7.3 Notes p.1

root in integral	Trig Substitution	Identity Used
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$	$1+\tan^2\theta=\sec^2\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$\sec^2\theta-1=\tan^2\theta$

1. $\displaystyle\int\frac{x^2}{\sqrt{9-x^2}}\,dx$ $\qquad x=3\sin\theta$, $\quad dx=3\cos\theta\,d\theta$

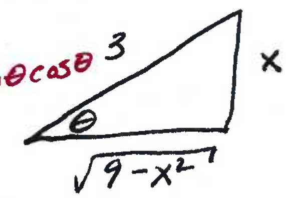
$\frac{1}{2}(1-\cos 2\theta)$

$$=\int\frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}3\cos\theta\,d\theta = 9\int\sin^2\theta\,d\theta = \frac{9}{2}\int(1-\cos 2\theta)\,d\theta$$

$\sqrt{9\cos^2\theta}=3\cos\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$=\frac{9}{2}\left[\theta-\frac{1}{2}\sin 2\theta\right]+C$$



$x=3\sin\theta$

$\frac{x}{3}=\sin\theta$

$\sin^{-1}\left(\frac{x}{3}\right)=\theta$

$$=\frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right)-\frac{1}{2}(2\sin\theta\cos\theta)\right]+C$$

$$=\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)-\frac{9(x\sqrt{9-x^2})}{2\cdot 3\,(3)}+C = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)-\frac{x\sqrt{9-x^2}}{\cancel{9}\,2}+C$$

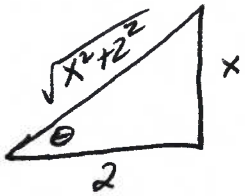
2. $\displaystyle\int\frac{1}{(4+x^2)^{3/2}}\,dx$ $\qquad x=2\tan\theta$, $\quad dx=2\sec^2\theta\,d\theta$

$$(4+x^2)^{3/2}=(4+4\tan^2\theta)^{3/2}=(4(1+\tan^2\theta))^{3/2}=(4\sec^2\theta)^{3/2}$$

$$=(2\sec\theta)^3=8\sec^3\theta$$

$$\int\frac{1}{(4+x^2)^{3/2}}\,dx=\int\frac{2\sec^2\theta\,d\theta}{8\sec^3\theta}=\frac{1}{4}\int\cos\theta\,d\theta=\frac{1}{4}\sin\theta+C$$

$$=\frac{1}{4}\left(\frac{x}{\sqrt{4+x^2}}\right)+C$$



$x=2\tan\theta$

$\frac{x}{2}=\tan\theta$

$$=\boxed{\frac{x}{4\sqrt{4+x^2}}+C}$$

3. $\displaystyle\int\frac{\sqrt{x^2-16}}{x}\,dx$ $\qquad x=4\sec\theta$, $\quad dx=4\sec\theta\tan\theta\,d\theta$

$\sqrt{16\sec^2\theta-16}=\sqrt{16(\sec^2\theta-1)}$

$=\sqrt{16\tan^2\theta}$

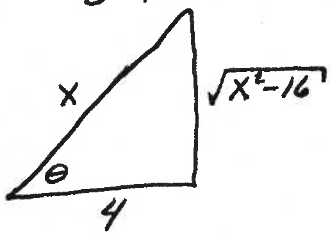
$=4\tan\theta$

$$=\int\frac{\sqrt{16\sec^2\theta-16}}{\cancel{4\sec\theta}}(\cancel{4\sec\theta}\tan\theta)\,d\theta=\int\frac{\tan\theta}{\sec\theta}(4\sec\theta\tan\theta)\,d\theta$$

$$=4\int\tan^2\theta\,d\theta$$

cont. →

$$= 4\int(\sec^2\theta - 1)\,d\theta = 4[\tan\theta - \theta] + C = 4\tan\theta - 4\theta + C$$



$x = 4\sec\theta$

$\frac{x}{4} = \sec\theta$

$\frac{4}{x} = \cos\theta$

$$= 4\left(\frac{\sqrt{x^2-16}}{4}\right) - 4\cos^{-1}\left(\frac{4}{x}\right) + C$$

$$= \sqrt{x^2-16} - 4\cos^{-1}\left(\frac{4}{x}\right) + C$$

4. $\int \frac{dx}{\sqrt{x^2+4x+7}}$

$\left(\frac{4}{2}\right)^2 = 4$

$(x^2+4x+4) + 7 - 4 = (x+2)^2 + 3$

let $u = x+2$, $du = dx$

$$= \int \frac{du}{\sqrt{u^2+3}}$$

$u = \sqrt{3}\tan\theta$

$du = \sqrt{3}\sec^2\theta\,d\theta$

$u^2+3 = 3\tan^2\theta + 3 = 3(\tan^2\theta+1)$
$= 3\sec^2\theta$

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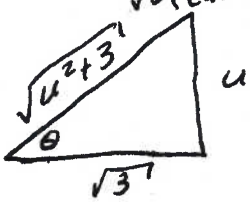
$$= \int \frac{\sqrt{3}\sec^2\theta\,d\theta}{\sqrt{3(\tan^2\theta+1)}} = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$



$u = \sqrt{3}\tan\theta$

$\frac{u}{\sqrt{3}} = \tan\theta$

$$= \ln\left|\frac{\sqrt{u^2+3}}{\sqrt{3}} + \frac{u}{\sqrt{3}}\right| + C$$

$$= \ln\left|\frac{\sqrt{x^2+4x+7}}{\sqrt{3}} + \frac{x+2}{\sqrt{3}}\right| + C$$

$$= \ln\left|\frac{\sqrt{x^2+4x+7} + x + 2}{\sqrt{3}}\right| + C$$

$$= \ln\left|\sqrt{x^2+4x+7} + x + 2\right| - \frac{1}{2}\ln 3 + C$$

$$= \boxed{\ln\left|\sqrt{x^2+4x+7} + x + 2\right| + C}$$

5. $\int \sqrt{12+4t^2}\,dt = 2\int\sqrt{3+t^2}\,dt$

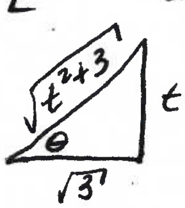
$t = \sqrt{3}\tan\theta$

$dt = \sqrt{3}\sec^2\theta\,d\theta$

$$= 2\int \underbrace{\sqrt{3+3\tan^2\theta}}_{\sqrt{3\sec^2\theta}}\,\sqrt{3}\sec^2\theta\,d\theta = 6\int \sec^3\theta\,d\theta$$

red. form → #48

$$6\left[\frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\int\sec\theta\,d\theta\right] \overset{\#13}{=} 3\sec\theta\tan\theta + 3\ln|\sec\theta + \tan\theta| + C$$



$t = \sqrt{3}\tan\theta$

$\frac{t}{\sqrt{3}} = \tan\theta$

$$= 3\left(\frac{\sqrt{t^2+3}}{\sqrt{3}}\right)\left(\frac{t}{\sqrt{3}}\right) + 3\ln\left|\frac{\sqrt{t^2+3}}{\sqrt{3}} + \frac{t}{\sqrt{3}}\right| + C$$

$$= t\sqrt{t^2+3} + 3\ln\left|\sqrt{t^2+3} + t\right| - 3\ln\sqrt{3} + C$$

$$= \boxed{t\sqrt{t^2+3} + 3\ln\left|\sqrt{t^2+3} + t\right| + C}$$

6. $\int x^3\sqrt{9-x^2}\,dx$ $\qquad x = 3\sin\theta$, $\quad dx = 3\cos\theta\,d\theta$

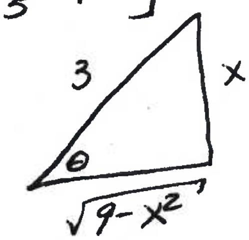
$\sqrt{9\cos^2\theta} = 3\cos\theta$ $\qquad (1-\cos^2\theta)\sin\theta$

$$= \int 27\sin^3\theta\sqrt{9-9\sin^2\theta}\;3\cos\theta\,d\theta = 243\int \sin^3\theta\cos^2\theta\,d\theta$$

$$= 243\int(\cos^2\theta-\cos^4\theta)\sin\theta\,d\theta$$

$u = \cos\theta$, $\quad du = -\sin\theta\,d\theta$

$$= -243\int(u^2-u^4)\,du = -243\left[\frac{u^3}{3}-\frac{u^5}{5}+C\right]$$



$x = 3\sin\theta$, $\quad \frac{x}{3} = \sin\theta$

$$= \frac{243\cos^5\theta}{5} - \frac{243\cos^3\theta}{3} + C$$

$$= \frac{243}{5}\left(\frac{\sqrt{9-x^2}}{3}\right)^5 - 81\left(\frac{\sqrt{9-x^2}}{3}\right)^3 + C$$

$$= \boxed{\frac{\left(\sqrt{9-x^2}\right)^5}{5} - 3\left(\sqrt{9-x^2}\right)^3 + C}$$

7. $\int \frac{dt}{(9t^2+4)^2}$ $\qquad (9t^2+4)^2 = \left(9\left(t^2+\frac{4}{9}\right)\right)^2 = 81\left(t^2+\frac{4}{9}\right)^2$

$$= \frac{1}{81}\int\frac{dt}{\left(t^2+\frac{4}{9}\right)^2}$$

let $t = \frac{2}{3}\tan\theta$, $\quad dt = \frac{2}{3}\sec^2\theta\,d\theta$ $\qquad \left(\frac{4}{9}(\tan^2\theta+1)\right)^2$ $\quad \frac{16}{81}\sec^4\theta$

$$= \frac{1}{81}\int\frac{\frac{2}{3}\sec^2\theta\,d\theta}{\left(\frac{4}{9}\tan^2\theta+\frac{4}{9}\right)^2} = \frac{1}{81}\int\frac{\frac{2}{3}\sec^2\theta\,d\theta}{\frac{16}{81}\sec^4\theta}$$

$\frac{2}{3}\cdot\frac{81}{16} = \frac{27}{8}$

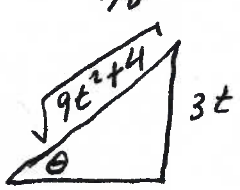
$\cos^2\theta = \frac{1}{2}(1+\cos2\theta)$

$$= \frac{1}{24}\int\cos^2\theta\,d\theta = \frac{1}{24}\int\left(\frac{1}{2}(1+\cos2\theta)\right)d\theta = \frac{1}{48}\left[\theta+\frac{\sin2\theta}{\cancel{2}}+C\right]$$

$\cancel{2}\sin\theta\cos\theta$

$$= \frac{1}{48}\theta + \frac{1}{96}\sin2\theta + C = \frac{1}{48}\theta + \frac{1}{48}(\sin\theta)(\cos\theta) + C$$

$\tan\theta = \frac{3t}{2}$



$$= \frac{1}{48}\tan^{-1}\left(\frac{3t}{2}\right) + \frac{1}{48}\left(\frac{3t}{\sqrt{9t^2+4}}\right)\left(\frac{2}{\sqrt{9t^2+4}}\right) + C$$

$$= \boxed{\frac{1}{48}\tan^{-1}\left(\frac{3t}{2}\right) + \frac{t}{8(9t^2+4)} + C}$$