Linear Quadratic Regulator: Optimal Control Dynamics

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Introduction

The Linear Quadratic Regulator (LQR) is an optimal control method that minimizes a quadratic cost function for a linear dynamical system, driving the state to a reference point.

Applications:

- Robotics and autonomous systems
- Control theory
- Optimization in engineering

Mathematical Definition

System:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \quad \mathbf{x} = [x, y, \dot{x}, \dot{y}]^T, \quad \mathbf{u} = [u_x, u_y]^T,$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Cost Function:

$$J = \int_0^\infty \Big((\mathbf{x} - \mathbf{x}_{\text{ref}})^T Q (\mathbf{x} - \mathbf{x}_{\text{ref}}) + \mathbf{u}^T R \mathbf{u} \Big)\, dt,$$

where $\mathbf{x}_{\text{ref}} = [2, 2, 0, 0]^T$, $Q = \text{diag}(1, 1, 0, 0)$, $R = 0.1 I_2$.
Penalizes position error and control effort.

Optimal Control Solution

The LQR solution is obtained by solving the continuous-time Riccati equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0.$$

The optimal control gain is:

$$K = R^{-1}B^T P,$$

yielding the control law:

$$\mathbf{u} = -K(\mathbf{x} - \mathbf{x}_{\text{ref}}).$$

Numerical Integration:

- Euler method with time step $\Delta t = 0.01$ s.
- Simulation over $T = 5$ s (500 points).
- Initial condition: $\mathbf{x}(0) = [0, 0, 0, 0]^T$.

Visualization

- Implemented in Python using `matplotlib.animation`.
- 2D trajectory plot over 500 time steps.
- Visual elements: Reference point at $(2, 2)$, secondary point at $(1, 1)$, and agent trajectory.
- lokiusandoLQR.gif

Conclusion

- The LQR demonstrates optimal control for linear systems, minimizing position error.
- Combines mathematical optimization with dynamic visualization.
- Valuable for education, research, and control system design.

Source code available at: `github.com/IsabelCasPe/Math-Dynamics`