A CONVERGENCE RESULT FOR THE PERIODIC UNFOLDING METHOD RELATED TO FAST DIFFUSION ON MANIFOLDS

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Abstract. Based on the periodic unfolding method in periodic homogenization, we deduce a convergence result for gradients of functions defined on connected, smooth and periodic manifolds. Under the assumption of certain a-priori estimates of the gradient, which are typical for fast diffusion, the sum of a term involving a gradient with respect to the slow variable and one with respect to the fast variable is obtained in the homogenization limit. In addition, we show in a brief example how to apply this result and find for a reaction–diffusion equation defined on a periodic manifold that the homogenized equation contains a term describing macroscopic diffusion.

Résumé. A l'aide de la méthode d'éclatement périodique, nous démontrons un résultat de convergence des gradients de fonctions définies sur des variétés connexes, différentiables et périodiques. Sous certaines conditions d'estimation du gradient, typiques de la diffusion rapide, nous obtenons á la limite d'homogénéisation la somme d'un gradient de la variable globale et d'un gradient de la variable locale. Un exemple illustre l'utilisation de ce résultat: Pour une équation de réaction et diffusion définie sur une variété périodique, nous démontrons que l'équation homogénéisée contient un terme dépeindrant diffusion globale.

1. Setting. The periodic unfolding method is a technique to homogenize partial differential equations. The main idea is the introduction of an operator $\mathcal{T}_\varepsilon$, which maps a function φ_ε defined on a finely structured periodic domain $\Omega_\varepsilon \subset \mathbb{R}^n$ to a function $\mathcal{T}_\varepsilon(\varphi_\varepsilon)$ defined on $\Omega \times Y$, where $Y = [0,1]^n$ is the periodicity cell. With $\Omega \subset \mathbb{R}^n$ being homogeneous, the domain of the function $\mathcal{T}_\varepsilon(\varphi_\varepsilon)$ is independent of ε and hence, we are able to use well-known convergence results from functional analysis.

The periodic unfolding method was developed in [3, 4, 5, 6] based on ideas of [2]. It is the purpose of this note to extend these results by a weak compactness result for H^1-functions defined on a periodic manifold satisfying certain bounds (Theorem 4 below). These arise in problems involving fast surface diffusion, cf. §4. For utilization in the proof of Theorem 4, we also show an extension lemma (Lemma 5), which may be useful in related contexts as well.

We briefly describe the setting and summarize important results required in what follows. Let $\Omega \subset \mathbb{R}^n$ be a domain, and further let $\Omega_\varepsilon = \bigcup_{k\in\mathbb{Z}^n} \varepsilon(k+Y) \cap \Omega$ and $\Gamma_\varepsilon = \bigcup_{k\in\mathbb{Z}^n} \varepsilon(k+\Gamma) \cap \Omega$ be sets with periodic fine-structure with unit cell $Y = [0,1]^n$ and a smooth manifold $\Gamma \subset Y$, such that Γ_ε is smooth and connected and Ω is representable by a finite union of axis-parallel cuboids, each of which is assumed to have corner coordinates in $\mathbb{Q}^n$. This last technical assumption is required in order to use a certain extension operator, cf. Remark 6. Note that there also exist recent works in the context of periodic unfolding and manifolds, where the manifold itself is not periodic but has a periodic pattern on its surface [7], which is different from the setting considered here.

Let $\Xi_\varepsilon := \{\xi \in \mathbb{Z}^n | \ \varepsilon(\xi + Y) \subset \Omega\}$ and $\hat{\Omega}_\varepsilon := \text{interior}\{\bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi + \overline{Y})\}$. For every $z \in \mathbb{R}^n$, we define $[z]_Y$ as the unique integer combination $\sum_{i=1}^n k_i e_i$ of the periods such that $\{z\}_Y = z - [z]_Y \in Y$. The periodic unfolding operator $\mathcal{T}_\varepsilon$ is then defined as follows [4]:

Definition 1 *Let $\varphi \in L^p(\Omega_\varepsilon)$, $p \in [1,\infty]$. For any $\varepsilon > 0$ we define $\mathcal{T}_\varepsilon : L^p(\Omega_\varepsilon) \to L^p(\Omega \times Y)$ such that*

$$[\mathcal{T}_\varepsilon(\varphi)](x,y) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) \text{ a.e. for } (x,y) \in \hat{\Omega}_\varepsilon \times Y, \qquad [\mathcal{T}_\varepsilon(\varphi)](x,y) = 0 \text{ a.e. for } (x,y) \in \Omega\backslash\hat{\Omega}_\varepsilon \times Y.$$

The main advantage of using the periodic unfolding operator is that $\mathcal{T}_\varepsilon(\varphi)$ is defined on the fixed domain $\Omega \times Y$ even for varying ε. Thus, we may use standard convergence results from functional analysis. For example, the following weak compactness result in H^1 is proven in [5]. It is the main ingredient in identifying the limit problem when homogenizing typical reaction–diffusion equations stated on Ω_ε.

Theorem 2 *For every $\varepsilon > 0$, let φ_ε be in $H^1(\Omega_\varepsilon)$ and let $\|\varphi_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ be bounded independently of ε. Then there exists $\varphi \in H^1(\Omega)$ and $\hat{\varphi} \in L^2(\Omega, H^1_{\text{per}}(Y))$ such that, up to a subsequence,*

$$\mathcal{T}_\varepsilon(\varphi_\varepsilon) \overset{\varepsilon\to 0}{\rightharpoonup} \varphi \text{ weakly in } L^2(\Omega, H^1_{\text{per}}(Y)), \qquad \mathcal{T}_\varepsilon(\nabla_x\varphi_\varepsilon) \overset{\varepsilon\to 0}{\rightharpoonup} \nabla_x\varphi + \nabla_y\hat{\varphi} \text{ weakly in } L^2(\Omega, L^2(Y)).$$

When internal boundary terms are to be homogenized, e.g. arising from interface conditions or surface concentrations, the boundary periodic unfolding operator $\mathcal{T}^b_\varepsilon$ is introduced. It is defined as follows, see [6].

Definition 3 *Let $\varphi \in L^p(\Gamma_\varepsilon)$, $p \in [1,\infty]$. Then the boundary periodic unfolding operator $\mathcal{T}^b_\varepsilon : L^p(\Gamma_\varepsilon) \to L^p(\Omega\times\Gamma)$ is defined as*

$$\mathcal{T}^b_\varepsilon(\varphi)(x,y) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) \text{ a.e. for } (x,y)\in\hat{\Omega}_\varepsilon\times\Gamma, \qquad \mathcal{T}^b_\varepsilon(\varphi)(x,y) = 0 \text{ a.e. for } (x,y)\in \Omega\backslash\hat{\Omega}_\varepsilon\times\Gamma.$$

It is well-known in periodic homogenization that different scalings with the homogenization parameter lead to different limit behaviour (see e.g. [13], where weak compactness results in the spirit of Theorem 2 are discussed for different scalings). The canonical scaling of surface terms is ε, that of surface gradients is ε^3, which is due to the fact that $|\Gamma_\varepsilon| \sim \varepsilon^{-1}$ in the limit. For these scalings, associated with slow diffusion, local (or microscopic) diffusion in the unit cell, i.e. with respect to the y-variable, is obtained in the homogenization limit [1, 12].

The purpose of this contribution is to extend the results to fast diffusion, associated with a scaling of the surface gradients with ε^1. It turns out that this leads to global (or macroscopic) diffusion, i.e. with respect to the x-variable, in the homogenization limit.

In what follows, we formulate the main result in §2, present the proof in §3 and apply it to homogenize a prototypical diffusion problem in §4.

2. Statement of the main result. The main result is the following weak compactness result for H^1-functions defined on a manifold Γ_ε.

Theorem 4 *Let $\varphi_\varepsilon \in H^1(\Gamma_\varepsilon)$ be a sequence of functions with*

$$\varepsilon\|\varphi_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} + \varepsilon\|\nabla_\Gamma\varphi_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} \le C,$$

where C is independent of ε. Let P_Γ be the orthogonal projection from $\mathbb{R}^n$ to the tangent space $T_y\Gamma$ for every $y\in\Gamma$. Then two assertions hold true:
1. There exists a function $\varphi_0 \in H^1(\Omega)$ such that, up to a subsequence,

$$\mathcal{T}^b_\varepsilon(\varphi_\varepsilon) \overset{\varepsilon\to 0}{\rightharpoonup} \varphi_0 \text{ weakly in } L^2(\Omega\times\Gamma) \text{ and } \varepsilon\int_{\Gamma_\varepsilon}\varphi_\varepsilon\psi\, d\sigma_x \overset{\varepsilon\to 0}{\to} \frac{|\Gamma|}{|Y|}\int_\Omega \varphi_0\psi\, dx \text{ for all } \psi\in C^\infty(\Omega).$$

2. There exists a $\hat{\varphi}\in L^2(\Omega, H^1_{\text{per}}(\Gamma))$ such that, up to a subsequence,

$$\mathcal{T}^b_\varepsilon(\nabla_x\varphi_\varepsilon) \overset{\varepsilon\to 0}{\rightharpoonup} P_\Gamma\nabla_x\varphi_0 + \nabla_\Gamma\hat{\varphi} \text{ weakly in } L^2(\Omega\times\Gamma).$$

3. Proof of the main result. For later use in the proof of Theorem 4, we first show an inverse trace lemma.

Lemma 5 *Let $\Omega\subset\mathbb{R}^n$ and $\Gamma\subset[0,1]^n = Y$ be a smooth and compact hypersurface such that $\Gamma_\varepsilon = \bigcup_{k\in\mathbb{Z}^n}\varepsilon(k+\Gamma)\cap\Omega$ is a smooth, periodic and connected hypersurface. Let $f_\varepsilon\in H^1(\Gamma_\varepsilon)$. Then, there exists a function $u_\varepsilon\in H^1(\Omega)$ with $u_{\varepsilon|_{\Gamma_\varepsilon}} = f_\varepsilon$ such that*

$$\|u_\varepsilon\|^2_{L^2(\Omega)} \le C_1\varepsilon\|f_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} \quad \text{and} \quad \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \le C_2\varepsilon\left(\|f_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} + \|\nabla_\Gamma f_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)}\right)$$

for constants $C_1, C_2 > 0$ independent of ε.

Proof Because Γ is smooth and compact, the normal in each point $y\in\Gamma$, n_y, is well-defined. For small $\delta>0$ we define $Y^* = \{y + dn_y |\ y\in\Gamma,\ d\in(-\delta,\delta)\}$ so that for every $z\in Y^*$ there exist unique $y\in\Gamma$ and $d\in(-\delta,\delta)$ with $y + dn_y = z$. On the tube Y^* we define a Riemannian metric by g_{ij}, $i,j = 1,\dots n$, such that the tangential vectors $\frac{\mathrm{d}}{\mathrm{d}y^i}$, $i = 1,\dots,n-1$ form a basis of the tangent space $T_y\Gamma$ and $\frac{\mathrm{d}}{\mathrm{d}y^n}$ treats the normal direction n_y. Because n_y is orthogonal to the tangent space $T_y\Gamma$, it follows $g_{in} = g_{ni} = g^{in} = g^{ni} = 0$ for $i = 1,\dots,n-1$.

Now we define $\Omega^*_\varepsilon = \bigcup_{k\in\mathbb{Z}^n}\varepsilon(k+Y^*)\cap\Omega$ and consider the scaled unit cell εY with scaled tube εY^*. The width of εY^* is now $2\delta\varepsilon$ and $d\in(-\varepsilon\delta,\varepsilon\delta)$. Analogously one finds for every $x\in\Omega^*_\varepsilon$ unique $y\in\Gamma_\varepsilon$ and

$d \in (-\varepsilon\delta, \varepsilon\delta)$ such that $y + dn_y = x$. Because the additional direction n_y is perpendicular to the tangent space of Γ_ε, it holds that $|\Omega_\varepsilon^*| \leq 2\varepsilon\delta c_1|\Gamma_\varepsilon|$ for a constant $c_1 > 0$ independent of ε, which can be seen by calculating the Lebesgue-measure of the manifold Γ_ε using its charts by means of integration by substitution.

We define a function $\tilde{u}_\varepsilon \in H^1(\Omega_\varepsilon^*)$ by $\tilde{u}_\varepsilon(x) = \tilde{u}_\varepsilon(y + dn_y) = f_\varepsilon(y)$ for every $x \in \Omega_\varepsilon^*$. Then, the following holds:

$$\|\tilde{u}_\varepsilon\|^2_{L^2(\Omega_\varepsilon^*)} = \int_{\Omega_\varepsilon^*} \tilde{u}_\varepsilon^2(y + dn_y)\mathrm{d}x \leq 2c_1\varepsilon\delta \int_{\Gamma_\varepsilon} f_\varepsilon^2(y)\mathrm{d}\sigma_y = 2c_1\delta\varepsilon\|f_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)}.$$

To estimate the gradient, we consider the gradient in the coordinates $\frac{\mathrm{d}}{\mathrm{d}y^{i,\varepsilon}}$, $i = 1, \dots, n-1$ on $T_y\Gamma_\varepsilon$, and exploit that $f_\varepsilon(y)$ is independent of d,

$$\nabla_x \tilde{u}_\varepsilon(x) = \sum_{ij=1}^{n} g^{ij,\varepsilon}(x)\frac{\partial \tilde{u}_\varepsilon}{\partial y^{j,\varepsilon}}(y + dn_y)\frac{\mathrm{d}}{\mathrm{d}y^{i,\varepsilon}} = \sum_{ij=1}^{n} g^{ij,\varepsilon}(x)\frac{\partial f_\varepsilon}{\partial y^{j,\varepsilon}}(y)\frac{\mathrm{d}}{\mathrm{d}y^{i,\varepsilon}} = \sum_{ij=1}^{n-1} g^{ij,\varepsilon}(x)\frac{\partial f_\varepsilon}{\partial y^{j,\varepsilon}}(y)\frac{\mathrm{d}}{\mathrm{d}y^{i,\varepsilon}}$$

for every $x \in \Omega_\varepsilon^*$ with $x = y + dn_y$, $y \in \Gamma_\varepsilon$. Since the Riemannian metric tensor $g^{ij,\varepsilon}$ is continuous and Γ_ε compact, there exists a constant c_2 such that for small $\delta > 0$

$$|\nabla_x \tilde{u}_\varepsilon(x)|^2 \leq c_2|\sum_{ij=1}^{n-1} g^{ij,\varepsilon}(y)\frac{\partial f_\varepsilon}{\partial y^{j,\varepsilon}}(y)\frac{\mathrm{d}}{\mathrm{d}y^{i,\varepsilon}}|^2 = c_2|\nabla_\Gamma f_\varepsilon(y)|^2.$$

Now the norm of the gradient u_ε can be estimated,

$$\|\nabla\tilde{u}_\varepsilon\|^2_{L^2(\Omega_\varepsilon^*)} \leq c_2 \int_{\Omega_\varepsilon^*} |\nabla_\Gamma f_\varepsilon(y)|^2\mathrm{d}x \leq c_1c_2 2\varepsilon\delta \int_{\Gamma_\varepsilon} |\nabla_\Gamma f_\varepsilon(y)|^2\mathrm{d}y = 2\delta\varepsilon c_1c_2\|\nabla_\Gamma f_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)}.$$

Therefore, we constructed an extension from Γ_ε to Ω_ε^* satisfying the estimates claimed. We continue by extending from Ω_ε^* to Ω by using the extension operator from the article [9] for connected sets Ω_ε, which leads to an extended function $u_\varepsilon \in H^1(\Omega)$ such that

$$\|u_\varepsilon\|^2_{L^2(\Omega)} \leq C\|\tilde{u}_\varepsilon\|^2_{L^2(\Omega_\varepsilon^*)} \quad \text{and} \quad \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \leq C\left(\|\tilde{u}_\varepsilon\|^2_{L^2(\Omega_\varepsilon^*)} + \|\nabla\tilde{u}_\varepsilon\|^2_{L^2(\Omega_\varepsilon^*)}\right).$$

This completes the proof with the constants $C_1 = 2c_1\delta C$ and $C_2 = 2\delta c_1 C \max\{1, c_2\}$. □

Remark 6 The statement of Lemma 5 can be strengthened slightly to the separate estimates $\|u_\varepsilon\|^2_{L^2(\Omega)} \leq C_1\varepsilon\|f_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)}$ and $\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \leq C_2\varepsilon\|\nabla_\Gamma f_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)}$, if an extension operator from $H^1(\Omega_\varepsilon^*)$ to $H^1(\Omega)$ with separate estimates $\|u_\varepsilon\|^2_{L^2(\Omega)} \leq C\|\tilde{u}_\varepsilon\|^2_{L^2(\Omega_\varepsilon^*)}$ and $\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \leq C\|\nabla\tilde{u}_\varepsilon\|^2_{L^2(\Omega_\varepsilon^*)}$ is available. For example, the extension operator described in [11] can be used in this way if the function u_ε vanishes at the exterior boundary of Ω_ε^* and, in this case, the technical assumption on the domain Ω being representable by cuboids can be dropped as well. We refer to [3] and [9] for further discussions on boundary behaviour and extensions in this context.

Proof of Theorem 4

1. We use Lemma 5 to deduce the existence of a function $\tilde{\varphi}_\varepsilon \in H^1(\Omega)$ such that $\gamma(\varphi_\varepsilon) = \tilde{\varphi}_{\varepsilon|_{\Gamma_\varepsilon}} = \varphi_\varepsilon$ and

$$\|\tilde{\varphi}_\varepsilon\|_{L^2(\Omega)} + \|\nabla\tilde{\varphi}_\varepsilon\|^2_{L^2(\Omega)} \leq c_1\varepsilon\left(\|\varphi_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} + \|\nabla_\Gamma\varphi_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)}\right) \leq C.$$

Hence, $\tilde{\varphi}_\varepsilon$ has a weak limit function φ_0 in $H^1(\Omega)$. We calculate with Theorem 2

$$\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\tilde{\varphi}_\varepsilon)(x, y)\psi(x, y)\mathrm{d}x\mathrm{d}y \overset{\varepsilon\to 0}{\longrightarrow} \int_{\Omega\times Y} \varphi_0(x)\psi(x, y)\mathrm{d}y\mathrm{d}x \tag{1}$$

weakly in $L^2(\Omega, H^1_{\mathrm{per}}(Y))$ for all $\psi \in C^\infty(\Omega \times Y)$.

The trace operator $\gamma_{\Omega\times\Gamma} : L^2(\Omega, H^1(Y)) \to L^2(\Omega, L^2(\Gamma))$ defined by $\gamma_{\Omega\times\Gamma}(\varphi) = \varphi_{|\Omega\times\Gamma}$ commutes with $\mathcal{T}_\varepsilon$ as follows. Let $\psi \in H^1(\Omega_\varepsilon)$, then

$$\mathcal{T}_\varepsilon^b(\gamma(\psi))(x, y) = \mathcal{T}_\varepsilon^b\left(\psi_{|\Gamma_\varepsilon}\right)(x, y) = \psi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon\underbrace{y}_{\in\Gamma}\right) = \mathcal{T}_\varepsilon(\psi)_{|\Omega\times\Gamma}(x, y) = \gamma_{\Omega\times\Gamma}(\mathcal{T}_\varepsilon(\psi))(x, y).$$

It holds that $\mathcal{T}_\varepsilon(\tilde{\varphi}_\varepsilon)$ converges weakly in $H^1_{\text{per}}(Y)$ in its second variable and the trace operator is linear and continuous. Using (1) it follows that

$$\begin{aligned}|Y|\varepsilon\int_{\Gamma_\varepsilon}\varphi_\varepsilon\psi\mathrm{d}\sigma_x &= \int_{\Omega\times\Gamma}\mathcal{T}^b_\varepsilon(\varphi_\varepsilon)(x,y)\psi(x)\mathrm{d}\sigma_y\mathrm{d}x = \int_{\Omega\times\Gamma}\mathcal{T}^b_\varepsilon(\gamma(\tilde{\varphi}_\varepsilon))(x,y)\psi(x)\mathrm{d}\sigma_y\mathrm{d}x \\ &= \int_{\Omega\times\Gamma}\gamma_{\Omega\times\Gamma}(\mathcal{T}_\varepsilon(\tilde{\varphi}_\varepsilon))(x,y)\psi(x)\mathrm{d}\sigma_y\mathrm{d}x \overset{\varepsilon\to 0}{\to} \int_{\Omega\times\Gamma}\gamma_{\Omega\times\Gamma}(\varphi_0)(x,y)\psi(x)\mathrm{d}\sigma_y\mathrm{d}x = |\Gamma|\int_\Omega\varphi_0\psi\mathrm{d}x\end{aligned}$$

for all $\psi\in C^\infty(\Omega)$. We used the integration formula of the operator $\mathcal{T}^b_\varepsilon$ in the first step and exploited that φ_0 is independent of y in the last step. This completes the proof of *1.*

2. To prove the second part of the theorem, we need some additional definitions and properties. We define for every $a\in\mathbb{R}^n$ the function $z_a:\Gamma\to\mathbb{R}$ by $z_a(y)=a^T\cdot y$ and its gradient by using the directional derivatives on Γ by $\langle\nabla_\Gamma z_a, v\rangle = dz_a(v) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} z_a(\gamma(t))$ for every $y\in\Gamma$ and for $\gamma:(-\delta,\delta)\to\Gamma$, $\gamma(0)=y$ and $\dot{\gamma}(0)=v$. We find that $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} z_a(\gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} a^T\cdot\gamma(t) = a^T\cdot\dot{\gamma}(t) = a^T\cdot v$ for all $v\in T_y\Gamma$. The only element of $T_y\Gamma$ satisfying $\langle\nabla_\Gamma z_a, v\rangle = a^T\cdot v$ for all $v\in T_y\Gamma$ is the orthogonal projection of a to $T_y\Gamma$. Hence, $\nabla_\Gamma z_a = P_\Gamma a$. We define $z^c_a(y) = z_a(y) - \frac{1}{|\Gamma|}\int_\Gamma z_a(y)\mathrm{d}\sigma_y$. Then for any vector $a\in\mathbb{R}^n$ it is true that

$$\int_\Gamma z^c_a(y)\mathrm{d}\sigma_y = \int_\Gamma z_a(y)\mathrm{d}\sigma_y - \int_\Gamma\frac{1}{|\Gamma|}\int_\Gamma z_a(y)\mathrm{d}\sigma_y\mathrm{d}\sigma_y = \int_\Gamma z_a(y)\mathrm{d}\sigma_y - \int_\Gamma z_a(y)\mathrm{d}\sigma_y = 0.$$

Further we define $M^b_\varepsilon(\varphi_\varepsilon) = \frac{1}{|\Gamma|}\int_\Gamma\mathcal{T}^b_\varepsilon(\varphi_\varepsilon)(x,y)\mathrm{d}\sigma_y$ and $Z^b_\varepsilon = \frac{1}{\varepsilon}\left(\mathcal{T}^b_\varepsilon(\varphi_\varepsilon) - M^b_\varepsilon(\varphi_\varepsilon)\right)$. Then we deduce $\int_\Gamma Z^b_\varepsilon\mathrm{d}\sigma_y = 0$ and $\nabla_\Gamma Z^b_\varepsilon = \frac{1}{\varepsilon}\nabla_\Gamma\mathcal{T}^b_\varepsilon(\varphi_\varepsilon) = \mathcal{T}^b_\varepsilon(\nabla_x\varphi_\varepsilon)$, since $M^b_\varepsilon(\varphi_\varepsilon)$ is independent of y.

Having finished these preparations, we now come to the main part of the proof of part 2. We consider $\|Z^b_\varepsilon(\varphi_\varepsilon) - z^c_{\nabla_x\varphi_0}\|^2_{L^2(\Gamma\times\Omega)}$. Note that for the y-component, the x-gradient $\nabla_x\varphi_0$ looks like a vector in $\mathbb{R}^n$ because φ_0 is independent of y. We use the Poincaré inequality on Riemannian manifolds (cf. [10]) to obtain

$$\begin{aligned}\|Z^b_\varepsilon(\varphi_\varepsilon) - z^c_{\nabla_x\varphi_0}\|^2_{\Omega\times\Gamma} &\le \int_\Omega C\|\nabla_\Gamma Z^b_\varepsilon(\varphi_\varepsilon) - \nabla_\Gamma z^c_{\nabla_x\varphi_0}\|^2_\Gamma\mathrm{d}x \\ &= C\int_\Omega\|\mathcal{T}^b_\varepsilon(\nabla_x\varphi_\varepsilon) - P_\Gamma\nabla_x\varphi_0\|^2_\Gamma\mathrm{d}x \le C\varepsilon|Y|\|\nabla_x\varphi_\varepsilon\|^2_{\Gamma_\varepsilon} + C|\Gamma|\underbrace{\|P_\Gamma\|_2}_{\le 1}\|\nabla_x\varphi_0\|^2_\Omega \le \tilde{C}\end{aligned}$$

for a constant $\tilde{C}>0$ independent of ε. Hence, $Z^b_\varepsilon(\varphi_\varepsilon) - z^c_{\nabla_x\varphi_0}$ converges weakly to a function $\hat{\varphi}\in L^2(\Omega,H^1(\Gamma))$, up to a subsequence, i.e. for $\varepsilon\to 0$

$$Z^b_\varepsilon(\varphi_\varepsilon)\rightharpoonup\hat{\varphi} + z^c_{\nabla_x\varphi_0},\quad \mathcal{T}^b_\varepsilon(\nabla_x\varphi_\varepsilon) = \nabla_\Gamma Z^b_\varepsilon(\varphi_\varepsilon)\rightharpoonup\nabla_\Gamma\hat{\varphi} + P_\Gamma\nabla_x\varphi_0.$$

Finally, we need to show that $\hat{\varphi}\in L^2(\Omega,H^1_{\text{per}}(\Gamma))$, this means that $\hat{\varphi}$ is Y-periodic in its second argument. For this purpose, we define $\partial_i Y := \{y\in Y|\ y_i = 0\}$ for $i=1,\dots,n$. We extend the functions from Ω to $\mathbb{R}^n$ by zero and compute

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\Gamma\cap\partial_i Y}[Z^b_\varepsilon(\varphi_\varepsilon)(x,y+e_i) - Z^b_\varepsilon(\varphi_\varepsilon)(x,y)]\psi(x,y)\mathrm{d}\sigma_y\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^n}\int_{\Gamma\cap\partial_i Y}\frac{1}{\varepsilon}[\mathcal{T}^b_\varepsilon(\varphi_\varepsilon)(x,y+e_i) - \mathcal{T}^b_\varepsilon(\varphi_\varepsilon)(x,y)]\psi(x,y)\mathrm{d}\sigma_y\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^n}\int_{\Gamma\cap\partial_i Y}\frac{1}{\varepsilon}\mathcal{T}^b_\varepsilon(\varphi_\varepsilon)(x,y)\psi(x-\varepsilon e_i,y) - \frac{1}{\varepsilon}\mathcal{T}^b_\varepsilon(\varphi_\varepsilon)(x,y)\psi(x,y)\mathrm{d}\sigma_y\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^n}\int_{\Gamma\cap\partial_i Y}\mathcal{T}^b_\varepsilon(\varphi_\varepsilon)(x,y)\frac{\psi(x-\varepsilon e_i,y) - \psi(x,y)}{\varepsilon}\mathrm{d}\sigma_y\mathrm{d}x \\ &\quad\overset{\varepsilon\to 0}{\to}\int_{\mathbb{R}^n}\int_{\Gamma\cap\partial_i Y}\gamma(\varphi_0)\left(-\frac{d\psi}{\mathrm{d}x_i}\right)\mathrm{d}\sigma_y\mathrm{d}x = \int_{\mathbb{R}^n}\int_{\Gamma\cap\partial_i Y}\frac{d\varphi_0}{\mathrm{d}x_i}\psi\mathrm{d}\sigma_y\mathrm{d}x = \int_{\mathbb{R}^n}\int_{\Gamma\cap\partial_i Y}e_i^T\cdot\nabla_x\varphi_0\psi\mathrm{d}\sigma_y\mathrm{d}x\end{aligned}$$

for all $\psi \in C^\infty(\mathbb{R}^n \times \Gamma)$. Because of the extension by zero on $\mathbb{R}^n$ we conclude for the domain Ω

$$\int_\Omega \int_{\Gamma\cap\partial_i Y} [Z_\varepsilon^b(\varphi_\varepsilon)(x, y + e_i) - Z_\varepsilon^b(\varphi_\varepsilon)(x, y)]\psi(x, y)\mathrm{d}\sigma_y \mathrm{d}x \overset{\varepsilon\to 0}{\to} \int_\Omega \int_{\Gamma\cap\partial_i Y} e_i^T \cdot \nabla_x \varphi_0 \psi \mathrm{d}\sigma_y \mathrm{d}x.$$

On the other hand we calculate

$$\begin{aligned}\int_\Omega \int_{\Gamma\cap\partial_i Y} &[z^c_{\nabla_x\varphi_0}(y + e_i) - z^c_{\nabla_x\varphi_0}(y)]\psi(x, y)\mathrm{d}\sigma_y \mathrm{d}x \\ &= \int_\Omega \int_{\Gamma\cap\partial_i Y} [\nabla_x\varphi_0 \cdot (y + e_i) - \nabla_x\varphi_0 \cdot y]\psi \mathrm{d}\sigma_y \mathrm{d}x = \int_\Omega \int_{\Gamma\cap\partial_i Y} e_i^T \cdot \nabla_x\varphi_0\psi \mathrm{d}\sigma_y \mathrm{d}x\end{aligned}$$

for all $\psi \in C^\infty(\Omega \times \Gamma)$. With $Z_\varepsilon^b(\varphi_\varepsilon) \rightharpoonup \hat{\varphi} + z^c_{\nabla_x\varphi_0}$ we conclude that

$$\int_\Omega \int_{\Gamma\times\partial_i Y} [\hat{\varphi}(x, y + e_i) - \hat{\varphi}(x, y)]\psi(x, y)\mathrm{d}\sigma_y \mathrm{d}x = 0$$

for all $\psi \in C^\infty(\Omega \times \Gamma)$ and $i = 1, \ldots, n$. So $\hat{\varphi} \in L^2(\Omega, H^1_{\mathrm{per}}(\Gamma))$ and the proof is completed. □

4. Example. Theorem 4 can be used to derive global diffusion on a manifold for a partial differential equation in the homogenization limit. For example, this arises in biomedical applications, where molecules diffuse fast on the surface of fine-structured membranes in human cells, cf. [8] and references therein. The following example illustrates how Theorem 4 can be used in this context.

Let $\Gamma \subset [0, 1]^n$ be a smooth, $n - 1$ dimensional manifold, such that $\Gamma_\varepsilon = \bigcup_{k\in\mathbb{Z}^n} \varepsilon(k + \Gamma)$ is periodic, connected and smooth. Let $\Omega \subset \mathbb{R}^n$ be bounded, $f \in C(\Omega, C_{\mathrm{per}}(Y))$ with $f_\varepsilon(x) := f(x, \frac{x}{\varepsilon})$, and $D_\varepsilon(x) = D(x, \frac{x}{\varepsilon})$ be an elliptic diffusion tensor on the tangent space of $T_y\Gamma_\varepsilon$, which is ε-periodic in its second argument and with $\lim_{\varepsilon\to 0}\|D_\varepsilon\|^2_{\Omega_\varepsilon} = \|D\|^2_{\Omega\times\Gamma}$ bounded. Further, for given $\varepsilon > 0$, let u_ε be the solution of the problem

$$\begin{aligned}\partial_t u_\varepsilon(x, t) - \nabla_\Gamma \cdot (D_\varepsilon(x)\nabla_\Gamma u_\varepsilon(x, t)) + u_\varepsilon(x, t) &= f_\varepsilon(x, t) && \text{on } \Gamma_\varepsilon,\\ u_\varepsilon(x, t) &= 0 && \text{on } \partial\Omega \cap \Gamma_\varepsilon.\end{aligned}$$

We multiply the weak formulation with ε and find with standard estimations that

$$\varepsilon\|u_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} + \varepsilon\|\sqrt{D_\varepsilon}\nabla_\Gamma u_\varepsilon\|^2_{L^2([0,t]\times\Gamma_\varepsilon)} + \varepsilon\|u_\varepsilon\|^2_{L^2([0,t]\times\Gamma_\varepsilon)} \le C.$$

Therefore, the conditions to use Theorem 4 are satisfied. Application of the boundary unfolding operator to the weak formulation, where the test function is denoted by ψ_ε, leads to

$$\begin{aligned}\int_\Omega \int_\Gamma \partial_t \mathcal{T}_\varepsilon^b(u_\varepsilon)(x, y, t)\mathcal{T}_\varepsilon^b(\psi_\varepsilon)(x, y)\mathrm{d}\sigma_y \mathrm{d}x + \int_\Omega \int_\Gamma \mathcal{T}_\varepsilon^b(D_\varepsilon)(x, y)\mathcal{T}_\varepsilon^b(\nabla_\Gamma u_\varepsilon)(x, y, t)\mathcal{T}_\varepsilon^b(\nabla_\Gamma\psi_\varepsilon)(x, y)\mathrm{d}\sigma_y \mathrm{d}x\\ + \int_\Omega \int_\Gamma \mathcal{T}_\varepsilon^b(u_\varepsilon)(x, y, t)\mathcal{T}_\varepsilon^b(\psi_\varepsilon)(x, y)\mathrm{d}\sigma_y \mathrm{d}x = \int_\Omega \int_\Gamma \mathcal{T}_\varepsilon^b(f_\varepsilon)(x, y, t)\mathcal{T}_\varepsilon^b(\psi_\varepsilon)(x, y)\mathrm{d}\sigma_y \mathrm{d}x.\end{aligned}$$

Now we use Theorem 4 and find for $\varepsilon \to 0$, noting that $|Y| = 1$,

$$\begin{aligned}|\Gamma| \int_\Omega \partial_t u_0(x, t)\psi_0(x)\mathrm{d}x + \int_\Omega \int_\Gamma D(x, y)[\nabla_\Gamma\hat{u}(x, y, t) + P_\Gamma\nabla_x u_0(x)][\nabla_\Gamma\hat{\psi}(x, y) + P_\Gamma\nabla_x\psi_0(x)]\mathrm{d}\sigma_y \mathrm{d}x\\ + |\Gamma| \int_\Omega u_0(x, t)\psi_0(x)\mathrm{d}x = \int_\Omega \int_\Gamma f_0(x, y, t)\psi_0(x)\mathrm{d}\sigma_y \mathrm{d}x.\end{aligned}$$

To determine the cell problem we first set $\psi_0 = 0$. Let $\hat{u}(x, y, t) = \sum_{i=1}^n \partial_{x_i} u_0(x)\chi_i(y, t)$ and $\nabla_\Gamma\hat{u}(x, y, t) = \sum_{i=1}^n \partial_{x_i} u_0(x)\nabla_\Gamma\chi_i(y, t)$ for some $\chi_i(y, t) : \Gamma \times [0, T] \to \mathbb{R}$, $i = 1, \ldots, n$. Further, we write $P_\Gamma\nabla_x u_0$ as $\sum_{i=1}^n \partial_{x_i} u_0 P_\Gamma e_i$ and obtain

$$\begin{aligned}\int_\Omega \int_\Gamma D(x, y) \left[\sum_{i=1}^n \partial_{x_i} u_0 \nabla_\Gamma\chi_i(y, t) + \sum_{i=1}^n \partial_{x_i} u_0 P_\Gamma e_i\right] \nabla_\Gamma\hat{\psi}(y, t)\mathrm{d}\sigma_y \mathrm{d}x\\ = \int_\Omega \sum_{i=1}^n \partial_{x_i} u_0 \int_\Gamma D(x, y)(\nabla_\Gamma\chi_i(y, t) + P_\Gamma e_i)\nabla_\Gamma\hat{\psi}(x, y)\mathrm{d}\sigma_y \mathrm{d}x = 0\end{aligned}$$

for all $\hat{\psi} \in C^\infty(\Omega, C^\infty_{\text{per}}(\Gamma))$. Hence, the strong formulation of the cell problem is given by

$$\begin{aligned} -\nabla_\Gamma \cdot D(x,y)(\nabla_\Gamma \chi_i(y,t) + P_\Gamma e_i) &= 0 \qquad \text{in } \Gamma, \\ D(x,y)(\nabla_\Gamma \chi_i(y,t) + P_\Gamma e_i) \cdot n &= 0 \qquad \text{on } \partial\Gamma, \end{aligned}$$

and χ_i Y-periodic for all $i = 1, \ldots, n$. This equation is well-defined since D maps elements of the tangent space $T_y\Gamma$ into the tangent space $T_y\Gamma$.

Now, we set $\hat{\psi} = 0$. Then, after a brief computation, we obtain

$$|\Gamma| \int_\Omega \partial_t u_0(x,t)\psi_0(x)\mathrm{d}x + \int_\Omega \sum_{i,j=1}^n \partial_{x_i} u_0 \int_\Gamma \left(P_\Gamma^T D(x,y)(\nabla_\Gamma \chi_i(y,t) + P_\Gamma e_i)\right)_j \mathrm{d}\sigma_y \partial_{x_j}\psi_0 \mathrm{d}x$$
$$+ |\Gamma| \int_\Omega u_0(x,t)\psi_0(x)\mathrm{d}x = \int_\Omega \int_\Gamma f_0(x,y,t)\psi_0(x)\mathrm{d}\sigma_y \mathrm{d}x,$$

for all $\psi_0 \in C^\infty(\Omega)$. Since the orthogonal projection is symmetric $P_\Gamma^T = P_\Gamma$ and $D(x,y)(\nabla_\Gamma \chi_i(y,t) + P_\Gamma e_i)$ is already in the tangent space $T_y\Gamma$, we drop the first P_Γ. Defining

$$s_{ij}(x) = \int_\Gamma (D(x,y)(\nabla_\Gamma \chi_j(y,t) + P_\Gamma e_j))_i \mathrm{d}\sigma_y, \qquad S = (s_{ij})_{i,j=1,\ldots,n},$$

we find the strong formulation of the homogenized limit problem

$$\begin{aligned} |\Gamma|\partial_t u_0 - \nabla_x \cdot (S(x)\nabla_x u_0) + |\Gamma| u_0 &= \textstyle\int_\Gamma f \mathrm{d}\sigma_y \qquad \text{in } \Omega, \\ u_0 &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

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References

[1] G. Allaire, A. Damlamian, and U. Hornung. Two-scale convergence on periodic surfaces and applications. In A. P. Bourgeat, C. Carasso, S. Luckhaus, and A. Mikelić, editors, *Proceedings of the international conference on mathematical modelling of flow through porous media*, pages 15–25. World Scientific, 1995.

[2] T. Arbogast, J. Douglas, and U. Hornung. Derivation of the double porosity model of single phase flow via homogenization theory. *SIAM J. Math. Anal.*, 21:823–836, 1990.

[3] D. Cioranescu, A. Damlamian, P. Donato, G. Griso, and R. Zaki. The periodic unfolding method in domains with holes. *SIAM J. Math. Anal.*, 44(2):718–760, 2012.

[4] D. Cioranescu, A. Damlamian, and G. Griso. Periodic unfolding and homogenization. *Comptes Rendus Mathematique*, 335(1):99–104, 2002.

[5] D. Cioranescu, A. Damlamian, and G. Griso. The periodic unfolding method in homogenization. *SIAM J. Math. Anal.*, 40(4):1585–1620, 2008.

[6] D. Cioranescu, P. Donato, and R. Zaki. The periodic unfolding method in perforated domains. *Portugaliae Mathematica*, 63(4):467–496, 2006.

[7] S. Dobberschütz and M. Böhm. A periodic unfolding operator on certain compact riemannian manifolds. *Comptes Rendus Mathmatique*, 350:1027–1030, 2012.

[8] I. Graf. *Multiscale modeling and homogenization of reaction-diffusion systems involving biological surfaces*. PhD dissertation, Universität Augsburg, 2013. Also: Logos Verlag Berlin, 2013.

[9] M. Höpker and M. Böhm. A note on the existence of extension operators for Sobolev spaces on periodic domains. *preprint*, 2013. Submitted for publication.

[10] P. Li. *Seminar on Differential Geometry.* Princeton University Press, 1982.

[11] M. Mabrouk and S. Hassan. Homogenization of a composite medium with a thermal barrier. *Mathematical Methods in the applied Sciences*, 27:405 – 425, 2004.

[12] M. Neuss-Radu. Some extensions of two-scale convergence. *C. R. Acad. Sci. Paris, Ser. I*, 322:899–904, 1996.

[13] M. A. Peter and M. Böhm. Different choises of scaling in homogenization of diffusion and interfacial exchange in a porous medium. *Mathematical Methods in the Applieds Sciences*, 31:1257–1282, 2008.