at the nozzle inlet that follows from the area ratio A_1 / A_{th} based on the choked throat $M_{th} = 1.0$ condition. Note the parameter trends inside a rocket engine shown in Figure 12.7.

EXAMPLE 12.1

The Space Shuttle Main Engine (SSME) produces 375,000 lb of thrust at sea level and 470,000 lb at altitude. Assuming constant momentum thrust with altitude, estimate the diameter D_2 of the SSME nozzle exit area A_2.

SOLUTION

The difference between the sea level and altitude thrust is primarily caused by the difference between the pressure thrusts, i.e.,

$$F_{\text{vacuum}} - F_{\text{sea level}} \equiv (p_{\text{sl}} - 0)A_2$$

Therefore, $A_2 \approx 95,000\,\text{lb}/(14.7\,\text{lb/in}^2)/(144\,\text{in}^2/\text{ft}^2) \approx 44.88\,\text{ft}^2$

Diameter $D_2 = (4A_2/\pi)^{1/2} \approx 7.56$ ft or 90.7 in.

The actual (SSME nozzle) exit diameter is listed at 94 in. There is a small discrepancy (i.e., ~3.5%) due to our simplified approach, e.g., assuming identical momentum thrust at sea level and altitude or neglecting pressure at altitude.

EXAMPLE 12.2

A rocket engine has a propellant mass flow rate of 1000 kg/s and the specific impulse is $I_s = 340$ s. Calculate

(a) rocket thrust F

(b) effective exhaust velocity c

SOLUTION

From the definition of specific impulse, we get

$$F = \dot{m}_p g_0 I_s = (1000\,\text{kg/s})(9.8\,\text{m/s}^2)(340\,\text{s}) \approx 3.332 \times 10^6\,\text{N}$$

The effective exhaust velocity is the ratio of thrust to propellant mass flow rate, $c = F/\dot{m}_p \approx 3332\,\text{m/s}$

12.6 Thrust Coefficient, C_F

It is customary to express the rocket thrust as a product of chamber pressure p_c, the throat area A_{th}, and a coefficient C_F, known as the *thrust coefficient*, that is,

$$F \equiv C_F p_c A_{th} \tag{12.10}$$

In this section, we will demonstrate that thrust coefficient is a function of the nozzle area expansion ratio A_2/A_{th}, the pressure ratio p_2/p_c, and the ratio of specific heats γ. Therefore, the thrust coefficient is explicitly independent of combustion temperature (thus propellant combination) and is treated as a nozzle-related parameter. The ratio of specific heats γ is the only link between thrust coefficient and the propellant combination or combustion temperature.

The mass flow rate through the nozzle may be written at the sonic throat, through the use of the continuity equation:

$$\dot{m}_p = \sqrt{\frac{\gamma}{R}}\frac{p_c}{\sqrt{T_c}}A_{th}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{12.11}$$

The exhaust velocity V_2 is written based on the isentropic expansion through the nozzle from the stagnation pressure of p_c to the exit static pressure p_2 through these simple steps:

$$T_c = T_2 + V_2^2/2c_p$$

$$V_2 = \sqrt{2c_p(T_c - T_2)} = \sqrt{2c_pT_c\left[1-\left(\frac{p_2}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{12.12}$$

Since the momentum thrust is the product of mass flow rate and the exhaust velocity (Equations 12.11 and 12.12), we may write the following expression for the thrust coefficient

$$C_F = \frac{\dot{m}_pV_2 + (p_2 - p_0)A_2}{p_cA_{th}} \tag{12.13a}$$

$$C_F = \sqrt{\left(\frac{2\gamma^2}{\gamma-1}\right)\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1-\left(\frac{p_2}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \frac{p_2-p_0}{p_c}\left(\frac{A_2}{A_{th}}\right) \tag{12.13b}$$

A perfectly expanded nozzle meets the condition of static pressure match $p_2 = p_0$, therefore the maximum thrust coefficient is a function of p_2/p_c and the ratio of specific heat γ. We may graph the optimum thrust coefficient, that is, C_F for a perfectly expanded nozzle with isentropic expansion using an Excel spreadsheet calculation and graph of the functionin.

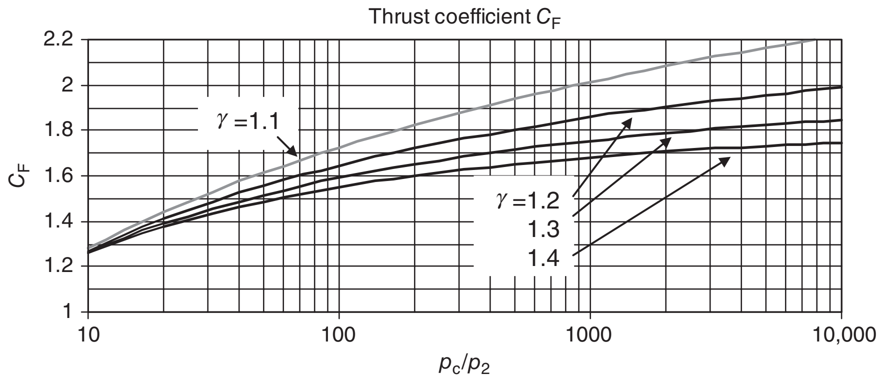
$$C_{F,opt} = \sqrt{\left(\frac{2\gamma^2}{\gamma-1}\right)\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1-\left(\frac{p_2}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{12.13c}$$

Figure 12.8 (with abscissa in logarithmic scale).

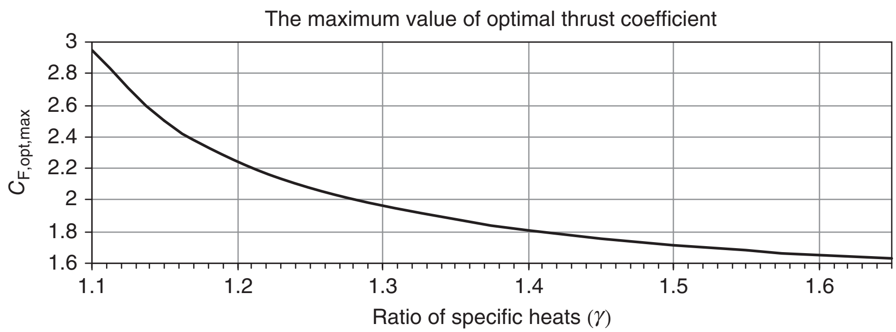
Before leaving the subject of thrust coefficient, we recognize the bracket term under the square root in Equations 12.13b or 12.13c as the ideal thermal efficiency of a Brayton cycle operating between the two pressures p_c and p_2, that is,

$$\eta_{th} = 1 - \left(\frac{p_2}{p_c}\right)^{\frac{\gamma-1}{\gamma}} \tag{12.14}$$

■ **FIGURE 12.8**
Variation of optimum thrust coefficient for different gas ratio of specific heats (note that p_c / p_2 is the familiar *nozzle pressure ratio* (NPR) for a perfectly expanded nozzle)



■ **FIGURE 12.9**
The variation of maximum, optimal thrust coefficient with ration of specific heats γ



For very large chamber pressures p_c and very large area ratio nozzles where the exit static pressure p_2 is low, the thermal efficiency approaches 1, which then points to the maximum attainable thrust coefficient $C_{F,\,opt,\,Max}$, that is,

$$C_{F,\,opt,\,Max} = \sqrt{\left(\frac{2\gamma^2}{\gamma - 1}\right)\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}} \tag{12.15}$$

The maximum value of thrust coefficient is a pure function of γ and is shown in Figure 12.9.

EXAMPLE 12.3

A rocket engine has a chamber pressure of $p_c = 200$ atm and the throat area is $A_{th} = 0.25\ m^2$. Assuming that the nozzle is perfectly expanded with the gas ratio of specific heats $\gamma = 1.3$ and the ambient pressure of $p_0 = 1$ atm, calculate

(a) optimum thrust coefficient $C_{F,opt}$
(b) thrust F in N or lbf
(c) nozzle exit Mach number M_2
(d) nozzle area expansion ratio A_2 / A_{th}

SOLUTION

We substitute for p_c/p_2 of 200/1 and $\gamma = 1.3$ in Equation 12.13c, to get

$$C_{F,\,opt.} = 1.650$$

Therefore, thrust is the product of thrust coefficient, chamber pressure, and throat area. The chamber pressure is first converted to N/m^2 (or Pa),

$$p_c = 200\,\text{atm} = 200(101\,\text{kPa}) = 20.2\,\text{MPa}$$

$$F = 8.332 \times 10^6\,\text{N} \quad (\text{or } 1.873 \times 10^6\,\text{lbf})$$

The nozzle exit Mach number for an isentropic flow in a nozzle is related to the ratio of p_c/p_2, according to

$$p_c/p_2 \approx p_{t2}/p_2 = \left[1 + (\gamma - 1)M_2^2/2\right]^{\gamma/(\gamma-1)}$$

Therefore, the exit Mach number is

$$M_2 = \sqrt{\frac{2}{\gamma - 1}\left[\left(\frac{p_c}{p_2}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]} \approx 4.0$$

From the continuity equation written between the exit and the throat, taking throat Mach number as unity, we express A_2/A^*, which is the same as A_2/A_{th} in terms of exit Mach number M_2 and a function of γ, according to

$$\frac{A_2}{A_{th}} = \frac{1}{M_2}\left[\frac{2}{\gamma + 1}\left(1 + \frac{\gamma - 1}{2}M_2^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \approx 15.89$$

12.7 Characteristic Velocity, c^*

A velocity parameter that is linked to the combustion chamber is called characteristic velocity c^*. We expect this velocity to be related to the speed of sound in the combustion chamber, as the characteristic speed. But, first its definition

$$c^* \equiv \frac{p_c A_{th}}{\dot{m}_p} \qquad \textbf{(12.16)}$$

In terms of specific impulse and thrust coefficient, or effective exhaust speed and thrust coefficient, the characteristic velocity may be written as

$$c^* = I_s g_0/C_F = c/C_F \qquad \textbf{(12.17)}$$

Returning to the definition of c^* in Equation 12.16, we may express the propellant mass flow rate for the sonic flow at the throat and gas total pressure and temperature, to get

$$c^* = \frac{\sqrt{\gamma R T_c}}{\gamma\left[\frac{2}{\gamma + 1}\right]^{\frac{\gamma+1}{2(\gamma-1)}}} \qquad \textbf{(12.18)}$$

As expected, the numerator in Equation 12.18 is the speed of sound in the combustion chamber. The propellant combination of fuel and oxidizer and their proportions establish the equivalence ratio, the products of combustion, and the mixture temperature for a

■ **TABLE 12.1**

Theoretical Performance of Liquid Propellant Combinations (data from Sutton and Biblarz, 2001)

Oxidizer	Fuel	Mixture ratio (Mass)	Chamber temperature (K)	Chamber c^* (m/s)
Oxygen	Methane	3.2	3526	1835
		3.0	3526	1853
	Hydrazine	0.74	3285	1871
		0.90	3404	1892
	Hydrogen	3.40	2959	2428
		4.02	2999	2432
	RP-1	2.24	3571	1774
		2.56	3677	1800
Fluorine	Hydrazine	1.83	4553	2128
		2.30	4713	2208
	Hydrogen	4.54	3080	2534
		7.60	3900	2549

combustion pressure p_c, as we studied in Chapter 7. From Equation 12.16, we note that the experimental estimation of characteristic velocity is rather straightforward, since the chamber pressure and propellant flow rate are easily measured and the throat area is a geometric parameter. The Table 12.1 shows some liquid propellant combinations and the characteristic speed c^*, also known as *chamber* c^* (from Sutton and Biblarz, 2001). The mixture ratio is the *oxidizer-to-fuel* mass ratio. The chamber pressure for the ideal performance values in Table 12.1 is taken to be 1000 psia (this corresponds to ~6.87 MPa). The 1000 psia chamber pressure is often taken as a reference value in rocket propulsion. RP-1 is a hydrocarbon fuel [$CH_{1.953}$] similar to kerosene.

A few examples of solid propellant combinations and the corresponding chamber c^* values are shown in Table 12.2 (from Sutton and Biblarz, 2001). Again, the reference chamber pressure is taken to be 1000 psia. The composition of solid propellant is characterized by the amount of binder (i.e., glue or epoxy), additives, and aluminum, which enhance combustion chamber temperature and chamber c^*.

■ **TABLE 12.2**

Theoretical Performance of Solid Propellant Combinations (data from Sutton and Biblarz, 2001)

Oxidizer	Fuel	Chamber temperature (K)	Chamber c^* (m/s)
Ammonium nitrate	11% binder and 7% additive	1282	1209
Ammonium Perchlorate 78–66%	18% organic polymer binder and 4–20% Al	2816	1590
Ammonium Perchlorate 84–68%	12% polymer binder and 4–20% Al	3371	1577

EXAMPLE 12.4

A liquid propellant rocket uses hydrogen fuel and oxygen for combustion. The oxidizer-to-fuel mass ratio is 4.02. The ratio of specific heats of the combustion gas is $\gamma = 1.26$ and the chamber pressure is 1000 psia. Use Equation 12.18 to estimate combustion gas constant R as well as the molecular weight of the mixture, MW.

SOLUTION

From Table 12.1, we read $T_c = 2999$ K and $c^* = 2432$ m/s corresponding to the liquid propellant described above.

$$c^* = \frac{\sqrt{\gamma R T_c}}{\gamma\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}}} = 2432\,\text{m/s} \Rightarrow \sqrt{\gamma R T_c} = 1801.5\,\text{m/s}$$

By substituting for γ and T_c, we estimate $R \approx 858.9\,\text{J/kg}\cdot\text{K}$ and since the universal gas constant is $\bar{R} = 8.3146\,\text{kJ/kmol}\cdot\text{K}$,

$$\text{MW} = \frac{\bar{R}}{R} \approx 9.68\frac{\text{kg}}{\text{kmol}}$$

EXAMPLE 12.5

Consider a fuel-rich combustion of hydrogen and oxygen in a liquid propellant rocket according to

$$4H_2 + O_2 \longrightarrow 2H_2O + 2H_2$$

Calculate

(a) the oxidizer-to-fuel mixture ratio

(b) the molecular weight of the mixture of gases in the products of combustion

SOLUTION

The oxidizer-to-fuel mass ratio is (32/8), which is 4, very close to the mixture ratio in Example 12.4.

The mixture molecular weight is

$$\text{MW} = \frac{2(18) + 2(2)}{4} = 10\frac{\text{kg}}{\text{kmol}}$$

This simple calculation of chemical reaction and the mixture of gases in the combustion products support our earlier calculations of molecular weight in Example 12.4 where we found that the average molecular weight of products of combustion was ~9.68 kg/kmol.

12.8 Flight Performance

Consider a rocket of instantaneous mass $m(t)$ flying at the instantaneous speed V. The flight path angle with respect to horizon is θ, where vertical is the direction of the gravitational force, as shown in Figure 12.10. The atmospheric drag D acts along the vehicle axis, as shown.

We may relate the net forces along the vehicle axis to the vehicle instantaneous acceleration according to

$$m\frac{dV}{dt} = F - D - mg\sin\theta \tag{12.19}$$