Elective Module 3: Mathematical Induction

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1 Module learning goals

1. Apply induction arguments to prove mathematical statements.

[1]

Requirements

1.1 Request for feedback

Hi Peter, I have really tried to utilise the knowledge I learned from the LaTeX advanced module to format the document into a clean and navigateable summary. Let me know if you like it.

1.2 Response to feedback

N/A

2 Introduction

As we have learnt about some types of proofs including of direct, indirect, contradiction, cases, and some others, we have developed a foundational understanding of some proofs, when to use them, where to use them, and how to use them. Now we will be looking at another type of proof, proof by induction. This type of proof is a sort of chain/domino effect, to do this, we prove that that $P(k)$ and $P(k+1)$ are both true and therefore will be true for all values (given the domain).

3 Chains of implications

Chains of implications refer to proving a chain of statements which allows us to prove that all of the statements are true. For example, if we wanted to prove that $n^3 + 3n^2 + 2n$ is divisible by three for all integers $n \leq 5$ and $n \geq 0$ we could demonstrate this in probably the most intuitive way you are thinking, by proving each individual value. This would mean that we prove that $P(0)$ is true, $P(1)$ is true all the way up to $P(5)$. This works well if the domain is small, however what if the question had the domain of $n \geq 0$? Well, we this would become impossible to prove as we wont be able to do this for every possible value for n as there are infinitely many. These type of problems can be proven via the method of mathematical induction, proving a base case is true, and then the proving the inductive step is also true.

4 What is meant by a 'chain'?

Suppose that we have a proposition $p \to q$ and we know that p is true, this means that q will also be true. However what if we added a layer to this and we had the propositions $p \to q$ and $q \to r$ knowing that p is also true? Well, this would mean that since $p \to q$ means that q is true and therefore in $q \to r$ the r is also true. Hopefully you see what is happening here by now, a sort of snake or domino effect structure. If we knew that $p_1 \to q_1$ and $p_2 \to q_2$ and .... and $p_{99} \to p_{100}$ where p is true, we can deduce that p_{100} is also true. These are the foundational building blocks of mathematical induction and is what I am referring to when I say "domino effect" or "snake like structure".

5 Defining chains of implications

Now since we have looked at chains of implications, let's take arbitrary values for our functions. To write this, we could use $P(k) \to P(k+1)$, there is absolutely no reason to pick k here, however it is just for clarity and seems to be the most preferred arbitrary value used. If we prove that the first proposition $P(1)$ is true (the base case) then we have created a chain reaction concluding that all propositions within the domain (or chain) are true, even if the chain is infinite (because k is arbitrary). Saying this, we can use a simple two step strategy to prove all the propositions are true, have a look below.

- Set up the chain reaction by showing that $P(k) \to P(k+1), \forall k$
- 'Trigger' the chain reaction by proving that $P(1)$ is true.

5.1 An example

1. The formula for the sum of the first n natural numbers is:

$$\sum_{i=0}^{n} i = 0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{1}$$

If we look back at the steps I gave you, we will start with step one: Setting up the chain reaction.

$$P(k) \equiv \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \tag{2}$$

After this, we now need to set up the inductive step by substituting in $P(k+1)$

$$P(k+1) \equiv \sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2} \tag{3}$$

After this, we then simplify to get an equation to work with, look below.

$$P(k+1) \equiv \sum_{i=1}^{k} i = \frac{(k+1)(k+2)}{2} \tag{4}$$

Now that we have seen that $P(k) \to P(k+1)$, we need to trigger the chain reaction by solving the first proposition. This looks like:

$$P(1) \equiv \sum_{i=1} i = \frac{1(1+1)}{2} \tag{5}$$

Now we just prove that it's true by solving! This may seem confusing at first, however after you do a few problems you will get into the routine and it will become super easy!

6 Mathematical induction

Now that we have understood how we can use chains of implications to prove whether a statement is true across the domain, we can now apply this concept to solve problems through mathematical induction. To do this, we will follow two instructions.

1. **Inductive step/Inductive hypothesis:** Show that $P(k) \to P(k+1)$ by assuming the statement for $P(k)$ is true for an arbitrary value k. Then using this assumption prove that $P(k+1)$ is also true.

2. **Base step:** Verify the statement $P(n_0)$ is true for the first value of n setting off the reaction.

Let's take one example, however, if I am coming back to review this myself, I suggest completing multiple questions in my own time rather then reading the explanation.

- Prove that $n^3 + 3n^2 + 2n$ **is divisible by three for all integers** $n \geq 0$

First we want to prove that it holds for the smallest value of n. So let's do that.

$$0^3 + 3(0)^2 + 2(0) \tag{6}$$

Since this obviously holds, we can then move on to assuming $k = n$. This would look like:

$$k^3 + 3k^2 + 2k \tag{7}$$

Now we assume that is also holds for $k + 1$ this would look like:

$$(k + 1)^3 + 3(k + 1)^2 + 2(k + 1) \tag{8}$$

Now we have to prove that this is still divisible by three, so let's expand this.

$$= k^3 + 3k^2 + 3k + 1 + 3(k^2 + 2k + 1) + 2k + 1 \tag{9}$$

Now after we have done this, we want to expand the brackets and group the terms which were from our original assumption.

$$(k^3 + 3k^2 + 2k) + 3k^2 + 9k + 6 \tag{10}$$

Now as you can probably see, in the second part of the equation above, it's actually also divisible by 3 (because we can take out a common factor of three). This means that both our original assumption and our inductive step are both divisible by three which successfully completes the proof.

7 Reflection

7.1 What is the most important thing I learnt in this module?

Personally, I think that the most important thing I learned within this module is what mathematical induction is, and how it can be used to prove statements to be true (given the initial statement is true) for all of the values within a certain domain. This becomes really useful when the domain is infinitive and we are unable to calculate for every case of the domain. Understanding how to prove a base case, and then proving that if the statement for $P(k)$ is true, then it also must be true for $P(k + 1)$ provides a way for me to solve problems relating to mathematics and computer science when the domain is infinite or very large (for example in a program I am making).

7.2 How do this relate to what I already know?

This module linked to what I understood about proofs such as direct, indirect, cases, contradiction, and others. I think that this module built upon my previous knowledge of understanding of how to represent ideas and solutions in a mathematical and standard way as I now know another proof in which I can do this. I think that it exists also on my previous knowledge of sets and domains. I also think that proofs relate to almost all areas of mathematics as now thinking about what I've learnt, I can develop proofs to prove core concepts in the core modules such as Euclid GCD algorithm.

7.3 Why do I think the course team wants me to learn this content for my degree?

I think that the course team at Deakin wants me to learn the content of this module as proving statements are a key skill to have when assessing if a solution will work as well as convincing others that it will work in a professional way. I think that understanding proofs, especially mathematical induction will assist me in improving my algorithm design, especially those in which involve recursive features. I think that learning this content allows me to tackle problems with mathematics and prepare for more advanced mathematical concepts which I will learn in the future.

8 FOLLOW UP QUESTION

- Show that $n^2 - n - 2$ is even for any n which is an element of the real numbers.

Base Case when $n = 1$

$$1 - 1 - 2 = -2 \tag{11}$$

Since -2 is obviously even, we have proved this to be true for our base case where $n = 1$. Now four our inductive hypothesis we assume $P(k)$ is also true.

$$P(k) = k^2 - k - 2 \tag{12}$$

Which we also assume to be even. Now we have to prove the inductive step, in other words that $P(k) \to P(k+1)$

$$= (k+1)^2 - (k+1) - 2 \tag{13}$$

Now after we expand and simplify we are left with.

$$= k^2 + k - 2 \tag{14}$$

Now notice that we can also re-write this for simplicity, I will do this.

$$= (k^2 - k - 2) + 2k \tag{15}$$

Since based upon our assumption, $k^2 - k - 2$ is even, therefore we can denote its simply as $2m$ (an even number is just a multiple of 2). Therefore:

$$= 2m + 2k = 2(m + k) \tag{16}$$

Since $2(m + k)$ is obviously divisible by two as it has a common factor of two, we have successfully proved that $n^2 - n - 2$ is even for all n that are elements of the real numbers with an inductive proof. We first proved a base case held, and then assumed $P(k)$ was also true and then proved that $P(k) \to P(k+1)$.

Other

References

[1] Deakin University. (n.d.). *Unit SIT192*. Deakin.edu.au