ANALYSIS I EXTENSION LECTUR
1. INTRODUCTION TO AXIOMATIC SET THEORY

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Calculus is typically done on $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}^n$, why not on $\mathbb{N}$ or $\mathbb{Q}$

- $\mathbb{N}$ is unsuitable because it is "discrete". Metric on $\mathbb{N}$ does not have arbitrarily small numbers. That is, if $d(m,n) < 1$, then $m = n$.
- $\mathbb{Q}$ is unsuitable because it has "holes" in it. There are unequal, arbitrarily close rational numbers. However, it is not a <u>complete</u> metric space. More about this later, but for example, $\{ x \in \mathbb{Q} \mid x^2 < 2 \}$ does not have a least upper bound in $\mathbb{Q}$ so there are bounded increasing sequences in $\mathbb{Q}$ that does not converge in $\mathbb{Q}$.

To do this properly, we should start at the very beginning: $\mathbb{R}$ is constructed from $\mathbb{Q}$, which is constructed from $\mathbb{Z}$, which is constructed from $\mathbb{N}$. Which is constructed from ...?

One approach (standard) is axiomatic set theory.

Definition. A set is an object S s. t. for each x, we either have $x \in S$ or $x \notin S$. That is, x is an element of S, or x is not an element of S, but not both.

A set can be specified by listing its elements, like $\{ 1, 2, 3 \}$, or by specification, like $\{ x \in \mathbb{N} \mid x \text{ is prime.} \}$. This definition works fine — until it doesn't!

Example (Russell's Paradox)**.** Let S be defined as follows:

$$S = \{ T \mid T \notin T \}$$

Question: Is $S \in S$? If yes, then by its definition, S is a set s. t. $S \notin S \implies S \notin S$, contradiction; If no, then $S \notin S$, matching the requirement for elements of $S \implies S \in S$, contradiction.

Zermelo-Fraenkel Set Theory (ZFC)

(Typically written in the language of first order logic, however we'll use a mix of logic and English.)

Axiom 1 (Axiom of Extension)**.** *Two sets are equal iff they have the same elements:*

$$X = Y \text{ iff } \forall z (z \in X \iff z \in Y)$$

Axiom 2 (Axiom of Existence)**.** *The "empty set" is a set. That is, $\varnothing$ exists and has the property that $\forall z (z \notin \varnothing)$.*

Axiom 3 (Axiom of Pairing)**.** *If X and Y are sets, then there is a set $\{ X, Y \}$.*

Example. $\{ 1, 2 \}$ and $\{ 2, 3 \}$ pair to $\{ \{ 1, 2 \}, \{ 2, 3 \} \}$.

Axiom 4 (Axiom of Union)**.** *If S is any set, then the "union over all elements of S" is a set. That is, there is a set whose elements are all x such that x belongs to some elements of S.*

Example.

$$\{ \{ 1,2 \}, \{ 2,3 \} \} \to \{ 1,2,3 \}$$

Axiom 5 (Axiom of Intersection)**.** *If S is a set, the "intersection over all elements of S" exists*

Axiom 6 (Axiom of Foundation)**.** *Every $X \neq \varnothing$ contains a member Y such that $X \cap Y = \varnothing$. (Roughly speaking, a sequence $z_1 \ni z_2 \ni \dots$ must always terminate.)*

Consequence of failure of the axiom:
$\forall Y_1 \in X, X \cap Y_1 \neq \varnothing$. So $\exists Y_2 \in X \cap Y_1 \implies X \cap Y_2 \neq \varnothing$ and so on!

Axiom 7 (Axiom Schema of Specification)**.** *We can build sets using "set builder" notation as subsets of a known set that satisfies certain predicate. A predicate can be seen as a kind of function outputting "TRUE" or "FALSE".*

Example. $\{ x \in \mathbb{N} \mid x > 2 \}$ is the set $\{ 3,4,5,\dots \}$ or $\{ x \in \{ \varnothing, \{ \varnothing \} \} \mid x \neq \varnothing \}$ is the set $\{ \{ \varnothing \} \}$.

Axiom 8 (Axiom of Power Set)**.** *If S is a set, then $\mathbb{P}(S)$ is the collection of all subsets of S, which is a set*

Based on previous axioms, we can construct the Cartesian Product, as follows:

Definition (Cartesian Product)**.** We define a ordered pair (a,b)

$$(a,b) := \{ \{ a \}, \{ a,b \} \}$$

Then we define

$$A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$$

Based on Cartesian Product, we can define the notation of Function

Definition (Function)**.** Let A, B be sets. A function $f : A \mapsto B$ is an element of $\mathbb{P}(A \times B)$ such that

(1) for every $a \in A$, there is some $b \in B$ such that $(a,b) \in f$.

(2) For every $a \in A$, if b_1 and b_2 such that $(a_1,b_1) \in f$ and $a_1, b_2 \in f$, then $b_1 = b_2$.

Axiom 9 (Axiom of Infinity)**.** *Let X be a set. Define* $\operatorname{succ}(X) = X^+ = X \cup \{ X \}$.
There is a set with the properties:

(1) $\varnothing \in S$

(2) If $X \in S$, *then* $\operatorname{succ}(X) \in S$

Example.
$\operatorname{succ}(\varnothing) = \{ \varnothing \}$
$\{ \varnothing \}^+ = \{ \varnothing \} \cup \{ \{ \varnothing \} \} = \{ \varnothing, \{ \varnothing \} \}$

In particular, S cannot have finitely many elements!

Definition (Choice Function). A Choice Function f defined on a collection X of none empty sets is a function with the property that if $a \in X$ then $f(a) \in a$

Example. $f(x) = \min(x)$ gives $f(\{ 1, 2, 3 \}) = 1 \rightarrow f$ is defined on $\mathbb{P}(\mathbb{N})$.

Axiom 10 (Axiom of Choice). *Choice functions always exist for any X*

The axiom states that if x is any set of nonempty sets, then there is a choice function on x. This is equivalent to the well-ordering principle:

Theorem (Well-ordering Pricinple). *For any set x, there is a binary relation $R(\leqslant)$ which well-orders x: every non-empty subset of x has a smallest element.*