Bridging Categories: Using the Singularity Category to Study Matrix Factorizations

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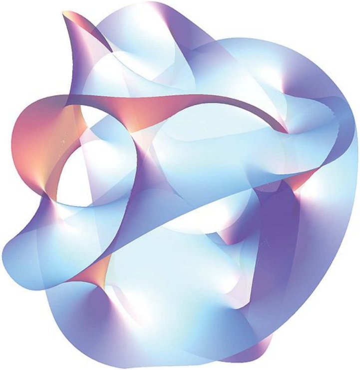
Why do we care?



Areas where this research could help:

- String theory
- Matrix algebra
- Factoring matrices is cool

What is Category Theory?

$$
\begin{tikzcd}
A \arrow[loop above, "1_A"] \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[loop above, "1_B"] \arrow[d, "g"] \\
& C \arrow[loop below, "1_C"]
\end{tikzcd}
$$

Examples of Categories:

- Rings and homomorphisms
- Sets and functions
- Vector spaces and linear transformations

Today's Topics:

- Category of matrix factorizations
- Singularity category

Category of Matrix Factorization

Let S be a commutative ring and $p \in S$.

Objects: Pairs of S-modules (F, G) with maps

$$F \xrightarrow{\phi} G \xrightarrow{\psi} F$$

satisfying $\psi \circ \phi = p \cdot 1_F$ and $\phi \circ \psi = p \cdot 1_G$.

Morphisms: Pairs of S-module homomorphisms (f, g) making the diagram commute:

$$\begin{array}{ccccc} F_1 & \xrightarrow{\phi_1} & G_1 & \xrightarrow{\psi_1} & F_1 \\ \downarrow f & & \downarrow g & & \downarrow f \\ F_2 & \xrightarrow{\phi_2} & G_2 & \xrightarrow{\psi_2} & F_2 \end{array}$$

Goal

Find an exceptional sequence of maximal length

- Analogous to finding a basis for a vector space, we aim to construct a full exceptional collection.

$$E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots \longrightarrow E_n$$

Building blocks for the category.

Singularity Category Objects

Objects: The objects are bounded complexes of graded modules

$$\cdots \to \overset{-2}{0} \to \overset{-1}{A} \to \overset{0}{B} \to \overset{1}{C} \to \overset{2}{0} \to \cdots$$

but every bounded complex is isomorphic to a complex with only one nonzero module,

$$\cdots \to \overset{-1}{0} \to \overset{0}{0} \to \mathbf{k}(\overset{1}{-}\vec{x}) \to \overset{2}{0} \to \cdots$$

so we'll write

$$\mathbf{k}(-\vec{x})[1]$$

Singularity Category Morphisms

Morphisms: The morphisms are called roofs. If $\mathbf{k}(-\vec{x})[1]$ and $\mathbf{k}(-\vec{x}-\vec{y})[2]$ are objects, the morphisms look like

$$\mathbf{k}(-\vec{x})[1] \xrightarrow{a} T \xleftarrow{s} \mathbf{k}(-\vec{x}-\vec{y})[2]$$

Where a and s are homomorphisms satisfying certain properties and T is a module satisfying other properties.

Computing the morphisms

To construct these hard to understand morphisms and build a full exceptional collection we have this recipe for simpler cases

· Find a sequence of projective modules that "approximately" represent M

$$P^* := \cdots \to A(-3) \to A(-2) \to A(-1) \to A \to M$$

· Construct the complex between P^* and N

$$Hom^*(P^*, N) := \cdots \leftarrow N(3) \leftarrow N(2) \leftarrow N(1) \leftarrow N \leftarrow 0$$

Which as P* represents M we can use to find Hom(M,N[i]) for N with different shifts in position

Computing the morphisms

Now we have two propostitions we use, the first says we can split these Homs for these different positions into Homs for different graded pieces

$$Hom(M, N[i]) = \bigoplus Hom(M, N(\ell)[i])$$

And then our final proposition helps us take this back to the singularity category

$$Hom_{sg}(M, N[i]) = Hom(M, N(-jw)[i + 2j])$$

Conclusion

- Use the formula to understand singularity category morphisms.
- Those are equivalent to morphisms in the category of matrix factorization.
- Given those morphisms, find a full exceptional sequence to construct the whole category.

Thank You!