Exercício 2: ACF e PACF em Modelos ARIMA

Seja um modelo ARIMA(1,1,2) para Y_t, onde $\phi_1 = 0{,}2$, $\theta_1 = 0{,}5$, $\theta_2 = -0{,}3$ e $u_t \sim N(0,1)$.

$$(1-\phi_1 B)(1-B)Y_t = (1-\theta_1 B-\theta_2 B^2)u_t$$

$$\Rightarrow (1-\phi_1 B)W_t = (1-\theta_1 B-\theta_2 B^2)u_t$$

$$\Rightarrow W_t = \phi_1 W_{t-1} + u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}$$

Além disto:

$$W_{t-1} = \phi_1 W_{t-2} + u_{t-1} - \theta_1 u_{t-2} - \theta_2 u_{t-3}$$

e

$$W_{t-2} = \phi_1 W_{t-3} + u_{t-2} - \theta_1 u_{t-3} - \theta_2 u_{t-4}$$

a) Sabendo que a autocovariância de ordem 1, γ_1, é igual a -0,36, calcule a variância da série estacionária W_t.

$$\gamma_0 = Cov(W_t, W_t) = Cov(W_t, \phi_1 W_{t-1} + u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2})$$

$$= \phi_1 Cov(W_t, W_{t-1}) + Cov(W_t, u_t) - \theta_1 Cov(W_t, u_{t-1}) - \theta_2 Cov(W_t, u_{t-2})$$

$$= \phi_1\gamma_1 + \sigma_u^2 - \theta_1 Cov(\phi_1 W_{t-1} + u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}, u_{t-1}) - \theta_2 Cov(\phi_1 W_{t-1} + u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}, u_{t-2})$$

$$= \phi_1\gamma_1 + \sigma_u^2 - \theta_1\phi_1 Cov(W_{t-1}, u_{t-1}) - \cancel{\theta_1 Cov(u_t, u_{t-1})}^{=0} + \theta_1^2 Cov(u_{t-1}, u_{t-1}) + \cancel{\theta_1\theta_2 Cov(u_{t-2}, u_{t-1})}^{=0} - \theta_2\phi_1 Cov(W_{t-1}, u_{t-2}) - \cancel{\theta_2 Cov(u_t, u_{t-2})}^{=0} + \cancel{\theta_2\theta_1 Cov(u_{t-1}, u_{t-2})}^{=0} + \theta_2^2 Cov(u_{t-2}, u_{t-2})$$

$$= \phi_1\gamma_1 + \sigma_u^2 - \theta_1\phi_1 Cov(\phi_1 W_{t-2} + u_{t-1} - \theta_1 u_{t-2} - \theta_2 u_{t-3}, u_{t-1}) + \theta_1^2\sigma_u^2 - \theta_2\phi_1 Cov(\phi_1 W_{t-2} + u_{t-1} - \theta_1 u_{t-2} - \theta_2 u_{t-3}, u_{t-2}) + \theta_2^2\sigma_u^2$$

$$= \phi_1\gamma_1 + \sigma_u^2 - \theta_1\phi_1^2 Cov(W_{t-2}, u_{t-1}) - \theta_1\phi_1 Cov(u_{t-1}, u_{t-1}) + \cancel{\theta_1^2\phi_1 Cov(u_{t-2}, u_{t-1})}^{=0} + \cancel{\theta_1\theta_2\phi_1 Cov(u_{t-3}, u_{t-1})}^{=0} + \theta_1^2\sigma_u^2 - \theta_2\phi_1 Cov(\phi_1 W_{t-2} + u_{t-1} - \theta_1 u_{t-2} - \theta_2 u_{t-3}, u_{t-2}) + \theta_2^2\sigma_u^2$$

$$= \phi_1\gamma_1 + \sigma_u^2 - \theta_1\phi_1\sigma_u^2 + \theta_1^2\sigma_u^2 - \theta_2\phi_1^2 Cov(W_{t-2}, u_{t-2}) - \cancel{\theta_2\phi_1 Cov(u_{t-1}, u_{t-2})}^{=0} + \theta_2\phi_1\theta_1 Cov(u_{t-2}, u_{t-2}) - \cancel{\theta_2^2\phi_1 Cov(u_{t-3}, u_{t-2})}^{=0} + \theta_2^2\sigma_u^2$$

$$= \phi_1\gamma_1 + \sigma_u^2 - \theta_1\phi_1\sigma_u^2 + \theta_1^2\sigma_u^2 - \theta_2\phi_1^2 Cov(\phi_1 W_{t-2} + u_{t-2} - \theta_1 u_{t-3} - \theta_2 u_{t-4}, u_{t-2}) + \theta_2\phi_1\theta_1\sigma_u^2 + \theta_2^2\sigma_u^2$$

$$= \phi_1\gamma_1 + \sigma_u^2 - \theta_1\phi_1\sigma_u^2 + \theta_1^2\sigma_u^2 - \theta_2\phi_1^2\sigma_u^2 + \theta_2\phi_1\theta_1\sigma_u^2 + \theta_2^2\sigma_u^2$$

$$= \boldsymbol{\phi_1\gamma_1 + (1 - \theta_1\phi_1 + \theta_1^2 - \theta_2\phi_1^2 + \theta_2\phi_1\theta_1 + \theta_2^2)\sigma_u^2}$$

$$= \mathbf{0,2} \times (\mathbf{-0,36}) + (\mathbf{1 - 0,5 \times 0,2 + 0,5^2 - (-0,3) \times 0,2^2} + (\mathbf{-0,3}) \times \mathbf{0,2 \times 0,5 + (-0,3)^2}) \times \mathbf{1}$$

$$= \mathbf{-0,072 + 1,222} \Rightarrow \boldsymbol{\gamma_0} = \mathbf{1,15}$$

b) Calcule o valor de ρ_1.

$$\boldsymbol{\rho_1} = \frac{\boldsymbol{\gamma_1}}{\boldsymbol{\gamma_0}} = \frac{\mathbf{-0,36}}{\mathbf{1,15}} = \mathbf{-0,31}$$

c) Calcule o valor de ρ_2.

$$\gamma_2 = Cov(W_t, W_{t-2}) = Cov(\phi_1 W_{t-1} + u_t - \theta_1 u_{t-1} - \theta_2 u_{t-2}, W_{t-2})$$

$$= \phi_1 Cov(W_{t-1}, W_{t-2}) + Cov(u_t, W_{t-2}) - \theta_1 Cov(u_{t-1}, W_{t-2}) - \theta_2 Cov(u_{t-2}, W_{t-2})$$

$$= \phi_1 \gamma_1 - \theta_2 Cov(u_{t-2}, \phi_1 W_{t-3} + u_{t-2} - \theta_1 u_{t-3} - \theta_2 u_{t-4})$$

$$= \phi_1 \gamma_1 - \theta_2 Cov(u_{t-2}, u_{t-2})$$

$$= \boldsymbol{\phi_1 \gamma_1 - \theta_2 \sigma_u^2}$$

$$= \mathbf{0,2} \times (\mathbf{-0,36}) - (\mathbf{-0,3}) \times \mathbf{1} = \mathbf{0,228}$$

$$\boldsymbol{\rho_2} = \frac{\boldsymbol{\gamma_2}}{\boldsymbol{\gamma_0}} = \frac{\mathbf{0,228}}{\mathbf{1,15}} = \mathbf{0,198}$$

d) Calcule as duas primeiras autocorrelações parciais.

$$\boldsymbol{\phi_{11}} = \boldsymbol{\rho_1} = \mathbf{-0,31}$$

$$\boldsymbol{\phi_{22}} = \frac{\boldsymbol{\rho_2 - \rho_1^2}}{\mathbf{1 - \rho_1^2}} = \mathbf{0,11}$$