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Spin Squeezing and Quantum Fisher Information in the Lipkin-Meshkov-Glick Model

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Abstract We investigate spin squeezing and quantum Fisher information in the Lipkin-Meshkov-Glick model. Approximate analytical expressions and the numerical analysis of spin squeezing and quantum Fisher information are derived. Spin squeezing and quantum Fisher information depend on the strength of the external transverse magnetic field and the anisotropic parameter

Keywords Spin squeezing · Quantum Fisher information · Lipkin-Meshkov-Glick model

1 Introduction

Quantum entanglement [1], one of the most interesting features in quantum mechanics, plays a significant role in quantum mechanics, as it not only holds the power for demonstration of the quantum nonlocality against local hidden variable theory [1], but also provides promising and wide applications in quantum information processing, such as teleportation [2], dense coding [3, 4], geometric quantum computation [5] and so on. Therefore, characterizing and manipulating entangled states of many-particle systems are far-reaching possibility of quantum information processing. However, how to characterize many-body entanglement is still a challenge because of its complexity.

Recently, it was found that both the quantum Fisher information and the spin squeezing are connected to quantum entanglement [6–12]. The spin squeezing refers to the minimum spin fluctuation of the plane perpendicular to the mean spin direction; while for pure states, the maximal quantum Fisher information refers to the maximal spin fluctuation over all directions; which is perpendicular to the mean spin direction and the spin squeezing direction [13]. Because quantum entanglement is still puzzling, especially many-body entanglement, one can use spin squeezing and quantum Fisher information to characterize entanglement.

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The Lipkin-Meshkov-Glick (LMG) model, originally introduced in nuclear physics, has been used in the description of many physical system. Up to now the LMG model has widely studied in quantum information processing [14–16]. Some entanglement properties have already been investigated in this model. Vidal *et al.* studied the entanglement properties in this model with concurrence [15]. Latorre *et al.* analyzed the entanglement entropy in the Lipkin-Meshkov-Glick model [17]. Kwok *et al.* studied the critical properties of the Lipkin-Meshkov-Glick Model in terms of the fidelity susceptibility [18].

In this paper, we investigate the entanglement dynamics in LMG model with spin squeezing and quantum Fisher information. For our model, we were not able to find a general analytic form of the time evolution operators, even at specific values of t. Since the coupling is strong, we can get some approximate solutions by adopting frozen-spin approximation [19–21]. Approximate analytical expressions and the numerical analysis of spin squeezing and quantum Fisher information are derived. In particular, we are interested in the dependence of spin squeezing parameter and quantum Fisher information on the strength of the external transverse magnetic field and the anisotropic parameter. Spin squeezing and quantum Fisher information influenced by the strength of the external transverse magnetic field and the anisotropic parameter. The bigger the values of the external transverse magnetic field and the anisotropic parameter are, the longer of the period of oscillation of spin squeezing and quantum Fisher information is. Therefore, one can control spin squeezing and quantum Fisher information in LMG model by adjusting the strength of the external transverse magnetic field and the anisotropic parameter.

2 Model and solution

The LMG model describes a set of N spin-1/2 particles mutually interacting through an (anisotropic) XY-like Hamiltonian and coupled to an external transverse magnetic field h [16]:

$$H = -\frac{\lambda}{N}\left(J_x^2 + \gamma J_y^2\right) - hJ_z, \tag{1}$$

where $J_\alpha = \Sigma_{i=1}^{N}\sigma_\alpha^i/2$ ($\alpha = x, y, z$) are the collective spin operators with σ_α being the Pauli matrices. In Eq. (1), γ is the anisotropic parameter, λ is the spin-spin interaction strength [14]. We focus here on the ferromagnetic case ($\lambda > 0$, without loss of generality we set $\lambda = 1$) and we mainly consider the case $0 \le \gamma \le 1$. The prefactor $1/N$ is necessary to get a finite free energy per spin in the thermodynamical limit [15].

Starting from the Hamiltonian (1), we now demonstrate how an appropriate choice of the external coupling field allows one to generate and control spin squeezing and quantum Fisher information. Suppose that the initial state of the system $|\frac{N}{2}, -\frac{N}{2}\rangle$, coherent spin state, is the lowest eigenstate of J_z,

$$J_z\left|\frac{N}{2}, -\frac{N}{2}\right\rangle = -\frac{N}{2}\left|\frac{N}{2}, -\frac{N}{2}\right\rangle. \tag{2}$$

Physically, the Dicke state $|\frac{N}{2}, -\frac{N}{2}\rangle$ represents all atoms occupying in the internal ground state. By applying a short $\pi/2$ pulse to the Dicke state, one can obtain the coherent spin state with each spin aligned along the negative z direction [22]. After that, one switches on the external field immediately, then the dynamics of the multi-qubit system is governed by the Hamiltonian (6). Then in order to investigate the spin dynamics, we first examine the Heisenberg equations of motion of the angular momentum operators in the x and y directions respectively,

$$\dot{J}_x = -\frac{\gamma}{N}(J_y J_z + J_z J_y) + h J_y, \tag{3}$$

$$\dot{J}_y = \frac{1}{N}(J_x J_z + J_z J_x) - h J_x. \tag{4}$$

In fact, for this case, there exist no exact analytic solutions; however, some approximate solutions are obtainable if the coupling is strong enough. Under this circumstance, we adopt frozen-spin approximation, i.e. replacing J_z by $-N/2$ in the Heisenberg equations. As a result, the mean spin $\langle J_z \rangle$ almost remains unchanged at $-N/2$. Using frozen-spin approximation, we can easily obtain the following formula

$$\ddot{J}_y = -(h+1)(h+\gamma)J_y = -\omega^2 J_y \tag{5}$$

with the frequency $\omega = \sqrt{(h+1)(h+\gamma)}$. Solving the above equations, we can easily obtain harmonic solutions of $J_x(t)$ and $J_y(t)$:

$$J_x(t) = -J_x(0)\cos(\omega t) + \sqrt{\frac{h+\gamma}{h+1}} J_y(0)\sin(\omega t) \tag{6}$$

and

$$J_y(t) = J_y(0)\cos(\omega t) - \sqrt{\frac{h+1}{h+\gamma}} J_x(0)\sin(\omega t). \tag{7}$$

Having known $J_x(t)$ and $J_y(t)$, then we can calculate spin squeezing and quantum Fisher information in the LMG model.

3 Spin Squeezing and Quantum Fisher Information

An important application of the spin squeezing parameter is to detect many-body entanglement. However, the definition of spin squeezing is not unique, in this paper, we employ the criteria of spin squeezing proposed by Ueda *et al.* and the spin squeezing parameter is defined as [6]

$$\xi^2 = \frac{4(\Delta J_\varphi)^2_{min}}{N}, \tag{8}$$

where $(\Delta J_\varphi)^2_{min}$ with is the minimal spin fluctuation in a plane perpendicular to the mean spin direction. The noncorrelated limit yields $\xi^2 = 1$; while the squeezing parameter $\xi^2 < 1$ is the mark of an entangled state [23].

Recently, Pezzé and Smerzi introduced a quantity

$$\chi^2 = \frac{N}{F(\rho_{in}, \hat{J}_{\vec{n}})} \tag{9}$$

and prove that $\chi^2 < 1$ implies multipartite entanglement. Here $F(\rho_{in}, \hat{J}_{\vec{n}}) = 4(\Delta \hat{R})^2$ is the quantum Fisher information [9, 12] and $\vec{n}$ is an arbitrary direction. The Hermitian operator $\hat{R}$ is the solution of the equation $\{\hat{R}, \rho_{in}\} = i[\hat{J}_{\vec{n}}, \rho_{in}]$ [24]. For pure states, this equation is solved by $\hat{R} = i[\hat{J}_{\vec{n}}, \rho_{in}]$ and we have $(\Delta \hat{R})^2 = (\Delta \hat{J}_{\vec{n}_\perp})^2$ [8], where $(\Delta \hat{J}_{\vec{n}_\perp})^2$ is the maximal variance of a spin component $\hat{J}_{\vec{n}_\perp} = \mathbf{J} \cdot \vec{n}_\perp$ in the plane perpendicular to mean spin direction, then Eq. (1) can be reduced to

$$\chi^2 = \frac{N}{4(\Delta \hat{J}_{\vec{n}_\perp})^2}. \tag{10}$$

Under our consideration, $\langle J_z(t)\rangle = N/2$, $\langle J_x(t)\rangle = \langle J_y(t)\rangle = 0$, i.e. the mean spin direction is along z axis, the minimal spin fluctuation and the maximal quantum Fisher information occur at (x, y) plane. Furthermore, if $\langle J_x(t)J_y(t)\rangle = \langle J_y(t)J_x(t)\rangle = 0$, i.e. there is no correlation between J_x and J_y, we have

$$\xi^2 = \frac{4\min(\langle J_x^2(t)\rangle, \langle J_y^2(t)\rangle)}{N}, \tag{11}$$

and

$$\chi^2 = \frac{N}{4\max(\langle J_x^2(t)\rangle, \langle J_y^2(t)\rangle)}. \tag{12}$$

However, straightforward calculation then gives

$$\big\langle J_x(t)J_y(t)\big\rangle = \big\langle J_y(t)J_x(t)\big\rangle = \frac{N}{8}\sin(2\omega t)\left(\sqrt{\frac{h+\gamma}{h+1}} + \sqrt{\frac{h+1}{h+\gamma}}\right). \tag{13}$$

Therefore, the minimal spin fluctuation and the maximal quantum Fisher information occur neither x nor y direction except $\omega t = n\pi$ with n being integer. In order to further determine the minimal spin fluctuation and the maximal Fisher information directions, we introduce a spin component as

$$J_\varphi = \vec{J}\cdot\vec{n}_\varphi = J_x\cos\varphi + J_y\sin\varphi, \tag{14}$$

where the unit vector $\vec{n}_\varphi = x\cos\varphi + y\sin\varphi$, with φ being an arbitrary angle between x axis and $\vec{n}_\varphi$. Since $\langle J_x(t)\rangle = \langle J_y(t)\rangle = 0$, the fluctuation of spin component J_φ reads [11]

$$(\Delta J_\varphi)^2 = \frac{1}{2}\big[\big\langle J_x^2 + J_y^2\big\rangle + \big\langle J_x^2 - J_y^2\big\rangle\cos(2\varphi)\big] + \frac{1}{2}\langle J_xJ_y + J_yJ_x\rangle\sin(2\varphi). \tag{15}$$

Optimally squeezed angle φ_{min} is obtained via minimizing $(\Delta J_\varphi)^2$ with respect to φ, yielding

$$\varphi_{min} = \frac{1}{2}\left[\pi + \tan^{-1}\left(\frac{\langle J_yJ_z + J_zJ_y\rangle}{\langle J_y^2 + J_z^2\rangle}\right)\right]; \tag{16}$$

while the maximal quantum Fisher information for the angle

$$\varphi_{max} = \pi + \frac{1}{2}\tan^{-1}\left(\frac{\langle J_yJ_z + J_zJ_y\rangle}{\langle J_y^2 + J_z^2\rangle}\right). \tag{17}$$

Substituting these results into Eqs. (8) and (10), we obtain

$$\begin{aligned}\xi^2 = {} & \frac{1}{2}\left[2\cos^2(\omega t) + \left(\frac{h+1}{h+\gamma} + \frac{h+\gamma}{h+1}\right)\sin^2(\omega t)\right] \\ & - \frac{1}{2}\left[\left(\frac{h+\gamma}{h+1} - \frac{h+1}{h+\gamma}\right)^2\sin^4(\omega t) + \sin^2(2\omega t)\left(\sqrt{\frac{h+\gamma}{h+1}} + \sqrt{\frac{h+1}{h+\gamma}}\right)^2\right]^{1/2},\end{aligned} \tag{18}$$

and

$$\begin{aligned}\chi^2 = 2\Bigg(& 2\cos^2(\omega t) + \left(\frac{h+1}{h+\gamma} + \frac{h+\gamma}{h+1}\right)\sin^2(\omega t) \\ & + \sqrt{\left(\frac{h+\gamma}{h+1} - \frac{h+1}{h+\gamma}\right)^2\sin^4(\omega t) + \sin^2(2\omega t)\left(\sqrt{\frac{h+\gamma}{h+1}} + \sqrt{\frac{h+1}{h+\gamma}}\right)^2}\,\Bigg)^{-1}.\end{aligned} \tag{19}$$

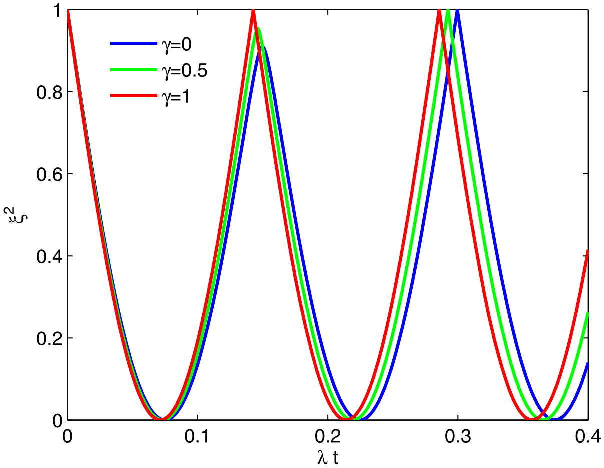

Fig. 1 ξ^2 versus λt for different γ with $h = 10\lambda$ (Color figure online)

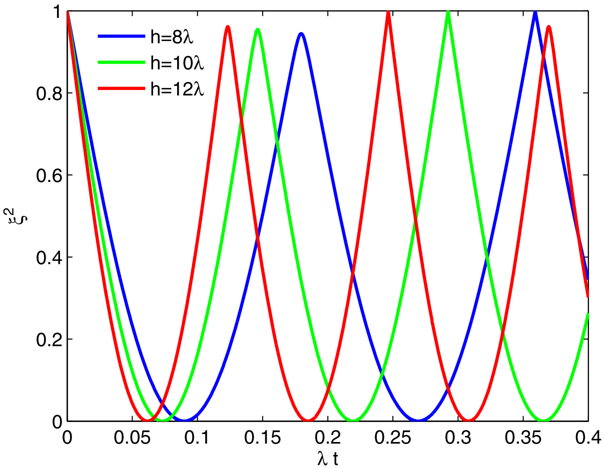

Fig. 2 ξ^2 versus λt for different h with $\gamma = 0.5\lambda$ (Color figure online)

From Eqs. (18) and (19) we see that the two parameters depend on the anisotropic parameter γ, the strength of the external transverse magnetic field h and the spin-spin interaction strength λ. Here we focus on γ and h's influence on ξ^2 and χ^2. In Fig. 1, we plot the spin squeezing parameter ξ^2 versus λt for different anisotropic parameter γ with $h = 10\lambda$. It illustrates quite clearly that initially, there is no spin squeezing as the initial state to be a special coherent spin state, as the dynamics evolves, spin squeezing occurs. Except some special cases, the system almost is squeezed. The squeezing parameter is a periodical function. With the increase of the strength of the external field, the period of oscillation of squeezing parameter decreases. However, the values of γ and h don't change the magnitude of ξ^2. Figure 2 plots ξ^2 versus λt for different strength of the external transverse magnetic field h, which shown that with the increase of the strength of the external transverse magnetic field, the period of oscillation of squeezing parameter increases. In Figs. 3 and 4, we plot χ^2 as functions of λt for various γ and h, respectively. Similar to spin squeezing parameter ξ^2, with the increase of the strength of the external transverse magnetic field, the period of oscillation of χ^2 increases.

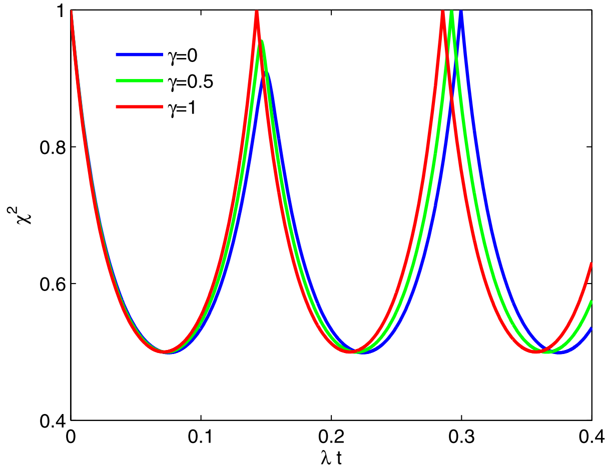

Fig. 3 χ^2 versus λt for different γ with $h = 10\lambda$ (Color figure online)

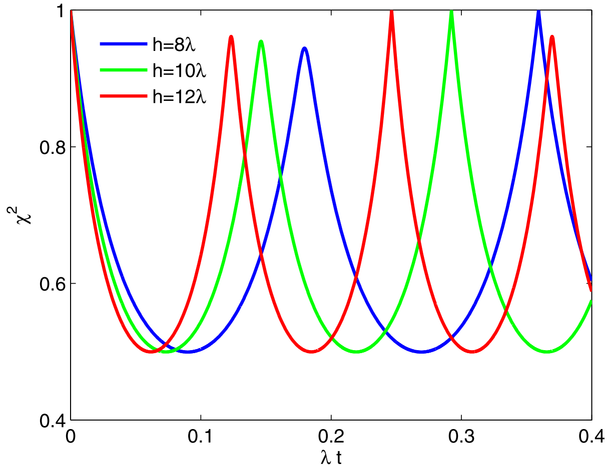

Fig. 4 χ^2 versus λt for different h with $\gamma = 0.5\lambda$ (Color figure online)

4 Summary

In summary, we investigated spin squeezing and quantum Fisher information dynamics in LMG model evolved from a lowest eigenstate of J_x, $|N/2, -N/2\rangle_x$ with frozen-spin approximation. Approximate analytical expressions and the numerical analysis of spin squeezing and quantum Fisher information are derived. In particular, we are interested in the dependence of spin squeezing parameter and quantum Fisher information on the strength of the external transverse magnetic field and the anisotropic parameter. Approximate analytical expressions and the numerical analysis shown that the bigger the values of h and γ are, the longer of the period of oscillation of ξ^2 and χ^2 are. That is to say, spin squeezing and quantum Fisher information in LMG model can be controlled by adjusting the strength of the external transverse magnetic field and the anisotropic parameter.

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