关系的运算

School of Computer
Wuhan University

1 关系的合成
- 关系的合成
- 关系的幂
- 关系的闭包
- 传递闭包的求解算法

关系上的运算

Remark

由于关系就是集合，因此集合上的运算也是关系的运算.

- "$\leqslant$" $-$ "$\mathbb{1}_A$" $=$ "$<$";
- $\mathbb{R}$上有: "$\leqslant$" $\cap$ "$\geqslant$" $=$ "$=$";
- "$\leqslant$" $\cup$ "$\geqslant$" $=$ $\mathbb{R}$上的全域关系;
- $\mathscr{P}(A)$上有: "$\subseteq$" $\cap$ "$\supseteq$" $=$ "$=$";
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由于关系的对象是n重组，因此还有些一般集合不具有的运算.

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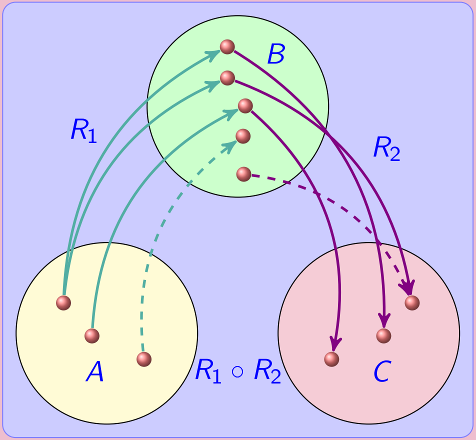
关系合成的图示



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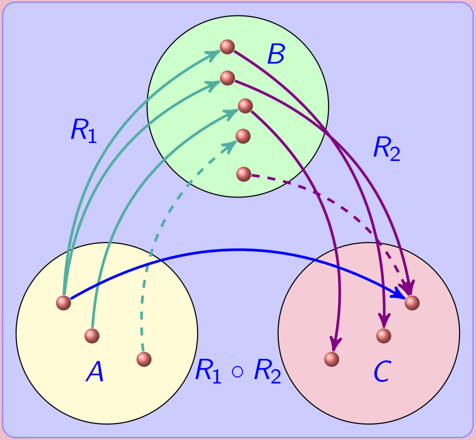

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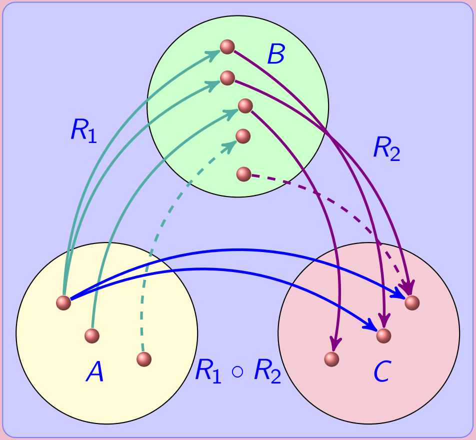

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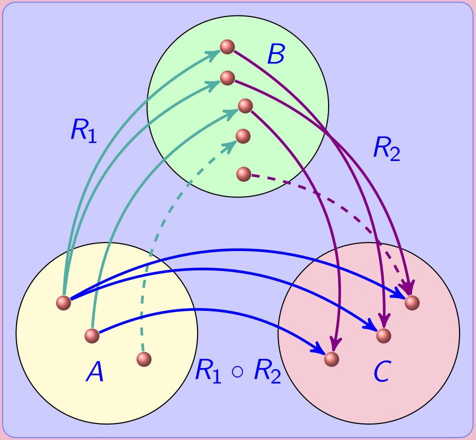

合成的定义

Definition (合成关系, Composite Relation)

设$\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2 \subseteq B \times C$, $\mathcal{R}_1$和$\mathcal{R}_2$的合成记为$\mathcal{R}_1 \circ \mathcal{R}_2$ $(\mathcal{R}_1\,\mathcal{R}_2)$定义为:

$$\mathcal{R}_1\,\mathcal{R}_2 \triangleq \{\langle a, c\rangle \mid a \in A,\ c \in C \wedge \exists b \in B \wedge a\,\mathcal{R}_1\,b \wedge b\,\mathcal{R}_2\,c\}$$

是A到C上的关系.

Remark

合成的条件: 第一个关系的陪域(codomain)和第二个关系的域(domain)是相同的集合.

Example

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- $\mathcal{R}_1$是兄弟关系；$\mathcal{R}_2$父子关系；$\mathcal{R}_1\mathcal{R}_2$是叔侄关系；
- $\mathcal{R}=\{\langle a,b\rangle \mid$ a和b间有直航航线$\}$, $\mathcal{R}\mathcal{R}$是城市之间经过一个城市转机的间接航线(记为$\mathcal{R}^2$)；
- $(=_4)^2 = =_4$;
- $\mathcal{R}\subseteq A\times B$; 则, $\mathbb{1}_A\mathcal{R}=\mathcal{R}\mathbb{1}_B=\mathcal{R}$;
- $\emptyset\mathcal{R}=\mathcal{R}\emptyset=\emptyset$;
- 合成对应的SQL语句：*SELECT* $\mathcal{R}_1$ *.first*, $\mathcal{R}_2$ *.second FROM* $\mathcal{R}_1$ *JOIN* $\mathcal{R}_2$ *ON* $\mathcal{R}_1$ *.second* = $\mathcal{R}_2$ *.first*.

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合成的运算性质(1/2)

Theorem

设$\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2, \mathcal{R}_3 \subseteq B \times C$, $\mathcal{R}_4 \subseteq C \times D$:

1. $\mathcal{R}_1(\mathcal{R}_2 \cup \mathcal{R}_3) = \mathcal{R}_1\mathcal{R}_2 \cup \mathcal{R}_1\mathcal{R}_3$ ($\circ$对$\cup$的分配律);
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合成的运算性质(2/2)

Proof.

②的证明:

1. $\quad\forall\langle a,c\rangle \in \mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3)$
2. $\iff \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2 \cap \mathcal{R}_3)$
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5. $\Longrightarrow \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2) \wedge \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_3)$
6. $\iff \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_2 \wedge \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_3$
7. $\iff \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_2 \cap \mathcal{R}_1\mathcal{R}_3$

$\square$

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合成的运算性质(2/2)

Proof.

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1. $\forall \langle a, c\rangle \in \mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3)$
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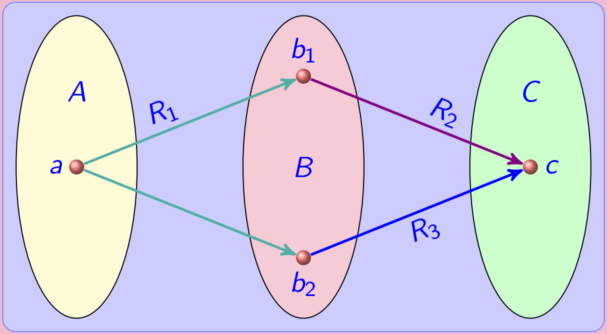
②的反例



$$\mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3) = \varnothing$$
$$\mathcal{R}_1\,\mathcal{R}_2 \cap \mathcal{R}_1\,\mathcal{R}_3 = \{\langle a, c\rangle\}$$

Example 1

Example

- 求所有可以间接通航的城市：

$$\begin{tikzcd} a \arrow[r, bend left, "1"] & a_1 \arrow[r, bend left, "2"] & a_2 \arrow[r, dotted, no head] & a_{n-1} \arrow[r, bend left, "n"] & b \end{tikzcd}$$

a和b可间接通航, iff:

$\exists a_1, a_2, \ldots, a_{n-1}\ (a\,\mathcal{R}\,a_1 \wedge a_1\,\mathcal{R}\,a_a \wedge \ldots \wedge a_{n-1}\,\mathcal{R}\,b)$

则：$\langle a, a_1\rangle \in \mathcal{R}$;

$\langle a, a_2\rangle \in \mathcal{R}^2$;

$\cdots\cdots$

$\langle a, a_{n-1}\rangle \in (\mathcal{R}^{n-2})\,\mathcal{R} \triangleq \mathcal{R}^{n-1}$;

$\langle a, b\rangle \in (\mathcal{R}^{n-1})\,\mathcal{R} \triangleq \mathcal{R}^{n}$;

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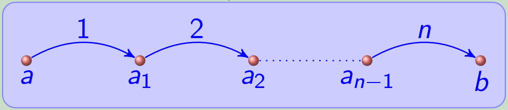

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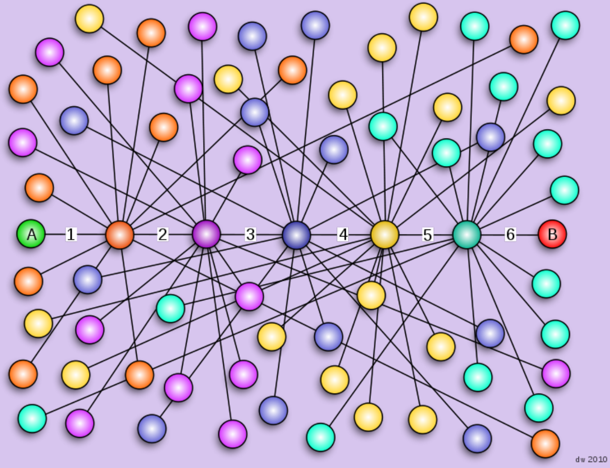
Example 2: Six Degrees of Separation (六度分隔)



见http://en.wikipedia.org/wiki/Six_degrees_of_separation.

关系的幂

Definition (关系的幂, Power of relation)

设$\mathcal{R}$是A上的关系，$n \in \mathbb{N}$, $\mathcal{R}$的乘幂递归定义如下：

1. $\mathcal{R}^0 = \mathbb{1}_A$;
2. $\mathcal{R}^{n+1} = \mathcal{R}^n \mathcal{R}$.

Example

$$\therefore \mathcal{R}^n = \begin{cases} \mathcal{R} & \text{if } n\text{是奇数} \\ \mathbb{1}_A & \text{if } n\text{是偶数} \end{cases}$$

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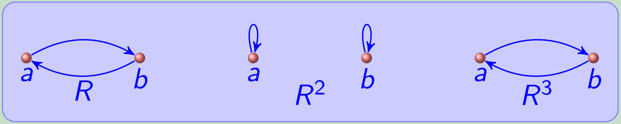

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1. $\mathcal{R}^m\mathcal{R}^n = \mathcal{R}^{m+n}$;
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对n用归纳法:

1. $n=0$时, $\mathcal{R}^m\mathcal{R}^0 = \mathcal{R}^m I_A = \mathcal{R}^m = \mathcal{R}^{m+0}$;
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1. 设 $|A| = n$，则存在 i, j $0 \leqslant i < j \leqslant 2^{n^2}$，使得：$\mathcal{R}^i = \mathcal{R}^j$.

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2. $\Rightarrow |\mathcal{P}(A \times A)| = 2^{n^2}$;
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4. 根据抽屉原则, $\exists i, j$ $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

$\square$

Corollary

$\forall m \in \mathbb{N}$ $\mathcal{R}^m \in \{\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}-1}\}$.

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Description (闭包，Closure)

数学上把包含某个给定的集合，并且具有某个性质的**最小**集合称为*闭包*.

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- 所有的可以间接通航的城市之间的关系是直接通航城市的传递闭包；

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Definition (关系的逆)

设$\mathcal{R} \subseteq A \times B$，关系$\mathcal{R}$的逆关系，记为$\widetilde{\mathcal{R}}$(读作tilde)，定义如下：

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相关性质

Theorem

$\mathcal{R}$是对称关系，iff, $\mathcal{R} = \tilde{\mathcal{R}}$.

Proof.

$\Longrightarrow$ $\forall \langle x, y \rangle \in \mathcal{R}$, $\therefore \langle y, x \rangle \in \mathcal{R}$; So $\langle x, y \rangle \in \tilde{\mathcal{R}}$

$\therefore \mathcal{R} \subseteq \tilde{\mathcal{R}}$, but $\tilde{\tilde{\mathcal{R}}} = \mathcal{R}$;

So, $\tilde{\mathcal{R}} \subseteq \tilde{\tilde{\mathcal{R}}} = \mathcal{R}$ ($\because \tilde{\mathcal{R}}$也是对称关系), $\therefore \mathcal{R} = \tilde{\mathcal{R}}$;

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Definition

设$\mathcal{R} \subseteq A^2$, $\mathcal{R}$的自反(对称、传递)闭包$\mathcal{R}'$是满足下述三条件的关系:

1. $\mathcal{R} \subseteq \mathcal{R}'$;
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分别记$\mathcal{R}$的自反、对称和传递闭包为: $r(\mathcal{R})$, $s(\mathcal{R})$和$t(\mathcal{R})$.

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$\mathcal{R}$是自反的(对称的、传递的), iff, $\mathcal{R} = r(\mathcal{R})$ $(s(\mathcal{R}), t(\mathcal{R}))$.

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闭包的构造(1/2)

Theorem

$$① \ r(\mathcal{R}) = \mathcal{R} \cup \mathbb{1}_A; \quad ② \ s(\mathcal{R}) = \mathcal{R} \cup \tilde{\mathcal{R}}\ ; \quad ③ \ t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i.$$

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   1. $n = 1$时，$\mathcal{R} \subseteq \mathcal{R}'$;
   2. 设$n = k$时结论成立，$n = k + 1$时：

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设$|A| = n$, $\mathcal{R} \subseteq A^2$, 则: $t(\mathcal{R}) = \bigcup_{i=1}^{n} \mathcal{R}^i$;

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1. 设$\langle x_0, x_{n+1} \rangle \in \mathcal{R}^{n+1}$;
2. $\exists x_1, x_2, \ldots, x_n \; x_0 \mathcal{R} x_1 \wedge x_1 \mathcal{R} x_2 \wedge \cdots x_n \mathcal{R} x_{n+1}$;
3. 即$\mathcal{R}$关系图中有从x_0到x_{n+1}长度为n的有向路径;
4. 而$x_1, x_2, \ldots, x_{n+1}$ $n+1$个元素只能在$|A| = n$个元素中选取;
5. 所以根据抽屉原则, $\exists 1 \leqslant i < j \leqslant n+1 \;\; x_i = x_j$;
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有限集合的传递闭包

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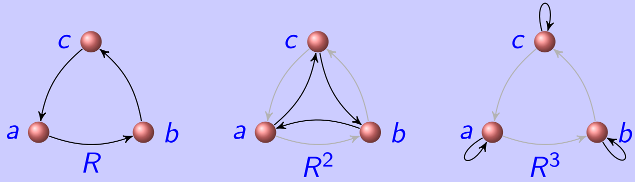
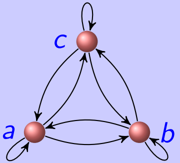
□

Examples

$\mathcal{R} = \{\langle a, b\rangle, \langle b.c\rangle, \langle c, a\rangle\}$的传递闭包



$t(R) = R \cup R^2 \cup R^3$



闭包之间的关系(1/3)

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设$\mathcal{R}$是自反关系，则，$t(\mathcal{R})$和$s(\mathcal{R})$也是自反关系；

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1. $\mathcal{R}$是自反的，iff，$\mathbb{1}_A \subseteq \mathcal{R}$
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闭包之间的关系(2/3)

Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^i; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i$$

用归纳法证明下述等式即可：

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

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闭包之间的关系(3/3)

Proof(continued).

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= & \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathbb{1}_A\right) && \text{(by 合成对并的分配率)} \\
= & \left(\bigcup_{i=1}^{k+1} \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) && \text{(by 合成对并的分配率)} \\
= & \bigcup_{i=0}^{k+1} \mathcal{R}^i
\end{aligned}
$$

闭包之间的关系(3/3)

Proof(continued).

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

1. $n = 0$时上述等式成立;
2. 设$n = k$时上述等式成立，则$n = k + 1$时:

$$\begin{aligned}
& (\mathbb{1}_A \cup \mathcal{R})^{k+1} \\
= & (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) && \text{(by 乘幂的定义)} \\
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关系的合成与关系矩阵乘积

Theorem

设$\mathcal{R} \subseteq A \times B$, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$和$|C| = p$，则：

$$M_{\mathcal{R}\,\mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}};$$

其中：$M_{\mathcal{R}} = (a_{ij})_{m \times n}$; $\quad M_{\mathcal{S}} = (b_{ij})_{n \times p}$

$$M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m \times p}; \quad c_{ij} \triangleq \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj};$$

Proof.

设$A = \{x_1, x_2, \ldots, x_m\}$, $B = \{y_1, y_2, \ldots, y_n\}$, $C = \{z_1, z_2, \ldots, z_p\}$

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1. $\quad c_{ij} = 1$
2. $\iff \exists k \quad a_{ik} = 1 \wedge b_{kj} = 1$
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$\square$

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□

Example

Example

设: $M_{\mathcal{R}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $M_{\mathcal{S}} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$;

则:

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$M_{\mathcal{R}\mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

传递闭包的求解算法

Description

$$M_{t(\mathcal{R})} = \sum_{i=1}^{n} M_{\mathcal{R}}^{i}$$

其中：$M_{\mathcal{R}}$是n阶方阵；

- 计算$M \cdot M$的每个元素$c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj}$ . . . . $\mathrm{O}(n)$;
- 计算$M \cdot M$ . . . . $\mathrm{O}(n^3)$;
- 计算$\sum_{i=1}^{n} M_{\mathcal{R}}^{i}$ . . . . $\mathrm{O}(n^4)$.

Warshall算法可降算法的复杂度为：$\mathrm{O}(n^3)$.

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Warshall算法

Definition

设$A = \{a_1, a_2, \ldots, a_n\}$, $\mathcal{R} \subseteq A^2$, M是$\mathcal{R}$的关系矩阵; n阶方阵W_k递归定义如下:

1. $W_0 = M$;
2. $W_k = (w_{ij}^k)_{n\times n}$, 其中: $w_{ij}^k = 1$, iff, 从a_i到a_j 有一条仅经过$a_1, a_2, \ldots, a_k$的有向路径.

Propostion

$$W_n = M_{t(\mathcal{R})}$$

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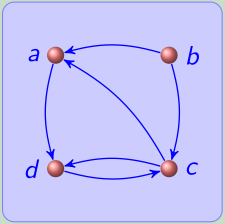
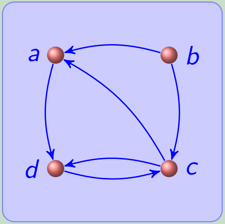


$$W_0 = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} a \\ b \\ c \\ d \end{array}$$

Example

Example



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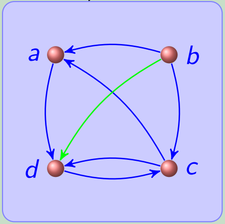
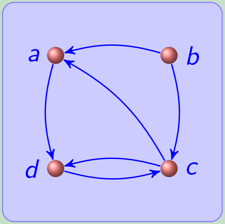
$k = 1$时，经过a点的路径：



$$W_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

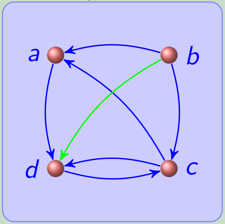
Example

Example



$$W_0 = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} a \\ b \\ c \\ d \end{array}$$

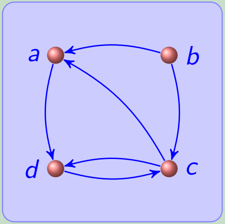
$k = 2$时，经过a, b点的路径：b没有引入的边，所以没有经过b的路径



$$W_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Example

Example



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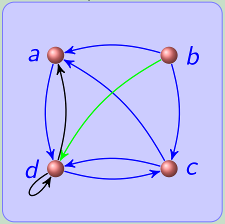
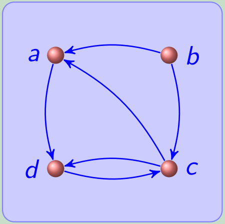
$k = 3$时，经过a, b, c点的路径：



$$W_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

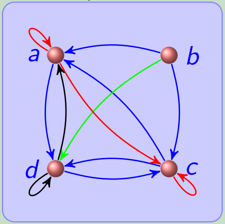
Example

Example



$$W_0 = \begin{array}{cc} \begin{matrix} a & b & c & d \end{matrix} & \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{matrix} a \\ b \\ c \\ d \end{matrix} \end{array}$$

$k=4$时，经过a，b，c，d点的路径：



$$W_4 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

W_k的计算

Description (W_k和W_{k+1}的关系)

$w_{ij}^{k+1} = 1$, iff, 下述两条件之一成立：

1. $w_{ij}^{k} = 1$，即从a_i到a_j有一条仅经过$a_1, a_2, \ldots, a_k$的有向路径；
2. 有一条仅经过$a_1, a_2, \ldots, a_{k+1}$，并且仅经过$a_{k+1}$一次的路径：如：

$$a_i, x_1, x_2, \ldots, x_p, a_{k+1}, y_1, y_2, \ldots, y_q, a_j$$

其中：$x_1, x_2, \ldots, x_p$和$y_1, y_2, \ldots, y_q$都在$\{a_1, a_2, \ldots, a_k\}$中；

$\therefore \quad w_{i(k+1)}^{k} = 1 \wedge w_{(k+1)j}^{k} = 1;$

故：

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Warshall算法

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```
procedure warshall(Matrix M_R)
{
  W := M_R;
  for k := 1 to n do {
    for i := 1 to n do {
      for j := 1 to n do {
        w_ij := w_ij ∨ (w_ik ∧ w_kj);
      }
    }
  }
}
```

本章小节