Appendix: Proof of Proposition 4

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Proposition. *Let H be a real symmetric random matrix satisfying that for any $X \in O(N)$, $P(X^THX) = P(H)$. Then, the most general p.d.f. satisfying the factorization property $P(H) = \prod_{1\leqslant j\leqslant k\leqslant N} f_{jk}(h_{jk})$ for f_{jk} differentiable is*

$$P(H) = Ce^{-a\,\mathrm{tr}(H^2)+b\,\mathrm{tr}(H)}.$$

Proof. First we begin by considering X to be a permutation matrix. For example, we could take X to be the matrix we obtain if we permute the j-th and k-th row (or column) of the identity matrix. Therefore, X^THX is the matrix we obtain by permuting the j-th and k-th row and column of H. Therefore, $P(X^THX) = P(H)$ together with the factorization property, implies that $f_{jl} = f_{kl}$ for all $l = 1, \ldots, N$ (here, we are abusing notation and taking $f_{jl} = f_{lj}$ if $j > l$). In other words, with these permutations we can permute any two elements of the diagonal and any two off-diagonal positions, and hence, they must respectively share the same distribution. We then have that for every $1 \leqslant j < k \leqslant N$, $f_{jj} = f$ and $f_{jk} = g$ for some f and g.

Now, let us take as our orthogonal matrix

$$X = \begin{pmatrix} \cos\varepsilon & \sin\varepsilon & 0 & \ldots & 0 \\ -\sin\varepsilon & \cos\varepsilon & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix},$$

with $|\varepsilon| \ll 1$. We can Taylor expand this matrix and get that

$$X = \begin{pmatrix} 1 & \varepsilon & 0 & \ldots & 0 \\ -\varepsilon & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} + O(\varepsilon^2).$$

Then, we have that

$$\tilde{H} = X^THX = \begin{pmatrix} h_{11} - 2\varepsilon h_{12} & h_{12} + \varepsilon(h_{11} - h_{22}) & h_{13} - \varepsilon h_{23} & \ldots & h_{1N} - \varepsilon h_{2N} \\ h_{12} + \varepsilon(h_{11} - h_{22}) & h_{22} + 2\varepsilon h_{12} & h_{23} + \varepsilon h_{13} & \ldots & h_{2N} + \varepsilon 1 h_{1N} \\ h_{13} - \varepsilon h_{23} & h_{23} + \varepsilon h_{13} & h_{33} & \ldots & h_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{1N} - \varepsilon h_{2N} & h_{2N} + \varepsilon 1 h_{1N} & h_{3N} & \ldots & h_{NN} \end{pmatrix} + O(\varepsilon^2)$$

As $P(\tilde{H}) = P(H)$, it must be

$$\prod_{j=1}^{N} f(h_{jj}) \prod_{1\leqslant j<k\leqslant N} g(h_{jk}) = \prod_{j=1}^{N} f(\tilde{h}_{jj}) \prod_{1\leqslant j<k\leqslant N} g(\tilde{h}_{jk}).$$

Since this should be true no matter the value of ε, if we expand the RHS of the equation, all the coefficients of the expansion but the constant term should vanish. In particular, the first order coefficient, which is given by

$$P(H)\left(-2\frac{h_{12}f'(h_{11})}{f(h_{11})}+2\frac{h_{12}f'(h_{22})}{f(h_{22})}+\frac{(h_{11}-h_{22})g'(h_{12})}{g(h_{12})}-\sum_{j=3}^{N}\left(\frac{h_{2j}g'(h_{1j})}{g(h_{1j})}-\frac{h_{1j}g'(h_{2j})}{g(h_{2j})}\right)\right),$$

vanishes. Hence (assuming $P(H)\neq 0$),

$$-2\frac{h_{12}f'(h_{11})}{f(h_{11})}+2\frac{h_{12}f'(h_{22})}{f(h_{22})}+\frac{(h_{11}-h_{22})g'(h_{12})}{g(h_{12})}-\sum_{j=3}^{N}\left(\frac{h_{2j}g'(h_{1j})}{g(h_{1j})}-\frac{h_{1j}g'(h_{2j})}{g(h_{2j})}\right)=0. \tag{1}$$

This holds for any H. Thus, by separation of variables the last term must vanish, which means

$$\frac{h_{2j}g'(h_{1j})}{g(h_{1j})}=\frac{h_{1j}g'(h_{2j})}{g(h_{2j})}.$$

Letting $x=h_{1j}$ and $y=h_{2j}$ we get

$$\frac{g'(x)}{xg(x)}=\frac{g'(y)}{yg(y)}.$$

By further separation of variables, there must exist $a\in\mathbb{R}$ such that

$$\frac{g'(x)}{xg(x)}=-4a.$$

Hence, solving the differential equation

$$g(x)=C_1e^{-2ax^2}.$$

Now, if we come back to (1) we get

$$h_{12}\left(-2\frac{f'(h_{11})}{f(h_{11})}+2\frac{f'(h_{22})}{f(h_{22})}-4(h_{11}-h_{22})a\right)=0.$$

Taking now $x=h_{11}$ and $y=h_{22}$, we have

$$\frac{f'(x)}{f(x)}+2xa=\frac{f'(y)}{f(y)}+2ya.$$

Once more, if we use separation of variables we can conclude that there exists $b\in\mathbb{R}$ with

$$\frac{f'(x)}{f(x)}+2xa=-b.$$

The solution to this ODE is

$$f(x)=C_2e^{-ax^2-bx}.$$

Then, using these expressions for f and g, we can conclude

$$P(H)=\prod_{1\leqslant j\leqslant N}f(h_{jj})\prod_{1\leqslant j<k\leqslant N}g(h_{jk})=Ce^{-a\sum_{1\leqslant j,k\leqslant N}h_{jk}^2-b\sum_{1\leqslant j\leqslant N}h_{ij}}=Ce^{-a\operatorname{tr}H^2-b\operatorname{tr}H}.$$

$\square$

References

[1] P.J. Forrester. *Log-Gases and Random Matrices (LMS-34)*. Princeton University Press, 2010.