

1 Preamble/Notation

The primary problem posed in this project, is to find a solution to the following LCP:

$$\begin{aligned} &\text{find } \lambda \in \mathbb{R}^n \text{ such that...} \\ &Q\lambda + b \geq 0, \lambda \geq 0, \lambda^T(Q\lambda + b) = 0 \\ &\text{With } \mathbf{Q} := G_{\mathbb{A}}^T M^{-1} G_{\mathbb{A}}, \mathbf{b} := G_{\mathbb{A}}^T \dot{q}^-(1 + c_r) \end{aligned}$$

2 Result: Relating Q matrix to angles

What does Q actually represent? Let's simplify for now and assume that $M = I = M^{-1}$ so that $Q = G_{\mathbb{A}}^T M^{-1} G_{\mathbb{A}} = G_{\mathbb{A}}^T G_{\mathbb{A}}$

$G_{\mathbb{A}} \in M_{3m \times n}(\mathbb{R})$ where n = number of collisions (size of "active set"), and m = the number of balls in the simulation. Here we assume that each ball has 3 coordinates: an x, y , and a rotation $\phi = 0^1$ (hence $3m$ rows).

Every column of $G_{\mathbb{A}}$ can be written as $(G_{\mathbb{A}}^T)_i = \nabla g_i(q)$. If $g_i(q)$ is the collision constraint between balls \mathbf{a} and \mathbf{b} , then, using a_x, a_y, b_x, b_y to represent the indices of the x and y coordinates of balls \mathbf{a} and \mathbf{b} respectively:

$$\begin{aligned} g(q) &= \text{dist}(a, b) \\ &= \sqrt{(q_{a_x} - q_{b_x})^2 + (q_{a_y} - q_{b_y})^2} - r_a - r_b \end{aligned}$$

Where r_a, r_b are the radii of balls \mathbf{a} and \mathbf{b} .²

The constraint gradient (i.e. the columns that make up $G_{\mathbb{A}}$) can now be written as:

¹We aren't yet concerned with friction effects, so balls with no rotation initially stay that way during and after collision.

²It's worth noting that this is the ONLY place where the radius of the balls will appear. Once we take the gradient of $g(q)$ the r constants will disappear.

$$\begin{aligned}
\nabla g_i(q) &= \left[\dots \frac{\partial \text{dist}(a, b)}{\partial q_{a_x}} \frac{\partial \text{dist}(a, b)}{\partial q_{a_y}} \dots \frac{\partial \text{dist}(a, b)}{\partial q_{b_x}} \frac{\partial \text{dist}(a, b)}{\partial q_{b_y}} \dots \right]^T \\
&= \left[0 \dots 0 \frac{\partial \text{dist}(a, b)}{\partial q_{a_x}} \frac{\partial \text{dist}(a, b)}{\partial q_{a_y}} 0 \dots 0 \frac{\partial \text{dist}(a, b)}{\partial q_{b_x}} \frac{\partial \text{dist}(a, b)}{\partial q_{b_y}} 0 \dots 0 \right]^T \\
&= \left[0 \dots 0 \frac{q_{a_x} - q_{b_x}}{\text{dist}(a, b)} \frac{q_{a_y} - q_{b_y}}{\text{dist}(a, b)} 0 \dots 0 \frac{q_{b_x} - q_{a_x}}{\text{dist}(a, b)} \frac{q_{b_y} - q_{a_y}}{\text{dist}(a, b)} 0 \dots 0 \right]^T
\end{aligned}$$

Notice that the sub-vector $\vec{n}_{ab} := \left[\frac{\partial \text{dist}(a, b)}{\partial q_{a_x}} \frac{\partial \text{dist}(a, b)}{\partial q_{a_y}} \right]^T = \left[\frac{q_{a_x} - q_{b_x}}{\text{dist}(a, b)} \frac{q_{a_y} - q_{b_y}}{\text{dist}(a, b)} \right]^T$

which is made up of the first 2 non-zero values of $\nabla g_i(q)$ is the collision normal vector \vec{n}_{ba} ! Similarly, the other 2 non-zero entries of $\nabla g_i(q)$ make up the opposing collision normal: $-\vec{n}_{ba} = \vec{n}_{ab}$

Now we know what the columns of $G_{\mathbb{A}}$ consist of, we can look at the individual elements of Q :

$$\begin{aligned}
Q_{ij} &= G_{\mathbb{A}i}^T \cdot G_{\mathbb{A}j}^T \\
&= \nabla g_i(q) \cdot \nabla g_j(q) \\
&= \dots
\end{aligned}$$

... 3 Cases

1: $i = j$ (2 balls)

In this case, $Q_{ij} = Q_{ii} = \nabla g_i(q) \cdot \nabla g_i(q) = \|\vec{n}_{ab}\|^2 + \|\vec{n}_{ba}\|^2 = 2$

2: $i \neq j$ with 2 separate collisions (4 balls)

Consider $Q_{ij} = \nabla g_i(q) \cdot \nabla g_j(q)$ where $g_i(q) = \text{dist}(a, b)$ and $g_j(q) = \text{dist}(c, d)$. i.e. we consider 2 collisions (i,j) involving 4 distinct balls (a,b,c,d). Here the 4 non-zero entries of $g_i(q)$ will occur at different indices than the non-zero elements of $g_j(q)$! ³ Therefore: $Q_{ij} = \nabla g_i(q) \cdot \nabla g_j(q) = 0$

³The k th element of $\nabla g_i(q) \neq 0$ means that the k th element is the partial of g w.r.t. either ball a or b. This implies that the k th element of $\nabla g_j(q) = \frac{\partial g_j(q)}{\partial q_k} = 0$ since $k \in \{a_x, a_y, b_x, b_y\}$ and constraint between balls c and d is independent of a or b's position.

3: $i \neq j$ with 2 interacting collisions (3 balls)

The most interesting case! Let's assume WLOG that 3 balls a, b, and c are colliding simultaneously. WLOG, assume $g_i(q) = \text{dist}(a, b)$, and $g_j(q) = \text{dist}(b, c)$ so that $i, j \in \mathbb{A}$.

$$\text{Then } Q_{ij} = \nabla g_i(q) \cdot \nabla g_j(q) = \vec{n}_{ab} \cdot \vec{n}_{cb} = |\vec{n}_{ab}| |\vec{n}_{cb}| \cos \theta = \cos \theta$$

Where $\theta := \angle abc$

Result: interpretation of b , $Q\lambda$, LCP criteria

b

b is the other constant value of interest in our LCP. If we assume once again that $g_i(q)$ is the constraint between balls a and b:

$$\begin{aligned} b &:= G_{\mathbb{A}}^T \dot{q}^- (1 + c_r) \\ \implies b_i &= (1 + c_r) \nabla g_i(q) \dot{q}^- \\ &= (1 + c_r) \frac{dg_i}{dt} \\ &= (1 + c_r) \frac{d(\text{dist}(a, b))}{dt} \end{aligned}$$

Interpretation: the b_i represents

$-(\mathbf{1} + \mathbf{c}_r) \times$ (relative speed ball a is approaching ball b).

(note the negative sign since $\frac{d(\text{dist}(a, b))}{dt} \leq 0$)

$Q\lambda$

From the Rosi paper, we know that $\lambda \in \mathbb{R}^{|\mathbb{A}|}$ is the vector of impulse coefficients (i.e. λ_i is the force that ball a exerts on ball b ***TODO: ensure this is *technically/semantically* correct!) Now consider the i th element of $Q\lambda$:

$$(Q\lambda)_i = \sum_{j=1}^{\mathbb{A}} Q_{ij} \lambda_j$$

For Q_{ij} there are 3 cases (see above). Firstly, collision constraint i could not be affected by either of the balls involved in collision j , in which case $Q_{ij} = 0$ and we can ignore those terms. Secondly, there will be a term where $i = j$, in this case $Q_{ij} = Q_{ii} = 2$. And finally, assume that constraint i is "concerned" with balls a and b, (i.e. $g_i(q) = \text{dist}(a, b)$) and j is "concerned" with *either* ball a or b *and* some 3rd ball c. In this case $Q_{ij} = \cos(\angle abc)$ or $Q_{ij} = \cos(\angle cab)$.

So, if we set:

$$A = \{x \in \mathbb{A} | s.t. g_x(q) = \text{dist}(a, c)\} \setminus \{i\}$$

$$\text{And } B = \{x \in \mathbb{A} | s.t. g_x(q) = \text{dist}(b, c)\} \setminus \{i\}$$

Where in both cases c is just a ball that is in the process of colliding with ball a or b respectively, then we can rewrite above as:

$$\begin{aligned} (Q\lambda)_i &= \sum_{j=1}^{\mathbb{A}} Q_{ij} \lambda_j \\ &= 2\lambda_i + \sum_{x \in A} \cos(\angle ba \text{ ball}(x)) \lambda_x + \sum_{x \in B} \cos(\angle ab \text{ ball}(x)) \lambda_x \\ &= 2\lambda_i + \left\| \sum_{x \in A} \text{proj}_{\overrightarrow{n_{ba}}}(\overrightarrow{n_{\text{ball}(x)a}}) \lambda_x \right\| + \left\| \sum_{x \in B} \text{proj}_{\overrightarrow{n_{ab}}}(\overrightarrow{n_{a \text{ ball}(x)}}) \lambda_x \right\| \\ &= \text{net force acting on balls a and b along direction: } \overrightarrow{n_{ab}} \end{aligned}$$

TODO: clean up the equation above... using "ball($()x$)" is confusing... maybe some of the ab should be ba... theres a little more explaining that could be done. ESPECIALLY relating to our disregard of mass - really, the above should be the net $\Delta \dot{q}$ once we bring back M^{-1} and divide each term by it's balls' masses.

LCP Criteria: $Q\lambda + b \geq 0$

This is saying for each collision $i \in \mathbb{A}$, we need $(Q\lambda)_i \geq -b_i$. Let $[\dot{q}]_{\overrightarrow{n_{ab}}}$ represent the speed of ball a wrt ball b.

As we have seen before, $b_i = -(1+c_r)[\dot{q}]_{\overrightarrow{n_{ab}}}$ so we can rewrite our lcp condition as:

$$\text{net force (speed?) acting on balls a and b} \geq (1 + c_r)[\dot{q}]_{\overrightarrow{n_{ab}}}$$

Or something like that... basically, the LCP condition is enforcing our solution (λ , the impulse coefficients) will result in exiting velocities that conserve momentum.

OHHH SHIT OKOKOKOKOK

when we factor in the complementary condition, the times 2 above makes sense!

either $(Q\lambda + b)_i = 0$, in this case $\lambda_i > 0$ and we get

$$\begin{aligned}
(Q\lambda)_i &= \sum_{j=1}^A Q_{ij} \lambda_j \\
&= 2\lambda_i + \sum_{x \in A} \cos(\angle ba \text{ ball}(x)) \lambda_x + \sum_{x \in B} \cos(\angle ab \text{ ball}(x)) \lambda_x \\
&= 2\lambda_i + \left\| \sum_{x \in A} \text{proj}_{\overrightarrow{n_{ba}}}(\overrightarrow{n_{\text{ball}(x)a}}) \lambda_x \right\| + \left\| \sum_{x \in B} \text{proj}_{\overrightarrow{n_{ab}}}(\overrightarrow{n_{a \text{ ball}(x)}}) \lambda_x \right\| \\
&= \left\| \sum_{x \in A} \text{proj}_{\overrightarrow{n_{ba}}}(\overrightarrow{n_{\text{ball}(x)a}}) \lambda_x \right\| + \left\| \sum_{x \in B} \text{proj}_{\overrightarrow{n_{ab}}}(\overrightarrow{n_{a \text{ ball}(x)}}) \lambda_x \right\|
\end{aligned}$$

basically, our LCP solution in this case requires that the relative exit velocity of balls a and b is ONLY a result of the forces of the OTHER balls colliding with balls a/b.

Or... wait no nvm lol

TODO: more investigation here perhaps?

result: Q is not generally an "M-matrix"

This was something we were concerned with in previous semesters of work. The off-diagonal entries of Q are $\cos \theta$ where $\theta \in [0, \pi]$! It's easy to imagine a scenario where 2 balls (a and b) simultaneously collide with a common 3rd ball (c) so that the angle connecting the center of the 3 balls $\angle acb < \pi/2$ and thus $\cos \theta > 0$, ($\implies Q$ cannot be an M-matrix)

TODO: illustration?

3 overlapping and collisions through balls

4 ”K3” Example

Generally speaking, when only 2 balls collide with each other, there isn't much special going on... $G_A \in M_{3m \times 1}(\mathbb{R})$ so $Q \in M_{1 \times 1}(\mathbb{R})$ and λ is really easy to find.

A more interesting scenario can be found when 3 balls collide with each other while all on the same axis.⁴ In this case, all angles are either 0 or π , so Q is:⁵

$$Q = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Which is singular! There are an infinite number of solutions to the LCP - and IPOPT does in fact find a different solution to policy iteration!

5 ”K4” Example

4 balls (all overlapping each other) configured in a square with velocities towards the center of the square makes a ”K4” type of graph where each of the 4 balls is colliding with the other 3 in the same instant. This also produces a singular Q matrix (rank = 5, nullity = 1), and IPOPT/PI yield different solutions.

in both K3 and K4 examples, there are 3 viable control sets.

⁴This does required that ball collides with ball *c* *through* ball *b*... but sadly, because of discrete time issues, this is a case that must be considered

⁵Order of values also depend on the order of collisions in the active set.