Activities Lesson Review: Euler’s theorem.

Please answer the question below. They ask you to summarise the class activities, referring to any learning evidence that you may provide, if necessary. The learning evidence can consist of your work on activities, (as well as notes taken during classes). As a rule of thumb, your summary should be half a page to one page long each (not including evidence). Your answers should be like your answers to the final section () of each activity.

The task is worth up to 3 marks. Your answer should relate to the questions in the activities, and the discussions that happened in class. Note that these questions do not refer to the content of the modules - just explaining how to apply the Euclidean algorithm, for example, is off-topic in this task (and has already been assessed in the Number Theory Lesson Review).

Question 1.

Explain the proof of Euler’s theorem on Euler paths and circuits using your answers to Tasks 32–37 of the activity.

Euler’s theorem on Euler Paths and Circuits is an interesting theorem that can seem like magic at first. However, it becomes interesting when you start to understand the why behind certain concepts. Euler’s theorem states that a finite graph has a Euler circuit if and only if all the graph’s nodes (or vertices) have an even degree. We will explore the why behind concepts and explain the proof of Euler's theorem.

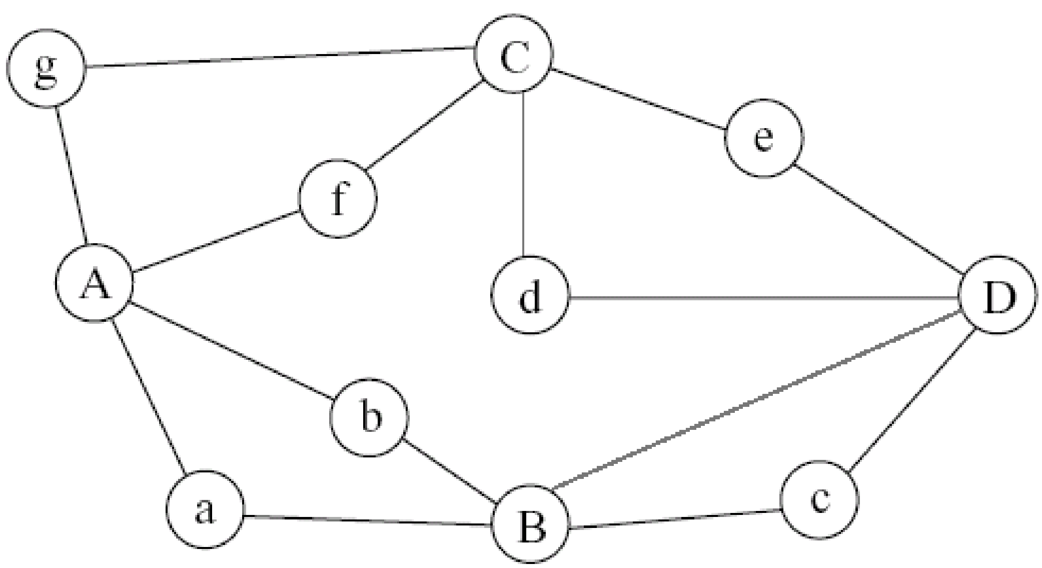
Task 32. Euler's theorem states that a graph is said to be connected if for every pair of vertices u,v in the graph there is a walk from u to v. This means that for every pair of vertices, there is an edge connecting them. A finite graph contains an Euler circuit if and only if G contains no vertices of an odd degree. All vertices must have an even degree as every time you enter a vertex (imagine you are a hiker again), you must also exit that vertex via another trail to continue your walking. This means that for each vertex, there needs to be an entry edge and an exit edge so you can continue your journey. For example, if ANY vertex has an odd degree, you will be able to walk in, walk out, and then walk in again, however now that you are inside you are stuck, and you cannot continue your journey any further as you have to backtrack on a path you have already walked over to get out. If any vertex in the connected graph has an odd degree, this same process will occur and you will not be able to complete an entire circuit, this leads on great to my next point. Euler also states that for an Euler path to occur, there needs to be at minimum two vertices with an odd degree, which will be the starting and the end. An Euler path is a path that traverses every edge of the graph once, however, does not have to start and end at the same point. Due to an Euler path having a different start and end point, they can have an odd number of degrees as they can include an entry, exit, and then the ending entry. This concept is like the intuition behind the previously explained paragraph.

In conclusion, a graph can have an Euler path if and only if it has a maximum of two vertices with an odd degree (addressing the if and only if part of the question). For a graph to have a Eulerian circuit every vertex must be of even degree.

Tasks 33 and 34.
Euler's theorem also depends greatly on the graphs being connected. This is important as we need to ensure that the walker can traverse each node without "jumping" over destinations. Consider a graph with two disjoint cycles (G = {A,B,C} {{a,b}}) we can see that A and B are connected, however C is isolated. This means that if you were a person walking along this track you could walk between the vertex A and B as much as you want, however, you could never get to C. After learning the importance of connectivity, we can then link it back to our new understanding of degree. If every vertex of a graph had an even degree but it wasn't connected, what do you think would happen? Well, it would look like an Euler path but it's not, you can't traverse from the connected nodes to the disconnected vertex. So far, we have explained Euler's Theorem of why a graph's vertex degree is important, as well as why connectivity is important. Let's continue.

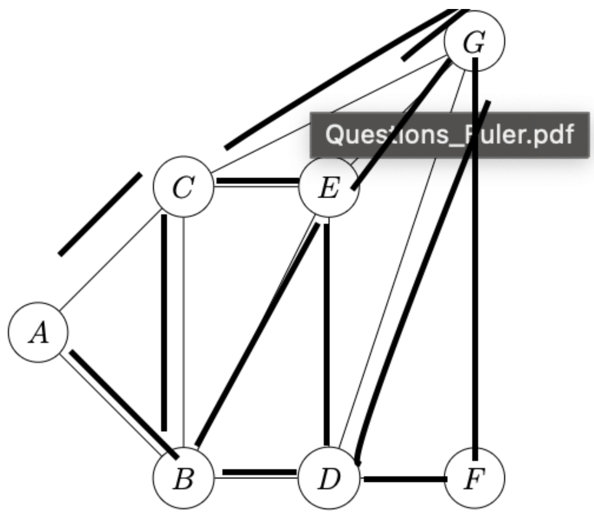
Tasks 36 & 37.
Here is a complex graph that contains an Euler Circuit (and therefore a Euler path)

(*CS109: Eulerian Circuits*, n.d.)



Here we can start at any vertex and walk over the edges ensuring that no edge is repeated. Since each vertex has an even degree, for every edge that you enter the edge from, you will always be able to exit the same vertex using another edge. For example, let's look at vertices B. We can enter from the bottom left, exit from the bottom right, re-enter on the top right and finally exit through the top left. This is very simple once you understand and do some practice problems. This means that you won't get "stuck" at any vertex and eventually, you will return at the starting vertex. This also highlights the importance of connectivity, if we disconnect the edge connecting C and D, we suddenly have vertices with odd degrees, meaning that you will get "stuck" at the vertices, meaning that no Euler Circuit is present.

As the questions asked, as we traverse along these edges, we remove the ones we have gone over and since the graph is finite (as Euler stated) eventually we will have no edges left meaning that we have a Euler circuit. Based on this, we can now see if a graph has a Euclid circle take this example from task 41 in the textbook.



This is the path we would take GFDGEDBECBACG. We know that a Euclid circuit is possible before even attempting to find the path based on our learnings on the degree of Vertexes. We can now understand why it's working based on my summary above.

References:

CS109: Eulerian Circuits.

(n.d.). https://www.cs.bu.edu/~best/courses/cs109/modules/euleriancircuits/