# ANALYSIS I EXTENSION LECTURE
# 7. TOPOLOGY(POINT-SET TOPOLOGY)

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## Introduction

So far the class has discussed metric spaces, such as $\mathbb{R}$ and $\mathbb{R}^n$. Recall some basic definitions [Say $X$ a metric space]

- The open ball of radius $\varepsilon$ around $p \in X$, denoted $B(p, \varepsilon)$, is
$$B(p, \varepsilon) = \{ x \in X \mid d(x, p) < \varepsilon \}$$
- A subset $Y \subseteq X$ is <u>open</u> if $\forall y \in Y$, there is some $\varepsilon \in \mathbb{R}$ such that $B(y, \varepsilon) \subseteq X$.
- A subset $Z \subseteq X$ is <u>closed</u> if $X \backslash Z$ is open.
- A <u>neighbourhood</u> of $x \in X$ is an open set $u$ containing $X$.

Notice the following facts (Of course there are many other observations).

- An arbitrary union of open subset of $X$ is open. (Also a finite $\cap$).
- Consequently, an arbitrary intersection of closed sets of $X$ is closed.

A <u>topological space</u> is supposed to generalise all above notions.

**Definition.** A <u>Topological Space</u> consists of a pair $(X, O)$, where $X$ is a set and $O$ is a subset of the power set of $X$, together with the following conditions:

(1) $\varnothing \in O$ and $X \in O$.

(2) An arbitrary union of elements of $O$ is also an element of $O$.

(3) A finite intersection of elements of $O$ is also an element of $O$

Given $(X, O)$, we say that $Y \subseteq X$ is <u>open</u> iff $Y \in O$. We say that $Z \subseteq X$ is <u>closed</u> iff $(X \backslash Y) \in O$.

*Remark.* $X$ need not have metric...

**Example.**

(1) Let $X = \{ 1, 2, 3 \}$; Then $O = \{ \varnothing, \{ 1, 2, 3 \} \}$ [or $X$ is any set, $O = \{ \varnothing, X \}$]

(2) Let $X$ be any set and $O = \mathbb{P}(X)$.

(3) $X = \{ 1, 2, 3 \}$; $O = \{ \varnothing \}, \{ 1 \}, \{ 1, 2 \}, \{ 1, 3 \}, \{ 1, 2, 3 \}$

(4) Let $X$ be any metric space; Let $O$ be the set of open subsets of $X$.

## Basis for Topological Space

**Definition.** Let $(X, O)$ be a topological space. A subset $B \subseteq O$ is called a basis for this topology if:

(1) Every $v \in O$ is a union of elements from $B$. Consequently,

(2) For every $u \in O$, there is some $v \in B$ s. t. $v \subseteq u$

**Example.** For $X = \mathbb{R}$ as before, we can take $B = \{ v \mid v \text{ is open interval in } \mathbb{R} \}$.

In fact we can define the topological space by specifying a basis $B$, the set $O$ then consists of all possible unions of elements of $B$.

**Example** (Lower-Limit Topology)**.** Once again take $X = \mathbb{R}$, but we take a different $O$: Set $B = \{ [a, b) \mid a, b \in \mathbb{R} \}$, so that $O$ consists of unions of these half-open intervals. This is called the lower-limit topology on $\mathbb{R}$.

**Proposition.** *Any interval $(a, b)$ is open in the lower limit topology. Therefore, any set open in the usual metric topology is open in the Lower Limit topology on $\mathbb{R}$.*

*Proof.* Consider the intervals $\left[a + \frac{1}{n}, b\right)$ for $n \geqslant 1$. These are open in $(\mathbb{R}, O_{\text{LL}})$. Their union is all points of $(a, b) \implies (a, b)$ is open in $R_{LL}$. □

$O_{\text{metric}} \subseteq O_{\text{LL}}$ because $B_{\text{metic}} \subseteq B_{\text{LL}}$. We say that the lower-limit topology is finer than the metric topology - it has more sets.

*Remark.* Is $O_{\text{metric}} = O_{\text{LL}}$? Specifically, IS $[a, b)$ open in $\mathbb{R}_{\text{metric}}$?
To answer this, we will need some more definitions and lemmas

**Definition.** Given $(X, O)$, we say that $Y \subseteq X$ is closed [in this topology] if $X - Y$ is open; i. e. $X - Y \in O$

We have the following analogies of the axioms in terms of closed sets:

**Axiom.** *If $\Phi$ is the set of all closed sets, then*

*(1) $\varnothing \in \Phi$ and $X \in \Phi$.*

*(2) Arbitrary intersections of closed sets are closed.*

*(3) Finite unions of closed sets are closed.*

**Definition** (Interior, Boundary, and Limit point)**.** Let $A \subseteq X$, and suppose that $x \in X$. Then there is the following trichotomy:

(1) $\exists u \in O$ s. t. $x \in u$ and $u \subseteq A$, or

(2) $\exists u \in O$ s. t. $x \in u$ and $u \subseteq X - A$, or

(3) $\forall u \in O$ s. t. $x \in u$, we have $u \cap A \neq \varnothing$ and $u \cap (X - A) \neq \varnothing$.

If (1) holds, we say $x$ is in the interior of $A$.
If (2) holds, we say $x$ is in the interior of $X - A$.
If (3) holds, we say $x$ is in the boundary of $X$ and $X - A$.
The set of all points where either (1) and (3) holds, i. e. interior$(A) \cup$ boundary$(A)$, is called the set of limit points of $A$.

**Example.** In $\mathbb{R}_{\text{metric}}$, let $A = [a, b)$, where $a < b$, then

(1) $\operatorname{int}(A) = (a, b)$

(2) $\text{boundary}(A) = \{ a, b \}$

(3) $\operatorname{int}(X - A) = (-\infty, a) \cup (b, \infty)$

**Definition** (Closure)**.** The set of limit points of $A \subseteq X$ is called the closure of $A$, denoted by $\overline{A}$.

**Proposition.** *Let $A \subset X$, then*

*(a)* $\operatorname{int}(A)$ *is open.*

*(b)* $\overline{A}$ *is closed.*

*(c) $A$ is open iff $A = \operatorname{int}(A)$.*

*(d) $A$ is closed iff $A = \overline{A}$.*

*Remark.* $O_{\text{LL}} \nsubseteq O_{\text{metric}}$ since $[a, b)$ is not the union of open intervals. What are some closed sets of $R_{\text{LL}}$? Does there exist a metric on $\mathbb{R}$ whose associated topology is $(\mathbb{R}, O_{\text{LL}})$?

**Example** (Finite-Complement Topology)**.** Let $X = \mathbb{R}$, and let

$$O = \{ Y \subseteq X \mid X - Y \text{ is finite} \}$$

Such topological space is called Complement-Finite Topology.

## Subspace Topology

If $(X, O)$ is a topological space and $Y \subseteq X$ is any set, we can define a topology on $Y$ as follows:

$$O_Y = \{ U \subset Y \mid \exists V \in O \text{ where } U = V \cap Y \}$$

In this case, we say $Y$ is a subspace of $X$.

**Lemma.** *Given $Y \subset X$ a subspace, and any $A \subseteq Y$, the closure of $A$ in $Y$ is $\overline{A} \cap Y$.*

*Proof.* Let $y$ be a limit point of $A$ in $Y$, then there exists an open set $U_y$ of $y$ s. t. $y \in Y$ and $U_y \cap A \neq \varnothing$. But $A \subseteq Y$, and $U_y = V \cap Y$ for some open $V$ of $X$. So $U_y \cap A = (V \cap Y) \cap A = V \cap A$. But $y$ is in $\overline{A}$, so $\forall V \in O$ s. t. $y \in V$, we have $V \cap A \neq \varnothing$. $\square$

## Continuity

**Definition** (Continuity)**.** Let $X$ and $Y$ be topological space. A function $f : X \mapsto Y$ is to be <u>continuous</u> if $\forall v$ open in $Y$, we have $f^{-1}(Y)$ is open in $X$.
Equivalently, $\forall Z \subset Y$ closed, $f^{-1}(Z)$ is closed in $X$. Equivalently, given $B_Y$, for all $v \subseteq Y$ s.t. $v \in B_Y$, $f^{-1}(v)$ is open in $X$.

**Lemma.** *If $f : X \mapsto Y$ and $g : Y \mapsto Z$ are continuous, then $(gf) : X \mapsto Z$ is also continuous.*

**Lemma.** *If $f : X \mapsto Y$ is continuous and $A \subseteq X$ is a subspace, then $f|_A : A \mapsto Y$ is continuous.*

**Definition** (Homeomorphism)**.** A continuous map $f : X \mapsto Y$ is a <u>homeomorphism</u> if it is one-to-one and onto, and also $f^{-1} : Y \mapsto X$ is continuous.

## Product topology

Let $X$ and $Y$ be topological spaces, with open sets $O_X$ and $O_Y$ respectively. We can define a topology on their Cartesian product, called the <u>product topology</u>, as follows:

**Definition.** A <u>basis</u> for the product topology on $X \times Y$ is $B_{X\times Y} := O_X \times O_Y$. So if $U \in O_X$ and $V \in O_Y$, then $U \times V$ is open in $X \times Y$.

*Remark.* THE CONVERSE NEED NOT HOLD.

**Example.** Let $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$,and define

$$\Delta = \{ (x, x) \mid x \in \mathbb{R} \}$$

$\Delta$ is closed, so its complement $\mathbb{R}^2 - \Delta$ is open in $\mathbb{R}^2$. However, the complement cannot be written as $U \times V$ for $U, V$ open in $\mathbb{R}$.

**Example.** Define a "cylinder" space inside $\mathbb{R}^3$. In $\mathbb{R}^2$, we can have a "annulus". These two spaces are homeomorphic, and they are homeomorphic to $(S' \times [0, 1])$.