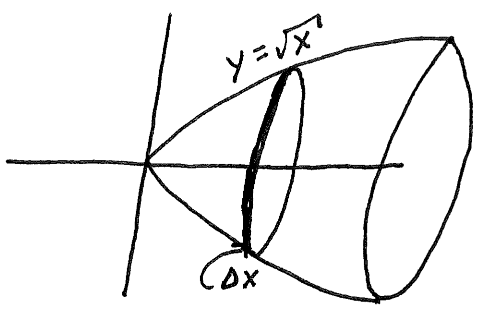
6.3 Disk and Washer Methods for finding Volumes (Slicing)

To find volume of an object we find volume of an arbitrary, infinitely thin, cross-section. We then sum all cross-sections.

1. Find volume of solid obtained by rotating $y=\sqrt{x}$ about the x-axis from $x=0$ to $x=1$.

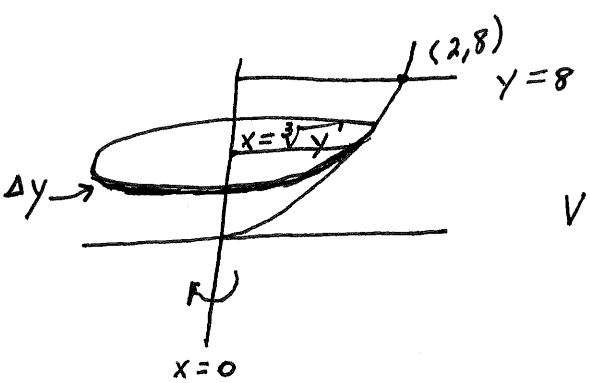


$$A_o = \pi(\sqrt{x})^2$$

$$V_o = \pi x \, \Delta x$$

$$V = \int_0^1 \pi x \, dx = \frac{\pi x^2}{2}\Big|_0^1 = \frac{\pi}{2}$$

2. Find volume of solid obtained by rotating region bound by $y=x^3$, $y=8$, $x=0$ about y-axis.

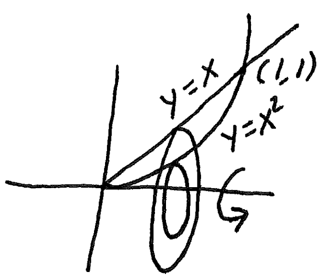


$$A_o = \pi(\sqrt[3]{y})^2$$

$$V_o = \pi y^{2/3} \, \Delta y$$

$$V = \int_0^8 \pi y^{2/3} \, dy = \frac{3\pi}{5} y^{5/3}\Big|_0^8 = \frac{96\pi}{5}$$

3. The region enclosed by $y=x$ and $y=x^2$ is rotated about the x-axis. Find volume of resulting solid.



$$A_{\circledcirc} = \pi(r_2^2 - r_1^2)$$

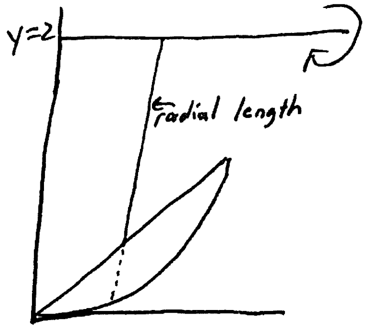
$$A_{slice} = \pi(x^2 - x^4)$$

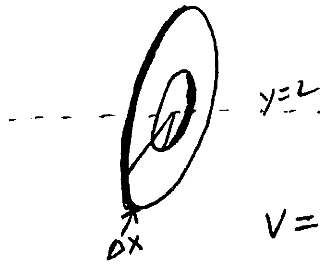
$$V_{slice} = \pi(x^2 - x^4)\Delta x$$

$$V = \pi \int_0^1 (x^2 - x^4)\, dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$V = \left(\frac{1}{3} - \frac{1}{5}\right)\pi = \frac{2\pi}{15}$$

4. Rotate enclosed region between $y=x^2$ and $y=x$ about $y=2$.
Find resulting volume.





$$A_\odot = \pi\left((2-x^2)^2-(2-x)^2\right)$$

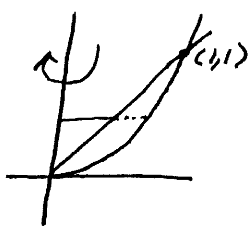
$$V_\odot = \pi\left[4-4x^2+x^4-(4-4x+x^2)\right]\Delta x$$

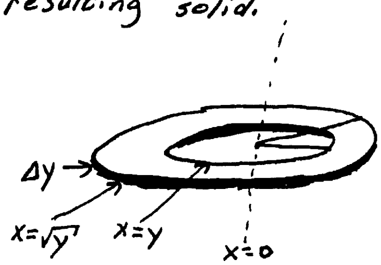
$$V_\odot = \pi\left[x^4-5x^2+4x\right]\Delta x$$

$$V = \pi\int_0^1 (x^4-5x^2+4x)\,dx = \pi\left[\frac{x^5}{5}-\frac{5x^3}{3}+2x^2\right]_0^1$$

$$V = \left(\frac{1}{5}-\frac{5}{3}+2\right)\pi = \frac{8\pi}{15}$$

5. Rotate region between $y=x^2$ and $y=x$ about the y-axis.
Find volume of resulting solid.



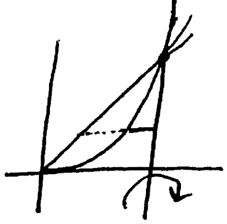


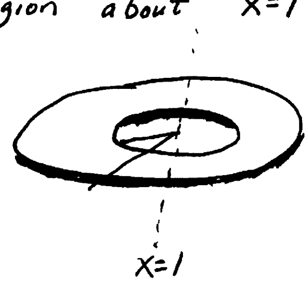
$$V_\odot = \pi\left[(\sqrt{y})^2 - y^2\right]\Delta y = \pi(y-y^2)\Delta y$$

$$V = \pi\int_0^1 (y-y^2)\,dy = \pi\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1$$

$$V = \left(\frac{1}{2}-\frac{1}{3}\right)\pi = \frac{\pi}{6}$$

6. Rotate same region about $x=1$





$r_1 = 1-y$, $r_2 = 1-\sqrt{y}$

$$V_\odot = \pi\left[(1-y)^2-(1-\sqrt{y})^2\right]\Delta y$$

$$V_\odot = \pi\left[1-2y+y^2-(1-2\sqrt{y}+y)\right]\Delta y$$

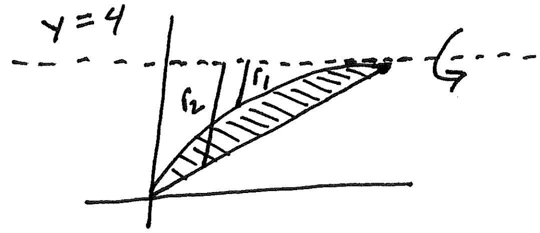
$$V_\odot = \pi\left[y^2-3y+2\sqrt{y}\right]\Delta y$$

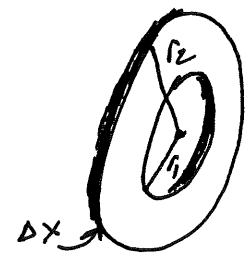
$$V = \pi\int_0^1 (y^2-3y+2\sqrt{y})\,dy = \pi\left[\frac{y^3}{3}-\frac{3y^2}{2}+\frac{4y^{3/2}}{3}\right]_0^1$$

$$V = \left(\frac{1}{3}-\frac{3}{2}+\frac{4}{3}\right)\pi = \frac{\pi}{9}$$

7. Rotate region enclosed by $y = 2\sqrt{x}$, $y = x$ about $y = 4$ and find resulting volume.

$2\sqrt{x} = x$
$4x = x^2$
$0 = x^2 - 4x$
$0 = x(x-4)$





$r_1 = 4 - 2\sqrt{x}$
$r_2 = 4 - x$

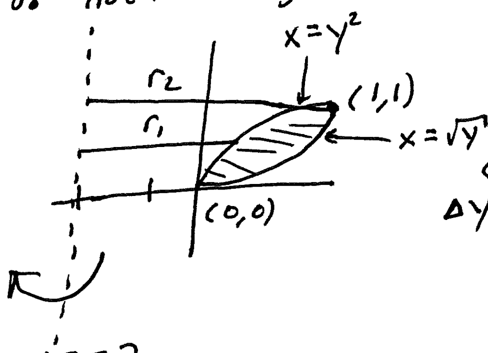
$$V_{slice} = \pi\left[(4-x)^2 - (4-2\sqrt{x})^2\right]\Delta x$$

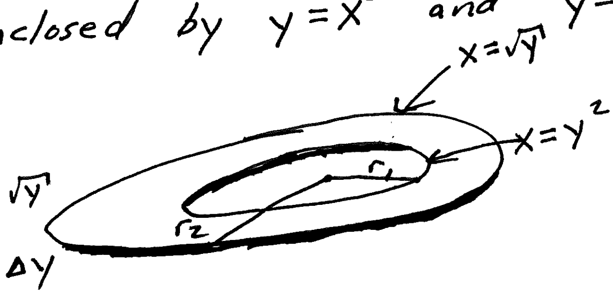
$$V_{slice} = \pi\left[16 - 8x + x^2 - 16 + 16\sqrt{x} - 4x\right]\Delta x = \pi(x^2 - 12x + 16\sqrt{x})\Delta x$$

$$V = \pi\int_0^4 (x^2 - 12x + 16x^{\frac{1}{2}})\,dx = \pi\left[\frac{x^3}{3} - 6x^2 + 32/3\, x^{3/2}\right]_0^4$$

$$V = \pi\left[\frac{64}{3} - 96 + \frac{256}{3}\right] = \left(\frac{320}{3} - \frac{288}{3}\right)\pi = \frac{32\pi}{3}$$

8. Rotate region enclosed by $y = x^2$ and $y = x^{\frac{1}{2}}$ about $x = -2$.





$r_1 = 2 + y^2$
$r_2 = 2 + \sqrt{y}$

$$V_{slice} = \pi\left[(2+\sqrt{y})^2 - (2-y^2)^2\right]\Delta y = \pi\left[(4 + 4\sqrt{y} + y) - (4 - 4y^2 + y^4)\right]\Delta y$$

$$V_{slice} = \pi(-y^4 + 4y^2 + y + 4\sqrt{y})\Delta y$$

$$V = \pi\int_0^1 (-y^4 + 4y^2 + y + 4y^{\frac{1}{2}})\,dy = \pi\left[\frac{-y^5}{5} + \frac{4y^3}{3} + \frac{y^2}{2} + \frac{8y^{\frac{3}{2}}}{3}\right]_0^1$$

$$V = \pi\left(\frac{-1}{5} + \frac{4}{3} + \frac{1}{2} + \frac{8}{3}\right) = \pi\left(\frac{-6 + 40 + 15 + 80}{30}\right) = \frac{129\pi}{30} = \frac{43\pi}{10} = 4.3\pi$$