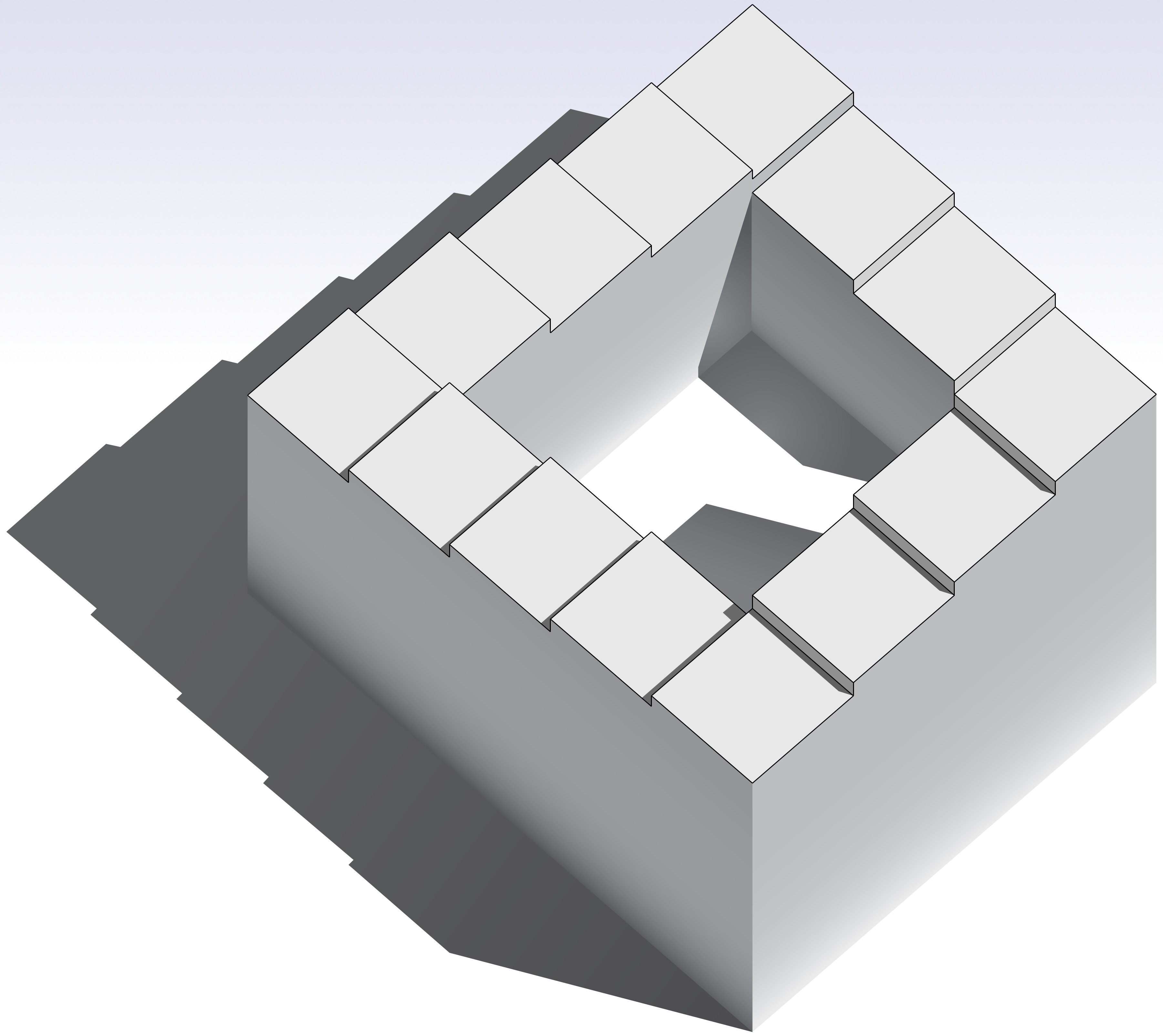
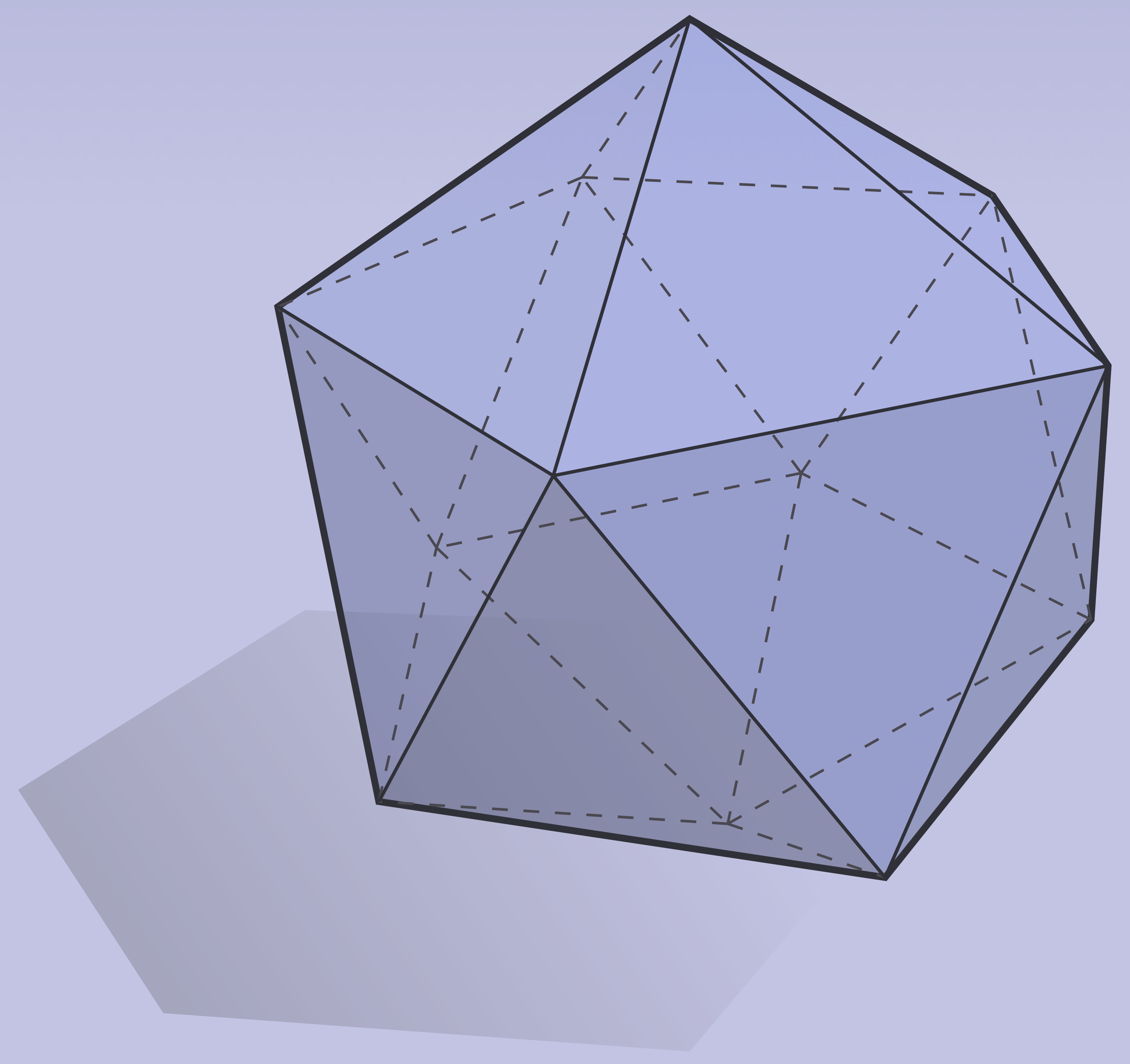


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858B • Fall 2017

LECTURE 2:
THE SIMPLICIAL COMPLEX

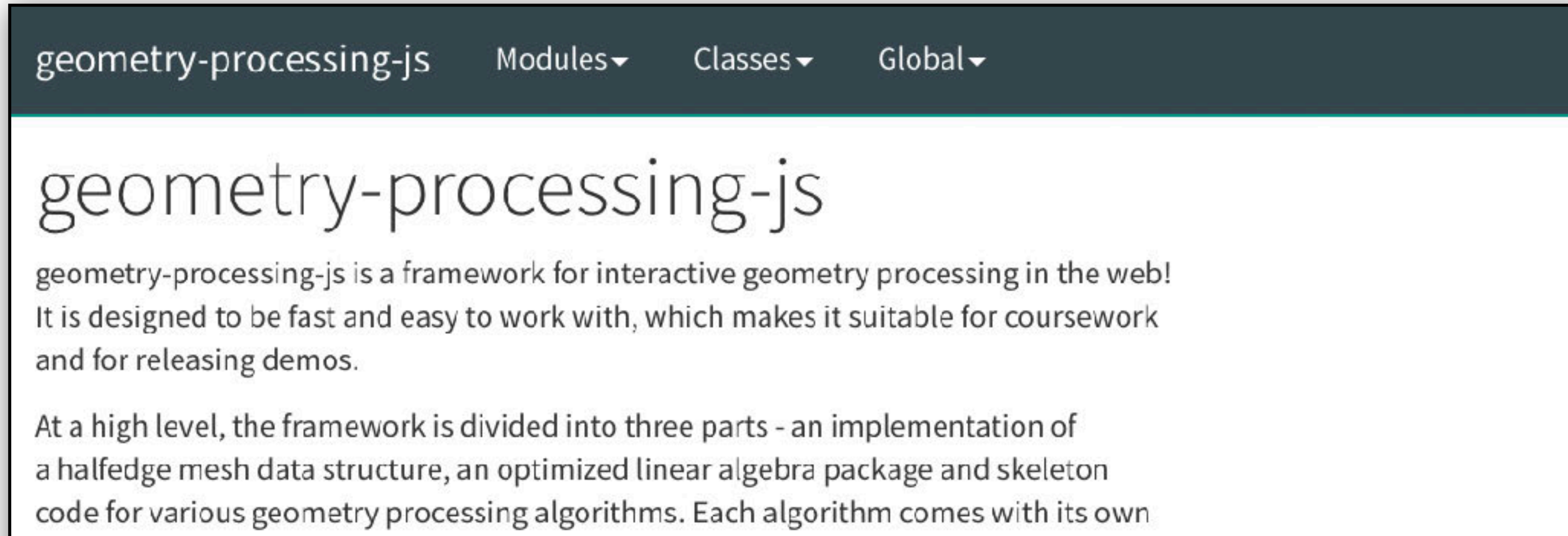


DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858B • Fall 2017

Administrivia

- First reading assignment was due 10am today! **(Please use Andrew ID)**
- **First homework assignment (A1)** — due 9/27
  - Basic tool we'll be using all term: (discrete) exterior calculus
  - Written part out today, coding part out next week
- Special *recitation on how to use code framework:* **9/8 @ 4–5pm, GHC 4215**



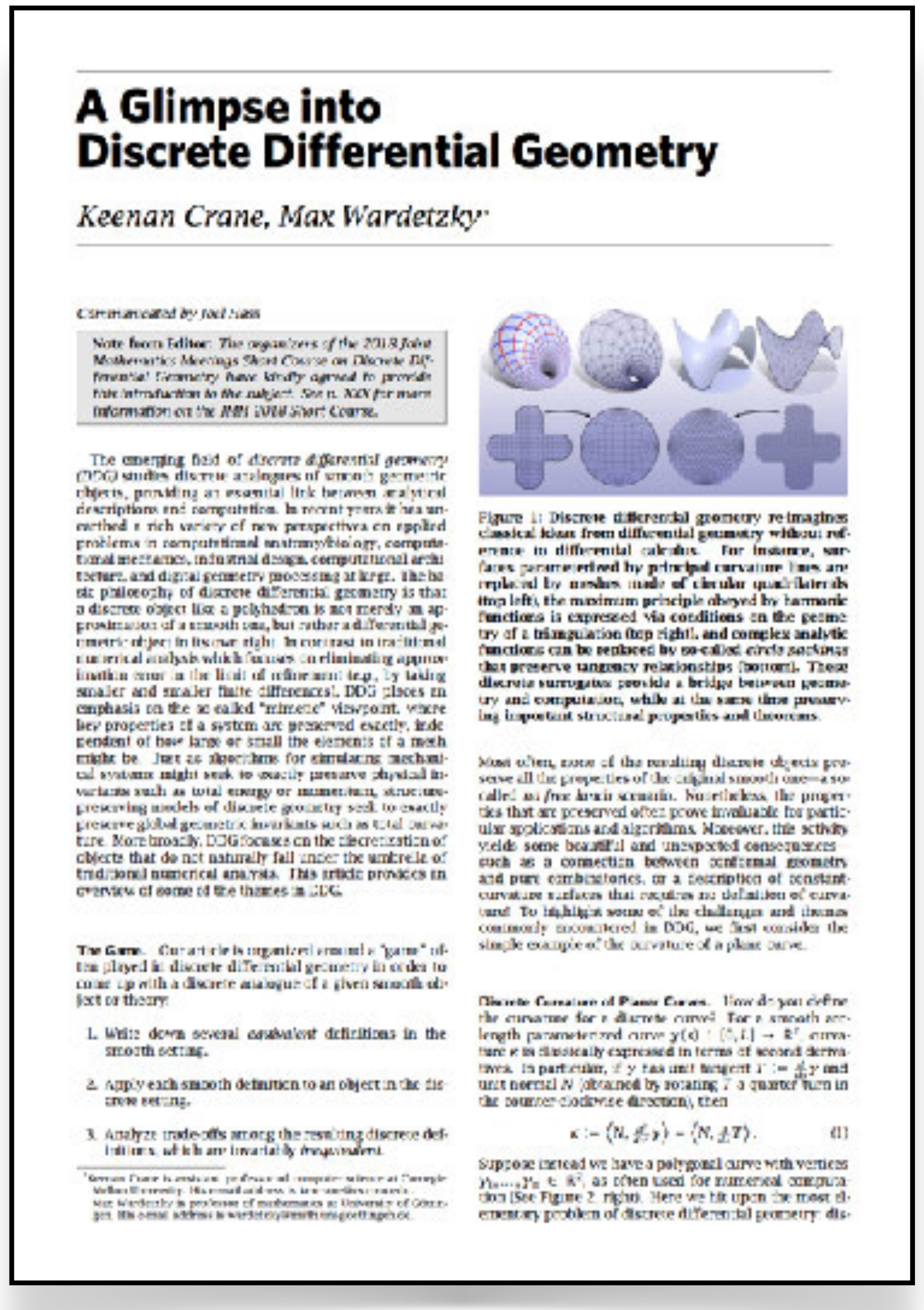
geometry-processing-js Modules Classes Global

geometry-processing-js

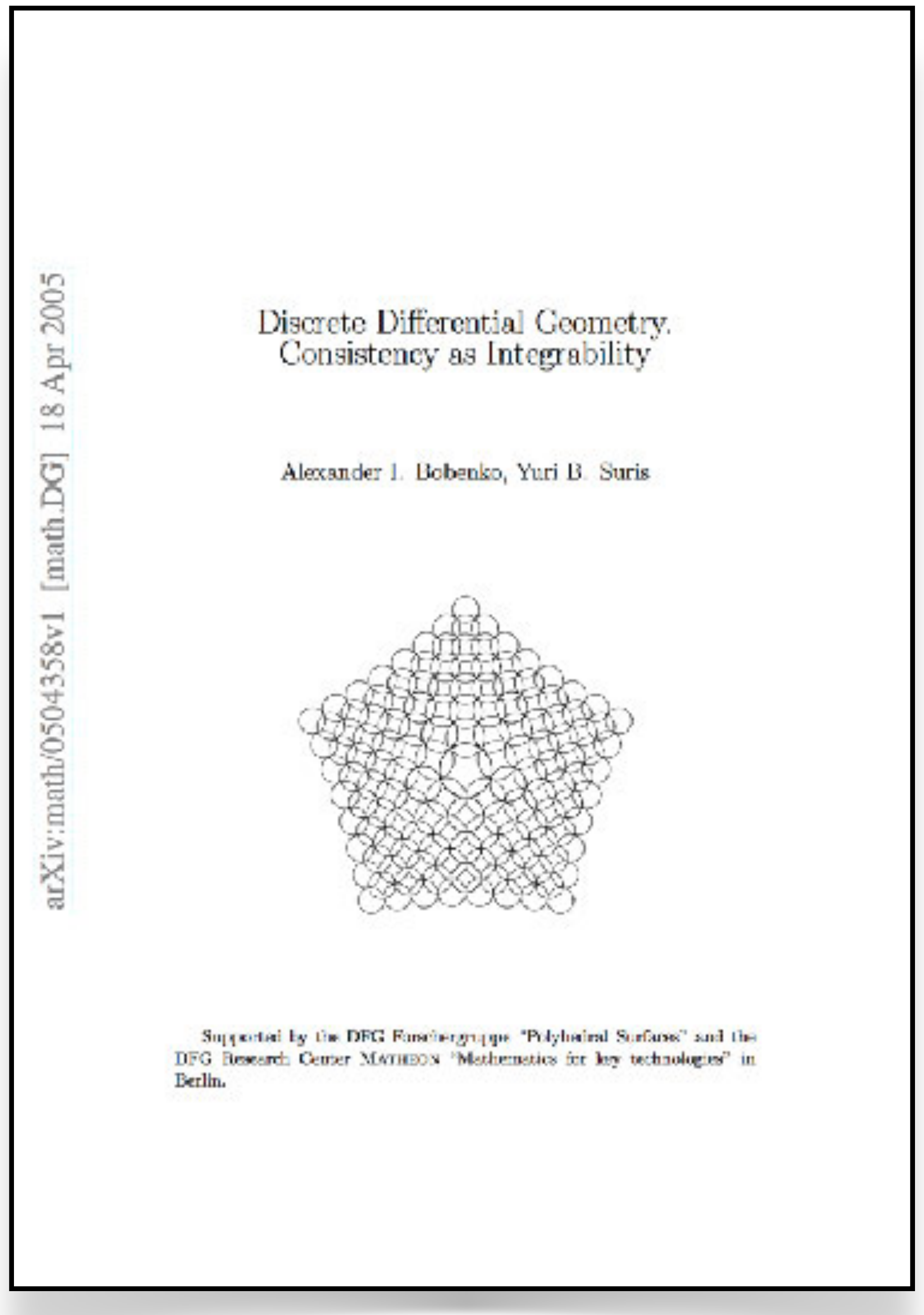
geometry-processing-js is a framework for interactive geometry processing in the web! It is designed to be fast and easy to work with, which makes it suitable for coursework and for releasing demos.

At a high level, the framework is divided into three parts - an implementation of a halfedge mesh data structure, an optimized linear algebra package and skeleton code for various geometry processing algorithms. Each algorithm comes with its own

Reading: Overview of DDG


A Glimpse into
Discrete Differential Geometry
Keenan Crane, Max Wardetzky


Architectural Geometry


arXiv:math/0504358v1 [math.DG] 18 Apr 2005
Discrete Differential Geometry.
Consistency as Integrability
Alexander I. Bobenko, Yuri B. Suris

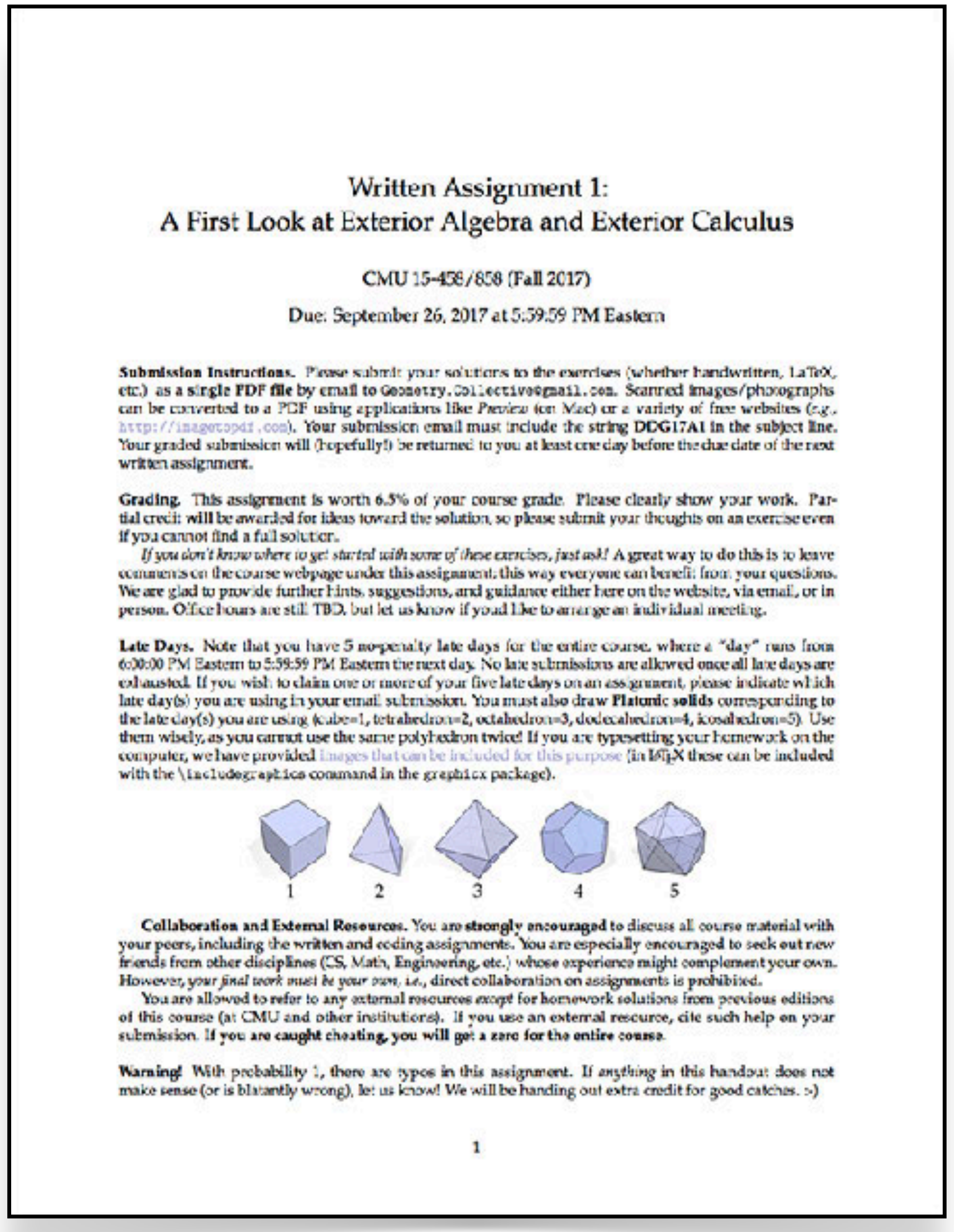
*"…I'm intimidated by the **math**…"*

*"…I'm intimidated by the **coding**…"*

DDG is by its very nature interdisciplinary—*everyone* will feel a bit uncomfortable!

We are aware of this fact! Everyone will be ok. :-) Lots of *details*; focus on the **ideas**.

Assignment 1—Written Out Later Today!



Written Assignment 1:
A First Look at Exterior Algebra and Exterior Calculus

CMU 15-458/858 (Fall 2017)

Due: September 26, 2017 at 5:59:59 PM Eastern

Submission Instructions. Please submit your solutions to the exercises (whether handwritten, LaTeX, etc.) as a **single PDF file** by email to Geometry.Collective@gmail.com. Scanned images/photographs can be converted to a PDF using applications like *Preview* (on Mac) or a variety of free websites (*e.g.*, http://imagetopdf.com). **Your submission email must include the string DDG17A1 in the subject line.** Your graded submission will (hopefully!) be returned to you at least one day before the due date of the next written assignment.

Grading. This assignment is worth 6.5% of your course grade. Please clearly show your work. Partial credit will be awarded for ideas toward the solution, so please submit your thoughts on an exercise even if you cannot find a full solution.

If you don't know where to get started with some of these exercises, just ask! A great way to do this is to leave comments on the course webpage under this assignment; this way everyone can benefit from your questions. We are glad to provide further hints, suggestions, and guidance either here on the website, via email, or in person. Office hours are still TBD, but let us know if you'd like to arrange an individual meeting.

Late Days. Note that you have 5 no-penalty late days for the entire course, where a "day" runs from 6:00:00 PM Eastern to 5:59:59 PM Eastern the next day. No late submissions are allowed once all late days are exhausted. If you wish to claim one or more of your five late days on an assignment, please indicate which late day(s) you are using in your email submission. You must also draw **Platonic solids** corresponding to the late day(s) you are using (cube=1, tetrahedron=2, octahedron=3, dodecahedron=4, icosahedron=5). Use them wisely, as you cannot use the same polyhedron twice! If you are typesetting your homework on the computer, we have provided images that can be included for this purpose (in LaTeX these can be included with the \includegraphics command in the graphicx package).

1 2 3 4 5

Collaboration and External Resources. You are **strongly encouraged** to discuss all course material with your peers, including the written and coding assignments. You are especially encouraged to seek out new friends from other disciplines (CS, Math, Engineering, etc.) whose experience might complement your own. However, *your final work must be your own, i.e.*, direct collaboration on assignments is prohibited.

You are allowed to refer to any external resources *except* for homework solutions from previous editions of this course (at CMU and other institutions). If you use an external resource, cite such help on your submission. **If you are caught cheating, you will get a zero for the entire course.**

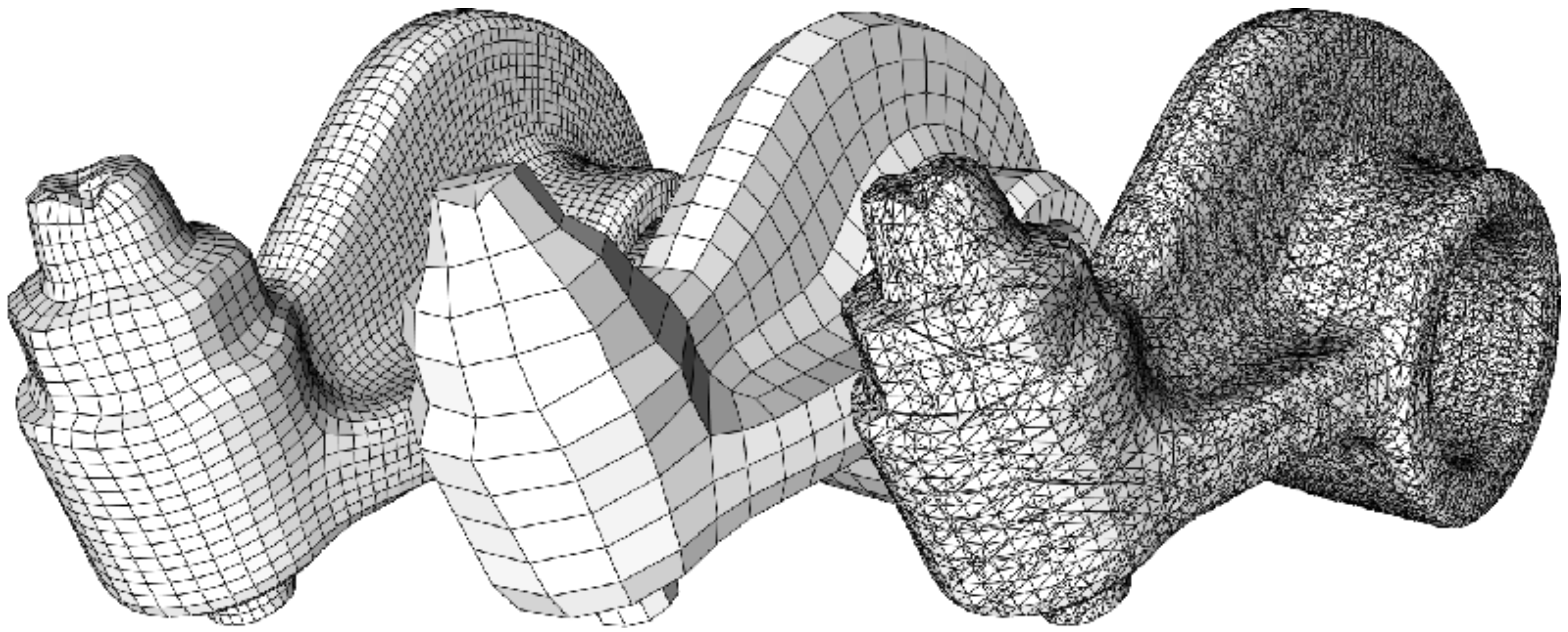
Warning! With probability 1, there are typos in this assignment. If *anything* in this handout does not make sense (or is blatantly wrong), let us know! We will be handing out extra credit for good catches. :-)

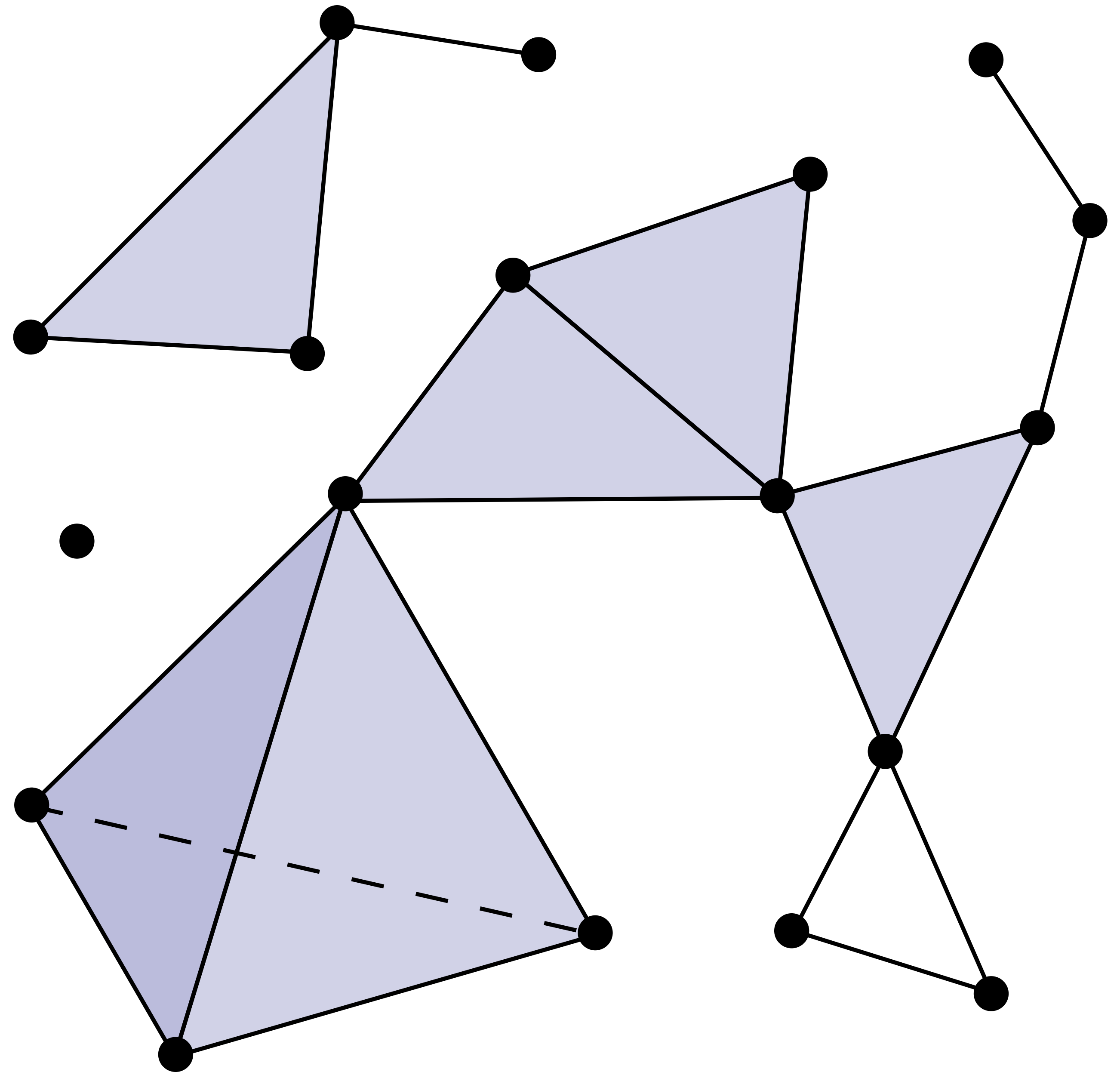
1

- First assignment
  - Written part out today
  - Coding part out next week
- **Topic:** *exterior algebra & exterior calculus*
  - Basic tools used throughout semester
  - Can't skip this one!
- Won't cover in class until next week
  - But, great intro available in our course notes
  - Good idea to get started now! (Read notes first.)
- All administrative details (handin, *etc.*) in assignment.

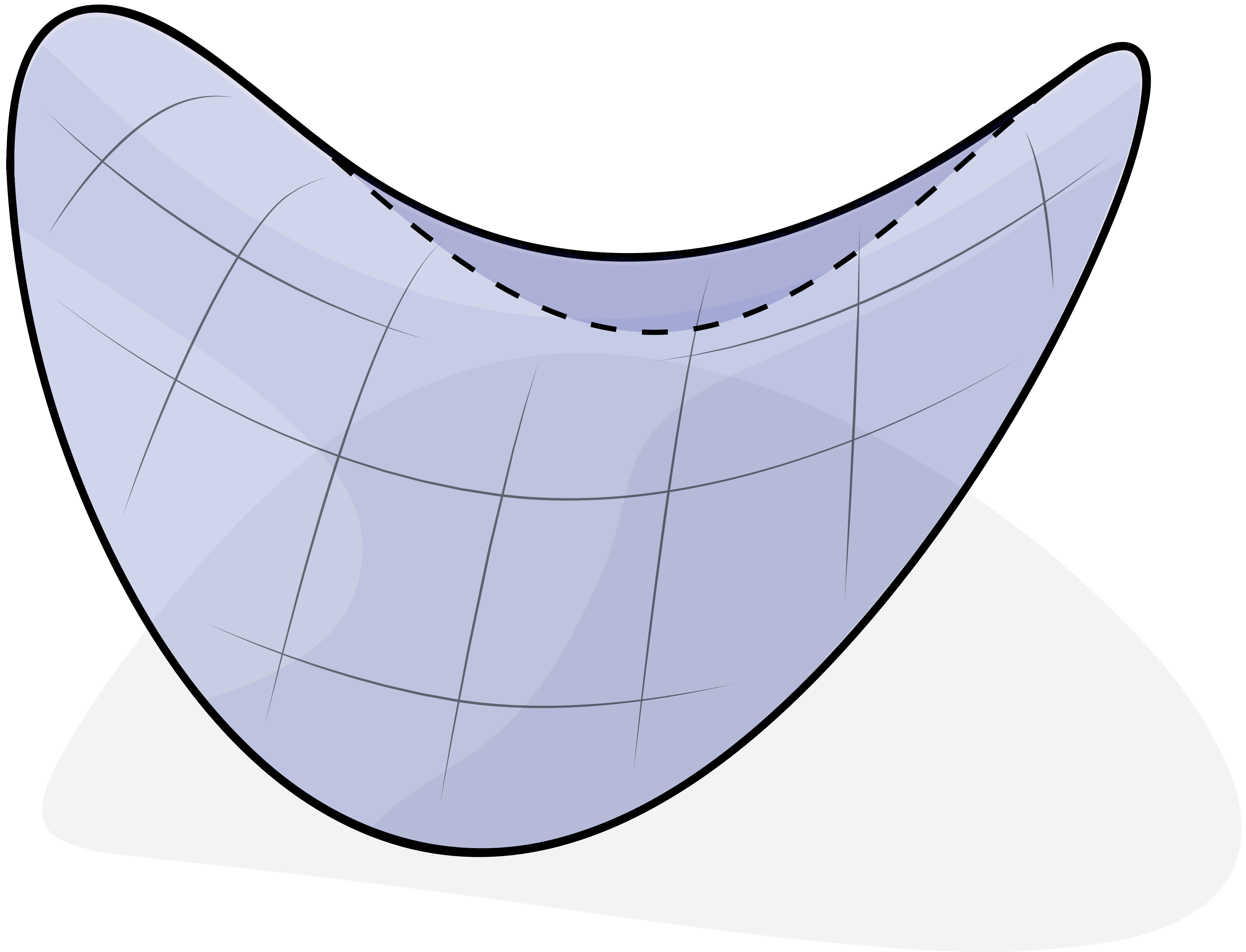
Today: What is a Mesh?

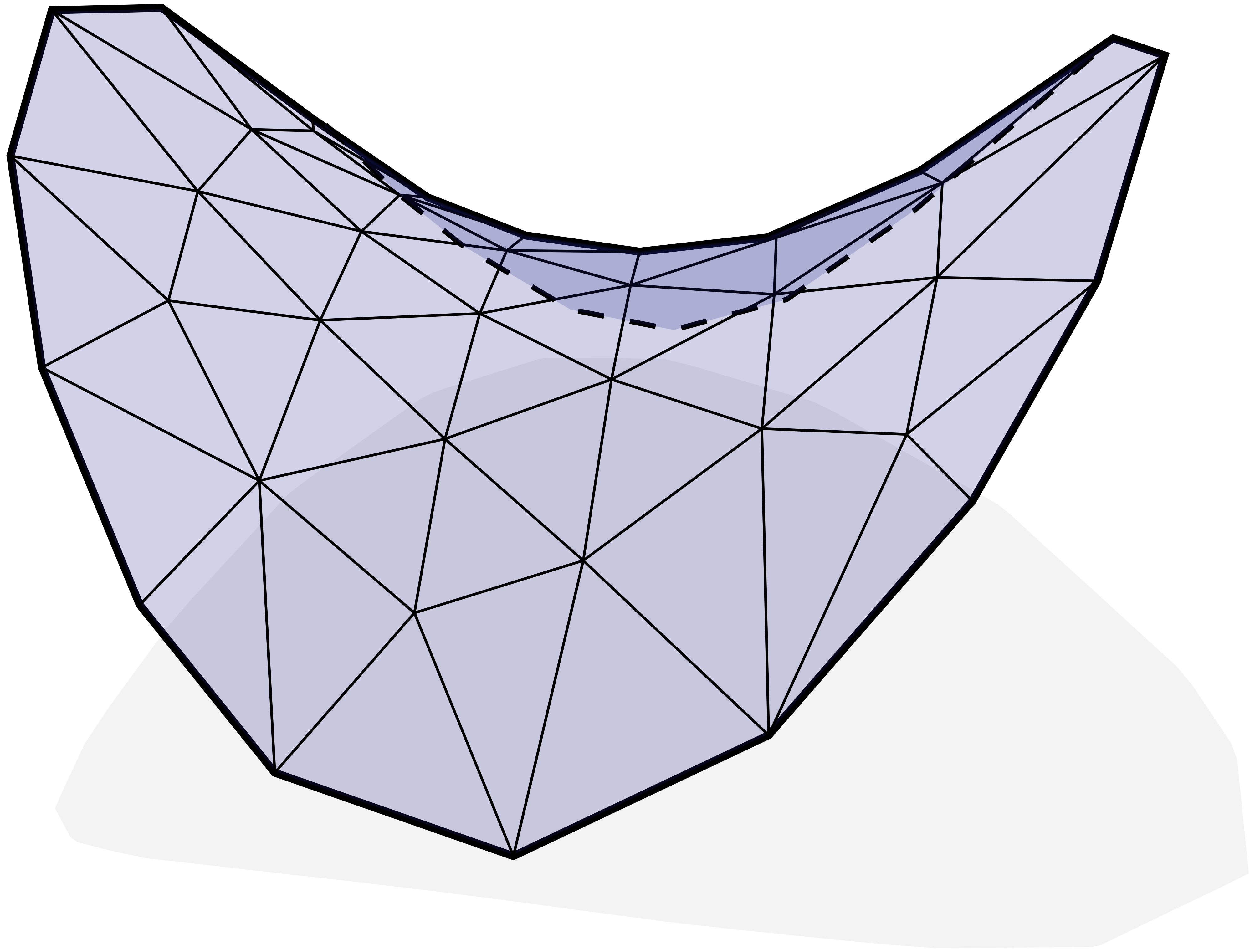
- Many possibilities…
- **Simplicial complex**
  - Abstract vs. geometric simplicial complex
  - Oriented, manifold simplicial complex
  - Application: *topological data analysis*
- **Cell complex**
  - Poincaré dual, discrete exterior calculus
- `Data structures:`
  - *adjacency list, incidence matrix, halfedge*





Connection to Differential Geometry?

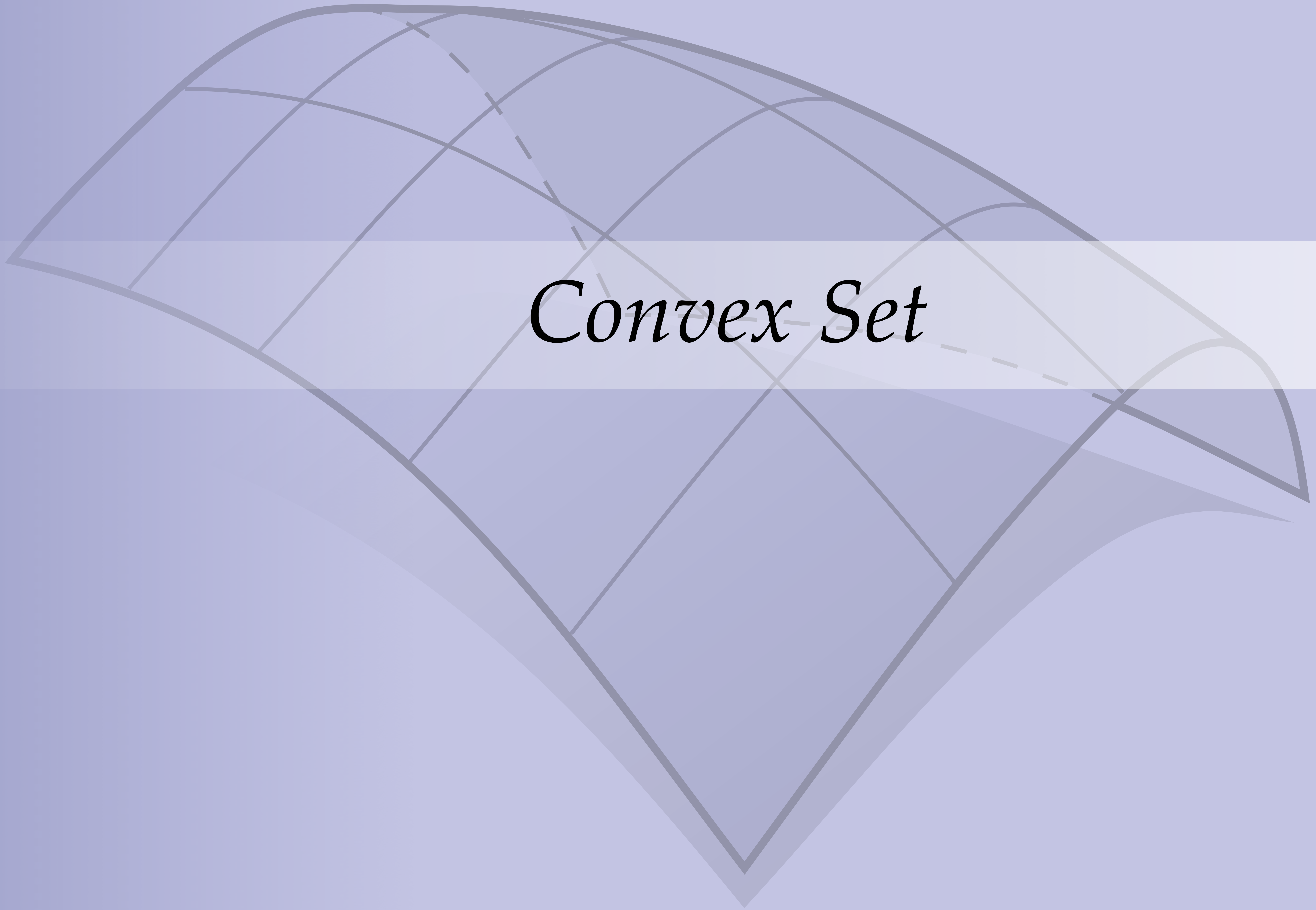




topological space* $\longleftrightarrow$ abstract simplicial complex

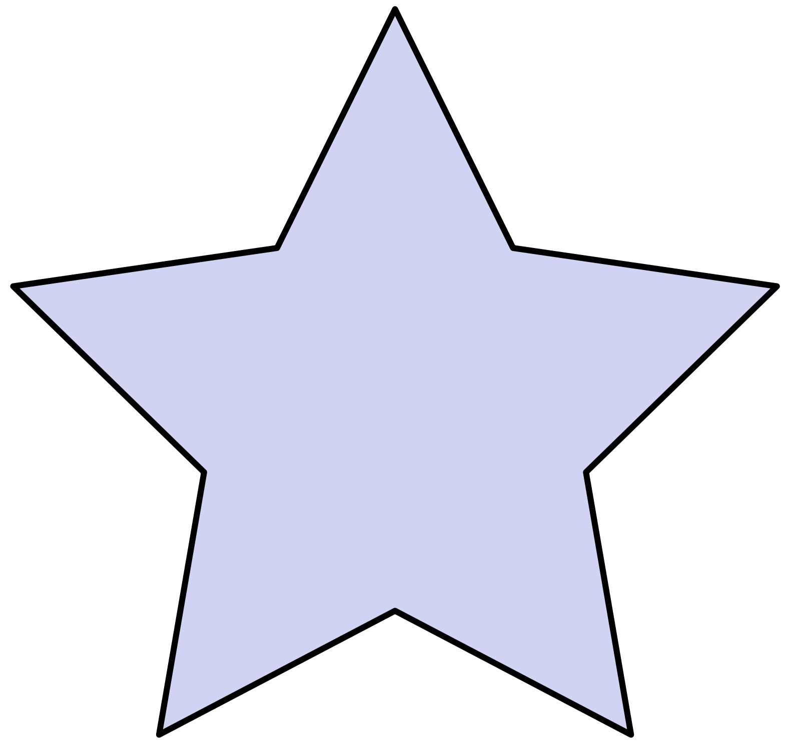
*We'll talk about this *later* in the course!

Convex Set

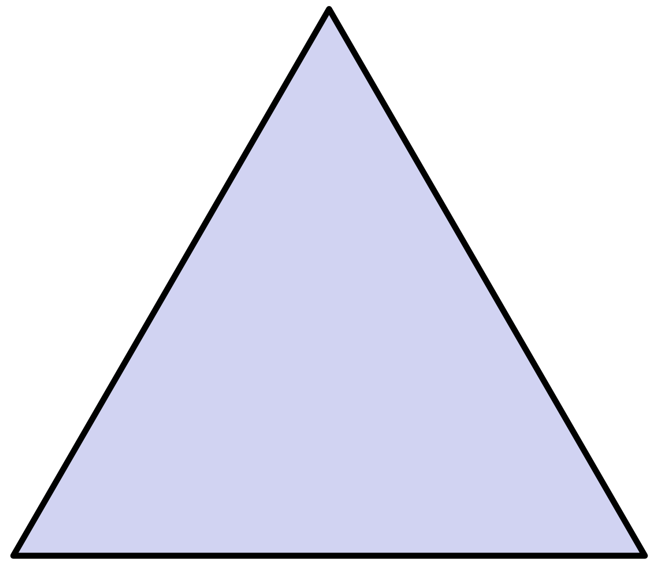


Convex Set—Examples

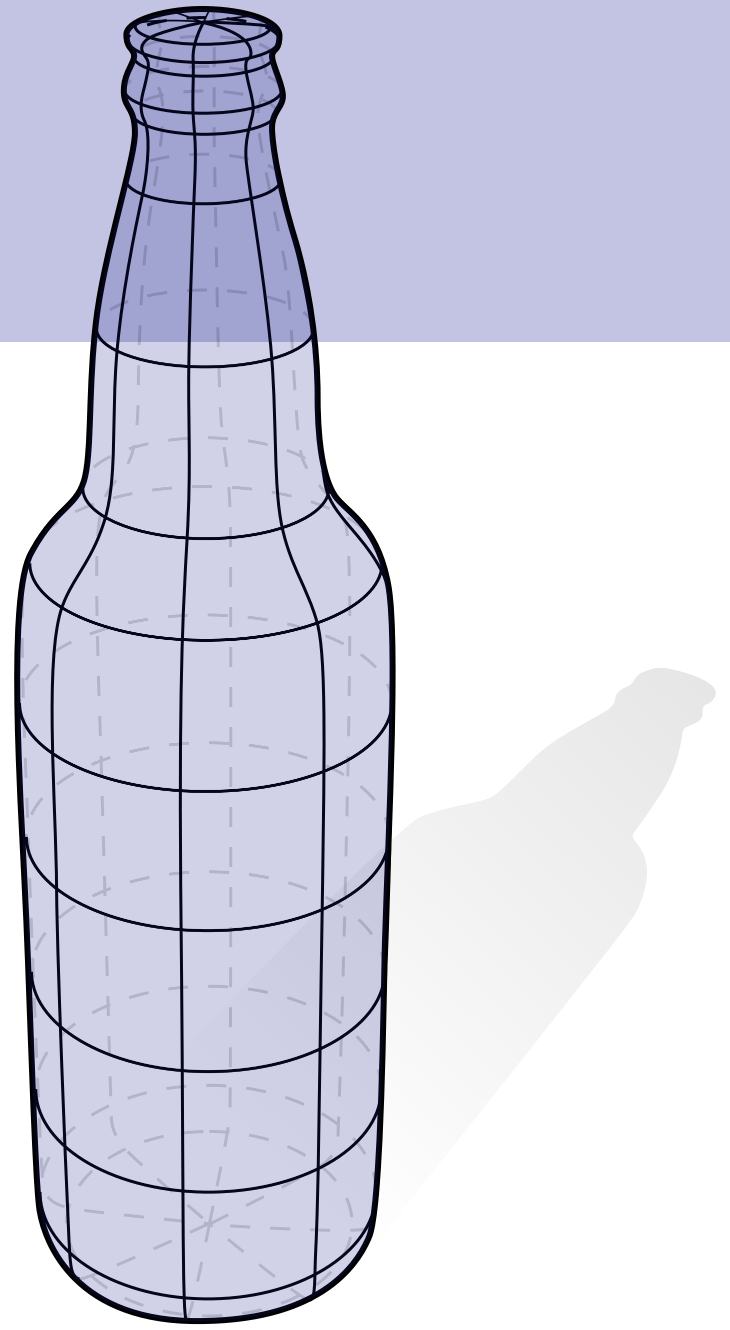
Which of the following sets are *convex*?



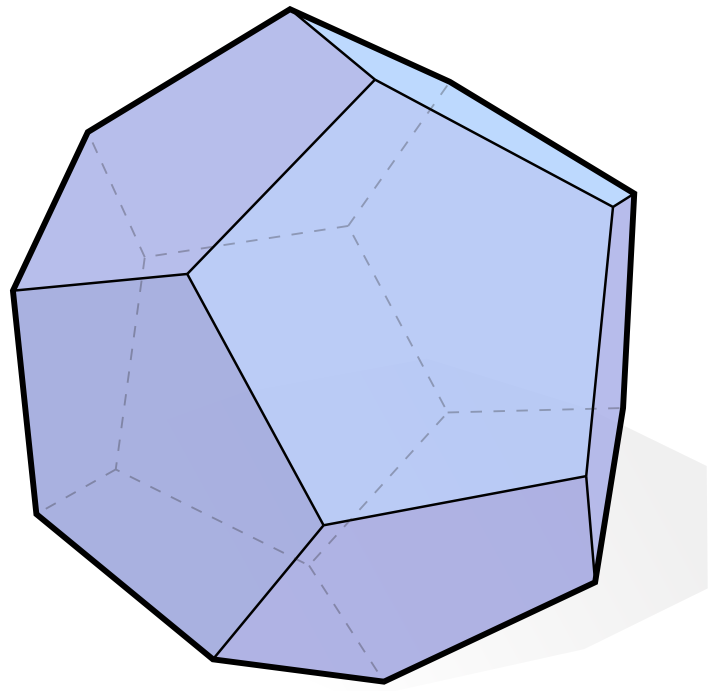
(A)



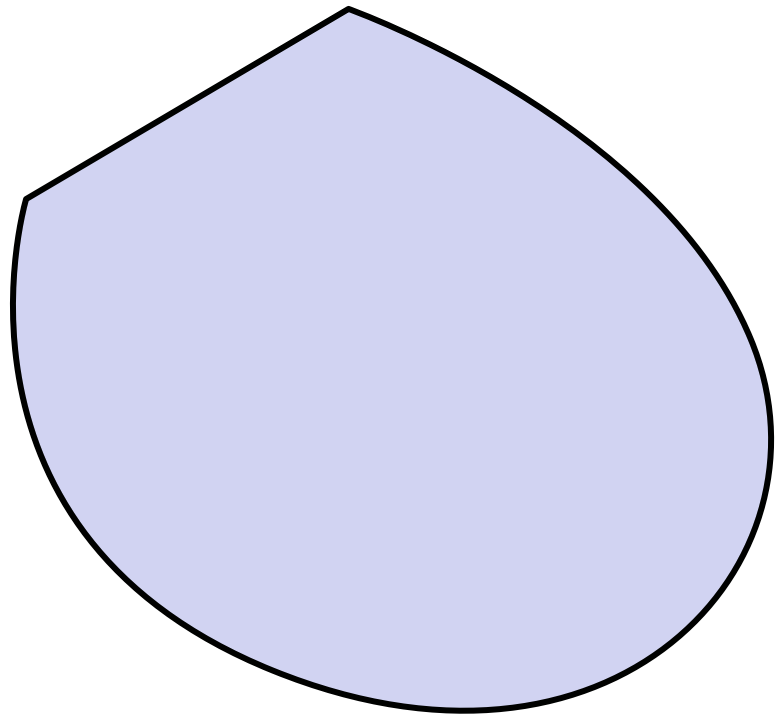
(B)



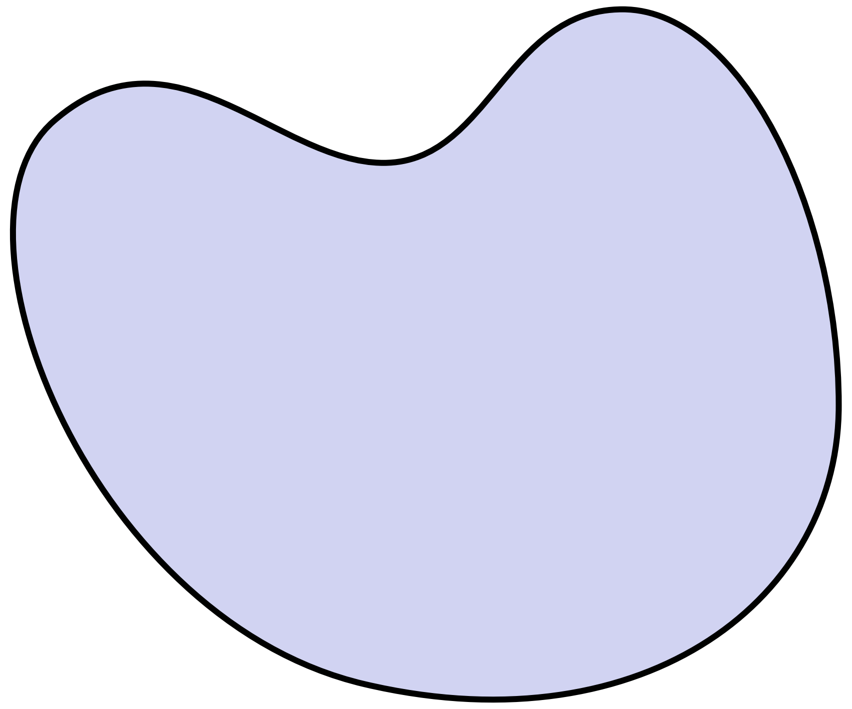
(C)



(D)



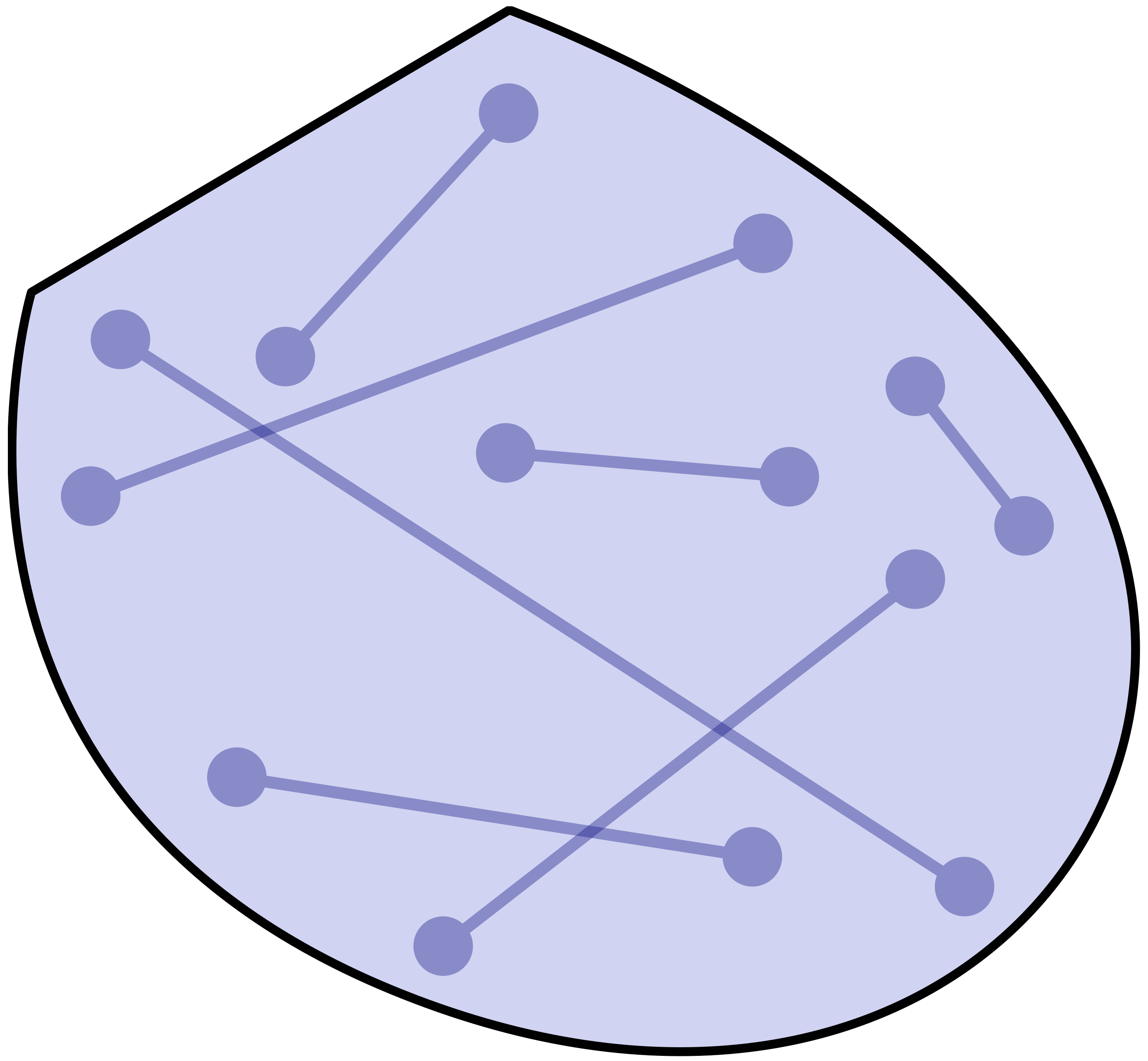
(E)



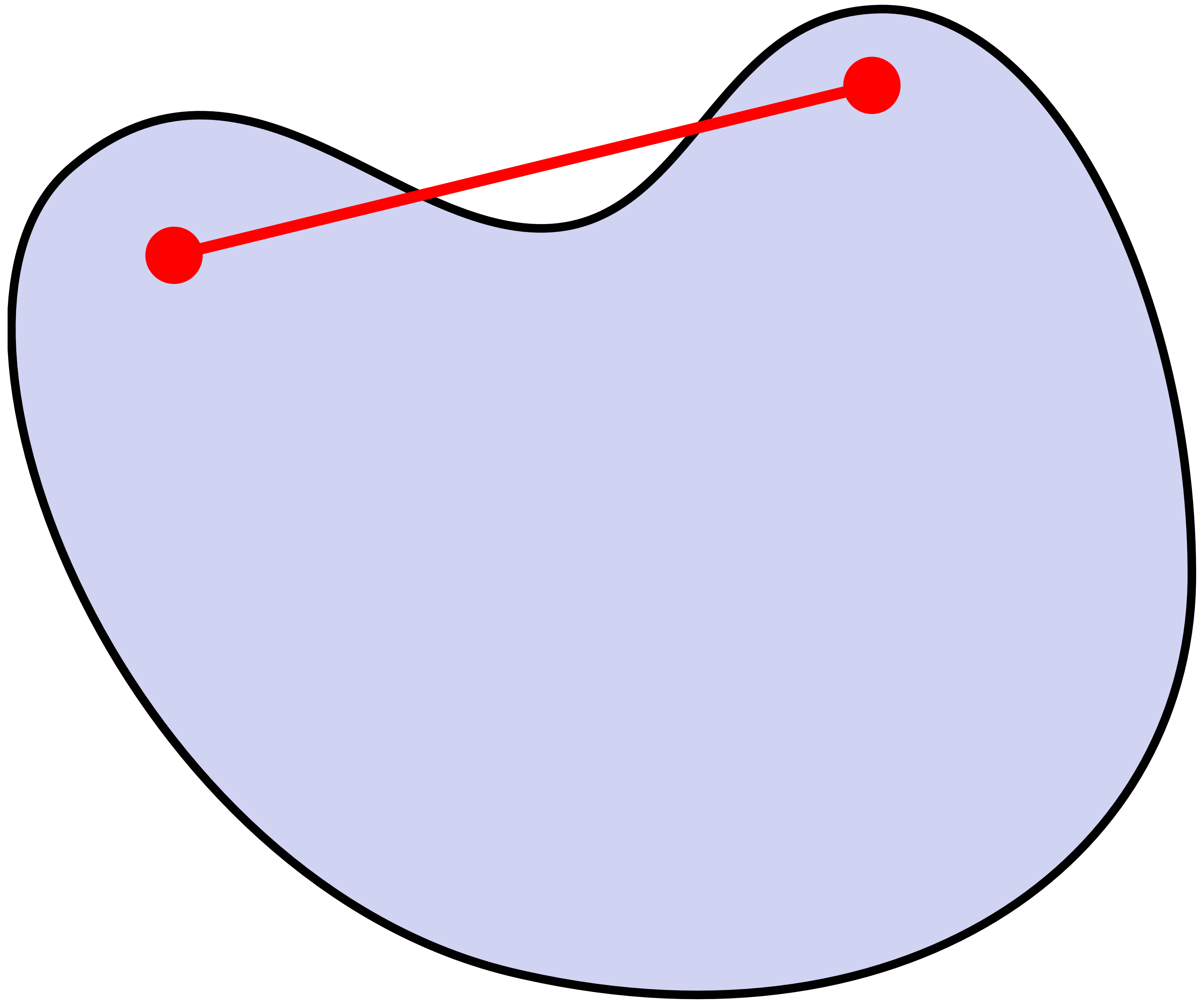
(F)

Convex Set

Definition. A subset $S \subset \mathbb{R}^n$ is *convex* if for every pair of points $p, q \in S$ the line segment between p and q is contained in S.

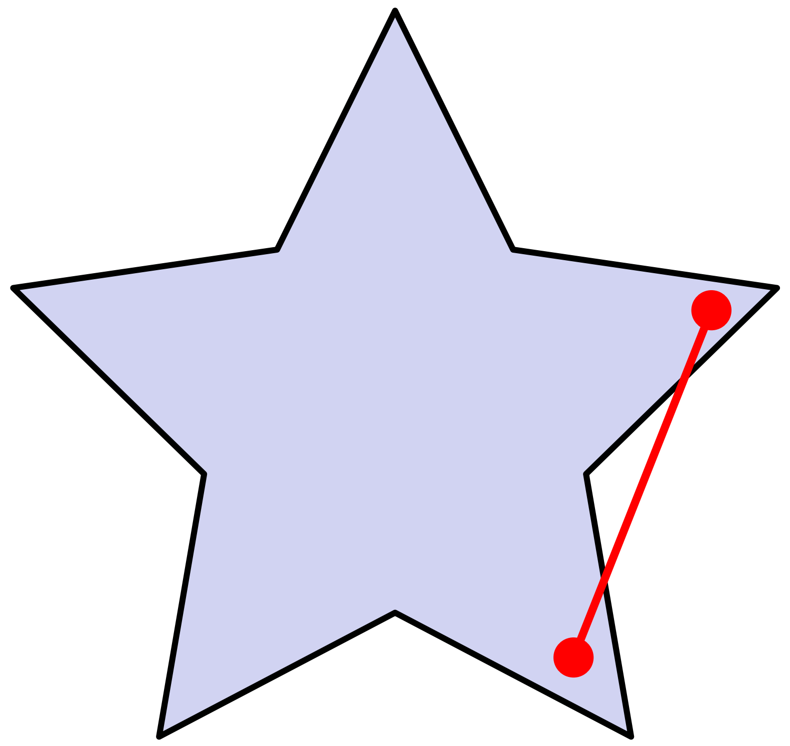


convex

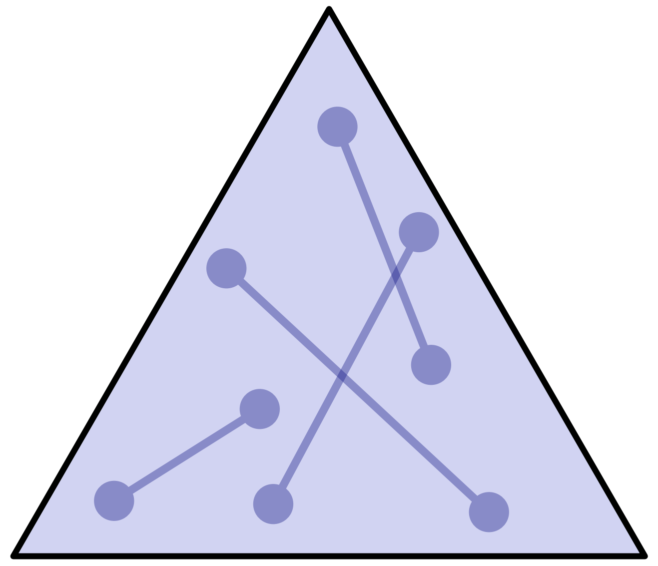


not convex

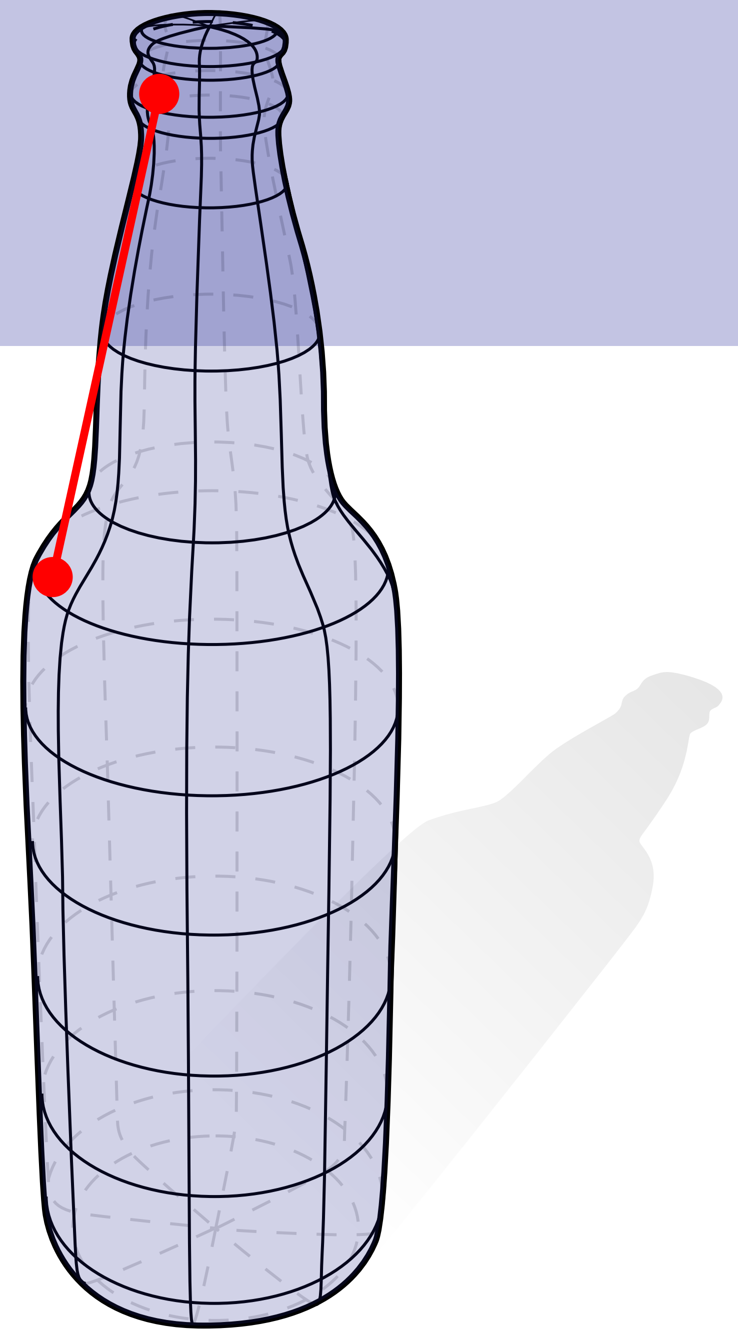
Convex Set — Examples



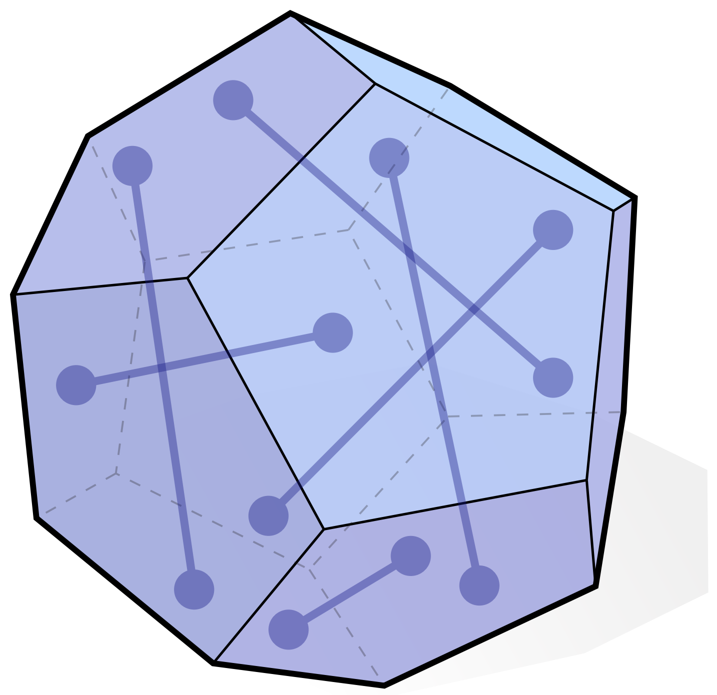
(A)



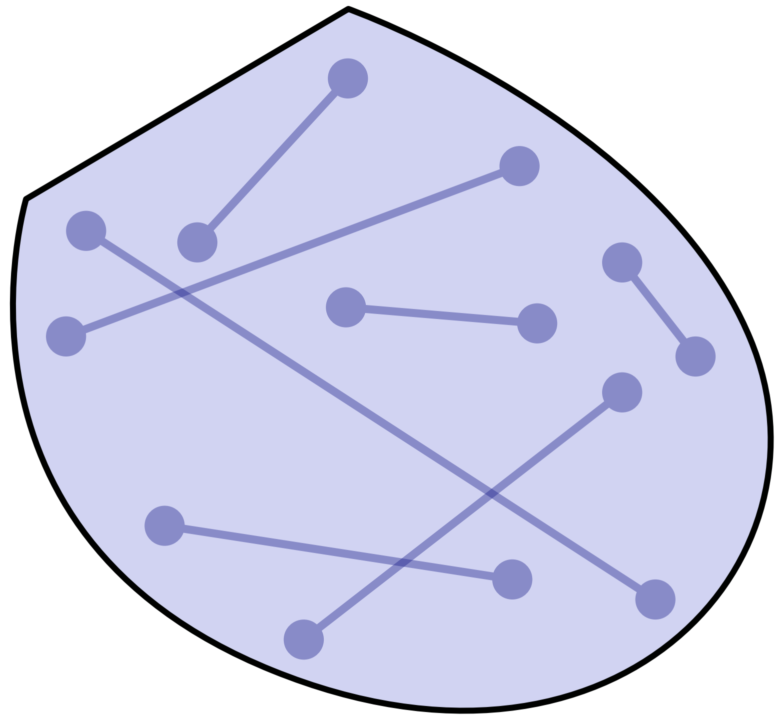
(B)



(C)



(D)



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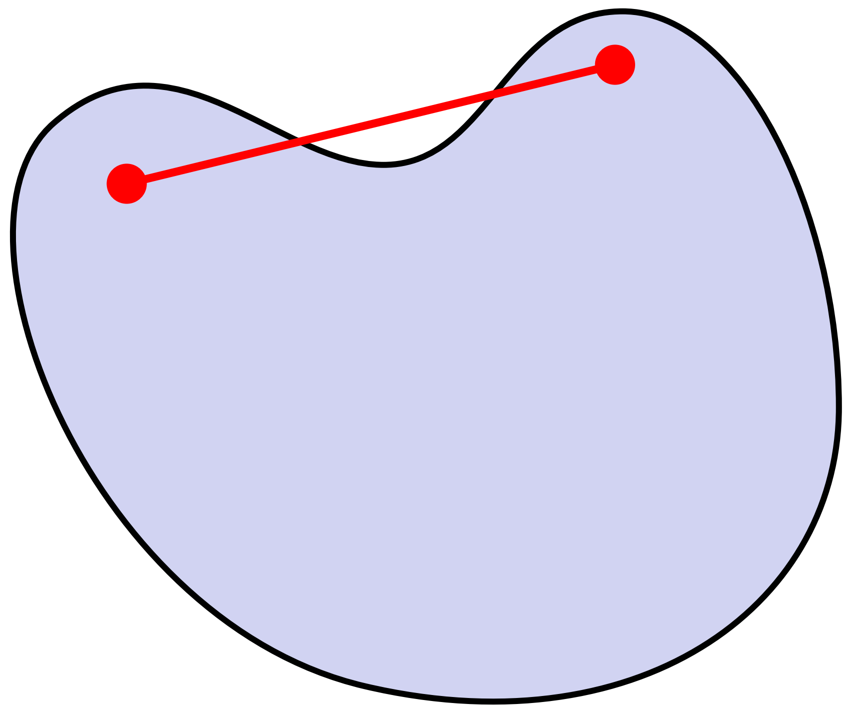
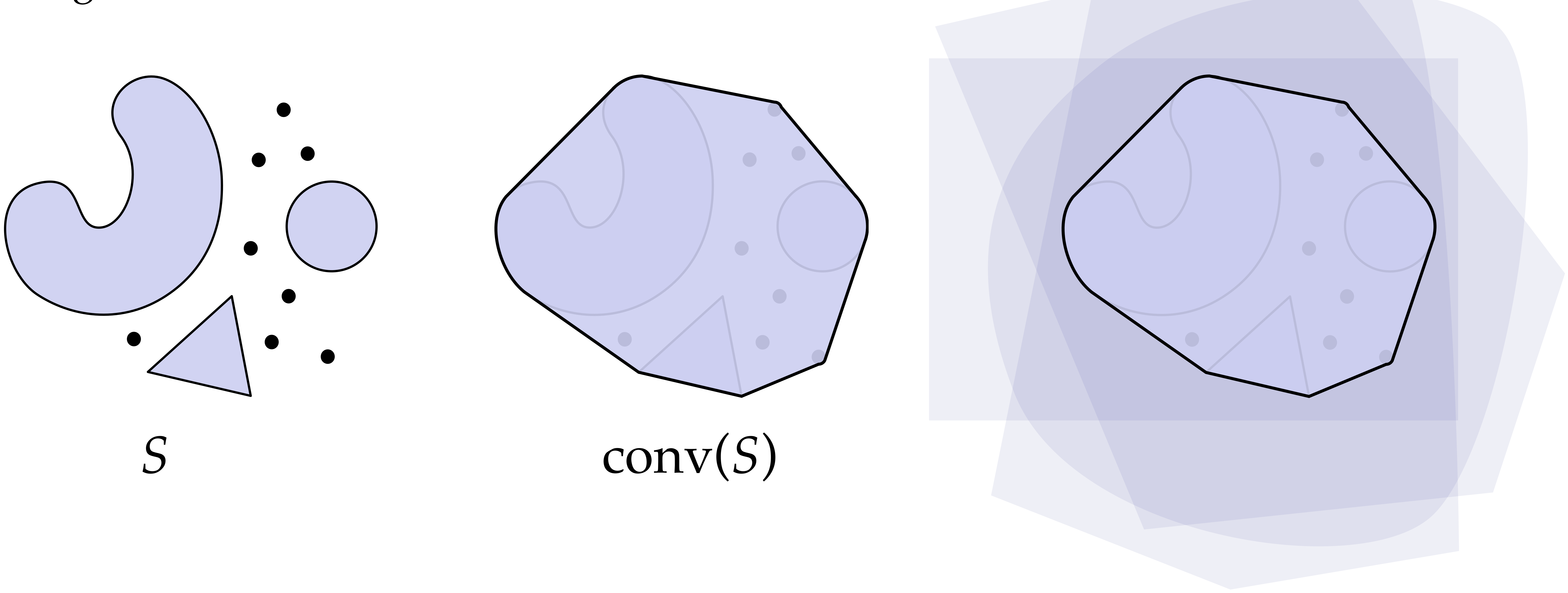


(F)

Convex Hull

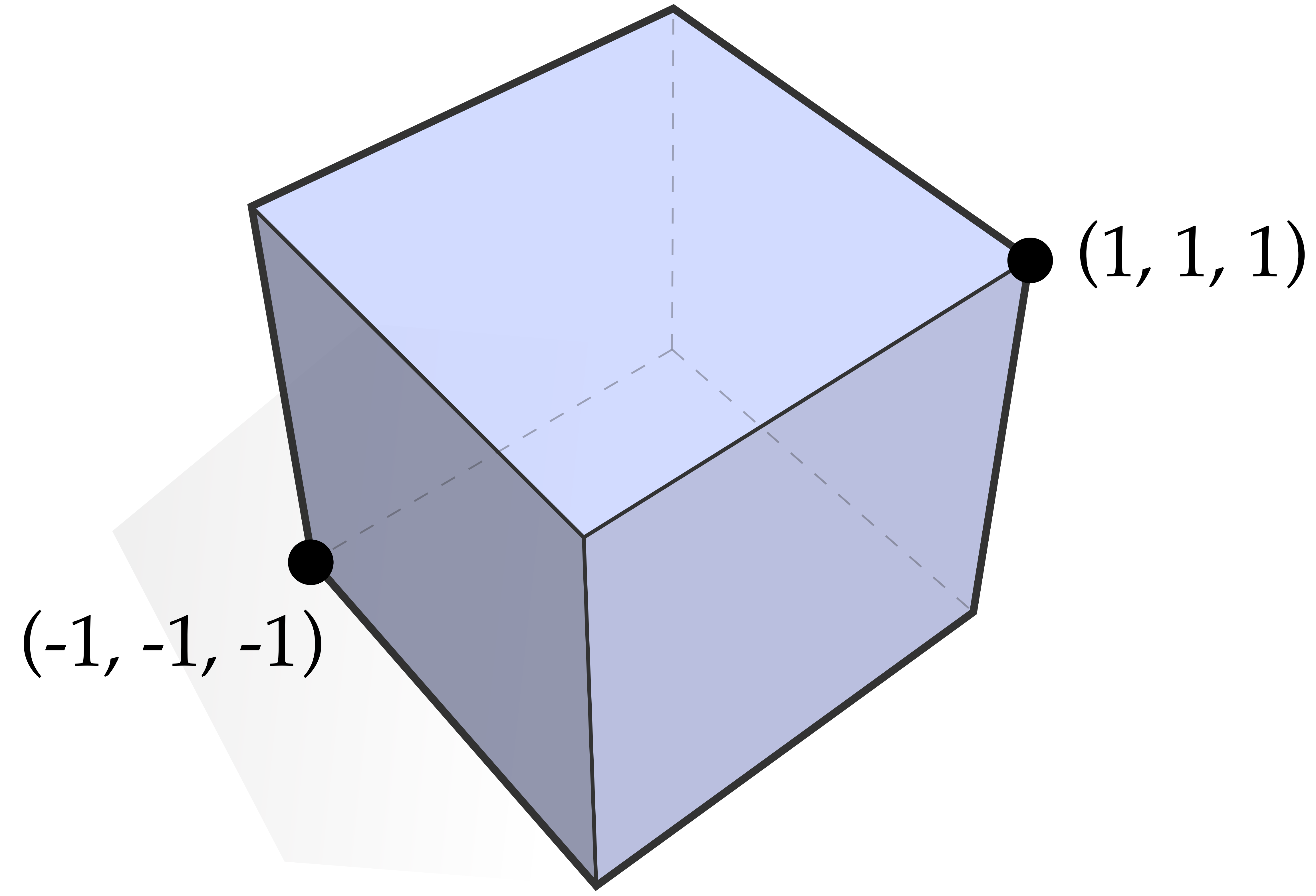
Definition. For any subset $S \subset \mathbb{R}^n$, its convex hull $\mathrm{conv}(S)$ is the smallest convex set containing S, or equivalently, the intersection of all convex sets containing S.

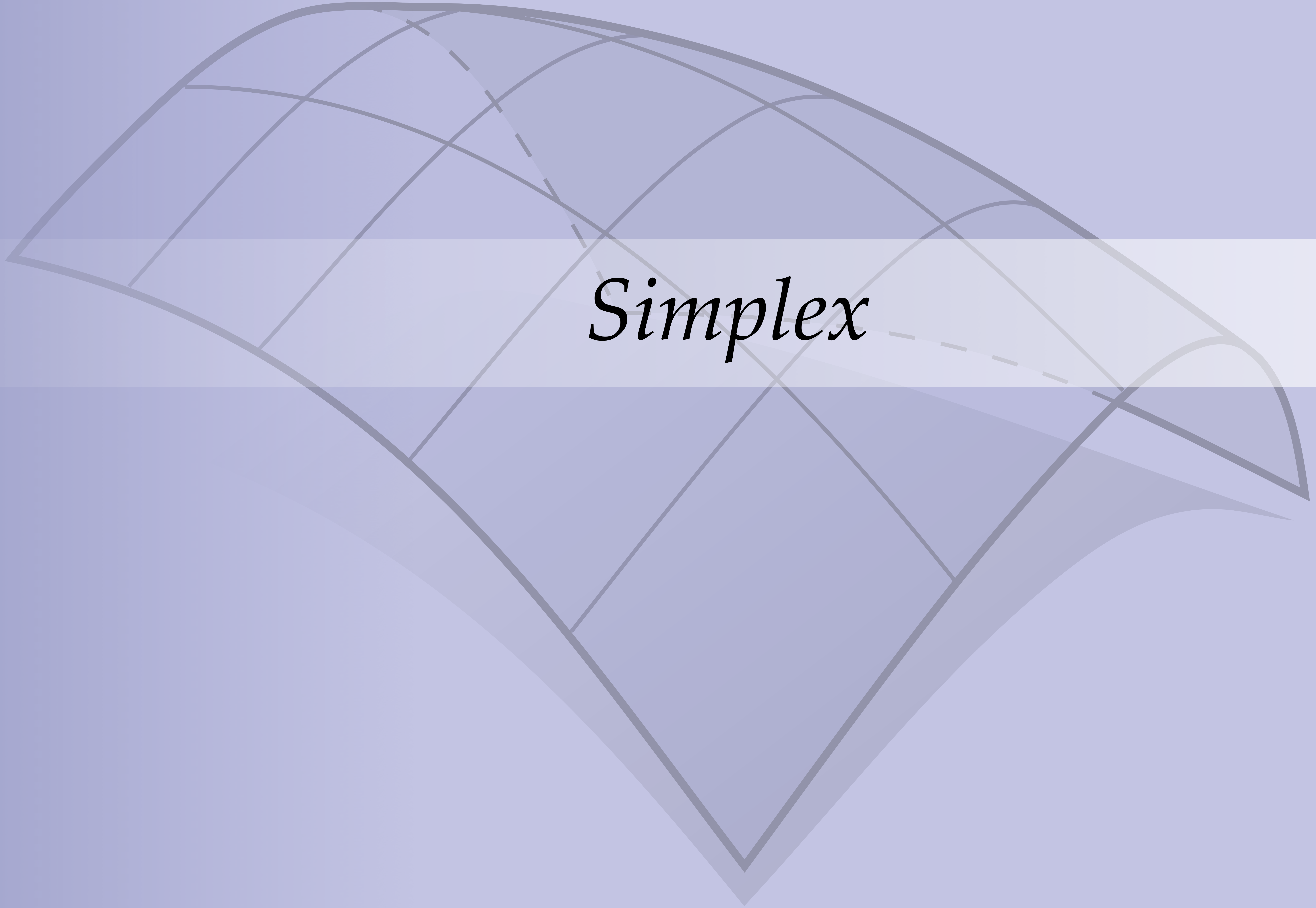


Convex Hull—Example

Q: What is the convex hull of the set $S := \{(\pm 1, \pm 1, \pm 1)\} \subset \mathbb{R}^3$?

A: A cube.

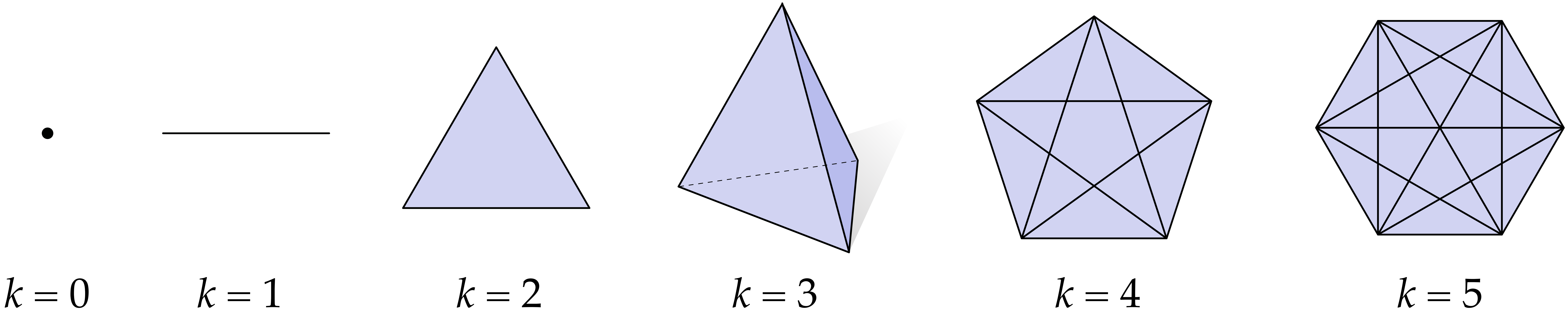




Simplex

Simplex — Basic Idea

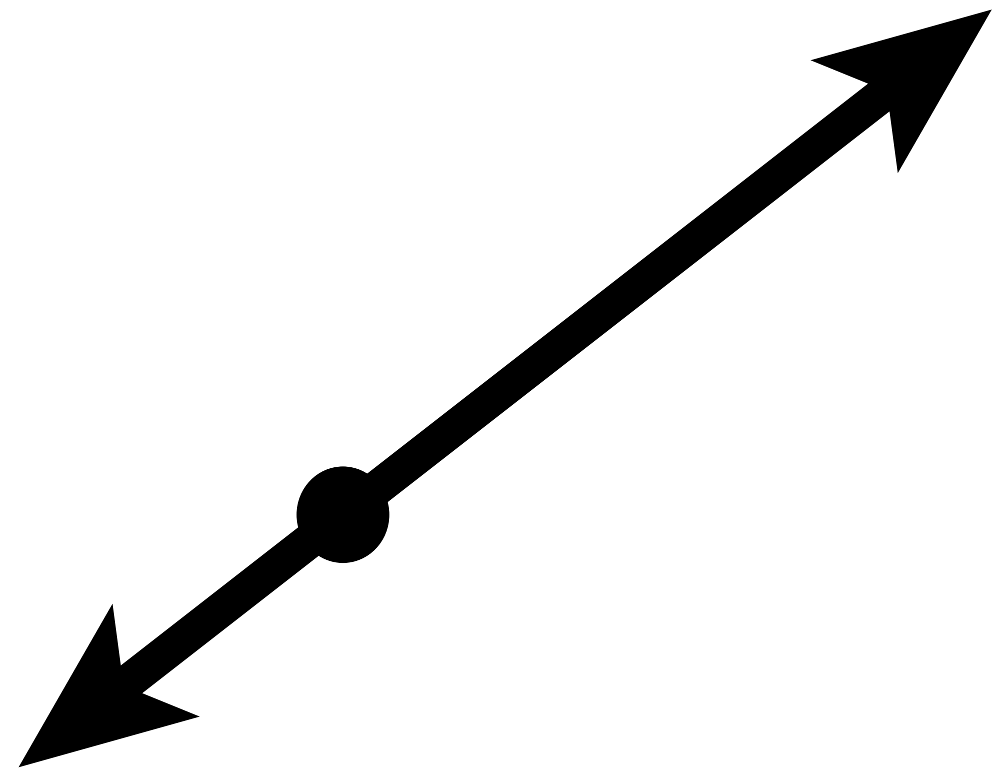
Roughly speaking, a *k-simplex* is a point, a line segment, a triangle, a tetrahedron…



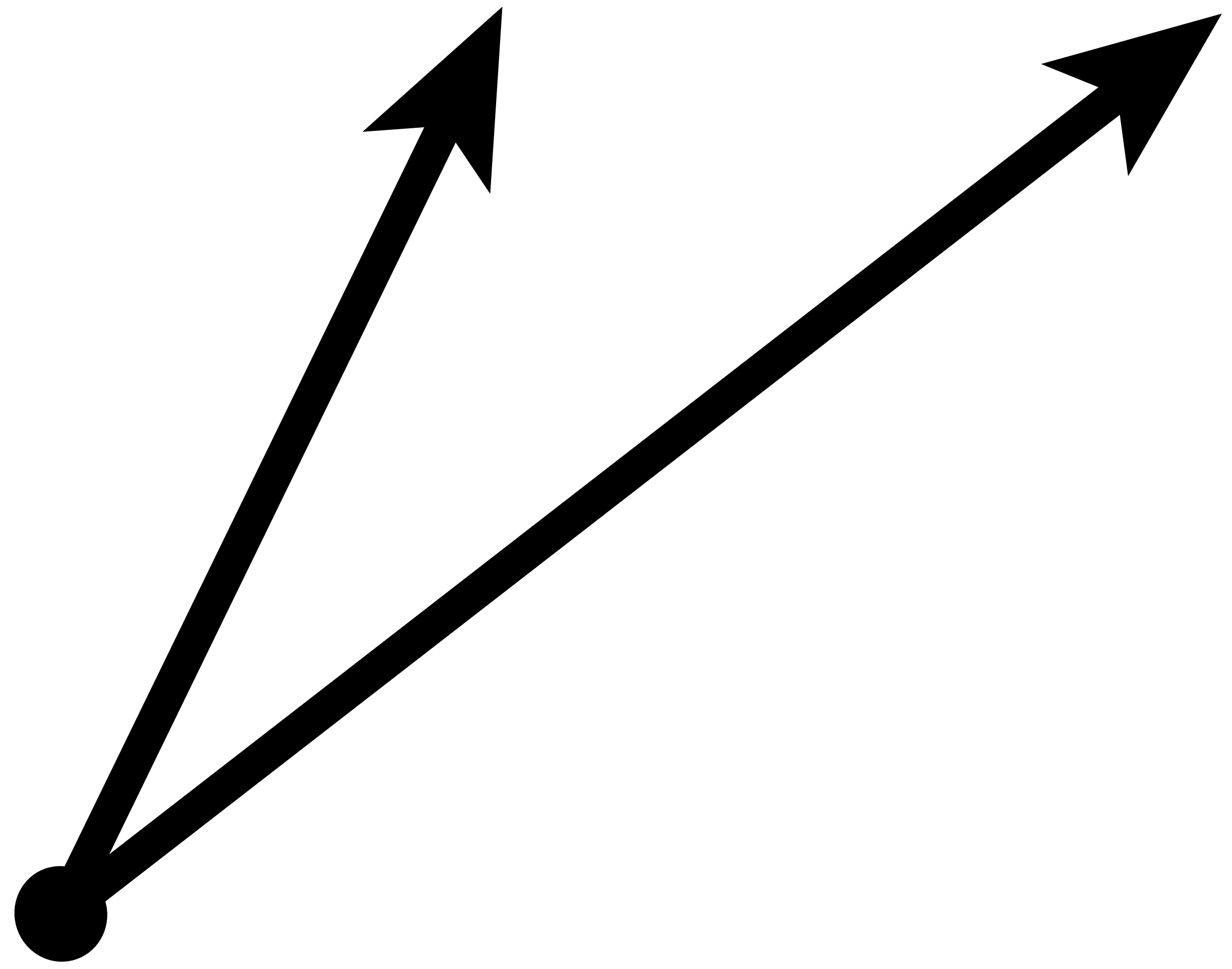
…much harder to draw for large k!

Linear Independence

Definition. A collection of vectors $v_1, \ldots, v_n$ is *linearly independent* if no vector can be expressed as a linear combination of the others, *i.e.*, if there is no collection of coefficients $a_1, \ldots, a_n \in \mathbb{R}$ such that $v_j = \sum_{i \neq j} a_i v_i$ (for any v_j).



linearly dependent



linearly independent

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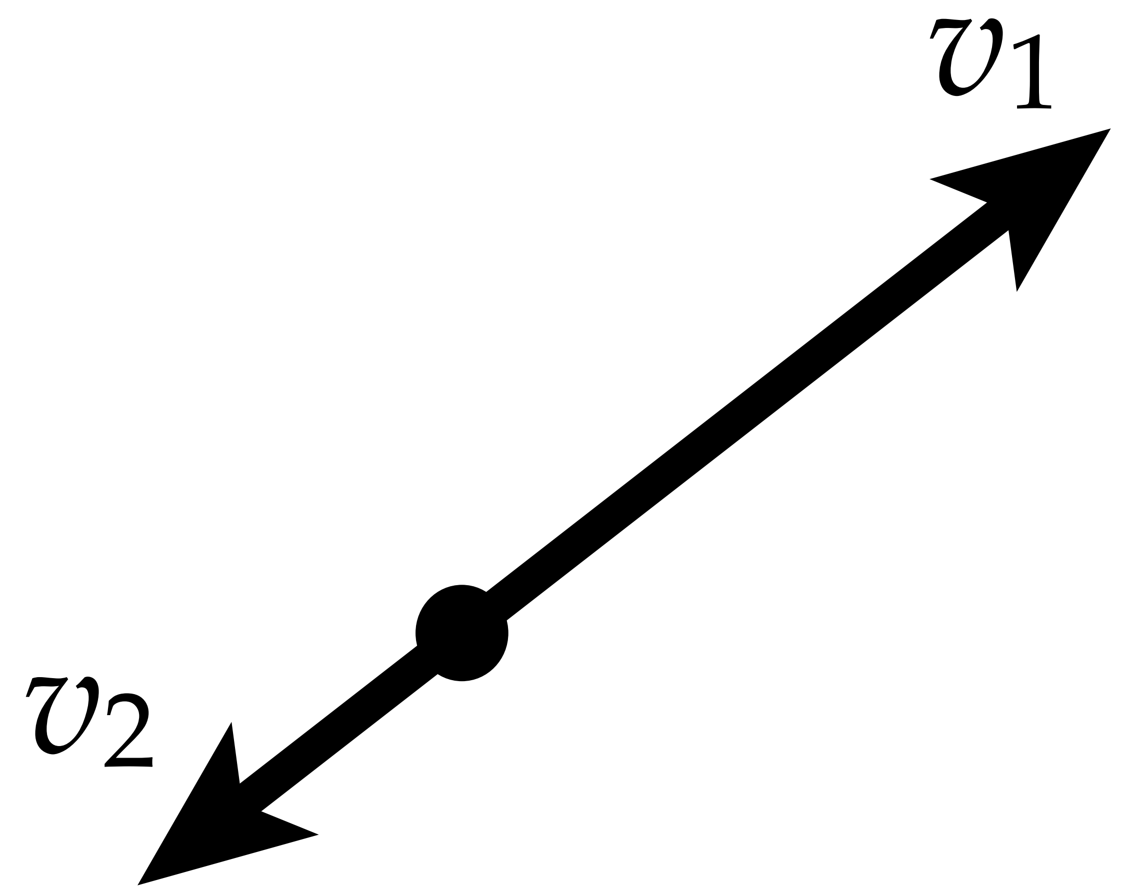
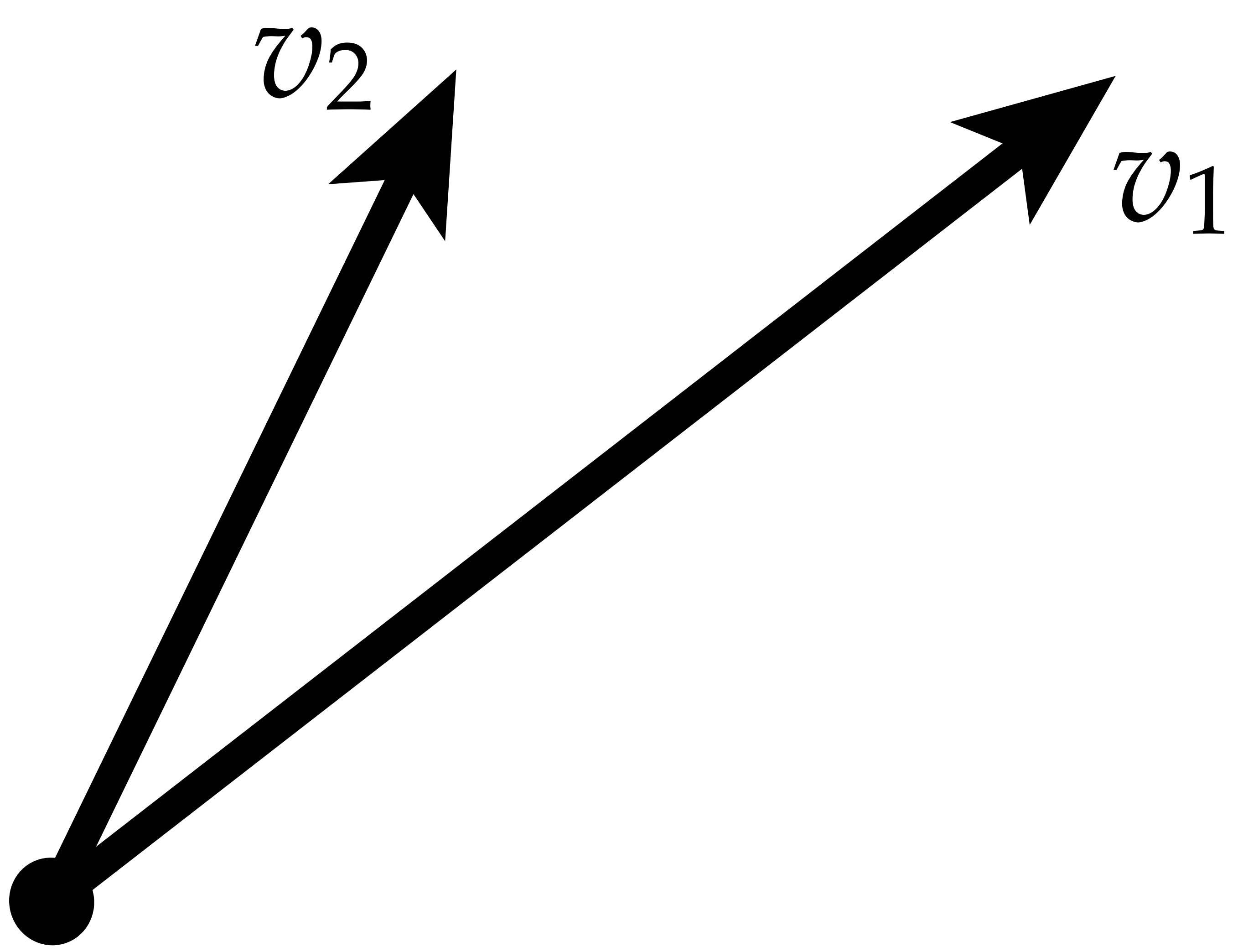


linearly dependent



linearly independent

Affine Independence

Definition. A collection of points $p_0, \ldots, p_k$ are *affinely independent* if the vectors $v_i := p_i - p_0$ are linearly independent.

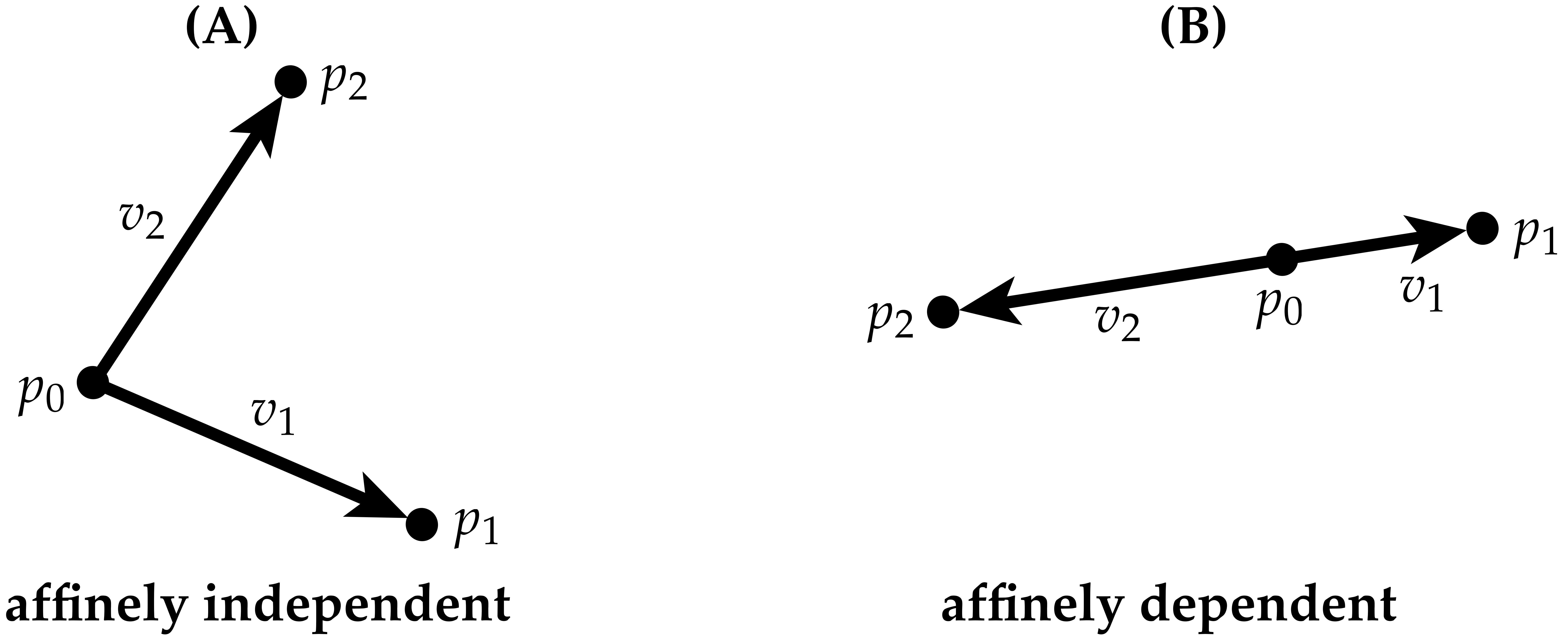
(A)

(B)

(Colloquially: might say points are in *"general position"*.)

Affine Independence

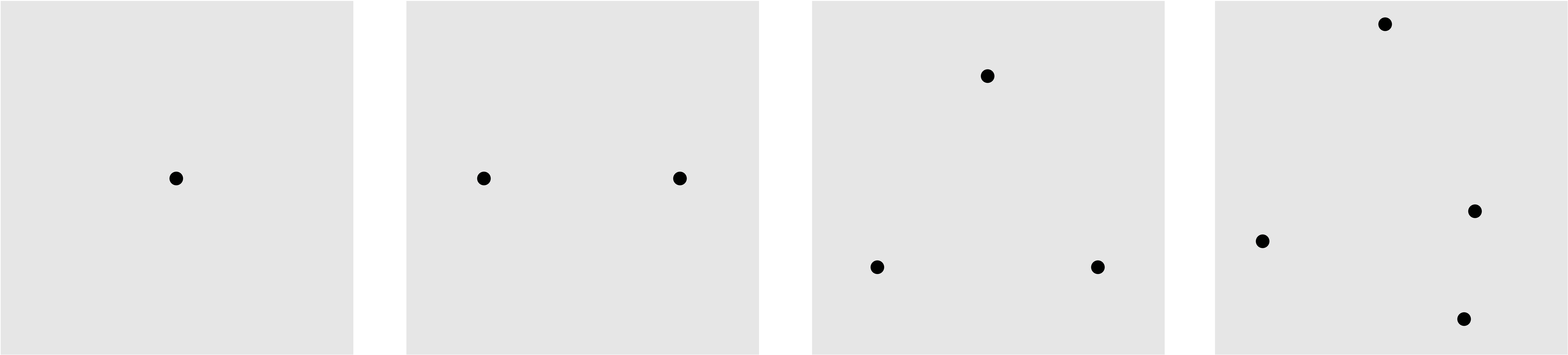
Definition. A collection of points $p_0, \ldots, p_k$ are *affinely independent* if the vectors $v_i := p_i - p_0$ are linearly independent.



(Colloquially: might say points are in *"general position"*.)

Simplex — Geometric Definition

Definition. A *k-simplex* is the convex hull of $k + 1$ affinely-independent points, which we call its *vertices*.



Simplex — Geometric Definition

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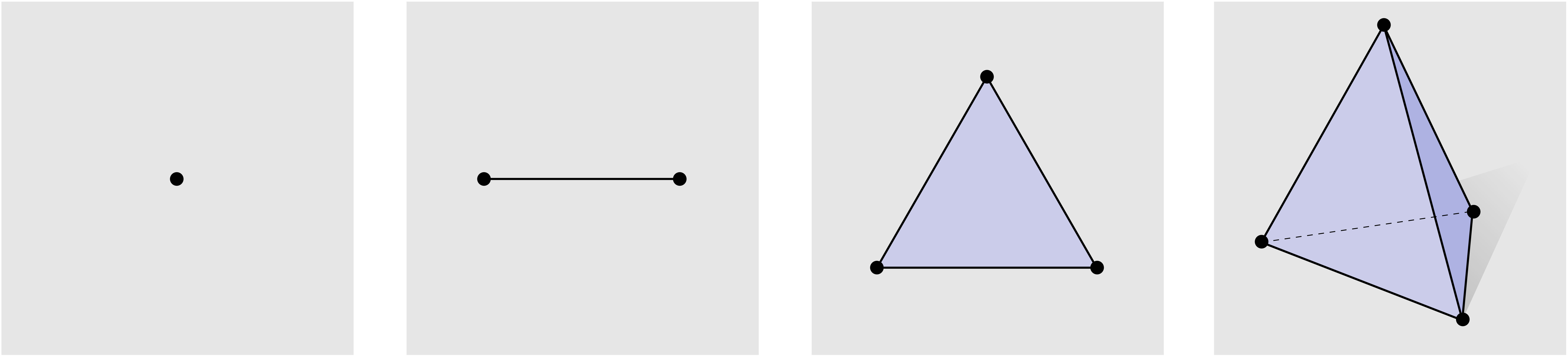
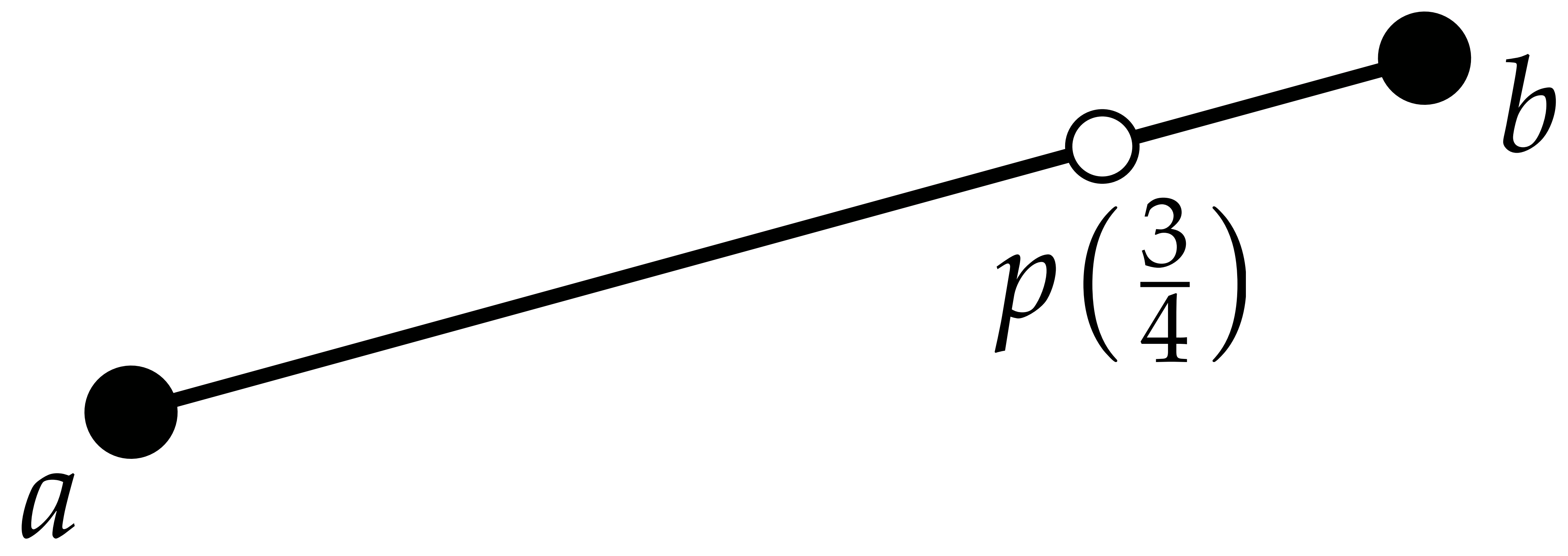


Q: How many affinely-independent points can we have in n dimensions?

Barycentric Coordinates—1-Simplex

- We can describe a *simplex* more explicitly using barycentric coordinates.
- For instance, a 1-simplex is comprised of all weighted combinations of the two points where the weights sum to 1:

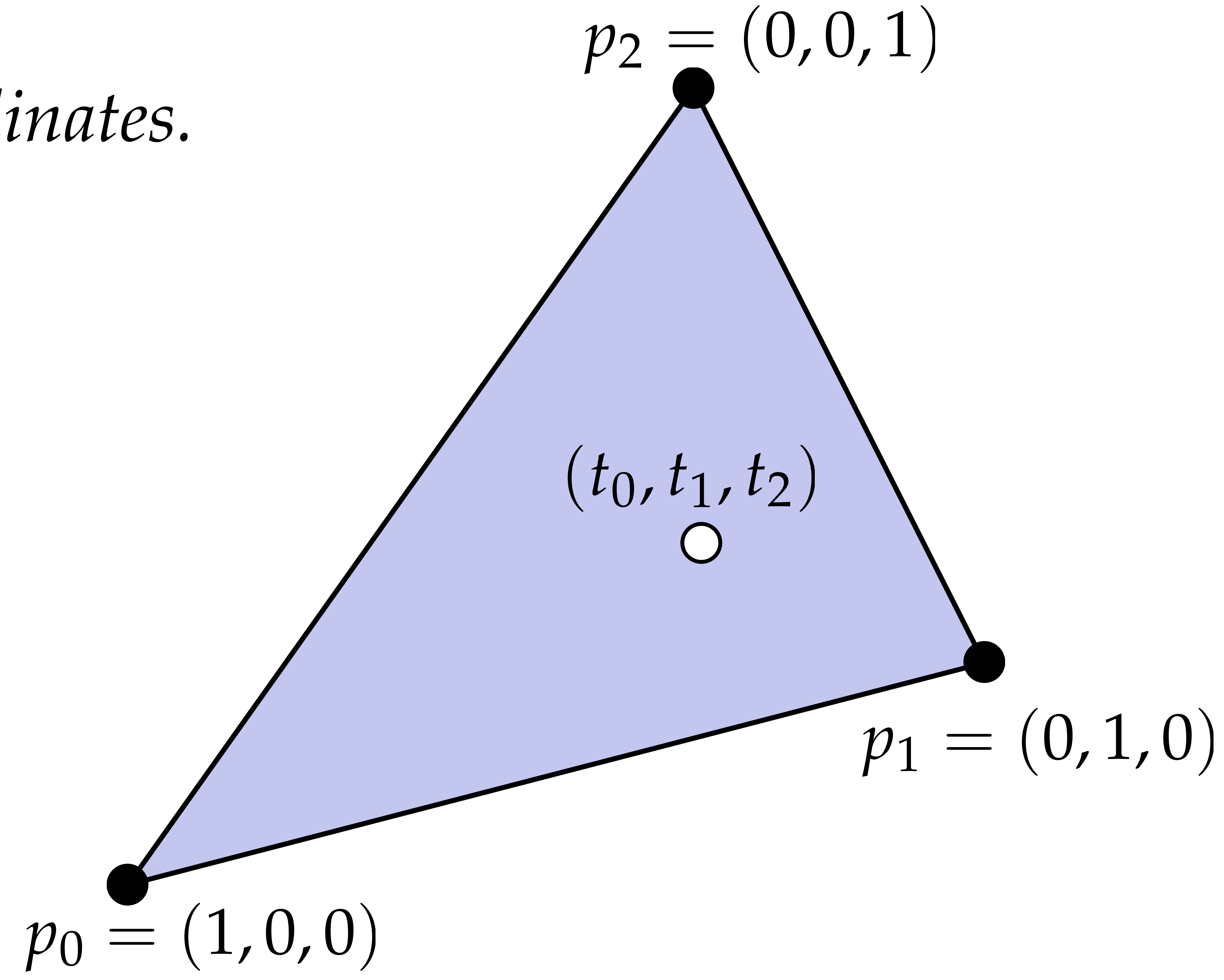


$$p(t) := (1-t)a + tb, \quad t \in [0,1]$$

Barycentric Coordinates—k-Simplex

- More generally, any point in a k-simplex can be expressed as a weighted combination of the vertices, where the weights sum to 1.
- The weights t_i are called the *barycentric coordinates*.

$$\sigma = \left\{ \sum_{i=0}^{k} t_i p_i \,\middle|\, \sum_{i=0}^{k} t_i = 1,\ t_i \geq 0\ \forall i \right\}$$

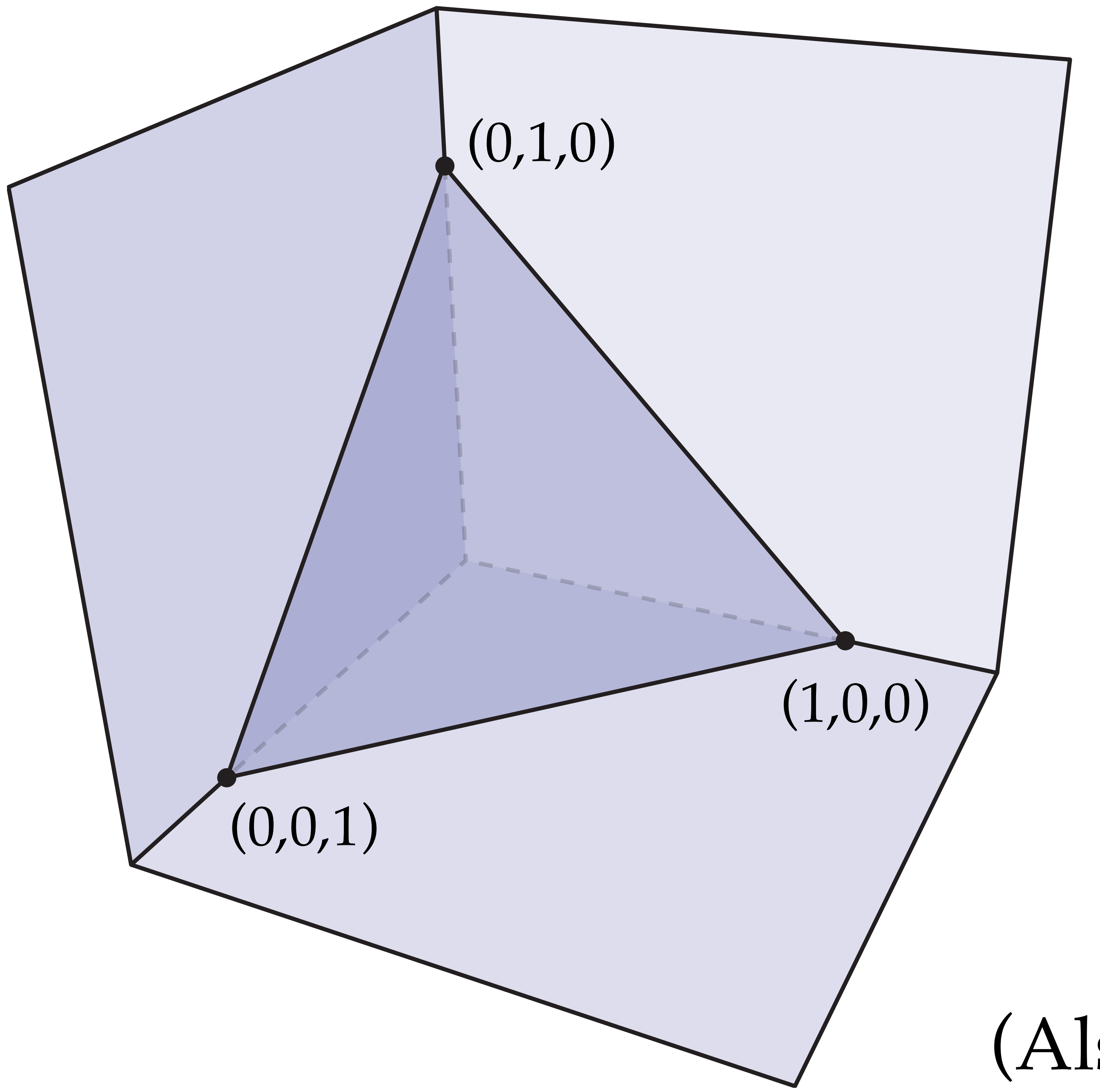


(Also called a "convex combination.")

Simplex—Example

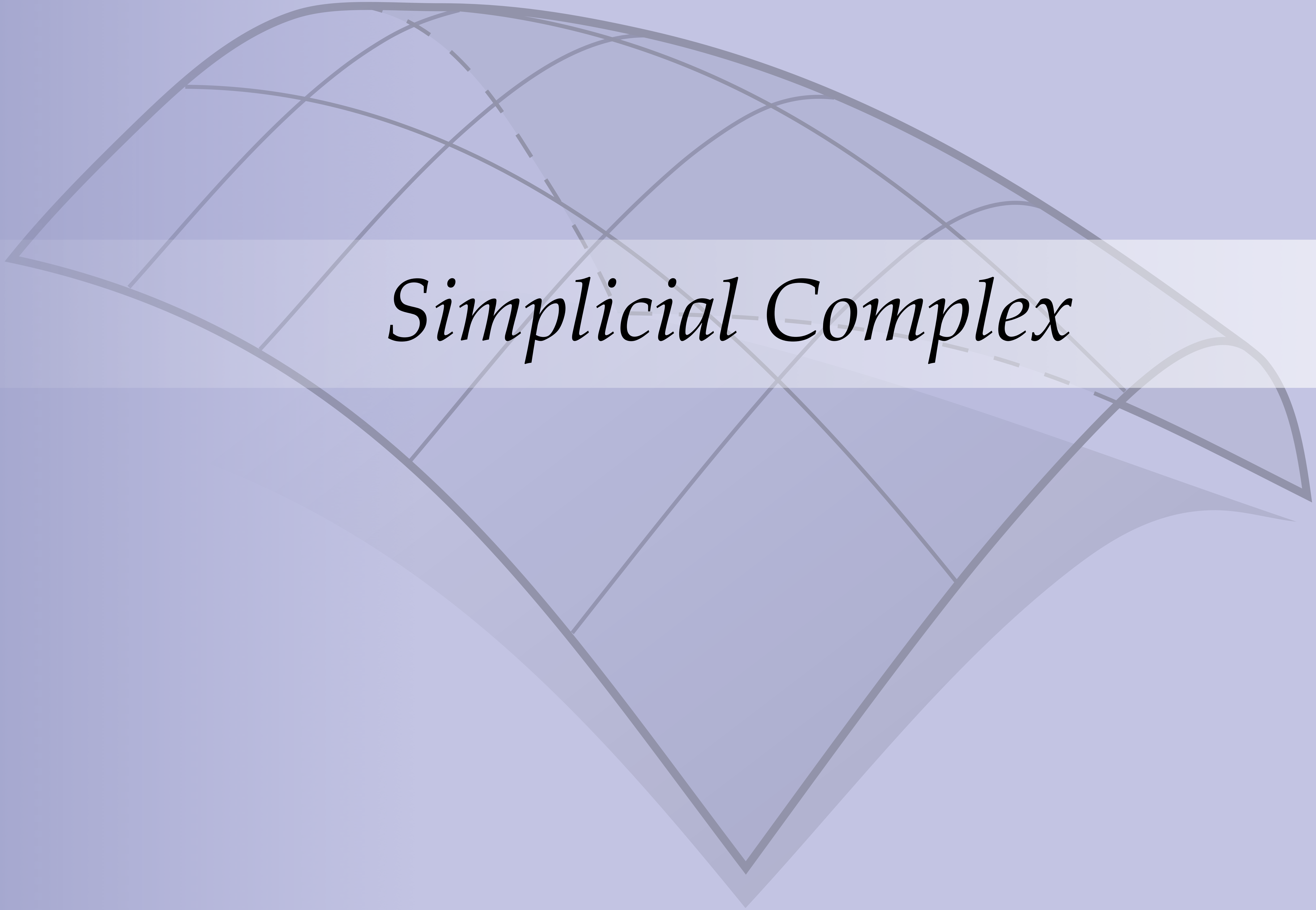
Definition. The *standard n-simplex* is the collection of points

$$\sigma := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0\ \forall i \right\}.$$



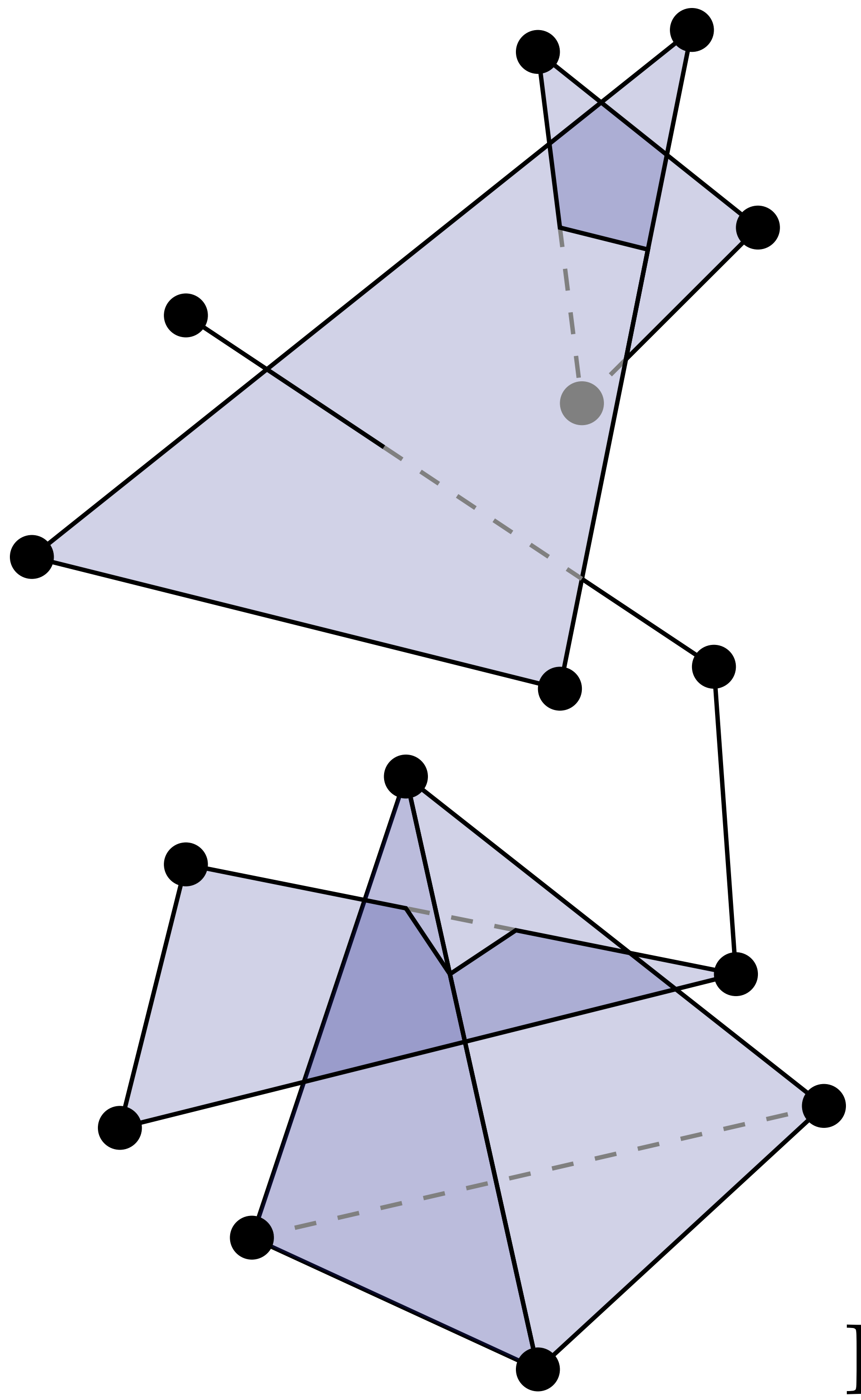
(Also known as the "probability simplex.")

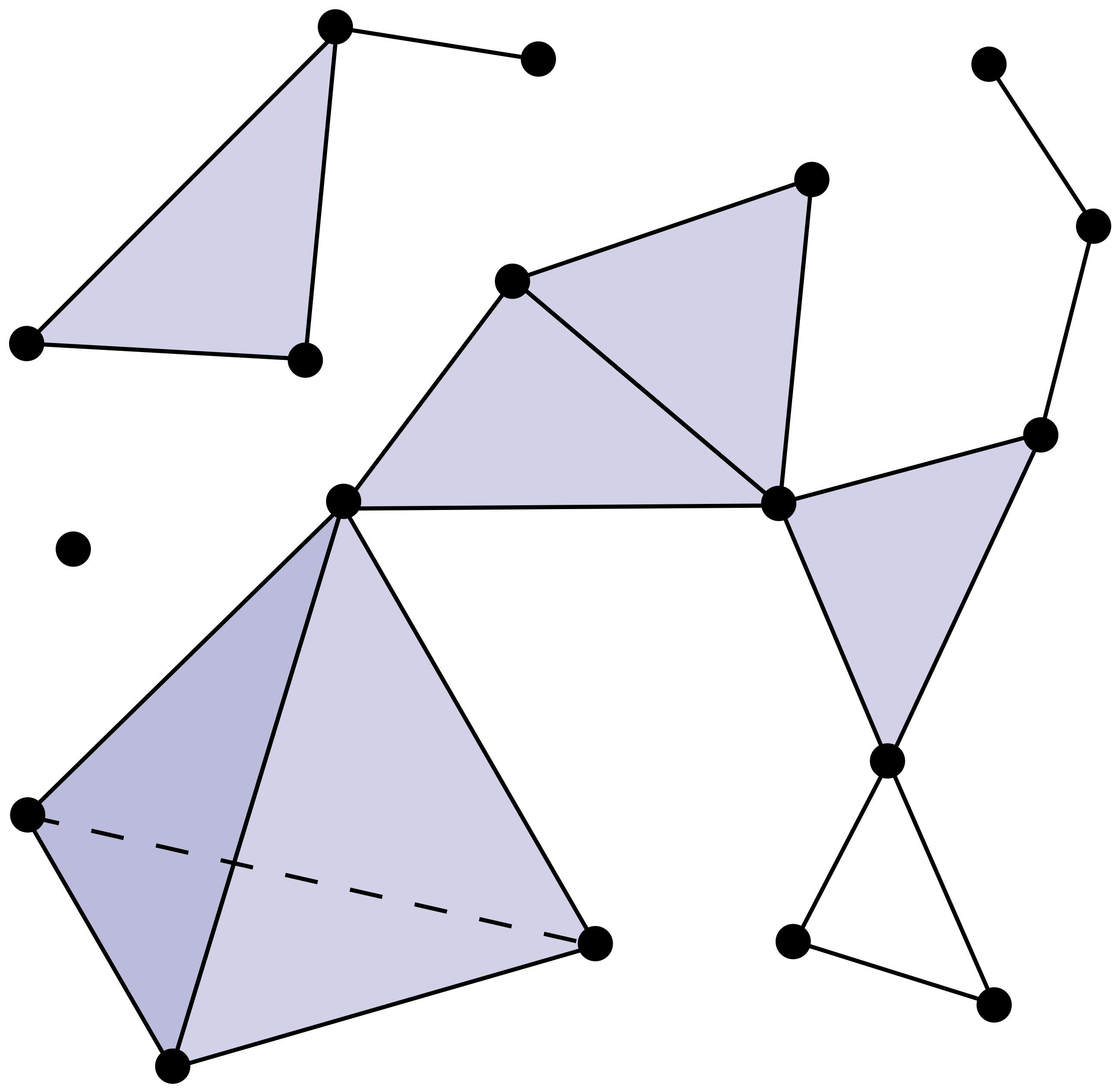
Simplicial Complex



Simplicial Complex — Rough Idea

- Roughly speaking, a *simplicial complex* is "a bunch of simplices*"
  - …but with some specific properties that make them easy to work with.
- Also have to resolve some basic questions—*e.g.*, how can simplices intersect?

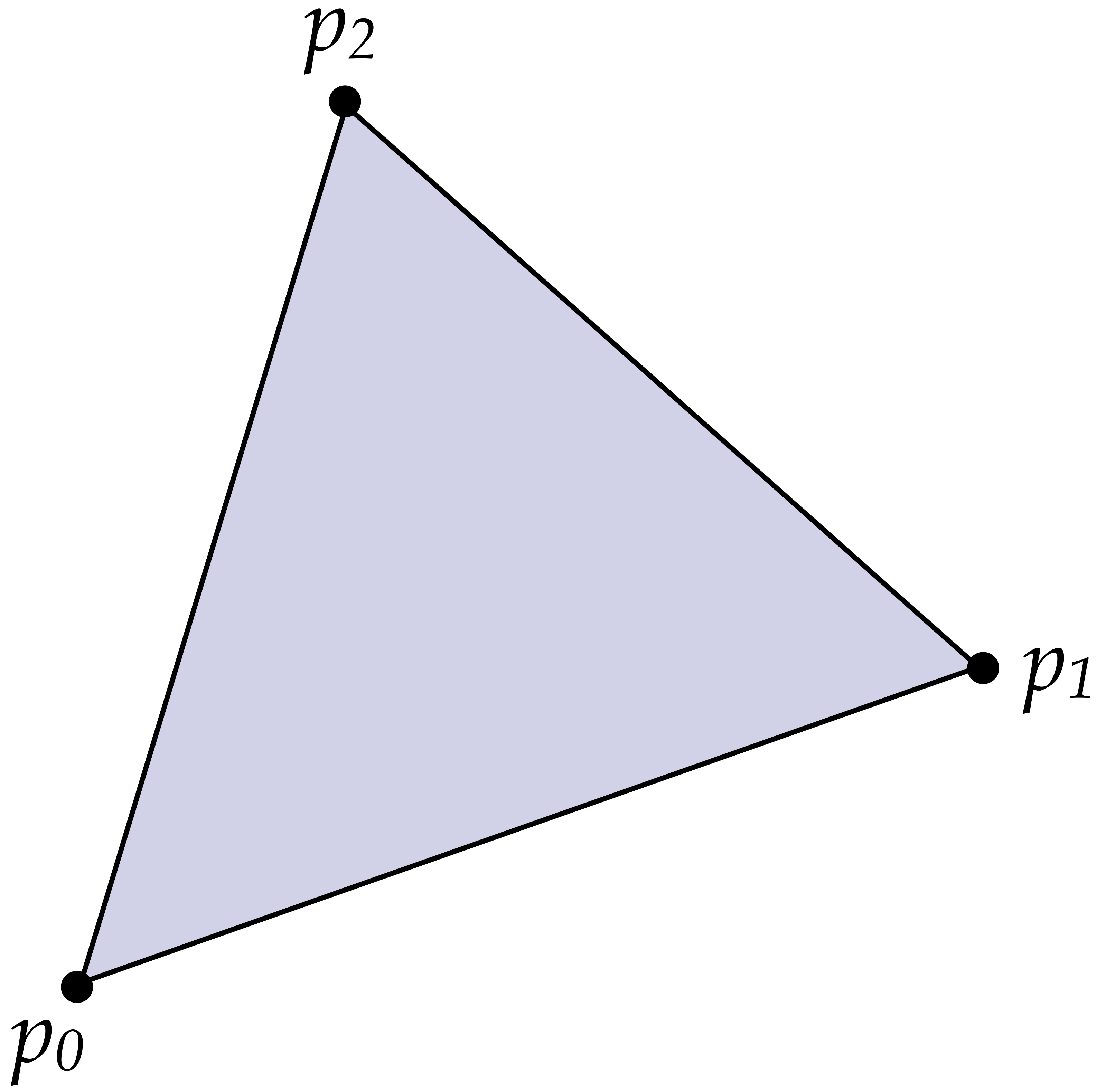


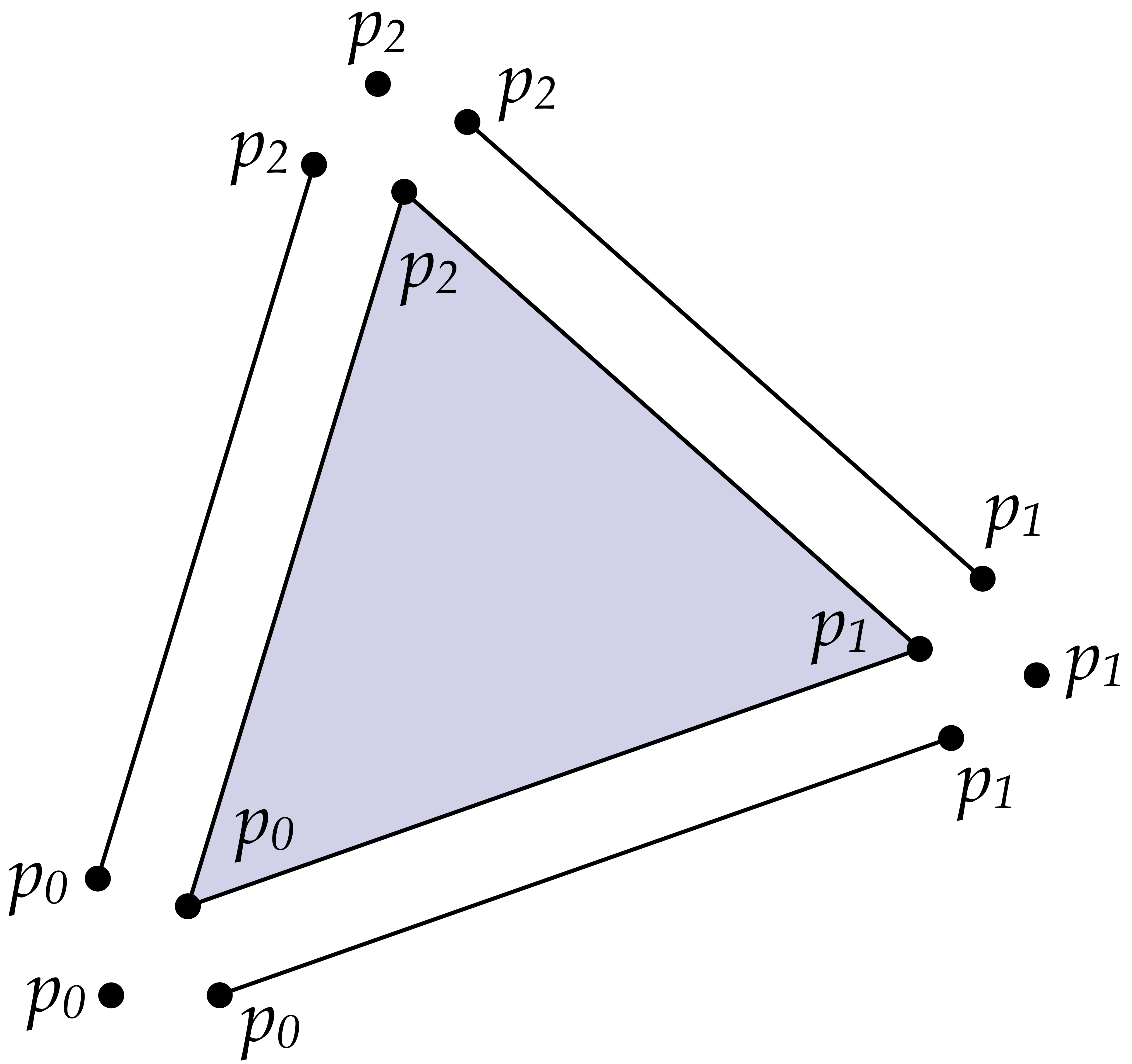


*Plural of simplex; not "simplexes." Pronounced like *vertices* and *vortices*.

Face of a Simplex

Definition. A *face* of a simplex σ is any simplex whose vertices are a subset* of the vertices of σ.





Q: Anything missing from this picture?

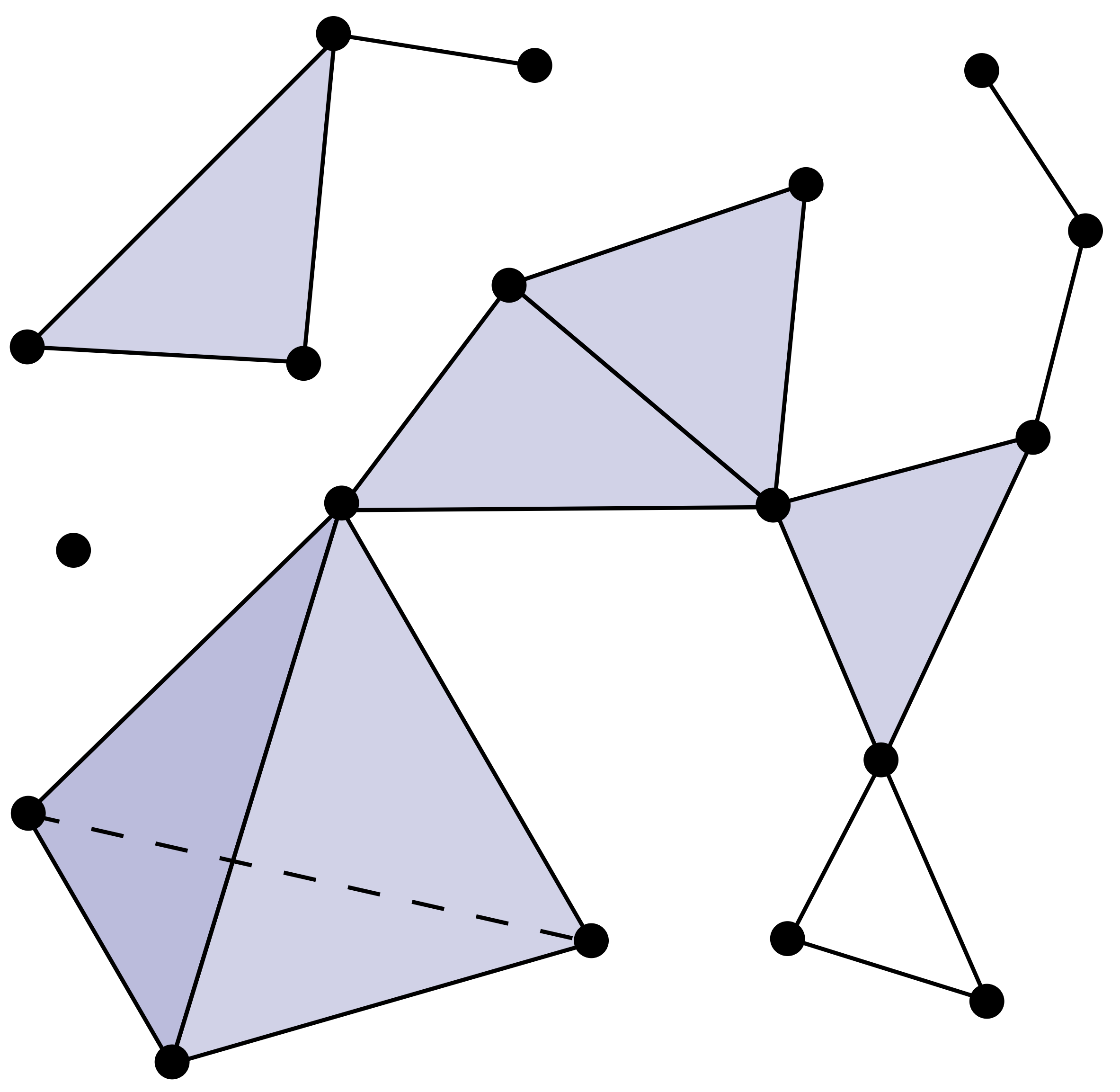
A: Yes—formally, the *empty set* ø.

*Doesn't have to be a *proper* subset, *i.e.,* a simplex is its own face.

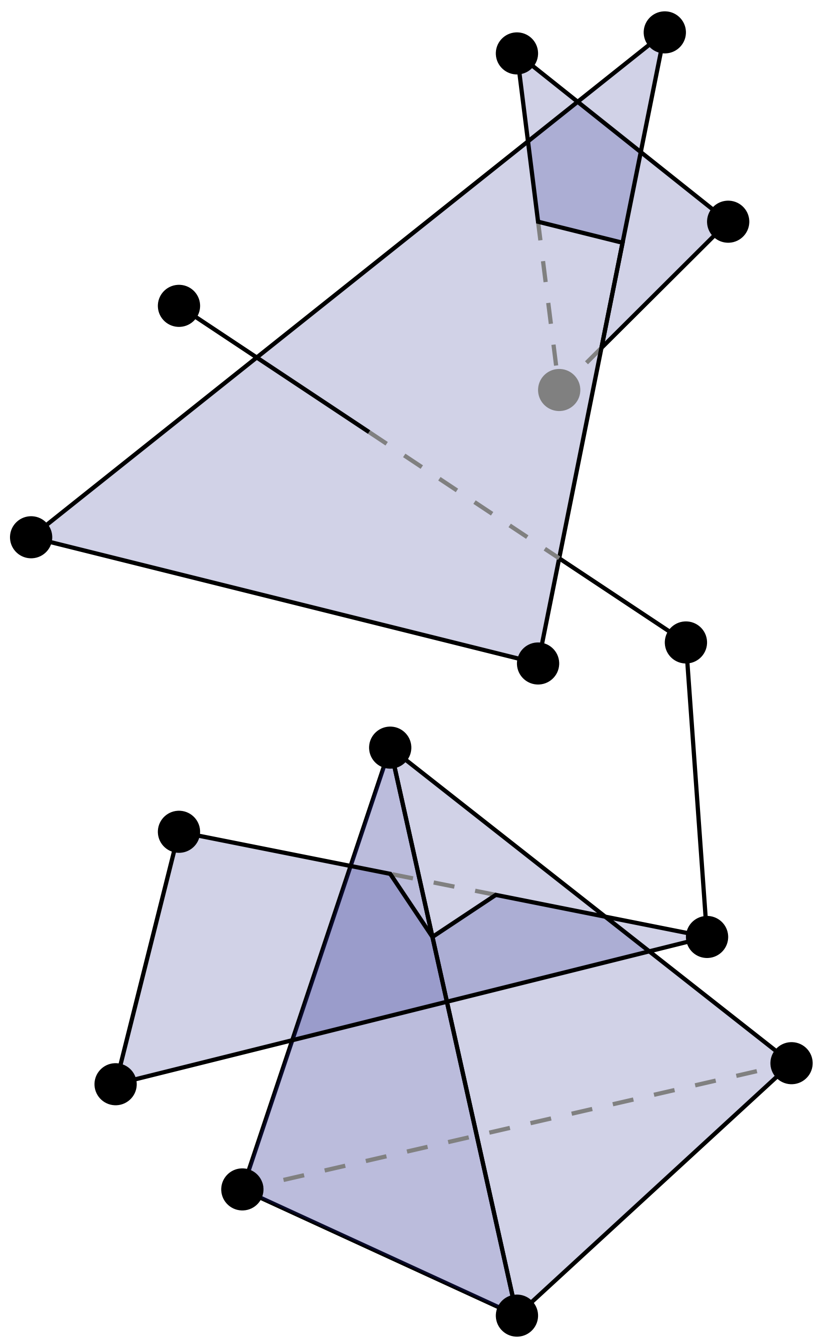
Simplicial Complex—Geometric Definition

Definition. A *(geometric) simplicial complex* is a collection of simplices where:

- the intersection of any two simplices is a simplex, and
- every face of every simplex in the complex is also in the complex.

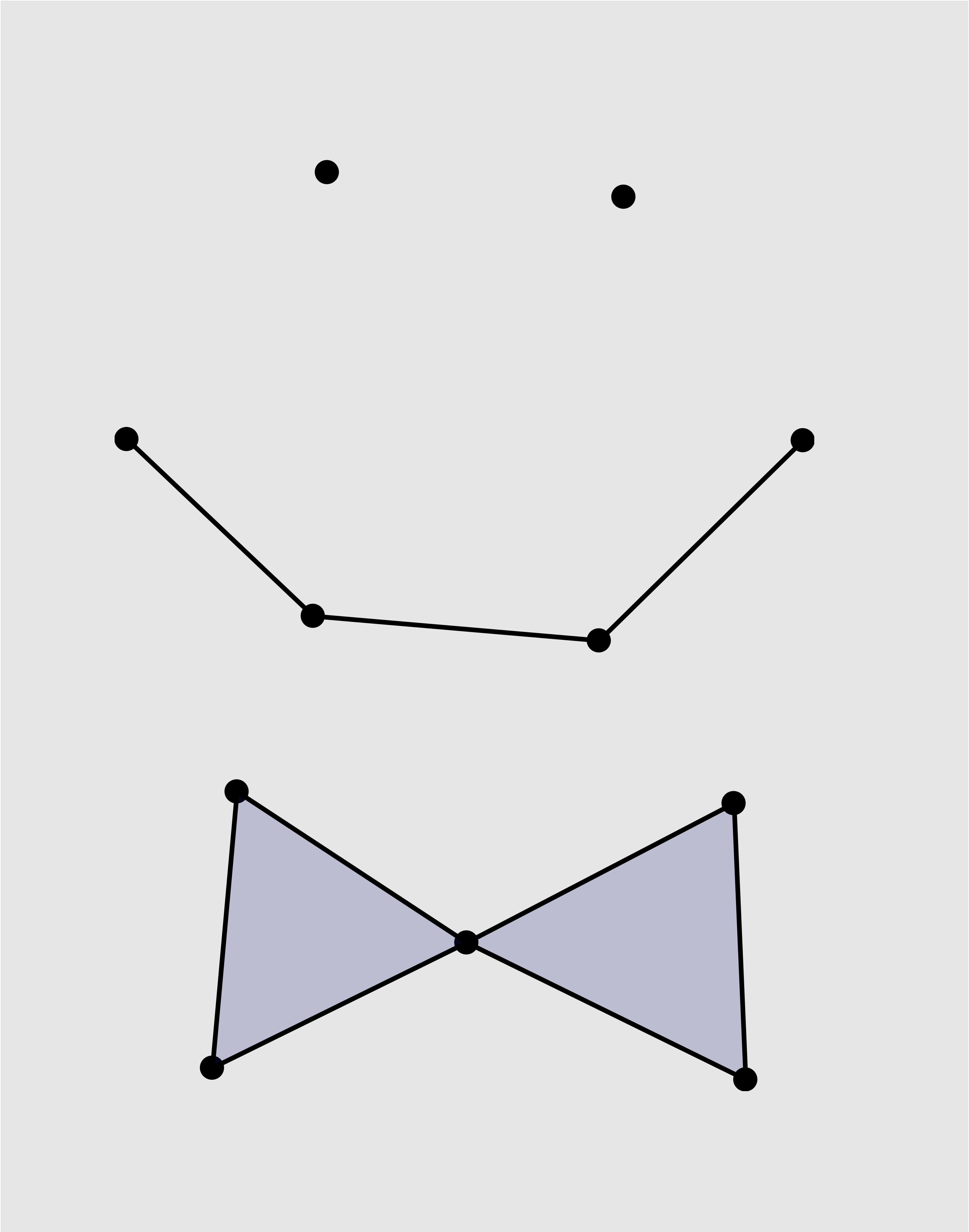


simplicial complex



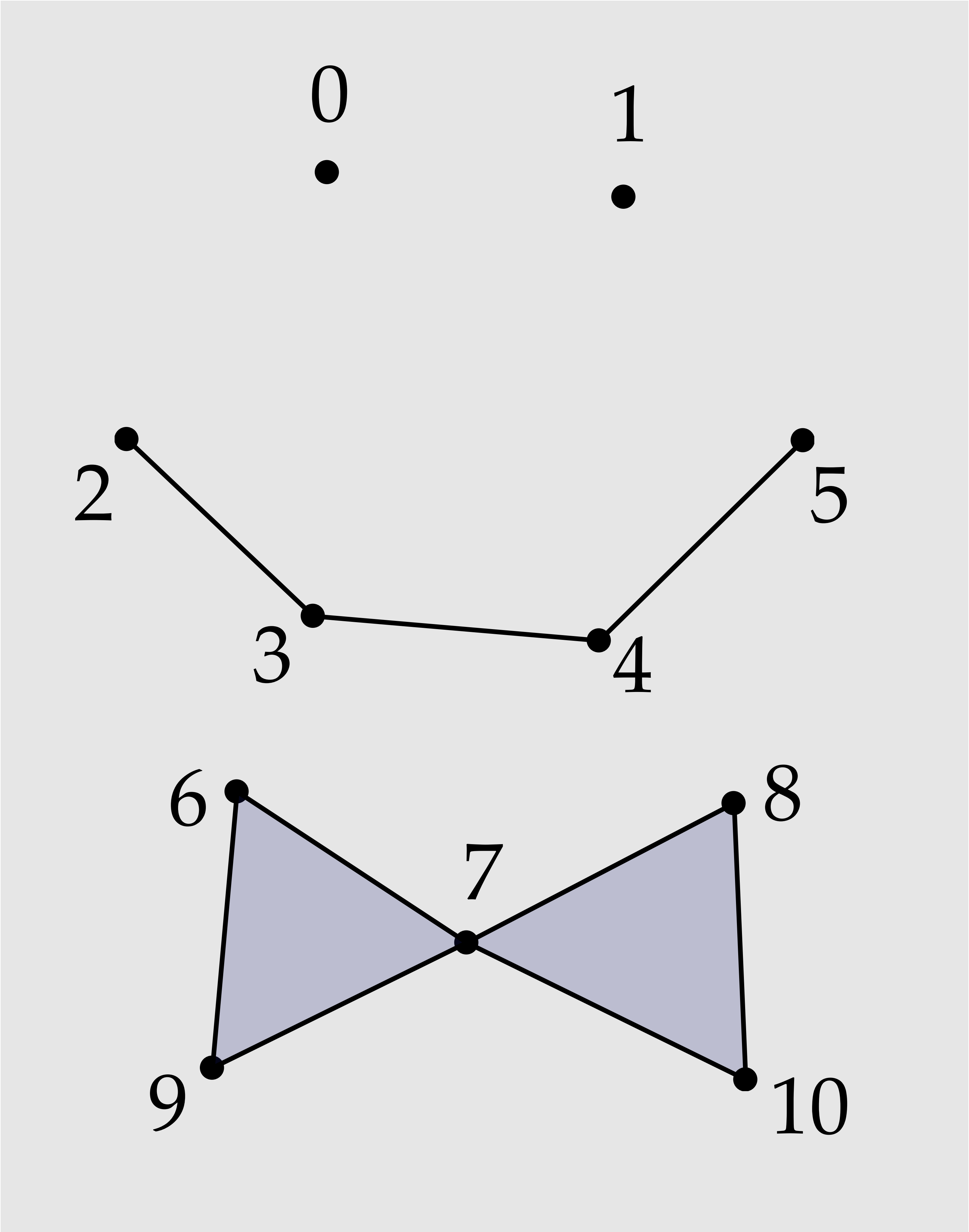
not a geometric simplicial complex…

Simplicial Complex — Example



Mr. GoPuff

Simplicial Complex — Example



Mr. GoPuff

Q: What are all the simplices in Mr. Go Puff?

A: {6,7,9} {7,10,8} {2,3} {3,4} {4,5} {0} {1}

{6,7} {7,9} {9,6} {7,8} {8,10} {10,7} {2} {3} {4} {5}

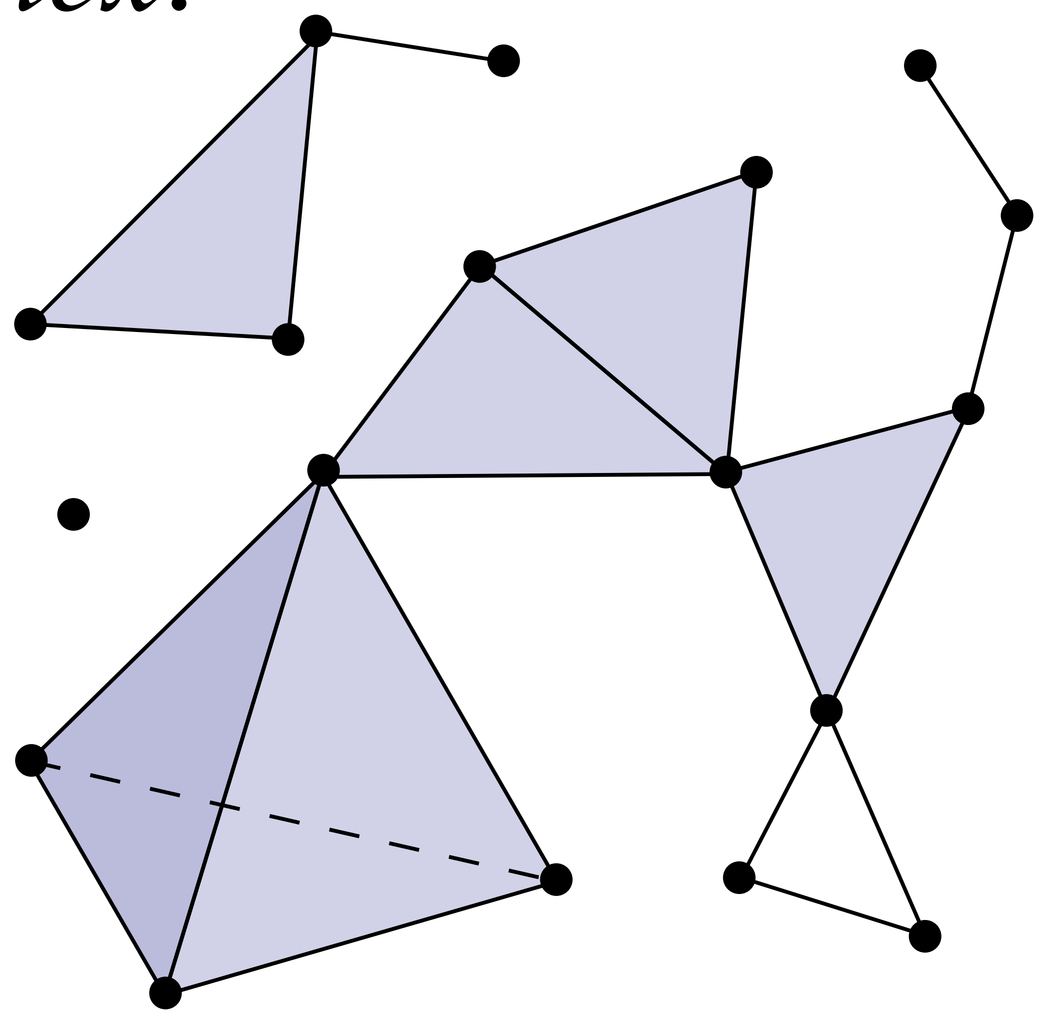
{6} {7} {8} {9} {10}

{ø}

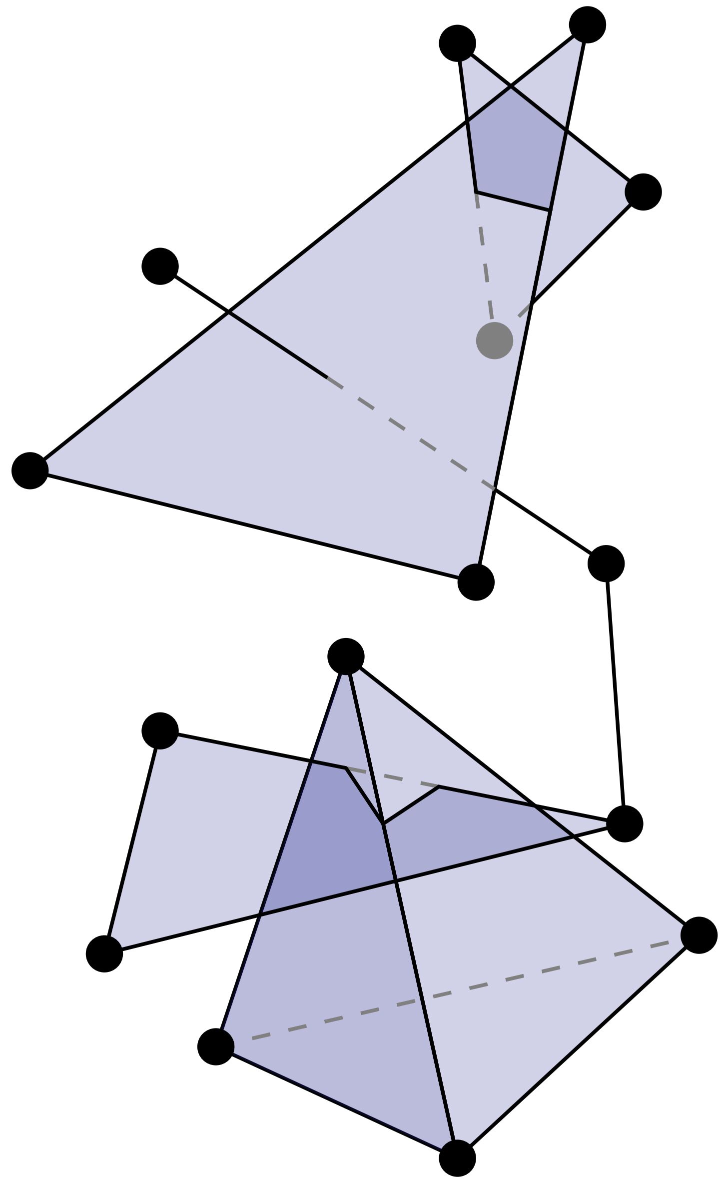
Didn't really use the geometry here…

Abstract Simplicial Complex

Definition. Let S be a collection of sets. If for each set $\sigma \in S$ all subsets of σ are contained in S, then S is an *abstract simplicial complex*. A set $\sigma \in S$ of size $k+1$ is an *(abstract) simplex*.



geometric simplicial complex



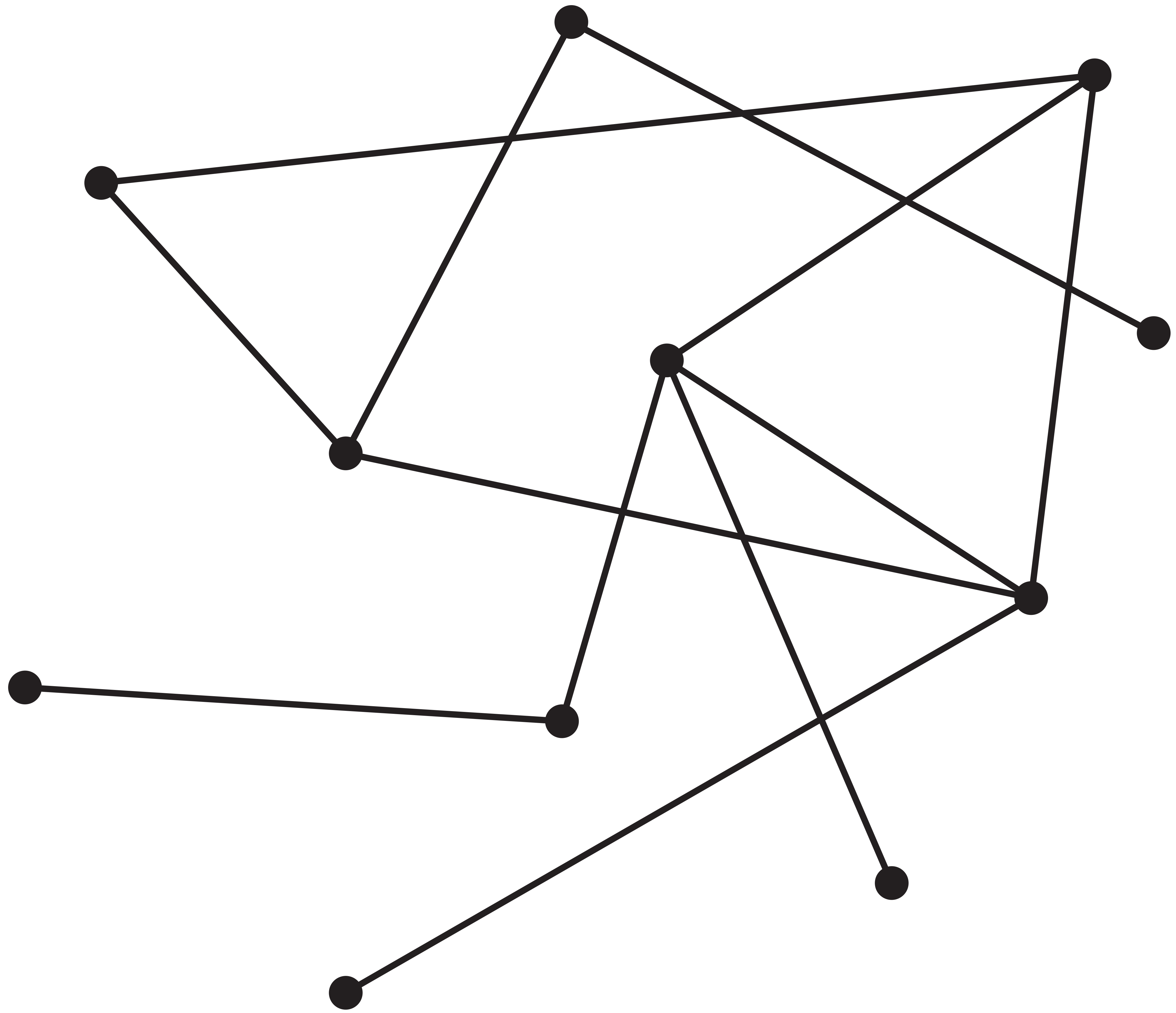
abstract simplicial complex*

- Only care about how things are *connected*, not how they are arranged geometrically.
- Later in class, will serve as our discretization of a *topological space*

*…visualized by mapping it into R^3.

Abstract Simplicial Complex — Graphs

- Any *(undirected) graph* $G = (V,E)$ is an abstract simplicial (1-)complex
  - 0-simplices are vertices
  - 1-simplices are edges

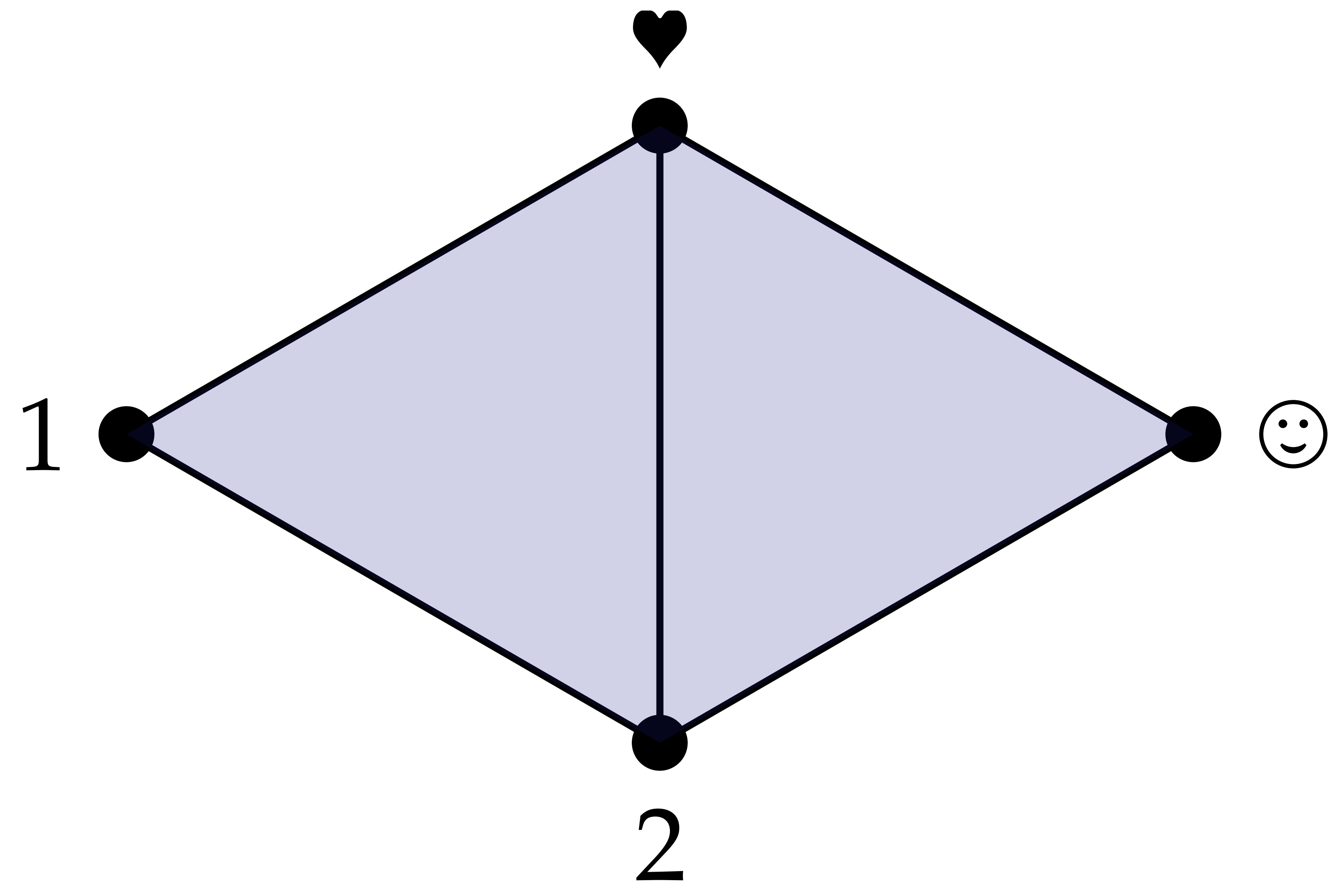


Abstract Simplicial Complex—Example

Example: Consider the set
$S := \{\{1,2,\heartsuit\},\{2,\heartsuit,☺\},\{1,2\},\{2,\heartsuit\},\{\heartsuit,1\},\{2,\heartsuit\},\{\heartsuit,☺\},\{1\},\{2\},\{\heartsuit\},\{☺\},\emptyset\}$

Q: Is this set an abstract simplicial complex? If so, what does it look like?

A: Yes—it's a pair of 2-simplices (triangles) sharing a single edge:

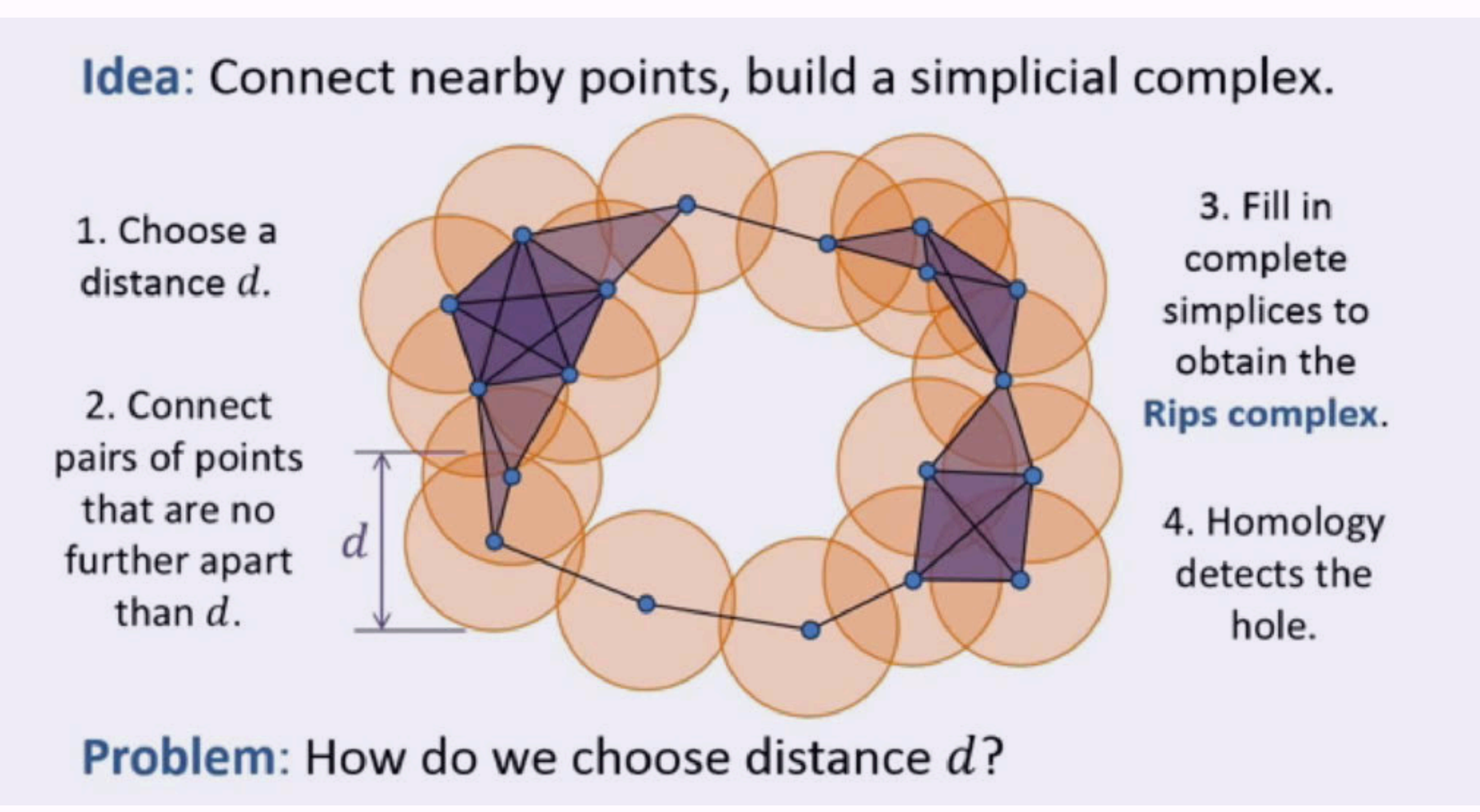


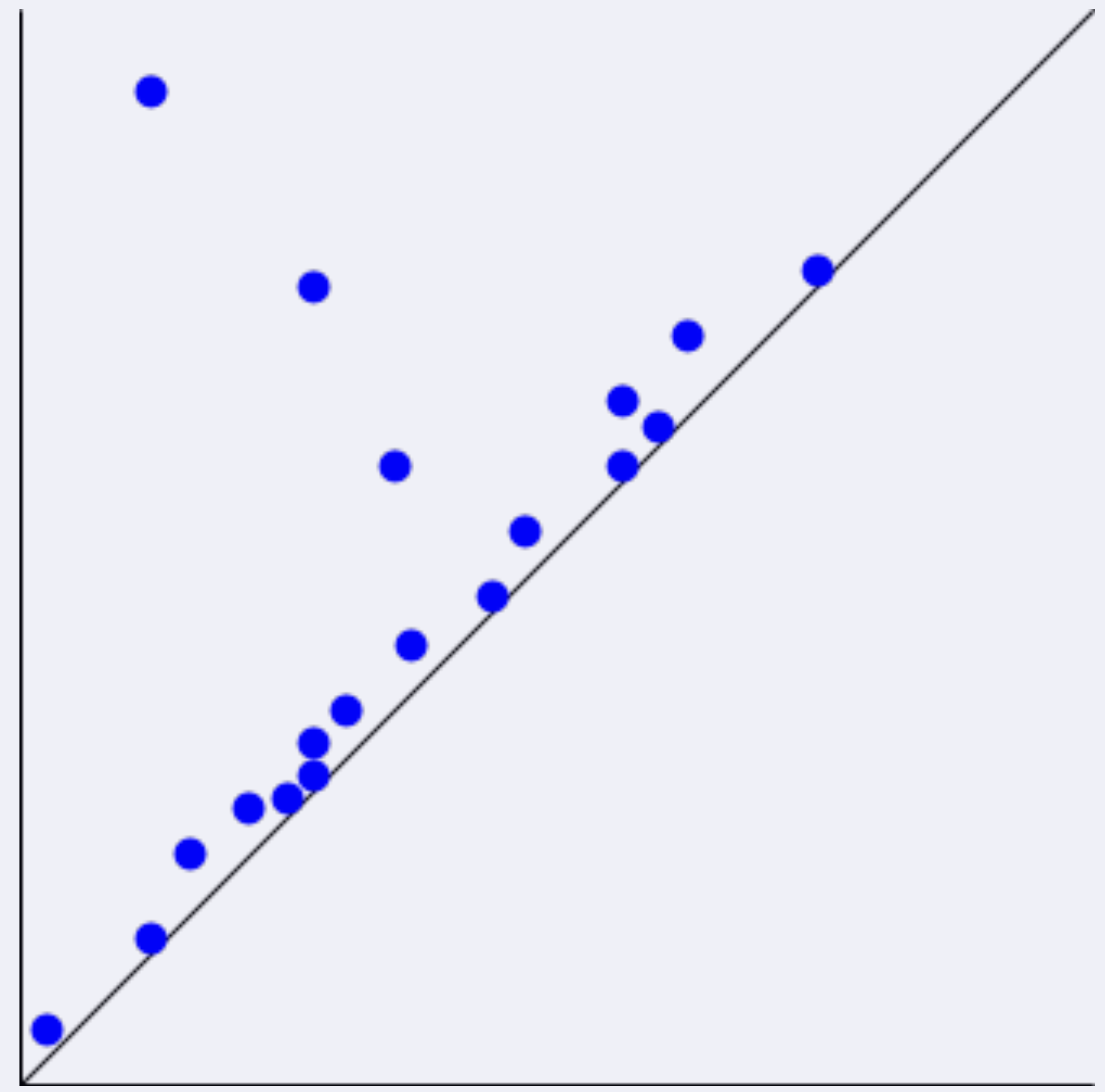
Vertices no longer have to be points in space; can represent anything at all.

Application: Topological Data Analysis

Forget (mostly) about geometry—try to understand data in terms of *connectivity*.

E.g., *persistent homology:*



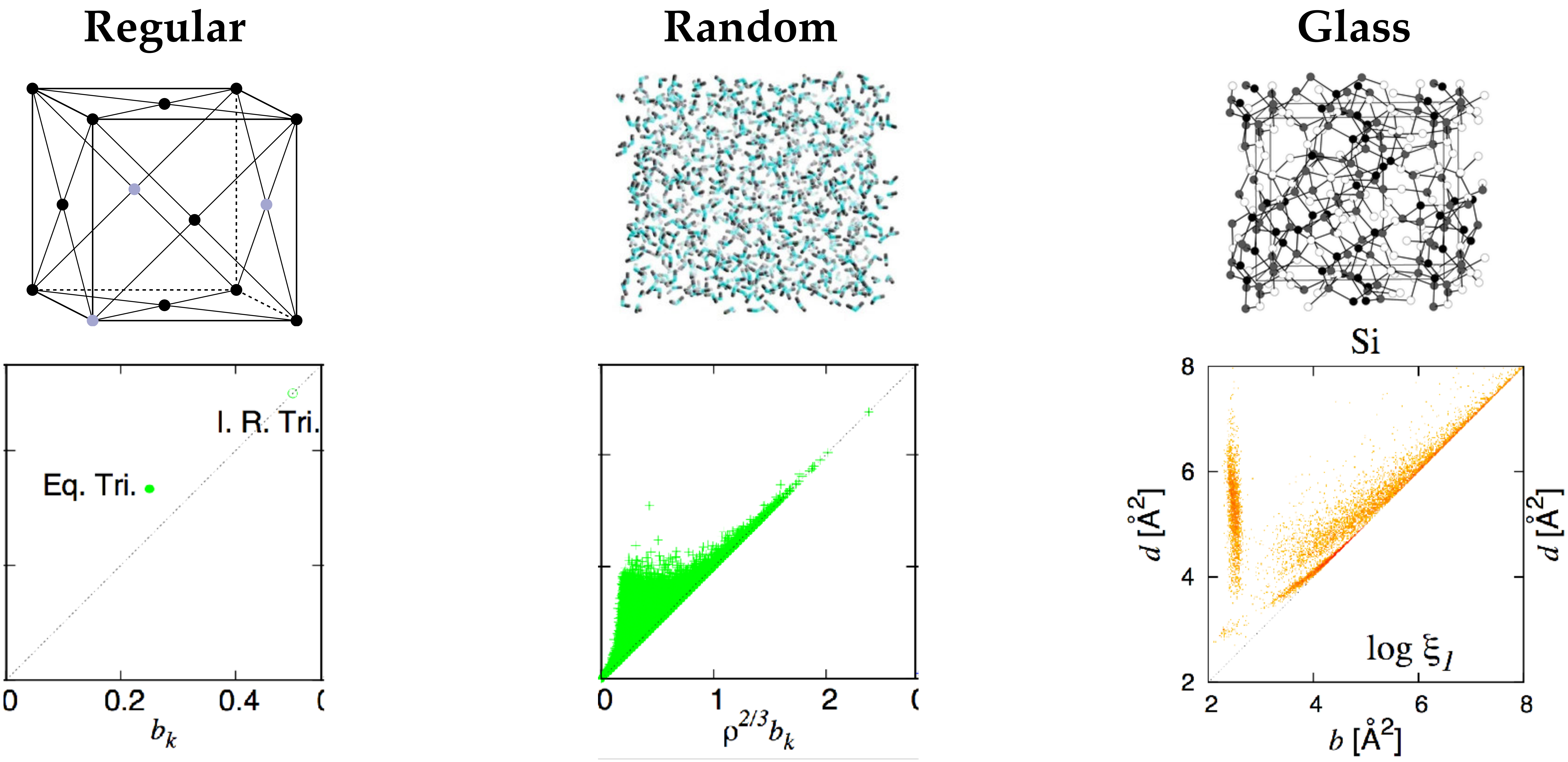


(persistence diagram)

https://youtu.be/h0bnG1Wavag

Material Characterization via Persistence

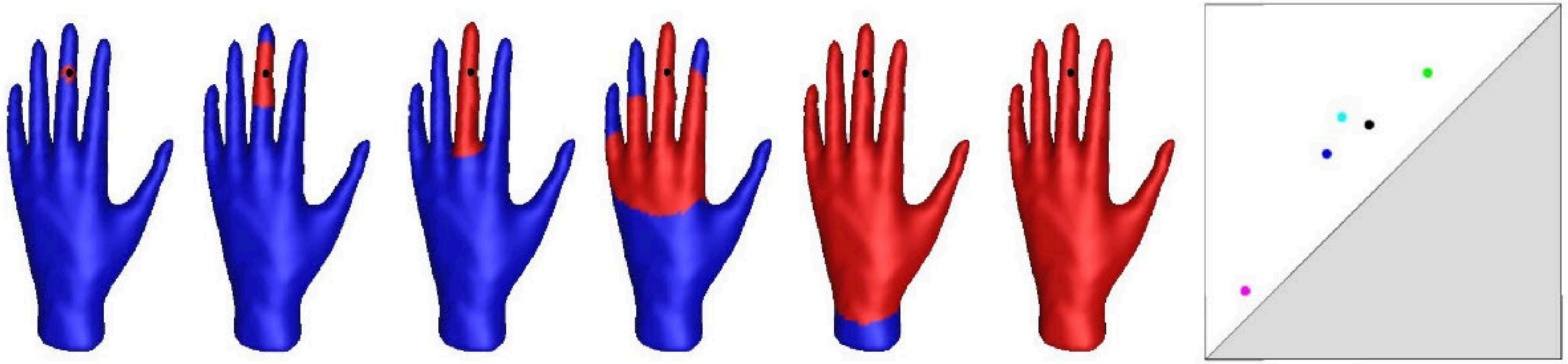
Regular | **Random** | **Glass**



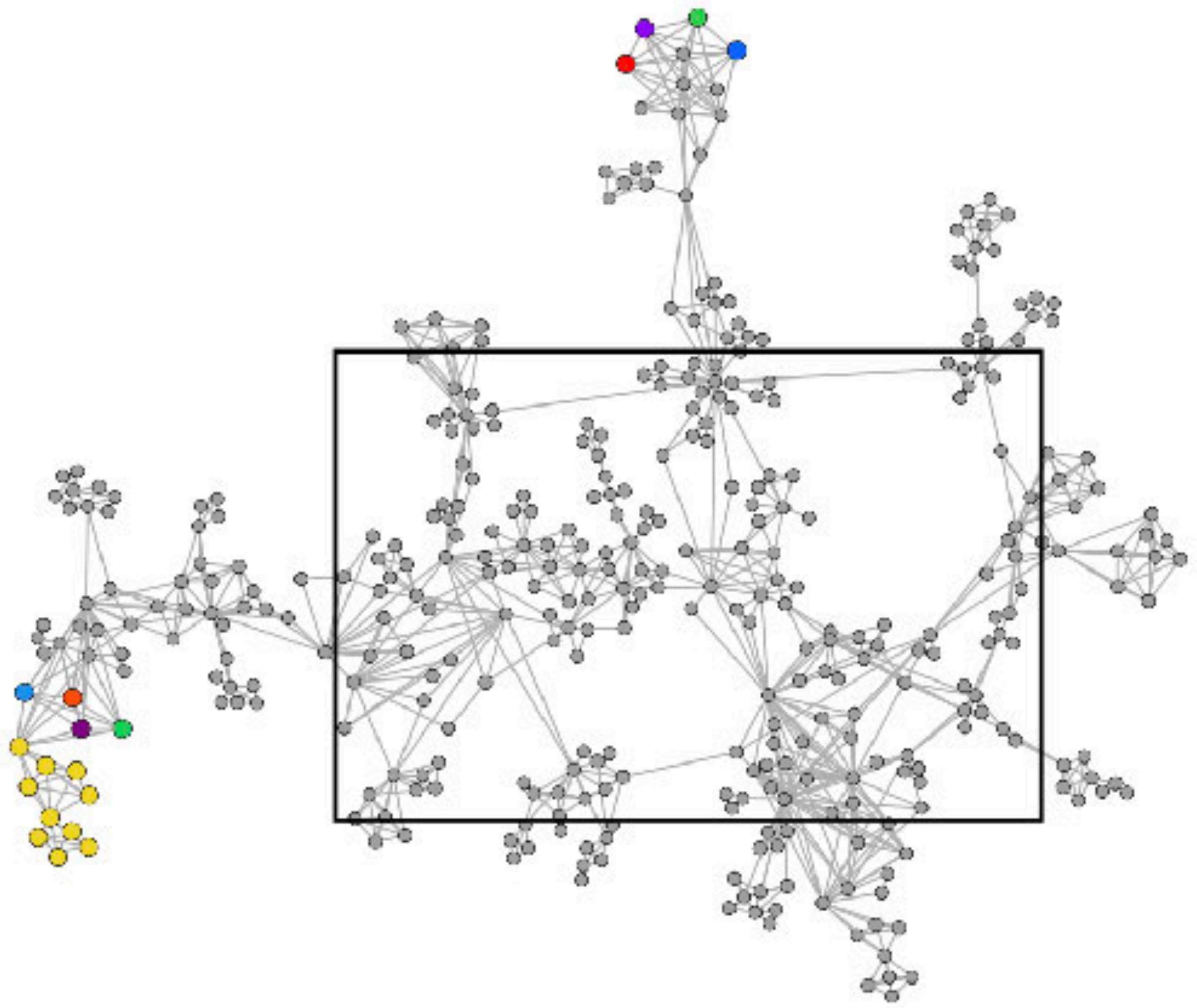
Nakamura et al, *"Persistent Homology and Many-Body Atomic Structure for Medium-Range Order in the Glass"*

Persistent Homology—More Applications

M. Carrière, S. Oudot, M. Ovsjanikov, *"Stable Topological Signatures for Points on 3D Shapes"*



C. Carstens, K. Horadam,
"Persistent Homology of Collaboration Networks "



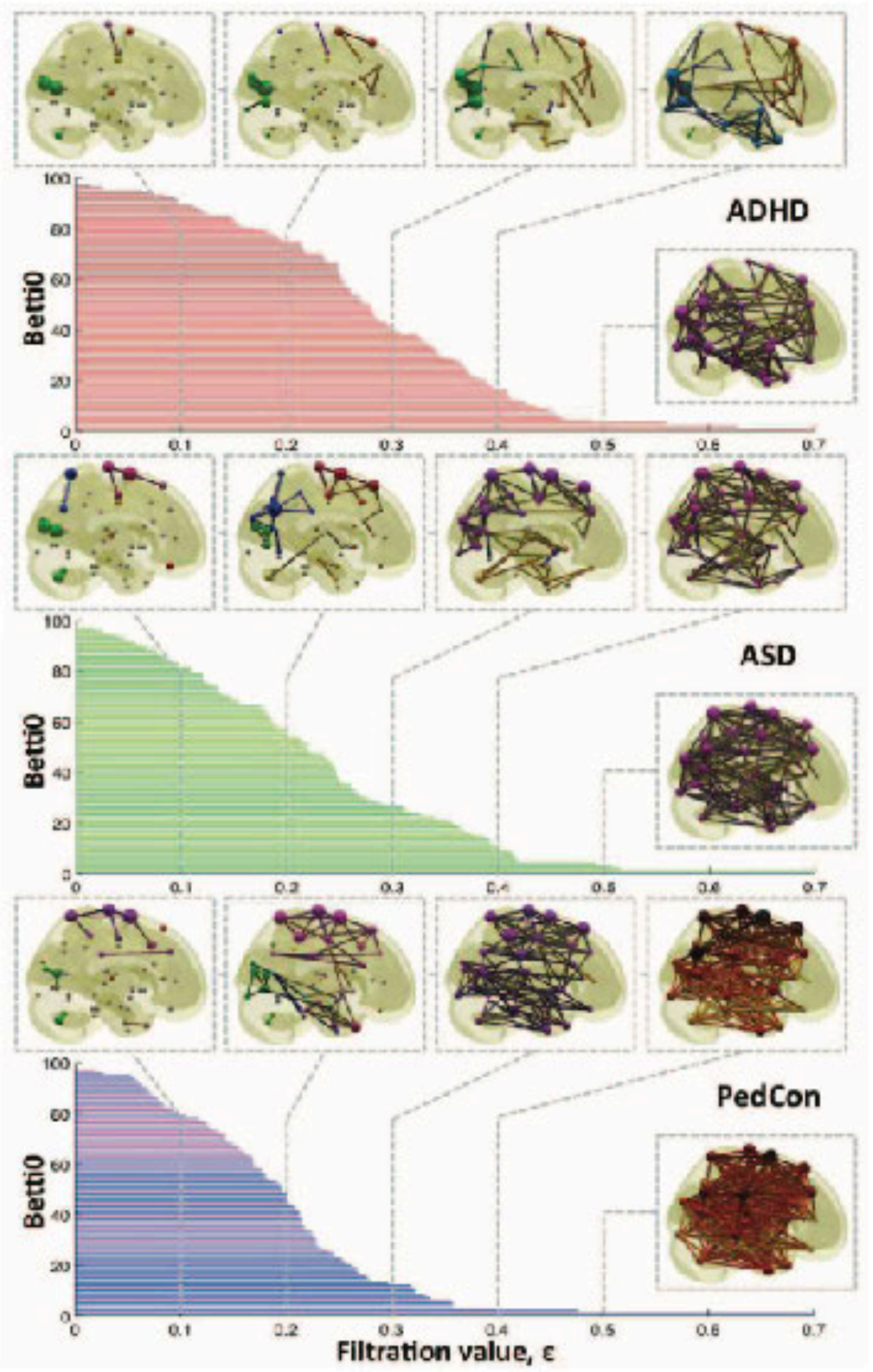


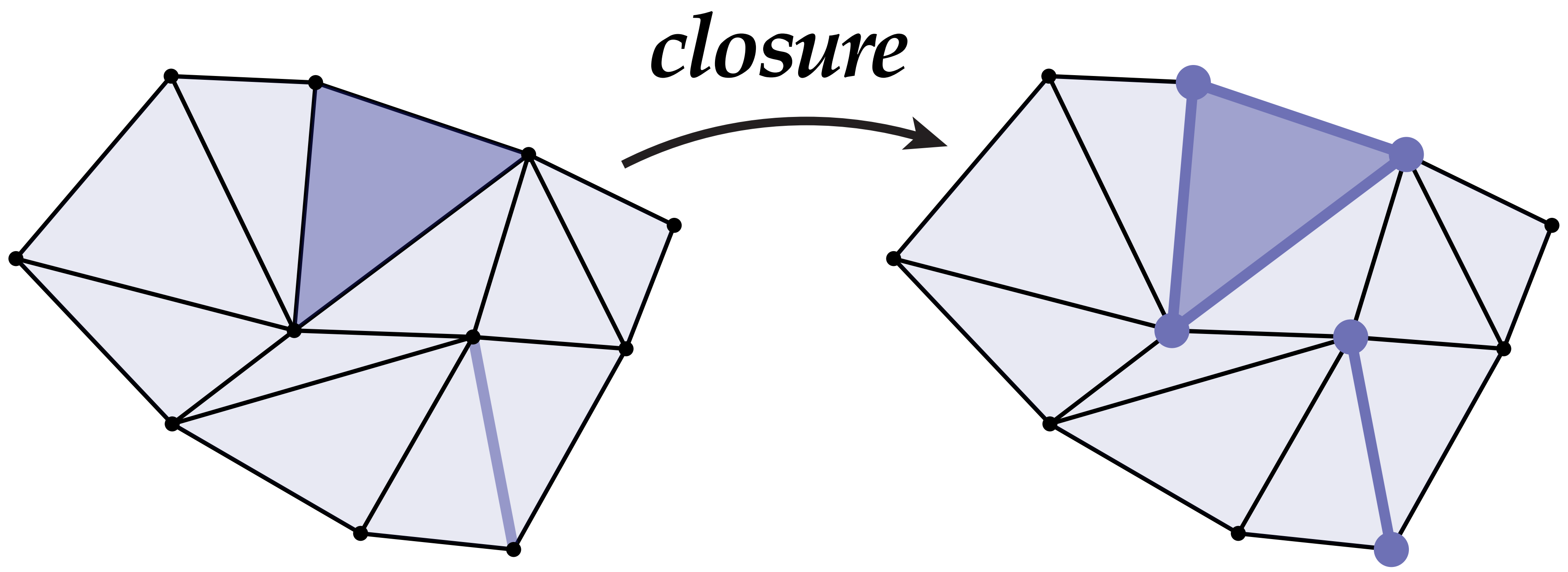
Fig. 4. Barcode of the 0-th Betti number.

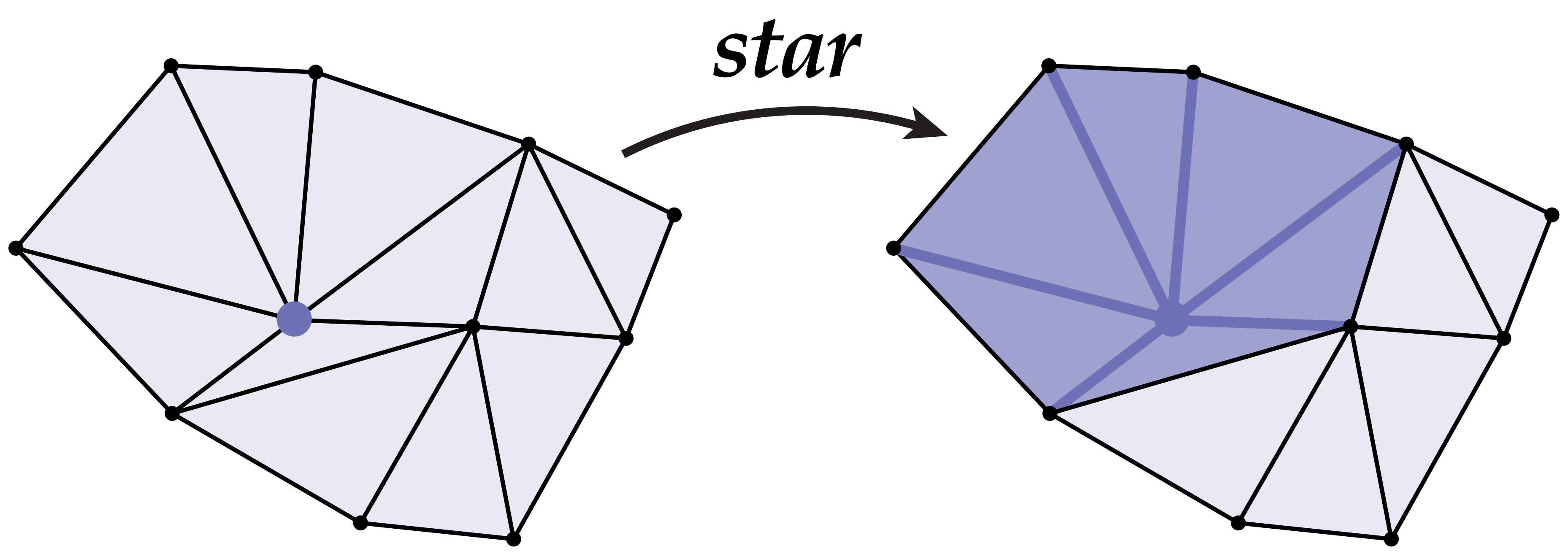
H. Lee, M. Chung, H. Kang, B. Kim, D. Lee
"Discriminative Persistent Homology of Brain Networks"

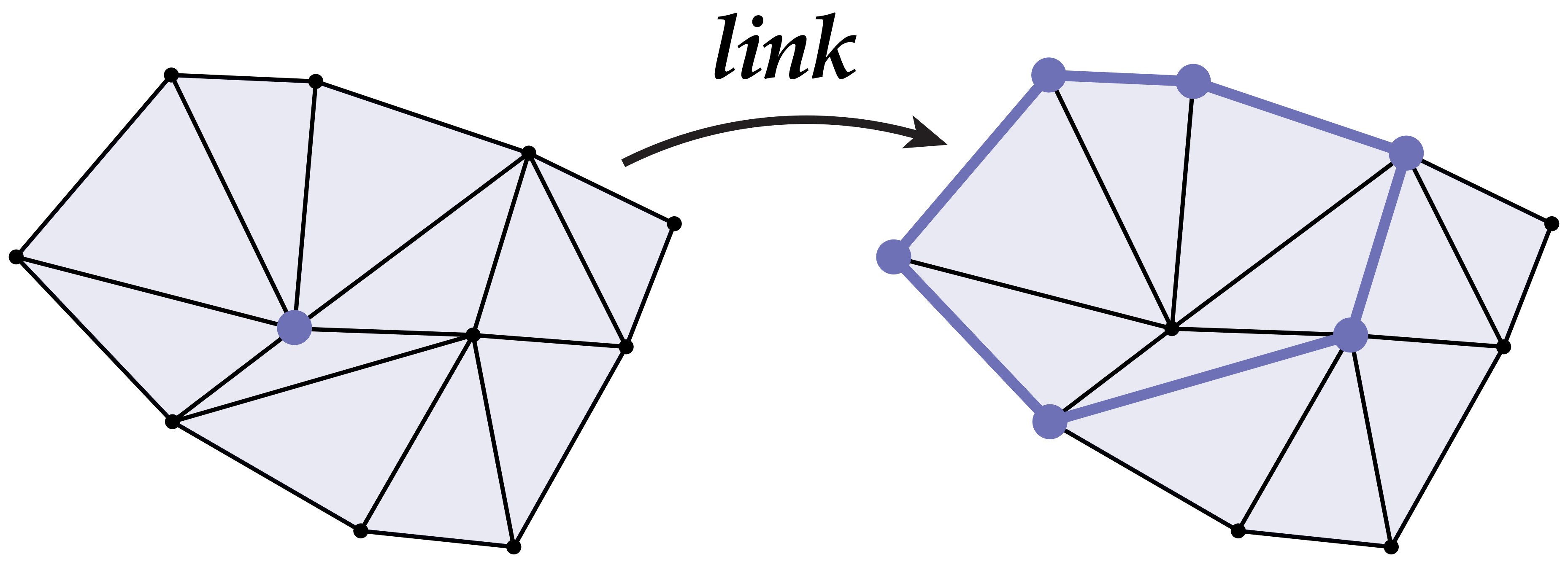
…and much much more (identifying patients with breast cancer, classifying players in basketball, new ways to compress images, *etc.*)

Anatomy of a Simplicial Complex

- **Closure:** smallest simplicial complex containing a given set of simplices
- **Star:** union of simplices containing a given subset of simplices
- **Link:** closure of the star minus the star of the closure

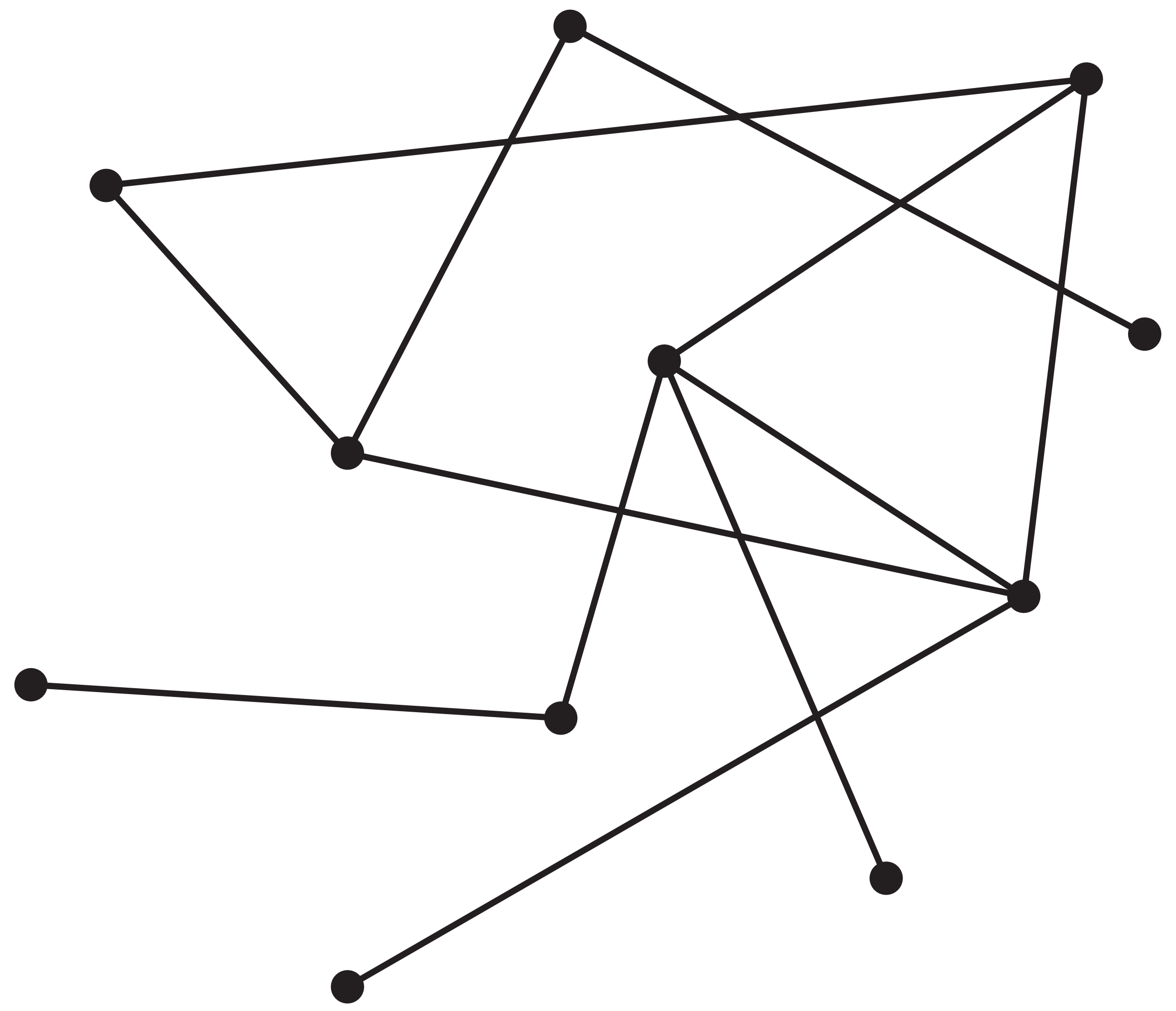




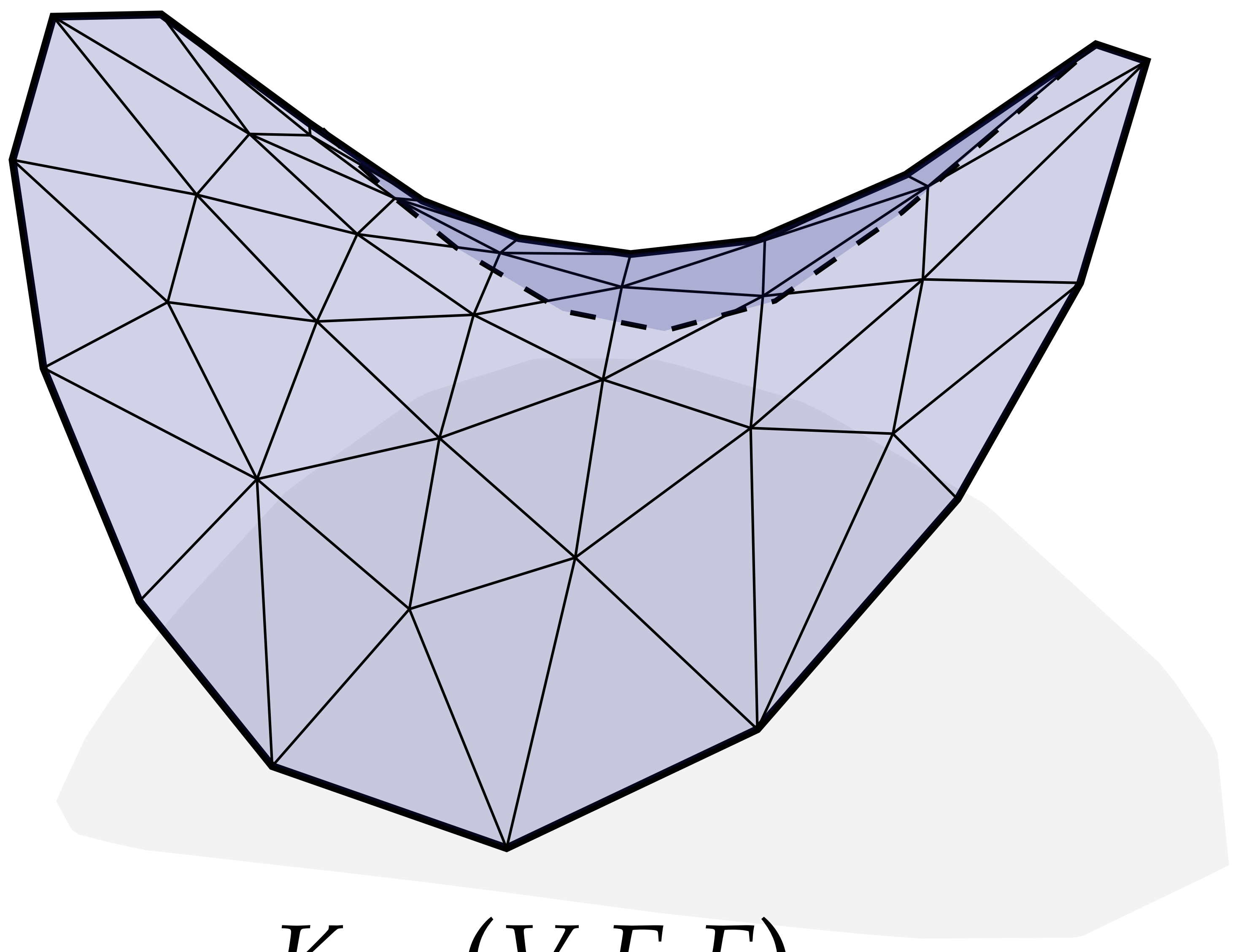


Vertices, Edges, and Faces

- Just a little note about notation:
  - For simplicial **1-complexes** (graphs) we often write $G = (V,E)$
  - Likewise, for simplicial **2-complexes** (triangle meshes) we write $K = (V,E,F)$
    - Vertices
    - Edges
    - Faces*
  - K is for *"Komplex!"*



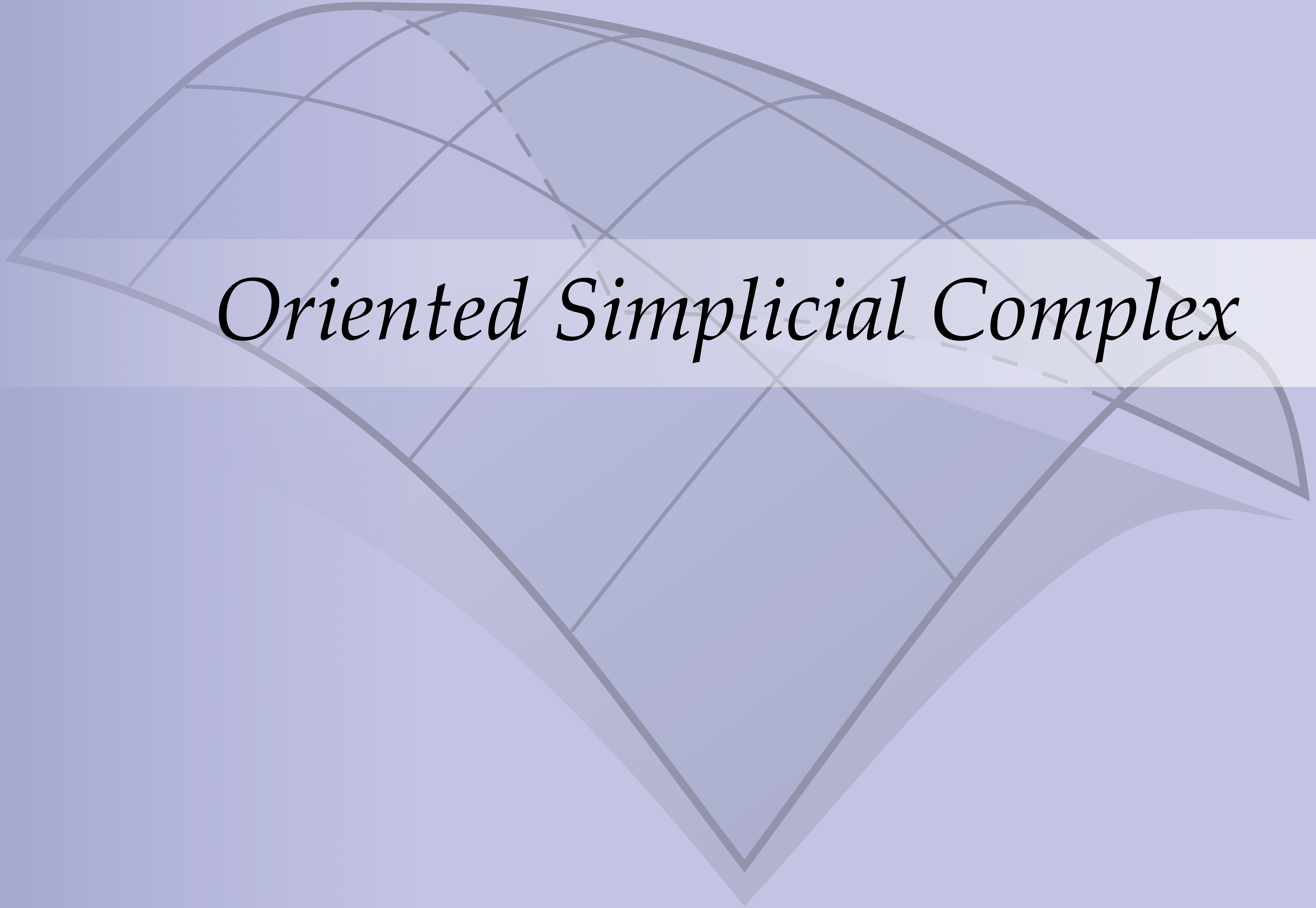
$G = (V,E)$



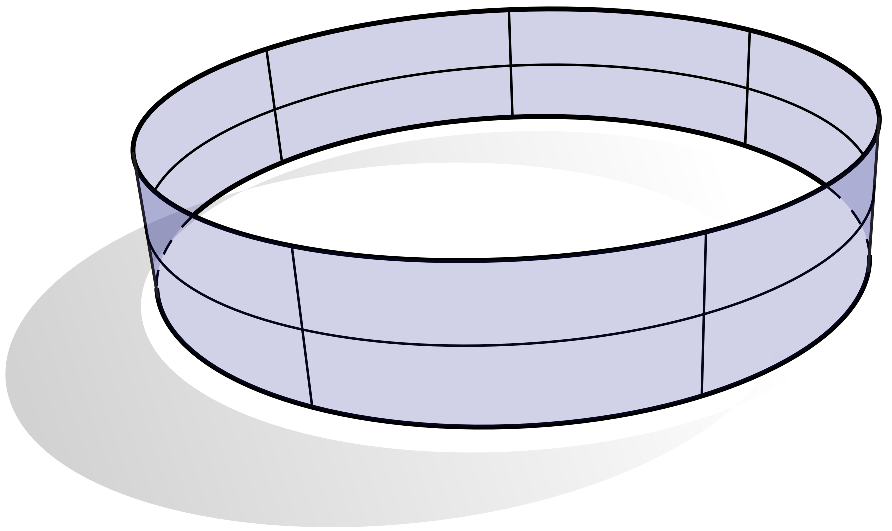
$K = (V,E,F)$

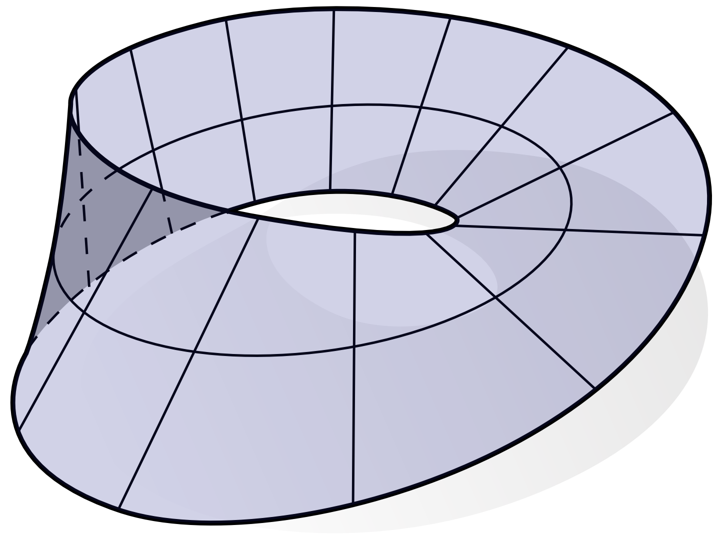
*Not to be confused with the generic *face* of a simplex…

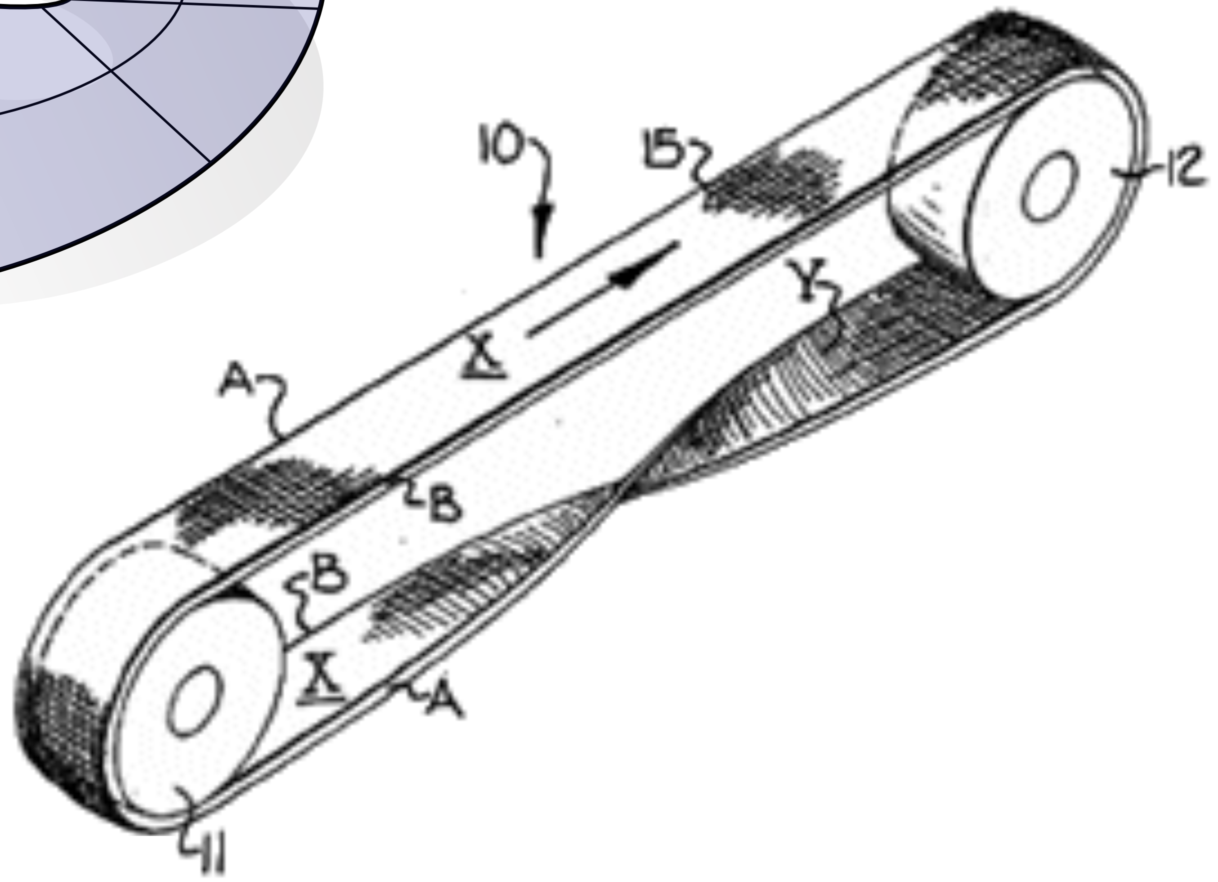
Oriented Simplicial Complex

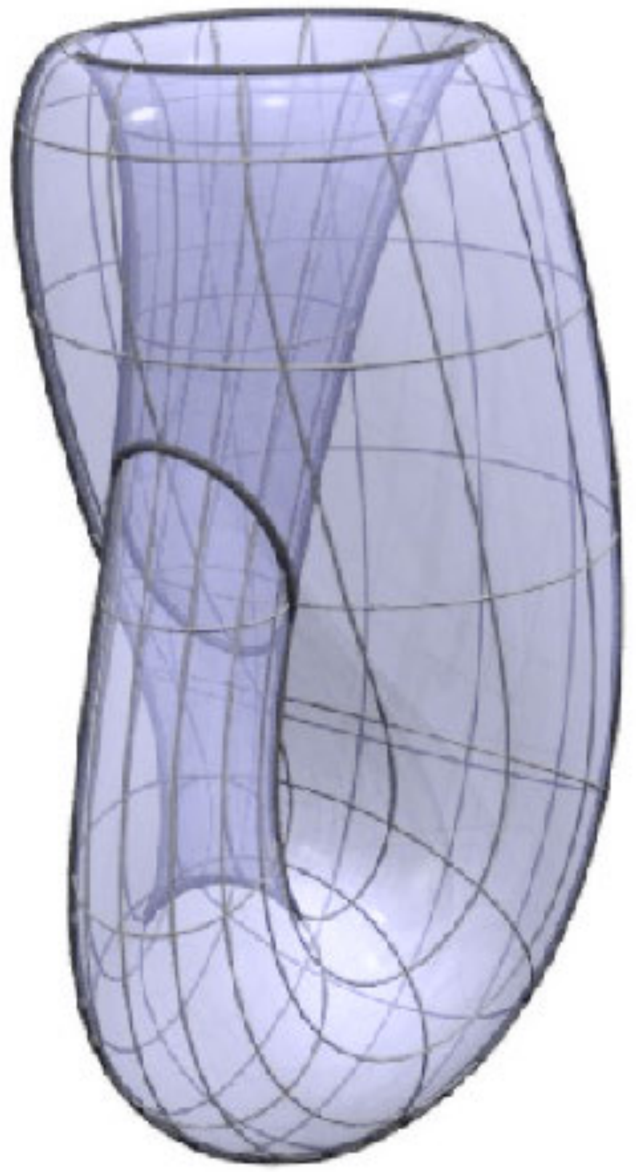


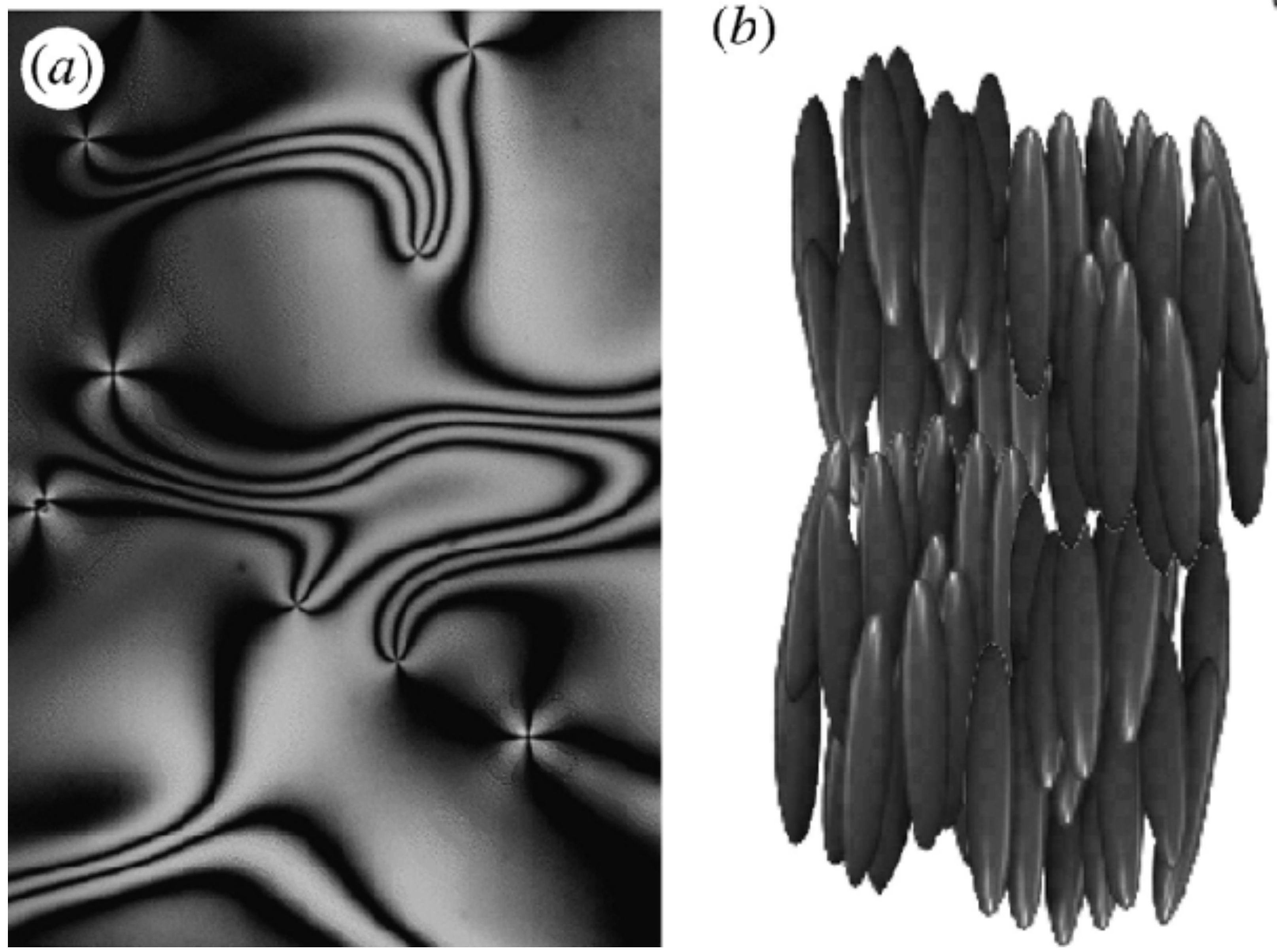
Orientation — Visualized

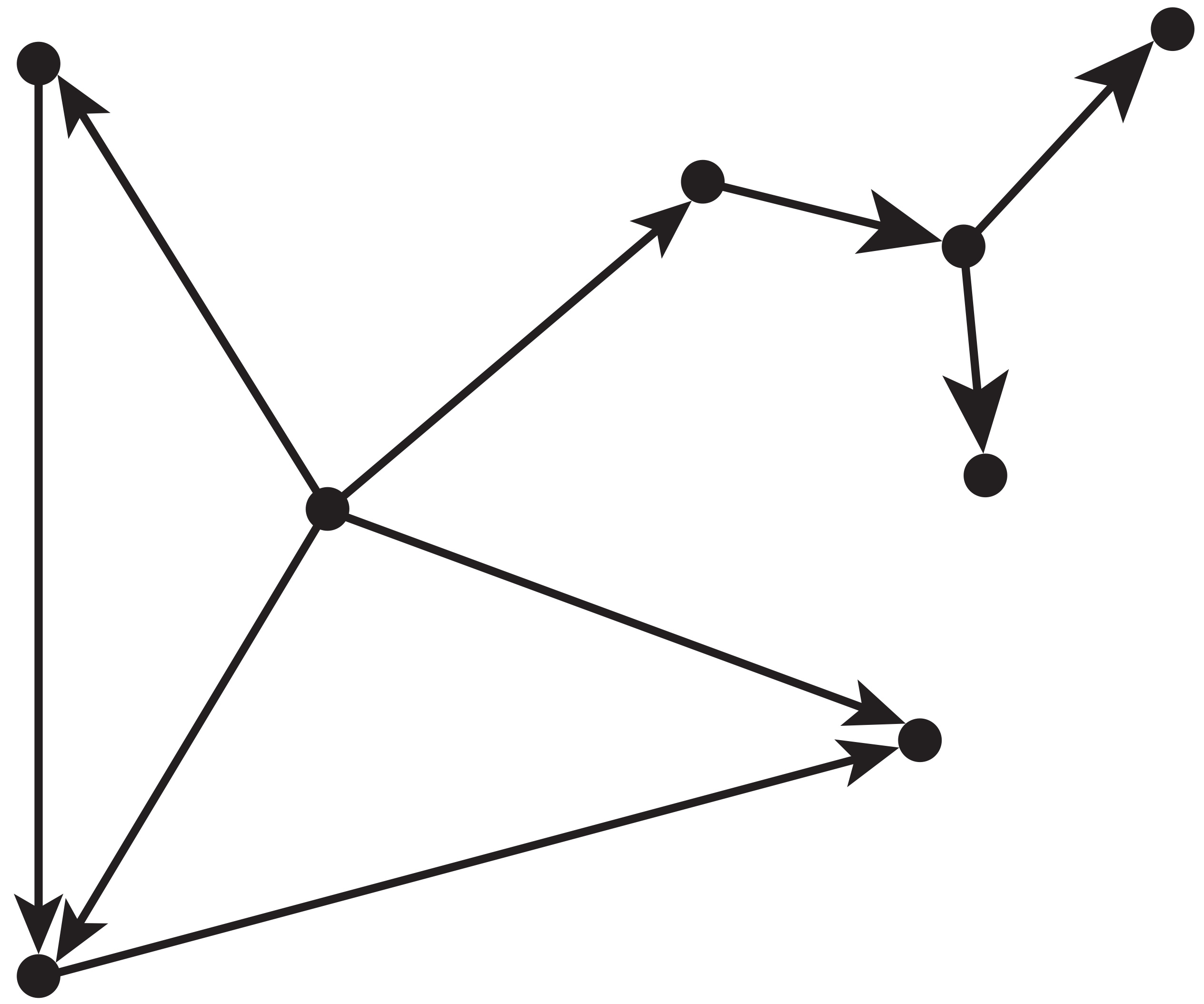






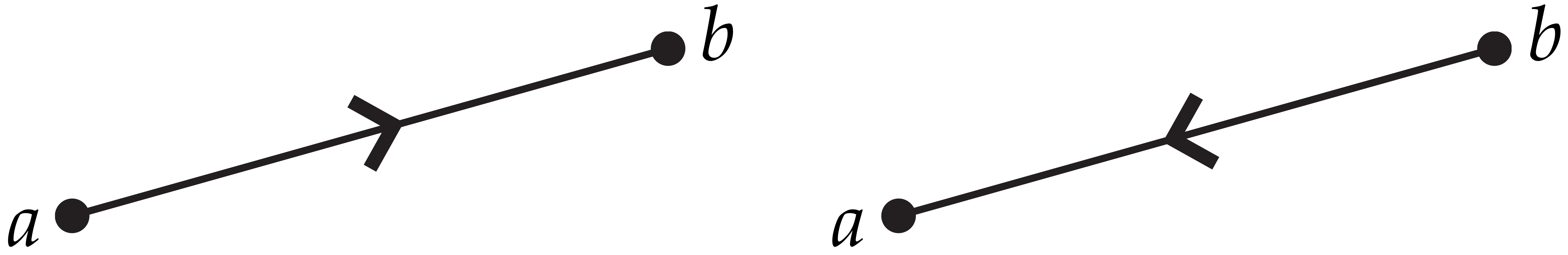






Orientation of a 1-Simplex

- Basic idea: does a 1-simplex $\{a,b\}$ go from a to b or from b to a?
- Instead of set $\{a,b\}$, now have *ordered tuple* (a,b) or (b,a)



- Why do we care? *Eventually* will have to do with integration…

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

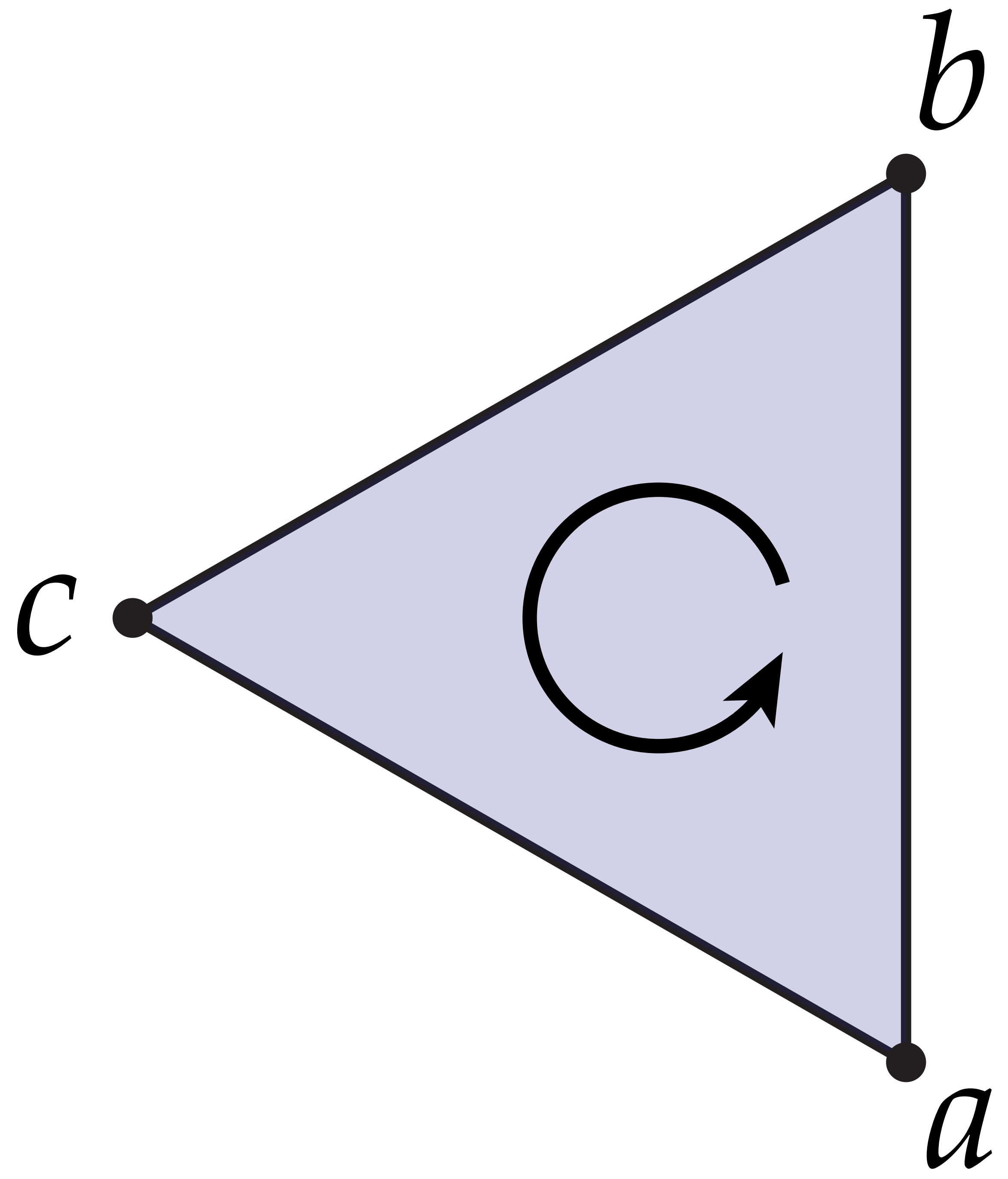
Orientation of a 2-Simplex

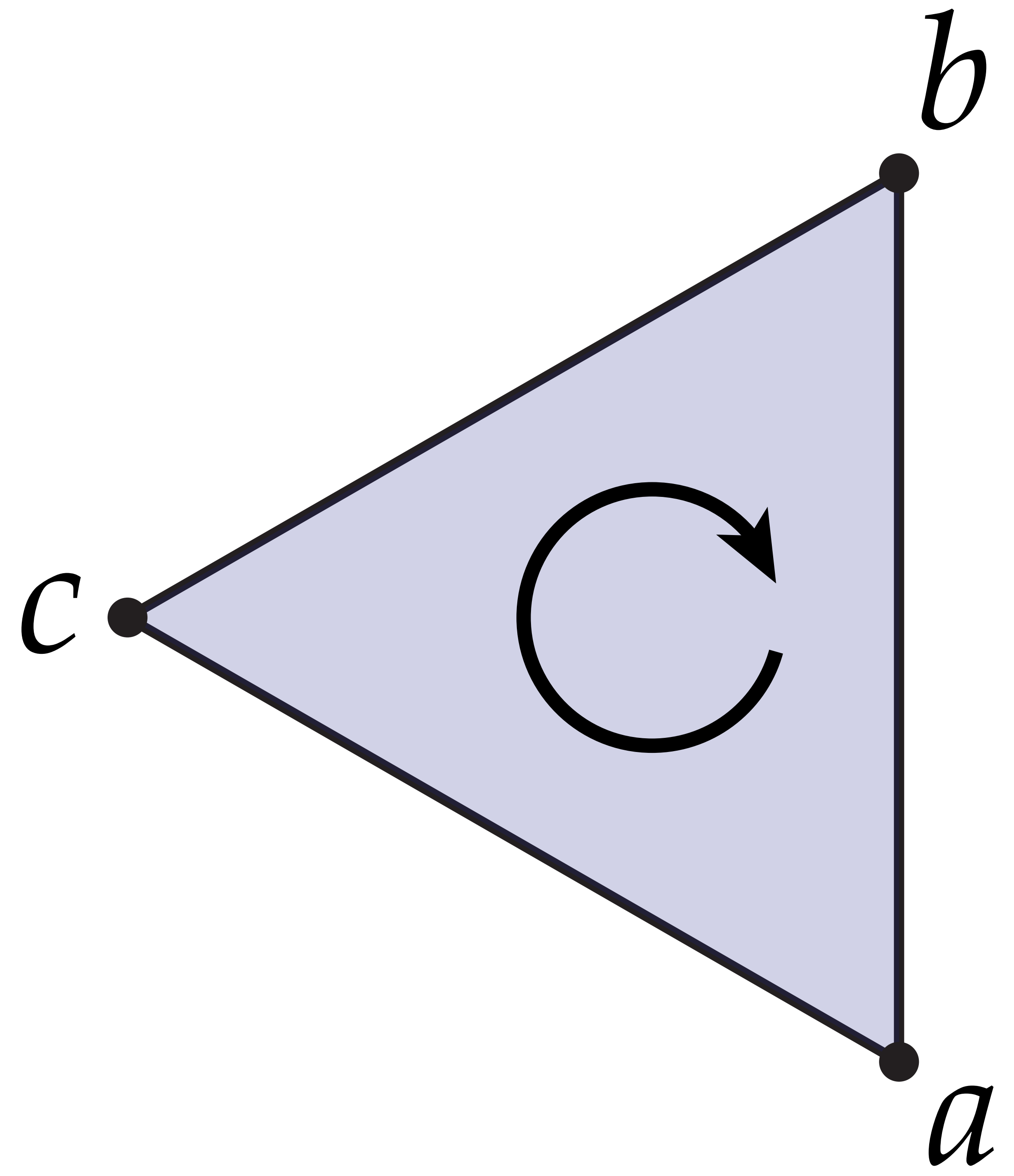
- For a 2-simplex, orientation given by "winding order" of vertices:

(a,b,c)

(b,c,a)

(c,a,b)





(a,c,b)

(c,b,a)

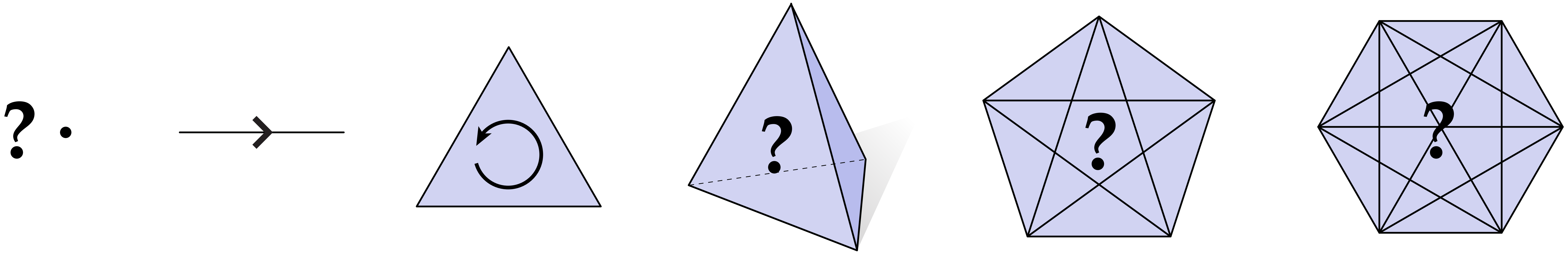
(b,a,c)

Q: How can we encode these *oriented 2-simplices?*

A: Oriented tuples, up to circular shift.

Oriented k-Simplex

How do we define orientation in general?



Similar idea to orientation for 2-simplex:

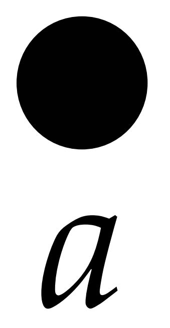
Definition. An oriented k-simplex is an ordered tuple, up to even permutation.

Hence, always* two orientations: *even* or *odd* permutations of vertices.

Call even permutations of (0,…,k) **"positive"**; otherwise **"negative."**

Oriented 0-Simplex?

What's the orientation of a single vertex?

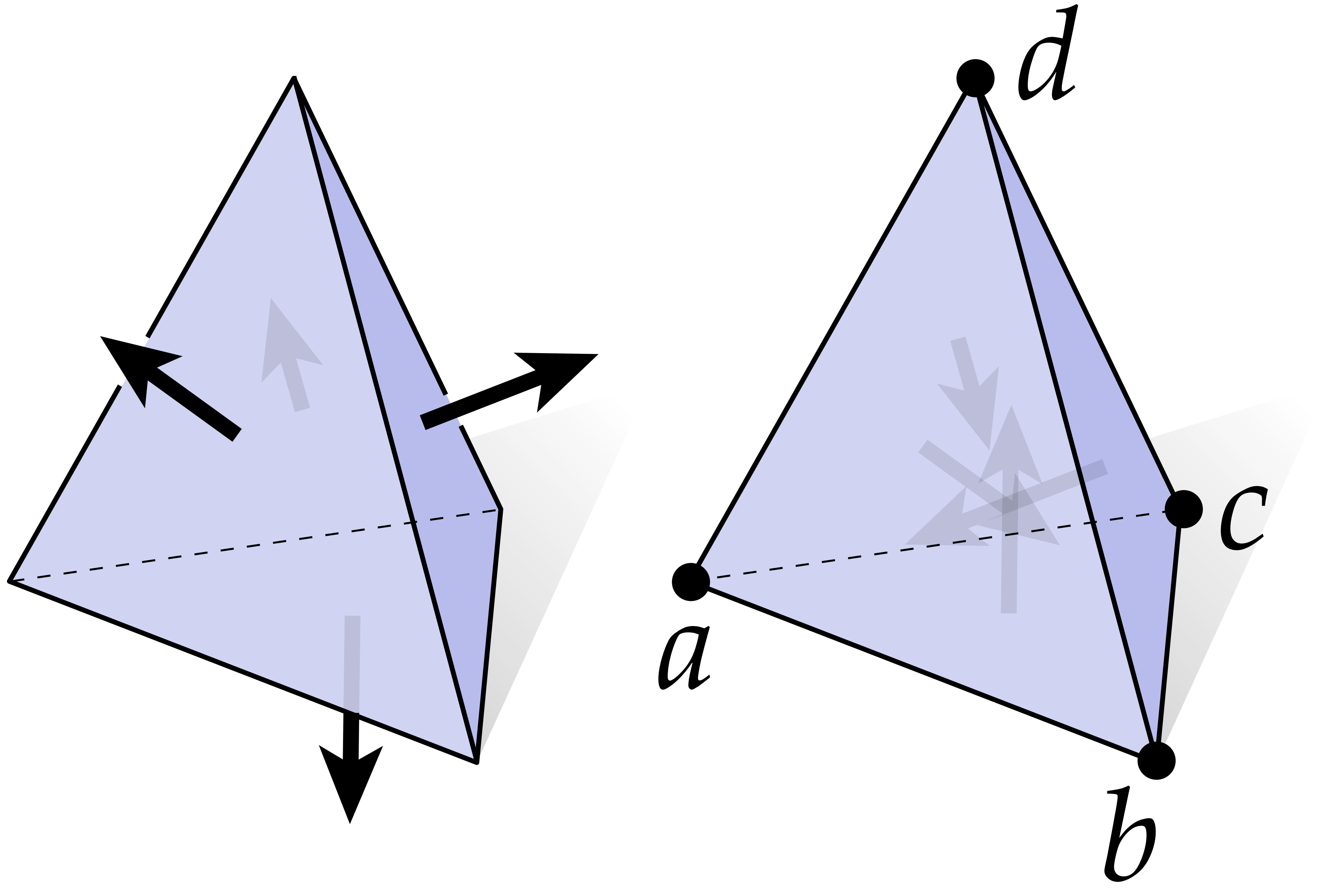


Only one permutation of vertices, so only one orientation! (Positive):

$$(a)$$

Oriented 3-Simplex

Hard to draw pictures as k gets large!



But still easy to apply definition:

even / positive		**odd / negative**	
$(1,2,3,4)$	$(3,1,2,4)$	$(1,2,4,3)$	$(3,1,4,2)$
$(1,3,4,2)$	$(3,2,4,1)$	$(1,3,2,4)$	$(3,2,1,4)$
$(1,4,2,3)$	$(3,4,1,2)$	$(1,4,3,2)$	$(3,4,2,1)$
$(2,1,4,3)$	$(4,1,3,2)$	$(2,1,3,4)$	$(4,1,2,3)$
$(2,3,1,4)$	$(4,2,1,3)$	$(2,3,4,1)$	$(4,2,3,1)$
$(2,4,3,1)$	$(4,3,2,1)$	$(2,4,1,3)$	$(4,3,1,2)$

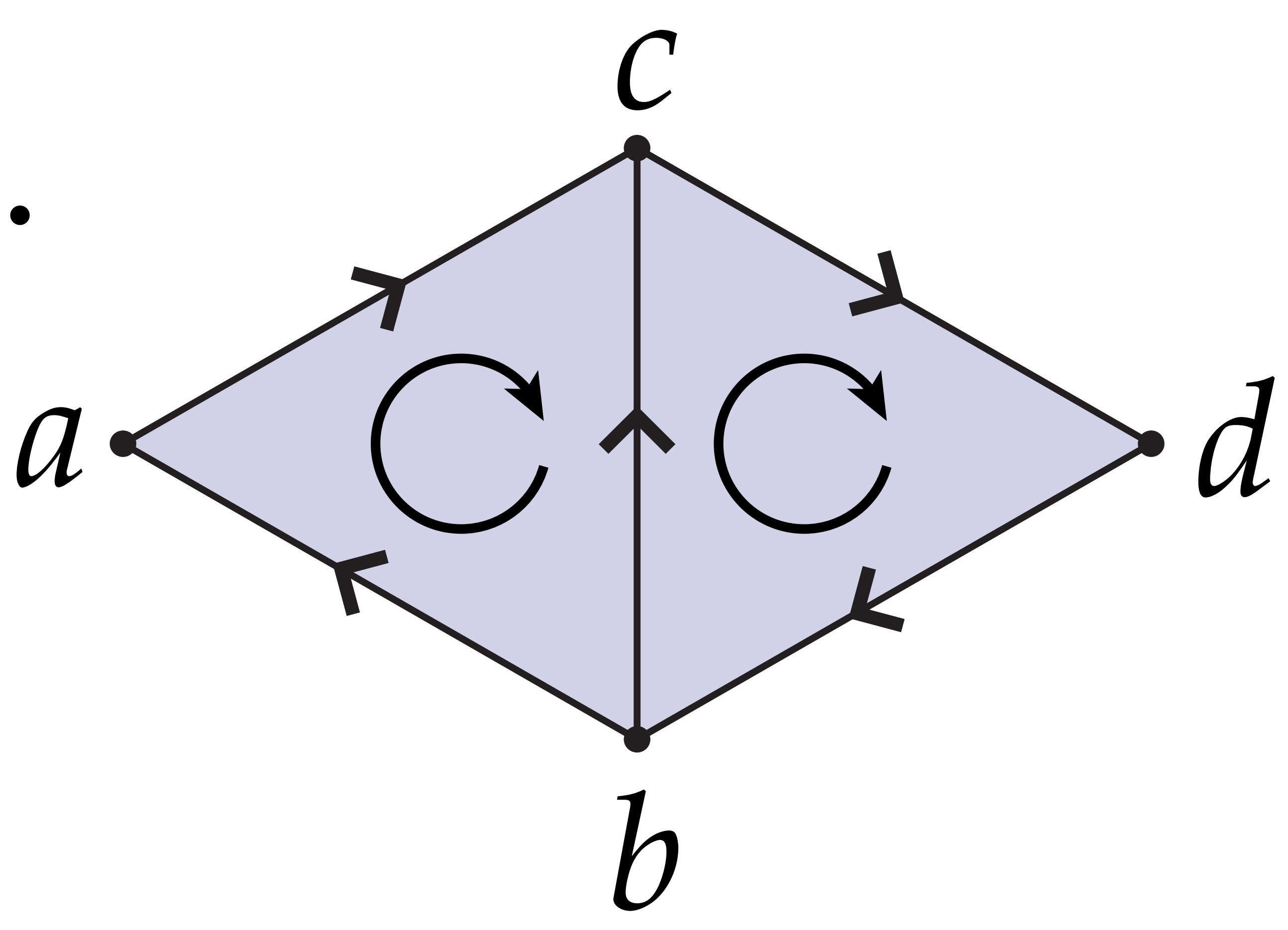
…much easier, of course, to just pick a single representative.

E.g., $+\sigma := (1,2,3,4)$, and $-\sigma := (1,2,4,3)$.

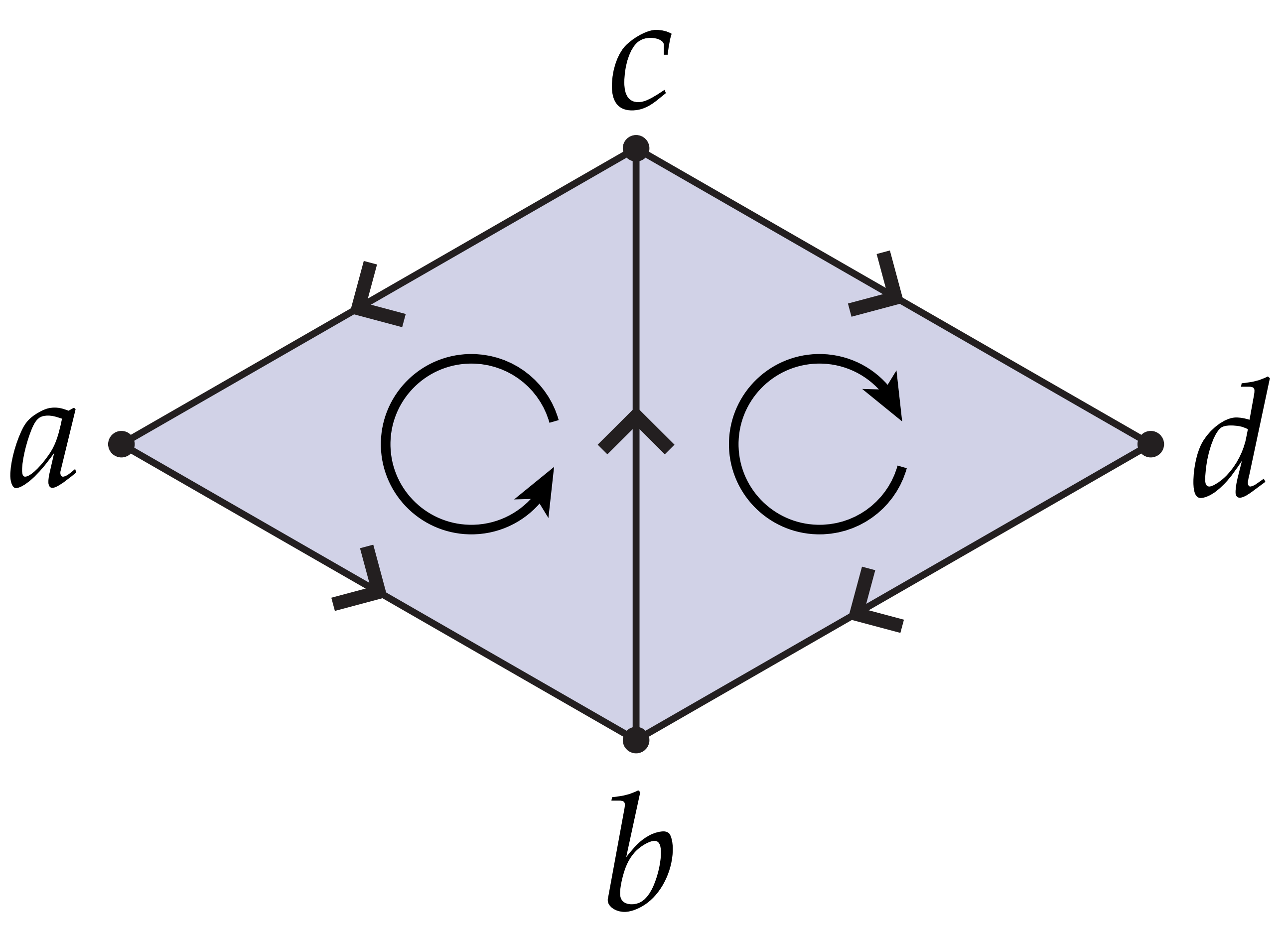
Oriented Simplicial Complex

Definition. An *orientation* of a simplex is an ordering of its vertices up to even permutation; one can specify an oriented simplex via one of its representative ordered tuples. An *oriented simplicial complex* is a simplicial complex where each simplex is given an ordering.

Example.



$$\{\emptyset, (a), (b), (c), (d), (a,c), (b,a), (b,c), (c,d), (d,b), (a,c,b), (b,c,d)\}$$

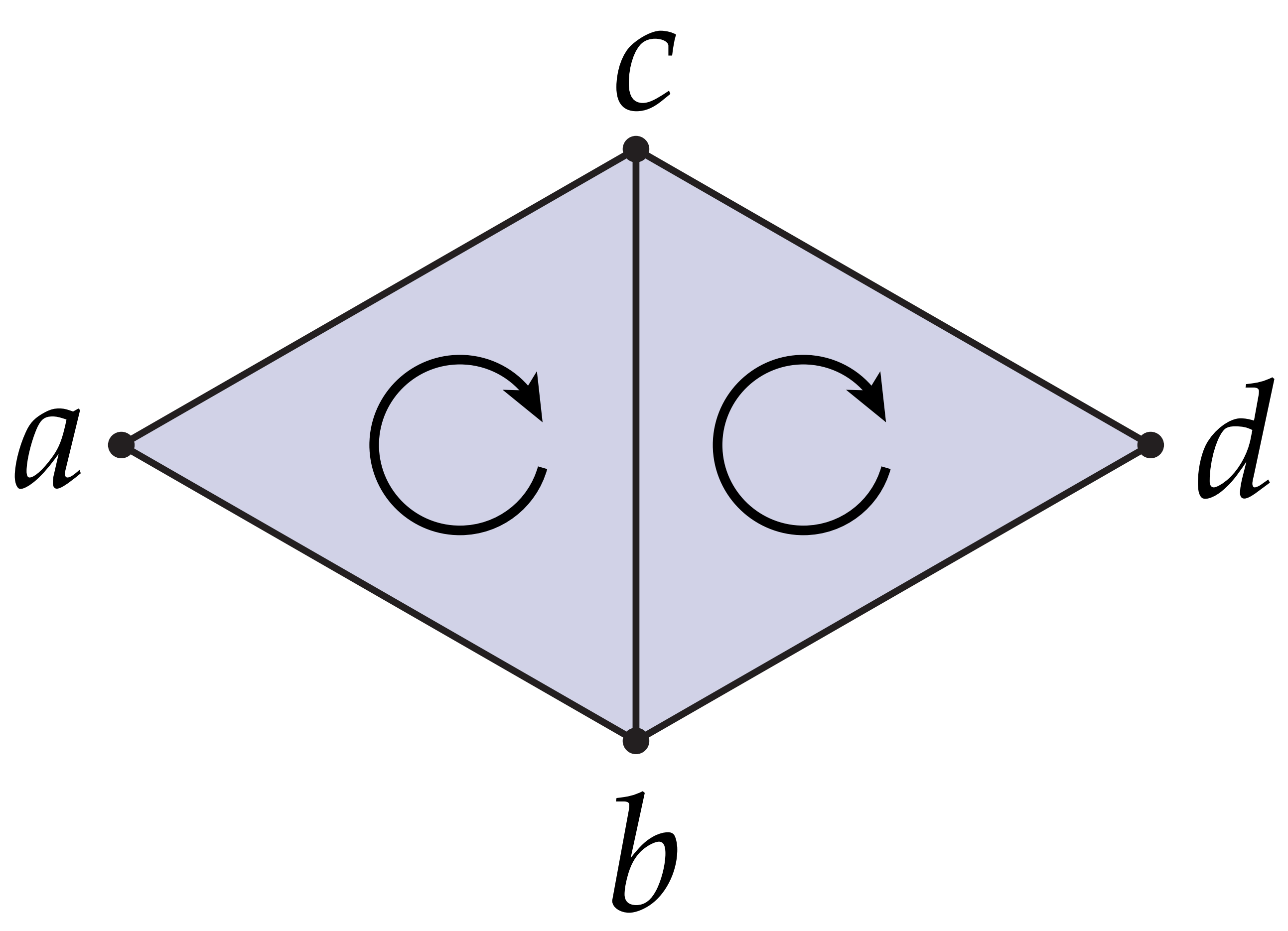


$$\{\emptyset, (a), (b), (c), (d), (c,a), (a,b), (b,c), (c,d), (d,b), (a,b,c), (b,c,d)\}$$

Relative Orientation

Definition. Two distinct oriented simplices have the same *relative orientation* if the two (maximal) faces in their intersection have **opposite** orientation.

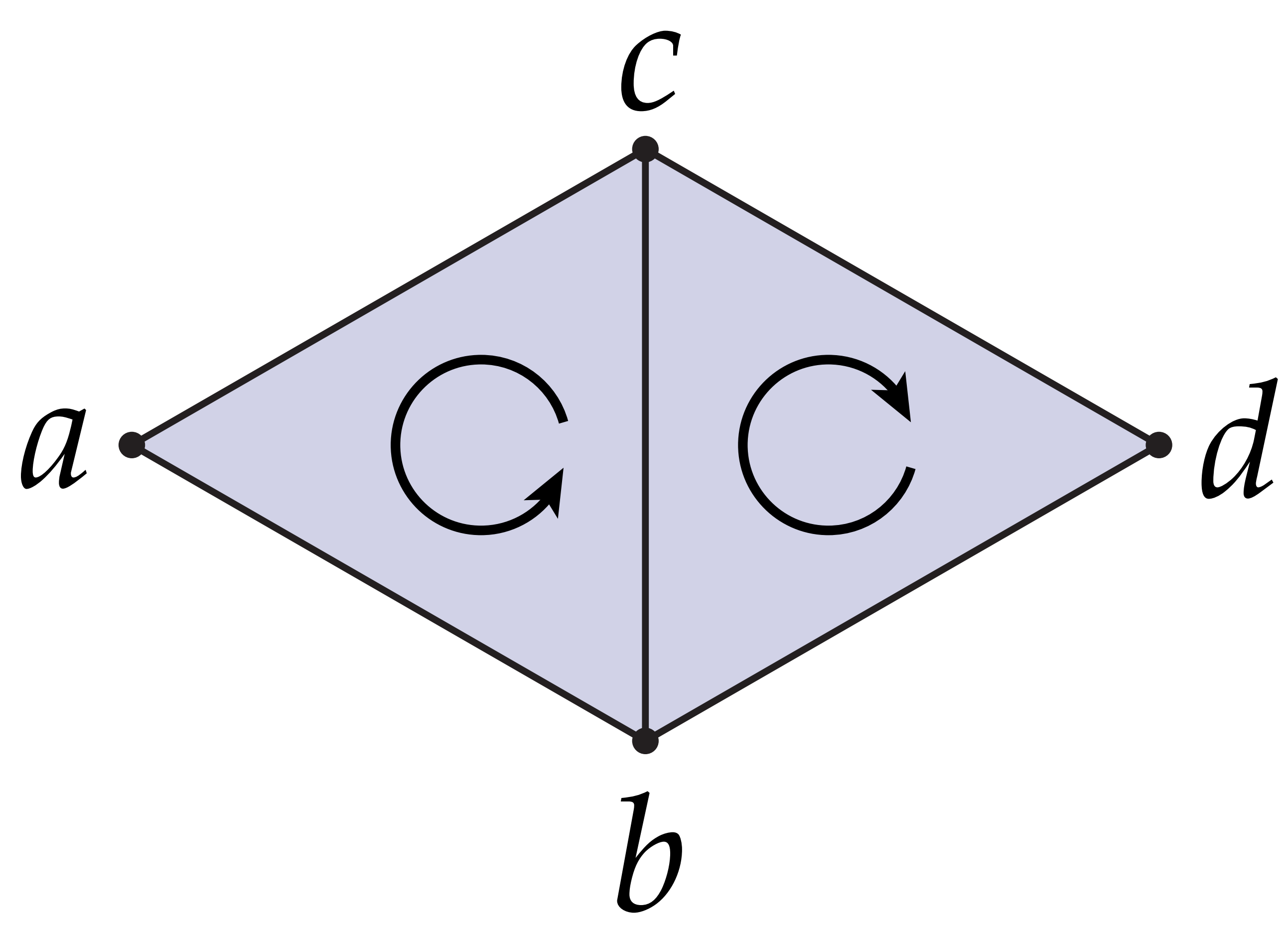
<u>**Example:**</u> Consider two triangles that intersect along an edge:



same relative orientation

$$(a,c,b) \quad \Rightarrow \quad (c,b)$$
$$(b,c,d) \quad \Rightarrow \quad (b,c)$$

$$(c,b) = -(b,c)$$

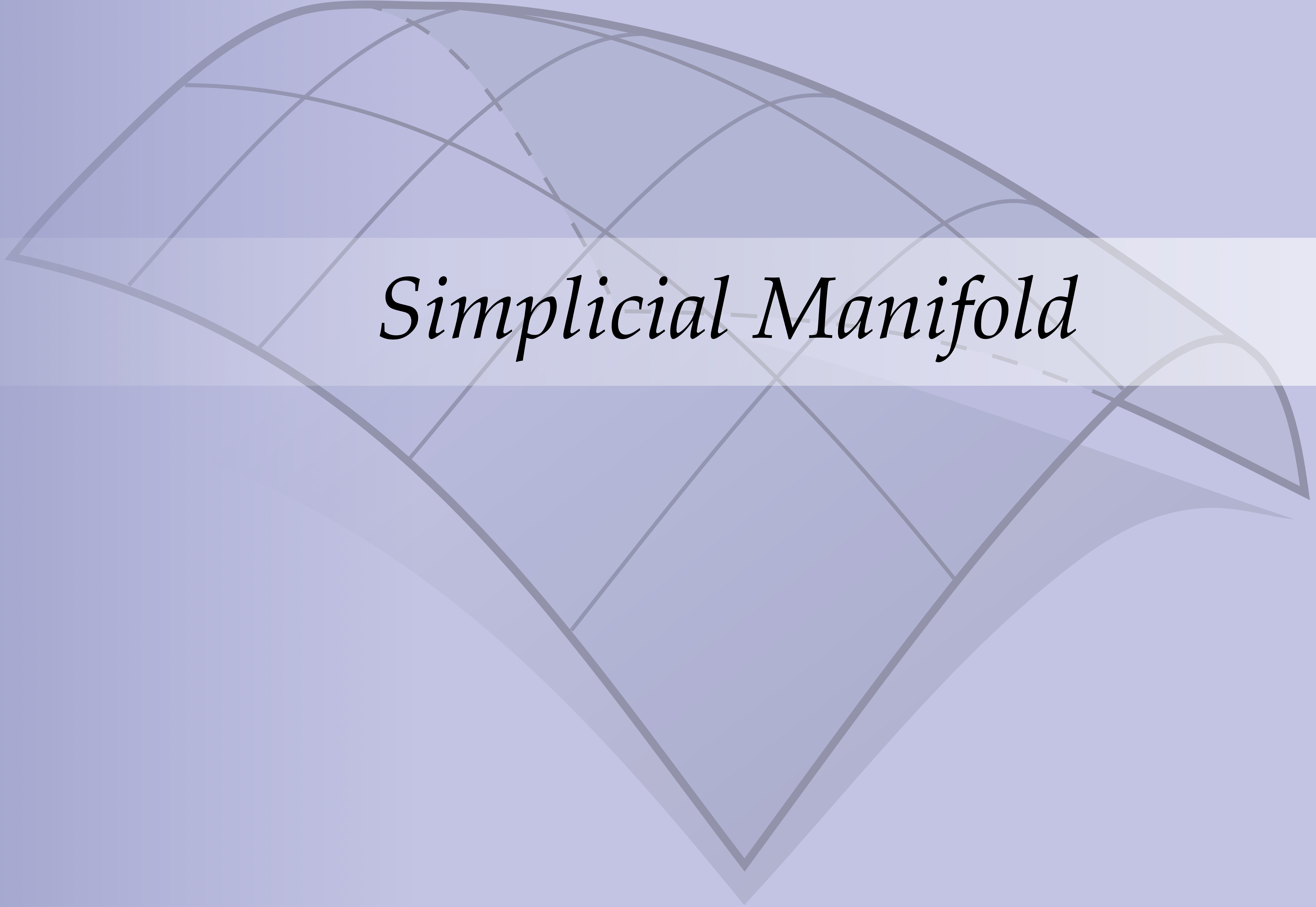


different relative orientation

$$(a,b,c) \quad \Rightarrow \quad (b,c)$$
$$(b,c,d) \quad \Rightarrow \quad (b,c)$$

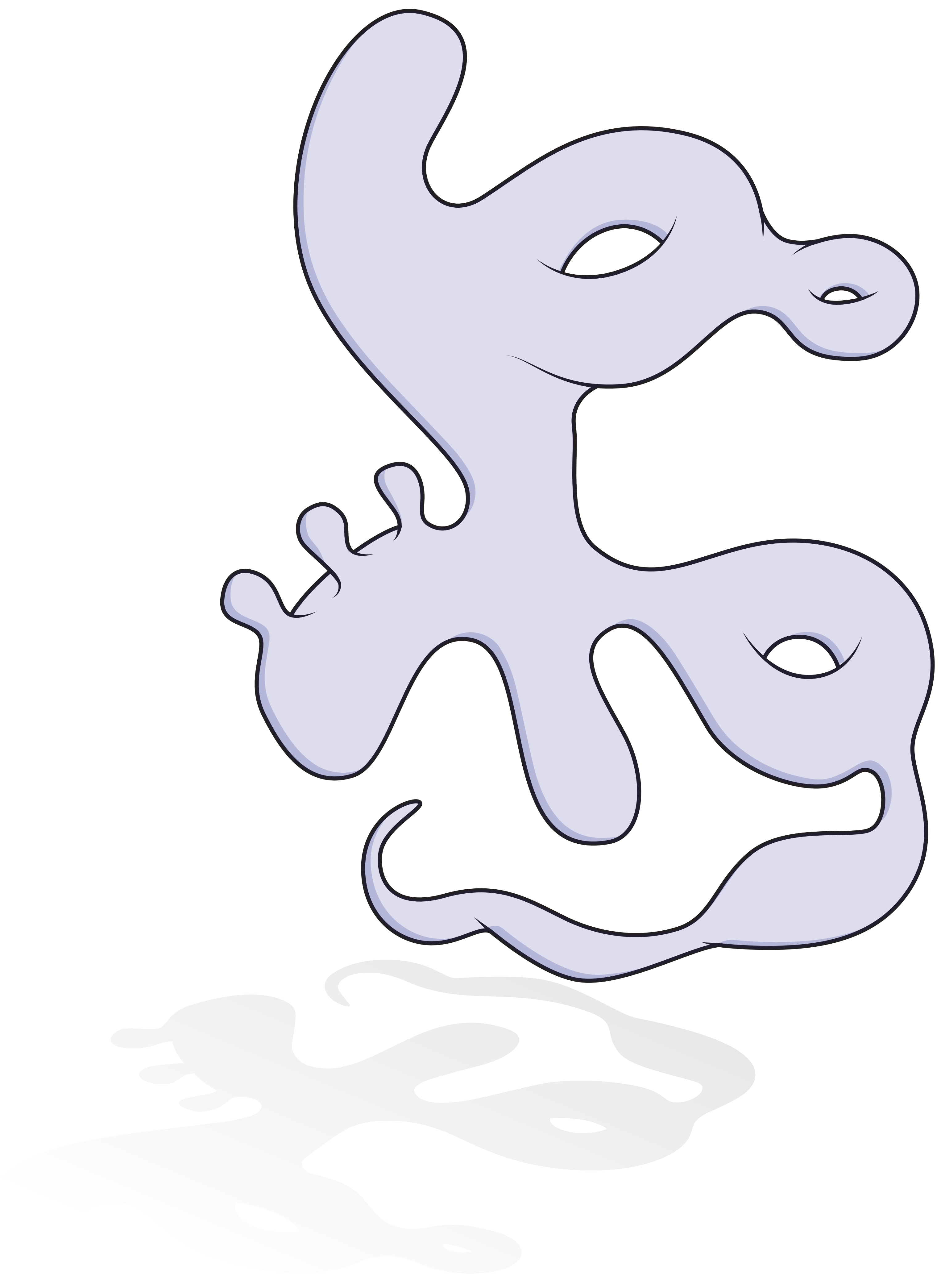
$$(b,c) = +(b,c)$$

Simplicial Manifold

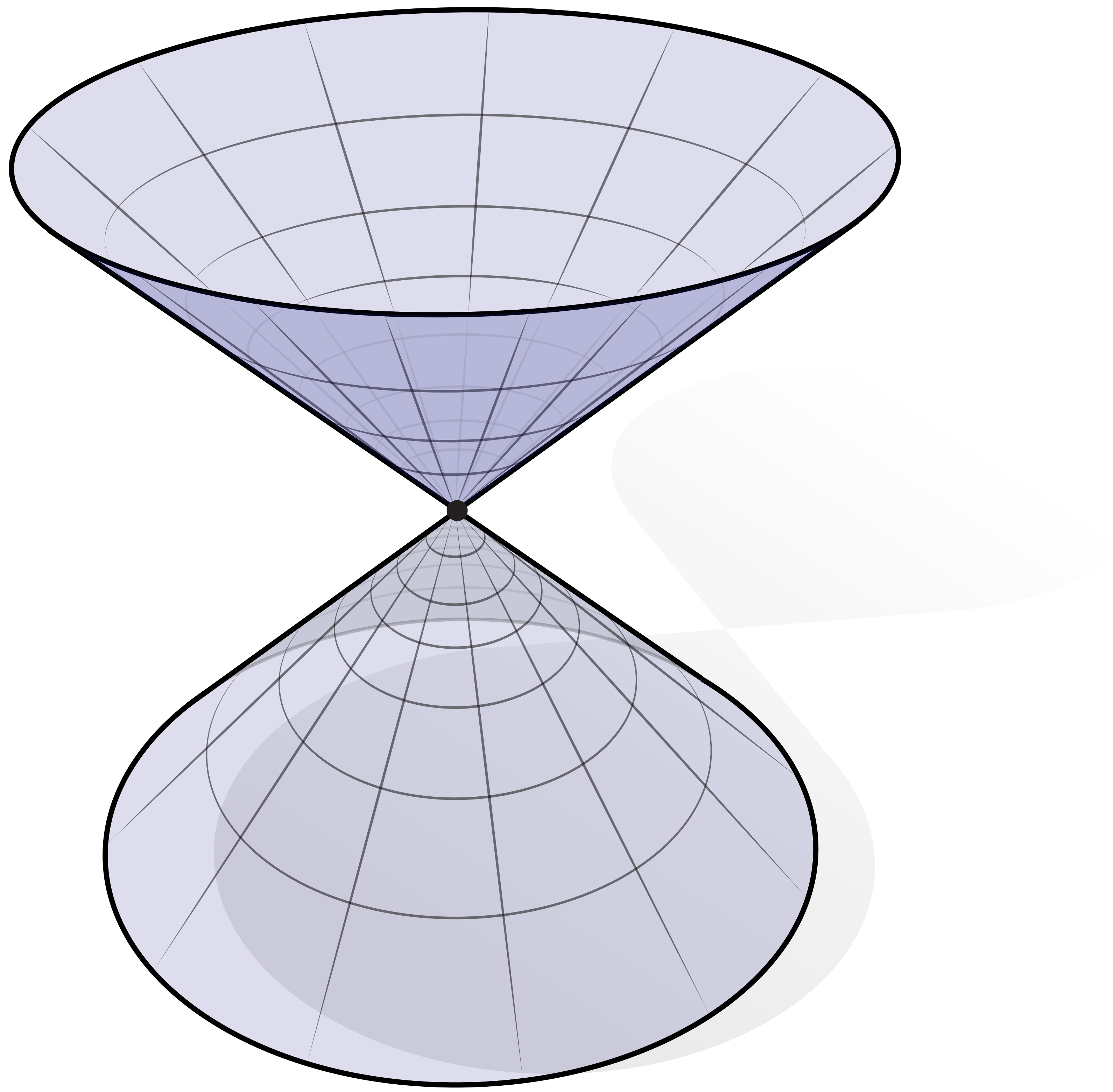


Manifold—First Glimpse

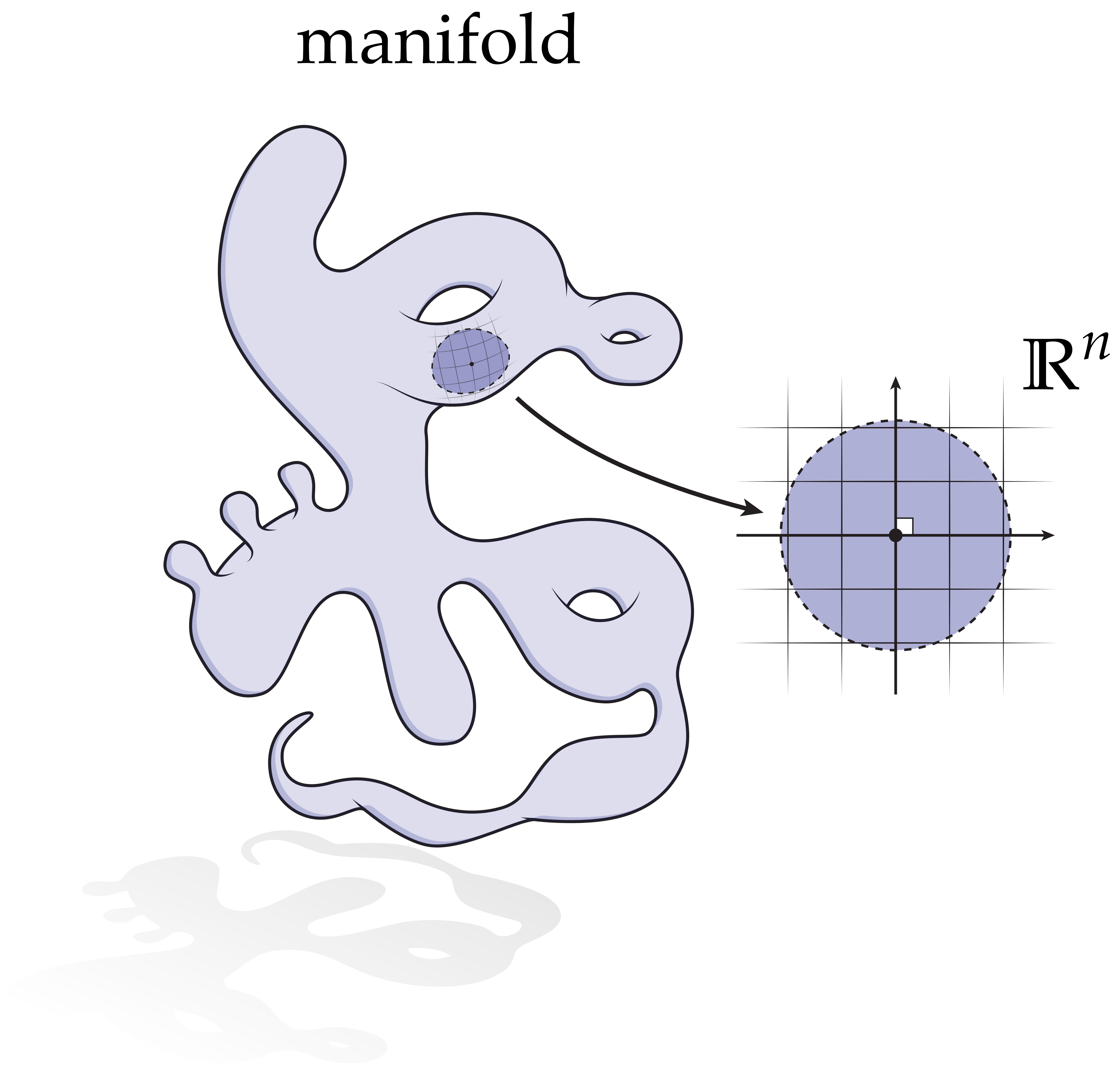
manifold

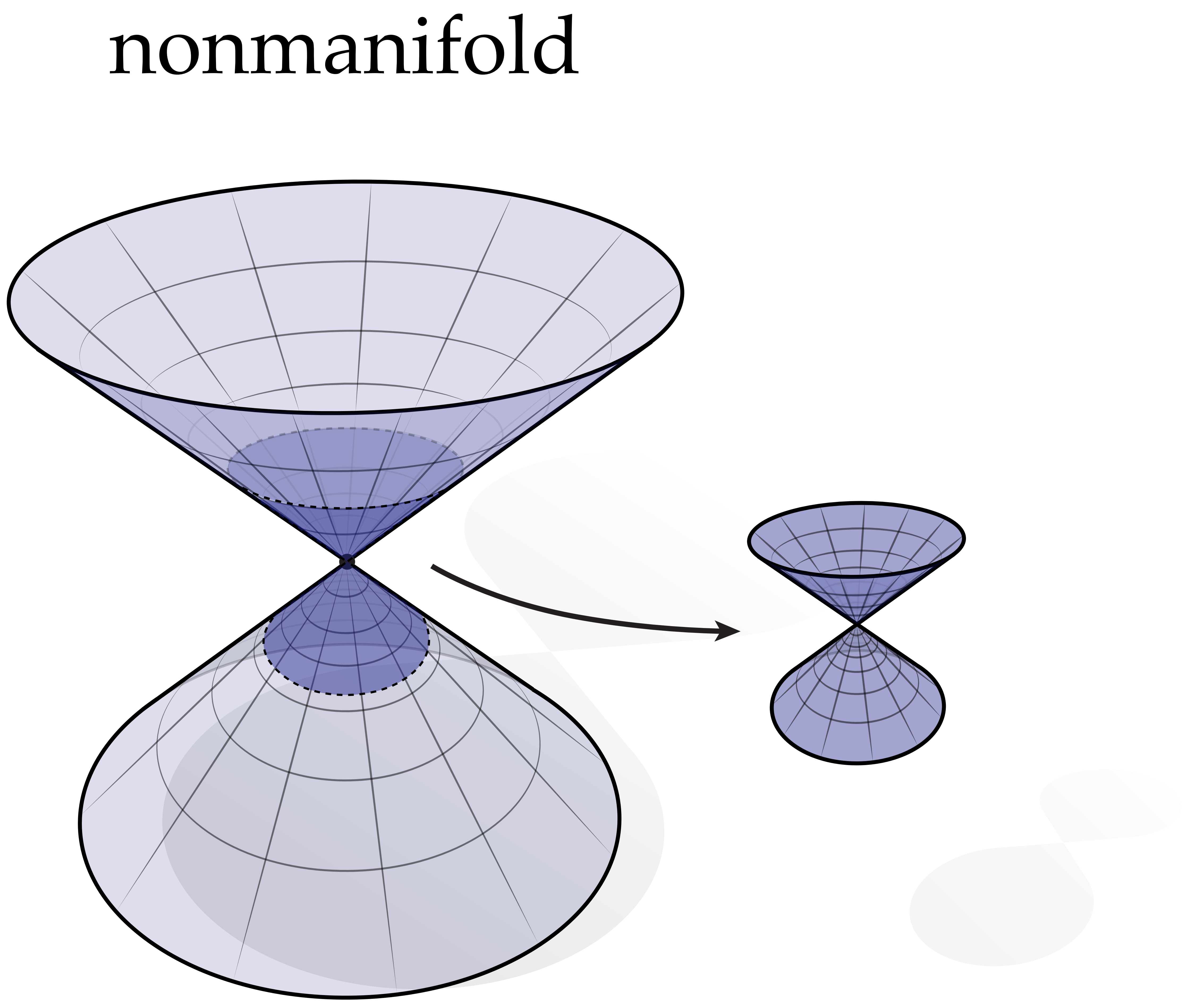


nonmanifold



Manifold—First Glimpse

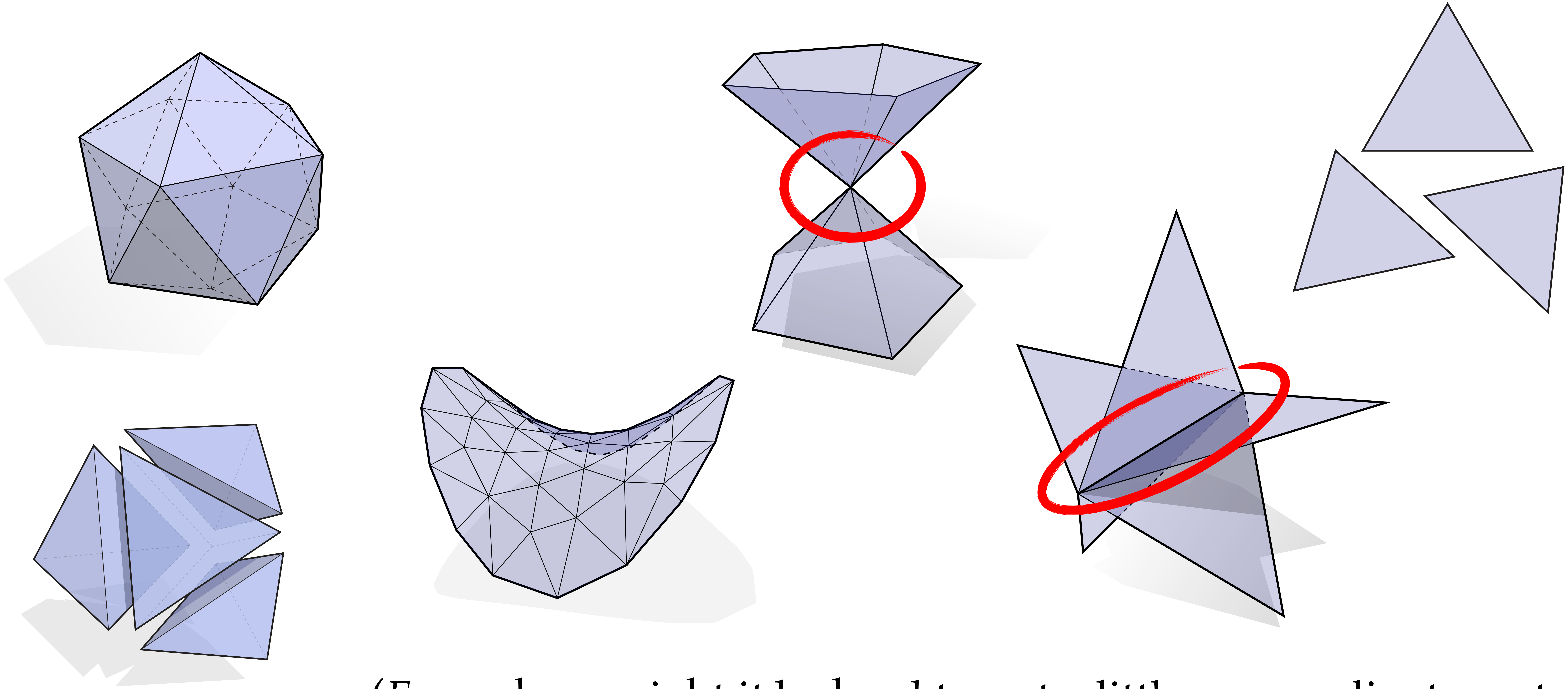




Key idea: "looks like R^n up close"

Simplicial Manifold—Visualized

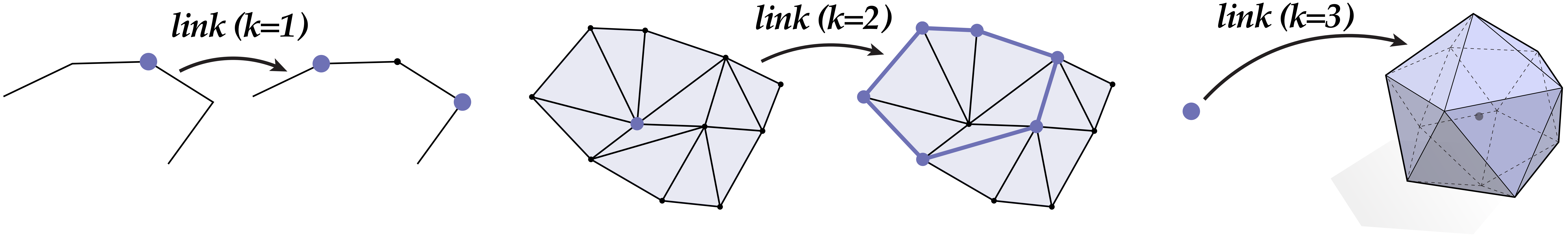
Which of these simplicial complexes look *"manifold?"*



(*E.g.*, where might it be hard to put a little *xy*-coordinate system?)

Simplicial Manifold—Definition

Definition. A simplicial k-complex is *manifold* if the **link** of every vertex looks like* an $(k-1)$-dimensional sphere.



__Aside:__ How hard is it to check if a given simplicial complex is manifold?

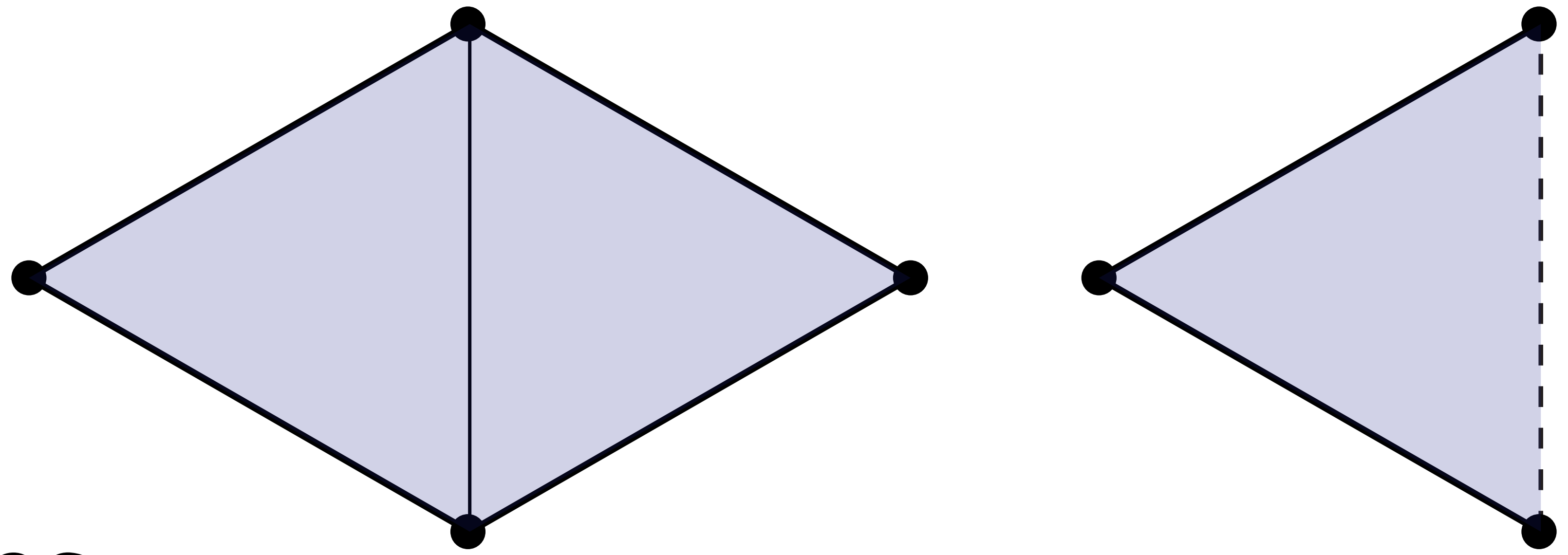
- $(k{=}1)$ *trivial*—is it a loop?
- $(k{=}2)$ *trivial*—is each link a loop?
- $(k{=}3)$ is each link a 2-sphere? Just check if $V\text{-}E{+}F = 2$ (Euler's formula)
- $(k{=}4)$ is each link a 3-sphere? …Well, it's known to be in NP! [S. Schleimer 2004]

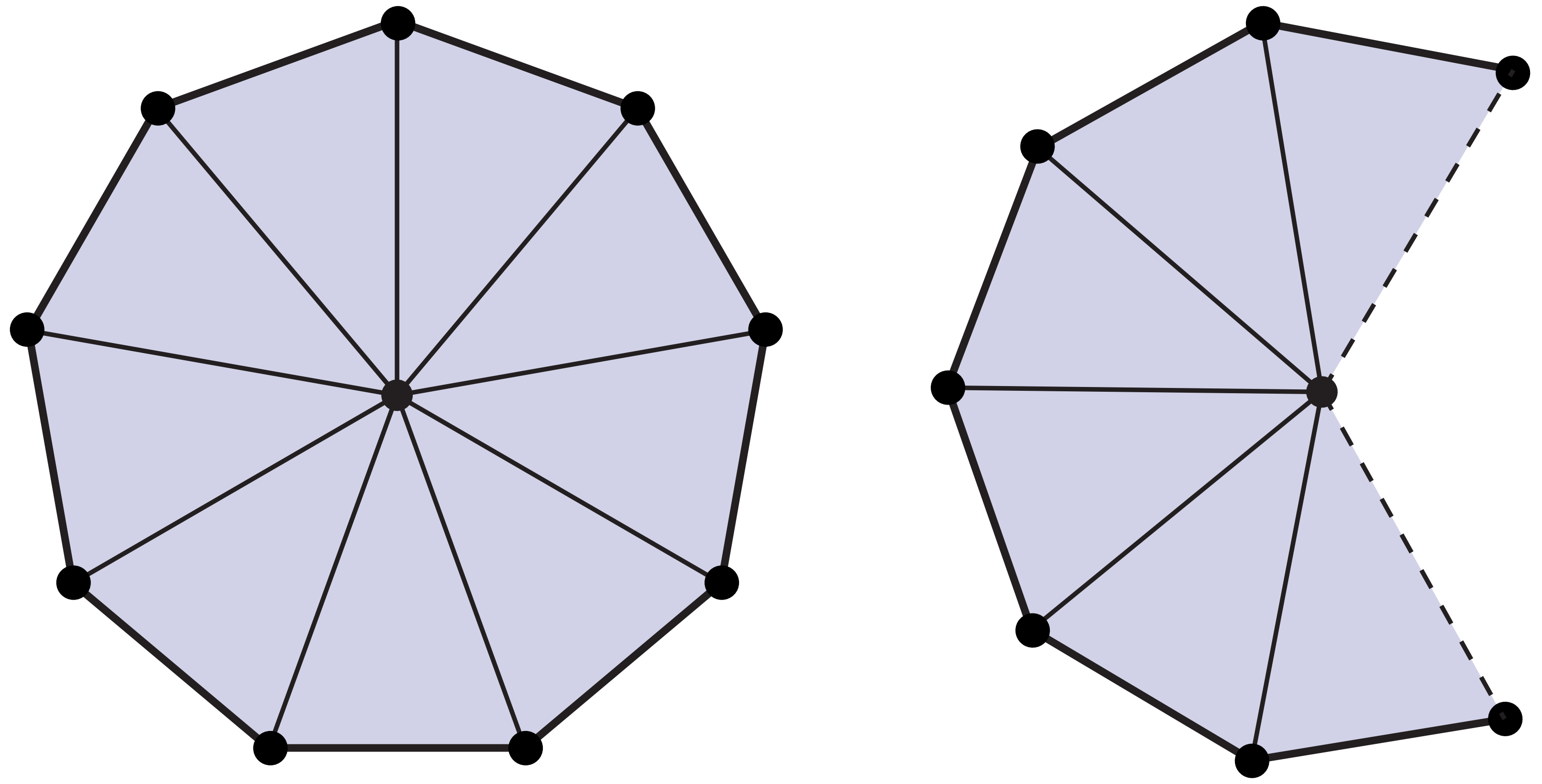
*I.e., is *homeomorphic* to.

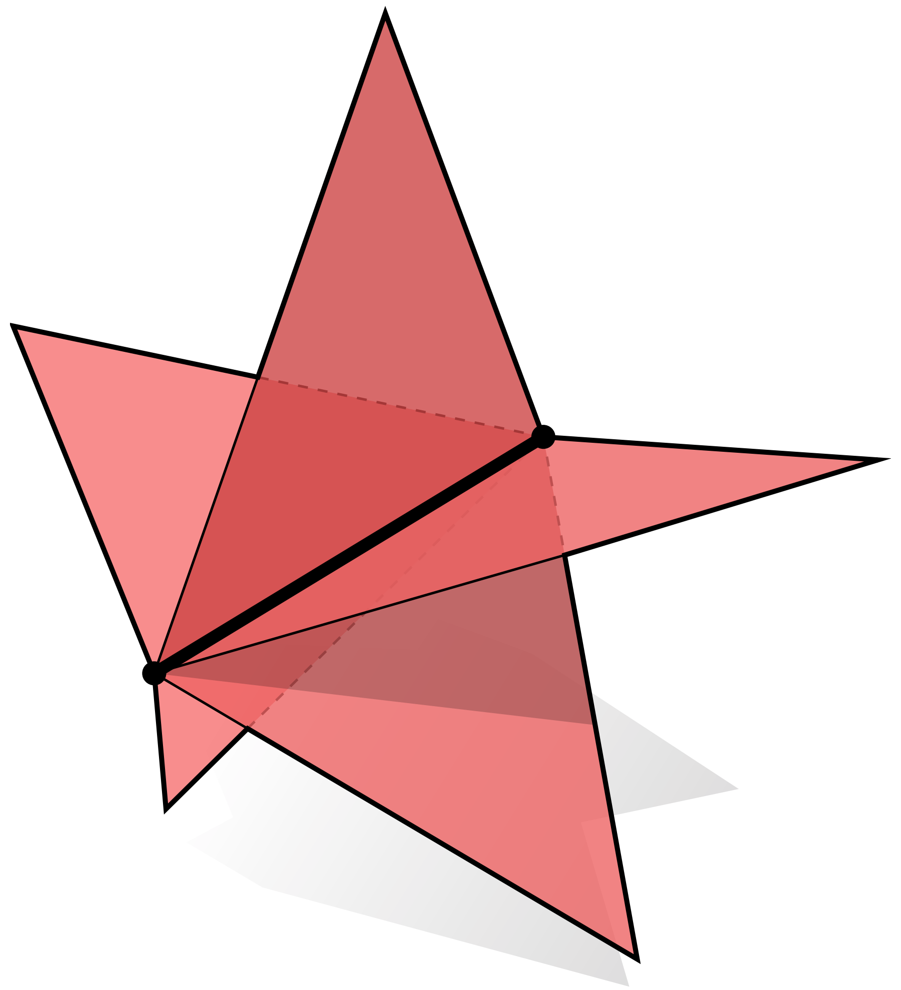
Manifold Triangle Mesh

<u>Key example:</u> For a triangle mesh (k=2):

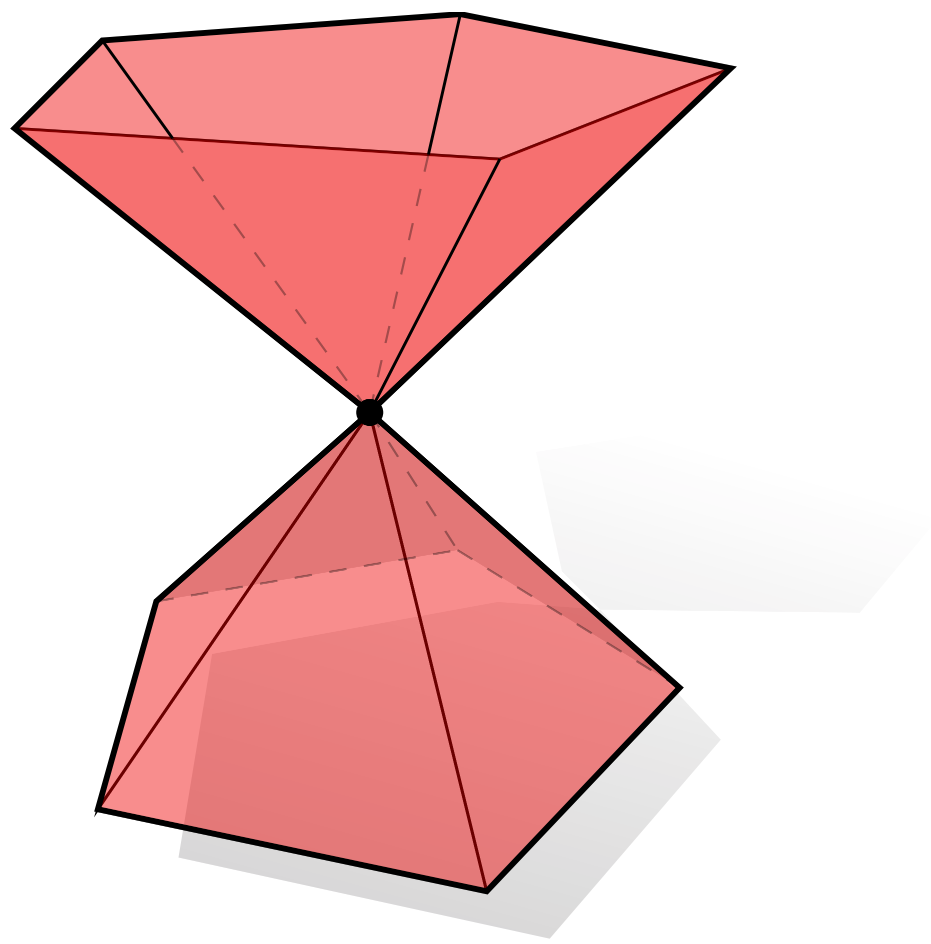
- every edge is contained in exactly two triangles
  - …or just one along the boundary
- every vertex is contained in a single "loop" of triangles
  - …or a single "fan" along the boundary





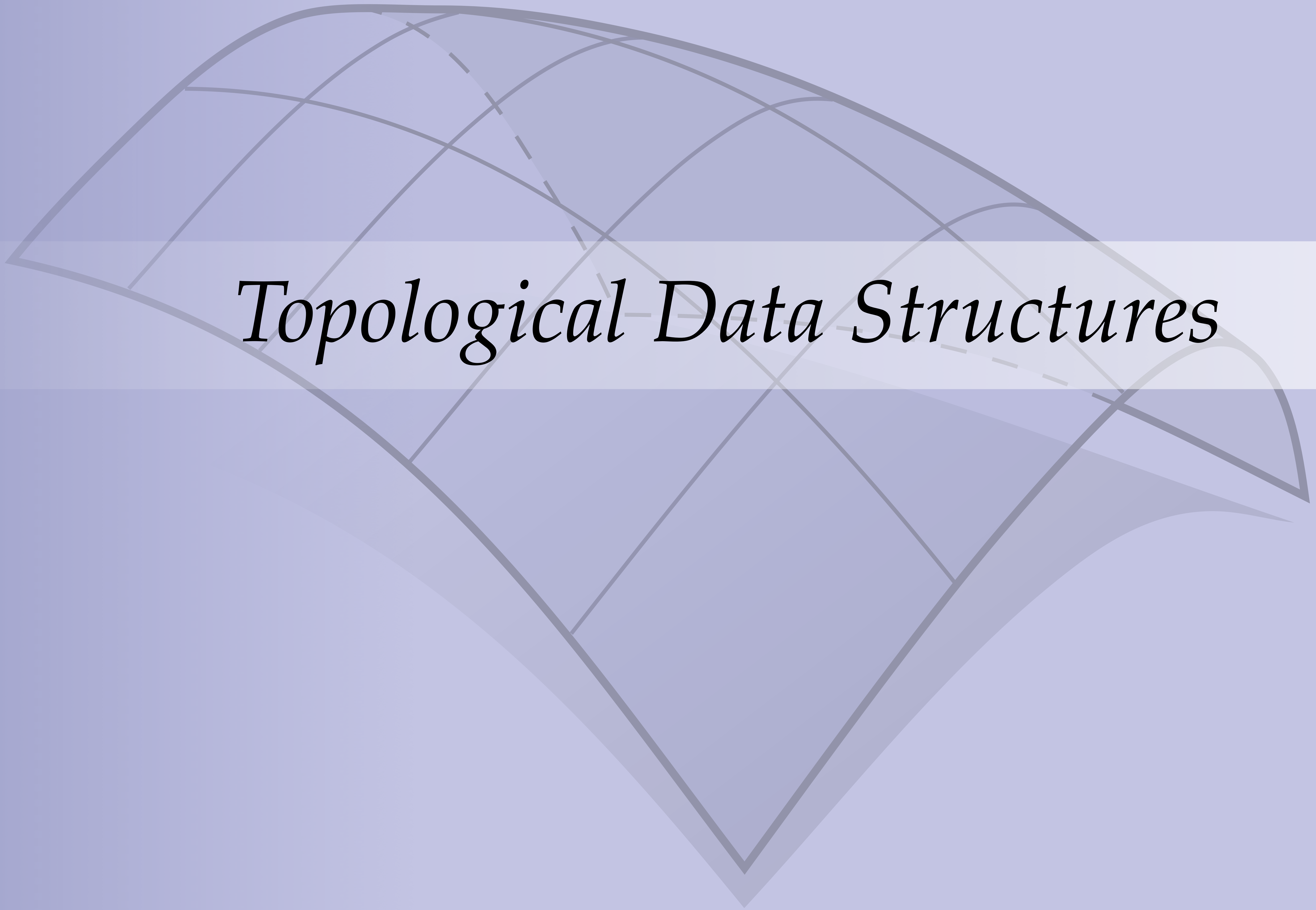


nonmanifold edge



nonmanifold vertex

Why? One reason: `data structures`…



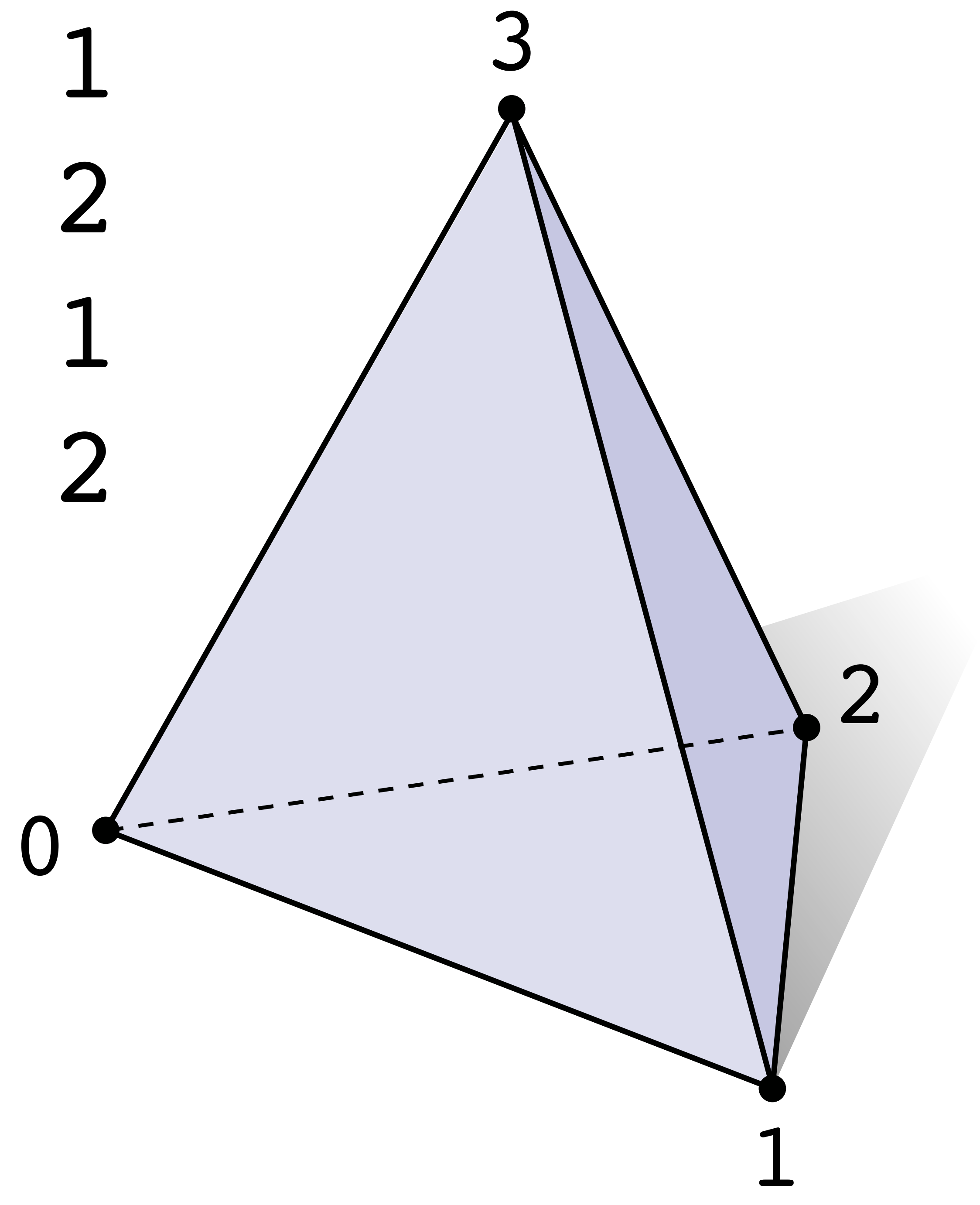
Topological Data Structures

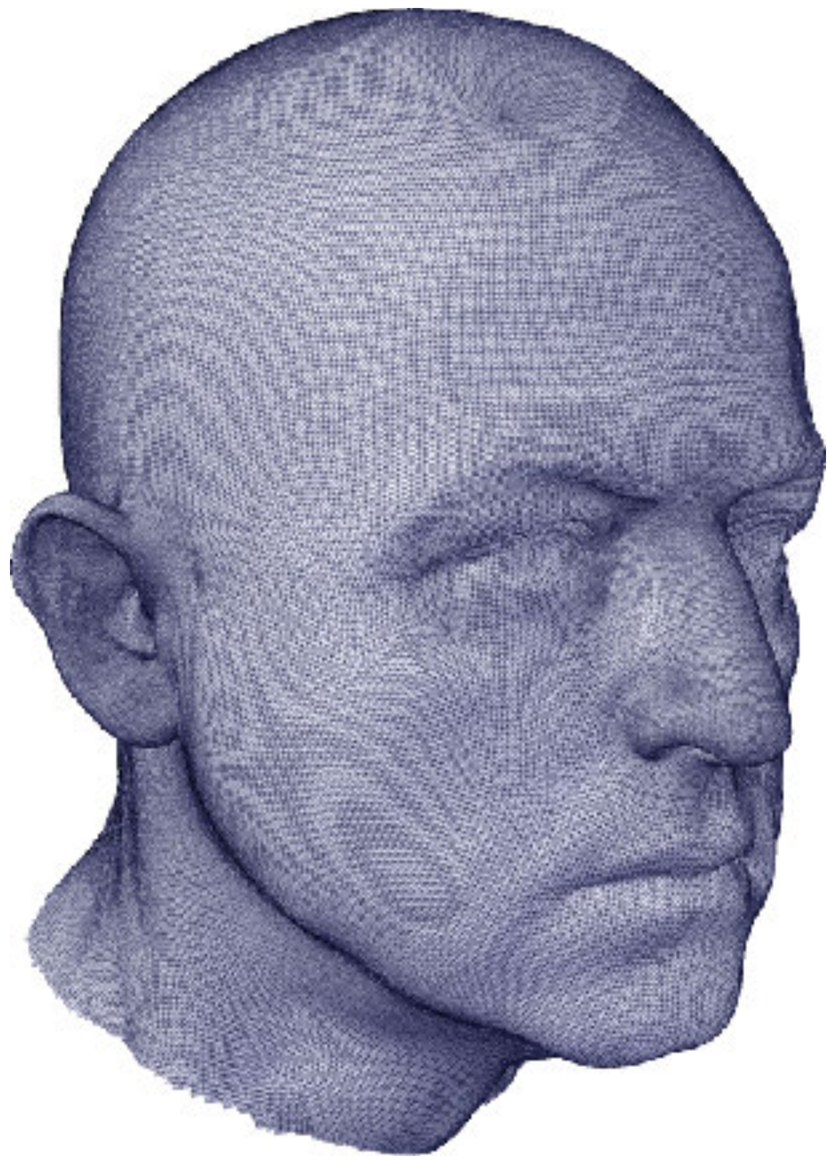
Topological Data Structures — Adjacency List

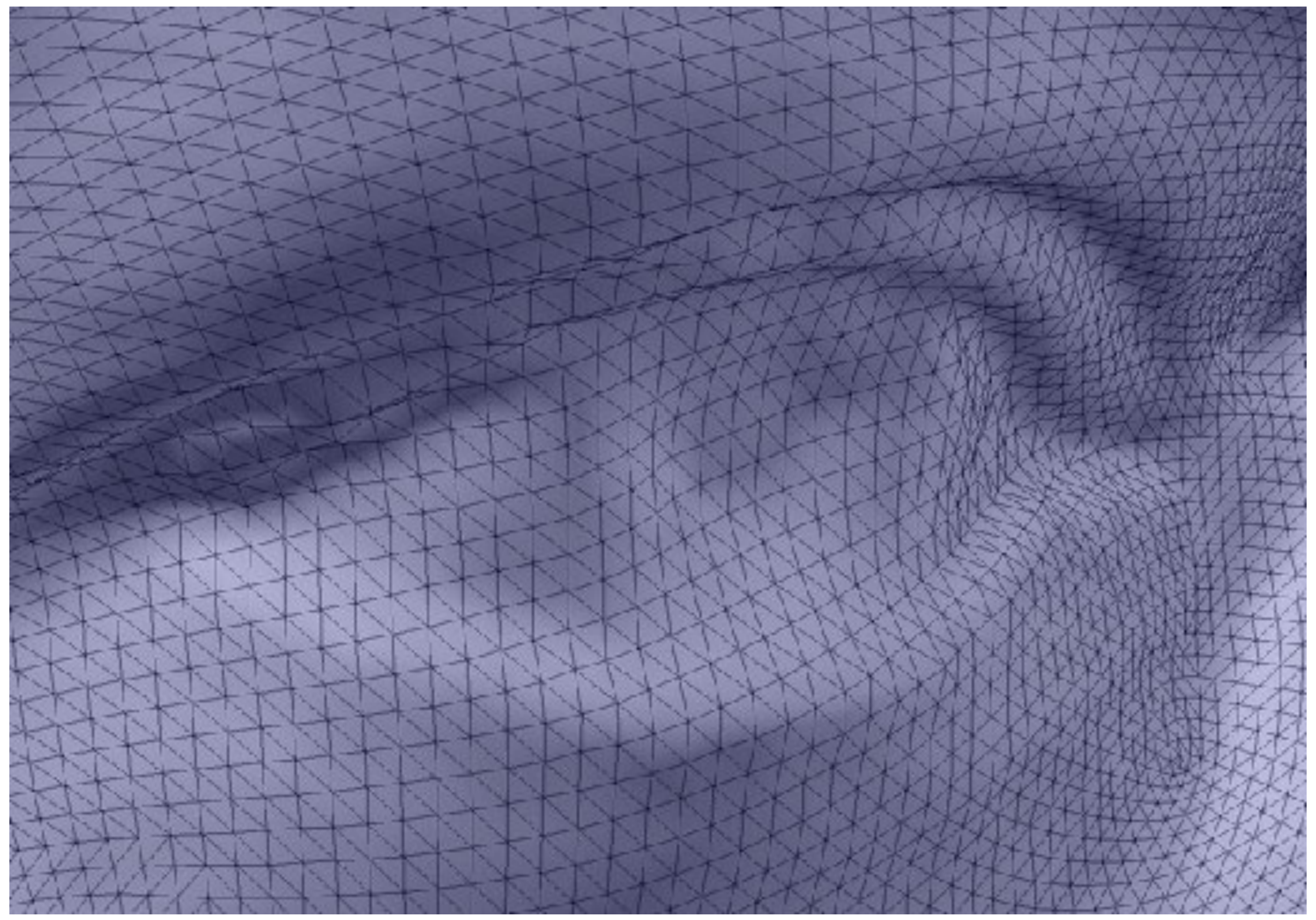
- Store only top-dimensional simplices
- Implicitly includes all facets
- Pros: simple, small storage cost
- Cons: hard to access neighbors

Example.

0	2	1
0	3	2
3	0	1
3	1	2







Q: How might you list all edges touching a given vertex? *What's the cost?*

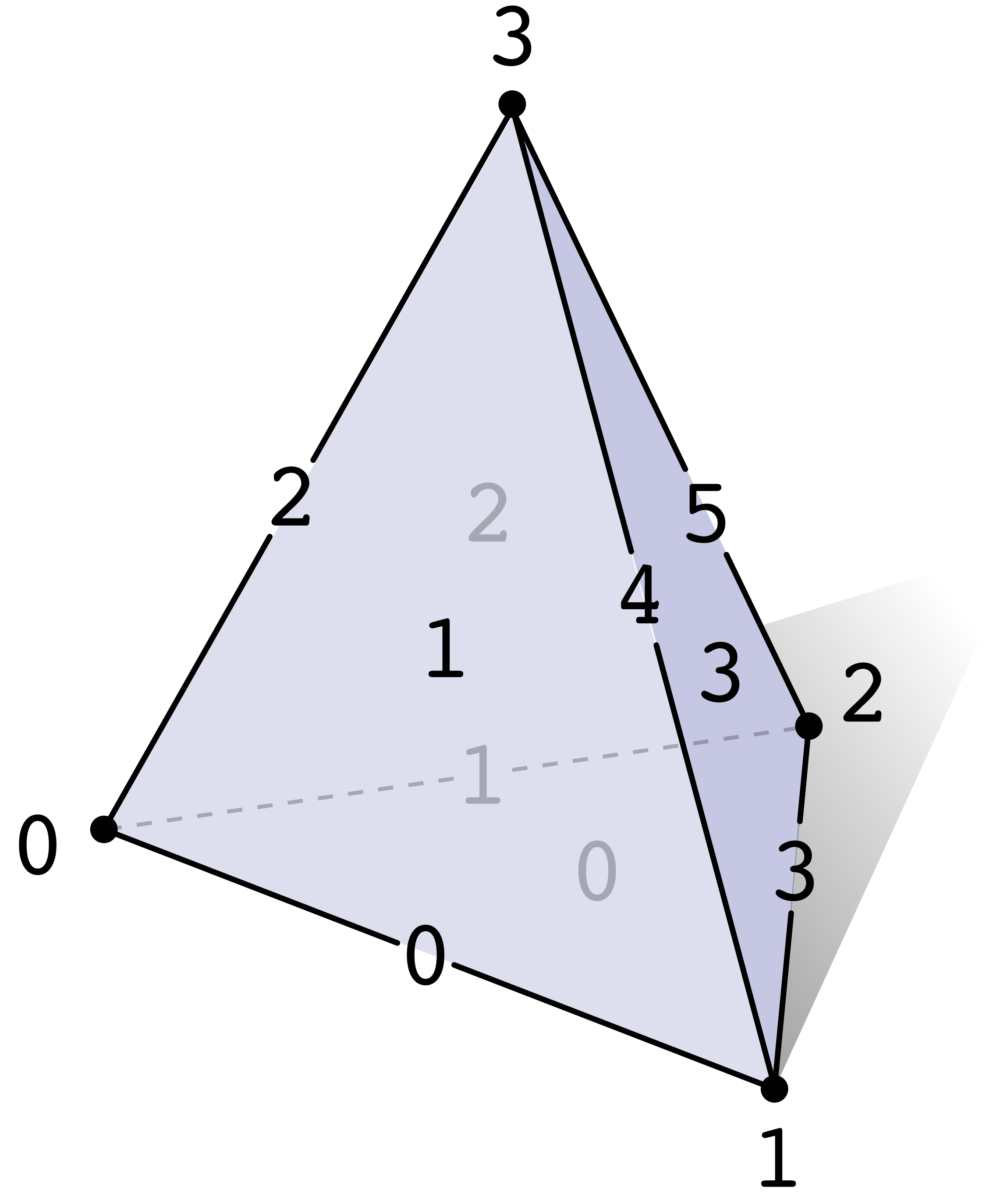
Topological Data Structures—Incidence Matrix

Definition. Let K be a simplicial complex, let n_k denote the number of k-simplices in K, and suppose that for each k we give the k-simplices a canonical ordering so that they can be specified via indices $1, \ldots, n_k$. The kth *incidence matrix* is then a $n_{k+1} \times n_k$ matrix E^k with entries $E^k_{ij} = 1$ if the jth k-simplex is contained in the ith $(k+1)$-simplex, and $E^k_{ij} = 0$ otherwise.

Example.

$$E^0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$E^1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

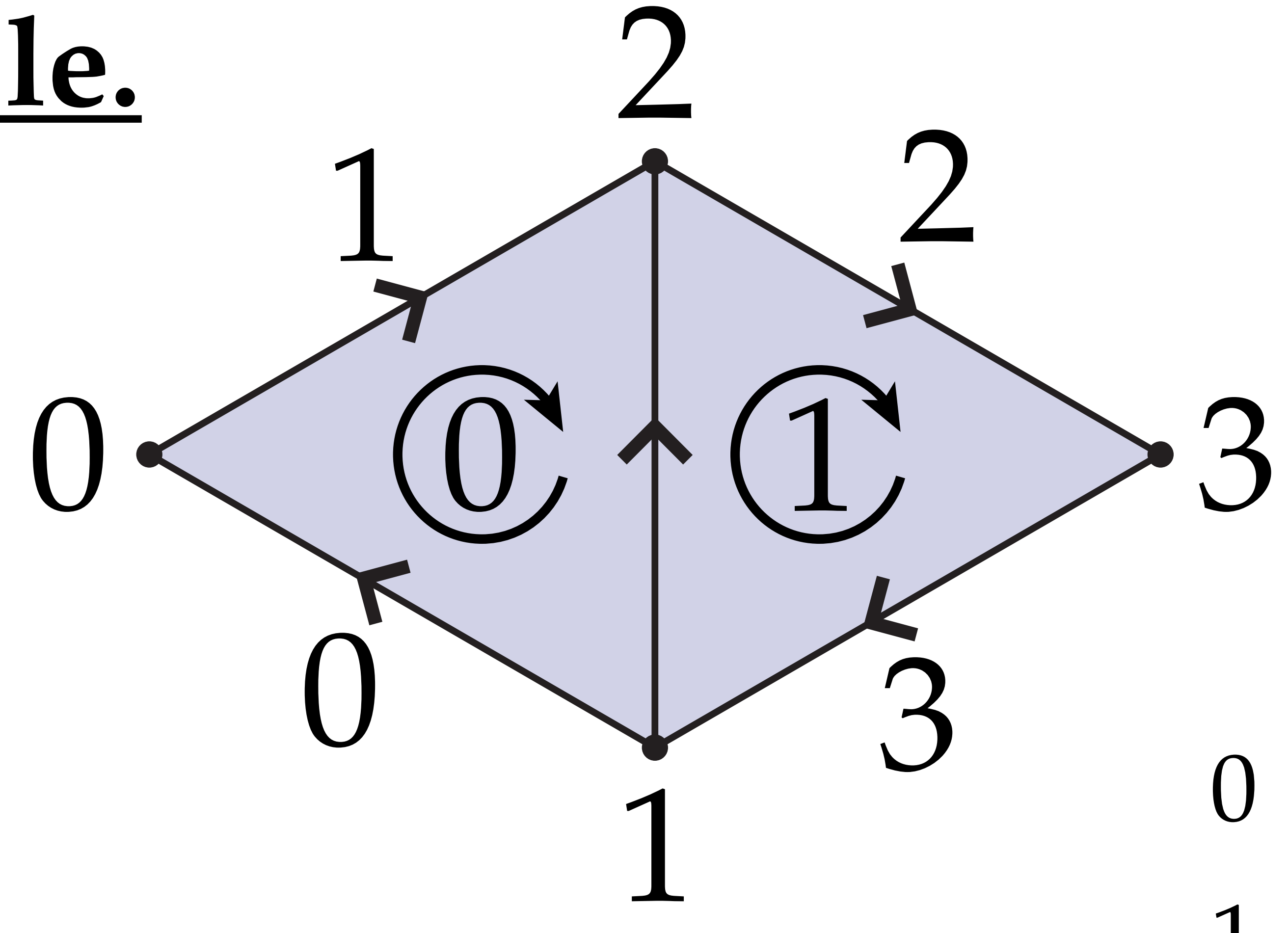


Q: Now what's the cost of finding edges incident on a given vertex?

Data Structures—Signed Incidence Matrix

A *signed incidence matrix* is an incidence matrix where the sign of each nonzero entry is determined by the relative orientation of the two simplices corresponding to that row/column.

Example.



$$E^0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \end{array} \qquad E^1 = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

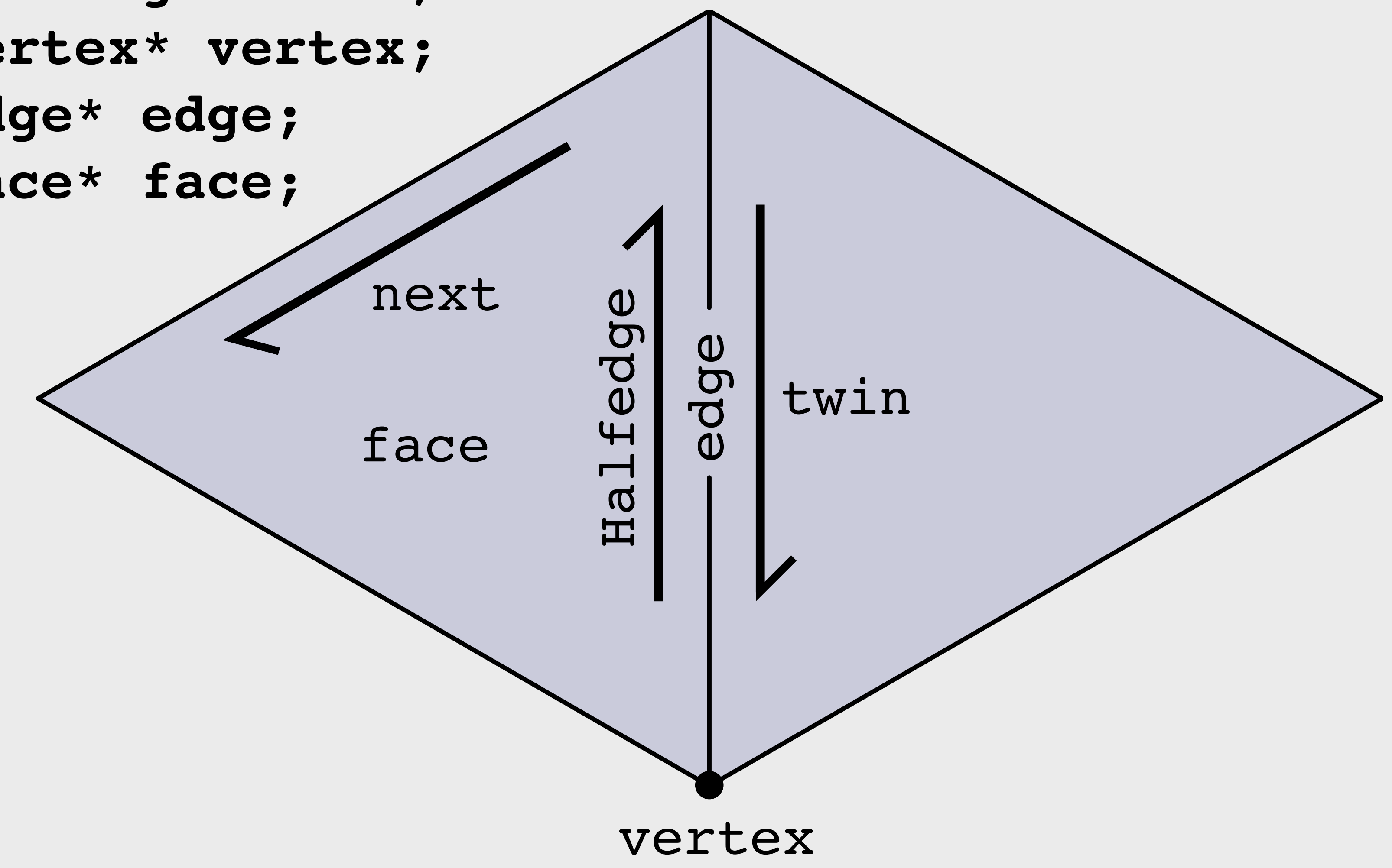
(Closely related to *discrete exterior calculus.)*

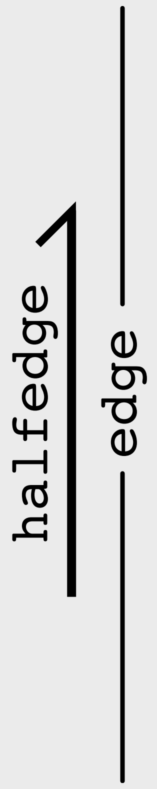
Topological Data Structures—Half Edge Mesh

Basic idea: each edge gets split into two *half edges.*

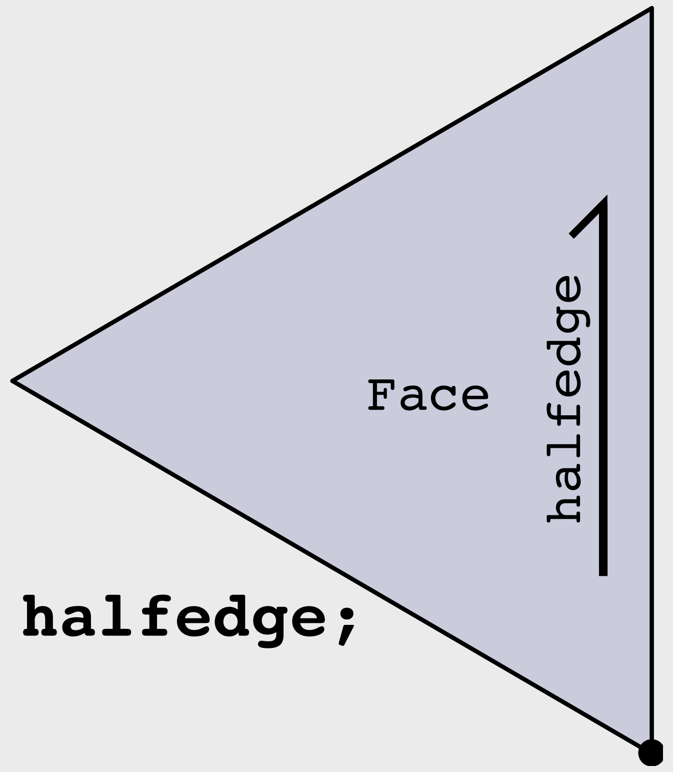
- Half edges act as "glue" between mesh elements.
- All other elements know only about a single half edge.

```
struct Halfedge
{
   Halfedge* twin;
   Halfedge* next;
   Vertex* vertex;
   Edge* edge;
   Face* face;
};
```

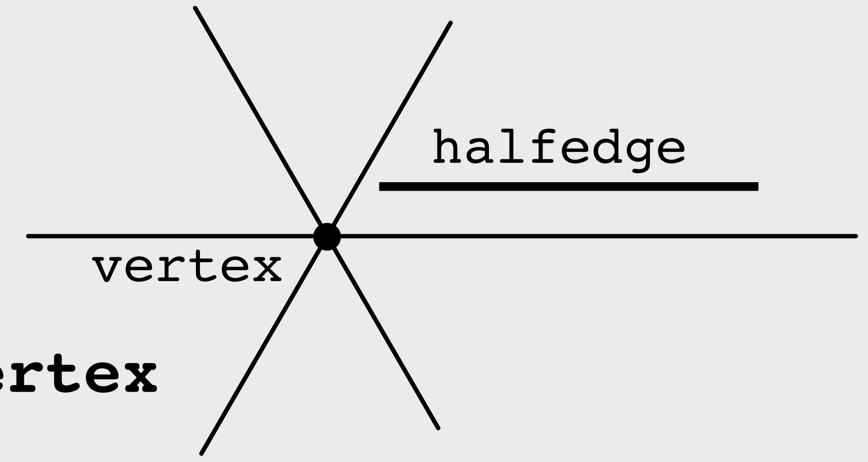





```
struct Edge
{
   Halfedge* halfedge;
};
```




```
struct Face
{
   Halfedge* halfedge;
};
```




```
struct Vertex
{
   Halfedge* halfedge;
};
```

(You'll use this one in your assignments!)

Half Edge—Algebraic Definition

Definition. Let H be any set with an even number of elements, let $\rho : H \to H$ be any permutation of H, and let $\eta : H \to H$ be an involution without any fixed points, *i.e.*, $\eta \circ \eta = \mathrm{id}$ and $\eta(h) \neq h$ for any $h \in H$. Then (H, ρ, η) is a *half edge mesh*, the elements of H are called *half edges*, the orbits of η are *edges*, the orbits of ρ are *faces*, and the orbits of $\eta \circ \rho$ are *vertices*.

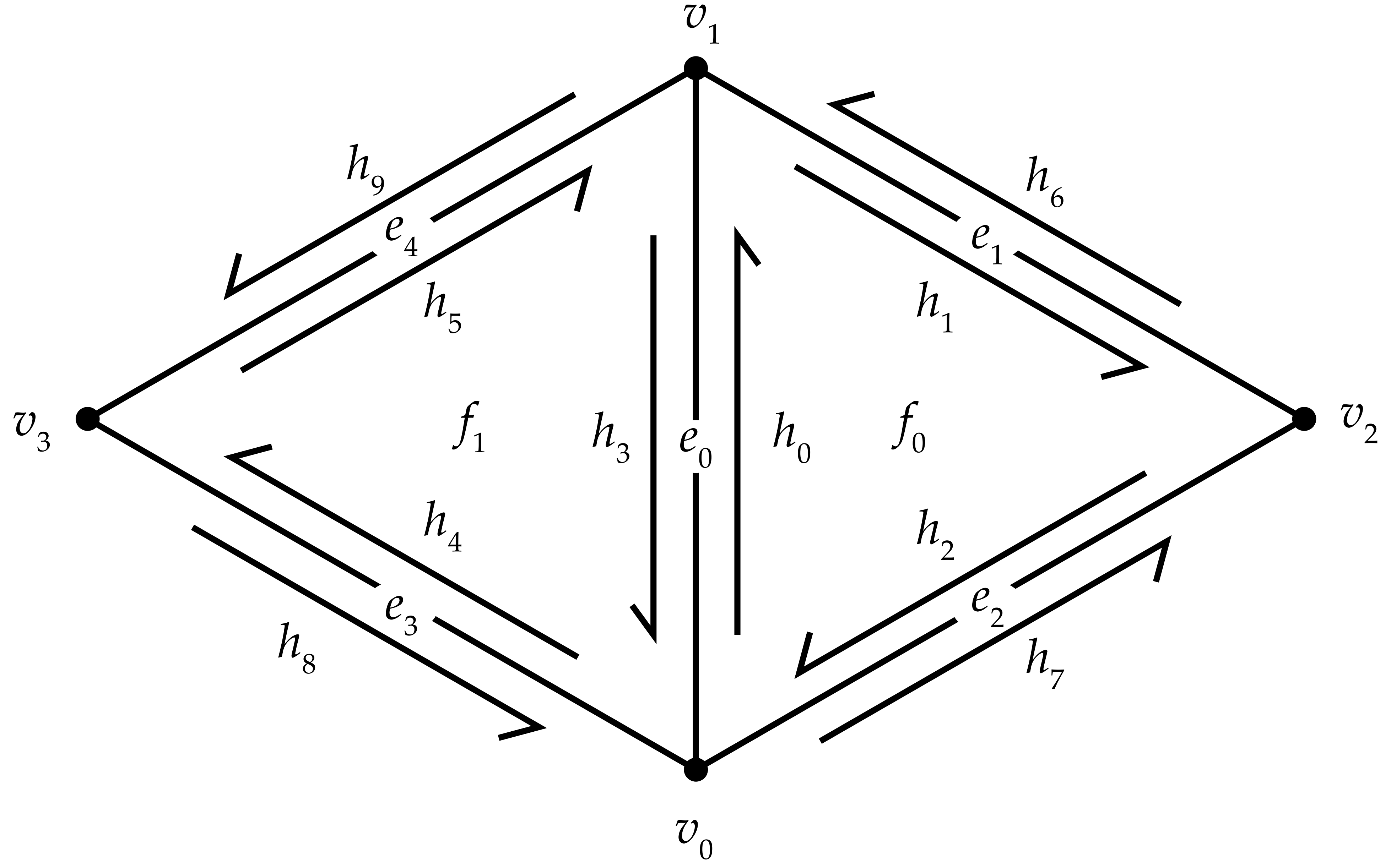
Fact. Every half edge mesh describes a compact oriented topological surface (without boundary).

$$(h_1, \cdots, h_{10}) \overset{\rho}{\mapsto} (h_1, h_2, h_0, h_4, h_5, h_3, h_9, h_6, h_7, h_8)$$

"next"

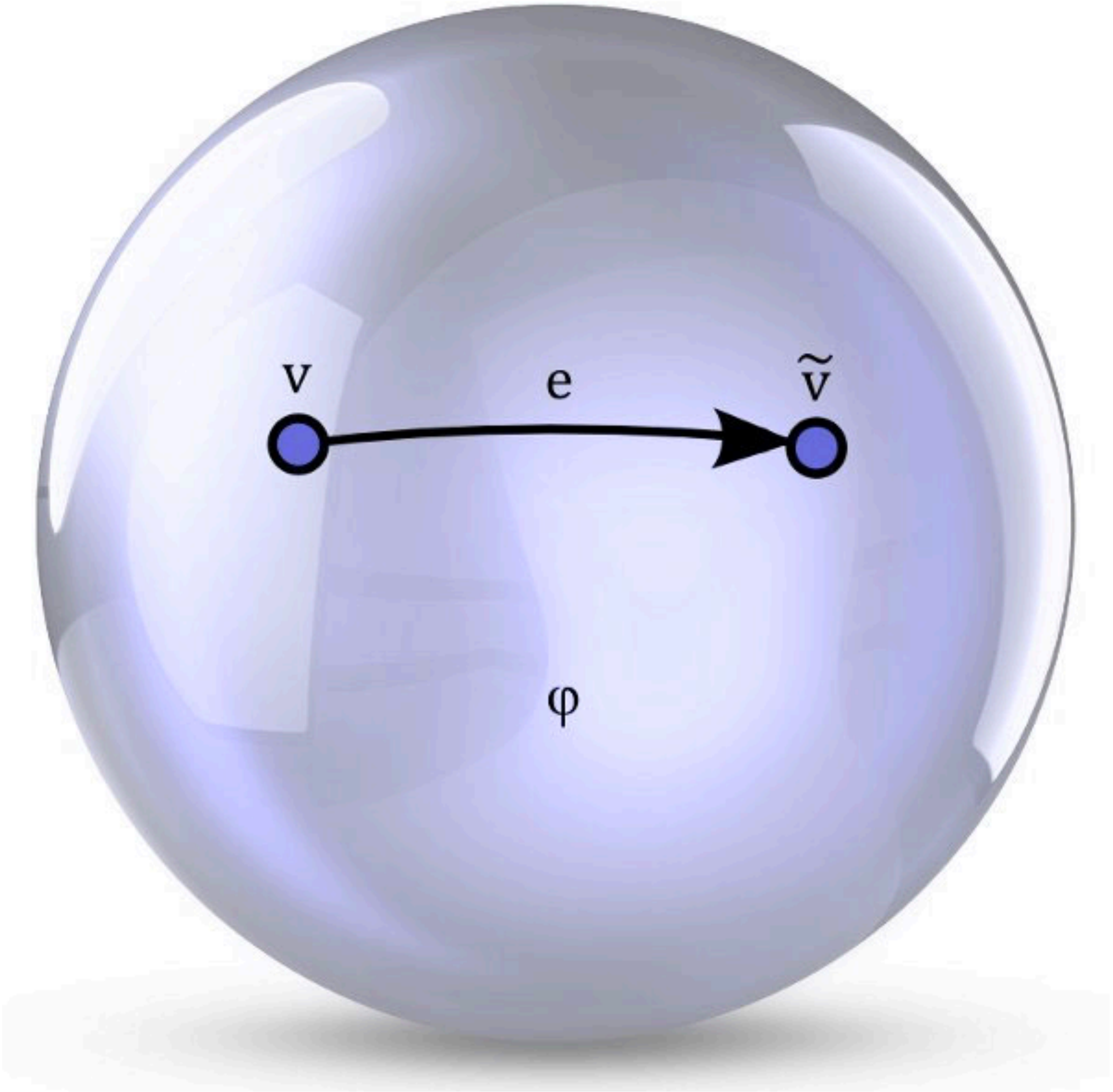
$$(h_1, \cdots, h_{10}) \overset{\eta}{\mapsto} (h_3, h_6, h_7, h_0, h_8, h_9, h_1, h_2, h_4, h_5)$$

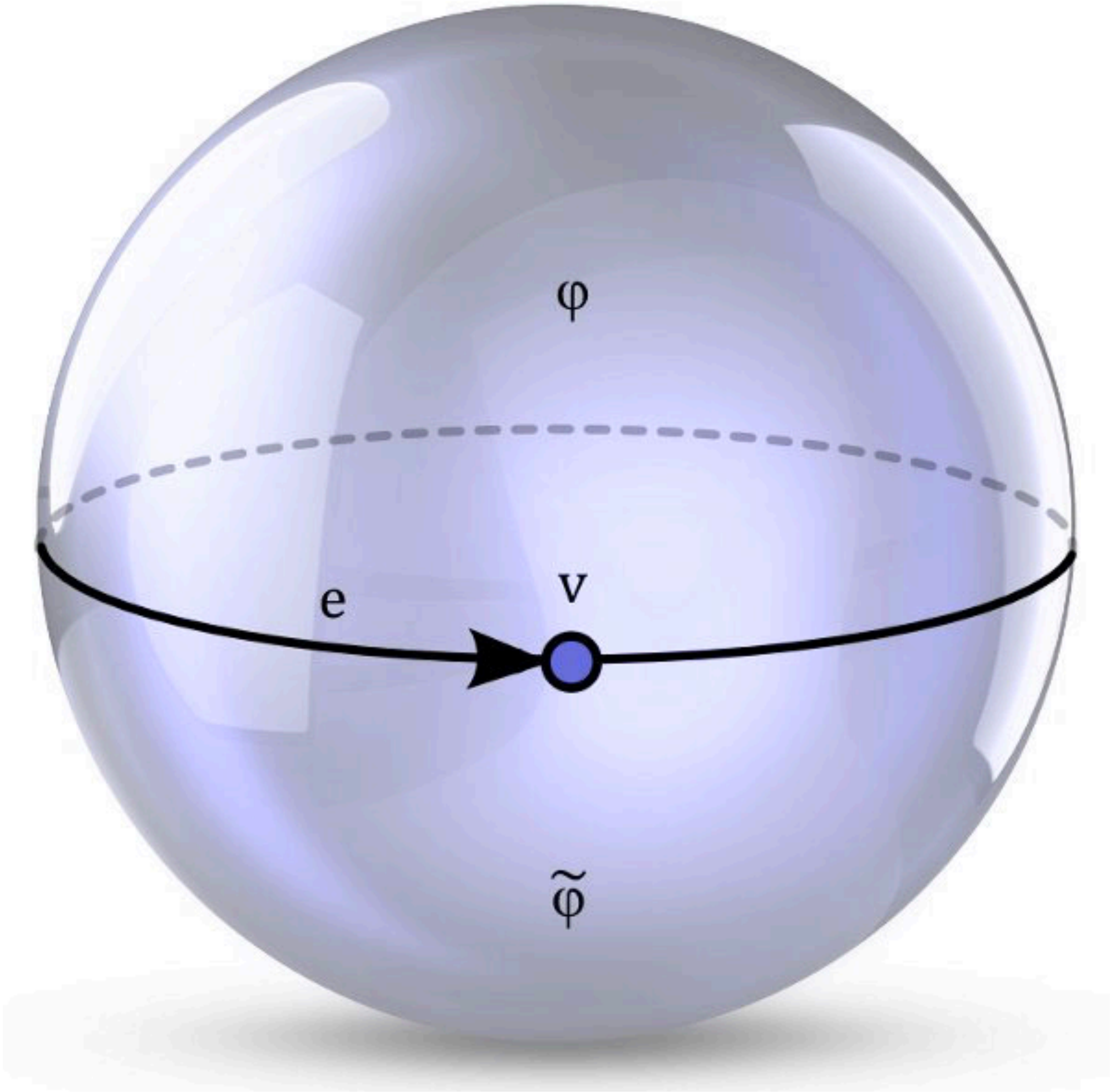
"twin"



Half Edge—Example

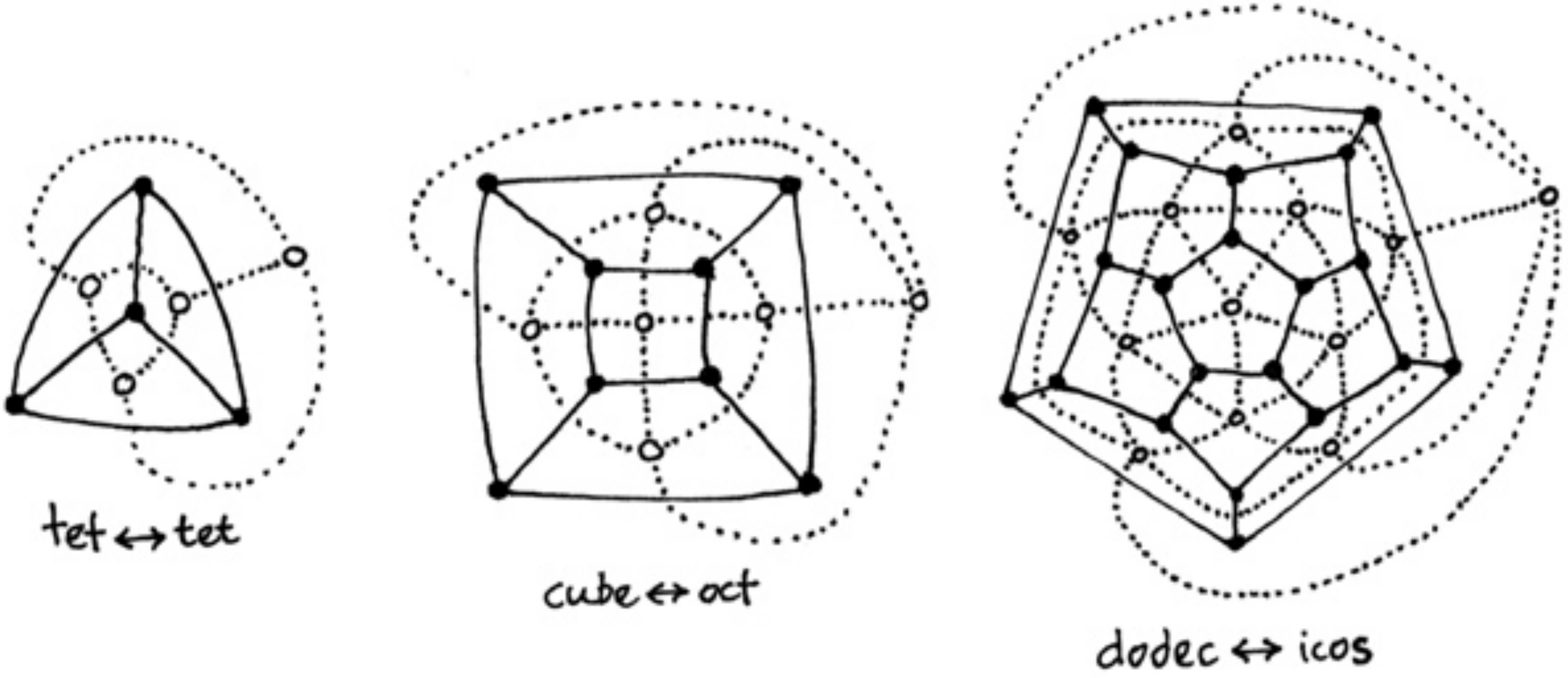
Smallest examples (two half edges):





(images courtesy U. Pinkall)

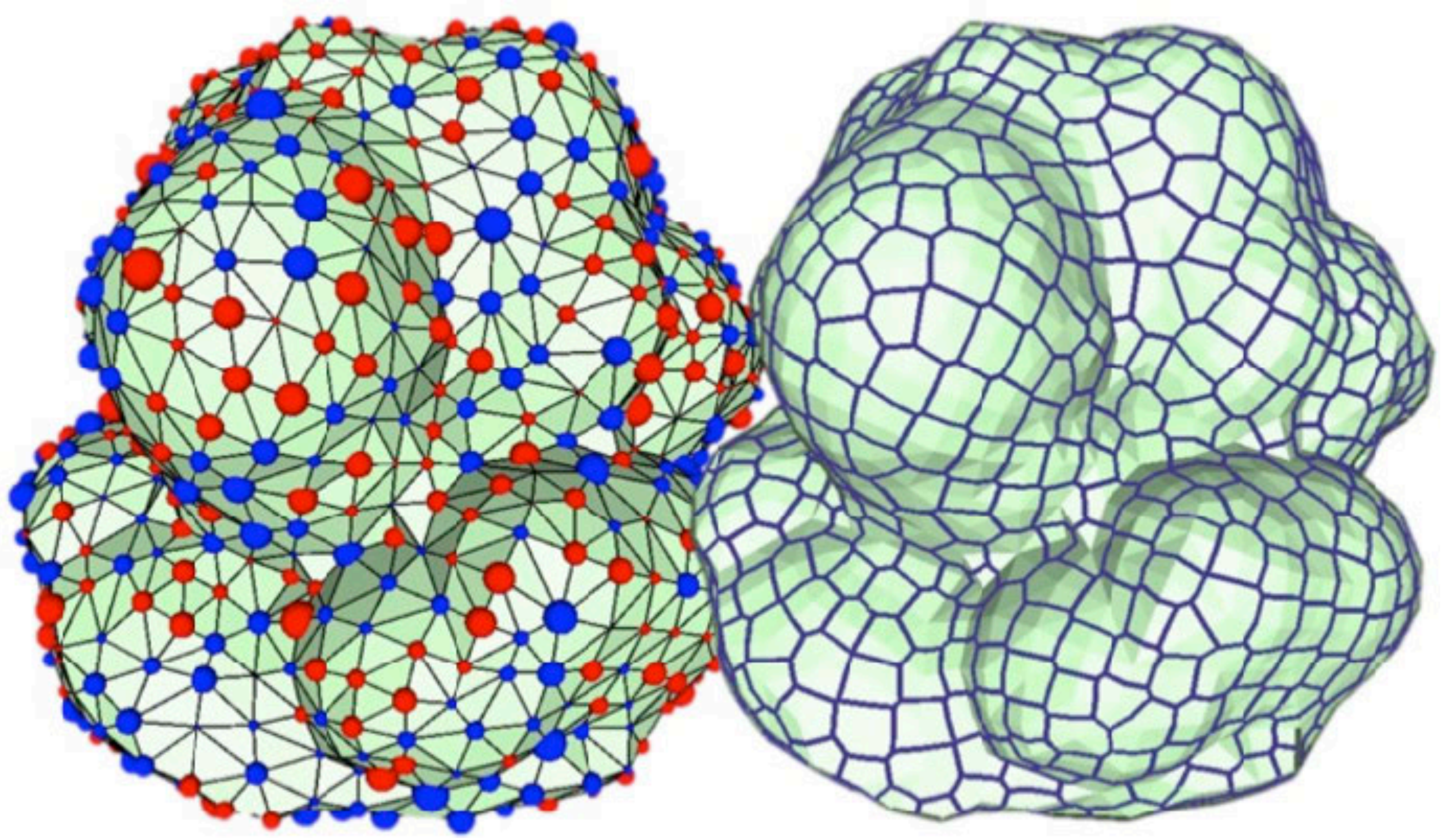
Data Structures—Quad Edge

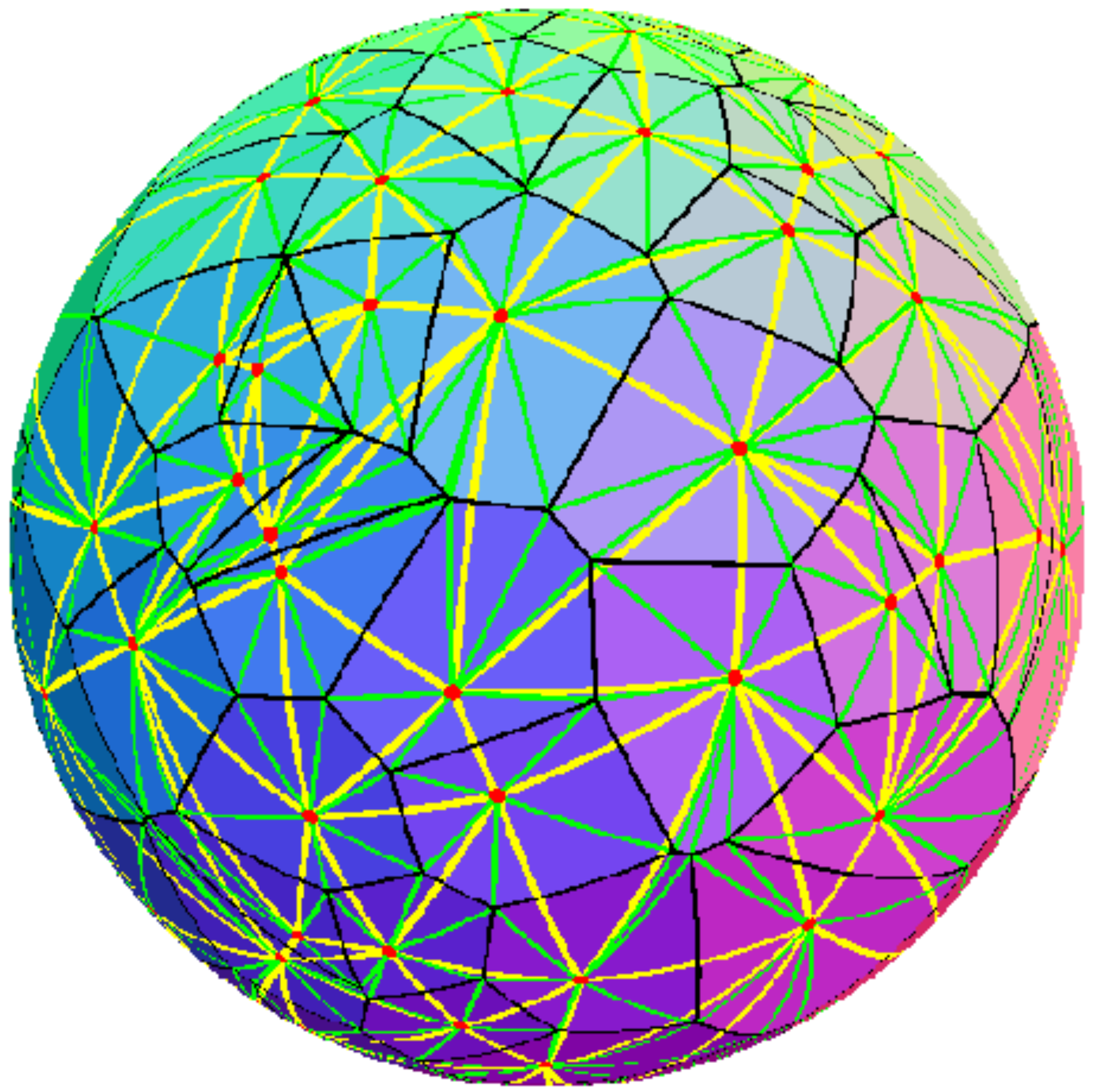


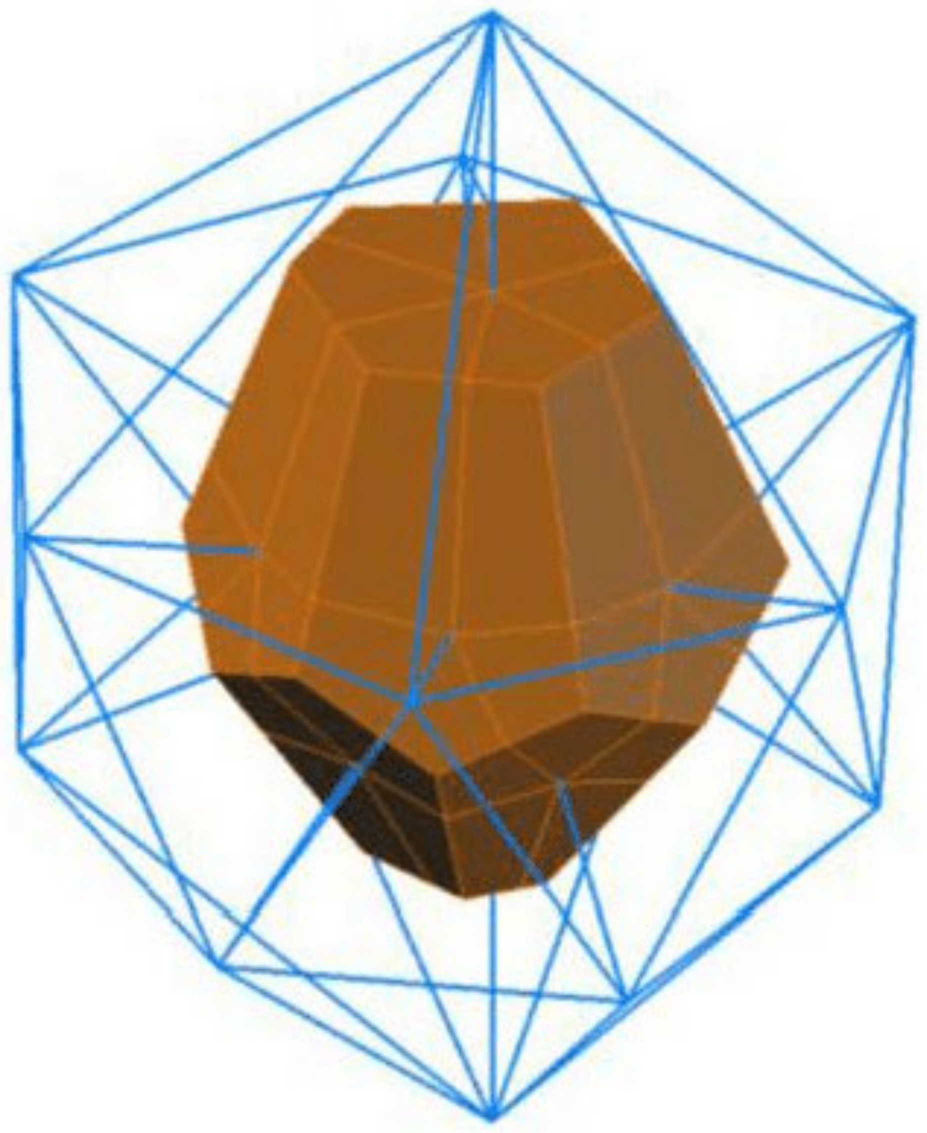
(L. Guibas & J. Stolfi, "Primitives for the Manipulation of General Subdivisions and the Computation of Voronoi Diagrams")

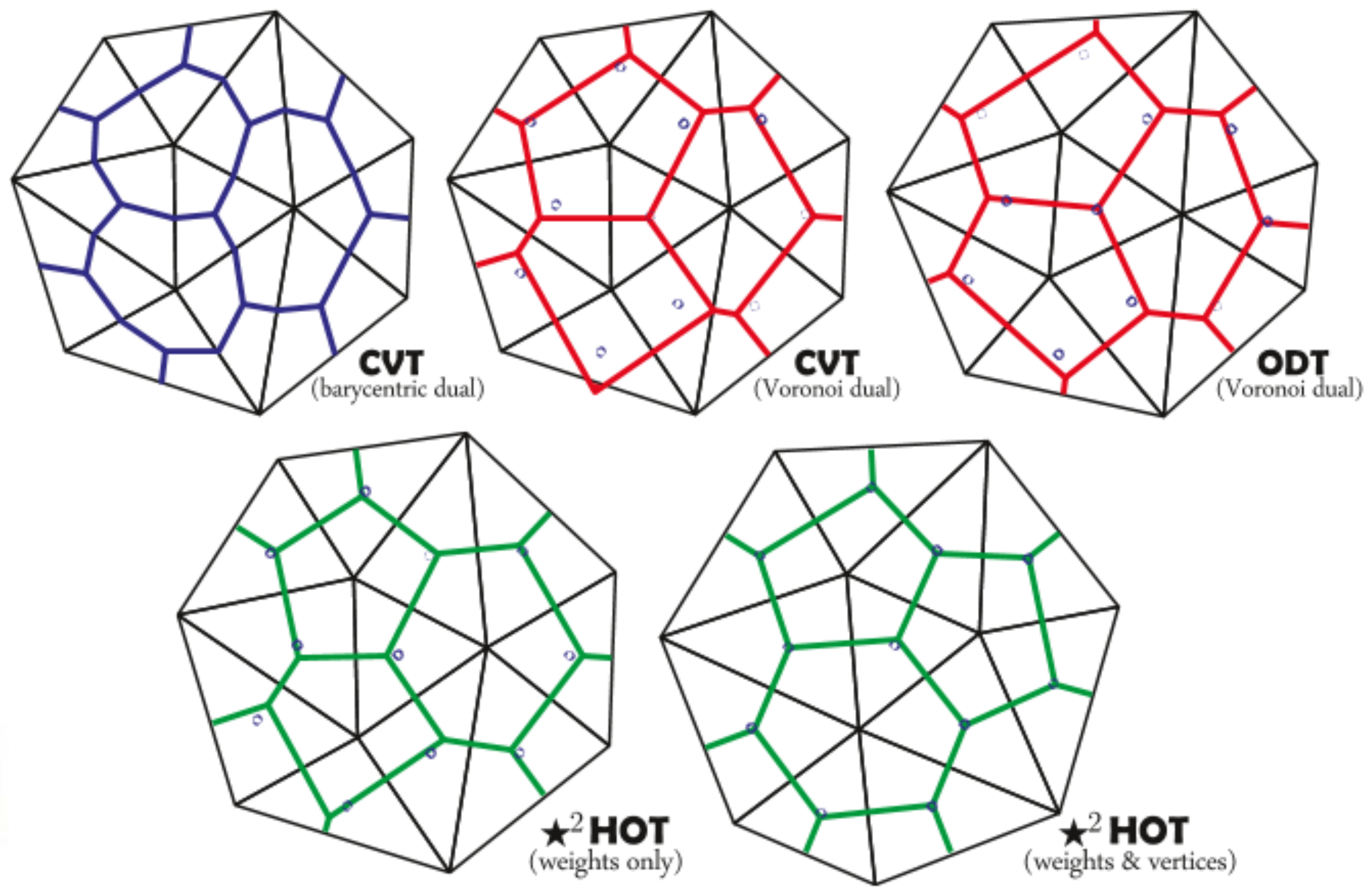
Dual Complex

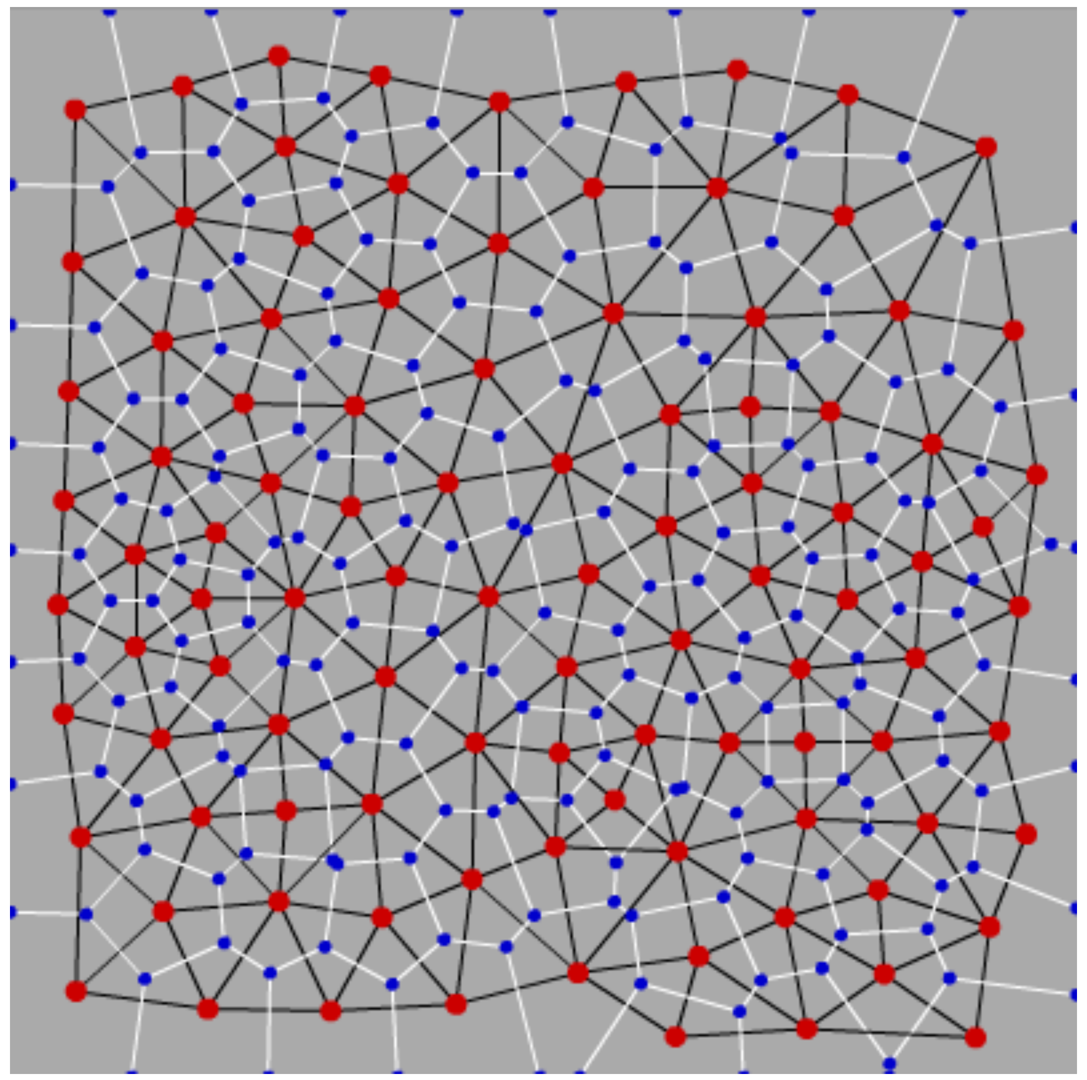
Dual Mesh — Visualized



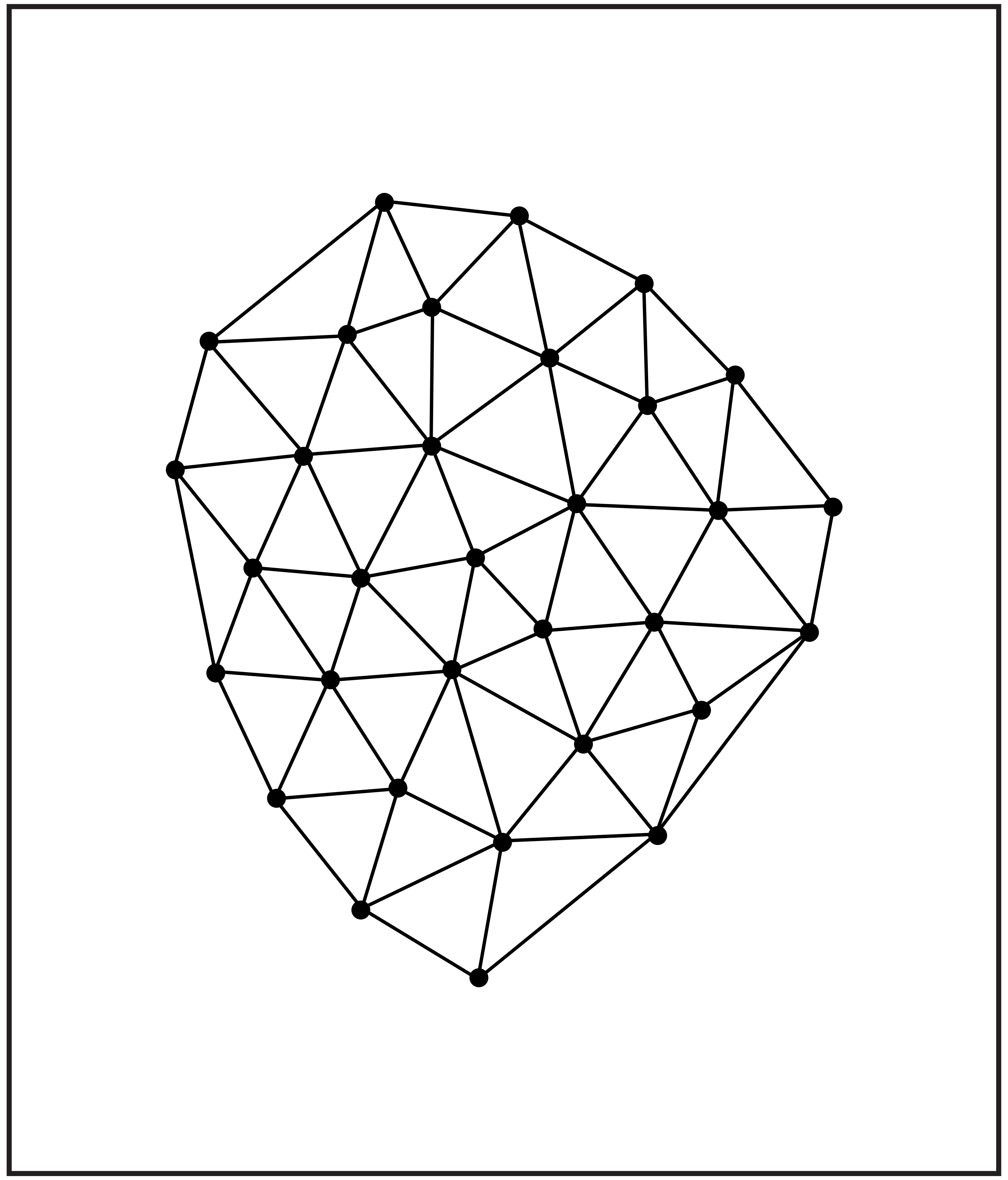




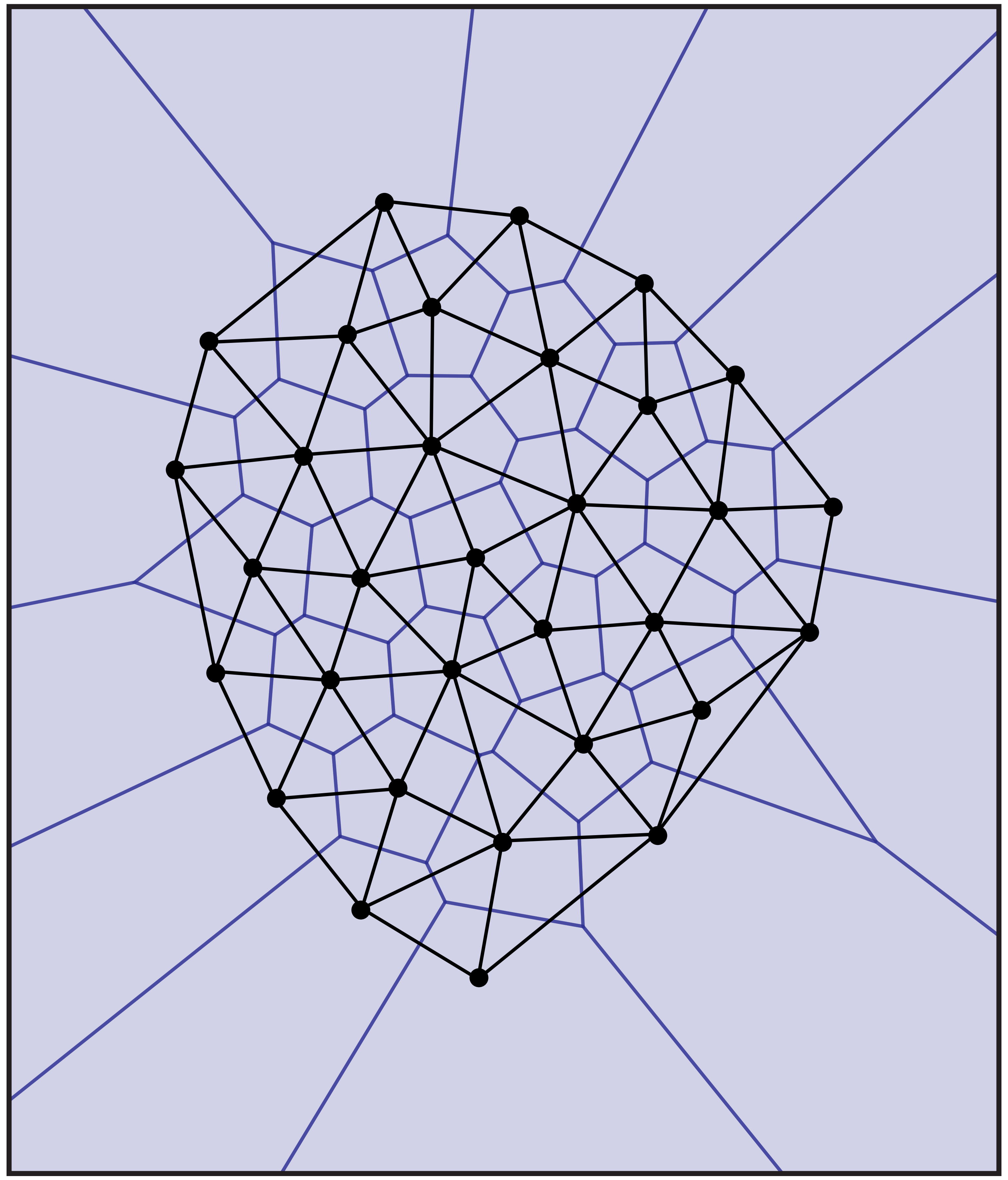




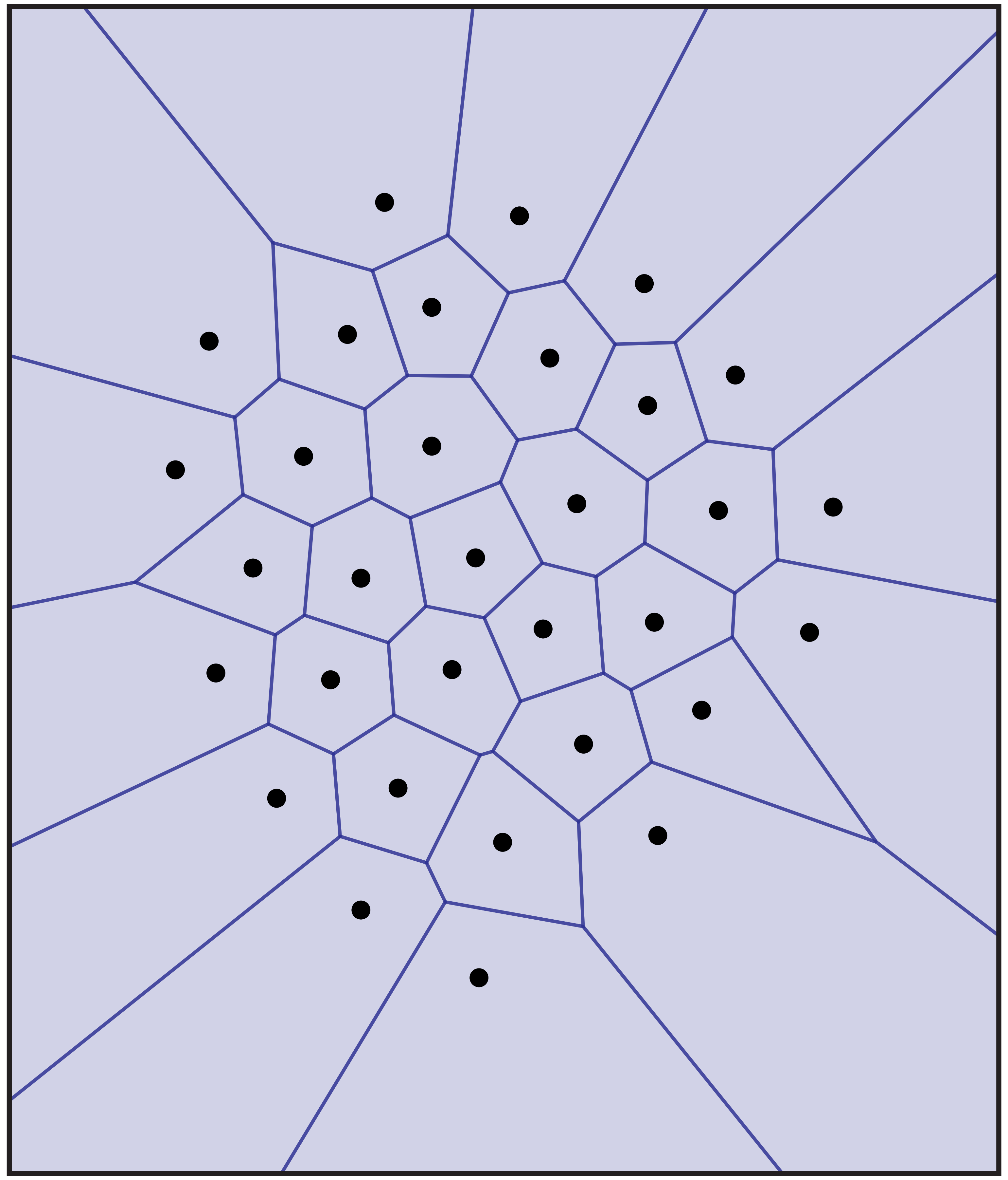
Poincaré Duality



simplicial complex

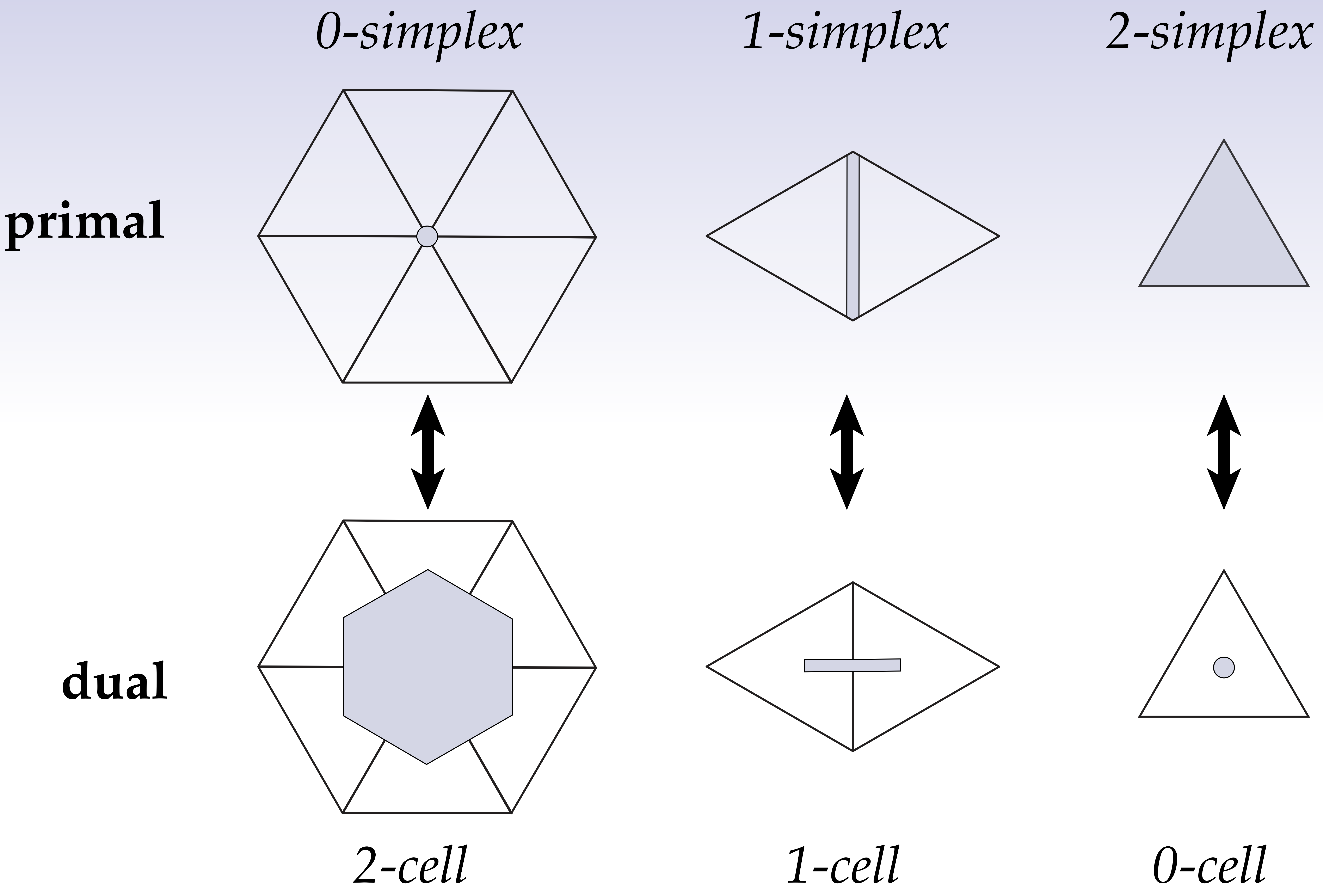


(Poincaré Duality)



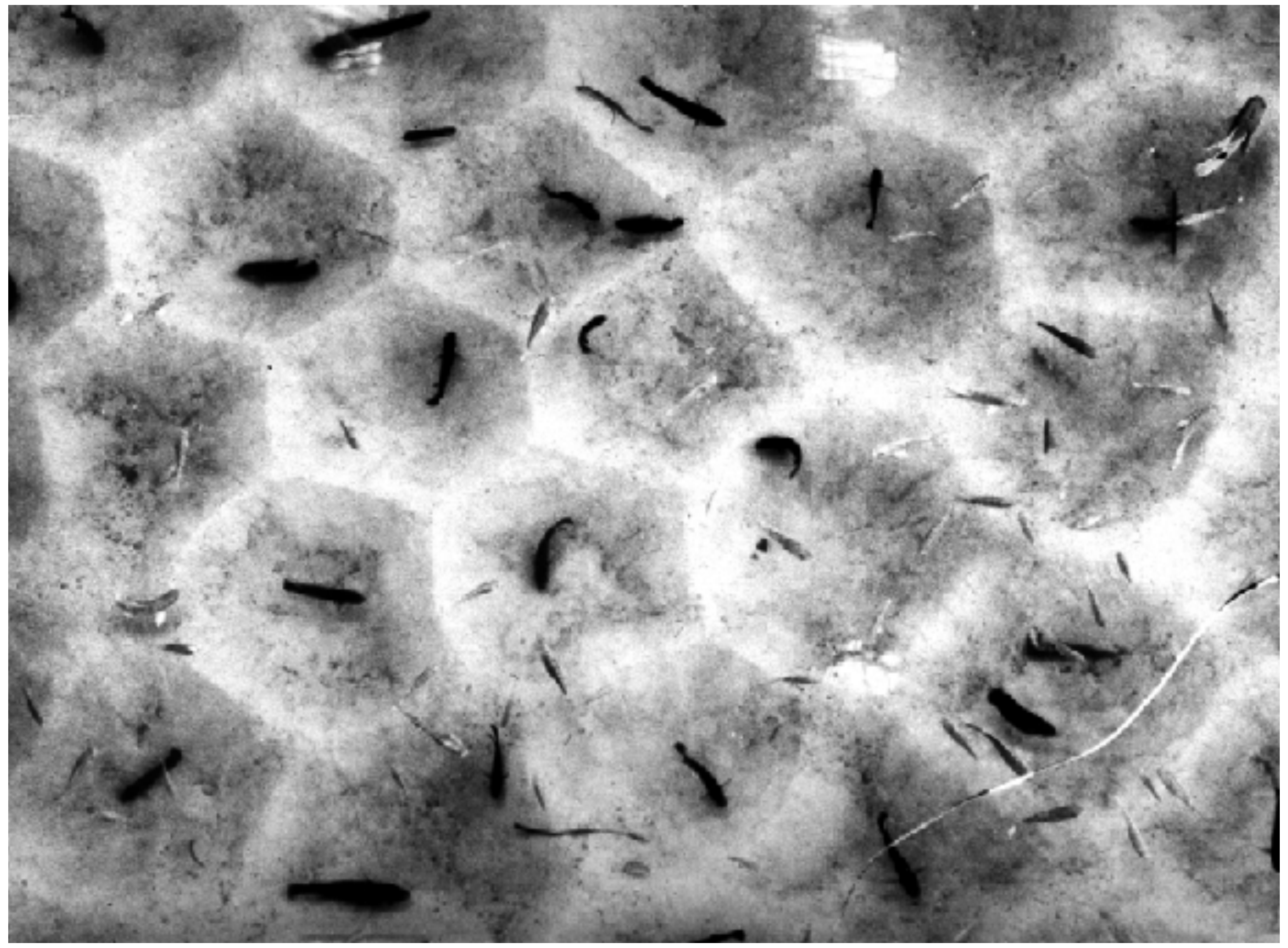
cell complex

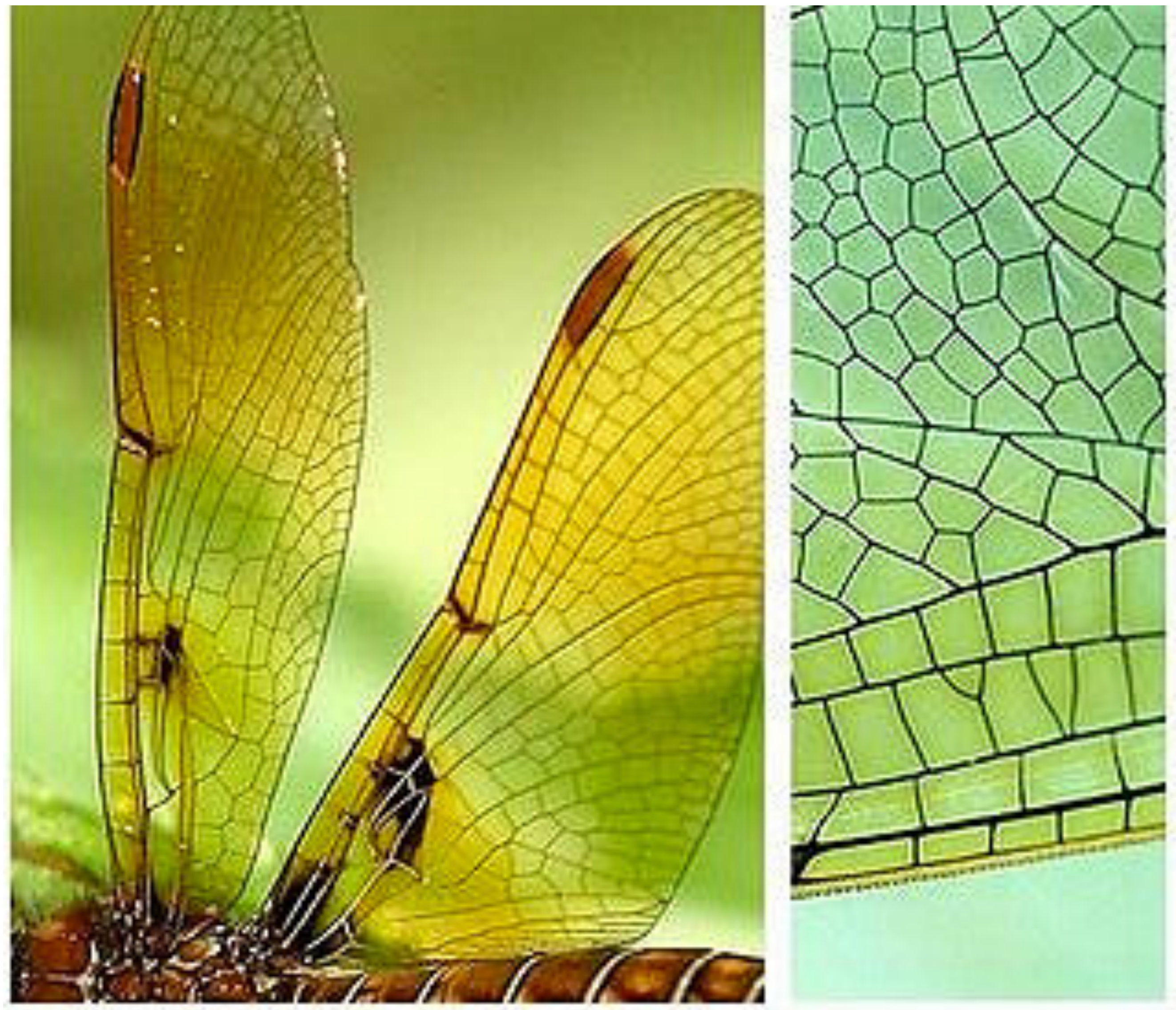
Primal vs. Dual

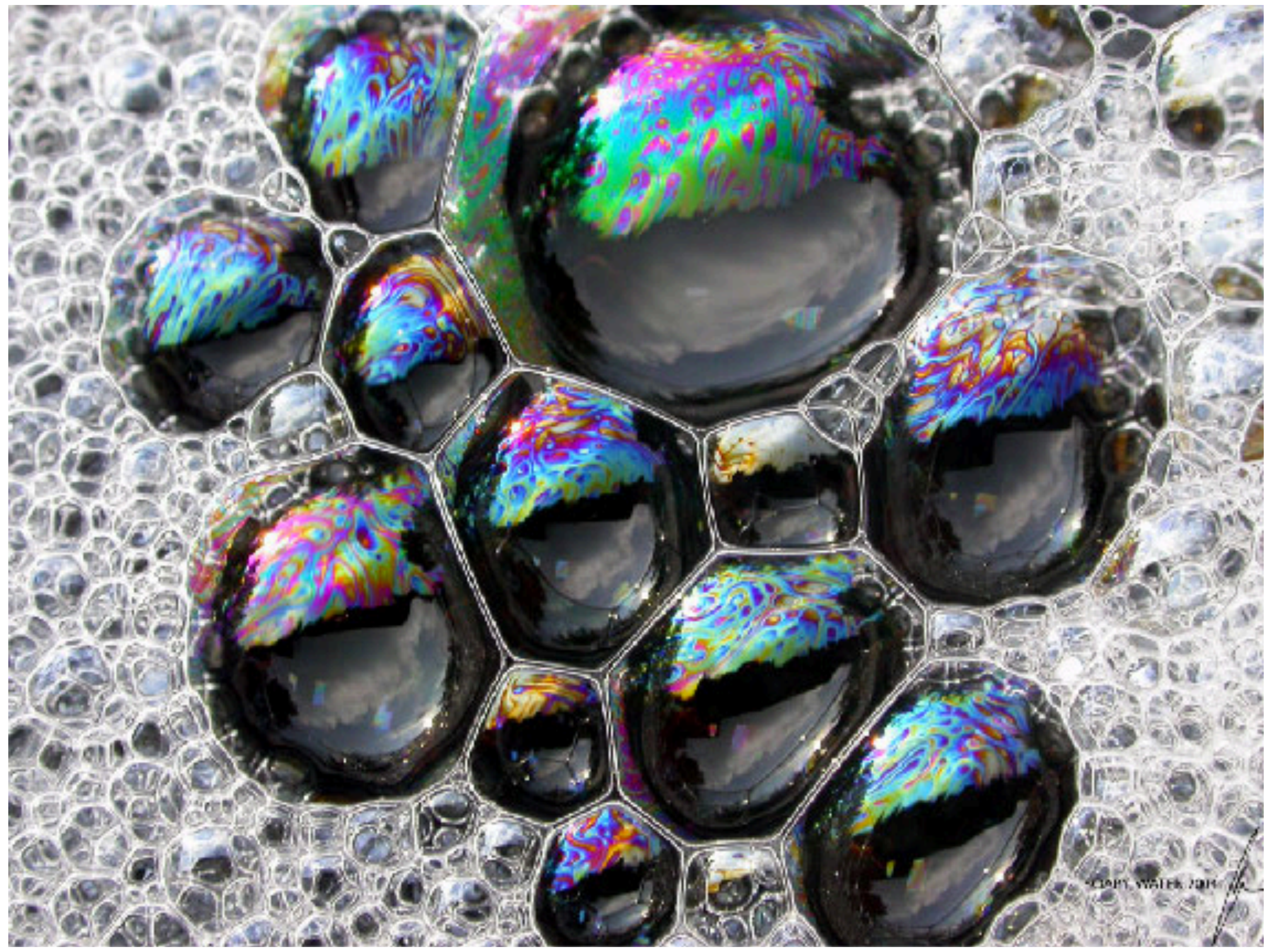


(Will say more when we talk about discrete exterior calculus!)

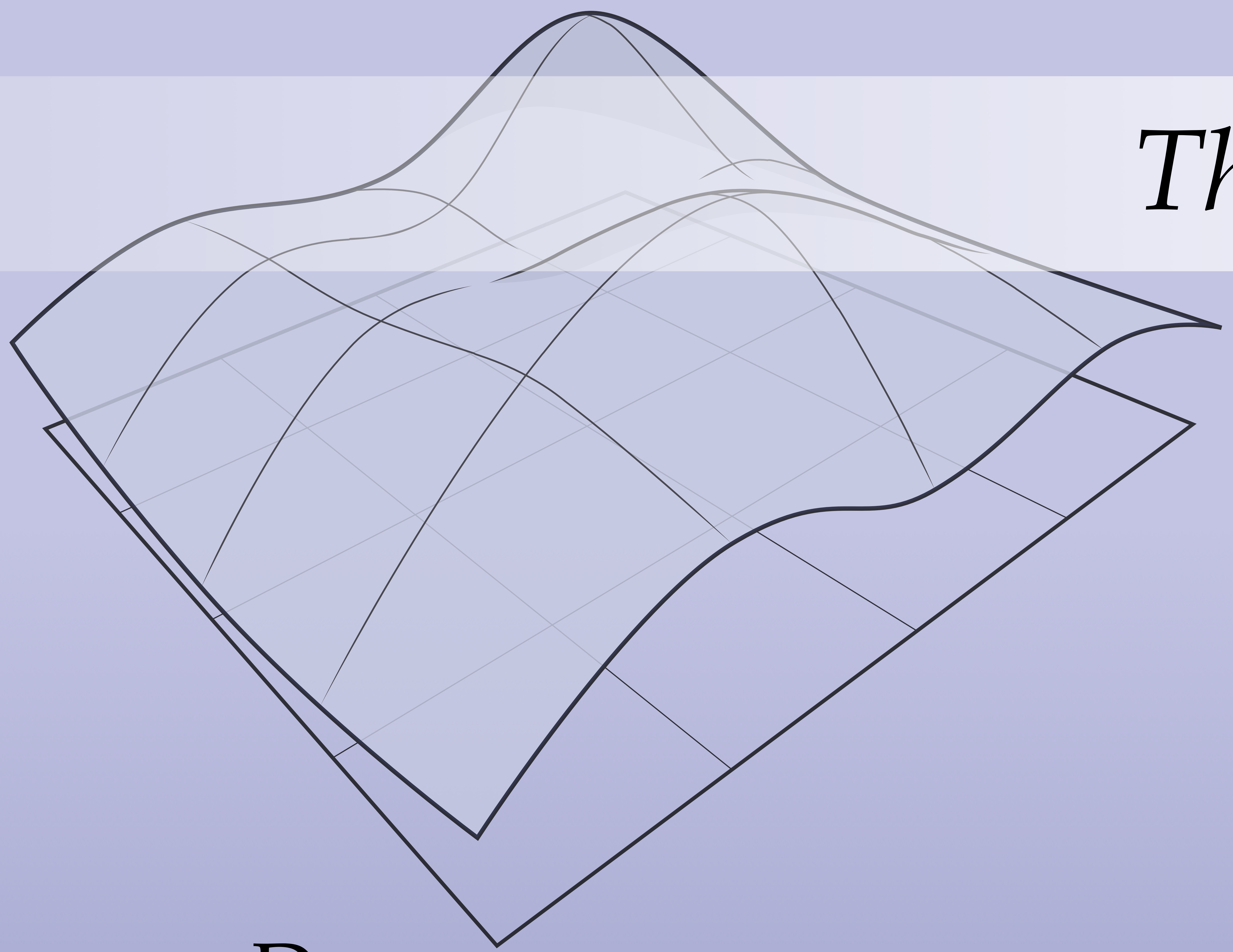
Poincaré Duality in Nature







Thanks!



DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858B • Fall 2017