10.5 Ratio Test

Ratio Test (probably the most useful convergence test)

If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ exists, then $\sum a_n$ converges absolutely if $L<1$ and diverges if $L>1$. If $L=1$, Ratio Test is inconclusive.

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{5^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1}{5^{n+1}} \cdot \frac{5^n}{n}\right| = \left|\frac{5^n}{5^{n+1}} \cdot \frac{n+1}{n}\right| = \frac{n+1}{5n}$$

$L = \lim_{n\to\infty} \frac{n+1}{5n} = \frac{1}{5} < 1 \quad \therefore \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{5^n}$ converges by the RT.

2. $\sum_{n=0}^{\infty} \frac{3n+2}{5n^3+1}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3(n+1)+2}{5(n+1)^3+1} \cdot \frac{5n^3+1}{3n+2} = \frac{3n+5}{3n+2} \cdot \frac{5n^3+1}{5(n+1)^3+1}$$

$L = \lim_{n\to\infty} \frac{3n+5}{3n+2} \cdot \frac{5n^3+1}{5(n+1)^3+1} = 1 \cdot 1 = 1 \rightarrow$ RT is inconclusive.

Limit Comparison Test with convergent $\sum \frac{1}{n^2}$

$L = \lim_{n\to\infty} \frac{\frac{3n+2}{5n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^3+2n^2}{5n^3+1} = \frac{3}{5} \quad \therefore \sum_{n=0}^{\infty} \frac{3n+2}{5n^3+1}$ also converges by LCT.

3. $\sum_{n=1}^{\infty} \frac{2^n}{n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{n+1} \cdot \frac{n}{2^n} = \frac{2^{n+1}}{2^n} \cdot \frac{n}{n+1} = \frac{2n}{n+1}$$

$L = \lim_{n\to\infty} \frac{2n}{n+1} = 2 \quad \therefore \sum_{n=1}^{\infty} \frac{2^n}{n}$ diverges by RT.

4. $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ $\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(2(n+1))!} \cdot \frac{(2n)!}{1} = \frac{(2n)!}{(2n+2)!} = \frac{1}{(2n+2)(2n+1)}$

$L = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0 \therefore \sum_{n=1}^{\infty} \frac{1}{(2n)!}$ converges by RT.

5. $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$ $\left|\frac{a_{n+1}}{a_n}\right| = \frac{((n+1)!)^3}{(3(n+1))!} \cdot \frac{(3n)!}{(n!)^3} = \frac{(n+1)(n+1)(n+1)}{(3n+3)(3n+2)(3n+1)}$

$(n+1)^3 \cdot (n!)^3$

$L = \lim_{n\to\infty} \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} = \lim_{n\to\infty} \frac{n^3}{27n^3} = \frac{1}{27} < 1 \therefore \sum \frac{(n!)^3}{(3n)!}$ converges by RT.

★ Root Test: If $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists, then $\sum a_n$ converges absolutely if $L<1$ and diverges if $L>1$. Root Test is inconclusive if $L=1$.

6. $\sum_{n=0}^{\infty} \frac{1}{10^n}$ $\rightarrow$ $\sqrt[n]{|a_n|} = \sqrt[n]{\frac{1}{10^n}} = \frac{1}{10}$

$L = \lim_{n\to\infty} \sqrt[n]{a^n} = \lim_{n\to\infty} \frac{1}{10} = \frac{1}{10} < 1 \therefore \sum_{n=0}^{\infty} \frac{1}{10^n}$ converges by Root Test.

7. $\sum_{k=1}^{\infty} \frac{(-2)^k}{k^{10}}$ $\rightarrow$ $\sqrt[k]{|a_k|} = \sqrt[k]{\left|\frac{(-2)^k}{k^{10}}\right|} = \frac{+2}{k^{10/k}}$

$L = \lim_{k\to\infty} \frac{+2}{k^{10/k}} = \frac{2}{k^0} = \frac{2}{1} = 2 > 1 \therefore \sum_{k=1}^{\infty} \frac{(-2)^k}{k^{10}}$ diverges by Root Test.