# Proofs Report

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## 1 Requirements

### 1.1 Request for feedback

I found this module to be pretty light compared to other modules such as graph theory. If you could please let me know if I have went in depth enough regarding the different types of proof. However, there doesn't seem like there is much to say, they are all pretty simple. I have also focused on being concise and to the point, shortening my summary greatly.

### 1.2 Response to feedback

I have successfully included an example for direct proofs, indirect proofs, proof by equivalence, proof by contradiction, proof by cases, and prove by counter example improving my summary descriptiveness while highlighting my understanding. I have also completed a follow up question provided by my tutor which can be found at the bottom of the document.

### 1.3 Module Learning Objective

Module learning objectives:

For completing this module, you should be able to:

- analyse proofs
- construct simple proofs for mathematical statements
- select and explain appropriate methods of proofs

## 2 Summary

### 2.1 What is a proof? Why do we use them?

Put simply, a proof is a sequence of logical arguments, starting from a proposition, and the leading to a conclusion with each step in this process being accounted for. Proofs are important as they allow you to effectively convey mathematical concepts and convince others why a proposition is either true or false. Saying this, proofs act as a sort of mathematical "language" where we can express ideas and thoughts.

## 2.2 Direct Proofs

Direct proofs directly prove a proposition, for example, if we had a statement $p \to q$ (which means if $p$ then $q$) we have to do two main steps. Assume that $p$ is true, and then use $p$ to show that $q$ is also true. As discussed before, we are showing the $q$ is the next logical sequence of $p$. Let's take an example to explore this concept, I will use the question "Prove that if $x = 2$ then $x^2 + 3x = 10$". First, we can identify that $p$ is $x = 2$ and the $q$ is $x^2 + 3x = 10$. Now, as highlighted before, we can show that $q$ is the next logical sequence of $p$. To do this, we can substitute $x = 2$ into our equation, when we do this, we are showing that the proposition $p \to q$ holds true. We have $2^2 + 3(2) = 10$ which simplifies to $6 + 4 = 10$. We have successfully proved the proposition $p \to q$

## 2.3 Indirect proofs

The second method of proof that was covered within the module was indirect proof which is also known as proof by contraposition. We use this type of proof because sometimes is easier to prove $\neg p \wedge \neg q$ is true rather than $p \wedge q$ is true. Remember that these two statements are logically equivalent so we are able to do this. Let's take the example: if $x^2+3x = 10$ then $x \neq 3$. We can then identify what $\neg p$ and $\neg q$ would be then solve based upon this. There are many example is in learning evidence and self assessment also attached with this submission. As stated before, we split up our statement into the propositions of $\neg p$ and $\neg q$, we then are saying that "if $x^2 + 3x \neq 10$ (which is our ¬ p), then $x = 3$ (which is our $\neg q$). We then prove this by substituting our $x$ value in and we are left with $3^2 + 3(3) = 10$ which simplifies to $18 = 10$ which is clearly not true. Here we have completed the proof using an indirect method.

## 2.4 Proof of Equivalence

The next proof we covered was the proof of equivalence. Based upon our understanding from propositions, we know that "if and only if" means that $p \to q$ and $q \to p$ at the same time. This can be denoted as $p \iff q$. Based on this, we would then need to prove that both $p \to q$ and $q \to p$ at the same time. We can do this by using different proofs (we don't have to prove both with just a direct proof for example). Proving by Equivalence is means we have to prove two things which results in double the amount of proofs required. For example if we were to prove $n$ is even if and only if $n^2$ is even, we would need to prove $n \to n^2$ AND $n^2 \to n$. Let's take the example "$n$ is odd if and only if $3n^2 - 2n + 5$ is even. First we will prove that $p \to q$ where $p$ is $n$ is odd and $q$ is $3n^2 - 2n + 5$ is even, I will use a direct approach for this. Take $n = 2k + 1$ as it is an odd number, we can then substitute this odd number into our equation which gives us $3(2k+1)^2 - 2(2k+1) + 5$ which should be even based upon our claim. This then simplifies to $3(4k^2 + 4k + 1) - 4k - 2 + 5$. Further simplifying we are left with $12k^2 + 8k + 6$. Since this is divisible by two, we have prove this statement to be true (as when $n$ is odd, $3n^2 - 2n + 5$ is even). Now, we cant stop there as we have only proved that $p \to q$, we also have to prove $q \to p$ so let's do that. First we say that $3n^2 - 2n + 5$ is odd if $n$ is odd. Since we know we can write $n$ as an odd number like $n = 2k + 1$ we then substitute that back into the equation. After simplifying, we are then left with $12k^2 + 8k + 6$ which is also even. Therefor, we have completed the proof as we have proved both $p \to q$ and $q \to p$

## 2.5 Proof by contradiction

To prove a proposition $p$ is true, some of the time proving $\neg p$ is false is easier. In this type of proof, we assume that the conclusion is false and then prove the impossibility. Let's take another example to highlight our understanding, we will work on the question "There is no largest integer". First, as this is a proof by contradiction, we assume that there is a largest integer which will be donated by $N$. If we say that $N$ is the largest integer, there will also be a larger integer which is $N + 1$ and then for this integer, there will be another larger integer $N + 1$ again. Since this is an infinite process, we have proved that there is no largest integer $N$ as there is always a larger integer $N + 1$. We proved this through contradiction as we assumed there was a largest integer, and then proved that this could never be true.

## 2.6 Proof by cases

Sometimes, it is much simpler to break down our proposition into different cases, or different possibilities rather than just trying to prove the whole proposition at once. To do this, we split our proposition into different possibilities and then solve each possibility. For example, if we were given the question "Prove that for any integer $n$, $n^2 + n$ is even" we could break this down into proving that the proposition holds true when $n$ is even, and then proving that it holds for when $n$ is odd which proves the entire proposition, lets complete this question. First we will prove case 1 when $n$ is odd, we know that an odd number can be denoted as $2k+1$ so therefore our $n$ becomes $n = 2k+1$. We can then use a direct proof by substituting this odd number into our $q$. We have $(2k+1)^2 + 2k + 1$ which we assume to be even as the question states. We then expand this algebraic equation out which results in the equation $4k^2 + 4k + 2k + 1 + 1$. We can then further simplify this into $4k^2 + 6k + 2$. As we know that $4k^2 + 6k + 2$ is even as it is divisible by 2, we have proved that when $n$ is odd, $n^2 + n$ even. Now for the second case, we have to prove that when $n$ is even, $n^2 + n$ is also even. We know that a even number can be denoted by $2k$ so we will say that $n = 2k$, we can then substitute this in to see if $n^2 + n$ is still even. After substitution, we have $(2k)^2 + 2k$ which simplifies to $4k^2 + 2k$. We can also see that this also even, completing the proof of this case. Here, we have successfully proved two cases, when $n$ is even and $n$ is odd which still result in $n^2 + n$ being even which successfully completes the proof.

### 2.7 Proof by counter example

Proving by counter example is very similar to how we have been doing this in our predicates module. We don't always need to prove that something holds for every case, we can simply just prove that it doesn't hold for one meaning that it does not hold for EVERY case. Let's take an example where we could do this, the question "$\forall x, 3n^2+n+1$ is even" is a perfect example. If we take that $n = 1$ that means that $3(1)^2+1+1$ should be even according to the question, however after simplification, we are left with $3 + 1 + 1$ which is 5 which is obviously not even. Proofs by counter examples are great because if we can visually see that the equation wont hold under a specific circumstance, we can prove this therefore proving that not every value of $x$ holds.

## 3 Reflection

### 3.1 What is the most important thing I learnt in this module?

The most important thing that I learned within this module was how to create and analyse different types of mathematical proofs. Understanding these core 6 different types of proofs allows me to formally verify my mathematical claims and improve my problem solving skills. Learning about how to use the different types of proofs also allows me to effectively communicate with others through a mathematical "language". I think that this contents learnt within the module also allow me to read proofs created by others to develop a more concrete example of why algorithms work, for example, Euler's formula.

### 3.2 How do this relate to what I already know?

This module build upon my previous understanding of propositional logic and propositional connective, as well as my logical reasoning and algebraic manipulation skills. This module required some background knowledge in simple algebra which built upon my previous understanding. Saying this, this module allowed me to find new ways to unitise algebra, specifically, to prove a proposition. Previously, I would be able to understand how to perform an algorithm, but now i can actually understand the "why" regarding that algorithm actually working (For example Ecluds GCD algorithm)

### 3.3 Why do I think the course team wants me to learn this content for my degree?

I think that the course team wants us to learn the contents of this module as being able to verify the correctness of your calculations are very important.

Alongside this, it also allows you to have the ability to convince others and prove why it works. I think I will need to use this in my degree when I encounter some more advanced programming concepts such as algorithms and data structures which will require proof-based reasoning before implementation. I think the course team also wants us to learning this as it builds a strong foundation for other more advanced mathematical concepts I will encounter in other units down the track.

# 4 FOLLOW UP QUESTION

Show that $n^2 - n - 2$ is even for any element of the Real Numbers.

Firstly, to tackle this question, I will prove that $n^2 - n - 2$ is even for two cases, the first one being where $n$ is even, and the second being $n$ is odd. CASE 1 where $n$ is even: Firstly, we know if $n$ is even, is can be denoted as $2k$ therefor $n = 2k$. We will then use a direct approach for this. Therefore after substitution, we have $(2k)^2 - 2k - 2$ which can be simplified to $4k^2 - 2k - 2$. Since we know $4k^2 - 2k - 2$ is even (as it is divisible by 2) we have effectively proved case 1 where $n$ is even and $n^2 - n - 2$ is also even. CASE 2 where $n$ is odd: As we know that an odd number can be denoted as $2k+1$, we can say that $n = 2k+1$, which I will use a direct approach once again. Therefore after substitution we have $(2k+1)^2 - (2k+1) - 2$ which the questions says should also be even. After expansion, we are left with $4k^2 + 4k + 1 - 2k - 1 - 2$ which after simplification results in $4k^2 + 2k - 2$. As $4k^2 + 2k - 2$ is also even, we have successfully proved that when $n$ is odd, $n^2 - n - 2$ is also even. Therefore as we have proved both cases proving the entire statement to be true!

# 5 References

Deakin University. (n.d.). Deakin. https://www.deakin.edu.au/