## ANALYSIS I EXTENSION LECTUR 5. CONSTRUCTION OF THE RATIONAL $\mathbb Q$

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The construction of the rational  $\mathbb{R}$  is very similar to that of of the Integers  $\mathbb{Z}$ .

**Definition.** We let Q be equivalence classes of ordered pairs of (certain) integers:

$$Q := \{ (a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b \neq 0 \}$$

Remark. Remember,  $0 = [(0,0)] \in \mathbb{Z} \dots$ 

**Definition.**  $R \in Q \times Q$  is defined as follows:

$$R = \{ ((a,b), (c,d)) \in Q \times Q \mid ad = bc \}$$

Claim. R is an equivalence relation

*Proof.* Exercise.  $\Box$ 

**Definition.** The set of rationals, i. e.  $\mathbb{Q}$ , as

$$\mathbb{Q} := Q / R = \{ \text{ equivalence classes of parts of integers } (a, b) \text{ with } b \neq 0 \}$$

Operations/Properties of  $\mathbb{Q}$ 

- (1) Write 0 = [(0, 1)] and 1 = [(1, 1, 1)], and see that  $0 \neq 1$ .
- (2) (Addition)

$$[(a,b)] +_{\mathbb{Q}} [(c,d)] := [(ad +_{\mathbb{Z}} bc, bd)]$$

(3) (Multiplication)

$$[(a,b)] \cdot [(b,c)] := [(ac,bd)]$$

This is commutative, associative, and distributes over addition.

(4) (Additive Inverse) For any  $[(a,b)] \in \mathbb{Q}$ , there is some  $[(c,d)] \in \mathbb{Q}$  s. t.

$$[(a,b)] + [(c,d)] = 0$$

In fact, [(a,b)] = [(-c,d)]

(5) (Multiplication Inverse) If  $[(a,b)] \neq 0$  then there exists a  $[(c,d)] \neq 0$  s. t.

$$[(a,b)] \cdot [(c,d)] = 1$$

In fact [(c,d)] = [(b,a)].

(6) 0 is the additive identity and 1 is the multiplicative identity

**Definition** (Field). Any set S together with elements  $0, 1 \in S$  and operations Addition(+) and Multiplication(·):  $S \times S \mapsto S$  satisfying the properties above is called a <u>field</u>. So  $\mathbb{Q}$  is a field.

There are some further properties of  $\mathbb{Q}$ :

(1) (Order) Say  $[(a,b)] \in \mathbb{Q}$ . We say that [(a,b)] > 0 if

$$b > 0$$
 and  $a > 0$ , or  $b < 0$  and  $a < 0$ 

This is a total order on  $\mathbb{Q}$  and satisfies trichotomy.

(2) (Distance) Write  $Q^{\geqslant 0}$  to be the set of rationals that are either 0 or positive. We have function  $d: \mathbb{Q} \times \mathbb{Q} \mapsto \mathbb{Q}^{\geqslant 0}$ , defined as

$$d([(a,b)],[(c,d)]) = \begin{cases} [(a,b)] - [(c,d)] & \text{if } [(a,b)] \geqslant [(c,d)] \\ [(c,d)] - [(a,b)] & \text{otherwise} \end{cases}$$

This function is a <u>metric</u> in the following sense:

a. 
$$d(x,y) \geqslant 0 \quad \forall x,y \text{ in } \mathbb{Q}$$

b. 
$$d(x,y) = 0$$
 iff  $x = y \quad \forall x, y \in \mathbb{Q}$ 

c. 
$$d(x, y) = d(y, x)$$
, and

d. 
$$d(x,z) \le d(x,y) + d(y,z)$$