ANALYSIS I EXTENSION LECTURE
5. CONSTRUCTION OF THE RATIONAL $\mathbb{Q}$

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The construction of the rational $\mathbb{R}$ is very similar to that of of the Integers $\mathbb{Z}$.

Definition. We let Q be equivalence classes of ordered pairs of (certain) integers:

$$Q := \{ (a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b \neq 0 \}$$

Remark. Remember, $0 = [(0, 0)] \in \mathbb{Z} \ldots$

Definition. $R \in Q \times Q$ is defined as follows:

$$R = \{ ((a, b), (c, d)) \in Q \times Q \mid ad = bc \}$$

Claim. *R is an equivalence relation*

Proof. Exercise. □

Definition. The set of rationals, i. e. $\mathbb{Q}$, as

$$\mathbb{Q} := Q \Big/ R = \{ \text{equivalence classes of parts of integers } (a, b) \text{ with } b \neq 0 \}$$

<u>Operations/Properties</u> of $\mathbb{Q}$

(1) Write $0 = [(0, 1)]$ and $1 = [(1, 1,)]$, and see that $0 \neq 1$.

(2) (Addition)

$$[(a, b)] +_{\mathbb{Q}} [(c, d)] := [(ad +_{\mathbb{Z}} bc, bd)]$$

(3) (Multiplication)

$$[(a, b)] \cdot [(b, c)] := [(ac, bd)]$$

This is commutative, associative, and distributes over addition.

(4) (Additive Inverse) For any $[(a, b)] \in \mathbb{Q}$, there is some $[(c, d)] \in \mathbb{Q}$ s. t.

$$[(a, b)] + [(c, d)] = 0$$

In fact, $[(a, b)] = [(-c, d)]$

(5) (Multiplication Inverse) If $[(a, b)] \neq 0$ then there exists a $[(c, d)] \neq 0$ s. t.

$$[(a, b)] \cdot [(c, d)] = 1$$

In fact $[(c, d)] = [(b, a)]$.

(6) 0 is the additive identity and 1 is the multiplicative identity

Definition (Field)**.** Any set S together with elements $0, 1 \in S$ and operations Addition(+) and Multiplication($\cdot$): $S \times S \mapsto S$ satisfying the properties above is called a <u>field</u>. So $\mathbb{Q}$ is a field.

There are some further properties of $\mathbb{Q}$:

(1) (Order) Say $[(a,b)] \in \mathbb{Q}$. We say that $[(a,b)] > 0$ if

$$b > 0 \text{ and } a > 0, \underline{\text{or}}\ b < 0 \text{ and } a < 0$$

This is a total order on $\mathbb{Q}$ and satisfies trichotomy.

(2) (Distance) Write $Q^{\geqslant 0}$ to be the set of rationals that are either 0 or positive. We have function $d : \mathbb{Q} \times \mathbb{Q} \mapsto \mathbb{Q}^{\geqslant 0}$, defined as

$$d([(a,b)],[(c,d)]) = \begin{cases} [(a,b)] - [(c,d)] & \text{if } [(a,b)] \geqslant [(c,d)] \\ [(c,d)] - [(a,b)] & \text{otherwise} \end{cases}$$

This function is a <u>metric</u> in the following sense:

a. $d(x,y) \geqslant 0 \quad \forall x,y$ in $\mathbb{Q}$

b. $d(x,y) = 0$ iff $x = y \quad \forall x,y \in \mathbb{Q}$

c. $d(x,y) = d(y,x)$, and

d. $d(x,z) \leqslant d(x,y) + d(y,z)$