NumDiff Project 1

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1 Introduction

In this project we have solved boundary value problems, both in 1-D linear and nonlinear case, for the Black-Scholes (BS) equation. These equations are PDEs that are used in mathematical finance. Black and Scholes is a pricing model used to determine the price for a call or a put option based on variables as volatility, type of option, underlying stock price, time, strike price, and risk-free rate. The Black-Scholes equations are as follows:

$$u_t - \frac{1}{2}\sigma^2 x^2 u_{xx} - rxu_x + cu = 0 \quad \textbf{(Linear 1-D)}$$

$$u_t - \frac{1}{2}\phi(u_{xx})u_{xx} = 0 \quad \textbf{(Nonlinear 1-D)}$$

$$\phi(x) = \sigma_1^2 + \frac{\sigma_2^2 - \sigma_1^2}{2}(1 + \frac{2}{\pi}\arctan x)$$

2 The linear Black-Scholes equation

2.1 Discretization

To solve any PDE numerically we must have numerical scheme, for the purposes of this report we discretize the equation with three different schemes; Forward Euler, Backward Euler and Crank-Nicolson. m and n will denote steps in space and time respectively, with h and k being our stepsizes in space and time, respectively. All spacial discretizations are done using the central difference approximation.

The schemes will work over a 2D-grid, with the grid points being enumerated (x_m, t_n) with $m = 0, 1, ..., M$ and $n = 0, 1, ..., N$. Along the axes $x = 0$, $x = R$ and $t = 0$, where R is the final space boundary, initial and boundary conditions are needed. These conditions will be explained in the next section. First, let us look at he discretizations.

Starting with central difference in space, forward Euler in time:

$$U_m^{n+1} = U_m^n + k(\frac{\sigma^2(mh)^2}{2}\frac{1}{h^2}\delta_x^2 U_m^n + rmh\frac{1}{2h}\delta_x U_m^n - cU_m^n) \tag{1}$$

$$U_m^{n+1} = \frac{km(\sigma^2 m + r)}{2}U_{m+1}^n + (1 - k\sigma^2 m^2 - kc)U_m^n + \frac{km(\sigma^2 m - r)}{2}U_{m-1}^n \tag{2}$$

Now central difference in space, backward Euler in time:

$$\frac{U_m^{n+1} - U_m^n}{k} = \frac{1}{2}\cdot\sigma^2(mh)^2\frac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{h^2} + rmh\cdot\frac{U_{m+1}^{n+1} - U_{m-1}^{n+1}}{2h} - cU_m^{n+1} \tag{3}$$

We rearrange the equation and isolate u_m^n on the left hand side and get:

$$U_m^n = U_{m-1}^{n+1}(\frac{-k(\sigma^2 m^2 - r)}{2}) + U_m^{n+1}(1 + k\sigma^2 m^2 + ck) + U_{m+1}^{n+1}(\frac{k(r - \sigma^2 m^2)}{2}) \tag{4}$$

Central difference in space, Crank-Nicolson in time:

$$\frac{1}{k}\nabla_t U_m^{n+1} = \frac{1}{2}\sigma^2(mh)^2(\frac{1}{2h^2}\delta_x^2 U_m^n + \frac{1}{2h^2}\delta_x^2 U_m^{n+1}) + rmh(\frac{1}{4h}\delta_x U_m^n + \frac{1}{4h}\delta_x U_m^{n+1}) - \frac{c}{2}(U_m^n + U_m^{n+1}) \tag{5}$$

We rearrange the equation and isolate the $u_{m-1,m,m+1}^{n+1}$ terms on the left hand side and the $u_{m-1,m,m+1}^n$ terms on the right hand side, then we get:

$$u_{m-1}^{n+1}\left(\frac{-km(\sigma^2 m - r)}{4}\right) + u_m^{n+1}\left(\frac{2+\sigma^2 km^2 + ck}{2}\right) + u_{m+1}^{n+1}\left(\frac{-km(\sigma^2 m + r)}{4}\right) =$$
$$u_{m-1}^{n}\left(\frac{km(\sigma^2 m - r)}{4}\right) + u_m^{n}\left(\frac{2-\sigma^2 km^2 - ck}{2}\right) + u_{m+1}^{n}\left(\frac{km(\sigma^2 m + r)}{4}\right)$$

From the above we can now isolate 3 coefficients which occur in all of the 3 methods which used to solve the linear PDE.

$$\alpha = k\frac{m^2\sigma^2 + mr}{2} \tag{6}$$

$$\beta = km^2\sigma^2 + kc \tag{7}$$

$$\gamma = k\frac{m^2\sigma^2 - mr}{2} \tag{8}$$

These coefficients arise in all the discretizations used here for Black-Scholes. Rearranging the discretizations as vector matrix problems, the coefficients make 2 needed matrices, one is used in forward Euler, and the other in backward Euler. Both of these matrices are used in the Crank-Nicolson method. Note that all of the coefficients are functions of m, and thus change from row to row in the matrices.

For explicit Euler we get a tridiagonal matrix on the form:

$$\mathrm{A} = \begin{bmatrix} \beta_1 & -\alpha_1 & 0 & \dots & 0 \\ -\gamma_2 & \beta_2 & -\alpha_2 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\alpha_{M-2} \\ 0 & \dots & 0 & -\gamma_{M-1} & \beta_{M-1} \end{bmatrix}$$

And similarly for implicit Euler we get

$$\mathrm{B} = \begin{bmatrix} \beta_1 & -\alpha_1 & 0 & \dots & 0 \\ -\gamma_2 & \beta_2 & -\alpha_2 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\alpha_{M-2} \\ 0 & \dots & 0 & -\gamma_{M-1} & \beta_{M-1} \end{bmatrix}$$

The iteration ideas for the two Euler methods are rather simple, for explicit Euler we do a simple matrix-vector multiplication:

$$\mathbf{u}^{n+1} = (I - A)\mathbf{u}^n + \mathbf{b}^n \tag{9}$$

Where $\mathbf{u}^n$ is a vector of all space coordinates at time n and I is the identity matrix. The vector $\mathbf{b} = [-\gamma_1 g_0^n, 0, \dots, 0, -\alpha_{M-1} g_1^n]^T$ handles the boundary conditions. The implicit method involves solving the linear system;

$$\mathbf{u}^n + \mathbf{b}^{n+1} = (I + B)\mathbf{u}^{n+1} \tag{10}$$

Where $\mathbf{b}$ is this time in g^{n+1} instead of g^n. Finally the Crank-Nicolson method involves both matrices, and we must solve the system:

$$(I - A/2)\mathbf{u}^n - \frac{\mathbf{b}^n}{2} = (I + B/2)\mathbf{u}^{n+1} + \frac{\mathbf{b}^{n+1}}{2} \tag{11}$$

All of the above matrices are of full rank, and thus our linear systems where simple to solve.

2.2 Boundary conditions

To numerically solve a PDE, we need values at the boundaries $x = 0$, $x = R$ and $t = 0$. In this assignment $t = 0$ is handled by the European put, butterfly spread and binary call. However, for $x = 0$ we must reduce the PDE to and ODE and solve a simple equation:

$$u_t + cu = 0 \tag{12}$$

Which solves by separability to

$$u(0,t) = u(0,0) \cdot e^{-ct} \tag{13}$$

Thus we can assign initial values along the t- and x-axes. For the last boundary, $x = R$ we will at first simply set it to 0, as a Dirichlet condtion. So finally we have a discretized grid with boundary values, and can begin solving.

2.3 Monotonicity and CFL-conditions

We know that a scheme is monotone if it can be written as follows:

$$\alpha_m^{n+1} U_m^{n+1} - \sum_{l,k \leq n+1} \sum_{(k,l) \neq (n+1,0)} \beta_{m,l}^k U_{m+l}^k + c_m^n = 0, \quad m = 1, ..., M-1, n = 1, ..., N-1$$

where

$$\alpha_m^n > 0, \quad \beta_m^n \geq 0, \quad \alpha_m^n \geq \sum_{l,m \leq n} \sum_{(k,l) \neq (n,0)} \beta_m^k, \quad c_m^n \in R$$

The Forward Euler scheme is monotone if the following inequalities are satisfied:

$$\begin{gathered}
1 - k\sigma^2 m^2 - kc > 0 \\
\frac{k\sigma^2 m^2}{2} + \frac{krm}{2} > 0 \\
\frac{k\sigma^2 m^2}{2} - \frac{krm}{2} > 0 \\
0 < \sum \beta \leq \alpha = 1
\end{gathered}$$

The second inequality is always true, as we sum two positive quantities. Under the condition that $\sigma^2 > r$, we can see that the third inequality holds as it reduces to $\sigma^2 m > r$, which must hold for all $m = 1, \ldots M-1$. The first inequality is not necessarily true, unless we impose a CFL-condition:

$$1 > k(\sigma^2 m^2 + c) \implies \frac{k}{h^2} < \frac{1}{\sigma^2 x^2} < \frac{1}{\sigma^2 R^2}.$$

If we sum up our β's, we get

$$\sum \beta = 1 - kc < 1 = \alpha.$$

The sum is obviously bigger than 0, as we are summing positive terms, and thus Forward Euler is stable under the stated CFL-condition.

Next we show that Crank-Nicolson is monotone. For it to be monotone the following inequalities must be satisfied:

$$\begin{gathered}
\frac{km(\sigma^2 m - r)}{4} > 0 \\
\frac{km(\sigma^2 m + r)}{4} > 0 \\
\frac{2 - k\sigma^2 m^2 - ck}{2} > 0 \\
0 < \sum \beta \leq \alpha = 1 + \frac{k\sigma^2 m^2}{2} + \frac{kc}{2}
\end{gathered}$$

The two first inequalities are true by the same argumentation as for Forward Euler. For the third inequality to hold, we need a CFL-condition.

$$2 > k(\sigma^2 m^2 + c) \implies \frac{k}{h^2} < \frac{2}{\sigma^2 x^2} < \frac{2}{\sigma^2 R^2}$$

Lastly, we must show that $\sum \beta \leq \alpha$:

$$\sum \beta = 1 + \frac{k\sigma^2 m^2}{2} - \frac{ck}{2} \leq 1 + \frac{k\sigma^2 m^2}{2} - \frac{ck}{2} + ck = 1 + \frac{k\sigma^2 m^2}{2} + \frac{ck}{2} = \alpha$$

Thus, our Crank-Nicolson scheme is monotone as long as the stated CFL-condition is met.

For the Backward Euler scheme the following inequalities must hold:

$$\begin{aligned} 1 &> 0 \\ \frac{k(\sigma^2 m^2 + r)}{2} &> 0 \\ \frac{k(\sigma^2 m^2 - r)}{2} &> 0 \\ 0 < \sum \beta &\leq \alpha = 1 + k\sigma^2 m^2 + ck \end{aligned}$$

The first inequality obviously holds, and the second and third inequalities both hold by the same argumentation as for the previous schemes. It becomes evident that Backward Euler does not need a CFL-condition for it to be monotone. Looking at the sum of our β's, we get:

$$\sum \beta = 1 + k\sigma^2 m^2 \leq 1 + k\sigma^2 m^2 + ck = \alpha.$$

Thus we have shown that our Backward Euler scheme is monotone.

2.4 Stability and consistency

We want to show L^∞ statbility for the Forward Euler case with Dirichlet B.C.'s. Here we use the matrix $C = I - A$ defined in the matrix formulation for Forward Euler. We need to take the infinity-norm of this matrix, which is simply the $\sum \beta$ from subsection 2.3:

$$||C||_\infty = 1 - ck \leq 1 \leq 1 + k\mu, \quad \mu \geq 0$$

The condition $\{||C||_\infty \leq 1 + k\mu, \quad \mu \geq 0\}$ is sufficient for L^∞-stability, and so we are done. This relates to consistency, which a scheme is if the following is true:

$$\tau_m^n \to 0 \quad \text{as} \quad h, k \to 0$$

If a scheme is both stable and consistent, one can conclude it is convergent. We want to show consistency for the Euler Forward scheme. To do this we insert a solution $u(x,t)$ into our numerical scheme, which will yield an error term $k\tau_m^n$:

$$u_m^{n+1} = u_{m+1}^n\left(\frac{k\sigma^2 m^2}{2} + \frac{krm}{2}\right) + u_m^n(1 - k\sigma^2 m^2 - kc) + u_{m-1}^n\left(\frac{k\sigma^2 m^2}{2} - \frac{krm}{2}\right) + k\tau_m^n$$

Then, we Taylor-expand our u_m^n's, where $u_m^n = u(x_m, t_n)$:

$$\begin{aligned} u_m^{n+1} &= u_m^n + k\partial_t u_m^n + \frac{k^2}{2!}\partial_t^2 u_m^n + O(k^3) \\ u_{m\pm 1}^n &= u_m^n \pm h\partial_x u_m^n + \frac{h^2}{2!}\partial_x^2 u_m^n \pm \frac{h^3}{3!}\partial_x^3 u_m^n + O(h^4) \end{aligned}$$

We need not Taylor-expand u_m^n. We will write u as shorthand for u_m^n in our.

$$\begin{aligned} k\tau_m^n &= (u + ku_t + \frac{k^2}{2}u_{tt} + O(k^3)) \\ &- \left(\frac{k\sigma^2 m^2}{2} + \frac{krm}{2}\right)(u + hu_x + \frac{h^2}{2}u_{xx} + \frac{h^3}{6}u_{xxx} + O(h^4)) \\ &- (1 - k\sigma^2 m^2 - kc)u \\ &- \left(\frac{k\sigma^2 m^2}{2} - \frac{krm}{2}\right)(u - hu_x + \frac{h^2}{2}u_{xx} - \frac{h^3}{6}u_{xxx} + O(h^4)) \\ &= (ku_t - kr(mh)u_x - k\frac{1}{2}\sigma^2(mh)^2 u_{xx} + kcu) \\ &+ \frac{k^2}{2}u_{tt} - \frac{kh^3 rm}{6}u_{xxx} + O(h^4 + k^3) \\ \implies \tau_m^n &= \frac{k}{2}u_{tt} - \frac{h^3 rm}{6}u_{xxx} + O(h^4 + k^2) \end{aligned}$$

Which, under the condition that $|u_{tt}|$ and $|u_{xxx}|$ are bounded, yields

$$||\tau||_\infty = \max_{m=0,\ldots,M,n=0,..,N} |\tau_m^n| \to 0 \quad \text{as} \quad h,k \to 0.$$

Thus, the Forward Euler scheme is consistent. Since we have shown L^∞ stability and consistency for the forward Euler scheme, we conclude that the Forward Euler scheme is L^∞-convergent. We got the following bound for τ in L^∞:

$$||\tau||_\infty \leq \frac{k}{2}\max|u_{tt}| + \frac{rRh^2}{6}\max|u_{xxx}|$$

We now need to find a bound for $|e_m^n|$. We insert $e_m^n = u_m^n - U_m^n$ into our scheme, where u and U are the exact and numerical solutions, respectively.

$$\begin{aligned} e_m^{n+1} &= \frac{km(\sigma^2 m + r)}{2}e_{m+1}^n + (1 - k\sigma^2 m^2 - kc)e_m^n + \frac{km(\sigma^2 m - r)}{2}e_{m-1}^n + k\tau_m^n \\ \implies |e_m^{n+1}| &\leq \max_l |e_l^n|(1-kc) + k|\tau_m^n| \end{aligned}$$

We denote $E^n = \max_l |e_l^n|$, and since $E^0 = 0$ we get:

$$\begin{aligned} E^{n+1} &\leq \left(\sum_{j=0}^{n}(1-kc)^j\right) k|\tau_m^n| \\ E^{n+1} &\leq \left(\sum_{j=0}^{n} 1^j\right) k|\tau_m^n| = nk|\tau_m^n| \leq T|\tau_m^n| \\ \implies ||e||_\infty &\leq T||\tau_m^n||_\infty \end{aligned}$$

2.5 Testing the schemes

Testing all the schemes with a European put with strike price, volatility, interest and dividends set to 1, convergence was achieved. Both implicit methods converged no matter the step lengths h and k, but Crank-Nicolson could run into some issues the time step got too large. The Forward Euler method converged as long as the CFL inequality

$$\frac{k}{h^2} < \frac{1}{x_{max}^2\sigma^2} \tag{14}$$

was fulfilled. Typically forward Euler converged for a little while after x_{max} was reached, before diverging extremely quickly. Below you will see such a case, where we demanded convergence until $x = 9$

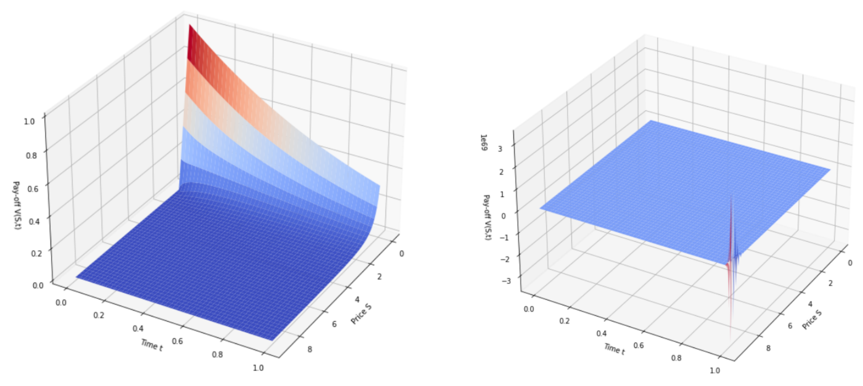

Figure 1: Left: Convergence until $x = 9$. Right: Sudden divergence as $x > 9.5$

Looking at the second figure we note that it managed to reach values on the order of 10^{69}, after breaching the CFL by $\Delta x = 0.5$. Thus it becomes evidently clear that the CFL is vital for explicit methods.

The error analysis proved tenuous, as using the expression for rate of convergence given by linear regression over the logarithms of the error and step sizes, yielded a rate of p which could be in the range $(0.3, 0.36)$ for all of the methods. These values were attained by imposing the solution $u_{ex} = e^{-t}cos(x)$. To do this we had to use the difference operator $L(f) = \frac{\partial f}{\partial t} - \frac{\sigma^2 x^2}{2}\frac{\partial^2 f}{\partial x^2} - rx\frac{\partial f}{\partial x} + cf$ on u_{ex}, and then set the numerical methods equal to $L(u_{ex})$ with boundaries given by $u_{ex}(x,t)$. We could thus find the error between the numerical and analytical solutions. Doing this we got the following logarithmic plots:

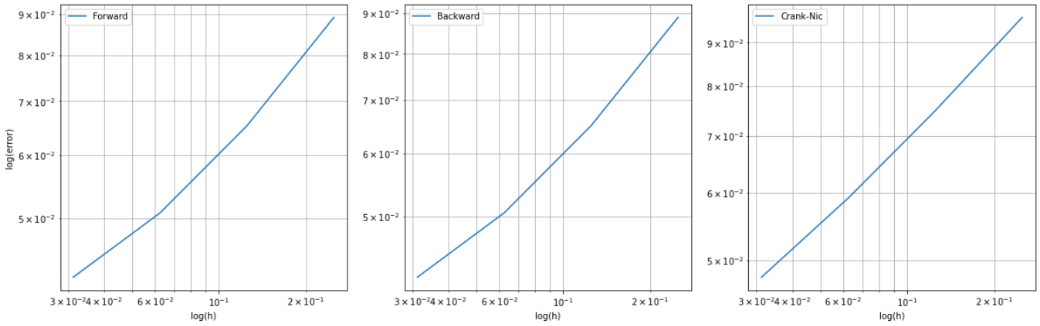

Figure 2: Log-Log plots of the errors against step sizes for all 3 numerical methods

From the above it is safe to say something has gone wrong. It is at this time however, too late to find the source of this error. The error given by differing step sizes in time were also attempted, but this turned out to be a mixed bag of results. Both Euler and Crank-Nicolson turned out to be nonsensical, while backwards Euler was found to have an order of 1. Below is a plot of these findings

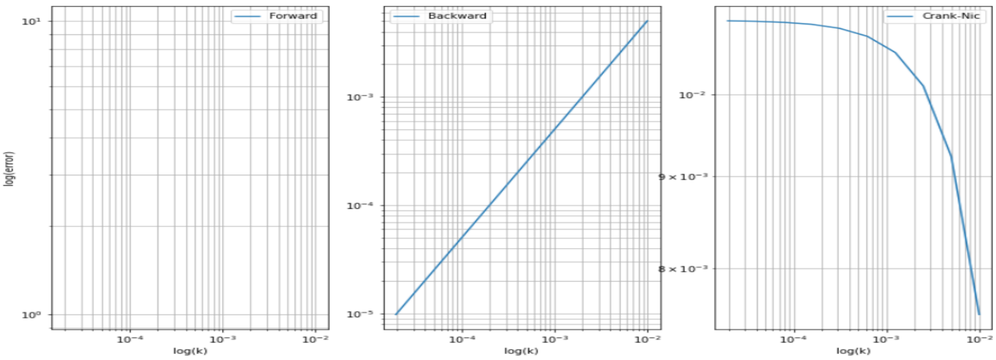

Figure 3: The results of temporal error analysis

Checking the computational time for the differing methods lead to the implicit methods losing out as the number of grid points increased. This is likely the result of needing to solve larger systems of equations, while forward Euler need only do matrix-vector multiplication. One specific example can be seen in the last cell of the Jupyter file, where forward Euler used 1.58 seconds, backwards 4.63 seconds, and Crank-Nicolson 4.75 seconds. Note that these times are subject to the processing power of the used computer, and might be more similar to each other on yours.

3 Nonlinear Black-Scholes

3.1 The IMEX scheme

When discretizing the IMEX scheme and building a matrix iteration we get the iteration matrix given by:

$$\text{A} = \begin{bmatrix} 1 + k\phi_1 & -\frac{k}{2}\phi_1 & & \\ -\frac{k}{2}\phi_2 & 1 + 4k\phi_2 & -\frac{k}{2}\phi_2 & \\ \ddots & \ddots & \ddots & \\ & -\frac{k}{2}\phi_{M-1} & 1 + k(M-1)^2\phi_{M_1} & -\frac{k}{2}\phi_{M_1} \end{bmatrix}$$

Where $\phi_i = \sigma_1^2 + \frac{\sigma_2^2 - \sigma_1^2}{2}(1 + \frac{2}{\pi}\frac{\delta_x^2 U_i^n}{k^2})$, with $\sigma_1 < \sigma_2$. Thus the iterative, implicit scheme is given by

$$\mathbf{u}^n = A\mathbf{u}^{n+1} \tag{15}$$

This system can be solved be inverting the matrix A. Using a butterfly spread with $K = 0$ and $H = 0.5$ we found the solution to be:

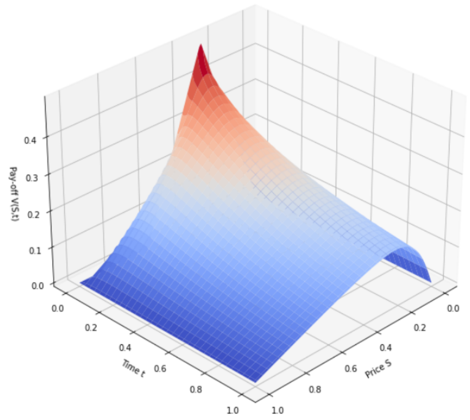

Figure 4: Buttefly spread with Dirichelt boundaries

3.2 Analyzing the nonlinear schemes

We start with showing that the schemes are monotone. The first scheme we are looking that is the IMEX scheme:

$$\frac{1}{k}\nabla_t U_m^{n+1} = \frac{1}{2}x_m^2\phi(\frac{1}{h^2}\delta_x^2 U_m^n)\frac{1}{h^2}\delta_x^2 U_m^{n+1}, \tag{16}$$

where the phi function is defined as:

$$\phi(r) = \sigma_1^2 + \frac{\sigma_2 - \sigma_1}{2}(1 + \frac{2}{\pi}\arctan(r)) \tag{17}$$

We write $\phi(\frac{1}{h^2}\delta_x^2 U_m^n)$ as ϕ. By using this notation and central differences, then we get:

$$U_m^{n+1} - U_m^n = \frac{k}{2}(mh)^h\phi \cdot \frac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{h^2}$$

Now we isolate the U_{m+1}-term in the left hand side and get:

$$(1 + km^2\phi)U_m^{n+1} = \frac{km^2\phi}{2}U_{m+1}^{n+1} + \frac{km^2\phi}{2}U_{m-1}^{n+1} + U_m^n$$

From this expression, we can see that the betas-terms correspond to:

$$\beta_0 = \frac{km^2\phi}{2} > 0.$$

$$\beta_1 = \frac{km^2\phi}{2} > 0.$$

$$\beta_2 = 1 > 0.$$

We have that $\alpha = 1 + km^2\phi > 0$. The sum of the betas is the following:

$$\sum \beta = 1 + km^2 = \alpha \leq \alpha,$$

then we have proved that the IMEX scheme is monotone. We are doing the same procedure for proving monotone for the Backward Euler scheme, but here is the ϕ-function defined as $\phi = \phi(\frac{1}{h^2}\delta_x^2 U_m^{n+1})$. Further we want to compute the truncation error for the schemes. We start with the IMEX scheme: We introduce the numerical solution as:

$$LU = \frac{1}{k}\nabla_t U_m^{n+1} - \frac{1}{2}x^2\phi\left(\frac{1}{h^2}\delta_x^2 U_m^{n+1}\right)\frac{1}{h^2}\delta_x^2 U_m^{n+1}$$

and the analytic solution as:

$$Lu = u_t - \frac{1}{2}x^2\phi(u_{xx})u_{xx}$$

Then we calculate the truncation error as:

$$\tau_m^n = Lu - LU$$

We let $\phi_1 = \phi(U_{xx})$ and $\phi_2 = \phi\left(\frac{1}{h^2}\delta_x^2 U_m^n\right)$. Then we get the truncation error for the IMEX scheme as:

$$\tau_m^n = -\frac{1}{2}(kh)^2\phi_1 \cdot (\delta_x^2 U_m^n - \frac{1}{3}h\delta_x^3 U_m^n + O(h^4)) + \frac{1}{2}k^2\phi_2 \cdot \delta_x^2 U_m^{n+1}.$$

We get the same expression for truncation error with the Backward Euler scheme, but with a different ϕ_2-function. For this scheme we have that $\phi_2 = \left(\frac{1}{h^2}\delta_x^2 U_m^{n+1}\right)$. Further we want to prove L^∞-stability when Dirichlet B.C.'s are used. We write the IMEX scheme as:

$$L_h U_m^n = -U_{m-1}^{n+1}\phi\left(\frac{1}{h^2}\delta_x^2 U_m^{n+1}\right)\frac{km^2}{2} + U_m^{n+1}(km^2 \cdot \phi\left(\frac{1}{h^2}\delta_x^2 U_m^{n+1}\right) + 1) - U_{m+1}^{n+1} \cdot \phi\left(\frac{1}{h^2}\delta_x^2 U_m^{n+1}\right) \cdot \frac{k}{2}.$$

We let the function $\phi_m^n = t$ such that we have that $L\phi_m^n = 1$. Then $W_m^n = V_m^n - ||\vec{f}||_{L^\infty}\phi_m^n$ satisfies:

$$L_h W_m^n = L_h V_m^n - ||\vec{f}||_{L^\infty} \cdot L_h \phi_m^n = f - ||\vec{f}||_{L^\infty} \leq 0.$$

Since we have that W_m^n is monotone, we have from discrete maximum principle that

$$W_m^n \leq \max_{m=0,\dots M} W_m^n \leq 0.$$

Then we can see that:

$$V_m^n \leq \phi_m^n ||\vec{f}||_{L^\infty}$$

We can from that conclude that:

$$max|V_m^n| \leq ||\vec{f}||_{L^\infty} max\phi_m^n.$$

Thus we have proved that the IMEX scheme is L^∞-stable.

4 Commentaries

Much more plotting and testing of our schemes is found in the Jupyter notebook belonging to this report, so it is advised to look at it. Please note that we implemented the methods to work based on step sizes and final coordinates, rather then desired grid points, this might not have been ideal in hindsight, but it was helpful for testing purposes.