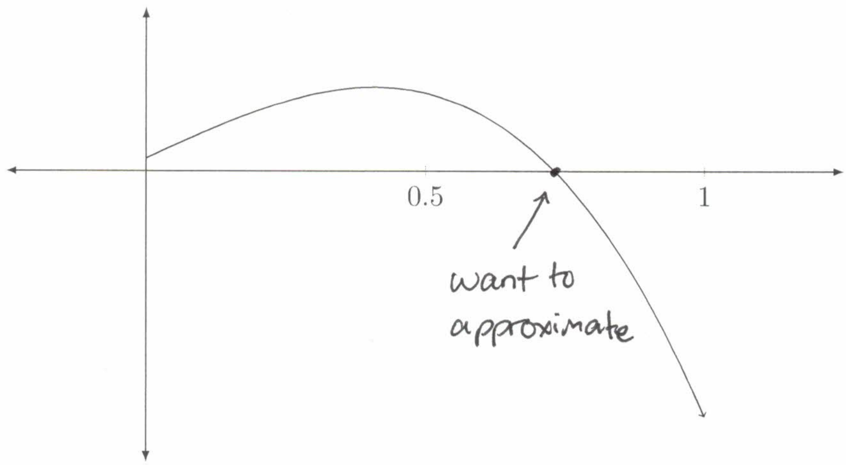
§4.8 Newton's Method

Goal: Use tangent lines to develop the algorithm "Newton's Method" and use Newton's Method to approximate zeros of functions.

Example 1. Given the following function in the graph below, how can we approximate the x-value where $f(x) = 0$?

"root"
"zero"
"x-intercept"



Newton's Method:

Step 1: Choose an initial guess and call it x_0.

$x_0 = .5$

Step 2: Find the equation of the tangent line to f at x_0.

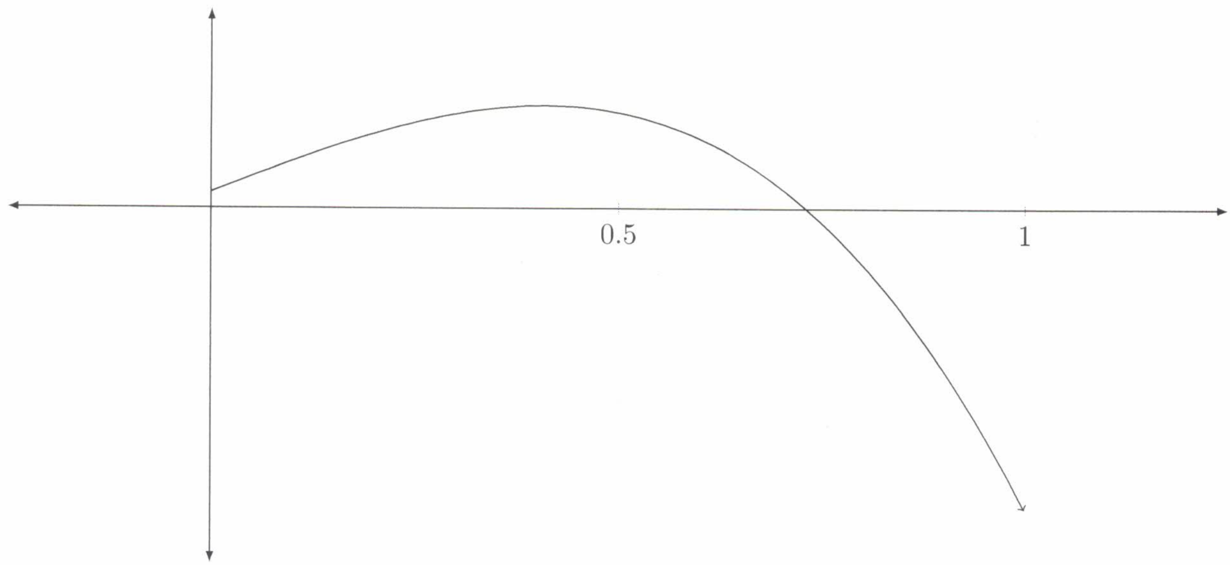
Step 3: Find the x-intercept of the tangent line and call this x_1.

Find the x-value where $y = 0$ on the tangent line.

Step 4: Repeat steps 1-3 using x_1 instead of x_0. New guess: x_1

We can do this as many times as we like.

Here's the same function f zoomed in closer to its root:



Instead of calculating the tangent line for each new guess and then finding its x-intercept, we can put together what we know about tangent lines to find a formula to make our calculations quicker.

Tangent line to f at x_n (our n^{th} guess):

point: $(x_n, f(x_n))$

slope: $f'(x_n)$

$\longrightarrow \quad y = f'(x_n)(x - x_n) + f(x_n)$

We want to find x when $y = 0$:

$$0 = f'(x_n)(x - x_n) + f(x_n)$$

$$\rightarrow \quad -f(x_n) = f'(x_n)(x - x_n)$$

$$\rightarrow \quad \frac{-f(x_n)}{f'(x_n)} = x - x_n \longrightarrow \quad x = \underbrace{x_n - \frac{f(x_n)}{f'(x_n)}}_{\text{x-intercept of tangent line}}$$

This x is our next guess, so it is x_{n+1}. That means the formula for Newton's Method is

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Example 2. For $f(x) = \cos(2x) - \sin(x)$, approximate the x-value in $\left[0, \frac{\pi}{2}\right]$ where $f(x) = 0$.
Note: This is the same function from the first page so we'll use the same x_0, but in general you can choose a different initial guess.

① Initial guess: $x_0 = .5$

② Update our guess using the formula we found.

$$f(x) = \cos(2x) - \sin(x)$$

$$f'(x) = -2\sin(2x) - \cos(x)$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = .5 - \frac{\cos(2(.5)) - \sin(.5)}{-2\sin(2(.5)) - \cos(.5)}$$

$$= \underbrace{.5237751158\ldots}_{\text{our next guess}}$$

③ Repeat. $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = \underbrace{.5235987846\ldots}_{\text{next guess}}$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \boxed{.5235987756\ldots}$$

Our new steps:

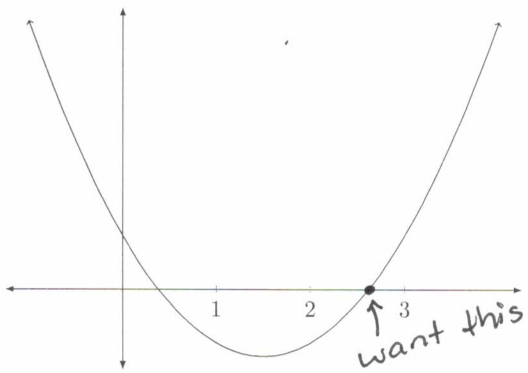
1. Choose initial guess and call it x_0.
2. Update our guess using our formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

   (So, if x_0 is our first guess at what the root is, our next guess should be $x_0 - \frac{f(x_0)}{f'(x_0)}$ which we will call x_1.)

3. Repeat.

Example 3. For $g(x) = x^2 - 3x + 1$, use Newton's method to approximate the largest x satisfying $g(x) = 0$.



① Choose x_0: $x_0 = 2$

② Update guess:

$g(x) = x^2 - 3x + 1$

$g'(x) = 2x - 3$

$\rightarrow \quad x_1 = x_0 - \frac{g(x_0)}{g'(x_0)} = 2 - \frac{2^2 - 3(2) + 1}{2(2) - 3} = 3$

new guess

③ Repeat. $x_2 = 3 - \frac{g(3)}{g'(3)} = 3 - \frac{3^2 - 3(3) + 1}{2(3) - 1} = 2.666666667$

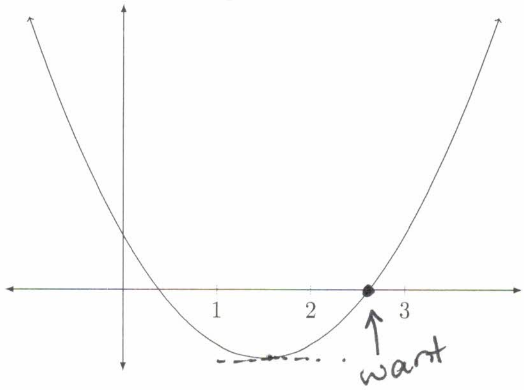
$x_3 = 2.619047619$

$x_4 = 2.618034449$

$x_5 = 2.618033989$

$\boxed{x_6 = 2.618033989}$

Example 4. In the previous example, what points would be a "bad" choice for x_0?



- Bad guess: any # where f' is 0 or where f' is undefined

  Why:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$\longleftarrow$ bad if this is 0 or undefined

- Bad guess: any x-value to the left of the smaller root

  Why: Newton's Method would take us toward the smaller root instead.

- Bad guess: any x-value b/w the horizontal tangent and smaller root

  Why: We go towards the smaller root instead.

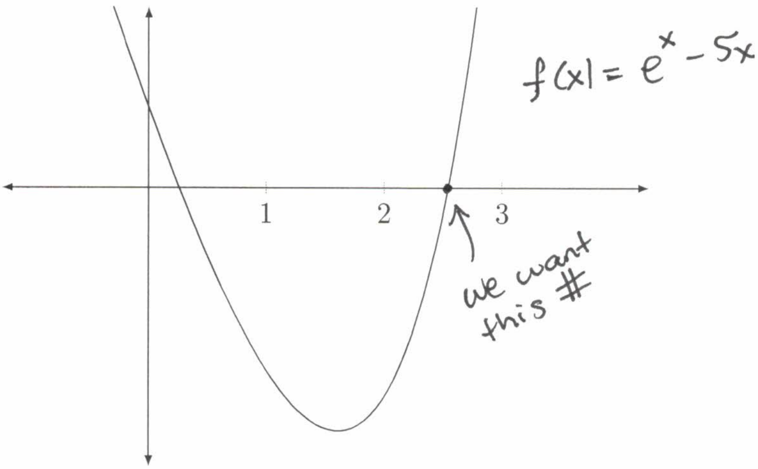
In general: We don't want there to be any points where f

- has a horizontal tangent line
- is non-differentiable

between our initial guess and the actual value of the root.

Example 5. Use Newton's Method to approximate the larger solution to $e^x = 5x$ to three decimal places.

$e^x - 5x = 0$



① Initial guess: $x_0 = 2$

② Update guess:

$f(x) = e^x - 5x$

$f'(x) = e^x - 5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \frac{e^2 - 5(2)}{e^2 - 5}$$

$$= 3.09288$$

③ Repeat. $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 2.70697$

$x_3 = 2.56184$

$x_4 = 2.54294$

$x_5 = 2.54264$

$x_6 = 2.54264$

larger root is approx: $\boxed{2.543}$

Extra Practice:

1. In a-b, apply Newton's Method to f and initial guess x_0 to calculate x_1, x_2, and x_3. (Use a calculator and round to 4 decimal places.)

   (a) $f(x) = x^3 - 10$, initial guess: $x_0 = 2$

   $f'(x) = 3x^2$

   $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2.1667$

   $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1545$

   $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.1544$

   (b) $f(x) = 1 - x\sin(x)$, initial guess: $x_0 = 7$

   $f'(x) = -x\cos(x) - \sin(x)$

   $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 7 - \frac{f(7)}{f'(7)} = 6.3935$

   $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 6.4393$

   $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 6.4391$

2. The first positive solution of $\sin(x) = 0$ is π. Use Newton's Method (and a calculator) to calculate π to four decimal places. Use $x_0 = 3$ as your initial guess.

   $f(x) = \sin(x)$, $f'(x) = \cos(x)$

   $x_0 = 3$

   $x_1 = 3 - \frac{\sin(3)}{\cos(3)} = 3.142546$

   $x_2 = 3.141593$

   $x_3 = 3.141593$

   So, $\boxed{\pi \approx 3.1416}$.