ANALYSIS I EXTENSION LECTUR
2. CONSTRUCTION OF $\mathbb{N}$

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Remark. Remember: In this world, everything is strictly a set.

Start with 0. We define $0 := \varnothing$. Recall $\operatorname{succ}(x) := x \cup \{ x \}$. Accordingly,

$$
\begin{aligned}
&1 := \varnothing \cup \{ \varnothing \} = \operatorname{succ}(\varnothing) = \{ \varnothing \} \\
&2 := \operatorname{succ}(1) = \operatorname{succ}(\{ \varnothing \}) = \{ \varnothing \} \cup \{ \{ \varnothing \} \} = \{ \varnothing, \{ \varnothing \} \} \\
&3 := \operatorname{succ}(2) = \{ \varnothing, \{ \varnothing \} \} \cup \{ \{ \varnothing, \{ \varnothing \} \} \} = \{ \varnothing, \{ \varnothing \}, \{ \varnothing, \{ \varnothing \} \} \} \\
&\vdots
\end{aligned}
$$

But what is $\mathbb{N}$? We can define $\mathbb{N}$ by using the Axiom of Infinity — We want $\mathbb{N}$ to contain $0 = \varnothing$, successor of 0, the successor of that, and so on, and nothing else.

Let S be a set such that $\varnothing \in S$ and if $x \in S$ then $\operatorname{sucss}(x) \in S$. However, S could be a lot bigger than $\mathbb{N}$, so we have to do some more work.

Definition.

$$I_S = \{ T \in \mathbb{P}(S) \mid \varnothing \in T \text{ and } x \in T, \operatorname{succ}(x) \in S \}$$

$$\mathbb{N} = \{ x \in S \mid \forall T \in I_S, x \in T \} = \bigcup_{u \in I_S} u$$

That is I_S is set of all inductive subset of S. $I_S \neq \varnothing$ because $S \in I_S$.

Theorem (Principle of Mathmetical Induction)**.** *Let p be a predicate (function that returns TRUE/FALSE defined on $\mathbb{N}$. Assume that $p(0)$ is true and $\forall k \in \mathbb{N}$, $p(k) \implies p(\operatorname{succ}(k))$, then $p(n)$ holds for all $n \in \mathbb{N}$.*

Proof. Fix p, with the above properties, and set $S := \{ n \in \mathbb{N} \mid p(n) \}$. We want to show $S = \mathbb{N}$. i.e. the element of S are exactly the element of $\mathbb{N}$.
We observe that S is inductive. Specifically,

(1) $0 \in S$ because $p(0) = p(\varnothing)$ holds.

(2) If $x \in S$, it means $p(x)$ holds. But p has the property that $p(x) \implies p(x^+)$. Then by definition of S, $x^+ \in S$.

By (1) and (2), S is inductive, and therefore $\mathbb{N} \subseteq S$. By the definition of S, it is a subset of $\mathbb{N}$, therefore $S \subseteq \mathbb{N}$.
$\therefore S = N$ $\square$

Theorem. *If m, n are two natural numbers such that $m^+ = n^+$, then $m = n$.*

Lemma. *Let* $x, n \in \mathbb{N}$. *If* $x \in n$, *then* $x \subset n$.

Proof. Define $p(n)$ as $p(n) = \forall x(x \in n \implies x \subset n)$ $\square$