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# Entanglement mean field theory: Lipkin–Meshkov–Glick Model

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**Abstract** Entanglement mean field theory is an approximate method for dealing with many-body systems, especially for the prediction of the onset of phase transitions. While previous studies have concentrated mainly on applications of the theory on short-range interaction models, we show here that it can be efficiently applied also to systems with long-range interaction Hamiltonians. We consider the (quantum) Lipkin–Meshkov–Glick spin model, and derive the entanglement mean field theory reduced Hamiltonian. A similar recipe can be applied to obtain entanglement mean field theory reduced Hamiltonians corresponding to other long-range interaction systems. We show, in particular, that the zero temperature quantum phase transition present in the Lipkin–Meshkov–Glick model can be accurately predicted by the theory.

**Keywords** Quantum information · Quantum many-body physics · Mean field theory · Entanglement mean field theory · Quantum spin models · Lipkin–Meshkov–Glick model

## 1 Introduction

During the last few years, extensive studies have been carried out at the interface of quantum information science with many-body physics [1,2]. In particular, entanglement properties of ground states and thermal equilibrium states have been investigated. It was shown that bipartite as well as multipartite entanglement of states of a large variety of interacting quantum many-body systems can indicate the onset of phase transitions of the many-body system. This includes singularities exhibited by different measures of entanglement calculated for the ground states of interacting quantum

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many-body systems, pointing to (zero-temperature) quantum phase transitions in the system [3]. While higher dimensional systems have been considered, the bulk of the research has been confined to one-dimensional systems. Among them, one-dimensional quantum spin models have been investigated for a large number of cases, and by using several measures of entanglement. In particular, it has been shown that the (zero-temperature) quantum phase transition in the nearest-neighbor (quantum) transverse Ising model [4] can be indicated by a singular behavior of (a derivative of) nearest-neighbor entanglement of the ground state of this model [5,6].

Exactly solvable many-body systems are few and far between [7]. It is therefore crucial to have approximate methods [3,7–21] as well as numerical techniques [22,23] to delve into their properties, and these approaches therefore play an important role in the study of many-body Hamiltonians, especially for higher dimensional models or for models with medium- or long-range interactions. A useful and simple approximate method that is used to deal with many-body Hamiltonians was presented a century ago by P. Weiss, and is known by "molecular field theory", "Weiss field theory", or more ocassionally, "mean field theory" [8–12]. The mean field theory (MFT) and its modifications [13–21] have been successfully used to obtain a variety of results in many-body physics. Although the prescription of the MFT is a simple one, it correctly reproduces the phase diagrams and indicates the critical exponents for a wide range of interacting many-body models [3,4,8–12].

The hallmark of a mean field-like theory is a self-consistency equation that is written in terms of a single-site physical quantity of the system under consideration. MFT starts off by reducing the many-body parent Hamiltonian into a single-body one, while still keeping some footprints of the parent. The modified versions of MFT usually reduces the parent Hamiltonian into one with a few sites. However, in all such cases, the self-consistency equation is written in terms of a single-site physical quantity.

Entanglement mean field theory (EMFT) reduces the many-body Hamiltonian into one with two sites, and then uses *two-site* physical quantities in its self-consistency equations [24]. While the self-consistency equations in MF-like theories consists of single-site properties like magnetization, the same in the EMFT is written in terms of two-body properties like two-body correlations. EMFT can be efficiently applied to study several paradigmatic quantum [24] as well as classical models [25]. However, the cases considered are mainly for systems with nearest-neighbor interactions. In this paper, we extend the EMFT to models with long-range interactions. We apply the technique to the Lipkin–Meshkov–Glick model [26–28] and show that EMFT can accurately predict the quantum phase transition at zero temperature present in this system [26–30].

## 2 Lipkin–Meshkov–Glick model

The Lipkin–Meshkov–Glick (LMG) model was first introduced in nuclear physics [26–28]. It has since been studied extensively in many-body physics from different perspectives. See e.g. [29–33], and references therein. Entanglement properties of the LMG model have also been studied recently [34].

The LMG model is a quantum spin model with long-range interactions (with a transverse field). The Hamiltonian that describes the LMG model for a system of $N$ spin-1/2 systems is given by

$$H_{LMG} = -\frac{J}{N}\left[S_x^2 + \gamma S_y^2\right] - hS_z, \tag{1}$$

where

$$S_\alpha = \sum_{i=1}^{N} \frac{\sigma_\alpha^i}{2}, \quad \alpha = x, y, z. \tag{2}$$

Here $\sigma_\alpha^i$ $(\alpha = x, y, z)$ are the Pauli spin-1/2 matrices of the $i$th spin. $J$ and $h$ are respectively the coupling strength and the transverse magnetic field. Both $J$ and $h$ are assumed to be positive. The Hamiltonian at zero temperature undergoes a quantum phase transition, when

$$h/J = 1, \tag{3}$$

for $0 \leq \gamma < 1$. On the positive $h/J$ axis, $h/J > 1$ corresponds to a symmetric phase while $h/J < 1$ is a broken phase [26–30].

## 3 Entanglement mean-field theory and the LMG model

Entanglement mean field theory was introduced in Ref [24] and applied to study properties of ground states as well as thermal states of several paradigmatic spin models with nearest neighbor interactions (cf. [35]). We now show that EMFT can be applied also for models with long-range interactions. As a specific and physically interesting example, we choose the Lipkin–Meshkov–Glick model for the demonstration. However, the procedure can similarly be applied to other long-range Hamiltonians.

We will now illustrate the EMFT for the LMG model. We begin by deriving the EMFT-reduced Hamiltonian for the LMG model.

### 3.1 EMFT-reduced Hamiltonian for the LMG model

Let us first consider the interaction term,

$$\frac{1}{N}S_x^2 \tag{4}$$

(ignoring the $-J$ for the moment), of the Hamiltonian $H_{LMG}$. In MF theory, as applied to a quantum spin model, one spin is assumed to be special, and to obtain the MF-reduced Hamiltonian, one replaces the remaining spin operators by their mean values in a chosen state. The chosen state can for example be the ground state or the thermal state. EMFT begins by assuming that there are *two* spins, say $k$ and $l$, which are special (in a sense to be clarified below). Let us rewrite the first interaction term in terms of the local Pauli spin operators:

$$\frac{1}{N} S_x^2 = \frac{1}{4N} \left[ \sum_{i=1}^{N} \left( \sigma_x^i \right)^2 + \sum_{i \neq j} \sigma_x^i \sigma_x^j \right]. \tag{5}$$

Since $\sigma_x^2$ is unity, the right hand side reduces to

$$\frac{1}{4} + \frac{1}{2N} \sum_{i>j} \sigma_x^i \sigma_x^j. \tag{6}$$

The additive constant, $\frac{1}{4}$, will have the effect of only shifting the energy levels, and will therefore be removed from further considerations.

We have assumed that the spins $k$ and $l$ are special. Consider now all the $N-2$ terms in

$$\sum_{i>j} \sigma_x^i \sigma_x^j, \tag{7}$$

where $\sigma_x^k$ is present, but $\sigma_x^l$ is not. So in particular, we are temporarily disregarding the term

$$\sigma_x^k \sigma_x^l. \tag{8}$$

We now multiply all these $N-2$ terms by

$$\left( \sigma_x^l \right)^2. \tag{9}$$

We can do that without changing the Hamiltonian by using the fact that $(\sigma_x^l)^2 = 1$. For example, consider the term

$$\sigma_x^k \sigma_x^m, \tag{10}$$

such that $m$ is neither $k$ nor $l$. It now reads

$$\sigma_x^k \sigma_x^m \sigma_x^l \sigma_x^l. \tag{11}$$

Let us rewrite the term as

$$\left( \sigma_x^k \sigma_x^l \right) \left( \sigma_x^m \sigma_x^l \right). \tag{12}$$

It now has a correlation term

$$\sigma_x^k \sigma_x^l \tag{13}$$

corresponding to the special pair $(k, l)$. And moreover, it has a correlation term

$$\sigma_x^m \sigma_x^l \tag{14}$$

corresponding to a non-special pair $(m, l)$. At this point, we begin our EMFT approximation, by ignoring the fluctuations (around the corresponding mean values) of all correlation terms that correspond to non-special pairs. In particular, the fluctuations of the correlation term

$$\sigma_x^m \sigma_x^l \tag{15}$$

in

$$\left(\sigma_x^k \sigma_x^l\right) \left(\sigma_x^m \sigma_x^l\right) \tag{16}$$

is replaced by its mean value $C_{xx}$ (in some state of the system) times a constant. Since $C_{xx}$ is appearing as a correlation between the *two* spins $m$ and $l$, only *one* of which is "special", we choose the constant to be $\frac{1}{2}$. By symmetry, all such correlation terms have the same mean value. Therefore, to obtain the EMFT-reduced Hamiltonian from the parent Hamiltonian $H_{LMG}$, all the terms in the parent Hamiltonian which contain $\sigma_x^k$ but not $\sigma_x^l$ will be approximated as follows:

$$\begin{aligned} &\sigma_x^k \sigma_x^m \\ &= \sigma_x^k \sigma_x^m \left(\sigma_x^l\right)^2 \\ &= \left(\sigma_x^k \sigma_x^l\right) \left(\sigma_x^m \sigma_x^l\right) \\ &\approx \frac{1}{2} C_{xx} \sigma_x^k \sigma_x^l, \end{aligned} \tag{17}$$

where the EMFT approximation is only in the last line.

We now do the same for all the terms that contain $\sigma_x^l$ but not $\sigma_x^k$. The only term that is as yet left out from the EMFT approximation, in this category (the ones obtained from the term $S_x^2$ in $H_{LMG}$), is $\sigma_x^k \sigma_x^l$. In the spirit of the procedure followed for the other terms, we should multiply this term by $\sigma_x^n$, where $n$ is neither $k$ nor $l$. The resulting term can be written as

$$\left(\sigma_x^k \sigma_x^n\right) \left(\sigma_x^n \sigma_x^l\right), \tag{18}$$

which consists of two non-special pairs, $(k, n)$ and $(n, l)$. This therefore will lead to the term $\frac{1}{4} C_{xx}^2$ in the EMFT approximation. Such a term will not appear in any further development of the theory, and will therefore be removed from the EMFT-reduced Hamiltonian.

A similar treatment is applied to the interaction terms that are obtained from the term $S_y^2$ in $H_{LMG}$. The interaction part of $H_{LMG}$ thus reduces to

$$-\frac{J}{2N}(2N-4)\left[\frac{1}{2}C_{xx}\sigma_x^k\sigma_x^l+\gamma\frac{1}{2}C_{yy}\sigma_y^k\sigma_y^l\right] \tag{19}$$

in the EMFT approximation.

Of the $N$ field terms in $H_{LMG}$, only two will contribute in a nontrivial way in the further developments. They are terms proportional to $\sigma_z^k$ and $\sigma_z^l$. The other field terms are therefore ignored. The nontrivial field terms "touches" only one of the two special spins, and so we multiply them by a factor $\frac{1}{2}$ in the EMFT approximation.

The EMFT-reduced Hamiltonian therefore reads

$$\begin{aligned}H_{LMG}^{EMFT} &= -\frac{J}{2N}(2N-4)\left[\frac{1}{2}C_{xx}\sigma_x^k\sigma_x^l+\gamma\frac{1}{2}C_{yy}\sigma_y^k\sigma_y^l\right]\\&\quad-\frac{h}{4}[\sigma_z^k+\sigma_z^l].\end{aligned} \tag{20}$$

Moreover, $\frac{2N-4}{2N}\approx 1$ for large $N$, and therefore the final form of the EMFT-reduced LMG Hamiltonian is given by

$$H_{LMG}^{EMFT}=-\frac{J}{2}\left[C_{xx}\sigma_x^k\sigma_x^l+\gamma C_{yy}\sigma_y^k\sigma_y^l\right]-\frac{h}{4}[\sigma_z^k+\sigma_z^l]. \tag{21}$$

### 3.2 EMFT equations for the LMG model

For self-consistency, the mean values introduced in deriving the EMFT Hamiltonian must be equal to the same mean values derived within the EMFT limit, i.e. by using the EMFT-reduced Hamiltonian. At this point, we have to specify the state of the system which was used in defining the mean values that have been employed in deriving in the EMFT-reduced Hamiltonian. Let us suppose that the state is $\rho$. In that case, the self-consistency equations (the EMFT equations) read

$$C_{xx}=\mathrm{tr}(\sigma_x^k\sigma_x^l\rho), \tag{22}$$
$$C_{yy}=\mathrm{tr}(\sigma_y^k\sigma_y^l\rho). \tag{23}$$

In particular, $\rho$ can be the thermal equalibrium state of the system, at a temperature $T$, so that

$$\rho=\frac{1}{Z_{EMFT}}\exp\left(-\beta H_{LMG}^{EMFT}\right), \tag{24}$$

with the EMFT partition function

$$Z_{EMFT}=\mathrm{tr}\left[\exp\left(-\beta H_{LMG}^{EMFT}\right)\right], \tag{25}$$

where $\beta = \frac{1}{k_B T}$, with $k_B$ being the Boltzmann constant, which could be used to investigate the properties of the system with changes in temperature, including temperature-driven transitions. Below however, we will investigate transitions driven by quantum fluctuations, at zero temperature.

It is worthwhile to stress here that the main difference of EMFT with the mean-field like theories is that whereas the self-consistency equations in MF-like theories are in terms of single site physical quantities like magnetization, in EMFT, they are in terms of two-site physical quantities.

## 4 EMFT predictions for the LMG Model

For $\gamma = 0$, there will be the single self-consistency equation

$$C_{xx} = \mathrm{tr}\left(\sigma_x^k \sigma_x^l \rho\right), \tag{26}$$

where $\rho$ can e.g. be the ground state or the thermal equalibrium state of the system in the EMFT limit.

For the EMFT ground state, the self-consistency equation reduces to

$$\begin{aligned} C_{xx} &= \sqrt{1 - \left(\frac{h}{J}\right)^2}, \quad \text{for} \quad \left|\frac{h}{J}\right| \leq 1, \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{27}$$

Clearly, the solution $C_{xx} = 0$ is also available for $\left|\frac{h}{J}\right| \leq 1$.

The (zero-temperature) quantum phase transition in the LMG model at $h/J = 1$ [26–30] is therefore clearly visible in the EMFT ground state.

We also compare the behavior of $C_{xx}$ of LMG model, obtained from numerical simulations of systems with relatively small system-size, with that predicted by EMFT (see Fig. 1).

## 5 Conclusions

Approximate and numerical methods play a crucial role in our quest for understanding the properties of many-body physics. The mean-field theory and its several extensions have been employed as useful and simple tools for providing qualitatively, and sometimes even quantitatively, correct predictions of several physical phenomena including quantum phase transtions, critical exponents, etc. Entanglement mean field theory is an approximate method inspired by the new developments in quantum information science, and is potentially useful in predicting cooperative phenomena in many-body systems, and in particular in predictions of quantum phase transitions. It opens up a new avenue to obtain entanglement as well as correlations and other two-body physical quantities for many-body systems, which are not amenable to exact diagonalizations for large system sizes. A crucial difference between entanglement mean field theory

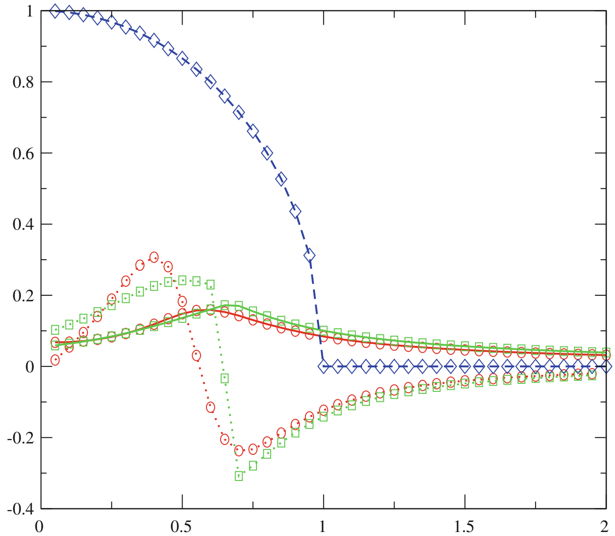


**Fig. 1** Correlation function in the ground state from entanglement mean field theory and from numerical simulations of relatively small systems. We plot the $xx$-correlation, $C_{xx}$, obtained by exact diagonalization for $N = 8$ (*squares*) and $N = 10$ (*circles*) spins governed by LMG Hamiltonian. The *bold connecting lines* are for $C_{xx}$ on the vertical axis, plotted against $h/J$ on the horizontal axis, while the *dotted connecting lines* are for the derivative of $C_{xx}$ with respect to $h/J$. The *diamonds*, connected with *dashed lines*, are for the $C_{xx}$ as predicted by EMFT. All quantities plotted on both the axes are dimensionless

and the MF class of theories is in their self-consistency equations. While the self-consistency equations in entanglement mean field theory is typically constructed in terms of correlation functions, the same in mean field-like theories is typically framed in terms of magnetization. We have presented the recipe for applying the entanglement mean field theory for long-range interaction Hamiltonians, and the method has been exemplified with an application to the Lipkin–Meshkov–Glick model. We have then demonstrated that the model, in the entanglement mean field limit, clearly signals the onset of the quantum phase transition of this model.

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