1 Preamble/Notation

The primary problem posed in this project, is to find a solution to the following LCP:

find $\lambda \in \mathbb{R}^n$ such that...

$Q\lambda + b \geq 0$, $\lambda \geq 0$, $\lambda^T(Q\lambda + b) = 0$

With $\mathbf{Q} := G_{\mathbb{A}}^T M^{-1} G_{\mathbb{A}}$, $\mathbf{b} := G_{\mathbb{A}}^T \dot{q}^-(1 + c_r)$

2 Result: Relating Q matrix to angles

What does Q actually represent? Let's simplify for now and assume that $M = I = M^{-1}$ so that $Q = G_{\mathbb{A}}^T M^{-1} G_{\mathbb{A}} = G_{\mathbb{A}}^T G_{\mathbb{A}}$

$G_{\mathbb{A}} \in M_{3m\times n}(\mathbb{R})$ where n = number of collisions (size of "active set"), and m = the number of balls in the simulation. Here we assume that each ball has 3 coordinates: an x, y, and a rotation $\phi = 0$ [1] (hence $3m$ rows).

Every column of $G_{\mathbb{A}}$ can be written as $(G_{\mathbb{A}}^T)_i = \nabla g_i(q)$. If $g_i(q)$ is the collision constraint between balls **a** and **b**, then, using a_x, a_y, b_x, b_y to represent the indices of the x and y coordinates of balls a and b respectively:

$$
\begin{aligned}
g(q) &= \mathrm{dist}(a,\, b) \\
&= \sqrt{(q_{a_x} - q_{b_x})^2 + (q_{a_y} - q_{b_y})^2} - r_a - r_b
\end{aligned}
$$

Where r_a, r_b are the radii of balls a and b.[2]

The constraint gradient (i.e. the columns that make up $G_{\mathbb{A}}$) can now be written as:

[1] We aren't yet concerned with friction effects, so balls with no rotation initially stay that way during and after collision.

[2] It's worth noting that this is the ONLY place where the radius of the balls will appear. Once we take the gradient of $g(q)$ the r constants will disappear.

$$
\begin{aligned}
\nabla g_i(q) &= \left[\dots \ \frac{\partial \text{dist}(a,\, b)}{\partial q_{a_x}} \ \frac{\partial \text{dist}(a,\, b)}{\partial q_{a_y}} \ \dots \ \frac{\partial \text{dist}(a,\, b)}{\partial q_{b_x}} \ \frac{\partial \text{dist}(a,\, b)}{\partial q_{b_y}} \ \dots \ \right]^T \\
&= \left[0 \dots 0 \ \frac{\partial \text{dist}(a,\, b)}{\partial q_{a_x}} \ \frac{\partial \text{dist}(a,\, b)}{\partial q_{a_y}} \ 0 \dots 0 \ \frac{\partial \text{dist}(a,\, b)}{\partial q_{b_x}} \ \frac{\partial \text{dist}(a,\, b)}{\partial q_{b_y}} \ 0 \dots 0 \ \right]^T \\
&= \left[0 \dots 0 \ \frac{q_{a_x} - q_{b_x}}{\text{dist}(a,\, b)} \ \frac{q_{a_y} - q_{b_y}}{\text{dist}(a,\, b)} \ 0 \dots 0 \ \frac{q_{b_x} - q_{a_x}}{\text{dist}(a,\, b)} \ \frac{q_{b_y} - q_{a_y}}{\text{dist}(a,\, b)} \ 0 \dots 0 \ \right]^T
\end{aligned}
$$

Notice that the sub-vector $\vec{n_{ab}} := \left[\frac{\partial \text{dist}(a,\, b)}{\partial q_{a_x}} \ \frac{\partial \text{dist}(a,\, b)}{\partial q_{a_y}}\right]^T = \left[\frac{q_{a_x} - q_{b_x}}{\text{dist}(a,\, b)} \ \frac{q_{a_y} - q_{b_y}}{\text{dist}(a,\, b)}\right]^T$ which is made up of the first 2 non-zero values of $\nabla g_i(q)$ is the collision normal vector $\vec{n_{ba}}$! Similarly, the other 2 non-zero entries of $\nabla g_i(q)$ make up the opposing collision normal: $-\vec{n_{ba}} = \vec{n_{ab}}$

Now we know what the columns of $G_{\mathbb{A}}$ consist of, we can look at the individual elements of Q:

$$
\begin{aligned}
Q_{ij} &= G^T_{\mathbb{A}i} \cdot G^T_{\mathbb{A}j} \\
&= \nabla g_i(q) \cdot \nabla g_j(q) \\
&= \dots
\end{aligned}
$$

... 3 Cases

1: $i = j$ (2 balls)

In this case, $Q_{ij} = Qii = \nabla g_i(q) \cdot \nabla g_i(q) = ||\vec{n_{ab}}||^2 + ||\vec{n_{ba}}||^2 = 2$

2: $i \neq j$ with 2 separate collisions (4 balls)

Consider $Q_{ij} = \nabla g_i(q) \cdot \nabla g_j(q)$ where $g_i(q) = \text{dist}(a,\, b)$ and $g_j(q) = \text{dist}(c,\, d)$. i.e. we consider 2 collisions (i,j) involving 4 distinct balls (a,b,c,d). Here the 4 non-zero entries of $g_i(q)$ will occur at different indices than the non-zero elements of $g_j(q)$! [3] Therefore: $Q_{ij} = \nabla g_i(q) \cdot \nabla g_j(q) = 0$

[3] The kth element of $\nabla g_i(q) \neq 0$ means that the kth element is the partial of g w.r.t. either ball a or b. This implies that the kth element of $\nabla g_j(q) = \frac{\partial g_j(q)}{\partial q_k} = 0$ since $k \in \{a_x, a_y, b_x, b_y\}$ and constraint between balls c and d is independent of a or b's position.

3: $i \neq j$ with 2 interacting collisions (3 balls)

The most interesting case! Let's assume WLOG that 3 balls a, b, and c are colliding simultaneously. WLOG, assume $g_i(q) = \text{dist}(a, b)$, and $g_j(q) = \text{dist}(b, c)$ so that $i, j \in \mathbb{A}$.

Then $Q_{ij} = \nabla g_i(q) \cdot \nabla g_j(q) = \vec{n_{ab}} \cdot \vec{n_{cb}} = |\vec{n_{ab}}||\vec{n_{cb}}| \cos\theta = \cos\theta$

Where $\theta := \angle abc$

Result: interpretation of b, $Q\lambda$, LCP criteria

b

b is the other constant value of interest in out LCP. If we assume once again that $g_i(q)$ is the constraint between balls a and b:

$$\begin{aligned} b &:= G_{\mathbb{A}}^T \dot{q}^-(1 + c_r) \\ \implies b_i &= (1 + c_r)\nabla g_i(q)\dot{q}^- \\ &= (1 + c_r)\frac{dg_i}{dt} \\ &= (1 + c_r)\frac{d(\text{dist}(a, b))}{dt} \end{aligned}$$

Interpretation: the b_i represents

$-(\mathbf{1} + \mathbf{c_r})\times$ (relative speed ball a is approaching ball b).

(note the negative sign since $\frac{d(\text{dist}(a, b))}{dt} \leq 0$)

$Q\lambda$

From the Rosi paper, we know that $\lambda \in \mathbb{R}^{|\mathbb{A}|}$ is the vector of impulse coefficients (i.e. λ_i is the force that ball a exerts on ball b ***TODO: ensure this is *technically/semantically* correct!) Now consider the ith element of $Q\lambda$:

$$(Q\lambda)_i = \sum_{j=1}^{\mathbb{A}} Q_{ij}\lambda_j$$

For Q_{ij} there are 3 cases (see above). Firstly, collision constraint i could not be affected by either of the balls involved in collision j, in which case $Q_{ij} = 0$ and we can ignore those terms. Secondly, there will be a term where $i = j$, in this case $Q_{ij} = Q_{ii} = 2$. And finally, assume that constraint i is "concerned" with balls a and b, (i.e. $g_i(q) = \text{dist}(a, b)$) and j is "concerned" with *either* ball a or b *and* some 3rd ball c. In this case $Q_{ij} = \cos(\angle abc)$ or $Q_{ij} = \cos(\angle cab)$.

So, if we set:
$A = \{x \in \mathbb{A} | s.t.\ g_x(q) = \text{dist}(a, c)\} \backslash \{i\}$
And $B = \{x \in \mathbb{A} | s.t.\ g_x(q) = \text{dist}(b, c)\} \backslash \{i\}$
Where in both cases c is just a ball that is in the process of colliding with ball a or b respectively, then we can rewrite above as:

$$
\begin{aligned}
(Q\lambda)_i &= \sum_{j=1}^{\mathbb{A}} Q_{ij}\lambda_j \\
&= 2\lambda_i + \sum_{x \in A} \cos(\angle ba\ \text{ball}(x))\lambda_x + \sum_{x \in B} \cos(\angle ab\ \text{ball}(x))\lambda_x \\
&= 2\lambda_i + || \sum_{x \in A} \text{proj}_{\overrightarrow{n_{ba}}}(\overrightarrow{n_{\text{ball}(x)a}})\lambda_x || + || \sum_{x \in B} \text{proj}_{\overrightarrow{n_{ab}}}(\overrightarrow{n_{a\ \text{ball}(x)}})\lambda_x || \\
&= \text{net force acting on balls a and b along direction: } \overrightarrow{n_{ab}}
\end{aligned}
$$

TODO: clean up the equation above... using "ball$(()x)$" is confusing... maybe some of the ab should be ba... theres a little more explaining that could be done. ESPECIALLY relating to our disregard of mass - really, the above should be the net $\Delta\dot{q}$ once we bring back M^{-1} and divide each term by it's balls' masses.

LCP Criteria: $Q\lambda + b \geq 0$

This is saying for each collision $i \in \mathbb{A}$, we need $(Q\lambda)_i \geq -b_i$. Let $[\dot{q}]_{\overrightarrow{n_{ab}}}$ represent the speed of ball a wrt ball b.
As we have seen before, $b_i = -(1+c_r)[\dot{q}]_{\overrightarrow{n_{ab}}}$ so we can rewrite our lcp condition as:

$$\text{net force (speed?) acting on balls a and b} \geq (1 + c_r)[\dot{q}]_{\overrightarrow{n_{ab}}}$$

Or something like that... basically, the LCP condition is enforcing our solution (λ, the impulse coefficients) will result in exiting velocities that conserve momentum.

when we factor in the complementary condition, the times 2 above makes sense!

either $(Q\lambda + b)_i = 0$, in this case $\lambda_i > 0$ and we get

$$
\begin{aligned}
(Q\lambda)_i &= \sum_{j=1}^{\mathbb{A}} Q_{ij}\lambda_j \\
&= 2\lambda_i + \sum_{x\in A} \cos(\angle ba \text{ ball}(x))\lambda_x + \sum_{x\in B} \cos(\angle ab \text{ ball}(x))\lambda_x \\
&= 2\lambda_i + || \sum_{x\in A} \text{proj}_{\overrightarrow{n_{ba}}}(\overrightarrow{n_{\text{ball}(x)a}})\lambda_x || + || \sum_{x\in B} \text{proj}_{\overrightarrow{n_{ab}}}(\overrightarrow{n_{a\text{ ball}(x)}})\lambda_x || \\
&= || \sum_{x\in A} \text{proj}_{\overrightarrow{n_{ba}}}(\overrightarrow{n_{\text{ball}(x)a}})\lambda_x || + || \sum_{x\in B} \text{proj}_{\overrightarrow{n_{ab}}}(\overrightarrow{n_{a\text{ ball}(x)}})\lambda_x ||
\end{aligned}
$$

basically, our LCP solution in this case requires that the relative exit velocity of balls a and b is ONLY a result of the forces of the OTHER balls colliding with balls a/b.

Or... wait no nvm lol

TODO: more investigation here perhaps?

result: Q is not generally an "M-matrix"

This was something we were concerned with in previous semesters of work. The off-diagonal entries of Q are $\cos\theta$ where $\theta \in [0, \pi]$! It's easy to imagine a scenario where 2 balls (a and b) simultaneously collide with a common 3rd ball (c) so that the angle connecting the center of the 3 balls $\angle acb < \pi/2$ and thus $\cos\theta > 0$, ($\implies Q$ cannot be an M-matrix)

TODO: illustration?

3 overlapping and collisions through balls

4 "K3" Example

Generally speaking, when only 2 balls collide with each other, there isn't much special going on... $G_{\mathbb{A}} \in M_{3m\times 1}(\mathbb{R})$ so $Q \in M_{1\times 1}(\mathbb{R})$ and λ is really easy to find.

A more interesting scenario can be found when 3 balls collide with each other while all on the same axis. [4] In this case, all angles are either 0 or π, so Q is: [5]

$$Q = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Which is singular! There are an infinite number of solutions to the LCP - and IPOPT does in fact find a different solution to policy iteration!

5 "K4" Example

4 balls (all overlapping each other) configured in a square with velocities towards the center of the square makes a "K4" type of graph where each of the 4 balls is colliding with the other 3 in the same instant. This also produces a singular Q matrix (rank = 5, nullity = 1), and IPOPT/PI yield different solutions.

in both K3 and K4 examples, there are 3 viable control sets.

[4] This does required that ball collides with ball c *through* ball b... but sadly, because of discrete time issues, this is a case that must be considered

[5] Order of values also depend on the order of collisions in the active set.