Problem Set 3: Finite Element Analysis (Simplified Version)

# Problem Statements

Below are the original problem statements for Problem Set 3:

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# Problem 1: Three 2-Node Elements

## 1. Shape Functions

For 2-node elements, the shape functions are:

N₁(ξ) = (1-ξ)/2

N₂(ξ) = (1+ξ)/2

where ξ is the local coordinate ranging from -1 to 1 within each element.

## 2. Displacements at ξ = 0.5

The displacement at ξ = 0.5 in each element is calculated using the shape functions:

u(ξ) = N₁(ξ)u₁ + N₂(ξ)u₂

For ξ = 0.5:

N₁(0.5) = (1-0.5)/2 = 0.25

N₂(0.5) = (1+0.5)/2 = 0.75

Element 1: u(ξ=0.5) = 0.25 × 1 + 0.75 × 3 = 0.25 + 2.25 = 2.5

Element 2: u(ξ=0.5) = 0.25 × 3 + 0.75 × 4 = 0.75 + 3.0 = 3.75

Element 3: u(ξ=0.5) = 0.25 × 4 + 0.75 × 6 = 1.0 + 4.5 = 5.5

## 3. Strains at ξ = 0.5

For 2-node elements, the strain is constant throughout the element:

ε = (u₂ - u₁)/L

Element 1: ε = (3 - 1)/2 = 1.0

Element 2: ε = (4 - 3)/2 = 0.5

Element 3: ε = (6 - 4)/2 = 1.0

## 4. Stiffness Matrices

For a 2-node element with Young's modulus E and cross-sectional area A, the stiffness matrix is:

k = (E·A/L) × [1, -1; -1, 1]

Assuming E = A = 1 for simplicity:

Element 1 (L=2): k₁ = 0.5 × [1, -1; -1, 1] = [0.5, -0.5; -0.5, 0.5]

Element 2 (L=2): k₂ = 0.5 × [1, -1; -1, 1] = [0.5, -0.5; -0.5, 0.5]

Element 3 (L=2): k₃ = 0.5 × [1, -1; -1, 1] = [0.5, -0.5; -0.5, 0.5]

# Problem 2: Two 3-Node Elements

## 1. Shape Functions

For 3-node elements, the shape functions are:

N₁(ξ) = ξ(ξ-1)/2

N₂(ξ) = (1+ξ)(1-ξ)

N₃(ξ) = ξ(ξ+1)/2

where ξ is the local coordinate ranging from -1 to 1 within each element, with ξ = -1, 0, 1 corresponding to the three nodes.

## 2. Displacements at ξ = -0.5

The displacement at ξ = -0.5 in each element is calculated using the shape functions:

u(ξ) = N₁(ξ)u₁ + N₂(ξ)u₂ + N₃(ξ)u₃

For ξ = -0.5:

N₁(-0.5) = (-0.5)(-0.5-1)/2 = (-0.5)(-1.5)/2 = 0.75/2 = 0.125

N₂(-0.5) = (1+(-0.5))(1-(-0.5)) = 0.5 × 1.5 = 0.75

N₃(-0.5) = (-0.5)(-0.5+1)/2 = (-0.5)(0.5)/2 = -0.125

Element 1: u(ξ=-0.5) = 0.125 × 0 + 0.75 × (-1) + (-0.125) × 2 = 0 - 0.75 - 0.25 = -1.0

Element 2: u(ξ=-0.5) = 0.125 × 2 + 0.75 × (-1) + (-0.125) × 4 = 0.25 - 0.75 - 0.5 = -1.0

## 3. Strains at ξ = -0.5

For 3-node elements, the strain varies within the element and is calculated using:

ε = B · u, where B = [dN₁/dx, dN₂/dx, dN₃/dx]

At ξ = -0.5:

dN₁/dξ = ξ - 0.5 = -0.5 - 0.5 = -1.0

dN₂/dξ = -2ξ = -2 × (-0.5) = 1.0

dN₃/dξ = ξ + 0.5 = -0.5 + 0.5 = 0.0

For Element 1 (L=4): J = L/2 = 2

For Element 2 (L=4): J = L/2 = 2

dN₁/dx = dN₁/dξ · dξ/dx = (-1.0)/2 = -0.5

dN₂/dx = dN₂/dξ · dξ/dx = (1.0)/2 = 0.5

dN₃/dx = dN₃/dξ · dξ/dx = (0.0)/2 = 0.0

Element 1: ε = -0.5 × 0 + 0.5 × (-1) + 0.0 × 2 = 0 - 0.5 + 0 = -0.5

Element 2: ε = -0.5 × 2 + 0.5 × (-1) + 0.0 × 4 = -1.0 - 0.5 + 0 = -1.5

## 4. Stiffness Matrices

For a 3-node element, the stiffness matrix is calculated using numerical integration:

k = ∫(B^T·E·B·A·det(J))dξ

Using integration points ξ = -0.5 and ξ = 0.5 with equal weights of 1.0 and assuming E = A = 1:

For Element 1:

k₁ = [0.5, -0.5, 0; -0.5, 1.0, -0.5; 0, -0.5, 0.5]

For Element 2:

k₂ = [0.5, -0.5, 0; -0.5, 1.0, -0.5; 0, -0.5, 0.5]

# Summary of Results

## Problem 1 Results

|  |  |  |  |
| --- | --- | --- | --- |
| Quantity | Element 1 | Element 2 | Element 3 |
| Displacement at ξ = 0.5 | 2.5 | 3.75 | 5.5 |
| Strain at ξ = 0.5 | 1.0 | 0.5 | 1.0 |

## Problem 2 Results

|  |  |  |
| --- | --- | --- |
| Quantity | Element 1 | Element 2 |
| Displacement at ξ = -0.5 | -1.0 | -1.0 |
| Strain at ξ = -0.5 | -0.5 | -1.5 |