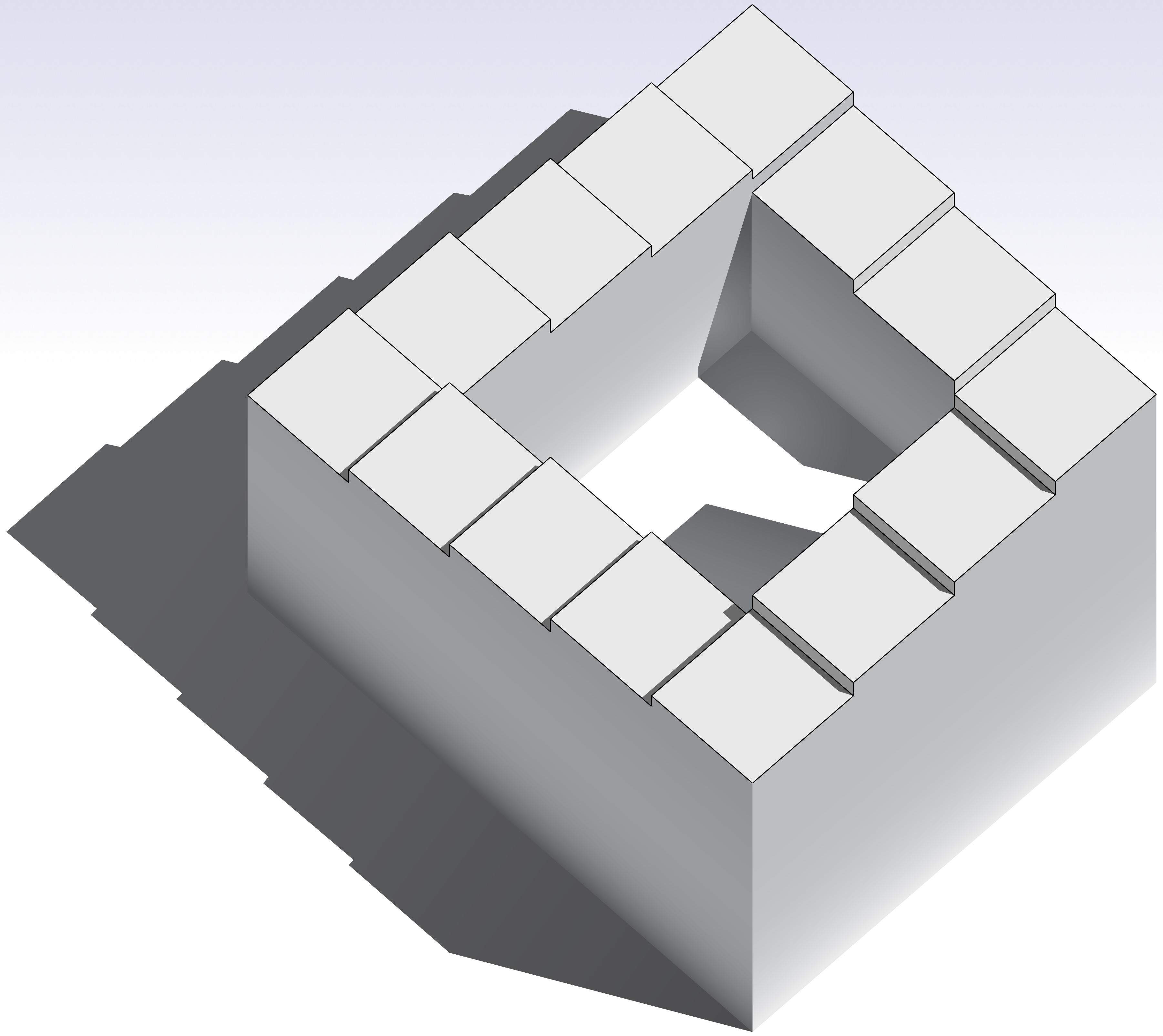
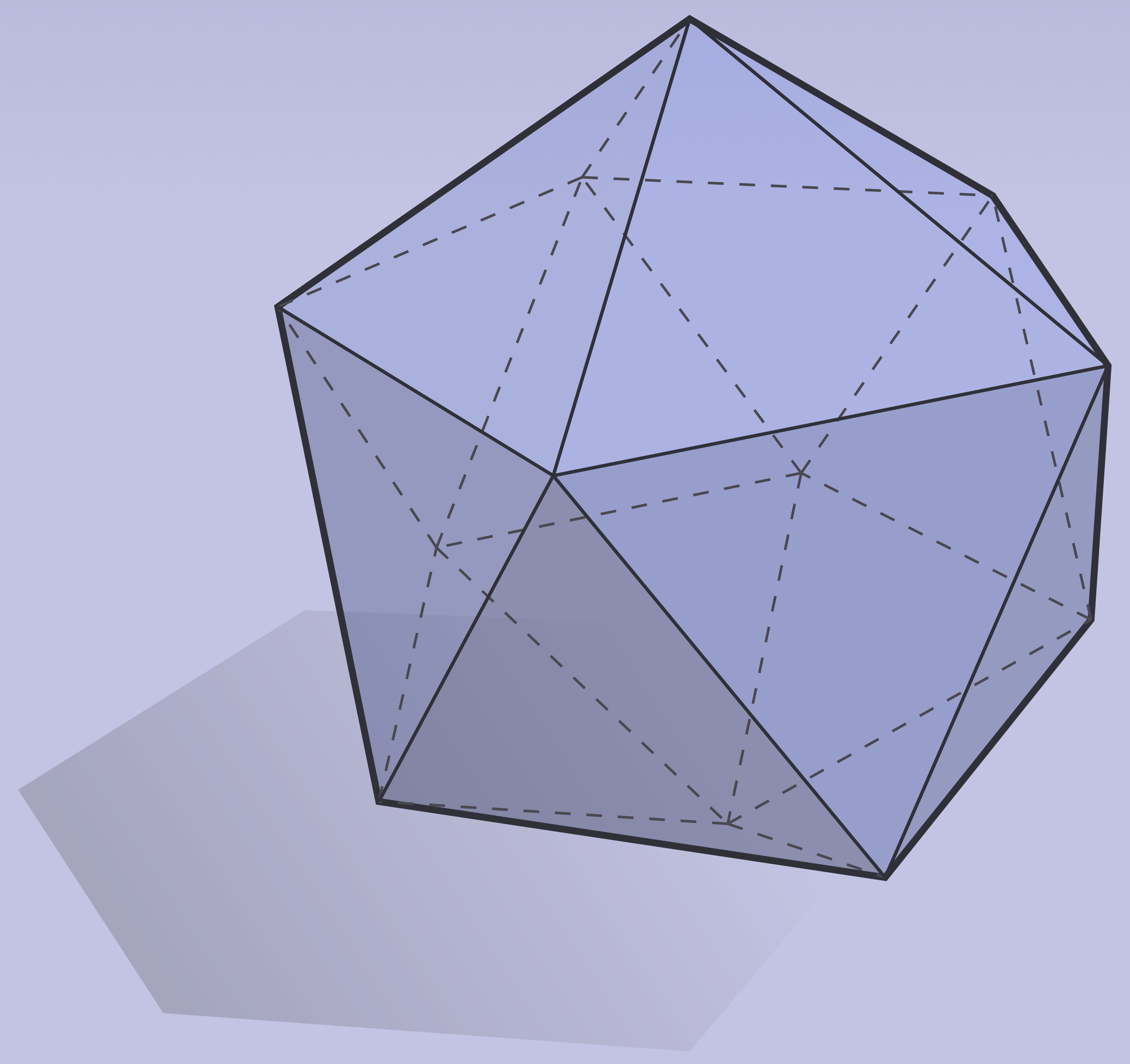


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858B • Fall 2017

LECTURE 4:
DIFFERENTIAL FORMS IN R^n



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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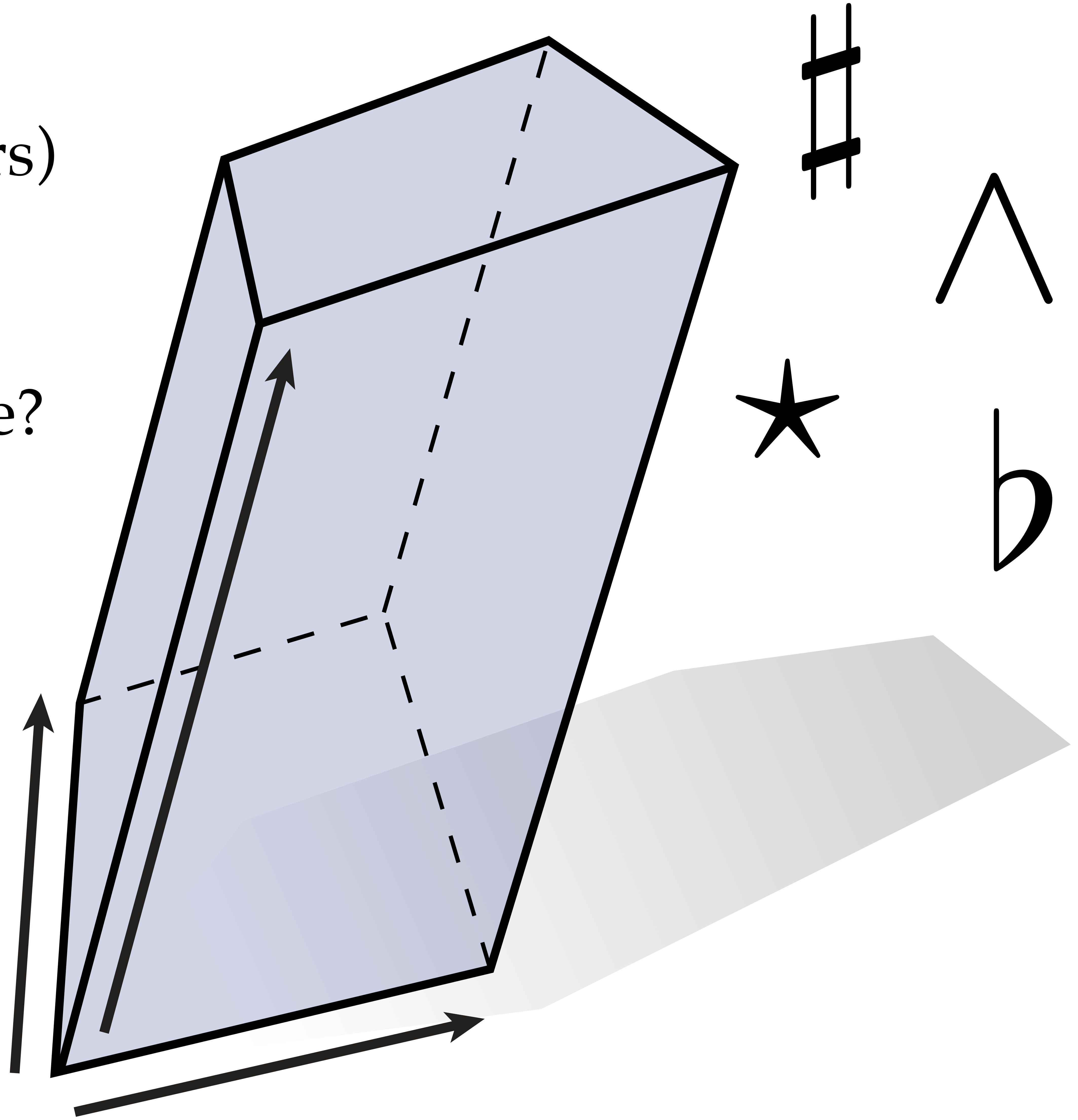
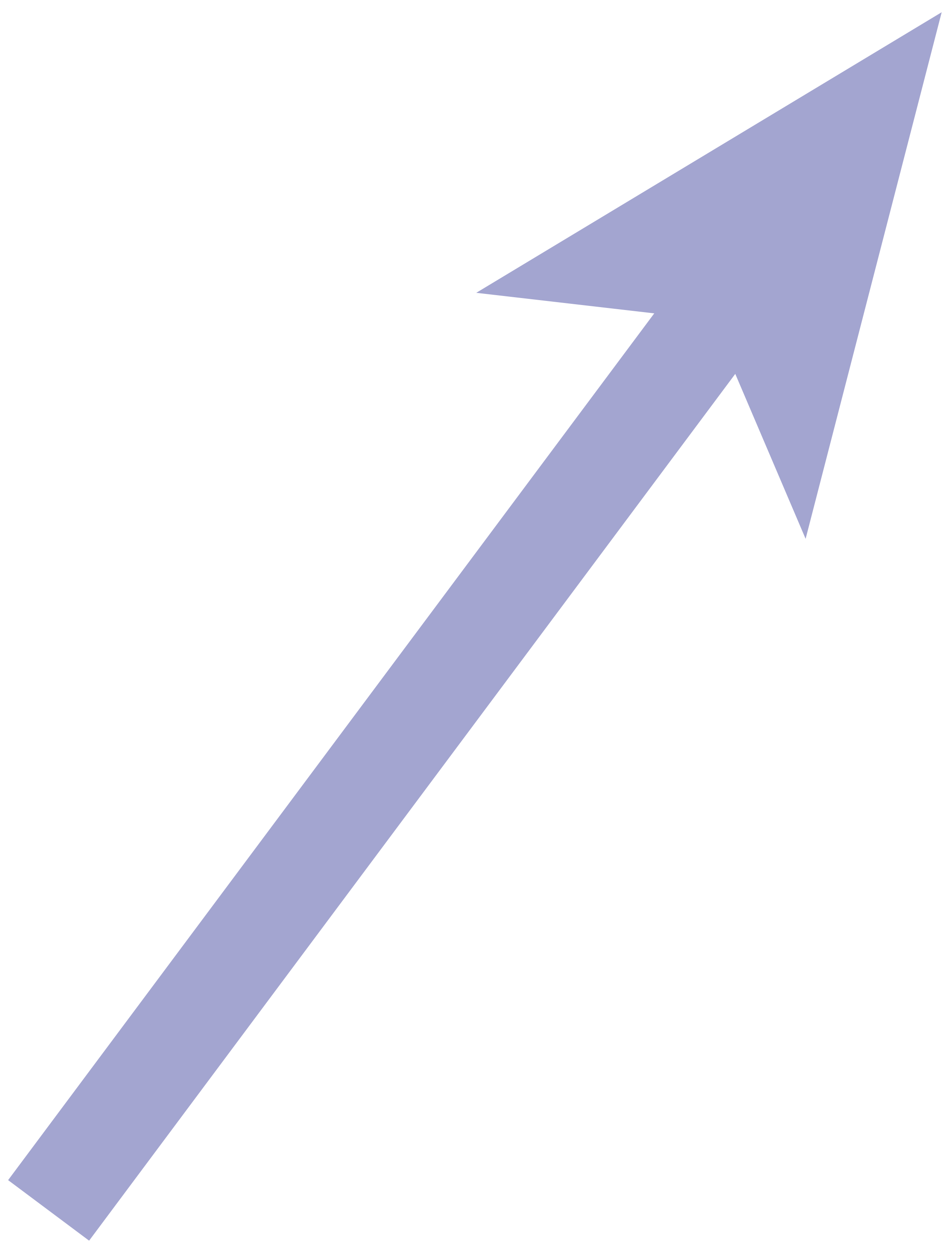
Exterior Calculus — Overview

- *Last time:*
  - **Exterior algebra**—"little volumes" (k-vectors)
- *This time:*
  - **Exterior calculus**—how do k-vectors change?
  - Before we considered just isolated *k*-vectors
  - Now we'll put a *k*-vector at each point
  - For now, stick to "flat" spaces (R^n)
  - Later, consider "curved" spaces (manifolds)

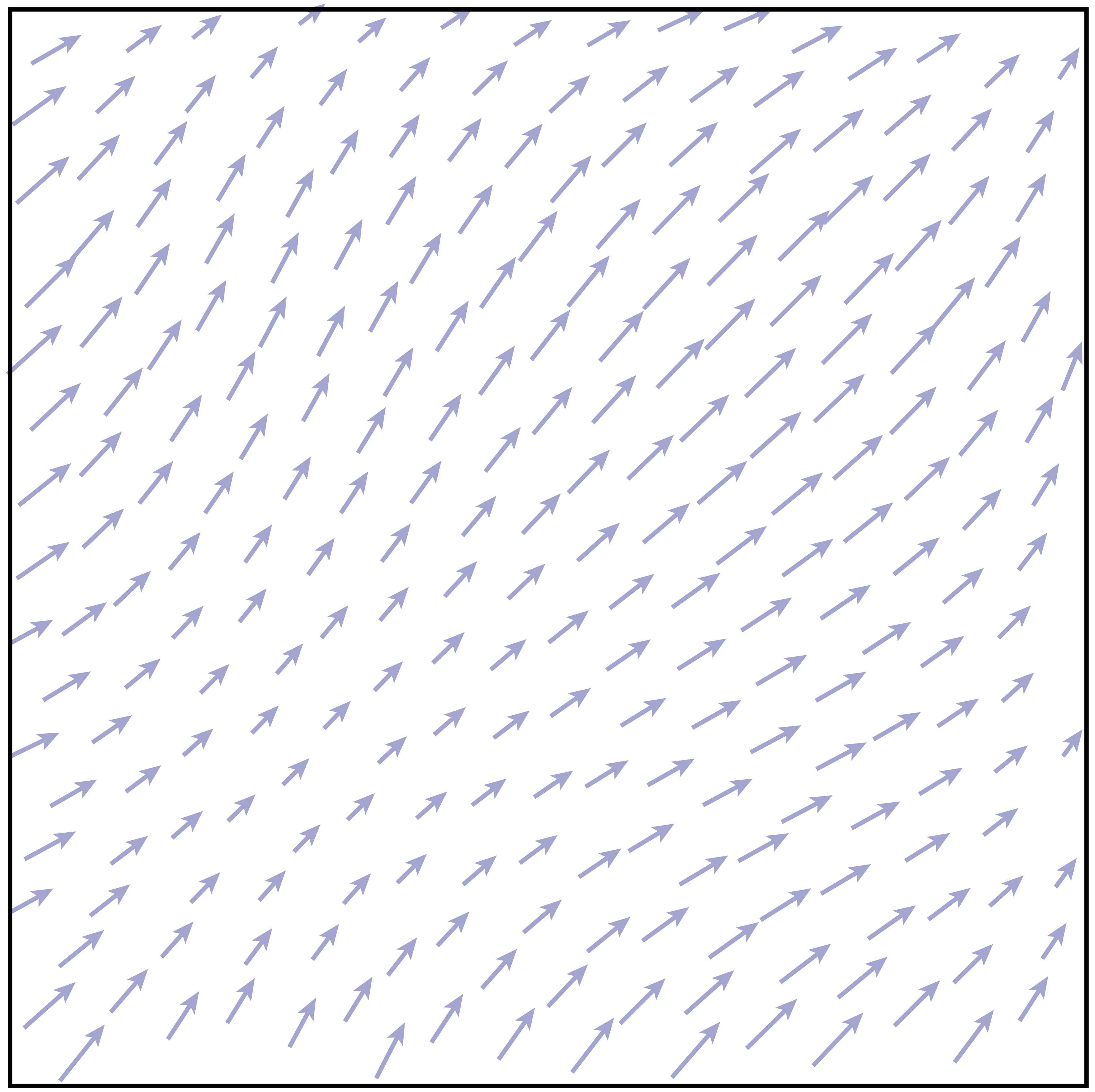


Review: Vector vs. Vector Field

- Recall that a vector *field* is an assignment of a vector to each point:



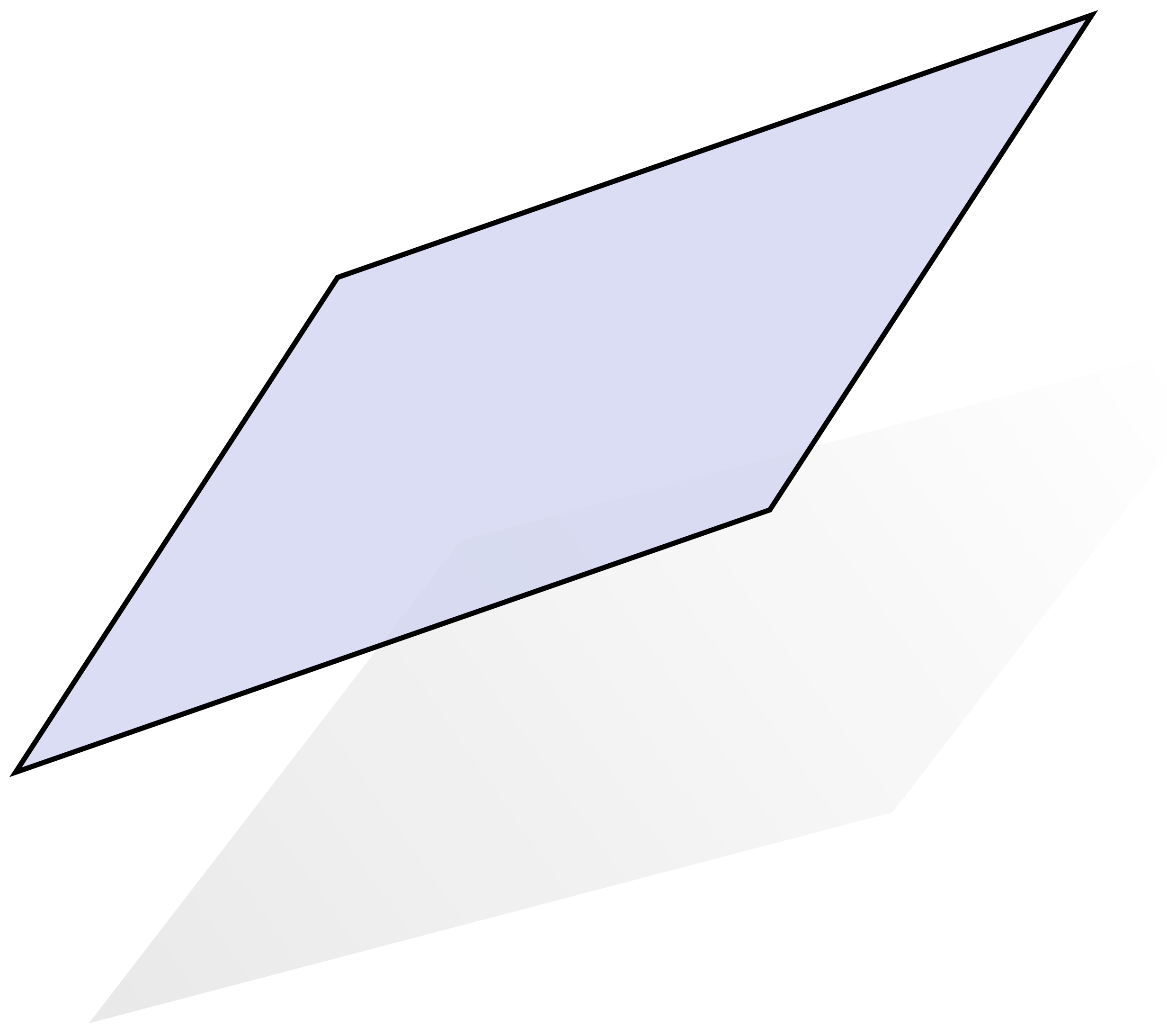
vector



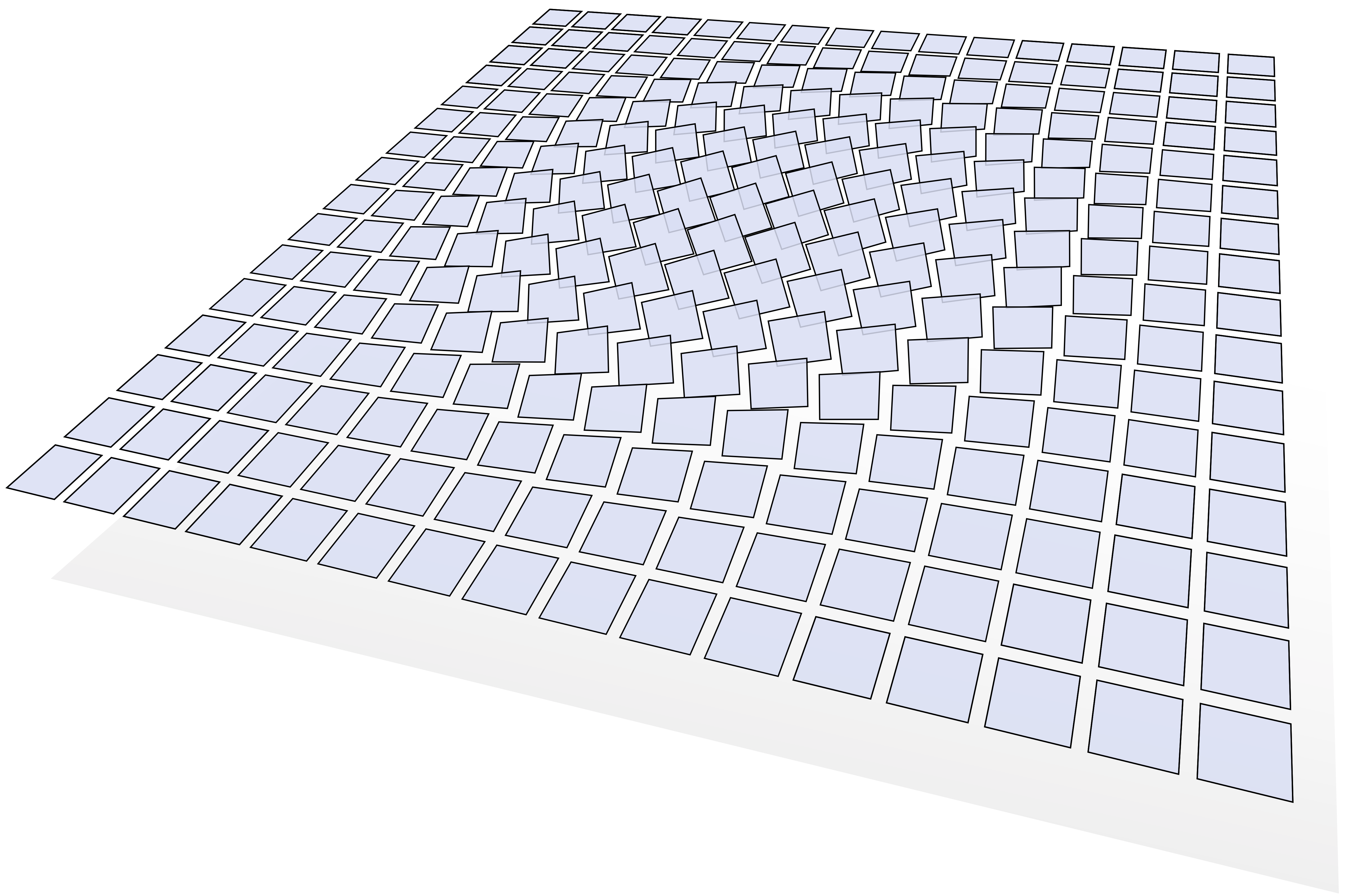
vector field

Preview: k-Vector vs. Differential Form

- A *differential k-form* will be an assignment of an object like a *k*-vector to each point:



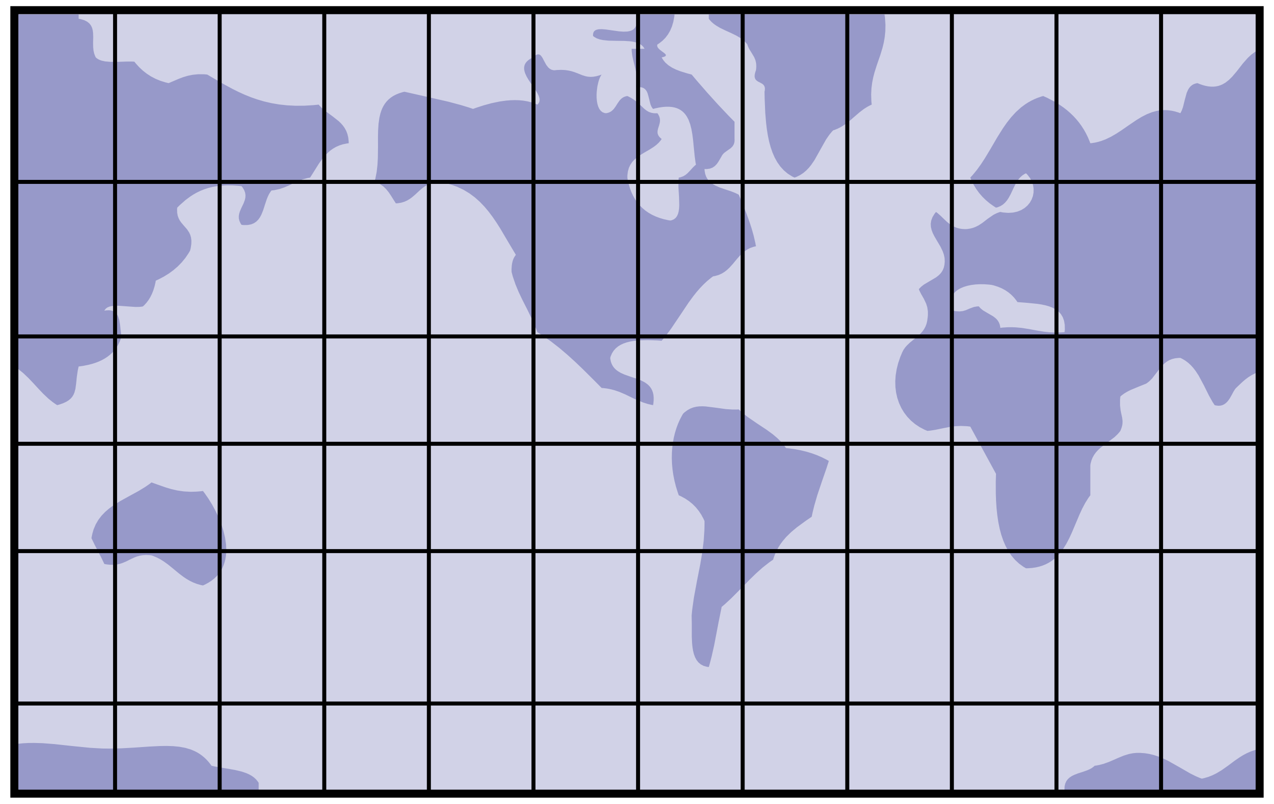
k-vector



differential form

Exterior Calculus: Flat vs. Curved Spaces

- For now, we'll only consider *flat* spaces like the 2D plane
  - Keeps all our calculations simple
  - Don't have to define *manifolds* (yet!)
- True power of exterior calculus revealed on *curved* spaces
  - In flat spaces, vectors and forms look *very* similar*
  - Difference will be less superficial on curved spaces
  - Close relationship to *curvature* (geometry)
  - Also close relationship to *mass* (physics)

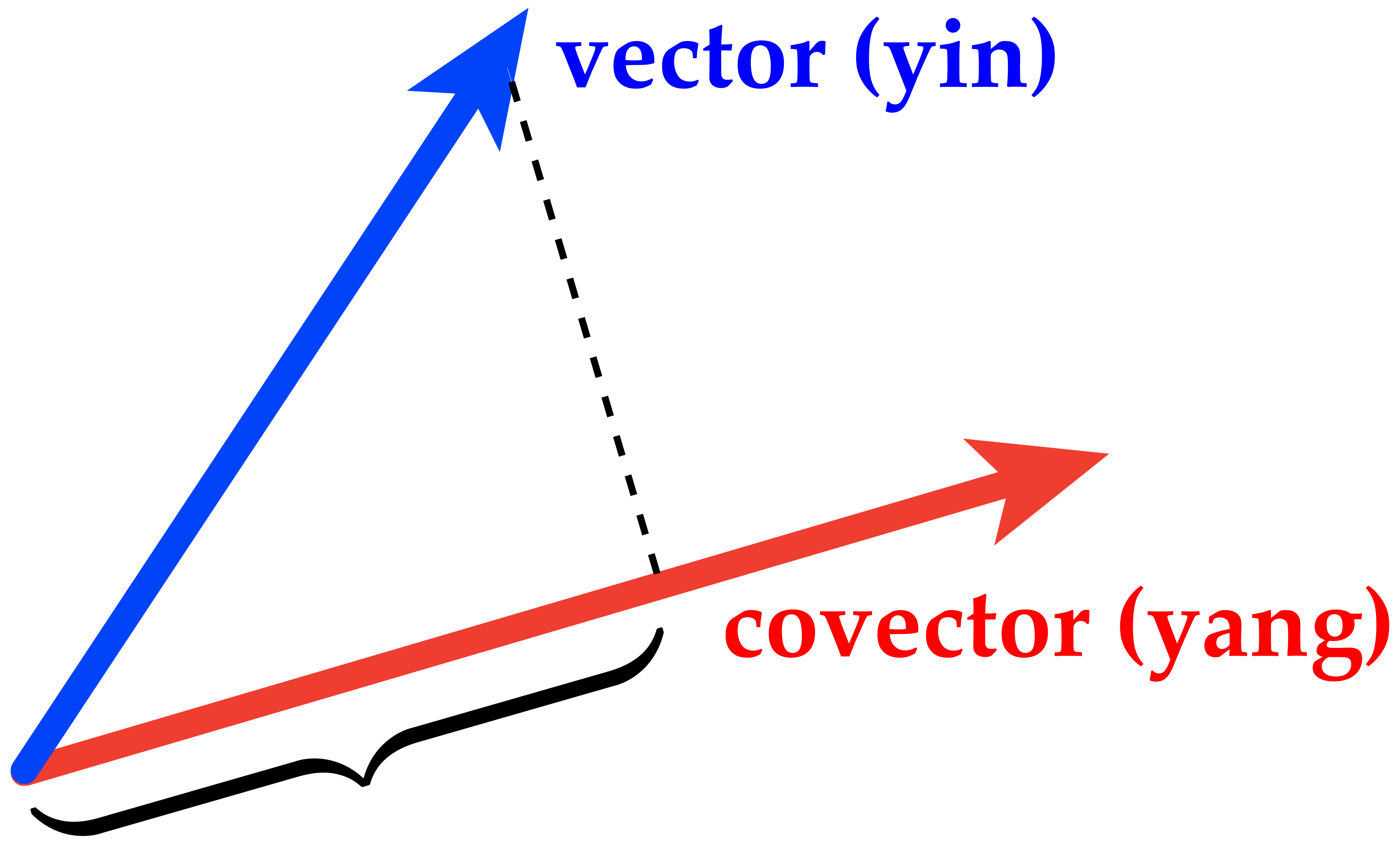


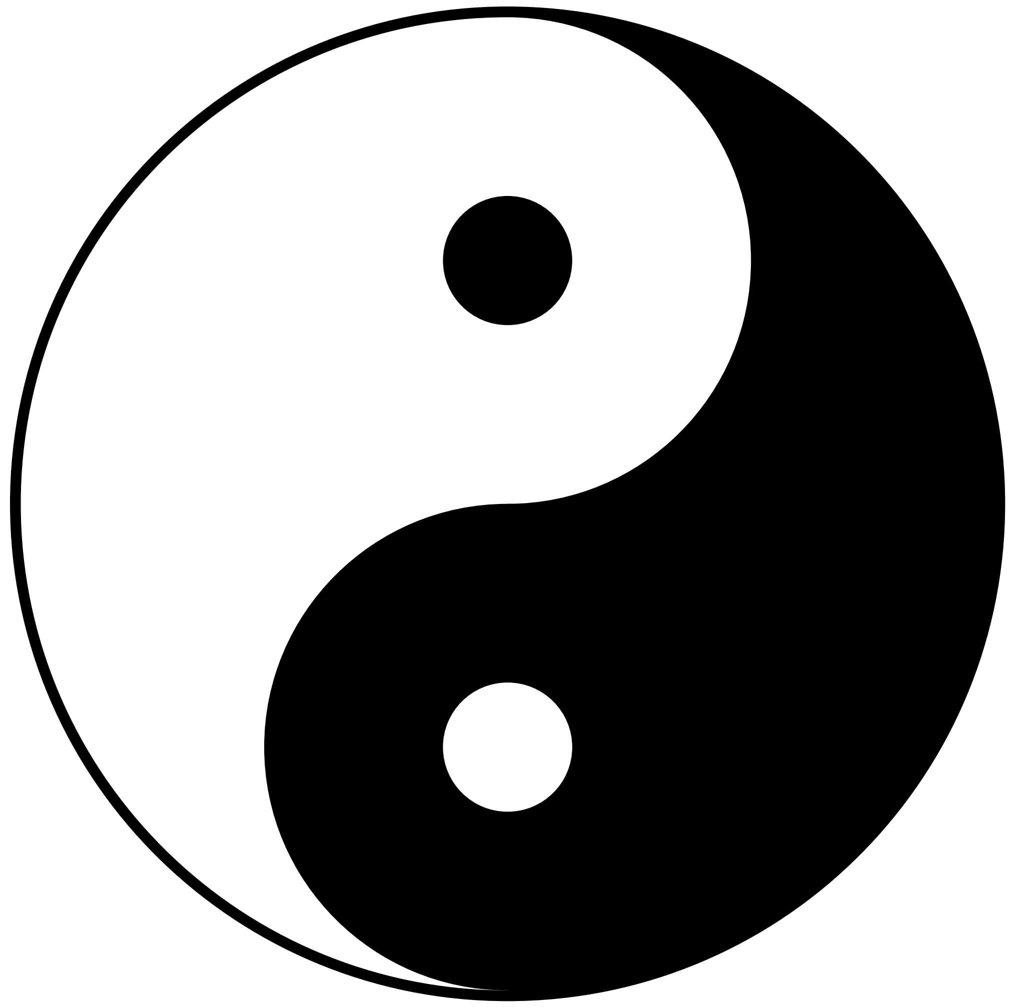
*So similar in fact you may wonder why we even bother to make a distinction! Well, stay tuned… :-)

Vectors & Covectors

Vector-Covector Duality

- Much of the expressivity of exterior calculus comes from the relationship between *vectors* and *covectors*.
- Loosely speaking:
  - **covectors** are objects that *"measure"*
  - **vectors** are objects that *"get measured"*
- We say that vectors and covectors are "dual"—roughly speaking, the are two sides of the same coin (but with important differences…).
- Ultimately, differential forms will be built up by taking wedges of covectors (rather than vectors).





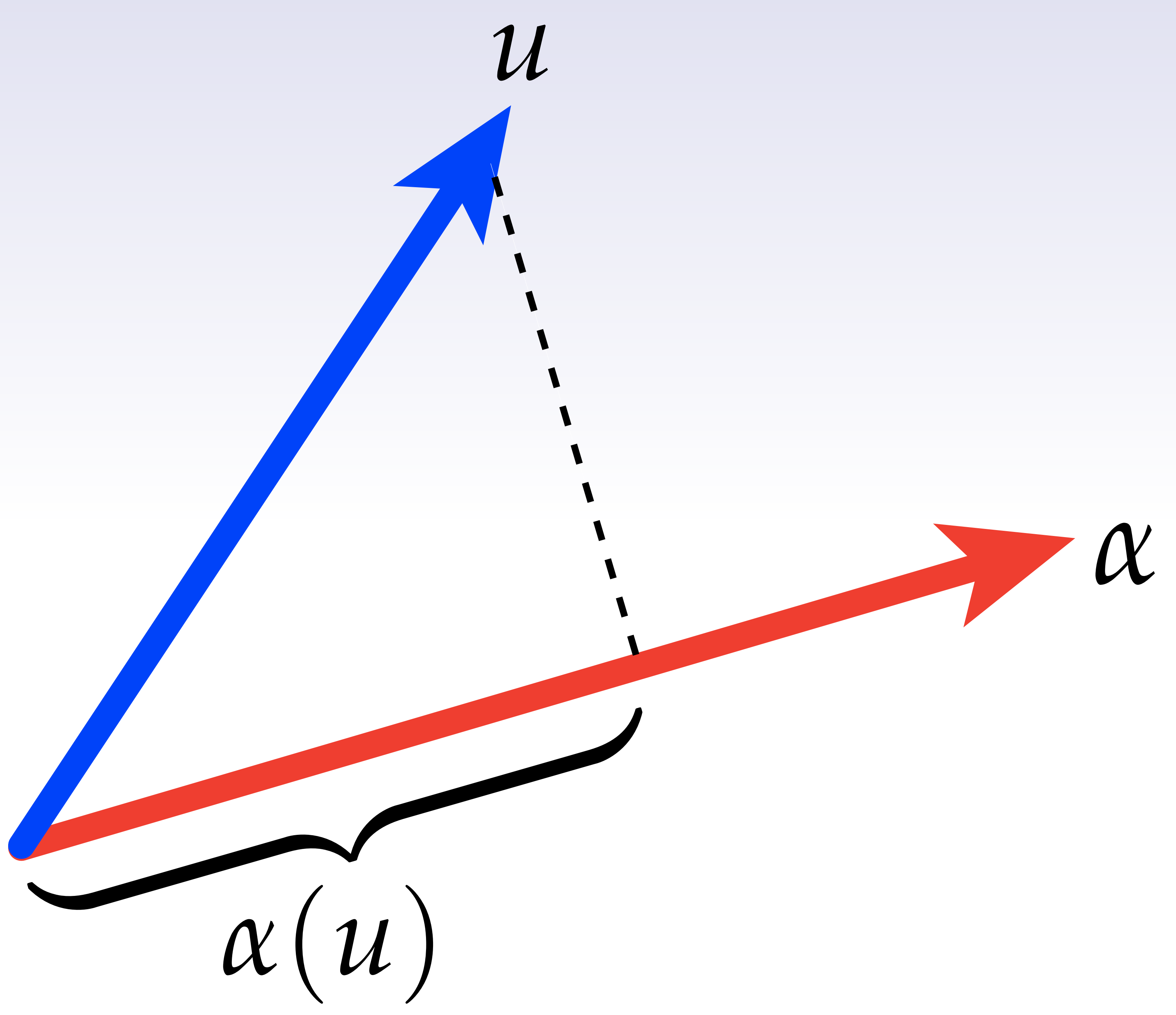
Analogy: Row & Column Vectors

In matrix algebra, we make a distinction between *row vectors* and *column vectors*:

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Q: Why do we make the distinction? What does it mean geometrically?
What does it mean as a *linear map*? (Is this convention useful?)

Vectors and Covectors



Analogy: *row vs. column vector*

Key idea: a covector *measures* length of vector along a particular direction

Dual Space & Covectors

Definition. Let V be any real vector space. Its *dual space* V^* is the collection of all linear functions $\alpha : V \to \mathbb{R}$ together with the operations of *addition*

$$(\alpha + \beta)(u) := \alpha(u) + \beta(u)$$

and *scalar multiplication*

$$(c\alpha)(u) := c(\alpha(u))$$

for all $\alpha, \beta \in V^*$, $u \in V$, and $c \in \mathbb{R}$.

Definition. An element of a dual vector space is called a *dual vector* or a *covector*.

(**Note:** unrelated to *Hodge dual*!)

Covectors — Example (R^3)

- As a concrete example, let's consider the vector space $V = R^3$
- Recall that a map f is *linear* if for all vectors $\mathbf{u}$, $\mathbf{v}$ and scalars a, we have

$$f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}) \qquad \textbf{and} \qquad f(a\mathbf{u}) = af(\mathbf{u})$$

- **Q:** What's an example of a *linear* map from R^3 to R?
  - Suppose we express our vectors in coordinates $\mathbf{u} = (x,y,z)$
- **A:** One of *many* possible examples: $f(x,y,z) = x + 2y + 3z$
- **Q:** What are *all* the possibilities?
- **A:** They all just look like f$(x,y,z) = ax + by + cz$ for constants a,b,c
- In other words in Euclidean R^3, a *covector* looks like just another 3-vector!

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

vector

$$\begin{bmatrix} a & b & c \end{bmatrix}$$

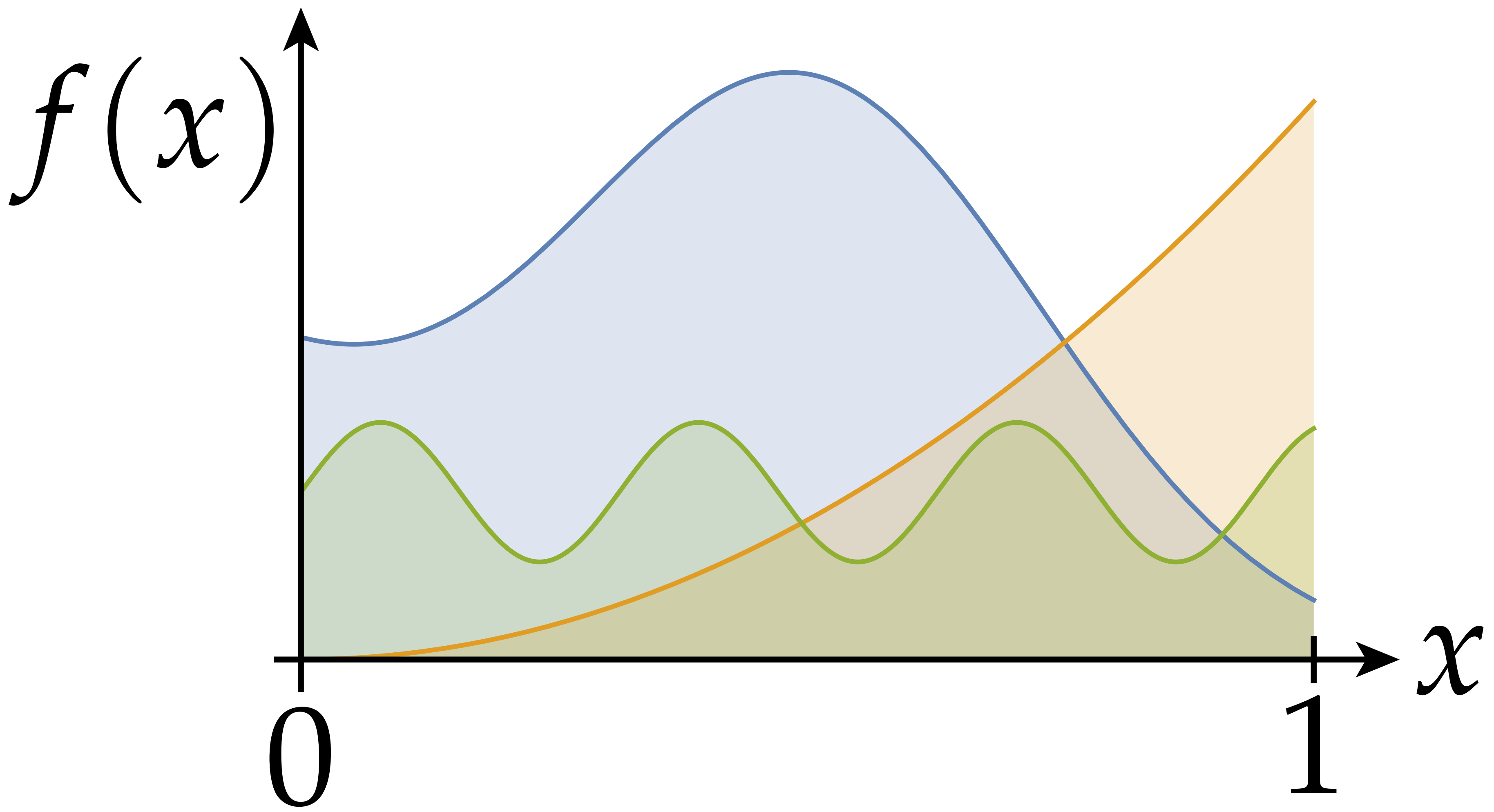
covector

Covectors—Example (Functions)

- If covectors are just the same as vectors, why even bother?
- Here's a more interesting example:

Example. Let V be the set of integrable functions $f : [0,1] \to \mathbb{R}$, and consider maps

- $\phi : V \to \mathbb{R}; f \mapsto \int_0^1 f(x)\,dx$
- $\delta : V \to \mathbb{R}; f \mapsto f(0)$



Is V a vector space? Are ϕ and δ covectors?

Key idea: the difference between primal & dual vectors is not merely superficial!

Sharp and Flat

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \xrightarrow{\mathsf{T}} \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$u, v \xrightarrow{\flat} u^\flat(v)$$

$$\alpha, \beta \xrightarrow{\sharp} \alpha(\beta^\sharp)$$

Analogy: *transpose*

(What's up with the musical symbols? Will see in a bit…)

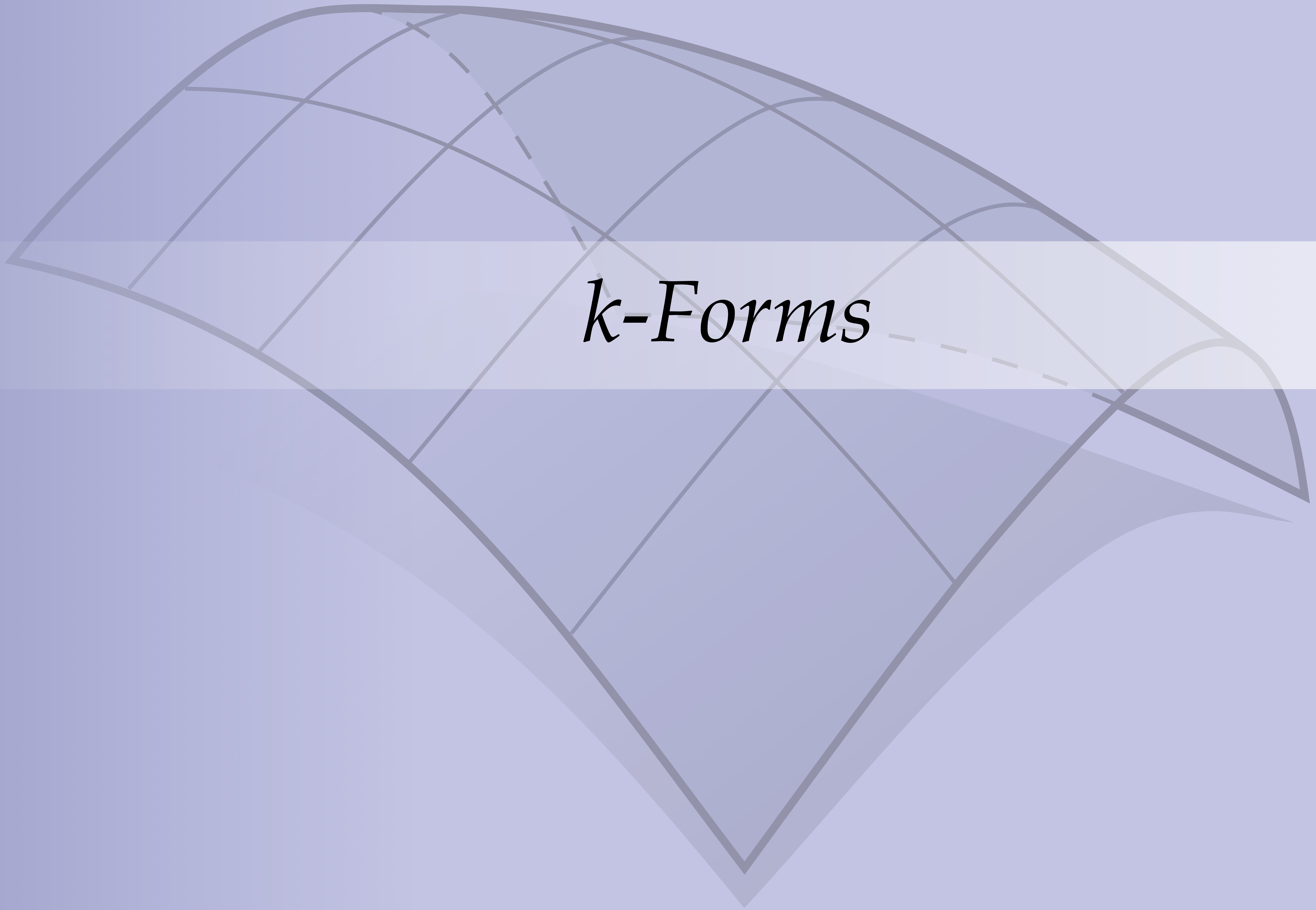
Sharp and Flat w/ Inner Product

$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$u^\flat(v) = u^\mathsf{T} M v \quad \Longleftrightarrow \quad u^\flat(\cdot) = \langle u, \cdot \rangle$$

$$\alpha(\beta^\sharp) = \alpha M^{-1} \beta^\mathsf{T} \quad \Longleftrightarrow \quad \langle \alpha^\sharp, \cdot \rangle = \alpha(\cdot)$$

Basic idea: applying the flat of a vector is the same as taking the inner product; taking the inner product w/ the sharp is same as applying the original 1-form.



k-Forms

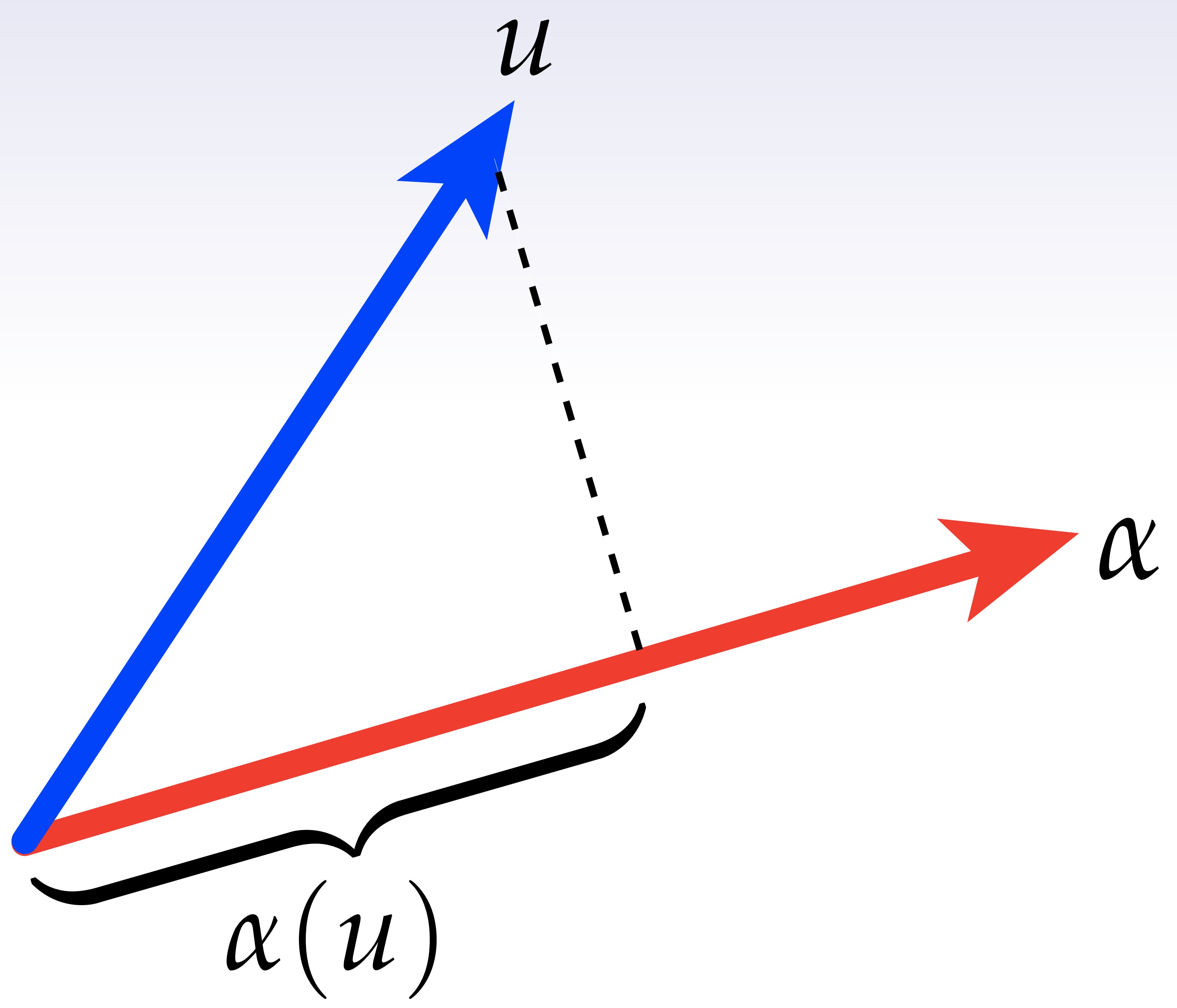
Covectors, Meet Exterior Algebra

- So far we've studied two distinct concepts:
  - **exterior algebra**—build up "volumes" from vectors
  - **covectors**—linear maps from vectors to scalars
- Combine to get an *exterior algebra of covectors*
  - Will call these objects *k-forms*
  - Just as a covector measures vectors…
  - …a k-form will *measure* k-vectors.
  - In particular, measurements will be **multilinear**, *i.e.*, linear in each 1-vector.

	primal	**dual**
vector space	vectors	covectors
exterior algebra	k-vectors	k-forms

Measurement of Vectors

Geometrically, what does it mean to take a **multilinear** measurement of a 1-vector?

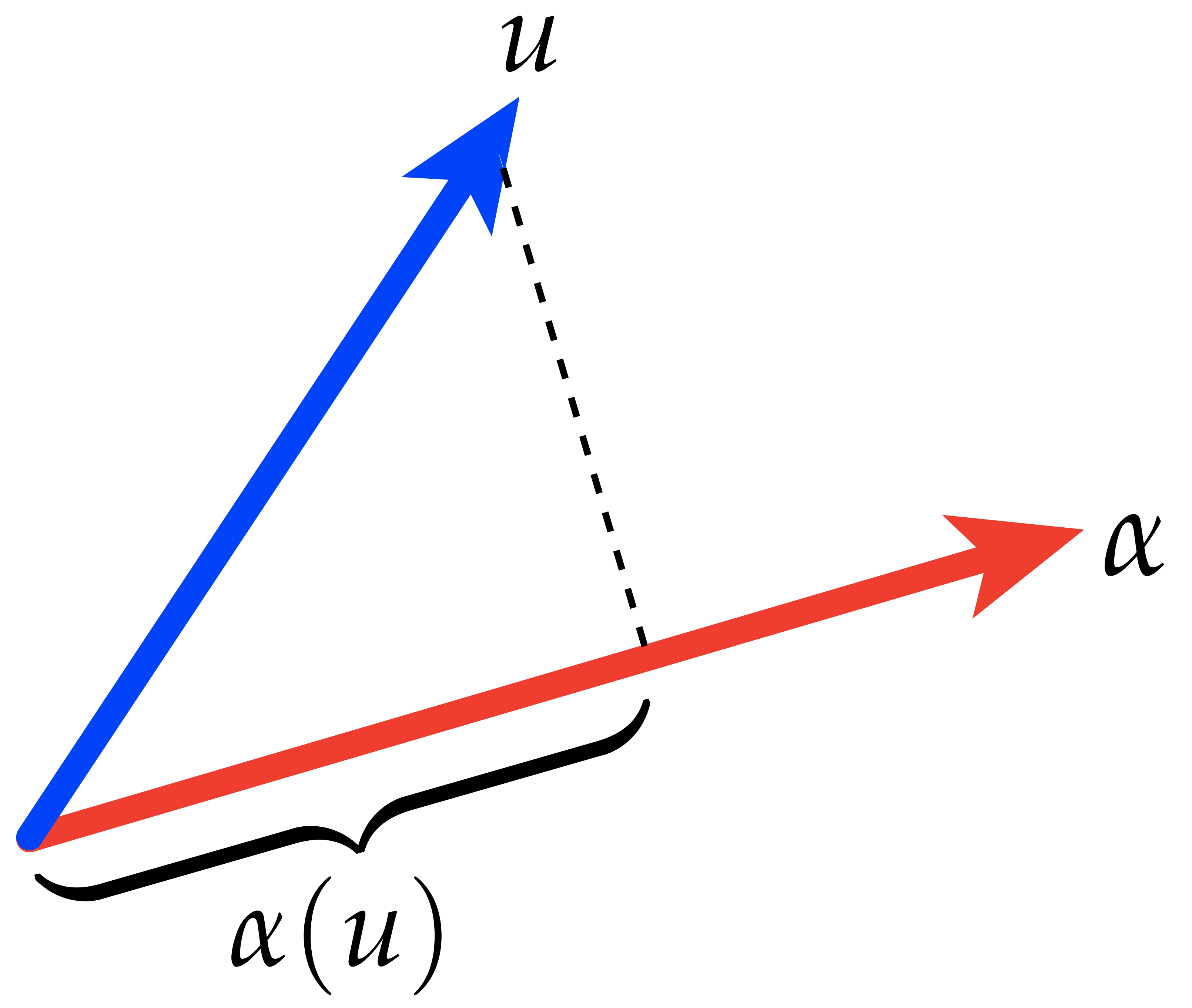


Observation: only thing we can do is measure extent along some other vector.

Computing the Projected Length

- Concretely, how do we compute projected length of one vector along another?
  - If α has unit norm then, we can just take the usual *dot product*
  - Since we think of u as the vector *"getting measured"* and α as the vector *"doing the measurement"*, we'll write this as a function $\alpha(u)$:

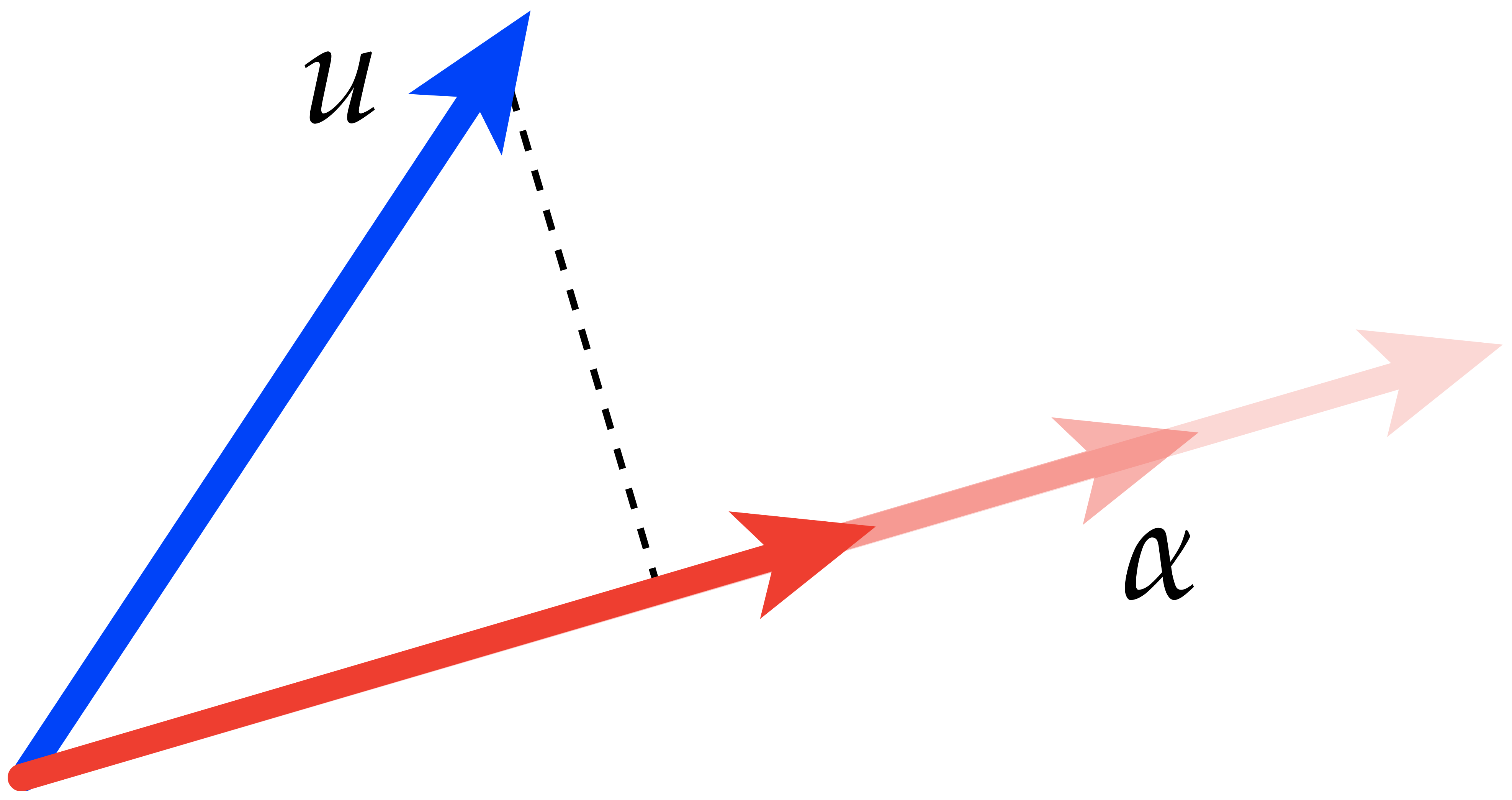
$$\alpha(u) = \sum_{i=1}^{n} \alpha_i u^i$$



1-form

We can of course apply this same expression when α does not have unit length:

$$\boxed{\alpha(u) := \sum_i \alpha_i u^i}$$

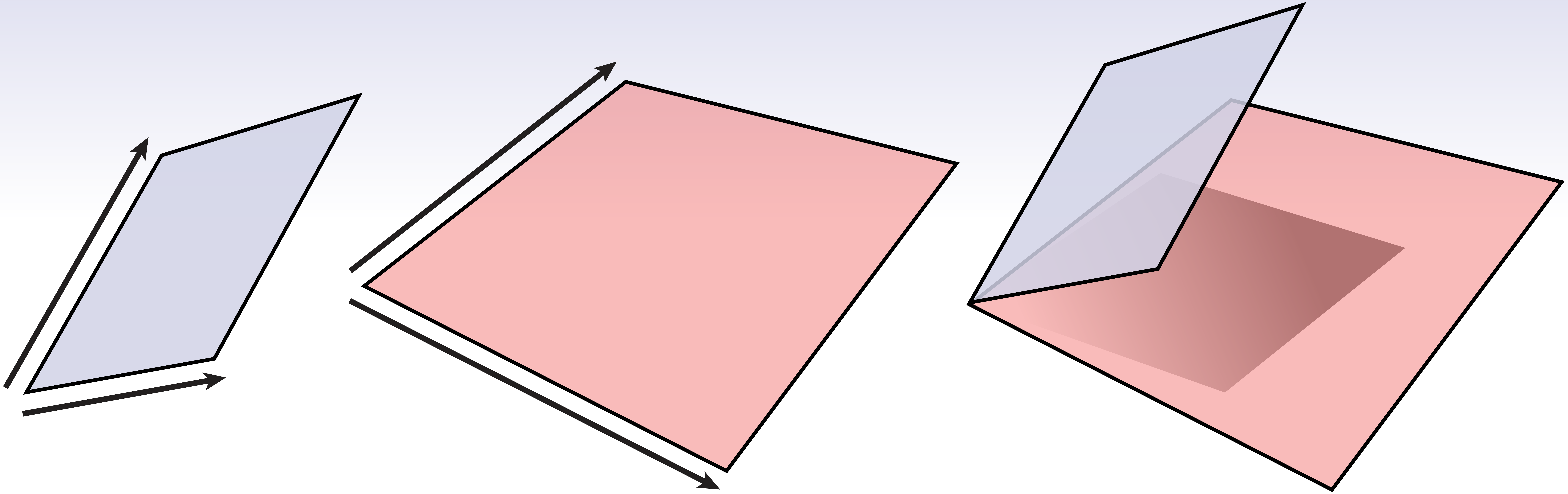


Interpretation?

Projected length gets scaled by magnitude of α.

Measurement of 2-Vectors

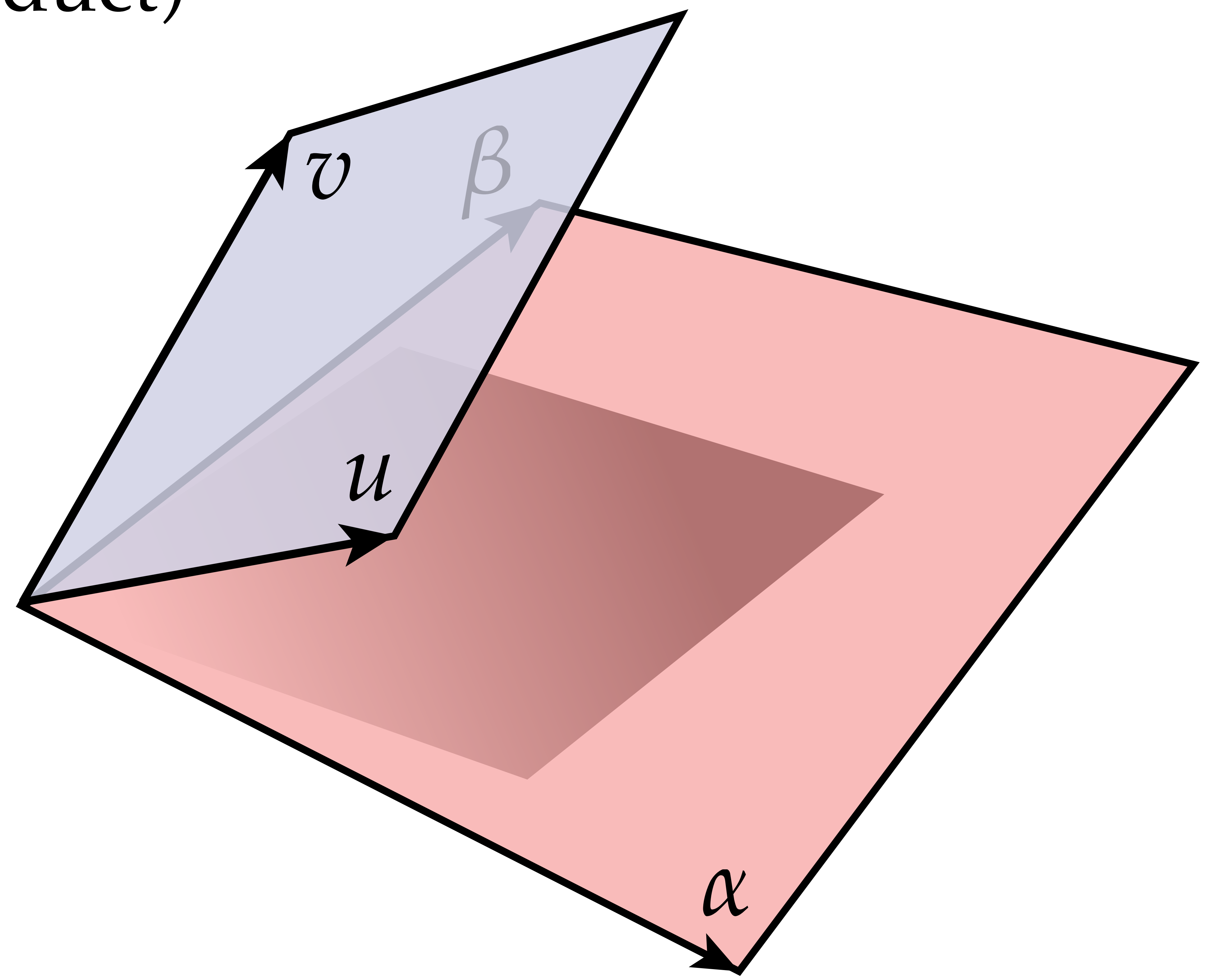
Geometrically, what does it mean to take a **multilinear** measurement of a 2-vector?



Intuition: size of "shadow" of one parallelogram on another.

Computing the Projected Area

- Concretely, how do we compute projected area of a parallelogram onto a plane?
  - First, project vectors defining parallelogram (u,v)
  - Then apply standard formula for area (cross product)
- Suppose for instance α,β are orthonormal:



Projection

$$u \mapsto (\alpha(u), \beta(u))$$
$$v \mapsto (\alpha(v), \beta(v))$$

Area

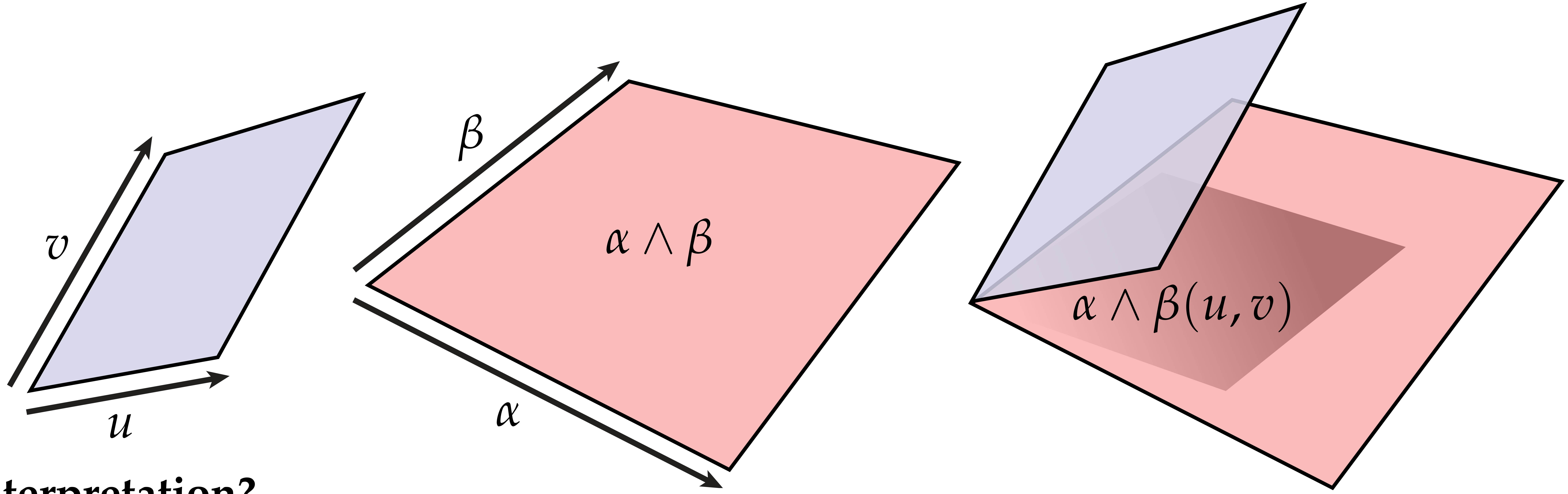
$$\alpha(u)\beta(v) - \alpha(v)\beta(u)$$

(Notice that in the projection we are treating α,β as 1-forms.)

2-form

We can of course apply this same expression when α,β are not orthonormal:

$$\boxed{(\alpha \wedge \beta)(u,v) := \alpha(u)\beta(v) - \alpha(v)\beta(u)}$$



Interpretation?

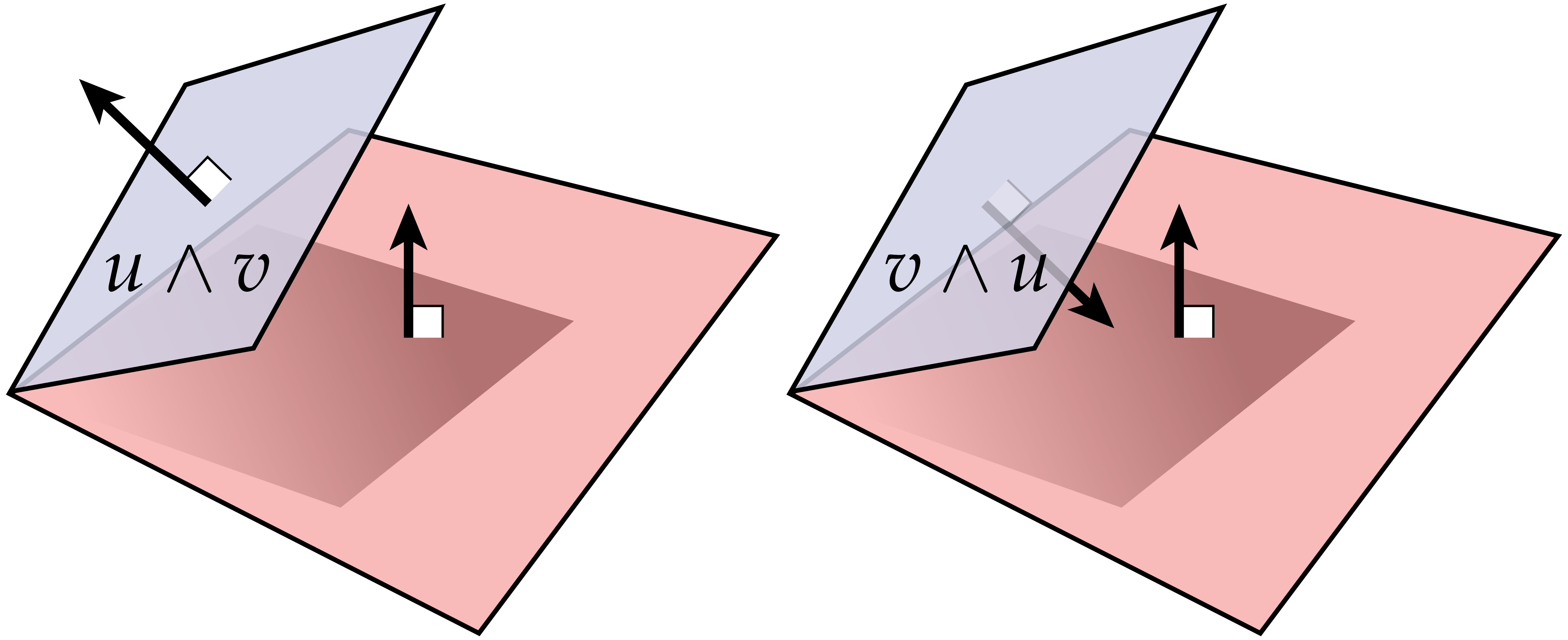
Projected area of u,v gets scaled by area of parallelogram with edges α,β.

Antisymmetry of 2-Forms

Notice that exchanging the arguments of a 2-form reverses sign:

$$\begin{aligned}(\alpha \wedge \beta)(v,u) &= \alpha(v)\beta(u) - \alpha(u)\beta(v)\\ &= -(\alpha(u)\beta(v) - \alpha(v)\beta(u))\\ &= -(\alpha \wedge \beta)(u,v)\end{aligned}$$

Q: What does this *antisymmetry* mean geometrically?



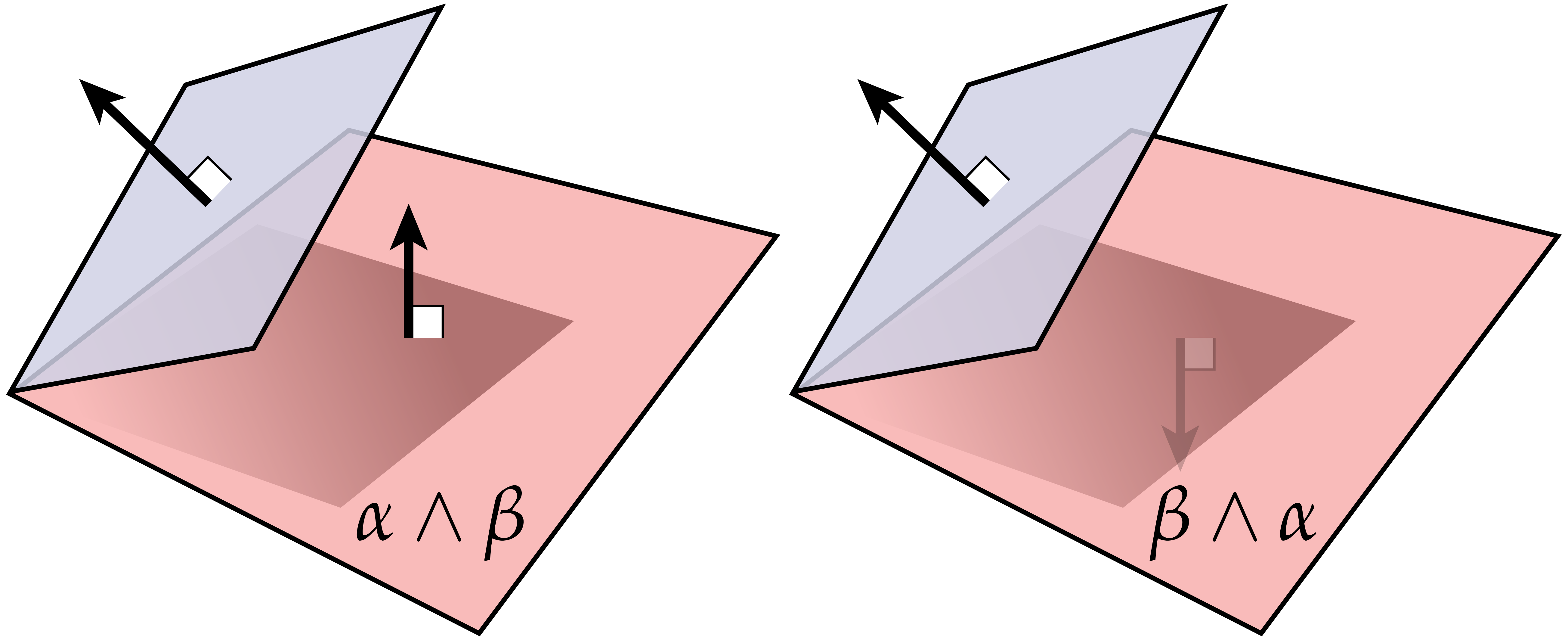
A: Opposite *orientations* of argument 2-vector.

Antisymmetry of 2-Forms

Remember that exchanging the arguments to a wedge product *also* reverses sign:

$$\begin{aligned}(\beta \wedge \alpha)(u,v) &= \beta(u)\alpha(v) - \beta(v)\alpha(u) \\ &= -(\alpha(u)\beta(v) - \alpha(v)\beta(u)) \\ &= -(\alpha \wedge \beta)(u,v)\end{aligned}$$

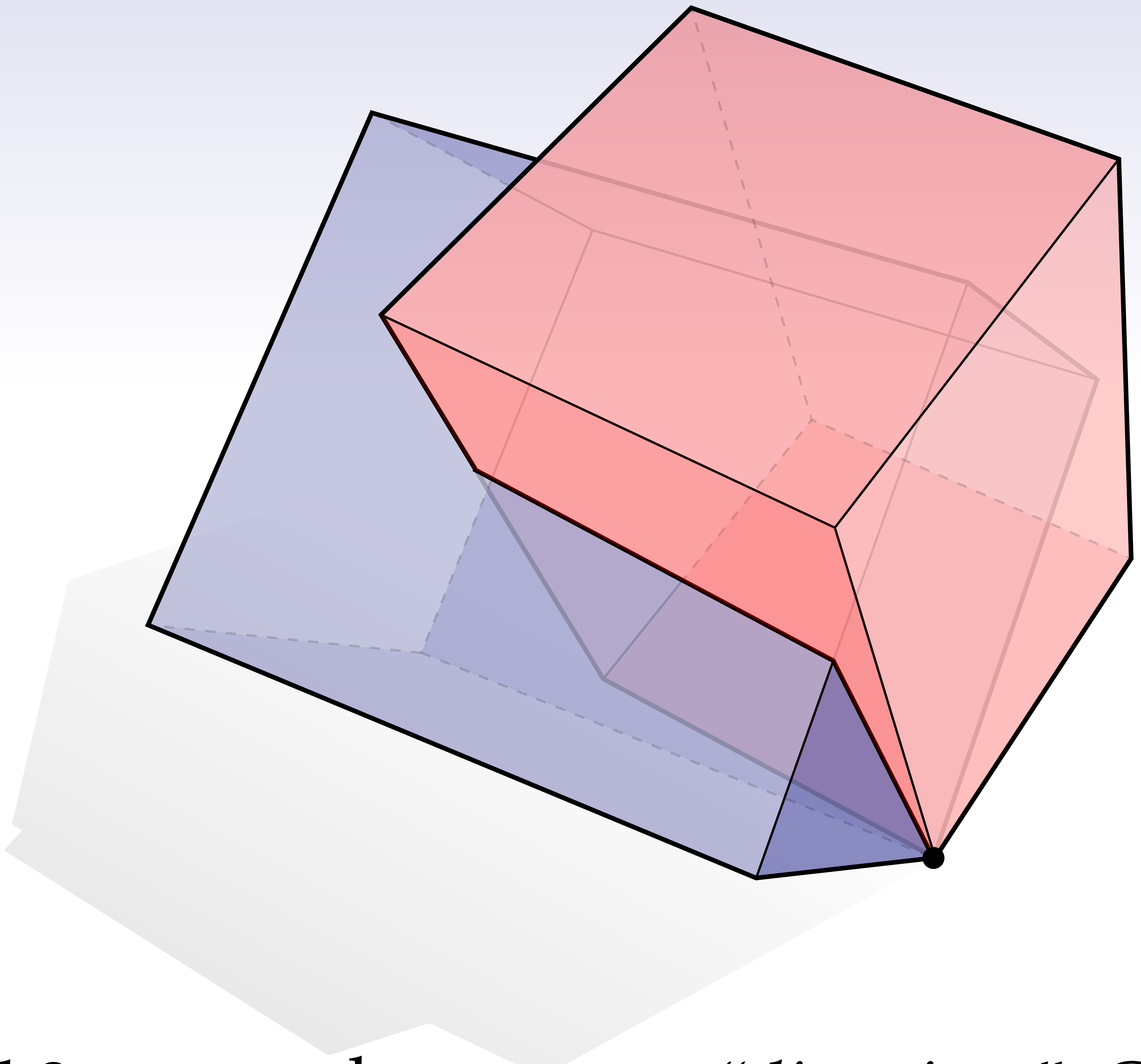
Q: What does this other kind of antisymmetry mean geometrically?



A: Opposite *orientations* of argument 2-vector.

Measurement of 3-Vectors

Geometrically, what does it mean to take a **multilinear** measurement of a 3-vector?



Observation: in R^3, all 3-vectors have same *"direction."* Can only measure *magnitude*.

Computing the Projected Volume

- Concretely, how do we compute the volume of a parallelepiped w/ edges u,v,w?
  - Suppose (α, β, γ) is an orthonormal basis
  - Project vectors u,v,w onto this basis
  - Then apply standard formula for volume (determinant)

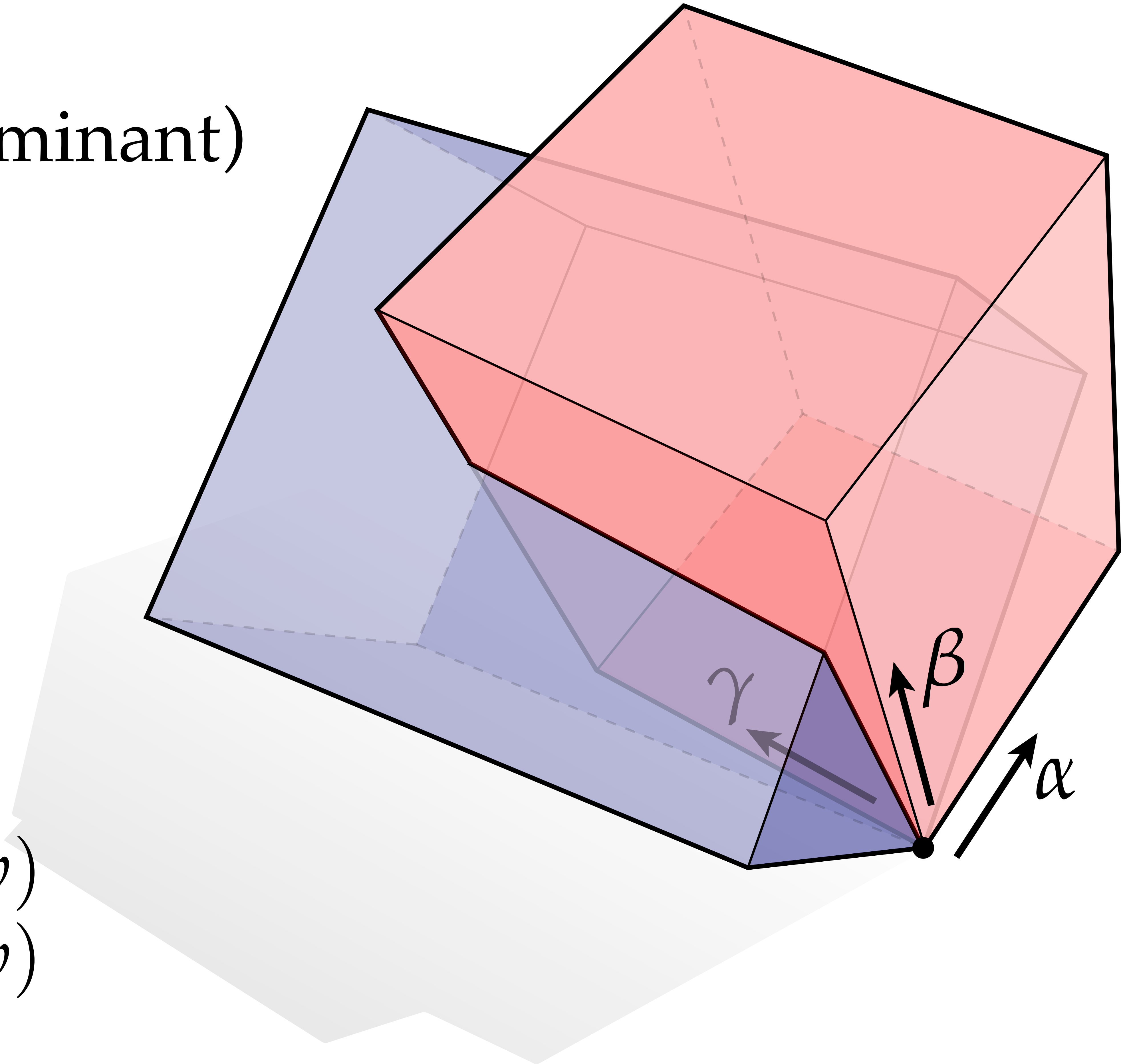
Projection

$$\begin{aligned} u &\mapsto (\alpha(u), \beta(u), \gamma(u)) \\ v &\mapsto (\alpha(v), \beta(v), \gamma(v)) \\ w &\mapsto (\alpha(w), \beta(w), \gamma(w)) \end{aligned}$$

Volume

$$\begin{vmatrix} \alpha(u) & \alpha(v) & \alpha(w) \\ \beta(u) & \beta(v) & \beta(w) \\ \gamma(u) & \gamma(v) & \gamma(w) \end{vmatrix}$$

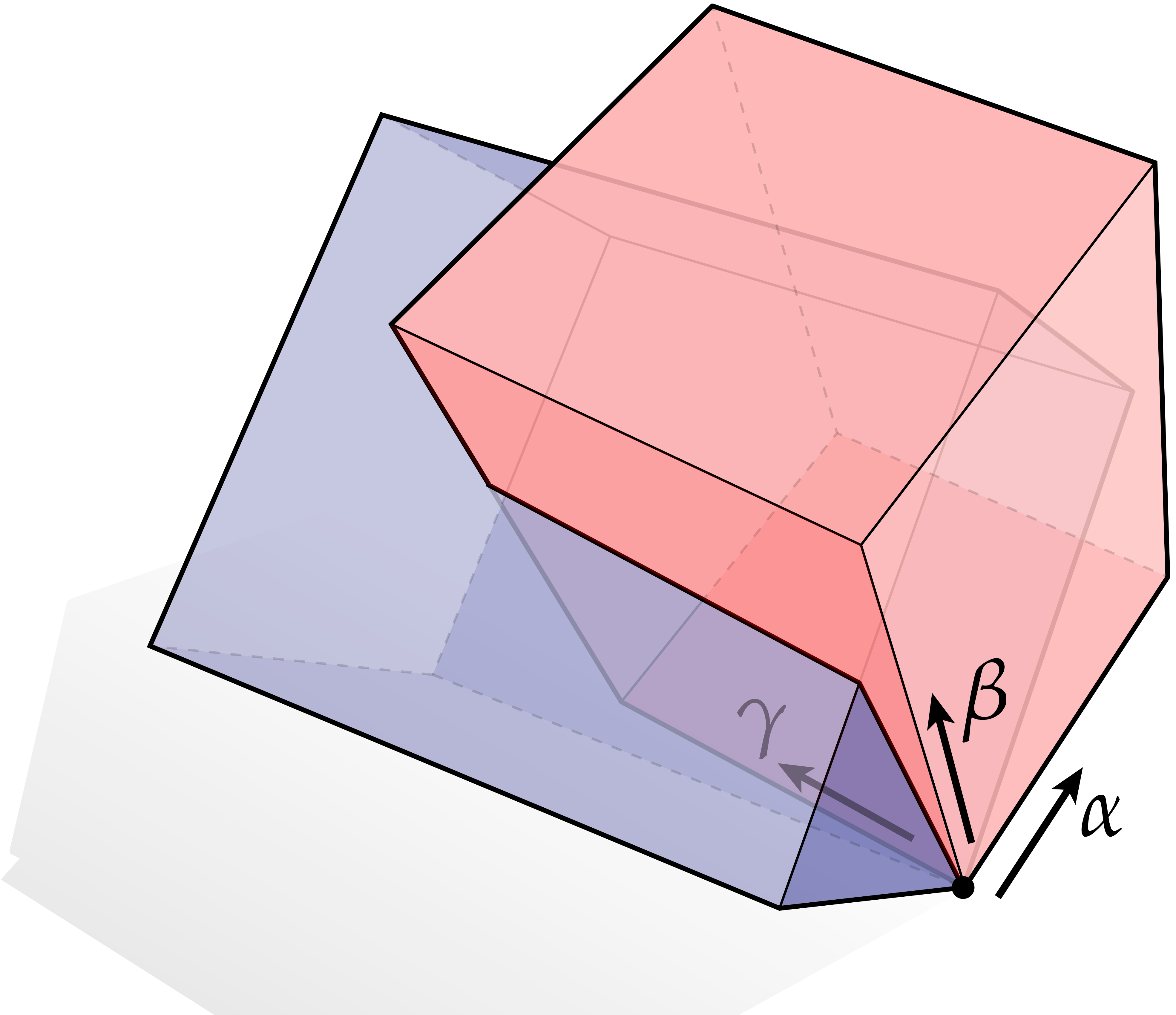
$$= \alpha(u)\beta(v)\gamma(w) + \alpha(v)\beta(w)\gamma(u) + \alpha(w)\beta(u)\gamma(v)$$
$$- \alpha(u)\beta(w)\gamma(v) - \alpha(w)\beta(v)\gamma(u) - \alpha(v)\beta(u)\gamma(w)$$



3-form

We can of course apply this same expression when α, β, γ are not orthonormal:

$$\boxed{\begin{aligned}(\alpha \wedge \beta \wedge \gamma)(u, v, w) := \alpha(u)\beta(v)\gamma(w) + \alpha(v)\beta(w)\gamma(u) + \alpha(w)\beta(u)\gamma(v) \\ - \alpha(u)\beta(w)\gamma(v) - \alpha(w)\beta(v)\gamma(u) - \alpha(v)\beta(u)\gamma(w)\end{aligned}}$$



Interpretation (in R^3)?

Volume of *u,v,w* gets scaled by volume of α, β, γ.

k-Form

- More generally, *k*-form is a *fully antisymmetric, multilinear* measurement of a *k*-vector.
- Typically think of this as a map from *k* vectors to a scalar:

$$\alpha : \underbrace{V \times \cdots V}_{k \text{ times}} \to \mathbb{R}$$

- *Multilinear* means "linear in each argument." *E.g.,* for a 2-form:

$$\begin{array}{rcl} \alpha(au + bv, w) & = & a\alpha(u, w) + b\alpha(v, w) \\ \alpha(u, av + bw) & = & a\alpha(u, v) + b\alpha(u, w) \end{array}, \quad \forall u, v, w \in V, a, b \in \mathbb{R}$$

- *Fully antisymmetric* means exchanging two arguments reverses sign. E.g., 3-form:

$$\alpha(u, v, w) = \alpha(v, w, u) = \alpha(w, u, v) =$$
$$-\alpha(u, w, v) = -\alpha(w, v, u) = -\alpha(v, u, w)$$

A Note on Notation

- A k-form effectively measures a k-vector
- However, *nobody* writes the argument k-vector using a wedge*
- Instead, the convention is to write a list of vectors:

$$\cancel{(\alpha \wedge \beta)(u \wedge v)} \qquad (\alpha \wedge \beta)(u, v)$$

*I have no idea why.

k-Forms and Determinants

- For 3-forms, saw that we could express a k-form via a *determinant*
- Captures the fact that k-forms are measurements of *volume*
- How does this work more generally?
  - **Conceptually**: "project" onto k-dimensional space and take determinant there

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(u_1, \ldots, u_k) := \begin{vmatrix} \alpha_1(u_1) & \cdots & \alpha_1(u_k) \\ \vdots & \ddots & \vdots \\ \alpha_k(u_1) & \cdots & \alpha_k(u_k) \end{vmatrix}$$

<u>k=1:</u>

$$\det\left(\left[\ \alpha_1(u_1)\ \right]\right) = \alpha_1(u_1)$$

(Determinant of a 1x1 matrix is just the entry of that matrix!)

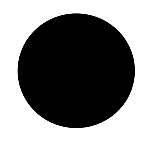
<u>k=2:</u>

$$\det\left(\begin{bmatrix} \alpha_1(u_1) & \alpha_1(u_2) \\ \alpha_2(u_1) & \alpha_2(u_2) \end{bmatrix}\right)$$

$$= \alpha_1(u_1)\alpha_2(u_2) - \alpha_1(u_2)\alpha_2(u_1)$$

0-Forms

- What's a 0-form?
  - In general, a *k*-form takes *k* vectors and produces a scalar
  - So a 0-form must take 0 vectors and produce a scalar
  - I.e., ***a 0-form is a scalar!***
- Basically looks like this:



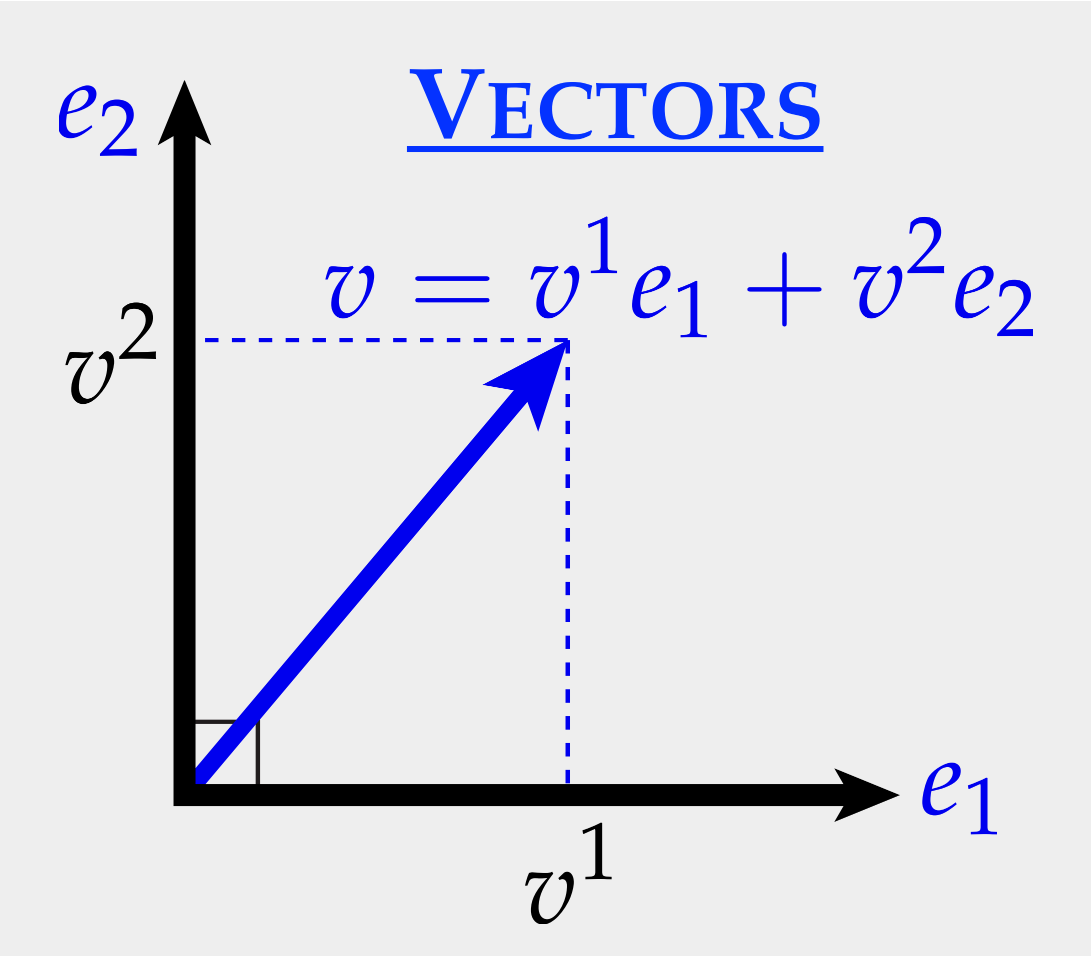
Note: still has *magnitude*, even though it has only one possible "direction."

k-Forms in Coordinates

Dual Basis

In an n-dimensional vector space V, can express vectors v in a basis $e_1, \ldots, e_n$:

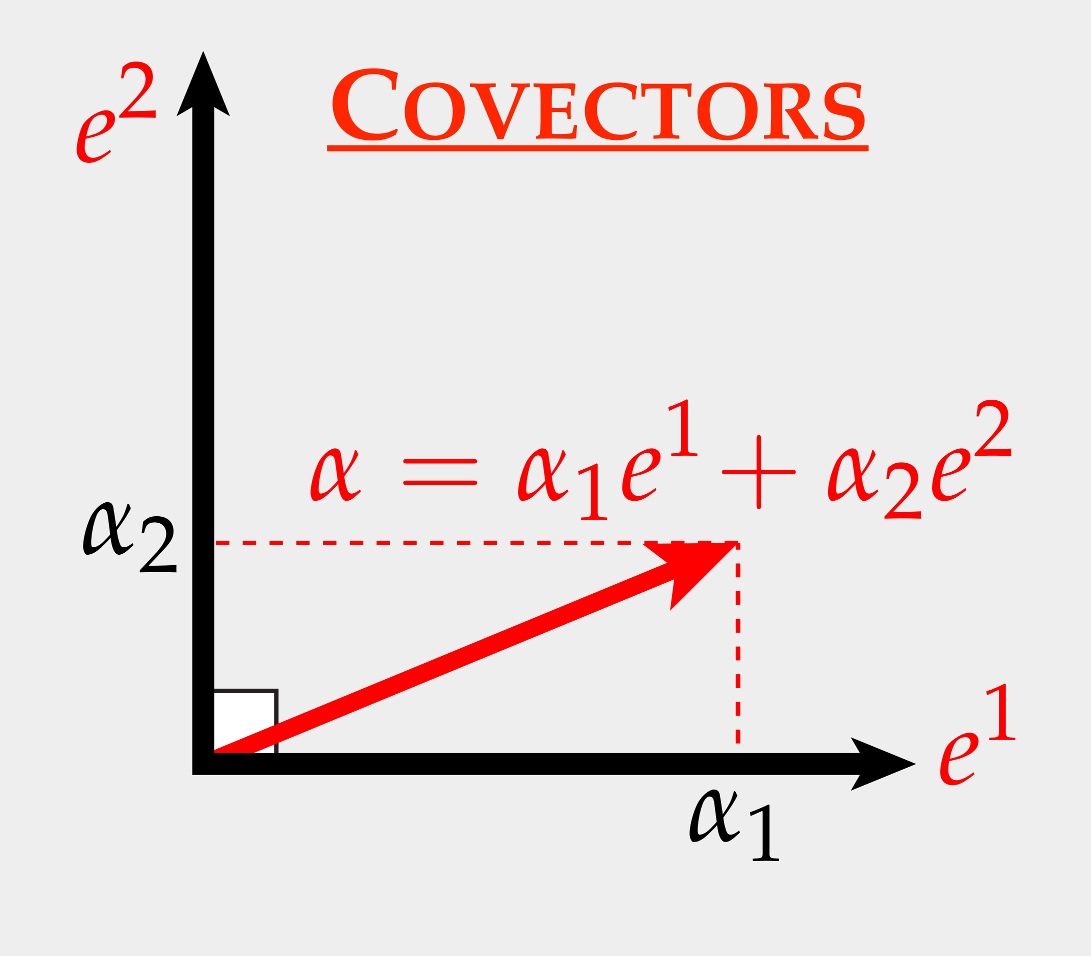
$$v = v^1 e_1 + \cdots + v^n e_n$$



The scalar values v^i are the *coordinates* of v.

We can also write *covectors* α in a so-called *dual basis* $e^1, \ldots, e^n$:

$$\alpha = \alpha_1 e^1 + \cdots + \alpha_n e^n$$



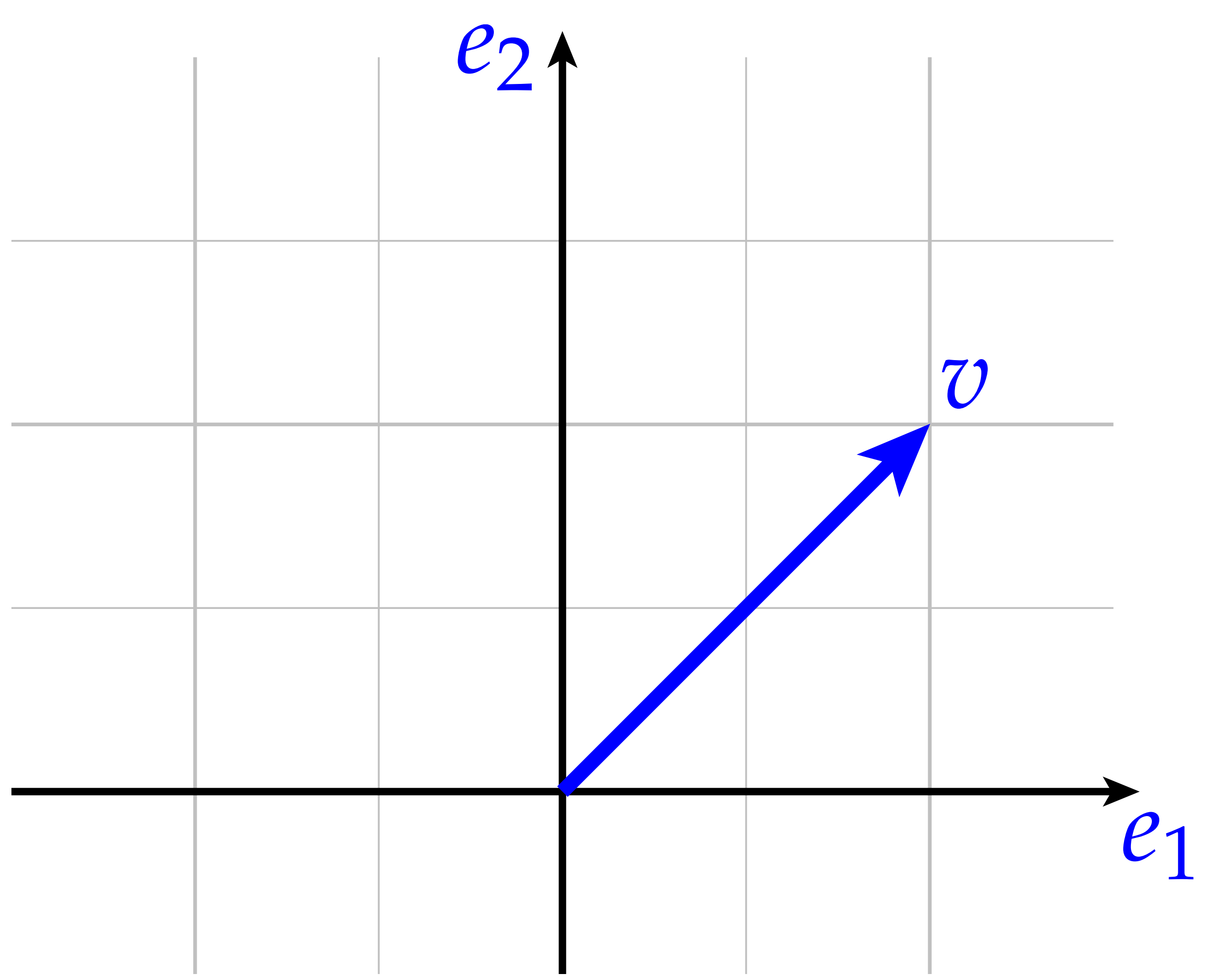
These bases have a special relationship, namely:

$$e^i(e_j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$$

(**Q:** What does e^i mean, geometrically?)

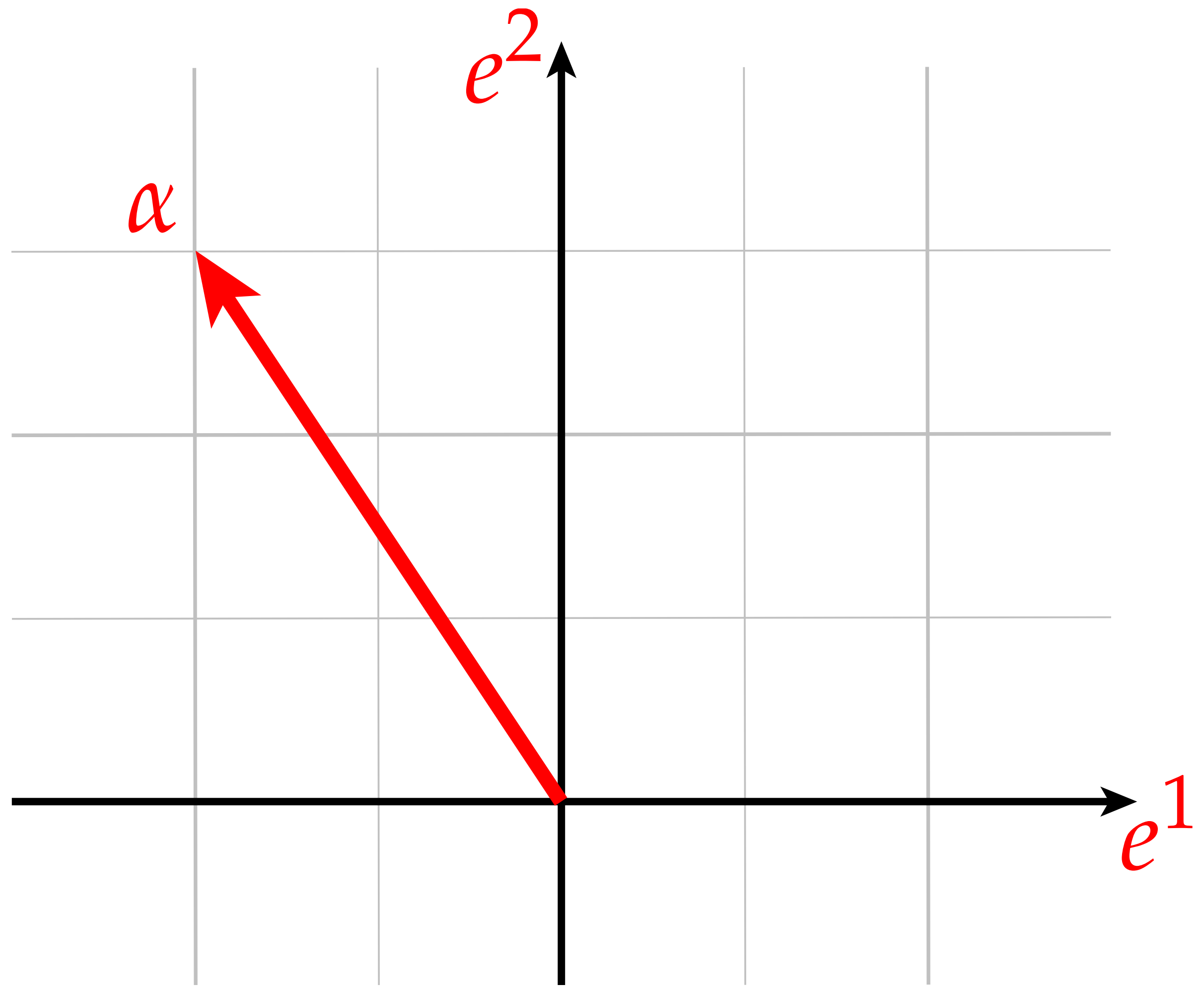
1-form — Example in Coordinates

- Some simple calculations in coordinates help to solidify understanding of k-forms.
- Let's start with a vector v and a 1-form α in the plane:



$$v = 2e_1 + 2e_2$$

$$\alpha = -2e^1 + 3e^2$$



$$\begin{aligned}
\alpha(v) &= (-2e^1 + 3e^2)(2e_1 + 2e_2) \\
&= -2e^1(2e_1 + 2e_2) + 3e^2(2e_1 + 2e_2) \\
&= -4\cancelto{1}{e^1(e_1)} - 4\cancelto{0}{e^1(e_2)} + 6\cancelto{0}{e^2(e_1)} + 6\cancelto{1}{e^2(e_2)} \\
&= -4 + 6 \qquad \text{(Just like a \textit{dot product!})} \\
&= 2.
\end{aligned}$$

2-form—Example in Coordinates

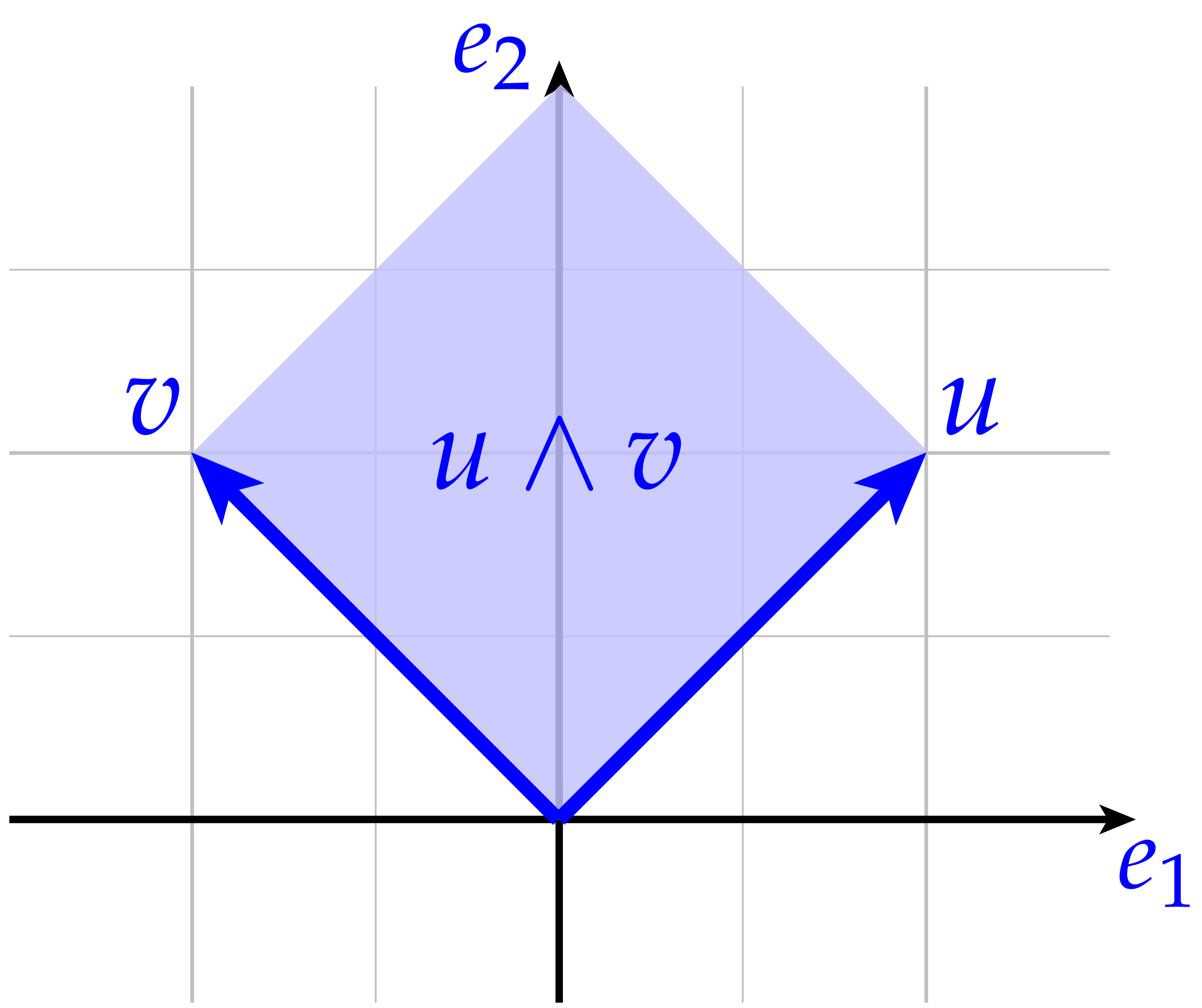
Consider the following vectors and covectors:

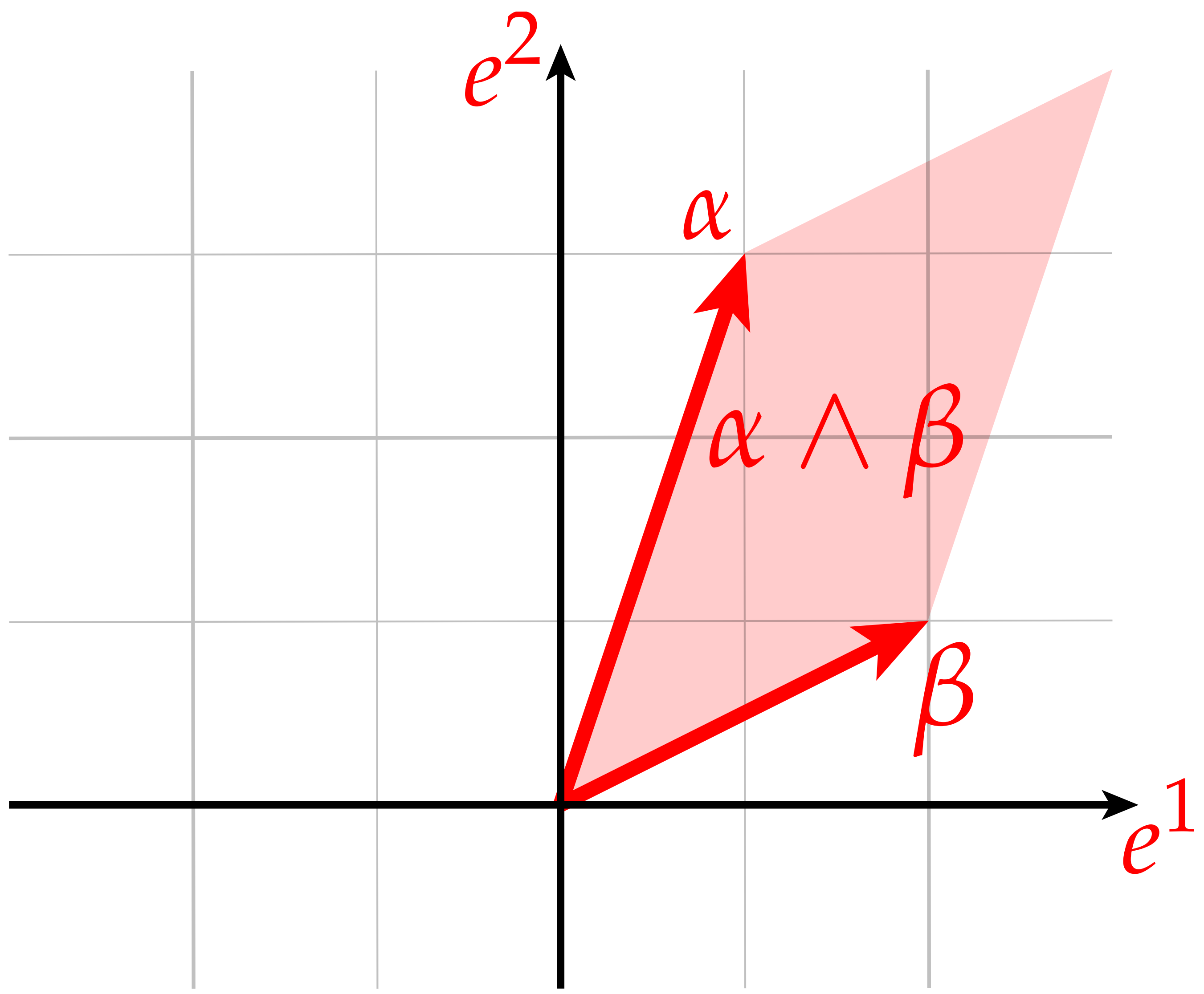
$$u = 2e_1 + 2e_2 \qquad \alpha = e^1 + 3e^2$$
$$v = -2e_1 + 2e_2 \qquad \beta = 2e^1 + e^2$$

We then have:

$$(\alpha \wedge \beta)(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$$
$$\alpha(u) = 1 \cdot 2 + 3 \cdot 2 = 8$$
$$\beta(v) = \cdots = -2$$
$$\alpha(v) = \cdots = 4$$
$$\beta(u) = \cdots = 6$$
$$\Rightarrow (\alpha \wedge \beta)(u, v) = 8 \cdot (-2) - 4 \cdot 6 = -40.$$

Q: What does this value mean, geometrically? Why is it *negative*?



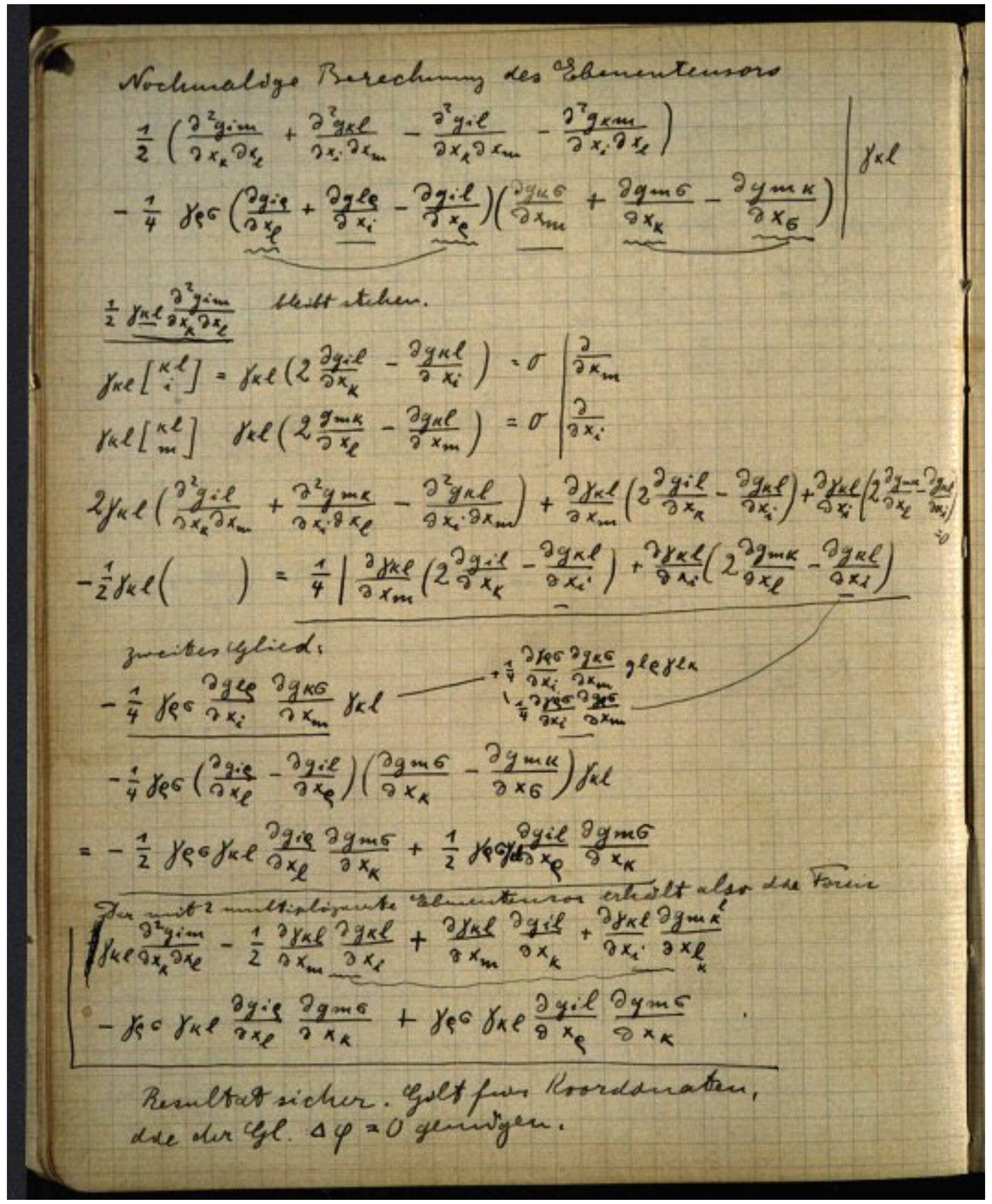


Einstein Summation Notation

Why are some indices *"up"* and others *"down"*?

Bemerkung zur Vereinfachung der Schreibweise der Ausdrücke.

Ein Blick auf die Gleichungen dieses Paragraphen zeigt, daß über Indizes, die zweimal unter einem Summenzeichen auftreten [z. B. der Index ν in (5)], stets summiert wird, und zwar *nur* über zweimal auftretende Indizes. Es ist deshalb möglich, ohne die Klarheit zu beeinträchtigen, die Summenzeichen wegzulassen. Dafür führen wir die Vorschrift ein: Tritt ein Index in einem Term eines Ausdruckes zweimal auf, so ist über ihn stets zu summieren, wenn nicht ausdrücklich das Gegenteil bemerkt ist.

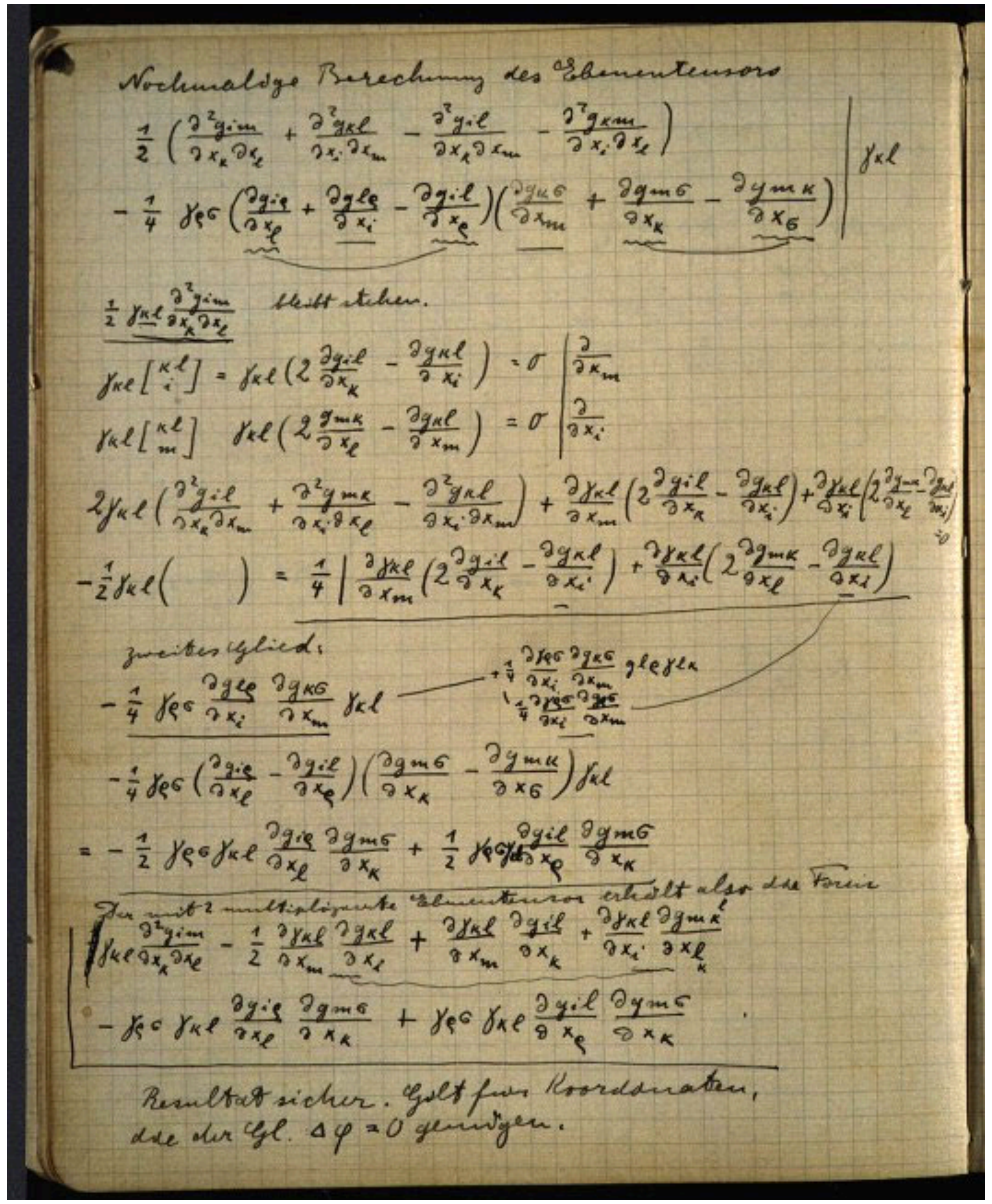


— Einstein, "Die Grundlage der allgemeinen Relativitätstheorie" (1916)

Einstein Summation Notation

Key idea: sum over repeated indices.

$$x^i y_i := \sum_{i=1}^{n} x^i y_i$$



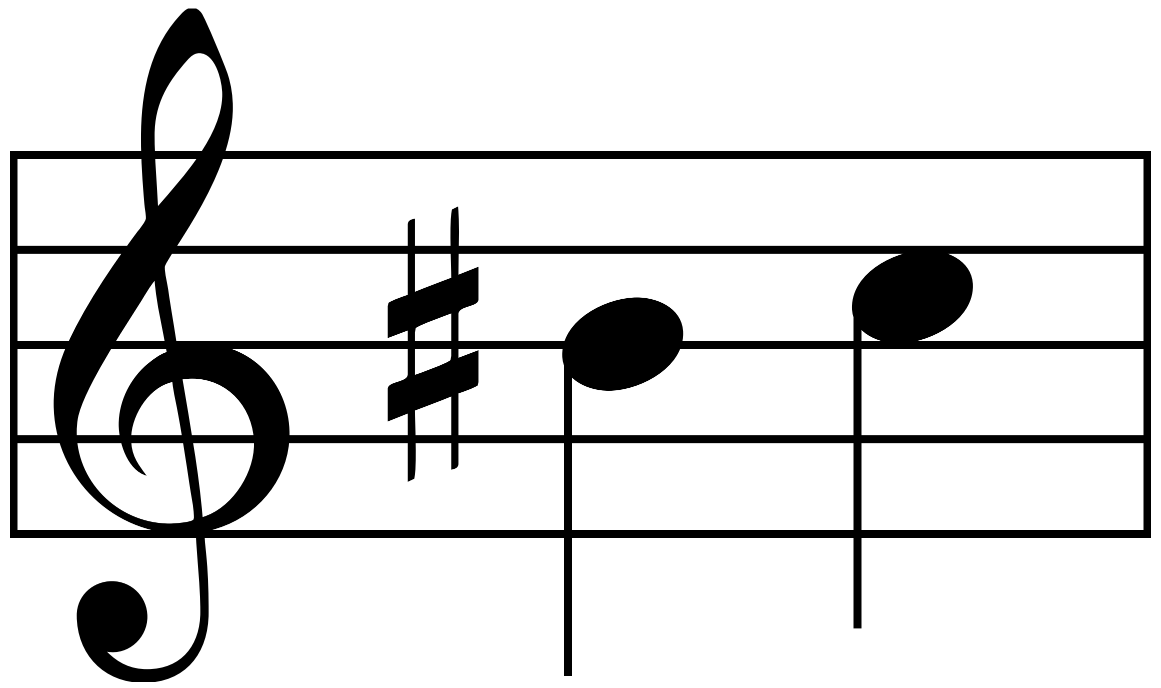
Note on a Simplified Way of Writing Expressions

A look at the equations of this paragraph show that there is always a summation over indices which occur twice, and only for twice-repeated indices. It is therefore possible, without detracting from clarity, to omit the sum sign. For this we introduce a rule: if an index in an expression appears twice, then a sum is implicitly taken over this index, unless specifically noted to the contrary.

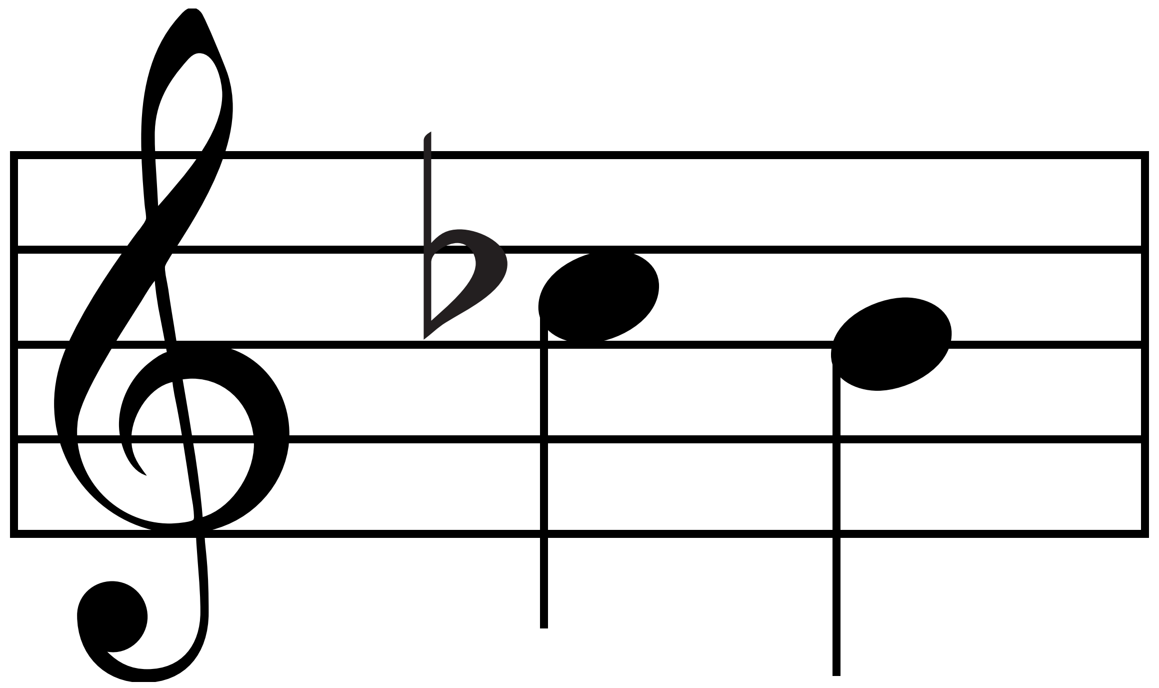
— *Einstein, "Die Grundlage der allgemeinen Relativitätstheorie" (1916)*

Sharp and Flat in Coordinates

Q: What do sharp and flat do on a musical staff?



(raise pitch)



(lower pitch)

Likewise, sharp and flat *raise* and *lower* indices of coefficients for 1-forms/vectors.

Suppose for instance that $\alpha(v) = \langle u, v \rangle$ for all $v \in V$. Then

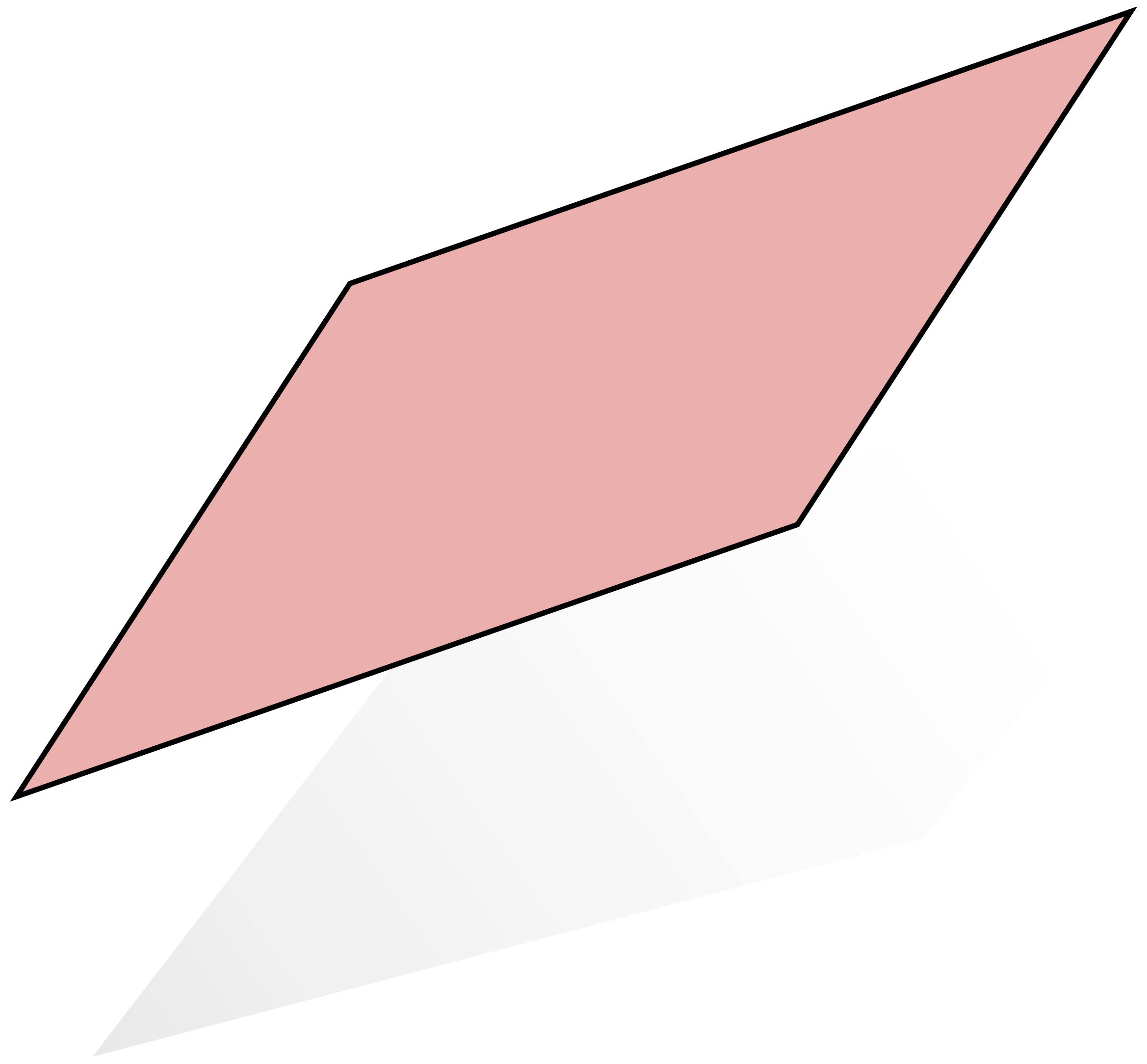
$$\alpha = \alpha_1 e^1 + \cdots + \alpha_n e^n \qquad \underset{\overset{\flat}{\Longleftarrow}}{\overset{\sharp}{\Longrightarrow}} \qquad u = u^1 e_1 + \cdots + u^n e_n$$

(Sometimes called the *musical isomorphisms.)*

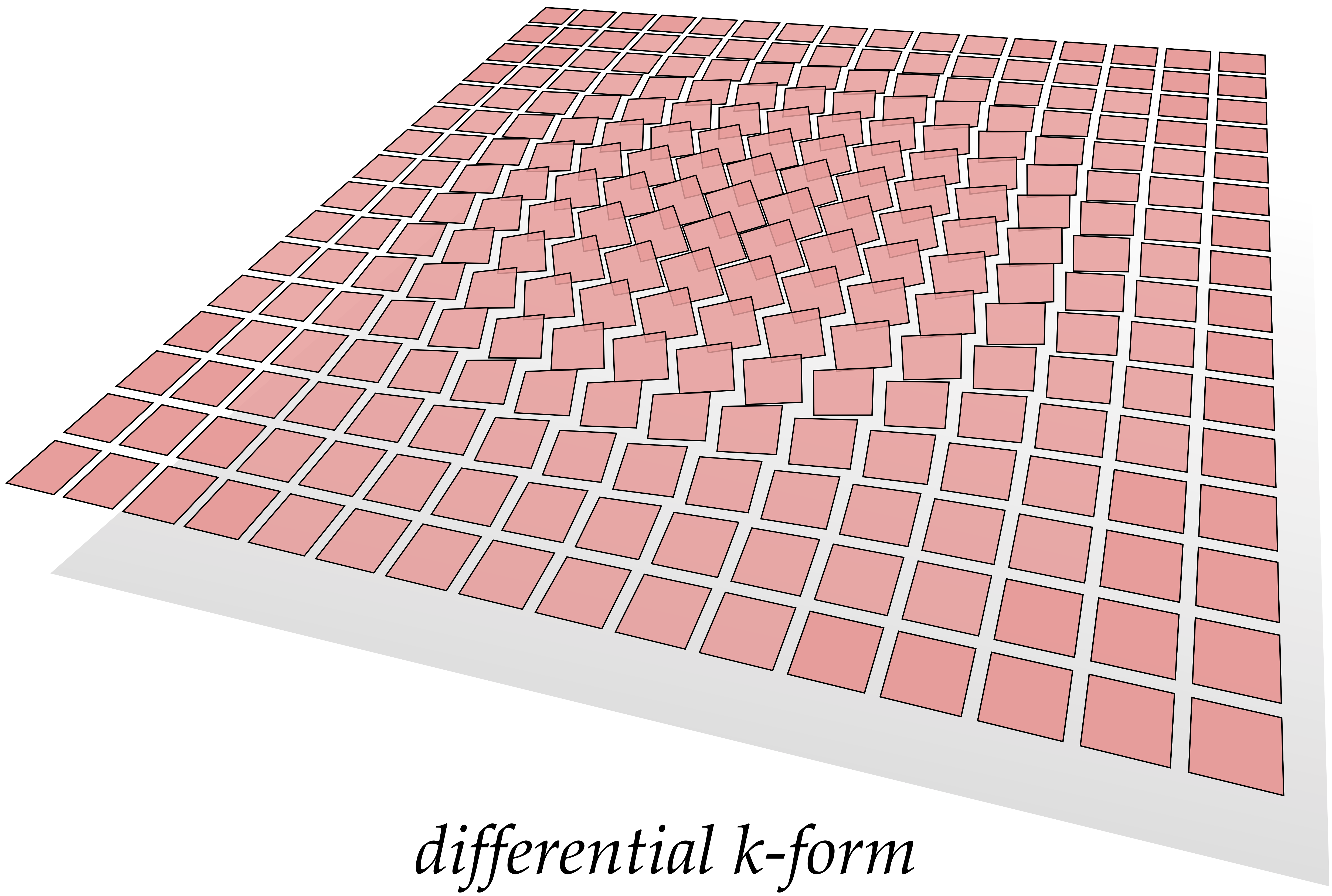
Differential k-Forms

Differential Form

- A *differential k-form* is an assignment of a k-form to each point*:



k-form

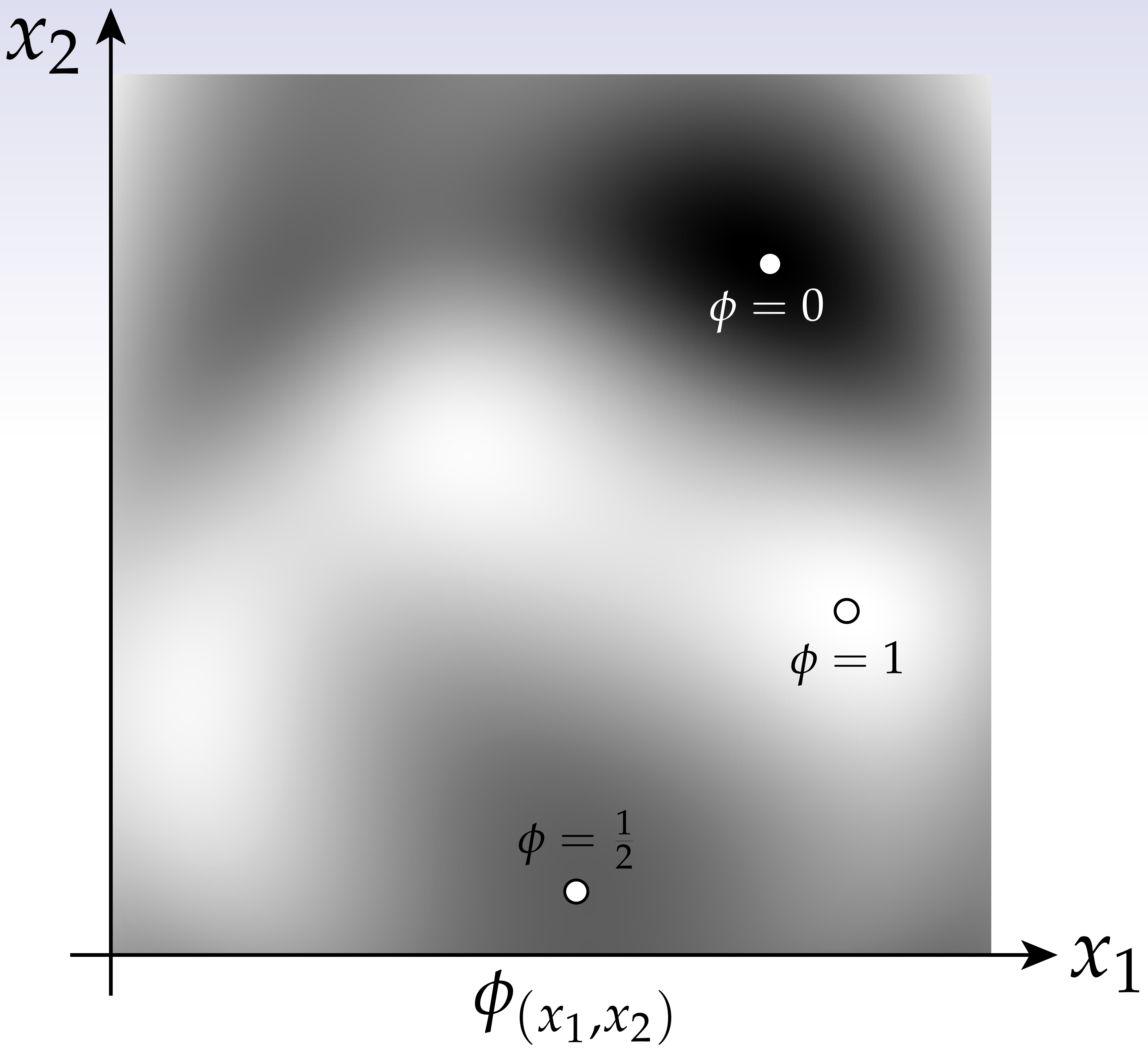


differential k-form

*Common (and confusing!) to abbreviate "differential k-form" as just "k-form"!

Differential 0-Form

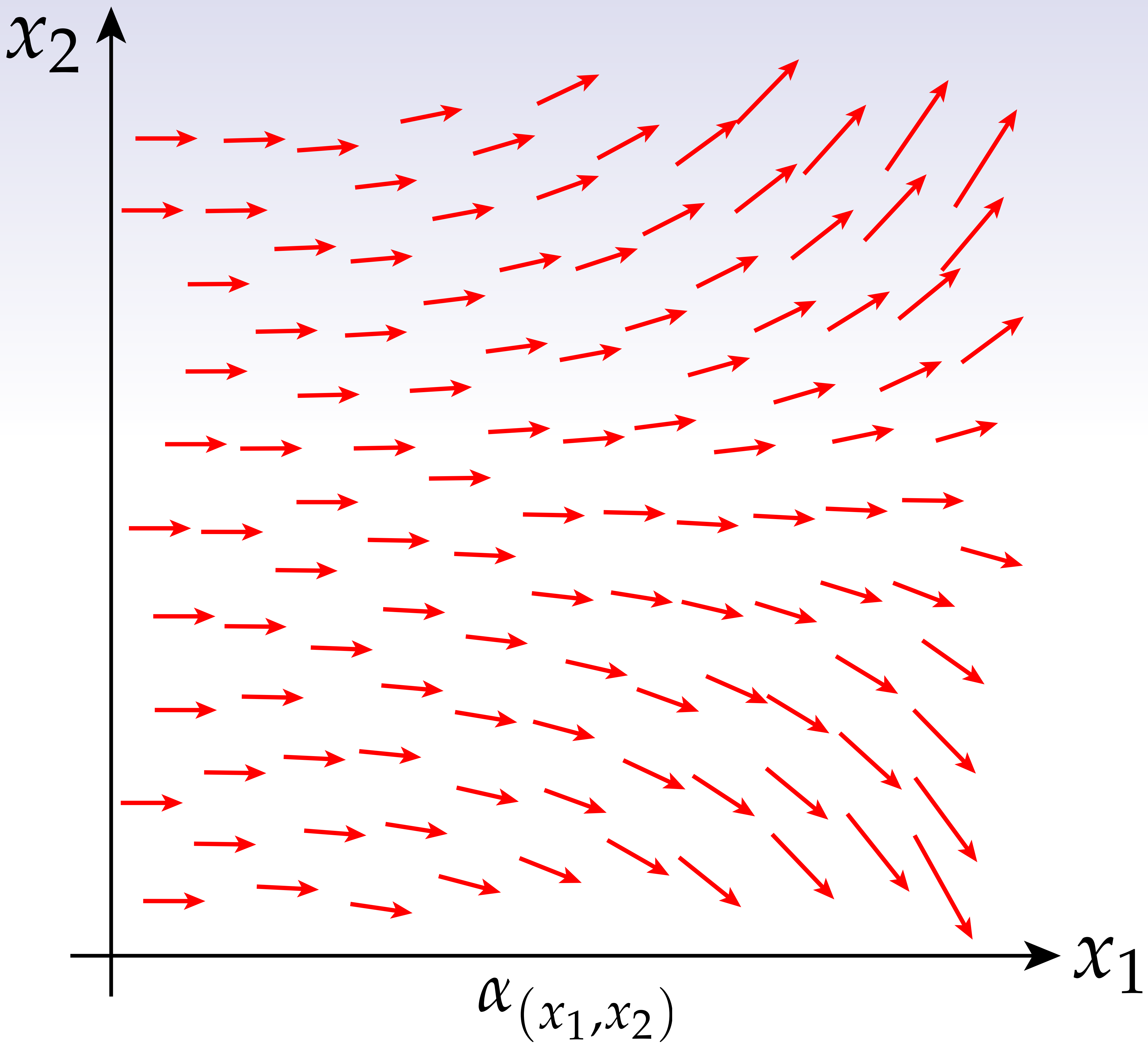
Assigns a scalar to each point. *E.g.*, in 2D we have a value at each point (x_1,x_2):



Note: exactly the same thing as a *scalar function!*

Differential 1-Form

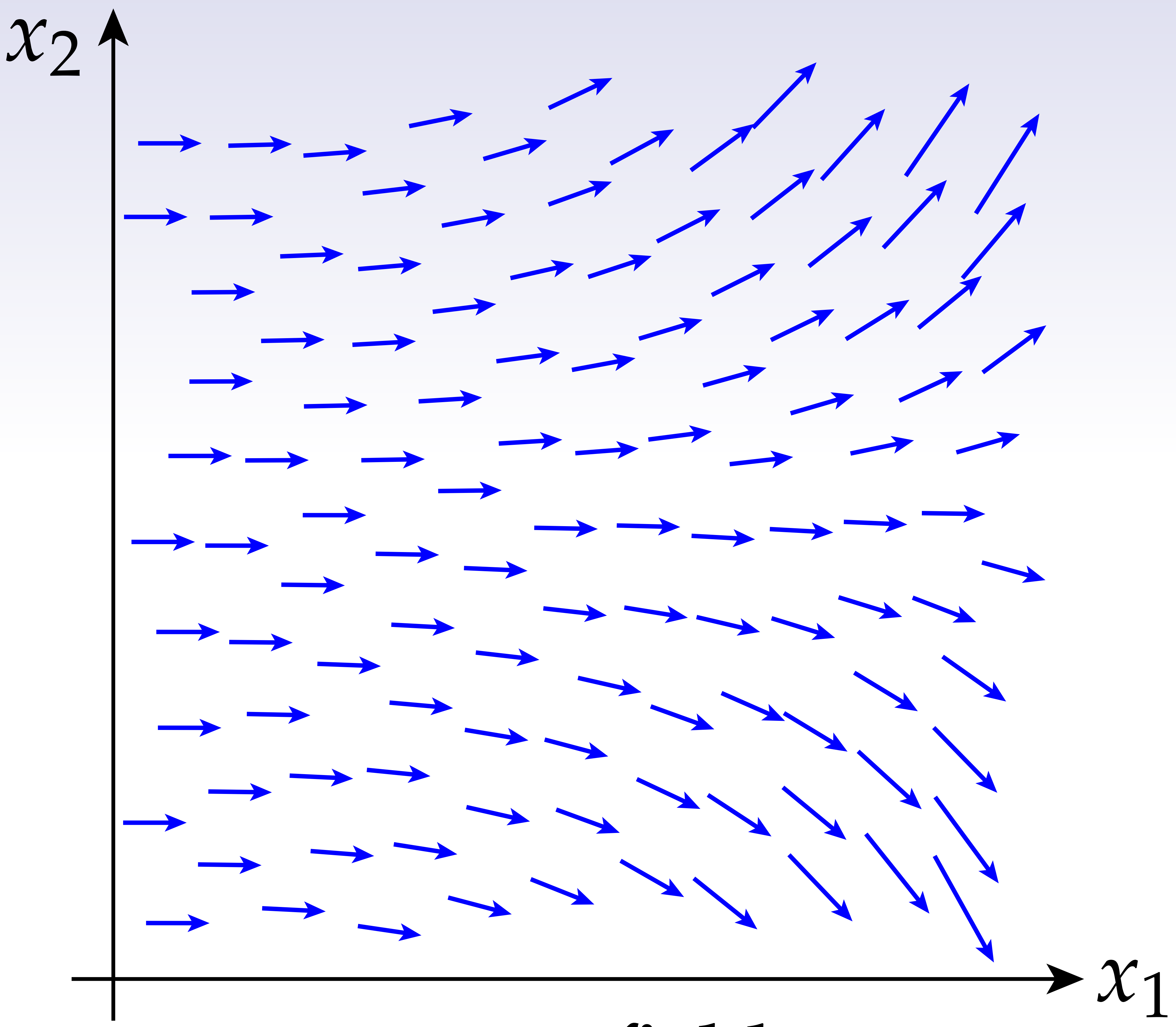
Assigns a 1-form each point. *E.g.*, in 2D we have a 1-form at each point (x_1, x_2):



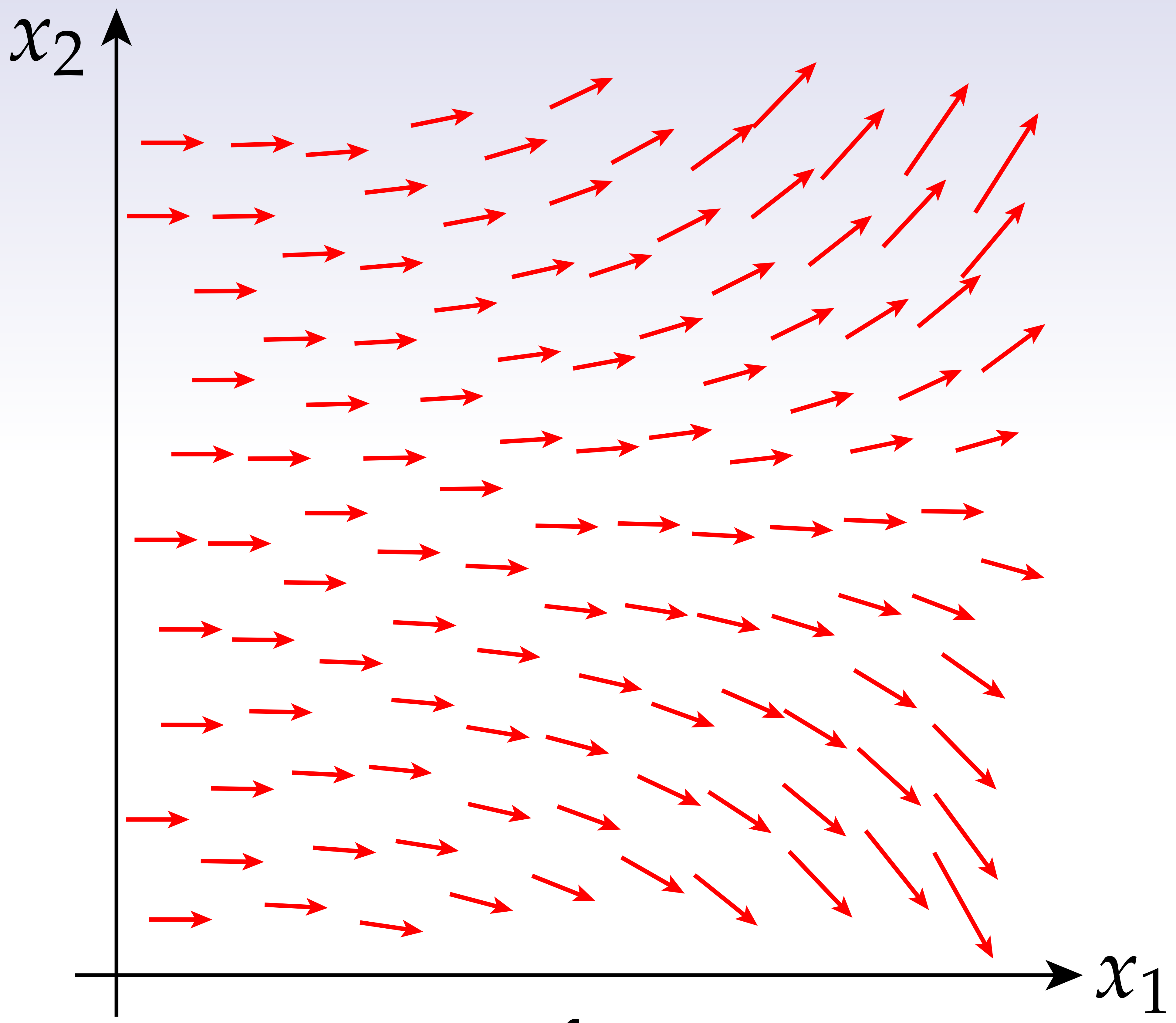
Note: NOT the same thing as a vector field!

Vector Field vs. Differential 1-Form

Superficially, vector fields and differential 1-forms look the same (in R^n):



vector field



1-form

But recall that a 1-form is a *linear function* from a vector to a scalar (here, at each point.)

Applying a Differential 1-Form to a Vector Field

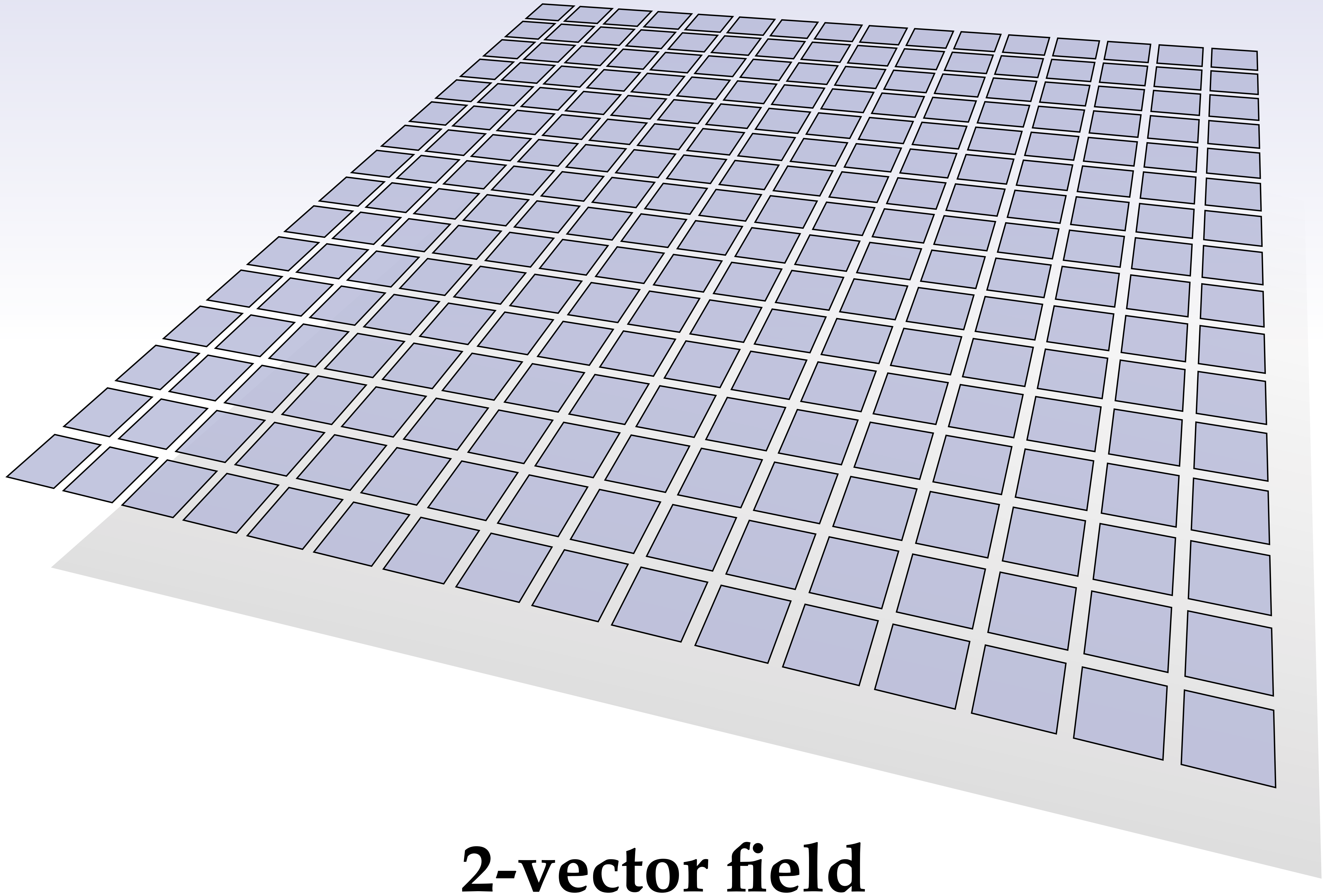
At each point (x_1,x_2), we can therefore use a 1-form to measure a vector field:

vector field $X_{(x_1,x_2)}$ | **differential 1-form** $\alpha_{(x_1,x_2)}$ | (superimposed) | **scalar function** $\alpha(X)_{(x_1,x_2)}$

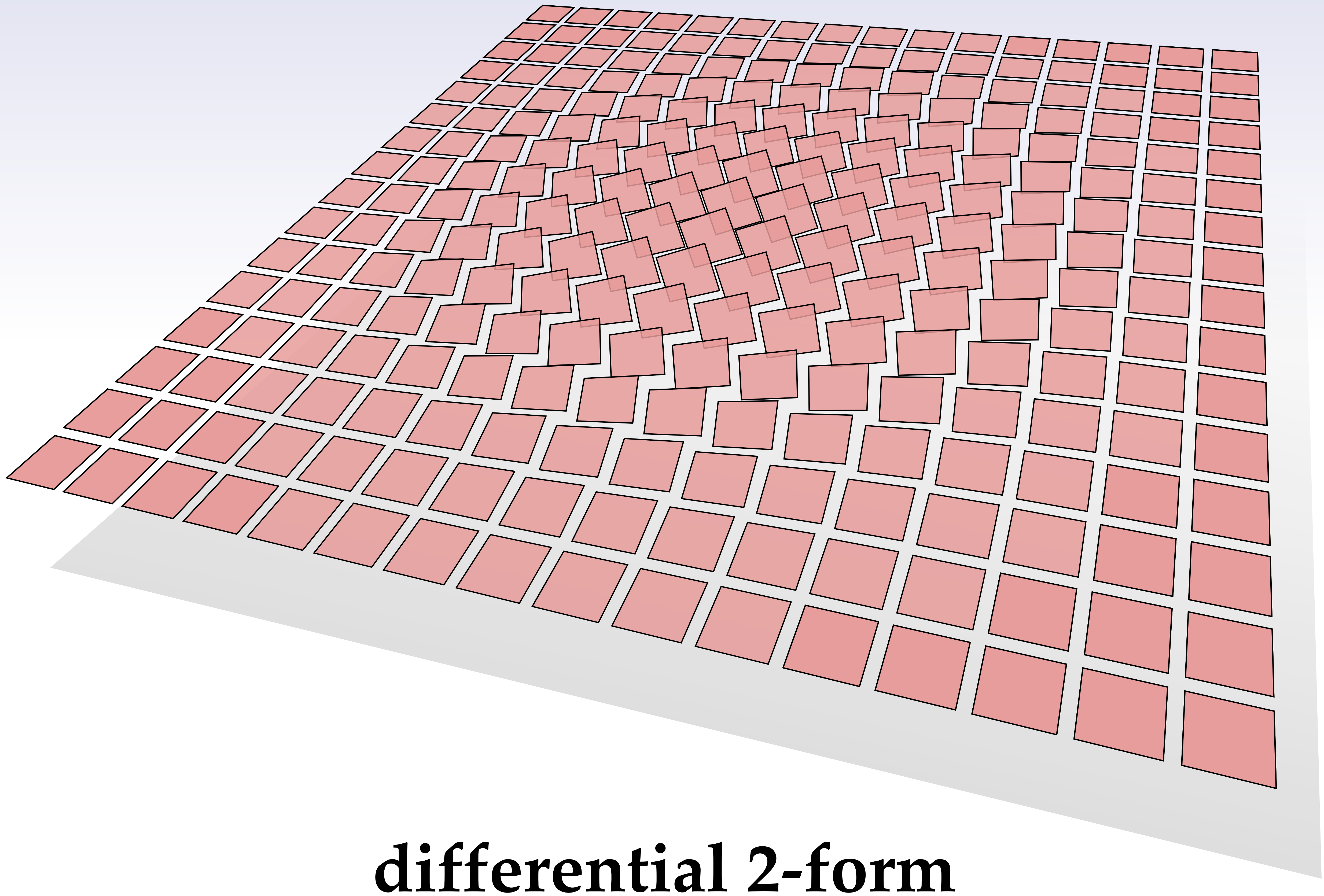
Intuition: resulting function indicates "how strong" X is along α.

Differential 2-Forms

Likewise, a differential 2-form is an area measurement at each point (x_1, x_2, x_3):

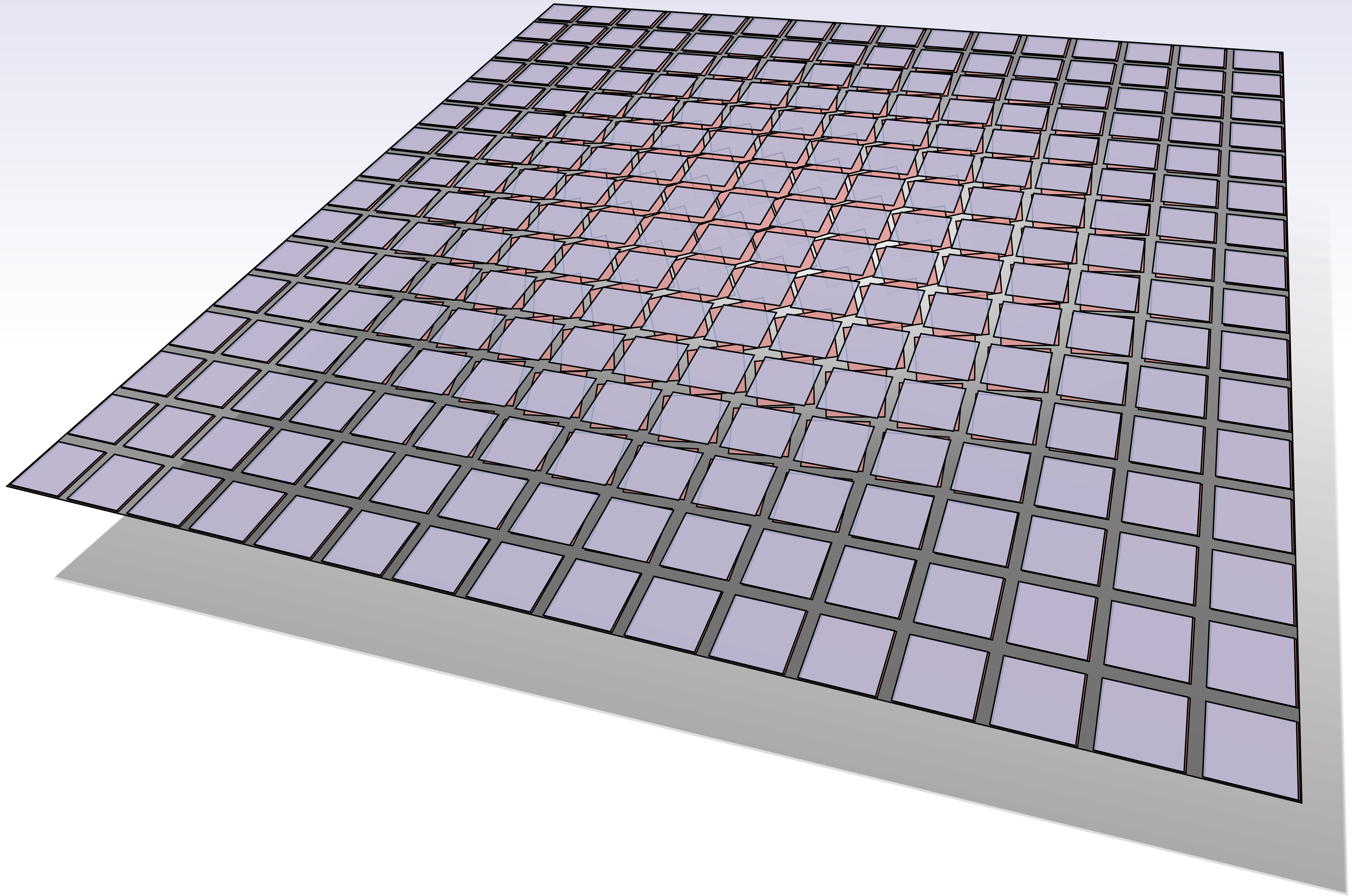


2-vector field
$(X \wedge Y)_{(x_1,x_2,x_3)}$

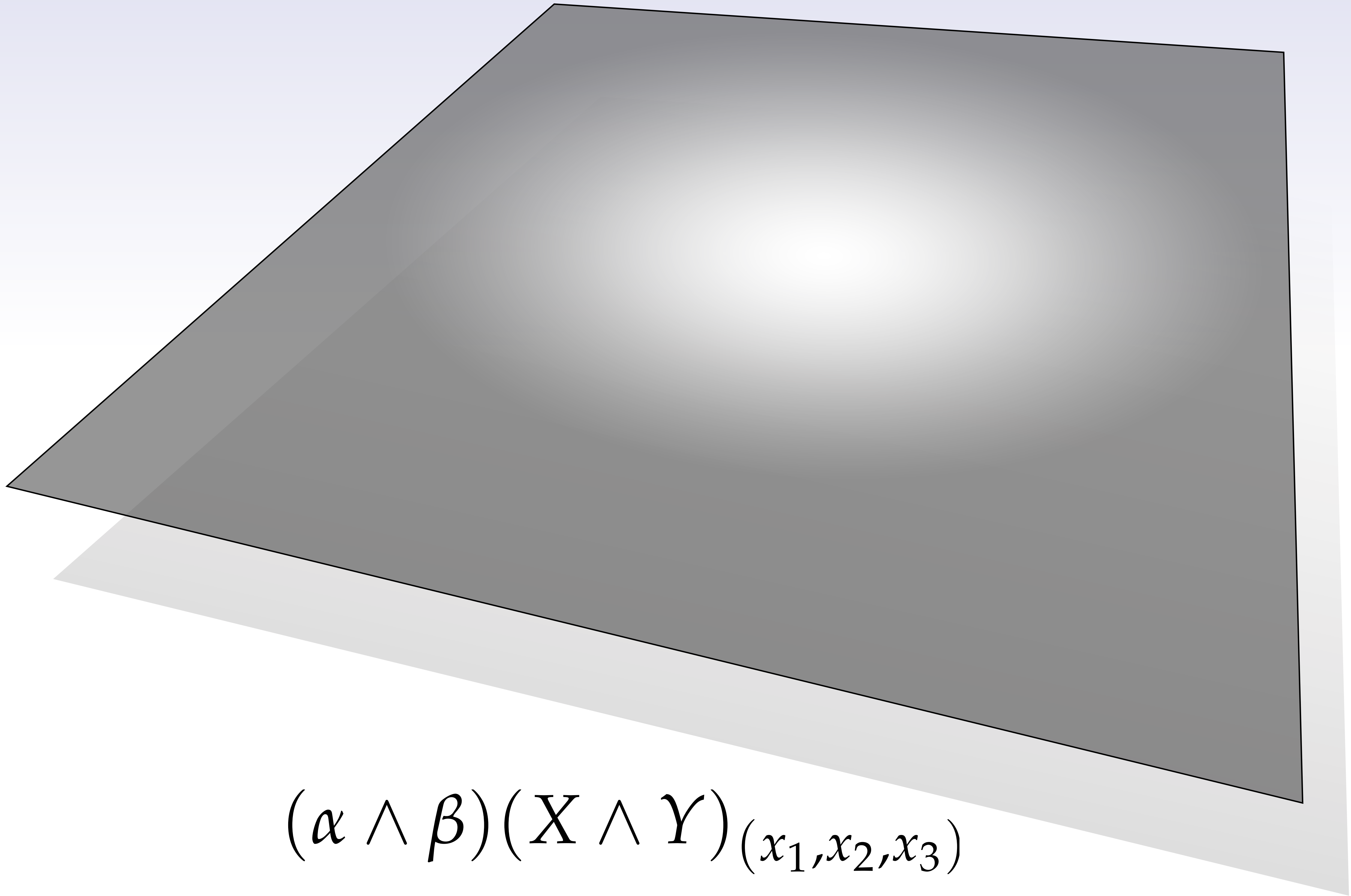


differential 2-form
$(\alpha \wedge \beta)_{(x_1,x_2,x_3)}$

Differential 2-Forms



Differential 2-Forms



$$(\alpha \wedge \beta)(X \wedge Y)_{(x_1,x_2,x_3)}$$

Resulting function says how much a 2-vector field “lines up” with a given 2-form.

Pointwise Operations on Differential k-Forms

- Most operations on differential k-forms simply apply that operation at each point.
- *E.g.*, consider two differential forms α, β on R^n. At each point $p := (x_1,\dots,x_n)$,

$$(\star\alpha)_p := \star(\alpha_p)$$

$$(\alpha \wedge \beta)_p := (\alpha_p) \wedge (\beta_p)$$

- In other words, to get the Hodge star of the *differential* k-form, we just apply the Hodge star to the individual k forms at each point p; to take the wedge of two differential k-forms we just wedge their values at each point.
- Likewise, if $X_1, \dots, X_k$ are vector fields on all of R^n, then

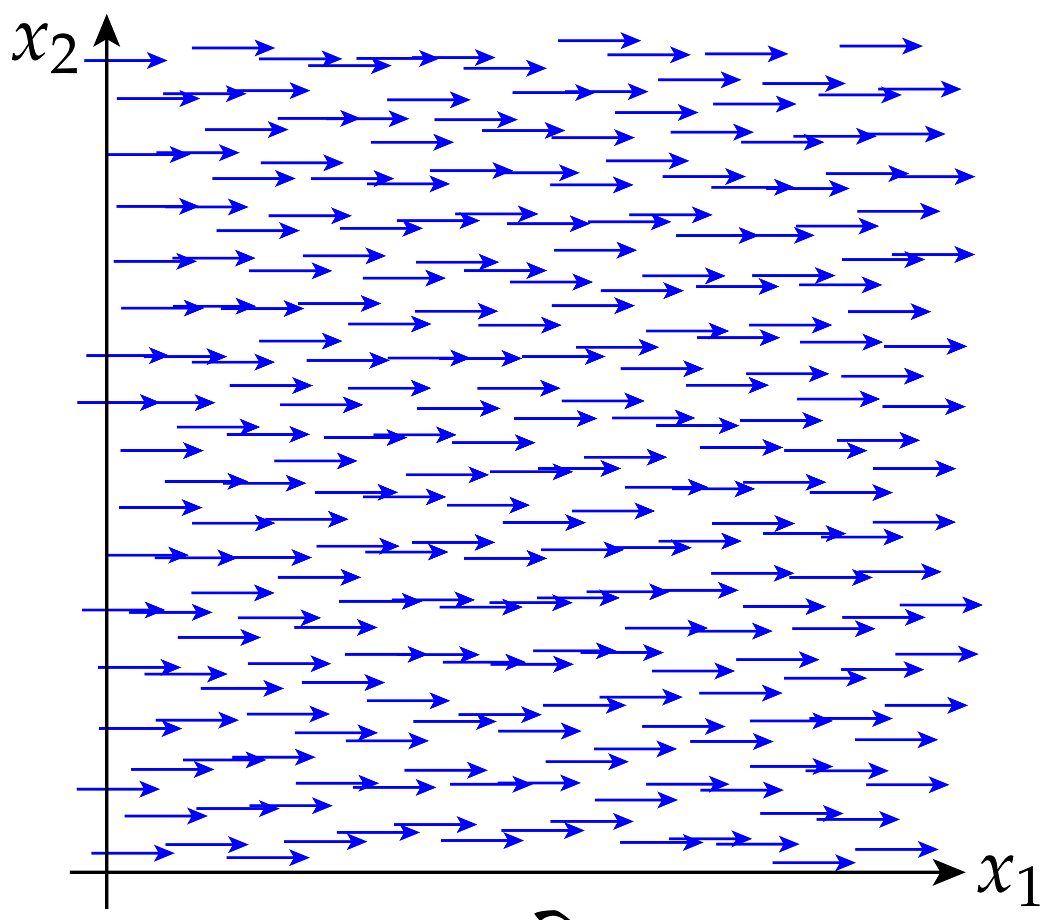
$$\alpha(X_1, \dots, X_k)_p := (\alpha_p)((X_1)_p, \dots, (X_k)_p)$$

Typically we just drop the p entirely and write $\star\alpha$, $\alpha \wedge \beta$, $\alpha(X, Y)$, *etc.*

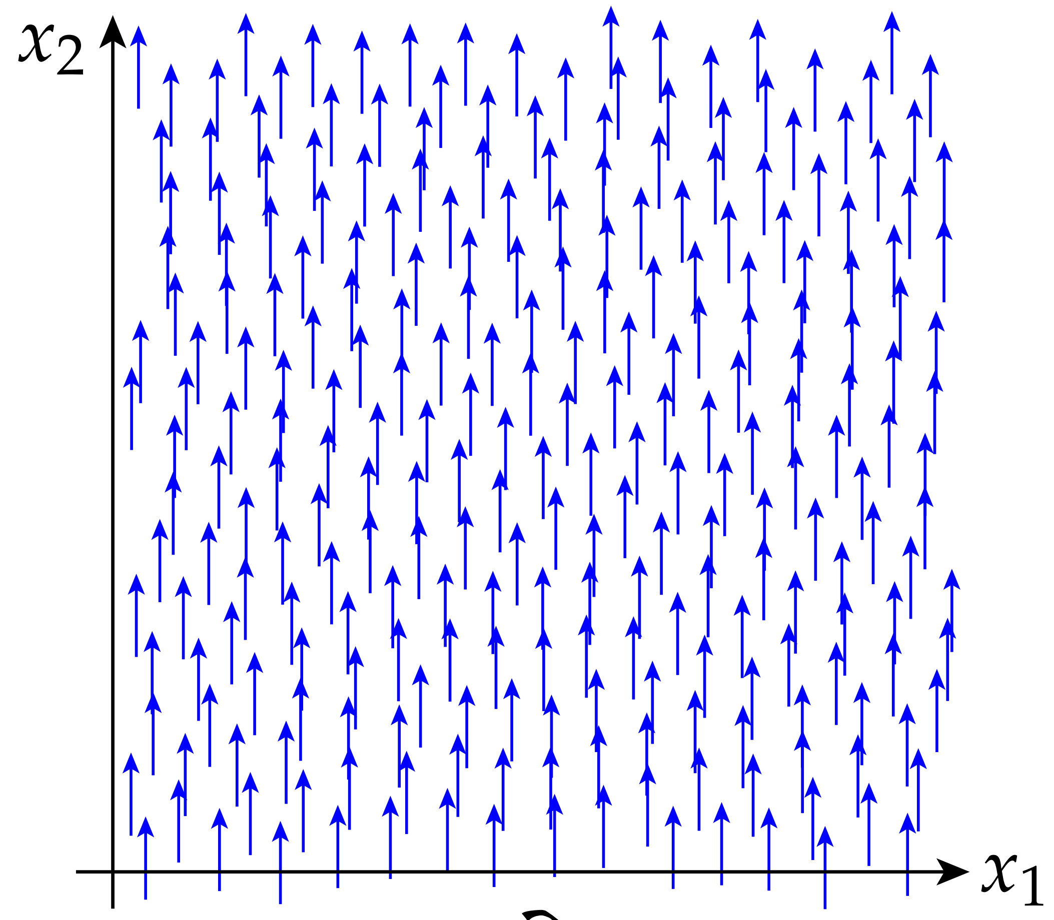
Differential k-Forms in Coordinates

Basis Vector Fields

- Just as we can pick a basis for *vectors*, we can also pick a basis for *vector fields*
- The standard basis for vector fields on R^n are just **constant** vector fields of unit magnitude pointing along each of the coordinate axes:



$$\frac{\partial}{\partial x^1}$$

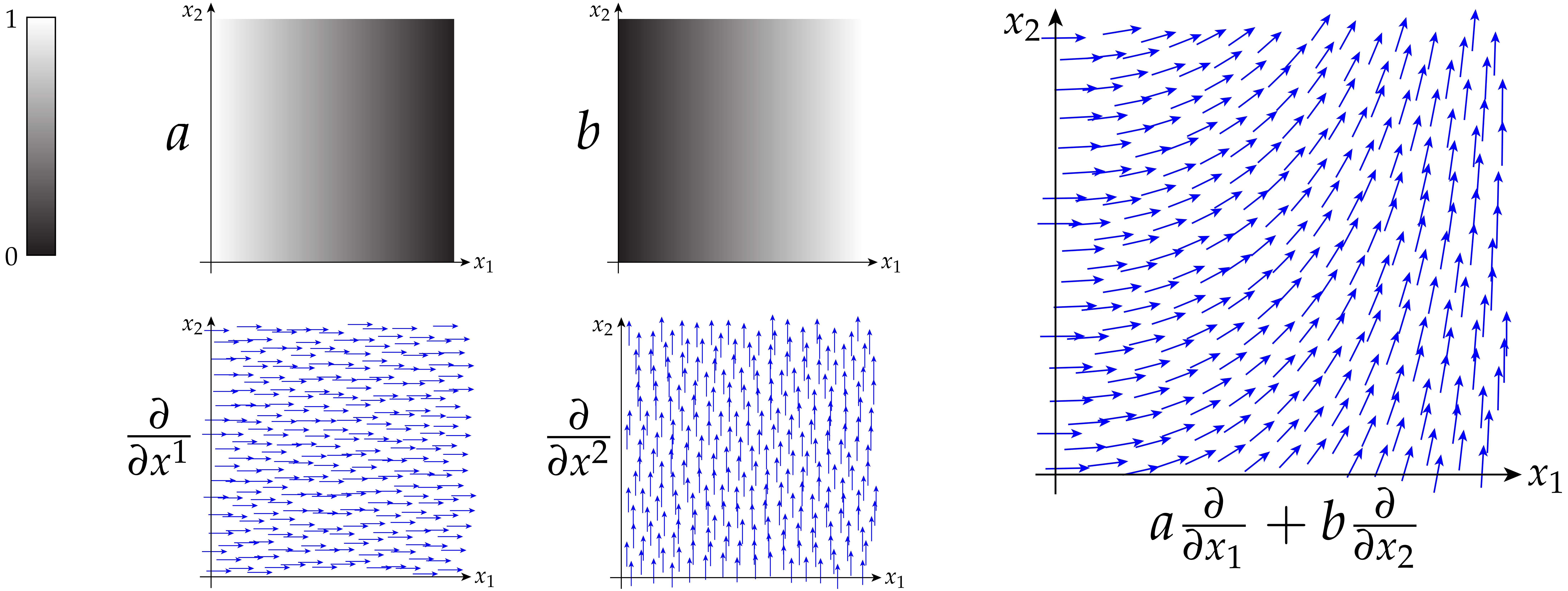


$$\frac{\partial}{\partial x^2}$$

- For historical reasons, these fields have funny-looking names that look like partial derivatives. But you will do yourself a *huge* favor by **forgetting that they have anything at all to do with derivatives!** (For now…)

Basis Expansion of Vector Fields

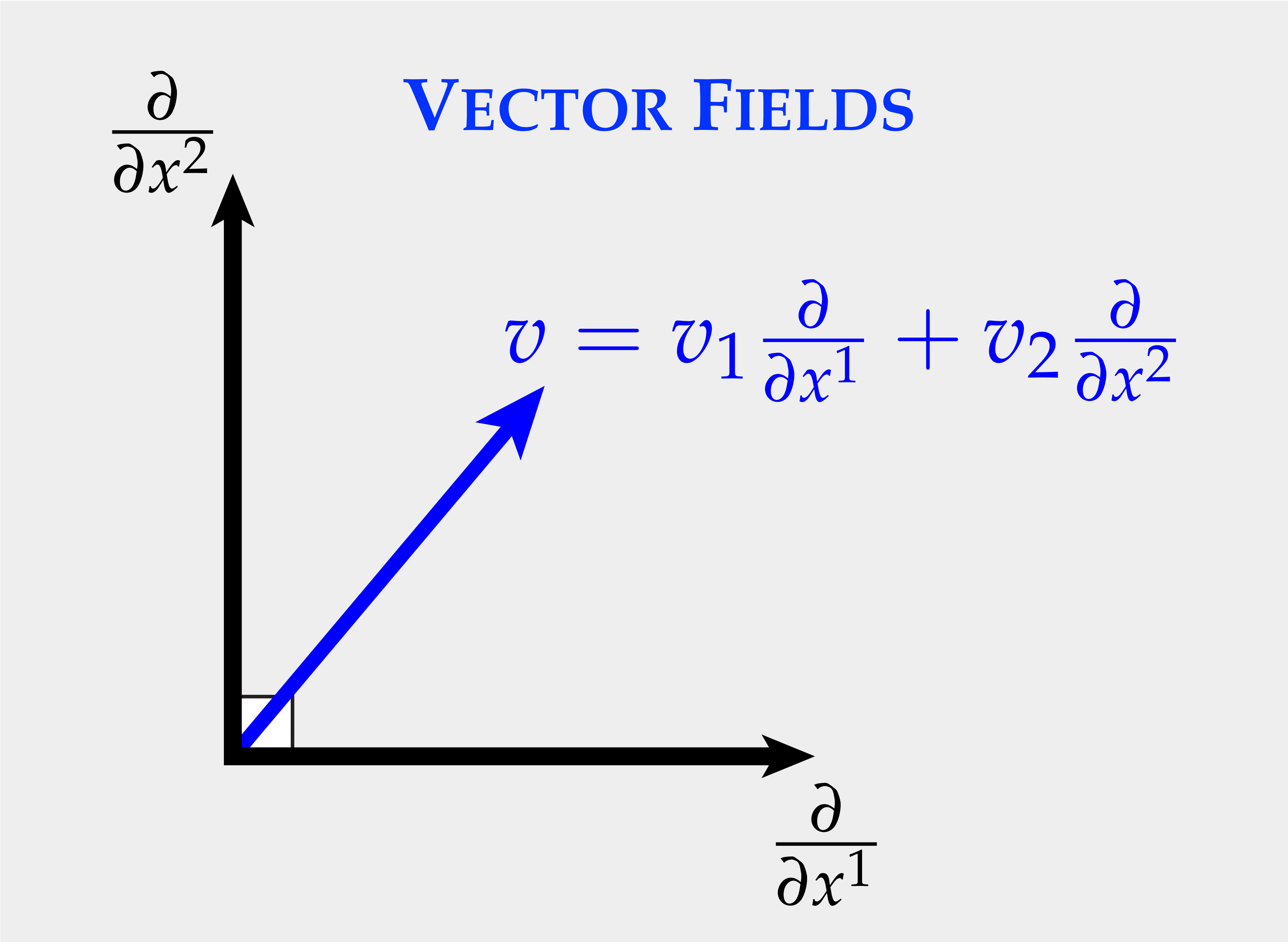
- Any other vector field is then a linear combination of the basis vector fields…
  - …*but*, coefficients of linear combination vary across the domain:

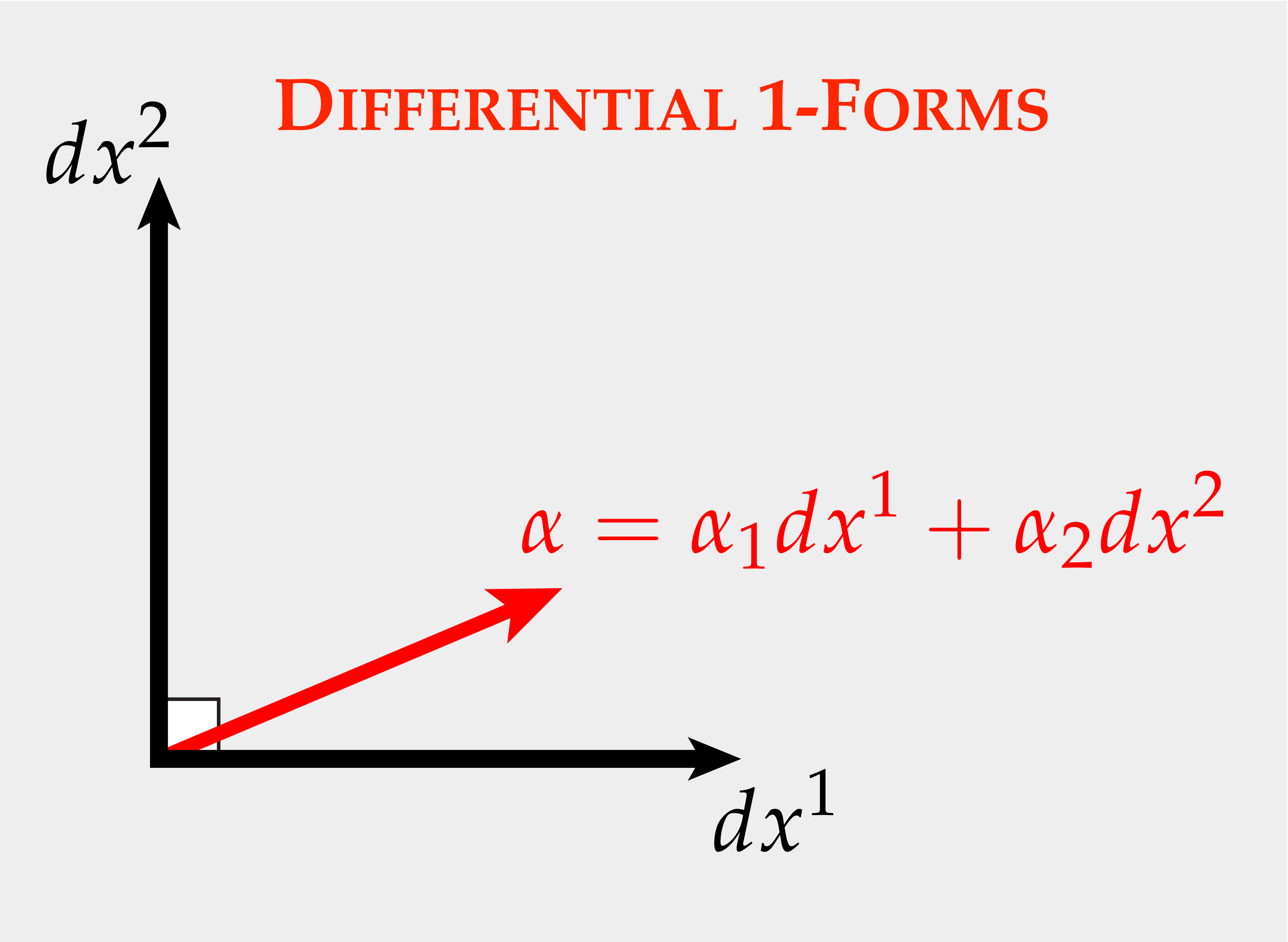


Q: What would happen if we didn't allow coefficients to vary?

Bases for Vector Fields and Differential 1-forms

The story is nearly identical for differential 1-forms, but with different bases:



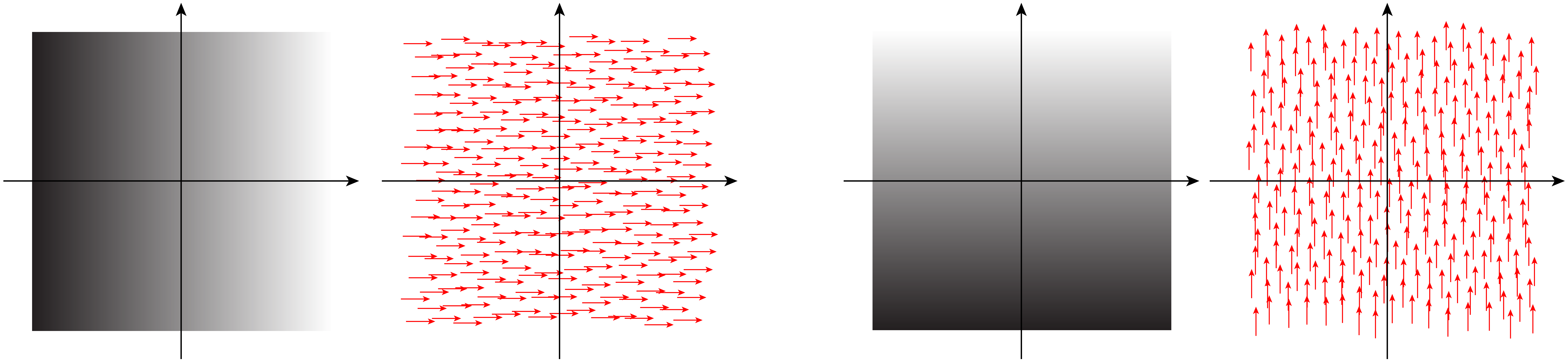


$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta^i_j := \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$$

Stay sane: think of these symbols as *bases*; forget they look like *derivatives!*

Coordinate Bases as Derivatives

Q: That being said, why the heck do we use symbols that look like *derivatives?*



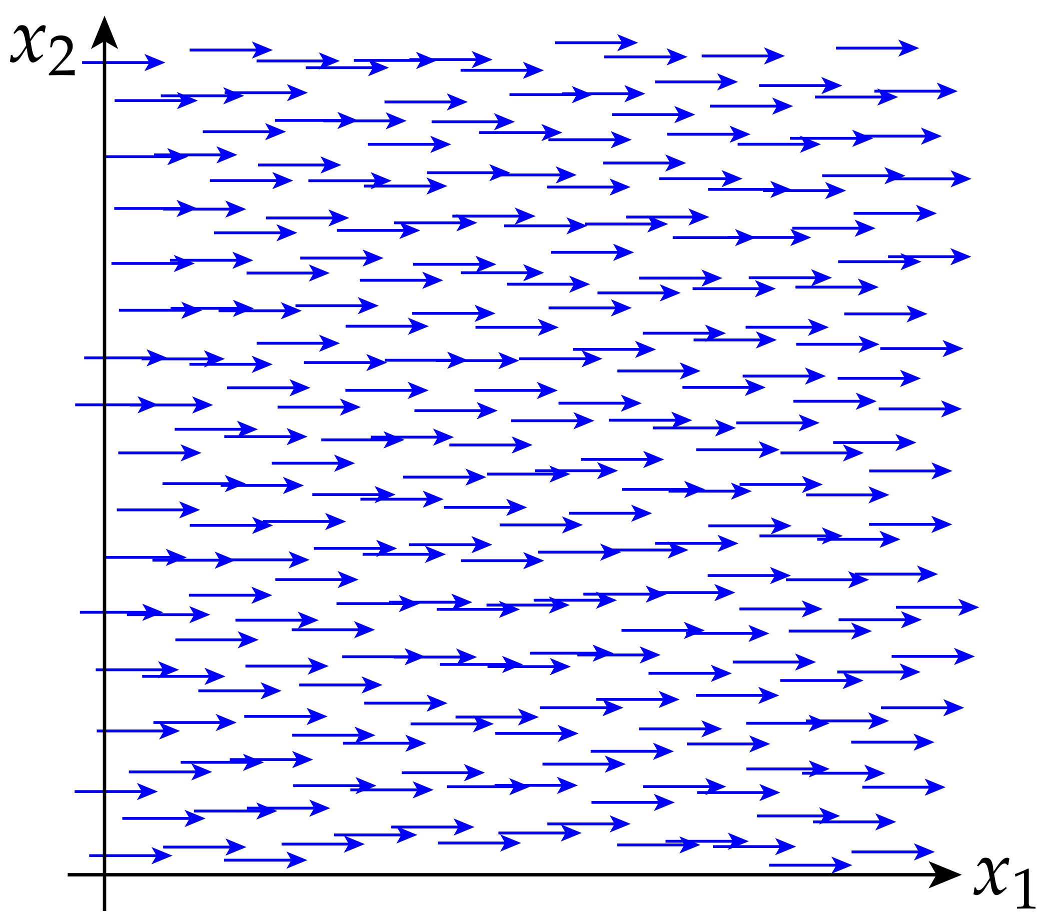
x_1 $\quad dx_1$ $\quad x_2$ $\quad dx_2$

Key idea: derivative of each coordinate function yields a constant basis field.

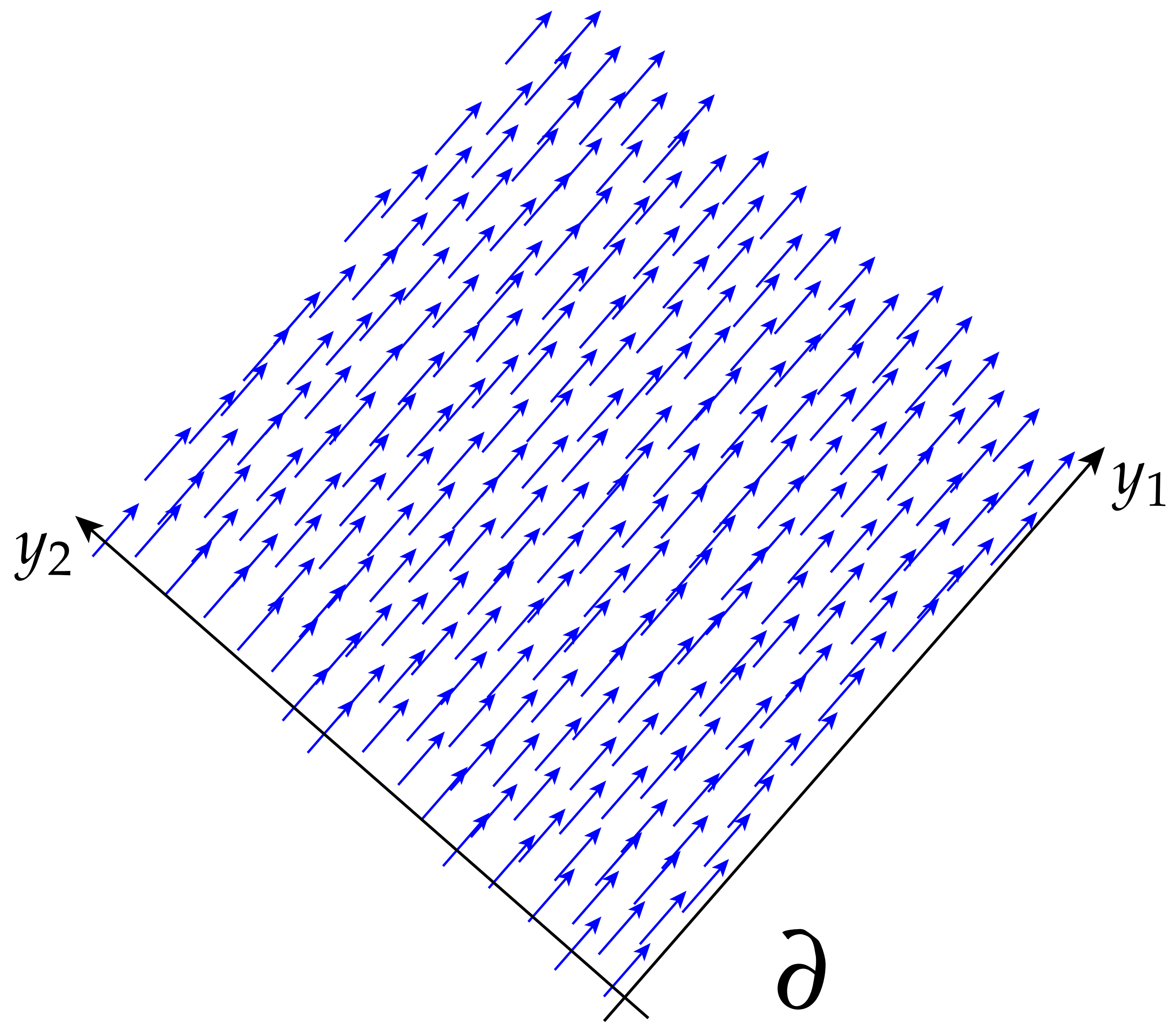
*We'll give a more precise meaning to "d" in a little bit.

Coordinate Notation—Further Apologies

- There is at least one good reason for using this notation for basis fields
- Imagine a situation where we're working with two different coordinate systems:



$$\frac{\partial}{\partial x^1}$$



$$\frac{\partial}{\partial y^1}$$

- Including the name of the coordinates in our name for the basis vector field (or basis differential 1-form) makes it clear which one we mean. Not true with e_i, X_i, etc.

Example: Hodge Star of Differential 1-form

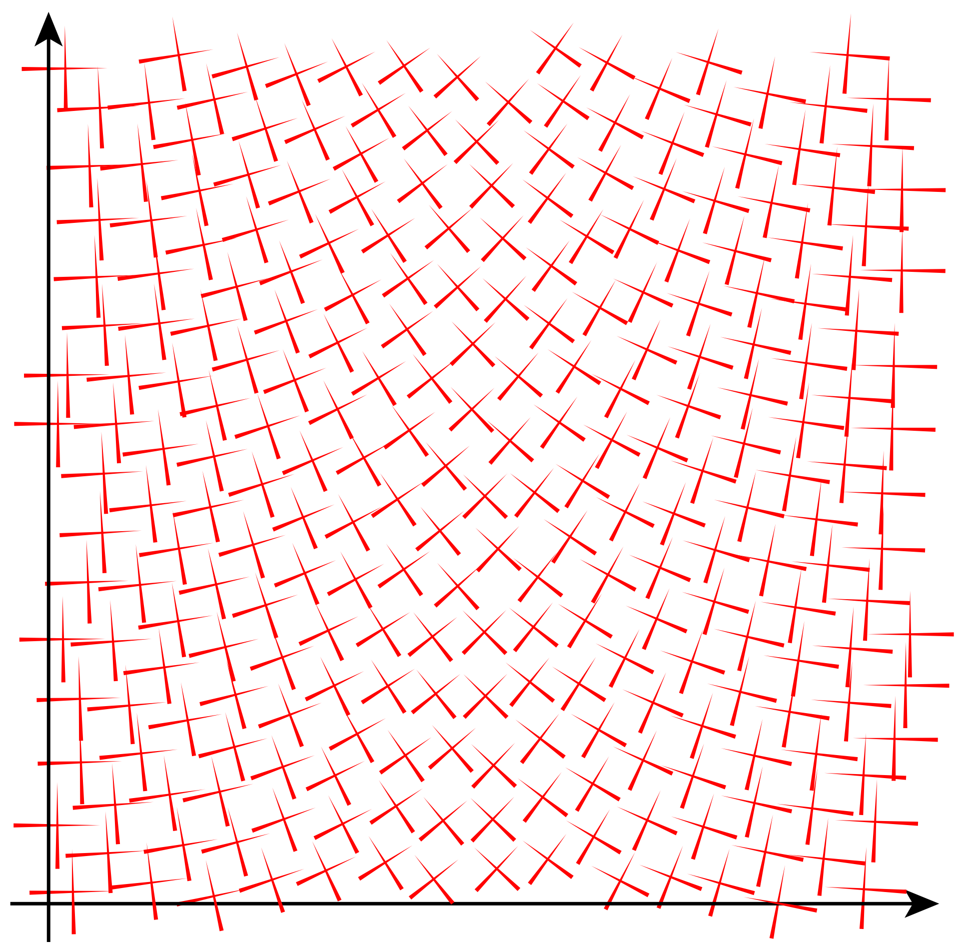
- Consider the differential 1-form $\alpha := (1-x)dx + xdy$
  - Use coordinates (x,y) instead of (x_1,x_2)
  - Notice this expression varies over space

Q: What's its Hodge star?

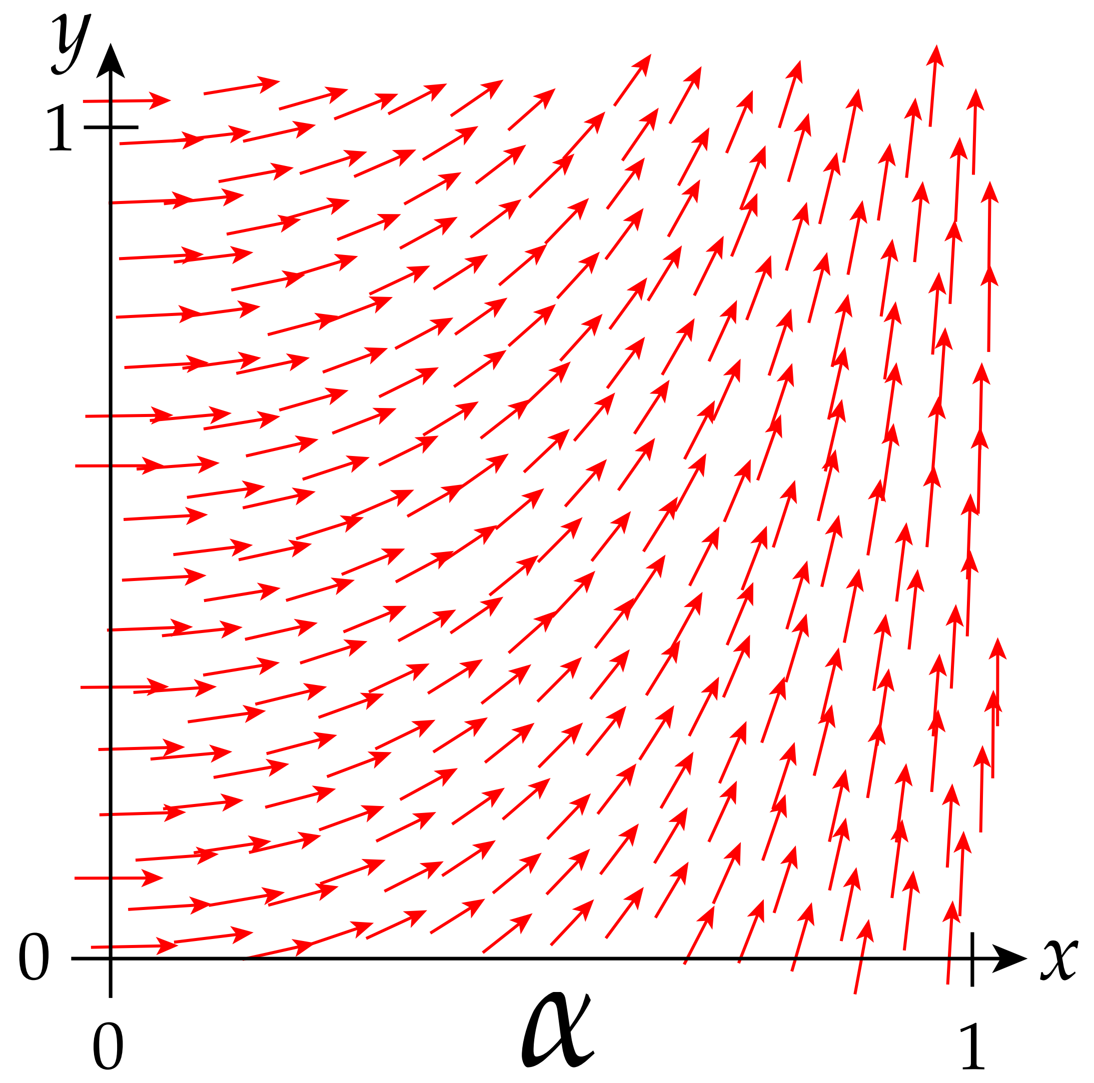
$$\begin{aligned}\star\alpha &= \star((1-x)dx) + \star(xdy) \\ &= (1-x)(\star dx) + x(\star dy) \\ &= (1-x)dy + -xdx\end{aligned}$$

Recall that in 2D, 1-form Hodge star is quarter-turn.

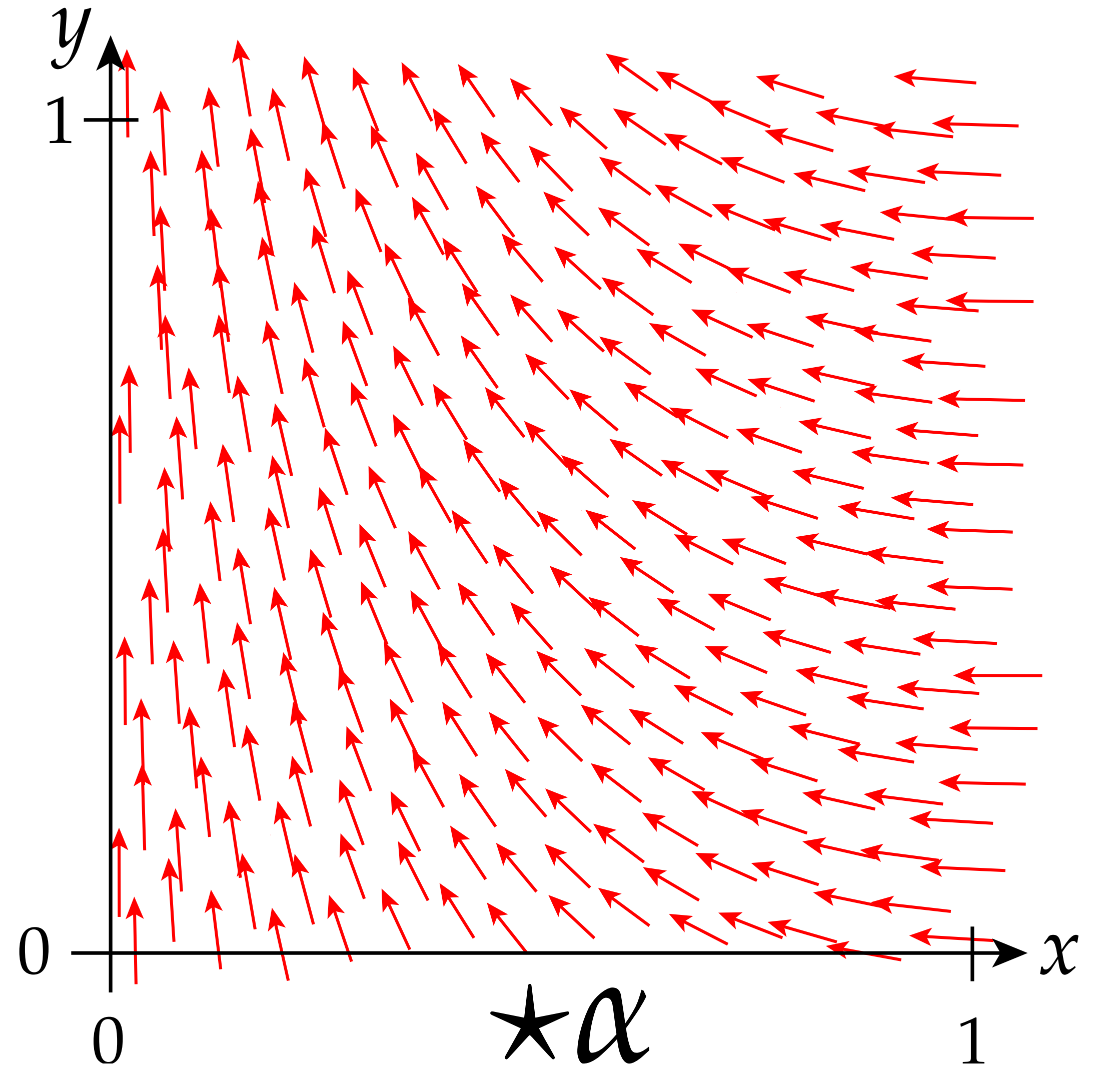
So, when we overlay the two we get little crosses…



$\alpha, \star\alpha$



α



$\star\alpha$

Example: Wedge of Differential 1-Forms

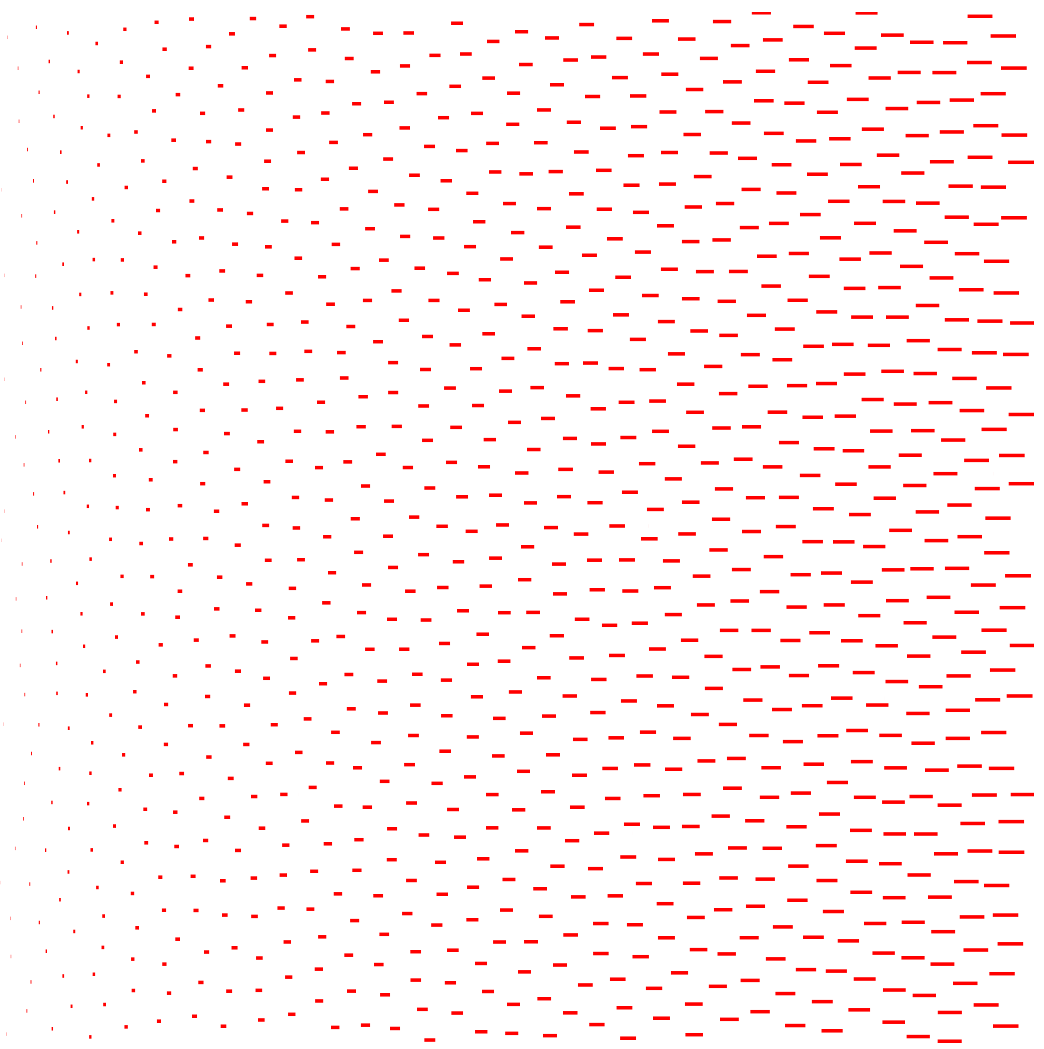
Consider the differential 1-forms*

$$\alpha := x dx, \qquad \beta := (1-x)dx + (1-y)dy$$

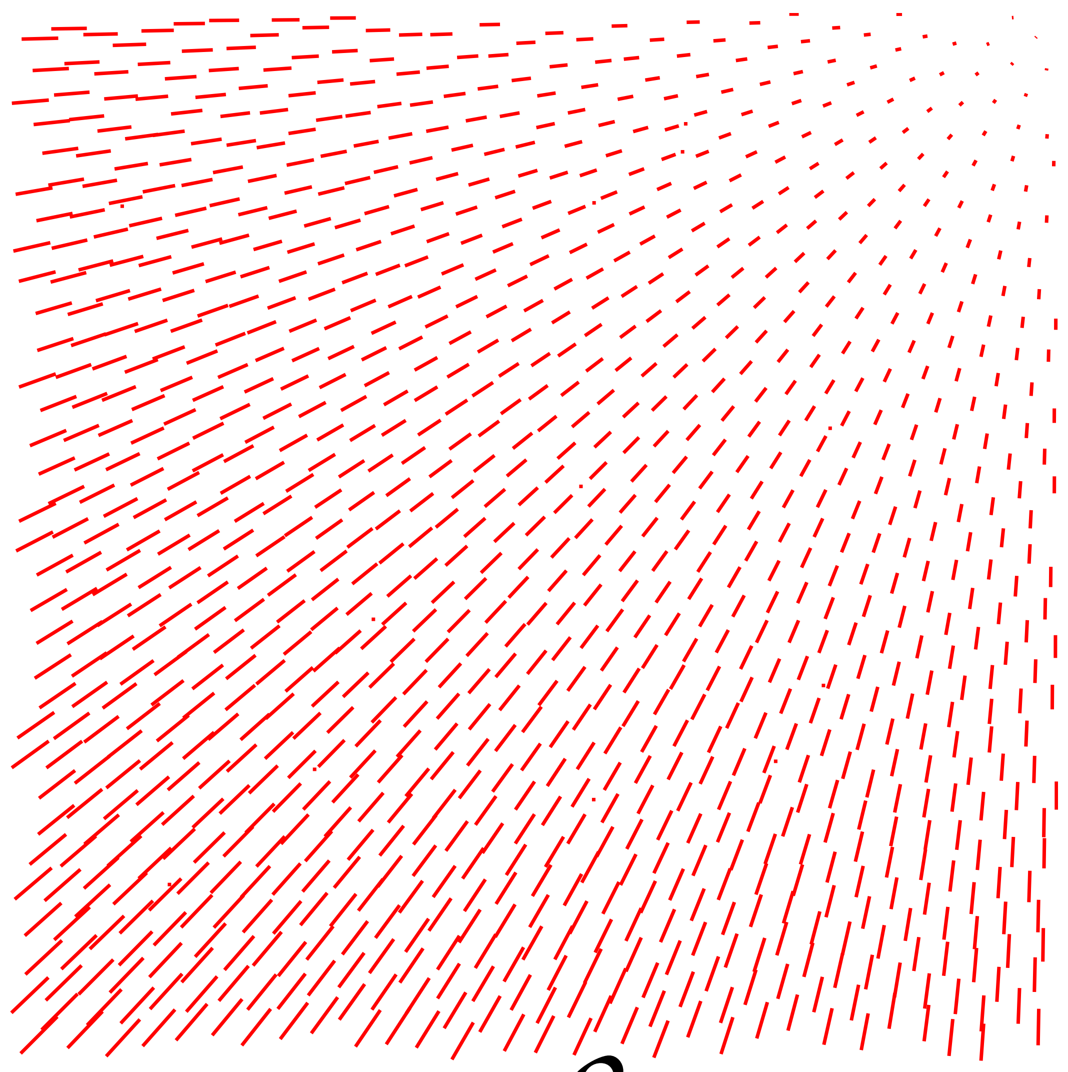
Q: What's their wedge product?

$$\begin{aligned}
\alpha \wedge \beta &= (x dx) \wedge ((1-x)dx + (1-y)dy) \\
&= (x dx) \wedge ((1-x)dx) + (x dx) \wedge ((1-y)dy) \\
&= x(1-x)\cancelto{0}{dx \wedge dx} + x(1-y)dx \wedge dy \\
&= (x - xy)dx \wedge dy
\end{aligned}$$

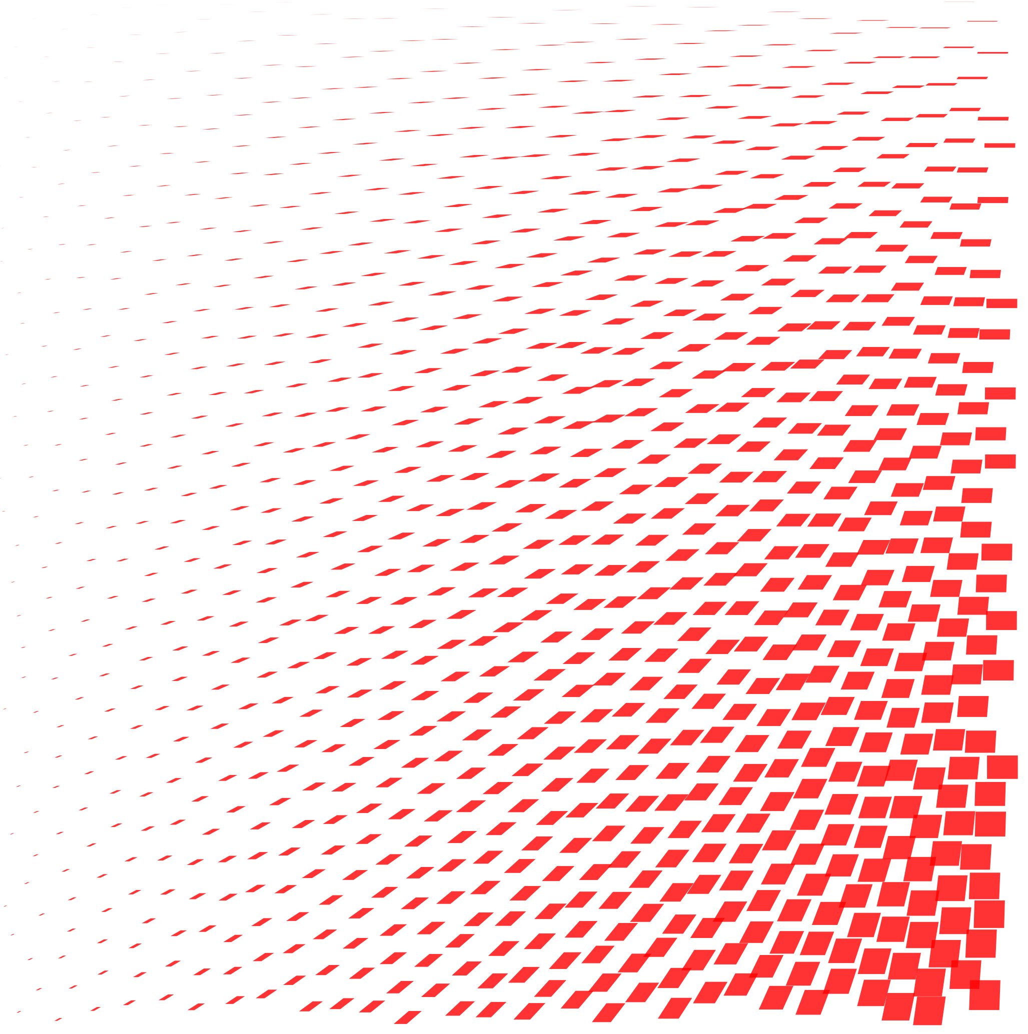
(What does the result **look** like?)



α



β



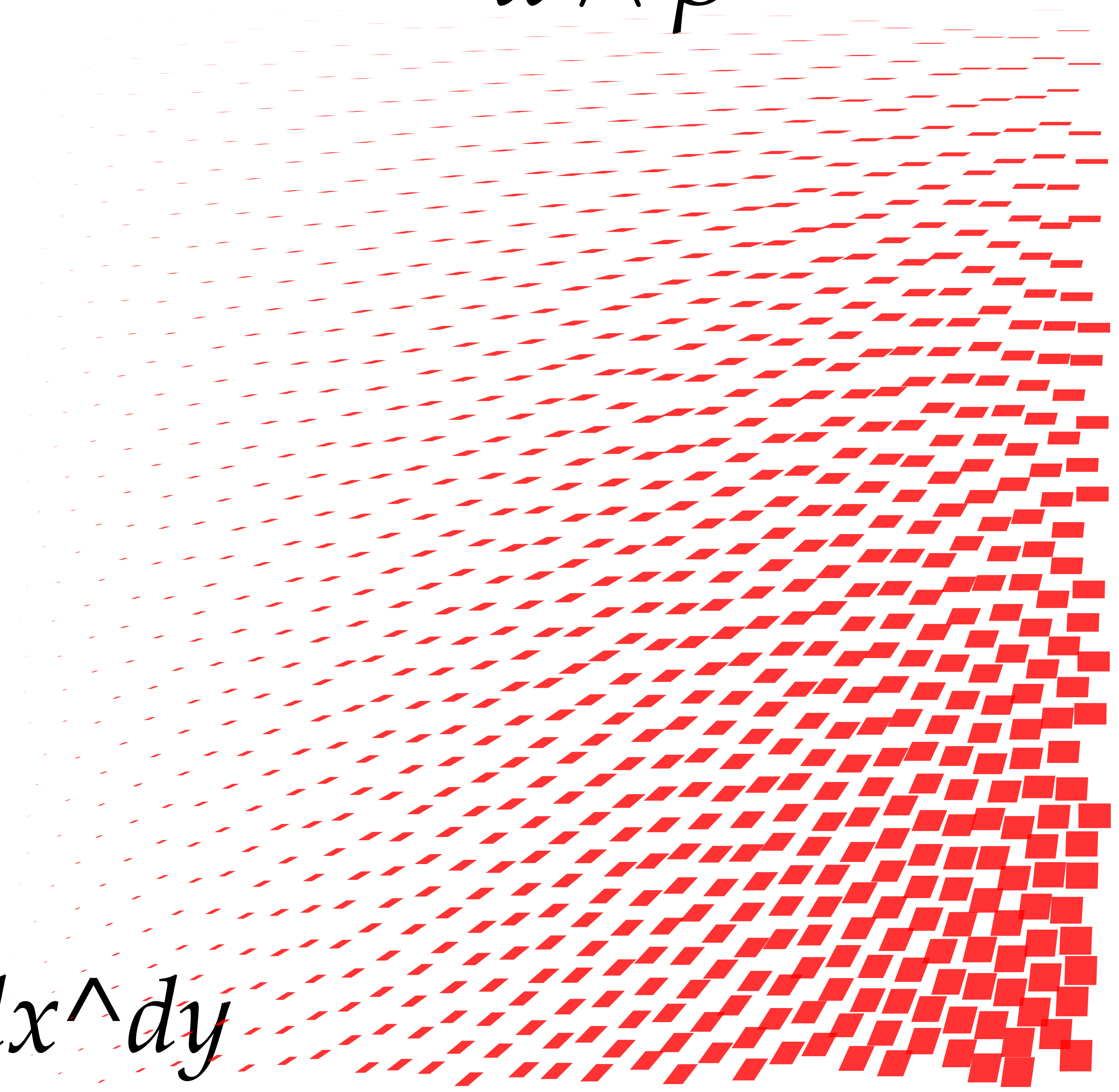
$\alpha \wedge \beta$

*All plots in this slide (and the next few slides) are over the unit square [0,1] x [0,1].

Volume Form / Differential n-form

- Our picture has little parallelograms
- But what information does our differential 2-form actually encode?

$$\alpha \wedge \beta = (x - xy)dx \wedge dy$$



- Has magnitude (x-xy), and “direction” dx^dy
- But in the plane, *every* differential 2-form will be a multiple of dx^dy!
  - More precisely, some scalar function times dx^dy, which measures *unit area*
- In n-dimensions, any *positive* multiple of $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ is called a *volume form*.
  - Provides some meaningful (i.e., nonzero, nonnegative) notion of volume.

Applying a Differential 1-Form to a Vector Field

- The whole point of a differential 1-form is to measure vector fields. So let's do it!

$$\alpha := xdx$$

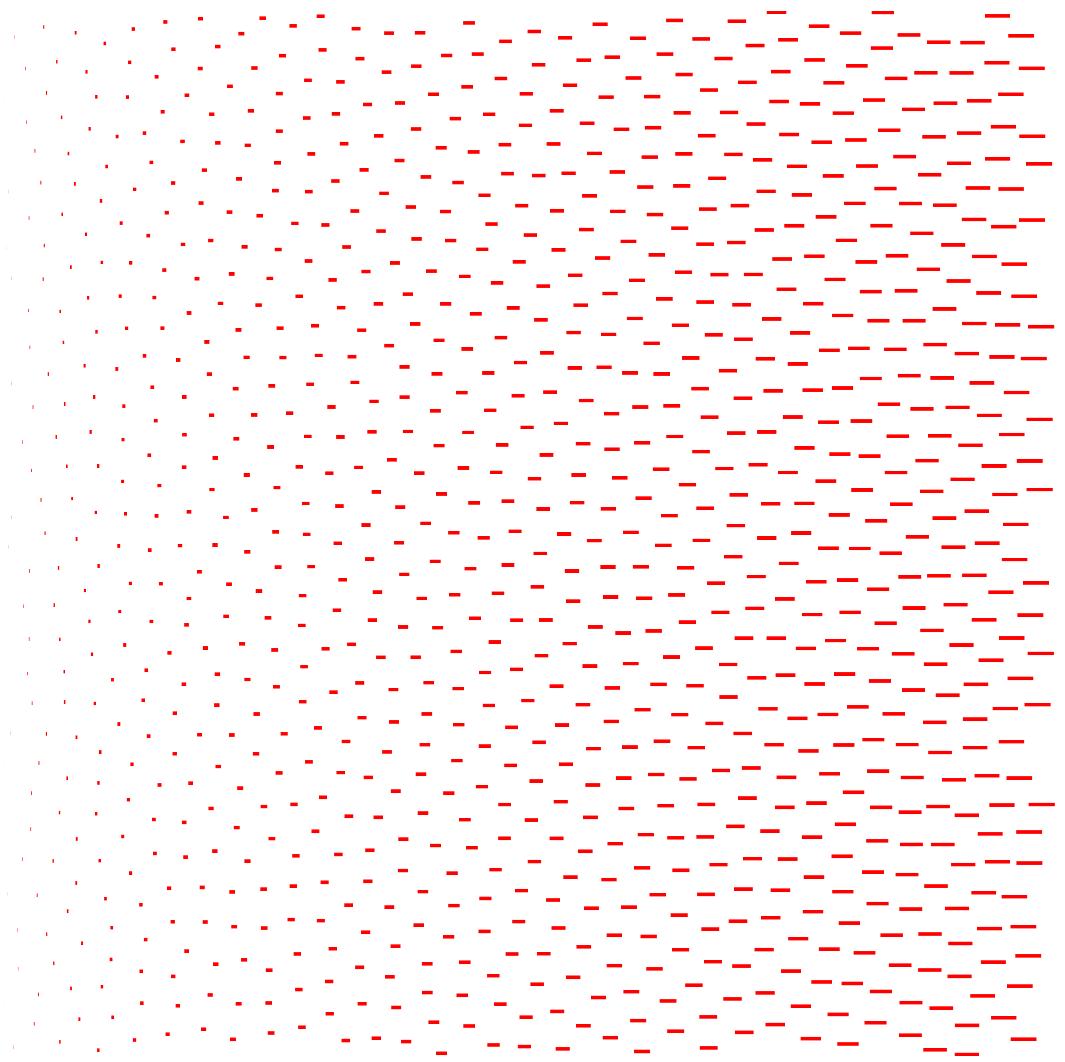
$$X := (1-x)\frac{\partial}{\partial x} + (1-y)\frac{\partial}{\partial y}$$

$$\alpha(X) = (xdx)\left((1-x)\frac{\partial}{\partial x} + (1-y)\frac{\partial}{\partial y}\right)$$

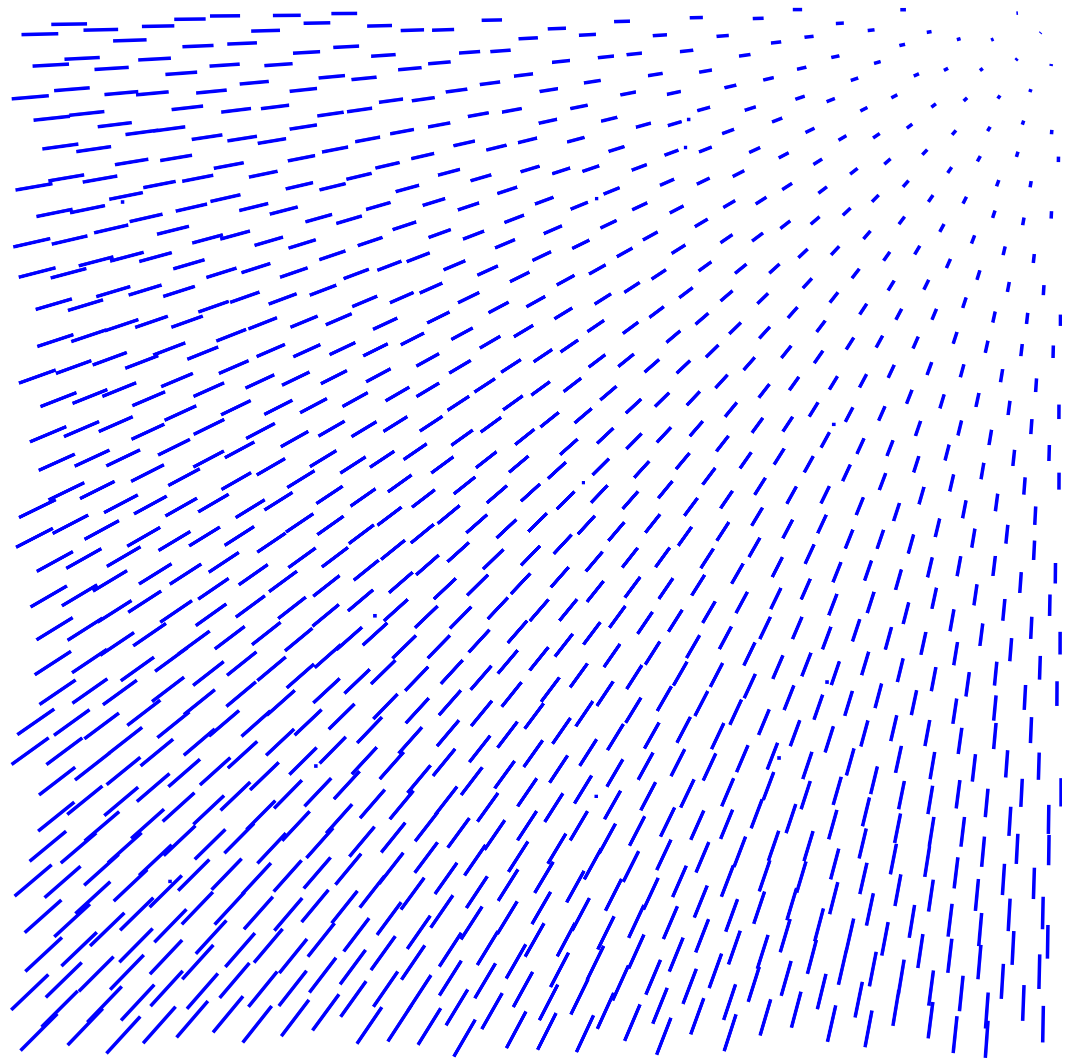
$$= (xdx)\left((1-x)\frac{\partial}{\partial x}\right) + (xdx)\left((1-y)\frac{\partial}{\partial y}\right)$$

$$= (x-x^2)\cancelto{1}{dx(\tfrac{\partial}{\partial x})} + (x-xy)\cancelto{0}{dx(\tfrac{\partial}{\partial y})}$$

$$= x - x^2$$



α



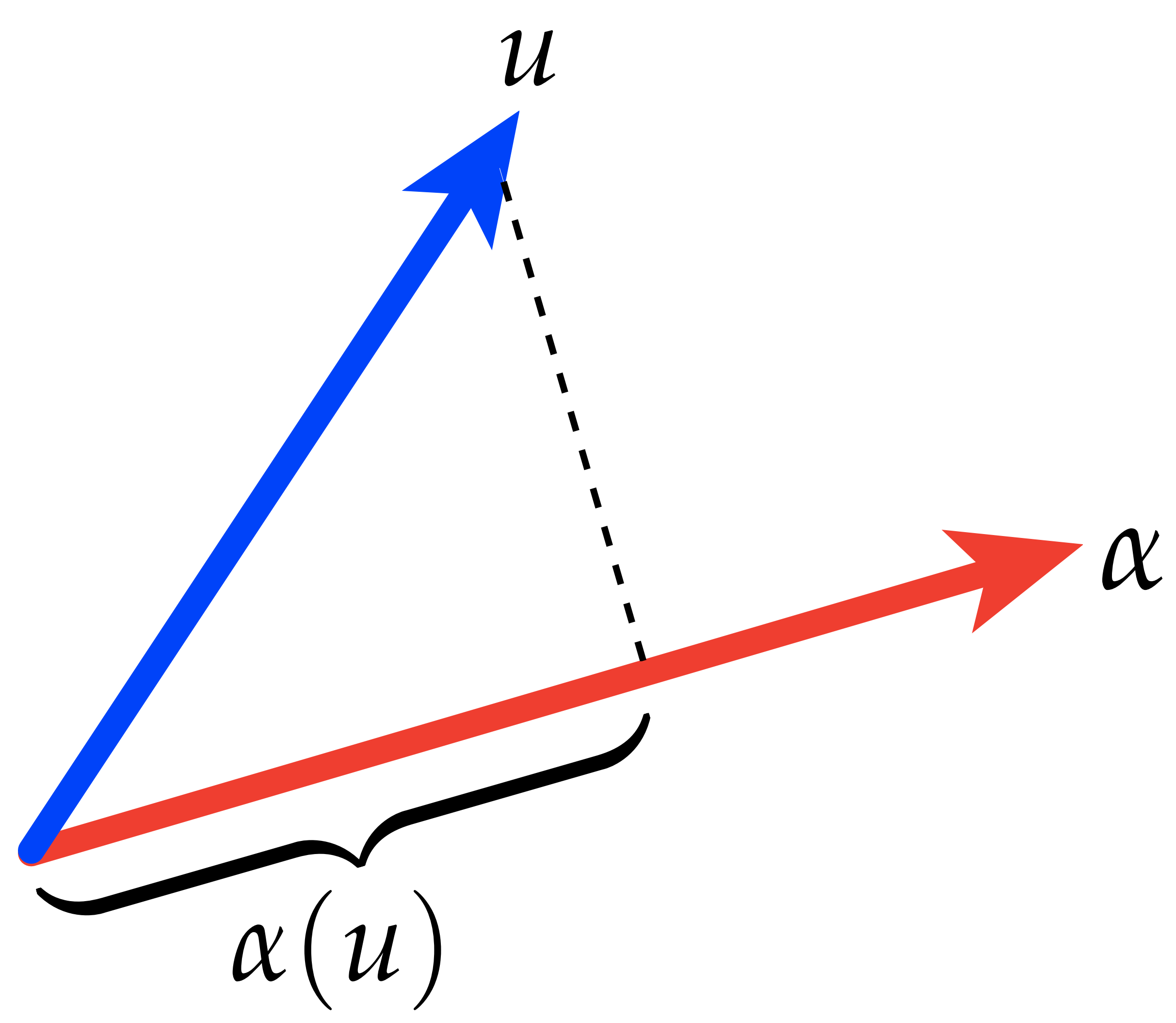
X

$\alpha(X)$

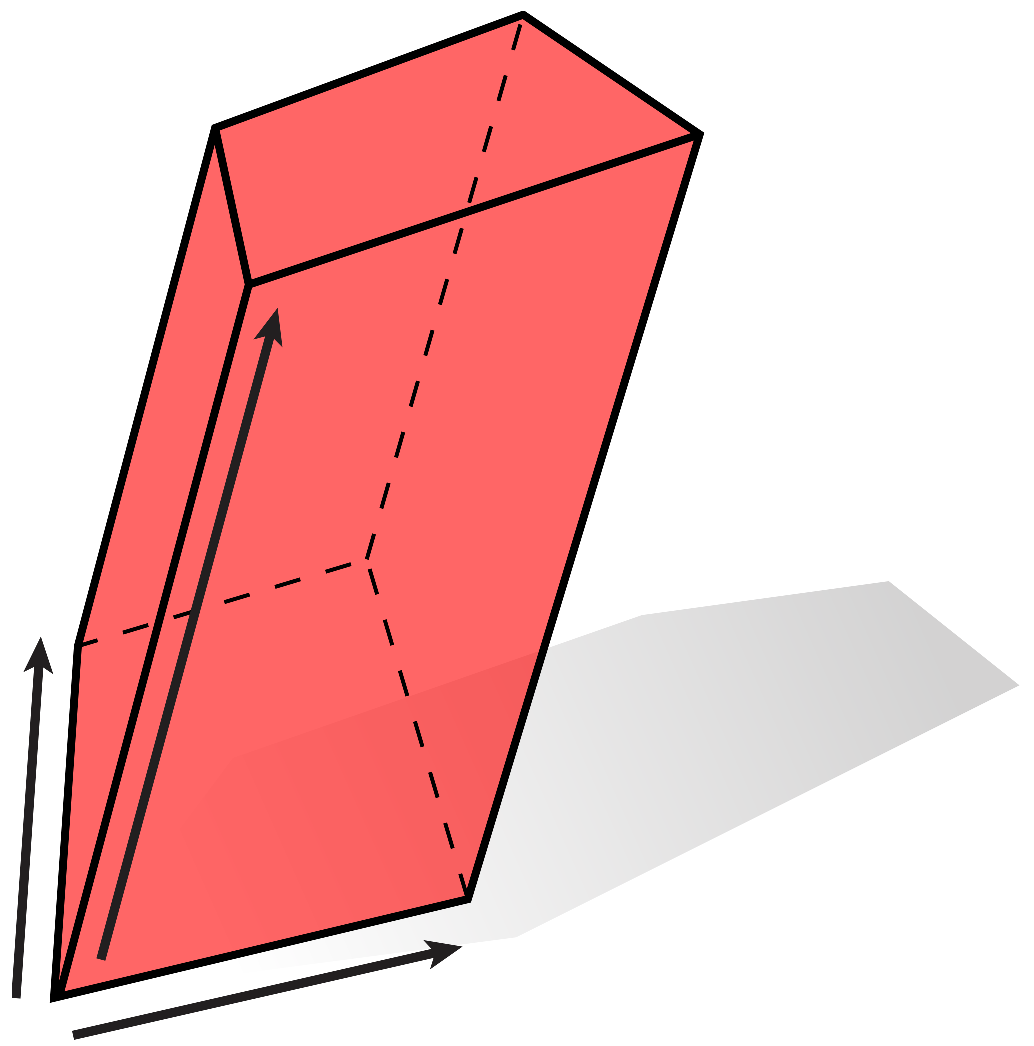
(Kind of like a dot product…)

Differential Forms in R^n - Summary

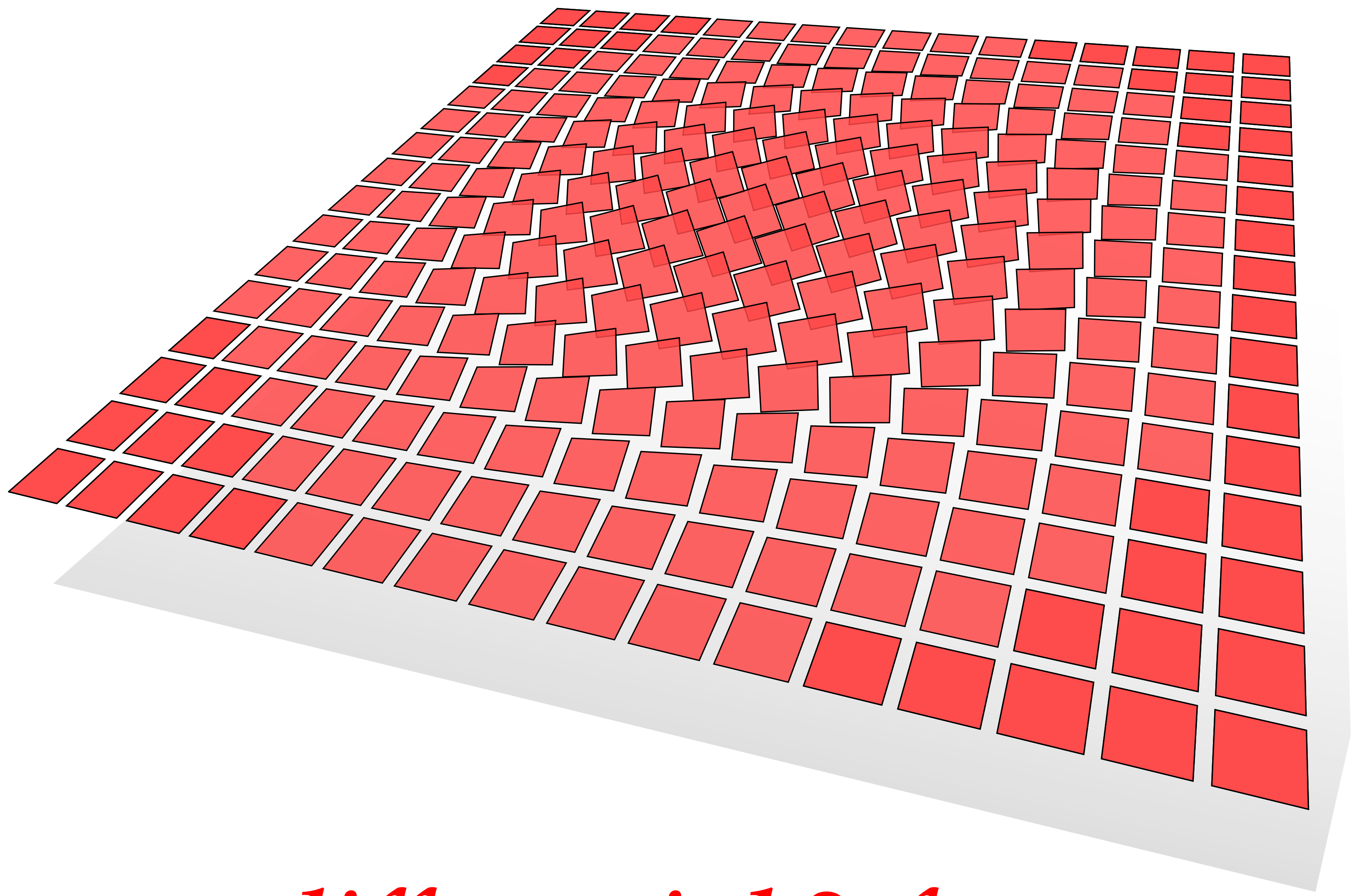
- Started with a vector space V (*e.g.*, R^n)
  - ***(1-forms)*** Dual space V^* of covectors, i.e., linear measurements of vectors
  - ***(k-forms)*** Wedge together k covectors to get a measurement of k-dim. volumes
  - ***(differential k-forms)*** Put a k-form at each point of space



1-form



3-form



differential 2-form

Exterior Algebra & Differential Forms—Summary

	primal	**dual**
vector space	vectors	covectors
exterior algebra	k-vectors	k-forms
spatially-varying	k-vector fields	differential k-forms

Where Are We Going Next?

GOAL: develop *discrete exterior calculus (DEC)*

Prerequisites:

Linear algebra: "little arrows" (vectors)

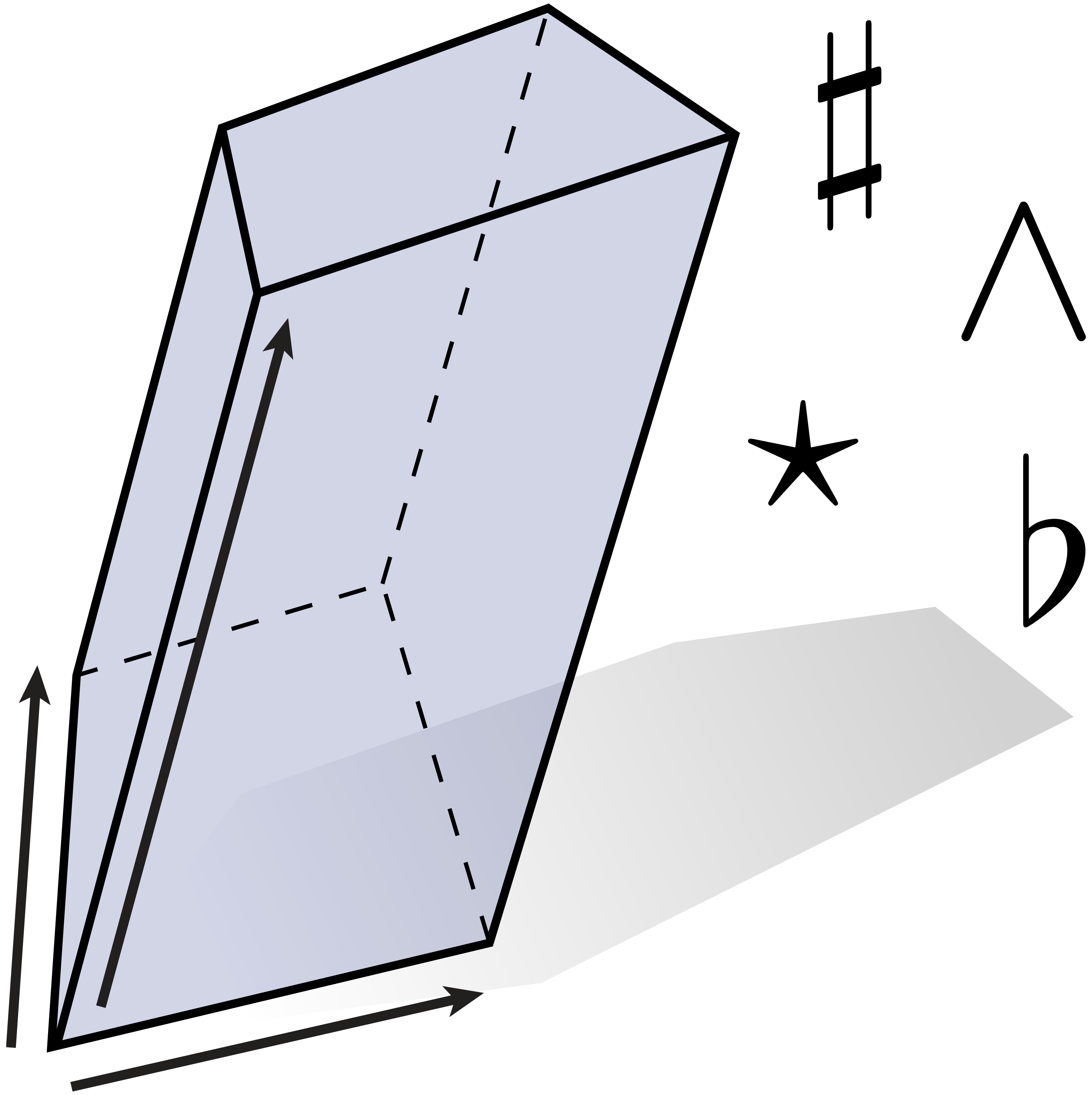
Vector Calculus: how do vectors *change?*

Next few lectures:

Exterior algebra: "little volumes" (k-vectors)

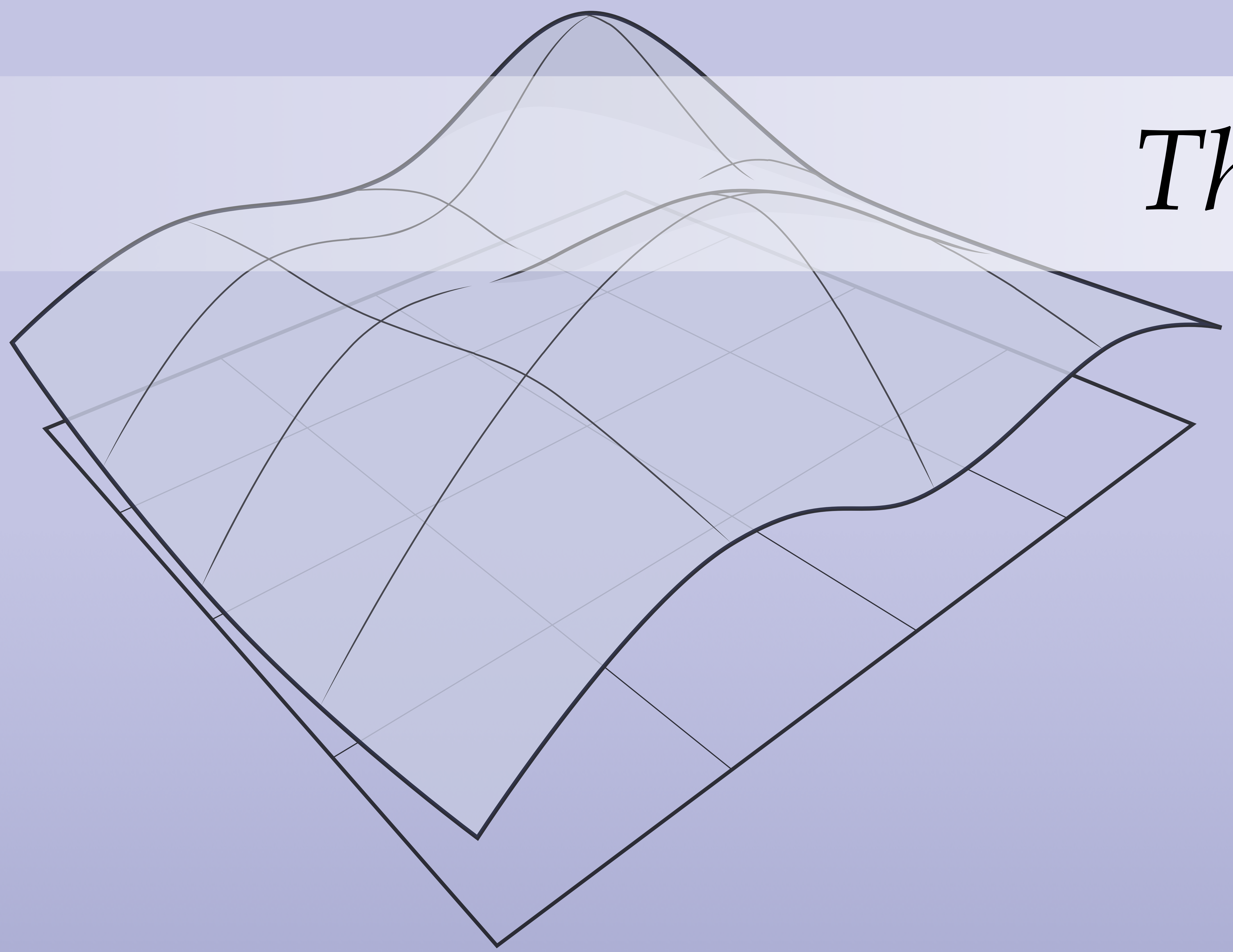
Exterior calculus: how do k-vectors change?

DEC: how do we do all of this on meshes?



Basic idea: replace vector calculus with computation on meshes.

Thanks!



DISCRETE DIFFERENTIAL GEOMETRY

AN APPLIED INTRODUCTION