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CHIZIQLI ALGEBRA

MTH 1234

MAVZU

CHIZIQLI OPERATORLAR VA ULARNING XOSSALARI

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DOTSENT

OLIY MATEMATIKA KAFEDRASI

CHIZIQLI OPERATOR

- Chiziqli operatorning matritsasi
- Chiziqli operatorlar ustida arifmetik amallar
- Chiziqli operatorlarning o‘tish matritsasi
- Chiziqli operatorning xos son va xos vektori

1. Chiziqli operator nima?

Matritsalar algebrasining asosiy tushunchalaridan biri – chiziqli operator tushunchasidir. Faraz qilaylik, ***L va L₁*** chiziqli fazolar bo‘lsin.

Agar biror $\tilde{A}$ qoida bo‘yicha har bir $x \in L$ elementga $y \in L_1$ element mos qo‘yilgan bo‘lsa, u holda *L* fazoni L_1 fazoga o‘tkazuvchi $\tilde{A}$ **operator (almashtirish, akslantirish)** aniqlangan deyiladi: $y = \tilde{A}(x)$

Agar ixtiyoriy $x, y \in L,\ \lambda \in R$ uchun:

1) $\tilde{A}(x+y) = \tilde{A}(x) + \tilde{A}(y)$ (operatorning additivligi);

2) $\tilde{A}(\lambda x) = \lambda \tilde{A}(x)$ (operatorning bir jinsliligi)

munosabatlar o‘rinli bo‘lsa, bunday operatorga **chiziqli operator** deyiladi.

Operatorning chiziqli operator ekanligini qanday tekshiramiz?

1-misol. $\tilde{A}: R^2 \to R^3$ operator $\tilde{A}(x, y) = (x, y, x + y)$ qoida asosida R^2 fazoni R^3 fazoga akslantirsa, bu operatorning chiziqli operator ekanligini ko‘rsating.

Yechilishi: $\vec{a}_1 = (x_1, y_1)$ va $\vec{a}_2 = (x_2, y_2)$ vektorlar uchun

$$\vec{a}_1 + \vec{a}_2 = (x_1 + x_2, y_1 + y_2)$$

Operatorni ta’sir qildiramiz. 1) Operatorning additivligini tekshiramiz:

$$\tilde{A}(\vec{a}_1 + \vec{a}_2) = \tilde{A}(x_1 + x_2, y_1 + y_2) = (x_1 + x_2, y_1 + y_2, x_1 + x_2 + y_1 + y_2) =$$
$$= (x_1, y_1, x_1 + y_1) + (x_2, y_2, x_2 + y_2) = \tilde{A}(\vec{a}_1) + \tilde{A}(\vec{a}_2).$$

2) Operatorning bir jinsliligini tekshiramiz: $ka_1 = (kx_1, ky_1)$

$$\tilde{A}(k\vec{a}_1) = \tilde{A}(kx_1, ky_1) = (kx_1, ky_1, kx_1 + ky_1) = k(x_1, y_1, x_1 + y_1) = k\tilde{A}(\vec{a}_1).$$

Demak, $\tilde{A}(x, y) = (x, y, x + y)$ operator chiziqli operatordir.

2. Chiziqli operatorning matritsasi

$y=\tilde{A}(x)\in L_1$ element $x\in L$ **elementning aksi**,

$x\in L$ elementning o‘zi esa $y\in L_1$ **elementning asli** deyiladi.

Agar $L=L_1$ bo‘lsa, u holda $\tilde{A}$ operator *L* fazoni o‘zini o‘ziga akslantiruvchi operator bo‘ladi.

1-teorema. Har bir $\tilde{A}:L^n\to L^n$ chiziqli operatorga berilgan bazisda *n*-tartibli matritsa mos keladi va aksincha, har bir *n*- tartibli matritsaga *n* o‘lchovli chiziqli fazoni *n* o‘lchovli chiziqli fazoga akslantiruvchi chiziqli $\tilde{A}$ operator mos keladi.

$A=(a_{ij})\ (i,j=1,2,...,n)$ matritsa $\tilde{A}$ operatorning $\{\vec{e}_1,\vec{e}_2,...,\vec{e}_n\}$ **bazisdagi matritsasi,**
$A=(a_{ij})\ (i,j=1,2,...,n)$ matritsaning rangi $\tilde{A}$ **operatorning rangi** deyiladi.

Chiziqli fazoda vektorning aksini topish

L fazoning barcha vektorlarini nol vektorga akslantiruvchi $\theta(x)=\theta$ operator **nol operator,** $\tilde{E}(x)=x$ tenglikni qanoatlantiruvchi operator **birlik operator** deyiladi.

2-misol. ***R*³** fazoda $\{e_1,e_2,e_3\}$ bazisda chiziqli operator matritsasi $A=\begin{pmatrix}3 & 2 & 4\\ -1 & 5 & 6\\ 1 & 8 & 2\end{pmatrix}$ berilgan bo‘lsin. $x=4\vec{e}_1-3\vec{e}_2+\vec{e}_3$ vektorning aksini toping: $y=\tilde{A}(x)$

Yechilishi:

$$x=\begin{pmatrix}4\\ -3\\ 1\end{pmatrix}$$

$$y=\tilde{A}(x)$$

$$\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}=\begin{pmatrix}3 & 2 & 4\\ -1 & 5 & 6\\ 1 & 8 & 2\end{pmatrix}\begin{pmatrix}4\\ -3\\ 1\end{pmatrix}=\begin{pmatrix}10\\ -13\\ -18\end{pmatrix}$$

$$y=10\vec{e}_1-13\vec{e}_2-18\vec{e}_3$$

Chiziqli operatorning matritsasini topish

3-misol. Berilgan operatorning matritsasini toping: $T:R^3\to R^4;\ T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1+x_2\\x_1-x_2\\x_3\\x_1\end{pmatrix}$

Yechilishi: $A=\begin{bmatrix}T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3)\end{bmatrix}$ matritsaning har bir elementini topamiz:

$$T(\vec{e}_1)=T\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1+0\\1-0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\\0\\1\end{pmatrix},$$

$$T(\vec{e}_2)=T\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0+1\\0-1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\-1\\0\\0\end{pmatrix},$$

$$T(\vec{e}_3)=T\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0+0\\0-0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}.$$

T operatorning matritsasi: $A=\begin{pmatrix}1 & 1 & 0\\1 & -1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{pmatrix}$

3. Chiziqli operatorlar ustida arifmetik amallar

L^n fazoda $\tilde{A}$ va $\tilde{B}$ operatorlar berilgan bo‘lsin.

$(\tilde{A}+\tilde{B})(x)=\tilde{A}(x)+\tilde{B}(x)$ tenglik bilan aniqlanadigan operatorni $\tilde{A}$ va $\tilde{B}$ **operatorlarning yig‘indisi** deyiladi.

2-teorema. Agar $\tilde{A}$ va $\tilde{B}$ operatorlar chiziqli operator bo‘lsa, $\tilde{A}+\tilde{B}$ operator ham chiziqli bo‘ladi.

$\left(\tilde{A}\tilde{B}\right)(x)=\tilde{B}\left(\tilde{A}(x)\right)$ tenglik bilan aniqlanadigan operatorni $\tilde{A}$ va $\tilde{B}$ **operatorlarning ko‘paytmasi** deyiladi.

3-teorema. Agar $\tilde{A}$ va $\tilde{B}$ operatorlar chiziqli operator bo‘lsa, $\tilde{A}\cdot\tilde{B}$ operator ham chiziqli bo‘ladi.

$\left(\alpha\tilde{A}\right)(\vec{x}) = \alpha\left(\tilde{A}(\vec{x})\right)$ tenglik bilan aniqlanadigan operatorni $\tilde{A}$ **operatorning α songa ko‘paytmasi** deyiladi.

4-teorema. Agar $\tilde{A}$ operator chiziqli operator bo‘lsa, $\alpha\tilde{A}$ operator ham chiziqli bo‘ladi.

$\tilde{A}(x)$ operator uchun $\tilde{A}\tilde{A}^{-1} = \tilde{A}^{-1}\tilde{A} = \tilde{E}$ munosabat o‘rinli bo‘lsa, u holda $\tilde{A}^{-1}$ operator $\tilde{A}(x)$ operatorga **teskari operator** deyiladi.

5-teorema. $\tilde{A}(x)$ operatorga teskari operator mavjud bo‘lishi uchun uning har qanday bazisdagi matritsasi xosmas bo‘lishi zarur va yetarlidir.

Matritsasi xosmas bo‘lgan operatorga **xosmas operator** deyiladi.

Ko‘paytma operator va uning matritsasi

4-misol. $\tilde{C}=\tilde{A}\cdot\tilde{B}$ operatorni va uning matritsasini toping:

$$\tilde{A}(x_1,x_2,x_3)=\left(2x_2,\,-2x_1+3x_2+2x_3,\,4x_1-x_2+5x_3\right)$$

$$\tilde{B}(x_1,x_2,x_3)=\left(-3x_1+x_2,\,2x_2+x_3,\,-x_2+3x_3\right)$$

Yechilishi: Dastlab $\tilde{A}$ va $\tilde{B}$ operatorlarning matritsalarini topib olamiz:

$$\tilde{A}(\vec{e}_1)=\begin{pmatrix}0\\-2\\4\end{pmatrix},\ \tilde{A}(\vec{e}_2)=\begin{pmatrix}2\\3\\-1\end{pmatrix},\ \tilde{A}(\vec{e}_3)=\begin{pmatrix}0\\2\\5\end{pmatrix}\qquad \tilde{B}(\vec{e}_1)=\begin{pmatrix}-3\\0\\0\end{pmatrix},\ B(\vec{e}_2)=\begin{pmatrix}1\\2\\-1\end{pmatrix},\ \tilde{B}(\vec{e}_3)=\begin{pmatrix}0\\1\\3\end{pmatrix}.$$

$$A=\begin{pmatrix}0&2&0\\-2&3&2\\4&-1&5\end{pmatrix},\quad B=\begin{pmatrix}-3&1&0\\0&2&1\\0&-1&3\end{pmatrix}.$$

C operator matritsasi:

$$C=AB=\begin{pmatrix}0&4&2\\6&2&9\\-12&-3&14\end{pmatrix}.$$

C operator matritsasi: $C = AB = \begin{pmatrix} 0 & 4 & 2 \\ 6 & 2 & 9 \\ -12 & -3 & 14 \end{pmatrix}.$

$\tilde{C}$ operatorni hosil qilamiz:

$$\tilde{C}(\vec{e}_1) = \begin{pmatrix} 0 \\ 6 \\ -12 \end{pmatrix}, \qquad \tilde{C}(\vec{e}_2) = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}, \qquad \tilde{C}(\vec{e}_3) = \begin{pmatrix} 2 \\ 9 \\ 14 \end{pmatrix}.$$

$$\tilde{C}(\vec{x}) = \left(4x_2 + 2x_3,\ 6x_1 + 2x_2 + 9x_3,\ -12x_1 - 3x_2 + 14x_3\right)$$

5. Chiziqli operatorlarning o‘tish matritsasi

Bitta chiziqli operatorning turli bazislardagi matritsalari orasidagi bog‘lanish haqidagi teorema:

6-teorema. Agar Ã chiziqli operatorning $\left\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\right\}$ va $\left\{\overrightarrow{e_1^*}, \overrightarrow{e_2^*}, \dots, \overrightarrow{e_n^*}\right\}$ bazislardagi matritsalari mos ravishda ***A*** va ***A**** matritsalar bo‘lsa, u holda $A^* = C^{-1}AC$ munosabat o‘rinli bo‘ladi.

Bu yerda ***C*** o‘tish matritsasi deyiladi.

Bir bazisdan boshqa bazisga o‘tganda operatorning matritsasi

5-misol. $\{\vec{e}_1, \vec{e}_2\}$ bazisda chiziqli operator matritsasi $A = \begin{pmatrix} 17 & 6 \\ 6 & 8 \end{pmatrix}$ berilgan bo‘lsin. Yangi $\begin{cases} \vec{e}_1^{\,*} = \vec{e}_1 - 2\vec{e}_2 \\ \vec{e}_2^{\,*} = 2\vec{e}_1 + \vec{e}_2 \end{cases}$ bazisdagi chiziqli operator matritsasini toping: ***A****

Yechilishi: $A^* = C^{-1}AC$

O‘tish matritsasini aniqlaymiz: $C = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

Unga teskari matritsani topamiz: $C^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

Yangi bazisda operatorning matritsasi quyidagi ko‘rinishda bo‘ladi:

$$A^* = C^{-1}AC = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 17 & 6 \\ 6 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 20 \end{pmatrix}.$$

6. Chiziqli operatorning xos son va xos vektori

Agar $\tilde{A}$ chiziqli operator va λ son uchun $\tilde{A}(x)=\lambda x$ tenglik o‘rinli bo‘lsa, u holda λ son $\tilde{A}(x)$ operatorning **xos soni**, unga mos $\vec{x}$ vektor operatorning **xos vektori** deyiladi.

$\tilde{A}(x)=\lambda x$ tenglikni operatorning matritsasidan foydalanib, yoyib yozamiz va tenglamalar sistemasini hosil qilamiz:

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=\lambda\cdot x_1 \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=\lambda\cdot x_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n=\lambda\cdot x_n \end{array}\right\} \Rightarrow \quad \left.\begin{array}{l} (a_{11}-\lambda)x_1+a_{12}x_2+\cdots+a_{1n}x_n=0 \\ a_{21}x_1+(a_{22}-\lambda)x_2+\cdots+a_{2n}x_n=0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{n1}x_1+a_{n2}x_2+\cdots+(a_{nn}-\lambda)x_n=0 \end{array}\right\}$$

Bundan $\left[A-\lambda E\right]\cdot X=0$

Bir jinsli tenglamalar sistemasining notrivial yechimlari mavjud bo‘lish sharti

Bilamizki, bir jinsli chiziqli tenglamalar sistemasi har doim trivial yechimga ega. **Bir jinsli chiziqli tenglamalar sistemasi notrivial yechimga ega bo‘lishi uchun** uning koeffitsiyentlaridan tuzilgan determinantning qiymati nolga teng bo‘lishi zarur va yetarli:

$$|A-\lambda E| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0$$

Ushbu determinant λ songa nisbatan n-darajali ko‘phaddir. Bu ko‘phad $\tilde{A}(x)$ **operatorning xarakteristik ko‘phadi** deyiladi.

Chiziqli operatorning xarakteristik ko‘phadi bazisni tanlashga bog‘liq emas.

Operatorning xos son va xos vektori qanday topiladi?

6-misol. $\tilde{A}(\vec{x}) = \left(2x_1 - x_2 + 2x_3,\ 5x_1 - 3x_2 + 3x_3,\ -x_1 - 2x_3\right)$ operatorning xos son va xos vektorini toping.

Yechilishi: Dastlab $\tilde{A}$ operatorning matritsasini tuzib olamiz: $A = \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$

Berilgan operatorga mos keluvchi bir jinsli tenglamalar sistemasi:
$$\begin{cases} (2-\lambda)x_1 - x_2 + 2x_3 = 0 \\ 5x_1 - (3+\lambda)x_2 + 3x_3 = 0 \\ -x_1 - (2+\lambda)x_3 = 0. \end{cases}$$

$$p(\lambda) \equiv \begin{vmatrix} 2-\lambda & -1 & 2 \\ 5 & -3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} = -(\lambda+1)^3.$$

Operatorning xos soni: $\lambda = -1$

$$\begin{cases} 3x_1 - x_2 + 2x_3 = 0, \\ 5x_1 - 2x_2 + 3x_3 = 0, \\ -x_1 - x_3 = 0. \end{cases} \qquad x_1 = x_2,\ x_1 = -x_3$$

Operatorning xos vektori:

$$X = \alpha \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Operatorning xos son va xos vektorlari qayerda ishlatiladi?

1) Chiziqli tenglamalar sistemasini notrivial yechimlarini topishda;
2) Kvadratik formalarni kanonik ko‘rinishga keltirishda ishlatiladi (14-mavzu);
3) Differensial tenglamalar sistemasining notrivial yechimlarini topishda;
4) va boshqa …

O‘z-o‘zini tekshirish uchun savollar:

1. L^n fazoda bir bazisdan ikkinchi bazisga o‘tish matritsasi qanday tuziladi?
2. Chiziqli fazoning chiziqli almashtirishi yoki operatori deb nimaga aytiladi?
3. Chiziqli operator ustida bajariladigan qanday amallarni bilasiz?
4. Chiziqli operatorning xos vektori va xos soni deb nimaga aytiladi?
5. Xos vektorlarning qanday xossalarini bilasiz?
6. Matritsaning xarakteristik ko‘phadi bazis tanlanishiga bog‘liqmi?

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E’TIBORINGIZ UCHUN RAHMAT!

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