lambda xy.\textbf{false} \\
&\rhd_\beta (\lambda f.f\,M\,N)\,\lambda xy.\textbf{false} \\
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the reduction does not use any list element the testing if the list is empty. so the **cons** and **pair** are lazy constructors. with this, we can infinite lists.

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Ackermann's function

- Most computable can be encoded by Church numerals with the power of inbuit repetition. e.g. Ackermann's function is not primitive recursive, but it can be encoded by Church numerals as:

$$\textbf{ack} \equiv \lambda m.m(\lambda fn.nf(f\underline{1}))\textbf{succ}$$

- we can see:

$$\begin{aligned}\textbf{ack}\,\underline{0}\,\underline{n} &\rhd_\beta \underline{0}(\lambda fn.nf(f\underline{1}))\textbf{succ}\,\underline{n}\\ &\rhd_\beta \textbf{succ}\,\underline{n} \rhd_\beta \underline{n+1}\end{aligned}$$

- and

$$\begin{aligned}\textbf{ack}\,\underline{m+1}\,\underline{n} &\rhd_\beta \underline{m+1}(\lambda fn.nf(f\underline{1}))\textbf{succ}\,\underline{n}\\ &\rhd_\beta (\lambda fn.nf(f\underline{1}))(\underline{m}(\lambda fn.nf(f\underline{1}))\,\textbf{succ})\underline{n}\\ &=_\beta (\lambda fn.nf(f\underline{1}))(\textbf{ack}\,\underline{m})\underline{n}\\ &\rhd_\beta \underline{n}(\textbf{ack}\,\underline{m})(\textbf{ack}\,\underline{m}\,\underline{1})\end{aligned}$$

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Ackermann's function (cont'd)

- $\mathbf{ack}\ \underline{m+1}\ \underline{n} \rhd_\beta \underline{n}(\mathbf{ack}\ \underline{m})(\mathbf{ack}\ \underline{m}\ \underline{1})$ as the lemma.
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which perfectly match the recursive definition of Ackermann's function.

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Recursion and fixed-points

- although it's possible encoding nearly all computable functions directly using Church numerals, but it's barely feasable with the complexity of recursions under composition. we must find the general method to express the recursions.
- recursion is the definition of a function using the function itself. e.g. the mathematical definition of factorial is
$$F\ N = \textbf{if}\ (\textbf{iszero}\ N)\ \underline{1}\ (\textbf{mult}\ N\ (F(\textbf{pred}\ N)))$$
the right hand side can be seen as a functional $(\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$:
$$G \equiv \lambda gn.\textbf{if}\ (\textbf{iszero}\ n)\ \underline{1}\ (\textbf{mult}\ n\ (g(\textbf{pred}\ n)))$$
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- so $\mathbf{Y}\, G =_\beta G(\mathbf{Y}\, G)$ is the fixed-point we expect.
- $\mathbf{Y}$ was discovered by Haskell B. Curry. it is defined as:
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Example of reductions with **Y** (normal order)

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$\mathbf{Y}\ G\ \underline{2}$

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$$\begin{aligned} & \mathbf{Y}\, G\, \underline{2} \\ \rhd_\beta\ & (\lambda x.G(xx))(\lambda x.G(xx))\, \underline{2} \end{aligned}$$

Example of reductions with $\mathbf{Y}$ (normal order)

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\begin{aligned}
& \mathbf{Y}\,G\,\underline{2} \\
\rhd_\beta\ & (\lambda x.G(xx))(\lambda x.G(xx))\,\underline{2} \\
\rhd_\beta\ & G((\lambda x.G(xx))\lambda x.G(xx))\,\underline{2} \quad (F \equiv (\lambda x.G(xx))\lambda x.G(xx))
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\end{aligned}
$$

Example of reductions with $\mathbf{Y}$ (normal order)

$$
\begin{aligned}
& \mathbf{Y}\, G\, \underline{2} \\
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Remarks

- $\mathbf{Y}$ will not work in the applicative order:

$$\begin{aligned} & \mathbf{Y}\, G\, \underline{0} \\ \rhd_\beta\ & \mathbf{Y}\, (\lambda gn.\mathbf{if}\, (\mathbf{iszero}\, n)\, \underline{1}\, (\mathbf{mult}\, n\, (g(\mathbf{pred}\, n))))\, \underline{0} \\ \rhd_\beta\ & (\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))\, F\, \underline{0} \\ \rhd_\beta\ & (\lambda f.f((\lambda x.f(xx))(\lambda x.f(xx))))\, F\, \underline{0} \\ \rhd_\beta\ & (\lambda f.f(f((\lambda x.f(xx))(\lambda x.f(xx)))))\, \underline{0} \\ \rhd_\beta\ & \cdots \end{aligned}$$

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$$\mathbf{Z} = \lambda f.(\lambda x.f(\lambda y.(xx)y))(\lambda x.f(\lambda y.(xx)y)))$$

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- in fact, the set of fixed-point combinators is recursively enumerable

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Examples of encoding recursions

- just place $\mathbf{Y}$ before the recursive definition to obtain the fixed-point:

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\mathbf{fact} &\equiv \mathbf{Y}\,(\lambda gn.\mathbf{if}\,(\mathbf{iszero}\,n)\,\underline{1}\,(\mathbf{mult}\,n\,(g(\mathbf{pred}\,n))))\\
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Exercises 6

- reimplement the function `lambdaToString` which return the most simplest of term replace the one with redundant parentheses. (e.g. `(@x.(@y.(xy)))` will simply output `@xy.xy`.
- show that the **sub** $\underline{m}\ \underline{n}$ will perform $m - n$.
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