# The Very Unfriendly Seating Arrangement Problem

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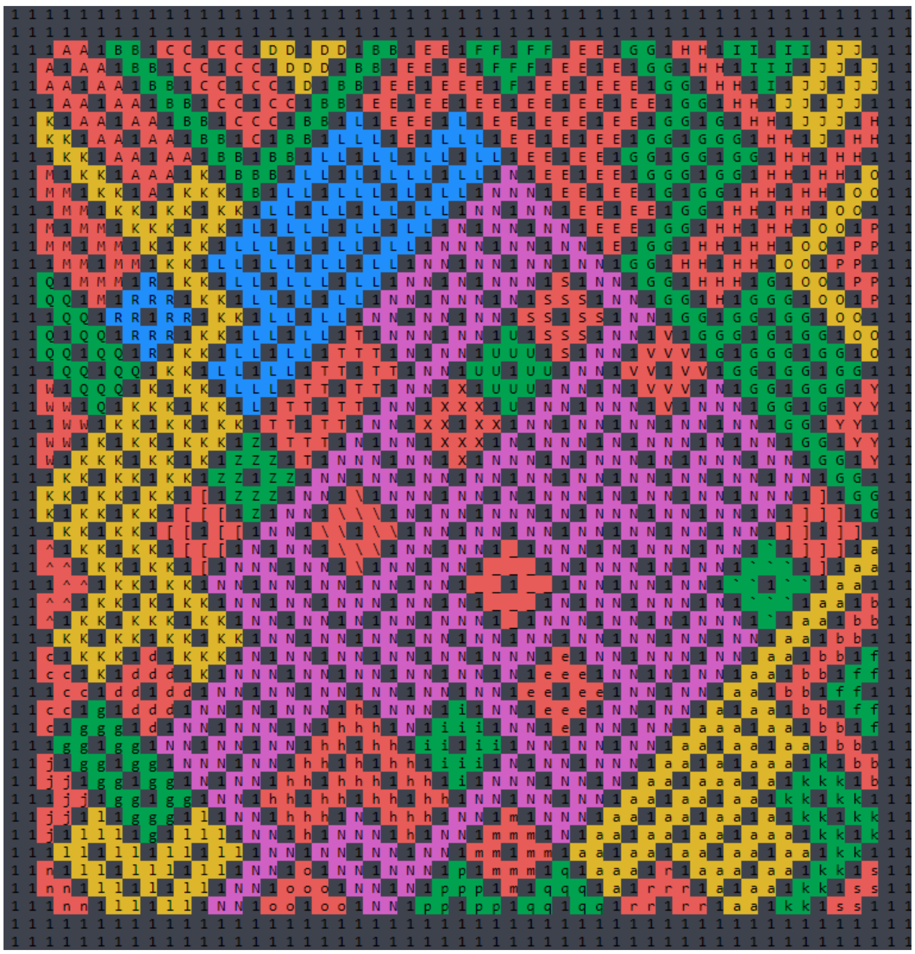


**Figure 1:** One random seating and greedy 5-coloring on the $49 \times 49$ grid. In this case, the black squares (1s) represent occupied spaces and the other spaces are unoccupied. Two extra layers of 1s surround the grid to simplify programming.

# Contents

# 1 Abstract

The unfriendly seating arrangement problem involves people randomly sitting down in a rectangular grid of seats, such that no new person will choose a seat adjacent to someone else, until no more people can be seated [4]. We require that the first arrival choose a corner seat, and that new arrivals choose seats that maximize their distance away from the nearest person (hence "very" unfriendly). Under this behavior, we show that $1 \times n$ grids are always invariant, meaning the same number of people will always be seated. For $n = 2^k + 1$, we show that $n \times n$ grids are invariant and both $1 \times n$ and $n \times n$ grids are optimally seated. For $n = 3 \cdot 2^k + 1$, we show that $1 \times n$ grids are minimally seated, and $n \times n$ grids are invariant but not minimally seated. We demonstrate how these $n = 3 \cdot 2^k + 1$ grids generate labyrinthine patterns, and classify the symmetries of these patterns for $k = 0$ and $k = 1$. We also define the construction of plane graphs corresponding to these patterns, and show that some of these graphs are not well-colored.

# 2 Introduction & literature review

Even before the coronavirus pandemic of 2020, social distancing has been practiced in certain scenarios. Consider a public seating space, such as a row of seats on a bus or a line of urinals in a restroom. In cases such as these, it is common for the first person who arrives at such a row to choose a seat on one of the row's ends (e.g., to be closest to the door or furthest from someone at the sink), and for each subsequent arrival to stay as far as possible from the others. This behavior can cause such spaces to fill up quickly, however. A well-known example is the case of seven urinals; the left- and right-most urinals are taken first and second, followed by the center. Now, the fourth person cannot choose a urinal which is not adjacent to another person, and is quite likely to just wait for the others to leave (see **Figure 2**). If the restroom had five urinals instead, people would still likely fill the first, last, and middle urinal, so the seven-urinal case is a waste of two urinals, in a sense.

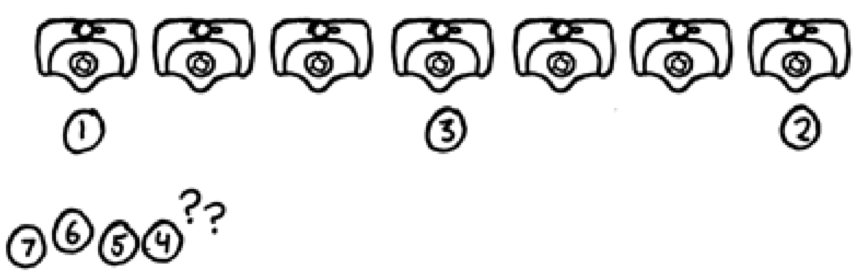



**Figure 2:** A row of 7 urinals will likely be used by only 3 people at once [8].

This paper will focus on finding which lengths of rows fill the highest or lowest percent of seats possible, assuming very unfriendly behavior. We will also generalize this behavior to two-dimensional seating spaces and arrive at similar results. In certain cases of square grids, we will see intricate patterns arise, such as in **Figure 1**, and we will use graph theory to classify and color these patterns.

The original unfriendly seating arrangement problem was posed by Freedman and Schepp in 1962 [4]. This version allowed arrivals to choose any spot at random, so long as no two people sat adjacent. Two years later, Friedman and Rothman showed that the expected fraction of entries filled in a $1 \times n$ grid goes to $\frac{1}{2} - \frac{1}{2e^2}$ as $n$ tends to infinity [5] . Georgiou, Kranakis & Krizanc showed that the fraction goes to $\frac{1}{2} - \frac{1}{4e}$ for a $2 \times n$ grid [6]. Chern, Hwang and Tsai investigated the application of Riccati equations to this problem [3]. Kranakis and Krizanc give a more in-depth analysis of the $1 \times n$ case, including a brief description of very unfriendly behavior (under the name "maximize-your-distance behavior"). Under this behavior, they describe which seats have optimal privacy for the longest duration, and mention that $n$ of the form $3 \cdot 2^k + 1$, $k \in N_0$, results in the most room for each person in the final arrangement [7]. Such a result also seats the lowest possible percent of people, so these numbers are called minimal in this paper. XKCD describes this case, as well as the optimal case $n = 2^k + 1$, and even gives a closed form for the number of people that will be seated [8]. This paper will focus on providing rigorous proofs to these previous observations, and new exploration into the $n \times n$ cases.

# 3 Initial definitions

**Definition 3.1.** *Let $m, n \in \mathbb{N}^+$. An $\underline{m \times n \text{ grid}}$ is a matrix, which is a function $G : X \to Y$, where*

$$X = \{(i, j) \mid i \in \{0, ..., m-1\}, j \in \{0, ..., n-1\}\}.$$

*For this paper, $Y = \{0, 1\}$ unless stated otherwise. The image of $(i, j)$ under $G$ is called the $\underline{\text{entry}}$ of $G$ at $(i, j)$, denoted $G(i, j)$.*

Note: This is the convention for array coordinates in Python. By this convention, the top row is the $0^{\text{th}}$ row ($i = 0$), and the leftmost column is the $0^{\text{th}}$ column ($j = 0$).

**Definition 3.2.** *Given a grid $G$ and two entries $a = G(x_1, y_1)$, $b = G(x_2, y_2)$ in $G$, the $\underline{\text{distance between a and b}}$, denoted $d(a, b)$ or $d((x_1, y_1), (x_2, y_2))$ is given by the Euclidean distance formula,*

$$d(a, b) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

**Definition 3.3.** *Two entries $a, b$ of a grid $G$ are called $\underline{\text{adjacent}}$ if $d(a, b) = 1$.*

**Definition 3.4.** *Given a grid $G$, Filling an entry $G(i, j)$, refers to replacing $G(i, j)$ with 1, i.e., editing the function $G$ so that the image of $(i, j)$ is 1. If $G(i, j) = 1$, then $G(i, j)$ is called filled. If $G(i, j)$ is not filled, then it is called empty.*

**Definition 3.5.** *A grid $G$ is called empty every entry of $G$ is empty, i.e.,*

$$\forall i \in \{0, ..., m-1\}, \forall j \in \{0, ..., n-1\},\ G(i, j) \neq 1$$

**Definition 3.6.** *Given an $m \times n$ grid $G$ and an entry $G(x, y)$, the room at $G(x, y)$, denoted $room_G(x, y)$, is equal to the distance between $G(x, y)$ and the nearest filled entry in $G$, given by*

$$room_G(x, y) = min\{d((x, y), (i, j)) \mid G(i, j) = 1\}$$

*If $G$ is empty, then each entry in $G$ is said to have infinite room.*

**Definition 3.7.** *An $m \times n$ grid $G$ is called saturated if every empty entry in $G$ has room 1, i.e.,*

$$\forall i \in \{0, ..., m-1\}, \forall j \in \{0, ..., n-1\},\ G(i, j) = 1 \text{ or } room_G(i, j) = 1$$

**Definition 3.8.** *An $m \times n$ grid $G$ is called comfortable if no two filled entries of $G$ are adjacent, i.e.,*

$$\forall i \in \{0, ..., m-1\}, \forall j \in \{0, ..., n-1\} \text{ s.t. } G(i, j) = 1,\ room_G(i, j) > 1$$

**Definition 3.9.** *The very unfriendly seating algorithm (VUSA) is an algorithm which fills entries of an $m \times n$ grid $G$ to give a saturated $m \times n$ grid $G'$. The algorithm operates as follows:*

1. *If $G$ is saturated, end the algorithm.*
2. *If $G$ is empty, let $C$ be the set of corners of $G$, i.e.,*

   $$C = \{(x, y) \mid x \in \{0, m-1\}, y \in \{0, n-1\}\}$$

   *Randomly choose one coordinate pair $(i_0, j_0)$ from $C$. Fill $G(i_0, j_0)$. This step is called the corner-first convention.*
3. *Let $r = max\{room_G(i, j) \mid i \in \{0, ..., m-1\}, j \in \{0, ..., n-1\}, G(i, j) \neq 1\}$.*
4. *Randomly choose one coordinate pair $(i, j)$ from the set $R = \{(x, y) \mid G(x, y) \neq 1,\ room_G(x, y) = r\}$. Fill $G(i, j)$.*
5. *If $G$ is saturated at this point, terminate the algorithm. Otherwise, there is at least one entry where a 0 is not adjacent to a 1, so return to step 3.*

In this paper, we will usually apply the VUSA to an empty grid. Otherwise, the originally filled entries of $G$ do influence the VUSA, but they are not considered filled by the VUSA.

**Theorem 3.1.** *Let $G$ be an $m \times n$ grid, and let $G'$ be any result of the VUSA applied to $G$. Then $G'$ is comfortable if and only if $G$ is comfortable.*

*Proof.* ($\Rightarrow$) Assume $G'$ is comfortable. Then the filled entries of $G$ are a subset of the filled entries of $G'$, so $G$ cannot contain two adjacent filled entries, therefore $G$ is comfortable.

($\Leftarrow$) Assume $G$ is comfortable. The VUSA cannot fill an entry with room 1, since step 1 will terminate the algorithm if the maximum room among all empty entries is 1. Thus, $G'$ will have no adjacent filled entries, i.e., $G'$ is comfortable. $\square$

**Definition 3.10.** *Given an $m \times n$ grid $G$, applying the VUSA to $G$ gives a finite random sequence $\{\psi_1, ..., \psi_u\}$ called the seating sequence of that outcome of the VUSA, where $u$ is the number of filled entries in $G'$, and $\psi_k$ is the $k^{th}$ entry $G(i_k, j_k)$ to be filled by the VUSA.*

**Definition 3.11.** *Given an $m \times n$ grid $G$, if the number of filled entries is the same for all possible outcomes of the VUSA, then we say $G$ is invariant.*

**Definition 3.12.** *Given an empty $m \times n$ grid $G$, the number of filled entries in an outcome of the VUSA on $G$ is a random variable called the unfriendly number of $G$, denoted $\Upsilon(m, n)$.*

**Definition 3.13.** *Given an empty $m \times n$ grid $G$, the maximum number of entries in $G$ that can be filled such that $G$ is comfortable is called the optimal number of $G$, denoted $\Omega(m, n)$.*

**Definition 3.14.** *Given an empty $m \times n$ grid $G$, the minimum number of entries in $G$ that can be filled such that $G$ is saturated is called the minimal number of $G$, denoted $\mu(m, n)$.*

Note: In the definitions for $\Upsilon, \Omega$, and $\mu$, it is very important that the initial grid $G$ be empty. Otherwise, these numbers would depend on which entries are chosen to be filled in the initial $G$.

# 4 One-Dimensional Grids

## 4.1 Invariance of $\Upsilon(n)$

The cases in which $G$ is a single row or column ($n = 1$ or $m = 1$) differ only by orientation, so we will illustrate such grids as a row, and express $\Upsilon(1, n)$ or $\Upsilon(n, 1)$ simply as $\Upsilon(n)$. We will also express entries of one-dimensional grids $G(0, i)$ or $G(j, 0)$ as $g_i$. In simulating the VUSA for these cases, it quickly becomes clear that every one-dimensional grid is invariant, even though the layout of the final arrangement may vary. Before we prove this property in **Theorem 4.2**, it is useful to first prove the following lemma about one-dimensional grids:

**Lemma 4.1** (**The split-row rule**). . *Let $n \geq 5$ and let $G$ be a $1 \times n$ grid $[g_0, g_1, g_2, ..., g_{n-1}]$ such that $g_k$ is filled for some $k \in \{1, ..., n-2\}$. Note that $g_k$ need not be the only filled entry of $G$. Let $H$ be the subrow $[g_0, ..., g_k]$ and let $I$ be the subrow $[g_k, ..., g_{n-1}]$. Then applying the VUSA to $G$ is equivalent to applying the VUSA to $H$ and $I$ independently.*

*Proof.* Since $g_k$ is filled in $H$, $I$, and $G$, each step of applying the VUSA to $G$ will fill an entry in $H$ or in $I$, until $G$, $H$ and $I$ are saturated. Let $g_h \in H, g_i \in I$. Then $h \leq k < i$, so $|h-k| < |h-i|$, i.e., $d(g_h, g_k) < d(g_h, g_i)$. Since $g_k$ is already filled, the room at any entry in $H$ is less than or equal to $d(g_h, g_k)$, and since $d(g_h, g_k) < d(g_h, g_i)\ \forall g_i \in I$, the room at each entry of $H$ does not depend on which entries are filled in $I$ (except for $g_k$ itself). Symmetrically, the room at each entry of $I$ does not depend on which entries are filled in $H$ (except for $g_k$ itself). Now we can take the seating sequence given by applying the VUSA to $G$, and delete each step that filled an entry of $I$. Since each step did not depend on the state of $I$, this subsequence gives the steps that the VUSA gives when applied to $H$ independently. The same is true for $I$, so the VUSA applied to $G$ fills the same entries, in the same relative order, as it does applied to $H$ and $I$ independently. $\square$

Note: When we talk about "dividing" rows, we do not mean that each subrow of the division is disjoint. In particular, we mean that each subrow of the division overlaps its neighbor (or neighbors) for exactly 1 entry per neighbor (usually a filled entry). For example, in the proof of the split-row rule, subrows $H$ and $I$ overlap only at $g_k$.

**Theorem 4.2.** *Let $G$ be an empty $1 \times n$ grid. Then $G$ is invariant.*

*Proof.* As in **Lemma 4.1**, denote $G$ as $[g_0, g_1, g_2, ..., g_{n-1}]$. If $n \leq 4$, we can manually show that $G$ is invaraint, since

$\Upsilon(1)$ is always 1: $[1]$
$\Upsilon(2)$ is always 1: $[1, 0]$ or $[0, 1]$
$\Upsilon(3)$ is always 2: $[1, 0, 1]$
$\Upsilon(4)$ is always 2: $[1, 0, 0, 1]$

So assume $n \geq 5$. In applying the VUSA to $G$, $\psi_1 = g_0$ or $\psi_1 = g_{n-1}$. Due to symmetry, we can assume $\psi_1 = g_0$ without loss of generality. After $\psi_1$ is filled, the entry of $G$ with the most room is $g_{n-1}$, so $\psi_2 = g_{n-1}$ is filled. At this point the result depends on the parity of $n$:

**Case 1** ($n$ is odd). The entry with the most room is $g_{\frac{1}{2}(n-1)}$, so $\psi_3 = g_{\frac{1}{2}(n-1)}$.

**Case 2** ($n$ is even). Two entries are tied for the most room: $g_{\frac{1}{2}n}$ and $g_{\frac{1}{2}n-1}$. Again by symmetry, we can assume $\psi_3 = g_{\frac{1}{2}n}$ without loss of generality.

In both cases, $G$ can be divided into two sub-rows $H_0 = [g_0, ..., \psi_3]$ and $H_1 = [\psi_3, ..., g_{n-1}]$. After the third step of the VUSA, $\psi_3$ is filled. Thus, by the split-row rule, the VUSA fills $H_0$ and $H_1$ independently. If $H_0$ and $H_1$ are both subrows of length 4 or less, we are done, since we manually checked that rows of this length are invariant. So assume $H_0$ and/or $H_1$ are rows of length at least 5. In this case, the next entry to be filled in $H_0$ or $H_1$ will divide that subrow into two smaller subrows, just as $G$ was divided by $\psi_3$.

The lengths of the subrows given by this dividing process do not depend on the result of random selection, since the odd subrows have only one midpoint (there is no choice) and the even subrows will have a symmetrical tie for the most room. Since $n$ is finite and this process produces sub-rows of strictly decreasing length, eventually $G$ will be divided into finitely many subrows of length 4 or less, which we know are each filled independently by the split-row rule. Due to this fact, and since we know that rows of length 4 or less are invariant, $G$ is invariant. □

In **Theorem 4.2**, we assumed that $G$ was initially empty, but this assumption is unnecessary. Now that we have proven invariance for empty grids, it is much easier to prove invariance for the general one-dimensional grid, which we will do in **Theorem 4.3**. Simply put, this proof rigorously shows that the "pockets" of empty entries in a one-dimensional grid are all invariant and filled independently.

**Theorem 4.3.** *Let $G$ be a $1 \times n$ grid. Then $G$ is invariant.*

*Proof.* If $G$ is empty, then **Theorem 4.2** states that $G$ is invariant, so we can assume $G$ is nonempty. If $G$ is saturated, then the VUSA will end without changing $G$, so $G$ is invariant. So, we can assume $G$ is not saturated, which means $G$ has at least one empty entry. In fact, $G$ contains finitely many sets of consecutive empty entries. So let $H = [g_h, ..., g_i]$ be any empty subrow of $G$, such that $g_{h-1}$ and $g_{i+1}$ are filled (if they exist). If $h = i$ then $H$ is a single empty entry with room 1, which cannot be filled by the VUSA, so $H$ is invariant. Otherwise, $H$ contains at least 2 empty entries, and there are three cases.

**Case 1** ($h = 0$). In this case, $g_{i+1}$ is the leftmost filled entry of $G$, and $H$ is the set of all empty entries to the left of it. By the split-row rule, the VUSA fills $H$ independently of the state of entries after $g_{i+1}$. Thus, the first entry of $H$ to be filled by the VUSA will be $g_0$. At this point, $H$ is contained in the subrow $[g_0, ..., g_{i+1}] = [1, 0, 0, ..., 0, 1]$. Our proof of **Theorem 4.2** shows that such a row is invariant, therefore $H$ is invariant.

**Case 2** ($i = n - 1$). In this case, $g_{h-1}$ is the rightmost filled entry of $G$, and $H$ is the set of all empty entries to the right of it. As such, this case is symmetric to case 1.

**Case 3** ($h \neq 0$ and $i \neq n - 1$). In this case, $g_{h-1}$ and $g_{i+1}$ are filled, so $H$ is contained in the subrow $[g_{h-1}, ..., g_{i+1}] = [1, 0, 0, ..., 0, 1]$. Our proof of **Theorem 4.2** shows that such a row is invariant, therefore $H$ is invariant.

Now we know that each empty entry in $G$ is part of a consecutive set of empty entries, each of which are invariant and filled independently by the VUSA. Thus, all of $G$ is invariant.

$\square$

### 4.2 Behavior of $\Upsilon(n)$

We have established that a $1 \times n$ grid is always invariant, i.e., $\Upsilon(n)$ is always the same. In other words, $\Upsilon : \mathbb{N}^+ \to \mathbb{N}^+$ is a function. **Table 1** shows the first 50 values for $\Upsilon(n)$, and **Figure 3** shows the first 130 values.

| $n$ | $\Upsilon(n)$ | $n$ | $\Upsilon(n)$ | $n$ | $\Upsilon(n)$ | $n$ | $\Upsilon(n)$ | $n$ | $\Upsilon(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 11 | 5 | 21 | 9 | 31 | 15 | 41 | 17 |
| 2 | 1 | 12 | 5 | 22 | 9 | 32 | 16 | 42 | 17 |
| 3 | 2 | 13 | 5 | 23 | 9 | 33 | 17 | 43 | 17 |
| 4 | 2 | 14 | 6 | 24 | 9 | 34 | 17 | 44 | 17 |
| 5 | 3 | 15 | 7 | 25 | 9 | 35 | 17 | 45 | 17 |
| 6 | 3 | 16 | 8 | 26 | 10 | 36 | 17 | 46 | 17 |
| 7 | 3 | 17 | 9 | 27 | 11 | 37 | 17 | 47 | 17 |
| 8 | 4 | 18 | 9 | 28 | 12 | 38 | 17 | 48 | 17 |
| 9 | 5 | 19 | 9 | 29 | 13 | 39 | 17 | 49 | 17 |
| 10 | 5 | 20 | 9 | 30 | 14 | 40 | 17 | 50 | 18 |

**Table 1:** The first 50 values of $\Upsilon(n)$

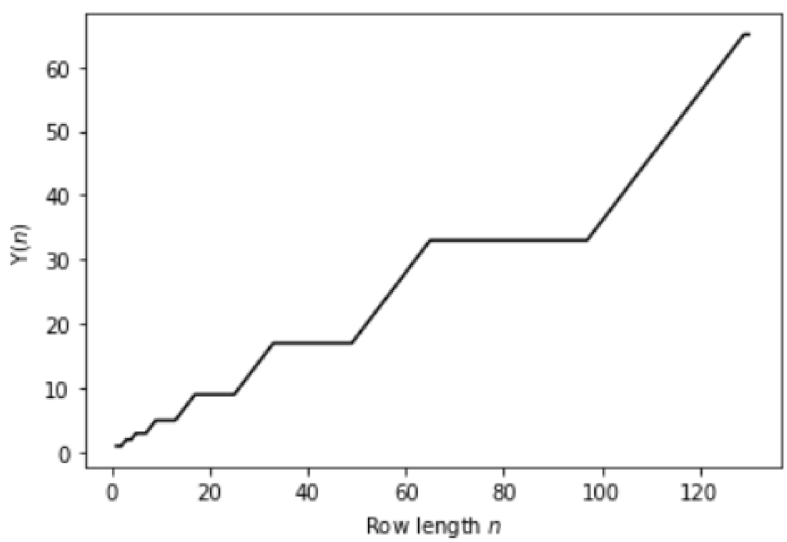



**Figure 3:** The first 130 values of $\Upsilon(n)$.

A peculiar pattern emerges for $n \geq 2$. $\Upsilon$ always increases with slope 1 (meaning $\Upsilon(n) = \Upsilon(n-1)+1$) until it achieves a value of the form $2^p+1$, $p \in \mathbb{N}_0$. Then, $\Upsilon$ stays constant (meaning $\Upsilon(n) = \Upsilon(n-1)$) for the next $2^p$ terms (three 3's in a row means $\Upsilon$ was constant for two terms). Note that the difference between two consecutive values of the form $2^p+1$ is $(2^{k+1}+1)-(2^k+1) = 2^k$. Since $\Upsilon$ increases with slope 1, each period of remaining constant for $2^p$ terms is followed by a period of increasing for $2^p$ terms.

The periods where $\Upsilon$ remains constant show an increase in row length without an increase in $\Upsilon$. In the context of trying to fill the most entries possible, e.g. an architect deciding how many urinals to build in a row, these added entries are a waste, since the same number of entries would be filled for a shorter row. So, values of $n$ where $\Upsilon$ stops being constant (where $\Upsilon(n-1) = \Upsilon(n) < \Upsilon(n+1)$) correspond with the row lengths that waste the most spaces, and are called optimal. On the other hand, values of $n$ where $\Upsilon$ stops increasing (where $\Upsilon(n-1) < \Upsilon(n) = \Upsilon(n+1)$) correspond with the row lengths that waste the fewest spaces, and are called minimal. This is the motivation for the discovery of the optimal and minimal values for $n$ described in **sections 3.3 & 3.4**.

## 4.3 Optimal values of $n$

For the same reason that $\Upsilon(1,n)$ can be expressed as $\Upsilon(n)$, the optimal number $\Omega(1,n)$ of a row $G$ can be expressed as $\Omega(n)$. The case where $G$ is a $1 \times 7$ row $[0,0,0,0,0,0,0]$ illustrates the fact that for some values of $n, \Upsilon(n) < \Omega(n)$. Specifically, the VUSA outputs [1,0,0,1,0,0,1], so $\Upsilon(7) = 3$. However, the seating arrangement [1,0,1,0,1,0,1] is the most optimal comfortable seating of $G$, so $\Omega(7) = 4$.

Given an empty $1 \times n$ grid $G$, it is quite easy to generate a comfortable seating which fills $\Omega(n)$ entries of $G$: simply fill every even-numbered entry of $G$ (and no others). This process serves as our optimal seating algorithm for a row.

Note: This algorithm is contained in the more general optimal seating algorithm in two dimensions, so it is not formally defined until **Section 5.3**.

Note: For odd $n$, our optimal seating algorithm gives the only possible seating arrangement with $\Omega(n)$ entries filled. For even $n$, there are multiple arrangements with $\Omega(n)$ entries filled. This is because the algorithm doesn't fill the last entry $g_{n-1}$. In this case, $g_{n-2}$, $g_{n-4}$, $g_{n-6}, ..., g_0$ will be filled, so we could pick any empty entry $g_k$ and shift all filled entries $g_p$ such that $p > k$ to the right by one, which would cause $g_{n-1}$ to be filled. This does not change the number of filled entries, and $G$ would still be saturated, since $g_k$ and $g_{k+1}$ are the only adjacent empty entries in $G$, and $g_{k-1}$ and $g_{k+2}$ are filled.

For even $n$, the optimal seating algorithm fills exactly half of the entries in $G$, so $\Omega(n) = \frac{1}{2}n$. For odd $n$, the optimal seating algorithm fills exactly half of the entries in $[g_0, g_1, ..., g_{n-2}]$, as well as the rightmost entry of $G$, $g_{n-1}$, so $\Omega(n) = \frac{1}{2}(n-1)+1 = \frac{1}{2}(n+1)$. **Figure 4** shows the plot of $\Omega(n)$ against $\Upsilon(n)$.

$$\Omega(n) = \left\lceil \frac{n}{2} \right\rceil = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even} \\ \frac{1}{2}(n+1) & \text{if } n \text{ is odd} \end{cases}$$

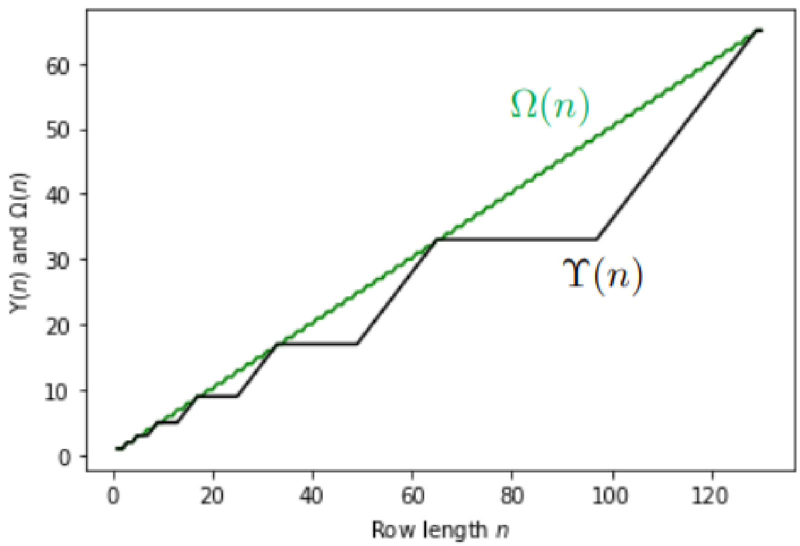



**Figure 4:** The first 130 values of $\Upsilon(n)$ and $\Omega(n)$.

As shown in **Figure 4**, $\Upsilon(n) = \Omega(n)$ for certain values of $n$. In these cases, this means that the VUSA fills as many entries as the optimal seating algorithm. The plot suggests that these points of equality occur at the values of $n$ where $\Upsilon(n)$ stops increasing (meaning $\Upsilon(n-1) < \Upsilon(n) = \Upsilon(n+1)$). We have observed that after $\Upsilon(3) = 2$, $\Upsilon$ is constant for 1 term, increases (with slope 1) for 1 term, is constant for 2 terms, increases for 2 terms, is constant for 4 terms, increases for 4 terms, ... , is constant for $2^p$ terms, increases for $2^p$ terms, and so on.

Observe: $n = 3$ is the first $n$ where $\Upsilon(n)$ stops increasing.

After $n = 3$, $\Upsilon$ remains constant for 1 term and increases for 1 term, so $n = 5$ is the second $n$ where $\Upsilon(n)$ stops increasing.

After $n = 3$, $\Upsilon$ remains constant for 1 + 2 terms and increases for 1 + 2 terms, so $n = 9$ is the third $n$ where $\Upsilon(n)$ stops increasing.

...

After $n = 3$, $\Upsilon$ remains constant for $(1 + 2 + 4 + ... + 2^{k-2})$ terms and increases for $(1 + 2 + 4 + ... + 2^{k-2})$ terms, so

$$n = 3 + 2\sum_{i=1}^{k-2} 2^i = 2^k + 1$$

is the $k^{\text{th}}$ $n$ where $\Upsilon(n)$ stops increasing.

This observation suggests that $n$ of the form $2^k+1$ implies that $\Upsilon(n) = \Omega(n)$. This is in fact true, and will be proven in **Theorem 4.4**. However, this is not the only such form of $n$. In particular, the only other such forms of $n$ are $n = 2^k$ and $n = 2^k + 2$, which can be observed in **Figure 4**. However, **Theorem 4.4** does not include these cases, since they are not as "efficient" at filling a maximal fraction of their entries as the $2^k + 1$ case is.

Specifically, the $2^k + 2$ case can be considered to waste one entry, since we observe that $\Upsilon(2^k+1) = \Upsilon(2^k+2)$. Meanwhile, the $2^k$ case cannot be said to waste spaces in the same sense, since it achieves a higher value for $\Upsilon$ than any shorter row. But, we observe $\Upsilon(2^k+1) = \Upsilon(2^k)+1$. So, the fraction of entries filled $\frac{\Upsilon(n)}{n}$ is less for the $2^k$ case than for the $2^k+1$ case. This is because

$$\tfrac{\Upsilon(2^k)}{2^k} < \tfrac{\Upsilon(2^k)+1}{2^k+1} = \tfrac{\Upsilon(2^k+1)}{2^k+1}$$

since $\frac{y}{x} < \frac{y+1}{x+1}$ if $0 < y < x$, and we know $0 < \Upsilon(n) < n \ \forall n > 1$.

**Theorem 4.4.** *Let $n = 2^k + 1$, where $k \in \mathbb{N}_0$. Then $\Upsilon(n) = \Omega(n)$.*

*Proof. (Induction on $k$).*

*Base Cases:* For $k \leq 2$, we manually check that $\Upsilon(n) = \Omega(n)$:

$k = 0 \Rightarrow n = 2$, and $\Upsilon(2) = 1 = \left\lceil \frac{2}{2} \right\rceil = \Omega(2)$
$k = 1 \Rightarrow n = 3$, and $\Upsilon(3) = 2 = \left\lceil \frac{3}{2} \right\rceil = \Omega(3)$
$k = 2 \Rightarrow n = 5$, and $\Upsilon(5) = 3 = \left\lceil \frac{5}{2} \right\rceil = \Omega(5)$

*Induction Hypothesis:* Let $k > 2$ and let $n_0 = 2^k + 1$ Assume $\Upsilon(n_0) = \Omega(n_0)$.

*Inductive Step* $(k \mapsto k+1)$: Let $n = 2^{k+1} + 1$. We want to show that $\Upsilon(n) = \Omega(n)$. Let $G = [g_0, ..., g_{n-1}]$, where each $g_k = 0$. Since $n$ is odd, $\Omega(n) = \frac{1}{2}(n+1)$. So we only need to show that $\Upsilon(n) = \frac{1}{2}(n+1)$.

To calculate $\Upsilon(n)$, we apply the VUSA to $G$. As in the proof of **Theorem 4.2**, we can assume without loss of generality that $\psi_1 = g_0$ and $\psi_2 = g_{n-1}$. Since $n$ is odd and $n > 5$, the next empty entry of $G$ with maximum room is $g_{\frac{1}{2}(n-1)}$, so $\psi_3 = g_{\frac{1}{2}(n-1)}$.

Satisfying the conditions for the split-row rule, $\psi_3$ divides $G$ into two subrows $H = [g_0, ..., g_{\frac{1}{2}(n-1)}]$ and $I = [g_{\frac{1}{2}(n-1)}, ..., g_{n-1}]$. $H$ and $I$ are both rows of length $l$, where

$$l = \frac{1}{2}(n-1) + 1 = 2^k + 1 = n_0$$

Therefore, by the induction hypothesis, $\Upsilon(l) = \Omega(l)$. Both $H$ and $I$ have length $l = \frac{1}{2}(n+1) = 2^k + 1$, which is odd. So, the number of entries of $H$ and the number of entries of $I$ filled by the VUSA are both given by

$$\Upsilon(l) = \Omega(l) = \tfrac{1}{2}(l+1)$$

Now, by the split-row rule, applying the VUSA to $G$ is equivalent to applying it to $H$ and $I$ separately. Thus, the number of entries filled by the VUSA in $G$ is equal to the number filled in $H$ plus the number filled in $I$ minus one, since $g_{\frac{1}{2}(n-1)}$ is counted in both $H$ and $I$. In other words,

$$\Upsilon(n) = \Upsilon(l) + \Upsilon(l) - 1 = 2\left[\frac{1}{2}(l+1)\right] - 1 = \frac{1}{2}(n+1)$$

□

## 4.4 Minimal values of $n$

$\Omega(n)$ and the optimal seating algorithm were fairly straightforward to establish. It is less clear to establish an algorithm for seating a row with its minimal number, $\mu(n) := \mu(1,n)$. Saturating a row with as few filled entries as possible correlates with wasting the most entries. Using the motivation from **Table 1** and **Figure 3**, it seems likely that the row lengths with the most wasted space (the values of $n$ where $\Upsilon(n)$ stops being constant) will align with $\mu(n)$. In these cases of the VUSA, every third entry is filled, including both endpoints (this can occur because it happens that each such case satisfies $n \equiv 1 \pmod 3$).

We will use this every-third-entry idea for our minimal seating algorithm. We could also begin with the endpoint $g_0$, but this would be a waste unless $n \equiv 1 \pmod 3$, which we will see shortly. Rather, we will begin with $g_1$, since this causes $g_0$ to have room 1 and this cannot increase the amount of filled entries it will take to saturate $G$. Imagine taking the case that starts with $g_0$ and simply shifting every filled entry one space to the right. If the row had ended in [...,1,0,0] or [...,0,1,0], it now ends in [...,0,1,0] or [...,0,0,1] (respectively), so we don't need to fill an extra entry at $g_{n-1}$. If the row had ended in [...1,0,0,1], it would cause the filled entry at $g_{n-1}$ to "slide off" the right end of the row so the row now ends with [...,1,0,0], but we can just fill $g_{n-1}$ again, and the amount of filled entries remains the same.

Now that we have justified how to saturate a row with as few filled entries as possible, we can define our minimal seating algorithm and find values for $\mu(n)$. Note: this algorithm is not analogous to the minimal seating algorithm in two-dimensions, so we will formally define it here.

**Definition 4.1.** *Let $G$ be a $1 \times n$ grid. Then the* <u>*minimal seating algorithm*</u> *operates as follows:*

If $n \leq 5$, we manually describe a minimal seating of $G$:

$$\text{Fcases}n = 1\text{: } [1],\ \mu(1) = 1$$
$$n = 2\text{: } [1, 0],\ \mu(2) = 1$$
$$n = 3\text{: } [0, 1, 0],\ \mu(3) = 1$$
$$n = 4\text{: } [0, 1, 0, 1],\ \mu(4) = 2$$
$$n = 5\text{: } [0, 1, 0, 0, 1],\ \mu(5) = 2$$

If $n > 5$, begin by filling every entry $g_t$ such that $t \equiv 1 \pmod 3$. Naturally, we must consider how this pattern ends at the rightmost end of the row:

- If $n \equiv 1 \pmod 3$, then $n - 3 \equiv 1 \pmod 3$, so the row ends with $[..., g_{n-3}, g_{n-2}, g_{n-1}] = [..., 1, 0, 0]$, and we will fill the last space, $g_{n-1}$ in order to saturate $G$. In this case, $G$ is composed of sections of $[0, 1, 0]$ for $n - 1$ entries, followed by the entry $g_{n-1} = 1$, so $\mu(n) = \frac{1}{3}(n + 2)$.

  In this case, we could shift every filled entry besides $g_{n-1}$ one space to the left, and $G$ will still be saturated, since it will now end with $[...1, 0, 0, 1, 0, 0, 1]$. This causes $g_1$ to be filled, which is why both endpoints can be filled in a minimal seating when $n \equiv 1 \pmod 3$.

- If $n \equiv 2 \pmod 3$, then $n - 1 \equiv 1 \pmod 3$, so the row ends with $[..., g_{n-4}, g_{n-3}, g_{n-2}, g_{n-1}] = [...1, 0, 0, 1]$, and $G$ is saturated. In this case, $G$ is composed of sections of $[0, 1, 0]$ for $n - 2$ entries, followed by the entries $g_{n-2} = 0, g_{n-1} = 1$, so $\mu(n) = \frac{1}{3}(n + 1)$.

  In this case, we could shift every filled entry one space to the left, but we must also shift $g_{n-1}$ to retain saturation, and this just turns the grid into its mirror image.

- If $n \equiv 0 \pmod 3$, then $n - 2 \equiv 1 \pmod 3$, so the row ends with $[..., g_{n-3}, g_{n-2}, g_{n-1}] = [..., 0, 1, 0]$, and $G$ is saturated. In this case, $G$ is composed of sections of $[0, 1, 0]$ for all $n$ terms, so $\mu(n) = \frac{1}{3}n$.

  In this case, $G$ is already minimally saturated, and the filled entries cannot be shifted either direction without causing $G$ to become unsaturated.

This algorithm gives us the following function for $\mu(n)$:

$$\mu(n) = \left\lceil \frac{n}{3} \right\rceil = \begin{cases} \frac{1}{3}n & \text{if } n \equiv 0 \pmod 3 \\ \frac{1}{3}(n+2) & \text{if } n \equiv 1 \pmod 3 \\ \frac{1}{3}(n+1) & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

With the optimal seating algorithm, it was straightforward to understand that filling every other seat gives an optimal seating. For a minimal seating, it is slightly more complicated, but we can justify that our minimal fillings are indeed minimal if we understand the two criteria that contribute to a row being minimally saturated.

**Theorem 4.5.** *Let $G$ be an empty $1 \times n$ grid. Our minimal seating algorithm fills $G$ with $\mu(n)$ filled entries. In other words, $\mu(n) = \lceil \frac{n}{3} \rceil$.*

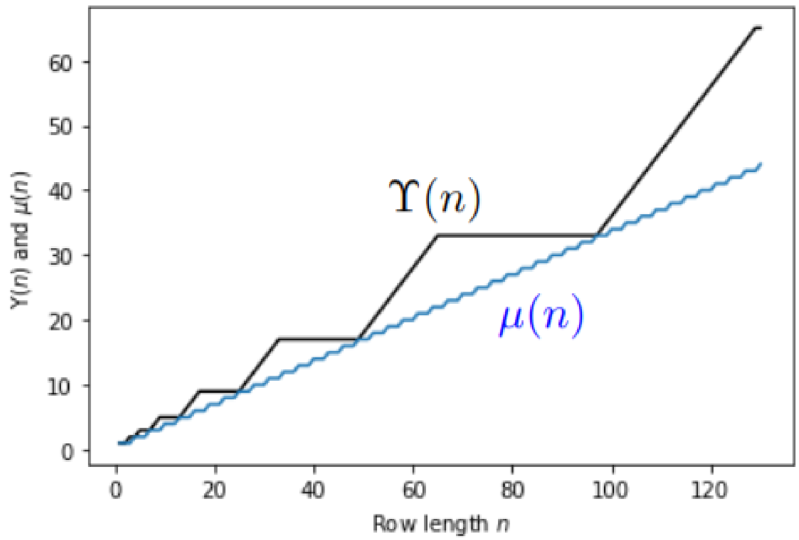



**Figure 5:** The first 130 values of $\Upsilon(n)$ and $\mu(n)$.

*Proof.* To saturate a row, each empty entry must be adjacent to at least one filled entry. To do this with as few filled entries as possible, we must fulfill as many of the following two conditions as possible: $(a)$ each filled entry to be adjacent to two empty entries, or $(b)$ each empty entry to be adjacent to only one filled entry.

If both $(a)$ and $(b)$ are fulfilled in a saturated $1 \times n$ grid $G$, then $g_0$ is empty, otherwise $(a)$ is unfulfilled. Thus, $g_1$ must be filled, otherwise $G$ is unsaturated. Then, $g_2$ must be empty, otherwise $(a)$ is unfulfilled. Then, $g_3$ must be empty, otherwise $(b)$ is unfulfilled. Finally, $g_4$ must be filled, otherwise $G$ is unsaturated. Continuing this pattern, every section $[g_{3k}, g_{3k+1}, g_{3k+2}]$ of $G$ (where $k \in \mathbb{N}$) must be [0,1,0]. Moreover, $G$ must consist of a whole number of these sections, otherwise $G$ ends with [...,0,1,0,1] ($(b)$ is unfulfilled) or [...,0,1,0,0,1] ($(a)$ is unfulfilled), and we cannot shift any entries of $G$ to avoid this, since our attempt to fulfill both $(a)$ and $(b)$ required us to fill exact entries [0,1,0,0,1,0,...]. Finally, this means that if $(a)$ and $(b)$ are both fulfilled, then $n \equiv 0 \pmod 3$. By contraposition, if $n \not\equiv 0 \pmod 3$, then $(a)$ and $(b)$ cannot both be fulfilled (for a saturated grid). Now we will use this fact to evaluate our algorithm for each case modulo 3.

If $n \equiv 0 \pmod 3$, our algorithm gives the only seating arrangement which fulfills both $(a)$ and $(b)$ [0,1,0,0,1,0,...,0,1,0].

If $n \equiv 1 \pmod 3$, $(a)$ and $(b)$ cannot both be fulfilled for a saturated grid. Our algorithm achieves neither at first, but by shifting every entry but $g_{n-1}$ to the left one space, we satisfy $(b)$. From there, we can shift any number of filled entries toward any central entry $g_k$, $0 < k < n-1$, which would allow the endpoints to be empty and satisfy $(a)$.

If $n \equiv 2 \pmod 3$, $(a)$ and $(b)$ cannot both be fulfilled for a saturated grid. As written, our algorithm satisfies $(b)$. Again, we can shift entries away from the endpoints to satisfy $(a)$.

We have shown that, given any minimal saturated filling $F$ of $G$, the filling given by our algorithm can be shifted (without changing the number of entries filled) to satisfy as many of $\{(a),(b)\}$ as $F$ satisfies. As such, no seating arrangement can saturate $G$ with fewer filled entries than our algorithm can. Since our algorithm can saturate $G$ with $\lceil \frac{n}{3} \rceil$ filled entries, we conclude that $\mu(n) = \lceil \frac{n}{3} \rceil$. $\square$

We have established the minimal seating algorithm and the function $\mu(n)$ on their own, and $\mu(n)$ will be used to prove the minimal numbers in **Theorem 4.6**. For this theorem, we could discover the pattern for which values of $n$ satisfy $\Upsilon(n) = \mu(n)$ by again examining the behavior of $\Upsilon$ to predictably increase and stop increasing. However, these findings were originally motivated by comparing the VUSA to the optimal seating algorithm, since the original goal of this paper was to inquire which row-lengths fell short of optimal seating, and by how much.

Looking at the plot of $\Omega(n)$ and $\Upsilon(n)$, it seems clear that the minimal numbers are the values of $n$ for which $\Upsilon(n)$ is furthest from $\Omega(n)$, locally speaking. Thus, it is useful to plot these values against their difference, $\Delta(n) := \Omega(n) - \Upsilon(n)$, as in **Figure 6**.

The local maxima of $\Delta(n)$ show the points where the VUSA falls short of optimal seating by the most filled entries. The peaks of $\Delta(n)$ seem to come exactly between its valleys, suggesting that a minimal number is the average of two consecutive optimal numbers, i.e.,

$$n = \frac{1}{2}\left(2^{k+1} + 1 + 2^{k+2} + 1\right) = 3 \cdot 2^k + 1$$

This form for the minimal numbers seems intuitively fitting, since the resulting saturated graph consists of $p$ sections [1,0,0] with length 3, followed by a single filled entry at the end, where $p$ must be a power of 2 so that the split-row rule will split the row into two subrows of this same form. As with **Theorem 4.4**, this splitting pattern is the key point to proving **Theorem 4.6**.

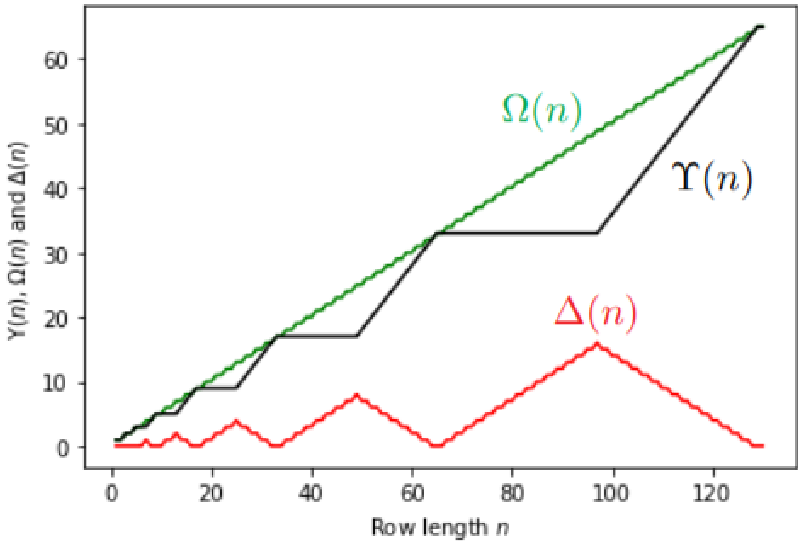



**Figure 6:** The first 130 values of $\Upsilon(n)$, $\Omega(n)$ and $\Delta(n)$.

**Theorem 4.6.** *Let $n = 3 \cdot 2^k + 1$, where $k \in \mathbb{N}_0$. Then $\Upsilon(n) = \mu(n)$.*

*Proof. (Induction on $k$).* Note: This proof will be very similar to the proof of optimal numbers (**Theorem 4.4**).

*Base Cases:* For $k \leq 1$, we manually show that $\Upsilon(n) = \mu(n)$:

$$k = 0 \Rightarrow n = 4, \text{ and } \Upsilon(4) = 2 = \left\lceil \tfrac{4}{3} \right\rceil = \mu(4)$$
$$k = 1 \Rightarrow n = 7, \text{ and } \Upsilon(7) = 3 = \left\lceil \tfrac{7}{3} \right\rceil = \mu(7)$$

*Induction Hypothesis:* Let $k > 1$ and let $n_0 = 3 \cdot 2^k + 1$ Assume $\Upsilon(n_0) = \mu(n_0)$.

*Inductive Step* ($k \mapsto k+1$): Let $n = 3 \cdot 2^{k+1} + 1$. We want to show that $\Upsilon(n) = \mu(n)$. Let $G = [g_0, ..., g_{n-1}]$, where each $g_k = 0$. Note that $n \equiv 1 \pmod 3$. Thus, $\mu(n) = \frac{1}{3}(n+2)$, so we only need to show that $\Upsilon(n) = \frac{1}{3}(n+2)$.

To calculate $\Upsilon(n)$, we apply the VUSA to $G$. As in the proof of **Theorem 4.2**, we can assume without loss of generality that $\psi_1 = g_0$ and $\psi_2 = g_{n-1}$. Note that $n$ is odd and $n > 7$. Thus, the next empty entry of $G$ with maximum room is $g_{\frac{1}{2}(n-1)}$, so $\psi_3 = g_{\frac{1}{2}(n-1)}$.

Satisfying the conditions for the split-row rule, $\psi_3$ divides $G$ into two subrows $H = [g_0, ..., g_{\frac{1}{2}(n-1)}]$ and $I = [g_{\frac{1}{2}(n-1)}, ..., g_{n-1}]$. $H$ and $I$ are both rows of length $l$, where

$$l = \frac{1}{2}(n+1) = 3 \cdot 2^k + 1 = n_0$$

Therefore, by the induction hypothesis, $\Upsilon(l) = \mu(l)$. Both $H$ and $I$ have length $l = \frac{1}{2}(n+1) = 3 \cdot 2^k + 1$, satisfying $l \equiv 1 \pmod 3$. So, the number of entries of $H$ and the number of entries of $I$ filled by the VUSA are given by

$$\Upsilon(l) = \mu(l) = \tfrac{1}{3}(l+2)$$

Now, by the split-row rule, applying the VUSA to $G$ is equivalent to appying it to $H$ and $I$ separately. Thus, the number of entries filled by the VUSA in $G$ is equal to the number filled in $H$ plus the number filled in $I$ minus one, since $g_{\frac{1}{2}(n-1)}$ is counted in both $H$ and $I$. In other words,

$$\Upsilon(n) = 2\Upsilon(l) - 1 = 2\left[\tfrac{1}{3}(l+2)\right] - 1 = \tfrac{1}{3}(n+2)$$

□

Recall that the optimal numbers, for which $\Upsilon(n) = \Omega(n)$, take not only the form $n = 2^k + 1$, but also $n = 2^k$ and $n = 2^k + 2$. By contrast, minimal numbers only occur for odd $n$, particularly $n = 3 \cdot 2^k + 1$. This difference can be seen in the plots, where $\Delta(n)$ has peaks for 1 term, while it has valleys at $y = 0$ for 3 terms in a row. Also, $\Upsilon(n)$ aligns with $\Omega(n)$ for 3 terms in a row, but aligns with $\mu(n)$ for only 1 term at a time. This difference is tied to the fact that $\Upsilon(n)$ and $\mu(n)$ are both ceiling functions of a constant fraction of $n$.

## 4.5 End behavior of $\frac{\Upsilon(n)}{n}$

**Theorem 4.7.** *The limit of $\frac{\Upsilon(n)}{n}$ as $n$ tends to infinity does not exist.*

*Proof.* We know that $n = 2^k + 1 \Rightarrow \Upsilon(n) = \Omega(n) = \frac{1}{2}(n+1)$. In these cases, the fraction of entries filled by the VUSA is given by

$$\frac{\Upsilon(n)}{n} = \frac{\frac{1}{2}(n+1)}{n} = \frac{1}{2} + \frac{1}{2n}$$

Therefore,

$$\lim_{k\to\infty} \frac{\Upsilon(2^k+1)}{2^k+1} = \frac{1}{2}$$

We also know that $n = 3 \cdot 2^k + 1 \Rightarrow \Upsilon(n) = \mu(n) = \frac{1}{3}(n+2)$. In these cases,

$$\frac{\Upsilon(n)}{n} = \frac{\frac{1}{3}(n+2)}{n} = \frac{1}{3} + \frac{2}{3n}$$

Therefore,

$$\lim_{k\to\infty} \frac{\Upsilon(3 \cdot 2^k+1)}{3 \cdot 2^k+1} = \frac{1}{3}$$

Thus, for any $\varepsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ such that $|\frac{\Upsilon(n)}{n} - \frac{1}{2}| < \varepsilon$, as well as infinitely many $n \in \mathbb{N}_0$ such that $|\frac{\Upsilon(n)}{n} - \frac{1}{3}| < \varepsilon$. As such, the limit of $\Upsilon(n)$ as $n$ tends to infinity does not exist.

□

# 5 Two-Dimensional Grids

In one dimension, the VUSA is easy to predict, due to the split-row rule and the symmetry of entries tied for the most room. The cases where $m > 1$ and $n > 1$ are significantly more complicated. Most notably, $\Upsilon(m, n)$ is not a function. For example, three different results of the VUSA on the empty $10 \times 10$ grid are shown in **Figure 7** (In Python, I denote filled spaces [X] and empty spaces [-]. In other words, the co-domain $Y$ of the matrix function is {-,X} rather than {0,1}).

```
X - X - X - X - - X
- X - X - - - - X -
- - X - - X - X - X
X - - X - - X - X -
- - X - - X - - - X
- X - X - - - X - -
- - - - - X - - - X
X - X - - - - X - -
- X - - X - - - X -
X - X - - - X - - X
```

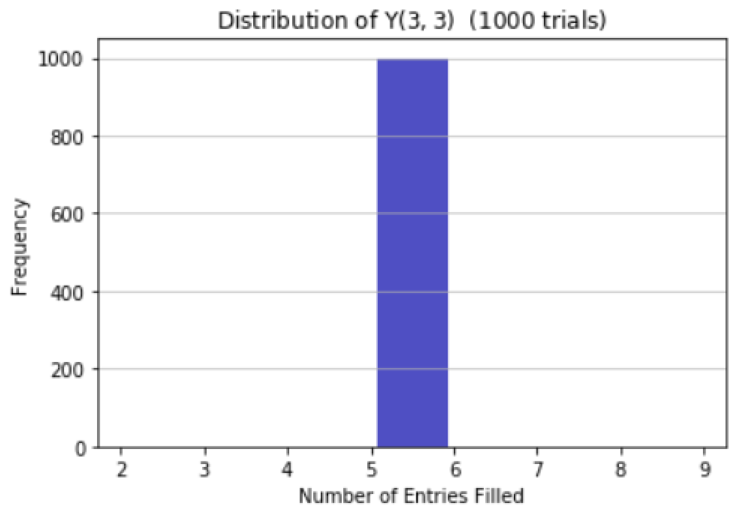
```
X - X - X - - X - X
- X - X - X - - X -
- - X - X - X - - -
X - - X - X - X - X
- X - - X - X - - -
X - X - - - - - X -
- X - X - X - - - -
X - X - X - - X - X
- X - - - X - - X -
X - - X - - - X - X
```

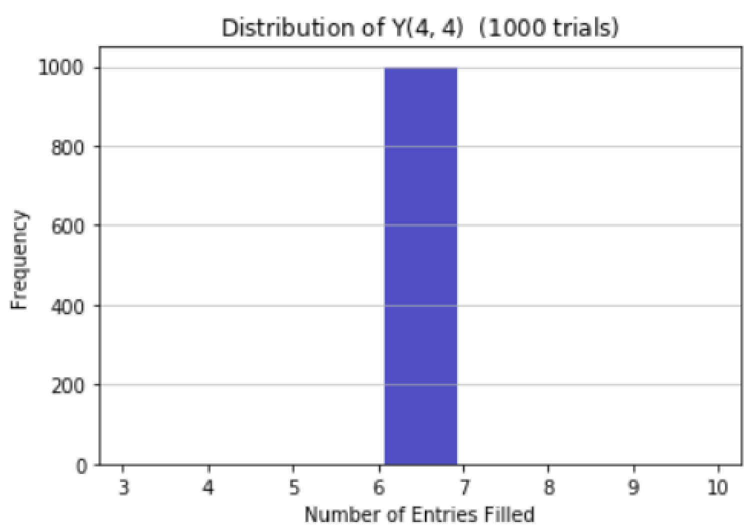
```
X - - X - X - X - X
- - X - X - X - X -
- X - X - X - X - X
X - X - X - X - X -
- - - X - X - X - X
- X - - X - X - X -
- - - X - X - X - X
X - X - - - X - X -
- X - - X - - X - -
X - X - - - X - - X
```

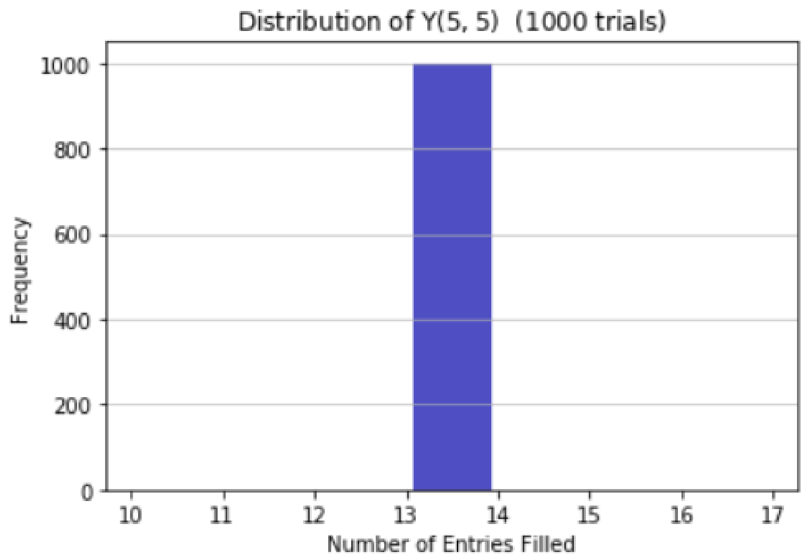
**Figure 7:** The VUSA can fill different numbers of entries for 2D grids. Here are three results of the algorithm applied to an empty $10 \times 10$ grid, which fill 34, 38, and 42 entries, respectively.

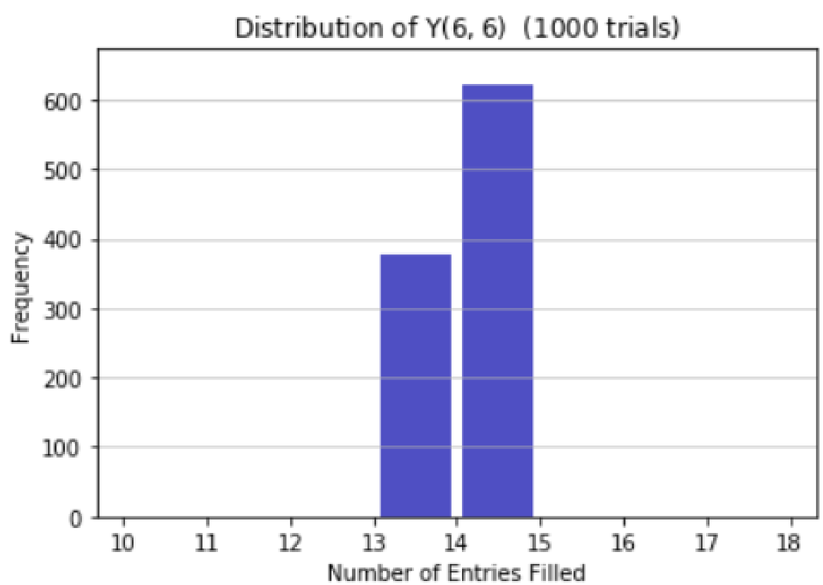
## 5.1 Probability distributions of $\Upsilon(n, n)$

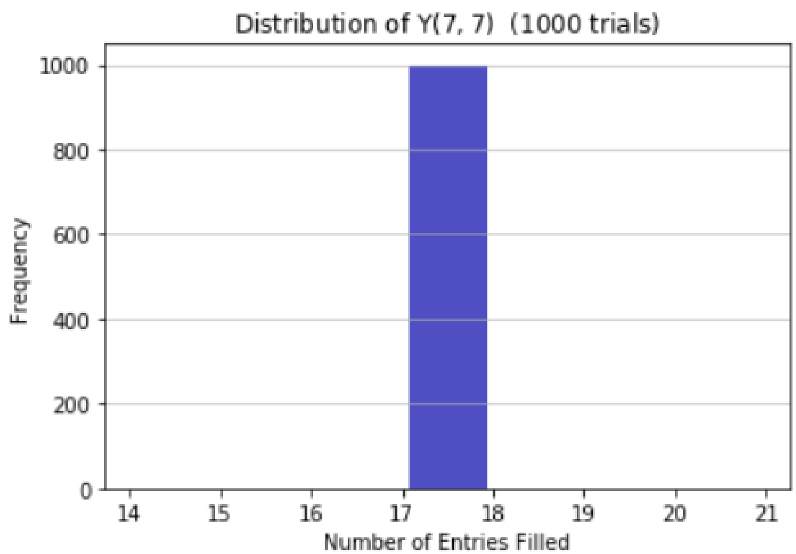
Due to the complex nature of 2D grids, I will only analyze square grids ($n = m$) in this paper. In simulating the VUSA on these squares, we can approximate the probability distributions of $\Upsilon(n, n)$ after 1000 trials. The results for $n \in \{3, ..., 13\}$ are shown here.

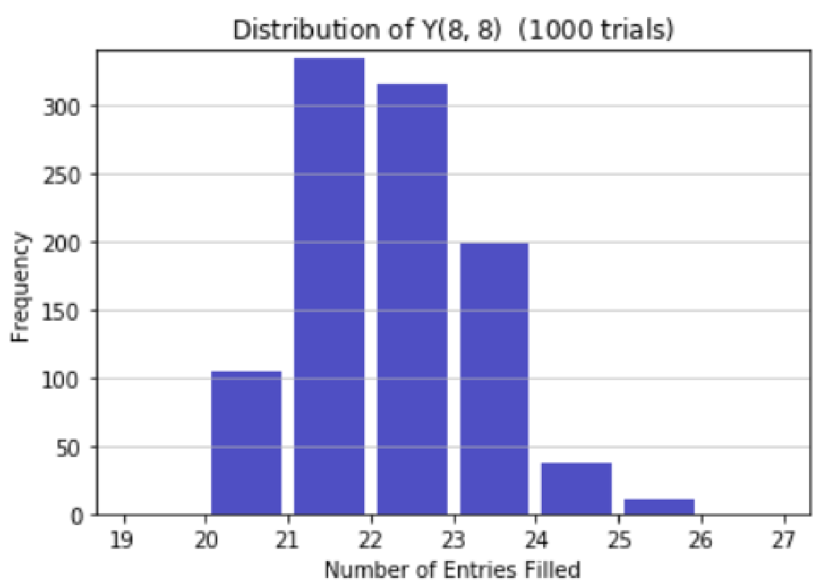

Distribution of Y(3, 3) (1000 trials)
Frequency
0
200
400
600
800
1000
2
3
4
5
6
7
8
9
Number of Entries Filled

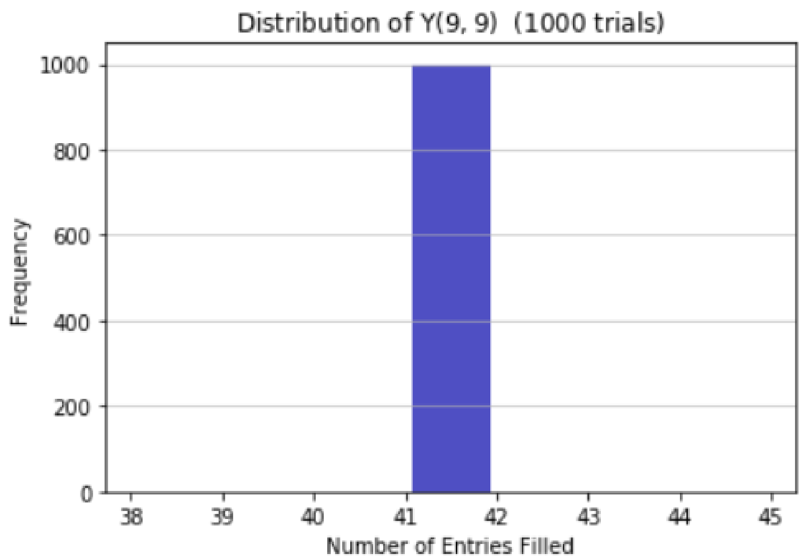

Distribution of Y(4, 4) (1000 trials)
Frequency
0
200
400
600
800
1000
3
4
5
6
7
8
9
10
Number of Entries Filled

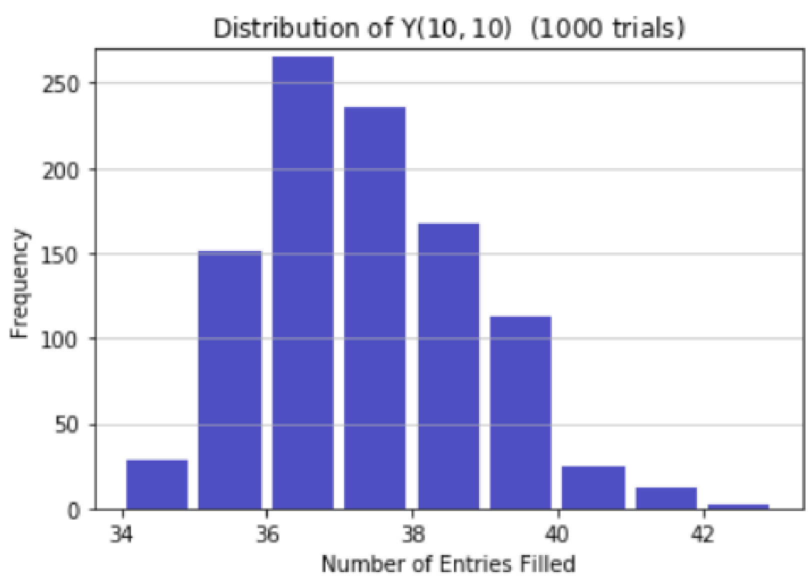

Distribution of Y(5, 5) (1000 trials)
Frequency
0
200
400
600
800
1000
10
11
12
13
14
15
16
17
Number of Entries Filled

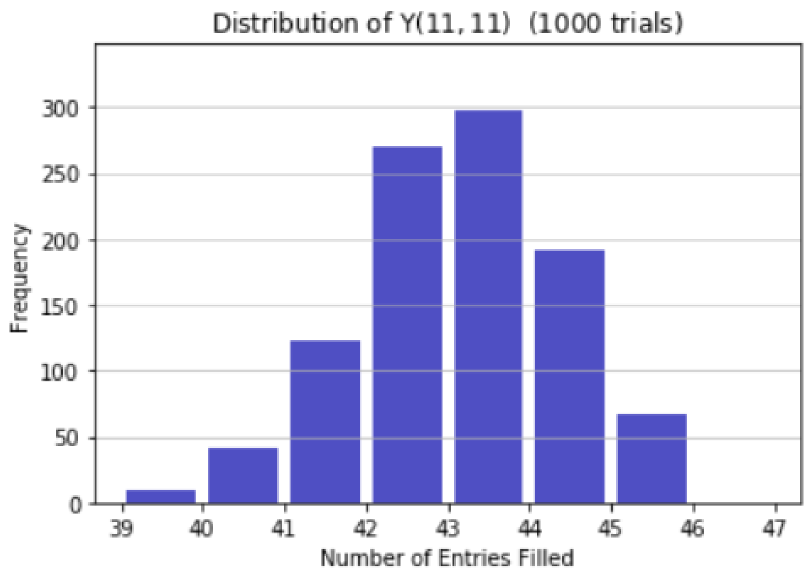

Distribution of Y(6, 6) (1000 trials)
Frequency
0
100
200
300
400
500
600
10
11
12
13
14
15
16
17
18
Number of Entries Filled

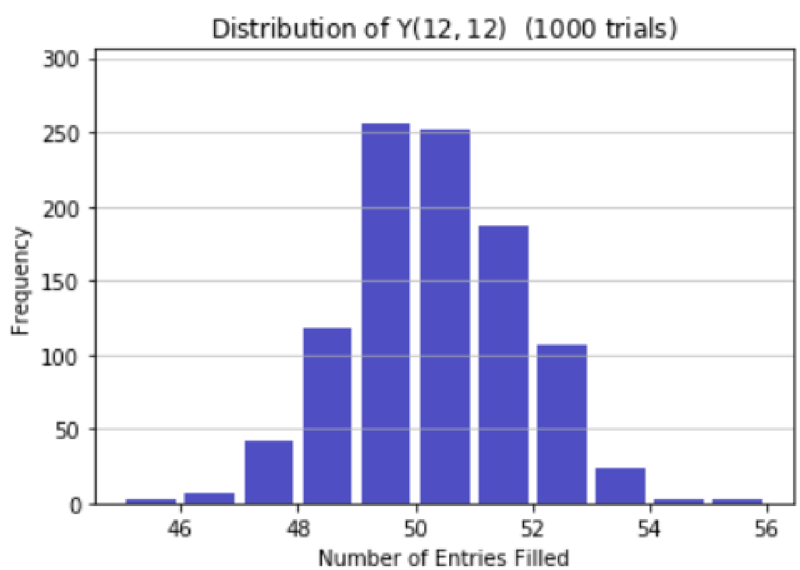

Distribution of Y(7, 7) (1000 trials)
Frequency
0
200
400
600
800
1000
14
15
16
17
18
19
20
21
Number of Entries Filled

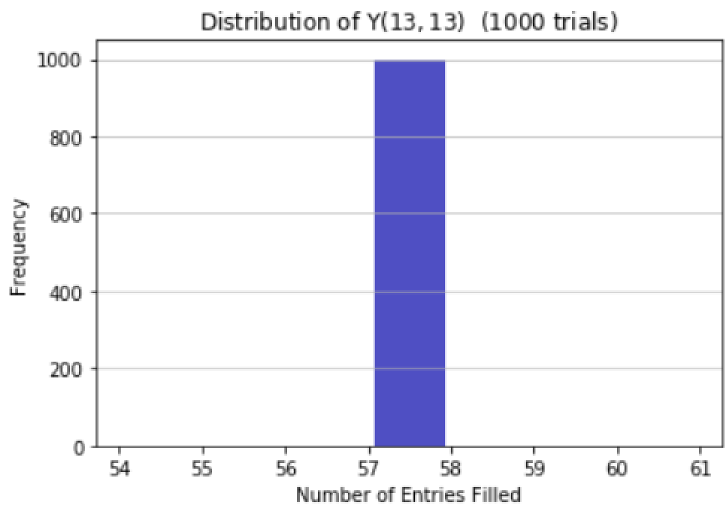

Distribution of Y(8, 8) (1000 trials)
Frequency
0
50
100
150
200
250
300
19
20
21
22
23
24
25
26
27
Number of Entries Filled


Distribution of Y(9, 9) (1000 trials)
Frequency
Number of Entries Filled
0
200
400
600
800
1000
38
39
40
41
42
43
44
45


Distribution of Y(10, 10) (1000 trials)
Frequency
Number of Entries Filled
0
50
100
150
200
250
34
36
38
40
42


Distribution of Y(11, 11) (1000 trials)
Frequency
0
50
100
150
200
250
300
39
40
41
42
43
44
45
46
47
Number of Entries Filled


Distribution of Y(12, 12) (1000 trials)
Frequency
0
50
100
150
200
250
300
46
48
50
52
54
56
Number of Entries Filled




## 5.2 Behavior of $\Upsilon(n, n)$

Examining the distributions of $\Upsilon(n, n)$ reveals that for certain values of $n$, an empty $n \times n$ grid is invariant. Closer examination shows that these such values correspond with the minimal and odd optimal numbers of $\Upsilon(n)$ in one dimension. In fact, the optimal case $n = 2^k + 1$ is true for two-dimensional grids as well. However, the minimal form $n = 3 \cdot 2^k + 1$ does not hold. In order to demonstrate these ideas, it is useful to have an analog to the split-row rule for two dimensions. While many cases of two-dimensional grids are not invariant, we can still predict the initial behavior of the VUSA for certain grids, which we will define as critical square grids.

**Definition 5.1.** *Let $n = a \cdot 2^k + 1$, where $a, k \in \mathbb{N}_0$ and $a \geq 2$. Let $G$ be an empty $n \times n$ grid. Then $G$ is called a critical square grid. Furthermore, the set $C = \{G(i, j) : a|i, a|j\}$ is called the critical set of $G$.*

**Theorem 5.1** (**The critical square grid theorem**). *Let $G$ be a critical square grid with critical set $C$. Then the VUSA fills every entry in $C$ before it fills any entries not in $C$.*

*Proof.* Let $G$ be an empty $n \times n$ grid, such that $n = a \cdot 2^k + 1$, $a, k \in \mathbb{N}^+$ and $a \geq 2$. Thus $n \geq 3$. Let $C$ be the critical set of $G$. Note that if $k = 0$, then the only elements of $C$ are the four corners of $G$, which are always filled first by the VUSA on an empty square grid, and we are done. So, assume $k \geq 0$, so $n \geq 5$.

To prove this theorem for $n \geq 5$, we will show how the VUSA fills entries that "cut" our square grid into sections, each half as long as the previous, and show that these cuts occur at the multiples of $a$. In order to describe the horizontal or vertical coordinates of these cuts, we will construct the cut sequence of G,

$c_G = \{c_0, c_1, ..., c_k\}$, where

$$c_t = \frac{n-1}{2^t} = \frac{a \cdot 2^k + 1 - 1}{2^t} = a \cdot 2^{k-t}$$

Note that $c_0 = n - 1$, and the final term is $c_k = a$. Moreover, each $c_t$ is $a$ times a power of 2, so they all divide $a$. Also note that this does not describe all of the coordinates of entries in $C$, only the points that mark the first whole ($c_0$), half ($c_1$), quarter ($c_2$), eighth ($c_3$), (and so on, until $c_k$), of $n - 1$. However, by including the points at three-quarters ($3c_2$), three-, five-, and seven-eighths ($3c_3$, $5c_3$, and $7c_3$), and every other odd multiple (less than $n$) of each $c_t$, then we describe every multiple of $c_k$ between 0 and $n-1$, which every multiple of $a$ between 0 and $n - 1$. Thus, letting $S = 0 \cup \{pc : p \text{ is odd}, \ c \in c_G, 0 \leq pc < n\}$,

$$C = \{G(i,j) : i, j \in S\}$$

The first four entries filled by the VUSA will be the corners, $G(0,0)$, $G(0,c_0)$, $G(c_0,0)$, and $G(c_0,c_0)$. Since $n$ is odd, each row and column of $G$ has a midpoint, and $G$ itself has the midpoint $G(c_1,c_1)$, which is the fifth entry filled by the VUSA.

The next four entries in $G$ to be filled will be the midpoints of rows and columns on the border of $G$, $G(0,c_1)$, $G(c_1,0)$, $G(c_0,c_1)$. Now, all nine entries with coordinate pairs in the set $\{0, c_0, c_1\} \times \{0, c_0, c_1\}$ are filled. If $k = 1$, then these are the only entries in $C$, and we are finished. Now, we will complete the proof using the principal of induction on $k$.

We have shown this theorem to be true for $k = 0$ and $k = 1$. So let $k - 1 > 1$ and let $G_0$ be a critical square grid with side length $n_0 = a \cdot 2^{k-1} + 1$ and critical set $C_0$. For our induction hypothesis, assume that the VUSA applied to $G_0$ fills every entry of $C_0$ before it fills any other entries of $G_0$.

Now, it remains to prove the inductive step ($k - 1 \mapsto k$). So let $n_1 = a \cdot 2^k + 1$, let $G_1$ be an empty $n_1 \times n_1$ grid, and let $C_1$ be the critical set of $G_1$. We want to show that the VUSA applied to $G_1$ fills every entry of $C_1$ before it fills any other entries. Let $c_{G_1} = \{c_0, c_1, c_2, ..., c_k\}$ be the cut sequence of $G_1$, i.e., $c_0 = n_1 - 1$ and $c_k = a$. Finally, let $S_1 = 0 \cup \{pc : p \text{ is odd}, \ c \in c_{G_1}, 0 \leq pc < n_1\}$.

As we previously showed, the first nine entries of $G_1$ to be filled will be the nine with coordinate pairs in the set $\{0, c_0, c_1\} \times \{0, c_0, c_1\}$. These entries split $G_1$ into four subgrids $H_1, ..., H_4$, whose length is $c_1 + 1 = \frac{1}{2}(n_1 - 1) + 1 = n_0$, and whose corners are filled entries of $G_1$. Note that any entry in the critical set of any $H_i$ has a coordinate pair in $S_1 \times S_1$, and all coordinate pairs in $S_1 \times S_1$ are entries in the critical set of some $H_i$. In other words, the critical set of $G$ is equal to the union of the critical set of each of these subgrids. Let $H$ be any of these subgrids.

Here, a peculiar thing happens. $H$ satisfies the same assumptions as $G_0$ (after the corners of $G_0$ are inevitably filled). However, we cannot immediately use this fact, since the VUSA could potentially be affected by the filled entries in the other subgrids. Instead, note that the midpoints of each subgrid, the four entries with coordinate pairs in the set $\{c_2, 3c_2\} \times \{c_2, 3c_2\}$, are all tied for the most room, and are each closer to the filled entry $G(c_1, c_1)$ than they are to each other. Thus, the next four entries to be filled will be these entries. The next set of ties for the most room are the midpoints of each border of each subgrid, which again are closer to a filled entry than they are to each other. Thus, these twelve entries will also be filled before any others.

As the VUSA continues, each entry in the current set of ties is in the middle of an odd square (or at the circumcenter of right isosceles triangle, if the entry is on a border of $G_1$) whose corners are filled and whose other entries are empty, so each such entry has less room than the distance to another such entry. As such, each set of ties will have all of its entries filled before any others. Thus, we can conclude that the VUSA fills $H$ independently of the other entries in $G$. Combining this with the fact that $H$ is a critical square grid with side length $n_0$, our induction hypothesis implies that each entry in the critical set of $H$ is filled before any others. Any quadrant of $G_1$ could have been chosen as $H$. Thus, the VUSA fills the critical set of $G_1$ before it fills any other entries.

□

This theorem is a powerful tool that we can apply to any odd $n \times n$ grid, toward the same purpose for which we used the split-row rule in one-dimension. Two-dimensional grids are far less well-behaved than one-dimensional rows, however. After the VUSA fills each entry of $C$, there are $(a-1) \times (a-1)$ squares of empty space between each filled entry. If $a$ is odd, then these squares are even, and each has a four-way tie for the most room in the middle. Even if $a$ is even, then $a$ can be divided by 2 and $k$ can be increased by 1 without changing $n$, until $a$ is not even. In other words, the VUSA will keep splitting the grid at the center of odd squares until even squares appear with a tie for the most room in each. This can only be avoided if $a$ is a power of 2, since $a$ won't become odd until $a = 1$, so these even squares have side length 0, in a sense. Otherwise, if $a > 3$ once it becomes odd, then the results of these ties will affect the VUSA in the neighboring squares. This motivates the conjecture that an empty $n \times n$ grid is invariant if and only if $n$ can be expressed as $2^k + 1$ or $3 \cdot 2^k + 1$. We further examine these cases and prove the "if" condition of this conjecture in **sections 4.3 & 4.4**.

## 5.3 Optimal values of $n$ in two dimensions

The optimal seating algorithm for empty 2D grids is very similar to the 1D case: the most efficient seating is to fill every other entry, and we can do this horizontally and vertically, in a checkerboard pattern. Rigorously speaking, we fill each entry $G(i, j)$ of an $m \times n$ grid $G$, such that $i + j$ is even. If $G$ contains an even number of entries, then it would be symmetric to fill the entries such that $i+j$ is odd instead, since these make up exactly half of the entries. However, if $G$ contains an odd number of entries, then there is one more entry such that $i+j$ is even than there are such that $i+j$ is odd, so we must choose even sums for our seating to be optimal. With this we can formally define our optimal seating algorithm:

**Definition 5.2.** *Let $G$ be an empty $m \times n$ grid. Then the* <u>*optimal set of $G$*</u> *is the set $O = \{G(i,j) : 2|(i+j)\}$, and the* <u>*optimal seating algorithm*</u> *on $G$ fills every entry in $O$ and no others.*

Note that this definition also serves to formally define our optimal seating algorithm for a one-dimensional row. Using this algorithm, we obtain a function for the optimal number of an $m \times n$ grid:

$$\Omega(m,n) = \left\lceil \frac{mn}{2} \right\rceil = \begin{cases} \frac{1}{2}mn & \text{if } mn \text{ is even} \\ \frac{1}{2}(mn+1) & \text{if } mn \text{ is odd} \end{cases}$$

From computation we can observe that when $n = 2^k + 1$, $\Upsilon(n,n) = \Omega(n,n)$ and $G$ is thus invariant. **Theorem 5.1** makes proving this is quite simple.

**Theorem 5.2.** *Let $n = 2^k + 1, k \in \mathbb{N}_0$ and let $G$ be an empty $n \times n$ grid with optimal set $O$. Then the VUSA will fill every entry in $O$ and no others. In particular, $G$ is invariant and $\Upsilon(n,n) = \Omega(n,n) = \frac{1}{2}(n+1)$.*

*Proof.* If $k = 0$, then $n = 2$ and the VUSA will fill either set of opposite corners of $G$ and nothing else, which are exactly the entries in $O$. If $k = 1$, then $n = 3$ and the VUSA will fill the four corners of $G$ and then the center, and nothing else, which are exactly the entries in $O$.

So, assume $k \geq 2$. Thus, $n = 2 \cdot 2^{k-1} + 1$. Since $k - 1 \in \mathbb{N}^+$, $G$ is a critical square grid with $a = 2$. Let $C$ be the critical set of $G$, i.e.,

$$C = \{G(i,j) : 2|i, 2|j\}$$

By **Theorem 5.1**, the VUSA fills every entry in $C$ before filling any other entries. Note that every entry $G(i,j)$ in $C$ satisfies $2|(i+j)$. Thus, $C \subset O$, so no entries not in $O$ have been filled. Furthermore, $O \setminus C = \{G(i,j) : 2 \nmid i, 2 \nmid j\}$. Note that each entry in $O \setminus C$ is at the center of a $3 \times 3$ square whose corners are filled entries of $C$. As such, each entry in $O \setminus C$ has room $\sqrt{2}$, while the minimum distance between any two such entries is 2. Thus, each of these entries will be filled by the VUSA without affecting the room at the others. Once each entry in $O$ has been filled, each empty entry will have room 1, so the VUSA will terminate. Thus, the VUSA gives us the same seating as the optimal seating algorithm, so $\Upsilon(n,n) = \Omega(n,n)$.

$\square$

### 5.4 Pseudo-minimal values of $n$ in two dimensions

In computing the behavior of $\Upsilon(n,n)$, we can observe that $\Upsilon(n,n)$ is always the same for $n = 3 \cdot 2^k + 1$. This proof is somewhat simpler than the proof of **Theorem 5.2**, since we do not aim to prove that such cases are minimal, though we will still prove the number of filled entries.

**Theorem 5.3.** *Let $n = 3 \cdot 2^k + 1$, where $k \in \mathbb{N}_0$. Let $G$ be an empty $n \times n$ grid. Then $G$ is invariant. In particular, $\Upsilon(n,n) = \frac{1}{3}(n^2 + 2)$.*

*Proof.* If $k = 0$, then $n = 4$, and the VUSA will first fill the corners of $G$, followed by either $G(1,1)$ and $G(2,2)$, or $G(1,2)$ and $G(2,1)$, which will saturate $G$. Thus $\Upsilon(n,n) = 6 = \frac{1}{3}(4^2 + 2)$, and we are done. So, assume $k \geq 1$.

Since $k \in \mathbb{N}^+$, $G$ is a critical square grid with $a = 3$. Let $C$ be the critical set of $G$, i.e.,

$$C = \{G(i,j) : 3|i, 3|j\}$$

By **Theorem 5.1**, the VUSA fills every entry in $C$ before filling any other entries. At this point, each $4 \times 4$ square of entries whose corners are filled entries of $C$ has a $2 \times 2$ cluster of entries in its center, each tied with room $\sqrt{2}$ (the other empty entries in the grid have room 1). See **Figure 8** for an example.

```
X - - X - - X - - X - - X
- H H - H H - H H - H H -
- H H - H H - H H - H H -
X - - X - - X - - X - - X
- H H - H H - H H - H H -
- H H - H H - H H - H H -
X - - X - - X - - X - - X
- H H - H H - H H - H H -
- H H - H H - H H - H H -
X - - X - - X - - X - - X
- H H - H H - H H - H H -
- H H - H H - H H - H H -
X - - X - - X - - X - - X
```

**Figure 8:** The $13 \times 13$ grid with its critical set filled. [H] denotes an entry tied for the most room.

Since each of the entries tied for most room has room $\sqrt{2}$ and the minimum distance between any two entries not in the same cluster is 2, each cluster will be filled by the VUSA without being affected by one another. Given any cluster, one of the four corners will be filled first. Then, its adjacent entries will have room 1, but the opposite corner still has room $\sqrt{2}$, so it will be filled as well. Once this happens in each cluster, the grid will be saturated, and the algorithm will terminate. Since the VUSA fills exactly two entries in each cluster, the VUSA always fills $G$ with the same number of entries, so $G$ is invariant.

In particular, the VUSA fills the entries of $C$, plus two entries for each cluster between entries of $C$. Given any $a$, there are $\frac{1}{a}(n-(n \mod a))+1$ multiples of $a$ from 0 to $n$. Thus, since $n-1 \equiv 0 \pmod 3$, there are $\frac{1}{3}(n-1)+1 = \frac{1}{3}(n+2)$ multiples of 3 from 0 to $n-1$, so there are $(\frac{1}{3}(n+2))^2$ entries in $C$. There are $\frac{1}{3}(n-1)$ spaces between these multiples of 3, so there are $(\frac{1}{3}(n-1))^2$ clusters. Together, the total entries filled by the VUSA is given by

$$\Upsilon(n,n) = \left(\frac{1}{3}(n+2)\right)^2 + 2\left(\frac{1}{3}(n-1)\right)^2 = \frac{1}{3}(n^2+2)$$

□

**Definition 5.3.** *Let $n = 3 \cdot 2^k + 1$, where $k \in \mathbb{N}_0$. Let $G$ be an empty $n \times n$ grid. Then $G$ is called a* <u>*pseudo-minimal square grid*</u>*.*

Though these cases of square graphs are invariant, they are not true minimal numbers, hence the term "pseudo-minimal". These cases do give a relatively low proportion of entries filled by the VUSA, but it is not the lowest proportion possible, even locally; the VUSA fills $17/49 \approx 0.3469$ of the entries of the $7 \times 7$ grid, while it fills at most $21/64 \approx 0.3281$ of the $8 \times 8$ grid.

More importantly, the VUSA does not give the most minimal seating possible in two-dimensions. For example, the VUSA fills the $7 \times 7$ grid with 17 entries, but this grid can be saturated with 14 entries, as shown in **Figure 9**.

```
X - - - X - -
- - X - - - X
X - - - X - -
- - X - - - X
X - - - X - -
- - X - - - X
X - - - X - -
```

**Figure 9:** A saturation of the $7 \times 7$ grid using only 14 filled entries.

Perhaps the VUSA would give a truly minimal value for certain forms of $n$ if we used a different distance metric or increased the room required in a saturated grid, but these are questions for future papers. Another goal for a future paper would be to determine the actual minimal seating algorithm. So far, it seems that the best approach is to fill entries in a tilted square pattern such that for each filled entry $G(i,j)$, $G(i+2,j+1)$ and $G(i-1,j+3)$ are filled as well (if they are in the grid). This seems to fill the middle of a large grid in the most minimal way possible, but it leaves unfortunate gaps around the border. In any case, if this is true for the center it could give the limit of the minimal fraction of seats at $n$ goes to infinity, even for rectangular grids, since the fraction of the grid taken up by the borders tends to zero.

### 5.5 End behavior of $\Upsilon(n,n)$

One thing that **Theorem 5.3** is currently useful for is showing the end behavior of $\Upsilon(n,n)$. If $n = 3 \cdot 2^k = 1$, this theorem gives us the fact that

$$\frac{\Upsilon(n,n)}{n^2} = \frac{\frac{1}{3}(n^2+2)}{n^2}$$

Therefore,

$$\lim_{k\to\infty} \frac{\Upsilon(3\cdot 2^k+1, 3\cdot 2^k+1)}{(3\cdot 2^k+1)^2} = \frac{1}{3}$$

However, **Theorem 5.2** shows that when $n = 2^k + 1$, then

$$\frac{\Upsilon(n,n)}{n^2} = \frac{\frac{1}{2}(n^2+1)}{n^2}$$

Therefore,

$$\lim_{k\to\infty} \frac{\Upsilon(2^k+1, 2^k+1)}{(2^k+1)^2} = \frac{1}{2}$$

Thus, for any $\varepsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ such that $|\Upsilon(n,n) - \frac{1}{2}| < \varepsilon$, as well as infinitely many $n \in \mathbb{N}_0$ such that $|\Upsilon(n,n) - \frac{1}{3}| < \varepsilon$. As such, the limit of $\Upsilon(n,n)$ as $n$ tends to infinity does not exist.

### 5.6 Variance of pseudo-minimal square arrangements

Though an $n \times n$ grid $G$ is invariant for $n = 3 \cdot 2^k + 1$, the actual arrangement of the filled entries in $G'$ varies. In the proof of **Theorem 5.3**, we showed that the critical set of $G$ will divide $G$ into $2 \times 2$ clusters of entries tied for the most room. The VUSA will fill either set of opposite corners of each cluster with equal likelihood, i.e., for each entry $G(i,j)$ such that $i \equiv 1 \pmod 3$ and $j \equiv 1$, there is a $\frac{1}{2}$ chance that $G(i,j)$ and $G(i+1,j+1)$ will be filled, and a $\frac{1}{2}$ chance that $G(i,j+1)$ and $G(i+1,j)$ will be filled. Equivalently, in each cluster either both entries such that $(i+j) \mid 2$ or both entries such that $(i+j) \nmid 2$ will be filled.

Note: this fact is used to minimize computation time when carrying out the VUSA on pseudo-minimal square grids, since we just need to fill the critical set and determine one random number for each cluster, which would be an algorithm with quadratic time.

Since there are $(\frac{1}{3}(n-1))^2 = (2^k)^2$ clusters, and each cluster can be filled two different ways, there are $2^{4^k}$ ways that the grid can be filled. Some of these are rotations and/or reflections of others, but this symmetry is less pronounced for higher values of $k$, and $2^{4^k}$ grows very quickly (for $k = 2$, there are 65,536 arrangements), so we will only classify the symmetries for $k = 0$ and $k = 1$ in this paper.

For $k = 0$, the $4 \times 4$ grid only has one cluster, so there are only two ways that the VUSA can fill $G$ (see **Figure 10**). Each result is a reflection of the other, so there is only one result up to symmetry.

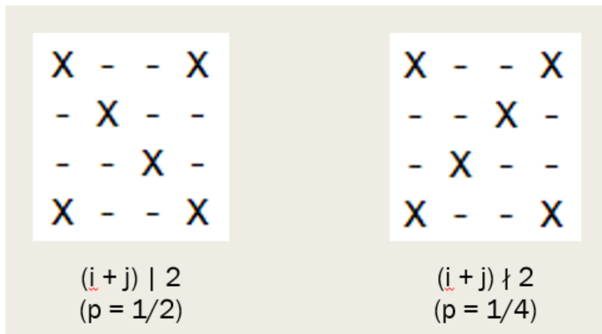



**Figure 10:** The two possible results for the VUSA on the empty $4 \times 4$ grid, classified by which cluster has entries that satisfy $2|(i + j)$.

For $k = 1$, the $7 \times 7$ grid has four clusters, so there are 16 possible results of the VUSA. We could classify the results of the clusters by the odd or even entries, but it is easier to follow the symmetry if we classify them by whether the diagonal of the filled entries "points" toward the center entry, i.e., a cluster is said to point to the center if the distance from that entry to $G(3, 3)$ is $\sqrt{2}$. There is only one of the 16 results in which zero clusters point to the center. Likewise, there is only one result in which all four clusters point to the center. These results thus occur with probability $\frac{1}{16}$. If only one cluster points toward the center, then there are four choices for this cluster, so this occurs with probability $\frac{1}{4}$. Similarly, the probability that three clusters will point toward the center (i.e., only one will *not*) is also $\frac{1}{4}$. If two clusters point toward the center, then these clusters can be opposite or adjacent to each other. If they are opposite, then there are only two unique results, so this occurs with probability $\frac{1}{8}$. If they are adjacent, then they can line up with any of the four edges of $G$, so there are four unique results, so the probability of exactly two adjacent clusters pointing to the center is $\frac{1}{4}$. Altogether, the 16 possible outcomes reduce to 6 unique patterns up to symmetry.

## 5.7 Labyrinthine patterns & colorings on pseudo-minimal square grids

For pseudo-minimal square grids, intricate patterns are generated by the independent clusters which appear once the critical set is filled generate. Specifically, the empty entries of the final grid can be partitioned into contiguous sections.

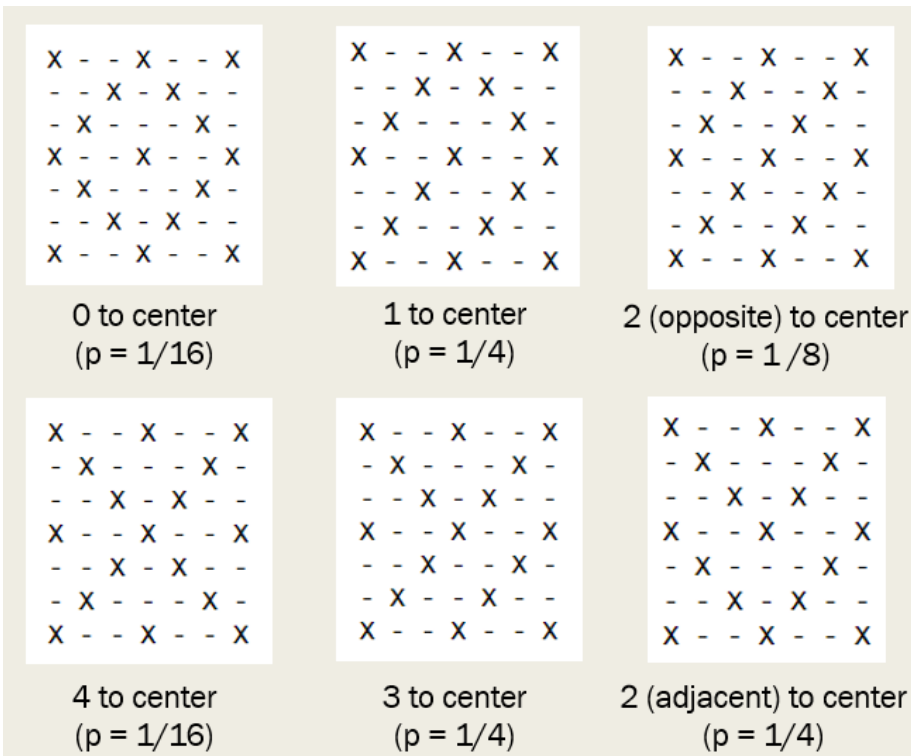



**Figure 11:** The six unique patterns (up to symmetry) for the VUSA on the empty $7 \times 7$ grid, classified by which clusters have entries filled that point toward the center.

**Definition 5.4.** *Let $G$ be a pseudo-minimal square grid which has been saturated by the VUSA. A section of $G$ is a set $A$ of empty entries with such that for any entry $G(i,j)$ in $A$, every empty entry adjacent to $G(i,j)$ is also in $A$.*

Note that the path along any section of empty spaces never forks or comes to a dead end; it only enters and exits the grid or returns to its starting point. As such, these patterns are not exactly mazes, but rather labyrinths. See **Figure 1** for an example.

If the contiguous sections of empty spaces in such grids are paths in labyrinth, then the diagonal lines of filled entries are the walls between paths. A wall may separate one part of a section from another part of the same section, but often these walls separate two separate sections, like borders on a political map. We refer to such sections as adjacent:

**Definition 5.5.** *Let $G$ be a pseudo-minimal grid saturated by the VUSA and let $A$ and $B$ be sections of $G$. $A$ and $B$ are said to be adjacent if there exists an entry $a$ in $A$ and an entry $b$ in $B$ such that $d(a,b) = \sqrt{2}$.*

**Figure 12** shows an example grid, and **Figure 13** shows how this grid would be partitioned into sections. Any labeling order would suffice, but for our purposes the sections are labeled in the order from left to right and then top to bottom.

```
X - - X - - X - - X - - X
- - X - X - - - X - - X -
- X - - - X - X - - X - -
X - - X - - X - - X - - X
- X - - X - - X - - - X -
- - X - - X - - X - X - -
X - - X - - X - - X - - X
- - X - X - - - X - - X -
- X - - - X - X - - X - -
X - - X - - X - - X - - X
- - X - - X - X - - X - -
- X - - X - - - X - - X -
X - - X - - X - - X - - X
```

**Figure 12:** One result of the VUSA on the empty $13 \times 13$ grid.

**Definition 5.6.** *A graph $\mathcal{G}$ is a set $\{V, E\}$, where $V$ is a set, called the vertex set of $\mathcal{G}$, whose elements are called the vertices of $\mathcal{G}$, and $E$ is a collection of two-element subsets of $V$, called the edge set of $\mathcal{G}$. For each element $(v_1, v_2)$ in $E$, $(v_1, v_2)$ is said to be an edge between $v_1$ and $v_2$, and $v_1$ and $v_2$ are said to be adjacent.*

**Definition 5.7.** *A graph $\mathcal{G}$ is said to be planar if $\mathcal{G}$ can be drawn in the plane with no edges crossing over each other. Once a planar graph $\mathcal{G}$ is drawn in the plane, $\mathcal{G}$ is called a plane graph.*

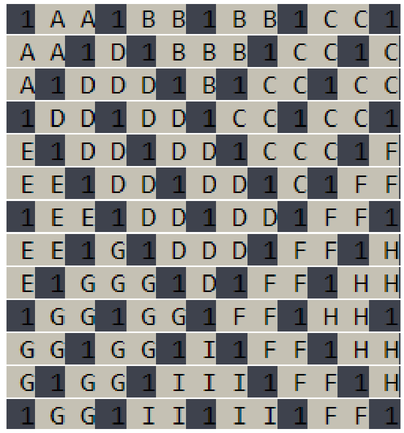



**Figure 13:** The empty entries of **Figure 14** are partitioned into sections, each labeled with a different letter (filled entries are colored black and labeled "1").

Drawing from graph theory, it feels only natural to construct a graph $\mathcal{G}$ to represent a given saturated pseudo-minimal square grid $G$, where the vertices of $\mathcal{G}$ are the sections of $G$, and two vertices are adjacent if and only if the corresponding sections are adjacent in $G$. Moreover, we can draw this graph on top of the given grid by placing the vertices at any entry inside their corresponding section, and drawing the edge between any two vertices so that it passes through the border between the two corresponding sections. This is the same process used to create a plane graph corresponding to a political map. As such, the graph resulting from this process is also a plane graph. In our example grid, section $A$ is adjacent only to section $D$ (note that sections which share a corner are not considered adjacent since the entries at these corners are distance 2 apart), while section $D$ is adjacent to sections $A, B, C, E, F$, and $G$. The full graph is shown in **Figure 14**.

**Definition 5.8.** *Given a graph $\mathcal{G}$, a coloring of $\mathcal{G}$ is an assignment of colors to the vertices of $G$. If a coloring does not assign the same color to any two adjacent vertices, then it is called a proper coloring.*

**Definition 5.9.** *Given a graph $\mathcal{G}$, the greedy coloring algorithm operates as follows:*

1. *Assign an order to the vertices and an order to the colors.*
2. *Assign the first color to the first vertex.*
3. *Assign the next vertex $v$ the lowest-ordered color such that no vertex adjacent to $v$ is already assigned this color.*
4. *Repeat step 3 for each remaining vertex, in order.*

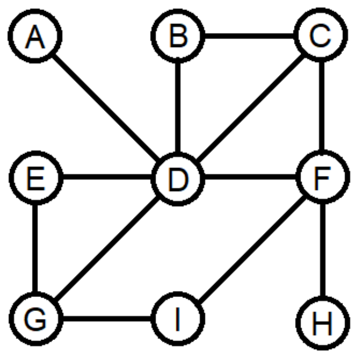



**Figure 14:** The plane graph corresponding to the grid in **Figure 13**

Given any graph, a common question to ask is how the graph can be colored. One of the simplest algorithms to find a proper coloring of any graph is called the greedy coloring algorithm (also known as the first-fit algorithm). In our example, let the color red have value 1, green have value 2, and yellow have value 3. We will use the same alphabetical order obtained by labeling the sections of the grid. Naturally, vertex $A$ is colored red. Since $B$ is not adjacent to $A$, it will be colored red as well. $C$, however, is adjacent to a red vertex at this point, so it must be colored green. $D$ is now adjacent to a red and a green vertex, so it must be colored yellow. The final result of this algorithm is shown in **Figure 15**. The resulting coloring can also be applied to the grid itself (see **Figure 16**).

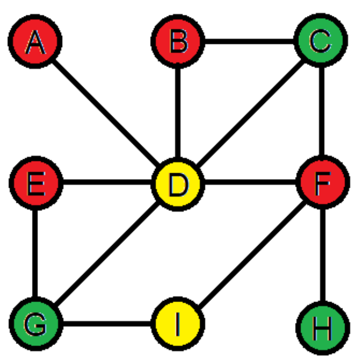



**Figure 15:** The result of the greedy coloring algorithm on the graph in **Figure 14**, with vertices ordered alphabetically.

**Definition 5.10.** *Given a graph $\mathcal{G}$, the minimum number of colors required to properly color $\mathcal{G}$ is known as the* <u>*chromatic number of $\mathcal{G}$*</u>*, denoted $\chi(\mathcal{G})$.*

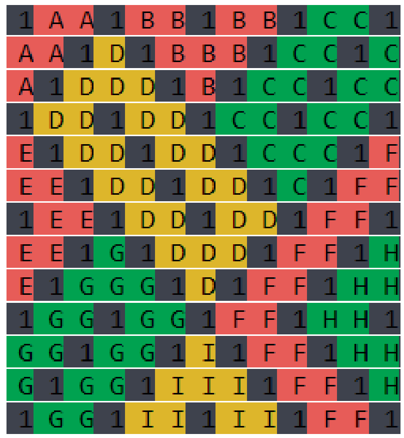



**Figure 16:** The coloring in **Figure 15** applied to the corresponding grid.

**Definition 5.11.** *Given a graph $\mathcal{G}$ with chromatic number $\chi(\mathcal{G})$, $\mathcal{G}$ is said to be* *well-colored* *if the greedy coloring algorithm colors $\mathcal{G}$ with $\chi(\mathcal{G})$ colors regardless of the ordering of vertices.*

**Theorem 5.4** (**The four-color map theorem [1]**). *Let $\mathcal{G}$ be a planar graph. Then $\chi(\mathcal{G}) \leq 4$.*

In our example graph, there are three vertices ($B, C$, and $D$) within the graph which are each adjacent to each other, so they must each be a different color, i.e., $\chi(\mathcal{G}) \geq 3$. Our ordering of the vertices caused the greedy algorithm to color $\mathcal{G}$ with three colors, so $\chi(\mathcal{G}) = 3$. In order to determine whether our graph is well-colored, however, we would need to check every possible ordering of vertices. By contrast, we only need to find a counter-example to show that a graph is not well-colored. Even so, it can be quite difficult to determine the chromatic number of a graph. Thankfully, the four-color map theorem gives us that any planar graph can be colored with four colors or fewer. Thus, if the greedy coloring algorithm used five or more colors to color the graph corresponding to a pseudo-minimal square grid, we could show that not all such graphs are well-colored. In fact, after simplifying the VUSA to be able to handle $49 \times 49$ grids in less than an hour, I was able to generate ten such grids, which contained such a counter-example (see **Figure 1**). In fact, such examples can occur in $25 \times 25$ grids, but I believe they are much less likely, since I discovered this after many $25 \times 25$ trials and after discovering the $49 \times 49$ case.

**Theorem 5.5.** *Let $G$ be a pseudo-minimal square grid which has been saturated by the VUSA, and let $\mathcal{G}$ be the plane grid corresponding to the sections of $G$. Then $\mathcal{G}$ is not necessarily well-colored.*

*Proof.* (Contradiction): Assume $\mathcal{G}$ is well-colored for any pseudo-minimal square grid. Then the greedy coloring algorithm colors $\mathcal{G}$ with four or fewer colors.

**Figure 1** shows a grid given by one result of the VUSA on the pseudo-minimal $49 \times 49$ grid. Thus, let $\mathcal{G}$ be the graph corresponding to this grid. Order the vertices according to the labels in **Figure 1** (there are more than 26 sections, so the labels use Python's convention for character index values). Then the greedy algorithm gives the same coloring on $\mathcal{G}$ as shown by the sections in **Figure 1**. Thus, the greedy algorithm colors $\mathcal{G}$ with 5 colors, which is a contradiction. $\square$

# 6 Discussion and open problems

This paper expands the unfriendly seating algorithm into two dimensions, but it focuses particularly on empty grids and square grids. As such, our findings propose many questions for future research. We conjectured that $n$ of the form $2^k + 1$ or $3 \cdot 2^k + 1$ gives the only invariant $n \times n$ grids, but this has not been proven. Additionally, a future paper could analyze the VUSA on rectangular grids or nonempty grids, and potentially classify all possible invariant grids. For non-invariant grids, the maximum, minimum, probability mass function, expected value, and variance of might be found for the random variable $\Upsilon(m, n)$.

We defined the sections of pseudo-minimal square grids and their corresponding plane graphs, but we did not delve very deep in this topic. The number of sections created and the chromatic number of these graphs are also random variables whose distributions could be analyzed. To simplify this, it might help to classify the symmetries of these grids beyond $k = 1$. Contiguous sections can even be defined for arbitrary grids, though these findings may not follow as clear a pattern.

One of the most practical results of our current findings is that bathroom designers should round the number of toilets in a row to the nearest $2^k + 1$, in order to avoid wasting space. In most real situations, however, public behavior is not so simple. Particularly, in two-dimensional seating arrangements such as theaters and auditoriums, corner seats are probably the least desirable seats, and are avoided even if they have the most room. As such, future research could examine the VUSA when the corner-first convention is removed, or when each seat is assigned a desirability value to be somehow compared against its room. Along this same track, findings could be more relevant if the minimum room required by each person is increased (e.g., call a grid saturated if no empty seat has room 6 instead of room 1).

For a more abstract approach, one could use a different distance metric to calculate the room at each entry, such as the taxicab metric or the king metric. Our paper uses grids, which are effectively grid graphs, but future papers could generalize the VUSA to triangular or hexagonal graphs. The VUSA could potentially be applied to arbitrary connected graphs, even with arbitrary edge weights, using Dijkstra's algorithm [2] for the distance metric.

A current limitation of our findings is the computing time required to carry out the VUSA. One way to minimize computing time would be to explore the idea of independently resolving set of entries (i.e., a set of entries tied for the most room $r$, such that the minimum distance between any two of these entries is greater than $r$) which could all be filled at once. In addition to individual entries being independent, sometimes entire clusters of entries will resolve independently of the rest of the grid, as in our pseudo-minimal squares. This idea was used intermittently to prove theorems in this paper, but it is a powerful tool for simplifying the algorithm and future proofs. Once this is achieved, higher-dimensional grids can be analyzed, and we can test whether $n \times n \times n \times ...$ grids follow the same optimal and pseudo-minimal patterns, or generate higher-dimensional labyrinths.

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