ANALYSIS I EXTENSION LECTURE
4. CONSTRUCTION OF THE INTEGERS $\mathbb{Z}$

ASILATA BAPAT

Construction of $\mathbb{Z}$

(Equivalence classes of pairs of naturals)
Let $Z = \mathbb{N} \times \mathbb{N}$. Define a relation R on $Z \times Z$ ($R \subseteq Z \times Z$) as follows:

$$R := \{ ((a,b),(c,d)) \in Z \times Z) \mid a + d = b + c \}$$

Example. $((3,1),(4,2)) \in R$ and $((1,5),(5,9)) \in R$

Claim. *R is an equivalence relations:*

(1) (Reflexivity) Since $a + b = b + a \quad \forall a, b \in \mathbb{N}$, we see that $\forall a, b \in \mathbb{N}$, the pair $((a,b),(a,b))$ in R.

(2) (Symmetry) If $((a,b),(c,d)) \in R$ then $a+d = b+c$, so $c+a = d+b$ and so $((c,d),(a,b)) \in R$

(3) (Transitivity) Suppose $((a+b),(c+d)) \in R$ and $((c,d),(p,g)) \in R$. Then $a + b = b + c$ and $c + q = d + p$. After adding these and some manipulations, we see that

$$(a + q) + (c + d) = (b + p) + (c + d)$$

By cancellation , we have $a + q = b + p$, so $((a,b),(p,q)) \in R$

If $x \in Z$, write

$$[x] := \{ y \in Z \mid (x,y) \in R \}$$

Then $[x] \subseteq Z$ and $x \in [x]$, so $[x] \neq \varnothing$. This is the <u>equivalence class</u> of x. $\mathbb{Z}$ is partitioned into disjoint, nonempty equivalence classes, so:

Definition.

$$\mathbb{Z} := Z \big/ R = \{ S \in \mathbb{P}(Z) \mid S = [x] \text{ for some } x \in Z \}$$
$$= \{ \text{all equivalence classes of } R \}$$

Example. $[(0,1)] = [(3,4)]$

There is an injective function $i : \mathbb{N} \mapsto \mathbb{Z}$ given by $n \mapsto [(n,0)]$.

THE PROPERTIES OF $\mathbb{Z}$

(1) (Addition) $[(a,b)] + [(c,d)] := [(a +_{\mathbb{N}} c, b +_{\mathbb{N}} d)]$

(2) (Negative) $-[(a,b)] := [(b,a)]$, and $[(a,b)] + [(b,a,)] = [(0,0)]$

(3) (Subtraction) $[(a,b)] - [(c,d)] := [(a,b)] + [(c,d)] = [(a+d, b+c)]$

(4) (Order Relation) We say $[(a,b)] < [(,d)]$ if $a+d < b+c$. Then this is a well-defined, total order. That is, if $[(a,b)]$ and $[(c,d)]$ are in $\mathbb{Z}$, then we have a trichotomy:

$$[(a,b)] < [(c,d)] \text{ or } [(c,d)] < [(a,b)] \text{ or } [(a,b)] = [(c,d)]$$

(5) (Multiplication) $[(a,b)] \cdot [(c,d)] := [(ac+bd, ad+bc)]$
This is commutative, associative, and distributes over addition.

(6) (Absolute Value)

$$|[a-b]| := \begin{cases} a-b & \text{if } a \geq b \\ b-a & \text{otherwise} \end{cases}$$

Henceforth, we write $[(a,b)]$ as follows:

- If $a \geq b$, we replace it by $(a-b) \rightarrow$ Subtraction in $\mathbb{N}$
- If $a < b$, we replace it by $-(b-a) \rightarrow$ subtraction in $\mathbb{N}$