10.6 Notes

Power Series: $\sum_{n=0}^{\infty} a_n (x-c)^n = F(x)$

ways $F(x)$ could converge

1. If $x=c$
2. For any x
3. For some $R>0$ where $F(x)$ converges absolutely if $|x-c|<R$ and diverges if $|x-c|>R$.

★ <u>Ratio Test</u> is normally best method to find R and the corresponding interval of convergence (IOC) that would be $(c-R, c+R)$.

1. $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$ $\qquad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}x^{n+1}}{n+1} \cdot \frac{n}{2^n x^n}\right| = \left|\frac{2x(n)}{n+1}\right|$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2xn}{n+1}\right| = |2x| \rightarrow$ so series converges if $|2x|<1$

$-1<2x<1$

$-\frac{1}{2}<x<\frac{1}{2}$

RT is inconclusive when $L=1$ so we must test endpoints ($x=\pm\frac{1}{2}$ in this problem) where $L=1$.

$x=-\frac{1}{2} \rightarrow \sum_{n=1}^{\infty} \frac{2^n}{n}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \rightarrow$ since $\lim_{n\to\infty} \frac{1}{n}=0$, series converges by AST.

$x=\frac{1}{2} \rightarrow \sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n} \rightarrow$ divergent p-series $(p=1\le 1)$

$\therefore$ IOC: $x = \left[-\frac{1}{2}, \frac{1}{2}\right)$

2. $\sum_{n=0}^{\infty} \frac{4^n}{(2n+1)!} x^{2n-1}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{4^{n+1} x^{2n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{4^n x^{2n-1}}\right| = \left|\frac{4^{n+1}}{4^n} \cdot \frac{(2n+1)!}{(2n+3)!} \cdot \frac{x^{2n+1}}{x^{2n-1}}\right|$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{4x^2}{(2n+3)(2n+2)}\right| = 0$

Since $L=0$, $\sum_{n=0}^{\infty} \frac{4^n}{(2n+1)!} x^{2n-1}$ converges absolutely for all x. (IOC: $x = (-\infty, \infty)$)

3. $\sum_{n=1}^{\infty} \frac{x^n}{n - 4\ln n}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n+1-4\ln(n+1)} \cdot \frac{n-4\ln n}{x^n}\right| = \left|x \cdot \frac{n-4\ln n}{n+1-4\ln(n+1)}\right|$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |x| \rightarrow |x|<1 \rightarrow -1<x<1$ (must check endpoints)

$x=-1 \rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n-4\ln n} \rightarrow$ Since $\lim_{n\to\infty} \frac{1}{n-4\ln n} = 0$, series converges by AST.

$x=1 \rightarrow \sum_{n=1}^{\infty} \frac{1}{n-4\ln n} \rightarrow$ Since $\frac{1}{n-4\ln n} \geq \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n-4\ln n}$ diverges.

$\therefore$ IOC is $x = [-1, 1)$

4. $\sum_{n=0}^{\infty} 27^n (x-1)^{3n+2}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{27^{n+1} (x-1)^{3n+5}}{27^n (x-1)^{3n+2}} \right| = \lim_{n\to\infty} \left| 27(x-1)^3 \right|$$

We know series converges if

$$|27(x-1)^3| < 1$$
$$-1 < 27(x-1)^3 < 1$$
$$\frac{-1}{27} < (x-1)^3 < \frac{1}{27}$$
$$-\frac{1}{3} < x-1 < \frac{1}{3}$$
$$\frac{2}{3} < x < \frac{4}{3}$$

Now, check $x = \frac{2}{3}$, $x = \frac{4}{3}$

$$x = \frac{2}{3} \rightarrow \sum_{n=0}^{\infty} 27^n \left(\frac{2}{3} - 1\right)^{3n+2} = \sum_{n=0}^{\infty} 27^n \left(\frac{-1}{3}\right)^{3n+2} = \sum_{n=0}^{\infty} 27^n \left(\left(-\frac{1}{3}\right)^3\right)^n \cdot \frac{1}{9} = \frac{1}{9} \sum_{n=0}^{\infty} (-1)^n$$

This series diverges since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \neq 0$ (Divergence Test).

$$x = \frac{4}{3} \rightarrow \sum_{n=0}^{\infty} 27^n \left(\frac{1}{3}\right)^{3n+2} = \frac{1}{9} \sum_{n=0}^{\infty} (1)^n$$

→ Again, diverges since $\lim_{n\to\infty} 1^n \neq 0$.

$\therefore$ $\boxed{\text{IOC is } x = \left(\frac{2}{3}, \frac{4}{3}\right)}$

5. $\sum_{n=10}^{\infty} n!\,(x+5)^n$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\,(x+5)^{n+1}}{n!\,(x+5)^n} \right| = \lim_{n\to\infty} \left| (n+1)(x+5) \right| = \infty$$

Since $L = \infty$, series diverges for all x except when $\boxed{x = -5}$ only.