7.6 Notes

1. $\int x\sec^{-1}x\,dx$ Inverse Trig. $\rightarrow$ IBP (usually)

$u=\sec^{-1}x \qquad dv = x\,dx$

$du=\dfrac{1}{x\sqrt{x^2-1}} \qquad v=\tfrac{1}{2}x^2$

$t=x^2-1$
$dt=2x\,dx$

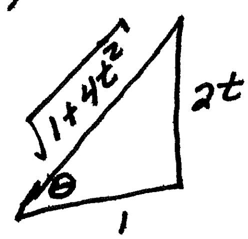
$$\int x\sec^{-1}x\,dx = \tfrac{1}{2}x^2\sec^{-1}x - \tfrac{1}{2}\int\frac{x}{\sqrt{x^2-1}}\,dx$$
$$= \tfrac{1}{2}x^2\sec^{-1}x - \tfrac{1}{4}\int\frac{1}{\sqrt{t}}\,dt$$
$$= \tfrac{1}{2}x^2\sec^{-1}x - \tfrac{1}{4}\left[2\sqrt{t}\right]+C = \boxed{\tfrac{1}{2}x^2\sec^{-1}x - \frac{\sqrt{x^2-1}}{2}+C}$$

Recognize "a^2+x^2" $\rightarrow$ Trig. Sub.
Let $2t=\tan\theta$
$t=\tfrac{1}{2}\tan\theta$
$dt=\tfrac{1}{2}\sec^2\theta\,d\theta$
$(1+(2t)^2)^{3/2} = (1+\tan^2\theta)^{3/2}=\sec^3\theta$

2. $$\int\frac{dt}{(1+4t^2)^{3/2}} = \int\frac{dt}{(1+(2t)^2)^{3/2}}$$

$$=\int\frac{\tfrac{1}{2}\sec^2\theta\,d\theta}{\sec^3\theta}$$

$$=\tfrac{1}{2}\int\cos\theta\,d\theta = \tfrac{1}{2}\left[\sin\theta+C\right]$$
$$=\tfrac{1}{2}\left[\frac{2t}{\sqrt{1+4t^2}}+C\right]$$
$$=\boxed{\frac{t}{\sqrt{1+4t^2}}+C}$$



$u=\tan x$
$du=\sec^2x\,dx$

3. $$\int\sec^6x\,dx = \int(\tan^2x+1)^2\sec^2x\,dx$$
$$=\int(u^2+1)^2\,du = \int(u^4+2u^2+1)\,du = \frac{u^5}{5}+\frac{2u^3}{3}+u+C$$
$$=\boxed{\tfrac{1}{5}\tan^5x+\tfrac{2}{3}\tan^3x+\tan x+C}$$

4. $\int x^3\sqrt{1+x^2}\,dx$ $\qquad u=1+x^2 \rightarrow x^2=u-1$
$du=2x\,dx$

$$=\tfrac{1}{2}\int(u-1)\sqrt{u}\,du = \tfrac{1}{2}\int\left(u^{3/2}-u^{\frac{1}{2}}\right)du = \tfrac{1}{2}\left[\tfrac{2}{5}u^{\frac{5}{2}}-\tfrac{2}{3}u^{3/2}\right]+C$$
$$=\tfrac{1}{5}\sqrt{(1+x^2)^5}-\tfrac{1}{3}\sqrt{(1+x^2)^3}+C$$

5. $\int \frac{x^5}{x^3-1}\,dx$

$u = x^3 - 1 \rightarrow x^3 = u - 1$

$du = 3x^2\,dx$

$$\int \frac{x^5}{x^3-1}\,dx = \frac{1}{3}\int \frac{x^3\,(3x^2\,dx)}{x^3-1} = \frac{1}{3}\int \frac{u-1}{u}\,du$$

$$= \frac{1}{3}\int \left(1 - \frac{1}{u}\right) du = \frac{1}{3}\left[u - \ln|u|\right] + C$$

$$= \frac{1}{3}\left[x^3 - 1 - \ln|x^3 - 1|\right] + C$$

6. $\int \frac{x^5}{x^4-1}\,dx$ sub. not helpful... use long division

$$\begin{array}{r} x \\ x^4 - 1 \;\overline{)\; x^5} \\ -(x^5 - x) \\ \hline x \end{array}$$

$$\rightarrow \int \frac{x^5}{x^4-1}\,dx = \int x\,dx + \underbrace{\int \frac{x}{(x^2+1)(x+1)(x-1)}\,dx}_{a}$$

a.) $\frac{x}{(x^2+1)(x+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$

$x = (Ax+B)(x+1)(x-1) + C(x^2+1)(x-1) + D(x^2+1)(x+1)$

$x = 1 \rightarrow 1 = D(2)(2) \rightarrow \underline{D = \frac{1}{4}}$

$x = -1 \rightarrow -1 = C(2)(-2) \rightarrow \underline{C = \frac{1}{4}}$

$x^3 \rightarrow 0 = A + C + D \rightarrow 0 = A + \frac{1}{2} \rightarrow \underline{A = -\frac{1}{2}}$

constant $\rightarrow 0 = -B - C + D \rightarrow 0 = -B - \frac{1}{4} + \frac{1}{4} \rightarrow \underline{B = 0}$

$$\int \frac{x\,dx}{(x^2+1)(x+1)(x-1)} = -\frac{1}{2}\int \frac{x}{x^2+1}\,dx + \frac{1}{4}\int \frac{1}{x+1}\,dx + \frac{1}{4}\int \frac{1}{x-1}\,dx$$

$$= -\frac{1}{4}\ln(x^2+1) + \frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + C$$

so...

$$\int \frac{x^5}{x^4-1}\,dx = \frac{1}{2}x^2 - \frac{1}{4}\ln|x^2+1| + \frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + C$$

7. $\int \sqrt{1+\sqrt{x}}\,dx$

$u = 1+\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx \rightarrow dx = 2\sqrt{x}\,du$

$= 2(u-1)\,du$

$$= \int \sqrt{u}\,(2)(u-1)\,du = 2\int (u^{3/2} - u^{\frac{1}{2}})\,du = 2\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right] + C$$

$$= \frac{4\sqrt{(1+\sqrt{x})^5}}{5} - \frac{4\sqrt{(1+\sqrt{x})^3}}{3} + C$$

8. $\int x \ln(x+12)\,dx$

log → IBP

$u = \ln(x+12)$ $\quad dv = x\,dx$

$du = \frac{1}{x+12}\,dx$ $\quad v = \frac{1}{2}x^2$

$$\int x\ln(x+12)\,dx = \tfrac{1}{2}x^2\ln(x+12) - \tfrac{1}{2}\int \frac{x^2}{x+12}\,dx$$

$$= \tfrac{1}{2}x^2\ln(x+12) - \tfrac{1}{2}\int(x-12)\,dx - \tfrac{1}{2}\int \frac{144}{x+12}\,dx$$

$$= \tfrac{1}{2}x^2\ln(x+12) - \tfrac{1}{4}x^2 + 6x - 72\ln|x+12| + C$$

$$\begin{array}{r} x-12 \\ x+12 \,\overline{)\, x^2 \qquad\quad} \\ -(x^2+12x) \\ \hline -12x \\ -(-12x-144) \\ \hline 144 \end{array}$$

9. $\int \frac{dx}{x(x^2-6x-7)} = \int \frac{1}{x(x-7)(x+1)}\,dx$

$$\frac{1}{x(x-7)(x+1)} = \frac{A}{x} + \frac{B}{x-7} + \frac{C}{x+1}$$

$1 = A(x-7)(x+1) + B(x)(x+1) + C(x)(x-7)$

$x=0 \rightarrow 1 = A(-7)(1) \rightarrow \underline{A = \frac{-1}{7}}$

$x=7 \rightarrow 1 = B(7)(8) \rightarrow \underline{B = \frac{1}{56}}$

$x=-1 \rightarrow 1 = C(-1)(-8) \rightarrow \underline{C = \frac{1}{8}}$

$$\int \frac{dx}{x(x^2-6x-7)} = \frac{-1}{7}\int \frac{1}{x}\,dx + \frac{1}{56}\int \frac{1}{x-7}\,dx + \frac{1}{8}\int \frac{1}{x+1}\,dx$$

$$= -\tfrac{1}{7}\ln|x| + \tfrac{1}{56}\ln|x-7| + \tfrac{1}{8}\ln|x+1| + C$$

10. $\int \frac{\sqrt{x}}{x^3+1}\,dx$

recognize "u^2+a^2"

$u^2 = x^3$

$u = x^{3/2}$

$du = \frac{3}{2}x^{\frac{1}{2}}dx$

$$\frac{2}{3}\int \frac{\frac{3}{2}\sqrt{x}\,dx}{x^3+1} = \frac{2}{3}\int \frac{du}{u^2+1} = \frac{2}{3}\tan^{-1}u + C$$

$$= \frac{2}{3}\tan^{-1}\left(\sqrt{x^3}\right) + C$$