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Homework 9
Electrodynamics

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1. Find the TE and TM modes for a rectangular cavity resonator. Compare the values of the frequencies ω_{mnp}

→ Como se dedujo en clase, a partir de la ecuación de Helmholtz y aplicando condiciones iniciales, las soluciones para las componentes del campo eléctrico están dadas por

$$E_x(x,y,z) = E_1 \cos(k_1 x)\, \text{sen}(k_2 y)\, \text{sen}(k_3 z)\, e^{-i\omega t} \quad \ldots (1)$$
$$E_y(x,y,z) = E_2\, \text{sen}(k_1 x) \cos(k_2 y)\, \text{sen}(k_3 z)\, e^{-i\omega t} \quad \ldots (2)$$
$$E_z(x,y,z) = E_3\, \text{sen}(k_1 x)\, \text{sen}(k_2 y) \cos(k_3 z)\, e^{-i\omega t} \quad \ldots (3)$$

donde $k_1 = \frac{m\pi}{a}$, $k_2 = \frac{n\pi}{b}$, $k_3 = \frac{p\pi}{c}$

a, b, c son las magnitudes de la cavidad, $m, n, p \in \mathbb{Z}$.

Además, consideramos
$$\boxed{\begin{aligned} \vec{k} \cdot \vec{E}_0 &= k_1 E_1 + k_2 E_2 + k_3 E_3 \\ \vec{H}_0 &= \frac{1}{\omega\mu} \vec{k} \times \vec{E}_0 = H_1 \hat{\imath} + H_2 \hat{\jmath} + H_3 \hat{k} \end{aligned}}$$

y se deduce que

$$(4) \ldots H_x(x,y,z) = -iH_1\, \text{sen}(k_1 x) \cos(k_2 y) \cos(k_3 z)\, e^{-i\omega t}$$
$$(5) \ldots H_y(x,y,z) = -iH_2 \cos(k_1 x)\, \text{sen}(k_2 y) \cos(k_3 z)\, e^{-i\omega t}$$
$$(6) \ldots H_z(x,y,z) = -iH_3 \cos(k_1 x) \cos(k_2 y)\, \text{sen}(k_3 z)\, e^{-i\omega t}$$

→ Para los modos TE se tiene que

$$E_z(x,y,z) = 0 \implies E_3 = 0,$$
$$\vec{k} \cdot \vec{E}_0 = 0 \implies k_1 E_1 = -k_2 E_2 \implies \boxed{E_2 = \frac{-k_1}{k_2} E_1}$$

y de la relación $\vec{H}_0 = \frac{1}{\omega\mu}(\hat{k} \times \vec{E}_0)$

$$\omega\mu \vec{H}_0 = \hat{k} \times \vec{E}_0$$

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$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ K_1 & K_2 & K_3 \\ E_1 & E_2 & 0 \end{vmatrix} = \hat{i}\left(-K_3 E_2\right) + \hat{j}\left(K_3 E_1\right) + \hat{k}\left(K_1 E_2 - K_2 E_1\right)$$
$$= \hat{i}\, \omega\mu H_1 + \hat{j}\, \omega\mu H_2 + \hat{k}\, \omega\mu H_3$$

$$\begin{cases} \omega\mu H_1 = -K_3 E_2 \\ \omega\mu H_2 = K_3 E_1 \\ \omega\mu H_3 = K_1 E_2 - K_2 E_1 \end{cases} \xrightarrow{\text{En términos de } E_1} \begin{cases} H_1 = +\dfrac{K_3 K_1}{\omega\mu K_2} E_1 \\ H_2 = \dfrac{K_3}{\omega\mu} E_1 \\ H_3 = -\dfrac{(K_1^2 + K_2^2)}{\omega\mu K_2} E_1 \end{cases}$$

Por lo qe los modos TE son:

$$\boxed{\begin{aligned}
E_x(x,y,z) &= E_1 \cos(K_1 x)\, \mathrm{sen}(K_2 y)\, \mathrm{sen}(K_3 z)\, e^{-i\omega t} \\
E_y(x,y,z) &= -\frac{K_1}{K_2} E_1\, \mathrm{sen}(K_1 x) \cos(K_2 y)\, \mathrm{sen}(K_3 z)\, e^{-i\omega t} \\
E_z(x,y,z) &= 0 \\
H_x(x,y,z) &= \frac{-i K_3 K_1}{\omega\mu K_2} E_1\, \mathrm{Sen}(K_1 x) \cos(K_2 y) \cos(K_3 z)\, e^{-i\omega t} \\
H_y(x,y,z) &= \frac{-i K_3}{\omega\mu} E_1 \cos(K_1 x)\, \mathrm{sen}(K_2 y) \cos(K_3 z)\, e^{-i\omega t} \\
H_z(x,y,z) &= +\frac{i(K_1^2 + K_2^2)}{\omega\mu\, K_2} E_1 \cos(K_1 x) \cos(K_2 y)\, \mathrm{sen}(K_3 z)\, e^{-i\omega t}
\end{aligned}}$$

donde $\omega_{mnp} = v_p \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2}$

Por otro lado, utilizando la ley de Ampère (no hay corriente) se tiene

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

$$\left(\frac{\partial}{\partial y}\cancel{H_z} - \frac{\partial}{\partial z}H_y\right)\hat{i} + \left(\frac{\partial}{\partial z}H_x - \frac{\partial}{\partial x}\cancel{H_z}\right)\hat{j} + \left(\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x\right) = \epsilon \frac{\partial \vec{E}}{\partial t}$$

$\longrightarrow$ Para los modos TM, sabemos que $H_z(x,y,z) = 0$

Componente x:

$$-i\omega\epsilon\, E_x(x,y,z) = -\frac{\partial}{\partial z}H_y \longrightarrow \text{Utilizamos } 5$$

$$i\omega\epsilon\, E_x(x,y,z) = iK_3 H_2 \cos(K_1 x)\,\text{sen}(K_2 y)\,\text{sen}(K_3 z)\, e^{-i\omega t}$$

$$\Longrightarrow E_x(x,y,z) = \frac{K_3}{\omega\epsilon} H_2 \cos(K_1 x)\,\text{sen}(K_2 y)\,\text{sen}(K_3 z)\, e^{-i\omega t}$$

Componente y:

$$-i\omega\epsilon\, E_y(x,y,z) = \frac{\partial}{\partial z}H_x \longrightarrow \text{Utilizamos } 4$$

$$-i\omega\epsilon\, E_y(x,y,z) = -iH_1 K_3\,\text{sen}(K_1 x)\cos(K_2 y)\,\text{sen}(K_3 z)\, e^{-i\omega t}$$

$$\Longrightarrow E_y(x,y,z) = \frac{K_3}{\omega\epsilon} H_1\,\text{sen}(K_1 x)\cos(K_2 y)\,\text{sen}(K_3 z)\, e^{-i\omega t}$$

Componente z:

$$-i\omega\epsilon\, E_z(x,y,z) = \frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x$$

$\frac{\partial}{\partial x}H_y \longrightarrow$ Ecuación 5

$\frac{\partial}{\partial y}H_x \longrightarrow$ Ecuación 4

$$-i\omega\epsilon\, E_z(x,y,z) = +iK_1 H_2\,\text{sen}(K_1 x)\,\text{sen}(K_2 y)\cos(K_3 z)\, e^{-i\omega t} + $$
$$- iH_1 K_2\,\text{sen}(K_1 x)\,\text{sen}(K_2 y)\cos(K_3 z)\, e^{-i\omega t}$$

$$E_z(x,y,z) = \frac{-i(H_1K_2 - H_2K_1)}{-i\omega\epsilon}\,\text{sen}(K_1 x)\,\text{sen}(K_2 y)\cos(K_3 z)\, e^{-i\omega t}$$

$$\Longrightarrow E_z(x,y,z) = \frac{(H_1K_2 - H_2K_1)}{\omega\epsilon}\,\text{sen}(K_1 x)\,\text{sen}(K_2 y)\cos(K_3 z)\, e^{-i\omega t}$$

Además: $K_1 H_1 + K_2 H_2 = 0 \Longrightarrow \boxed{H_2 = \frac{-K_1}{K_2} H_1}$

Por lo cual, los modos TM son:

$$\boxed{\begin{aligned}
H_x(x,y,z) &= -i H_1 \operatorname{sen}(K_1 x)\cos(K_2 y)\cos(K_3 z)\, e^{-i\omega t}\\
H_y(x,y,z) &= +i\frac{K_1}{K_2} H_1 \cos(K_1 x)\operatorname{sen}(K_2 y)\cos(K_3 z)\, e^{-i\omega t}\\
H_z(x,y,z) &= 0\\
E_x(x,y,z) &= \frac{-K_1 K_3}{\omega \epsilon K_2} H_1 \cos(K_1 x)\operatorname{sen}(K_2 y)\operatorname{sen}(K_3 z)\, e^{-i\omega t}\\
E_y(x,y,z) &= \frac{K_3}{\omega \epsilon} H_1 \operatorname{sen}(K_1 x)\cos(K_2 y)\operatorname{sen}(K_3 z)\, e^{-i\omega t}\\
E_z(x,y,z) &= \frac{(K_2^2+K_1^2)}{\omega \epsilon K_2} H_1 \operatorname{sen}(K_1 x)\operatorname{sen}(K_2 y)\cos(K_3 z)\, e^{-i\omega t}
\end{aligned}}$$

donde

$$\omega_{mnp} = v_p \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2}$$

$\longrightarrow$ Las frecuencias son iguales (modos degenerados).

2. Write the particular TM_{110} and TE_{011} modes.

Para los TM_{110} $m=1$, $n=1$, $p=0$
tomando las expresiones de la caja roja.

$$K_1 = \frac{\pi}{a}, \quad K_2 = \frac{\pi}{b}, \quad K_3 = 0$$

$$H_x(x,y,z) = -i H_1 \operatorname{Sen}\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right) e^{-i\omega t}$$

$$H_y(x,y,z) = -i \frac{b}{a} H_1 \cos\left(\frac{\pi x}{a}\right)\operatorname{sen}\left(\frac{\pi y}{b}\right) e^{-i\omega t}$$

$$H_z(x,y,z) = 0 \quad ; \quad E_y(x,y,z) = 0$$

$$E_x(x,y,z) = 0$$

$$E_z(x,y,z) = \frac{\pi b\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}{\omega \epsilon} H_1 \operatorname{Sen}\left(\frac{\pi x}{a}\right)\operatorname{sen}\left(\frac{\pi y}{b}\right) e^{-i\omega t}$$

$$\omega_{110} = \frac{1}{\sqrt{\epsilon\mu}}\sqrt{\frac{\pi^2}{a^2}+\frac{\pi^2}{b^2}} = \frac{\pi}{\sqrt{\epsilon\mu}}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}$$

Para los TE_{011}, $m=0$, $n=1$, $p=1$
tomando las expresiones de la caja naranja

$$K_1 = 0, \quad K_2 = \frac{\pi}{b}, \quad K_3 = \frac{\pi}{c}$$

$$E_x(x,y,z) = E_1 \operatorname{Sen}\left(\frac{\pi y}{b}\right)\operatorname{Sen}\left(\frac{\pi z}{c}\right) e^{-i\omega t}$$

$$E_y(x,y,z) = 0, \quad E_z(x,y,z) = 0$$

$$H_x(x,y,z) = 0$$

$$H_y(x,y,z) = \frac{-i\pi}{c\omega\mu} E_1 \operatorname{Sen}\left(\frac{\pi}{b} y\right)\cos\left(\frac{\pi z}{c}\right) e^{-i\omega t}$$

$$H_z(x,y,z) = \frac{i\pi}{\omega\mu b} E_1 \cos\left(\frac{\pi y}{b}\right)\operatorname{Sen}\left(\frac{\pi z}{c}\right) e^{-i\omega t}$$

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3. Depict the modes TM_{110} and TE_{011}

→ Se grafican las siguientes expresiones TM_{110}

$$H_x(x,y) = -H_1 \operatorname{sen}\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)\operatorname{sen}(\omega t)$$

$$H_y(x,y) = -\frac{b}{a} H_1 \cos\left(\frac{\pi x}{a}\right)\operatorname{sen}\left(\frac{\pi y}{b}\right)\operatorname{sen}(\omega t)$$

$$H_z = 0 \qquad E_y = 0\,, \quad E_x = 0$$

$$E_z(x,y) = \frac{\pi b\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}{\omega \epsilon} H_1 \operatorname{sen}\left(\frac{\pi x}{a}\right)\operatorname{sen}\left(\frac{\pi y}{b}\right)\cos(\omega t)$$

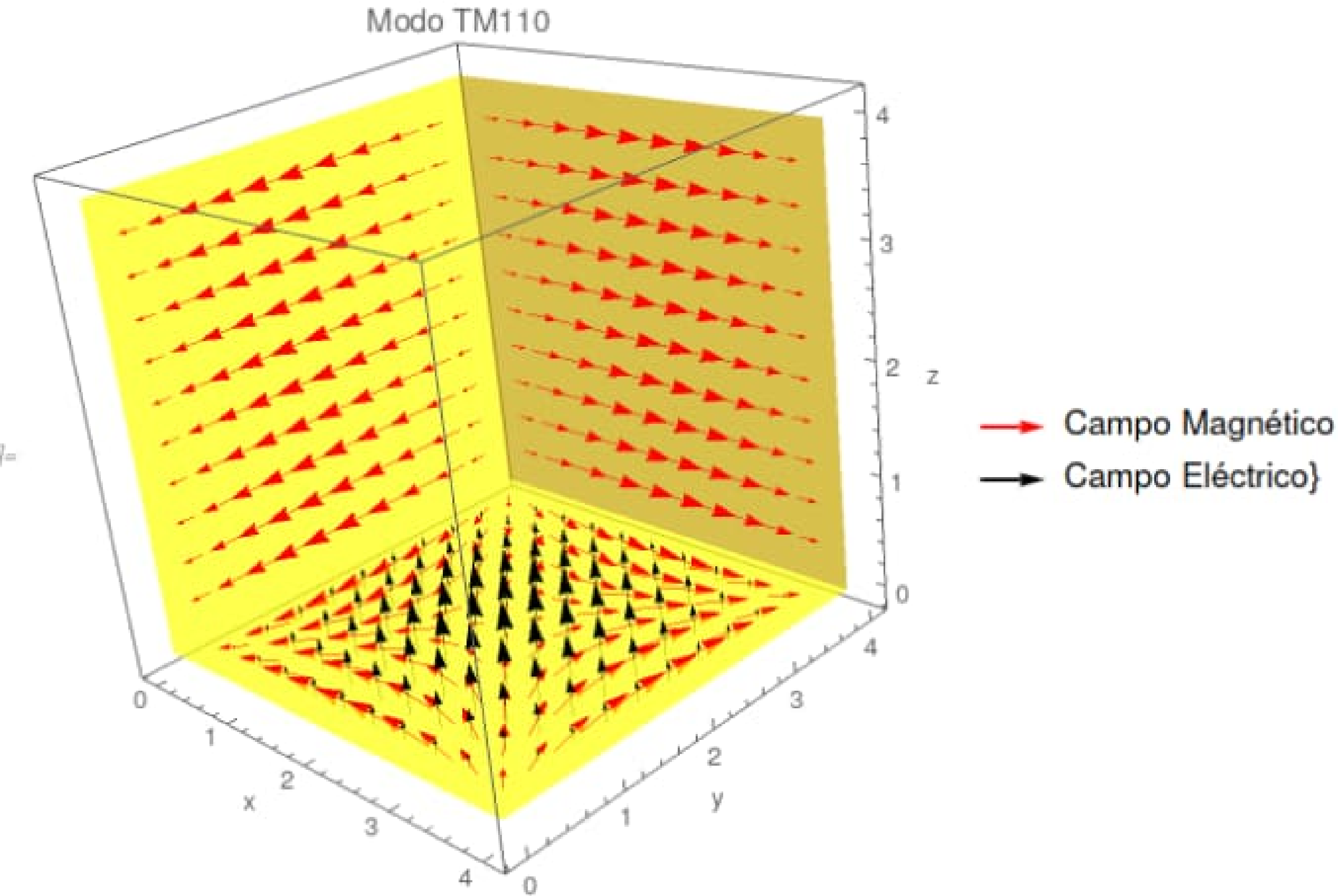
→ Para TE_{011}

$$E_x(y,z) = E_1 \operatorname{sen}\left(\frac{\pi y}{b}\right)\operatorname{sen}\left(\frac{\pi z}{c}\right)\cos(\omega t)$$

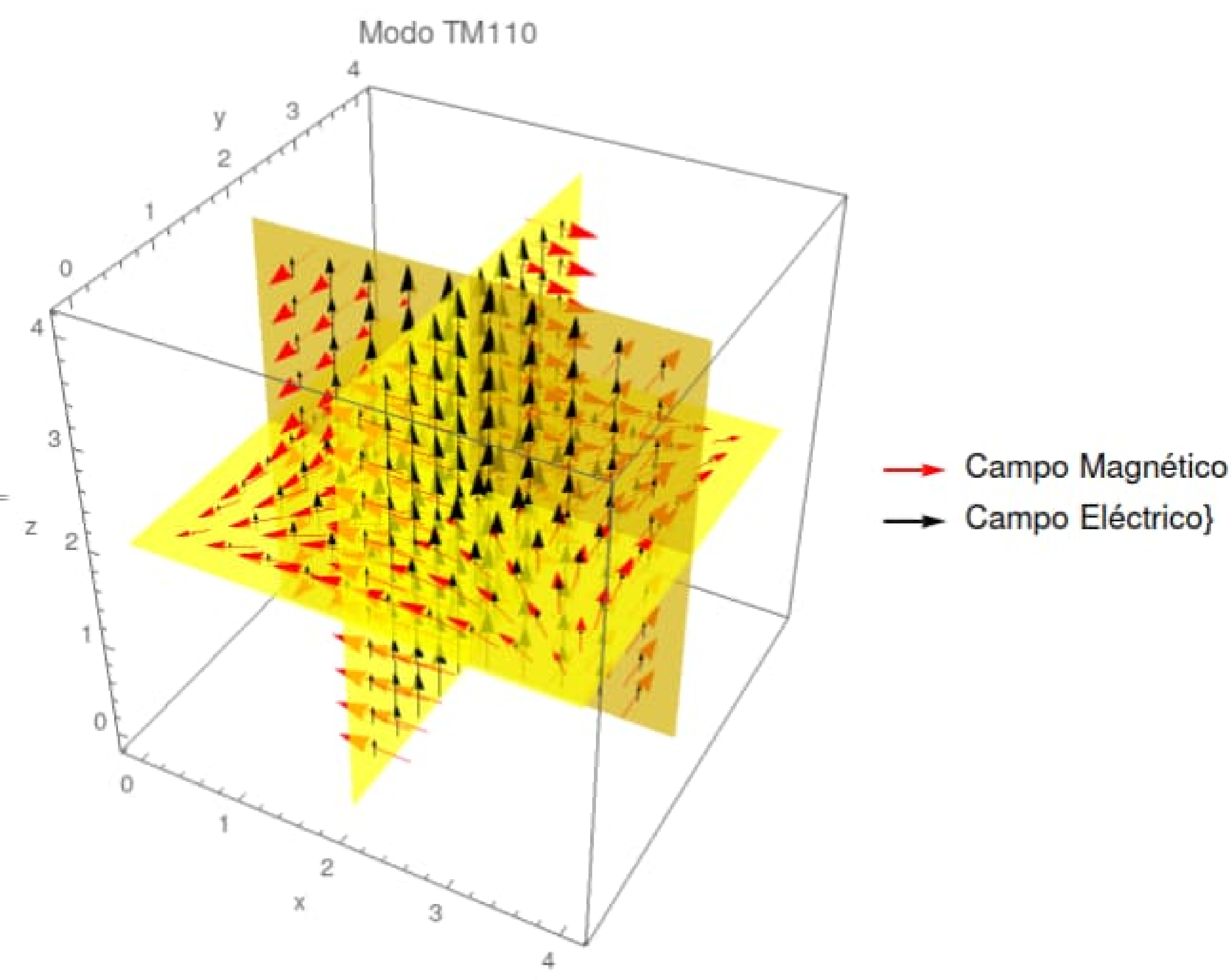
$$E_y = 0\,, \quad E_z = 0 \qquad H_x = 0$$

$$H_y(y,z) = -\frac{\pi}{c\omega\mu} E_1 \operatorname{sen}\left(\frac{\pi y}{b}\right)\cos\left(\frac{\pi z}{c}\right)\operatorname{sen}(\omega t)$$

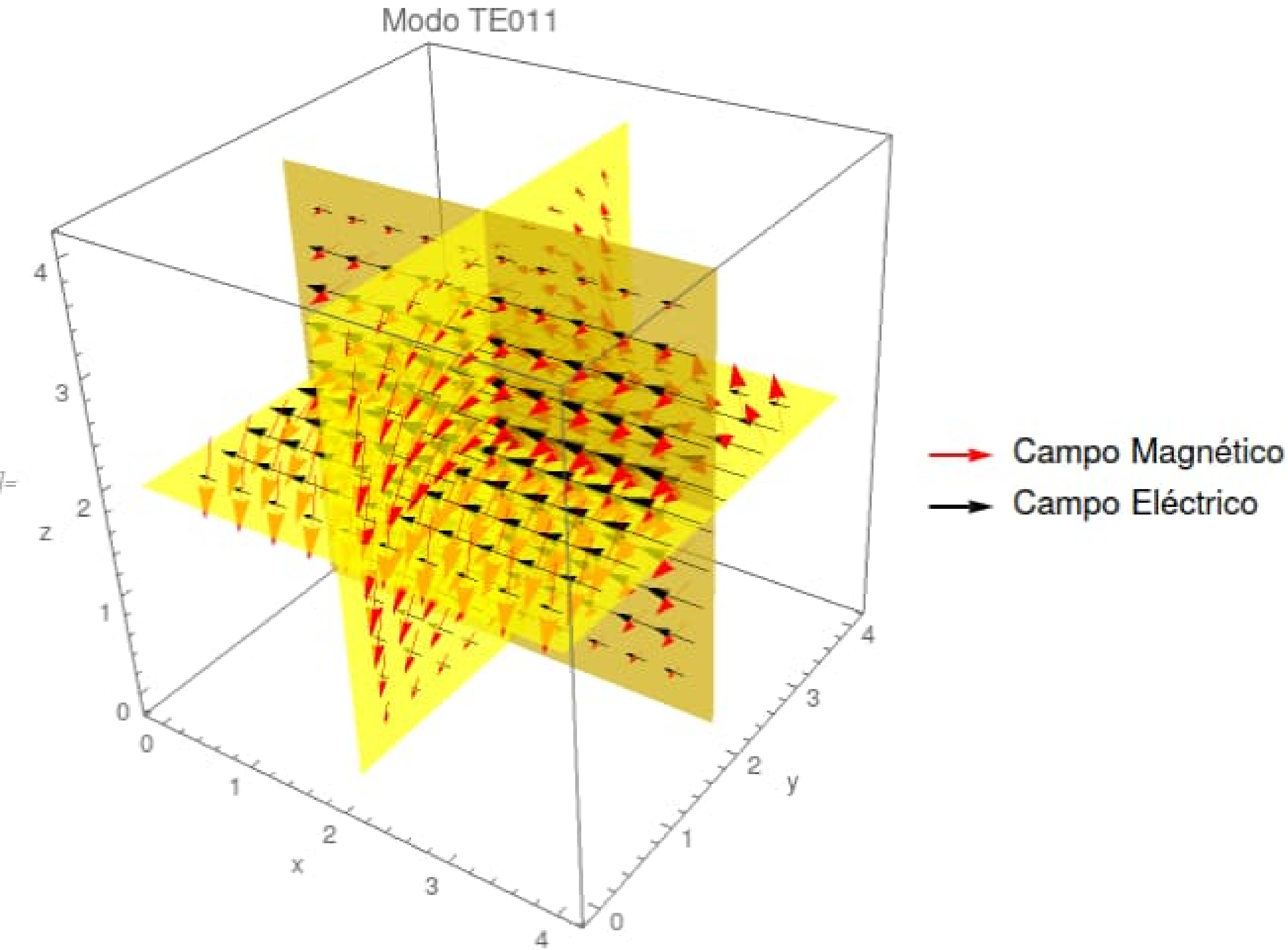
$$H_z(y,z) = -\frac{\pi}{\omega\mu b} E_1 \cos\left(\frac{\pi y}{b}\right)\operatorname{sen}\left(\frac{\pi z}{c}\right)\operatorname{sen}(\omega t)$$

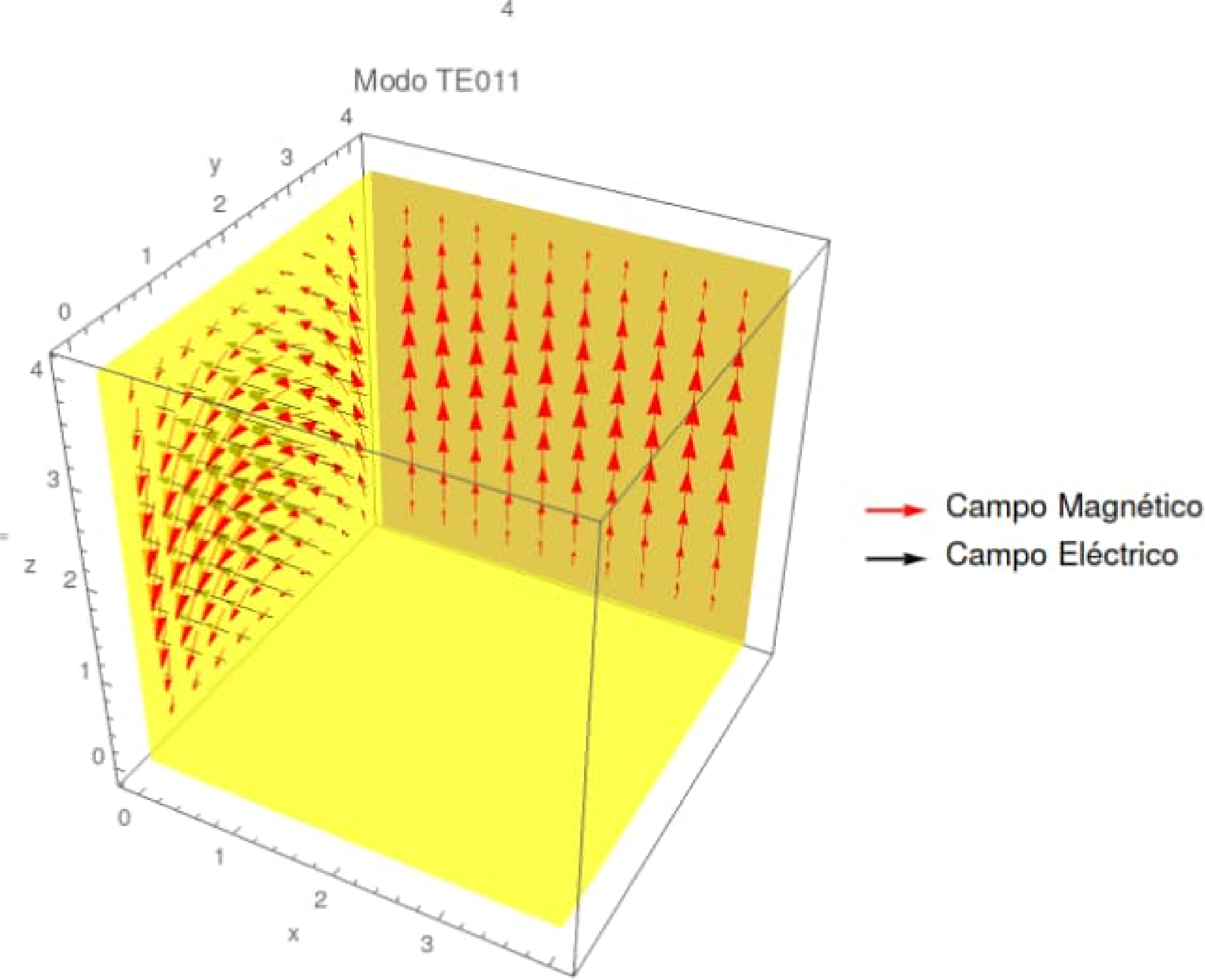


Modo TM110
Campo Magnético
Campo Eléctrico}
x
y
z
0
1
2
3
4



Modo TM110
x
y
z
0
1
2
3
4
Campo Magnético
Campo Eléctrico}


Modo TE011
Campo Magnético
Campo Eléctrico
x
y
z
0
1
2
3
4


Modo TE011
x
y
z
0
1
2
3
4
Campo Magnético
Campo Eléctrico