

Problem Statements

Below are the original problem statements for Problem Set 3:

Problem 1

See below the structure consisting of three 2-node elements. The displacements at nodes 1, 2, 3, and 4 are 1, 3, 4, and 6.

1. Derive shape functions for all three elements.
2. Use reference element to find the displacements in each element at $\xi = 0.5$.
3. Use reference element to find the strains in each element at $\xi = 0.5$.
4. Calculate the stiffness matrix of all elements using Gauss integration of reference element.

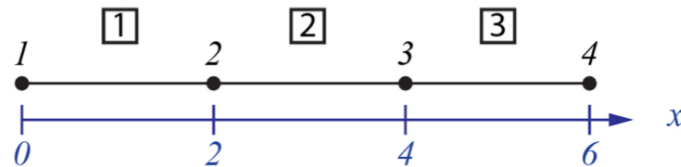


Figure 1: Problem 1 Statement

Problem 2

See below the structure consisting of two 3-node elements. The displacements at nodes 1, 2, 3, 4 and 5 are 0, -1, 2, -1, and 4.

1. Derive shape functions for both elements.
2. Use reference element to find the displacements in each element at $\xi = -0.5$.
3. Use reference element to find the strains in each element at $\xi = -0.5$.
4. Calculate the stiffness matrix of all elements using reference elements using integration points of $\xi = -0.5$ and $\xi = 0.5$.

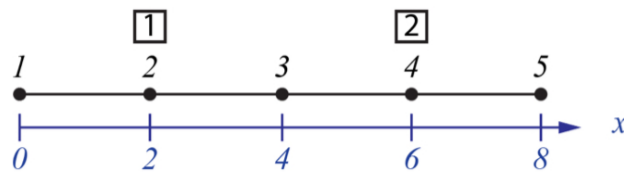


Figure 2: Problem 2 Statement

1 Problem 1: Three 2-Node Elements

1.1 Part 1: Shape Functions

For 2-node elements, the shape functions are:

$$N_1(\xi) = \frac{1 - \xi}{2} \quad (1)$$

$$N_2(\xi) = \frac{1 + \xi}{2} \quad (2)$$

where ξ is the local coordinate ranging from -1 to 1 within each element.

1.2 Part 2: Displacements at $\xi = 0.5$

The displacement at $\xi = 0.5$ in each element is calculated using the shape functions:

$$u(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2 \quad (3)$$

For $\xi = 0.5$:

$$N_1(0.5) = \frac{1 - 0.5}{2} = 0.25 \quad (4)$$

$$N_2(0.5) = \frac{1 + 0.5}{2} = 0.75 \quad (5)$$

Element 1 (nodes 1-2):

$$u(\xi = 0.5) = 0.25 \times 1 + 0.75 \times 3 \quad (6)$$

$$= 0.25 + 2.25 = 2.5 \quad (7)$$

Element 2 (nodes 2-3):

$$u(\xi = 0.5) = 0.25 \times 3 + 0.75 \times 4 \quad (8)$$

$$= 0.75 + 3.0 = 3.75 \quad (9)$$

Element 3 (nodes 3-4):

$$u(\xi = 0.5) = 0.25 \times 4 + 0.75 \times 6 \quad (10)$$

$$= 1.0 + 4.5 = 5.5 \quad (11)$$

1.3 Part 3: Strains at $\xi = 0.5$

For 2-node elements, the strain is constant throughout the element:

$$\varepsilon = \frac{u_2 - u_1}{L} \quad (12)$$

We need to calculate the strains for each element:

Element 1 ($L = 2$):

$$\varepsilon = \frac{3 - 1}{2} = 1.0 \quad (13)$$

Element 2 ($L = 2$):

$$\varepsilon = \frac{4 - 3}{2} = 0.5 \quad (14)$$

Element 3 ($L = 2$):

$$\varepsilon = \frac{6 - 4}{2} = 1.0 \quad (15)$$

1.4 Part 4: Stiffness Matrices

For a 2-node element with Young's modulus E and cross-sectional area A , the stiffness matrix is:

$$\mathbf{k} = \frac{E \cdot A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (16)$$

Assuming $E = A = 1$ for simplicity:

Element 1 ($L = 2$):

$$\mathbf{k}_1 = \frac{1 \cdot 1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (17)$$

$$= 0.5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (18)$$

$$= \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \quad (19)$$

Element 2 ($L = 2$):

$$\mathbf{k}_2 = 0.5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (20)$$

$$= \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \quad (21)$$

Element 3 ($L = 2$):

$$\mathbf{k}_3 = 0.5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (22)$$

$$= \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \quad (23)$$

2 Problem 2: Two 3-Node Elements

2.1 Part 1: Shape Functions

For 3-node elements, the shape functions are:

$$N_1(\xi) = \frac{\xi(\xi - 1)}{2} \quad (24)$$

$$N_2(\xi) = (1 + \xi)(1 - \xi) \quad (25)$$

$$N_3(\xi) = \frac{\xi(\xi + 1)}{2} \quad (26)$$

where ξ is the local coordinate ranging from -1 to 1 within each element, with $\xi = -1, 0, 1$ corresponding to the three nodes.

2.2 Part 2: Displacements at $\xi = -0.5$

The displacement at $\xi = -0.5$ in each element is calculated using the shape functions:

$$u(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2 + N_3(\xi)u_3 \quad (27)$$

For $\xi = -0.5$:

$$N_1(-0.5) = \frac{-0.5(-0.5 - 1)}{2} = \frac{-0.5 \times (-1.5)}{2} = \frac{0.75}{2} = 0.125 \quad (28)$$

$$N_2(-0.5) = (1 + (-0.5))(1 - (-0.5)) = 0.5 \times 1.5 = 0.75 \quad (29)$$

$$N_3(-0.5) = \frac{-0.5(-0.5 + 1)}{2} = \frac{-0.5 \times 0.5}{2} = -0.125 \quad (30)$$

Element 1 (nodes 1-2-3):

$$u(\xi = -0.5) = 0.125 \times 0 + 0.75 \times (-1) + (-0.125) \times 2 \quad (31)$$

$$= 0 - 0.75 - 0.25 = -1.0 \quad (32)$$

Element 2 (nodes 3-4-5):

$$u(\xi = -0.5) = 0.125 \times 2 + 0.75 \times (-1) + (-0.125) \times 4 \quad (33)$$

$$= 0.25 - 0.75 - 0.5 = -1.0 \quad (34)$$

2.3 Part 3: Strains at $\xi = -0.5$

For 3-node elements, the strain varies within the element and is calculated using:

$$\varepsilon = \mathbf{B} \cdot \mathbf{u}, \text{ where } \mathbf{B} = \left[\frac{dN_1}{dx}, \frac{dN_2}{dx}, \frac{dN_3}{dx} \right] \quad (35)$$

At $\xi = -0.5$:

$$\frac{dN_1}{d\xi} = \xi - 0.5 = -0.5 - 0.5 = -1.0 \quad (36)$$

$$\frac{dN_2}{d\xi} = -2\xi = -2 \times (-0.5) = 1.0 \quad (37)$$

$$\frac{dN_3}{d\xi} = \xi + 0.5 = -0.5 + 0.5 = 0.0 \quad (38)$$

The Jacobian for both elements ($L = 4$):

$$J = \frac{L}{2} = \frac{4}{2} = 2 \quad (39)$$

Converting to derivatives with respect to x :

$$\frac{dN_1}{dx} = \frac{dN_1}{d\xi} \cdot \frac{d\xi}{dx} = \frac{-1.0}{2} = -0.5 \quad (40)$$

$$\frac{dN_2}{dx} = \frac{dN_2}{d\xi} \cdot \frac{d\xi}{dx} = \frac{1.0}{2} = 0.5 \quad (41)$$

$$\frac{dN_3}{dx} = \frac{dN_3}{d\xi} \cdot \frac{d\xi}{dx} = \frac{0.0}{2} = 0.0 \quad (42)$$

Element 1:

$$\varepsilon = -0.5 \times 0 + 0.5 \times (-1) + 0.0 \times 2 \quad (43)$$

$$= 0 - 0.5 + 0 = -0.5 \quad (44)$$

Element 2:

$$\varepsilon = -0.5 \times 2 + 0.5 \times (-1) + 0.0 \times 4 \quad (45)$$

$$= -1.0 - 0.5 + 0 = -1.5 \quad (46)$$

2.4 Part 4: Stiffness Matrices

For a 3-node element, the stiffness matrix is calculated using numerical integration:

$$\mathbf{k} = \int \mathbf{B}^T \cdot E \cdot \mathbf{B} \cdot A \cdot \det(J) d\xi \quad (47)$$

Using integration points $\xi = -0.5$ and $\xi = 0.5$ with equal weights of 1.0 and assuming $E = A = 1$:

For Element 1 at $\xi = -0.5$:

$$\mathbf{B}^T = \begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix} \quad (48)$$

$$\mathbf{k}_{contribution} = \begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -0.5 & 0.5 & 0 \end{bmatrix} \cdot 1 \cdot 1 \cdot 2 \cdot 1 \quad (49)$$

$$= \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (50)$$

For Element 1 at $\xi = 0.5$:

$$\mathbf{B}^T = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix} \quad (51)$$

$$\mathbf{k}_{contribution} = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0.5 \end{bmatrix} \cdot 1 \cdot 1 \cdot 2 \cdot 1 \quad (52)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \quad (53)$$

Combining both contributions for Element 1:

$$\mathbf{k}_1 = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix} \quad (54)$$

$$= \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1.0 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix} \quad (55)$$

Similarly for Element 2:

$$\mathbf{k}_2 = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1.0 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix} \quad (56)$$

3 Summary of Results

3.1 Problem 1 Results

Quantity	Element 1	Element 2	Element 3
Displacement at $\xi = 0.5$	2.5	3.75	5.5
Strain at $\xi = 0.5$	1.0	0.5	1.0

Table 1: Summary of results for Problem 1

3.2 Problem 2 Results

Quantity	Element 1	Element 2
Displacement at $\xi = -0.5$	-1.0	-1.0
Strain at $\xi = -0.5$	-0.5	-1.5

Table 2: Summary of results for Problem 2