ANALYSIS I EXTENSION LECTURE
6. CONSTRUCTION OF $\mathbb{R}$ THROUGH $\mathbb{Q}$

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Definition of $\mathbb{R}$

Henceforth we write rationals as usual: $[(-3,5)] \to -\dfrac{3}{5}$.

Definition. A <u>sequence</u> s of numbers in $\mathbb{Q}$ is a function

$$s : \mathbb{N} \mapsto \mathbb{Q}$$

Definition. A sequence s of numbers in $\mathbb{Q}$ is called <u>convergent to $q \in \mathbb{Q}$</u> if for every $n \in \mathbb{N}$ where $n \neq 0$, there is $m \in \mathbb{N}$ such that for each $M > m$ we have

$$d(s_M, q) < \frac{1}{n}$$

Especially, we say that a sequence s is <u>null</u> if $s \to 0$.

Definition. A sequence s of $\mathbb{Q}$ is called <u>Cauchy</u> if for every $n \in \mathbb{N}$ where $n \neq 0$, there is $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ where $n \neq 0$, there is a $m \in \mathbb{N}$ such that for every $P, Q > m$, we have

$$d(s_P, s_Q) < \frac{1}{n}$$

Theorem. *If a sequence s_n converges to $q \in \mathbb{Q}$, then it is Cauchy.*

Proposition. *There exists Cauchy sequences in $\mathbb{Q}$ that are not convergent.*

We would like a enlargement of $\mathbb{Q}$, in which $Cauchy \iff convergent$

Definition. We will say a field F is <u>complete</u> if every Cauchy sequence converge.

<u>Idea</u>: Given any $q \in \mathbb{Q}$, say $q = \dfrac{1}{2}$; consider the sequence $(q, q, q, \cdots)$. This is a Cauchy sequence with limit q. Also, $S_n = (q + \dfrac{1}{n})$, then $s_n \to q$.

Now let $s_n = 3, 3.1, 3.14, 3.141, 3.1415, \cdots$. This is a Cauchy sequence, but it probably does not converge in $\mathbb{Q}$. But we <u>want</u> it to converge.

Let

$$R = \{ \text{ all Cauchy sequences of } \mathbb{Q} \}$$

and let $\mathbb{P}(\mathbb{N} \times \mathbb{Q})$ be the power set of $\mathbb{N} \times \mathbb{Q} =$ all relations $\mathbb{N} \to \mathbb{Q}$. We look at

$$\{ F \in \mathbb{P}(\mathbb{N} \times \mathbb{Q}) \mid F \text{ is a function that defines a Cauchy sequence} \}$$

We want all element of R to be the representative for its "limit", even if it does not exist. However there is a problem: $(1, 2, 2, 2, \cdots)$ and $(2, 2, 2, 2, \cdots)$ go to the same limit, but they are different sequences. The way to deal with it is to include equivalence relations. We say two sequence (a_n) and (b_n) are equivalent (written as $(a_n) \sim (b_n)$) if their difference $(a_n - b_n)$ converges to 0.

Definition. We define real number $\mathbb{R}$ as

$$\mathbb{R} := R\Big/\sim = \text{ equivalence classes under this relation}$$

Lemma. *Every Cauchy sequence is bounded. This means there is some M s. t. $|a_n| \leqslant M$.*

Lemma. *Every Cauchy sequence (a_n) that does not converge to 0 is bounded away from 0. Explicitly, this means that $\exists \varepsilon > 0$ and $N \in \mathbb{N}$ s. t. $\forall n \in \mathbb{N}$ and $n > N$, $|a_n| > \varepsilon$*

Proposition. *$\sim$ is an equivalence relation.*

Proof.

(1) (Reflexivity) $(a_n) \sim (a_n)$ because $(a_n - a_n)$ is the zero sequence.

(2) (Symmetry) If $(a_n) \sim (b_n)$ then $(a_n - b_n) \to 0$. From the definition, $(b_n - a_n) \to -0 = 0$. So $(b_n) \sim (a_n)$

(3) (Transitivity) Say $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$. Then $\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ such that

- $\forall n > N_1$ we have $|a_n - b_n| < \dfrac{\varepsilon}{2}$.

- $\forall \varepsilon > 0$, $\exists N_2 \in \mathbb{N}$ s. t. $\forall n > N_2$, we have $|b_n - c_n| < \dfrac{\varepsilon}{2}$

Now let $N = \max\{N_1, N_2\}$. So if $n > N$, then

$$|a_n - c_n| \leqslant |a_n - b_n| + |b_n - c_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since ε is arbitrary, we see $(a_n - c_n) \to 0$. So $(a_n) \sim (c_n)$.

$\square$

We now show that $\mathbb{R}$ is well-defined.

Proof. $\mathbb{Q}$ is embedded into $\mathbb{R}(\mathbb{Q} \overset{i}{\hookrightarrow} \mathbb{R})$. That is, given any rational number q we can construct a sequence converging to q as follows:

$$q \mapsto [(q, q, q, \dots)]$$

If $p \neq q$, $i(p) \neq i(q)$ then $i(p) \neq i(q)$ because $[(p - q, p - q, \dots)] \neq [(0, 0, 0, \dots)]$ $\square$

In particular, in $\mathbb{R}$ we have a $0 = [(0, 0, 0, \dots)]$ and a $1 = [(1, 1, 1, 1, \dots)]$, as shown above, therefore $0 \neq 1$.

Properties of $\mathbb{R}$

(1) (Addition) We say that:

$$[(a_n)] + [(b_n)] := [(a_n + b_n)]$$

Then addition is well-defined, commutative, and associative. Also $[(a_n)] + 0 = [(a_n)]$

(2) (Multiplication) We say

$$[(a_n)] \cdot [(b_n)] = [(a_n b_n)]$$

Claim. *Multiplication of $\mathbb{R}$ is well defined.*

Proof. Suppose that $[(a_n)] = [(c_n)]$ and $[(b_n)] = [(d_n)]$. We will show that $[(a_n b_n) = [(c_n d_n)]$, meaning that the sequence $(a_n b_n - c_n d_n) \to 0$.
We know

$$(a_n - c_n) \to 0 \text{ and } (b_n - d_n) \to 0$$

Further, these are Cauchy sequences. Recall that if (s_n) is Cauchy, then $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall m, n > N$, we have $|s_m - s_n| < \varepsilon$
Fix some ε, the corresponding N, and some $n > N$. Then $\forall m > N$, we have

$$s_m \leqslant |s_m - s_n| + |s_n| \leqslant \varepsilon + |s_n|$$

So the sequence is bonded above by the max of $\{ \ | s_1|, |s_2|, \ldots, |s_N|, \ldots, |s_{n_1}|, |s_n| + \varepsilon \}$.
Consider

$$\begin{aligned} |a_n b_n - c_n d_n| &= |a_n b_n - b_n c_n + b_n c_n - c_n d_n| = |b_n(a_n - c_n) + c_n(b_n - d_n)| \\ &\leqslant |b_n||a_n - c_n| + |c_n||b_n - d_n| \end{aligned}$$

Let M be a positive upper bound for both b_n and c_n. Then $\forall \varepsilon > 0$,

- $\exists N_1$ s.t. if $n > N_1$, then

$$|a_n - c_n| < \frac{\varepsilon}{2M}$$

- $\exists N_2$ s.t. if $n > N_2$, then

$$|b_n - d_n| < \frac{\varepsilon}{2M}$$

So if $N = \max\{ N_1, N_2 \}$ and $n > N$, then

$$|a_n b_n - c_n d_n| < M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon$$

So $|a_n - b_n| \to 0$, indicating $(a_n b_n) \sim (c_n d_n)$.
So multiplication is well-defined. $\square$

Further, multiplication distributes over addition; and $[(a_n)] \cdot 1 = [(a_n)]$.

(3) (Division) If $[(a_n)] \neq 0$, then $\exists$ some (b_n) s.t. $[(a_n b_n)] = 1$.

Proof. Suppose $[(a_n)] \neq 0$, which means that (a_n) is Cauchy but does not converge to 0. So, by definition, $\exists \varepsilon > 0$ s. t. $\forall N \in \mathbb{N}, \exists n > N$ s. t. $|a_n| > \varepsilon$ Now fix an $\varepsilon > 0$. So $\exists N \in \mathbb{N}$ s. t. $\forall m, n > N$, we have $|a_m - a_n| < \dfrac{\varepsilon}{2}$.
Fix n to be some natural number larger than N, s. t. $|a_n| > \varepsilon$. Observe that

$$|a_n| < |a_n - a_m| + |a_m| < \frac{\varepsilon}{2}$$

This means that $\exists \varepsilon > 0$ and $N_1 \in \mathbb{N}$ s. t. $\forall m > N_1$, we have $a_m > \dfrac{\varepsilon}{2}$; in particular, $a_m \neq 0$ for sufficiently large m.
Define the sequence b_n as follows:

- If $n \leqslant N_1$, set $b_n = 1$.
- If $n > N_1$, set $b_n = \dfrac{1}{a_n}$.

Notice that b_n is Cauchy because the sequence $\dfrac{1}{a_n}$ is Cauchy. Moreover, we can compute $(a_n b_n)$

- If $n \leq N$, $a_n b_n = a_n$
- If $n > N$, $a_n b_n = 1$

So $\forall n > N, a_n b_n = 1$. This means that $[(a_n b_n)] = 1$. Therefore every nonzero element of $\mathbb{R}$ has an inverse. □

(4) (Order) Let $[(a_n)]$ and $[(b_n)] \in \mathbb{R}$. We say $[(a_n)] \geqslant [(b_n)]$ iff either $[(a_n)] = [(b_n)]$, or $\exists N \in \mathbb{N}$ s. t. $\forall n > N$, $a_n \geqslant b_n$.

Claim. *This is a totoal order on* $\mathbb{R}$

Proof. Suppose $[(a_n) \neq (b_n)]$, so $a_n - b_n \nrightarrow 0$. So $\exists \varepsilon > 0$ s. t. $\forall N \in \mathbb{N}$, $\exists n > N$ s. t. $|a_n - b_n| > \varepsilon$.
Fix this ε. As before, we see that $\exists N_1 \in \mathbb{N}$ s. t. $\forall m, n > N_1$, $|a_m - a_n| < \dfrac{\varepsilon}{2}$ and $\exists N_2 \in \mathbb{N}$ s. t. $\forall m, n > N_2, |b_m - b_n| < \dfrac{\varepsilon}{2}$.
Let $M = \max\{ N_1, N_2 \}$. Fix $n > M$ s. t. $|a_n - b_n| > \varepsilon$. Suppose WLOG $a_n > b_n$, so $a_n - b_n > \varepsilon$. Then $a_m \geqslant a_n - \dfrac{\varepsilon}{2}$, and $b_m \leqslant b_n + \dfrac{\varepsilon}{2}$.
Therefore,

$$a_m - b_m \geqslant a_n - \frac{\varepsilon}{2} - b_n - \frac{\varepsilon}{2} \geqslant 0$$

So, $\forall m > M$, $a_m \geqslant b_m$. □

(5) (Completeness) We first introduce the definition of Least Upper Bound, as follows:

Definition (Least Upper Bound)**.** Let $S \subseteq \mathbb{R}$ be non-empty.

A number $x \in \mathbb{R}$ is an upper bound for S if $\forall s \in S$, we have $x \geqslant s$.

A number $x \in \mathbb{R}$ is the least upper bound for S if $\forall y$ upper bound of S, we have $x \leqslant y$.

Theorem. *Every bounded set of reals that has a upper bound has a least upper bound.*

Proof. Let $S \subseteq \mathbb{R}$ be nonempty. Let $u \in \mathbb{R}$ be an upper bound for S. Assume that u is rational (by increasing if required). Choose some $l \in \mathbb{Q}$ s. t. $l < s$ for <u>some</u> $s \in S$. We will define sequence u_n and l_n, of rationals as follows:

(i) $u_0 = u$, and $l_0 = l$.

(ii) For each $n \in \mathbb{N}$, $m_n = \dfrac{u_n + l_n}{2}$.

(iii) If m_n is an upper bound for S, set
$$u_{n+1} = m_n \text{ and } l_{n+1} = l_n$$
Otherwise, set
$$u_{n+1} = u_n \text{ and } l_{n+1} = m_n$$

Observations,

(a) $\forall n \in \mathbb{N}$, $u_{n+1} \leqslant u_n$.

(b) $\forall n \in \mathbb{N}$, $l_{n+1} \geqslant l_n$.

(c) $\forall n \in \mathbb{N}$, $u_n \geqslant l_n$.

(d) By induction, $u_{n+1} - l_{n+1} = \dfrac{u-l}{2^n}$.

(e) If $m > n$, then $|u_m - u_n| \leqslant \dfrac{u-l}{2^{n-1}}$. This shows u_n is Cauchy.

(f) Similiarly, l_n is Cauchy.

(g) $\forall s \in S$, $u_n \geqslant s$, so $[(u_n)] \geqslant s$ for all $s \in S$.
$\implies [(u_n)]$ is an upper bound.

(h) $[(l_n)]$ is never a upper bound for S.

(i) $[(l_n)] = [(u_n)]$ because the difference goes to 0.

Now we claim that $[(u_n)]$ is a least upper bound for S. Suppose now, $\exists b \in \mathbb{N}$ which is an upper bound for S and $[(u_n)] > b$. So $[(l_n)] > b$. So $\exists n \in \mathbb{N}$ s. t. $l_n > b$. However, since l_n is not an upper bound for S, b cannot be one.

$\square$