10.2 Notes

Infinite Series: An infinite series, $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots a_n$ (as $n \to \infty$) converges to L if $\lim_{n\to\infty} \sum_{n=1}^{n} a_n = L$, otherwise it diverges.

Lets practice series (summation) notation

1.) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \rightarrow a_n = \frac{1}{n^2} \rightarrow S_n = \lim_{n\to\infty} \sum_{n=1}^{n} \frac{1}{n^2}$

2.) $\frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots \rightarrow a_n = \frac{1}{(n+2)^2} \rightarrow S_n = \lim_{n\to\infty} \sum_{n=1}^{n} \frac{1}{(n+2)^2}$

note: $S_n = \lim_{n\to\infty} \sum_{n=3}^{n} \frac{1}{n^2}$ would also work.

3.) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \rightarrow a_n = (-1)^{n-1}\left(\frac{1}{2n-1}\right) \rightarrow S_n = \lim_{n\to\infty} \sum_{n=1}^{n} \frac{(-1)^{n-1}}{2n-1}$

4. $\frac{125}{9} + \frac{625}{16} + \frac{3125}{25} + \frac{15625}{36} \rightarrow a_n = \frac{5^{n+2}}{(n+2)^2} \rightarrow S_n = \lim_{n\to\infty} \sum_{n=1}^{n} \frac{5^{n+2}}{(n+2)^2}$

or $S_n = \lim_{n\to\infty} \sum_{n=3}^{n} \frac{5^n}{n^2}$

5.) Find S_2, S_4, S_6 for $\sum_{k=1}^{\infty} (-1)^k k^{-1}$

$S_2 = (-1)^1(1)^{-1} + (-1)^2(2)^{-1} = -1 + \frac{1}{2} = \frac{-1}{2}$

$S_4 = S_2 + a_3 + a_4 = \frac{-1}{2} + (-1)^3(3)^{-1} + (-1)^4(4)^{-1} = \frac{-1}{2} - \frac{1}{3} + \frac{1}{4} = \frac{-7}{12}$

$S_6 = S_4 + a_5 + a_6 = \frac{-7}{12} + (-1)^5(5)^{-1} + (-1)^6(6)^{-1} = \frac{-7}{12} - \frac{1}{5} + \frac{1}{6} = \frac{-37}{60}$

Divergence Test - If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ <u>diverges</u>.

If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ may <u>converge</u> <u>or</u> <u>diverge</u>.

6.) Show $\sum \frac{n}{\sqrt{n^2+1}}$ diverges

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$$

Since $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} \neq 0$, $\sum \frac{n}{\sqrt{n^2+1}}$ <u>diverges</u>.

* Sum of a telescoping series

7. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \cdots$ Clearly $\lim_{n\to\infty} a_n = 0$ so series could <u>converge</u> or <u>diverge</u>.

Let's show it converges...

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} \rightarrow 1 = A(n+2) + Bn \rightarrow \begin{array}{l} n=0 \rightarrow 1 = 2A \rightarrow A = \frac{1}{2} \\ n=-2 \rightarrow 1 = -2B \rightarrow B = \frac{-1}{2} \end{array}$$

So... $$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{2}\left[\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)\right] = \frac{1}{2}\left[\left(1-\cancel{\tfrac{1}{3}}\right) + \left(\tfrac{1}{2}-\cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{3}}-\tfrac{1}{5}\right) + \cdots + \left(\tfrac{1}{n}-\tfrac{1}{n+2}\right)\right]$$

$$= \frac{1}{2} \cdot \lim_{n\to\infty}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}$$

Geometric Series: $c + cr + cr^2 + cr^3 + \dots = \sum_{n=0}^{\infty} cr^n = \sum_{n=1}^{\infty} cr^{n-1}$

How can we find S_n of any convergent $(-1<r<1)$ geometric series?
$|r|<1$

$S_n = c + cr + cr^2 + cr^3 + \dots cr^n$

$rS_n = cr + cr^2 + cr^3 + cr^4 + \dots cr^{n+1}$

$S_n - rS_n = c - cr^{n+1}$

$S_n(1-r) = c - cr^{n+1}$

$S_n = \frac{c(1-r^{n+1})}{1-r}$ As $n \to \infty$, since $|r|<1 \to S_n = \frac{c}{1-r}$

8. Find sum of $S_n = 5 + \frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \dots$

$c=5,\ r=\frac{1}{2} \to S_n = \frac{5}{1-\frac{1}{2}} \cdot \frac{2}{2} = \frac{10}{1} = 10$

9. $\sum_{n=0}^{\infty} \left(\frac{\pi}{e}\right)^n \to c = \left(\frac{\pi}{e}\right)^0 = 1,\ r = \frac{\pi}{e} \to \boxed{S_n = \frac{1}{1-\frac{\pi}{e}} \cdot \frac{e}{e} = \frac{e}{\cancel{e-\pi}} \approx \cancel{-6.42}}$

Not valid... Remember this is only valid if $|r|<1$

so we say... S_n diverges

10. $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n \to c=1,\ r=\frac{e}{\pi} \to S_n = \frac{1}{1-\frac{e}{\pi}} \cdot \frac{\pi}{\pi} = \frac{\pi}{\pi - e} \approx 7.421$

11. $\sum_{n=0}^{\infty} \frac{3(-2)^n + 5^n}{8^n} = \sum_{n=0}^{\infty} 3\left(\frac{-1}{4}\right)^n + \sum_{n=0}^{\infty}\left(\frac{5}{8}\right)^n \to S_n = \frac{3}{1-\left(\frac{-1}{4}\right)} + \frac{1}{1-\frac{5}{8}}$

$= \frac{12}{4+1} + \frac{8}{8-5} = \frac{12}{5} + \frac{8}{3} = \frac{76}{15}$