7.5 Notes - Partial Fractions

We have done problems like this in algebra...

$$\frac{2}{x-3} - \frac{1}{x+1} = \frac{2(x+1) - 1(x-3)}{(x-3)(x+1)} = \frac{x+5}{(x-3)(x+1)}$$

When we integrate, we would prefer the two separate fractions so we need to be able to do this problem in reverse.

Partial Fractions - Case 1 - Denominator has linear factors

1. $\int \frac{x+5}{(x-3)(x+1)}\,dx$

$$\frac{x+5}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$x+5 = A(x+1) + B(x-3)$$

$x=3 \rightarrow \quad 3+5 = A(3+1) + B(0) \rightarrow \quad 8 = 4A \rightarrow \quad A=2$

$x=-1 \rightarrow \quad -1+5 = A(0) + B(-1-3) \rightarrow \quad 4 = -4B \rightarrow \quad B=-1$

So... $\int \frac{x+5}{(x-3)(x+1)}\,dx = \int \frac{2}{x-3}\,dx + \int \frac{-1}{x+1}\,dx$

$$= 2\ln|x-3| - \ln|x+1| + C$$

$$= \ln\left|\frac{(x-3)^2}{x+1}\right| + C$$

Case 1 (but more fun)

2. $\int \frac{3x^2+7x-2}{x^3-x^2-2x}\,dx \qquad x^3-x^2-2x = x(x^2-x-2) = x(x-2)(x+1)$

$$\frac{3x^2+7x-2}{x^3-x^2-2x} = \frac{3x^2+7x-2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$$

$$3x^2+7x-2 = A(x-2)(x+1) + B(x)(x+1) + C(x)(x-2)$$

$x=0 \rightarrow \quad -2 = A(-2)(1) \rightarrow -2 = -2A \rightarrow \underline{A=1}$

$x=2 \rightarrow 3(2)^2+7(2)-2 = B(2)(3) \rightarrow 24 = 6B \rightarrow \underline{B=4}$

$x=-1 \rightarrow 3(-1)^2+7(-1)-2 = C(-1)(-3) \rightarrow -6 = 3C \rightarrow \underline{C=-2}$

$$I = \int \frac{1}{x}\,dx + \int \frac{4}{x-2}\,dx + \int \frac{-2}{x+1}\,dx$$

$$I = \ln|x| + 4\ln|x-2| - 2\ln|x+1| + C = \ln\left|\frac{x(x-2)^4}{(x+1)^2}\right| + C$$

Case 2

Denominator contains linear factors raised to powers

3. $\int \frac{5x^2-3x+2}{x^3-2x^2}\,dx = \int \frac{5x^2-3x+2}{x^2(x-2)}\,dx$

must have separate fraction for each power.

$$\frac{5x^2-3x+2}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$5x^2-3x+2 = A(x)(x-2) + B(x-2) + C(x^2)$$

$x=0 \rightarrow \quad 2 = B(-2) \rightarrow \underline{B=-1}$

$x=2 \rightarrow 5(2)^2-3(2)+2 = C(2^2) \rightarrow 16 = 4C \rightarrow \underline{C=4}$

$x^2 \rightarrow \quad 5 = A+C \rightarrow 5 = A+4 \rightarrow \underline{A=1}$

3 cont.) $I = \int \frac{1}{x}\,dx + \int \frac{-1}{x^2}\,dx + \int \frac{4}{x-2}\,dx$

$I = \ln|x| - \int x^{-2}\,dx + 4\ln|x-2| + C$ $\quad \rightarrow \frac{x^{-1}}{-1}$

$I = \ln|x(x-2)^4| + \frac{1}{x} + C$

4. $\int \frac{3\,dx}{x(x+1)^2}$

$\frac{3}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$3 = A(x+1)^2 + B(x)(x+1) + C(x)$

$x=0 \rightarrow 3 = A(1)^2 \rightarrow \underline{3 = A}$

$x=-1 \rightarrow 3 = C(-1) \rightarrow \underline{C = -3}$

$x^2 \rightarrow \quad 0 = A + B \rightarrow B = -A \rightarrow \underline{B = -3}$

$I = \int \frac{3}{x}\,dx + \int \frac{-3}{x+1}\,dx + \int \frac{-3}{(x+1)^2}\,dx$ $\quad \rightarrow \frac{-3(x+1)^{-1}}{-1}$

$I = 3\ln|x| - 3\ln|x+1| + \frac{3}{x+1} + C = \ln\left|\frac{x^3}{(x+1)^3}\right| + \frac{3}{x+1} + C$

Case 3

Denominator contains an irreducible quadratic

5. $\int \frac{x+1}{x(x^2+4)}\,dx \qquad \frac{x+1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$ ← Always 1 degree lower

$x+1 = A(x^2+4) + (Bx+C)(x)$

$x=0 \rightarrow 1 = A(4) \rightarrow \underline{A = \frac{1}{4}}$

$x^2 \rightarrow \quad 0 = A + B \rightarrow B = -A \rightarrow \underline{B = \frac{-1}{4}}$

$x \rightarrow \quad \underline{1 = C}$

5 cont.) $I = \int \frac{\frac{1}{4}}{x}\,dx + \int \frac{-\frac{1}{4}x+1}{x^2+4}\,dx$

$I = \frac{1}{4}\ln|x| - \frac{1}{4}\int \frac{x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$

#16 $\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C$

$I = \frac{1}{4}\ln|x| - \frac{1}{8}\ln|x^2+4| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$

$I = \ln\left|\frac{\sqrt[4]{x}}{\sqrt[8]{x^2+4}}\right| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$

Case 4

Denominator has an irredecible quadratic to a power

6. $\int \frac{2\,dx}{x(x^2+1)^2}$ $\qquad \frac{2}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

$2 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x)$

$2 = A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex$

$x=0 \to 2 = A(1) \to \underline{A=2}$

$x^4 \to 0 = A+B \to \underline{B=-2}$

$x^3 \to \underline{0=C}$

$x^2 \to 0 = 2A+B+D \to 0 = 4-2+D \to \underline{D=-2}$

$x \to 0 = C+E \to \underline{E=0}$

$u=x^2+1$
$du = 2x\,dx$

so... $I = \int \frac{2}{x}\,dx + \int \frac{-2x}{x^2+1}\,dx + \int \frac{-2x}{(x^2+1)^2}\,dx$

$I = 2\ln|x| - \ln|x^2+1| - \int \frac{1}{u^2}\,du \to \frac{u^{-1}}{-1}+C = \frac{-1}{u}+C$

$I = 2\ln|x| - \ln|x^2+1| + \frac{1}{x^2+1}+C = \ln\left|\frac{x^2}{x^2+1}\right| + \frac{1}{x^2+1}+C$

Case 4 <u>can</u> get gorgeous!

Case 5

You should use long division first anytime degree of numerator $\geq$ degree of denominator

7. $\int \frac{x^4+1}{x^3+9x}\,dx$

$$\begin{array}{r} x \phantom{{}+9x^2)} \\ x^3+9x \overline{\smash{)}\, x^4+1 } \\ -(x^4+9x^2) \\ \hline -9x^2+1 \end{array}$$

$$I = \int x\,dx + \int \frac{-9x^2+1}{x(x^2+9)}\,dx$$

$$\frac{-9x^2+1}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9} \rightarrow -9x^2+1 = A(x^2+9) + (Bx+C)(x)$$

$x=0 \rightarrow 1 = A(9) \rightarrow A = \frac{1}{9}$

$x^2 \rightarrow -9 = A+B \rightarrow \frac{-81}{9} = \frac{1}{9} + B$

$B = \frac{-82}{9}$

$x \rightarrow 0 = C$

$$I = \frac{1}{2}x^2 + \int \frac{\frac{1}{9}}{x}\,dx + \int \frac{\frac{-82}{9}x}{x^2+9}\,dx$$

$$I = \frac{1}{2}x^2 + \frac{1}{9}\ln|x| - \frac{41}{9}\int \frac{2x}{x^2+9}\,dx = \frac{1}{2}x^2 + \frac{1}{9}\ln|x| - \frac{41}{9}\ln|x^2+9| + C$$

$$= \boxed{\frac{1}{2}x^2 + \ln\left|\frac{\sqrt[9]{x}}{\sqrt[9]{(x^2+9)^{41}}}\right| + C}$$