Cahn-Hilliard Inpainting and a Generalization for Grayvalue Images

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CAHN-HILLIARD INPAINTING AND A GENERALIZATION FOR GRAYVALUE IMAGES

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Abstract. The Cahn-Hilliard equation is a fourth order reaction diffusion equation originating in material science for modeling phase separation and phase coarsening in binary alloys. The inpainting of binary images using the Cahn-Hilliard equation is a new approach in image processing. In this paper we discuss the stationary state of the proposed model and introduce a generalization for grayvalue images of bounded variation. This is realized by using subgradients of the total variation functional within the flow, which leads to structure inpainting with smooth curvature of level sets.

Key words. Cahn-Hilliard equation, TV minimization, image inpainting

AMS subject classifications. 49J40

1. Introduction. An important task in image processing is the process of filling in missing parts of damaged images based on the information obtained from the surrounding areas. It is essentially a type of interpolation and is referred to as inpainting. Given an image f in a suitable Banach space of functions defined on $\Omega \subset \mathbb{R}^2$, an open and bounded domain, the problem is to reconstruct the original image u in the damaged domain $D \subset \Omega$, called inpainting domain. In the following we are especially interested in so called non-texture inpainting, i.e., the inpainting of structures, like edges and uniformly colored areas in the image, rather than texture.

In the pioneering works of Caselles et al. [11] (with the term disocclusion instead of inpainting) and Bertalmio et al. [5] partial differential equations have been first proposed for digital non-texture inpainting. The inpainting algorithm in [5] extends the graylevels at the boundary of the damaged domain continuously in the direction of the isophote lines to the interior via anisotropic diffusion. The resulting scheme is a discrete model based on the nonlinear partial differential equation

$$u_t = \nabla^{\perp} u \cdot \nabla \Delta u,$$

to be solved inside the inpainting domain D using image information from a small strip around D. The operator $\nabla^{\perp}$ denotes the perpendicular gradient $(-\partial_y, \partial_x)$. In subsequent works variational models, originally derived for the tasks of image denoising, deblurring and segmentation, have been adopted to inpainting. In contrast to former approaches (like [5]) the proposed variational algorithms are applied to the image on the whole domain Ω. This procedure has the advantage that inpainting can be carried out for several damaged domains in the image simultaneously and that possible noise outside the inpainting domain is removed at the same time. The general form of such a variational inpainting approach is

$$\hat{u}(x) = argmin_{u \in H_1} \left(J(u) = R(u) + \frac{1}{2} \left\| \lambda (f(x) - u(x)) \right\|_{H_2}^2 \right),$$

where $f \in H_2$ (or $f \in H_1$ depending on the approach) is the given damaged image and $\hat{u} \in H_1$ is the restored image. H_1, H_2 are Banach spaces on Ω and $R(u)$ is the so called regularizing term $R : H_1 \to \mathbb{R}$. The function λ is the characteristic function of $\Omega \setminus D$ multiplied by a (large) constant, i.e., $\lambda(x) = \lambda_0 >> 1$ in $\Omega \setminus D$ and 0 in D. Depending on the choice of the regularizing term $R(u)$ and the Banach spaces H_1, H_2 various approaches have been developed. The most

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famous model is the total variation (TV) model, where $R(u) = |Du|(\Omega)$, is the total variation of u, $H_1 = BV(\Omega)$ the space of functions of bounded variation (see Appendix A for the definition of functions of bounded variation) and $H_2 = L^2(\Omega)$, cf. [16, 14, 31, 32]. A variational model with a regularizing term of higher order derivatives, i.e., $R(u) = \int_\Omega (1 + \left(\nabla \cdot (\frac{\nabla u}{|\nabla u|})\right)^2)|\nabla u|\ dx$, is the Euler elastica model [13, 25]. Other examples are the active contour model based on the Mumford and Shah segmentation [33], and the inpainting scheme based on the Mumford-Shah-Euler image model [19].

Second order variational inpainting methods (where the order of the method is determined by the derivatives of highest order in the corresponding Euler-Lagrange equation), like TV inpainting (cf. [32], [14], [15]) have drawbacks as in the connection of edges over large distances or the smooth propagation of level lines (sets of image points with constant grayvalue) into the damaged domain. In an attempt to solve both the connectivity principle and the so called staircasing effect resulting from second order image diffusions, a number of third and fourth order diffusions have been suggested for image inpainting.

A variational third order approach to image inpainting is the CDD (Curvature Driven Diffusion) method [15]. While realizing the Connectivity Principle in visual perception, (i.e., level lines are connected also across large inpainting domains) the level lines are still interpolated linearly (which may result in corners in the level lines along the boundary of the inpainting domain). This has driven Chan, Kang and Shen [13] to a reinvestigation of the earlier proposal of Masnou and Morel [25] on image interpolation based on Eulers elastica energy. In their work the authors present the fourth order elastica inpainting PDE which combines CDD and the transport process of Bertalmio et. al [5]. The level lines are connected by minimizing the integral over their length and their squared curvature within the inpainting domain. This leads to a smooth connection of level lines also over large distances. This can also be interpreted via a second boundary condition, necessary for an equation of fourth order. Not only the grayvalues of the image are specified on the boundary of the inpainting domain but also the gradient of the image function, namely the directions of the level lines are given. Further, also combinations of second and higher order methods exist, e.g. [24]. The combined technique is able to preserve edges due to the second order part and at the same time avoids the staircasing effect in smooth regions. A weighting function is used for this combination.

The main challenge in inpainting with higher order flows is to find simple but effective models and to propose stable and fast discrete schemes to solve them numerically. To do so also the mathematical analysis of these approaches is an important point, telling us about solvability and convergence of the corresponding equations. This analysis can be very hard because often these equations do not admit a maximum or comparison principle and sometimes do not even have a variational formulation.

A new approach in the class of fourth order inpainting algorithms is inpainting of binary images using a modified Cahn-Hilliard equation, as proposed in [6], [7] by Bertozzi, Esedoglu and Gillette. The inpainted version u of $f \in L^2(\Omega)$ assumed with any (trivial) extension to the inpainting domain is constructed by following the evolution of

$$u_t = \Delta(-\epsilon\Delta u + \frac{1}{\epsilon}F'(u)) + \lambda(f - u) \quad \text{in } \Omega, \tag{1.1}$$

where $F(u)$ is a so called double-well potential, e.g., $F(u) = u^2(u-1)^2$, and - as before:

$$\lambda(x) = \begin{cases} \lambda_0 & \Omega \setminus D \\ 0 & D \end{cases}$$

is the characteristic function of $\Omega \setminus D$ multiplied by a constant $\lambda_0 >> 1$. In [7] the authors prove global existence and uniqueness of a weak solution of (1.1). Moreover, they derive properties of possible stationary solutions in the limit $\lambda_0 \to \infty$. Nevertheless the existence of a solution of the stationary equation

$$\Delta(-\epsilon\Delta u + \frac{1}{\epsilon}F'(u)) + \lambda(f - u) = 0 \quad \text{in } \Omega, \tag{1.2}$$

remains unaddressed. The difficulty in dealing with the stationary equation is the lack of an energy functional for (1.1), i.e., the modified Cahn-Hilliard equation (1.1) cannot be represented by a gradient flow of an energy functional over a certain Banach space. One challenge of this paper is to extend the analysis from [7] by partial answers to questions concerning the stationary equation (1.2) using alternative methods, namely by fixed point arguments. We shall prove

THEOREM 1.1. *Equation* (1.2) *admits a weak solution in* $H^1(\Omega)$ *provided* $\lambda_0 \geq O(\frac{1}{\epsilon^3})$.

We will see in our numerical examples that the condition $\lambda_0 \geq O(\frac{1}{\epsilon^3})$ in Theorem 1.1 is naturally fulfilled, since in order to obtain good visual results in inpainting approaches λ_0 has to be chosen rather large in general. Note that the same condition also appears in [7] where it is needed to prove the global existence of solutions of (1.1).

The second goal of this paper is to generalize the Cahn-Hilliard inpainting approach to grayvalue images. This is realized by using subgradients of the TV functional within the flow, which leads to structure inpainting with smooth curvature of level sets. To build the connection to Cahn-Hilliard inpainting we shall see that solutions of an appropriate time-discrete Cahn-Hilliard inpainting approach Γ-converge, as $\epsilon \to 0$, to solutions of an optimization problem regularized with the TV norm. A similar form of this approach already appeared in the context of decomposition and restoration of grayvalue images, see for example [30] and [23]. We shall call this new inpainting approach $TV - H^{-1}$ inpainting and define it in the following way: The inpainted image u of $f \in L^2(\Omega)$, shall evolve via

$$u_t = \Delta p + \lambda(f-u), \quad p \in \partial TV(u), \tag{1.3}$$

with

$$TV(u) = \begin{cases} |Du|(\Omega) & \text{if } |u(x)| \leq 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise.} \end{cases} \tag{1.4}$$

The inpainting domain D and the characteristic function $\lambda(x)$ are defined as before for the Cahn-Hilliard inpainting approach. The space $BV(\Omega)$ is the space of functions of bounded variation on Ω and $|Du|(\Omega)$ denotes the total variation of the distributional derivative Du (cf. Appendix B). Further $\partial TV(u)$ denotes the subdifferential of the functional $TV(u)$ (cf. Appendix C for the definition).

The L^∞ bound in the definition of the TV functional (1.4) is quite natural as we are only considering digital images u whose grayvalue can be scaled to $[0,1]$. It is further motivated by the $\Gamma-$ convergence result of the Cahn-Hilliard inpainting approach in Section 3.1.

Using a similar methodology as in the proof of Theorem 1.1 we obtain the following existence theorem,

THEOREM 1.2. *Let* $f \in L^2(\Omega)$. *The stationary equation*

$$\Delta p + \lambda(f-u) = 0, \quad p \in \partial TV(u) \tag{1.5}$$

admits a solution $u \in BV(\Omega)$.

We shall also give a characterization of elements in the subdifferential $\partial TV(u)$ for $TV(u)$ defined as in (1.4), i.e., $TV(u) = |Du|(\Omega) + \chi_1(u)$, where $|Du|(\Omega)$ is the total variation of Du and

$$\chi_1(u) = \begin{cases} 0 & \text{if } |u| \leq 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise.} \end{cases}$$

For (1.3) we additionally give error estimates for the inpainting error and stability information in terms of the Bregman distance (defined later).

Finally we present numerical results for the proposed binary- and grayvalue inpainting approaches and briefly explain the numerical implementation using convexity splitting methods.

Organization of the paper. In Section 2 a fixed point approach is proposed to prove the existence of a stationary solution for the modified Cahn-Hilliard equation (1.1) with Dirichlet boundary conditions. In Section 3 we discuss the new $TV - H^{-1}$ inpainting approach. We show in subsection 3.1 that the $\Gamma-$limit, as $\epsilon \to 0$, of the corresponding optimization approach gives a fourth order problem with a subgradient of the total variation within its flow. This $\Gamma-$limit is generalized to an inpainting approach for grayvalue images, called $TV - H^{-1}$ inpainting (cf. (1.3)). Similarly to the existence proof in Section 2 we prove in subsection 3.2 the existence of a stationary solution of this new inpainting approach for grayvalue images. In Section 3.3 we additionally give a characterization of elements in the subdifferential of the corresponding regularizing functional (1.4). In addition we present error estimates for both the error in inpainting the image by means of (1.3) and for the stability of solutions of (1.3) in terms of the Bregman distance. Section 4 is dedicated to the numerical solution of Cahn-Hilliard- and $TV - H^{-1}-$inpainting and the presentation of numerical examples. Finally in Appendix A the space H_∂^{-1} is defined and its elements are characterized in order to analyze (1.1) for Neumann boundary conditions. In Appendix B basic facts about functions of bounded variation are presented.

Notation. Before we begin with the discussion of our results let us introduce some notations. By $\|.\|$ we always denote the norm in $L^2(\Omega)$ with corresponding inner product $\langle .,. \rangle$ and by $\|.\|_{-1} := \left\|\nabla\Delta^{-1}.\right\|$ the norm in $H^{-1}(\Omega) = (H_0^1(\Omega))^*$ with corresponding inner product $\langle .,. \rangle_{-1} := \left\langle \nabla\Delta^{-1}., \nabla\Delta^{-1}. \right\rangle$ where Δ^{-1} denotes the inverse of the Laplacian on H_0^1.

2. Cahn-Hilliard inpainting - Proof of Theorem 1.1. In this chapter we prove the existence of a weak solution of the stationary equation (1.2). Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $f \in L^2(\Omega)$ given. Instead of Neumann boundary data as in the original Cahn-Hilliard inpainting approach (cf. [7]) we use Dirichlet boundary conditions for our analysis, i.e., we consider

$$\begin{cases} u_t = \Delta\left(-\epsilon\Delta u + \frac{1}{\epsilon}F'(u)\right) + \lambda(f-u) & \text{in } \Omega \\ u = f, -\epsilon\Delta u + \frac{1}{\epsilon}F'(u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

This change from a Neumann- to a Dirichlet problem makes it easier to deal with the boundary conditions in our proofs but does not have a significant impact on the inpainting process as long as we assume that $\bar{D} \subset \Omega$. In Appendix A we nevertheless propose a setting to extend the presented analysis for (1.1) to the originally proposed model with Neumann boundary data. In our new setting we define a weak solution $u \in H_0^1(\Omega)$ of equation (1.2) by

$$\langle \epsilon\nabla u, \nabla\phi \rangle + \left\langle \frac{1}{\epsilon}F'(u), \phi \right\rangle - \langle \lambda(f-u), \phi \rangle_{-1} = 0, \quad \forall \phi \in H_0^1(\Omega). \tag{2.2}$$

Remark 2.1. *With $u \in H^1(\Omega)$ and the compact embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for every $1 \le q < \infty$ and $\Omega \subset \mathbb{R}^2$ the weak formulation is well defined.*

To see that (2.2) defines a weak formulation for (1.2) with Dirichlet boundary conditions we integrate by parts in (2.2) and get

$$\begin{array}{l} \int_\Omega \left(-\epsilon\Delta u + \frac{1}{\epsilon}F'(u) - \Delta^{-1}\left(\lambda(f-u)\right)\right)\phi\, dx \\ -\int_{\partial\Omega} \Delta^{-1}\left(\lambda(f-u)\right)\nabla\Delta^{-1}\phi \cdot \nu\, d\mathcal{H}^1, \quad \forall \phi \in H_0^1(\Omega), \end{array}$$

where $\mathcal{H}^1$ denotes the one dimensional Hausdorff measure. This yields

$$\begin{cases} \epsilon\Delta u - \frac{1}{\epsilon}F'(u) + \Delta^{-1}\left(\lambda(f-u)\right) = 0 & \text{in } \Omega \\ \Delta^{-1}\left(\lambda(f-u)\right) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Assuming sufficient regularity on u and using the assumption that the inpainting domain fulfills $\bar{D} \subset \Omega$ we can use the definition of Δ^{-1} to see that u solves

$$\begin{cases} -\epsilon\Delta\Delta u + \frac{1}{\epsilon}\Delta F'(u) + \lambda(f-u) = 0 & \text{in } \Omega \\ u = f, -\Delta u + \frac{1}{\epsilon}F'(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

i.e., u solves (1.2) with Dirichlet boundary conditions.

For the proof of existence of a solution to (2.2) we follow the subsequent strategy. We consider the fixed point operator $\mathcal{A} : L^2(\Omega) \to L^2(\Omega)$ where $\mathcal{A}(v) = u$ fulfills for a given $v \in L^2(\Omega)$ the equation

$$\begin{cases} \frac{1}{\tau}\Delta^{-1}(u-v) = \epsilon\Delta u - \frac{1}{\epsilon}F'(u) + \Delta^{-1}\left[\lambda(f-u) + (\lambda_0 - \lambda)(v-u)\right] & \text{in } \Omega, \\ \Delta^{-1}\left(\frac{1}{\tau}(u-v) + \lambda(f-u) + (\lambda_0-\lambda)(v-u)\right) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $\tau > 0$ is a parameter. We define the weak formulation of (2.4) as before by

$$\begin{aligned} &\left\langle \tfrac{1}{\tau}(u-v), \phi \right\rangle_{-1} + \langle \epsilon\nabla u, \nabla\phi\rangle + \left\langle \tfrac{1}{\epsilon}F'(u), \phi\right\rangle \\ &- \langle \lambda(f-u) + (\lambda_0-\lambda)(v-u), \phi\rangle_{-1} \qquad \forall \phi \in H_0^1(\Omega). \end{aligned} \tag{2.5}$$

A fixed point of the operator $\mathcal{A}$, provided it exists, then solves the stationary equation with Dirichlet boundary conditions as in (2.3).

Note that in (2.4) the characteristic function λ in the fitting term $\lambda(f-u)+(\lambda_0-\lambda)(v-u) = \lambda_0(v-u)+\lambda(f-v)$ only appears in combination with given functions f, v and is not combined with the solution u of the equation. For equation (2.4) we can therefore state a variational formulation, i.e., for a given $v \in L^2(\Omega)$ equation (2.4) is the Euler-Lagrange equation of the minimization problem

$$u^* = \operatorname{argmin}_{u \in H^1(\Omega)} J^\epsilon(u,v)$$

with

$$J^\epsilon(u,v) = \int_\Omega \left(\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}F(u)\right)\, dx + \frac{1}{2\tau}\|u-v\|_{-1}^2 + \frac{\lambda_0}{2}\left\|u - \frac{\lambda}{\lambda_0}f - \left(1-\frac{\lambda}{\lambda_0}\right)v\right\|_{-1}^2. \tag{2.6}$$

We are going to use the variational formulation (2.6) to prove that (2.4) admits a weak solution in $H^1(\Omega)$. This solution is unique under additional conditions.

THEOREM 2.2. *Equation* (2.4) *admits a weak solution in* $H^1(\Omega)$. *For* $\tau \le O(\epsilon^3)$ *the weak solution of* (2.4) *is unique.*

Further we prove that the operator $\mathcal{A}$ admits a fixed point under certain conditions.

THEOREM 2.3. *Set* $\mathcal{A} : L^2(\Omega) \to L^2(\Omega)$, $\mathcal{A}(v) = u$, *where* $u \in H^1(\Omega)$ *is the unique weak solution of* (2.4). *Then* $\mathcal{A}$ *admits a fixed point* $\hat{u} \in H^1(\Omega)$ *if* $\tau \le O(\epsilon^3)$ *and* $\lambda_0 \ge O(\frac{1}{\epsilon^3})$.

Hence the existence of a stationary solution of (1.1) follows under the condition $\lambda_0 \ge O(1/\epsilon^3)$.

We begin with considering the fixed point equation (2.4), i.e., the minimization problem

$$u^* = \operatorname{argmin}_{u \in H^1(\Omega)} J^\epsilon(u,v)$$

with J^ϵ defined as in (2.6). In the following we prove the existence of a unique weak solution of (2.4) by showing the existence of a unique minimizer for (2.6).

Proof. (Proof of Theorem 2.2) We want to show that $J^\epsilon(u,v)$ has a minimizer in $H^1(\Omega)$. For this we consider a minimizing sequence $u^n \in H^1(\Omega)$ of $J^\epsilon(u,v)$. To see that u^n is uniformly bounded in $H^1(\Omega)$ we show that $J^\epsilon(u,v)$ is coercive in $H^1(\Omega)$. With $F(u) \ge C_1u^2 - C_2$ for two positive constants $C_1, C_2 > 0$ and the triangular inequality in the $H^{-1}(\Omega)$ space, we obtain

$$\begin{aligned} J^\epsilon(u,v) &\ge \frac{\epsilon}{2}\|\nabla u\|_2^2 + \frac{C_1}{\epsilon}\|u\|_2^2 - \frac{C_2}{\epsilon} + \frac{1}{2\tau}\left(\frac{1}{2}\|u\|_{-1}^2 - \|v\|_{-1}^2\right) \\ &\quad + \frac{\lambda_0}{2}\left(\frac{1}{2}\|u\|_{-1}^2 - \left\|\frac{\lambda}{\lambda_0}f + \left(1-\frac{\lambda}{\lambda_0}\right)v\right\|_{-1}^2\right) \\ &\ge \frac{\epsilon}{2}\|\nabla u\|_2^2 + \frac{C_1}{\epsilon}\|u\|_2^2 + \left(\frac{\lambda_0}{4} + \frac{1}{4\tau}\right)\|v\|_{-1}^2 - C_3 \end{aligned}$$

Therefore a minimizing sequence u^n is bounded in $H^1(\Omega)$ and it follows that $u^n \rightharpoonup u^*$ in $H^1(\Omega)$. To finish the proof of existence for (2.4) we have to show that $J^\epsilon(u,v)$ is weakly lower semicontinuous in $H^1(\Omega)$. For this we divide the sequence $J^\epsilon(u^n, v)$ of (2.6) in two parts. We denote the first term

$$a^n = \underbrace{\int_\Omega \left(\frac{\epsilon}{2} |\nabla u^n|^2 + \frac{1}{\epsilon} F(u^n) \right) dx}_{CH(u^n)}$$

and the second term

$$b^n = \underbrace{\frac{1}{2\tau} \|u^n - v\|_{-1}^2}_{D(u^n,v)} + \underbrace{\frac{\lambda_0}{2} \left\| u^n - \frac{\lambda}{\lambda_0} f - \left(1 - \frac{\lambda}{\lambda_0}\right) v \right\|_{-1}^2}_{FIT(u^n,v)}.$$

Since $H^1 \hookrightarrow\hookrightarrow L^2$ it follows $u^n \to u^*$ in $L^2(\Omega)$. Further we know that if b^n converges strongly, then

$$\liminf (a^n + b^n) = \liminf a^n + \lim b^n. \tag{2.7}$$

We begin with the consideration of the last term in (2.6). We denote $\tilde{f} := \frac{\lambda}{\lambda_0} f + (1 - \frac{\lambda}{\lambda_0}) v$. We want to show

$$\begin{array}{ccc} \left\| u^n - \tilde{f} \right\|_{-1}^2 & \longrightarrow & \left\| u^* - \tilde{f} \right\|_{-1}^2 \\ & \Longleftrightarrow & \\ \langle \Delta^{-1}(u^n - \tilde{f}), u^n - \tilde{f} \rangle & \longrightarrow & \langle \Delta^{-1}(u^* - \tilde{f}), u^* - \tilde{f} \rangle. \end{array}$$

For this we consider the absolute difference of the two terms,

$$\begin{aligned} & |\langle \Delta^{-1}(u^n - \tilde{f}), u^n - \tilde{f} \rangle - \langle \Delta^{-1}(u^* - \tilde{f}), u^* - \tilde{f} \rangle| \\ &= |\langle \Delta^{-1}(u^n - u^*), u^n - \tilde{f} \rangle - \langle \Delta^{-1}(u^* - \tilde{f}), u^n - u^* \rangle| \\ &\leq |\langle u^n - u^*, \Delta^{-1}(u^n - \tilde{f}) \rangle| + |\langle \Delta^{-1}(u^* - \tilde{f}), u^* - u^n \rangle| \\ &\leq \underbrace{\|u^n - u^*\|}_{\to 0} \cdot \left\| \Delta^{-1}(u^n - \tilde{f}) \right\| + \underbrace{\|u^n - u^*\|}_{\to 0} \cdot \left\| \Delta^{-1}(u^* - \tilde{f}) \right\| \end{aligned}$$

Since the operator $\Delta^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ is a linear and continuous operator it follows that

$$\left\| \Delta^{-1} F \right\| \leq \left\| \Delta^{-1} \right\| \cdot \|F\| \quad \text{for all } F \in H^{-1}(\Omega).$$

Thus

$$\begin{aligned} & |\langle \Delta^{-1}(u^n - \tilde{f}), u^n - \tilde{f} \rangle - \langle \Delta^{-1}(u^* - \tilde{f}), u^* - \tilde{f} \rangle| \\ &\leq \underbrace{\|u^n - u^*\|}_{\to 0} \underbrace{\left\| \Delta^{-1} \right\|}_{const} \underbrace{\left\| u^n - \tilde{f} \right\|}_{bounded} + \underbrace{\|u^n - u^*\|}_{\to 0} \underbrace{\left\| \Delta^{-1} \right\|}_{const} \underbrace{\left\| u^* - \tilde{f} \right\|}_{const} \\ &\longrightarrow 0 \text{ as } n \to \infty, \end{aligned}$$

and we conclude that $FIT(u^n, v)$ converges strongly to $FIT(u^*, v)$. With the same argument it follows that $D(u^n, v)$ converges strongly and in sum that the sequence b^n converges strongly in $L^2(\Omega)$. Further $CH(.)$ is weakly lower semicontinuous, which follows from the lower semicontinuity of the Dirichlet integral and from the continuity of F by applying Fatou's Lemma. Hence we obtain

$$J^\epsilon(u^*, v) \leq \liminf J^\epsilon(u^n, v).$$

Therefore J^ϵ has a minimizer in H^1, i.e.,

$$\exists u^* \text{ with } u^* = argmin_{u \in H^1(\Omega)} J^\epsilon(u, v).$$

For simplicity let in the following $u = u^*$. To see that the minimizer u is a weak solution of (2.4) we compute the corresponding Euler-Lagrange equation to the minimization problem. For this sake we choose any testfunction $\phi \in H_0^1(\Omega)$ and compute the first variation of J^ϵ, i.e.,

$$\left(\frac{d}{d\delta} J^\epsilon(u + \delta\phi, v)\right)_{\delta=0},$$

which has to be zero for a minimizer u. Thus we have

$$\epsilon \langle \nabla u, \nabla \phi \rangle + \frac{1}{\epsilon} \langle F'(u), \phi \rangle + \left\langle \frac{1}{\tau}(u - v) + \lambda_0 \left[u - \frac{\lambda}{\lambda_0} f - \left(1 - \frac{\lambda}{\lambda_0}\right) v \right], \phi \right\rangle_{-1} = 0.$$

Integrating by parts in both terms we get

$$\left\langle -\epsilon \Delta w + \frac{1}{\epsilon} F'(w) - \Delta^{-1} \left(\frac{1}{\tau}(u - v) + \lambda_0 \left[u - \frac{\lambda}{\lambda_0} f - \left(1 - \frac{\lambda}{\lambda_0}\right) v \right] \right), \phi \right\rangle$$
$$+ \int_{\partial\Omega} \nabla u \cdot \nu \phi \, ds + \int_{\partial\Omega} \Delta^{-1} \left(\frac{1}{\tau}(u - v) + \lambda_0 \left[u - \frac{\lambda}{\lambda_0} f - \left(1 - \frac{\lambda}{\lambda_0}\right) v \right] \right) \nabla \Delta^{-1} \phi \cdot \nu \, ds = 0.$$

Since ϕ is an element in $H_0^1(\Omega)$ the first boundary integral vanishes and we obtain that u fulfills the weak formulation (2.5) of (2.4).

For the uniqueness of the minimizer, we need to prove that J^ϵ is strictly convex. To do so, we prove that for any $u_1, u_2 \in H^1(\Omega)$,

$$J^\epsilon(u_1, v) + J^\epsilon(u_2, v) - 2J^\epsilon\left(\frac{u_1 + u_2}{2}, v\right) > 0, \tag{2.8}$$

based on an assumption that $F(.)$ satisfies $F(u_1) + F(u_2) - 2F(\frac{u_1+u_2}{2}) > -C(u_1 - u_2)^2$, with the constant $C > 0$. For example, when $F(u) = \frac{1}{8}(u^2 - 1)^2$, $C = \frac{1}{8}$. Denote $u = u_1 - u_2$, we have

$$J^\epsilon(u_1, v) + J^\epsilon(u_2, v) - 2J^\epsilon\left(\frac{u_1 + u_2}{2}, v\right) > \frac{\epsilon}{4} \|u\|_{H^1}^2 + \left(\frac{1}{4\tau} + \frac{\lambda_0}{4}\right) \|u\|_{-1}^2 - \frac{C}{\epsilon} \|u\|_2^2$$

By using the inequality

$$\|u\|_2^2 \leq \|u\|_{H^1} \|u\|_{-1}, \tag{2.9}$$

and the Cauchy-Schwarz inequality, for (2.8) to be fulfilled, we need

$$2\sqrt{\frac{\epsilon}{4}\left(\frac{1}{4\tau} + \frac{\lambda_0}{4}\right)} \geq \frac{C}{\epsilon}.$$

i.e.,

$$\epsilon^3 \left(\lambda_0 + \frac{1}{\tau}\right) \geq C^2.$$

Therefore $J^\epsilon(u, v)$ is strictly convex in u and our minimization problem has a unique minimizer if τ is chosen smaller than $O(\epsilon^3)$. Because of the convexity of J^ϵ in ∇u and u, every weak solution of the Euler-Lagrange equation (2.4) is in fact a minimizer of J^ϵ. This proves the uniqueness of a weak solution of (2.4) provided $\tau << O(\epsilon^3)$. □

Next we want to prove Theorem 2.3, i.e., the existence of a fixed point of (2.4) and with this the existence of a stationary solution of (1.1). To do so we are going to apply Schauder's fixed point theorem.

Proof. (Proof of Theorem 2.3) We consider a solution $\mathcal{A}(v) = u$ of (2.4) with $v \in L^2(\Omega)$ given. In the following we will prove the existence of a fixed point by using Schauder's fixed point theorem. For this we will show

$$\|u\|^2 \leq \beta \|v\|^2 + \alpha, \tag{2.10}$$

with $\beta < 1$. Having this we have shown that $\mathcal{A}$ is a map from the closed ball $K = B(0, M) = \{u \in L^2(\Omega) : \|u\| \leq M\}$ into itself for an appropriate constant $M > 0$. Then K is a compact and convex subset of $L^2(\Omega)$, because K is the ball with radius M around 0.

By inverting Δ^{-1} in (2.4), i.e., by applying the operator $-\Delta$ to the equation, we obtain

$$\frac{1}{\tau}(u - v) = -\Delta(\epsilon \Delta u - \frac{1}{\epsilon} F'(u)) + [\lambda(f - u) + (\lambda_0 - \lambda)(v - u)] .$$

With $\lambda(f - u) + (\lambda_0 - \lambda)(v - u) = \lambda(f - v) + \lambda_0(v - u)$, and by multiplying the above equation by u we conclude

$$\int_\Omega u \cdot \frac{1}{\tau}(u - v) = -\epsilon \|\Delta u\|^2 - \frac{1}{\epsilon} \int_\Omega F''(u) |\nabla u|^2 \, dx$$
$$+ \lambda_0 \left(\int_{\Omega \setminus D} u(f - u) \, dx + \int_D u(v - u) \, dx \right) .$$

With $F''(u) \geq C_1 u^2 - C_2$ for some constants $C_1, C_2 > 0$ and for all $u \in \mathbb{R}$, and by further applying the Cauchy-Schwarz inequality to terms connected to λ_0 we obtain

$$\int_\Omega u \cdot \frac{1}{\tau}(u - v) \leq -\epsilon \|\Delta u\|^2 - \frac{C_1}{\epsilon} \|u |\nabla u|\|^2 + \frac{C_2}{\epsilon} \|\nabla u\|^2$$
$$+ \lambda_0 \left[-\left(1 - \frac{\delta}{2}\right) \int_{\Omega \setminus D} u^2 \, dx + \left(\frac{\delta_1}{2} - 1\right) \int_D u^2 \, dx \right.$$
$$\left. + \frac{1}{2\delta_1} \int_D v^2 \, dx + C(|\Omega \setminus D|, f) \right] .$$

Setting $\delta = 1$ and $\delta_1 = 2$ we see that

$$\int_\Omega u \cdot \frac{1}{\tau}(u - v) \leq -\epsilon \|\Delta u\|^2 - \frac{C_1}{\epsilon} \|u |\nabla u|\|^2 + \frac{C_2}{\epsilon} \|\nabla u\|^2$$
$$+ \lambda_0 \left[-\frac{1}{2} \int_{\Omega \setminus D} u^2 \, dx + \frac{1}{4} \int_D v^2 \, dx + C(|\Omega \setminus D|, f) \right] .$$

We follow the argumentation of the proof of existence for (1.1) in [7] by observing the following property: A standard interpolation inequality for ∇u reads

$$\|\nabla u\|^2 \leq \delta_1 \|\Delta u\|^2 + \frac{C_3}{\delta_1} \|u\|^2 . \tag{2.11}$$

The domain of integration in the second integral of the equation above can be taken to be smaller than Ω by taking a larger constant C_3. By further using the L^1 version of Poincare's inequality, i.e., Theorem C.2 in Appendix C, we obtain

$$\|u\|^2 \leq C_4 \left\|\nabla u^2\right\|_{L^1(\Omega)} + C_4 \int_{\Omega \setminus D} u^2 \, dx,$$

where C_4 depends on the size of D compared to Ω. By Hölders inequality we also have that

$$\left\|\nabla u^2\right\|_{L^1(\Omega)} \leq \frac{\alpha}{2} \|u |\nabla u|\|^2 + \frac{C_5}{2\alpha}.$$

Putting the last three inequalities together we obtain

$$\|\nabla u\|^2 \leq \delta_1 \|\Delta u\|^2 + \frac{C_3 C_4 \alpha}{2\delta_1} \|u |\nabla u|\|^2 + \frac{C_3 C_4}{\delta_1} \int_{\Omega \setminus D} u^2 \, dx + \frac{C_3 C_4 C_5}{2\alpha\delta_1}.$$

We now use the last inequality to bound the gradient term in our estimates from above to get

$$\begin{aligned}\int_\Omega u \cdot \tfrac{1}{\tau}(u-v) \leq \quad & (\tfrac{C_2\delta_1}{\epsilon} - \epsilon)\|\Delta u\|^2 + (\tfrac{C_2C_3C_4\alpha}{2\delta_1\epsilon} - \tfrac{C_1}{\epsilon})\|u\,|\nabla u|\|^2 \\ & +(\tfrac{C_2C_3C_4}{\delta_1\epsilon} - \tfrac{C_4\lambda_0}{2})\|u\|^2 + \tfrac{\lambda_0}{4}\int_D v^2\,dx + C(\lambda_0, f, \epsilon, \Omega, D).\end{aligned} \tag{2.12}$$

With $\delta_1 < \frac{\epsilon^2}{C_2}$ and α small enough the first two terms can be estimated from above by zero. Applying the Cauchy-Schwarz inequality on the left-hand side and rearranging the terms on both sides of the inequality we conclude

$$\left(\frac{1}{2\tau} + \frac{C_4\lambda_0}{2} - \frac{CC_2}{\delta_1\epsilon}\right)\|u\|^2 \leq \left(\frac{\lambda_0}{4} + \frac{1}{2\tau}\right)\|v\|^2 + C(\lambda_0, |\Omega \setminus D|, f).$$

Choosing $\lambda_0 \geq O(\frac{1}{\epsilon^3})$ the solution u and v fulfill

$$\|u\|^2 \leq \beta\|v\|^2 + C, \tag{2.13}$$

with $\beta < 1$ and a constant C independent of v. Hence u is bounded in $L^2(\Omega)$.

In addition the operator $\mathcal{A}$ is continuous. Indeed if $v_k \to v$ in $H^1(\Omega)$ then $\mathcal{A}(v_k) = u_k$ is bounded in $H^1(\Omega)$ for all $k = 0, 1, 2, \ldots$. To see this we consider (2.12) with appropriate constants δ_1 and α as specified in the paragraph below (2.12). But now we only estimate the second term on the right side by zero and keep the first term. By applying the Cauchy-Schwarz inequality and rearranging the terms as before we obtain

$$\left(\frac{1}{2\tau} + \frac{C_4\lambda_0}{2} - \frac{CC_2}{\delta_1\epsilon}\right)\|u\|^2 + \left(\epsilon - \frac{C_2\delta_1}{\epsilon}\right)\|\Delta u\|^2 \leq \left(\frac{\lambda_0}{4} + \frac{1}{2\tau}\right)\|v\|^2 + C(\lambda_0, |\Omega \setminus D|, f),$$

with the coefficient $\epsilon - \frac{C_2\delta_1}{\epsilon} \geq 0$ due to our choice of δ_1. Therefore not only the L^2- norm of u is uniformly bounded but also the L^2- norm of Δu. By the standard interpolation inequality (2.11) the boundedness of u in $H^1(\Omega)$ follows. Thus, we can consider a weakly convergent subsequence $u_{k_j} \rightharpoonup u$ in $H^1(\Omega)$. Because $H^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, $1 \leq q < \infty$ the sequence u_{k_j} converges also strongly to u in $L^q(\Omega)$. Hence, a weak solution $\mathcal{A}(v_k) = u_k$ of (2.4) weakly converges to a weak solution u of

$$\frac{1}{\tau}(-\Delta^{-1})(u-v) = \epsilon\Delta u - \frac{1}{\epsilon}F'(u) - \Delta^{-1}\left[\lambda(f-u) + (\lambda_0 - \lambda)(v-u)\right],$$

where u is the weak limit of $\mathcal{A}(v_k)$ as $k \to \infty$. Because the solution of (2.4) is unique provided $\tau \leq O(\epsilon^3)$ (cf. Theorem 2.2), $u = \mathcal{A}(v)$, and therefore $\mathcal{A}$ is continuous. Applying Schauder's Theorem we have shown that the fixed point operator $\mathcal{A}$ admits a fixed point $\hat{u}$ in $L^2(\Omega)$ which fulfills

$$\langle\epsilon\nabla\hat{u}, \nabla\phi\rangle + \left\langle\frac{1}{\epsilon}F'(\hat{u}), \phi\right\rangle - \langle\lambda(f-\hat{u}), \phi\rangle_{-1} + \int_{\partial\Omega}\Delta^{-1}\left(\lambda(f-\hat{u})\right)\nabla\Delta^{-1}\phi\cdot\nu\, d\mathcal{H}^1 = 0, \quad \forall\phi \in H_0^1(\Omega).$$

Because the solution of (2.4) is an element of $H^1(\Omega)$ also the fixed point $\hat{u} \in H^1(\Omega)$. □

Following the arguments from the beginning of this section we conclude with the existence of a stationary solution for (1.1).

By modifying the setting and the above proof in an appropriate way one can prove the existence of a stationary solution for (1.1) also under Neumann boundary conditions, i.e.,

$$\nabla u \cdot \nu = \nabla\Delta u \cdot \nu = 0, \quad \text{on}\partial\Omega.$$

A corresponding reformulation of the problem is given in Appendix A.

3. Total Variation - H^{-1} inpainting. In this section we discuss our newly proposed inpainting scheme (1.3), i.e., the inpainted image u of $f \in L^2(\Omega)$ evolves via

$$u_t = \Delta p + \lambda(f-u), \quad p \in \partial TV(u),$$

with

$$TV(u) = \begin{cases} |Du|(\Omega) & \text{if } |u(x)| \leq 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise.} \end{cases}$$

Before starting this section we suggest readers who are unfamiliar with the space $BV(\Omega)$ to first read Appendix B and maybe recall the definition of the subdifferential of a function in Definition C.1.

3.1. Γ-Convergence of the Cahn-Hilliard energy. In the following we want to see what happens if we send the parameter ϵ in (1.1) to zero. In other words we want to know how the Γ-limit 'of the equation' looks like. Before starting our discussion lets recall the definition of Γ-convergence and its impact within the study of optimization problems. For more details on Γ-convergence we refer to [26].

Definition 3.1. *Let $X = (X,d)$ be a metric space and (F_h), $h \in \mathbb{N}$ be family of functions $F_h : X \to [0,+\infty]$. We say that (F_h)* **Γ-converges** *to a function $F : X \to [0,+\infty]$ on X if $\forall x \in X$ we have*

(i) for every sequence x_h with $d(x_h, x) \to 0$ we have

$$F(x) \leq \liminf_h F_h(x_h);$$

(ii) there exists a sequence $\bar{x_h}$ such that $d(\bar{x_h}, x) \to 0$ and

$$F(x) = \lim_h F_h(\bar{x_h})$$

(or, equivalently, $F(x) \geq \limsup_h F_h(\bar{x_h})$).

We write $F(x) = \Gamma - \lim_h F_h(x)$, $x \in X$, is the Γ-limit of (F_h) in X. The formulation of the Γ-limit for $\epsilon \to 0$ is analogous by defining a sequence ϵ_h with $\epsilon_h \to 0$ as $h \to \infty$.

The important property of Γ-convergent sequences of functions F_h is that its minima converge to minima of the Γ-limit F. In fact we have the following theorem

Theorem 3.2. *Let (F_h) be like in Definition 3.1 and additionally equicoercive, that is there exists a compact set $K \subset X$ (independent of h) such that*

$$\inf_{x \in X} \{F_h(x)\} = \inf_{x \in K} \{F_h(x)\}.$$

If F_h Γ-converges on X to a function F we have

$$\min_{x \in X} \{F(x)\} = \lim_h \inf_{x \in X} \{F_h(x)\}.$$

In fact Modica and Mortola have shown in [28] and [29] that the sequence of Cahn-Hilliard functionals

$$CH(u) = \int_\Omega \left(\frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} F(u) \right) \, dx$$

Γ-converges in the topology $L^1(\Omega)$ to

$$TV(u) = \begin{cases} C_0 |Du|(\Omega) & \text{if } |u(x)| = 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise} \end{cases}$$

as $\epsilon \to 0$, where $C_0 = 2\int_{-1}^{1} \sqrt{F(s)}\, ds$. (The space $BV(\Omega)$ and the total variation $|Du|(\Omega)$ are defined in Appendix B.)

Because of the lack of an energy functional for the evolution equation (1.1) we consider the functional J^ϵ for our fixed point approach (2.4):

$$J^\epsilon(u,v) = \underbrace{\int_\Omega \left(\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}F(u)\right)\, dx}_{:=CH(u)} + \underbrace{\frac{1}{2\tau}\|u-v\|_{-1}^2}_{:=D(u,v)} + \underbrace{\frac{\lambda_0}{2}\left\|u - \frac{\lambda}{\lambda_0}f - (1-\frac{\lambda}{\lambda_0})v\right\|_{-1}^2}_{:=FIT(u,v)}, \quad (3.1)$$

for a given function $v \in L^2(\Omega)$. $J^\epsilon(u)$ is the sum of the regularizing term $CH(u)$, the damping term $D(u,v)$ and the fitting term $FIT(u,v)$. We recall the following fact,

THEOREM 3.3. *[Dal Maso, [26], Prop. 6.21.] Let $G : X \to \mathbb{R}$ be a continuous function and (F_h) $\Gamma-$ converges to F in X, then $(F_h + G)$ $\Gamma-$ converges to $F + G$ in X.*

Since $D(u,v)$ and $FIT(u,v)$ are continuous with respect to u and due to Theorem 3.3 the modified Cahn-Hilliard functional J^ϵ can be seen as a regularized approximation in the sense of Γ-convergence of the TV-functional

$$J(u,v) = TV(u) + D(u,v) + FIT(u,v),$$

for functions $u \in BV(\Omega)$ with $|u(x)| = 1$ a.e. in Ω.

This property leads us from the Cahn-Hilliard inpainting approach for binary images to a generalization for grayvalue images $u \in BV(\Omega)$ with $|u(x)| \le 1$, namely our so called $TV - H^{-1}$ inpainting equation (1.3).

3.2. Existence of a stationary solution - proof of Theorem 1.2. Our strategy for proving the existence of a stationary solution for $TV - H^{-1}$ inpainting (1.3) is similar to our existence proof for a stationary solution of the modified Cahn-Hilliard equation (1.1) in Section 2. Similarly as in our analysis for (1.1) in Section 2 we consider equation (1.3) with Dirichlet boundary conditions, namely

$$\begin{aligned} u_t &= \Delta p + \lambda(f-u) && \text{in } \Omega \\ u &= f && \text{on } \partial\Omega, \end{aligned}$$

for $p \in \partial TV(u)$.

Now let $f \in L^2(\Omega)$, $|f| \le 1$ be the given grayvalue image. For $v \in L^r(\Omega)$, $1 < r < 2$, we consider the minimization problem

$$u^* = \arg\min_{u \in BV(\Omega)} J(u,v),$$

with functionals

$$J(u,v) := TV(u) + \underbrace{\frac{1}{2\tau}||u-v||_{-1}^2}_{D(u,v)} + \underbrace{\frac{\lambda_0}{2}||u - \frac{\lambda}{\lambda_0}f - (1-\frac{\lambda}{\lambda_0})v||_{-1}^2}_{FIT(u,v)}, \quad (3.2)$$

with $TV(u)$ defined as in (1.4), i.e.,

$$TV(u) = \begin{cases} |Du|(\Omega) & \text{if } |u(x)| \le 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $L^r(\Omega)$ can be continuously embedded in $H^{-1}(\Omega)$. Hence the functionals in (3.2) are well defined.

First we will show that for a given $v \in L^r(\Omega)$ the functional $J(.,v)$ attains a unique minimizer $u^* \in BV(\Omega)$ with $|u^*(x)| \le 1$ a.e. in Ω.

THEOREM 3.4. *Let $f \in L^2(\Omega)$ be given with $|f(x)| \le 1$ a.e. in Ω and $v \in L^r(\Omega)$. Then the functional $J(.,v)$ has a unique minimizer $u^* \in BV(\Omega)$ with $|u^*(x)| \le 1$ a.e. in Ω.*

Proof. Let $(u^n)_{n\in\mathbb{N}}$ be a minimizing sequence for $J(u,v)$, i.e.,

$$J(u^n,v) \to \inf_{u\in BV(\Omega)} J(u,v).$$

Then $u^n \in BV(\Omega)$ and $|u^n(x)| \le 1$ in Ω (because otherwise $TV(u^n)$ would not be finite). Therefore

$$|Du^n|(\Omega) \le M, \quad \text{for an } M \ge 0 \text{ and for all } n \ge 1,$$

and, because of the uniform boundedness of $|u(x)|$ for every point $x \in \Omega$,

$$\|u^n\|_{L^p(\Omega)} \le \tilde{M}, \text{ for an } M \ge 0, \ \forall n \ge 1, \text{ and } 1 \le p \le \infty.$$

Thus u^n is uniformly bounded in $L^p(\Omega)$ and in particular in $L^1(\Omega)$. Together with the boundedness of $|Du^n|(\Omega)$, the sequence u^n is also bounded in $BV(\Omega)$ and there exists a subsequence, still denoted u^n, and a $u \in BV(\Omega)$ such that $u^n \rightharpoonup u$ weakly in $L^p(\Omega)$, $1 \le p \le \infty$ and weakly* in $BV(\Omega)$. Because $L^2(\Omega) \subset L^2(\mathbb{R}^2) \subset H^{-1}(\Omega)$ (by zero extensions of functions on Ω to $\mathbb{R}^2$) $u^n \rightharpoonup u$ also weakly in $H^{-1}(\Omega)$. Because $|Du|(\Omega)$ is lower semicontinuous in $BV(\Omega)$ and by the lower semicontinuity of the H^{-1} norm we get

$$\begin{aligned} J(u,v) &= TV(u) + D(u,v) + FIT(u,v) \\ &\le \liminf_{n\to\infty}(TV(u_n) + D(u_n,v) + FIT(u_n,v)) \\ &= \liminf_{n\to\infty} J(u_n,v). \end{aligned}$$

So u is a minimizer of $J(u,v)$ over $BV(\Omega)$.

To prove the uniqueness of the minimizer we (similarly as in the proof of Theorem 2.2) show that J is strictly convex. Namely we prove that for all $u_1, u_2 \in BV(\Omega)$, $u_1 \ne u_2$

$$J(u_1,v) + J(u_2,v) - 2J\left(\frac{u_1+u_2}{2}, v\right) > 0.$$

We have

$$\begin{aligned} J(u_1,v) + J(u_2,v) - 2J\left(\frac{u_1+u_2}{2}, v\right) &= \left(\frac{1}{2\tau} + \frac{\lambda_0}{2}\right)\left(\|u_1\|_{-1}^2 + \|u_2\|_{-1}^2 - 2\left\|\frac{u_1+u_2}{2}\right\|_{-1}^2\right) \\ &\quad + TV(u_1) + TV(u_2) - 2TV\left(\frac{u_1+u_2}{2}\right) \\ &\ge \left(\frac{1}{4\tau} + \frac{\lambda_0}{4}\right)\|u_1 - u_2\|_{-1}^2 > 0. \end{aligned}$$

This finishes the proof. □

Next we shall prove the existence of stationary solution for (1.3). For this sake we consider the corresponding Euler-Lagrange equation to (3.2), i.e.,

$$\Delta^{-1}\left(\frac{u-v}{\tau}\right) + p - \Delta^{-1}\left(\lambda(f-u) + (\lambda_0 - \lambda)(v-u)\right) = 0,$$

with weak formulation

$$\left\langle \tfrac{1}{\tau}(u-v), \phi \right\rangle_{-1} + \langle p, \phi \rangle - \langle \lambda(f-u) + (\lambda_0-\lambda)(v-u), \phi \rangle_{-1} \quad \forall \phi \in H_0^1(\Omega).$$

A fixed point of the above equation, i.e., a solution $u = v$, is then a stationary solution for (1.3). Thus, to prove the existence of a stationary solution of (1.3), i.e., to prove Theorem 1.2, we as before are going to use a fixed point argument. Let $\mathcal{A} : L^r(\Omega) \to L^r(\Omega)$, $1 < r < 2$, be the operator which maps a given $v \in L^r(\Omega)$ to $\mathcal{A}(v) = u$ under the condition that $\mathcal{A}(v) = u$ is the minimizer of the functional $J(.,v)$ defined in (3.2). The choice of the fixed point operator $\mathcal{A}$ over $L^r(\Omega)$ was made in order to obtain the necessary compactness properties for the application of Schauder's theorem.

Since here the treatment of the boundary conditions is similar as in Section 2 we will leave this part of the analysis in the upcoming proof to the reader and just carry out the proof without explicitly taking care of the boundary.

Proof. Let $\mathcal{A} : L^r(\Omega) \to L^r(\Omega)$, $1 < r < 2$, be the operator that maps a given $v \in L^r(\Omega)$ to $\mathcal{A}(v) = u$, where u is the unique minimizer of the functional $J(.,v)$ defined in (3.2). Existence and uniqueness follow from Theorem 3.4. Since u minimizes $J(.,v)$ we have $u \in L^\infty(\Omega)$ hence $u \in L^r(\Omega)$. Additionally we have $J(u,v) \leq J(0,v)$, i.e.,

$$\begin{aligned}\frac{1}{2\tau}||u-v||_{-1}^2 + \frac{\lambda_0}{2}||u - \frac{\lambda}{\lambda_0}f - (1-\frac{\lambda}{\lambda_0})v||_{-1}^2 + TV(u) &\leq \frac{1}{2\tau}||v||_{-1}^2 + \frac{\lambda_0}{2}||\frac{\lambda}{\lambda_0}f + (1-\frac{\lambda}{\lambda_0})v||_{-1}^2 \\ &\leq \frac{|\Omega|}{2\tau} + \lambda_0(|\Omega| + |D|).\end{aligned} \tag{3.3}$$

Here the last inequality was obtained since $L^r(\Omega) \hookrightarrow H^{-1}(\Omega)$ and hence $||v||_{-1} \leq C$ and $||\lambda v||_{-1} \leq C$ for a $C > 0$. (In fact, since $H^1(\Omega) \hookrightarrow L^{r'}(\Omega)$ for all $1 \leq r' < \infty$ from duality it follows that $L^r(\Omega) \hookrightarrow H^{-1}(\Omega)$ for, $1 < r < \infty$.) By the last estimate we obtain $u \in BV(\Omega)$. Since $BV(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ compactly for $1 \leq r \leq 2$ and $\Omega \subset \mathbb{R}^2$ (cf. Theorem B.7), the operator $\mathcal{A}$ maps $L^r(\Omega) \to BV(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$, i.e., $\mathcal{A} : L^r(\Omega) \to K$, where K is a compact subset of $L^r(\Omega)$. Thus, for $v \in B(0,1)$ (where $B(0,1)$ denotes the ball in $L^\infty(\Omega)$ with center 0 and radius 1), the operator $\mathcal{A} : B(0,1) \to B(0,1) \cap K = \tilde{K}$, where $\tilde{K}$ is a compact and convex subset of $L^r(\Omega)$.

Next we have to show that $\mathcal{A}$ is continuous in $L^r(\Omega)$. Let $(v_k)_{k\geq 0}$ be a sequence which converges to v in $L^r(\Omega)$. Then $u_k = A(v_k)$ solves

$$\Delta p_k = \frac{u_k - v_k}{\tau} - (\lambda(f-u_k) + (\lambda_0 - \lambda)(v_k - u_k)),$$

where $p_k \in \partial TV(u_k)$. Thus u_k is uniformly bounded in $BV(\Omega) \cap L^\infty(\Omega)$ (and hence in $L^r(\Omega)$) and, since the right-hand side of the above equation is uniformly bounded in $L^r(\Omega)$, also Δp_k is bounded in $L^r(\Omega)$. Thus there exists a subsequence p_{k_l} such that $\Delta p_{k_l} \rightharpoonup \Delta p$ in $L^r(\Omega)$ and a subsequence u_{k_l} that converges weakly $*$ to a u in $BV(\Omega) \cap L^\infty(\Omega)$. Since $BV(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ we have $u_{k_l} \to u$ strongly in $L^r(\Omega)$. Therefore the limit u solves

$$\Delta p = \frac{u - v}{\tau} - (\lambda(f-u) + (\lambda_0 - \lambda)(v - u)). \tag{3.4}$$

If we additionally apply Poincare's inequality to Δp_k we conclude

$$\|\nabla p_k - (\nabla p_k)_\Omega\|_{L^r(\Omega)} \leq C \|\nabla \cdot (\nabla p_k - (\nabla p_k)_\Omega)\|_{L^r(\Omega)},$$

where $(\nabla p_k)_\Omega = \frac{1}{|\Omega|}\int_\Omega \nabla p_k \, dx$. In addition, since $p_k \in \partial TV(u_k)$, it follows that $(p_k)_\Omega = 0$ and $\|p_k\|_{BV^*(\Omega)} \leq 1$. Thus $(\nabla p_k)_\Omega < \infty$ and p_k is uniformly bounded in $W^{1,r}(\Omega)$. Thus there exists a subsequence p_{k_l} such that $p_{k_l} \rightharpoonup p$ in $W^{1,r}(\Omega)$. In addition $L^{r'}(\Omega) \hookrightarrow BV^*(\Omega)$ for $2 < r' < \infty$ (this follows again from Theorem B.7 by a duality argument) and $W^{1,r}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for $1 \leq q < \frac{2r}{2-r}$ (cf. Theorem C.3). By choosing $2 < q < \frac{2r}{2-r}$ we have in sum $W^{1,r}(\Omega) \hookrightarrow\hookrightarrow BV^*(\Omega)$. Thus $p_{k_l} \to p$ strongly in $BV^*(\Omega)$. Hence the element p in (3.4) is an element in $\partial TV(u)$.

Because the minimizer of (3.2) is unique, $u = \mathcal{A}(v)$, and therefore $\mathcal{A}$ is continuous in $L^r(\Omega)$. From Schauder's fixed point theorem the existence of a stationary solution follows.
□

3.3. Characterization of Solutions. Finally we want to compute elements $\hat{p} \in \partial TV(\hat{u})$. Like in [10] the model for the regularizing functional is the sum of a standard regularizer plus the indicator function of the L^∞ constraint. Especially we have $TV(u) = |Du|(\Omega) + \chi_1(u)$, where $|Du|(\Omega)$ is the total variation of Du and

$$\chi_1(u) = \begin{cases} 0 & \text{if } |u| \leq 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise.} \end{cases} \tag{3.5}$$

We want to compute the subgradients of TV by pretending $\partial TV(u) = \partial\left|Du\right|(\Omega) + \partial\chi_1(u)$. This means we can separately compute the subgradients of χ_1. To guarantee that the splitting above is allowed we have to consider a regularized functional of the total variation, like $\int_\Omega \sqrt{\left|\nabla u\right|^2+\delta}\, dx$. This is sufficient because both $\left|D.\right|(\Omega)$ and χ_1 are convex and $\left|D.\right|(\Omega)$ is continuous (compare [18] Proposition 5.6., pp. 26).

The subgradient $\partial\left|Du\right|(\Omega)$ is already well described, as, for instance, in [4] or [34]. We will just shortly recall its characterization. Thereby we do not insist on the details of the rigorous derivation of these conditions, and we limit ourself to mention the main facts.

It is well known [34, Proposition 4.1] that $p \in \partial|Du|(\Omega)$ implies

$$\begin{cases} p = -\nabla\cdot(\frac{\nabla u}{|\nabla u|}) & \text{in } \Omega \\ \frac{\nabla u}{|\nabla u|}\cdot\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

The previous conditions do not fully characterize $p \in \partial|Du|(\Omega)$, additional conditions would be required [4, 34], but the latter are, unfortunately, hardly numerically implementable. Since we anyway consider a regularized version of $\left|Du\right|(\Omega)$ the subdifferential becomes a gradient which reads

$$\begin{cases} p = -\nabla\cdot(\frac{\nabla u}{\sqrt{|\nabla u|^2+\delta}}) & \text{in } \Omega \\ \frac{\nabla u}{\sqrt{|\nabla u|^2+\delta}}\cdot\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

The subgradient of χ_1 is computed like in the following Lemma.

Lemma 3.5. *Let $\chi_1 : L^r(\Omega) \to \mathbb{R}\cup\{\infty\}$ be defined by (3.5), and let $1 \le r \le \infty$. Then $p \in L^{r^*}(\Omega)$, for $r^* = \frac{r}{r-1}$, is a subgradient $p \in \partial\chi_1(u)$ for $u \in L^r(\Omega)$ with $\chi_1(u) = 0$, if and only if*

$$\begin{aligned} &p = 0 \text{ a.e. on } supp(\{|u| < 1\}) \\ &p \le 0 \text{ a.e. on } supp(\{u = -1\}) \\ &p \ge 0 \text{ a.e. on } supp(\{u = 1\}). \end{aligned}$$

Proof. Let $p \in \partial\chi_1(u)$. Then we can choose $v = u + \epsilon w$ for w being any bounded function supported in $\{|u| < 1-\alpha\}$ for arbitrary $0 < \alpha < 1$. If ϵ is sufficiently small we have $|v| \le 1$. Hence

$$0 \ge \langle v-u, p\rangle = \epsilon\int_{\{|u|<1-\alpha\}} wp\, dx.$$

Since we can choose ϵ both positive and negative, we obtain

$$\int_{\{|u|<1-\alpha\}} wp\, dx = 0.$$

Because $0 < \alpha < 1$ and w are arbitrary we conclude $p = 0$ on the support of $\{|u| < 1\}$. If we choose $v = u + w$ with w is an arbitrary bounded function with

$$\begin{cases} 0 \le w \le 1 & \text{on } supp(\{-1 \le u \le 0\}) \\ w = 0 & \text{on } supp(\{0 < u \le 1\}). \end{cases}$$

Then v is still between -1 and 1 and

$$\begin{aligned} 0 \ge \langle v-u, p\rangle &= \int_{\{u=-1\}} wp\, dx + \int_{\{u=1\}} wp\, dx \\ &= \int_{\{u=-1\}} wp\, dx. \end{aligned}$$

Because w is arbitrary and positive on $\{u = -1\}$ it follows that $p \leq 0$ a.e. on $\{u = -1\}$. If we choose now $v = u + w$ with w is an arbitrary bounded function with

$$\begin{cases} w = 0 & \text{on } supp(\{-1 \leq u \leq 0\}) \\ -1 \leq w \leq 0 & \text{on } supp(\{0 < u \leq 1\}). \end{cases}$$

Then v is still between -1 and 1 and

$$\begin{aligned} 0 \geq \langle v - u, p \rangle &= \int_{\{u=-1\}} wp\, dx + \int_{\{u=1\}} wp\, dx \\ &= \int_{\{u=1\}} wp\, dx. \end{aligned}$$

Analogue to before, since w is arbitrary and negative on $\{u = 1\}$ it follows that $p \geq 0$ a.e. on $\{u = 1\}$.
On the other hand assume that

$$\begin{aligned} &p = 0 \text{ a.e. on } supp(\{|u| < 1\}) \\ &p \leq 0 \text{ a.e. on } supp(\{u = -1\}) \\ &p \geq 0 \text{ a.e. on } supp(\{u = 1\}). \end{aligned}$$

We need to verify the subgradient property

$$\langle v - u, p \rangle \leq \chi_1(v) - \chi_1(u) = \chi_1(v) \text{ for all } v \in L^r(\Omega)$$

only for $\chi_1(v) = 0$, since it is trivial for $\chi_1(v) = \infty$. So let $v \in L^r(\Omega)$ be a function between -1 and 1 almost everywhere on Ω. Then with p as above we obtain

$$\begin{aligned} \langle v - u, p \rangle &= \int_{\{u=-1\}} p(v - u)\, dx + \int_{\{u=1\}} p(v - u)\, dx \\ &= \int_{\{u=-1\}} p(v + 1)\, dx + \int_{\{u=1\}} p(v - 1)\, dx. \end{aligned}$$

Since $-1 \leq v \leq 1$ the first and the second term are always ≤ 0 since $p \leq 0$ for $\{u = -1\}$ and $p \geq 0$ for $\{u = 1\}$ respectively. Therefore $\langle v - u, p \rangle \leq 0$ and we are done. □

3.4. Error estimation and stability analysis with the Bregman distance. In the following analysis we want to present estimations for both the error we actually make in inpainting an image with our $TV - H^{-1}$ approach (1.3) (see (3.11)) and for the stability of solutions for this problem (see (3.12)) in terms of the Bregman distance. Let $f_{dam} \in L^2(\Omega)$ be the given damaged image with inpainting domain $D \subset \Omega$ and f_{true} the original image. We consider the stationary equation to (1.3), i.e.,

$$-\Delta p + \lambda(u - f_{dam}) = 0, \quad p \in \partial TV(u), \tag{3.6}$$

where we define $TV(u)$ as a functional over $L^2(\Omega)$ as

$$TV(u) = \begin{cases} |Du|(\Omega) & \text{if } u \in BV(\Omega), \|u\|_{L^\infty} \leq 1 \\ +\infty & \text{otherwise.} \end{cases}$$

In the subsequent analysis we want to characterize the error we make by solving (3.6) for u, i.e., how large do we expect the distance between the restored image u and the original image f_{true} to be. We are going to determine the inpainting error of our approach (3.6) in terms of the Bregman distance (cf. [10]). In [10] the general equation

$$p + \lambda_0 A^*(Au - f_{dam}) = 0,$$

is considered, where A is a bounded linear operator and A^* its adjoint. Let Δ^{-1} be the inverse operator to $-\Delta$ with zero Dirichlet boundary conditions as before. In our case the operator A is the embedding operator from $L^2(\Omega)$ into $H^{-1}(\Omega)$ and the adjoint operator $A^* = \Delta^{-1}$ which maps $H^{-1}(\Omega)$ into $H_0^1(\Omega)$. We assume that the given image f_{dam} coincides with f_{true} outside of the inpainting domain, i.e.,

$$\begin{array}{ll} f_{dam} = f_{true} & \text{in } \Omega \setminus D \\ f_{dam} = 0 & \text{in } D. \end{array} \tag{3.7}$$

Further we assume that f_{true} satisfies the so called source condition:

There exists $\xi \in \partial TV(f_{true})$ such that $\xi = A^* q = \Delta^{-1} q$ for a source element $q \in H^{-1}(\Omega)$. (3.8)

It can be shown (cf. [9]) that this is equivalent to require from f_{true} to be a minimizer of

$$TV(u) + \frac{\lambda_0}{2} \|Au - f_{dam}\|^2 ,$$

for arbitrary $f_{dam} \in H^{-1}(\Omega)$ and $\lambda_0 \in \mathbb{R}$.

For the following analysis we first rewrite (3.6). For $\hat{u}$, a solution of (3.6), we get

$$\hat{p} + \lambda_0 \Delta^{-1}(\hat{u} - f_{true}) = \Delta^{-1}\left[(\lambda_0 - \lambda)(\hat{u} - f_{true})\right], \quad \hat{p} \in \partial TV(\hat{u}).$$

Here we replaced f_{dam} by f_{true} using assumption (3.7). By adding a $\xi \in \partial TV(f_{true})$ from (3.8) to the above equation we obtain

$$\hat{p} - \xi + \lambda_0 \Delta^{-1}(\hat{u} - f_{true}) = -\xi + \lambda_0 \Delta^{-1}\left[\left(1 - \frac{\lambda}{\lambda_0}\right)(\hat{u} - f_{true})\right]$$

Taking the duality product with $\hat{u} - f_{true}$ (which is just the inner product in $L^2(\Omega)$ in our case) we get

$$D_{TV}^{symm}(\hat{u}, f_{true}) + \lambda_0 \|\hat{u} - f_{true}\|_{-1}^2 = \left\langle \nabla\xi, \nabla\Delta^{-1}(\hat{u} - f_{true})\right\rangle + \lambda_0 \left\langle \left(1 - \frac{\lambda}{\lambda_0}\right)(\hat{u} - f_{true}), \hat{u} - f_{true}\right\rangle_{-1},$$

where

$$D_{TV}^{symm}(\hat{u}, f_{true}) = \langle \hat{u} - f_{true}, \hat{p} - \xi\rangle, \quad \hat{p} \in \partial TV(\hat{u}),\ \xi \in \partial TV(f_{true}),$$

is the symmetric Bregman distance (cf. [10]). An application of Young's inequality yields

$$D_{TV}^{symm}(\hat{u}, f_{true}) + \frac{\lambda_0}{2} \|\hat{u} - f_{true}\|_{-1}^2 \le \frac{1}{\lambda_0} \|\xi\|_1^2 + \lambda_0 \left\| \left(1 - \frac{\lambda}{\lambda_0}\right)(\hat{u} - f_{true})\right\|_{-1}^2 \tag{3.9}$$

For the last term we obtain

$$\begin{aligned} &\left\|\left(1 - \tfrac{\lambda}{\lambda_0}\right) v\right\|_{-1} = \sup_{\phi, \|\phi\|_{-1}=1} \left\langle \phi, \left(1 - \tfrac{\lambda}{\lambda_0}\right) v\right\rangle_{-1} = \sup_{\phi, \|\phi\|_{-1}=1} -\left\langle \Delta^{-1}\phi, \left(1 - \tfrac{\lambda}{\lambda_0}\right) v\right\rangle \\ &= \sup_{\phi, \|\phi\|_{-1}=1} -\left\langle \left(1 - \tfrac{\lambda}{\lambda_0}\right)\Delta^{-1}\phi, v\right\rangle \le_{\text{Hölder}} \|v\|_2 \cdot \sup_{\phi, \|\phi\|_{-1}=1} \left\|\left(1 - \tfrac{\lambda}{\lambda_0}\right)\Delta^{-1}\phi\right\|_2. \end{aligned}$$

With $\Delta^{-1} : H^{-1} \to H^1 \hookrightarrow L^r$, $2 < r < \infty$ we get

$$\begin{aligned} &\int_\Omega \left(\left(1 - \tfrac{\lambda}{\lambda_0}\right)\Delta^{-1}\phi\right)^2 dx = \int_D \left(\Delta^{-1}\phi\right)^2 dx \le_{\text{Hölder}} |D|^{\frac{1}{q'}} \cdot \left(\int_\Omega \left(\Delta^{-1}\phi\right)^{2q}\right)^{\frac{1}{q}} \\ &=_{\text{choose } q=\frac{r}{2}} |D|^{\frac{r-2}{r}} \cdot \left\|\Delta^{-1}\phi\right\|_p^2 \le_{H^1 \hookrightarrow L^r} C|D|^{\frac{r-2}{r}} \|\phi\|_{-1}^2 = C|D|^{\frac{r-2}{r}}, \end{aligned}$$

i.e.,

$$\left\|\left(1 - \frac{\lambda}{\lambda_0}\right) v\right\|_{-1}^2 \le C|D|^{\frac{r-2}{r}} \|v\|^2 . \tag{3.10}$$

Applying (3.10) to (3.9) we see that

$$D_{TV}^{symm}(\hat{u}, f_{true}) + \frac{\lambda_0}{2} \|\hat{u} - f_{true}\|_{-1}^2 \leq \frac{1}{\lambda_0} \|\xi\|_1^2 + C\lambda_0 \, |D|^{(r-2)/r} \, \|\hat{u} - f_{true}\|^2$$

To estimate the last term we use some error estimates for $TV-$ inpainting computed in [12]. First we have

$$\|\hat{u} - f_{true}\|^2 = \int_{\Omega \setminus D} (\hat{u} - f_{true})^2 \, dx + \int_D (\hat{u} - f_{true})^2 \, dx.$$

Since $\hat{u} - f_{true}$ is uniformly bounded in Ω (this follows from the L^∞ bound in the definition of $TV(u)$) we estimate the first term by a positive constant K_1 and the second term by the L^1 norm over D. We obtain

$$\|\hat{u} - f_{true}\|^2 \leq K_1 + K_2 \int_D |\hat{u} - f_{true}| \, dx.$$

Now let $\hat{u} \in BV(\Omega)$ be given by $\hat{u} = u^s + u^d$, where u^s is a smooth function and u^d is a piecewise constant function. Following the error analysis in [12] (Theorem 8.) for functions $\hat{u} \in BV(\Omega)$ we have

$$\begin{aligned} \|\hat{u} - f_{true}\|^2 &\leq K_1 + K_2 \mathrm{err}(D) \\ &\leq K_1 + K_2 \left(|D| \, C \left(M(u^s), \beta \right) + 2 \left| R(u^d) \right| \right), \end{aligned}$$

where $M(u^s)$ is the smoothness bound for u^s, β is determined from the shape of D, and the error region $R(u^d)$ is defined from the level lines of u^d. Note that in general the error region from higher-order inpainting models including the TV seminorm is smaller than that from $TV - L^2$ inpainting (cf. Section 3.2. in [12]).

Finally we end up with

$$D_J^{symm}(\hat{u}, f_{true}) + \frac{\lambda_0}{2} \|\hat{u} - f_{true}\|_{-1}^2 \leq \frac{1}{\lambda_0} \|\xi\|_1^2 + C\lambda_0 \, |D|^{(r-2)/r} \, \mathrm{err}_{\mathrm{inpaint}}, \tag{3.11}$$

with

$$\mathrm{err}_{\mathrm{inpaint}} := K_1 + K_2 \left(|D| \, C \left(M(u^s), \beta \right) + 2 \left| R(u^d) \right| \right).$$

The first term in (3.11) depends on the regularizer TV, and the second term on the size of the inpainting domain D.

Remark 3.6. *From inequality* (3.11) *we derive an optimal scaling for λ_0, i.e., a scaling which minimizes the inpainting error. It reads*

$$\begin{aligned} &\lambda_0^2 \, |D|^{\frac{r-2}{r}} \sim 1 \\ &\lambda_0 \sim |D|^{-\frac{r-2}{2r}}. \end{aligned}$$

In two space dimensions r can be chosen arbitrarily big, which gives $\lambda_0 \sim 1/\sqrt{|D|}$ as the optimal order for λ_0.

Stability estimates for (3.6) can also be derived with an analogous technique. For u_i being the solution of (3.6) with $f_{dam} = f_i$ (again assuming that $f_i = f_{true}$ in $\Omega \setminus D$), the estimate

$$D_J^{symm}(u_1, u_2) + \frac{\lambda_0}{2} \|u_1 - u_2\|_{-1}^2 \leq \frac{\lambda_0}{2} \int_D (f_1 - f_2)^2 \, dx \tag{3.12}$$

holds.

4. Numerics. In the following numerical results for the two inpainting approaches (1.1) and (1.3) are presented. For both approaches we used convexity splitting algorithms, proposed by Eyre in [21], for the discretization in time. For more details to the application of convexity splitting algorithms in higher order inpainting compare [8].

For the space discretization we used the cosine transform to compute the finite differences for the derivatives in a fast way and to preserve the Neumann boundary conditions in our inpainting approaches (also cf. [8] for a detailed description).

4.1. Convexity splitting scheme for Cahn-Hilliard inpainting. For the discretization in time we use a convexity splitting scheme applied by Bertozzi et al. [7] to Cahn-Hilliard inpainting. The original Cahn-Hilliard equation is a gradient flow in H^{-1} for the energy

$$E_1[u] = \int_\Omega \frac{\epsilon}{2}\left|\nabla u\right|^2 + \frac{1}{\epsilon}F(u)\ dx,$$

while the fitting term in (1.1) can be derived from a gradient flow in L^2 for the energy

$$E_2[u] = \frac{1}{2}\int_\Omega \lambda(f-u)^2\ dx.$$

We apply convexity splitting for both E_1 and E_2 separately. Namely we split E_1 as $E_1 = E_{11} - E_{12}$ with

$$E_{11} = \int_\Omega \frac{\epsilon}{2}\left|\nabla u\right|^2 + \frac{C_1}{2}\left|u\right|^2\ dx,$$

and

$$E_{12} = \int_\Omega -\frac{1}{\epsilon}F(u) + \frac{C_1}{2}\left|u\right|^2\ dx.$$

A possible splitting for E_2 is $E_2 = E_{21} - E_{22}$ with

$$E_{21} = \frac{1}{2}\int_\Omega \frac{C_2}{2}\left|u\right|^2\ dx,$$

and

$$E_{22} = \frac{1}{2}\int_\Omega -\lambda(f-u)^2 + \frac{C_2}{2}\left|u\right|^2\ dx.$$

For the splittings discussed above the resulting time-stepping scheme is

$$\frac{u_{k+1}-u_k}{\tau} = -\nabla_{H^{-1}}(E_{11}^{k+1} - E_{12}^{k}) - \nabla_{L^2}(E_{12}^{k+1} - E_{22}^{k}),$$

where $\nabla_{H^{-1}}$ and ∇_{L^2} represent gradient descent with respect to the H^{-1} inner product and the L^2 inner product respectively. This translates to a numerical scheme of the form

$$\frac{u_{k+1}-u_k}{\tau} + \epsilon\Delta\Delta u_{k+1} - C_1\Delta u_{k+1} + C_2 u_{k+1} = \frac{1}{\epsilon}\Delta F'(u_k) - C_1\Delta u_k + \lambda(f-u_k) + C_2 u_k.$$

To make sure that E_{11}, E_{12} and E_{21}, E_{22} are convex the constants $C_1 > \frac{1}{\epsilon}$, $C_2 > \lambda_0$.

4.2. Convexity splitting scheme for $TV - H^{-1}$ inpainting. We consider equation (1.3) where $\hat{p} \in \partial TV(u)$ is replaced by the formal expression $\nabla \cdot (\frac{\nabla u}{|\nabla u|})$, namely

$$u_t = -\Delta(\nabla \cdot (\frac{\nabla u}{|\nabla u|})) + \lambda(f-u). \tag{4.1}$$

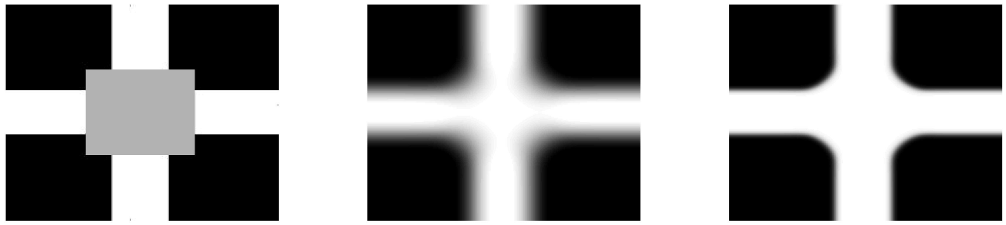

FIG. 4.1. *Destroyed binary image and the solution of Cahn-Hilliard inpainting with switching ϵ value: u(1200) with $\epsilon = 0.1$, u(2400) with $\epsilon = 0.01$*

FIG. 4.2. *Destroyed binary image and the solution of Cahn-Hilliard inpainting with switching ϵ value: u(800) with $\epsilon = 0.8$, u(1600) with $\epsilon = 0.01$*

Similar to the convexity splitting for the Cahn-Hilliard inpainting we propose the following splitting for the TV-H^{-1} inpainting equation. The regularizing term in (4.1) can be modeled by a gradient flow in H^{-1} of the energy

$$E_1 = \int_\Omega |\nabla u|\; dx.$$

We split E_1 in $E_{11} - E_{12}$ with

$$\begin{aligned} E_{11} &= \int_\Omega \frac{C_1}{2}|\nabla u|^2\; dx \\ E_{12} &= \int_\Omega -|\nabla u| + \frac{C_1}{2}|\nabla u|^2\; dx. \end{aligned}$$

The fitting term is a gradient flow in L^2 of the energy

$$E_2 = \frac{1}{2}\int_\Omega \lambda(f-u)^2 dx$$

and is splitted into $E_2 = E_{21} - E_{22}$ with

$$\begin{aligned} E_{21} &= \int_\Omega \frac{C_2}{2}|u|^2\; dx \\ E_{22} &= \frac{1}{2}\int_\Omega -\lambda(f-u)^2 + C_2|u|^2\; dx. \end{aligned}$$

Analogous to above the resulting time-stepping scheme is

$$\frac{u_{k+1}-u_k}{\tau} + C_1\Delta\Delta u_{k+1} + C_2 u_{k+1} = C_1\Delta\Delta u_k - \Delta(\nabla\cdot(\frac{\nabla u_k}{|\nabla u_k|})) + C_2 u_k + \lambda(f-u_k).$$

In order to make the scheme unconditionally stable, the constants C_1 and C_2 have to be chosen so that $E_{11}, E_{12}, E_{21}, E_{22}$ are all convex. The choice of C_1 depends on the regularization of the total variation we are using. Using the square regularization $|\nabla u|$ is replaced by $\sqrt{|\nabla u|^2 + \delta^2}$ the condition turns out to be $C_1 > \frac{1}{\delta}$ and $C_2 > \lambda_0$.

FIG. 4.3. *Destroyed binary image and the solution of Cahn-Hilliard inpainting with switching ϵ value: u(800) with $\epsilon = 0.8$, u(1600) with $\epsilon = 0.01$*

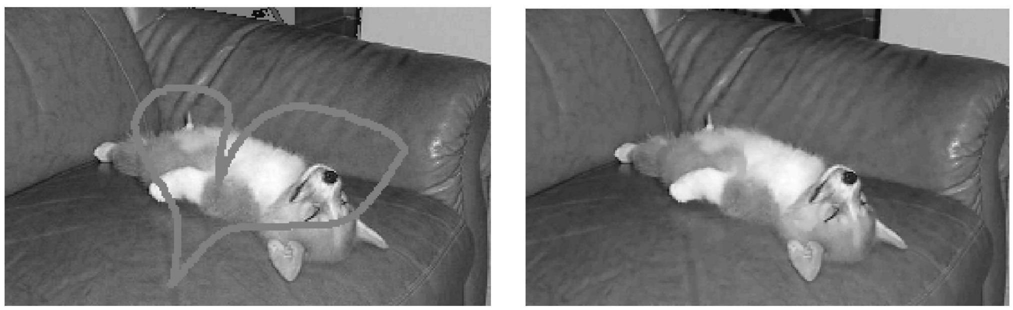

FIG. 4.4. *$TV - H^{-1}$ inpainting: u(1000) with $\lambda = 10^3$*

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Appendix A. Neumann boundary conditions and the space $H_\partial^{-1}(\Omega)$.

In this section we want to pose the Cahn-Hilliard inpainting problem with Neumann boundary conditions in a way such that the analysis from Section 2 can be carried out in a similar way. Namely we consider

$$\begin{cases} u_t = \Delta(-\epsilon\Delta u + \frac{1}{\epsilon}F'(u)) + \lambda(f-u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

For the existence of a stationary solution of this equation we consider again a fixed point approach similar to (2.4) in the case of Dirichlet boundary conditions, i.e.,

$$\begin{cases} \frac{u-v}{\tau} = \Delta(-\epsilon\Delta u + \frac{1}{\epsilon}F'(u)) + \lambda(f-u) + (\lambda_0 - \lambda)(v-u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial\left(\epsilon\Delta u - \frac{1}{\epsilon}F'(u)\right)}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.1}$$

To reformulate the above equation in terms of the operator Δ^{-1} with Neumann boundary conditions we first have to introduce the space $H_\partial^{-1}(\Omega)$ in which the operator Δ^{-1} is now the inverse of $-\Delta$ with Neumann boundary conditions.

Thus we define the non-standard Hilbert space

$$H_\partial^{-1}(\Omega) = \left\{ F \in H^1(\Omega)^* \mid \langle F, 1\rangle_{(H^1)^*, H^1} = 0 \right\}.$$

FIG. 4.5. *$TV - H^{-1}$ inpainting: u(1000) with $\lambda = 10^3$*

Since Ω is bounded we know $1 \in H^1(\Omega)$, hence $H_\partial^{-1}(\Omega)$ is well defined. Before we define a norm and an inner product on $H_\partial^{-1}(\Omega)$ we have to define more spaces. Let

$$H_\phi^1(\Omega) = \left\{ \psi \in H^1(\Omega) : \int_\Omega \psi \, dx = 0 \right\},$$

with norm $\|u\|_{H_\phi^1} := \|\nabla u\|_{L^2}$ and inner product $\langle u, v\rangle_{H_\phi^1} := \langle \nabla u, \nabla v\rangle_{L^2}$. This is a Hilbert space and the norms $\|.\|_{H^1}$ and $\|.\|_{H_\phi^1}$ are equivalent on $H_\phi^1(\Omega)$. Let $(H_\phi^1(\Omega))^*$ denote the dual of $H_\phi^1(\Omega)$. We will use $(H_\phi^1(\Omega))^*$ to induce an inner product on $H_\partial^{-1}(\Omega)$. Given $F \in (H_\phi^1(\Omega))^*$ with associate $u \in H_\phi^1(\Omega)$ (from the Riesz representation theorem) we have by definition

$$\langle F, \psi\rangle_{(H_\phi^1)^*, H_\phi^1} = \langle u, \psi\rangle_{H_\phi^1} = \langle \nabla u, \nabla \psi\rangle_{L^2} \quad \forall \psi \in H_\phi^1(\Omega).$$

Lets now define a norm and an inner product on $H_\partial^{-1}(\Omega)$.

DEFINITION A.1.

$$\begin{aligned} H_\partial^{-1}(\Omega) &:= \left\{ F \in H^1(\Omega)^* \mid \langle F, 1\rangle_{(H^1)^*, H^1} = 0 \right\} \\ \|F\|_{H_\partial^{-1}} &:= \left\| F|\, H_\phi^1 \right\|_{(H_\phi^1)^*} \\ \langle F_1, F_2\rangle_{H_\partial^{-1}} &:= \langle \nabla u_1, \nabla u_2\rangle_{L^2}, \end{aligned}$$

where $F_1, F_2 \in H_\partial^{-1}(\Omega)$ and where $u_1, u_2 \in H_\phi^1(\Omega)$ are the associates of $F_1|\, H_\phi^1$, $F_2|\, H_\phi^1 \in (H_\phi^1(\Omega))^*$.

At this point it is not entirely obvious that for a given $F \in H_\partial^{-1}(\Omega)$ we have $F|\, H_\phi^1 \in (H_\phi^1(\Omega))^*$. That this is the case though is explained in the following theorem.

THEOREM A.2.

1. $H_\partial^{-1}(\Omega)$ is closed in $(H^1(\Omega))^*$.

2. The norms $\|.\|_{H_\partial^{-1}}$ and $\|.\|_{(H^1)^*}$ are equivalent on $H_\partial^{-1}(\Omega)$.

Theorem A.2 can be easily checked just by the application of the definitions and the fact that the norms $\|.\|_{H^1}$ and $\|.\|_{H_\phi^1}$ are equivalent on $H_\phi^1(\Omega)$. From point 1. of the theorem we have that $H_\partial^{-1}(\Omega)$ is a Hilbert space w.r.t. the $(H^1(\Omega))^*$ norm and point 2. tells us that the norms $\|.\|_{H_\partial^{-1}}$ and $\|.\|_{(H^1)^*}$ are equivalent on $H_\partial^{-1}(\Omega)$. Therefore the norm in Definition A.1 is well defined and $H_\partial^{-1}(\Omega)$ is a Hilbert space w.r.t. $\|.\|_{H_\partial^{-1}}$.

In the following we want to characterize elements $F \in H_\partial^{-1}(\Omega)$. By the above definition we have for each $F \in H_\partial^{-1}(\Omega)$, there exists a unique element $u \in H_\phi^1(\Omega)$ such that

$$\langle F, \psi\rangle_{(H^1)^*, H^1} = \int_\Omega \nabla u \cdot \nabla \psi \, dx, \quad \forall \psi \in H_\phi^1(\Omega). \tag{A.2}$$

Since $\langle F,1\rangle_{(H^1)^*,H^1}=0$, we see that $\langle F,\psi+K\rangle_{(H^1)^*,H^1}=\langle F,\psi\rangle_{(H^1)^*,H^1}$ for all constants $K\in\mathbb{R}$ and therefore (A.2) extends to all $\psi\in H^1(\Omega)$. We define

$$\Delta^{-1}F := u \tag{A.3}$$

the unique solution to (A.2).

Now suppose $F\in L^2(\Omega)$ and assume $u\in H^2(\Omega)$. Set $\langle F,\psi\rangle := \int_\Omega F\psi\, dx$. Because $L^2(\Omega)\subset H_\partial^{-1}(\Omega)$ an element F is also an element in $H_\partial^{-1}(\Omega)$. Thus there exists a unique element $u\in H_\phi^1(\Omega)$ such that

$$\int_\Omega(-\Delta u - F)\psi\, dx + \int_{\partial\Omega}\nabla u\cdot\nu\psi\, ds = 0,\quad \forall\psi\in H_\phi^1(\Omega).$$

Therefore $u\in H_\phi^1(\Omega)$ is the unique weak solution of the following problem:

$$\begin{cases} -\Delta u - F = 0 & \text{in } \Omega\\ \nabla u\cdot\nu = 0 & \text{on } \partial\Omega.\end{cases} \tag{A.4}$$

Remark A.3. *With the above characterization of elements $F\in H_\partial^{-1}(\Omega)$ and the notation* (A.3) *for its associates the inner product and the norm can be written as*

$$\langle F_1,F_2\rangle_{H_\partial^{-1}} := \int_\Omega \nabla\Delta^{-1}F_1\cdot\nabla\Delta^{-1}F_2\, dx,\quad \forall F_1,F_2\in H_\partial^{-1}(\Omega),$$

and norm

$$\|F\|_{H_\partial^{-1}} := \sqrt{\int_\Omega(\nabla\Delta^{-1}F)^2\, dx}.$$

Throughout the rest of this appendix we will write the short forms $\langle .,.\rangle_{-1}$ and $\|.\|_{-1}$ for the inner product and the norm in $H_\partial^{-1}(\Omega)$ respectively.

Now its important to notice that in order to rewrite (A.1) in terms of Δ^{-1} we require the "right hand side" of the equation, i.e., $\frac{u-v}{\tau}+\lambda(u-f)+(\lambda_0-\lambda)(u-v)$ to be an element of our new space $H_\partial^{-1}(\Omega)$ (cf. Definition A.1). In other words the "right hand side" has to have zero mean over Ω. Because we cannot guarantee this property for solutions of the fixed point equation (A.1) we are going to modify the right hand side by subtracting its mean. Let

$$\begin{aligned} F_\Omega &= \tfrac{1}{\tau}F_\Omega^1+\lambda_0 F_\Omega^2\\ F_\Omega^1 &= \tfrac{1}{|\Omega|}\textstyle\int_\Omega(u-v)\, dx\\ F_\Omega^2 &= \tfrac{1}{|\Omega|}\textstyle\int_\Omega \frac{\lambda}{\lambda_0}(u-f)+\left(1-\frac{\lambda}{\lambda_0}\right)(u-v)\, dx,\end{aligned}$$

and consider instead of (A.1) the equation

$$\begin{cases} \epsilon\Delta u-\frac{1}{\epsilon}F'(u)=\Delta^{-1}\left(\frac{u-v}{\tau}-\lambda(f-u)-(\lambda_0-\lambda)(v-u)-F_\Omega\right) & \text{in } \Omega,\\ \frac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega,\end{cases}$$

where the second Neumann boundary condition $\frac{\partial\left(\epsilon\Delta u-\frac{1}{\epsilon}F'(u)\right)}{\partial\nu}=0$ on $\partial\Omega$ is included in the definition of Δ^{-1}. The functional of the corresponding variational formulation then reads

$$\begin{aligned} J^\epsilon(u,v) = &\int_\Omega\left(\frac{\epsilon}{2}|\nabla u|^2+\frac{1}{\epsilon}F(u)\right)dx+\frac{1}{2\tau}\left\|(u-v)-F_\Omega^1\right\|_{-1}^2\\ &+\frac{\lambda_0}{2}\left\|\left(u-\frac{\lambda}{\lambda_0}f-\left(1-\frac{\lambda}{\lambda_0}\right)v\right)-F_\Omega^2\right\|_{-1}^2.\end{aligned}$$

With these definitions the proof for the existence of a stationary solution for the modified Cahn-Hilliard equation with Neumann boundary conditions can be carried out similarly to the proof in

Section 2. Note that every solution of (1.1) fulfills $\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega \lambda(f-u)\,dx$. This means that for a stationary solution $\hat{u}$ the integral $C\int_\Omega \lambda(f-u)\,dx = 0$ for every constant $C \in \mathbb{R}$ (,i.e., the "right hand side" has zero mean and therefore $F^1_\Omega = F^2_\Omega = 0$).

Appendix B. Functions of bounded variation.

The following results can be found in [3]. Let $\Omega \subset \mathbb{R}^2$ be an open and bounded Lipschitz domain. As in [3] the space of functions of bounded variation $BV(\Omega)$ in two space dimensions is defined as follows:

DEFINITION B.1. *($BV(\Omega)$) Let $u \in L^1(\Omega)$. We say that u is a* **function of bounded variation in** Ω *if the distributional derivative of u is representable by a finite Radon measure in Ω, i.e., if*

$$\int_\Omega u \frac{\partial \phi}{\partial x_i}\,dx = -\int_\Omega \phi d D_i u \quad \forall \phi \in C_c^\infty(\Omega),\ i = 1,2,$$

for some $\mathbb{R}^2-$ valued measure $Du = (D_1 u, D_2 u)$ in Ω. The vector space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$. Further, the space $BV(\Omega)$ can be characterized by the total variation of Du. For this we first define the so called variation $V(u,\Omega)$ of a function $u \in L^1_{loc}(\Omega)$.

DEFINITION B.2. *(Variation) Let $u \in L^1_{loc}(\Omega)$. The* **variation** *$V(u,\Omega)$ of u in Ω is defined by*

$$V(u,\Omega) := \sup\left\{ \int_\Omega u\, div\phi\, dx :\ \phi \in \left(C_c^1(\Omega)\right)^2, \|\phi\|_\infty \leq 1 \right\}.$$

A simple integration by parts proves that

$$V(u,\Omega) = \int_\Omega |\nabla u|\ dx,$$

if $u \in C^1(\Omega)$. By a standard density argument this is also true for functions $u \in W^{1,1}(\Omega)$. Before we proceed with the characterization of $BV(\Omega)$ let us recall the definition of the total variation of a measure:

DEFINITION B.3. *(Total variation of a measure) Let $(X,\mathcal{E})$ be a measure space. If μ is a measure, we define its* **total variation** *$|\mu|$ as follows:*

$$|\mu|(E) := \sup\left\{ \sum_{h=0}^\infty |\mu(E_h)| : E_h \in \mathcal{E} \text{ pairwise disjoint },\ E = \bigcup_{h=0}^\infty E_h \right\},\ \forall E \subset \mathcal{E}.$$

With Definition B.2 the space $BV(\Omega)$ can be characterized as follows

THEOREM B.4. *Let $u \in L^1(\Omega)$. Then, u belongs to $BV(\Omega)$ if and only if $V(u,\Omega) < \infty$. In addition, $V(u,\Omega)$ coincides with $|Du|(\Omega)$, the total variation of Du, for any $u \in BV(\Omega)$ and $u \mapsto |Du|(\Omega)$ is lower semicontinuous in $BV(\Omega)$ with respect to the $L^1_{loc}(\Omega)$ topology.*

Note that $BV(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega).$$

Now we introduce so called weak* convergence in $BV(\Omega)$ which is useful for its compactness properties. Note that this convergence is much weaker than the norm convergence.

DEFINITION B.5. *(Weak* convergence) Let $u, u_h \in BV(\Omega)$. We say that (u_h)* **weakly* converges in** $BV(\Omega)$ **to** u *(in signs $u_h \overset{*}{\rightharpoonup} u$) if (u_h) converges to u in $L^1(\Omega)$ and (Du_h) weakly* converges to Du in all (Ω), i.e.,*

$$\lim_{h\to\infty} \int_\Omega \phi\, dDu_h = \int_\Omega \phi\, dDu \quad \forall \phi \in C_0(\Omega).$$

A simple criterion for weak* convergence is the following:

Theorem B.6. *Let $(u_h) \subset BV(\Omega)$. Then (u_h) weakly* converges to u in $BV(\Omega)$ if and only if (u_h) is bounded in $BV(\Omega)$ and converges to u in $L^1(\Omega)$.*

Further we have the following compactness theorem:

Theorem B.7. *(Compactness for $BV(\Omega)$)*

- *Let Ω be a bounded domain with compact Lipschitz boundary. Every sequence $(u_h) \subset BV_{loc}(\Omega)$ satisfying*

$$sup\left\{\int_A |u_h|\ dx + |Du_h|(A) :\ h \in \mathbb{N}\right\} < \infty \quad \forall A \subset\subset \Omega \text{ open},$$

  admits a subsequence (u_{h_k}) converging in $L^1_{loc}(\Omega)$ to $u \in BV_{loc}(\Omega)$. If the sequence is further bounded in $BV(\Omega)$ then $u \in BV(\Omega)$ and a subsequence converges weakly* to u.
- *Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary. Then, every uniformly bounded sequence $(u_k)_{k\geq 0}$ in $BV(\Omega)$ is relatively compact in $L^r(\Omega)$ for $1 \leq r < \frac{d}{d-1}$, $d \geq 1$. Moreover, there exists a subsequence u_{k_j} and u in $BV(\Omega)$ such that $u_{k_j} \rightharpoonup u$ weakly* in $BV(\Omega)$. In particular for $d = 2$ this compact embedding holds for $1 \leq r < 2$.*

Let $u \in L^1(\Omega)$. We introduce the mean value u_Ω of u as

$$u_\Omega := \frac{1}{|\Omega|}\int_\Omega u(x)\ dx.$$

A generalization of the Poincare inequality gives the so called Poincare-Wirtinger inequality for functions in $BV(\Omega)$.

Theorem B.8. *(Poincare-Wirtinger inequality) If $\Omega \subset \mathbb{R}^2$ is a bounded, open and connected domain with compact Lipschitz boundary, we have*

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C\,|Du|(\Omega) \quad \forall u \in BV(\Omega), \quad 1 \leq p \leq 2$$

for some constant C depending only on Ω.

Appendix C. Theorems. The following definitions and results can be found in [20].

We introduce the notion of the subdifferential of a function.

Definition C.1. *Let X be a locally convex space, X^* its dual, $\langle .,. \rangle$ the bilinear pairing over $X \times X^*$ and F a mapping of X into $\mathbb{R}$. The* **subdifferential** *of F at $u \in X$ is defined as*

$$\partial F(u) = \{p \in X^* |\, \langle v - u, p\rangle \leq F(v) - F(u), \forall v \in X\}.$$

Every normed vector space is a locally convex space and therefore the theory of subdifferentials applies to our framework (with $X = BV(\Omega)$).

In addition we recall the Poincare inequality and the Rellich-Kondrachov Compactness theorem (cf. [2], Theorem 8.7, p. 243).

Theorem C.2. *(Poincare's inequality). Assume that $1 \leq p \leq \infty$ and that Ω is precompact open subset of n-dimensional Euclidean space $\mathbb{R}^n$ having Lipschitz boundary (i.e., Ω is an open, bounded Lipschitz domain). Then there exists a constant C, depending only on Ω and p, such that, for every function u in the Sobolev space $H^p(\Omega)$,*

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)},$$

where $u_\Omega = \frac{1}{|\Omega|}\int_\Omega u(y)\,\mathrm{d}y$ is the average value of u over Ω, with $|\Omega|$ denoting the Lebesgue measure of the domain Ω.

Theorem C.3. *(Rellich-Kondrachov Compactness Theorem) Assume Ω is a bounded and open subset of $\mathbb{R}^d$ with Lipschitz boundary. Suppose $1 \leq r < d$. Then*

$$W^{1,r}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega),$$

for each $1 \leq q < \frac{dr}{d-r}$.

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