Guidelines for Evaluating $\int \sin^m x \cos^n x \, dx$

1.) If m is odd rewrite $\sin^m x$ as $(\sin^{m-1} x)(\sin x)$ then as $(1-\cos^2 x)^{\frac{m-1}{2}}(\sin x)$

2.) If n is odd rewrite $\cos^n x$ as $(\cos^{n-1} x)(\cos x)$

3.) If both m and n are even use $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

1. $\int \sin^5 x \cos^2 x \, dx \longrightarrow$ $\sin^5 x = \sin^4 x(\sin x) = (\sin^2 x)^2(\sin x)$

$= (1-\cos^2 x)^2(\sin x)$

$= (1-2\cos^2 x + \cos^4 x)(\sin x)$

$= \int \cos^2 x(1-2\cos^2 x + \cos^4 x)(\sin x)\, dx$

$= \int (\cos^2 x - 2\cos^4 x + \cos^6 x)(\sin x)\, dx$

$u = \cos x \qquad du = -\sin x \, dx$

$= -\int (u^2 - 2u^4 + u^6)\, du = -\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right] + C$

$= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$

2. $\int \sin^4 x \cos^3 x \, dx$ $\qquad \cos^3 x = \cos^2 x(\cos x) = (1-\sin^2 x)\cos x$

$= \int \sin^4 x(1-\sin^2 x)\cos x \, dx = \int (\sin^4 x - \sin^6 x)\cos x \, dx$ $\qquad u = \sin x$, $du = \cos x\, dx$

$= \int (u^4 - u^6)\, du = \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$

3. $\int \sin^4 x \, dx$ $\qquad (\sin^2 x)^2 = \left(\frac{1}{2}(1-\cos 2x)\right)^2 = \frac{1}{4}(1-2\cos 2x + \cos^2 2x)$

$\cos^2 2x = \frac{1}{2}(1+\cos 4x)$

$\sin^4 x = \frac{1}{4}(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x)$

$\int \sin^4 x \, dx = \frac{1}{4}\int\left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$

$= \frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right] + C = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

4. $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^{\frac{1}{2}} x \, dx = \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cos^{\frac{1}{2}} x \sin x \, dx$ $\quad u = \cos x$, $du = -\sin x dx$

$= -\int_1^0 (u^{\frac{1}{2}} - u^{5/2}) du = \left[\frac{2}{3}u^{3/2} - \frac{2}{7}u^{7/2}\right]_0^1 = \frac{2}{3} - \frac{2}{7} = \frac{8}{21}$

* Guidelines for evaluating $\int \tan^m x \sec^n x \, dx$

1. If power of $\tan x$ is odd use $\tan^2 x = \sec^2 x - 1$ to rewrite all but $(\tan x)$! Use $u = \sec x$.
2. If $\sec x$ has even power, use $\sec^2 x = 1 + \tan^2 x$ to rewrite all but $\sec^2 x$. Use $u = \tan x$.

5. $\int \tan^3 x \sec^7 x \, dx$ $\quad \tan^3 x = (\sec^2 x - 1)\tan x$

$= \int (\sec^2 x - 1)\sec^7 x \tan x \, dx$ $\quad u = \sec x$, $du = \sec x \tan x \, dx$

$= \int (u^2 - 1) u^6 du = \int (u^8 - u^6) du = \frac{1}{9}\sec^9 x - \frac{1}{7}\sec^7 x + C$

6. $\int_0^{\frac{\pi}{4}} \sqrt{\tan x} \sec^6 x \, dx$ $\quad \sec^6 x = (1+\tan^2 x)^2 \sec^2 x = (1 + 2\tan^2 x + \tan^4 x)\sec^2 x$

$\int_0^{\pi/4} (\tan^{\frac{1}{2}} x + 2\tan^{5/2} x + \tan^{9/2} x) \sec^2 x \, dx$ $\quad u = \tan x$, $du = \sec^2 x \, dx$, $u(0) = 0$, $u(\frac{\pi}{4}) = 1$

$= \int_0^1 (u^{\frac{1}{2}} + 2u^{5/2} + u^{9/2}) du = \left[\frac{2}{3}u^{3/2} + \frac{4}{7}u^{\frac{7}{2}} + \frac{2}{11}u^{\frac{11}{2}}\right]_0^1$

$= \frac{2}{3} + \frac{4}{7} + \frac{2}{11} = \frac{154}{231} + \frac{132}{231} + \frac{42}{231} = \frac{328}{231}$

7. $\int \cot^5 x \csc^5 x \, dx$ $\quad \cot^5 x = (\csc^2 x - 1)^2 \cot x = (\csc^4 x - 2\csc^2 x + 1)\cot x$

$= \int (\csc^4 x - 2\csc^2 x + 1)\csc^5 x \cot x \, dx$

$u = \csc x \quad du = -\csc x \cot x$

$= -\int (u^8 - 2u^6 + u^4) du = \frac{-1}{9}\csc^9 x + \frac{2}{7}\csc^7 x \quad \frac{1}{5}\csc^5 x + C$

Other options... Convert to sines and cosines

8. $\int \frac{\tan x}{\sec^2 x}\,dx = \int \left(\frac{\sin x}{\cos x}\right)(\cos^2 x)\,dx = \int \sin x \cos x\,dx$ $\quad u = \sin x$, $du = \cos x\,dx$

$= \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$

SHOW GRAPHICALLY WHY BOTH ARE CORRECT.

or...

$\int \frac{\tan x}{\sec^2 x}\,dx$ $\quad u = \sec x$, $du = \sec x \tan x\,dx$ $\rightarrow \int \frac{\sec x \tan x}{\sec^3 x}\,dx = \int \frac{1}{u^3}\,du$

$= \frac{u^{-2}}{-2} + C = -\frac{1}{2}\left(\frac{1}{\sec^2 x}\right) + C = -\frac{1}{2}\cos^2 x + C$

☆ Prod. to Sum $\rightarrow \sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$

9. $\int \sin 4x \cos 5x\,dx = \frac{1}{2}\int \left[\sin -x + \sin 9x\right]dx$

$= \frac{1}{2}\left[\cos(-x) - \frac{1}{9}\cos(9x)\right] + C$

$= \frac{1}{2}\left(\cos x - \frac{1}{9}\cos 9x\right) + C$

More Trig Substitution Practice

1. $\int \cos^3(2-x)\sin(2-x)\,dx$ $\qquad u = \cos(2-x)$, $du = \sin(2-x)\,dx$

$= \int u^3\,du = \frac{u^4}{4} + C = \frac{1}{4}\cos^4(2-x) + C$

Use reduction formula $\int \cos^n x\,dx = \frac{\cos^{n-1}x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$

2. $\int \cos^7 3x\,dx$ $\qquad u = 3x$, $du = 3dx$ $\Rightarrow \frac{1}{3}\int \cos^7 u\,du$

$= \frac{1}{3}\left[\frac{\cos^6 u \sin u}{7} + \frac{6}{7}\int \cos^5 x\,dx\right]$

$= \frac{\cos^6 u \sin u}{21} + \frac{2}{7}\left[\frac{\cos^4 u \sin u}{5} + \frac{4}{5}\int \cos^3 u\,du\right]$

$= \frac{\cos^6 u \sin u}{21} + \frac{2\cos^4 u \sin u}{35} + \frac{8}{35}\int \cos^3 u\,du$

$= \quad '' \qquad '' \quad + \frac{8}{35}\left[\frac{\cos^2 u \sin u}{3} + \frac{2}{3}\int \cos u\,du\right]$

$= \frac{\cos^6 3x \sin 3x}{21} + \frac{2\cos^4 3x \sin 3x}{35} + \frac{8\cos^2 3x \sin 3x}{105} + \frac{16\sin 3x}{105} + C$

Use $\int \sec^m x\,dx = \frac{\tan x \sec^{m-2}x}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2}x\,dx$

and $\int \sec x\,dx = \ln|\sec x + \tan x| + C$ to evaluate

3. $\int \tan^4 x \sec x\,dx$ $\qquad \tan^4 x = (\sec^2 x - 1)^2 = \sec^4 x - 2\sec^2 x + 1$

$= \int (\sec^5 x - 2\sec^3 x + \sec x)\,dx$

$= \frac{\tan x \sec^3 x}{4} + \frac{3}{4}\int \sec x\,dx - 2\int \sec^3 x\,dx + \ln|\sec x + \tan x| + C$

$= \frac{\tan x \sec^3 x}{4} - \frac{5}{4}\left[\frac{\tan x \sec x}{2} + \frac{1}{2}\int \sec x\,dx\right] + \ln|\sec x + \tan x| + C$

$= \frac{\tan x \sec^3 x}{4} - \frac{5\tan x \sec x}{8} + \frac{3\ln|\sec x + \tan x|}{8} + C$