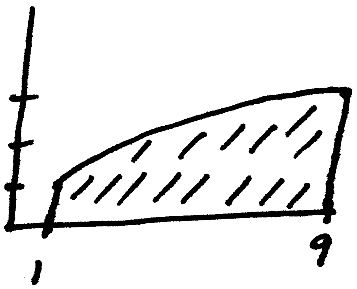
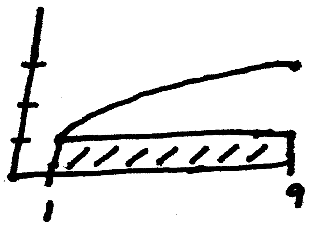
Graphically an integral from $x=a$ to $x=b$ $\left(\int_a^b f(x)\,dx \right)$ gives the area between the x-axis and $f(x)$.

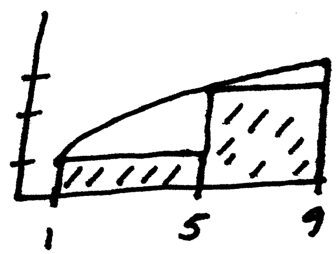
1.) Find area under $y=\sqrt{x}$ from $x=1$ to $x=9$



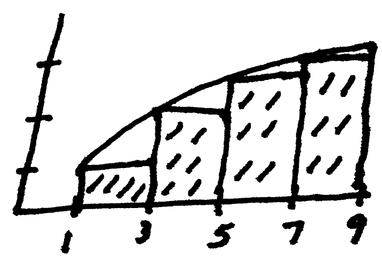
Since area is irregular we can use rectangles to approximate it.



$L_1 = 8 \cdot f(1) = 8(1) = 8$



$L_2 = 4\left[f(1) + f(5)\right]$
$= 4\left[1 + \sqrt{5}\right] \approx 12.944$



$\Delta x = \frac{9-1}{4}$

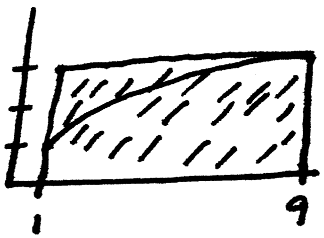
$\Delta x = 2$

$L_4 = 2\left[f(1) + f(3) + f(5) + f(7)\right]$

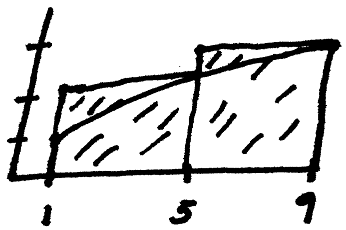
$L_4 = 2\left[1 + \sqrt{3} + \sqrt{5} + \sqrt{7}\right] \approx 15.228$

L_n gives exact answer as $n \to \infty$.

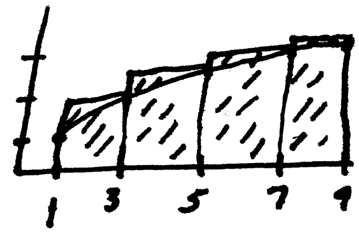
Notice, though, each L_n approximation $<$ Area (for any increasing curve)



$R_1 = 8 \cdot f(9)$
$= 8[3] = 24$



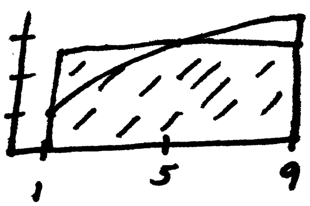
$R_2 = 4\,[f(5) + f(9)]$
$R_2 = 4[\sqrt{5} + 3] \approx 20.944$



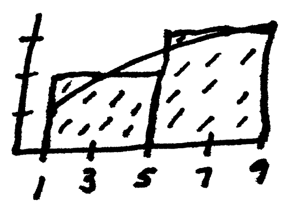
$R_4 = 2\,[f(3) + f(5) + f(7) + f(9)]$
$R_4 = 2[\sqrt{3} + \sqrt{5} + \sqrt{7} + 3] \approx 19.228$

R_n gives <u>exact</u> answer as $n \to \infty$

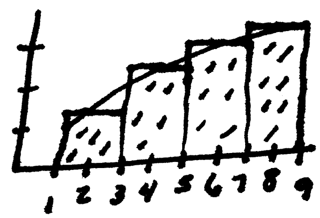
Each R_n approximation > Area (for any increasing curve)



$M_1 = 8 \cdot f(5)$
$= 8\sqrt{5} \approx 17.889$



$M_2 = 4\,[f(3) + f(7)]$
$M_2 = 4[\sqrt{3} + \sqrt{7}] \approx 17.511$

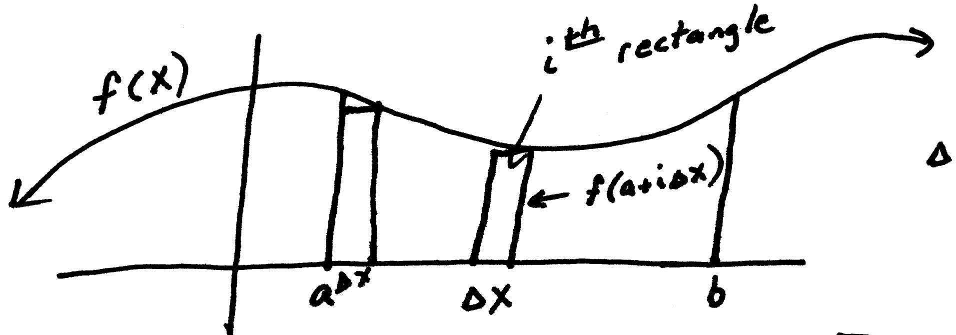


$M_4 = 2\,[f(2) + f(4) + f(6) + f(8)]$
$M_4 = 2\,[\sqrt{2} + 2 + \sqrt{6} + \sqrt{8}] \approx 17.384$

Again, M_n gives <u>exact</u> answer as $n \to \infty$

As $n \to \infty$, $L_n = M_n = R_n = \underset{\text{sum}}{\int_1^9} \underset{\text{height}}{\sqrt{x}}\, \underset{\text{width}}{dx} = \frac{2}{3} x^{3/2} \Big|_1^9 = \frac{2}{3}\left[9^{3/2} - 1^{3/2}\right] = \frac{2}{3}(26) = \frac{52}{3}$
≈ 17.333

General Case of finding area under a curve, $f(x)$, on $[a, b]$



$\Delta x = \frac{b-a}{n}$

$L_n = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(a+i\Delta x)\,\Delta x$

$R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\,\Delta x$

$M_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+(i-\tfrac{1}{2})\Delta x)\,\Delta x$

All give $\int_a^b f(x)\,dx$

In order to find infinite summations we need some formulas...

Carl Gauss (at age 10 in 1787) proved the following:

Prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\rightarrow$ Let $S = 1+2+3+\dots(n-2)+(n-1)+n$

$S = n+(n-1)+(n-2)+\dots 3+2+1$

then $2S = (n+1)+(n+1)+\dots (n+1)$

hence $2S = n(n+1)$

$S = \frac{n(n+1)}{2}$

So... $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Similarly we can show $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

2.) Working with summations

a.) $\sum_{i=1}^{5} 3 = 3+3+3+3+3 = 3(5) = \boxed{15}$

b.) $\sum_{j=0}^{5} 3 = 3+3+3+3+3+3 = 3(6) = \boxed{18}$

c.) $\sum_{k=2}^{4} k^3 = 2^3 + 3^3 + 4^3 = \boxed{99}$

d.) $\sum_{j=3}^{4} \sin\left(\frac{j\pi}{2}\right) = \sin\frac{3\pi}{2} + \sin\frac{4\pi}{2} = \boxed{-1}$

e.) $\sum_{i=2}^{4} \frac{1}{i-1} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \boxed{\frac{11}{6}}$

f.) $\sum_{j=0}^{3} 3^j = 3^0 + 3^1 + 3^2 + 3^3 = \boxed{40}$

g.) $\sum_{j=101}^{200} j = \sum_{j=1}^{200} j - \sum_{j=1}^{100} j = \frac{200(201)}{2} - \frac{100(101)}{2} = \boxed{15{,}050}$

3.) Write in summation notation

a.) $3^5 + 4^5 + 5^5 + 6^5 = \sum_{i=3}^{6} i^5$

b.) $\sqrt{1+1^3} + \sqrt{2+2^3} + \dots + \sqrt{n+n^3} = \sum_{i=1}^{n} \sqrt{i+i^3}$

c.) $e^{\pi} + e^{\frac{\pi}{2}} + e^{\frac{\pi}{3}} + \dots + e^{\frac{\pi}{n}} = \sum_{i=1}^{n} e^{\frac{\pi}{i}}$

Find formula for R_n and compute area under graph

may drop index (assumed to be 1, to n)

4. $f(x) = 2x+7$, $[3,6]$ $\rightarrow$ $\Delta x = \frac{6-3}{n} = \frac{3}{n}$

$R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(3+\frac{3i}{n})(\frac{3}{n}) = \lim_{n\to\infty} \frac{3}{n}\sum f(3+\frac{3i}{n})$

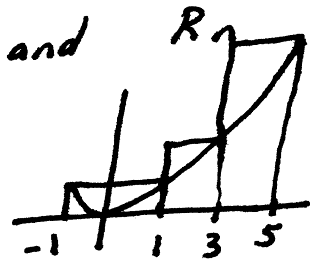
$R_n = \lim_{n\to\infty} \frac{3}{n}\sum [2(3+\frac{3i}{n})+7] = \lim_{n\to\infty} \frac{3}{n}\sum [13+\frac{6i}{n}] = \lim_{n\to\infty} \frac{3}{n}\sum 13 + \frac{3}{n}\sum\frac{6i}{n}$

$R_n = \lim_{n\to\infty} \frac{3}{n}(13n) + \frac{3}{n}(\frac{6}{n})\sum i = \lim_{n\to\infty} 39 + \frac{18}{n^2}(\frac{n(n+1)}{2})$

$R_n = \lim_{n\to\infty} 39 + \frac{18n^2+\cdots}{2n^2} = 39 + 9 = 48$

5. $f(x) = x^2$, $[-1,5]$ Find R_3 and R_n

For R_3, $\Delta x = \frac{5-(-1)}{3} = \frac{6}{3} = 2$



$R_3 = 2\cdot[f(1)+f(3)+f(5)]$

$R_3 = 2[1+9+25] = 70$

$R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(-1+\frac{6i}{n})(\frac{6}{n})$

$R_n = \lim_{n\to\infty} \frac{6}{n}\sum(-1+\frac{6i}{n})^2 = \lim_{n\to\infty} \frac{6}{n}\sum[1-\frac{12i}{n}+\frac{36i^2}{n^2}]$

$R_n = \lim_{n\to\infty} \frac{6}{n}\sum 1 - \frac{6}{n}\sum\frac{12i}{n} + \frac{6}{n}\sum\frac{36i^2}{n^2} = \lim_{n\to\infty} \frac{6}{n}(n) - \frac{72}{n^2}\sum i + \frac{216}{n^3}\sum i^2$

$R_n = \lim_{n\to\infty} 6 - \frac{72}{n^2}(\frac{n(n+1)}{2}) + \frac{216}{n^3}(\frac{n(n+1)(2n+1)}{6}) = \lim_{n\to\infty} 6 - \frac{72n^2+\cdots}{2n^2} + \frac{432n^3+\cdots}{6n^3}$

$R_n = 6 - 36 + \frac{432}{6} = 42$

6. $f(x) = x^3 + 2x^2$, $[0,3]$ $\quad \Delta x = \frac{3-0}{n} = \frac{3}{n}$

$$R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(0 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \lim_{n\to\infty} \frac{3}{n} \sum f\left(\frac{3i}{n}\right)$$

$$R_n = \lim_{n\to\infty} \frac{3}{n} \sum \left[\left(\frac{3i}{n}\right)^3 + 2\left(\frac{3i}{n}\right)^2\right] = \lim_{n\to\infty} \frac{3}{n} \sum \left[\frac{27i^3}{n^3} + \frac{18i^2}{n^2}\right]$$

$$R_n = \lim_{n\to\infty} \frac{3}{n}\left(\frac{27}{n^3}\right)\sum i^3 + \frac{3}{n}\left(\frac{18}{n^2}\right)\sum i^2$$

$$R_n = \lim_{n\to\infty} \frac{81}{n^4}\left(\frac{n^2(n+1)^2}{4}\right) + \frac{54}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \lim_{n\to\infty} \frac{81n^4 + \cdots}{4n^4} + \frac{108n^3 + \cdots}{6n^3}$$

$$R_n = \frac{81}{4} + \frac{108}{6} = \frac{153}{4} \text{ or } 38.25$$