$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx$$

1. $$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx \overset{\lim_{b\to\infty}}{=} \ln x\Big|_1^b = \lim_{b\to\infty} \ln b - \ln 1$$
$$= \infty$$

- An integral is convergent if the limit exists.
- An integral is divergent if the limit goes to ∞ or cannot be determined.

2. Find values of p for which $\int_1^\infty \frac{1}{x^p}$ is convergent.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \frac{1}{1-p} \lim_{b\to\infty} \left[\frac{1}{x^{p-1}}\right]_1^b = \frac{1}{1-p}\left[\lim_{b\to\infty}\left(\frac{1}{b^{p-1}} - 1\right)\right]$$

- If $p>1$, $\lim_{b\to\infty} \frac{1}{b^{p-1}}$ converges to 0, so integral converges to $\frac{1}{1-p}(-1) = \frac{-1}{1-p} = \frac{1}{p-1}$
- If $p<1$, $\lim_{b\to\infty} \frac{1}{b^{p-1}} = \infty$, so integral diverges.

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \text{diverges} & p\le 1 \end{cases}$$

3. $\int_{-1}^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_{-1}^{b} e^{-x}dx = \lim_{b\to\infty} \left[-e^{-x}\right]_{-1}^{b}$

$= \lim_{b\to\infty} \frac{-1}{e^{b}} + e^{1} = e$

4. $\int_{0}^{\infty} \cos x\,dx = \lim_{b\to\infty} \int_{0}^{b} \cos x dx = \lim_{b\to\infty} \left[\sin x\right]_{0}^{b}$

$= \lim_{b\to\infty} \sin b - \sin 0 \rightarrow$ Divergent since $\lim_{b\to\infty} \sin b$ DNE

5. $\int_{-\infty}^{0} x e^{x}\,dx = \lim_{a\to-\infty} \int_{a}^{0} x e^{x} dx$ $\quad u = x \quad dv = e^{x}dx$ $\quad du = dx \quad v = e^{x}$

$= \lim_{a\to-\infty} \left[xe^{x} - e^{x}\right]_{a}^{0} = (-1) - \lim_{a\to-\infty} \left[ae^{a} - e^{a}\right]$

$= -1 + \lim_{a\to\infty} \frac{a}{e^{a}} + \lim_{a\to\infty} \frac{1}{e^{a}} = -1$

6. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}dx$

$= \lim_{a\to-\infty} \left[\tan^{-1}x\right]_{a}^{0} + \lim_{b\to\infty} \left[\tan^{-1}x\right]_{0}^{b}$

$= -\left(\lim_{a\to-\infty} \tan^{-1}a\right) + \lim_{b\to\infty} \tan^{-1}b = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$

7. $\int_{2}^{4} \frac{1}{\sqrt{4-x}}\,dx = \lim_{b\to4^-} \int_{2}^{b} \frac{1}{\sqrt{4-x}}dx$

$= \lim_{b\to4^-} \left[-2\sqrt{4-x}\right]_{2}^{b} = \lim_{b\to4^-} \left[-2\sqrt{4-b} + 2\sqrt{4-2}\right] = 2\sqrt{2}$

8. $\displaystyle\int_0^1 \frac{dx}{x^2} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{x^2} = \lim_{a\to 0^+} \left[\frac{-1}{x}\right]_a^1 = -1 + \lim_{a\to 0^+} \frac{1}{a} = \infty$

9. $\displaystyle\int_0^1 \ln x\, dx = \lim_{a\to 0^+} \int_a^1 \ln x\, dx = \lim_{a\to 0^+} \left[x\ln x - x\right]_a^1$

$\displaystyle = -1 - \lim_{a\to 0^+} \left[a\ln a - a\right] = -1 - \lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a}}$

$\displaystyle = -1 - \lim_{a\to 0^+} \frac{\frac{1}{a}}{\frac{-1}{a^2}} = -1 - \frac{a}{-1} = -1$