ANALYSIS I EXTENSION LECTURE
9. COMPACTNESS

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Compactness is a property (like connectedness) that some topological spaces have but others don't.

Definition (Open Cover)**.** An open cover of a space X is a collection of open sets whose union is X.

Definition (Connectness)**.** A topological space X is called compact if <u>every</u> open cover of X has a <u>finite</u> subcover.

Example. $\mathbb{R}$ is not compact. Consider the open covers $(n, n+2)$ for $n \in \mathbb{Z}$.

Remark. If $Y \subseteq X$, sometimes we specify an open cover of Y by giving a set $\{ u_\alpha \}_{\alpha \in I}$ of open sets of X,whose union contains Y.

This suggests that a noncompact space is one that can "escape out to infinity". But seemingly finite spaces can also be noncompact.

Example. $(0, 1)$ is not compact; consider the open cover $(\frac{1}{n}, \frac{1}{n+2})$ for $n \in \mathbb{N}$ and $n \geqslant 1$. This has no finite subcover.

Theorem. *In the metric topology on $\mathbb{R}$, any closed interval $[a, b]$ is compact.*

Proof. If $a = b$, then $[a, b] = a$; this is compact because any finite set is compact.

If $a \neq b$, consider an open cover $\{u_\alpha\}_{\alpha \in I}$ of $[a, b]$. Let $a \in u_\alpha$ for some α. Since u_α is open, $\exists$ some $c \leqslant b$ s. t. $[a, c] \subseteq u_\alpha$. If $c = b$, then we are done; Otherwise, let

$$C = \{ x \in (a, b] \mid [a, x] \text{ is contained in a finite union of the sets } u_\beta \}$$

Notice that $c \in C$ so $C \neq \varnothing$. Clearly C is bounded above by b. Let $L = \sup(C)$.

Pf: $L \in C$.
Let $L \in u_\beta$, and suppose $L \notin C$. Then $[a, L]$ cannot be covered by finitely many sets $u_\beta \in \{ u_\alpha \}_{\alpha \in I}$. But if $L \in u_\beta$ and u_β is open, there is some ε s. t. $[L - \varepsilon, L] \subseteq u_\beta$. Since $L = \sup(C)$, $L - \varepsilon$ is not an upper bound of C, so $[a, L - \varepsilon]$ is contained in a finite union of sets u_β. So $[a, L]$ is contained in the above union of u_β, still finite. Contradiction.
$\therefore L \in C$ $\square$

Pf: $L = b$.
Similar argument: If $L \in u_\beta$ and $L \neq b$, then $\exists \varepsilon > 0$ s. t. $[L, L + \varepsilon] \subseteq u_\beta$. Then $L + \varepsilon \in C$, contradiction.
$\therefore L = b$. $\square$

Therefore, $[a, b]$ is compact in $\mathbb{R}$. □

Proposition. *A closed subset of a compact space is compact in the subspace topology.*

Proof. Let $Y \subseteq X$, where X is compact and Y is closed. Consider any open cover of Y, written as $\{ u_\alpha \cap Y \}_{\alpha \in I}$ where u_α is open in X. Consider the cover $\{u_\alpha\}_{\alpha\in I} \cup (X - Y)$ which is an open cover in X as Y is closed. Since X is compact, it has a finite subcover, say v_1, v_2, $\dots v_n$. Then

$$\{v_1 \cap Y, v_2 \cap Y, \dots v_n \cap Y\}$$

is a finite subcover of the original open cover of Y.
$\therefore Y$ is also a compact set. □

Proposition. *If $f : X \mapsto Y$ is continuous and surjective, and X is compact, then Y also is compact.*

Proof. Let $Y = \bigcup_{\alpha \in I} u_\alpha$. Then

$$X = f^{-1}\left(\bigcup_{\alpha\in I} u_\alpha\right) = \bigcup_{\alpha\in I} f^{-1}(u_\alpha)$$

Since f is continuous and surjective. This has a finite subcover, say $f^{-1}(v_1), f^{-1}(v_2), \dots, f^{-1}(v_n)$. Then

$$\{v_1, v_2, \dots v_n\}$$

is a finite open cover of Y. $\therefore Y$ is compact. □

Example. A circle is compact since it is the continuous image of $[0, 1]$.

Proposition. *If X and Y are compact, then $X \times Y$ is compact.*

Proof. Let $\bigcup_{\alpha\in I} u_\alpha$ be open cover of $X \times Y$. If $(x, y) \in X \times Y$, then $(x, y) \in u_\alpha$ for some $\alpha \in I$. Therefore $\exists$ some $A_{xy} \times B_{xy}$, containing (x, y), lying in u_α. Fix $x \in X$, and consider $\{ B_{xy} \mid y \in Y \}$, which is a open cover of Y.

Take a finite subcover, say $\{ B_{xy_1}, B_{xy_2}, \dots, B_{xy_n} \}$. Set $A_x := \bigcap_{i=1}^{n} B_{xy_i}$. Then

$$A_x \times B_{xy_1}, A_x \times B_{xy_2}, \dots, A_x \times B_{xy_n}$$

cover $A_x \times Y$, and each $A_x \times B_{xy_i}$ is contained in some u_α.

Now consider $\{ A_x \mid x \in X \}$, which has a finite cover, say $A_{x_1}, A_{x_2}, \dots, A_{x_m}$. Then $\{ A_{x_i} \}$ cover $X \times Y$, so the corresponding $u'_\alpha s$ cover $X \times Y$. □

Theorem (Heine-Borel)**.** *A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.*

Proof. Let $X \subseteq \mathbb{R}^n$ be closed and bounded. Then $\exists r \in \mathbb{R}$ s. t. $X \subseteq [-r, r] \times [-r, r] \cdots \times [-r, r]$, which is compact.
As X is closed in an compact subset,it therefore is compact.
Now suppose X is compact.

X **is bounded**: Consider the open ball of radius $n \in \mathbb{N}$ around 0 $B(n, 0)$. Then

$$\bigcup_{n \in \mathbb{N}} B(n, 0) = \mathbb{R}$$

Since X is compact, there exists a finite sequence

$$(n_i)_{i=1}^k \text{ s. t. } X \subseteq \bigcup_{(n_i)} B(n_i, 0)$$

$\therefore \max(n_i)$ is an upper bound of X, implying X is bounded.

X **is closed**: Assume, for sake of contradiction, X is compact but not closed.
Let x be a limit point of X s. t. $x \notin X$. Accordingly $\forall r > 0, \overline{B}(r, x) \cap X \neq \varnothing$. Then $\left\{ \mathbb{R}^n - \overline{B}(r, x) \right\}$ is an open cover of X. That is,

$$\bigcup_{r \in \mathbb{R}} \left(\mathbb{R}^n - \overline{B}(r, x)\right) = \mathbb{R} - \{x\} \supseteq X$$

By compactness of X, there exists finite $\{r_i\}_{1 \leqslant i \leqslant n}$ s. t. $X \subseteq \bigcup_{i=1}^{n} \left(\mathbb{R}^n - \overline{B}(r_i, x)\right) = Y$.
However, it is easy to verify that $Y \cap B(\min(r_i), x) = \varnothing$, contradicting with assumption that x is a limit point of X.
Therefore, X is closed.

$\square$