Optimal Sampling-Based Planning for General Kinodynamic Systems

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***Abstract*—We introduce a method for automatically deriving the cost metric and steering method— critical components of the RRT* algorithm—for general kinodynamic systems with differentiable dynamics and twice differentiable cost functions. We first introduce a version of the RRT* algorithm that uses an LQR-based extension to plan for problems with affine dynamics and second order costs, and prove that it almost surely converges to the optimal plan. We then show how to automatically derive a cost metric and extension method for problems with nonlinear dynamics and non-quadratic costs by locally approximating both functions. This removes the need for domain-specific extension methods, which are a major obstacle to the general applicability of RRT*.**

I. Introduction

The RRT* algorithm [1] offers a practical and effective method for probabilistically complete optimal motion planning. Like all members of the RRT family of planning algorithms [2] it incrementally builds a tree by using a cost measure to determine the "closest" vertices in the tree to a random sample, and a steering method to find an extension trajectory from a given node in the tree to the sample. Designing such procedures is straightforward for kinematic systems, but it is much more challenging for kinodynamic systems because it is non-trivial to calculate good local kinodynamic trajectories, even in free space. Hence, designing the distance measure and steering method often represents the major obstacle to applying RRT* to kinodynamic systems.

Since control theory offers methods for exactly computing optimal policies and the corresponding cost-to-go functions for linear systems with quadratic costs (using the linear quadratic regulator, or LQR, family of solution methods), several researchers have applied LQR-based methods to estimate distance and to calculate locally optimal trajectories, as originally suggested by LaValle and Kuffner [2]. Glassman and Tedrake used an affine version of LQR to estimate kinodynamic distance between a random sample and vertices in an RRT [3]. Perez et al. [4] extended that idea to the RRT* setting using an infinite-horizon LQR controller that both estimated the cost and calculated trajectories for extending the tree. Unfortunately, due to its use of an infinite horizon LQR controller, this steering method does not always yield locally optimal trajectories for the finite-time extensions (even for linear systems). Webb and van den Berg [5] used a finite-horizon optimal controller to calculate the extension trajectories, obtaining the optimal finite time-horizon by finding the roots of a high-order polynomial. In conjunction with RRT*, this steering method was proven to converge to optimal solutions for linear systems problems with indefinite time horizons. Unfortunately, this method requires a very specific type of quadratic cost function. Moreover, it is not clear how it could be extended to non-linear problems, where the root-finding method would not apply.

We propose a new method for automatically deriving the cost metric and steering method for general kinodynamic systems using affine LQR. We avoid the problem of determining the optimal time horizon to use for LQR when extending the tree by including time as an additional dimension of the space in which the tree grows. This approach is sometimes used to solve problems with dynamic obstacles environments [**?**], [6]. In our case, the purpose is to sample from the space of possible trajectory durations. Although this can increase the asymptotic computational complexity of the algorithm, it eliminates the need to calculate an optimal extension time horizon using special-purpose tools, while remaining applicable to We show that this method trivially satisfies Karaman and Frazzoli's requirements for probabilitic optimality of the ¡¡¡¡¡¡¡ Updated upstream resulting trajectory, when the system has linear dynamics and quadratic costs. We also show how to use locally linear dynamic and locally quadratic cost approximations to apply the method to general kinodynamic systems. We provide simulations illustrating the behavior of the algorithms for both linear systems with quadratic costs, and in more general settings.

II. Background

III. Background

A. Problem Statement

Given a deterministic system with known process dynamics, the optimal kinodynamic motion planning problem is to find a trajectory that begins in a given start state, terminates in a goal region, avoids obstacles, and satisfies differential dynamics constraints. Let the state space and control input space of the system be described by compact sets, $X \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^m$, respectively. The system process dynamics are described by a continuously differentiable function,

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

where $x(t) \in X$ and $u(t) \in U$. Given a time horizon, $T \in \mathbb{R}_{>0}$, a dynamically feasible trajectory is a continuous function, $x : [0, T] \to X$, for which there exists a control function, $u : [0, T] \to U$, that satisifies Equation 8. The optimal kinodynamic motion planning problem is to find a dynamically feasible trajectory starts in an initial state, $x_0 \in X$, avoids obstacles, $X_{obs} \subset X$, and terminates in a goal region, $x_{goal} \subset X$ while minimizing a cost functional of Equation 9.

Problem 1: (Optimal kinodynamic motion planning [7])

Given a state space, X, obstacle region, X_{obs}, goal region, X_{goal}, control input space, U, and function f describing the system process dynamics, find a control, $u : [0, T] \rightarrow U$, for some $T \in \mathbb{R}_{>0}$ such that the corresponding dynamically feasible trajectory, $x : [0, T] \rightarrow X$, avoids the obstacle region, reaches the goal region, and minimizes the cost functional

$$J(x, u) = \int_{t=0}^{T} g(x(t), u(t)). \tag{2}$$

B. RRT*

The Optimal Rapidly-Exploring Random Tree (RRT*) [1] is a version of the RRT algorithm [2] that has the *asymptotic optimality* property, i.e., almost-sure convergence to an optimal solution.

The algorithmic primitives are:

- *Random sampling*: The `Sample` procedure provides independent uniformly distributed random samples of states from the configuration space.
- *Nearest nodes*: Given a set V of vertices in the tree and a state x, the $\texttt{Nearest}(V, x)$ procedure provides the state in V that is closest to x.
- *Near nodes*: Given a set V of vertices in the tree and a state x, the $\texttt{Near}(V, x)$ procedure provides a set of states in V that are close to x. Here, closeness is measured with respect to some metric as follows:

  $$\texttt{Near}(V, x) := \left\{ x' \in V \ : \ \|x - x'\| \leq \gamma \left(\frac{\log n}{n} \right)^{1/d} \right\},$$

  where $\| \cdot \|$ is the distance using the metric, n is the number of vertices in the tree, d is the dimension of the space, and γ is a constant.
- *Steering*: Given two states x, x', the $\texttt{Steer}(x, x')$ procedure returns a path σ that connects x and x'. Most implementations connect two given states with a straight path in configuration space.
- *Collision checking*: Given a path σ, $\texttt{CollisionFree}(\sigma)$ returns true if the path lies in the obstacle-free portion of configuration space.

Algorithm 1: $\texttt{RRT}^*((V, E), N)$

```
1  for i = 1, ..., N do
2      x_rand ← Sample;
3      x_nearest ← Nearest(V, x_rand);
4      (x_new, σ_new) ← Steer(x_nearest, x_rand) ;
5      if CollisionFree(σ_new) then
6          X_near ← Near(V, x_new);
7          c_min ← ∞; x_min ← NULL; σ_min ← NULL;
8          for x_near ∈ X_near do
9              σ ← Steer(x_near, x_new);
10             if Cost(x_near) + Cost(σ) < c_min then
11                 c_min ← Cost(x_near) + Cost(σ);
12                 x_min ← x_near; σ_min ← σ;
13         V ← V ∪ {x_new};
14         E ← E ∪ {(x_min, x_new) };
15         (V, E) ← Rewire( (V, E), X_near, x_new );
16 return G = (V, E);
```

Algorithm 2: $\texttt{Rewire}((V, E), X_{\text{near}}, x_{\text{new}})$

```
1  for x_near ∈ X_near do
2      σ ← Steer(x_new, x_near) ;
3      if Cost(x_new) + Cost(σ) < Cost(x_near) then
4          if CollisionFree(σ) then
5              x_parent ← Parent(x_near);
6              E ← E \ {x_parent, x_near};
7              E ← E ∪ {x_new, x_near};
8  return (V, E);
```

The algorithm proceeds as follows. Firstly, a state is sampled, denoted as x_{rand}, from the configuration space (Line 1), then, the nearest vertex is extended torwards this sample (Lines 2-3). The resulting trajectory is denoted as σ_{new} and its final state as x_{new} (Line 2). If no collision is found in this trajectory, the `Near` function is invoked to calculate the set of vertices close to x_{new} (Line 6). Then, x_{min}, the vertex in the set X_{near} that reaches x_{new} with minimum cost, is returned along with the path (Lines 7-12). The algorithm then connects x_{min} to x_{new}, and attempts to "rewire" the vertices in X_{near} using the `Rewire` procedure (Algorithm 2). The `Rewire` procedure attempts to connect x_{new} to each vertex in X_{near}. The x_{new} vertex is made the parent of a vertex in X_{near} if the trajectory connecting x_{new} to the vertex does so by incurring less cost than that of its current parent.

C. Affine LQR

Linear quadratic regulation (LQR) is an optimal control technique that efficiently calculates an optimal policy for linear systems with quadratic cost functions. This paper uses finite-horizon LQR that solves the following problem: given deterministic linear process dynamics,

$$\dot{x}(t) = Ax(t) + Bu(t)), \tag{3}$$

find a state and control trajectory, x and u, that minimizes the cost functional,

$$\begin{aligned} J_{LQR}(x, u) =& x(T)^T Q_F x(T) \\ &+ \int_{t=0}^{t=T} x(t)^T Q x(t) + u(t)^T R u(t), \end{aligned} \tag{4}$$

where we have initial and final value constraints on the state trajectory, $x(0) = x_0$ and $x(T) = x_F$. The linear quadratic regulator solves for the optimal solution to the above problem in two steps. First, it calculates the optimal value function by integrating the differential Riccati equation backward in time

starting at the final time T with $P(T) = Q_F$ and integrating toward 0:

$$-\dot{P}(t) = A^T P(t) + P(t)A - P(t)BR^{-1}B^T P(t) + Q.$$

Second, LQR calculates the optimal action to take at time $t \leq T$ using:

$$u(t) = -R^{-1}B^T Px.$$

The standard LQR formulation requires process dynamics to be linear and the goal state to be at the origin (*i.e.* the state cost function to be measured with respect to the origin). However, our RRT application requires goals in locations different than the point about which the system is linearized. Also, we allow for affine process dynamics. Specifically, suppose that the process dynamics are now

$$\dot{x}(t) = Ax(t) + Bu + c, \tag{5}$$

and instead of minimizing Equation 11, suppose that our goal is now to minimize

$$J_{aff}(x,u) = (x(T) - x^*)^T Q_F (x(T) - x^*) + \int_{t=0}^{t=T} (x(t) - x^*)^T Q(x(t) - x^*) + u(t)^T Ru(t). \tag{6}$$

The difficulty is that the cost function is no longer a quadratic form. It turns out that this problem can easily be solved by reformulating the problem using a change of coordinates in state space and an augmentation of the state vector. Redefine state to be $\bar{x}(t) = x(t) - x^*$, and augment the new state vector with an additional coordinate that will always be equal to one. The new process dynamics are:

$$\begin{pmatrix} \dot{\bar{x}}(t) \\ 1 \end{pmatrix} = \begin{pmatrix} A & c + Ax^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x}(t) \\ 1 \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u.$$

It may be verified that these process dynamics are the same as those in Equation 10 except that they are now expressed in terms of the new variable. We can now express the offset cost functional in Equation 14 as:

$$J(\bar{x},u) = \begin{pmatrix} \bar{x}(T) \\ 1 \end{pmatrix} \bar{Q}_F \begin{pmatrix} \bar{x}(T) \\ 1 \end{pmatrix} + \int_{t=0}^{t=T} \begin{pmatrix} \bar{x}(t) \\ 1 \end{pmatrix} \bar{Q} \begin{pmatrix} \bar{x}(t) \\ 1 \end{pmatrix} + u(t)^T Ru(t), \tag{7}$$

where $\bar{Q}_F = \begin{pmatrix} Q_F & 0 \\ 0 & 1 \end{pmatrix}$, and $\bar{Q} = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}$ and we solve the affine version of the problem by applying LQR to the augmented system in the standard way.

IV. Kinodynamic Planning for Systems with Affine Dynamics and Quadratic Costs

======= resulting trajectory. Most importantly, we believe that the method can be extended directly to non-linear systems with similar optimality guarantees. We provide simulations illustrating the behavior of the algorithms for both linear and non-linear systems.

V. Background

A. Problem Statement

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$$J(x,u) = \int_{t=0}^{T} g(x(t), u(t)). \tag{9}$$

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  $$\texttt{Near}(V,x) := \left\{ x' \in V \ : \ \|x - x'\| \leq \gamma \left(\frac{\log n}{n} \right)^{1/d} \right\},$$

  where $\|\cdot\|$ is the distance using the metric, n is the number of vertices in the tree, d is the dimension of the space, and γ is a constant.
- *Steering*: Given two states x, x', the $\texttt{Steer}(x,x')$ procedure returns a path σ that connects x and x'. Most

implementations connect two given states with a straight path in configuration space.

- *Collision checking*: Given a path σ, $\texttt{CollisionFree}(\sigma)$ returns true if the path lies in the obstacle-free portion of configuration space.

Algorithm 3: $\texttt{RRT}^*((V, E), N)$

1 **for** $i = 1, \ldots, N$ **do**
2 $x_{\text{rand}} \leftarrow \texttt{Sample}$;
3 $x_{\text{nearest}} \leftarrow \texttt{Nearest}(V, x_{\text{rand}})$;
4 $(x_{\text{new}}, \sigma_{\text{new}}) \leftarrow \texttt{Steer}(x_{\text{nearest}}, x_{\text{rand}})$;
5 **if** $\texttt{CollisionFree}(\sigma_{\text{new}})$ **then**
6 $X_{\text{near}} \leftarrow \texttt{Near}(V, x_{\text{new}})$;
7 $c_{\min} \leftarrow \infty$; $x_{\min} \leftarrow \texttt{NULL}$; $\sigma_{\min} \leftarrow \texttt{NULL}$;
8 **for** $x_{\text{near}} \in X_{\text{near}}$ **do**
9 $\sigma \leftarrow \texttt{Steer}(x_{\text{near}}, x_{\text{new}})$;
10 **if** $\texttt{Cost}(x_{\text{near}}) + \texttt{Cost}(\sigma) < c_{\min}$ **then**
11 $c_{\min} \leftarrow \texttt{Cost}(x_{\text{near}}) + \texttt{Cost}(\sigma)$;
12 $x_{\min} \leftarrow x_{\text{near}}$; $\sigma_{\min} \leftarrow \sigma$;
13 $V \leftarrow V \cup \{x_{\text{new}}\}$;
14 $E \leftarrow E \cup \{(x_{\min}, x_{\text{new}})\ \}$;
15 $(V, E) \leftarrow \texttt{Rewire}(\ (V, E), X_{\text{near}}, x_{\text{new}}\ )$;
16 **return** $G = (V, E)$;

Algorithm 4: $\texttt{Rewire}(\ (V, E), X_{\text{near}}, x_{\text{new}}\ )$

1 **for** $x_{\text{near}} \in X_{\text{near}}$ **do**
2 $\sigma \leftarrow \texttt{Steer}(x_{\text{new}}, x_{\text{near}})$;
3 **if** $\texttt{Cost}(x_{\text{new}}) + \texttt{Cost}(\sigma) < \texttt{Cost}(x_{\text{near}})$ **then**
4 **if** $\texttt{CollisionFree}(\sigma)$ **then**
5 $x_{\text{parent}} \leftarrow \texttt{Parent}(x_{\text{near}})$;
6 $E \leftarrow E \setminus \{x_{\text{parent}}, x_{\text{near}}\}$;
7 $E \leftarrow E \cup \{x_{\text{new}}, x_{\text{near}}\}$;
8 **return** (V, E);

The algorithm proceeds as follows. Firstly, a state is sampled, denoted as x_{rand}, from the configuration space (Line 1), then, the nearest vertex is extended torwards this sample (Lines 2-3). The resulting trajectory is denoted as σ_{new} and its final state as x_{new} (Line 2). If no collision is found in this trajectory, the `Near` function is invoked to calculate the set of vertices close to x_{new} (Line 6). Then, $x_{\min}$, the vertex in the set X_{near} that reaches x_{new} with minimum cost, is returned along with the path (Lines 7-12). The algorithm then connects $x_{\min}$ to x_{new}, and attempts to "rewire" the vertices in X_{near} using the `Rewire` procedure (Algorithm 2). The `Rewire` procedure attempts to connect x_{new} to each vertex in X_{near}. The x_{new} vertex is made the parent of a vertex in X_{near} if the trajectory connecting x_{new} to the vertex does so by incurring less cost than that of its current parent.

C. Affine LQR

Linear quadratic regulation (LQR) is an optimal control technique that efficiently calculates an optimal policy for linear systems with quadratic cost functions. This paper uses finite-horizon LQR that solves the following problem: given deterministic linear process dynamics,

$$\dot{x}(t) = Ax(t) + Bu(t)), \tag{10}$$

find a state and control trajectory, x and u, that minimizes the cost functional,

$$J_{LQR}(x, u) = x(T)^T Q_F x(T) + \int_{t=0}^{t=T} x(t)^T Q x(t) + u(t)^T R u(t), \tag{11}$$

where we have initial and final value constraints on the state trajectory, $x(0) = x_0$ and $x(T) = x_F$. The linear quadratic regulator solves for the optimal solution to the above problem in two steps. First, it calculates the optimal value function by integrating the differential Riccati equation backward in time starting at the final time T with $P(T) = Q_F$ and integrating toward 0:

$$-\dot{P}(t) = A^T P(t) + P(t)A - P(t)BR^{-1}B^T P(t) + Q.$$

Second, LQR calculates the optimal action to take at time $t \leq T$ using:

$$u(t) = -R^{-1}B^T P x.$$

The standard LQR formulation requires process dynamics to be linear and the goal state to be at the origin (*i.e.* the state cost function to be measured with respect to the origin). However, our RRT application requires goals in locations different than the point about which the system is linearized. Also, we allow for affine process dynamics. Specifically, suppose that the process dynamics are now

$$\dot{x}(t) = Ax(t) + Bu + c, \tag{12}$$

and instead of minimizing Equation 11, suppose that our goal is now to minimize

$$J_{aff}(x, u) = (x(T) - x^*)^T Q_F (x(T) - x^*) + \int_{t=0}^{t=T} (x(t) - x^*)^T Q (x(t) - x^*) + u(t)^T R u(t). \tag{13}$$

The difficulty is that the cost function is no longer a quadratic form. It turns out that this problem can easily be solved by reformulating the problem using a change of coordinates in state space and an augmentation of the state vector. Redefine state to be $\bar{x}(t) = x(t) - x^*$, and augment the new state vector with an additional coordinate that will always be equal to one. The new process dynamics are:

$$\begin{pmatrix} \dot{\bar{x}}(t) \\ 1 \end{pmatrix} = \begin{pmatrix} A & c + Ax^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x}(t) \\ 1 \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u.$$

It may be verified that these process dynamics are the same as those in Equation 10 except that they are now expressed in terms of the new variable. We can now express the offset cost functional in Equation 14 as:

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where $\bar{Q}_F = \begin{pmatrix} Q_F & 0 \\ 0 & 1 \end{pmatrix}$, and $\bar{Q} = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}$ and we solve the affine version of the problem by applying LQR to the augmented system in the standard way.

VI. LQR-RRT* FOR AFFINE SYSTEMS

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The key idea of this paper is to solve kinodynamic RRT* problems using LQR by sampling in time as well as in state space. A tree is constructed where each vertex has an associated state *and* time. Each vertex is also associated with the one-step cost associated with reaching that vertex from its parent.

A. The Kinodynamic Planning Problem with Affine Dynamics and Quadratic Costs

Consider the version of Problem 2 where the system process dynamics (Equation 8) are linear or affine and where the cost functional (Equation 9) is quadratic. Notice that except for the presence of obstacles, this would be a linear problem solvable using LQR. Specifically, we require the process dynamics to be affine (of the form in Equation 12) and the cost function to be of the form,

$$J(x,u) = \int_{t=0}^{T} (x(t)-x^*)^T Q (x(t)-x^*) + u(t)^T R u(t). \tag{15}$$

The obstacle region, X_{obs}, may be of any shape. We consider two instances of the problem: a) where the time horizon, T, is fixed and given as input, and b) where the time horizon is free to vary in order to minimize cost (*i.e.* T is an optimization variable).

B. Extensions to the Tree

First, suppose that we are given a final time constraint, T, at which the trajectory must end. We sample points from state-time, $(x_{rand}, t_{rand}) \sim X \times T$. This happens on each iteration of the FOR loop in Algorithm 3. Given this sample, we calculate the affine LQR solution by integrating the differential Riccati equation for the affine dynamics backward in time to 0 starting from t_{rand} with the quadratic cost function centered at the sample point, $x^* = x_{rand}$ (see Section V-C). The LQR cost function used in the Riccati integration has Q and R equal to their values in the global quadratic cost function. Q_F is set to a multiple of the identity matrix that is sufficiently large to cause LQR to find trajectories that terminate very close to the random sample [1]. The result of this integration will be written as a function, $P : [0,T] \rightarrow S_+^{n+1}$ (the result of the Riccati integration is always a positive semi-definite matrix).

RRT* begins with the standard RRT tree extension. We calculate the cost of traveling to the sampled state by the sampled time from each of the vertices in the tree (this is implemented in the **Near** function of Algorithm **??**). This can be accomplished by evaluating the LQR cost-to-go function for each vertex, $v_i = (x_i, t_i) \in V$, in the tree:

$$c_1(v_i, x_{rand}) = (x_{rand} - x_i)^T P(t_i)(x_{rand} - x_i).$$

We define the cost to any vertex with a vertex time larger than the sample time, $t_i \geq t_{rand}$, to be infinite. Without loss of generality, we assume the time associated with the root vertex to be 0 (non-zero start times can be handled by shifting all times forward). As a result, there is always at least one vertex with a time smaller than the sample time. Once the vertex "closest" to the sample is found, we generate a trajectory between (x_i, t_i) and (x_{rand}, t_{rand}) using LQR. We check whether this trajectory intersections the obstacle region or not. If it does intersect, then we discard the trajectory and re-sample.

C. Tree Rewiring

In the context of a kinodynamic problem, once the tree has been extended, RRT* perform two re-wiring steps [7]. In the first, we identify the set of vertices for which

$$c_1(v_i, x_{rand}) \leq \gamma \left(\frac{\log n}{n} \right)^{-\frac{1}{d}}. \tag{16}$$

For each vertex in this set, we calculate the total cost to the root by tracing the tree back to the root and summing the costs. After ranking these vertices in order of cost, we start at the top of this list and attempt to connect the vertex to the random sample. If the resulting trajectory intersects an obstacle or if LQR does not terminate sufficiently close to the sample point, that connection is discarded and the algorithm moves down the list to the vertex associated with the next smallest cost. Once a successful trajectory is found, the tree is correspondingly re-wired.

Similarly, RRT* searches for vertices that can be reached *from* the sample and re-wires as necessary. Costs are calculated in the same way as above. This time, it is necessary to integrate the differential Riccati equation backward from each vertex in the tree. Instead of requiring this integration to occur for the entire tree on each re-wire step, we store the backward Riccati integration for each vertex in the tree when it is originally added and access the appropriate P function as necessary. Only vertices within the cost threshold of Equation 16 are evaluated.

[1] In principle, it is possible to integrate an inverse form of the differential Riccati equation that allows setting Q_F to infinity (*i.e.* $Q_F^{-1} = 0$). However, we have found this approach to be numerically troublesome and recommend setting Q_F to a large value that is determined to work well in practice.

D. Optimality

Recognizing the optimality of the proposed approach for affine systems with quadratic costs is straightforward. Karaman and Frazzoli [7] proved that the optimality guarantee for RRT* in kinodynamic systems holds under two conditions.

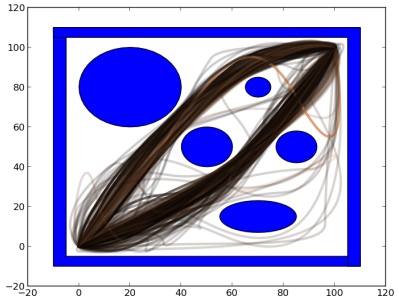
First, the estimate of cost used during re-wiring as well as the trajectory generation itself must be optimal in the absence of obstacles, for all trajectories below some length $\bar{\epsilon}$ [7]. Since our affine LQR formulation solves this problem exactly for any time horizon and any length of trajectory, this requirement is trivially satisfied.

Second, the domain must satisfy a weakened local controllability criterion (Assumption 3 in Karaman and Frazzoli [7]) and an ϵ-collision-free approximate trjectories criterion (Assumption 4 in Karaman and Frazzoli [7]). The addition of time as a dimension does not invalidate these criteria if they already hold for the un-augmented system.

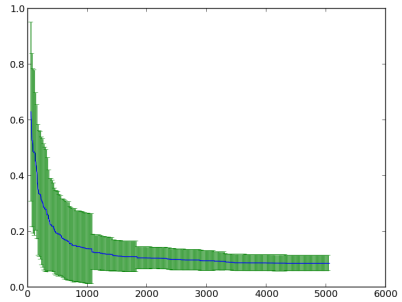
E. Experiments

We evaluated our approach to affine systems control using a two-dimensional double integrator (four dimensional state space). This system has a unit mass and a damping factor of 0.1. The planning problem is to reach a goal state at $(8, 0)$ with zero velocity 15 seconds after starting at $(0, 0)$ with zero velocity while minimizing a cost function (Equation 15) with $Q = 0$ and $R = I$. Similar evaluations of RRT* performance for the double integrator have appeared in [7] and [5]. Figure 1 illustrates the working of the algorithm. Initially, the algorithms finds a trajectory (Figure 1(a)) with cost 209. The algorithm found seven more trajecories to the goal with successively smaller costs: 145, 65, 17, 15, 16, 11, 9.

We also evaluated average performance over multiple runs of the algorithm (Figure 2). Figure 2(a) shows the set of best paths during 50 different runs of the algorithm for the double integrator domain shown. Notice that the different runs of the algorithm tend to identify two different homotopies that have nearly equal costs. Figure 2(b) shows the mean and standard deviations of the best-cost paths as a function of algorithm iteration.



(a)



(b)

Fig. 2. (a) Best-cost trajectories found during 50 separate runs of the algorithm. Darkness of the line is inversely proportional to trajectory cost. (b) Average and standard deviations of the costs of the best trajectories as a function of algorithm iteration number.

VII. Application to General Dynamical Systems

Consider a discrete-time dynamical system in the form

$$x_{k+1} = f(x_k, u_k) \tag{17}$$

with additive cost function

$$J(\mathbf{u}, \mathbf{x}) = \sum_{k=0}^{T} g(x_k, u_k) \tag{18}$$

and starting point x_0. The state vector, x, is n-dimensional and the control vector, u, is m-dimensional. We aim to find a sequence $\mathbf{u} = \{u_0, \cdots, u_T\}$ which induces a trajectory $\mathbf{x} = \{x_1, \cdots, x_T\}$ satisfying the dynamics (17) such that C is minimized according to (18).

The real cost of moving from point x' to x'' is

$$C(x, x') = \min_{\mathbf{u}} J(\mathbf{u}, \mathbf{x})$$
$$\text{subject to (17), } x_0 = x,\ x_T = x'$$

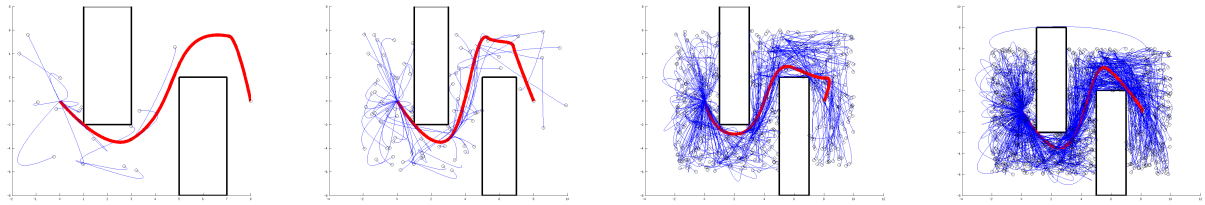
Note that the minimization happens over control sequences $\mathbf{u}$ of a fixed time lengths, according to

We approximate $C(x', x'')$ by taking a first-order approximation of the dynamics and a second-order approximation of the cost and applying LQR control. In general, the approximated dynamics and cost are of the following form

$$x_{k+1} \approx Ax_k + Bu_k + c \tag{19}$$

$$J(\mathbf{u}, \mathbf{x}) \approx \sum_{k=0}^{T} x_k^T Q x_k + u_k^T R u_k + 2q^T x_k + 2r^T u_k + d \tag{20}$$

A and Q are $n \times n$, B is $m \times n$, R is $n \times n$. c and q are $n \times 1$, r is $m \times 1$ and d is a scalar.



(a) (b) (c) (d)

Fig. 1. Example of using LQR-RRT* to find an optimal solution to the kinodynamic motion planning for the double integrator in the presence of obstacles. The four images show the tree at different points during growing. The blue lines show the tree. The red path denotes the current best path to the goal. The tree in (a) contains 21 vertices; in (b) contains 85 vertices; in (c) contains 485 vertices; and in (d) contains 1049 vertices. The costs of the four best trajectories, from left to right, are 209, 17, 16, and 9.

$$A = \left.\frac{\partial f}{\partial x}\right|_{x^*,u^*}$$
$$B = \left.\frac{\partial f}{\partial u}\right|_{x^*,u^*}$$
$$c = -Ax^* - Bu^* + f(x^*, u^*)$$

x^*, u^* is the point about which the linearization is performed. Typically u^* is taken to be $\mathbf{0}$ and $x^* = x'$

Equations 19 and 20 are the truncated Taylor expansions of f and g. The dynamics f must be once-differentiable and addition cost g must be twice-differentiable.

A. Reduction to the Previous Problem

It is possible to transform the problem specified with 19 and 20 into LQR form (where there is only an A, B, Q, R matrix) using the following:

$$\underbrace{\begin{bmatrix} x_{k+1} \\ 1 \end{bmatrix}}_{\hat{x}_{k+1}} = \underbrace{\begin{pmatrix} A & c - R^{-1}r \\ 0 & 1 \end{pmatrix}}_{\hat{A}} \underbrace{\begin{bmatrix} x_k \\ 1 \end{bmatrix}}_{\hat{x}_k} + \underbrace{\begin{pmatrix} B \\ 0 \end{pmatrix}}_{\hat{B}} \hat{u}_k$$

$$C(\mathbf{u}, \mathbf{x}) = \sum_{k=0}^{T} \hat{x}_k^T \hat{Q} \hat{x}_k + \hat{u}_k^T R \hat{u}_k$$

with $\hat{u}_k = u_k + R^{-1}r$ and $\hat{Q} = \begin{pmatrix} Q & q \\ q^T & d \end{pmatrix}$.

The $\hat{A}$, $\hat{B}$, $\hat{Q}$, and R matrices specify a linear dynamical system with quadratic costs to which an optimal solution can be found with LQR.

B. Nuances and Subtleties

1) Non-exact steering: rewiring and propagating dynamics

2) Uncontrollable Dynamics: The linearized system may be uncontrollable – the A and B matrices are such that it's not possible to control all the modes of the system. This is the case, for example, for a cart with two inverted pendulums of the same length linearized about the upward-pointing fixed point. The control input to the system affects both linearized pendulums in the same way, so it's not possible to independently stabilize them. For the infinite-horizon LQR control problem, there is no solution. For the finite-horizon problem, there is a solution, though it might not be possible to go to any arbitrary location. If the system linearized at x' cannot reach x'', then $C(x', x'')$ needs to be defined in another way.

is Therefore using the LQR cost metric cannot approximate the cost

3) Indefinite Cost:

4) Actuation Constraints: The LQR framework does not permit actuation constraints.

5) Asymmetric Cost:

C. Non-linear Domain

spaceship orientation

D. Results

Quick mention of performance (or not). Picture of tree, cost over iteration

VIII. Discussion and Conclusion and Future Work

Code available. Spatial Data structure future

IX. Acknowledgments

X. References

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