If you can't solve a problem, then
there is an easier problem you
can't solve: find it.
*Conway misquoting Pólya*

# TRIVIAL PROBLEMS AND MATHEMATICAL TRIVIA

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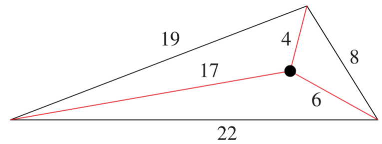
Abstract. Open problems in mathematics drive research and motivate the discovery of new tools and ideas. In this talk, we will run through open problems chosen from diverse areas of mathematics and give some context for why they are interesting or important. We focus on problems that are perhaps underappreciated outside their immediate field, and avoid all the usual suspects with which you are already familiar.

A key skill any aspiring mathematician must develop is a healthy level of scepticism. To this end, mixed in with the open problems we discuss will be a scattering of known results with short, not-too-technical proofs. It is up to the audience to identify and provide proofs for these – dare I say trivial? – results.

These problems were largely compiled by scouring the internet. Some were suggested by various people in and around the department. Ollie Thakar and Dhilan Lahoti each contributed significant problem lists. I take full responsibility for any inaccuracies or out-of-date claims.

## 1. Quick intro problems

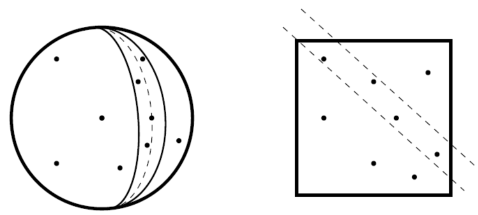
1. Does there exist a nontrivial element of the mapping class group of $S^4$? (i.e. Is there a diffeomorphism of $S^4$ which is not smoothly isotopic to the identity?)
2. Write down a coordinate expression for a Ricci-flat Kahler metric on a K3 surface (Yau's proof of the Calabi conjecture and deformation theory tell us there's a non-empty 58-dimensional space of such metrics!)
3. Let $R$ be a UFD of of finite Krull dimension. Is $\dim R[x] = \dim R + 1$?
4. (Euler bricks) Is there a rectangular prism whose edges, face diagonals, and space/interior diagonal all have integer lengths? The smallest prism which satisfies all the conditions except the interior diagonal is (44, 117, 240) discovered by Halcke in 1719.
   - (A couple more open problems about rational distances)
   - ** Is there a point in the unit square with integer distances to all four vertices? (Progress: Conway-Guy found an infinite family where 3 of the 4 distances are rational, $(s^2+b^2-a^2)^2+(s^2+b^2-c^2)^2 = (2bs)^2$, with $a, b, c$ three integer distances and $s$ the side length of the square. Then scale down to the unit square.)
   - (2001, Pegg) There is a scalene triangle of side lengths 8, 19, 22 with a point whose distances to the three vertices are all integers: 4, 6, 17.
   - ** (Klee-Wagon, 1991) Is there a dense subset of the plane where all the pairwise distances between points are rational?
5. Is it possible to find three corners of a square that form an equilateral triangle? (Ok, this one is fake. In two dimensions obviously impossible. In 3 dimensions there is a tetrahedron formed from 4 vertices of a cube. In general, is there a subset of vertices of the $n$-cube which forms a regular $n$-simplex? Conjecturally, this holds iff $n \equiv 3 \mod 4$. Easy to see that the congruence condition is necessary, existence of solutions has been shown up to $n = 663$ thus far. Equivalent to the Hadamard conjecture, which posits existence of Hadamard matrices of every order $4k$. Smallest order for which no Hadamard matrix is yet known is 668. Hadamard matrix: all entries are $\pm 1$, any two columns are orthogonal.)
6. Is the positive root of $x^{x^x} = 2$ algebraic? Is the positive root of $x^{x^{x^x}} = 2$ rational? (Rationality also unknown for all higher iterated exponentiations.) Is $e^{e^{e^{e^e}}}$ an integer?
7. (codim 1 manifold factor problem) If $X \times Y$ is a manifold, must $X \times \mathbb{R}^1$ be a manifold? (Note if $X \times Y$ is a manifold, $X$ need not be a manifold. The dogbone space (Bing, 1957) is a non-manifold whose product with $\mathbb{R}^1$ is $\mathbb{R}^4$.)
8. Compute the unknotting number of every 8-crossing knot. (In particular, the connected sum of the positive (2,3)-torus knot and the negative (2,5)-torus knot has an unknown unknotting number: is it 2 or 3?)
9. (Brocard's problem) How many integers $n$ are there such that $n! + 1$ is a perfect square? Only 3 are known, $n = 4, 5, 7$. Computational searches up to one quadrillion ($10^{15}$) have found no additional solutions.
10. How many isomorphism classes of graphs on 20 vertices are there? (The answer for 19 vertices was recently calculated – it has 35 digits.)




## 2. Problems with more context

11. Does every triangle have a periodic billiard ball trajectory?
    - Label the edges 1, 2, 3 in increasing order of length. Then a periodic billiard path has a combinatorial type described by the order in which it touches the edges.
    - (1775, Fagnano) Every acute triangle has a periodic billiard path of combinatorial type 123. (It's the orthic triangle: namely, drop a perpendicular from each vertex to the opposite side, and these three points define a triangle which is a periodic billiard path. To prove, note that the 3 vertices plus orthocentre plus bases of altitudes define 3 cyclic quadrilaterals, then move angles around.)
    - Every right triangle has a periodic billiard path of type 312321.
    - (1927, Birkhoff) A smooth convex curve has periodic billiard paths of every prime order. (Order = number of line segments.)
    - A periodic billiard path is called stable if after slightly perturbing the triangle it retains a periodic billiard path of the same type. Stability turns out to be a property of the combinatorial type.
    - (2007, Hooper) No right triangle has a stable periodic billiard path, acute triangles do.
    - A sufficiently small perturbation of an isosceles triangle has a periodic billiard path.
    - (2008, Schwartz) An obtuse triangle whose maximum angle is at most 100 degrees has a periodic billiard path. The authors think that their computational methods could stretch to 105 or perhaps even 110 degrees, but there is a difficult barrier to cross at 112.5 degrees ($5\pi/8$ radians).
    - (2018, Tokarsky-Garber-Marinov-Moore) Every obtuse triangle with maximal angle at most 112.3 degrees has a periodic billiard path.
12. (tensor-nilpotence) Call an object $R$ *tensor-nilpotent* of order $n$ if $R^{\otimes n} \simeq 0$ but $R^{\otimes(n-1)} \not\simeq 0$. In Spectra, what orders of nilpotence are possible?
    - (connective) Spectra are supposed to be analogous to abelian groups, so let's try to understand what happens there first.
    - We have $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \cong 0$ but $\mathbb{Q}/\mathbb{Z} \neq 0$, so there is an abelian group of nilpotence order 2.
    - There are no abelian groups of nilpotence order $\geq 3$; if $G^{\otimes n} \cong 0$ for $n \geq 3$ then actually $G \otimes G \cong 0$.
    - For the proof: Let $R$ be nilpotent. After basechanging to a field, $R \otimes F$ is a nilpotent vector space so must be 0. Since $R \otimes \mathbb{Q} = 0$, $R$ is torsion, so splits as a direct sum of its $p$-localisations $R_{(p)}$, each nilpotent. Working one prime at a time, $R_{(p)} \otimes \mathbb{F}_p = 0$ so every element is divisible by $p$, and thus $R_{(p)} \otimes R_{(p)} = 0$. In Spectra, $X \otimes H\mathbb{Q} = 0$ so the arithmetic fracture square tells us nilpotent $X$ decomposes as the product of all its $p$-localisations. But now the condition $\mathbb{F}_p \otimes X = 0$ doesn't produce anything new, and we don't get a convenient characterisation of nilpotence for $p$-local spectra the way we do for $p$-groups.
    - In Spectra, $H\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{S}} H\mathbb{Q}/\mathbb{Z} \not\simeq 0$ (its $\pi$ is $\mathbb{Q}/\mathbb{Z}$, not 0). However, there is a nonzero spectrum $I$, the Brown-Comenetz dual of the sphere, such that $I \otimes I \simeq 0$.
    - No-one knows whether there exist nilpotent spectra of order $\geq 3$.
    - Note: $I$ is defined by the property that $[X, I] \simeq \hom(\pi_0 X, \mathbb{Q}/\mathbb{Z})$ for any spectrum $X$. Its existence is guaranteed by Brown representability.
13. (Motzkin-Schmidt problem) Take $n$ points in a square $[0,1]^2$. What is the minimal $\varepsilon$ such that there is guaranteed to exist a line $\ell$ with $\geq 3$ points lying in an $\varepsilon$-neighbourhood of $\ell$ (i.e. a strip of diameter $\varepsilon$, not radius)? There is an easy pigeonholing argument that shows $\varepsilon = 3/n$ works, but this trivial bound has never been improved.
    - Ok, actually that's a lie.
    - Pigeonhole principle says if we divide the square vertically into fewer than $n/2$ strips, at least one must contain 3 or more points. So need $n/2 - 1$ or fewer strips, each width is $\frac{2}{n-2}$.
    - The real conjecture is that $\varepsilon = o(1/n)$ is possible.
    - This result has been proven under the assumption that the points are somewhat evenly distributed (J. Beck) but no progress has been made in the general case.
    - Can pose the same problem on $S^2$ with neighbourhoods of great circles.
14. (Frank) There are infinitely many elliptic curves over $\mathbb{Q}$ of rank 2. What about rank 3?

- See David Zywina paper arXiv:2502.01957, the introduction gives sufficient context for this problem
- Ranks 0 and 1 it is known that there are infinitely many
- Loosely 'most' elliptic curves over $\mathbb{Q}$ are expected to have rank 0 or 1
- It has been known for a while that there are infinitely many of rank $\geq 2$
- 2 is the first specific integer $> 1$ for which it is known there are infinitely many



## 3. Small parts of big theorems

Let's talk about some conjectures which used to have important consequences, but which were circumvented by other results. These are all open problems which were initially posed because an affirmative answer would give a proof of some much more famous problem which was open at the time. However, all of those much more famous problems have since been solved using a different simplification and so these stepping-stone problems were mostly discarded and forgotten.

15. For primes $p$ and $q$, the expressions $\frac{p^q-1}{p-1}$ and $\frac{q^p-1}{q-1}$ are always coprime.
    - This is false, $p = 17$ and $q = 3313$ provides a counterexample. But there is a real version of this open problem:
    - (Feit-Thompson conjecture) Do there exist primes $p$ and $q$ so that $\frac{p^q-1}{p-1}$ divides $\frac{q^p-1}{q-1}$? Conjecturally no.
    - (Feit-Thompson theorem) Every finite group of odd order is solvable.
    - If The Feit-Thompson conjecture could be answered in the negative, it would allow a significant shortening of the proof of the theorem. (The original paper is 255 pages long.)
16. (Alice) Given a number field $F$, does there always exist an elliptic curve $E$ over $F$ such that $E(F)$ has rank one as an abelian group?
    - Context: Hilbert's 10th problem asks if there is an algorithm to determine whether an integer polynomial in several variables has solutions in the integers. We could ask the same thing for the ring of integers $O_K$ of any number field $K$.
    - Has a negative answer over $\mathbb{Q}$ by work of Davis-Putnam-Robinson and Matiyasevich.
    - Poonen and independently Cornelissen-Pheidas-Zahidi showed that if there is an elliptic curve defined over $\mathbb{Q}$ with
    $$\operatorname{rank} E(\mathbb{Q}) = \operatorname{rank} E(K) = 1,$$
    then Hilbert's 10th problem has a negative answer for $O_K$.
    - Some strengthenings, due to Shlapentokh, of this result: can replace $= 1$ with $> 0$, can restrict to the case of cyclic extensions of number fields and then to quadratic extensions
    - This eventually made it possible to show that Hilbert's 10th problem has a negative answer over every number field. (The problem was reduced to proving that for a quadratic extension of number fields $K/F$ there exists an abelian variety $A/F$ with $\operatorname{rank} A(F) = \operatorname{rank} A(K) > 1$.)
    - However, the question about elliptic curves of rank 1 remains open.
    - The paper arxiv:2501.18774 has all the context, it finishes the disproof of Hilbert's 10th problem over number fields
17. (Zeeman conjecture) Given a finite contractible 2-dimensional CW complex $K$, is $K \times [0, 1]$ necessarily collapsible?
    - Collapsible means: collapsing is a combinatorial operation you can perform on a simplicial complex to delete some faces, and a complex is collapsible if there is a sequence of collapses reducing it to a point. Collapsible implies contractible, but the converse is not true.
    - The Zeeman conjecture implies both the Poincaré conjecture (known to be true) and the Andrews-Curtis conjecture (open and widely(?) believed to be false).

- Andrews-Curtis conjecture is about balanced presentations of the trivial group. (Every presentation of the trivial group with the same number of generators as relations can be transformed into the trivial presentation via Nielsen transformations.)
- (Poincaré conjecture) Every 3-manifold which is closed, connected, and simply connected is homeomorphic to $S^3$.

## 4. Weird and wonderful

18. (Köthe conjecture) Let $R$ be a non-commutative ring. The sum of two nil left ideals is nil. A nil ideal is one whose elements are all nilpotent. Has been proven for right Noetherian rings.
19. (Singmaster's conjecture) There is a finite upper bound on the multiplicity of entries in Pascal's triangle, other than the number 1.
    - If such a bound exists, it could be as low as 8 – the number
    $$\binom{3003}{1} = \binom{78}{2} = \binom{15}{5} = \binom{14}{6}$$
    is the only one known to appear $\geq 8$ times.
    - There are infinitely many numbers which appear with each of the multiplicities $2, 3, 4$, and 6. It is not known whether any number appears 5 or 7 times. 2 is the only number which appears exactly once.
20. (Hopf) Every degree 1 self-map of a closed orientable manifold is a homotopy equivalence.
    - Since the map is degree 1 it is a homology equivalence (Poincaré duality gives injectivity, finite-dimensionality). We also know $\pi_1 f$ must be a surjection because otherwise $f$ factors through a non-trivial covering of $M$ and would then have degree $> 1$.
21. (Agoh-Guiga conjecture, $\sim$1950) $p > 1$ is prime if and only if
    $$\sum_{k=1}^{p-1} k^{p-1} \equiv -1 \mod p.$$
    The relation follows immediately from Fermat's Little Theorem when $p$ is prime, but the converse direction is unknown.
22. (Casas-Alvero) Let $k$ be a field (of characteristic 0). If a monic polynomial $f \in k[x]$ of degree $n$ has a divisor in common with each of its first $n-1$ derivatives, then $f(x) = (x-a)^n$. (Skip the hypothesis 'of characteristic zero' at first so it's false.)
23. (Schinzel-Sierpinski) Conjecture: Every $x \in \mathbb{Q}_{>0}$ can be represented in the form $x = \frac{p+1}{q+1}$ with $p$ and $q$ prime. (It is known that expressions of the form on the RHS generate a subgroup of $\mathbb{Q}_{>0}$ of index $\leq 3$.)
24. Is it possible to raise a transcendental number to a transcendental power and get a non-rational algebraic answer? (Set e.g. $x^y = \sqrt{2}$ and let $x$ range over all transcendental positive reals. Each equation has a solution in $y$, so we get a corresponding family of uncountably many choices of $y$. These cannot all be algebraic since there are countably many algebraic numbers, so some pair $(x, y)$ has both numbers transcendental.)
25. If $R$ is a commutative ring, a finite flat $R$-module is projective. (This one is false: for an elementary counterexample take $R = C^\infty(\mathbb{R})$ and $I$ to be the ideal of smooth functions vanishing on some open neighbourhood of 0. Then $R/I$ is finite flat but not projective. And $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ is a flat closed immersion which is not open.)
26. (Borsuk) Can every bounded $S \subseteq \mathbb{R}^n$ be partitioned into $n+1$ subsets each of strictly smaller diameter than $S$? What if $S$ is finite?
27. The *Lyusternik-Schnirelmann category* of a topological space $X$ is the smallest integer $k$ for which there exists an open covering of $X$ of size $k+1$ with each inclusion $U \hookrightarrow X$ being nullhomotopic, e.g. $cat(S^2) = 2 - 1 = 1$. Conjecturally, the (symplectic) Lie group $Sp(n)$ has $cat(Sp(n)) = 2n - 1$. This is true for $n = 1, 2, 3$ and open for $n \geq 4$. The conjecture was made after the $n = 2$ case was proven in the 1960s. I don't know whether anyone really thinks this is true.

## 5. Leave these until the end

28. Every scheme has a closed point. (This is false. It is true for any quasi-compact scheme.)

29. (Erdős distinct subset sums) Let $S$ be a set of $n$ integers where all the sums of subsets of $S$ are distinct. Give a lower bound on the largest element of $S$.
    - Powers of two produce a valid set $S$ of size $n$ with largest element $2^{n-1}$.
    - This is not the best you can do; heuristically, such sets have too many small numbers and it is more optimal to have several of the largest numbers in $S$ be close together.
    - Can anyone provide an example that is better than powers of 2? (e.g. 3, 5, 6, 7 is better than 1, 2, 4, 8).
    - Erdős conjectured that $a_n \geq c2^n$ for some fixed constant $c$. Best known bound is $a_n \geq c\frac{2^n}{\sqrt{n}}$ with $c \sim \sqrt{2/\pi}$.
    - Elkies reformulated the problem in terms of Fourier analysis.
30. What is the rational homology of the moduli space $M_{g,n}$ of curves of genus $g$ with $n$ marked points?
31. (Hopf conjecture) Does there exist a Riemannian metric on $S^2 \times S^2$ with positive scalar curvature?
32. (Chern's conjecture) Does every closed manifold with a flat connection on its tangent bundle have Euler characteristic zero?
33. Does every 6-manifold that has an almost complex structure also have an integrable almost complex structure?
34. A topological space is called homogeneous if there exists a homeomorphism sending any point to any other point. Here are two open problems:
    - (Rudin, 1956) Is it true that every infinite homogeneous compact space contains a nontrivial convergent sequence?
    - (1980s) Is it possible to represent an arbitrary compact space as the image of a homogeneous compact space under a continuous mapping?
35. (triangle areas) Given $n$ distinct points on the unit circle, look at the areas of all the triangles formed with these points as their vertices. How many distinct areas should we expect to see?
    - One conjecture: the number of distinct areas for triangles on $n$ points should be $\gtrsim n^2$, i.e. there should be an asymptotic lower bound given by $cn^2$ for some constant $c$.
    - Equispacing $n$ points around the circle gives on the order of $n^2$ distinct areas, but it is not known whether this construction is minimal.
    - Something we do know: if you just take $n$ points in the plane, unrestricted to the unit circle, then the number of distinct triangle areas is at least $\lfloor \frac{n-1}{2} \rfloor$ (due to Erdős-Purdy). This bound is sharp by taking equispaced points on a pair of parallel lines. (Audience can easily prove the sharpness result, area of any triangle is determined by the distance between its two base vertices, i.e. the two of its vertices which lie on the same line.)
36. (Ulam sequence) Start with 1, 2 and then define the next term to be the smallest integer which can be written uniquely as a sum of two distinct previous terms. The sequence begins

    $$1, 2, 3, 4, 6, 8, 11, 13, \ldots$$

    but very little is known about it.
    - Since the sequence is increasing we have $a_{n+1} \leq a_n + a_{n-1}$, and thus the growth rate is at most exponential. No better bound on growth rate is known.
    - There is a constant $\alpha \approx 2.5714474995$ such that $\cos(\alpha a_n) < 0$ for at least the first $10^7$ terms of the series (except $a_n = 2, 3, 47, 69$), and $\alpha a_n \mod 2\pi$ has a strange bimodal distribution.
    - If you change the initial constants 1, 2: sometimes the sequence produced is a union of arithmetic sequences, but when it is not it has the same property as above for a choice of $\alpha$ depending on the given constants.
37. (Ulam binary) Another version of the problem: say 0 and 1 are both Ulam. Call a binary string Ulam if it can be written uniquely as the concatenation of two smaller distinct Ulam binary strings. How many Ulam strings of length $n$ are there (say, asymptotically)?
    - Some people (Adutwum, Clark, Emerson, Sheydvasser, Sheydvasser, Tougouma arXiv:2410.01217) counted up to $n = 30$ and conjectured that

    $$\frac{\text{Ulam strings of length n}}{2^n} \sim \frac{0.526}{n^{3/10}}.$$

38. Let $P \in \mathbb{Z}[x]$ be nonconstant. For which $n$ does there exist an integer $m$ such that $P(m)$ has at least $n$ prime factors? (This one is known, it's true for every $n$. It's sufficient to find $n$ outputs of $P$ each divisible by a distinct prime, via CRT. Take $p \mid P(m)$ a prime divisor and then $p \mid P(kp+m)$. Under the condition $M \equiv m_i \mod p_i$ for a family of distinct primes $p_i \mid P(m_i)$, we have $\prod_i p_i \mid P(M)$. By CRT we can always find such an $M$. Now for any $n-1$ primes, the size of the $k$th integer divisible by only those primes grows exponentially, hence faster than any polynomial. So we can always find a larger output with a new prime divisor.)

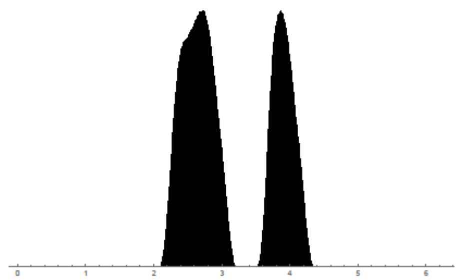



FIGURE 4. The empirical density of $2.571\ldots a_n \mod 2\pi$ seems to be compactly supported.

*Postnote:* During the talk, I discussed problems 1-7, 11, 12, 15, 16, 18, 19, 21, and 23 (in total, 15 problems). If I were giving a talk like this again, I could easily use problems 8-10, 13, 14, 20, 22, 24-27, 29, and 38.