# EEC161, Applied Probability for Electrical & Computer Engineers
## Project 2

This project is related to the plinko game with the plinko board shown in Project 1. The objective of the assignment is to compute the probabilities that a chip dropped in slot $i$, where $i = 1, 2, \ldots, 9$, at level $t = 1$ at the top of the board ends up in slot $j$, where $j = 1, 2, \ldots, 9$, at level $t = 7$ at the bottom of the board. In Project 1 you obtained estimates of these probabilities using MATLAB simulations. We will also consider these probabilities in a generalized plinko game as the number of levels, $t$, is not restricted to be 7.

As in the simulations, we will make the following assumptions. When a chip hits a peg, it is equally probable to go to the left or to the right and in either case it falls in the first available space. If a chip hits a wall, it has only one direction to go to and it falls in the first available space.

The project is composed of three parts. In the first part we determine the required probabilities for the standard plinko board using a simple argument based on binomial random variables. However, the argument is not applicable to all pairs of $i$ and $j$. In the second part, we determine the probabilities for all $i$ and $j$ using an argument based on Markov chains.

In the third part, we obtain by simulations the probabilities that a chip dropped in slot $i$, where $i = 1, 2, \ldots, 9$, at level $t = 1$ at the top of the board ends up in slot $j$, where $j = 1, 2, \ldots, 9$, at level $\ell$ for increasing values of $\ell$. We also verify the simulation results using the theory of Markov chains.

**Binomial Random Variables** We number the slots at the top of the plinko board as $1, 2, \ldots, 9$ from left to right. We also number the slots at the bottom of the plinko board as $1, 2, \ldots, 9$ from left to right. Suppose that *every* chip dropped in slot $i$ at the top of the plinko board ends up in slot $j$ at the bottom of the board *without hitting a wall* that forces it to move in one direction. Then every chip encounters 12 pegs and at each peg it immediately falls either to the right or to the left of the peg with equal probability. If a chip encountering a peg falls to the right then call this a success. The probability that a chip dropped in slot $i$ ends up in slot $j$ is the probability of $j-i+6$ successes in 12 independent trials. This is the probability that the random variable Binomial$(12, 0.5)$ assumes the value $j - i + 6$.

**Task 1** Specify all pairs $i$ and $j$ for which all chips dropped in slot $i$ at the top of the board end up in slot $j$ at the bottom of the board at level $t = 7$ without hitting a wall that forces them to move in one direction. For example, not all chips dropped in slot 2 end up in slot 3 without hitting a wall that forces them to move in one direction. Indeed, a chip dropped in slot $i = 2$ can end up in slot $j = 3$ by the sequence of moves

$$R, R, L, L, L, L, \mathbf{R}, R, R, R, R, L,$$

where $R$ stands for Right and $L$ stands for Left. The boldface **R** is a forced move to the right as the chip hits a wall. We conclude that $i = 2$ and $j = 3$ is not a pair for which all chips dropped in slot $i$ end up in slot $j$ without hitting a wall that forces them to move in one direction. However, notice the $i = 2$ and $j = 7$ is a pair for which all chips dropped in slot $i$ end up in slot $j$ without hitting a wall.

**Task 2** Form a $9 \times 9$ matrix in which the $(i, j)$ entry is the probability that a chip dropped in slot $i$ ends up in slot $j$ at level $t = 7$ provided that $i$ and $j$ are one of the pairs specified in Task 1. If $i$ and $j$ are not one of the pairs specified in Task 1, then leave the $(i, j)$ entry blank. The probabilities have to be computed using binomial random variables.

**Markov Chains** In this approach, we model the movement of a chip as a Markov chain. Indeed, given the current position of a chip, its following positions are independent of its preceding positions. Notice that the number of available slots in which the chip can be alternates between nine and eight. In order to avoid this difficulty, we take the states of the Markov chain to be the positions of the chip when it is in a level with nine slots. A chip in one of these levels either encounters two pegs or a wall and a peg before reaching the next level with nine slots. Thus, we have a nine-state Markov chain. Attached is a sketch of the plinko board where the levels with the nine slots are numbered from 1 to 7 and the slots at each one of these levels are numbered from 1 to 9.

**Task 3** Find the transition matrix, $P$, of the Markov chain.

**Task 4** Using the theory of Markov chains, form a $9 \times 9$ matrix with rows and columns indexed by $1, 2, \ldots, 9$. The entry in position $(i, j)$ should be the probability that a chip dropped in slot $i$ at the top of the board at level $t = 1$ ends up in slot $j$ at the bottom of the board at level $t = 7$.

The results you get in Task 2 should agree with those you get in Task 4 for the pairs $i$ and $j$ considered in Task 2. Also, the results you get in Task 4 should be close to those you got using simulations in Project 1 assuming that the results and simulations are correct.

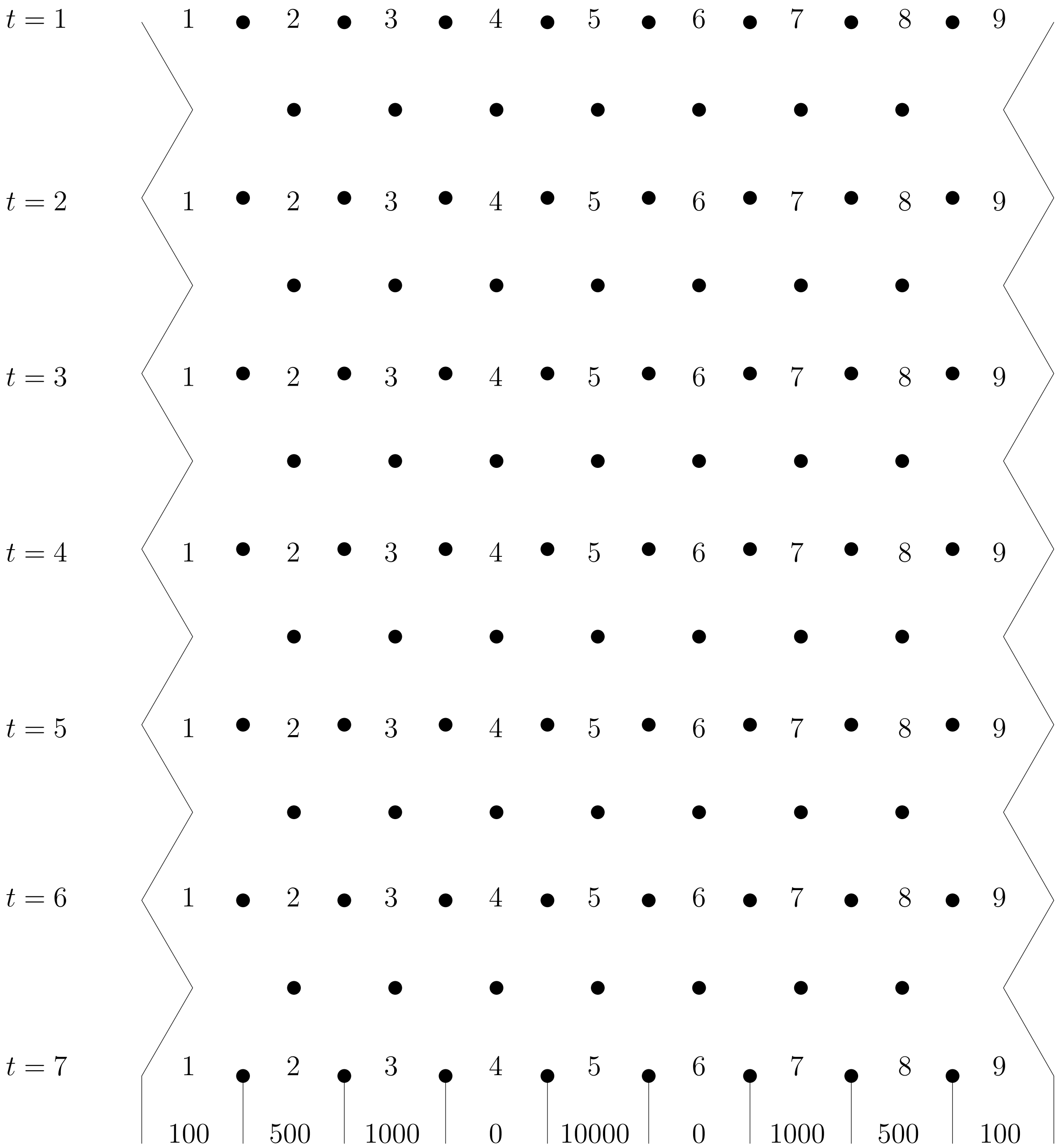
**Generalized Plinko Board** Suppose that the plinko board shown with levels $1, 2 \ldots, 7$ is replaced by a board with levels $1, 2, \ldots, \ell$. Otherwise, the shape of the board is the same.

**Task 5** Write a matlab program to find the probability that a chip dropped in slot $i$ at the top of the board at level $t = 1$ ends up in slot $j$ at the bottom of the board at level $t = \ell$. The program can be a simple modification of the program you wrote in Project 1 assuming that it is correct.

**Task 6** Use the matlab program to find the probabilities in case $\ell = 20$, $\ell = 40$, $\ell = 80$, and $\ell = 160$. In each case, list the probabilities as a $9 \times 9$ matrix as in Project 1.

**Task 7** Is the Markov chain regular? Either prove that $P^n$ has zero entries for every positive integer $n$ to show that it is not regular or specify the smallest positive integer $n$ such that all entries of $P^n$ are positive. If the Markov chain is regular, use the theory of Markov chain to find theoretically the limit of the rows of the matrices in Task 6 as $\ell$ goes to infinity. This should be the solution of $\vec{\pi}P = \vec{\pi}$. Matlab can be used to give $\vec{\pi}$ as an eigenvector associated with the eigenvalue 1.

Submit the list of pairs of Task 1, the matrix of Task 2, the matrix of Task 3, the matrix of Task 4, the matlab program of Task 5, the four matrices of Task 6, and the solution of Task 7.



t = 1
t = 2
t = 3
t = 4
t = 5
t = 6
t = 7
1 2 3 4 5 6 7 8 9
100 500 1000 0 10000 0 1000 500 100