Double Pendulum: Celestial Chaos with Poincaré Section

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Introduction

The double pendulum is a classic chaotic system, showcasing extreme sensitivity to initial conditions. The Poincaré section reduces its 4D phase space to a 2D map, revealing fractal patterns.

Applications:

- Chaos theory
- Dynamical systems
- Celestial mechanics

Inspired by Poincaré's work and M.V.'s lectures at IMPA.

Mathematical Definition

Parameters:

- $m_1 = m_2 = 1.0$: Masses
- $l_1 = l_2 = 1.0$: Lengths
- $g = 9.8$: Gravity

Equations:

$$\dot{\theta}_1 = \frac{p_1}{m_1 l_1^2},$$
$$\dot{\theta}_2 = \frac{p_2}{m_2 l_2^2},$$
$$\dot{p}_1 = -\frac{m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)}{2(m_1 + m_2 \sin^2(\theta_1 - \theta_2))} - (m_1 + m_2) g l_1 \sin(\theta_1),$$
$$\dot{p}_2 = \frac{m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)}{2(m_1 + m_2 \sin^2(\theta_1 - \theta_2))} - m_2 g l_2 \sin(\theta_2).$$

Nonlinear coupling drives chaotic motion.

Poincaré Section

The Poincaré section is taken at $\theta_1 = 0$, with $\dot{\theta}_1 > 0$, mapping the 4D phase space $(\theta_1, \theta_2, p_1, p_2)$ to a 2D plane (θ_2, p_2).

$$\text{Condition: } |\theta_1(t) - 0| < 0.02, \quad p_1(t) > 0$$

Reveals fractal structure of the chaotic attractor.

- `scipy.integrate.odeint`: Solves the differential equations.
- **Initial Conditions**: Two nearby trajectories:

$$(\theta_1, \theta_2, p_1, p_2) = \left(\frac{\pi}{2}, \frac{\pi}{2}, 0, 0\right), \quad \left(\frac{\pi}{2} + 0.01, \frac{\pi}{2}, 0, 0\right)$$

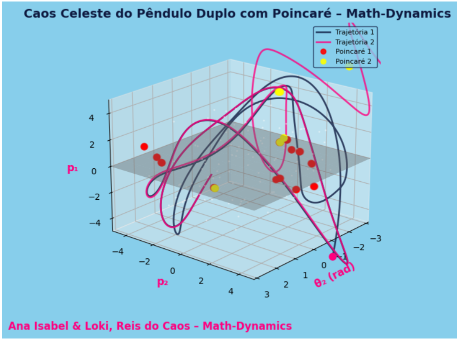
- **Time Span**: $t \in [0, 20]$, 10,000 points.
- **3D Visualization**: Trajectories (θ_2, p_2, p_1) with Poincaré points.

Shows divergence due to chaos.

Visualization

- Python with `matplotlib`.
- 3D plot: Trajectories in *financeblue* and *neonpink*, Poincaré points in *neonred* and *neonyellow* with white glow.
- Background: *skyblue* with starry specks.

Below: Animation of chaotic trajectories and Poincaré section.



Conclusion

- The double pendulum exhibits rich chaotic behavior.
- Poincaré section reveals fractal patterns, echoing celestial mechanics.
- A visually stunning tool for studying chaos, inspired by Poincaré and M.V.

Source code: `github.com/IsabelCasPe/Math-Dynamics`