5.7 Integrals of Transcendentals

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \longrightarrow \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1} \longrightarrow \int \frac{dx}{x^2+1} = \tan^{-1}x + C$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} \longrightarrow \int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1}x + C$$

$$\frac{d}{dx}b^x = b^x \cdot \ln b \longrightarrow \int b^x\,dx = \frac{b^x}{\ln b} + C$$

1. $\displaystyle\int_2^4 \frac{dt}{3t+4}$ $\qquad u = 3t+4$, $du = 3dt$ $\qquad u(2) = 10$, $u(4) = 16$

$$= \frac{1}{3}\int_2^4 \frac{3\,dt}{3t+4} = \frac{1}{3}\int_{10}^{16} \frac{1}{u}\,du = \frac{1}{3}\Big[\ln|u|\Big]_{10}^{16}$$

$$= \frac{1}{3}\left[\ln 16 - \ln 10\right] = \ln\sqrt[3]{1.6}$$

2. $\displaystyle\int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\Big|_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$

$$= \frac{\pi}{3} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{2}$$

3. $\displaystyle\int \frac{dx}{4+x^2} = \frac{1}{4}\int \frac{dx}{1+\frac{1}{4}x^2}$ $\qquad u^2 = \frac{1}{4}x^2$, $u = \frac{1}{2}x$, $du = \frac{1}{2}dx$

$$= \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

4. $\displaystyle\int_{\frac{4}{\sqrt{3}}}^{\frac{4}{\sqrt{2}}} \frac{dx}{x\sqrt{x^2-4}} = \frac{1}{2}\int_{\frac{4}{\sqrt{3}}}^{\frac{4}{\sqrt{2}}} \frac{dx}{x\sqrt{\frac{1}{4}x^2-1}}$ $\qquad u^2 = \frac{1}{4}x^2$, $u = \frac{1}{2}x$, $du = \frac{1}{2}dx$

$$= \int_{\frac{4}{2\sqrt{3}}}^{\frac{2}{\sqrt{2}}} \frac{du}{2u\sqrt{u^2-1}} = \frac{1}{2}\int_{\frac{2}{\sqrt{3}}}^{2/\sqrt{2}} \frac{du}{u\sqrt{u^2-1}} = \frac{\sec^{-1}u}{2}\Big|_{2/\sqrt{3}}^{2/\sqrt{2}} = \frac{\cos^{-1}u}{2}\Big|_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{2}}{2}}$$

$$= \frac{\pi}{8} - \frac{\pi}{12} = \frac{\pi}{24}$$

5. $\int \frac{\ln(\cos^{-1}x)\,dx}{\cos^{-1}x\sqrt{1-x^2}}$

$u = \ln(\cos^{-1}x)$

$du = \frac{1}{\cos^{-1}x}\left(\frac{-1}{\sqrt{1-x^2}}\right)dx$

$= -\int u\,du = -\left[\frac{u^2}{2} + C\right] = \frac{-1}{2}\left(\ln(\cos^{-1}x)\right)^2 + C$

6. $\int_{-2}^{2} x\,10^{x^2}\,dx$

$u = x^2$

$du = 2x\,dx$ $\rightarrow$ $\frac{1}{2}\int_4^4 10^u\,du = \frac{1}{2}\left[\frac{10^u}{\ln 10}\right]_4^4 = \boxed{0}$

7. $\int \frac{dx}{\sqrt{5^{2x}-1}}$

NOTE: $u = 5^{2x}-1$ will NOT help!

factor 5^{2x} out of root

$\rightarrow \int \frac{dx}{\sqrt{5^{2x}(1-5^{-2x})}} = \int \frac{dx}{5^x\sqrt{1-5^{-2x}}}$

Notice form $\sqrt{1-u^2}$ in denominator $\rightarrow$

$u^2 = 5^{-2x}$

$u = 5^{-x}$

$du = 5^{-x}(-1)\ln 5\,dx$

$= \frac{-1}{\ln 5}\int \frac{(-1)(\ln 5)\,dx}{5^x\sqrt{1-5^{-2x}}} = \frac{-1}{\ln 5}\int \frac{1}{\sqrt{1-u^2}}\,du$

$= \frac{-1}{\ln 5}\left[\sin^{-1}u + C\right] = \frac{-1}{\ln 5}\sin^{-1}\left(\frac{1}{5^x}\right) + C$

8. $\int \frac{4x\,dx}{x^2+1}$

$u = x^2+1$

$du = 2x\,dx$ $\rightarrow$ $2\int \frac{2x\,dx}{x^2+1} = 2\int \frac{1}{u}\,du$

$= 2\ln|u| + C = 2\ln(x^2+1) + C$

9. $\int \frac{dx}{\sqrt{9-16x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{1-\frac{16}{9}x^2}}$

$u^2 = \frac{16}{9}x^2$

$u = \frac{4}{3}x$

$du = \frac{4}{3}dx$

$= \frac{1}{3}\cdot\frac{3}{4}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{4}\sin^{-1}u + C = \frac{1}{4}\sin^{-1}\left(\frac{4x}{3}\right) + C$

10. $\int \frac{e^{2x} - e^{4x}}{e^x}\,dx = \int (e^x - e^{3x})\,dx = e^x - \frac{1}{3}e^{3x} + C$

11. $\int e^x (e^{2x}+1)^4\,dx$ $\quad u = e^x$, $du = e^x\,dx$

1 1
1 2 1
1 3 3 1
1 4 6 4 1

$= \int (u^2+1)^4\,dx$

$= \int (u^8 + 4u^6 + 6u^4 + 4u^2 + 1)\,du$

$= \frac{u^9}{9} + \frac{4u^7}{7} + \frac{6u^5}{5} + \frac{4u^3}{3} + u + C$

$= \frac{1}{9}e^{9x} + \frac{4}{7}e^{7x} + \frac{6}{5}e^{5x} + \frac{4}{3}e^{3x} + e^x + C$

12. $\int \frac{(3x-1)}{9-2x+3x^2}\,dx$ $\quad u = 3x^2 - 2x + 9$, $du = (6x-2)\,dx$, $du = 2(3x-1)\,dx$

$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|3x^2 - 2x + 9| + C$

13. $\int \frac{(\ln x)^2}{x}\,dx$ $\quad u = \ln x$, $du = \frac{1}{x}\,dx$ $\rightarrow \int u^2\,du = \frac{1}{3}u^3 + C = \frac{(\ln x)^3}{3} + C$

14. $\int \left(\frac{1}{2}\right)^{3x+2} dx$ $\quad u = 3x+2$, $du = 3\,dx$

$= \frac{1}{3}\int \left(\frac{1}{2}\right)^u du = \frac{1}{3}\left[\frac{\left(\frac{1}{2}\right)^u}{\ln\frac{1}{2}} + C\right] = \frac{\left(\frac{1}{2}\right)^{3x+2}}{3\ln\frac{1}{2}} + C$

15. $\int \frac{5x+3}{x^2+1}\,dx$ Cannot easily obtain factor of $2x$ in numerator so split.

$= \int \frac{5x}{x^2+1}\,dx + \int \frac{3}{x^2+1}\,dx$

$u = x^2+1$
$du = 2x\,dx$

$= \frac{5}{2}\int \frac{1}{u}\,du + 3\tan^{-1}x + C = \frac{5}{2}\ln(x^2+1) + 3\tan^{-1}x + C$