Sequences

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Personal quick cheat sheet.

- **Level 1 (First Difference Constant)? ⇒ Arithmetic Sequence**
  - Formula: $a_n = an + b$
  - Grows by **addition/subtraction** each step
- **Level 2 (Second Difference Constant)? ⇒ Quadratic Sequence**
  - Formula: $a_n = an^2 + bn + c$
  - Grows in a **polynomial pattern** (squared terms)
- **Constant Ratio Between Terms? ⇒ Geometric Sequence**
  - Formula: $a_n = a_0 \cdot r^n$
  - Grows by **multiplication (exponential growth)**

Quick Trick:

- **Addition/subtraction ⇒ Arithmetic**
- **Increasing second difference ⇒ Quadratic**
- **Multiplication pattern ⇒ Geometric**

1 Requirements

1.1 Request for feedback

Have I made this too long? I tried to include examples of each concept to strengthen my report based on your previous feedback. In saying this, I hope you can see that I've tried to be straight to the point however this module was pretty content heavy so it was hard to avoid a lengthy summary.

1.2 Response to feedback

N/A

1.3 Module Learning Objective

Module learning objectives:

For completing this module, you should be able to:

- Find the elements of a sequence given in closed form or in recurrence form
- Evaluate partial sums of sequences
- Solve linear recurrence of order 1 and of order 2.

2 Sequences and their terminology

Firstly, let's understand what a sequence is; a sequence is a function defined on the domain of natural numbers ($x \in \mathbb{N}$) and is an ongoing list of numbers, for example: $(1, 2, 3, 4, 5....)$, $(1, 3, 5, 7, 9....)$ and $(2, 4, 8, 16, 32, 64....)$ are all sequences. One important thing to remember is that not all sequences have to be common, widely known or easily identifiable. As we explored the sequence of numbers to the power of 2 in the example which you probably could identify, in most cases you wont recognise the sequence straight away. The numbers appearing in a sequence are referred to as its elements, for example $3, 5$, and 7 are all elements of the sequence of odd numbers. The index of a sequence refers to the position of an element within the sequence, this is EXACTLY like an array in C++ (remember index starts at 0). For example in the sequence $(1, 3, 5, 7, 9....)$, the element 1 is at index 0 whereas the element 7 is at index 3, like I said, exactly like a Array. We can denote the sequence's name and index as a_n where a is the name of the sequence and n is the index. For example in the sequence of odd numbers, $a_0 = 1, a_2 = 3, a_4 = 9, a_{230} = 461$ and so on. We can define a sequence in two simple ways (so far. More on this later) through a closed form or a recurrence form. We need to mathematically declare a sequence as otherwise its actually impossible to distinguish the sequence, for example I will ask you what is the next number is this sequence: $(1, 2, 3, 4, 5, 6, 7, 8, 9....)$. Did you say -1? Probably not, which is exactly why we need to define the sequence.

2.1 Closed form

In a closed form of a sequence, the elements are given as the function on the domain: $\mathbb{N} \cup \{0\} : a_n = f(n)$ (Deakin University. (n.d.)). This may look intimidating at first, however if a sequence was defined by the function $a_n = 2n + 2$ it becomes very simple as we have learnt about functions in our previous modules. In this specific form we don't need to know any previous index's of elements, we can simply just plug in the index we want to know and solve. For example if I wanted to know what element was at index 25 in this sequence I can plug it into the closed form, this would look like $a_{25} = 2(25) + 2$ which is 52. As you can see, you don't need to know any previous information about the sequence which makes this the most intuitive and simple form in my opinion.

2.2 Recurrence form

A recurrence form is a little more involved then the closed form as if depends on its preceding values within the sequence. This means that each element is given as a function of the elements that have came before it; for example $a_n = f(a_{n-1}, a_{n-2,}, a_{n-3}, ...)$. Unlike in our closed form sequences, we need to define the initial state of the sequence (because then we would run into the same problem of not knowing the sequence) this could look like defining $a_0, a_1, ...$ Another key thing to remember about this form is a term called the recurrence order; the recurrence order is the number of preceding elements in the function that need to be defined. For example if we had the function $a_n = f(a_{n-1})$ we have to give the value of a_0 as otherwise we have no idea what a_1 would be, since we only need to give one value, (a_0) this function is said to have an order of 1. On the other hand the function $a_n = f(a_{n-1}, a_{n-2})$ has an order of 2 because we have to give a_0 AND a_1 to be able to define the rest of the sequence. You will see that this process is like a ladder, solving the previous index before the next.

2.3 Quick example

Closed form: Identify the elements at index 2 and 3 for $a_n = 2n$. First we want to find the element at index 2, to do this substitute it into the function therefore $a_2 = 4$. This means that the element at index 2 is 4. Now we can just do the same thing with the index 3 element. $a_3 = 6$. Recurrence form: Find elements 0-2: $a_0 = 0$ and $a_n = a_{n-1} + 2$. We have already been given element 0 so we can move on to finding the element at index 1. The element at index 1 is found through the function $a_1 = a_{1-1} + 2$ which just simplifies to $a_1 = a_0 + 2$. Since we know the element at index 0 was 0 we then have $a_1 = 0 + 2$ which is just 2, pretty simple. Now lets solve for a_2, since we know $a_1 = 2$ then $a_2 = 2 + 2$ which is just 4.

2.4 Partial Sum

Sometimes we need to define a sequence based upon other sequences. For example, what if we wanted to calculate the sum of elements within a sequence? We can define a partial sum as the sum of the first elements of a sequence all the way up to index k. This can be denoted with Sigma, lets analyse this:

$$\sum_{i=0}^{k} a_i$$

The big E means sum, the i is the starting index, and the k is the ending index. The big funny E is the sum of all a_i elements. This is VERY similar to a for loop in C++ which we are used too. This means we are just adding up $a_0, a_1, a_2, a_3....a_k$ which is exactly what we have already been doing but without the addition. Very simple.

3 Arithmetic sequences

A arithmetic sequence is a sequence of numbers which has a fixed difference from the previous number to the n^{th} number. This means that $n_1 = n_0 + k$ where k is a fixed constant. Let's take the example sequence $3, 7, 11, 15, 19....$ to highlight the relationship between each index (a_n). We know an arithmetic sequence has a constant relationship between the a_n and the a_{n-1} which we will call k. This means that $a_n = a_{n-1} + k$, we know that $a_0 = 3$ so let's try to find a_1. This means that $a_1 = 3 + 4$ where 3 is our a_0 and 4 is our constant k. Based on this, we know that our recurrence form is $a_n = a_{n-1} + 4$. Now, to get a closed form expression, we want to reflect a_n where it is not dependent on the previous term. This means that $a_n = 4(n) + 3$ is the correct closed form expression as it's not dependent on the previous term. To the test this we could find any element based on the index by just substituting the n in.

3.1 Difference equations

Sometimes its not as easy to determine weather a sequence is growing linearly or in another way, this is where difference equations come in to help us identify weather its growing from the a_n to the a_{n+1th} term through addition or some other type of acceleration like multiplication. To do this we take the difference of the a_n to a_{n+1} terms to see if its growing through addition (i.e. its arithmetic). We then take the difference of these differences to see if its growing in some other type of way (like a quadratic sequence). Let's take the example sequence $1, 4, 9, 16, 25...$ so you can see what I am taking about. First, we want to see if there it is a arithmetic sequence which we can do by finding the difference with $a_n = a_{n_1} + k$ We can see that $a_1 = 1 + 3$ where k is the increase of 3, I will not do every difference here as its a summary, however I know we will be left with the sequence $3, 5, 7, 9$. We can see that the sequence is not growing arithmetically as there is no common constant k between any $a_{n^{th}}$ and $a_{n-1^{th}}$ term. However, just because there is no arithmetic relation ship doesn't mean there is no relationship at all, let's inspect the difference of these differences. We are left with $2, 2, 2$ which is definently a relationship in growth (It is quadratic growth)

3.2 Quadratic sequence

As discussed before, we have a quadratic sequence if the second difference is a constant. Based on this constant we can then determine a closed form of a quadratic sequence. This type of sequence is often much more difficult to solve then others as because it is a quadratic it involves steps such a solving systems of equations. I will provide an example for this in the next section due to length and complexity.

3.3 Quadratic sequence

Find the closed form of the sequence starting with -4,1,2,-1,-8,-19,-34,-53.... First we can calculate the first difference as the sequence $5, 1, -3, -7, -11, -15, -19$ and therefor the second difference as $-4, -4, -4$.... Okay so far we have identified that it is a quadratic sequence. Now we know the equation for a quadratic sequence is $a_n = An^2 + Bn + C$. Since we know C is a constant we are left with $An^2 + Bn - 4$. Now we know our n_1 value is 1 therefore $1 = A + B - 4$ which simplifies to $A + B = 5$ which is our equation one. We then re-arrange this to $A = -B + 5$ for easy substitution later. Then we know our a_2 is 2 therefor we have $2 = 4A + 2B - 4$ which after simplification leaves us with $3 = 2A + B$. Now we solve this systems of equation by substituting our first equation into the second. We then have $3 = 2(-B + 5) + B$.. After solving we are left with $B = 7$ which we then substitute back into out equation 1 which is $A = -7 + 5$ which leaves us with $A = -2$. Now putting A and B back into our quadratic we are left with $a_n = -2n^2 + 7n - 4$. Substituting a_1 back in to check this is working we have $a_1 = -2(1)^2 + 7(1) - 4$ which is indeed 1. Great Job!

3.4 Partial sums of arithmetic sequences.

As we learnt previously, the partial sum is the addition of each element within a sequences. If we wanted to add up 1000 elements for example it would take a long time to do this as we first have to define the sequence and then add every element up within that sequence, there must be a faster way to do this right? Luckily there is, if we find a pattern between the sequence we can simply just use multiplication to solve this problem (this is because an arithmetic sequence its just repeated addition which is exactly what multiplication is!). Let's do a practise problem to demonstrate exactly how we can do this.

3.5 Practise problem

As we previously defined, a partial sum can be denoted with Sigma, keep that in mind when working through the question.

$$\sum_{i=0}^{k} a_i$$

Lets try to problem $a_0 = 2, a_{n-1} + 3$ trying to find the sum of the sequence which can be denoted as

$$\sum_{i=0}^{5} a_i$$

First we actually have to understand what the sequence is showing us, therefore we will solve for the first few values of the sequence. We know that $a_0 = 2, a_1 = 5, a_2 = 8, a_3 = 11, a_4 = 14, a_5 = 17$ and so on, let's denote this as $x_1 = 2, 5, 8, 11, 14, 17$. We know that from primary school math that the order we add the numbers does not matter, therefore we can write this same sequence as $x_2 = 17, 14, 11, 8, 5, 2$. You may not notice a pattern yet, however what if i re-arrange the formatting a little bit.

$$x_1 = 2, 5, 8, 11, 14, 17$$

$$x_2 = 17, 14, 11, 8, 5, 2$$

Can you see the pattern? If we look at the columns we can see that each index from x_1 added to the same index of x_2 all sum to 19 we have successfully found a pattern. We will denote this as $2x = 19, 19, 19, 19, 19....$, Note that this is $2x$ because we have added the sum twice (like a bijective function). Now as we said previously, multiplication is just repeated addition and since we are looking for the sum of the first 6 terms (remember index starts at 0)we know that can be denoted by $6 * 19$ which is 114. We then divide this by two since we added the sum twice which leaves us with 57. Feel free to add up every element in the sequence and you will get the exact same result. This is very useful when we want to find the sum of lots and lots of elements. We can also do the same thing no matter the form, we just need to find the first few elements.

4 Geometric sequence

A Geometric sequence may sound intimidating, however it is very similar to an arithmetic sequence we have been learning with one minor difference. As we explored earlier, an arithmetic sequence is just the constant difference k between the $a_n th$ and the a_{n-1th} term in the sequence through addition. A geometric sequence is the exact same however the fixed constant k being an increase or decrease by a multiplier instead of addition which can be defined by the recurrence relation $a_n = ka_{n-1}$. We can see that the sequence $2, 4, 8, 16, 32$ is geometric as a_n is $2a_{n-1}$, in other words, $k = 2$. Similarly for the sequence $5, 15, 45, 135...$ shows that $k = 3$. We can define the closed form of a sequence by running through the first few terms and expressing it as $a_n = k^n a_0$. This process is similar to what we have been doing to find the closed form of a arithmetic sequence.

5 Partial sums of geometric sequences

For any geometric sequence defined as $a_n = k^n a_0$, we can calculate the partial sum by multiplying every element of the sequence by k, then shift the multiplied sequence one step to the right, subtract the original sequence from the multiplied sequence and add up all of the results of the subtraction. Let's take an example to highlight these steps, the question I will be solving is: "for the sequence $a_0 = 2, a_n = 4a_{n-1}$ calculate,

$$\sum_{i=0}^{4} a_i$$

" First, we solve the first few elements of the index, this leaves us with $b_4 = 2 + 8 + 32 + 128 + 512$, we then multiple this by 4 as the steps say, we get $4b_4 = 4+8+32+128+512+2048$. Let's then shift these to the right and put them in a more readable format.

$$4b_4 = 8 + 32 + 128 + 512 + 2048$$

$$b_4 = 2 + 8 + 32 + 128 + 512$$

Now after subtracting these we are left with the sequence $-2+0+0+0+0+2048$ which added up is 2046. Remember we need to divide by 3 therefore the answer is 682. Great job.

6 Linear recurrence of order 1.

A linear recurrence of order one is defined by the equation denoted as $a_n = ka_{n-1} + d$. We know that it is a linear recurrence as it's growing based on the fixed values k and d. It's obviously an order one as it refers to only the previous element. We must also know that if $k = 1$, then it is arithmetic and if $d = 0$ then it's a genetic sequence. To solve find the closed form of an order one linear recurrence we need to do three main steps.

1. Compute the first difference of the sequence a_n.
2. Find the closed form of the first difference
3. using this closed form of $a_{n-1} - a_n$ and the original formula, find the closed form of a_n

I am going to generate a NUMBAS practice question to so we can highlight this process. The question I was generated was "solve the closed form of the relations $a_0 = 3, a_n = 2a_{n-1} + 1$ First we want to find the value of a_1 which we should be comfortable in doing by now, this looks like $a_1 = 2(3) + 1$ which results in 7. We then calculate the first difference of the sequences which we can simply do by subtracting 7 from 4 which gives us a difference of 4, therefore $b_0 = 4$. We then find the closed form of the first difference which can be done by following $b_n = a_{n+1} - a_n$, after substitution you should have $b_n = 2a_n + 1 - (2a_{n_1} + 1)$, we can then expand these brackets and are left with $b_n = 2a_n + 1 - 2a_{n_1} - 1)$ We can then collect the the like terms (-1 and 1) and then factor out a 2, this should leave you with $b_n = 2(a_n - a_{n-1})$. We now know our coefficient will be 2, recall out b_0 being 4 therefore we have our closed form expression $b_n = 4 * 2^n$. Now we use what we have solved to evaluate the expression, to do this, we will use the formula $b_n = a_{n+1} - a_n$. We know our b_n therefore we are left with $4 * 2^n = 2a_n + 1 - a_n$ solving for a_n. If we subtract 1 from both sides we are left with $4 * 2^n - 1 = 2a_n - a_n$. We the collect the like terms on the left side and are left with $a_n = 4 * 2^n - 1$. Here we have now solved for a_n, good job!.

7 Linear recurrence of order 2.

Now, you might think that this is pretty simple as there cant be much difference between solving an order one and an order 2 equation, however the way we solve them is very different. There are some simple steps to follow to solve these difficult order 2 equations which have many different algebraic skills required.

1. First re-arrange to have all of the terms in the recurrence on the left hand side (which means the right side = 0)
2. Replace a_n with x^n (and then a_{n-1} to $a^{x-1} etc$)
3. Factorise x^{n-2} out of the equation

After doing this, we are then left with a polynomial to solve. Have a look at the next subsection to see how we would go about doing something like this.

7.1 A order 2 linear recurrence problem

1. $b_0 = 1, b_1 = 2$
2. $b_n = -5b_{n-1} - 4b_{n-2}$

First as we declared in our steps, we move all of the terms from the right hand side to the left had so we end up with the equation being equal to 0. This will look like $b_n + 5b_{n-1} + 4b_{n-2} = 0$ if you have done it correctly, we then replace all of the b_n's with x^n's, if you have done this correctly you should be left with $x^n + 5x^{n-1} + 4x^{n-2} = 0$. We then take out the common factor x^{n-2} which will result in the equation $x^{n-2}(x^2 + 5x + 4) = 0$. Now at this stage we are interested in the polynomial inside of the brackets as we can solve this, therefor we disregard the common factor x^{n-2}. After solving this with the quadratic formula, we are left with x being two values, -1 and -2. This means that something multiplied by $(-1)^n$ + something multiplied by $(-4)^n$ should be equal to the b_n we are trying to solve for, we will denote these "somethings" as s_1 and s_2. As you may of noticed this can be solved as a systems of equations, we have the two equations b_0 (which is one) = $s_1 + s_2$ and b_1 (which is 2) = $-s_1 - 4s_2$. After solving this, we are left with $s_1 = 2$ and $s_2 = -1$ which we can sub back into out equation which we determined to be b_n therefor $b_n = 2(-1)^n - 1(-4)^n$. Great job.

8 Reflection

8.1 What is the most important thing I learnt in this module?

In this specific module, I think the most important thing that i learned was how to solve linear recurrence relations and started to develop an understanding of how they relate to real world problems. The recurrence relations and their closed form representation are important regarding some complex problems is CS and Engineering which is why I found this module particularly interesting as I enjoy putting my theoretical knowledge to good use by solving problems (which I can hopefully do with what I learned here). I think another thing that was important when studying this module is where we can see recurrences in the real world and how we can then identify patterns based on them, this could include of disease cases or even numbers of cars in a parking lot (if one car comes the parking lot has +1 cars).

8.2 How do this relate to what I already know?

This related heavily to the algebra I already had a somewhat sound knowledge of from my year 11 math methods three class, this module explored how I could use these algebraic techniques in an abstract way to solve interesting and cool problems. From my SIT 102 programming unit I have done some learning on some recursive functions function which this module allowed me to gain some deeper understanding on how these are actually iterated through the computer.

8.3 Why do I think the course team wants me to learn this content for my degree?

I think the course team wants me to learn this content as sequences are the key to many efficient algorithms, particularly algorithms with feature recursion or problem solving over many consistent, consecutive iterations. I think that this module also developed my mathematical reasoning and how to like mathematical concepts to real world problems, as well as building some key skills which are useful to understand before learning some more complex mathematical concepts. The course team likely wanted me to lean this content as its useful in career paths like Data Science which is very interesting to me.

9 References

Deakin University. (n.d.). Deakin. https://www.deakin.edu.au/