Arclength and Surface Area

$y = f(x)$

An infintesimally small portion of arclength would be $\sqrt{(dx)^2+(dy)^2}$

factor out $(dx)^2$

$$\sqrt{(dx)^2+(dy)^2} = \sqrt{(dx)^2\left(1+\left(\frac{dy}{dx}\right)^2\right)} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$\therefore$ Arclength from $x=a$ to $x=b$ is $\boxed{s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$

Basic example...

1. $y = 9-3x$, $[1,3]$ $\rightarrow$ $s = \int_1^3 \sqrt{1+9}\,dx = \sqrt{10}\,x\Big|_1^3$

$\frac{dy}{dx} = -3 \rightarrow \left(\frac{dy}{dx}\right)^2 = 9$ $\qquad s = 3\sqrt{10} - \sqrt{10} = \boxed{2\sqrt{10}}$

More Fun (and tricky) example...

2. $y = \frac{1}{3}x^{3/2} - x^{\frac{1}{2}}$, $[2,8]$

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}x - \frac{1}{2} + \frac{1}{4}x^{-1}$$

$$1+\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}x + \frac{1}{2} + \frac{1}{4}x^{-1} = \left(\frac{1}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}\right)^2$$

Must recognize to work problem!

$$s = \int_2^8 \sqrt{\left(\frac{1}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}\right)^2}\,dx = \int_2^8 \left(\frac{1}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}\right)dx$$

$$s = \left[\frac{1}{3}x^{3/2} + x^{\frac{1}{2}}\right]_2^8 = \left(\frac{1}{3}(2\sqrt{2})^3 + 2\sqrt{2}\right) - \left(\frac{1}{3}(\sqrt{2})^3 + \sqrt{2}\right)$$

$$= \frac{16\sqrt{2}}{3} + \frac{6\sqrt{2}}{3} - \frac{2\sqrt{2}}{3} - \frac{3\sqrt{2}}{3} = \frac{17\sqrt{2}}{3}$$

Arclength and Surface Area p.2

3. $y = \ln(\cos x)$, $[0, \frac{\pi}{4}]$ $\rightarrow$ $s = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx$

$\frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) = -\tan x$

$s \overset{\#13}{=} \left[\ln|\sec x + \tan x|\right]_0^{\pi/4}$

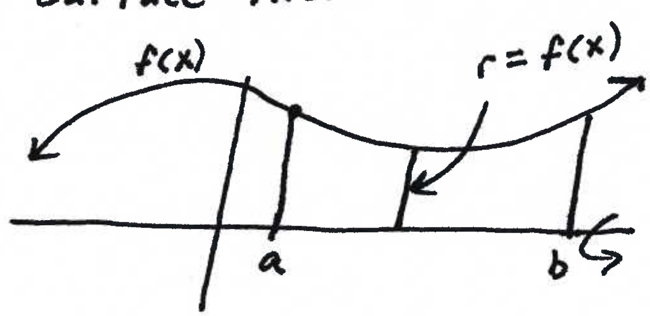
$\left(\frac{dy}{dx}\right)^2 = \tan^2 x$

$s = \ln|\sqrt{2}+1| - \ln|1+0|$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$

$s = \ln(\sqrt{2}+1)$

Surface Area of revolution



Surface Area of rotated solid would be $2\pi r$ (arc length)

$$SA = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx$$

Find surface area of revolution

4. $y = 4x + 3$, $[0, 1]$ $\rightarrow$ $SA = 2\pi \int_0^1 (4x+3)\sqrt{1+4^2}\, dx$

$\frac{dy}{dx} = 4$

$SA = 2\sqrt{17}\,\pi \int_0^1 (4x+3)\, dx = 2\sqrt{17}\,\pi \left[2x^2 + 3x\right]_0^1$

$= 2\sqrt{17}\,\pi\,[5 - 0] = 10\pi\sqrt{17}$

5. $y = x^3$, $[-1, 7]$ $\rightarrow$ $SA = 2\pi \int_{-1}^{7} x^3\sqrt{1+9x^4}\, dx$

$u = 1 + 9x^4$

$du = 36x^3\, dx$

$u(-1) = 10$

$u(7) = 21610$

$y' = 3x^2$

$(y')^2 = 9x^4$

$SA = \frac{2\pi}{36} \int_{10}^{2610} u^{\frac{1}{2}}\, du$

$SA = \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_{10}^{21610} = \frac{\pi}{27} u^{3/2}\Big|_{10}^{21610} = \frac{\pi}{27}\left[(21610)^{3/2} - (10)^{3/2}\right]$

≈ 369627.211