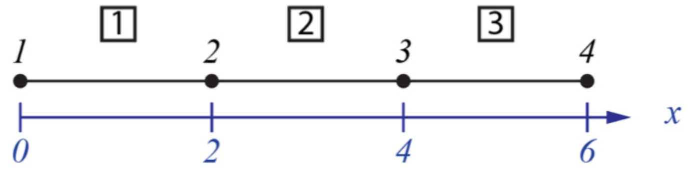
Problem 1

See below the structure consisting of three 2-node elements. The displacements at nodes 1, 2, 3, and 4 are 1, 3, 4, and 6.

1. Derive shape functions for all three elements.
2. Use reference element to find the displacements in each element at $\xi = 0.5$.
3. Use reference element to find the strains in each element at $\xi = 0.5$.
4. Calculate the stiffness matrix of all elements using Gauss integration of reference element.



1 Problem 1

1.1 Shape Functions

For 2-node elements, the shape functions are:

$$N_1(\xi) = \frac{1-\xi}{2} \tag{1}$$

$$N_2(\xi) = \frac{1+\xi}{2} \tag{2}$$

where ξ is the local coordinate ranging from -1 to 1 within each element.

1.2 Displacements at $\xi = 0.5$

The displacement at $\xi = 0.5$ in each element is calculated using the shape functions:

$$u(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2 \tag{3}$$

For $\xi = 0.5$:

$$N_1(0.5) = \frac{1-0.5}{2} = 0.25 \tag{4}$$

$$N_2(0.5) = \frac{1+0.5}{2} = 0.75 \tag{5}$$

Element 1 (nodes 1-2):

$$u(\xi = 0.5) = 0.25 \times 1 + 0.75 \times 3 \tag{6}$$

$$= 0.25 + 2.25 = 2.5 \tag{7}$$

Element 2 (nodes 2-3):

$$u(\xi = 0.5) = 0.25 \times 3 + 0.75 \times 4 \tag{8}$$

$$= 0.75 + 3.0 = 3.75 \tag{9}$$

Element 3 (nodes 3-4):

$$u(\xi = 0.5) = 0.25 \times 4 + 0.75 \times 6 \tag{10}$$
$$= 1.0 + 4.5 = 5.5 \tag{11}$$

1.3 Strains at $\xi = 0.5$

For 2-node elements, the strain is constant throughout the element:

$$\varepsilon = \frac{u_2 - u_1}{L} \tag{12}$$

We need to calculate the strains for each element:

Element 1 (L = 2):

$$\varepsilon = \frac{3 - 1}{2} = 1.0 \tag{13}$$

Element 2 (L = 2):

$$\varepsilon = \frac{4 - 3}{2} = 0.5 \tag{14}$$

Element 3 (L = 2):

$$\varepsilon = \frac{6 - 4}{2} = 1.0 \tag{15}$$

1.4 Stiffness Matrices

For a 2-node element with Young's modulus E and cross-sectional area A, the stiffness matrix is:

$$\mathbf{k} = \frac{E \cdot A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{16}$$

Assuming $E = A = 1$ for simplicity:

Element 1 (L = 2):

$$\mathbf{k_1} = \frac{1 \cdot 1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{17}$$
$$= 0.5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{18}$$
$$= \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \tag{19}$$

Element 2 (L = 2):

$$\mathbf{k_2} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \tag{20}$$

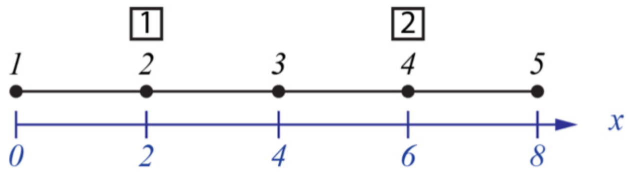
Element 3 (L = 2):

$$\mathbf{k_3} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \tag{21}$$

Problem 2

See below the structure consisting of two 3-node elements. The displacements at nodes 1, 2, 3, 4 and 5 are 0, -1, 2, -1, and 4.

1. Derive shape functions for both elements.
2. Use reference element to find the displacements in each element at $\xi = -0.5$.
3. Use reference element to find the strains in each element at $\xi = -0.5$.
4. Calculate the stiffness matrix of all elements using reference elements using integration points of $\xi = -0.5$ and $\xi = 0.5$.



2 Problem 2

2.1 Shape Functions

For 3-node elements, the shape functions are:

$$N_1(\xi) = \frac{\xi(\xi - 1)}{2} \tag{22}$$

$$N_2(\xi) = (1 + \xi)(1 - \xi) \tag{23}$$

$$N_3(\xi) = \frac{\xi(\xi + 1)}{2} \tag{24}$$

where ξ is the local coordinate ranging from -1 to 1 within each element, with $\xi = -1, 0, 1$ corresponding to the three nodes.

2.2 Displacements at $\xi = -0.5$

The displacement at $\xi = -0.5$ in each element is calculated using the shape functions:

$$u(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2 + N_3(\xi)u_3 \tag{25}$$

For $\xi = -0.5$:

$$N_1(-0.5) = \frac{-0.5(-0.5 - 1)}{2} = \frac{-0.5 \times (-1.5)}{2} = \frac{0.75}{2} = 0.125 \tag{26}$$

$$N_2(-0.5) = (1 + (-0.5))(1 - (-0.5)) = 0.5 \times 1.5 = 0.75 \tag{27}$$

$$N_3(-0.5) = \frac{-0.5(-0.5 + 1)}{2} = \frac{-0.5 \times 0.5}{2} = -0.125 \tag{28}$$

Element 1 (nodes 1-2-3):

$$u(\xi = -0.5) = 0.125 \times 0 + 0.75 \times (-1) + (-0.125) \times 2 \tag{29}$$

$$= 0 - 0.75 - 0.25 = -1.0 \tag{30}$$

Element 2 (nodes 3-4-5):

$$u(\xi = -0.5) = 0.125 \times 2 + 0.75 \times (-1) + (-0.125) \times 4 \tag{31}$$
$$= 0.25 - 0.75 - 0.5 = -1.0 \tag{32}$$

2.3 Strains at $\xi = -0.5$

For 3-node elements, the strain varies within the element and is calculated using:

$$\varepsilon = \mathbf{B} \cdot \mathbf{u}, \text{ where } \mathbf{B} = \left[\frac{dN_1}{dx}, \frac{dN_2}{dx}, \frac{dN_3}{dx}\right] \tag{33}$$

At $\xi = -0.5$:

$$\frac{dN_1}{d\xi} = \xi - 0.5 = -0.5 - 0.5 = -1.0 \tag{34}$$
$$\frac{dN_2}{d\xi} = -2\xi = -2 \times (-0.5) = 1.0 \tag{35}$$
$$\frac{dN_3}{d\xi} = \xi + 0.5 = -0.5 + 0.5 = 0.0 \tag{36}$$

The Jacobian for both elements (L = 4):

$$J = \frac{L}{2} = \frac{4}{2} = 2 \tag{37}$$

Converting to derivatives with respect to x:

$$\frac{dN_1}{dx} = \frac{dN_1}{d\xi} \cdot \frac{d\xi}{dx} = \frac{-1.0}{2} = -0.5 \tag{38}$$
$$\frac{dN_2}{dx} = \frac{dN_2}{d\xi} \cdot \frac{d\xi}{dx} = \frac{1.0}{2} = 0.5 \tag{39}$$
$$\frac{dN_3}{dx} = \frac{dN_3}{d\xi} \cdot \frac{d\xi}{dx} = \frac{0.0}{2} = 0.0 \tag{40}$$

Element 1:

$$\varepsilon = -0.5 \times 0 + 0.5 \times (-1) + 0.0 \times 2 \tag{41}$$
$$= 0 - 0.5 + 0 = -0.5 \tag{42}$$

Element 2:

$$\varepsilon = -0.5 \times 2 + 0.5 \times (-1) + 0.0 \times 4 \tag{43}$$
$$= -1.0 - 0.5 + 0 = -1.5 \tag{44}$$

2.4 Stiffness Matrices

For a 3-node element, the stiffness matrix is calculated using numerical integration:

$$\mathbf{k} = \int \mathbf{B}^T \cdot E \cdot \mathbf{B} \cdot A \cdot \det(J)\, d\xi \tag{45}$$

Using integration points $\xi = -0.5$ and $\xi = 0.5$ with equal weights of 1.0 and assuming $E = A = 1$:

For Element 1 at $\xi = -0.5$:

$$\mathbf{B}^T = \begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix} \tag{46}$$

$$\mathbf{k}_{contribution} = \begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -0.5 & 0.5 & 0 \end{bmatrix} \cdot 1 \cdot 1 \cdot 2 \cdot 1 \tag{47}$$

$$= \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{48}$$

For Element 1 at $\xi = 0.5$:

$$\mathbf{B}^T = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix} \tag{49}$$

$$\mathbf{k}_{contribution} = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0.5 \end{bmatrix} \cdot 1 \cdot 1 \cdot 2 \cdot 1 \tag{50}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \tag{51}$$

Combining both contributions for Element 1:

$$\mathbf{k_1} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix} \tag{52}$$

$$= \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1.0 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix} \tag{53}$$

Similarly for Element 2:

$$\mathbf{k_2} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1.0 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix} \tag{54}$$

Quantity	**Element 1**	**Element 2**	**Element 3**
Displacement at $\xi = 0.5$	2.5	3.75	5.5
Strain at $\xi = 0.5$	1.0	0.5	1.0

Table 1: Results for Problem 1

Quantity	**Element 1**	**Element 2**
Displacement at $\xi = -0.5$	-1.0	-1.0
Strain at $\xi = -0.5$	-0.5	-1.5

Table 2: Results for Problem 2