ANALYSIS I EXTENSION
8. CONNECTEDNESS AND PASS-CONECTEDNESS

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Connectedness

Definition. A topological space X is connected if it cannot be expressed as the union of two non-empty, disjoint open sets.

Example. $X = [0,1] \cup (3,4)$ is disconnected (subspace topology).
$[0,1]$ is open because $[0,1] = (-\frac{1}{2}, \frac{1}{2}+1)$ is intersection of X is intersection of X and an open set in $\mathbb{R}$. Therefore X is union of $[0,1]$ and $(3,4)$, which are both open and mutually disjoint.

Equivalently, X is connected if it cannot be expressed as the union of two nonempty, disjoint closed sets. Equivalently, X is connected iff the only subsets of X that are both open and closed are $\varnothing$ and X.

Example. $\mathbb{Q} \subseteq \mathbb{R}$ is disconnected, because

$$\mathbb{Q} = (\mathbb{Q} \cap (-\infty, \sqrt{2})) \cup (\mathbb{Q} \cap (\sqrt{2}, \infty))$$

Remark. Is $[0,1]$ connected in $\mathbb{R}_{\text{metric}}$? What about $(0,1)$?

Theorem. *Every single interval (open, closed, or half-open) is connected in* $\mathbb{R}_{metric}$

Proof. We will prove it for closed intervals for simplicity.
Let $X = [a,b]$. If $a = b$, X is a singleton, so is connected. Suppose $a \neq b$ and $a < b$. Suppose that $X = A \cup B$, where A and B are both clopen and A is not empty. We need to show B has to be $\varnothing$.

Let's also suppose, WLOG, $a \in A$. First of all, A is open in X, so

$$A = [a,b] \cap U \text{ , where } U \text{ is open in } \mathbb{R}$$

Also, $a \in A \implies a \in U$. Since U is open, $\exists\varepsilon$ sufficiently small s. t. $(a-\varepsilon, a+\varepsilon) \subseteq U$. Then $(a-\varepsilon, a+\varepsilon) \cap [a,b] \subseteq A \implies [a, a+\varepsilon) \subseteq A$.
Define the set

$$C = \{\, c \in [a,b] \mid [a,c] \subseteq A \,\}$$

Clearly C has an upper bound, namely b. Therefore c has a least upper bound $L \leqslant b$.

Now we are going to show that $L \in C$.
L is a limit point of A. $L \in X$ because $a < L \leqslant b$. A is clopen in X so A is closed in X, so $L \in A \implies [a,L] \subseteq A$.

Now we show that $L = b$. Suppose, for contradiction, $L < b$. As $L \in A$ and A is open in X, $\exists\varepsilon > 0$ s. t. $[L, \varepsilon) \subseteq A$. Therefore,$[L, \varepsilon] \subseteq A$ since A is closed, indicating $L + \varepsilon \in C$.

However, $L = \sup(C)$. Contradiction. $\therefore L = b$. Therefore, the whole proof implies $A = X$ and $B = \varnothing$. Accordingly, any closed interval is connected in R_{metric}. $\square$

Path-connectedness

Now that we know that $[0, 1]$ is connected, we use it to define another notion.

Definition (Path-Connectedness)**.** A space X is path-connected if $\forall a, b \in X$, there is a "path" from a to b. That is, if there is a continuous function $f : [0, 1] \mapsto X$ with $f(0) = a$ and $f(1) = b$.

Remark. $[0, 1]$ is a "stand-in" for any other closed interval because $[0, 1]$ and $[a, b]$ are homeomorphic for any $p \neq q$. (Exercise)

Proposition. *If X is path-connected, then it is connected.*

Proof. Suppose for contradiction, that X is path-connected but not connected so $X = A \cup B$, $A \neq \varnothing$, $B \neq \varnothing$, $A \cap B = \varnothing$, and A, B is open. Let $a \in A$ and $b \in B$. There is some $f : [0, 1] \implies X$ s. t.

$$f(0) = a \text{ and } f(1) = b$$

So, consider $f^{-1}(A)$ and $f^{-1}(B)$. Not that $f^{-1}(A)$ and $f^{-1}(B)$ are open, because f is continuous. They are also nonempty and disjoint (because if $x \in f^{-1}(A)$ and if $x \in f^{-1}(B)$, then $f(x) \in A \cup B$).

Also, $f^{-1}(A) \cup f^{-1}(B) = [0, 1]$. This contradicts the fact that $[0, 1]j$ is connected. $\square$

This gives an easy chance to see if spaces are connected.

Example.

- Circle in $\mathbb{R}^2$ is connected.
- $\forall n \in \mathbb{N}^+$, $\mathbb{R}^n$ is connected.
- Any interval is connected because it is path connected. We can consider

$$f(t) := (1 - t)a + tb \qquad (0 \leqslant t \leqslant 1)$$

Proposition. *If $X \subseteq \mathbb{R}^n$ is <u>convex</u>, it is connected.*

Remark. There exist connected spaces that are not path-connected! A counterexample is sine curve,

$$\left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \,\middle|\, x \in (0, 1] \right\} \cup \{ (0, 0) \}$$

Proposition. *Let $f : X \mapsto Y$ be continuous and surjective. Then:*

(1) If X is connected, so is Y.

(2) If X is path-connected, so is Y.

Now we can compare spaces and distinguish them.

Corollary. *If X and Y are homeomorphic, then*

(1) X is connected iff Y is connected.

(2) X is path-connected iff Y is path-connected.

This can be used to deduce (for example) that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n > 1$.

Definition (Connected Component)**.** Let X be a space and $x \in X$. The connected component of $x \in X$ is the union of all connected subspace of X containing x, denoted by C_x. Especially, X is itself connected.

Definition (Path Components)**.** Let X be a space and $x \in X$. The path component of x is the set of all $y \in X$ s. t. there is a path from x to y, called P_x.

Remark. Say $x \sim_c y$ if $C_x = C_y$; $x \sim_p y$ if $P_x = P_y$. Then these are equivalent relations.

Proposition. *Every path-connected space is connected. In particular, every interval in $\mathbb{R}$ is connected.*

These notions let us distinguish spaces from each others. Recall:

Theorem. *Let $f : X \mapsto Y$ be continuous and surjective,*

(1) If X is connected, then Y is connected.

(2) If X is path-connected, then Y is path connected.

In particular, if $f : X \mapsto Y$ is a homeomorphism, then X is connected (resp. path-connected) iff Y is connected (resp. path-connected)

Proposition. *$\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any $n > 1$*

Proof. Suppose $f : \mathbb{R} \mapsto \mathbb{R}^n$ is a homeomorphism. Then $f : \mathbb{R} - \{ 0 \} \mapsto \mathbb{R}^n - \{ f(0) \}$ is a homeomorphism. $\mathbb{R} - \{ 0 \}$ is neither connected nor path-connected. But $\mathbb{R}^n - \{ f(0) \}$ is path-connected. Contradiction. □