Bridging Categories: Using the Singularity Category to Study Matrix Factorizations

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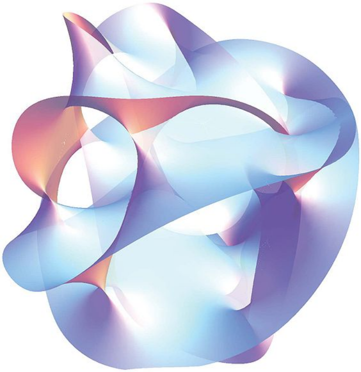
Why do we care?



Areas where this research could help:

- String theory
- Matrix algebra
- Factoring matrices is cool

What is Category Theory?

$$\begin{tikzcd}
A \arrow[loop above, "1_A"] \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[loop above, "1_B"] \arrow[d, "g"] \\
& C \arrow[loop below, "1_C"]
\end{tikzcd}$$

Examples of Categories:

- Rings and homomorphisms
- Sets and functions
- Vector spaces and linear transformations

Today's Topics:

- Category of matrix factorizations
- Singularity category

Goal

Find an exceptional sequence of maximal length

- Analogous to finding a basis for a vector space

$$E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots \longrightarrow E_n$$

Building blocks for the category

Matrix Factorizations

$$3 \cdot 2 = 6$$

Matrix Factorizations

$$x^2 + 2x + 1 = (x + 1)^2$$

$$\begin{bmatrix} -5 & 2 & 2 \\ 0 & 5 & 3 \\ 6 & 7 & 1 \end{bmatrix} = ?$$

Matrix Factorizations

$$\begin{bmatrix} -5 & 2 & 2 \\ 0 & 5 & 3 \\ 6 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 2 & 3 & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

Matrix Factorizations

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Category of Matrix Factorization

Let S be a commutative ring and $p \in S$.

Objects: Pairs of S-modules (F, G) with maps

$$F \xrightarrow{\phi} G \xrightarrow{\psi} F$$

satisfying $\psi \circ \phi = p \cdot 1_F$ and $\phi \circ \psi = p \cdot 1_G$.

Morphisms: Pairs of S-module homomorphisms (f, g) making the diagram commute:

$$\begin{array}{ccccc} F_1 & \xrightarrow{\phi_1} & G_1 & \xrightarrow{\psi_1} & F_1 \\ \downarrow f & & \downarrow g & & \downarrow f \\ F_2 & \xrightarrow{\phi_2} & G_2 & \xrightarrow{\psi_2} & F_2 \end{array}$$

Singularity Category Objects

Objects: bounded complexes of graded modules

$$\cdots \to \overset{-2}{0} \to \overset{-1}{A} \to \overset{0}{B} \to \overset{1}{C} \to \overset{2}{0} \to \cdots$$

or

$$\cdots \to \overset{-1}{0} \to \overset{0}{0} \to \overset{1}{\mathbf{k}(-\vec{x})} \to \overset{2}{0} \to \cdots$$

or

$$\mathbf{k}(-\vec{x})[1]$$

Singularity Category Morphisms

Morphisms: roofs

$$\mathbf{k}(-\vec{x})[1] \xrightarrow{a} T \xleftarrow{s} \mathbf{k}(-\vec{x}-\vec{y})[2]$$

Where a and s are homomorphisms satisfying certain properties and T is a module satisfying other properties

Computing the morphisms

To find morphisms between M and N, we find

$$P^* := \cdots \to A(-3) \to A(-2) \to A(-1) \to A \to M$$

Then we construct the complex between P^* and N

$$Hom^*(P^*, N) := \cdots \leftarrow N(3) \leftarrow N(2) \leftarrow N(1) \leftarrow N \leftarrow 0$$

to get Hom(M, N[i])

Computing the morphisms

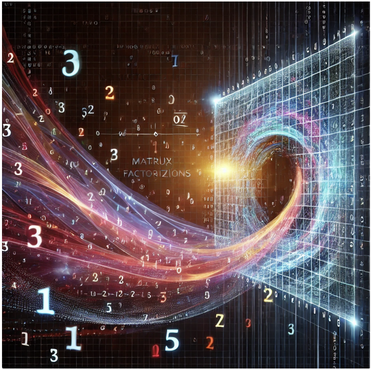
We can split Hom(M, N[i]) into

$$Hom(M, N[i]) = \bigoplus Hom(M, N(\ell)[i])$$

And we can send this back to the singularity category

$$Hom_{sg}(M, N[i]) = Hom(M, N(-jw)[i + 2j])$$

Conclusion



Use formula

$\downarrow$

Singularity
Category

$\downarrow$

Category of
Matrix Factorizations

$\downarrow$

Find full
exceptional sequence

Thank You!