# Problem 2: Heaviside Calculus

**The second approximations for the numerical derivative of a function f(x) is found using a 2-point centered difference.**

## Contents



## Original Approximation

$$\frac{df(x)}{dx} = Af(x+\frac{h}{2}) + Bf(x-\frac{h}{2}) + Err[f(x)]$$

## The $\hat{D}$ Operator

We apply the $\hat{D}$ operator to the equation.

$$\hat{D}[\frac{df(x)}{dx}] = Ae^{\frac{1}{2}h\hat{D}}f(x) + Be^{-\frac{1}{2}h\hat{D}}f(x) + Err[f(x)]$$

Here, we can divide both sides of the equation by f(x). Also, we recognize the $e^{\frac{1}{2}h\hat{D}}$ and $e^{-\frac{1}{2}h\hat{D}}$ terms with Taylor-Series expansions, as shown below.

$$e^{\frac{1}{2}h\hat{D}} = 1 + \frac{1}{2}h\hat{D} + \frac{1}{2}(\frac{1}{2}h)^2(\hat{D})^2 + \frac{1}{6}(\frac{1}{2}h)^3(\hat{D})^3 + ...$$

$$e^{-\frac{1}{2}h\hat{D}} = 1 - \frac{1}{2}h\hat{D} + \frac{1}{2}(\frac{1}{2}h)^2(\hat{D})^2 - \frac{1}{6}(\frac{1}{2}h)^3(\hat{D})^3 + ...$$

## Solving for Coefficients

With this equation, we can solve for A and B.

Since we have a $\hat{D}$ term on the left hand side of the equation, we must solve for A and B such that the $D^0$ term is zero, and the $D^1$ term is 1.

## Solving for $D^0$

$$A + B = 0$$

$$A = -B$$

## Solving for $D^1$

$$(A-B)\frac{h}{2} = 1$$

$$(A-(-A))\frac{h}{2} = 1$$

$2A\frac{h}{2} = 1$

$A = \frac{1}{h}$

$B = -\frac{1}{h}$

## Error Factor

The next term in the expansion relates to the leading error term. We can exlude the factors used in solving for coefficients.

To find the error estimate, we plug coefficients back into the original function, other than the first two terms we used to solve A,B.

$\hat{D} = (\frac{1}{h})(\frac{1}{2}(\frac{1}{2}h)^2(\hat{D})^2 + \frac{1}{6}(\frac{1}{2}h)^3(\hat{D})^3) + (-\frac{1}{h})(\frac{1}{2}(\frac{1}{2}h)^2(\hat{D})^2 - \frac{1}{6}(\frac{1}{2}h)^3(\hat{D})^3) + Err$

$\hat{D} = \frac{1}{8}(h)(\hat{D})^2 + \frac{1}{48}(h)^2(\hat{D})^3) - \frac{1}{8}h(\hat{D})^2 + \frac{1}{48}(h)^2(\hat{D})^3) + Err$

$\hat{D} = \frac{1}{48}(h)^2(\hat{D})^3) + \frac{1}{48}(h)^2(\hat{D})^3) + Err$

Since the second order factor is zero, the leading error term of this function is determined by the $O(h^4)$ factor.

$\hat{D} = \frac{1}{24}(h)^2(\hat{D})^3) + Err$

We find the leading order in the truncation error to be:

$Error = -\frac{1}{24}(h)^2(f(x))^3)$

## Conclusion

Thus, the derivative can be approximated by

$f'(x) = (\frac{1}{h})(f(x + \frac{1}{2}h) - f(x - \frac{1}{2}h)) - \frac{1}{24}(h)^2(f'''(x))) + O(h^4)$

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