Substitution Notes

Functions like $\int 2x\sqrt{1+x^2}\,dx$ are not integrable with our basic rules. We introduce the variable "u" to represent a part of the integrand (usually the inner-most part of function) in an effort to make integral simpler.

Substitution Method → If $u=f(x)$ is differentiable

then $\int g(f(x))\,f'(x)\,dx = \int g(u)\,du$

1. $\int 2x\sqrt{1+x^2}\,dx = \int \underbrace{\sqrt{1+x^2}}_{g(f(x))}\,\underbrace{2x\,dx}_{f'(x)\,dx}$

let $u=1+x^2 \rightarrow \frac{du}{dx}=2x \rightarrow du=2x\,dx$

so... $\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2}+C = \frac{2(\sqrt{1+x^2})^3}{3}+C$

Check → $\frac{d}{dx}\left[\frac{2}{3}(1+x^2)^{3/2}+C\right] \stackrel{\checkmark}{=} (1+x^2)^{\frac{1}{2}}\cdot 2x$

2. $\int \sqrt{2x+1}\,dx$ $\qquad u=2x+1,\ du=2dx$

$= \frac{1}{2}\int \sqrt{2x+1}\cdot 2dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}+C\right) = \frac{1}{3}u^{3/2}+C$

$= \frac{1}{3}\sqrt{(2x+1)^3}+C$

3. $\int x^3\cos(x^4+2)\,dx$ $\qquad u=x^4+2 \rightarrow du=4x^3dx$

$= \frac{1}{4}\int 4x^3\cos(x^4+2)\,dx = \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C$

$= \frac{1}{4}\sin(x^4+2)+C$

4. $\int \frac{x}{\sqrt{1-4x^2}}\,dx$ $\qquad u = 1-4x^2 \rightarrow du = -8x\,dx$

$= \frac{1}{-8}\int \frac{-8x\,dx}{\sqrt{1-4x^2}} = \frac{-1}{8}\int \frac{1}{\sqrt{u}}\,du = \frac{-1}{8}\left[2u^{\frac{1}{2}} + C\right]$

$= \frac{-1}{4}\sqrt{u} + C = \frac{-\sqrt{1-4x^2}}{4} + C$

☺ Tricky one...

5. $\int x(x+5)^7\,dx$ $\qquad u = x+5 \rightarrow du = dx$

$\hookrightarrow$ Note $x = u-5$

$= \int (u-5)\,u^7\,du = \int (u^8 - 5u^7)\,du = \frac{u^9}{9} - \frac{5u^8}{8} + C$

$= \frac{(x+5)^9}{9} - \frac{5(x+5)^8}{8} + C$

6. $\int 5^{\pi x}\,dx$ $\qquad u = \pi x \rightarrow du = \pi\,dx$

$= \frac{1}{\pi}\int 5^{\pi x}\,\pi\,dx = \frac{1}{\pi}\int 5^u\,du = \frac{1}{\pi}\left[\frac{5^u}{\ln 5} + C\right]$

$= \frac{5^{\pi x}}{\pi \ln 5} + C$

7. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$ $\qquad u = \cos x \rightarrow du = -\sin x\,dx$

$= -\int \frac{-\sin x\,dx}{\cos x} = -\int \frac{1}{u}\,du = -\left[\ln|u| + C\right]$

$= -\ln|\cos x| + C = \ln|\sec x| + C$

8. $\displaystyle\int_{\frac{13}{2}}^{16} \sqrt[3]{2x-5}\, dx$ $\qquad u = 2x-5 \rightarrow du = 2dx$

$u(\tfrac{13}{2}) = 2(\tfrac{13}{2}) - 5 = 8 \quad , \quad u(16) = 2(16) - 5 = 27$

$$= \frac{1}{2}\int_8^{27} \sqrt[3]{u}\, du = \frac{1}{2}\left[\frac{3}{4}u^{\frac{4}{3}}\right]_8^{27} = \frac{3}{8}\left(\sqrt[3]{u}\right)^4 \Big|_8^{27}$$

$$= \frac{3}{8}\left[81 - 16\right] = \frac{195}{8}$$

9. $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$ $\qquad u = 3-5x \rightarrow du = -5dx$

$u(1) = 3 - 5(1) = -2 \qquad u(2) = 3 - 5(2) = -7$

$$= \frac{1}{-5}\int_1^2 \frac{-5dx}{(3-5x)^2} = \frac{-1}{5}\int_{-2}^{-7} \frac{du}{u^2} = \frac{-1}{5}\left[\frac{-1}{u}\right]_{-2}^{-7} = \frac{1}{5u}\Big|_{-2}^{-7}$$

$$= \frac{-1}{35} + \frac{1}{10} = \frac{-2}{70} + \frac{7}{70} = \frac{5}{70} = \frac{1}{14}$$

10. $\displaystyle\int_1^e \frac{\ln x}{3x}\, dx$ $\qquad u = \ln x \rightarrow du = \frac{1}{x}dx$

$u(1) = \ln 1 = 0 \quad , \quad u(e) = \ln e = 1$

$$= \frac{1}{3}\int_1^e \frac{\ln x}{x}\, dx = \frac{1}{3}\int_0^1 u\, du = \frac{1}{3}\left[\frac{u^2}{2}\right]_0^1$$

$$= \frac{1}{3}\left[\frac{1}{2} - 0\right] = \frac{1}{6}$$

11. $\displaystyle\int x^5\sqrt{x^3+1}\, dx$ $\qquad u = x^3+1 \rightarrow du = 3x^2 dx$

$$= \frac{1}{3}\int 3x^2 \cdot x^3\sqrt{x^3+1}\, dx$$ but we know $x^3 = u-1$

$$= \frac{1}{3}\int (u-1)\sqrt{u}\, du = \frac{1}{3}\int \left(u^{3/2} - u^{1/2}\right) du = \frac{1}{3}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C\right]$$

$$= \frac{2\sqrt{(x^3+1)^5}}{15} - \frac{2\sqrt{(x^3+1)^3}}{9} + C$$

12. $\int \frac{x^2\,dx}{(x+5)^3}$ $\qquad u = x+5 \rightarrow du = dx$
$\hookrightarrow x = u-5 \rightarrow x^2 = u^2 - 10u + 25$

$= \int \frac{u^2-10u+25}{u^3}\,du = \int \left(\frac{1}{u} - 10u^{-2} + 25u^{-3}\right)du$

$= \ln|u| + 10u^{-1} - \frac{25}{2}u^{-2} + C = \ln|x+5| + \frac{10}{x+5} - \frac{25}{2(x+5)^2} + C$

13. $\int \frac{\tan(x^{\frac{4}{5}})}{x^{1/5}}\,dx$ $\qquad u = x^{4/5} \rightarrow du = \frac{4}{5}x^{-\frac{1}{5}}dx$

$= \frac{5}{4}\int \frac{4}{5}x^{-\frac{1}{5}}\tan(x^{\frac{4}{5}})dx = \frac{5}{4}\int \tan u\,du = \frac{5}{4}\int \frac{\sin u}{\cos u}\,du$

$w = \cos u \rightarrow dw = -\sin u\,du$

$= -\frac{5}{4}\int \frac{-\sin w\,dw}{\cos w} = -\frac{5}{4}\int \frac{dw}{w} = -\frac{5}{4}\left[\ln|w| + C\right]$

$= -\frac{5}{4}\ln|\cos u| + C$

$= -\frac{5}{4}\ln|\cos x^{\frac{4}{5}}| + C$

14. $\int \cos t \cos(\sin t)\,dt$ $\qquad u = \sin t \rightarrow du = \cos t\,dt$

$= \int \cos u\,du = \sin u + C = \sin(\sin t) + C$

15. $\int_e^{e^2} \frac{dx}{x(\ln x)^2}$ $\qquad u = \ln x \rightarrow du = \frac{1}{x}dx$
$u(e) = \ln e = 1, \quad u(e^2) = \ln e^2 = 2$

$= \int_1^2 \frac{du}{u^2} = \left[\frac{-1}{u}\right]_1^2 = \frac{-1}{2} - (-1) = \frac{1}{2}$

16. $\int (\cot x) \ln(\sin x)\, dx \qquad u = \ln(\sin x) \to du = \frac{\cos x}{\sin x}\, dx$

$$= \int u\, du = \frac{u^2}{2} + C = \frac{1}{2}\left(\ln(\sin x)\right)^2 + C$$

17. $\int_0^{\sqrt{e-1}} \frac{x^3}{x^2+1}\, dx \qquad u = x^2+1 \to du = 2x\, dx$

$u(0) = 1, \quad u(\sqrt{e-1}) = e-1+1 = e$

$\to x^2 = u-1$

$$= \frac{1}{2}\int_0^{\sqrt{e-1}} \frac{2x \cdot x^2}{x^2+1}\, dx = \frac{1}{2}\int_1^e \frac{u-1}{u}\, du = \frac{1}{2}\int_1^e \left(1 - \frac{1}{u}\right) du$$

$$= \frac{1}{2}\Big[u - \ln|u|\Big]_1^e = \frac{1}{2}\Big[(e-1) - (1-0)\Big] = \frac{1}{2}(e-2)$$

$$= \frac{1}{2}e - 1$$

18. $\int_0^7 x\sqrt{15 - \sqrt{3+4x^2}}\, dx \qquad u = 3+4x^2 \to du = 8x\, dx$

$u(0) = 3 \qquad u(7) = 3 + 4(49) = 199$

$$= \frac{1}{8}\int_3^{199} \sqrt{15 - \sqrt{u}}\, du$$

$w = 15 - \sqrt{u} \qquad dw = \frac{-1}{2\sqrt{u}}\, du$

$\sqrt{u} = 15 - w \qquad du = -2\sqrt{u}\, dw$

$w(3) = 15 - \sqrt{3}$

$w(199) = 15 - \sqrt{199}$

$$= \frac{-1}{8}\int_{u=3}^{u=199} \sqrt{w}\,(2\sqrt{u})\, dw =$$

$$= \frac{-1}{4}\int_{15-\sqrt{3}}^{15-\sqrt{199}} \sqrt{w}\,(15-w)\, dw = \frac{-1}{4}\int_{15-\sqrt{3}}^{15-\sqrt{199}} \left(15w^{\frac{1}{2}} - w^{3/2}\right) dw$$

$$= \frac{-1}{4}\left[10w^{\frac{3}{2}} - \frac{2}{5}w^{\frac{5}{2}}\right]_{15-\sqrt{3}}^{15-\sqrt{199}}$$

$$= \frac{-1}{4}\left[8.14082 - 226.79821\right]$$