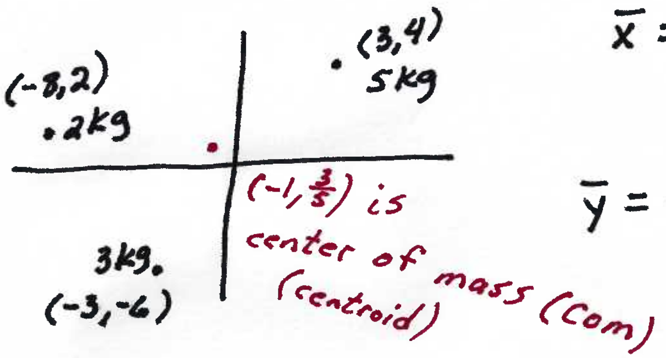
Center of Mass

Finite example from physics



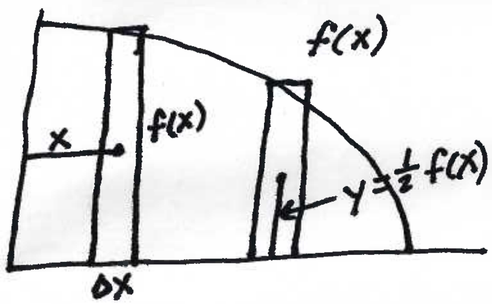
$$\bar{x} = \frac{3(5)+(-8)(2)+(-3)(3)}{5+2+3} = \frac{-10}{10} = -1$$

moments about y-axis (M_y)

$$\bar{y} = \frac{4(5)+2(2)+(-6)(3)}{5+2+3} = \frac{6}{10} = \frac{3}{5}$$

Moments about x-axis (M_x)

Centers of mass in Calculus...

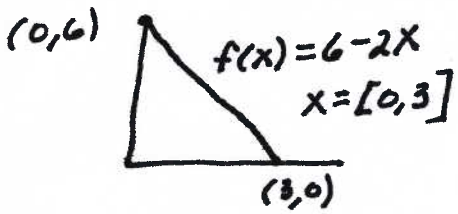


Find center of mass of region bound by $f(x)$.

$$\bar{x} = \frac{M_y}{m} = \frac{\sum_{i=1}^{n} x \cdot f(x)\Delta x}{\sum_{i=1}^{n} f(x)\Delta x} = \frac{\int_a^b x \cdot f(x)\,dx}{\int_a^b f(x)\,dx}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\sum_{i=1}^{n} \frac{1}{2} f(x)\cdot f(x)\Delta x}{\sum_{i=1}^{n} f(x)\,dx} = \frac{\frac{1}{2}\int_a^b (f(x))^2\,dx}{\int_a^b f(x)\,dx}$$

1. Find centroid



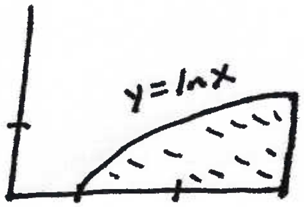
$$m = \int_0^3 (6-2x)\,dx = \left[6x - x^2\right]_0^3 = 9$$

$$M_y = \int_0^3 x(6-2x)\,dx = \int_0^3 (6x-2x^2)\,dx = \left[3x^2 - \tfrac{2}{3}x^3\right]_0^3 = 27-18 = 9$$

$$M_x = \frac{1}{2}\int_0^3 (6-2x)^2\,dx = \frac{1}{2}\int_0^3 (36-24x+4x^2)\,dx = \frac{1}{2}\left[36x - 12x^2 + \frac{4x^3}{3}\right]_0^3 = \frac{1}{2}(36) = 18$$

$$\bar{x} = \frac{M_y}{m} = \frac{9}{9}, \quad \bar{y} = \frac{M_x}{m} = \frac{18}{9} = 2 \rightarrow \text{centroid} = (\bar{x}, \bar{y}) = \boxed{(1,2)}$$

2. Find centroid for $y = \ln x$, $[1,3]$



$$m = \int_1^3 \ln x\,dx = \Big[x\ln x - x\Big]_1^3 = (3\ln 3 - 3) - (-1) = 3\ln 3 - 2$$

$$m_y = \int_1^3 x\ln x\,dx = \Big[\tfrac{1}{2}x^2\ln x - \tfrac{1}{2}\int x\,dx\Big]_1^3 = \Big[\tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2\Big]_1^3$$

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{x^2}{2}$

$$m_y = \left(\tfrac{9}{2}\ln 3 - \tfrac{9}{4}\right) - \left(\tfrac{-1}{4}\right) = \tfrac{9}{2}\ln 3 - 2$$

$$m_x = \tfrac{1}{2}\int_1^3 (\ln x)^2\,dx = \tfrac{1}{2}\Big[x(\ln x)^2 - 2\int \ln x\,dx\Big]_1^3 = \tfrac{1}{2}\Big[x(\ln x)^2 - 2(x\ln x - x)\Big]_1^3$$

$u = (\ln x)^2 \qquad dv = dx$

$du = 2(\ln x)\left(\frac{1}{x}\right)dx \qquad v = x$

$$m_x = \Big[\tfrac{1}{2}x(\ln x)^2 - x\ln x + x\Big]_1^3$$

$$m_x = \left(\tfrac{3}{2}(\ln 3)^2 - 3\ln 3 + 3\right) - 1 = \tfrac{3}{2}(\ln 3)^2 - 3\ln 3 + 2$$

$$\bar{x} = \frac{m_y}{m} = \frac{\frac{9}{2}\ln 3 - 2}{3\ln 3 - 2} \qquad \bar{y} = \frac{m_x}{m} = \frac{\frac{3}{2}(\ln 3)^2 - 3\ln 3 + 2}{3\ln 3 - 2}$$