

ANALYSIS I EXTENSION LECTURE

1. INTRODUCTION TO AXIOMATIC SET THEORY

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Calculus is typically done on \mathbb{R} , \mathbb{R}^n , \mathbb{C}^n , why not on \mathbb{N} or \mathbb{Q}

- \mathbb{N} is unsuitable because it is “discrete”. Metric on \mathbb{N} does not have arbitrarily small numbers. That is, if $d(m, n) < 1$, then $m = n$.
- \mathbb{Q} is unsuitable because it has “holes” in it. There are unequal, arbitrarily close rational numbers. However, it is not a complete metric space. More about this later, but for example, $\{x \in \mathbb{Q} \mid x^2 < 2\}$ does not have a least upper bound in \mathbb{Q} so there are bounded increasing sequences in \mathbb{Q} that does not converge in \mathbb{Q} .

To do this properly, we should start at the very beginning: \mathbb{R} is constructed from \mathbb{Q} , which is constructed from \mathbb{Z} , which is constructed from \mathbb{N} . Which is constructed from ...?

One approach (standard) is axiomatic set theory.

Definition. A set is an object S s.t. for each x , we either have $x \in S$ or $x \notin S$. That is, x is an element of S , or x is not an element of S , but not both.

A set can be specified by listing its elements, like $\{1, 2, 3\}$, or by specification, like $\{x \in \mathbb{N} \mid x \text{ is prime.}\}$. This definition works fine — until it doesn’t!

Example (Russell’s Paradox). Let S be defined as follows:

$$S = \{T \mid T \notin T\}$$

Question: Is $S \in S$? If yes, then by its definition, S is a set s.t. $S \notin S \implies S \notin S$, contradiction; If no, then $S \notin S$, matching the requirement for elements of $S \implies S \in S$, contradiction.

Zermelo-Fraenkel Set Theory (ZFC)

(Typically written in the language of first order logic, however we’ll use a mix of logic and English.)

Axiom 1 (Axiom of Extension). *Two sets are equal iff they have the same elements:*

$$X = Y \text{ iff } \forall z(z \in X \iff z \in Y)$$

Axiom 2 (Axiom of Existence). *The “empty set” is a set. That is, \emptyset exists and has the property that $\forall z(z \notin \emptyset)$.*

Axiom 3 (Axiom of Pairing). *If X and Y are sets, then there is a set $\{X, Y\}$.*

Example. $\{1, 2\}$ and $\{2, 3\}$ pair to $\{\{1, 2\}, \{2, 3\}\}$.

Axiom 4 (Axiom of Union). *If S is any set, then the “union over all elements of S ” is a set. That is, there is a set whose elements are all x such that x belongs to some elements of S .*

Example.

$$\{ \{ 1, 2 \}, \{ 2, 3 \} \} \rightarrow \{ 1, 2, 3 \}$$

Axiom 5 (Axiom of Intersection). *If S is a set, the “intersection over all elements of S ” exists*

Axiom 6 (Axiom of Foundation). *Every $X \neq \emptyset$ contains a member Y such that $X \cap Y = \emptyset$. (Roughly speaking, a sequence $z_1 \ni z_2 \ni \dots$ must always terminate.)*

Consequence of failure of the axiom:

$\forall Y_1 \in X, X \cap Y_1 \neq \emptyset$. So $\exists Y_2 \in X \cap Y_1 \implies X \cap Y_2 \neq \emptyset$ and so on!

Axiom 7 (Axiom Schema of Specification). *We can build sets using “set builder” notation as subsets of a known set that satisfies certain predicate. A predicate can be seen as a kind of function outputting “TRUE” or “FALSE”.*

Example. $\{ x \in \mathbb{N} \mid x > 2 \}$ is the set $\{ 3, 4, 5, \dots \}$ or $\{ x \in \{ \emptyset, \{ \emptyset \} \} \mid x \neq \emptyset \}$ is the set $\{ \{ \emptyset \} \}$.

Axiom 8 (Axiom of Power Set). *If S is a set, then $\mathbb{P}(S)$ is the collection of all subsets of S , which is a set*

Based on previous axioms, we can construct the Cartesian Product, as follows:

Definition (Cartesian Product). We define a ordered pair (a, b)

$$(a, b) := \{ \{ a \}, \{ a, b \} \}$$

Then we define

$$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

Based on Cartesian Product, we can define the notation of Function

Definition (Function). Let A, B be sets. A function $f : A \mapsto B$ is an element of $\mathbb{P}(A \times B)$ such that

- (1) for every $a \in A$, there is some $b \in B$ such that $(a, b) \in f$.
- (2) For every $a \in A$, if b_1 and b_2 such that $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$.

Axiom 9 (Axiom of Infinity). *Let X be a set. Define $\text{succ}(X) = X^+ = X \cup \{ X \}$. There is a set with the properties:*

- (1) $\emptyset \in S$
- (2) If $X \in S$, then $\text{succ}(X) \in S$

Example.

$$\begin{aligned} \text{succ}(\emptyset) &= \{ \emptyset \} \\ \{ \emptyset \}^+ &= \{ \emptyset \} \cup \{ \{ \emptyset \} \} = \{ \emptyset, \{ \emptyset \} \} \end{aligned}$$

In particular, S cannot have finitely many elements!

Definition (Choice Function). A Choice Function f defined on a collection X of none empty sets is a function with the property that if $a \in X$ then $f(a) \in a$

Example. $f(x) = \min(x)$ gives $f(\{1, 2, 3\}) = 1 \rightarrow f$ is defined on $\mathbb{P}(\mathbb{N})$.

Axiom 10 (Axiom of Choice). *Choice functions always exist for any X*

The axiom states that if x is any set of nonempty sets, then there is a choice function on x . This is equivalent to the well-ordering principle:

Theorem (Well-ordering Principle). *For any set x , there is a binary relation $R(\leq)$ which well-orders x : every non-empty subset of x has a smallest element.*