Integration by Parts

Assume u and v are functions of x.

Product rule : $(uv)' = u'v + uv'$

Integrate both sides $\rightarrow$ $uv = \int u'v\,dx + \int uv'\,dx$

Rearrange $\rightarrow$ $\int uv'\,dx = uv - \int vu'\,dx$

Note : $\frac{dv}{dx} = v'(x)$ $\qquad$ $\frac{du}{dx} = u'(x)$

$dv = v'(x)\,dx$ $\qquad$ $du = u'(x)\,dx$

so... $\int u\,dv = uv - \int v\,du$

I think "uv v du" to remember order.

This method is helpful for integration of products where one term can be called u and the rest can represent dv.

Best choices for $u \rightarrow$ L I A T E

L: Logs
I: Inverse Trig
A: Algebraic
T: Trig
E: Exponential

1. $\int x e^x\,dx$

$u = x \qquad dv = e^x\,dx$

$du = dx \qquad v = e^x$

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

2. $\int x \sin x \, dx$

$u = x \qquad dv = \sin x \, dx$
$du = dx \qquad v = -\cos x$

$$\int x \sin x \, dx = x(-\cos x) - \int -\cos x \, dx$$
$$= -x\cos x + \int \cos x \, dx$$
$$= -x\cos x + \sin x + C$$

Repeated Use

3. $\int x^2 e^x \, dx$

$u = x^2 \qquad dv = e^x dx$
$du = 2x\,dx \qquad v = e^x$

$$\int x^2 e^x \, dx = x^2 e^x - \int e^x 2x \, dx$$
$$= x^2 e^x - 2\int x e^x \, dx$$

$u = x \qquad dv = e^x dx$
$du = dx \qquad v = e^x$

$$= x^2 e^x - 2\left[x e^x - \int e^x dx\right]$$
$$= x^2 e^x - 2x e^x + 2e^x + C$$
$$= e^x(x^2 - 2x + 2) + C$$

Cycle Problem

4. $\int e^{2x} \sin x \, dx$

$u = \sin x \qquad dv = e^{2x} dx$
$du = \cos x \, dx \qquad v = \frac{1}{2} e^{2x}$

$$\int e^{2x} \sin x \, dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{2}\int e^{2x} \cos x \, dx$$

$u = \cos x \qquad dv = e^{2x} dx$
$du = -\sin x \, dx \qquad v = \frac{1}{2} e^{2x}$

$$\int e^{2x} \sin x \, dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{2}\left[\frac{1}{2} e^{2x} \cos(x) + \frac{1}{2}\int e^{2x} \sin x \, dx\right]$$
$$\int e^{2x} \sin x \, dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x - \frac{1}{4}\int e^{2x} \sin x \, dx$$
$$\frac{5}{4}\int e^{2x} \sin x \, dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x + C$$
$$\therefore \int e^{2x} \sin x \, dx = \frac{1}{5} e^{2x}(2\sin x - \cos x) + C$$

5. $\int_1^2 \ln x \, dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$$\int_1^2 \ln x \, dx = \left[x \ln x - \int dx\right]_1^2 = \left[x \ln x - x\right]_1^2$$

$$= (2\ln 2 - 2) - (1 \ln 1 - 1) = 2\ln 2 - 1$$

Repeated Use

6. $\int x^2 \ln^2 x \, dx$

$u = (\ln x)^2 \qquad dv = x^2 dx$

$du = \frac{2(\ln x)}{x} dx \qquad v = \frac{x^3}{3}$

$$\int x^2 \ln^2 x \, dx = \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\int x^2 \ln x \, dx$$

$u = \ln x \qquad dv = x^2 dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^3}{3}$

$$\int x^2 \ln^2 x \, dx = \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\left[\frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx\right]$$

$$\int x^2 \ln^2 x \, dx = \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{9}\left(\frac{x^3}{3}\right) + C$$

$$= \frac{1}{27}x^3\left(9(\ln x)^2 - 6\ln x + 2\right) + C$$

7. $\int \sin^{-1} x \, dx$

$u = \sin^{-1} x \qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}} dx \qquad v = x$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

substitution let $t = 1 - x^2$

$dt = -2x\,dx$

$$= x \sin^{-1} x + \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$= x \sin^{-1} x + \frac{1}{2}\left(2t^{\frac{1}{2}} + c\right)$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + C$$

check answer by taking derivative

8. $\int x\tan^{-1}(x^2)\,dx$

$u=\tan^{-1}(x^2)$ $\qquad dv = x\,dx$

$du = \frac{1}{1+(x^2)^2}(2x)\,dx$ $\qquad v=\frac{1}{2}x^2$

$du = \frac{2x}{1+x^4}\,dx$

$$\int x\tan^{-1}(x^2)\,dx = \tfrac{1}{2}x^2\tan^{-1}(x^2) - \int \frac{x^3}{1+x^4}\,dx$$

$t=1+x^4$
$dt=4x^3dx$

$$= \tfrac{1}{2}x^2\tan^{-1}(x^2) - \tfrac{1}{4}\int\frac{dt}{t}$$

$$= \tfrac{1}{2}x^2\tan^{-1}(x^2) - \tfrac{1}{4}\ln|t| + C$$

$$= \tfrac{1}{2}x^2\tan^{-1}(x^2) - \tfrac{1}{4}\ln(1+x^4) + C$$

9. $\int_0^{\frac{\pi}{2}} x\cos 2x\,dx$

$u=x$ $\qquad dv=\cos 2x\,dx$

$du=dx$ $\qquad v=\frac{1}{2}\sin 2x$

$$\int_0^{\frac{\pi}{2}} x\cos 2x\,dx = \left[\tfrac{1}{2}x\sin 2x - \tfrac{1}{2}\int \sin 2x\,dx\right]_0^{\frac{\pi}{2}}$$

$$= \left[\tfrac{1}{2}x\sin 2x + \tfrac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left(0-\tfrac{1}{4}\right) - \left(0+\tfrac{1}{4}\right) = -\tfrac{1}{2}$$

10. $\int_{\frac{2}{\sqrt{3}}}^{2} z\sec^{-1}z\,dz$

$u=\sec^{-1}z$ $\qquad dv = z\,dz$

$du=\frac{1}{z\sqrt{z^2-1}}$ $\qquad v=\frac{1}{2}z^2$

$$\int_{\frac{2}{\sqrt{3}}}^{2} z\sec^{-1}z\,dz = \left[\tfrac{1}{2}z^2\sec^{-1}z - \tfrac{1}{2}\int\frac{z}{\sqrt{z^2-1}}\,dz\right]_{\frac{2}{\sqrt{3}}}^{2}$$

$t=z^2-1$
$dt=2z\,dz$

$$= \left[\tfrac{1}{2}z^2\sec^{-1}z - \tfrac{1}{4}\int\frac{dt}{\sqrt{t}}\right]_{z=\frac{2}{\sqrt{3}}}^{z=2}$$

$$= \left[\tfrac{1}{2}z^2\sec^{-1}z - \tfrac{1}{2}\sqrt{z^2-1}\right]_{\frac{2}{\sqrt{3}}}^{2} = \left(2\left(\tfrac{\pi}{3}\right)-\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{2}{3}\left(\tfrac{\pi}{6}\right)-\tfrac{1}{2\sqrt{3}}\right)$$

$$= \frac{5\pi}{9} + \frac{-2}{2\sqrt{3}} = \frac{5\pi}{9} - \frac{1}{\sqrt{3}}$$

11. $\int x \tan x \sec x \, dx$ $\quad u = x \quad du = dx \quad dv = \sec x \tan x \, dx \quad v = \sec x$

$$\int x \tan x \sec x \, dx = x\sec x - \int \sec x \, dx$$

Formula #13 on Integral Table (Know It!)

$$= x\sec x - \ln|\sec x + \tan x| + C$$

Prove It! $\rightarrow \frac{d}{dx}\left[x\sec x - \ln|\sec x + \tan x| + C\right]$

$$= \sec x + x\sec x \tan x - \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x)$$

$$= \sec x + x\sec x \tan x - \frac{\sec x(\tan x + \sec x)}{(\tan x + \sec x)}$$

$$\stackrel{\checkmark}{=} x\sec x \tan x$$

cycle

12. $\int \sin(\ln x)\,dx$ $\quad u = \sin(\ln x) \quad du = \frac{\cos(\ln x)}{x}dx \quad dv = dx \quad v = x$

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx$$

$u = \cos(\ln x) \quad du = \frac{-\sin(\ln x)}{x}dx \quad dv = dx \quad v = x$

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \left[x\cos(\ln x) + \int \sin(\ln x)\,dx\right]$$

$$2\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) + C$$

$$\int \sin(\ln x)\,dx = \frac{1}{2}x\left(\sin(\ln x) - \cos(\ln x)\right) + C$$