Notes for High-Dimensional Probability: Random Vectors in High Dimensions

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1 Preliminaries

Proposition 1. *Let X be a real-values random variable and $Y = \sqrt{|X|}$. Then X is sub-exponential if and only if Y is sub-Gaussian, and in such case $\|X\|_{\psi_1} = \|Y\|_{\psi_2}^2$*

Theorem 2 (Bernstein's inequality)**.** *Let $(X_i)_{i=1}^n$ be a sequence of independent real-valued zero-mean random variables such that $\|X_i\|_{\psi_2} < \infty$. Then, for every $t > 0$*

$$\mathbb{P}(|\sum_{i=1}^n X_i| > t) \leq 2\exp(-\frac{1}{2}\min(\frac{t^2}{\sum_{i=1}^n \|X_i\|_{\psi_1}}, \frac{t}{\max_i \|X_i\|_{\psi_1}})). \tag{1}$$

2 Concentration of the Norm

Theorem 3 (Concentration of the L_2 norm)**.** *Let $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ be a random vector with independent sub-gaussian coordinates X_i that satisfy $\mathbb{E}X_i^2 = 1$. Then*

$$\big\| \|X\|_2 - \sqrt{n} \big\|_{\psi_2} \leq CK^2, \tag{2}$$

where $K = \max_i \|X_i\|_{\psi_2}$ and C is an absolute constant.

Proof. We first note that $K \geq 1$. By Jensen's Inequality we have that $\mathbb{E}\exp(\frac{X_i^2}{t^2}) \geq \exp(\frac{\mathbb{E}X_i^2}{t^2}) = \exp(t^{-2})$ and using $t = 1$ we see that $\mathbb{E}\exp(X_i^2) \geq e > 2$ so $\|X_i\|_{\psi_2} \geq 1$ for all $i \in \{1, \ldots, n\}$. Since $K = \max_i \|X_i\|_{\psi_2} \geq 1$ we are done.

Now consider the quantity $\frac{1}{n}\|X\|_2^2 - 1$ which we can write as

$$\frac{1}{n}\|X\|_2^2 - 1 = \frac{1}{n}\sum_{i=1}^{n}(X_i^2 - 1) = \frac{1}{n}\sum_{i=1}^{n} Y_i, \tag{3}$$

where $Y_i = X_i^2 - 1$. Since $\mathbb{E}X_i^2 = 1$ for any i, $(Y_i)_{i=1}^n$ is a vector of zero-centred random variables. Since $\|X_i\|_{\psi_2} < \infty$ we can show that $\|Y_i\|_{\psi_1} < \infty$ since

$$\begin{aligned}\left\|X_i^2 - 1\right\|_{\psi_1} &\leq C\left\|X_i^2\right\|_{\psi_1} &(4)\\ &= C\|X_i\|_{\psi_2}^2 &(5)\\ &\leq CK^2, &(6)\end{aligned}$$

by use of centring property of sub-exponential norm, proposition 1 and definition of K. Since we showed that Y_i's are sub-exponential we can apply Bernstein's Inequality Thm. 2 □