8AM

§2.6 **Limits at Infinity; Horizontal Asymptotes**

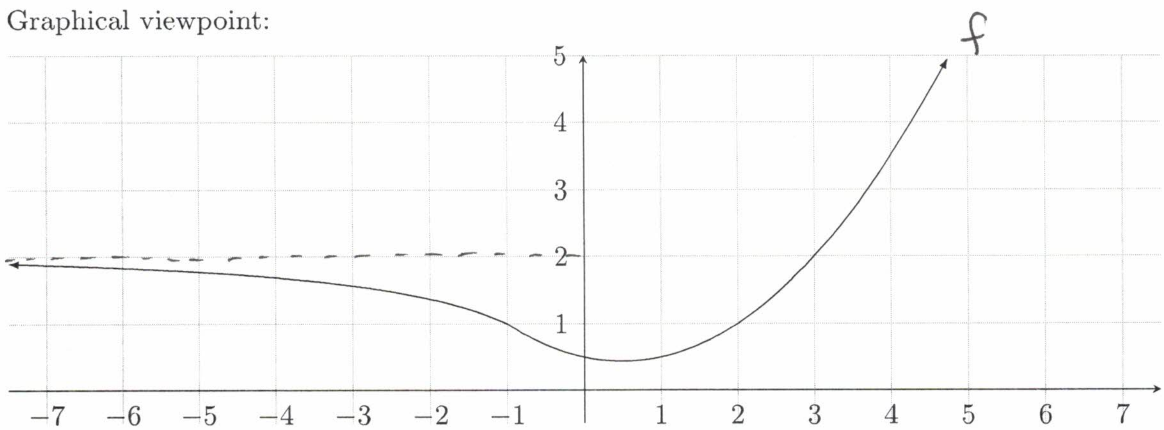
Goal: Compute limits when $x \to \infty$ or $x \to -\infty$ and interpret these as horizontal asymptotes.

Definition: $\lim_{x \to \infty} f(x) = L$ means the y-values of f can be made arbitrarily close to L by taking x-values to be large enough.

$\lim_{x \to -\infty} f(x) = L$ means the y-values of f can be made arbitrarily close to L by taking x-values to be a large enough negative number.

In either case, we call the line $y = L$ a horizontal asymptote of f.

Graphical viewpoint:

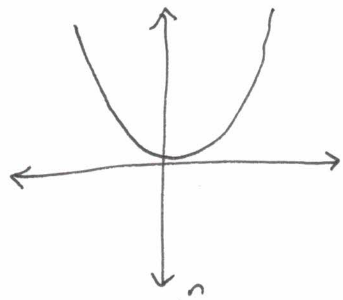


$\lim_{x \to -\infty} f(x) = 2$ $\qquad$ $\lim_{x \to \infty} f(x) = \infty$

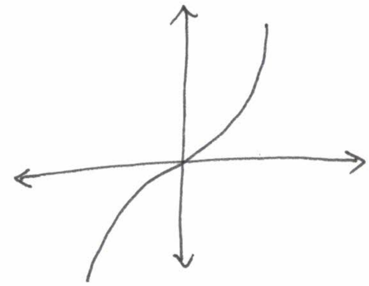
Example 1. Building-block examples (aka tools for later problems)
Assume n is an integer with $n > 0$.

$\lim_{x\to\infty} x^n = \infty$

$\lim_{x\to-\infty} x^n = \begin{cases} \infty & n \text{ even} \\ -\infty & n \text{ odd} \end{cases}$



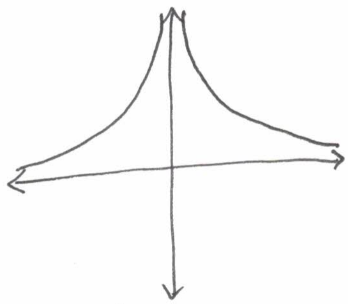
$y = x^n$, n even



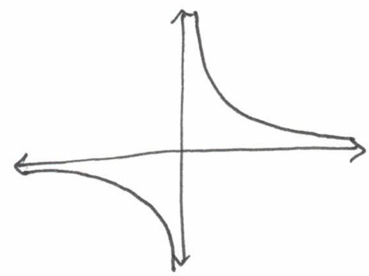
$y = x^n$, n odd

* $\lim_{x\to\infty} \frac{1}{x^n} = 0$

* $\lim_{x\to-\infty} \frac{1}{x^n} = 0$



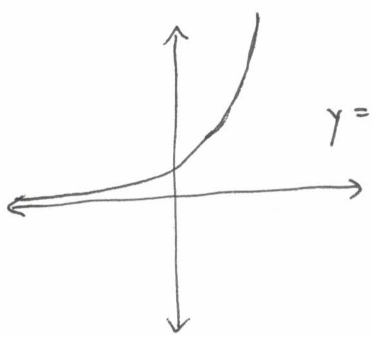
$y = \frac{1}{x^n}$, n even



$y = \frac{1}{x^n}$, n odd

$\lim_{x\to\infty} e^x = \infty$

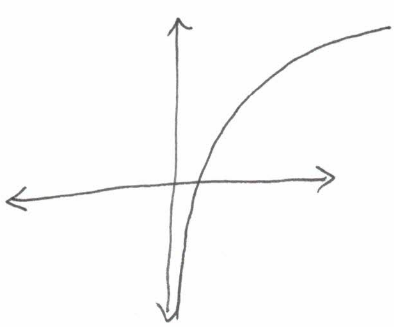
$\lim_{x\to-\infty} e^x = 0$



$y = e^x$

* We often rely on the graphs of parent functions to compute limits at infinity.

$\lim_{x\to\infty} \ln(x) = \infty$

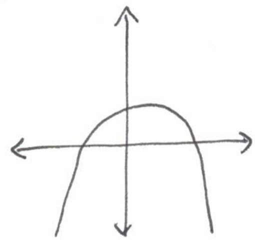


$y = \ln(x)$

Example 2. Compute $\lim_{x\to-\infty}(x-3x^2)$.

Parent function: $y = x^2$

- flipped over x-axis

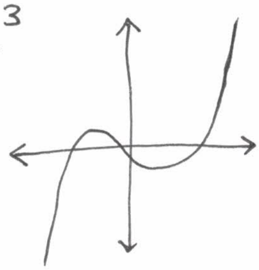


So, $\boxed{\lim_{x\to-\infty}(x-3x^2) = -\infty.}$

Example 3. Compute $\lim_{x\to\infty}(5x^3-2x^2)$.

Parent function : $y = x^3$

- no flips

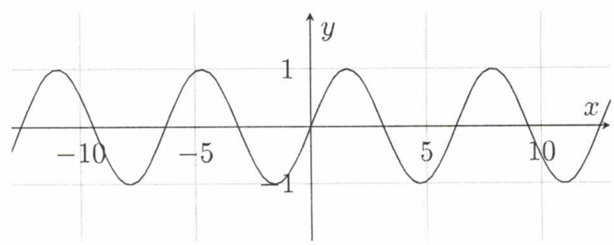


So, $\boxed{\lim_{x\to\infty}(5x^3-2x^2) = \infty.}$

Example 4.

$\lim_{x\to\infty}\sin(x) =$ DNE

b/c the y-values oscillate between -1 and 1 forever.



Note: On some problems, WebAssign uses the following directions: *Find the limit, if it exists. (If an answer does not exist, enter DNE.)* On these problems, WebAssign will accept "DNE" or the appropriate choice of "∞" or "$-\infty$". If this problem was on a test or quiz, for full credit you should answer with the appropriate "∞" or "$-\infty$".

Important Fact: If $\lim_{x\to\infty} g(x) = \pm\infty$ and $\lim_{x\to\infty} f(x) = L$ for some constant L, then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = 0$$

In other words:

$$\frac{\text{goes to a \#}}{\text{goes to } \infty \text{ or } -\infty} = \text{goes to } 0$$

$\frac{\text{goes to } \pm\infty}{\text{goes to } \pm\infty}$ or $\frac{\text{goes to } 0}{\text{goes to } 0}$ means we have to do algebra!

Example 5. Compute $\displaystyle\lim_{x\to-\infty}\frac{x^2}{x-1}$. $\left(\frac{\infty}{-\infty} \leftarrow \text{have to do more work.}\right)$

* Factor out the largest power of x from both the numerator + denominator.

$$\lim_{x\to-\infty}\frac{x^2}{x-1} = \lim_{x\to-\infty}\frac{x^2}{x\left(1-\frac{1}{x}\right)} = \lim_{x\to-\infty}\frac{\overbrace{x}^{\to-\infty}}{1-\underbrace{\frac{1}{x}}_{\to 0}} = \boxed{-\infty}.$$

(Instead of factoring, we could have multiplied by $\frac{\frac{1}{x}}{\frac{1}{x}}$.)

Example 6. Find the horizontal asymptote(s) of $f(x) = \dfrac{3x^2+20x}{4x^2+9}$.

* Need to compute $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.

① $$\lim_{x\to-\infty}\frac{3x^2+20x}{4x^2+9} = \lim_{x\to-\infty}\frac{x^2\left(3+\frac{20}{x}\right)}{x^2\left(4+\frac{9}{x^2}\right)} = \lim_{x\to-\infty}\frac{3+\overbrace{\frac{20}{x}}^{\to 0}}{4+\underbrace{\frac{9}{x^2}}_{\to 0}}$$

$$= \frac{3}{4}$$

Using other way for $\lim_{x\to\infty} f(x)$.

② $$\lim_{x\to\infty}\frac{3x^2+20x}{4x^2+9}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty}\frac{3+\overbrace{\frac{20}{x}}^{\to 0}}{4+\underbrace{\frac{9}{x^2}}_{\to 0}} = \frac{3}{4}$$

HAs of f: $\boxed{y=\frac{3}{4}}$

Example 7. Compute $\displaystyle\lim_{x\to\infty}\frac{2-7x}{3x^4+2x^2+1}$.

$$= \lim_{x\to\infty}\frac{x\left(\frac{2}{x}-7\right)}{x^4\left(3+\frac{2}{x^2}+\frac{1}{x^4}\right)} = \lim_{x\to\infty}\frac{\overbrace{\frac{2}{x}}^{\to 0}-7}{\underbrace{x^3}_{\to\infty}\left(3+\underbrace{\frac{2}{x^2}}_{\to 0}+\underbrace{\frac{1}{x^4}}_{\to 0}\right)} = \boxed{0}$$

Example 8. Compute $\lim\limits_{x\to-\infty} \dfrac{12x+25}{\sqrt{16x^2+100x+500}}$.

$$= \lim_{x\to-\infty} \frac{x\left(12+\frac{25}{x}\right)}{\sqrt{x^2\left(16+\frac{100}{x}+\frac{500}{x^2}\right)}} = \lim_{x\to-\infty} \frac{x\left(12+\frac{25}{x}\right)}{\sqrt{x^2}\sqrt{16+\frac{100}{x}+\frac{500}{x^2}}}$$

* Note: $\sqrt{x^2} = |x|$, So if $x<0$, then $\sqrt{x^2} = -x$.

Since $x\to-\infty$, we can assume $x<0$.

$$= \lim_{x\to-\infty} \frac{x\left(12+\frac{25}{x}\right)}{-x\sqrt{16+\frac{100}{x}+\frac{500}{x^2}}} = \lim_{x\to-\infty} \frac{12+\frac{25}{x}}{-\sqrt{16+\frac{100}{x}+\frac{500}{x^2}}}$$

$\frac{25}{x}\to 0$, $\frac{100}{x}\to 0$, $\frac{500}{x^2}\to 0$

$$= \frac{12}{-\sqrt{16}}$$

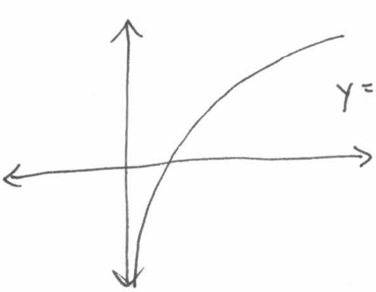
$$= \frac{12}{-4}$$

$$= \boxed{-3}$$

Example 9. Compute $\lim\limits_{x\to\infty} \ln\left(\dfrac{x^2}{x-1}\right)$.

$$= \lim_{x\to\infty} \ln\left(\frac{x^2}{x\left(1-\frac{1}{x}\right)}\right) = \lim_{x\to\infty} \ln\left(\frac{x}{1-\frac{1}{x}}\right) = \boxed{\infty}$$

$x\to\infty$, $\frac{1}{x}\to 0$, $\underbrace{1-\frac{1}{x}}_{\to\infty}$



Horizontal asymptotes vs. vertical asymptotes

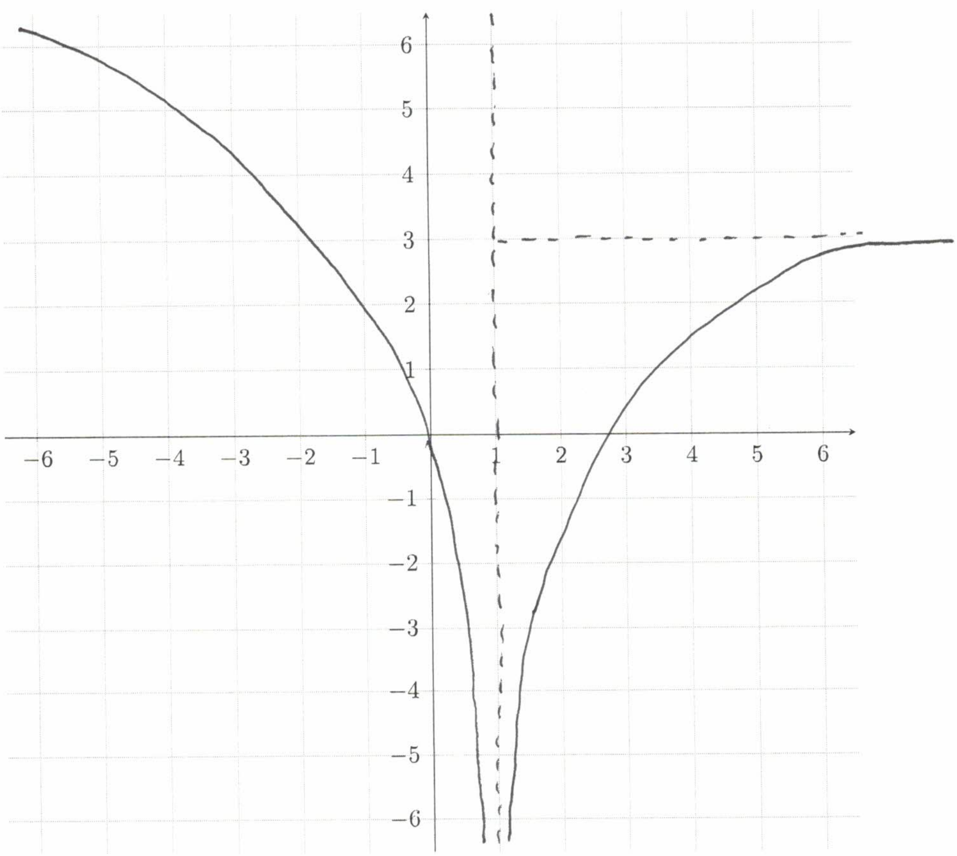
$\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$	$\lim_{x\to a} f(x) = \pm\infty$ (or DNE)
$y = L$	$x = a$

If $\lim_{x\to 3} f(x) = \infty$, then f has a VA at $x = 3$.

If $\lim_{x\to\infty} f(x) = 7$, then f has a HA at $y = 7$.

Example 10. Sketch the graph of a function f that satisfies all of the following conditions.

- $\lim_{x\to-\infty} f(x) = \infty$ left side goes up
- $\lim_{x\to 1} f(x) = -\infty$ VA at 1 (both sides down)
- $\lim_{x\to\infty} f(x) = 3$ HA at 3 (on right side)



Extra Practice:

1. Assume that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to L} g(x) = \infty$.

   Which of the following statements are correct?

   (a) $x = L$ is a vertical asymptote of g.
   (b) $y = L$ is a horizontal asymptote of g.
   (c) $x = L$ is a vertical asymptote of f.
   (d) $y = L$ is a horizontal asymptote of f.

2. Compute the following limits.

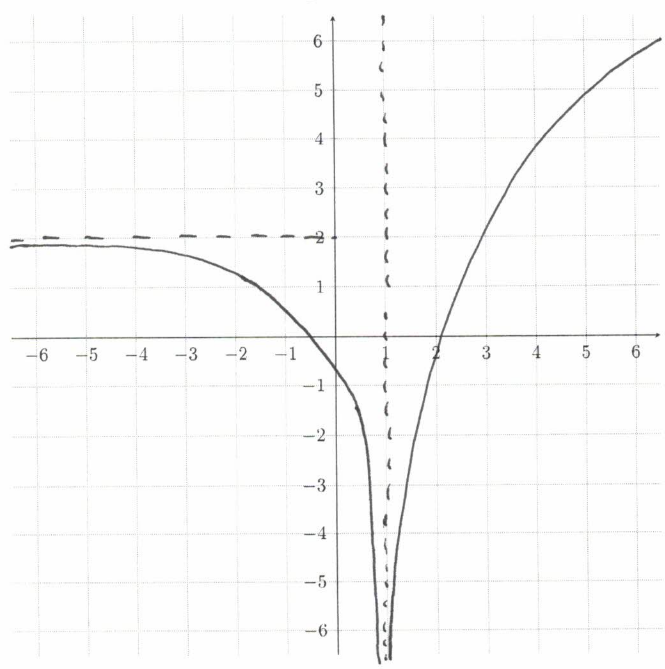
   - $\lim_{x\to\infty} \cos\left(\underbrace{\frac{1}{x}}_{\to 0}\right) = \cos(0) = \boxed{1}$

   - $\lim_{x\to 0} \cos\left(\underbrace{\frac{1}{x}}_{\to \pm\infty}\right) = \boxed{\text{DNE}}$

     ($-\infty$ from left, ∞ from right)

3. Sketch the graph of a function f that satisfies all of the following conditions.

   - $\lim_{x\to -\infty} f(x) = 2$ HA at 2 on left
   - $\lim_{x\to 1} f(x) = -\infty$ VA at 1 (both sides down)
   - $\lim_{x\to \infty} f(x) = \infty$ right side goes up



4. Compute the following limits.

- $$\lim_{x\to\infty} \frac{3x^5+x^2-2}{6x^2-5x^5+1} = \lim_{x\to\infty} \frac{x^5\left(3+\frac{1}{x^3}-\frac{2}{x^5}\right)}{x^5\left(\frac{6}{x^3}-5+\frac{1}{x^5}\right)} = \lim_{x\to\infty} \frac{3+\underbrace{\frac{1}{x^3}}_{\to 0}-\underbrace{\frac{2}{x^5}}_{\to 0}}{\underbrace{\frac{6}{x^3}}_{\to 0}-5+\underbrace{\frac{1}{x^5}}_{\to 0}}$$

$$= \boxed{-\frac{3}{5}}$$

- $$\lim_{x\to\infty} \frac{\sqrt{4x^4+5x+2}}{6x^2+5} = \lim_{x\to\infty} \frac{\sqrt{x^4\left(4+\frac{5}{x^3}+\frac{2}{x^4}\right)}}{x^2\left(6+\frac{5}{x^2}\right)} = \lim_{x\to\infty} \frac{x^2\sqrt{4+\frac{5}{x^3}+\frac{2}{x^4}}}{x^2\left(6+\frac{5}{x^2}\right)} =$$

$$= \lim_{x\to\infty} \frac{\sqrt{4+\underbrace{\frac{5}{x^3}}_{\to 0}+\underbrace{\frac{2}{x^4}}_{\to 0}}}{6+\underbrace{\frac{5}{x^2}}_{\to 0}} = \frac{\sqrt{4}}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

- $$\lim_{x\to-\infty} \frac{x^4+7}{\sqrt{5x^6+4x+9}} = \lim_{x\to-\infty} \frac{x^4\left(1+\frac{7}{x^4}\right)}{\sqrt{x^6\left(5+\frac{4}{x^5}+\frac{9}{x^6}\right)}} = \lim_{x\to-\infty} \frac{x^4\left(1+\frac{7}{x^4}\right)}{-x^3\sqrt{5+\frac{4}{x^5}+\frac{9}{x^6}}} =$$

$$= \lim_{x\to-\infty} \frac{\underbrace{x}_{\to -\infty}\left(1+\underbrace{\frac{7}{x^4}}_{\to 0}\right)}{-\sqrt{5+\underbrace{\frac{4}{x^5}}_{\to 0}+\underbrace{\frac{9}{x^6}}_{\to 0}}} = \frac{-\infty}{-\sqrt{5}} = \boxed{\infty}$$