Sets and Logic.

Isak Oswald

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1 Introduction to sets.

1.1 What are sets and their key properties

A set is just a collection of objects. The objects within a set are defined as elements. A set is represented by curly braces $\{\}$, and $\in$ represents that the element exists within a set. On the other hand, the $\notin$ represents ab element not being in the set. In this specific case, upper case letters (A) denote sets, whereas elements are denoted by low case letters (a). One thing to take into account with sets is that they are not ordered, and duplicates are not counted. For example the set $A = \{1, 3, 3, 2\}$ is exactly the same as the set $B = \{3, 1, , 2\}$. We can say that these are equivalent ($=$).

1.2 Examples of sets

Sets could include of items in a grocery store which could look like $I = \{nutella, bread, chocolate, etc\}$. Sets don't have to be items, they could also be people, numbers, car brands, shop names, anything you want. For example, a set of all prime numbers less than 10 could be represented as $P = \{2, 3, 5, 7\}$. We can also use something called set builder notation to represent sets of elements, this could look like $A = \{a \in \mathbb{R} : a < 10 is even\}$

1.3 Number sets

As we have just seen, i used something that looked this $\mathbb{R}$. Well what is this? This is something called a number set, there are many different number sets including a range of numbers. $\mathbb{N}$ refers to all natural numbers ($\{1, 2, 3, 4, 5...\}$), $\mathbb{Z}$ is the set of integers ($\{... - 2, -1, 0, 1, 2...\}$), $\mathbb{Q}$ is the set of rational numbers ($\{1/2, 1/4, 1/8, 1/10, 1/11...\}$), $\mathbb{R}$ refers to all of the previous number sets, and includes irrational numbers. Finally, $\mathbb{C}$ represents complex numbers (i).

1.4 Set properties.

Two sets are equal if they contain the same elements (remember order and duplicates are ignored). This is denoted by $A = B$. The cardinality of a set refers to how many elements it has, denoted by $|A|$. The null set is denoted by $\emptyset$ as means there are no elements within the set. Finally, the universal set, denoted by U, is the set that contains all elements under consideration of the problem.

1.5 Sub section and proper subsets

A is a subset of B, if all elements of A are within B, this is denoted as $A \subset B$. Conversely, we can say that B is a superset of a A ($B \supset A$). This may get confusing but we can just look at what the opening of the $\subset$ is pointing to.

2 Applying Set Operations

2.1 Set operations

The union of A and B means that all of the elements in A OR B combined. This is denoted as $A \cup B$. Oppositely, intersection means refers to elements shared by both sets, denoted as $A \cap B$. The complement of A, denoted as $\overline{A}$ refers to all elements in the Universal set that are NOT in the set A. Lastly, the difference is all elements in A, excluding the elements that are also in B. This is denoted as $A \setminus B$

2.2 Relation to propositions

Set operations are awfully similar to the laws of logical equivalence.

$$\vee \text{ corresponds to } \cup$$

$$\wedge \text{ corresponds to } \cap$$

$$\neg \text{ corresponds to } ^c$$

$$T \text{ corresponds to } U$$

$$F \text{ corresponds to } \emptyset$$

As we know we can simplify these propositions with laws, we can also do the same here. Due to being a summary, I'm not going to go into all laws, but just know you can.

2.3 Venn diagrams

Venn diagrams are a great way to visualise different set operators. For example, if we have sets A and B, we can draw two circles A and B and ten visually see the different set operations. For example, $\cap$ is the overlap of A and B, whereas $\cup$ everything in both of the circles. You get the point.

3 Functions and sets.

3.1 What are functions?

Sometimes, we want to assign labels to elements within a set. This can be done by "mapping" labels to elements within a specific set. Deakin University defined a function perfectly in my opinion as "Let X and Ybe sets. A function f from X to Y assigns each element of $X$to exactly one element in Y" (Deakin University, n.d.). A function $f : X \to Y$ reads as the function f is a mapping that transforms elements from the set X to the set Y. This means that the function f maps every value of x to exactly one value of y. We refer to X as the domain, and Y as the co-domain, similarly, we can also say that y is the image of x, and x is the pre-image of y. Let's provide an example to reinforce our understanding here, take set $X = \{a, b, c, d\}$ and set $Y = \{1, 2, 3, 4\}$. A function f maps each element of X to one element of Y, this means that $f(a) = 1, f(b) = 2, f(c) = 3, f(d) = 4$.

3.2 Floor, and ceiling functions

The floor function takes an input of x, and rounds up to the next greatest integer less than or equal to x. Pretty much, we are rounding down. This can be expressed mathematically as $f : \mathbb{R} \to \mathbb{Z}, \quad f(x) = \lfloor x \rfloor$ For example, if $x = 2.4$ the floor function $f(2.4) = \lfloor 2.4 \rfloor = 2$. Similar to this, there is also a ceiling function, denoted as $f : \mathbb{R} \to \mathbb{Z}, \quad f(x) = \lceil x \rceil$. If we take $x = 2.1$, we have $f(2.1) = \lceil 2.1 \rceil = 3$

4 Injective, Surjective, and Bijective

4.1 Injective

An injective function means that every value of X maps uniquely to a value of Y. We can show this in a mathematical way:

$$f : A \to B \text{ is injective if } \forall x_1, x_2 \in A,\ f(x_1) = f(x_2) \implies x_1 = x_2.$$

Let's take the example to make sure that a function is injective. Remember, if we sense injectivity means $x_1 = x_2$, we can set values to the different x's and see if they yield a different result, if they are, it's injective. Remember, we can also do this without setting values for x, I'm just doing this for clarity. Consider this:

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x$$

$$\text{Let } f(1) = f(-1). \text{ Then, } 2(1) = 2(-1) \implies 2 = -2,$$

We can see that $2 = -2$, which is a contradiction. This means that for each

$$\text{Therefore, } f(1) \neq f(-1), \text{ confirming that } f \text{ is injective.}$$

If we know if a function is non injective however, all we need to do its provide a example where $x_1 = x_2$ is true. For example:

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$$

$$\text{Let } x_1 = 1 \text{ and } x_2 = -1.$$

$$f(1) = 1^2 = 1 \quad \text{and} \quad f(-1) = (-1)^2 = 1.$$

$$\text{Since } f(1) = f(-1) \text{ but } 1 \neq -1, \text{ the function is not injective.}$$

4.2 Surjective

Surjective means that every element of Y is mapped to X. This means that $\forall X$, there $\exists Y$ Lets take an example.

$$f(x) = 2x + 3$$

$$\text{Let } y \in \mathbb{R} \text{ be any real number. We want to find } x \in \mathbb{R} \text{ such that } f(x) = y.$$

$$f(x) = 2x + 3$$

$$\text{Set } f(x) = y : \quad 2x + 3 = y$$

$$2x = y - 3$$

$$x = \frac{y-3}{2}$$

$$\text{Thus, for every } y \in \mathbb{R}, \text{ we can find } x = \frac{y-3}{2} \text{ such that } f(x) = y.$$

4.3 Bijective

Bijective is simple, the way i like to remember it is that a bijective function is both injective and surjective at the same time. I don't want to make this summary too long, however, bijectivity just means bijective AND surjective. Like a logical connector.

5 Personal Reflection

5.1 What was the most important thing I learned?

I think the most important thing I learned was Sets, and what they were. Sets are just like an array in programming, however, order and duplicates don't matter. These allow us to group common elements and identify relationships between the two, where i think that the absence of something is the most important. Just take class attendance, for example, it's not as important who came, but more important who did NOT. I think that functions and their properties were also within some of the key things that I learned within this module. I think learning about surjective, injective, and bijective functions will be important going ahead as they allow us to see if something is mapped uniquely or not (for example cars and number plates). Being able to prove the function properties was also important, if a function is is not injective, we can prove with a counterexample, if we want to prove its injective, we can prove algebraically. Similar to surjective we can prove that a function is surjective with an example, and prove it is not with a general statement.

5.2 How does this link to what I already know?

When proving or disproving function properties, I noticed that they are closely linked with quantifiers. For example, injectivity means that for all x there exists some unique y. This expanded my knowledge of quantifiers as I could see them being used in many spaces, not only predicates. I also had some previous knowledge of functions, however, the diagrams in the module really helped me improve my understanding of what functions are doing (mapping between two sets) which I found really interesting. Having completed a few tasks in my AI unit, I can see how this could possibly be linked to AI decision-making?I also knew a little about sets when gaining my Python certification online, this just took thinking about sets outside of the coding world and more into the mathematical realm.

5.3 Why does the course team want us to learn this?

I believe that the course team wants us to learn this content in SIT192 as it allows me to have a foundational understanding of more advanced topics in Computer Science such as graph theory, databases, and machine learning. It is obvious that this module builds mathematical reasoning which increases problem solving skills and the ability to think abstractly about problems. I think this knowledge will be important for building efficient complex programs, as well as being able to come at problems with a more open, abstract way of thinking.

6 References

Deakin University. (n.d.). Deakin. https://www.deakin.edu.au/