关系的运算

School of Computer Wuhan University

- 1/145 -

- 1 关系的合成
 - 关系的合成
 - 关系的幂
 - 关系的闭包
 - 传递闭包的求解算法

关系上的运算

Remark

由于关系就是集合,因此集合上的运算也是关系的运算.

- " \leq " " $\mathbb{1}_A$ " = "<";
- ℝ上有: "≤"∩"≥"= "=";
- "≤"」"≥"= ℝ 上的全域关系:
- a "□"」"□" → 人瑞光系
- 由于关系的对象是n重组,因此还有些一般集合不具有的运算.

关系上的运算

Remark

由于关系就是集合,因此集合上的运算也是关系的运算.

- "\le " "\la \" = " \le ":
- ℝ上有: "≤" ∩ "≥" = "=";
- "≤"∪"≥"= ℝ上的全域关系;
- "□"∪"□"≠ 全域关系.
- 由于关系的对象是11重组、因此还有些一般集合不具有的运算.

由于关系就是集合,因此集合上的运算也是关系的运算.

- "\le " "\la \" = " \le ":
- ℝ上有: "≤" ∩ "≥" = "=":
- "≤"∪"≥"= ℝ上的全域关系;
- 𝒯(A)上有: "⊆"∩"⊇"= "=";
- "⊂"∪"⊃"≠ 全域关系.

由于关系的对象是n重组、因此还有些一般集合不具有的运算。

由于关系就是集合,因此集合上的运算也是关系的运算.

- "\le " "\la \" = " \le ":
- ℝ上有: "≤" ∩ "≥" = "=";
- "≤"∪"≥"= ℝ上的全域关系;
- "□"∪"□"≠全域关系.

由于关系的对象是10重组、因此还有些一般集合不具有的运算.

由于关系就是集合,因此集合上的运算也是关系的运算.

- "\le " "\la \" = " \le ":
- ℝ上有: "≤" ∩ "≥" = "=";
- "≤"∪"≥"= ℝ上的全域关系;
- "□"∪"□"≠ 全域关系.

由干关系的对象是17重组,因此还有些一般集合不具有的运算,

由于关系就是集合,因此集合上的运算也是关系的运算.

- " \leq " " $\mathbb{1}_A$ " = "<";
- ℝ上有: "≤" ∩ "≥" = "=";
- "≤"∪"≥"= ℝ上的全域关系;
- "⊆"∪"⊇"≠ 全域关系.

由于关系的对象是n重组,因此还有些一般集合不具有的运算.

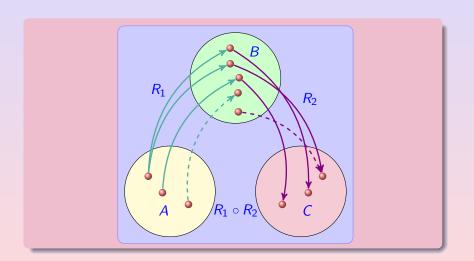
关系上的运算

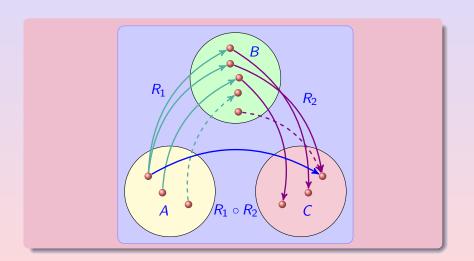
Remark

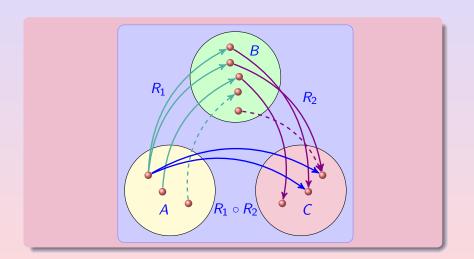
由于关系就是集合,因此集合上的运算也是关系的运算.

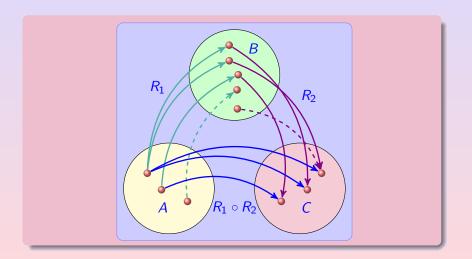
- "\le " "\la \" = " \le ":
- ℝ上有: "≤" ∩ "≥" = "=";
- "≤"∪"≥"= ℝ上的全域关系;
- "⊆"∪"⊇"≠ 全域关系.

由于关系的对象是n重组,因此还有些一般集合不具有的运算.









合成的定义

Definition (合成关系, Composite Relation)

设 $\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2 \subseteq B \times C$, \mathcal{R}_1 和 \mathcal{R}_2 的合成记为 $\mathcal{R}_1 \circ \mathcal{R}_2$ ($\mathcal{R}_1 \mathcal{R}_2$)定义为:

 $\mathcal{R}_1 \mathcal{R}_2 \triangleq \{ \langle a, c \rangle \mid a \in A, c \in C \land \exists b \in B \land a \mathcal{R}_1 b \land b \mathcal{R}_2 c \}$ 是A到C上的关系.

Remark

合成的条件:第一个关系的陪域(codomain)和第二个关系的域(domain)是相同的集合.

- R1 是兄弟关系; R2 父子关系; R1 R2 是叔侄关系
- $\mathcal{R} = \{\langle a, b \rangle \mid a \Rightarrow b \in \mathbb{R} \}$ 和 b 间有直航航线 $\}$, \mathcal{R} \mathcal{R} 是城市之间经过一个城市转机的间接航线(记为 \mathcal{R}^2);
- $\bullet \ (=_4)^2 = =_4;$
- $\mathcal{R} \subseteq A \times B$; M, $\mathbb{1}_{A}\mathcal{R} = \mathcal{R}\mathbb{1}_{B} = \mathcal{R}$;
- $\emptyset \mathcal{R} = \mathcal{R} \emptyset = \emptyset$;
- 合成对应的SQL语句: SELECT R₁ .first, R₂ .second FROM R₃ .JOIN R₃ ON R₄ .second = R₂ .first.

- R1是兄弟关系; R2父子关系; R1 R2是叔侄关系;
- $\mathcal{R} = \{\langle a, b \rangle \mid a \Rightarrow b \in \mathbb{R} \}$ 和 b 间有直航航线 $\}$, \mathcal{R} 况是城市之间经过一个城市转机的间接航线(记为 \mathcal{R}^2);
- $(=_4)^2 = =_4;$
- $\bullet \varnothing \mathcal{R} = \mathcal{R} \varnothing = \varnothing;$
- 合成对应的SQL语句: SELECT R₁ .first, R₂ .second FROM R₂ .IOIN R₂ ON R₃ second = R₂ .first

- R1是兄弟关系; R2父子关系; R1 R2是叔侄关系;
- \bullet $(=_4)^2 = =_4$;
- $\mathcal{R} \subseteq A \times B$; M, M_A $\mathcal{R} = \mathcal{R} M$ _B = \mathcal{R} ;
- $\bullet \varnothing \mathcal{R} = \mathcal{R} \varnothing = \varnothing;$
- 合成对应的SQL语句: SELECT R₁ .first, R₂ .second FROM R₂ . IOIN R₂ ON R₃ second = R₂ first

- R_1 是兄弟关系; R_2 父子关系; R_1 R_2 是叔侄关系;
- $\mathcal{R} = \{\langle a, b \rangle \mid \text{amb间有直航航线} \}$, \mathcal{R} \mathcal{R} 是城市之间经过一个城市转机的间接航线(记为 \mathcal{R}^2);
- \bullet $(=_4)^2 = =_4$;
- $\mathcal{R} \subseteq A \times B$; M, M_A $\mathcal{R} = \mathcal{R} M$ _B = \mathcal{R} ;
- $\varnothing \mathcal{R} = \mathcal{R} \varnothing = \varnothing;$
- 合成对应的SQL语句: SELECT R₁ .first, R₂ .second FROM
 R₁ .JOIN R₂ ON R₃ .second = R₂ .first.

- R1是兄弟关系; R2父子关系; R1 R2是叔侄关系;
- \bullet $(=_4)^2 = =_4;$
- $\mathcal{R} \subseteq A \times B$; M, M_A $\mathcal{R} = \mathcal{R} M$ _B = \mathcal{R} ;
- $\bullet \varnothing \mathcal{R} = \mathcal{R} \varnothing = \varnothing;$
- 合成对应的SQL语句: $SELECT \mathcal{R}_1$.first, \mathcal{R}_2 .second FROM \mathcal{R}_1 JOIN \mathcal{R}_2 ON \mathcal{R}_1 .second = \mathcal{R}_2 .first.

- R1是兄弟关系; R2父子关系; R1 R2是叔侄关系;
- $\mathcal{R} = \{\langle a, b \rangle \mid \text{amb间有直航航线} \}$, \mathcal{R} \mathcal{R} 是城市之间经过一个城市转机的间接航线(记为 \mathcal{R}^2);
- $\bullet \ (=_4)^2 = =_4;$
- $\mathcal{R} \subseteq A \times B$; \mathbb{N} , $\mathbb{1}_A \mathcal{R} = \mathcal{R} \mathbb{1}_B = \mathcal{R}$;
- $\bullet \varnothing \mathcal{R} = \mathcal{R} \varnothing = \varnothing;$
- 合成对应的SQL语句: SELECT \mathcal{R}_1 .first, \mathcal{R}_2 .second FROM \mathcal{R}_1 JOIN \mathcal{R}_2 ON \mathcal{R}_1 .second = \mathcal{R}_2 .first.



- R1是兄弟关系; R2父子关系; R1 R2是叔侄关系;
- $\mathcal{R} = \{\langle a, b \rangle \mid \text{anb间有直航航线} \}$, \mathcal{R} \mathcal{R} 是城市之间经过一个城市转机的间接航线(记为 \mathcal{R}^2);
- $\bullet (=_4)^2 = =_4;$
- $\mathcal{R} \subseteq A \times B$; \mathbb{N} , $\mathbb{1}_A \mathcal{R} = \mathcal{R} \mathbb{1}_B = \mathcal{R}$;
- $\varnothing \mathcal{R} = \mathcal{R} \varnothing = \varnothing$;
- 合成对应的SQL语句: SELECT \mathcal{R}_1 .first, \mathcal{R}_2 .second FROM \mathcal{R}_1 JOIN \mathcal{R}_2 ON \mathcal{R}_1 .second = \mathcal{R}_2 .first.



Theorem

- ① $\mathcal{R}_1(\mathcal{R}_2 \cup \mathcal{R}_3) = \mathcal{R}_1 \mathcal{R}_2 \cup \mathcal{R}_1 \mathcal{R}_3$ ($\circ \mathcal{M} \cup \mathfrak{h} \rightarrow \mathfrak{m} \neq \mathfrak{k}$);
- ③ $(R_2 \cup R_3) R_4 = R_2 R_4 \cup R_3 R_4$ (○对∪的分配律);

Theorem

- $\mathcal{R}_1(\mathcal{R}_2 \cup \mathcal{R}_3) = \mathcal{R}_1 \mathcal{R}_2 \cup \mathcal{R}_1 \mathcal{R}_3 \ (\circ \mathcal{M} \cup \ \mathcal{M} \cap \mathcal{M});$

Theorem

- ① $\mathcal{R}_1(\mathcal{R}_2 \cup \mathcal{R}_3) = \mathcal{R}_1 \mathcal{R}_2 \cup \mathcal{R}_1 \mathcal{R}_3$ (\circ 对 \cup 的分配律);
- ③ $(\mathcal{R}_2 \cup \mathcal{R}_3) \mathcal{R}_4 = \mathcal{R}_2 \mathcal{R}_4 \cup \mathcal{R}_3 \mathcal{R}_4 \ (\circ \ \ \ \ \)$ 的分配律);
- ③ $(\mathcal{R}_1 \mathcal{R}_2) \mathcal{R}_4 = \mathcal{R}_1(\mathcal{R}_2 \mathcal{R}_4)$ (结合率).

Theorem

- ① $\mathcal{R}_1(\mathcal{R}_2 \cup \mathcal{R}_3) = \mathcal{R}_1 \mathcal{R}_2 \cup \mathcal{R}_1 \mathcal{R}_3$ (\circ 对 \cup 的分配律);
- ③ $(\mathcal{R}_2 \cup \mathcal{R}_3) \mathcal{R}_4 = \mathcal{R}_2 \mathcal{R}_4 \cup \mathcal{R}_3 \mathcal{R}_4$ (\circ 对 \cup 的分配律);



Theorem

- ① $\mathcal{R}_1(\mathcal{R}_2 \cup \mathcal{R}_3) = \mathcal{R}_1 \mathcal{R}_2 \cup \mathcal{R}_1 \mathcal{R}_3$ (\circ 对 \cup 的分配律);

- ⑤ $(\mathcal{R}_1 \mathcal{R}_2) \mathcal{R}_4 = \mathcal{R}_1(\mathcal{R}_2 \mathcal{R}_4)$ (结合率).



Theorem

- ① $\mathcal{R}_1(\mathcal{R}_2 \cup \mathcal{R}_3) = \mathcal{R}_1 \mathcal{R}_2 \cup \mathcal{R}_1 \mathcal{R}_3$ (\circ 对 \cup 的分配律);
- ③ $(\mathcal{R}_2 \cup \mathcal{R}_3) \mathcal{R}_4 = \mathcal{R}_2 \mathcal{R}_4 \cup \mathcal{R}_3 \mathcal{R}_4$ (○对∪的分配律);
- **③** $(\mathcal{R}_1 \mathcal{R}_2) \mathcal{R}_4 = \mathcal{R}_1(\mathcal{R}_2 \mathcal{R}_4)$ (结合率).



Proof.



Proof.



Proof.

- $\forall \langle a, c \rangle \in \mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3)$



Proof.

- $\exists b(\langle a,b\rangle \in \mathcal{R}_1 \land \langle b,c\rangle \in \mathcal{R}_2) \land \exists b(\langle a,b\rangle \in \mathcal{R}_1 \land \langle b,c\rangle \in \mathcal{R}_3)$



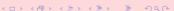
Proof.

- $\exists b(\langle a,b\rangle \in \mathcal{R}_1 \land \langle b,c\rangle \in \mathcal{R}_2) \land \exists b(\langle a,b\rangle \in \mathcal{R}_1 \land \langle b,c\rangle \in \mathcal{R}_3)$



Proof.





Proof.



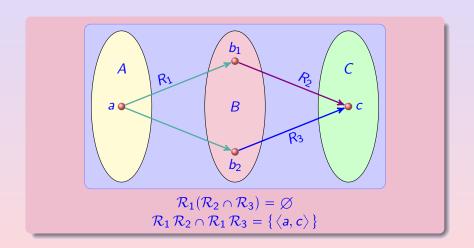


Proof.





②的反例



Example 1

Example

• 求所有可以间接通航的城市:



a和b可间接通航, iff:

$$\exists a_1, a_2, \ldots, a_{n-1} \ (a \,\mathcal{R} \, a_1 \, \wedge \, a_1 \,\mathcal{R} \, a_a \, \wedge \ldots \wedge a_{n-1} \,\mathcal{R} \, b)$$

则:
$$\langle a, a_1 \rangle \in \mathcal{R}$$
; $\langle a, a_2 \rangle \in \mathcal{R}^2$;

$$\langle a, a_{n-1} \rangle \in (\mathcal{R}^{n-2}) \mathcal{R} \triangleq \mathcal{R}^{n-1}$$

$$\langle a,b\rangle\in(\mathcal{R}^{n-1})\,\mathcal{R}\triangleq\mathcal{R}^n$$
;

:, a和b可间接通航, iff, $\exists n \langle a, b \rangle \in \mathbb{R}^n$

Example 1

Example

• 求所有可以间接通航的城市:



a和b可间接通航, iff:

$$\exists a_1, a_2, \ldots, a_{n-1} \ (a \mathcal{R} a_1 \wedge a_1 \mathcal{R} a_a \wedge \ldots \wedge a_{n-1} \mathcal{R} b)$$

则:
$$\langle a, a_1 \rangle \in \mathcal{R};$$

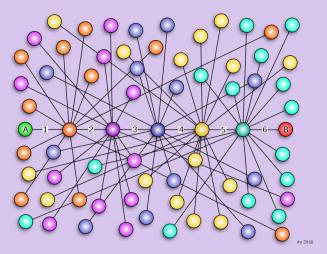
 $\langle a, a_2 \rangle \in \mathcal{R}^2;$

$$\langle a, a_{n-1} \rangle \in (\mathcal{R}^{n-2}) \mathcal{R} \triangleq \mathcal{R}^{n-1};$$

 $\langle a, b \rangle \in (\mathcal{R}^{n-1}) \mathcal{R} \triangleq \mathcal{R}^{n};$

∴ $a \pi b$ 可间接通航, iff, $\exists n \langle a, b \rangle \in \mathbb{R}^n$.

Example 2: Six Degrees of Separation (六度分隔)



见http://en.wikipedia.org/wiki/Six_degrees_of_separation.

设 \mathcal{R} 是A上的关系, $n \in \mathbb{N}$, \mathcal{R} 的乘幂递归定义如下:

- $\mathbb{R}^0 = \mathbb{1}_A;$

```
Example

R = \begin{cases} \mathcal{R} & \text{if } n
R^3 & \text{if } n
```

设R是A上的关系, $n \in \mathbb{N}$, R的乘幂递归定义如下:

- **1** $\mathcal{R}^0 = \mathbb{1}_A$;



设R是A上的关系, $n \in \mathbb{N}$, R的乘幂递归定义如下:

- **1** $\mathcal{R}^0 = \mathbb{1}_A$;

设R是A上的关系, $n \in \mathbb{N}$, R的乘幂递归定义如下:

- $\mathbf{0} \ \mathcal{R}^0 = \mathbb{1}_A;$



$$\therefore \mathcal{R}^n = \begin{cases} \mathcal{R} & \text{if } n \neq 3 \\ \mathbb{1}_A & \text{if } n \neq 3 \end{cases}$$

- $(\mathcal{R}^m)^n = \mathcal{R}^{mn};$

①的证明.

- $(\mathcal{R}^m)^n = \mathcal{R}^{mn};$

①的证明.

Theorem

- $(\mathcal{R}^m)^n = \mathcal{R}^{mn};$

①的证明

Theorem

- $(\mathcal{R}^m)^n = \mathcal{R}^{mn};$

①的证明.

- ① n = 0时, $\mathcal{R}^m \mathcal{R}^0 = \mathcal{R}^m \mathbb{1}_A = \mathcal{R}^m = \mathcal{R}^{m+0}$;
- ② $\mathfrak{F}_n = k \mathfrak{H}, \, \mathcal{R}^m \mathcal{R}^k = \mathcal{R}^{m+k};$
- ③ n = k + 1时:

$$\mathcal{R}^{m}\mathcal{R}^{k+1}$$

$$=\mathcal{R}^{m}(\mathcal{R}^{k}\mathcal{R}) \quad \text{(def)}$$

$$=(\mathcal{R}^{m}\mathcal{R}^{k})\mathcal{R} \quad \text{(结合律)}$$

$$=\mathcal{R}^{m+k+1} \quad \text{(归纳假设)}$$

Theorem

- $(\mathcal{R}^m)^n = \mathcal{R}^{mn};$

①的证明.

- **1** n = 0 H, $\mathcal{R}^m \mathcal{R}^0 = \mathcal{R}^m \mathbb{1}_A = \mathcal{R}^m = \mathcal{R}^{m+0}$;
- ③ n = k + 1时:

$$\mathcal{R}^{m}\mathcal{R}^{k+1}$$

$$=\mathcal{R}^{m}(\mathcal{R}^{k}\mathcal{R}) \quad \text{(def)}$$

$$=(\mathcal{R}^{m}\mathcal{R}^{k})\mathcal{R} \quad \text{(结合律)}$$

$$=\mathcal{R}^{m+k}\mathcal{R} \quad \text{(归纳假设)}$$

Theorem

- $(\mathcal{R}^m)^n = \mathcal{R}^{mn};$

①的证明.

- **①** n = 0时, $\mathcal{R}^m \mathcal{R}^0 = \mathcal{R}^m \mathbb{1}_A = \mathcal{R}^m = \mathcal{R}^{m+0}$;
- ② 设n = k时, $\mathcal{R}^m \mathcal{R}^k = \mathcal{R}^{m+k}$;
- **③** n = k + 1时:

$$\mathcal{R}^{m}\mathcal{R}^{k+1}$$
 $=\mathcal{R}^{m}(\mathcal{R}^{k}\mathcal{R})$ (def)
 $=(\mathcal{R}^{m}\mathcal{R}^{k})\mathcal{R}$ (结合律)
 $=\mathcal{R}^{m+k}\mathcal{R}$ (归纳假设)
 $=\mathcal{R}^{m+k+1}$ (def)

Theorem

 $(\mathcal{R}^m)^n = \mathcal{R}^{mn};$

①的证明.

- **①** n = 0时, $\mathcal{R}^m \mathcal{R}^0 = \mathcal{R}^m \mathbb{1}_A = \mathcal{R}^m = \mathcal{R}^{m+0}$;
- ② 设n = k时, $\mathcal{R}^m \mathcal{R}^k = \mathcal{R}^{m+k}$;
- ③ n = k + 1时:

$$\mathcal{R}^{m}\mathcal{R}^{k+1}$$
 $=\mathcal{R}^{m}(\mathcal{R}^{k}\mathcal{R})$ (def)
 $=(\mathcal{R}^{m}\mathcal{R}^{k})\mathcal{R}$ (结合律)
 $=\mathcal{R}^{m+k}\mathcal{R}$ (归纳假设)
 $=\mathcal{R}^{m+k+1}$ (def)

① 设|A| = n, 则存在 $i, j \in \{1, 2\}$ 0 $\{1, 2\}$ 0 $\{1, 2\}$ 0 $\{2\}$ 1 $\{2\}$ 2 $\{2\}$ 2 $\{2\}$ 3 $\{2\}$ 4 $\{2\}$ 3 $\{2\}$ 4 $\{2\}$ 5 $\{2\}$ 6 $\{2\}$ 9 $\{$

Proof.

Corollary

- ① 设|A| = n, 则存在i, j $0 \le i < j \le 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.
- Proof.
- $\bigcirc |A| = n, \quad |A \times A| = n$

- Corollary

Theorem

① 设|A| = n, 则存在 $i, j \in \{0\}$ 0 $\{i < j \le 2^{n^2}\}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Proof.

- \bullet 而 \mathcal{R}^0 , \mathcal{R}^1 , ..., $\mathcal{R}^{2''}$ 共有 $2^{n''}$ + 1项;
- 根据抽屉原则, $\exists i, j \ 0 \leq i < j \leq 2^{n^2}$, 使得: $\mathcal{R}' = \mathcal{R}'$.

Corollary

Theorem

① 设|A| = n, 则存在 $i, j \in \{0\}$ 0 $\{i < j \le 2^{n^2}\}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Proof.

- ⑤ 而 \mathbb{R}^0 , \mathbb{R}^1 , ..., $\mathbb{R}^{2^{n^2}}$ 共有 2^{n^2} + 1项;
- 根据抽屉原则, $\exists i, j \ 0 \leq i < j \leq 2^{n^{\alpha}}$, 使得: $\mathcal{R}' = \mathcal{R}'$.

Corollary

Theorem

① 设|A| = n, 则存在 $i, j \in \{0\}$ 0 $\{i < j \le 2^{n^2}\}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Proof.

- ③ 而 \mathbb{R}^0 , \mathbb{R}^1 , ..., $\mathbb{R}^{2^{n^2}}$ 共有 2^{n^2} + 1项;
- ① 根据抽屉原则, $\exists i, j \ 0 \leq i < j \leq 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Corollary

Theorem

① 设|A| = n, 则存在 $i, j \in \{0\}$ 0 《 $i < j \in \{2\}^n$ 》,使得: $\mathbb{R}^i = \mathbb{R}^j$.

Proof.

- ③ 而 $\mathcal{R}^0, \mathcal{R}^1, \dots, \mathcal{R}^{2^{n^2}}$ 共有 2^{n^2} +1项;
- ④ 根据抽屉原则, $\exists i, j \in I < j \leq 2^{n^2}$, 使得: $\mathcal{R}' = \mathcal{R}^J$.

Corollary

Theorem

① 设|A| = n, 则存在 $i, j \in \{0\}$ 0 $\leq i < j \leq 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Proof.

- ③ 而 $\mathcal{R}^0, \mathcal{R}^1, \dots, \mathcal{R}^{2^{n^2}}$ 共有 2^{n^2} +1项;
- ④ 根据抽屉原则, $\exists i, j \ 0 \leq i < j \leq 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Corollary

Theorem

① 设|A| = n, 则存在 $i, j \in \{0\}$ 0 $\{i < j \le 2^{n^2}\}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Proof.

- $2 : |\mathscr{P}(A \times A)| = 2^{n^2};$
- **③** 而 \mathcal{R}^0 , \mathcal{R}^1 , ..., $\mathcal{R}^{2^{n^2}}$ 共有 2^{n^2} + 1项;
- ④ 根据抽屉原则, $\exists i, j \ 0 \leq i < j \leq 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Corollary

Description (闭包, Closure)

数学上把包含某个给定的集合,并且具有某个性质的最小集合称为闭包.

Description (闭包, Closure)

·········

数学上把包含某个给定的集合,并且具有某个性质的最**小**集合称 为闭包.

- 所有的可以间接通航的城市之间的关系是直接通航城市的传递闭包;
- ② 极限的闭包:包含某个集合并且所有该集合上的极限都在该集合中的最小集合,如:]0,1[=[0,1].

Description (闭包, Closure)

数学上把包含某个给定的集合,并且具有某个性质的最小集合称 为闭包.

- 所有的可以间接通航的城市之间的关系是直接通航城市的传递闭包;
- ② 极限的闭包:包含某个集合并且所有该集合上的极限都在该集合中的最小集合,如: [0,1] = [0,1].

设 $\mathcal{R} \subseteq A \times B$, 关系 \mathcal{R} 的逆关系, 记为 $\widetilde{\mathcal{R}}$ (读作tilde), 定义如下: $\widetilde{\mathcal{R}} = \{\langle y, x \rangle \mid \langle x, y \rangle \in \mathcal{R}\} \subseteq B \times A$

设 $\mathcal{R} \subseteq A \times B$, 关系 \mathcal{R} 的逆关系, 记为 $\widetilde{\mathcal{R}}$ (读作tilde), 定义如下: $\widetilde{\mathcal{R}} = \{\langle y, x \rangle \mid \langle x, y \rangle \in \mathcal{R}\} \subseteq B \times A$

Example

 $\bullet \leqslant = \geqslant$; $\widetilde{\mathbb{1}}_A = \mathbb{1}_A$; $\widetilde{\subseteq} = \supseteq$;

- 关系的逆是关系的对偶概念;如果尺具有五性,则尺也相应的具有;
- 关系的逆与关系的补是不同的概念:
 - $\overline{\mathcal{R}} = \{\langle x, y \rangle \mid \langle x, y \rangle \notin \mathcal{R} \} \subseteq A \times B$

设 $\mathcal{R} \subseteq A \times B$, 关系 \mathcal{R} 的逆关系, 记为 $\widetilde{\mathcal{R}}$ (读作tilde), 定义如下: $\widetilde{\mathcal{R}} = \{\langle y, x \rangle \mid \langle x, y \rangle \in \mathcal{R}\} \subseteq B \times A$

Example

 $\bullet \ \widetilde{\leqslant} \ = \geqslant; \ \widetilde{\mathbb{1}_A} = \mathbb{1}_A; \ \widetilde{\subseteq} \ = \supseteq;$

- 关系的逆是关系的对偶概念;如果尺具有五性,则尺也相应的具有;
- 关系的逆与关系的补是不同的概念: $\overline{\mathcal{R}} = \{\langle x, v \rangle | \langle x, v \rangle \notin \mathcal{R} \} \subseteq A \times \mathbb{R}$

设 $\mathcal{R} \subseteq A \times B$, 关系 \mathcal{R} 的逆关系, 记为 $\widetilde{\mathcal{R}}$ (读作tilde), 定义如下: $\widetilde{\mathcal{R}} = \{\langle y, x \rangle \mid \langle x, y \rangle \in \mathcal{R}\} \subseteq B \times A$

- $\bullet \leqslant = \geqslant ; \widetilde{\mathbb{1}_A} = \mathbb{1}_A; \cong = \supseteq ;$
- ◆ 关系的逆是关系的对偶概念;如果尺具有五性,则Ĉ也相应的具有;
- 关系的逆与关系的补是不同的概念:
 - $\mathcal{R} = \{ \langle x, y \rangle \mid \langle x, y \rangle \notin \mathcal{R} \} \subseteq A \times B$

设 $\mathcal{R} \subseteq A \times B$, 关系 \mathcal{R} 的逆关系, 记为 $\widetilde{\mathcal{R}}$ (读作tilde), 定义如下: $\widetilde{\mathcal{R}} = \{\langle y, x \rangle \mid \langle x, y \rangle \in \mathcal{R}\} \subseteq B \times A$

- $\bullet \leqslant = \geqslant$; $\widetilde{\mathbb{1}}_A = \mathbb{1}_A$; $\widetilde{\subseteq} = \supseteq$;
- ◆ 关系的逆是关系的对偶概念;如果尺具有五性,则Ĉ也相应的具有;
- 关系的逆与关系的补是不同的概念: $\overline{\mathcal{R}} = \{\langle x, y \rangle \mid \langle x, y \rangle \notin \mathcal{R} \} \subseteq A \times B$

 \mathcal{R} 是对称关系, iff, $\mathcal{R} = \tilde{\mathcal{R}}$.

Proof

《□》《圖》《圖》《圖》 圖:

 \mathcal{R} 是对称关系, iff, $\mathcal{R} = \tilde{\mathcal{R}}$.

Proof.

- ⇒ $\forall \langle x, y \rangle \in \mathcal{R}, \therefore \langle y, x \rangle \in \mathcal{R}; So \langle x, y \rangle \in \tilde{\mathcal{R}}$ ∴ $\mathcal{R} \subseteq \tilde{\mathcal{R}}, \text{ but } \tilde{\mathcal{R}} = \mathcal{R};$ $So. \tilde{\mathcal{R}} \subseteq \tilde{\mathcal{R}} = \mathcal{R} (\because \tilde{\mathcal{R}} \text{ d. e.g.}), \therefore \mathcal{R} = \tilde{\mathcal{I}}$



 \mathcal{R} 是对称关系, iff, $\mathcal{R} = \tilde{\mathcal{R}}$.

Proof.

⇒ $\forall \langle x, y \rangle \in \mathcal{R}, \therefore \langle y, x \rangle \in \mathcal{R}; So \langle x, y \rangle \in \widetilde{\mathcal{R}}$ ∴ $\mathcal{R} \subseteq \widetilde{\mathcal{R}}, but \overset{\widetilde{\mathcal{R}}}{\mathcal{R}} = \mathcal{R};$ $So, \widetilde{\mathcal{R}} \subseteq \overset{\widetilde{\mathcal{R}}}{\mathcal{R}} = \mathcal{R} \ (: \widetilde{\mathcal{R}}$ 也是对称关系), ∴ $\mathcal{R} = \widetilde{\mathcal{R}};$

= ∀⟨x,y⟩∈R∴⟨x,y⟩∈R; So⟨y,x⟩∈R 所以况是对称关系.

 \mathcal{R} 是对称关系, iff, $\mathcal{R} = \tilde{\mathcal{R}}$.

Proof.

 $\iff \forall \langle x, y \rangle \in \mathcal{R} : \langle x, y \rangle \in \widetilde{\mathcal{R}}; So \langle y, x \rangle \in \mathcal{R}$ 所以尺是对称关系.

特性关系的闭包

Definition

设 $\mathcal{R} \subseteq A^2$, \mathcal{R} 的自反(对称、传递)闭包 \mathcal{R}' 是满足下述三条件的关系:

- ② R'是自反的(对称的、传递的);
- ④ 设 \mathbb{R}'' 是满足上述两条件的关系,则 $\mathbb{R}' \subseteq \mathbb{R}''$.

分别记R的自反、对称和传递闭包为: r(R), s(R)和t(R).

Propostion

R是自反的(对称的、传递的), iff, R = r(R) (s(R)), t(R)).

特性关系的闭包

Definition

设 $\mathcal{R} \subseteq A^2$, \mathcal{R} 的自反(对称、传递)闭包 \mathcal{R}' 是满足下述三条件的关系:

- ② 尺'是自反的(对称的、传递的);
- ③ 设 \mathbb{R}'' 是满足上述两条件的关系,则 $\mathbb{R}' \subseteq \mathbb{R}''$

分别记R的自反、对称和传递闭包为: r(R), s(R)和t(R).

Propostion

R是自反的(对称的、传递的), iff, R = r(R) (s(R)), t(R))

特性关系的闭包

Definition

设 $\mathcal{R} \subseteq A^2$, \mathcal{R} 的自反(对称、传递)闭包 \mathcal{R}' 是满足下述三条件的关系:

- ② 尺'是自反的(对称的、传递的);
- ③ 设 \mathbb{R}'' 是满足上述两条件的关系,则 $\mathbb{R}' \subseteq \mathbb{R}''$.

分别记R的自反、对称和传递闭包为: r(R), s(R)和t(R).

Propostion

R是自反的(对称的、传递的), iff, R = r(R) (s(R), t(R)).

特性关系的闭包

Definition

设 $\mathcal{R} \subseteq A^2$, \mathcal{R} 的自反(对称、传递)闭包 \mathcal{R}' 是满足下述三条件的关系:

- ② 尺'是自反的(对称的、传递的);
- ③ 设 \mathbb{R}'' 是满足上述两条件的关系,则 $\mathbb{R}' \subseteq \mathbb{R}''$.

分别记R的自反、对称和传递闭包为: r(R), s(R)和t(R).

Propostion

R是自反的(对称的、传递的), iff, R = r(R) (s(R), t(R)).

特性关系的闭包

Definition

设 $\mathcal{R} \subseteq A^2$, \mathcal{R} 的自反(对称、传递)闭包 \mathcal{R}' 是满足下述三条件的关系:

- ② R'是自反的(对称的、传递的);
- ③ 设 \mathbb{R}'' 是满足上述两条件的关系,则 $\mathbb{R}' \subseteq \mathbb{R}''$.

分别记R的自反、对称和传递闭包为: r(R), s(R)和t(R).

Propostion

R是自反的(对称的、传递的), iff, R = r(R) (s(R)), t(R)).

Theorem

①
$$r(\mathcal{R}) = \mathcal{R} \cup \mathbb{1}_A$$
; ② $s(\mathcal{R}) = \mathcal{R} \cup \widetilde{\mathcal{R}}$; ③ $t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i$.

③的证明.

- 78/145 -

闭包的构造(1/2)

Theorem

①
$$r(\mathcal{R}) = \mathcal{R} \cup \mathbb{1}_A$$
; ② $s(\mathcal{R}) = \mathcal{R} \cup \widetilde{\mathcal{R}}$; ③ $t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i$.

③的证明.

- $\mathbf{0} \ \mathcal{R} \subseteq \bigcup_{i=1}^{\infty} \mathcal{R}^i;$

 $\exists m, n \ \langle x, y \rangle \in \mathcal{R}^m, \ \langle y, z \rangle \in \mathcal{R}^n; \ \therefore \langle x, z \rangle \in \mathcal{R}^m \mathcal{R}^n = \mathcal{R}^{m+n} \subseteq \bigcup_{i=1}^m \mathcal{R}^i;$

所以 $\bigcup \mathcal{R}^i$ 是传递的.

闭包的构造(1/2)

00000**00000000000**000000000000000

Theorem

①
$$r(\mathcal{R}) = \mathcal{R} \cup \mathbb{1}_A$$
; ② $s(\mathcal{R}) = \mathcal{R} \cup \widetilde{\mathcal{R}}$; ③ $t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i$.

(3)的证明.

- $\forall \langle x, y \rangle \in \bigcup_{i=1}^{\infty} \mathcal{R}^{i}, \ \langle y, z \rangle \in \bigcup_{i=1}^{\infty} \mathcal{R}^{i};$

 $\exists m, n \ \langle x, y \rangle \in \mathcal{R}^m, \ \langle y, z \rangle \in \mathcal{R}^n; \ \therefore \langle x, z \rangle \in \mathcal{R}^m \mathcal{R}^n = \mathcal{R}^{m+n} \subseteq \bigcup_{i \to j} \mathcal{R}^i;$

所以 $\bigcup \mathcal{R}^i$ 是传递的.

闭包的构造(1/2)

Theorem

①
$$r(\mathcal{R}) = \mathcal{R} \cup \mathbb{1}_A$$
; ② $s(\mathcal{R}) = \mathcal{R} \cup \widetilde{\mathcal{R}}$; ③ $t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i$.

③的证明.

- $\mathbf{0} \ \mathcal{R} \subseteq \bigcup_{i=1}^{\infty} \mathcal{R}^{i};$

$$\exists m, n \ \langle x, y \rangle \in \mathcal{R}^m, \ \langle y, z \rangle \in \mathcal{R}^n; \ \therefore \langle x, z \rangle \in \mathcal{R}^m \mathcal{R}^n = \mathcal{R}^{m+n} \subseteq \bigcup_{i=1}^{\infty} \mathcal{R}^i;$$

所以 $\bigcup_{i=1}^{\infty} \mathcal{R}^i$ 是传递的.

③的证明.

③ 设传递关系 $\mathbb{R}' \supseteq \mathbb{R}$,则要证明: $\bigcup_{i=1} \mathbb{R}' \subseteq \mathbb{R}'$;
用归纳法证明: $\forall n \ \mathbb{R}^n \subseteq \mathbb{R}'$.

(3)的证明.

③ 设传递关系 $\mathcal{R}'\supseteq\mathcal{R}$,则要证明: $\bigcup_{i=1}^{n}\mathcal{R}^i\subseteq\mathcal{R}'$;

用归纳法证明: $\forall n \ \mathcal{R}^n \subseteq \mathcal{R}'$.

- n = 1 时, $\mathcal{R} \subseteq \mathcal{R}'$;
- ② $\dot{\mathbf{Q}}_n = k$ 时结论成立,n = k + 1 时:
 - $\therefore \exists v \langle x, v \rangle \in \mathcal{R}^k \land \langle v, z \rangle \in \mathcal{R}^k$
 - S_{α}/x \sqrt{x} C_{α}/x \sqrt{x} C_{α}/x \sqrt{x}
 - So $\langle x, y \rangle \in \mathcal{R}' \land \langle y, z \rangle \in \mathcal{R}';$
 - $\therefore \langle x, z \rangle \in \mathcal{R}'(\mathcal{R}'$ 是传递的).

③的证明.

③ 设传递关系 $\mathcal{R}' \supseteq \mathcal{R}$, 则要证明: $\bigcup_{i=1}^{\infty} \mathcal{R}^i \subseteq \mathcal{R}'$;

用归纳法证明: $\forall n \ \mathcal{R}^n \subseteq \mathcal{R}'$.

- n = 1 时, $\mathcal{R} \subseteq \mathcal{R}'$;
 - ② 设n = k时结论成立,n = k + 1时: 设 $\langle x, z \rangle \in \mathcal{R}^{k+1} = \mathcal{R}^k \mathcal{R};$ ∴ $\exists y \langle x, y \rangle \in \mathcal{R}^k \land \langle y, z \rangle \in \mathcal{R};$ $So \langle x, y \rangle \in \mathcal{R}' \land \langle y, z \rangle \in \mathcal{R}';$

(3)的证明.

③ 设传递关系 $\mathcal{R}'\supseteq\mathcal{R}$,则要证明: $\bigcup_{i=1}^{\infty}\mathcal{R}^i\subseteq\mathcal{R}'$;

用归纳法证明: $\forall n \ \mathcal{R}^n \subseteq \mathcal{R}'$.

- n = 1 时, $\mathcal{R} \subseteq \mathcal{R}'$;
- ② 设n = k时结论成立, n = k + 1时: 设 $\langle x, z \rangle \in \mathcal{R}^{k+1} = \mathcal{R}^k \mathcal{R}$; ∴ $\exists y \langle x, y \rangle \in \mathcal{R}^k \land \langle y, z \rangle \in \mathcal{R}$;
 - So $\langle x, y \rangle \in \mathcal{R}' \land \langle y, z \rangle \in \mathcal{R}';$
 - $\therefore \langle x, z \rangle \in \mathcal{R}'(\mathcal{R}'$ 是传递的).

Example

- $r(<) = \le : s(<) = "\neq" :$
- s(≤) = 全域关系; r(≠) = 全域关系;
- 设尺是城市之间有直接航线的关系,则城市之间有间接航线∞□

的关系等于 $\bigcup \mathcal{R}^{i}$.

Example

• $r(<) = \le; s(<) = "\neq";$

- s(≤) = 全域关系; r(≠) = 全域关系;
- $oldsymbol{v}$ 设 \mathcal{R} 是城市之间有直接航线的关系,则城市之间有间接航线的关系等于 $oldsymbol{ec{v}}$ \mathcal{R}^i .

Example

• $r(<) = \le; s(<) = "\neq";$

- $s(\leq) =$ 全域关系; $r(\neq) =$ 全域关系;
- 设 \mathbb{R} 是城市之间有直接航线的关系,则城市之间有间接航线的关系等于 $| \mathcal{R}^i$.

Examples

Example

• $r(<) = \le; s(<) = "\neq";$

- $s(\leq) =$ 全域关系; $r(\neq) =$ 全域关系;
- 设 \mathcal{R} 是城市之间有直接航线的关系,则城市之间有间接航线的关系等于 $\bigcap_{\mathcal{R}^i}$.

◆ロ → ◆個 → ◆ 重 → ◆ 重 ・ 夕 ♀ ○

Theorem

$$\mathcal{L}[A]=\mathbf{n},\;\mathcal{R}\subseteq A^2,\;\mathbf{M}\colon\;\mathbf{t}(\mathcal{R})=igcup_{i=1}^n\mathcal{R}^i;$$

Theorem

$$\mathcal{E}|A|=n,\ \mathcal{R}\subseteq A^2,\ M\colon\ t(\mathcal{R})=\bigcup_{i=1}^n\mathcal{R}^i;$$

Proof.

- ① 设 $\langle x_0, x_{n+1} \rangle \in \mathbb{R}^{n+1}$;
- $\exists x_1, x_2, \ldots, x_n \ x_0 \ \mathcal{R} \ x_1 \land x_1 \ \mathcal{R} \ x_2 \land \cdots x_n \ \mathcal{R} \ x_{n+1};$
- ③ 即尺关系图中有从x0到xn+1长度为n的有向路径;
- ⑥ 而x₁, x₂,...,x_{n+1} n+1个元素只能在|A| = n个元素中选取;
- ⑤ 所以根据抽屉原则, $\exists 1 \leq i < j \leq n+1 \ x_i = x_j$;
- $\bigcirc \therefore \underbrace{x_0 \mathcal{R} x_1 \wedge x_1 \mathcal{R} x_2 \wedge \cdots \wedge x_i \mathcal{R} x_{j+1} \wedge \cdots \wedge x_n \mathcal{R} x_{n+1}}_{};$

n+1-(j-i)/4

 $\bigcirc :: \langle x_0, x_{n+1} \rangle \in \mathcal{R}^{n+1-(j-i)} \subseteq \bigcup^n \mathcal{R}^i.$

Theorem

$$\mathcal{E}|A|=n,\ \mathcal{R}\subseteq A^2,\ M\colon\ t(\mathcal{R})=\bigcup_{i=1}^n\mathcal{R}^i;$$

Proof.

- ① 设 $\langle x_0, x_{n+1} \rangle \in \mathbb{R}^{n+1}$;
- $\exists x_1, x_2, \ldots, x_n x_0 \mathcal{R} x_1 \wedge x_1 \mathcal{R} x_2 \wedge \cdots \times_n \mathcal{R} x_{n+1};$
- ③ 即R关系图中有从X0到Xn+1长度为n的有向路径;
- ① 而 $x_1, x_2, ..., x_{n+1}$ n+1个元素只能在|A| = n个元素中选取;
- ⑤ 所以根据抽屉原则, $\exists 1 \leq i < j \leq n+1 \ x_i = x_j$;

 $n+1-(j-i)\wedge$

Theorem

$$\mathcal{E}|A|=n,\ \mathcal{R}\subseteq A^2,\ M\colon\ t(\mathcal{R})=\bigcup_{i=1}^n\mathcal{R}^i;$$

Proof.

- ① 设 $\langle x_0, x_{n+1} \rangle \in \mathbb{R}^{n+1}$;
- $\exists x_1, x_2, \ldots, x_n \ x_0 \ \mathcal{R} \ x_1 \ \wedge \ x_1 \ \mathcal{R} \ x_2 \ \wedge \cdots \times_n \mathcal{R} \ x_{n+1};$
- ③ 即R关系图中有从X0到Xn+1长度为n的有向路径;
- ① 而 $x_1, x_2, \ldots, x_{n+1}$ n+1个元素只能在|A|=n个元素中选取;
- ⑤ 所以根据抽屉原则, $\exists 1 \leq i < j \leq n+1 \ x_i = x_j$;

n+1-(j-i)

 $\bigcirc :: \langle x_0, x_{n+1} \rangle \in \mathcal{R}^{n+1-(j-i)} \subseteq \bigcup^n \mathcal{R}^i.$

Theorem

$$\mathcal{E}|A|=n,\ \mathcal{R}\subseteq A^2,\ M\colon\ t(\mathcal{R})=\bigcup_{i=1}^n\mathcal{R}^i;$$

- ① 设 $\langle x_0, x_{n+1} \rangle \in \mathbb{R}^{n+1}$;
- $\exists x_1, x_2, \ldots, x_n \ x_0 \ \mathcal{R} \ x_1 \land x_1 \ \mathcal{R} \ x_2 \land \cdots x_n \ \mathcal{R} \ x_{n+1};$
- ③ 即 \mathbb{R} 关系图中有从 x_0 到 x_{n+1} 长度为n的有向路径;
- ⑤ 所以根据抽屉原则, ∃1 ≤ $i < j ≤ n + 1 \times_i = x_i$;
- $\bigcirc \therefore \underbrace{x_0 \mathcal{R} x_1 \wedge x_1 \mathcal{R} x_2 \wedge \cdots \wedge x_i \mathcal{R} x_{j+1} \wedge \cdots \wedge x_n \mathcal{R} x_{n+1}}_{};$
 - n+1=(j-1)

Theorem

$$\mathcal{E}|A|=n,\ \mathcal{R}\subseteq A^2,\ M\colon\ t(\mathcal{R})=\bigcup_{i=1}^n\mathcal{R}^i;$$

Proof.

- ① 设 $\langle x_0, x_{n+1} \rangle \in \mathbb{R}^{n+1}$;
- $\exists x_1, x_2, \ldots, x_n x_0 \mathcal{R} x_1 \wedge x_1 \mathcal{R} x_2 \wedge \cdots \times_n \mathcal{R} x_{n+1};$
- ③ 即尺关系图中有从x0到xn+1长度为n的有向路径;
- ④ 而 $x_1, x_2, ..., x_{n+1}$ n+1个元素只能在|A| = n个元素中选取;
- ⑤ 所以根据抽屉原则, $\exists 1 \leq i < j \leq n+1 \ x_i = x_j$;
- $\underbrace{\bullet} \ \therefore x_0 \mathcal{R} x_1 \wedge x_1 \mathcal{R} x_2 \wedge \cdots \wedge x_i \mathcal{R} x_{j+1} \wedge \cdots \wedge x_n \mathcal{R} x_{n+1};$

n+1-(j-i)

Theorem

$$\mathcal{E}|A|=n,\ \mathcal{R}\subseteq A^2,\ M\colon\ t(\mathcal{R})=\bigcup_{i=1}^n\mathcal{R}^i;$$

- ① 设 $\langle x_0, x_{n+1} \rangle \in \mathbb{R}^{n+1}$;
- $\exists x_1, x_2, \ldots, x_n x_0 \mathcal{R} x_1 \wedge x_1 \mathcal{R} x_2 \wedge \cdots \times_n \mathcal{R} x_{n+1};$
- ③ 即尺关系图中有从x0到xn+1长度为n的有向路径;
- ④ 而 $x_1, x_2, \ldots, x_{n+1}$ n+1个元素只能在|A|=n个元素中选取;
- ⑤ 所以根据抽屉原则, $∃1 ≤ i < j ≤ n+1 x_i = x_j$;
- - n+1-(j-i)个

Theorem

$$\mathcal{E}|A|=n,\ \mathcal{R}\subseteq A^2,\ M\colon\ t(\mathcal{R})=\bigcup_{i=1}^n\mathcal{R}^i;$$

- ① 谈 $\langle x_0, x_{n+1} \rangle \in \mathbb{R}^{n+1}$;
- $\exists x_1, x_2, \ldots, x_n \ x_0 \ \mathcal{R} \ x_1 \land x_1 \ \mathcal{R} \ x_2 \land \cdots x_n \ \mathcal{R} \ x_{n+1};$
- ③ 即尺关系图中有从x0到xn+1长度为n的有向路径;
- ④ 而 $x_1, x_2, ..., x_{n+1}$ n+1个元素只能在|A| = n个元素中选取;
- ⑤ 所以根据抽屉原则, $∃1 ≤ i < j ≤ n+1 x_i = x_j$;
- $\bullet : \underbrace{x_0 \mathcal{R} x_1 \wedge x_1 \mathcal{R} x_2 \wedge \cdots \wedge x_j \mathcal{R} x_{j+1} \wedge \cdots \wedge x_n \mathcal{R} x_{n+1}}_{G};$

$$n+1-(j-i)$$

Theorem

$$\mathcal{E}|A|=n,\ \mathcal{R}\subseteq A^2,\ M:\ t(\mathcal{R})=\bigcup_{i=1}^n\mathcal{R}^i;$$

- ① 设 $\langle x_0, x_{n+1} \rangle \in \mathbb{R}^{n+1}$;
- $\exists x_1, x_2, \ldots, x_n \ x_0 \ \mathcal{R} \ x_1 \land x_1 \ \mathcal{R} \ x_2 \land \cdots x_n \ \mathcal{R} \ x_{n+1};$
- ③ 即尺关系图中有从x0到xn+1长度为n的有向路径;
- ④ 而 $x_1, x_2, \ldots, x_{n+1}$ n+1个元素只能在|A|=n个元素中选取;
- ⑤ 所以根据抽屉原则, ∃1 ≤ i < j ≤ n+1 $x_i = x_j$;
- $\underbrace{x_0 \mathcal{R} x_1 \wedge x_1 \mathcal{R} x_2 \wedge \cdots \wedge x_i \mathcal{R} x_{j+1} \wedge \cdots \wedge x_n \mathcal{R} x_{n+1}}_{n+1-(i-i) \uparrow };$

$$\bigcirc \ \, \therefore \left\langle x_0, x_{n+1} \right\rangle \in \mathcal{R}^{n+1-(j-i)} \subseteq \bigcup_{i=1}^n \mathcal{R}^i.$$

Examples

Propostion

设R是自反关系,则,t(R)和s(R)也是自反关系;

Propostion

设R是自反关系,则, t(R)和s(R)也是自反关系;

- ① \mathcal{R} 是自反的, iff, $\mathbb{1}_A \subseteq \mathcal{R}$
- ② 而 $t(\mathcal{R}) = \bigcup_{i=1} \mathcal{R}^i \supseteq \mathbb{1}_A;$
- ③ 所以, R是自反的.



Propostion

设 \mathcal{R} 是自反关系,则, $t(\mathcal{R})$ 和 $s(\mathcal{R})$ 也是自反关系;

- ① \mathcal{R} 是自反的, iff, $\mathbb{1}_A \subseteq \mathcal{R}$
- ② 而 $t(\mathcal{R}) = \bigcup_{i=1} \mathcal{R}^i \supseteq \mathbb{1}_A;$
- ③ 所以, R是自反的.



Propostion

设 \mathcal{R} 是自反关系,则, $t(\mathcal{R})$ 和 $s(\mathcal{R})$ 也是自反关系;

- ① \mathcal{R} 是自反的, iff, $\mathbb{1}_A \subseteq \mathcal{R}$
- **②** $\operatorname{Hom}(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i \supseteq \mathbb{1}_A;$
- ③ 所以, R是自反的.





Propostion

设R是自反关系,则, t(R)和s(R)也是自反关系;

- ① \mathcal{R} 是自反的, iff, $\mathbb{1}_A \subseteq \mathcal{R}$
- **②** $\operatorname{Hom}(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i \supseteq \mathbb{1}_A;$
- ⑤ 所以, R是自反的.





Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^i; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i \in \mathcal{R}} \mathcal{R}^i;$$

由此:

$$\forall n \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup \mathcal{R}^i \subseteq \bigcup \mathcal{R}^i;$$

所以:

$$\bigcup_{i=1}^{\infty} \mathcal{R}^i \subseteq \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i \subseteq \bigcup_{i=1}^{\infty} \mathcal{R}^i;$$

$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^{i}; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_{A} \cup \mathcal{R})^{i}$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

由此:

$$\forall n \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup_{i=0}^n \mathcal{R}^i \subseteq \bigcup_{i=0}^n \mathcal{R}^i;$$

所以:

$$\bigcup_{i=1}^{\infty} \mathcal{R}^i \subseteq \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i \subseteq \bigcup_{i=1}^{\infty} \mathcal{R}^i;$$

$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

ooooo**ooooooooo**

Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^i; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

由此:

$$\forall n \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup^n \mathcal{R}^i \subseteq \bigcup^\infty \mathcal{R}^i;$$

所以:

$$\bigcup_{i=0}^{\infty} \mathcal{R}^i \subseteq \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

即:

 $rt(\mathcal{R}) = tr(\mathcal{R}).$

Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^{i}; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_{A} \cup \mathcal{R})^{i}$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

由此:

$$\forall n \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup_{i=0}^n \mathcal{R}^i \subseteq \bigcup_{i=0}^\infty \mathcal{R}^i;$$

所以:

$$\bigcup_{i=0}^{\infty} \mathcal{R}^i \subseteq \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^i; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

由此:

$$\forall n \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup_{i=0}^n \mathcal{R}^i \subseteq \bigcup_{i=0}^\infty \mathcal{R}^i;$$

所以:

$$\bigcup_{i=0}^{\infty} \mathcal{R}^{i} \subseteq \bigcup_{i=0}^{\infty} (\mathbb{1}_{A} \cup \mathcal{R})^{i} \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^{i};$$

$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^i; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

由此:

$$\forall n \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup_{i=0}^n \mathcal{R}^i \subseteq \bigcup_{i=0}^\infty \mathcal{R}^i;$$

所以:

$$\bigcup_{i=0}^{\infty} \mathcal{R}^{i} \subseteq \bigcup_{i=0}^{\infty} (\mathbb{1}_{A} \cup \mathcal{R})^{i} \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^{i};$$

$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

(c) hfwang

闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

- n = 0时上述等式成立;
- ② 设n = k时上述等式成立,则n = k + 1时:

(c) hfwang

闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

- □ n = 0时上述等式成立;
- ② 设n = k时上述等式成立,则n = k + 1时:

闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

- ① n = 0时上述等式成立;
- ② 设n = k时上述等式成立, 则n = k + 1时:

$$\left(\bigcup_{i=0}^{\mathcal{R}^i}\right)\mathcal{R}\left(\bigcup_{i=0}^{\mathcal{R}^i}\right)\mathbb{1}_A\right)$$
 $\bigcup_{i=1}^{k+1}\mathcal{R}^i$
 $\bigcup_{i=0}^{k}\mathcal{R}^i$

Chfwang

闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

- ① n = 0时上述等式成立;
- ② 设n = k时上述等式成立,则n = k + 1时:

$$\begin{array}{l} & \textbf{($\mathbb{1}_A \cup \mathcal{R}$)^{k+1}} \\ = (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) & \text{(by 乘幂的定义)} \\ = \left(\bigcup_{i=0}^k \mathcal{R}^i\right) (\mathbb{1}_A \cup \mathcal{R}) & \text{(by 归纳假设)} \\ = \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \mathcal{R} \right) \bigcup \left(\left(\bigcup_{i=0}^k \mathcal{R}^i\right) \mathbb{1}_A\right) & \text{(by 合成对并的分配率)} \\ = \left(\bigcup_{i=1}^{k+1} \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^k \mathcal{R}^i\right) & \text{(by 合成对并的分配率)} \end{array}$$

Chfwang

闭包之间的关系(3/3)

Proof(continued).

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

- □ n = 0时上述等式成立;
- ② 设n = k时上述等式成立,则n = k + 1时:

$$(\mathbb{1}_A \cup \mathcal{R})^{k+1}$$

= $(\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R})$

(by 乘幂的定义)

$$= \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^{i} \right) \mathcal{R} \right) \bigcup_{k} \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^{i} \right) \mathbb{1}_{A} \right)$$

$$=\bigcup \mathcal{R}^i$$

C)hfwarig-

闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

- ① n = 0时上述等式成立;
- ② 设n = k时上述等式成立,则n = k + 1时:

$$\begin{split} & (\mathbb{1}_A \cup \mathcal{R})^{k+1} \\ &= (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) \\ &= \left(\bigcup_{i=0}^k \mathcal{R}^i\right) (\mathbb{1}_A \cup \mathcal{R}) \\ &= \left(\left(\bigcup_{i=0}^k \mathcal{R}^i\right) \mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^k \mathcal{R}^i\right) \mathbb{1}_A\right) \\ &= \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \mathbb{1}_A \end{aligned} \text{ (by 合成对并的分配率)}$$

Chfwang

闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=1}^n \mathcal{R}^i;$$

- □ n = 0时上述等式成立;
- ② 设n = k时上述等式成立,则n = k + 1时:

$$\begin{array}{l} \left(\mathbb{1}_{A} \cup \mathcal{R}\right)^{k+1} \\ = \left(\mathbb{1}_{A} \cup \mathcal{R}\right)^{k} (\mathbb{1}_{A} \cup \mathcal{R}) \\ = \left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) (\mathbb{1}_{A} \cup \mathcal{R}) \\ = \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathbb{1}_{A}\right) \\ = \left(\bigcup_{i=0}^{k+1} \mathcal{R}^{i}\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathbb{1}_{A} \end{array} \right) \text{ (by 合成对并的分配率)} \\ = \left(\bigcup_{i=1}^{k+1} \mathcal{R}^{i}\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathbb{1}_{A} \\ = \left(\bigcup_{i=1}^{k+1} \mathcal{R}^{i}\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathbb{1}_{A} \end{aligned} \right)$$

Chfwang

闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

- ① n = 0时上述等式成立;
- ② 设n = k时上述等式成立, 则n = k + 1时:

$$\begin{split} & (\mathbb{1}_A \cup \mathcal{R})^{k+1} \\ & = (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) \\ & = \left(\bigcup_{i=0}^k \mathcal{R}^i\right) (\mathbb{1}_A \cup \mathcal{R}) \\ & = \left(\left(\bigcup_{i=0}^k \mathcal{R}^i\right) \mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^k \mathcal{R}^i\right) \mathbb{1}_A\right) \\ & = \left(\bigcup_{i=1}^{k+1} \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \\ & = \left(\bigcup_{i=1}^k \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \end{aligned} \tag{by 合成对并的分配率)$$

闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

- □ n = 0时上述等式成立;
- ② 设n = k时上述等式成立,则n = k + 1时:

$$\begin{split} &(\mathbb{1}_A \cup \mathcal{R})^{k+1} \\ &= (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) \\ &= \left(\bigcup_{i=0}^k \mathcal{R}^i\right) (\mathbb{1}_A \cup \mathcal{R}) \\ &= \left(\left(\bigcup_{i=0}^k \mathcal{R}^i\right) \mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^k \mathcal{R}^i\right) \mathbb{1}_A\right) \\ &= \left(\bigcup_{i=1}^{k+1} \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \\ &= \left(\bigcup_{i=1}^k \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \\ &= \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \\ &= \left(\bigcup_{i=1}^k \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \\ &= \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \\ &= \left(\bigcup_{i=1}^k \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \\ &= \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^k \mathcal{R}^i\right) \\$$

设
$$\mathcal{R} \subseteq A \times B$$
, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$ 和 $|C| = p$, 例: $M_{\mathcal{R} \mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}}$; 其中: $M_{\mathcal{R}} = (a_{ij})_{m \times n}$; $M_{\mathcal{S}} = (b_{ij})_{n \times p}$ $M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m \times p}$; $c_{ij} \triangleq \bigvee_{i} a_{ik} \wedge b_{kj}$;

设
$$A = \{x_1, x_2, \dots, x_m\}, B = \{y_1, y_2, \dots, y_n\}, C = \{z_1, z_2, \dots, z_p\}$$

设
$$\mathcal{R} \subseteq A \times B$$
, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$ 和 $|C| = p$, 则: $M_{\mathcal{R} \mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}}$; 其中: $M_{\mathcal{R}} = (a_{ij})_{m \times n}$; $M_{\mathcal{S}} = (b_{ij})_{n \times p}$ n $M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m \times p}$; $c_{ij} \triangleq \bigvee_{i} a_{ik} \wedge b_{kj}$;

设
$$A = \{x_1, x_2, \dots, x_m\}, B = \{y_1, y_2, \dots, y_n\}, C = \{z_1, z_2, \dots, z_n\}$$

- $c_{ij} =$

设
$$\mathcal{R} \subseteq A \times B$$
, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$ 和 $|C| = p$, 则: $M_{\mathcal{R} \mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}}$; 其中: $M_{\mathcal{R}} = (a_{ij})_{m \times n}$; $M_{\mathcal{S}} = (b_{ij})_{n \times p}$ n $M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m \times p}$; $c_{ij} \triangleq \bigvee_{i} a_{ik} \wedge b_{kj}$;

设
$$A = \{x_1, x_2, \dots, x_m\}, B = \{y_1, y_2, \dots, y_n\}, C = \{z_1, z_2, \dots, z_p\}$$

- $c_{ij} = 1$
- $2 \iff \exists k \ a_{ik} = 1 \land b_{ki} = 1$

设
$$\mathcal{R} \subseteq A \times B$$
, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$ 和 $|C| = p$, 例: $M_{\mathcal{R} \mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}}$; 其中: $M_{\mathcal{R}} = (a_{ij})_{m \times n}$; $M_{\mathcal{S}} = (b_{ij})_{n \times p}$ n $M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m \times p}$; $c_{ij} \triangleq \bigvee_{i} a_{ik} \wedge b_{kj}$;

设
$$A = \{x_1, x_2, \dots, x_m\}, B = \{y_1, y_2, \dots, y_n\}, C = \{z_1, z_2, \dots, z_p\}$$

$$c_{ii} = 1$$

设
$$\mathcal{R} \subseteq A \times B$$
, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$ 和 $|C| = p$, 例: $M_{\mathcal{R} \mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}}$; 其中: $M_{\mathcal{R}} = (a_{ij})_{m \times n}$; $M_{\mathcal{S}} = (b_{ij})_{n \times p}$ n $M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m \times p}$; $c_{ij} \triangleq \bigvee_{i} a_{ik} \wedge b_{kj}$;

设
$$A = \{x_1, x_2, \dots, x_m\}, B = \{y_1, y_2, \dots, y_n\}, C = \{z_1, z_2, \dots, z_p\}$$

$$c_{ii} = 1$$

设
$$\mathcal{R} \subseteq A \times B$$
, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$ 和 $|C| = p$, 例: $M_{\mathcal{R} \mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}}$; 其中: $M_{\mathcal{R}} = (a_{ij})_{m \times n}$; $M_{\mathcal{S}} = (b_{ij})_{n \times p}$ n $M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m \times p}$; $c_{ij} \triangleq \bigvee_{i} a_{ik} \wedge b_{kj}$;

设
$$A = \{x_1, x_2, \dots, x_m\}, B = \{y_1, y_2, \dots, y_n\}, C = \{z_1, z_2, \dots, z_p\}$$

$$c_{ij} = 1$$

设:
$$M_{\mathcal{R}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
; $M_{\mathcal{S}} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$;

则:

$$\mathbf{M}_{\mathcal{R}\mathcal{S}} = \mathbf{M}_{\mathcal{R}} \cdot \mathbf{M}_{\mathcal{S}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



- 126/145 -

Description

$$M_{t(\mathcal{R})} = \sum_{i=1}^{n} M_{\mathcal{R}}^{i}$$

其中: MR是n阶方阵;

• 计算
$$M \cdot M$$
的每个元素 $c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj} \dots \dots \dots \dots \cap (n)$

•
$$\mathcal{H}$$
 $M_{\mathcal{R}}$ $O(n^4)$

Warshall算法可降算法的复杂度为: $O(n^3)$.

传递闭包的求解算法

Description

$$M_{t(\mathcal{R})} = \sum_{i=1}^{n} M_{\mathcal{R}}^{i}$$

其中: MR 是n阶方阵;

• 计算
$$M \cdot M$$
的每个元素 $c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj} \dots O(n);$

• 计算
$$\sum_{i=1}^{n} M_{\mathcal{R}}^{i}$$
 $O(n^4)$.

Warshall算法可降算法的复杂度为: $O(n^3)$.

传递闭包的求解算法

Description

$$M_{t(\mathcal{R})} = \sum_{i=1}^{n} M_{\mathcal{R}}^{i}$$

其中: MR是n阶方阵;

• 计算
$$M \cdot M$$
的每个元素 $c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj} \dots O(n);$

• 计算
$$\sum_{i=1}^{\infty}M_{\mathcal{R}}^{i}$$
 $O(n^{4})$.

Warshall算法可降算法的复杂度为: $O(n^3)$.

传递闭包的求解算法

Description

$$M_{t(\mathcal{R})} = \sum_{i=1}^{n} M_{\mathcal{R}}^{i}$$

其中: MR 是n阶方阵;

• 计算
$$M \cdot M$$
的每个元素 $c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj} \dots O(n);$

• 计算
$$\sum_{i=1}^{n} M_{\mathcal{R}}^{i}$$
 $O(n^{4})$.

Warshall算法可降算法的复杂度为: ○(n³).

Definition

设 $A = \{a_1, a_2, \dots, a_n\}, \mathcal{R} \subseteq A^2, M$ 是 \mathcal{R} 的关系矩阵; n阶方阵 W_k 递归定义如下:

- $\mathbf{0} \ W_0 = M_1$
- ② $W_k = (w_{ij}^k)_{n \times n}$, 其中: $w_{ij}^k = 1$, iff, $A_i = 1$, iff, $A_i = 1$, A_i

Propostion

 $W_n = M_{t(\mathcal{R})}$

Definition

设 $A = \{a_1, a_2, \dots, a_n\}, \mathcal{R} \subseteq A^2, M$ 是 \mathcal{R} 的关系矩阵; n阶方阵 W_k 递归定义如下:

- **1** $W_0 = M$;
- ② $W_k = (w_{ij}^k)_{n \times n}$, 其中: $w_{ij}^k = 1$, iff, $A_i = 1$, iff, $A_i = 1$, A_i

Propostion

$$W_n = M_{t(\mathcal{R})}$$

Definition

设 $A = \{a_1, a_2, \dots, a_n\}, \mathcal{R} \subseteq A^2, M$ 是 \mathcal{R} 的关系矩阵; n阶方阵 W_k 递归定义如下:

- **1** $W_0 = M$;
- ② $W_k = (w_{ij}^k)_{n \times n}$, 其中: $w_{ij}^k = 1$, iff, $A_i =$

Propostion

$$W_n = M_{t(\mathcal{R})}$$



Definition

设 $A = \{a_1, a_2, \dots, a_n\}, \mathcal{R} \subseteq A^2, M$ 是 \mathcal{R} 的关系矩阵; n阶方阵 W_k 递归定义如下:

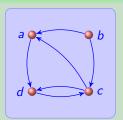
- **1** $W_0 = M$;
- ② $W_k = (w_{ij}^k)_{n \times n}$, 其中: $w_{ij}^k = 1$, iff, $A_i =$

Propostion

$$W_n = M_{t(\mathcal{R})}$$

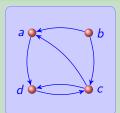


Example



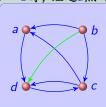
$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Example



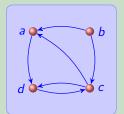
$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_d^a$$

k = 1时,经过a点的路径:



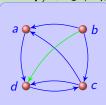
$$W_1 = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Example



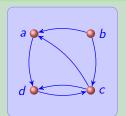
$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

k=2时,经过a,b点的路径:b没有引入的边,所以没有经过b的路径



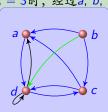
$$W_2 = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Example

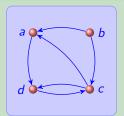


$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{d}^{a}$$

k = 3时, 经过a, b, c点的路径:

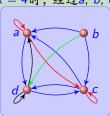


$$W_3 = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right)$$



$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

k = 4时, 经过a, b, c, d点的路径:



$$W_4 = \left(\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right)$$

Description (W_k 和 W_{k+1} 的关系)

$$w_{ii}^{k+1} = 1$$
, iff, 下述两条件之一成立:

- ① $w_{ii}^{k} = 1$, 即从 a_{i} 到 a_{i} 有一条仅经过 $a_{1}, a_{2}, ..., a_{k}$ 的有向路径;
- ② 有一条仅经过a₁, a₂,..., a_{k+1}, 并且仅经过a_{k+1}一次的路径: 如:

 $a_i, x_1, x_2, \dots, x_p, a_{k+1}, y_1, y_2, \dots, y_q, a_j$ 其中: x_1, x_2, \dots, x_p 和 y_1, y_2, \dots, y_q 都在 $\{a_1, a_2, \dots, a_k\}$ 中; \vdots $w_{i(k+1)}^k = 1 \land w_{(k+1)i}^k = 1;$

故:

$$w_{ij}^{k+1} = w_{ij}^k \vee (w_{i(k+1)}^k \wedge w_{(k+1)j}^k)$$

Description (W_k 和 W_{k+1} 的关系)

$$w_{ii}^{k+1} = 1$$
, iff, 下述两条件之一成立:

- ① $w_{ij}^k = 1$, 即从 a_i 到 a_j 有一条仅经过 a_1, a_2, \ldots, a_k 的有向路径;
- ② 有一条仅经过 $a_1, a_2, \ldots, a_{k+1}$, 并且仅经过 a_{k+1} 一次的路径: 如:

 $w_{i(k+1)}^{n} = 1 \wedge w_{(k+1)j}^{n} = 1;$

故:

$$w_{ij}^{k+1} = w_{ij}^k \vee (w_{i(k+1)}^k \wedge w_{(k+1)j}^k)$$

Wk的计算

Description (W_k 和 W_{k+1} 的关系)

$$w_{ii}^{k+1} = 1$$
, iff, 下述两条件之一成立:

- $w_{ij}^k = 1$, $p_{ij} M_{ai} = 1$, $p_{ij} M_{$
- ② 有一条仅经过a₁, a₂,..., a_{k+1}, 并且仅经过a_{k+1}一次的路径: 如:

$$a_i, x_1, x_2, \dots, x_p, a_{k+1}, y_1, y_2, \dots, y_q, a_j$$

其中: x_1, x_2, \dots, x_p 和 y_1, y_2, \dots, y_q 都在 $\{a_1, a_2, \dots, a_k\}$ 中;
 $w_{i(k+1)}^k = 1 \land w_{(k+1)j}^k = 1;$

故:

$$w_{ij}^{k+1} = w_{ij}^k \vee (w_{i(k+1)}^k \wedge w_{(k+1)j}^k)$$

```
Warshall算法.
procedure warshall (Matrix M_R)
  W := M_{\mathcal{R}};
  for k := 1 to n do {
     for i := 1 to n do {
       for j := 1 to n do {
          w_{ii} := w_{ii} \vee (w_{ik} \wedge w_{ki});
```

- ① 关系的合成
 - 关系的合成
 - 关系的幂
 - 关系的闭包
 - 传递闭包的求解算法