10.3 Convergence Tests

Integral Test: If f is positive, decreasing, and continuous then

$$S = \sum_{n=1}^{\infty} f(n) \text{ converges/diverges if } \int_{m}^{\infty} f(x)\,dx \text{ converges/diverges.}$$

1. Use IT to determine if $\sum_{n=1}^{\infty} n e^{-n^2}$ converges or diverges.

$f(x) = x e^{-x^2}$ is continuous and positive for $x > 0$

decreasing? $\rightarrow$ $f'(x) = e^{-x^2} + x\left(e^{-x^2} \cdot -2x\right) = e^{-x^2}(1-2x^2) = 0 \leftarrow CP$

$$1 = 2x^2$$
$$\tfrac{1}{2} = x^2$$
$$\pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2} = x$$

x	Test Point	$e^{-x^2}(1-2x^2)$
$\left(\frac{\sqrt{2}}{2}, \infty\right)$	2	$(+)\ (-) = -$

$\therefore$ $f(x) = x e^{-x^2}$ is continuous, positive, and decreasing for all integers $x \geq 1$.

$$IT \rightarrow \int_{1}^{\infty} x e^{-x^2}\,dx = \lim_{b\to\infty} \int_{1}^{b} x e^{-x^2}\,dx \qquad u = -x^2, \quad du = -2x\,dx \qquad = \lim_{b\to\infty} \frac{-1}{2}\int_{x=1}^{x=b} e^{u}\,du$$

$$I = \lim_{b\to\infty} \frac{-1}{2} e^{u}\Big|_{x=1}^{x=b} = \lim_{b\to\infty} \frac{-1}{2} e^{-x^2}\Big|_{1}^{b} = \lim_{b\to\infty} \frac{-1}{2}e^{-b^2} + \frac{1}{2}e^{-1} = 0 + \frac{1}{2e} = \boxed{\frac{1}{2e}}$$

Since $\int_{1}^{\infty} x e^{-x^2}\,dx$ converges, $\sum_{n=1}^{\infty} n e^{-n^2}$ also converges. (though __NOT__ to $\frac{1}{2e}$)

p-series : $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$, diverges if $p \le 1$

Verify with <u>integral</u> <u>test</u>.

$p=.9 \rightarrow \int_1^\infty \frac{1}{x^{.9}}\,dx = \lim_{b\to\infty} \int_1^b x^{-.9}\,dx = \lim_{b\to\infty} \left[10x^{.1}\right]_1^b = \infty$

$p=1 \rightarrow \int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x \Big|_1^b = \infty$ (called harmonic series... smallest divergent series)

$p=1.1 \rightarrow \int_1^\infty \frac{1}{x^{1.1}}\,dx = \lim_{b\to\infty} \int_1^b x^{-1.1}\,dx = \lim_{b\to\infty} \left[\frac{-10}{x^{.1}}\right]_1^b = 0 + 10 = 10$ (convergent!)

* p-series is <u>very</u> <u>helpful</u> for the <u>Comparison</u> <u>Test</u>

If $0 \le a_n \le b_n$ then if $\sum b_n$ converges, $\sum a_n$ converges
if $\sum a_n$ diverges, $\sum b_n$ diverges.

2.) Use CT to determine convergence of $\sum_{n=1}^{\infty} \frac{n^3}{n^5+4n+1}$

For $n \ge 1$ $\frac{n^3}{n^5+4n+1} \le \frac{n^3}{n^5} = \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ is a convergent p-series $(p=2>1)$

Since $\frac{n^3}{n^5+4n+1} \le \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{n^3}{n^5+4n+1}$ must also converge.

3. Use CT to determine convergence of $\sum_{n=4}^{\infty} \frac{\sqrt{n}}{n-3}$ p-series $p=\frac{1}{2}$

For $n \geq 4$ $\frac{\sqrt{n}}{n-3} \geq \frac{\sqrt{n}}{n} = \frac{1}{n^{1/2}}$ $\rightarrow$ we know $\sum \frac{1}{n^{1/2}}$ is divergent $(p=\frac{1}{2}\leq 1)$

Since $\frac{\sqrt{n}}{n-3} \geq \frac{1}{n^{1/2}}$, $\sum_{n=4}^{\infty} \frac{\sqrt{n}}{n-3}$ must also diverge.

★ Sometimes the CT is used in combination with the Geometric Series $\sum c r^{n-1}$ converges when $|r|<1$.

4. Use CT to show convergence of $\sum_{n=1}^{\infty} 2^{-n^2}$

For $n \geq 1$, $2^{-n^2} = \frac{1}{2^{n^2}} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$

$\sum \left(\frac{1}{2}\right)^n$ is a convergent geometric series $(r=\frac{1}{2})$

Since $\frac{1}{2^{n^2}} \leq \frac{1}{2^n}$, $\sum_{n=1}^{\infty} 2^{-n^2}$ also converges.

5. Determine convergence of $\sum_{n=1}^{\infty} \frac{n!}{n^3}$ As $n\to\infty$ we know $(n-2)(n-1) \geq n$

$\frac{n!}{n^3} \geq \frac{\cancel{n}\,\cancel{(n-1)}\,\cancel{(n-2)}\,(n-3)}{\cancel{n}\cdot\cancel{n}\cdot n} \geq \frac{n-3}{n} \geq \frac{1}{n}$ $\rightarrow$ $\sum \frac{1}{n}$ is divergent $(p=1\leq 1)$

Since $\frac{n!}{n^3} \geq \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{n!}{n^3}$ also diverges.

A better/easier Comparison Test

Limit Comparison Test (LCT): If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ $(L \neq 0)$ $(L \neq \pm\infty)$,

then $\sum b_n$ converges/diverges iff (if and only if) $\sum a_n$ converges/diverges

6. Use LCT to determine convergence of $\sum_{n=2}^{\infty} \frac{1}{n^2-\sqrt{n}}$

Compare with $b_n = \frac{1}{n^2}$ (Note: $\frac{1}{n^2-\sqrt{n}} \not< \frac{1}{n^2}$ so CT would fail)

$$L = \lim_{n\to\infty} \frac{\frac{1}{n^2-\sqrt{n}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-\sqrt{n}} = 1$$

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p=2>1)$. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, by the LCT we can conclude $\sum_{n=2}^{\infty} \frac{1}{n^2-\sqrt{n}}$ also converges.

7. $\sum_{n=1}^{\infty} \frac{5(e^n+n)}{e^{2n}-n^2} \to \frac{5(e^n+n)}{e^{2n}-n^2} = \frac{5(e^n+n)}{(e^n-n)(e^n+n)} = \frac{5}{e^n-n} \approx \frac{1}{e^n}$ ← use for comparison

$\left(\frac{1}{e}\right)^n$ $r = \frac{1}{e} < 1$

$$L = \lim_{n\to\infty} \frac{\frac{5(e^n+n)}{e^{2n}-n^2}}{\frac{1}{e^n}} = \lim_{n\to\infty} \frac{5e^n(e^n+n)}{e^{2n}-n^2} = \lim_{n\to\infty} \frac{5e^{2n}+5ne^n}{e^{2n}-n^2} = 5$$

$\sum \frac{1}{e^n}$ is a convergent geometric series $\left(|r| = \frac{1}{e} < 1\right)$. Since $L = 5$, by the LCT we can conclude that $\sum_{n=1}^{\infty} \frac{5(e^n+n)}{e^{2n}-n^2}$ also converges.