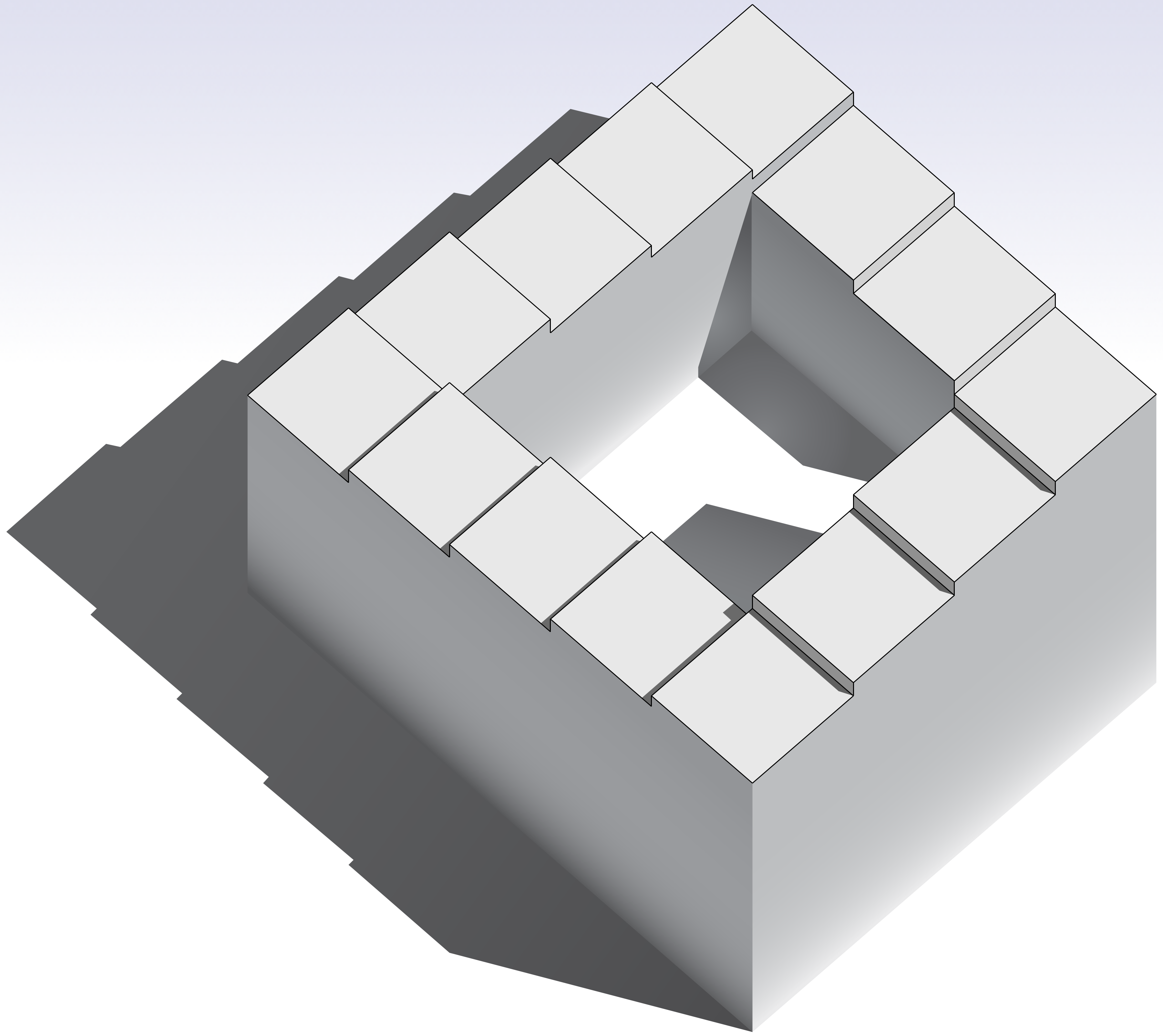
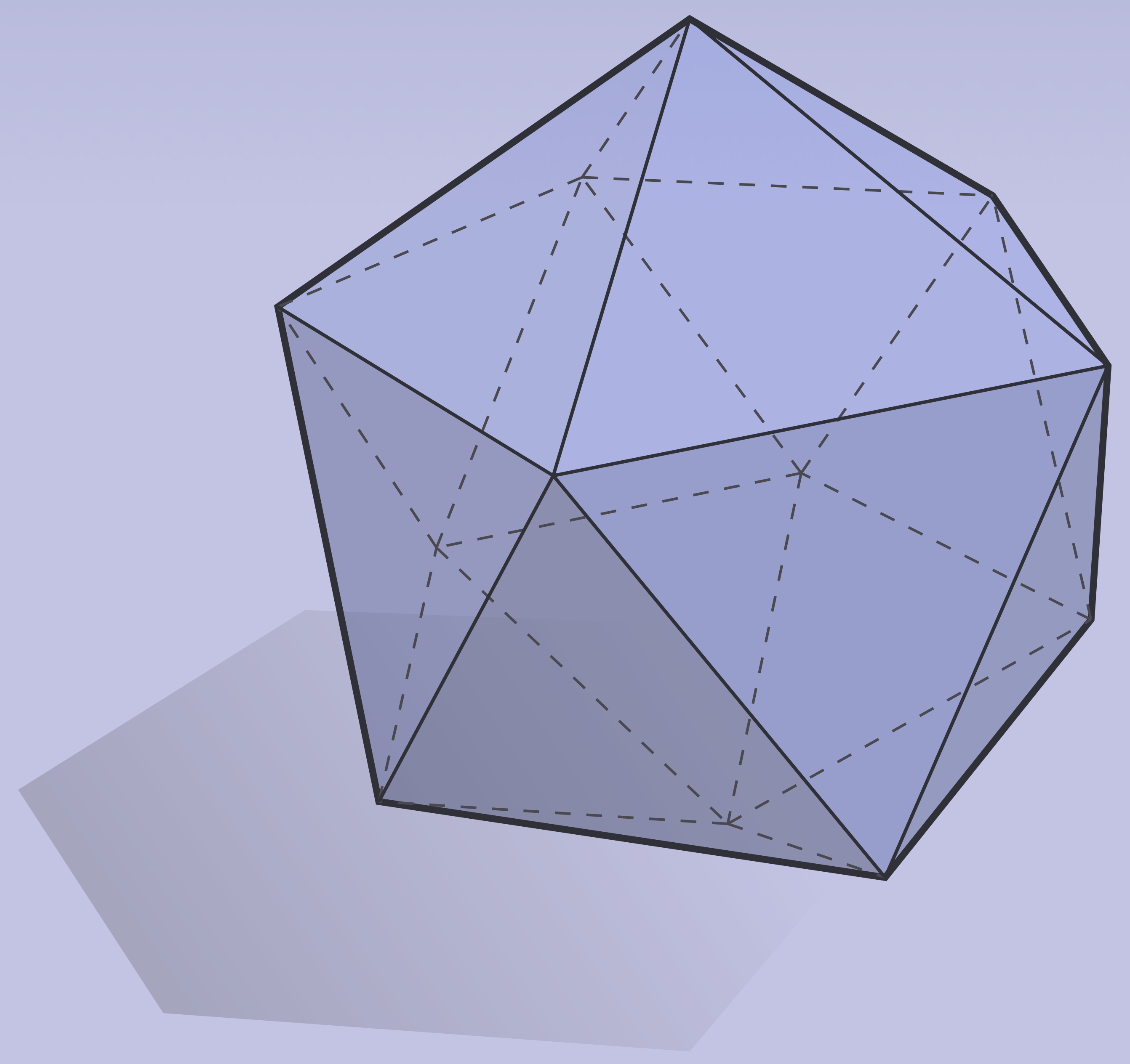


# Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858B • Fall 2017

# LECTURE 2:
# EXTERIOR ALGEBRA



## DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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# *Where Are We Going Next?*

**GOAL:** develop *discrete exterior calculus (DEC)*

<u>*Prerequisites:*</u>

**Linear algebra:** "little arrows" (vectors)

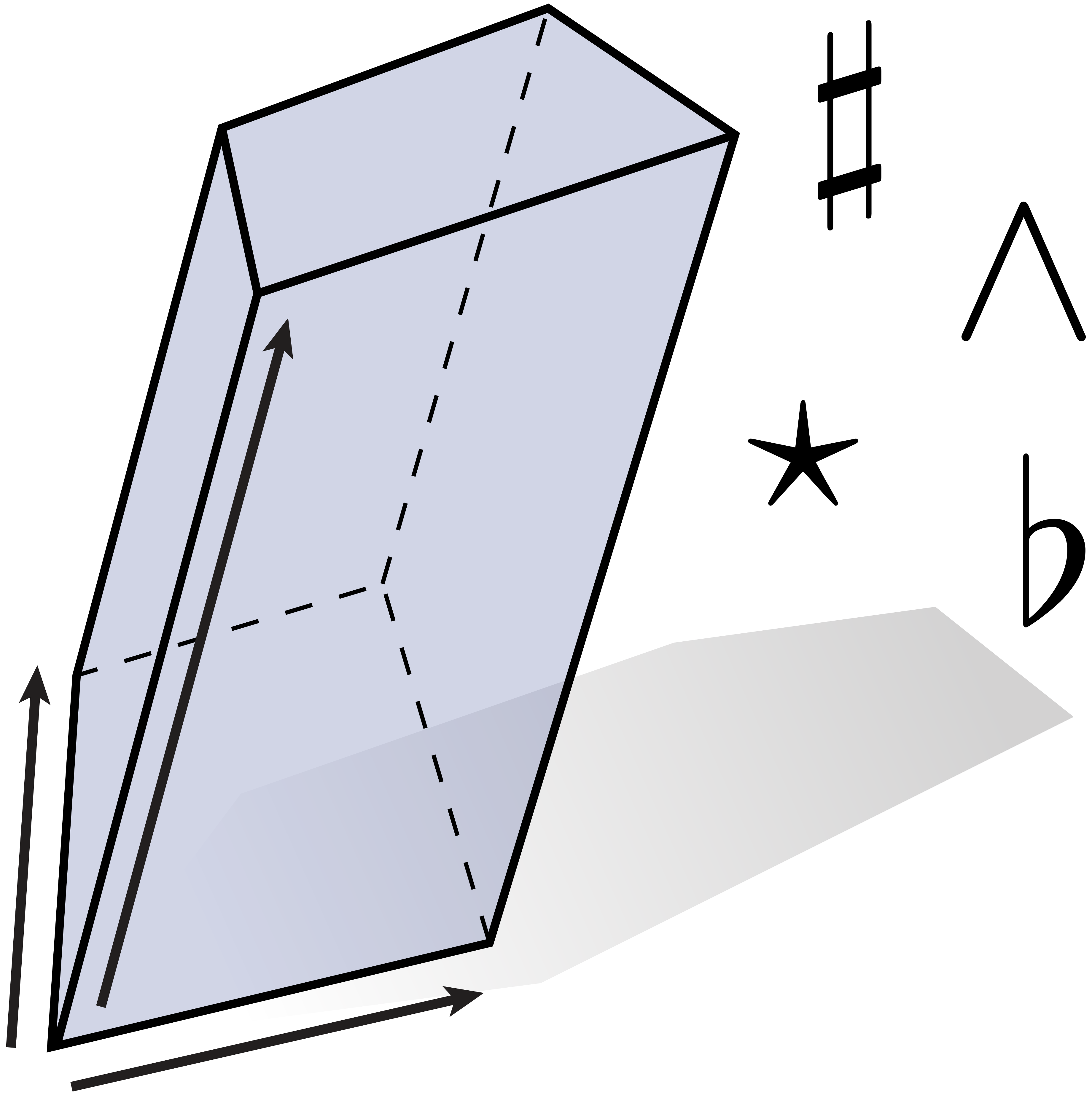
**Vector Calculus:** how do vectors *change?*

<u>*Next few lectures:*</u>

**Exterior algebra**: "little volumes" ($k$-vectors)

**Exterior calculus**: how do $k$-vectors change?

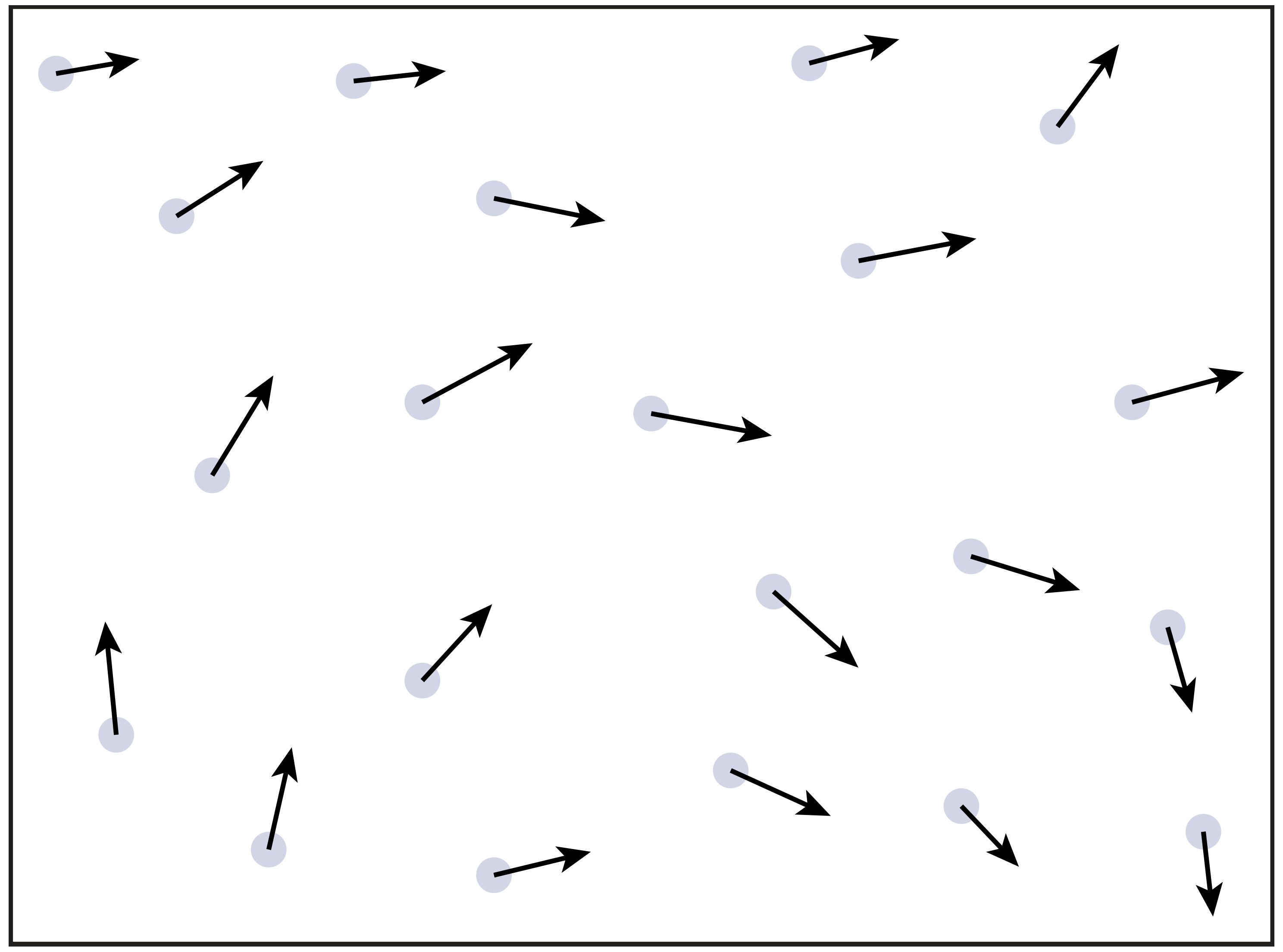
**DEC:** how do we do all of this on meshes?



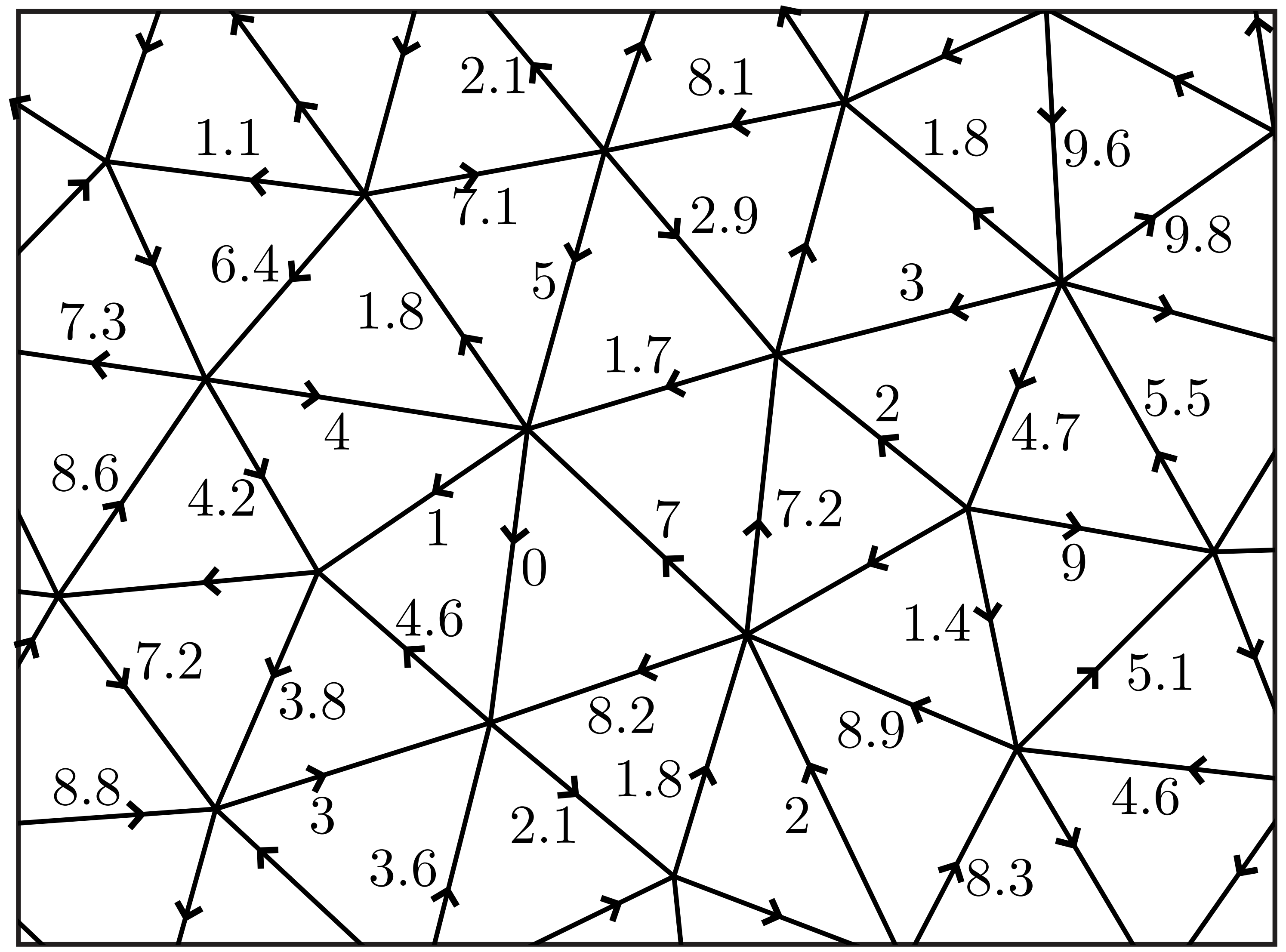

**Basic idea:** replace vector calculus with computation on meshes.

# *Why Are We Going There?*

- **TLDR:** *So that we can solve equations on meshes!*



**integrate**

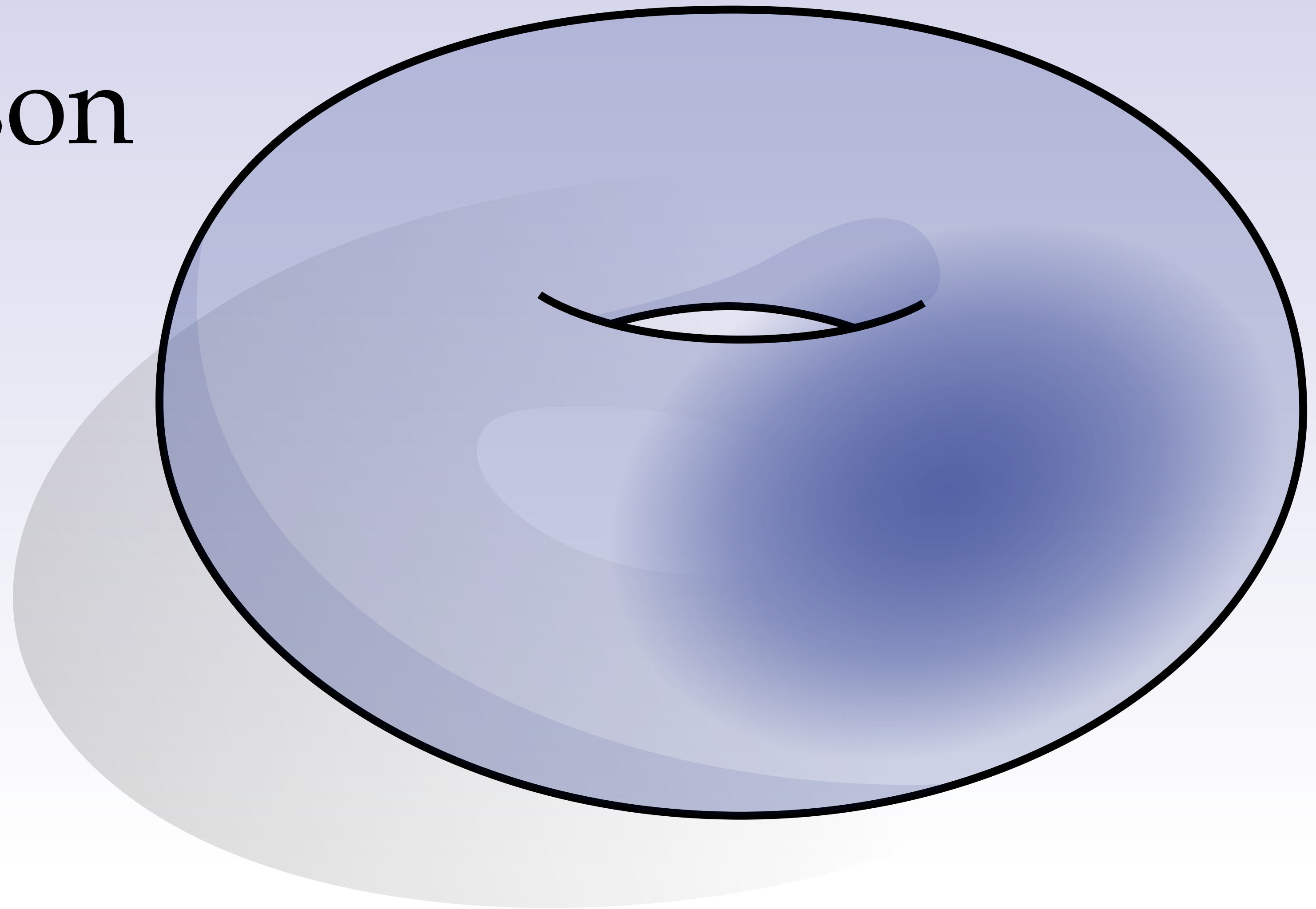



- Geometry processing algorithms solve *equations* on *meshes*
- Meshes are made up of little *volumes*

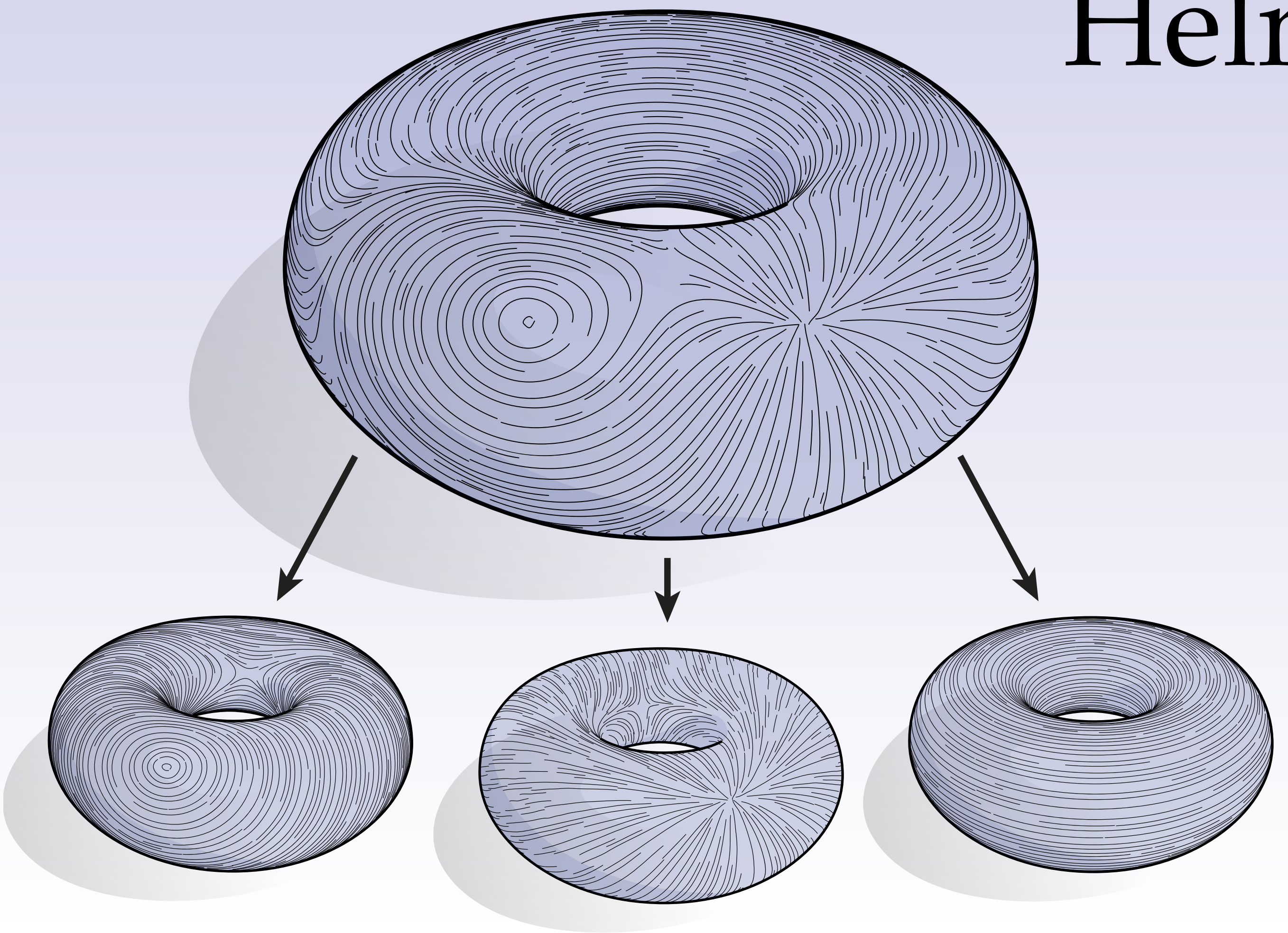
$\Rightarrow$ Need to learn to *integrate equations over little volumes* to do computation!

# *Basic Computational Tools*

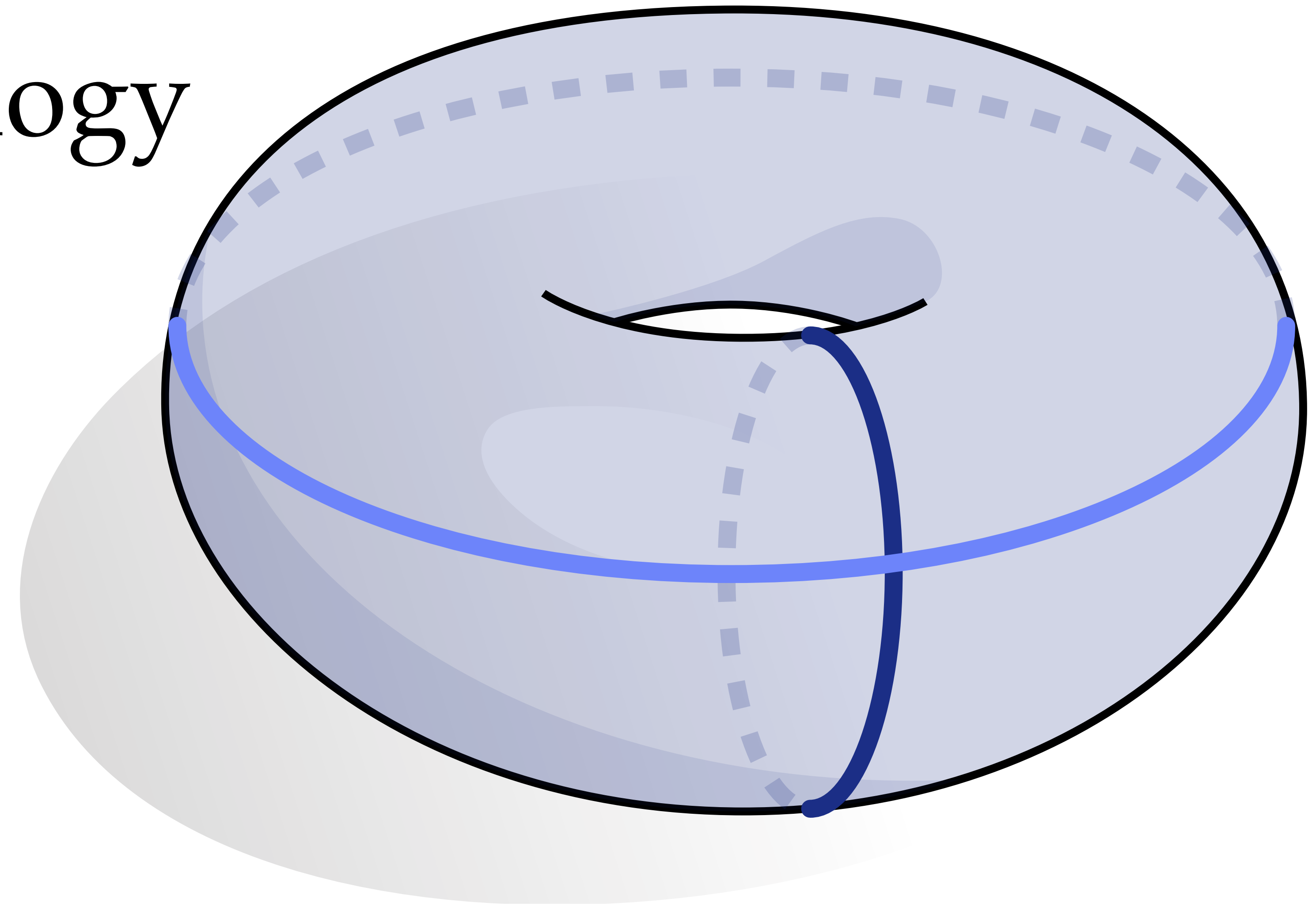
Poisson



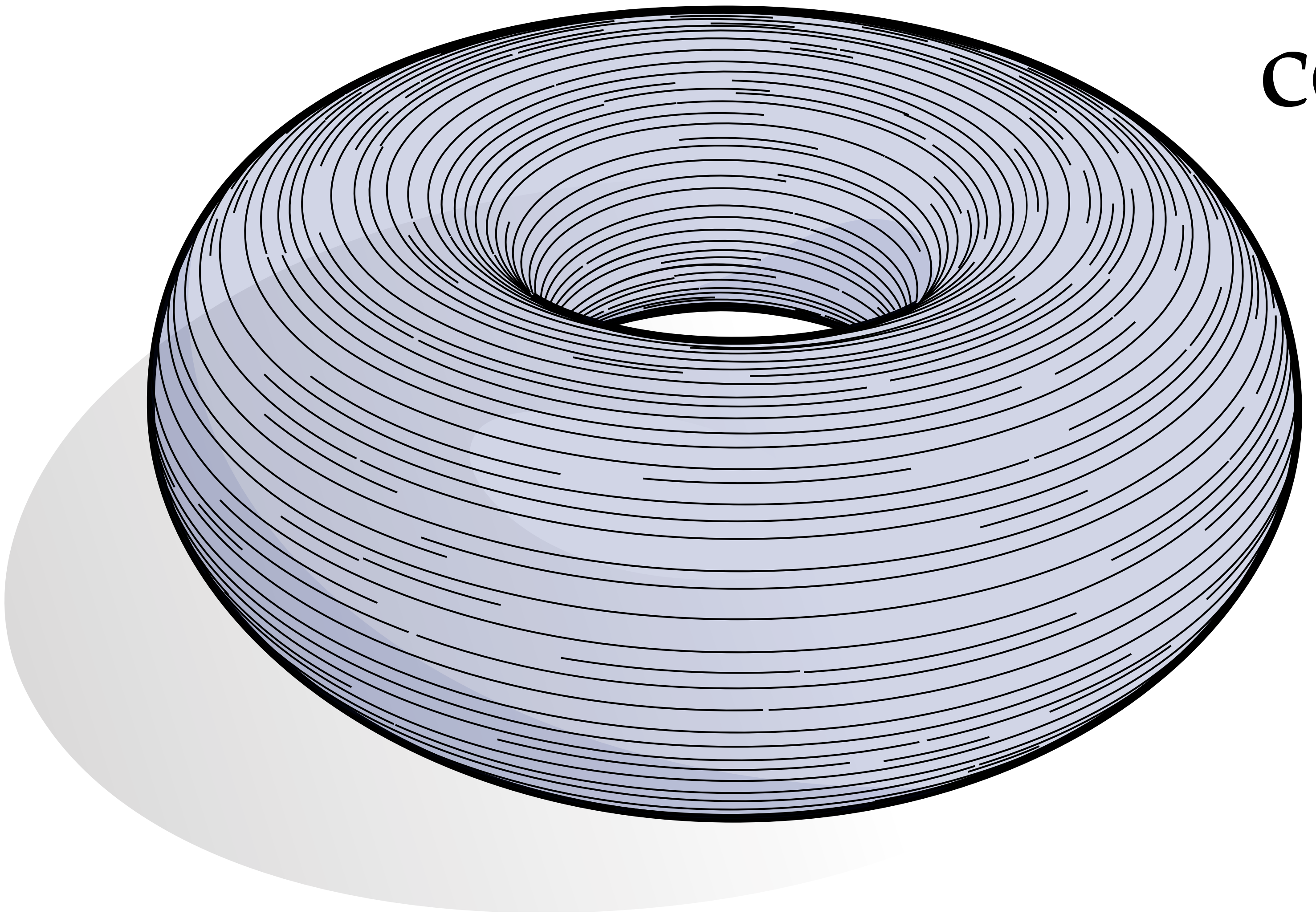
Helmholtz-Hodge



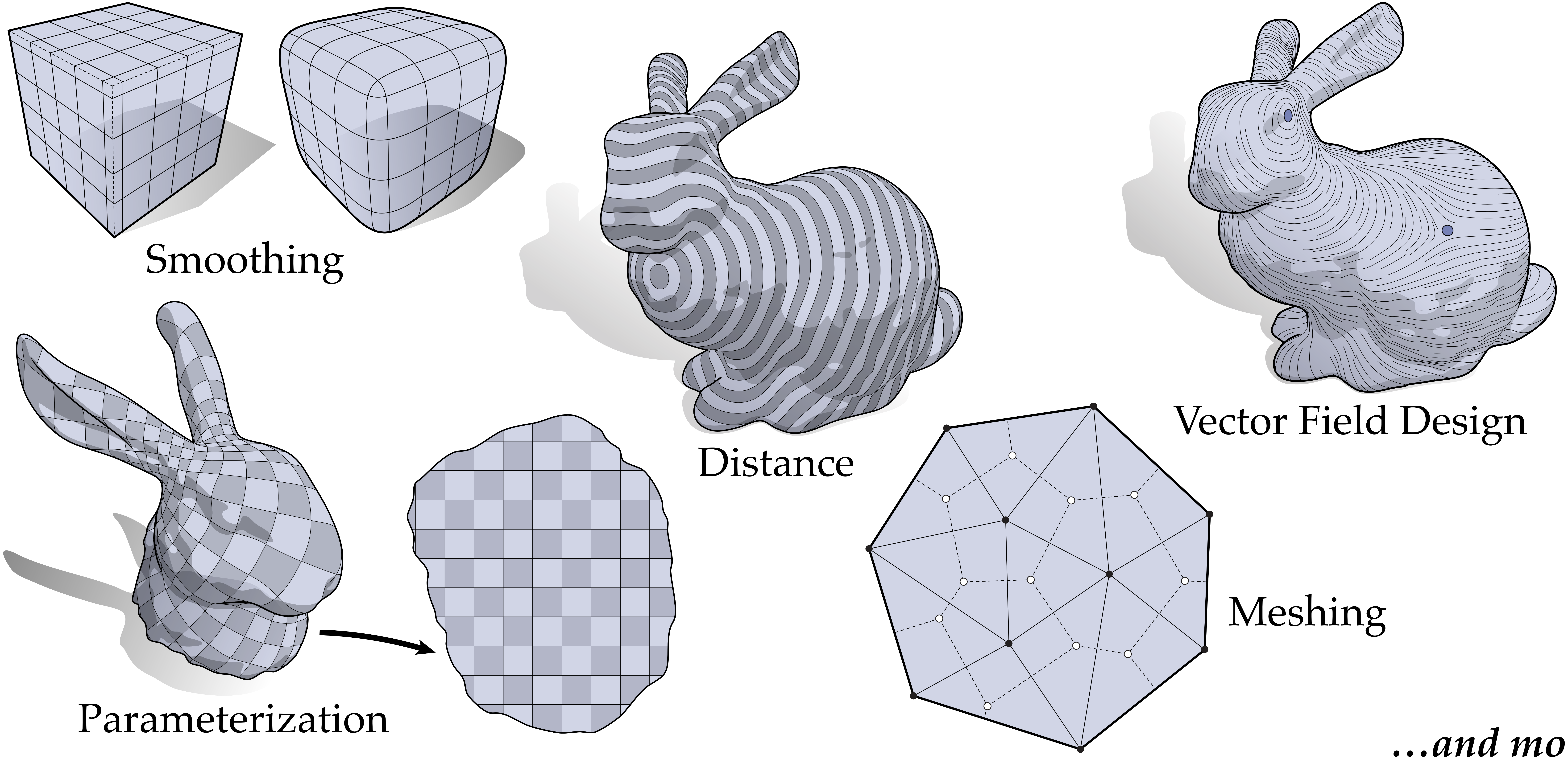
homology



cohomology

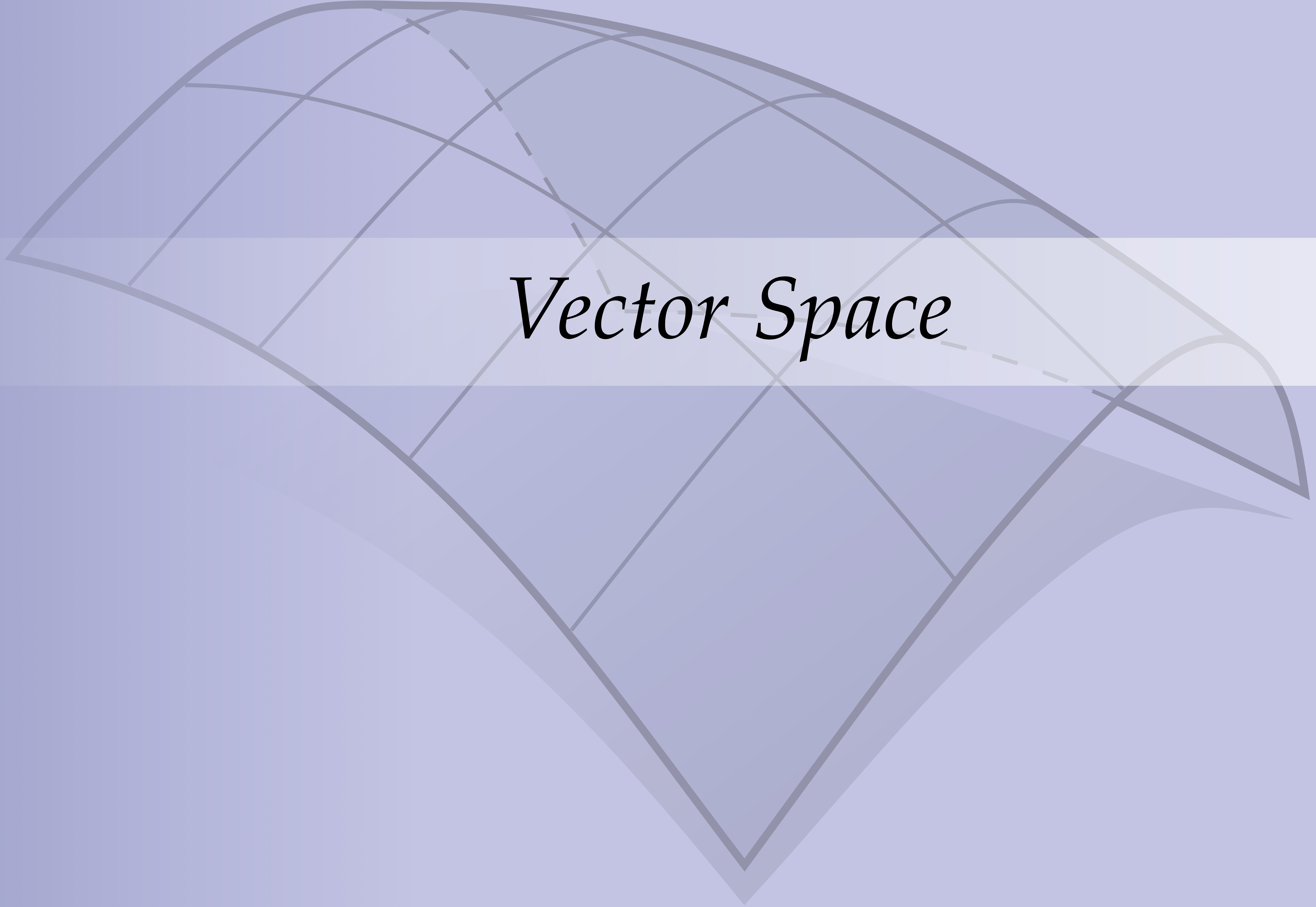


# *Applications*



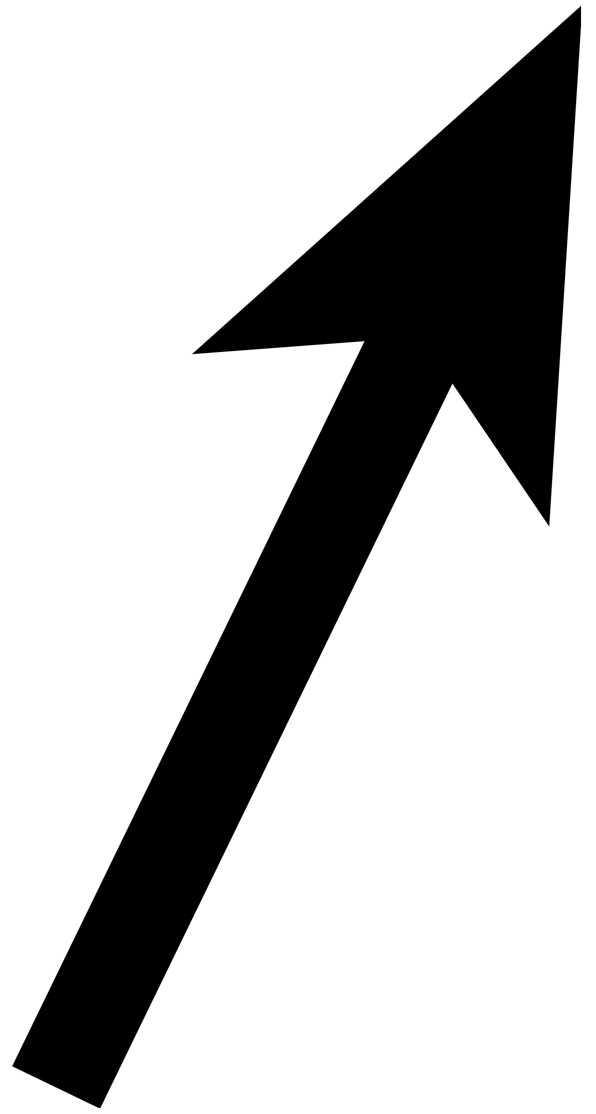

***...and more!***

# Vector Space

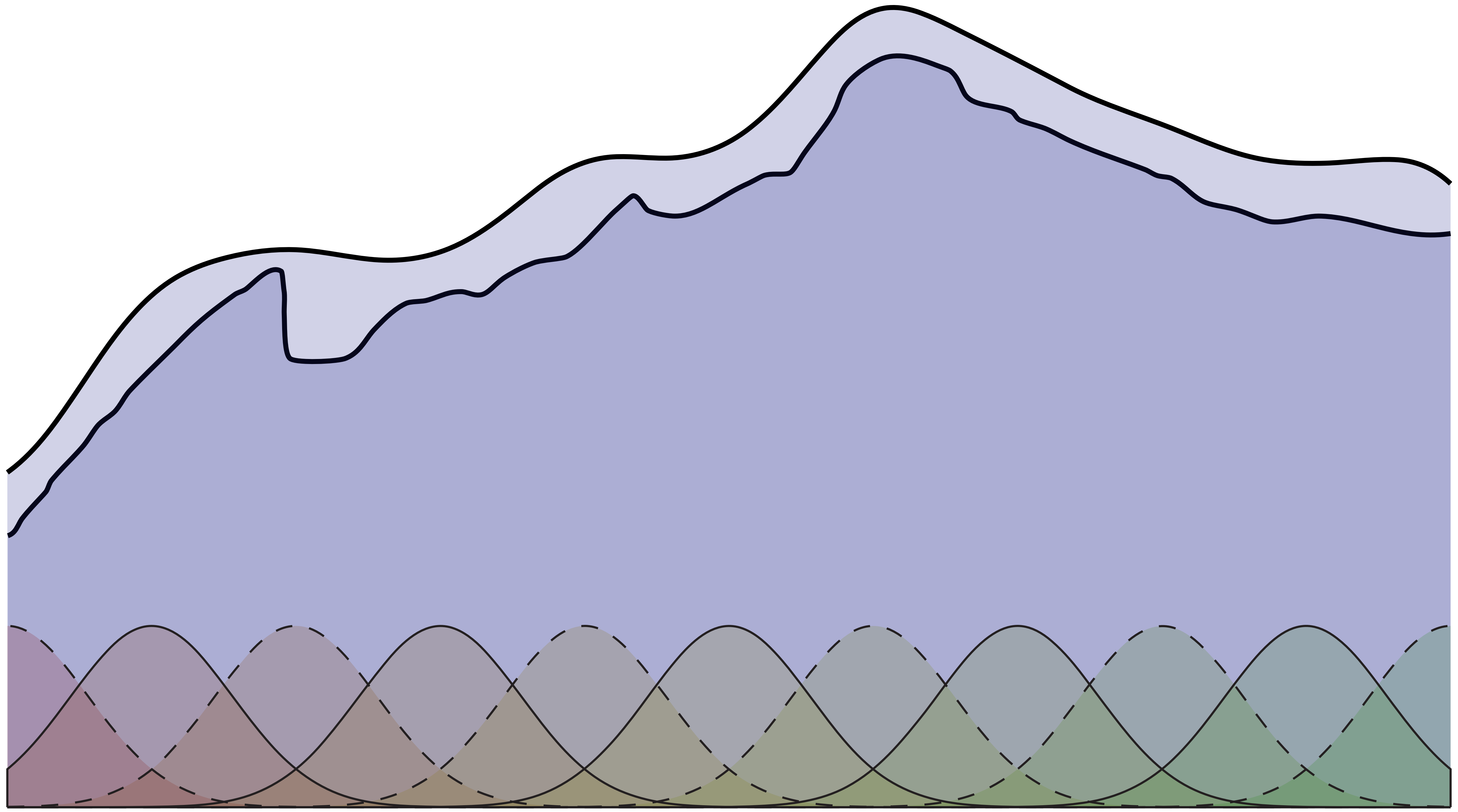


# *Review: Vector Spaces*

- What is a vector? *(Geometrically?)*



**finite-dimensional**



**infinite-dimensional**

For geometric computing, often care most about dimensions 1, 2, 3, …and $\infty$!

# *Review: Vector Spaces*

- Formally, a *vector space* is a set V together with a binary operations*

$$+ : V \times V \to V \qquad \text{“addition”}$$

$$\cdot : \mathbb{R} \times V \to V \qquad \text{“scalar multiplication”}$$

- Must satisfy the following properties for all vectors *x,y,z* and scalars *a,b*:

$$x + y = y + x$$

$$(x + y) + z = x + (y + z)$$

$$\exists 0 \in V \text{ s.t. } x + 0 = 0 + x = x$$

$$\forall\, x, \exists \tilde{x} \in V \text{ s.t. } x + \tilde{x} = 0$$

$$(ab)x = a(bx)$$

$$1x = x$$

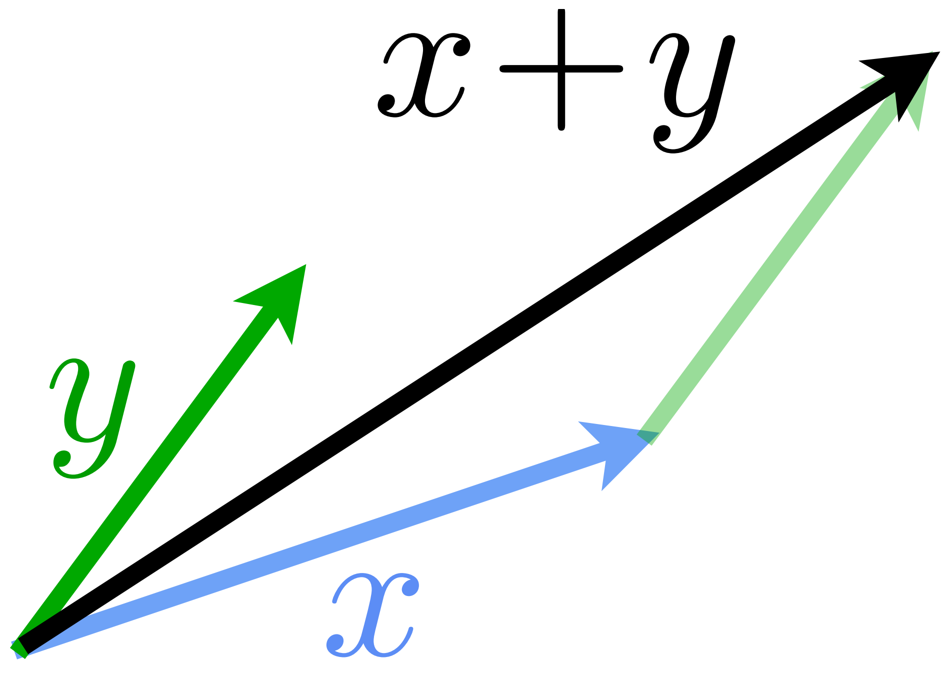
$$a(x + y) = ax + ay$$

$$(a + b)x = ax + bx$$

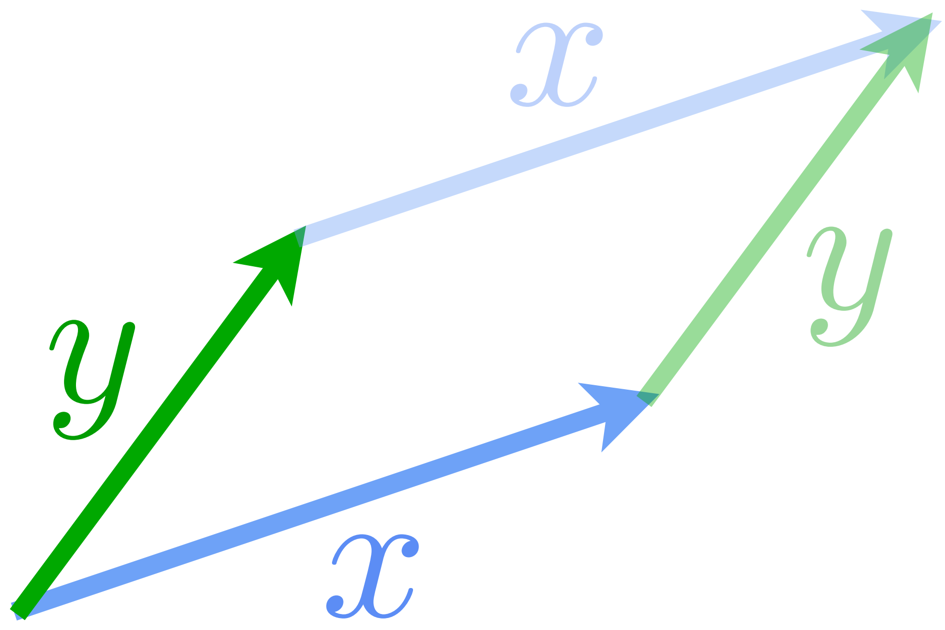
*Note: in general, could use something other than *reals* here.

# *Vector Spaces—Geometric Reasoning*

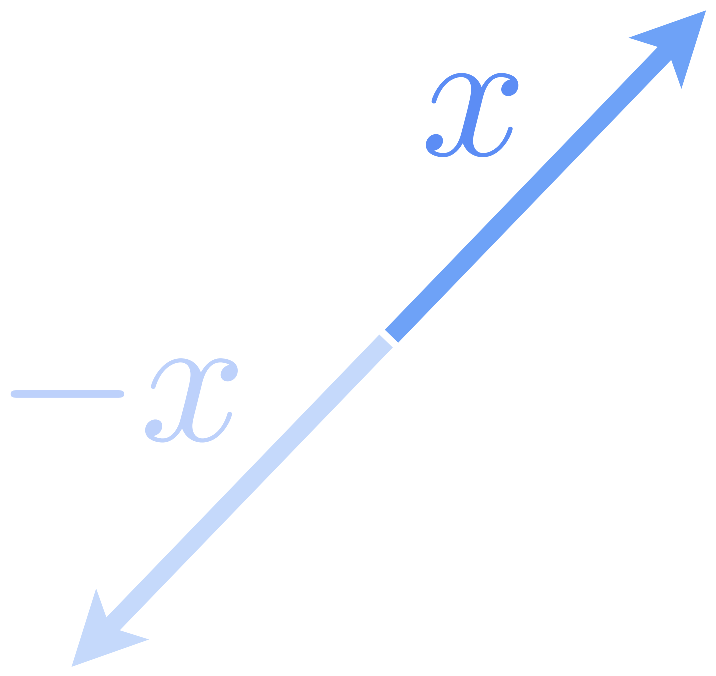
- Where do these rules come from?
- As with numbers, reflect how *oriented lengths* (vectors) behave in nature.



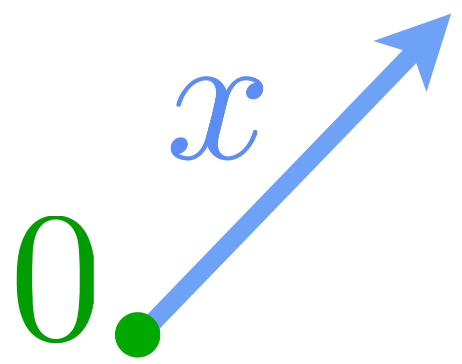

$$\boxed{x+y \in V}$$



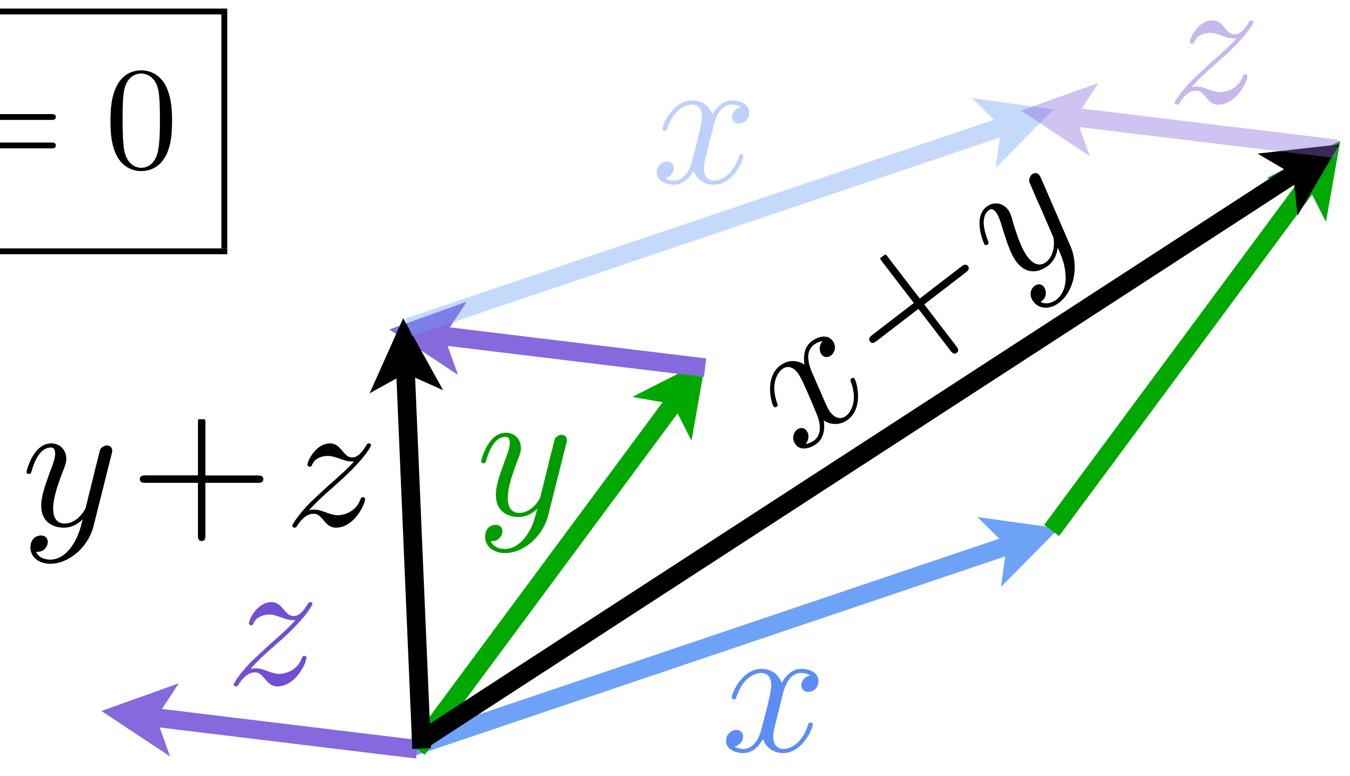

$$\boxed{x+y=y+x}$$




$$\boxed{x+(-x)=0}$$




$$\boxed{x+0=x}$$

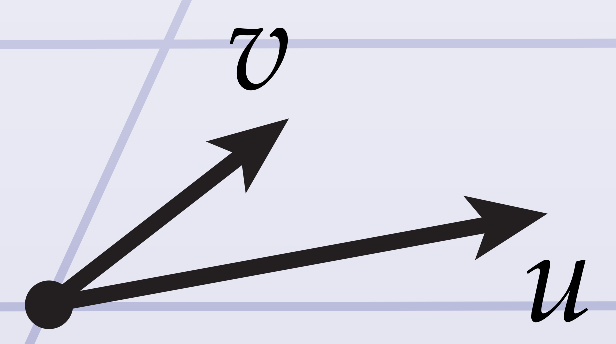



$$\boxed{(x+y)+z=x+(y+z)}$$

# *Wedge Product*

# *Review: Span*

**Q:** Geometrically, what is the *span* of two vectors?




$$u, v \in V, \quad \mathrm{span}(\{u, v\}) := \{x \in V \mid x = au + bv,\ a, b \in \mathbb{R}\}$$

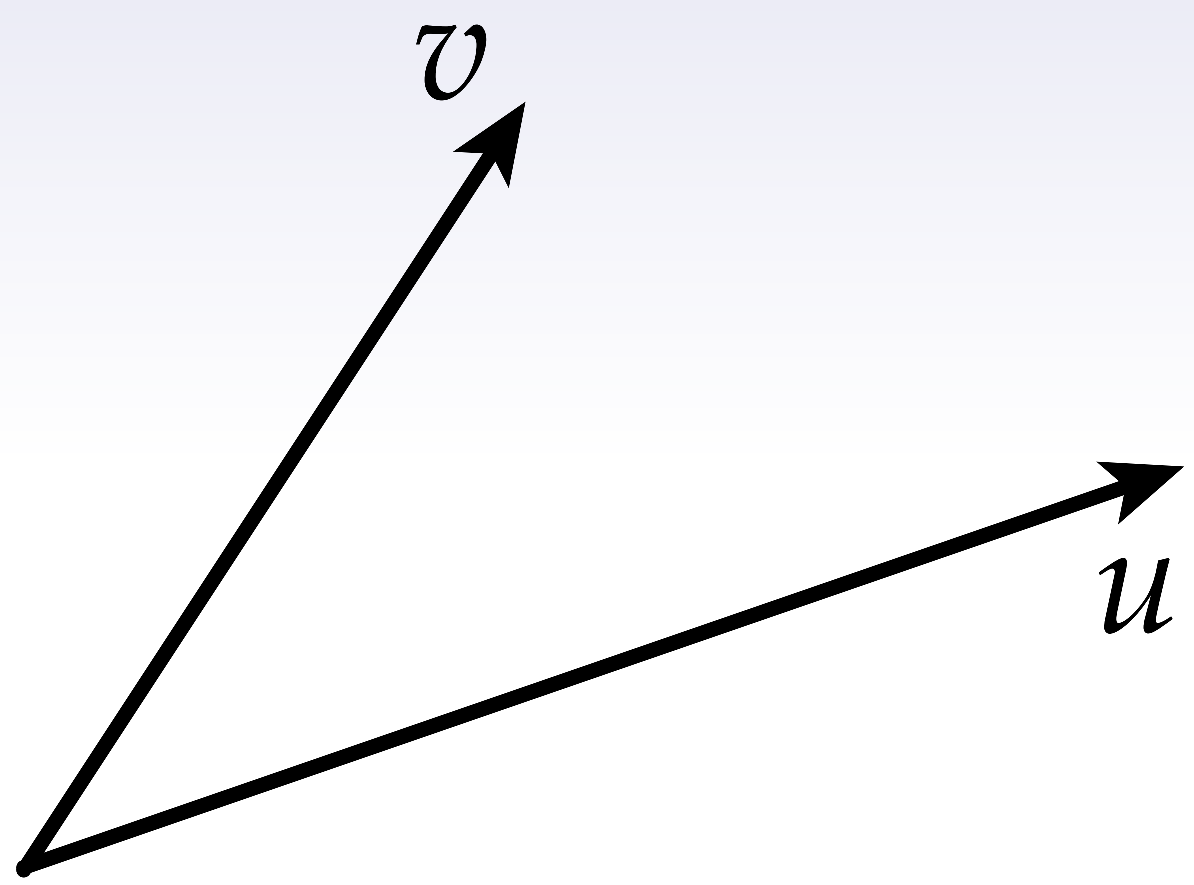
# *Span*

**Definition.** In any vector space $V$, the *span* of a finite collection of vectors $\{v_1, \ldots, v_k\}$ is the set of all possible linear combinations

$$\mathrm{span}(\{v_1, \ldots, v_n\}) := \left\{ x \in V \;\middle|\; x = \sum_{i=1}^{k} a_i v_i, \quad a_i \in \mathbb{R} \right\}.$$

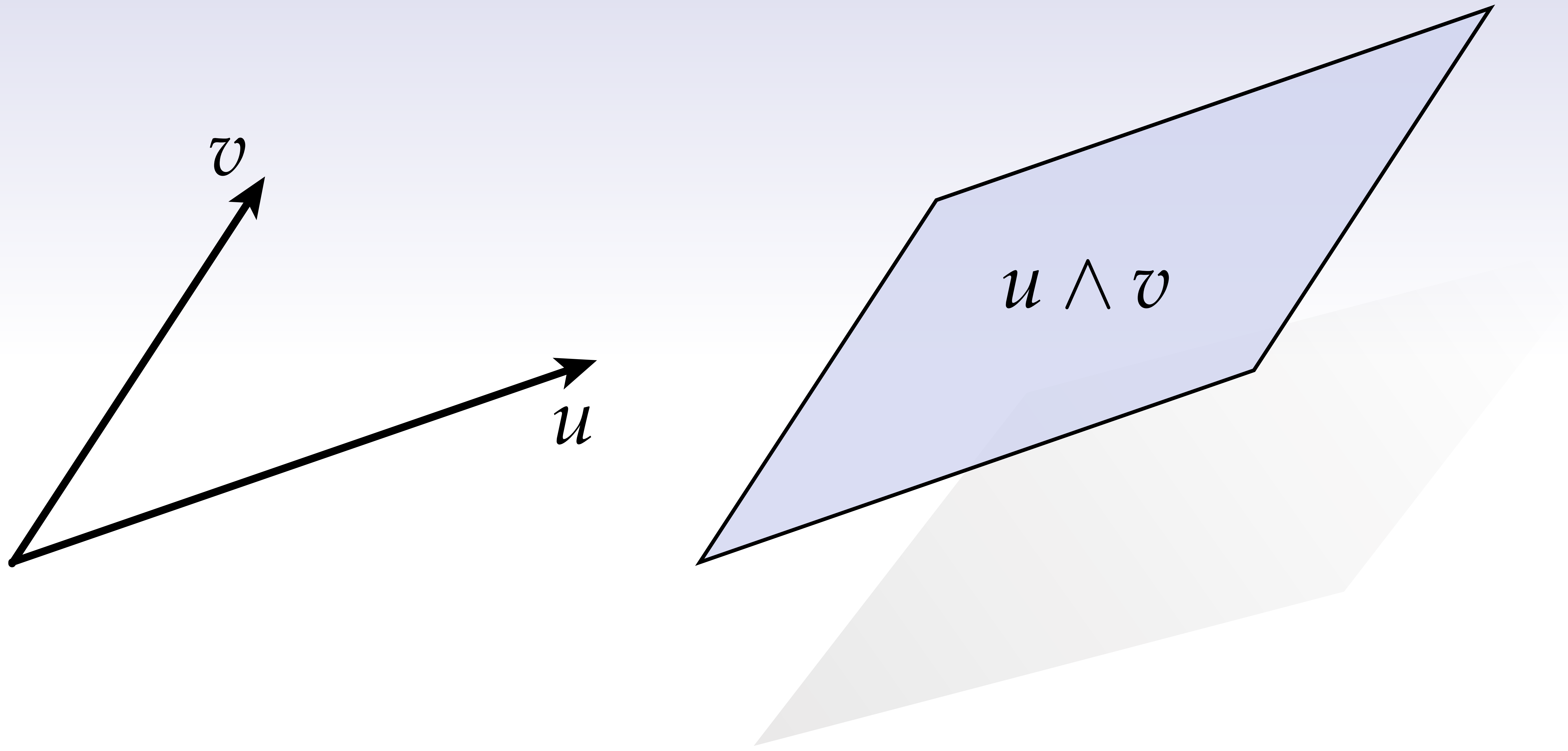
(*Note:* one cannot extend this definition to infinite sums without additional assumptions about $V$.) The span of a collection of vectors is a *linear subspace*, *i.e.*, a subset that forms a vector space with respect to the original vector space operations.

# *Wedge Product* $(\wedge)$



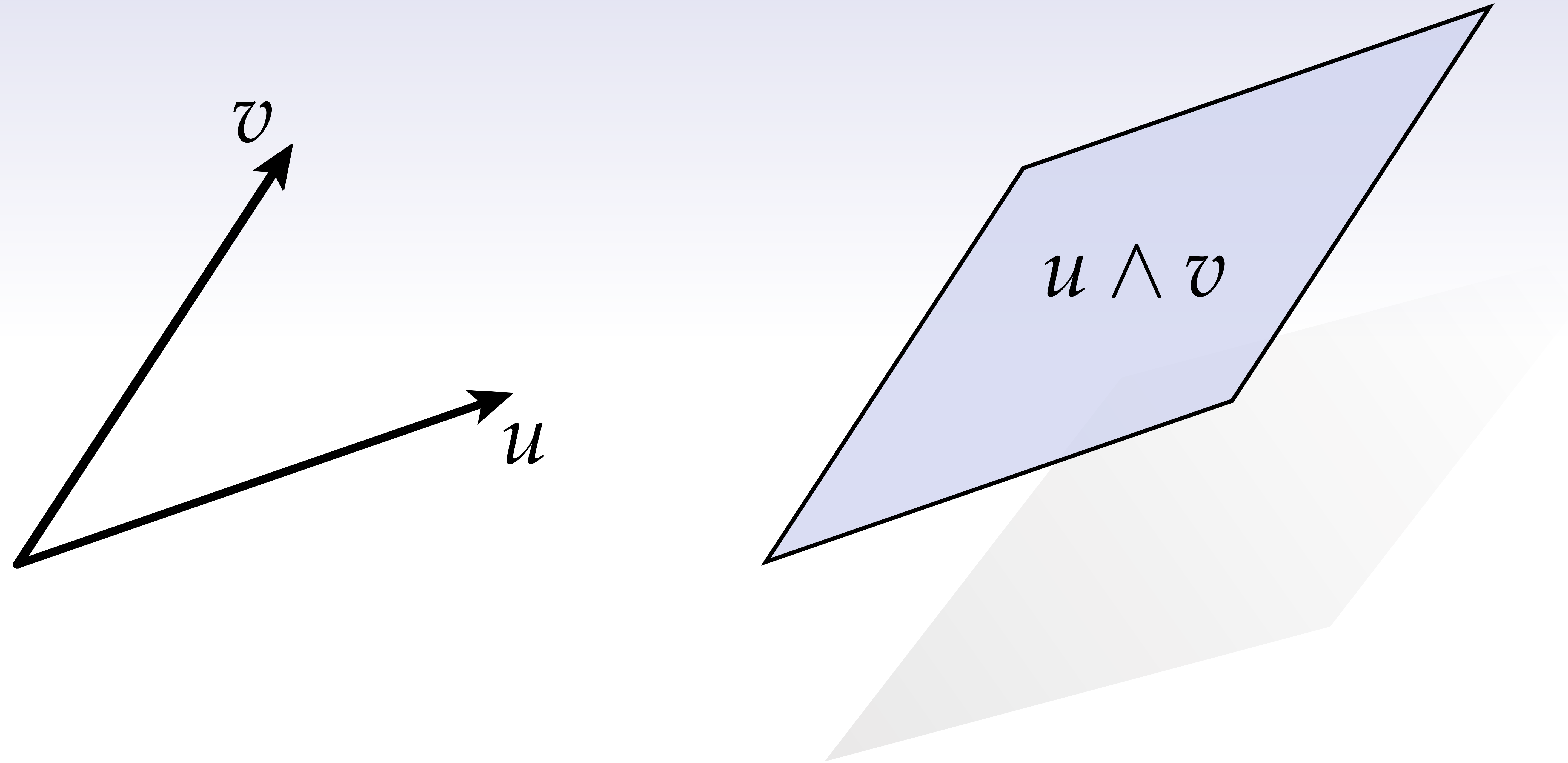

**Analogy:** *span*

# *Wedge Product* $(\wedge)$




**Analogy:** *span*

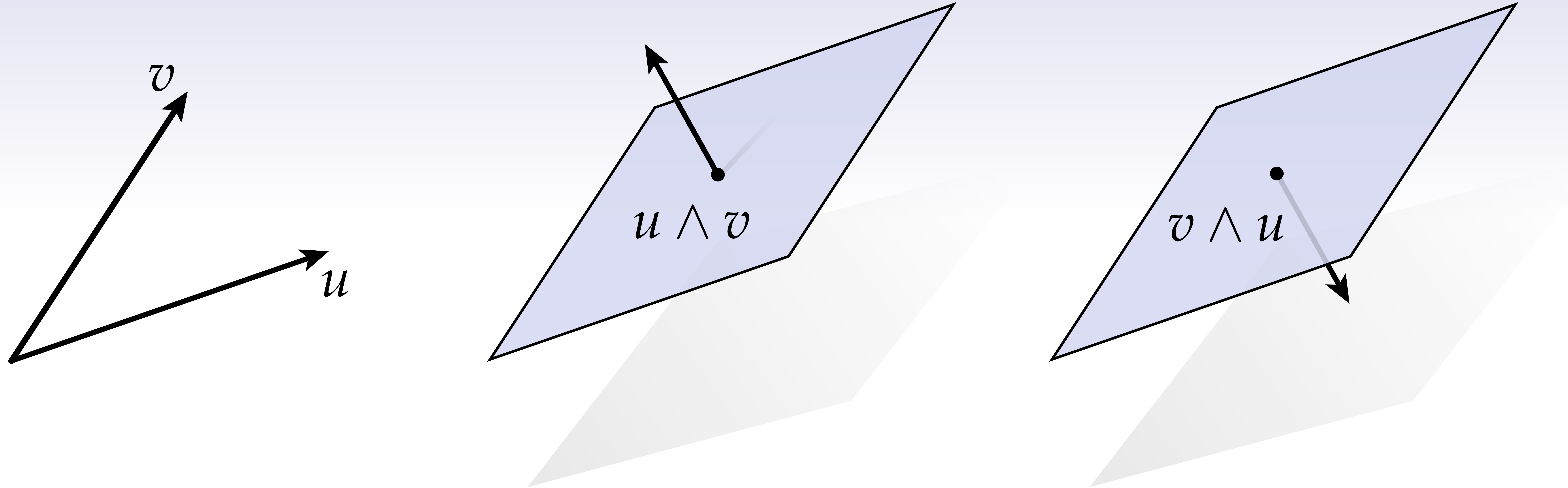
# *Wedge Product* $(\wedge)$




**Analogy:** *span*

# *Wedge Product* $(\wedge)$

$$u \wedge v = -v \wedge u$$



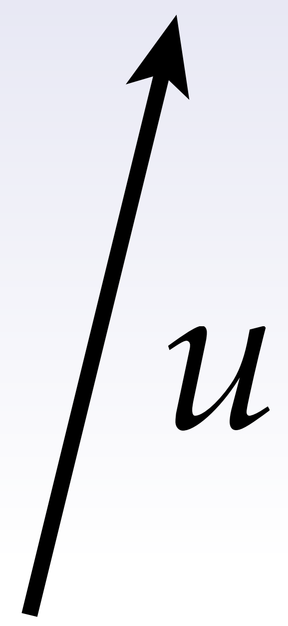

**Analogy:** *span*

**Key differences:** orientation & "finite extent"

**Key property:** *antisymmetry*

# *Wedge Product—Degeneracy*

**Q:** What is the wedge product of a vector with itself?

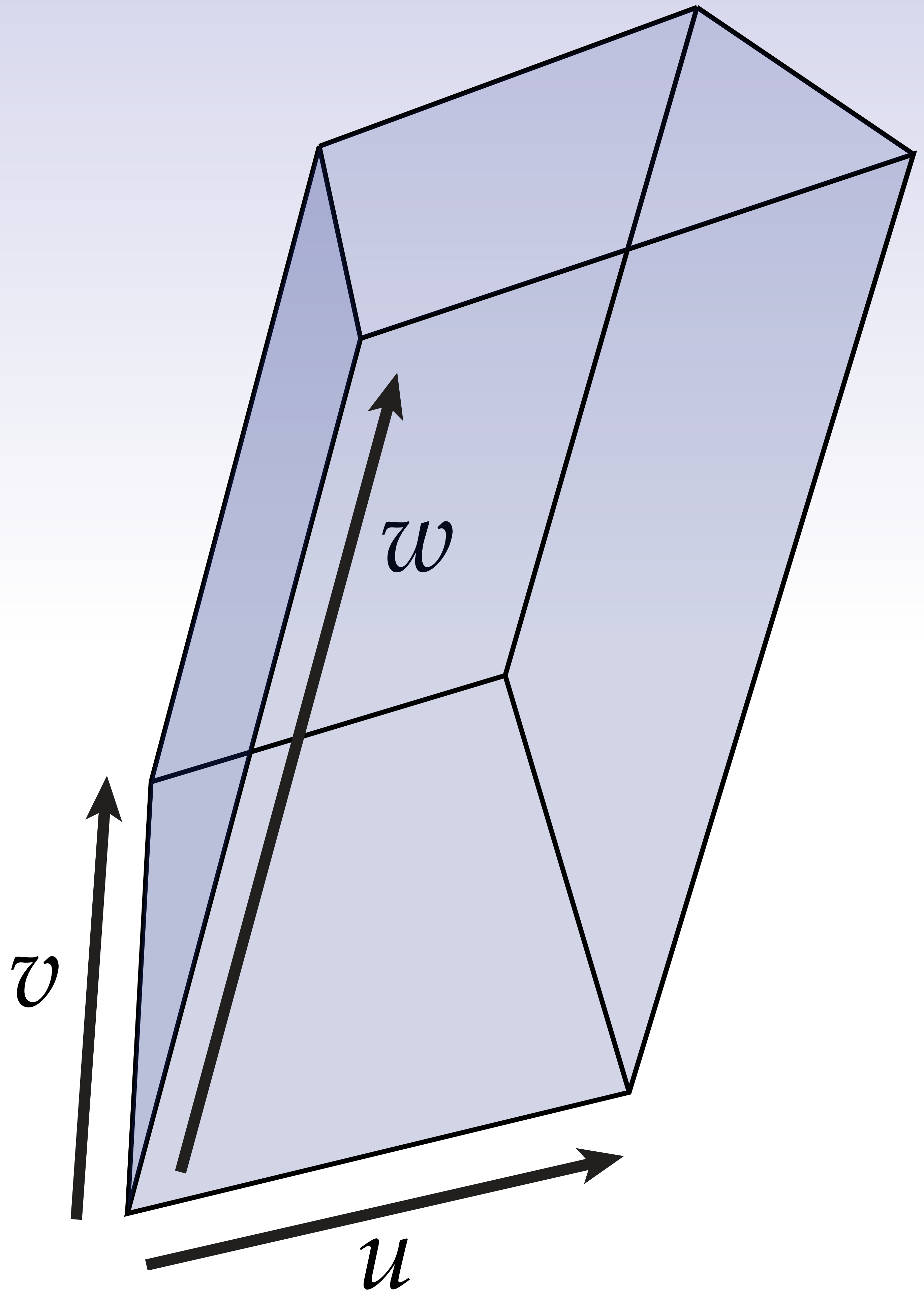



**A:** Geometrically, spans a region of *zero area*.

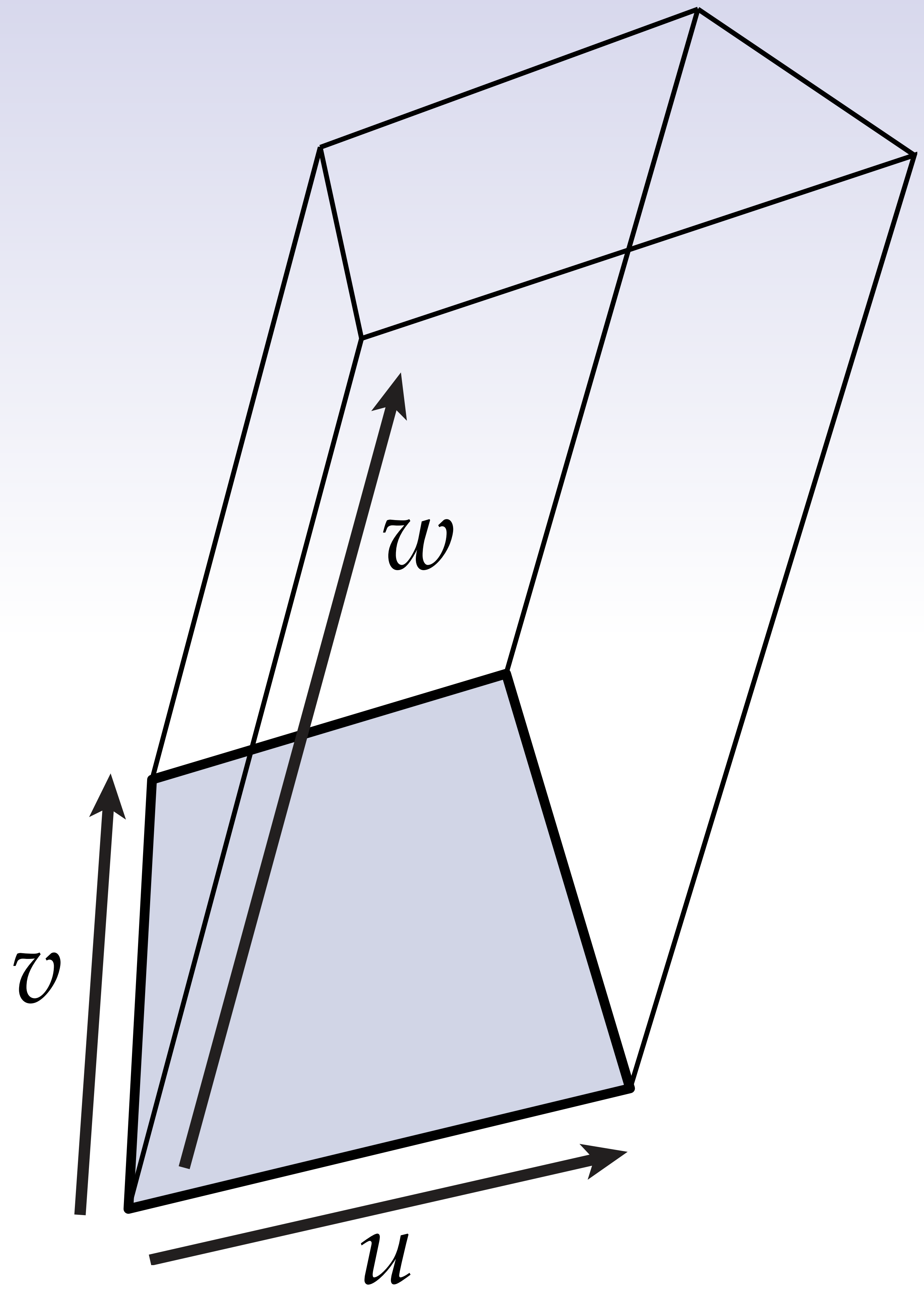
$$\boxed{u \wedge u = 0}$$

(*Slight oversimplification. More later...)

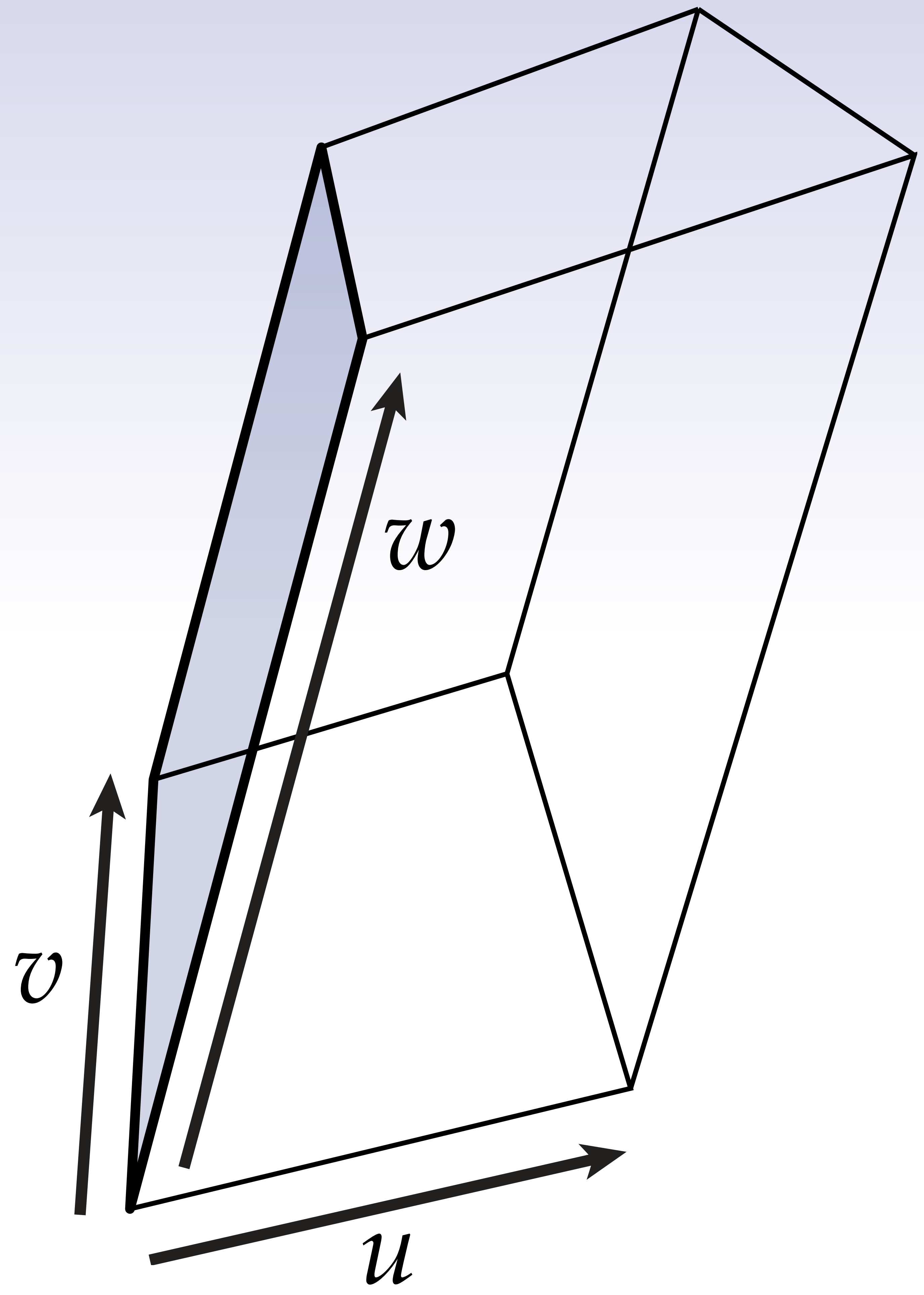
# *Wedge Product - Associativity*




$u \wedge v \wedge w$

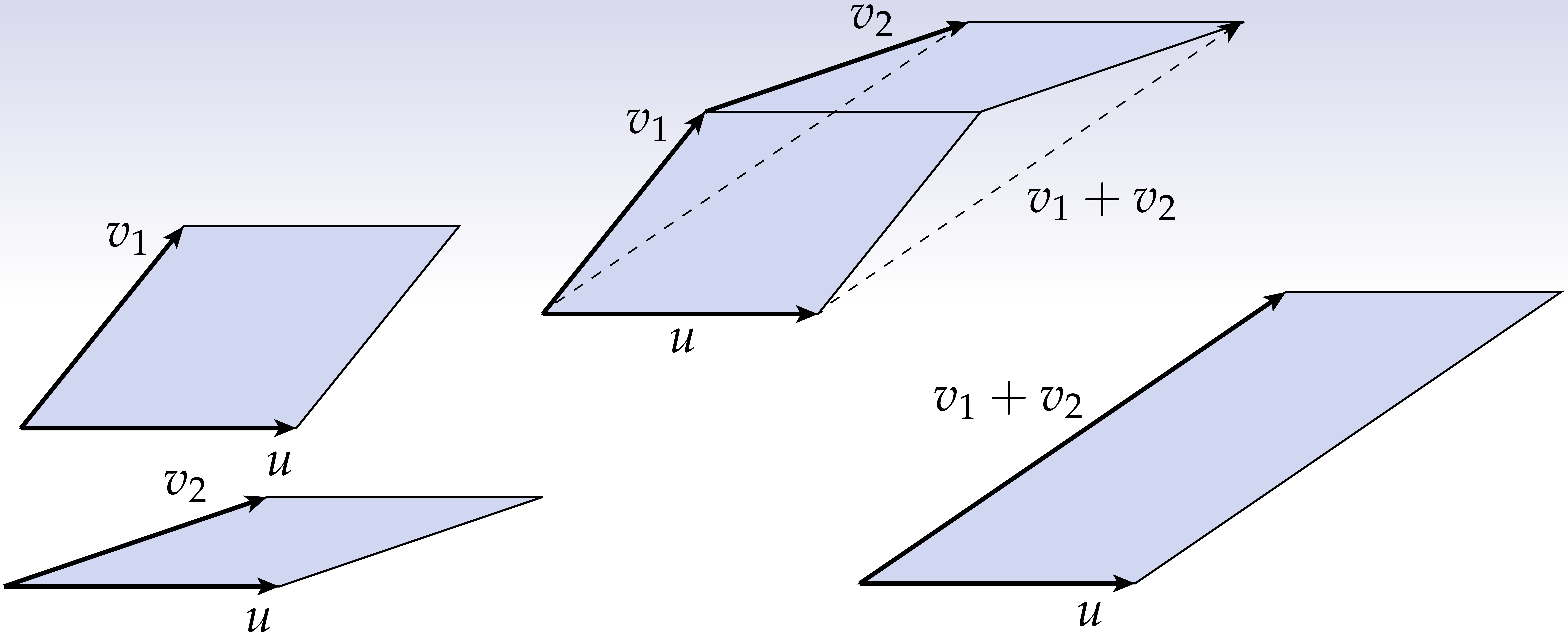



$(u \wedge v) \wedge w$




$u \wedge (v \wedge w)$

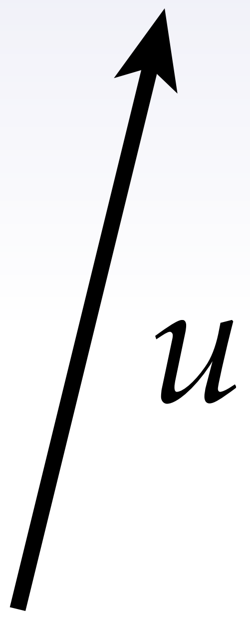
# *Wedge Product - Distributivity*

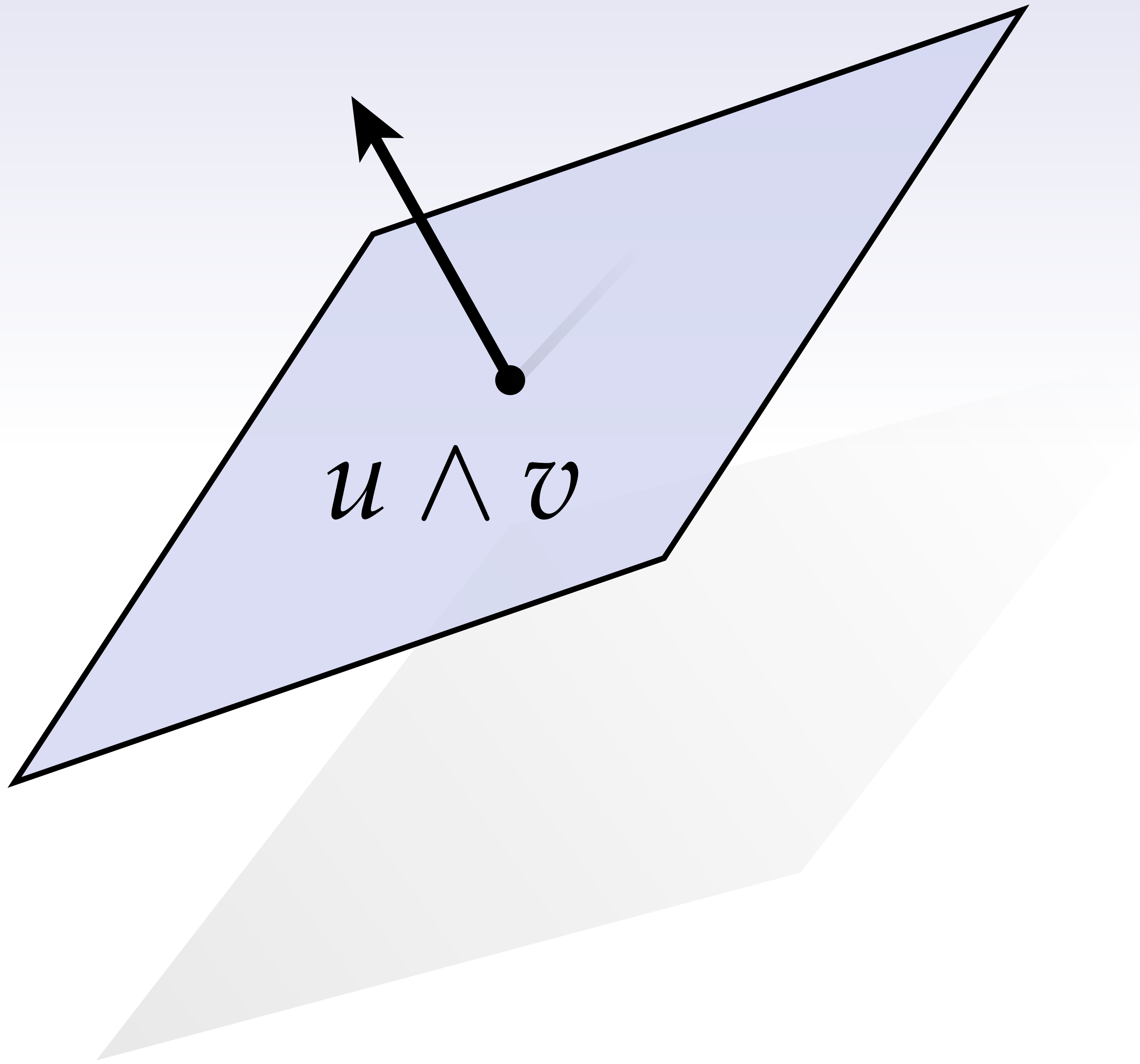


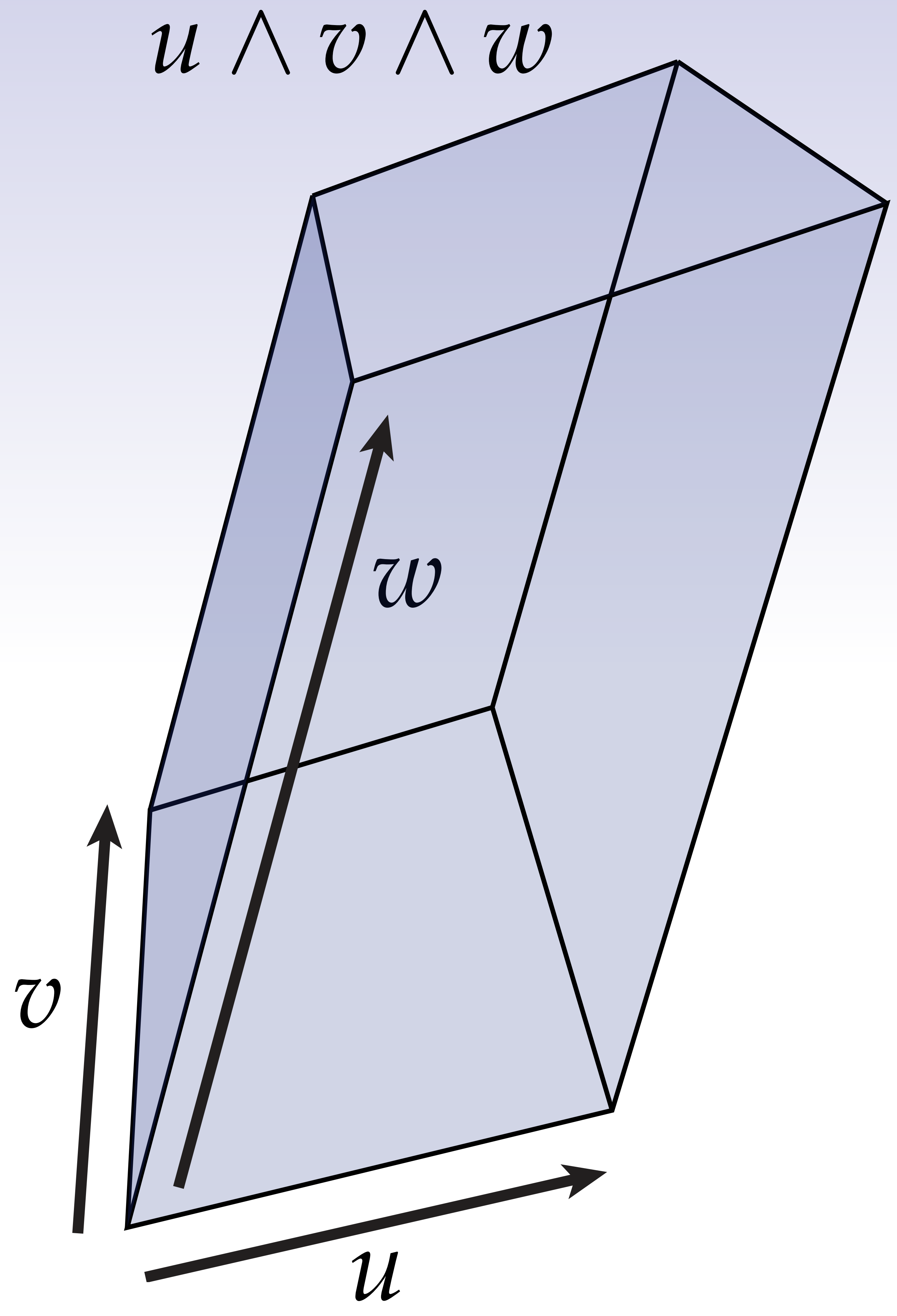

$$u \wedge v_1 + u \wedge v_2 = u \wedge (v_1 + v_2)$$

# *k-Vectors*

The wedge of $k$ vectors is called a *“k-vector.”*



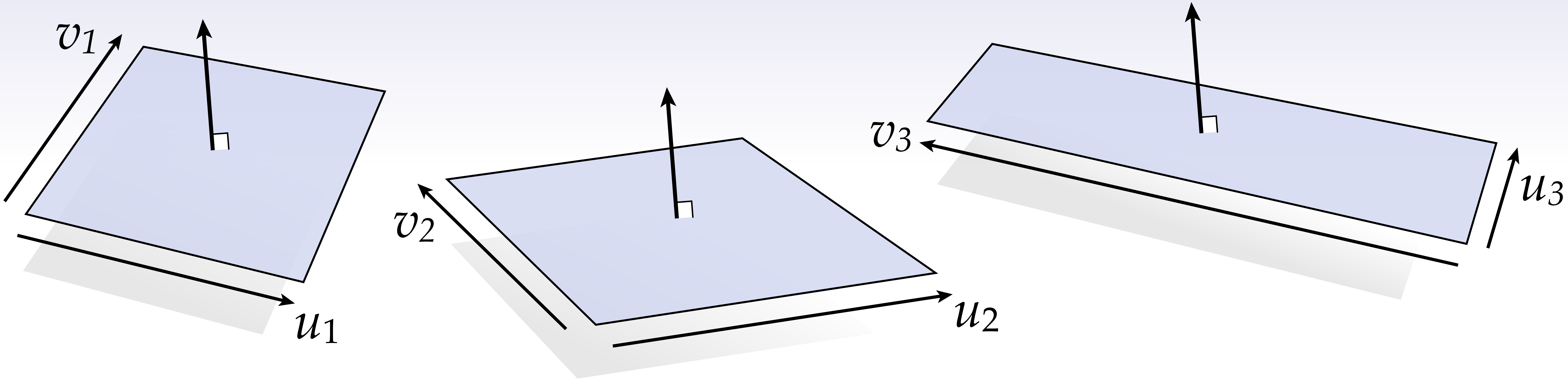




**0-vector** **1-vector** **2-vector** **3-vector**

# *Visualization of k-Vectors*

Our visualization is a little misleading: $k$-vectors only have *direction* & *magnitude*.

*E.g.*, parallelograms w/ same plane, orientation, and area represent same 2-vector:




$$u_1 \wedge v_1 = u_2 \wedge v_2 = u_3 \wedge v_3$$

(Could say a 2-form is an *equivalence class* of parallelograms…)

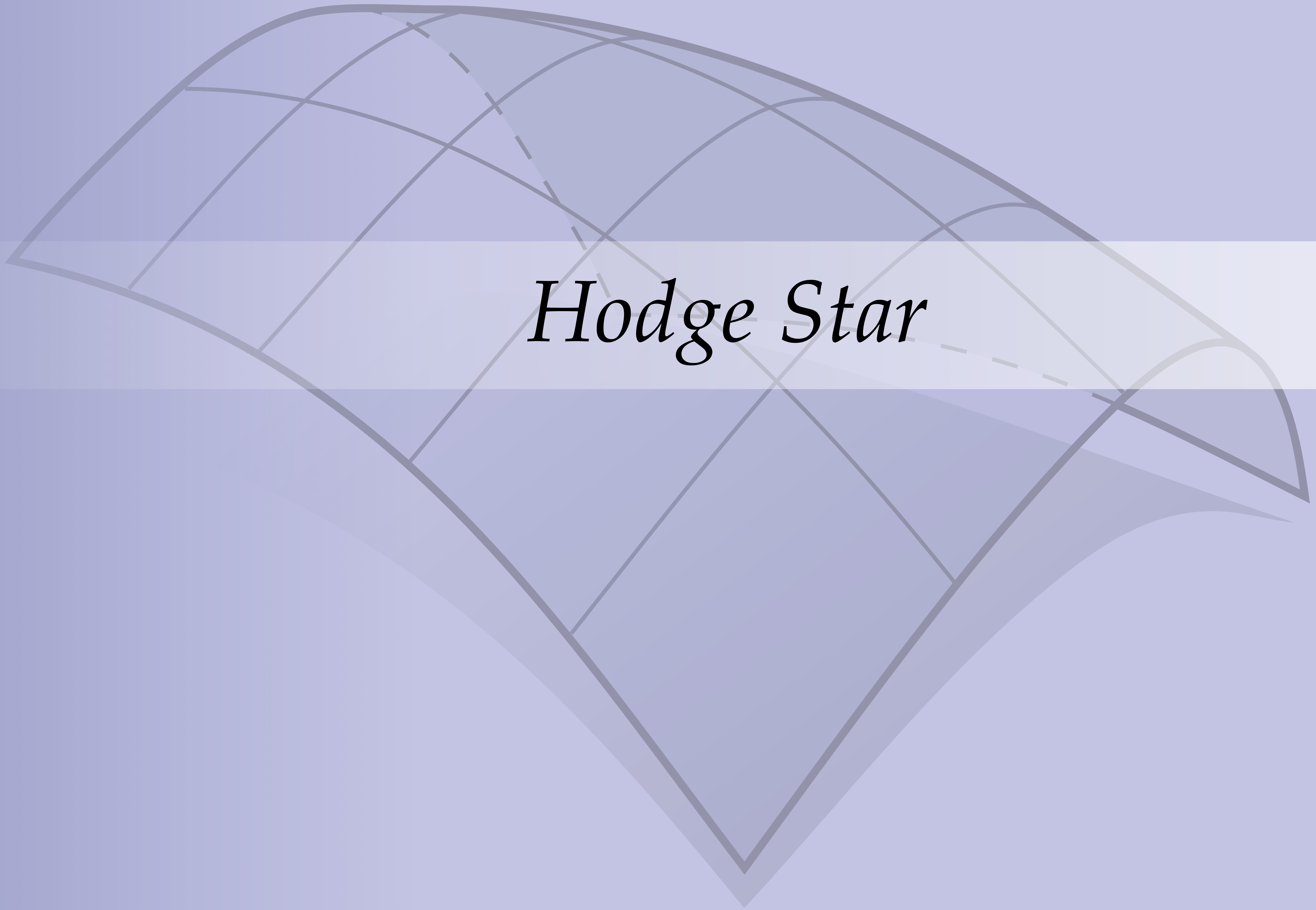
# *0-vectors as Scalars*

**Q:** What do you get when you wedge *zero* vectors together?

**A:** You get this:

For convenience, however, we will say that a "0-vector" is a *scalar value* (e.g., a real number). This treatment becomes extremely useful later on...

**Key idea:** *magnitude*, but no *direction* (scalar).



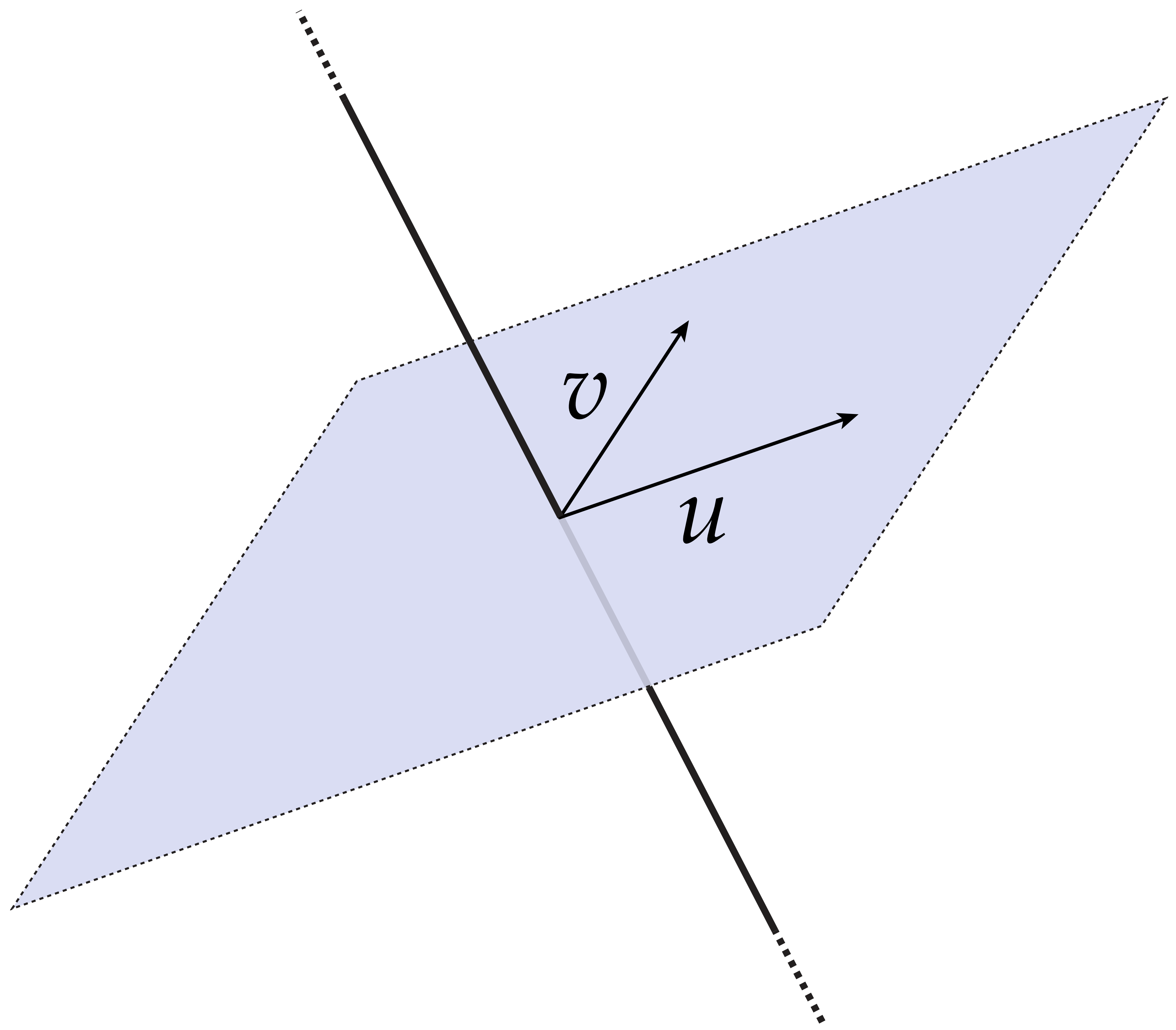
# *Hodge Star*

# *Review: Orthogonal Complement*

**Q:** Geometrically, what is the *orthogonal complement* of a linear subspace?

**Example:** *orthogonal complement of a span*

$$V := \mathrm{span}(\{u, v\})$$
$$V^\perp := \{x \in V \mid \langle x, w \rangle = 0 \;\; \forall\, w \in V\}$$

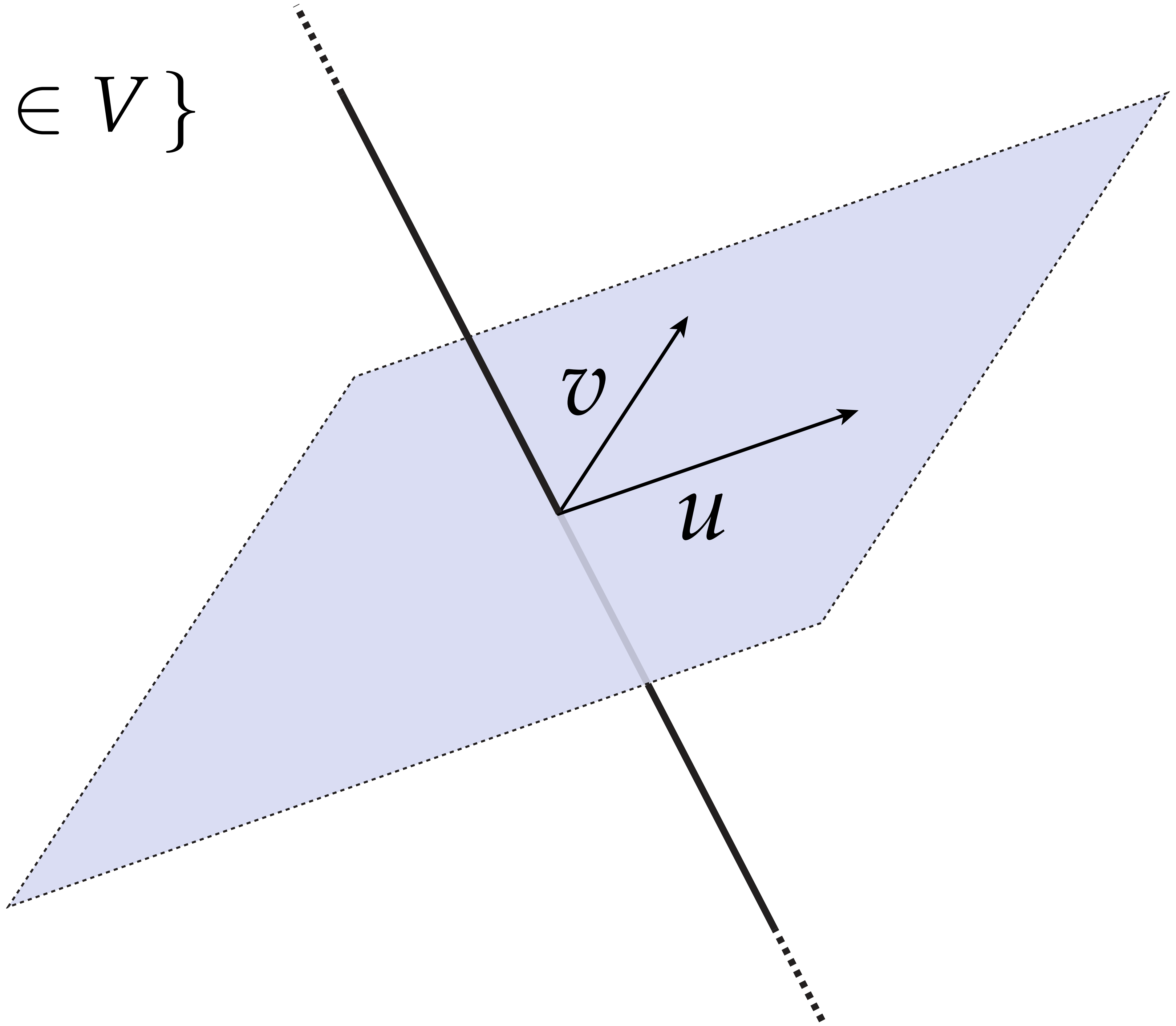



**Notice:** orthogonal complement meaningful only if we have an *inner product!*

# *Orthogonal Complement*

**Definition.** Let $(V, \langle\cdot,\cdot\rangle)$ be a real inner product space, and let $U \subseteq V$ be a linear subspace. The *orthogonal complement* of $V$ is the collection of vectors

$$U^\perp := \{u \in V \,|\, \langle u, v\rangle = 0 \;\; \forall v \in V\}$$

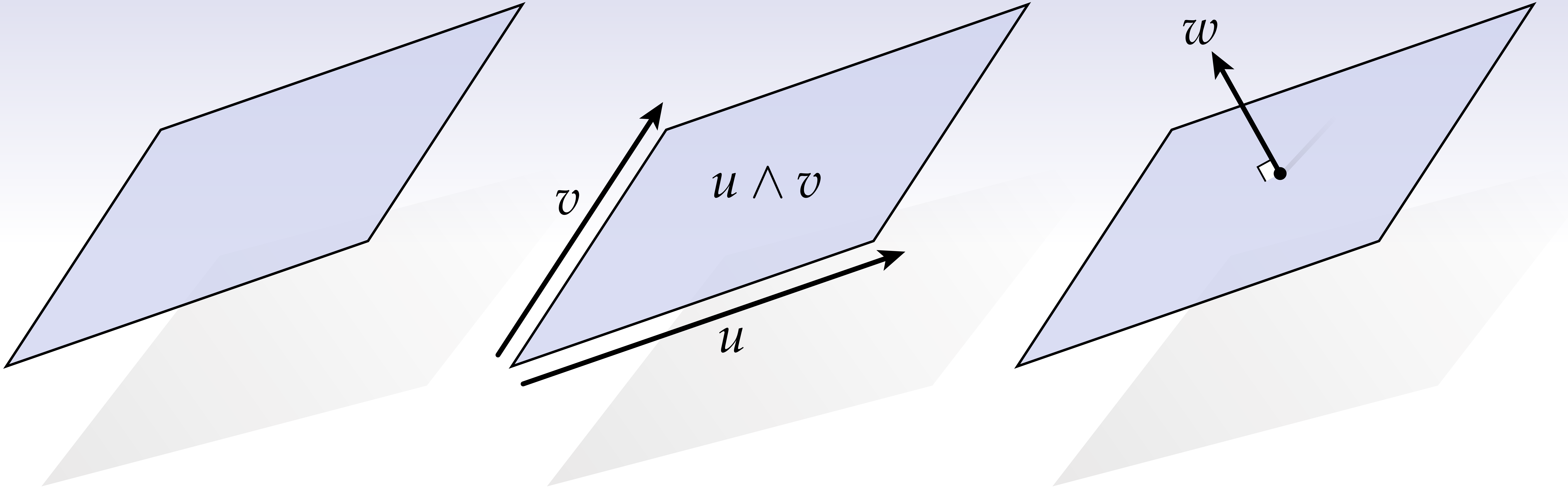



**Example.** "What kind of cuisine do you like?"
*Option 1:* "I like Vietnamese, Italian, Ethiopian, ..."
*Option 2:* "I like everything but Bavarian food!"

**Key idea:** often it's easier to specify a set by saying what it *doesn't* contain.

# *Hodge Star* $(\star)$

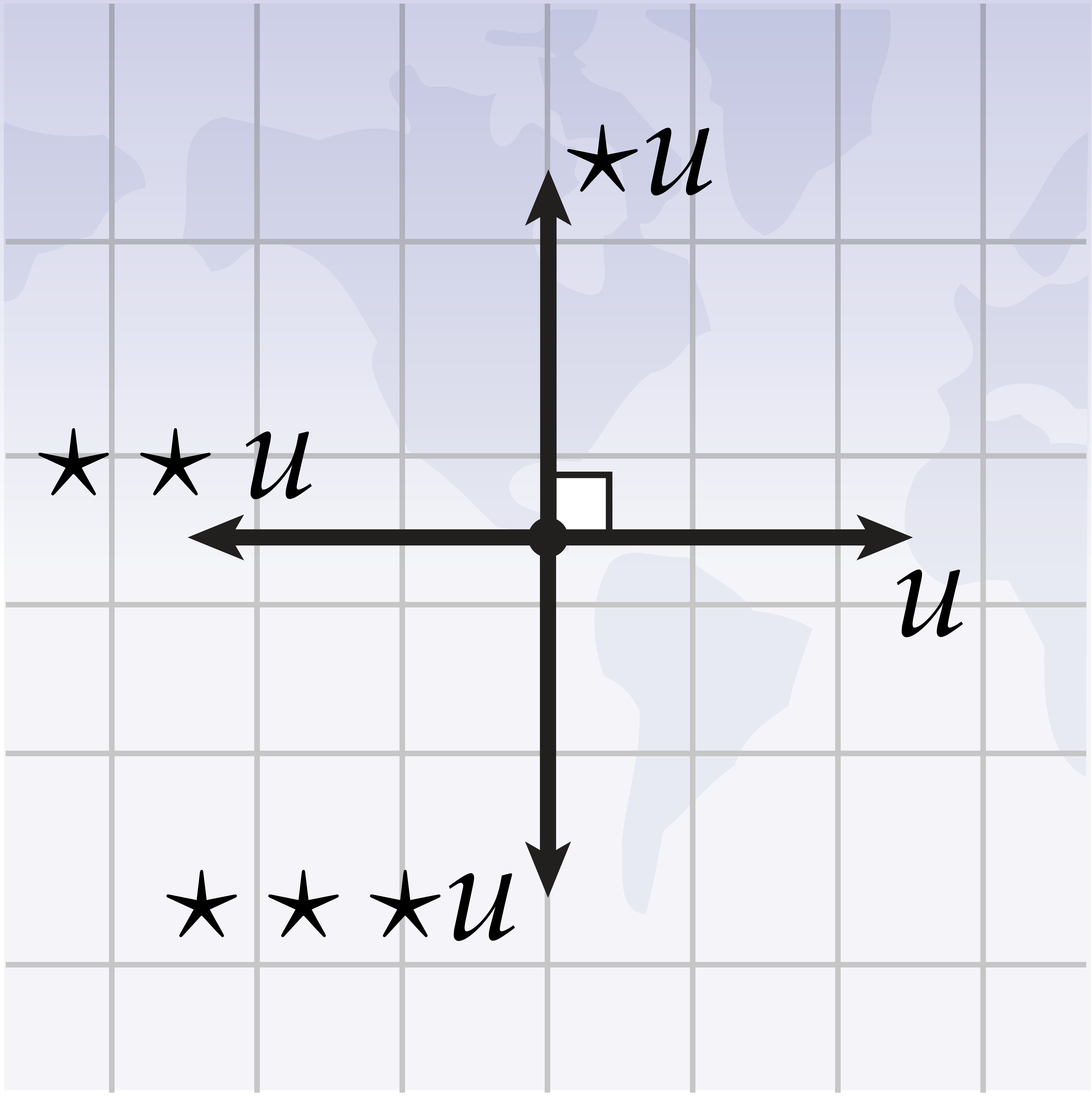
$$\star(u \wedge v) = w$$




**Analogy:** *orthogonal complement*
**Key differences:** orientation & "finite extent"
**Small detail:** $z \wedge \star z$ is *positively oriented*

$$k \mapsto (n-k)$$

# *Hodge Star - 2D*

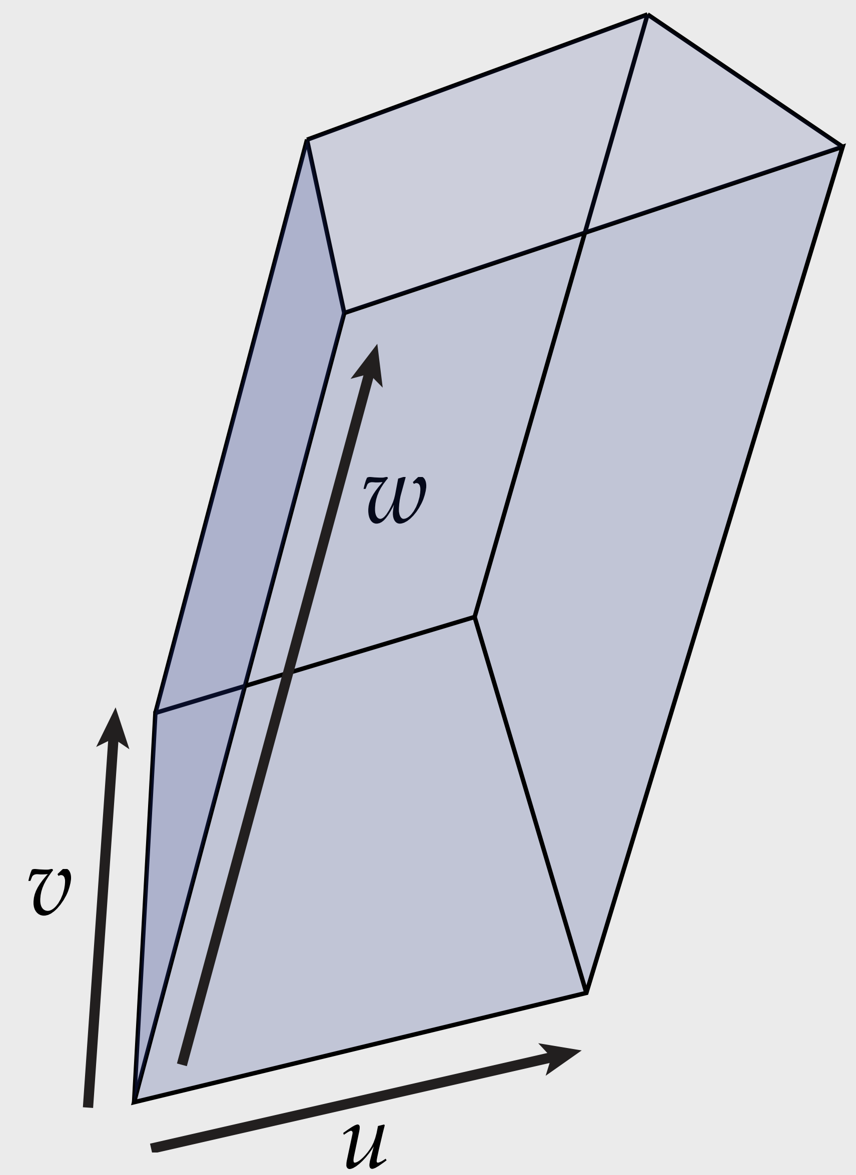



Analogy: *90-degree rotation*

# *Exterior Algebra—Recap*

Let $V$ be an $n$-dimensional vector space, consisting of vectors or *1-vectors*.

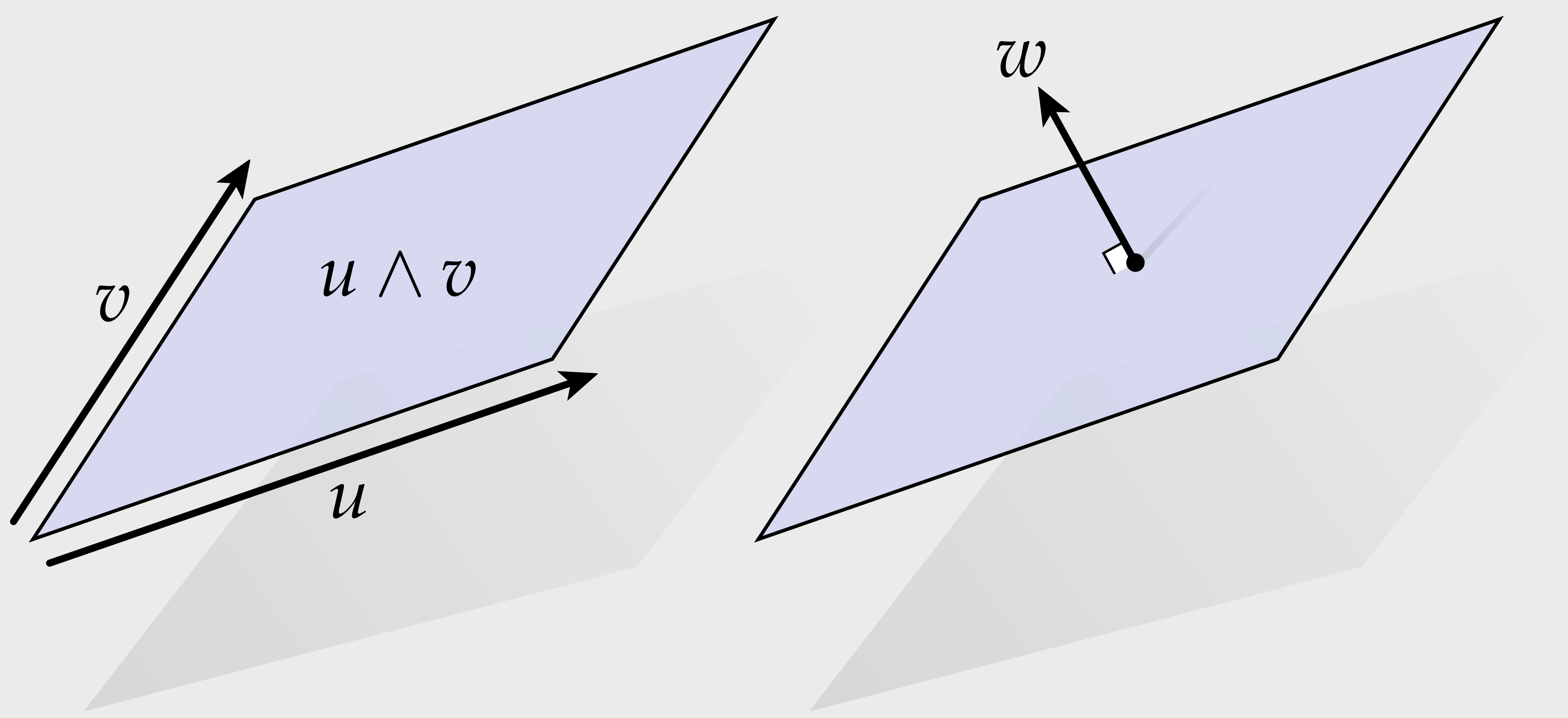
Can "wedge together" $k$ vectors to get a *k-vector* (signed volume).




$u \wedge v \wedge w$

Can apply the Hodge star to get the "complementary" $k$-vector.

$\star(u \wedge v) = w$



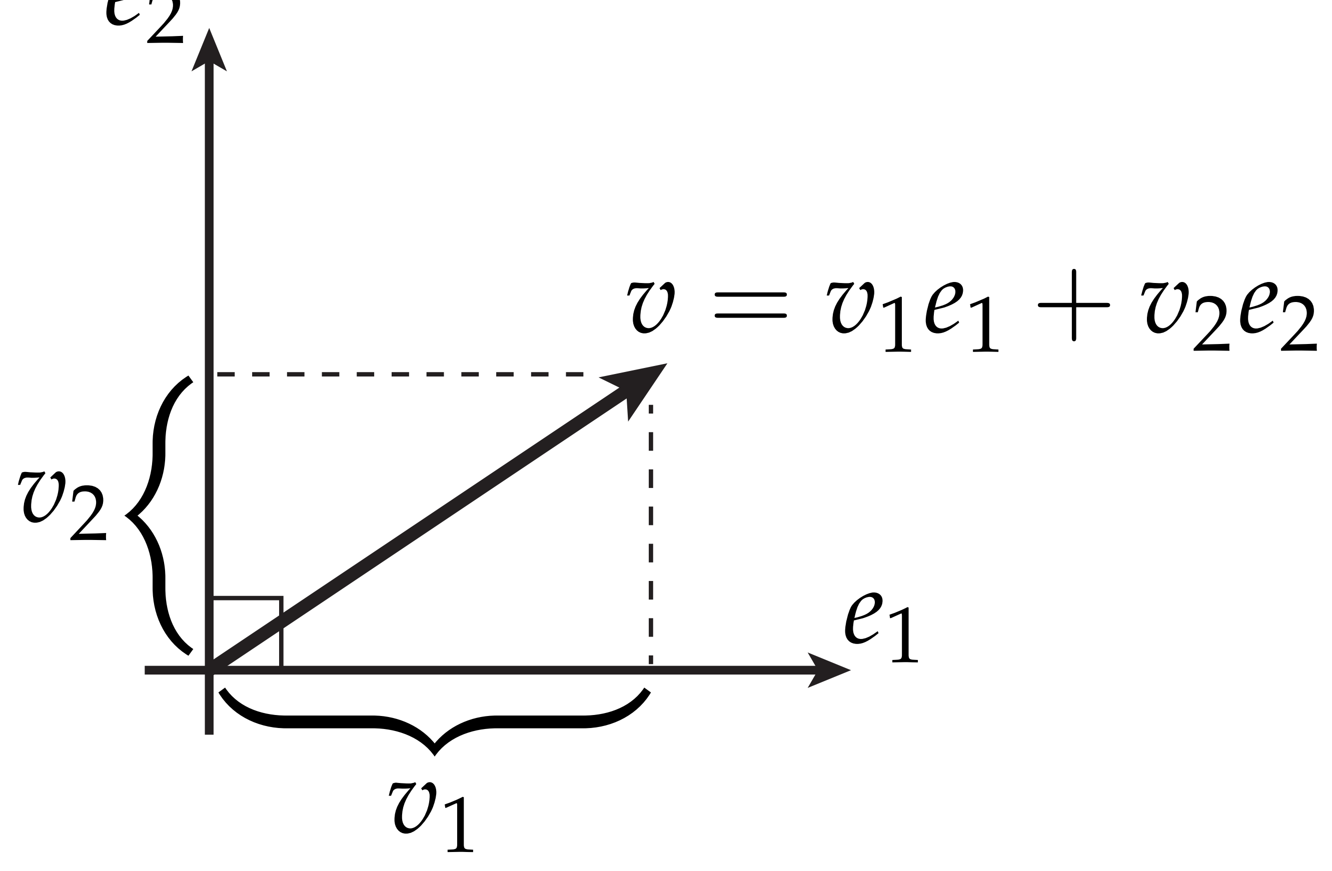

(Also have the usual vector space operations: sum, scalar multiplication, ...)

# *Basis*

**Definition.** Let $V$ be a vector space. A collection of vectors is *linearly independent* if no vector in the collection can be expressed as a linear combination of the others. A linearly independent collection of vectors $\{e_1, \ldots, e_n\}$ is a *basis* for $V$ if every vector $v \in V$ can be expressed as

$$v = v_1 e_1 + \cdots + v_n e_n$$

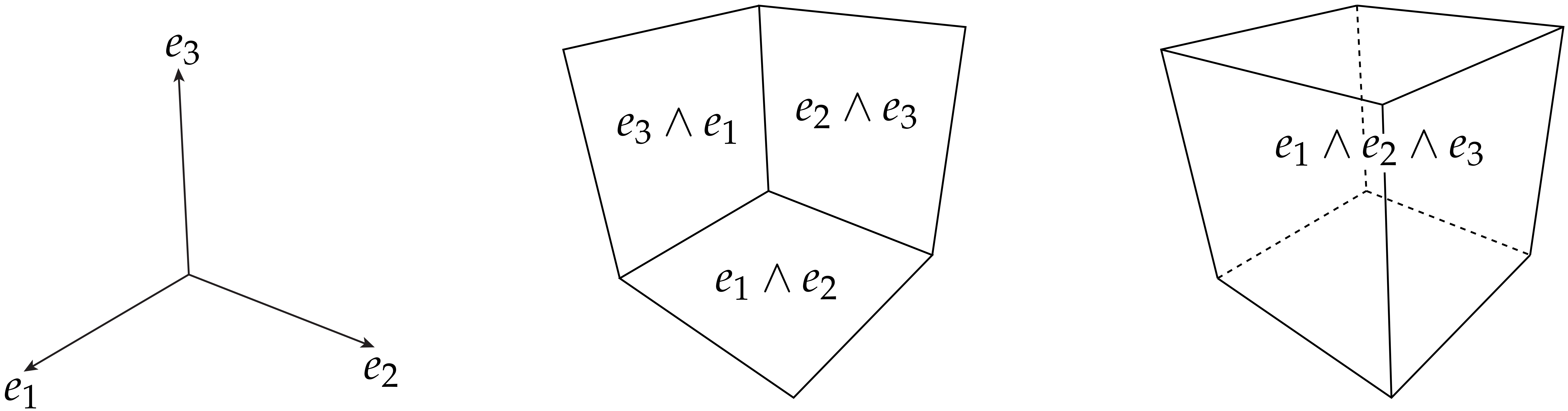
for some collection of coefficients $v_1, \ldots, v_n \in \mathbb{R}$, i.e., if every vector can be uniquely expressed as a linear combination of the *basis vectors* $e_i$. In this case, we say that $V$ is *finite dimensional*, with dimension $n$.



# *Basis k-Vectors — Visualized*

$(V = \mathbb{R}^3)$

**basis 1-vectors** **basis 2-vectors** **basis 3-vectors**




**Key idea:** signed volumes can be expressed as linear combinations of "basis volumes" or basis *k-vectors*.

# *Basis k-Vectors—How Many?*

Consider $V = \mathbb{R}^4$ with basis $\{e_1, e_2, e_3, e_4\}$.

**Q:** How many basis 2-vectors?

$$\begin{array}{lll} e_1 \wedge e_2 & & \\ e_1 \wedge e_3 & e_2 \wedge e_3 & \\ e_1 \wedge e_4 & e_2 \wedge e_4 & e_3 \wedge e_4 \end{array}$$

**Q:** How many basis 3-vectors?

$$\begin{array}{l} e_1 \wedge e_2 \wedge e_3 \\ e_1 \wedge e_2 \wedge e_4 \\ e_1 \wedge e_3 \wedge e_4 \\ e^2 \wedge e^3 \wedge e^4 \end{array}$$

Why not $e_3 \wedge e_2$? $e_4 \wedge e_4$?
*What do these bases represent* ***geometrically?***

**Q:** How many basis 4-vectors?

$$e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

**Q:** How many basis 1-vectors?

**Q:** How many basis 0-vectors?

**Q:** Notice a pattern?

| $\mathbb{R}^3$ | $\mathbb{R}^4$ |
|---|---|
| 1 | 1 |
| 3 | 4 |
| 3 | 6 |
| 1 | 4 |
| | 1 |

# *Hodge Star — Basis k-Vectors*

Consider $V = \mathbb{R}^3$ with basis $\{e_1, e_2, e_3\}$.

**Q:** How does the Hodge star map basis $k$-vectors to basis $(n-k)$-vectors (n=3)?

**A:** *Defining* property of Hodge star—for any $k$-vector $\alpha := e_{i1} \wedge \cdots \wedge e_{i_k}$, must have $\det(\alpha \wedge \star\alpha) = 1$, *i.e.*, if we start with a "unit volume," wedge with its Hodge star must also be a unit, positively-oriented volume. For example:

Given $\alpha := e_2$, find $\star\alpha$ such that $\det(e_2 \wedge \star e_2) = 1$.

$\Rightarrow$ Must have $\star\alpha = e_3 \wedge e_1$, since then

$$e_2 \wedge \star e_2 = e_2 \wedge e_3 \wedge e_1,$$

which is an even permutation of $e_1 \wedge e_2 \wedge e_3$.

$$\begin{aligned}
\star 1 &= e_1 \wedge e_2 \wedge e_3 \\
\star e_1 &= e_2 \wedge e_3 \\
\star e_2 &= e_3 \wedge e_1 \\
\star e_3 &= e_1 \wedge e_2 \\
\star(e_2 \wedge e_3) &= e_1 \\
\star(e_3 \wedge e_1) &= e_2 \\
\star(e_1 \wedge e_2) &= e_3 \\
\star(e_1 \wedge e_2 \wedge e_3) &= 1
\end{aligned}$$

# *Exterior Algebra—Formal Definition*

**Definition.** Let $e_1, \ldots, e_n$ be the basis for an $n$-dimensional inner product space $V$. For each integer $0 \leq k \leq n$, let $\bigwedge^k$ denote an $\binom{n}{k}$-dimensional vector space with basis elements denoted by $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for all possible sequences of indices $1 \leq i_1 < \cdots < i_k \leq n$, corresponding to all possible "axis-aligned" $k$-dimensional volumes. Elements of $\bigwedge^k$ are called *k-vectors*. The *wedge product* is a bilinear map

$$\wedge_{k,l} : \bigwedge^k \times \bigwedge^l \to \bigwedge^{k+l}$$

uniquely determined by its action on basis elements; in particular, for any collection of *distinct* indices $i_1, \ldots, i_{k+l}$,

$$(e_{i_1} \wedge \cdots \wedge e_{i_k}) \wedge_{k,l} (e_{i_{k+1}} \wedge \cdots \wedge e_{i_{k+l}}) := \mathrm{sgn}(\sigma)\, e_{\sigma(i_1)} \wedge \cdots \wedge e_{\sigma(i_{k+l})},$$

where $\sigma$ is a permutation that puts the indices of the two arguments in canonical (lexicographic) order. Arguments with repeated indices are mapped to $0 \in \bigwedge^{k+l}$. For brevity, one typically drops the subscript on $\wedge_{k,l}$. Finally, the *Hodge star on k-vectors* is a linear isomorphism

$$\star : \bigwedge^k \to \bigwedge^{n-k}$$

uniquely determined by the relationship

$$\det(\alpha \wedge \star\alpha) = 1$$

where $\alpha$ is any $k$-vector of the form $\alpha = e_{i_1} \wedge \cdots \wedge e_{i_k}$ and det denotes the determinant of the constituent 1-vectors (treated as column vectors) with respect to the inner product on $V$. The collection of vector spaces $\bigwedge^k$ together with the maps $\wedge$ and $\star$ define an *exterior algebra* on $V$, sometimes known as a *graded algebra*.

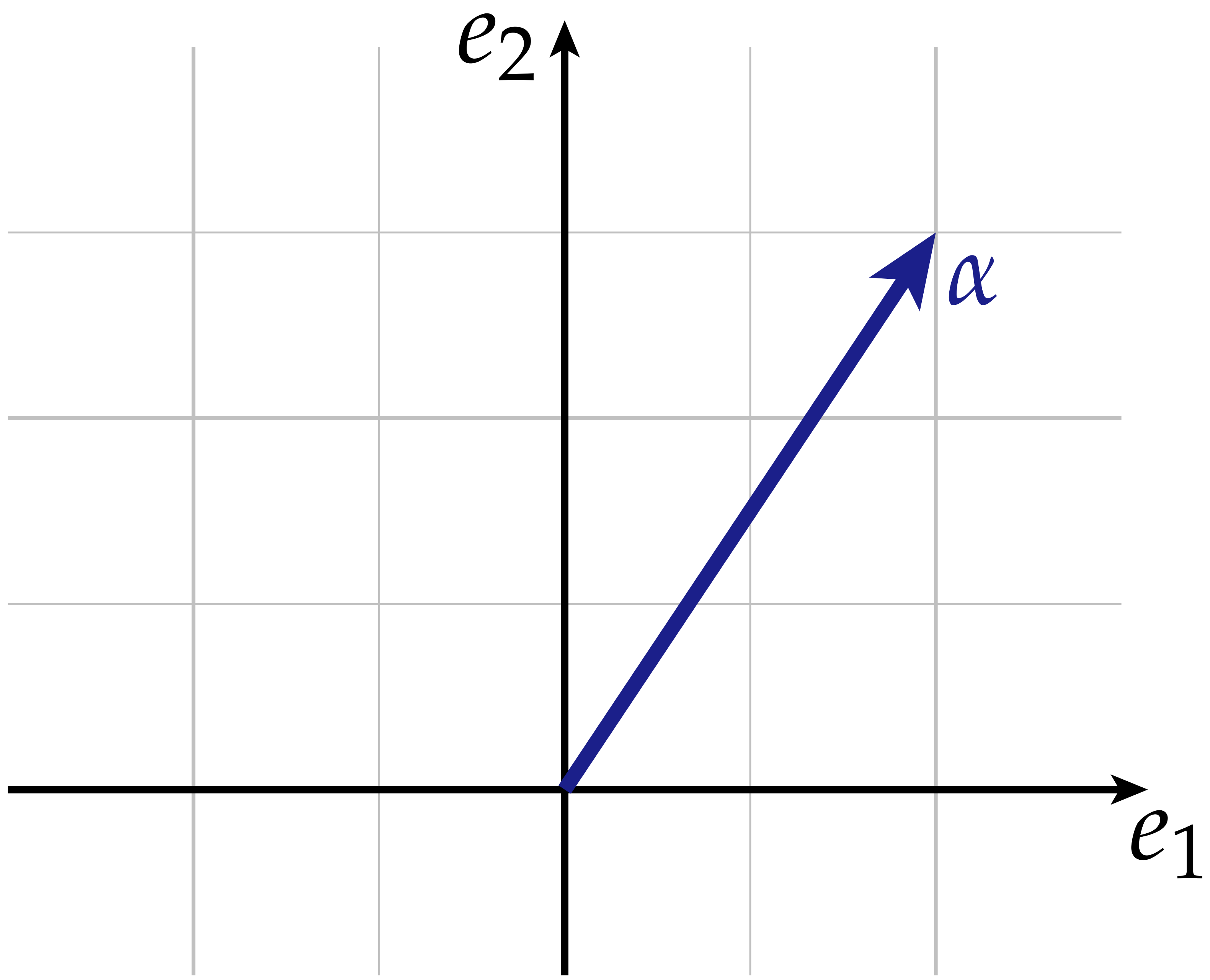
***(...don't worry too much about this!)***

# *Sanity Check*

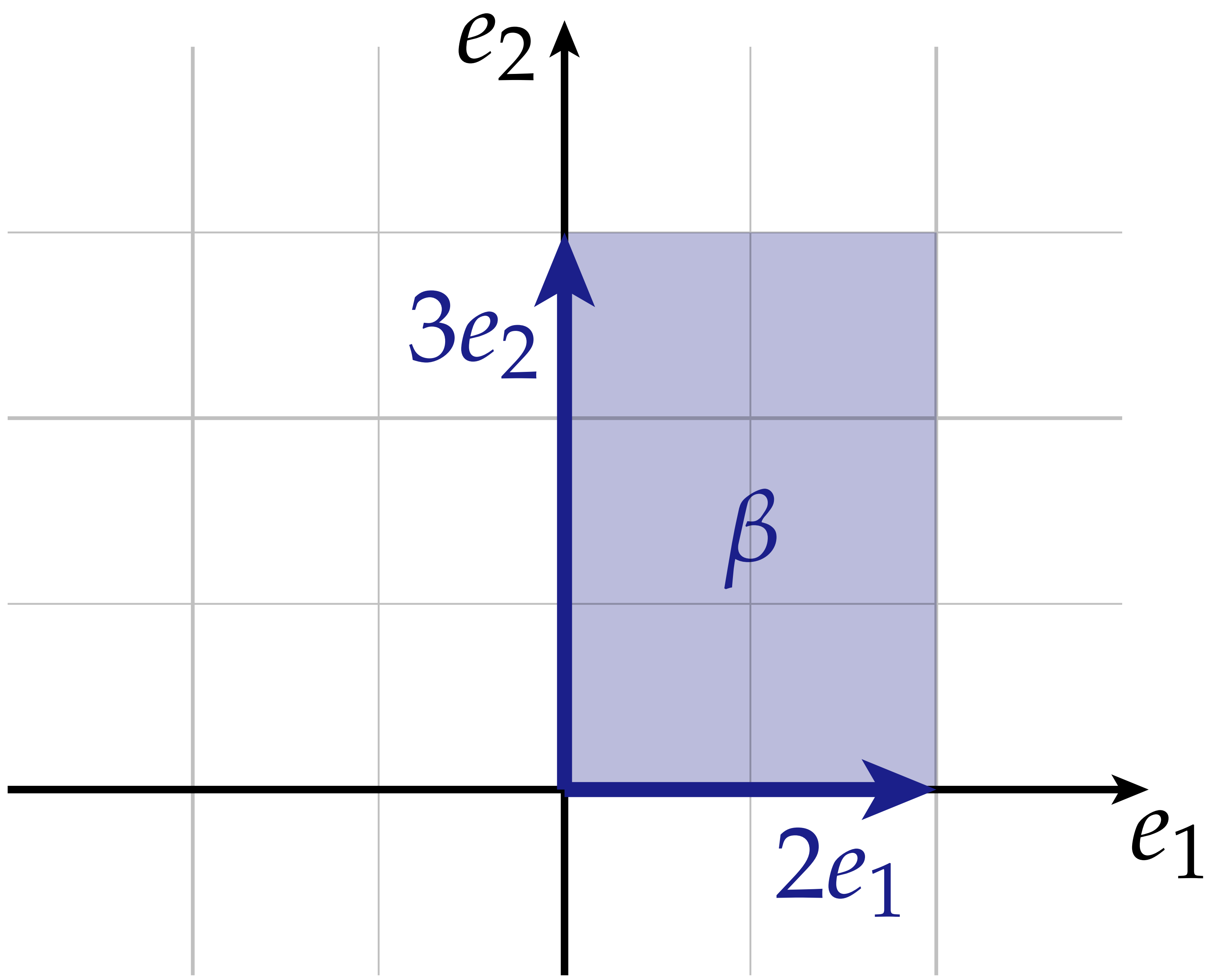
**Q**: What's the difference between

$$\alpha = 2e_1 + 3e_2 \qquad \text{and} \qquad \beta = 2e_1 \wedge 3e_2?$$

**A:**




(vector)




(2-vector)

# *Exterior Algebra—Example*

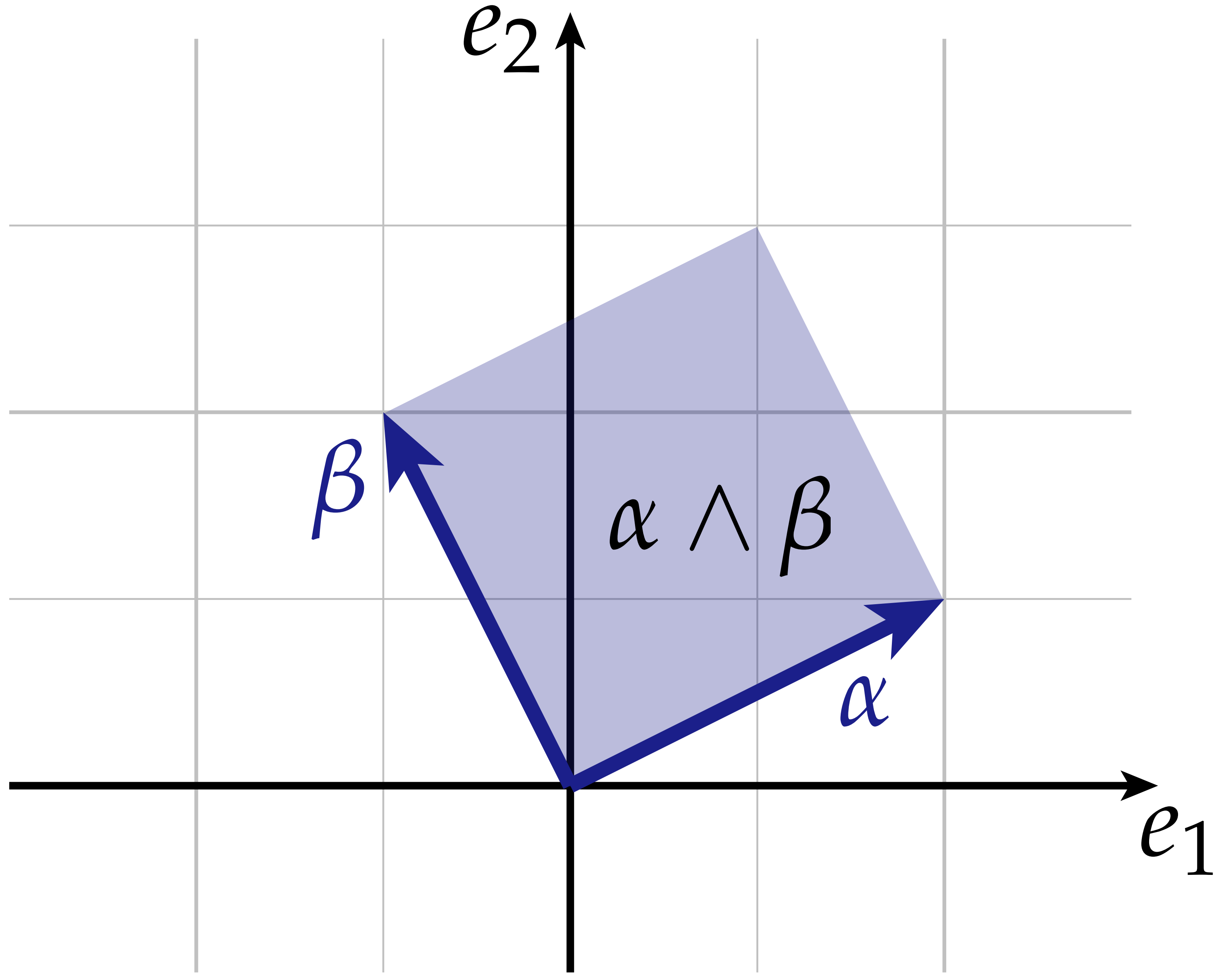
$V = \mathbb{R}^2$

$\alpha = 2e_1 + e_2$

$\beta = -e_1 + 2e_2$

**Q:** What is the value of $\alpha \wedge \beta$?

**A:**
$$\begin{aligned}\alpha \wedge \beta &= (2e_1 + e_2) \wedge (-e_1 + 2e_2) \\ &= (2e_1 + e_2) \wedge (-e_1) + (2e_1 + e_2) \wedge (2e_2) \\ &= -\cancelto{0}{2e_1 \wedge e_1} - e_2 \wedge e_1 + 4e_1 \wedge e_2 + \cancelto{0}{2e_2 \wedge e_2} \\ &= e_1 \wedge e_2 + 4e_1 \wedge e_2 \\ &= \boxed{5e_1 \wedge e_2}\end{aligned}$$




**Q:** What does the result *mean*, geometrically?

# *Exterior Algebra—Example*

$V = \mathbb{R}^3$

$\alpha = 2e_1 \wedge e_2$

$\beta = 3e_3$

$\gamma = e_2 \wedge e_1$

**Q:** What is $\star(\alpha \wedge \beta + \beta \wedge \gamma)$?

**A:**

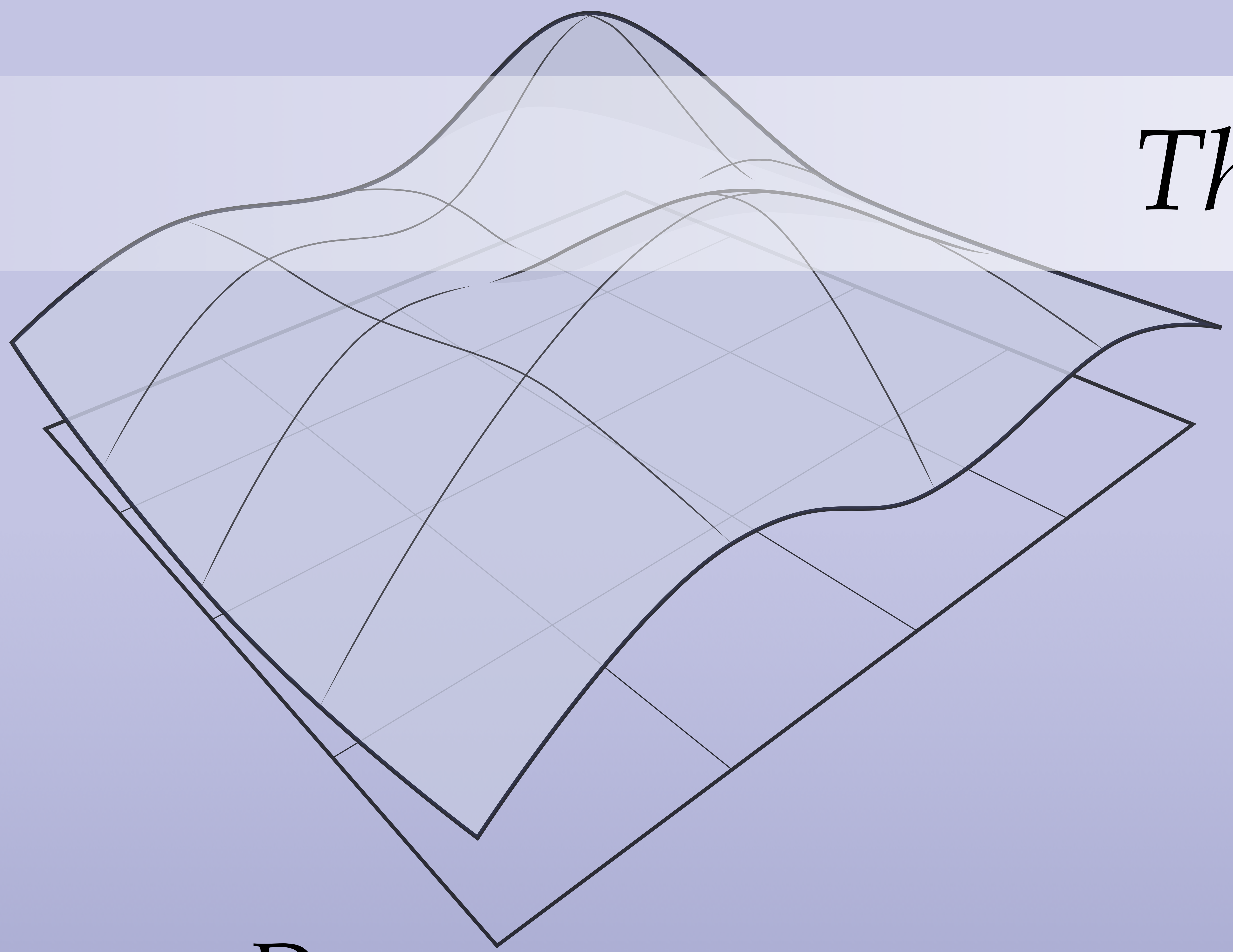
$$\begin{aligned}
\star(\alpha \wedge \beta + \beta \wedge \gamma) &= \star((2e_1 \wedge e_2) \wedge 3e_3 + 3e_3 \wedge (e_2 \wedge e_1)) \\
&= \star(6e_1 \wedge e_2 \wedge e_3 + 3e_3 \wedge e_2 \wedge e_1) \\
&= \star(6e_1 \wedge e_2 \wedge e_3 - 3e_1 \wedge e_2 \wedge e_3) \\
&= \star(3e_1 \wedge e_2 \wedge e_3) \\
&= 3.
\end{aligned}$$

**Key idea:** in this example, it would have been fairly hard to reason about the answer geometrically. Sometimes the algebraic approach is (*incredibly!*) useful.

# *Exterior Algebra - Summary*

- **Exterior algebra**
  - language for manipulating *signed volumes*
  - length matters (magnitude)
  - order matters (orientation)
  - behaves like a vector space (e.g., can add two volumes, scale a volume, ...)
- **Wedge product**—analogous to *span* of vectors
- **Hodge star**—analogous to *orthogonal complement* (in 2D: 90-degree rotation)
- Coordinate representation—encode vectors in a *basis*
  - Basis $k$-forms are all possible wedges of basis vectors

# *Thanks!*



# Discrete Differential Geometry: An Applied Introduction

**Keenan Crane • CMU 15-458/858B • Fall 2017**