Counting

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1 Requirements

1.1 Request for feedback

Based on you previous feedback, you wanted some example of concepts to strengthen my summary and portray my understanding, let me know if you like the format and whether or not I have done a good job implementing it.

1.2 Response to feedback

N/A

1.3 Module Learning Objective

Module learning objectives:
For completing this module, you should be able to:

- Apply basic principles of counting
- Select the appropriate principle when solving real-world problems
- Identify and count the different ways items can be combined.

2 Introduction to counting

2.1 Principles of counting

Counting is a core concept in mathematics and CS used to evaluate the number of ways certain events can occur. For example, the question "If I went to a restaurant and could pick one main and one beverage considering there were 8 mains and 20 beverage options, how many different combinations of main and beverage could I chose?".

2.2 A quick example

Just say I was prompted to set a four digit PIN with each entry being a number from 0-9. This means that for the first entry of the pin I could pick 10 numbers, for the second digit of the pin I could also pick 10 numbers, with the same being for the third and fourth entry. This means that there are $10 \times 10 \times 10 \times 10$ or 10^4 possibilities. If I was trying to guess someone's PIN trying random numbers, at worst case I would have to try $10,000$ combinations before I am guaranteed to get it right. These are some questions we can answer with counting, based on this, how many different possibilities are there for unique number plates?

3 Product rule and sum rule

3.1 Product rule

- Product rule with two sets: Suppose that we have an object which is split up into two parts, A and B. If there are k possible values for A and z possible values for B, then there are $k \times z$ possible values for A and B. (Deakin University. (n.d.))
- Product rule with multiple sets: Just like if we had two sets, the product rule states that if we had n sets, then all of the combinations are the number of elements of each set multiplied. Just like we saw in the example above with the PIN number, we have four different sets with S_1 containing the natural numbers between 0-9, S_2 being the natural numberers between 0-9 and so on until we reach our S_4. Since each set has 10 elements, we then have 10 possibilities for the first set, 10 for the second, 10 for the third, and 10 for the fourth, and therefore 10^4 different PINs.

In order to apply the product rule, we need to determine how many parts compose the object and for each part, how many different options there are. As we said before, a PIN number is composed of 4 digits (4 parts) with 10 numbers to choose from.

3.2 Sum rule

The sum rule tells us that id the objects have only one part, which can come for either A_m then:

- $N = n_1 + n_2 + n_3 .... + n_m$

This means that if we are only able to chose one beverage out of 30 option OR one main from 10 option, we can pick 1 out of 40 options.

3.3 A quick example

Let's take the exercise on the module page to explore this. "Given there are 26 letters of the alphabet, and 10 single-digit numbers (0 to 9), how many ways are there of:

- Picking a letter and a number?
- Picking a letter or a number?

Since there are our object is broken into 2 sets, one for the letters and one for the numbers, we have 26×10 or 260 different combinations of letters and numbers. On the other hand, If we were to choose a letter OR a number, we have 36 options to pick from.

4 Functions

4.1 Non-Injective

As we recall a function is when the elements of set X are mapped the elements of set Y. Since this is not injective, elements of X can be mapped to multiple elements of Y. Lets say X and Y are sets and the elements for the set X is denoted as n and the elements for set Y are denoted as m. If we consider all functions $f : X \to Y$ how many functions are there in total? In other words how many different mappings of X to Y are there? Since each element of X can be mapped to any element in Y we can link it back to the product rule we learnt earlier. Have a look in the section below "Injective" to see how we would go about solving this.

4.2 Injective

As we know, and injective function does no map to one to one between sets. This means that if there are sets X and Y that means X can initially map to any element of Y which we are denoted as n. After this, it cant be mapped to that same element therefor has the option to map to $n - 1$ elements and so on, then $n - 2$, $n - 3$ elements and so on.

4.3 Function examples

Lets take a look at a the function which is using set X which has 5 elements and set Y which has 3 options. This means that element one from X can map to any 3 elements of Y, then the second element can map to any element of Y and so on. This means there are 3^5 different options available. However if the function was injective, what would happen? Well since X is greater then Y in this case no injective function exits.

5 Applications of counting

Now that we have look at the basic principles of counting, we then therefore want to have a look at how these can be applied to real world problems. We will look at question like "How many numbers of 100 are also a factor of 2" as well as the inclusion and exclusion principle.

6 Divisibility explained with an example

One problem to consider is how many positive integers less then or equal to 100 are divisible by two? We know that there are 50 numbers in 100 that are divisible by two (as all numbers are composed of even and odd). What about how many numbers that are divisible by five? Well we know that there are 20 numbers divisible by 5. We can just divide the number to see the amount of solutions. If you encounter a number that is not perfectly divisible by the divisor we just use the floor function which works because of the same reasons as highlighted above. However what if we wanted to know the amount of numbers between 12 and 20 are divisible by three, we would find out how many numbers are divisible by 3 in 20 and how many numbers are divisible by 3 in 12 and take the difference. we know that 6 numbers divide 20 by 3 (using the floor function) and 3 numbers divide 12 by three). Therefore we have 3 numbers.

7 Inclusion exclusion principle

In the second application we look at how to count two sets that have overlapping elements, we do this with the inclusion-exclusion principle and follow these steps. The inclusion-exclusion principle looks like this:

- $|A \cup B| = |A| + |B| - |A \cap B|$

To count the stets that have over lapping elements we first find the cardinality of the first set A followed by the cardinality of B. We then find the cardinality of $|A \cap B|$, $|A \cup B|$. Have a look at the next section to see how we would solve a problem using this principle.

7.1 A quick example of the principle

"How many 5-letter passwords start with MA or finish with HS (assuming only letters are used)?" Since we have the two characters defined already it means we have $26 \times 26 \times 26$ remaining, this is the same for the ending HS as we can only pick the first 3 characters. Therefore, the cardinality of A in this case is 26^3 with B sharing the same cardinality. We then find the cardinality of $A \cup B$ which is 26. We apply the inclusion exclusion principle which leaves us with $A \cap B = 26^3 + 26^3 - 26 = 2 \times 26^3 - 26$ The ends up to be 35,126 which is the the number of passwords that either start with "MA" or end with "HS". One important thing to note is that we are subtracting 26 here as it was counted in both cases.

8 Pigeonhole principle

Despite the funny name, the principle states that if m pigeons occupy n pigeonholes with $m > n$ there must be at least one pigeonhole with at least two pigeons inside. The easiest way to summaries this is with an example.

9 Pigeonhole principle

Let's take the question "There are 4.5 million people, how many people share the same birthday?" First we identify the amount of pigeons, in this case its people therefore $m = 4.5$ million. There are 365 birthdays in a year which represent the pigeonholes as every person must fall within one of these 365 days. Since m is greater then n then it must mean that people share the same birthday. To calculate the exact amount of people who share the same birthday we can simply divide 4.5 million by the pigeons holes. This means that around 12,329 pigeons are in each pigeon hole! The wholes represent groups of people with the same birthday.

10 Permutations and combinations

10.1 Permutation

Initially, a permutation and a combination seem similar however they have one key identifier which allows us to use them. A permutation is a set of distinct objects in an ordered arrangement of its objects, this means that permutations care about the order so the word 'HELLO' is different from the word 'HELOL' which is exactly what you would expect. We can calculate the amount of permutations a set has by using our knowledge gained in the product rule section, have a look below where I highlight an example.

10.2 Permutations example

Just say we have the set $W = S, I, T, 1, 9, 2$, as you can probably see, this spells out SIT192, however that is only one permutation, there are many different ways of re-arranging these letters are order does not matter here. For example you could have '291TIS' or whatever you want, in saying this, no matter how you pick your letters you will always have 5 choices to begin with, then 4 choices, then 3, then 2, then you final choice. This can be denoted as $6 \times 5 \times 4 \times 3 \times 2 \times 1$ or simply 6!. However, what if i wanted to find out how many 2 letter permutations are in the set A we can use this handy formula

- $P(n, r) = \frac{n!}{(n-r)!}$

I like to think of $P(n, r)$ as $P(\text{number, chose amount})$ So back to our question we have 5 characters, and we are choosing two, therefore,

- $P(6, 2) = \frac{6!}{(6-2)!}$

This turns out to be 30, cool!

11 Combinations

As we learnt just before, permutations are an ordered collection of elements. Combinations are similar however they are un-ordered, this means that 'HELLO' and 'OLLOH' are the same as they are still the same elements (or letters in this case). One way to think about this is just say my bus ticket had the numbers 7,4, and 2 which represented my bus ID or something. If I go to get on the bus and the ticket man asks 'What is your bus ID?' i could say '7,4, and 2' or '2,4, and 7', or '4,2 and 7', it doesn't matter as in the end its still the same ID. The formula we use to calculate combinations are very similar to the one we use to find the permutations, have a look:

- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Let's use a question to explain how this works.

11.1 Combinations example

"How many ways are there to select 8 players from a 10 member tennis squad?". The first step when solving questions like these are to identify whether it's a permutation or a combination. If with think about a squad of tennis players being 'Isak', 'John', 'Peter', 'Juan' (all the way up to 10 players), it doesn't matter the order in which I pick the players as regardless I'll end up with the same team (that is if I'm picking the same players of-course). We then use our formula which will look something like this after you plug in the correct values.

- $\binom{10}{8} = \frac{10!}{8!(10-8)!}$

This turns out to be 45. This means that there are 45 different combinations I could choose from to pick 8 players.

12 Expressing combinations with repetition

We use the '*' and '—' characters to represent repetitive combinations. We put a '—' to symbolise the change of the repetitive pattern, and a '*' do denote actual elements. For example if we had 'ABBC', this would look like `'*|**|*'` and BBBC would look like `'|***|*'`.

13 Solutions to integer equations

Following along with what we have been doing so far, how would we solve something like $x_1 + x_2 + x_3 = 4$? (where all of the x's are non negative integers). If we think about this, the question is asking us to make four choices among three objects (which are the x's) where order does not matter, this sounds a-lot like a combination doesn't it? To solve this we will have the equation $C(4 + 3 - 1, 2)$, we get the two here from the amount of '+' signs (or borders between the objects), we then get $C(6, 2)$. This is just $6 \times (5/2!)$ which is 15.

14 Reflection

14.1 What is the most important thing I learnt in this module?

I think that the most important thing I learned while studying this module was permutations and combinations as they apply to a lot of real world problems. Learning about permutations and combinations will hopefully allow me to approve my ability to tackle real world problems with mathematical thinking in the future, as well as being able to apply mathematical concepts while problem solving in general. I think that learning this module has enabled me to improve my critical thinking surrounding counting and arrangements of objects and the real world.

14.2 How do this relate to what I already know?

This knowledge tied into some of my existing knowledge in basic algebra and arithmetic. While working through the module I also noticed some links back to sequences and recurrences which I thought was cool. I was already familiar with some basic counting concepts such as the concept of "5 choose 2" and figuring out if something was a permutation or a combination, however this content expanded upon that and build upon this understanding which will hopefully allow me to solve real world problems.

14.3 Why do I think the course team wants me to learn this content for my degree?

I think that the course team at Deakin wants me to learn this content as an understanding of the basics counting are foundational knowledge for things like algorithms and acts as a pathway to more advanced topics in the future. I also think that the course team wants me to learn this content as it builds a mindset of coming at real world challenges with critical thinking which allows for effective solutions to be developed.

15 References

Deakin University. (n.d.). Deakin. https://www.deakin.edu.au/