1. a) $A_n = n(\sqrt{n^2+3} - n) = n^2\left(\sqrt{1+\frac{3}{n^2}} - 1\right)$

$L = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \dfrac{\sqrt{1+\frac{3}{n^2}}-1}{\frac{1}{n^2}} \overset{H}{=} \lim_{n\to\infty} \dfrac{\frac{1}{2}\left(1+\frac{3}{n^2}\right)^{-\frac{1}{2}}(-6n^{-3})}{-2n^{-3}} = \lim_{n\to\infty} \dfrac{3}{2\sqrt{1+\frac{3}{n^2}}}$

$L = \dfrac{3}{2\sqrt{1+0}} = \dfrac{3}{2}$ $\therefore$ A_n converges to $\frac{3}{2}$.

b) $B_n = e^{4-n^2}$

$L = \lim_{n\to\infty} B_n = \lim_{n\to\infty} e^{4-n^2} = \lim_{n\to\infty} \dfrac{1}{e^{n^2-4}} = \dfrac{1}{\infty} = 0$ $\therefore$ B_n converges to 0.

2. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n-1)(n+1)}$ $\qquad \dfrac{1}{(n-1)(n+1)} = \dfrac{A}{n-1} + \dfrac{B}{n+1} \rightarrow$ $1 = A(n+1) + B(n-1)$

$n=1 \rightarrow 1 = A(2) \rightarrow A = \frac{1}{2}$

$n=-1 \rightarrow 1 = B(-2) \rightarrow B = \frac{-1}{2}$

$\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n-1)(n+1)} = \frac{1}{2}\sum_{n=3}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$

$S = \frac{1}{2}\lim_{n\to\infty}\left[\left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \dots \left(\cancel{\frac{1}{n-2}} - \frac{1}{n}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right)\right]$

$S = \frac{1}{2}\lim_{n\to\infty} \frac{1}{2} + \frac{1}{3} - \frac{1}{n} - \frac{1}{n+1} = \frac{1}{2}\left(\frac{5}{6}\right) = \frac{5}{12}$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{3}{4^n} \rightarrow c = 3,\ r = \frac{1}{4}$ (Since $|r| = \frac{1}{4} < 1$ series converges.)

$S = \dfrac{c}{1-r} = \dfrac{3}{1-\frac{1}{4}} \cdot \dfrac{4}{4} = \dfrac{12}{4-1} = 4$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{7+6^n}{7^n} = \sum_{n=0}^{\infty} 7\left(\frac{1}{7}\right)^n + \sum_{n=0}^{\infty}\left(\frac{6}{7}\right)^n$ ← Each converge since $|r| < 1$

$S = \dfrac{7}{1-\frac{1}{7}} + \dfrac{1}{1-\frac{6}{7}} = \dfrac{49}{7-1} + \dfrac{7}{7-6} = \dfrac{49}{6} + 7 = \dfrac{91}{6}$

5. $\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^3}$ Does $\lim_{k\to\infty} A_k = 0$?

$L = \lim_{k\to\infty} \frac{k(k+2)}{(k+3)^3} = \lim_{k\to\infty} \frac{k^2+\cdots}{k^3} = 0$ $\therefore$ series could <u>converge</u> or <u>diverge</u>.

6. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2-5}}$ Compare with divergent p-series $\sum \frac{1}{n^{2/3}}$ $(p = \frac{2}{3} \le 1)$

$$\frac{1}{\sqrt[3]{n^2-5}} > \frac{1}{\sqrt[3]{n^2}} = \frac{1}{n^{2/3}}$$

Since $\frac{1}{\sqrt[3]{n^2-5}} > \frac{1}{n^{2/3}}$, $\sum \frac{1}{\sqrt[3]{n^2-5}}$ also <u>diverges</u>.

7. $\sum \frac{(-1)^n}{\sqrt{n}}$ Since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, series <u>converges</u> by the LT.

8. $\sum \frac{3n+7}{n^3+5n+1}$ Compare with convergent p-series $\sum \frac{1}{n^2}$ $(p=2>1)$

$$L = \lim_{n\to\infty} \frac{\frac{3n+7}{n^3+5n+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^3+7n^2}{n^3+5n+1} = 3$$

Since $L=3$, $\sum \frac{3n+7}{n^3+5n+1}$ also <u>converges</u>.

9. $\sum_{n=4}^{\infty} \frac{15}{2^n-9}$ Compare with $\sum \left(\frac{1}{2}\right)^n$ $\leftarrow$ convergent geometric series $(|r| = \frac{1}{2} < 1)$

$$L = \lim_{n\to\infty} \frac{\frac{15}{2^n-9}}{\frac{1}{2^n}} = \lim_{n\to\infty} 15\left(\frac{2^n}{2^n-9}\right) = 15\cdot 1 = 15$$

Since $L=15$, $\sum_{n=4}^{\infty} \frac{15}{2^n-9}$ also <u>converges</u>.

10. $\sum \frac{n^{400}}{n!}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{400}}{(n+1)!} \cdot \frac{n!}{n^{400}} = \frac{n!}{(n+1)\,n!} \cdot \frac{(n+1)^{400}}{n^{400}}$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n^{400}+\ldots}{n^{400}} = \lim_{n\to\infty} \frac{n^{400}+\ldots}{n^{401}+\ldots} = 0$

Since $L = 0 < 1$, $\sum \frac{n^{400}}{n!}$ converges.

11. $\sum \frac{(-1)^n (x-5)^n}{4^n n}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-5)^{n+1}}{4^{n+1}(n+1)} \cdot \frac{4^n n}{(x-5)^n}\right| = \left|\frac{(x-5)\,n}{4(n+1)}\right|$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-5)\,n}{4(n+1)}\right| = \left|\frac{x-5}{4}\right|$

Series converges when $\left|\frac{x-5}{4}\right| < 1 \rightarrow -1 < \frac{x-5}{4} < 1 \rightarrow -4 < x-5 < 4$

$1 < x < 9$

Now, we check endpoints...

If $x = 1 \rightarrow \sum \frac{(-1)^n (1-5)^n}{4^n n} = \sum \frac{(-1)^n (-4)^n}{4^n n} = \sum \frac{4^n}{4^n n} = \sum \frac{1}{n}$ This is a divergent p-series $(p=1)$ so $x \neq 1$.

If $x = 9 \rightarrow \sum \frac{(-1)^n (9-5)^n}{4^n n} = \sum \frac{(-1)^n 4^n}{4^n n} = \sum \frac{(-1)^n}{n}$

This series converges by Leibniz Test since $\lim_{n\to\infty} \frac{1}{n} = 0$.

so $x = 9$.

$\therefore$ IOC is $(1, 9]$

12. $\sum \frac{5^n (x+4)^n}{4n}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{5^{n+1}(x+4)^{n+1}}{4(n+1)} \cdot \frac{4n}{5^n (x+4)^n}\right| = \left|\frac{5 \cdot 5^n}{5^n} \cdot \frac{4n}{4(n+1)} \cdot \frac{(x+4)(x+4)^n}{(x+4)^n}\right|$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|5 \cdot \frac{4n}{4(n+1)} \cdot (x+4)\right| = \left|5(x+4)\right|$

12 cont.) Series converges when $|5(x+4)|<1 \rightarrow -1<5(x+4)<1$

$$\frac{-1}{5}<x+4<\frac{1}{5}$$

$$\frac{-21}{5}<x<\frac{-19}{5}$$

Now, we check endpoints...

If $x=\frac{-21}{5} \rightarrow \sum \frac{5^n\left(\frac{-21}{5}+4\right)^n}{4n} = \sum \frac{5^n\left(\frac{-1}{5}\right)^n}{4n} = \sum \frac{(-1)^n}{4n}$

Since $\lim_{n\to\infty} \frac{1}{4n}=0$, series converges by Leibniz Test. So $x=\frac{-21}{5}$.

If $x=\frac{-19}{5} \rightarrow \sum \frac{5^n\left(\frac{-19}{5}+4\right)^n}{4n} = \sum \frac{5^n\left(\frac{1}{5}\right)^n}{4n} = \sum \frac{1}{4n} = \frac{1}{4}\sum \frac{1}{n}$

This is a divergent p-series $(p=1)$. So $x \neq \frac{-19}{5}$.

IOC is $x=\left[\frac{-21}{5}, \frac{-19}{5}\right)$

13. $y=x^{3/2}$, $x=[1,3]$ $\quad \frac{dy}{dx}=\frac{3}{2}x^{\frac{1}{2}} \rightarrow \left(\frac{dy}{dx}\right)^2=\frac{9}{4}x$

$$s=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^3 \left(1+\frac{9}{4}x\right)^{\frac{1}{2}}dx = \left[\frac{\left(1+\frac{9}{4}x\right)^{3/2}}{\frac{3}{2}\left(\frac{9}{4}\right)}\right]_1^3 = \left[\frac{8}{27}\left(\sqrt{1+\frac{9}{4}x}\right)^3\right]_1^3$$

$$s=\frac{8}{27}\left[\left(\frac{31}{4}\right)^{3/2}-\left(\frac{13}{4}\right)^{3/2}\right] \approx 4.657$$

14. $y=2x^3$ $\quad x=[0,2]$ $\quad y'=6x^2 \rightarrow (y')^2=36x^4$

$$SA=2\pi\int_a^b y\sqrt{1+(y')^2}\,dx = 2\pi\int_0^2 2x^3\sqrt{1+36x^4}\,dx = \frac{\pi}{36}\int_1^{577} u^{\frac{1}{2}}du = \frac{\pi}{36}\left[\frac{2}{3}u^{3/2}\right]_1^{577}$$

$u=1+36x^4 \quad u(0)=1$

$du=144x^3dx \quad u(2)=577$

$$SA=\frac{\pi}{54}\left[u^{3/2}\right]_1^{577} = \frac{\pi}{54}\left(577^{3/2}-1\right) \approx 806.285$$