6.2 Average Value

The average value of $f(x)$ on $[a,b]$ is given by

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

1. Find the average value of $f(x) = 1+x^2$ on $[-1,2]$

$$f_{avg} = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)\,dx = \frac{1}{3}\left[x+\frac{x^3}{3}\right]_{-1}^{2}$$

$$f_{avg} = \frac{1}{3}\left[\left(2+\frac{8}{3}\right)-\left(-1-\frac{1}{3}\right)\right] = \frac{1}{3}(6) = \boxed{2}$$

2. $f(x) = \dfrac{\sin\left(\frac{\pi}{x}\right)}{x^2}$, $[1,2]$

$$f_{avg} = \frac{1}{2-1}\int_1^2 \frac{\sin\left(\frac{\pi}{x}\right)}{x^2}\,dx = \frac{1}{-\pi}\int_1^2 \frac{-\pi}{x^2}\sin\left(\frac{\pi}{x}\right)dx$$

$u = \frac{\pi}{x} = \pi x^{-1}$ $\qquad u(1) = \pi$

$du = -\pi x^{-2}dx = \frac{-\pi}{x^2}dx \qquad u(2) = \frac{\pi}{2}$

$$f_{avg} = \frac{-1}{\pi}\int_{\pi}^{\frac{\pi}{2}} \sin u\,du = \frac{1}{\pi}\cos u\Big|_{\pi}^{\frac{\pi}{2}} = \frac{1}{\pi}[0+1] = \boxed{\frac{1}{\pi}}$$

3. $f(x) = \dfrac{1}{x^2+1}$, $[-1,1]$

$$f_{avg} = \frac{1}{1-(-1)}\int_{-1}^{1}\frac{1}{x^2+1}\,dx = \frac{1}{2}\left[\tan^{-1}x\right]_{-1}^{1} = \frac{1}{2}\left[\frac{\pi}{4}-\left(\frac{-\pi}{4}\right)\right]$$

$$= \boxed{\frac{\pi}{4}}$$