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MTH 1234

CHIZIQLI ALGEBRA

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OLIY MATEMATIKA KAFEDRASI

CHIZIQLI FAZO. EVKLID FAZOSI

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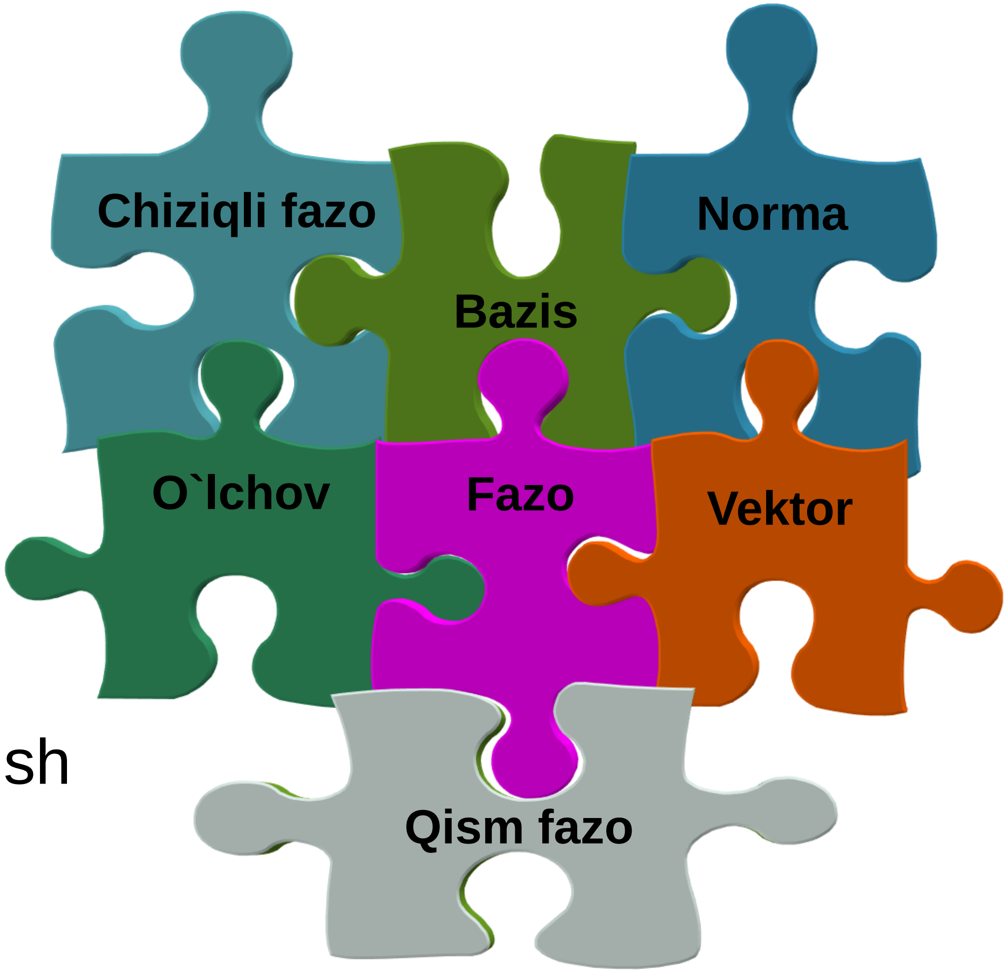

1. Chiziqli fazo

R^n vektor fazoda vektorlarni qo‘shish va skalyar songa ko‘paytirish amallari aniqlangan bo‘lsa, bu amallar 8 ta xossani qanoatlantirsa, vektor fazo **chiziqli fazo** deyiladi.

Vektorlarni qo‘shish va skalyar songa ko‘paytirish amallari 8 ta xossani qanoatlantirishi kerak.

Buning ma’nosi shuki – natijaviy vektor yana shu fazoga tegishli bo‘lishi kerak.

1. $X+Y=Y+X;$
2. $X+(Y+Z)=(X+Y)+Z;$
3. $X+\Theta=X,$ bunda $\Theta=(0,0,...,0)^T;$
4. $X+(-X)=\Theta;$
5. $1\cdot X=X;$
6. $(\alpha+\beta)X=\alpha X+\beta X;$
7. $\alpha(X+Y)=\alpha X+\alpha Y;$
8. $\alpha(\beta X)=(\alpha\beta)X.$

Bu yerda α va β - ixtiyoriy sonlar;

X, Y va Z lar n o'lchovli vektorlar.

Chiziqli fazoni aniqlovchi aksiomalar (xossalar)

Agar chiziqli fazodagi vektorlar uchun faqat haqiqiy songa ko‘paytirish amali aniqlangan bo‘lsa, bu fazoga **haqiqiy chiziqli fazo** deyiladi.

Agar chiziqli fazodagi vektorlar uchun kompleks songa ko‘paytirish amali aniqlangan bo‘lsa, bu fazoga **kompleks chiziqli fazo** deyiladi.

1-xossa. Har qanday chiziqli fazo uchun yagona nol vektor mavjud: $\Theta = (0,0,...,0)^T;$

2-xossa. Har qanday chiziqli fazoda har bir vektorga qarama-qarshi vektor mavjud: $X + (-X) = \Theta;$

3-xossa. Har qanday chiziqli fazoda har bir vektorni nol soniga ko‘paytirganda nol vektor hosil bo‘ladi: $0 \cdot X = \Theta;$

4-xossa. Har qanday haqiqiy sonni nol vektorga ko‘paytirilsa, nol vektor hosil bo‘ladi: $\lambda \cdot \Theta = \Theta;$

5-xossa. Har qanday chiziqli fazoda $\lambda \cdot X = \Theta; \quad \Rightarrow \lambda = 0 \quad \text{yoki} \quad X = \Theta$

Matritsalar to‘plami chiziqli fazo tashkil qiladimi?

1-misol. *M* – barcha 2x2 o‘lchamli matritsalar to‘plami chiziqli fazo tashkil qiladimi?

Yechilishi:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \text{ va } \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$1)\quad \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$$

$$2)\quad \lambda \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} \lambda a_1 & \lambda b_1 \\ \lambda c_1 & \lambda d_1 \end{bmatrix}$$

Matritsalarni qo‘shish va songa ko‘paytirish amallari aniqlangan.
8 ta xossani qanoatlantiradi.
M – barcha 2x2 o‘lchamli matritsalar chiziqli fazo tashkil qiladi.

2. Chiziqli fazoning bazisi va o‘lchamini topish

2- misol. Barcha 2-tartibli matritsalar chiziqli fazosining bazisi va o‘lchamini toping:

$$M^2 = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : \; a_{11}, a_{12}, a_{21}, a_{22} \in R \right\}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Yechilishi: Bu fazoning bazisi sifatida e_1, e_2, e_3, e_4 matritsalar sistemasini olish mumkin: Ixtiyoriy 2-tartibli matritsani bu matritsalarning chiziqli kombinatsiyasi orqali yozish mumkin:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}e_1 + a_{12}e_2 + a_{21}e_3 + a_{22}e_4$$

Bazis matritsalar sistemasining chiziqli erkliligini ko‘rsatamiz.

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Bu tenglik faqat va faqat $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ bo‘lganda o‘rinli. Demak, e_1, e_2, e_3, e_4 matritsalar sistemasi M^2 fazoning bazisi ekani kelib chiqadi. Fazo 4 o‘lchovli.

Chiziqli fazoning bazisi

3- misol. $P_1(x)=x^3+x^2-1,\quad P_2(x)=x^2-2x,\ P_3(x)=x^3+x,\quad P_4(x)=x^2-3$

ko‘phadlar chiziqli fazoda bazis tashkil etadimi?

Yechilishi: Ushbu ko ‘phadlar fazosida standart bazis – bu: $\{x^3,x^2,x,1\}$

$P_1,\ P_2,\ P_3,\ P_4$ ko‘phadlar bazis hosil qiladimi? Buning uchun ular chiziqli erkli bo‘lishi kerak. Quyidagi chiziqli kombinatsiyani quramiz:

$$c_1P_1(x)+c_2P_2(x)+c_3P_3(x)+c_4P_4(x)=0$$

$$c_1(x^3+x^2-1)+c_2(x^2-2x)+c_3(x^3+x)+c_4(x^2-3)=0$$

$$x^3(c_1+c_3)+x^2(c_1+c_2+c_4)+x(-2c_2+c_3)+(-c_1-3c_4)=0$$

$$\begin{cases} c_1+c_3=0 \\ c_1+c_2+c_4=0 \\ -2c_2+c_3=0 \\ -c_1-3c_4=0 \end{cases} \Rightarrow \begin{cases} c_1=-c_3 \\ c_1+c_2+c_4=0 \\ c_3=2c_2 \\ c_1=-3c_4 \end{cases}$$

$$\Rightarrow\ c_1=c_3=c_2=c_4=0$$

Demak, berilgan ko‘phadlar **chiziqli erkli** va chiziqli fazoda **bazis hosil qiladi**.

3. Chiziqli fazo elementini bazis elementlari bo‘yicha yoyish

4- misol. $P(t)=5-2(t+1)+3(t+1)^2+(t+1)^3$ ko‘phadning quyidagi bazislarga nisbatan koordinatalarini toping:

a) $\begin{cases} e_1=1 \\ e_2=t \\ e_3=t^2 \\ e_4=t^3 \end{cases}$ b) $\begin{cases} e_1=1 \\ e_2=t+1 \\ e_3=(t+1)^2 \\ e_4=(t+1)^3 \end{cases}$

Yechilishi: $P(t)=t_1e_1+t_2e_2+t_3e_3+t_4e_4$

a) $e_1=1,\ e_2=t,\ e_3=t^2,\ e_4=t^3$ bazisga ko‘ra,

$$P(t)=5-2(t+1)+3(t+1)^2+(t+1)^3$$

$$P(t)=5-2t-2+3t^2+6t+3+t^3+3t^2+3t+1=7+7t+6t^2+t^3$$

$$P(t)=\{7;\ 7;\ 6;\ 1\}$$

b) $e_1=1, e_2=t+1, e_3=(t+1)^2,\ e_4=(t+1)^3$ bazisga ko‘ra,

$$P(t+1)=5-2(t+1)+3(t+1)^2+(t+1)^3$$

$$P(t)=\{5;\ -2;\ 3;\ 1\}$$

4. Chiziqli fazoning qism fazolari

L chiziqli fazoning *V* qism to‘plami ham *L* da aniqlangan elementlarni qo‘shish va elementlarni songa ko‘paytirish amallariga nisbatan chiziqli fazo tashkil qilsa, u holda *V* fazo *L* fazoning **chiziqli qism fazosi** deyiladi.

5-misol. $V=\{(x_1,x_2,x_3,x_4): x_1-2x_3=0\}$ to‘plam R^4 fazoning qism fazosi bo‘ladimi?

Yechilishi: $x_1-2x_3=0$ yoki $x_1=2x_3$ bo'lsa, u holda $V=\{2x_3,x_2,x_3,x_4\}$

To‘plamning ixtiyoriy 2 ta elementini olamiz va ular orasida qo‘shish va skalyar songa ko‘paytirish amallari o‘rinlimi? Tekshirib ko‘ramiz:

$$x=\left(2x_3,x_2,x_3,x_4\right),\quad y=\left(2y_3,y_2,y_3,y_4\right).$$

$$x+y=\left(2\left(x_3+y_3\right),x_2+y_2,x_3+y_3,x_4+y_4\right)\in R^4,$$

$$\lambda x=(2\lambda x_3,\lambda x_2,\lambda x_3,\lambda x_4)\in R^4$$

Demak, V to‘plam R^4 fazoning qism fazosi bo‘ladi.

5. Evklid fazosi - E^n

Agar chiziqli fazo elementlari orasida skalyar ko‘paytma aniqlangan bo‘lsa, bu fazoga **Evklid fazosi** deyiladi.

Har qanday R^n fazoda skalyar ko‘paytmani aniqlash bilan uni Evklid fazosiga aylantirish mumkin.

Agar $\vec{x}, \vec{y} \in E^n$ elementlar uchun $(\vec{x}, \vec{y}) = 0$ bo‘lsa, bu $\vec{x}$ *va* $\vec{y}$ vektorlar **ortogonal vektorlar** deyiladi.

Noldan farqli $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n \in E^n$ elementlardan tashkil topgan vektorlar sistemasidagi vektorlar jift - juft ortogonal bo‘lsa, bu sistema **ortogonal vektorlar sistemasi** deyiladi.

Agar $\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\} \subset E^n$ ortogonal vektorlar sistemasi bo‘lib, $|\vec{a}_i| = 1 \ (i = 1, 2, ..., n)$ bo‘lsa, bu vektorlar sistemasiga **ortonormal vektorlar sistemasi** deyiladi.

Evklid fazosining bazisi

Agar $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, ..., \vec{e}_n\} \subset E^n$ vektorlar sistemasi E^n fazoning bazisi bo‘lib, ortonormal vektorlar sistemasini tashkil qilsa, ularga **ortonormal bazis** deyiladi va quyidagi munosabat o‘rinli bo’ladi:

$$\left(\vec{e}_i, \vec{e}_k\right) = \begin{cases} 1, & agar\ i = k \quad bo'lsa \\ 0, & agar\ i \neq k \quad bo'lsa \end{cases}$$

Teorema. Har qanday *n* o‘lchovli haqiqiy Evklid fazosida ortonormallangan bazis mavjud.

Agar bazis ortonormallanmagan bo‘lsa, uni **Gramm Shmidt formulalari** bilan ortonormallash mumkin.

$$1)\ \vec{e}' = \vec{e}_1$$

$$2)\ \vec{e}'_t = \vec{e}_t - \sum_{i=1}^{t-1} \frac{\left(\vec{e}'_i \cdot \vec{e}_t\right)}{\left(\vec{e}'_i \cdot \vec{e}'_i\right)} \vec{e}'_i, \quad t = 2, 3, ..., k$$

6. Evklid fazosida ortonormallangan bazis qurish

6-misol. $\vec{a}_1=(2,1,3,-1),\ \vec{a}_2=(7,4,3,-3),\ \ \vec{a}_3=(1,1,-6,0)\ ,\ \ \vec{a}_4=(5,7,7,8)$ vektorlarning ortonormallangan bazis vektorlarini toping.

Yechilishi: 1) Bu vektorlar bazis tashkil qiladimi?

Agar vektorlar bazis tashkil qilsa, ulardan tuzilgan determinant nolga teng bo‘ladi.

$$\begin{pmatrix} 2 & 1 & 3 & -1 \\ 7 & 4 & 3 & -3 \\ 1 & 1 & -6 & 0 \\ 5 & 7 & 7 & 8 \end{pmatrix} \begin{matrix} -3\cdot I+II,8\cdot I+IV \\ \\ \\ \\ \end{matrix} \Rightarrow \begin{pmatrix} 2 & 1 & 3 & -1 \\ 1 & 1 & -6 & 0 \\ 1 & 1 & -6 & 0 \\ 21 & 15 & 31 & 0 \end{pmatrix} -1\cdot II+III,-21\cdot II+IV \Rightarrow \begin{pmatrix} 2 & 1 & 3 & -1 \\ 1 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -6 & 157 & 0 \end{pmatrix}$$

Yuqoridagi tasdiq o‘rinli, demak $\vec{a}_1,\vec{a}_2,\vec{a}_4$ vektorlar bazis tashkil qilar ekan.

$$\vec{a}_1=(2,1,3,-1),\quad \vec{a}_2=(7,4,3,-3),\quad \vec{a}_4=(5,7,7,8)$$

2) Bu vektorlarni G.Shmidt formulalaridan foydalanib, ortonormallashtiramiz.

$$1)\ \vec{\boldsymbol{e}}'=\vec{\boldsymbol{e}}_1 \qquad 2)\vec{\boldsymbol{e}}'_t=\vec{\boldsymbol{e}}_t-\sum_{i=1}^{t-1}\frac{\left(\vec{\boldsymbol{e}}'_i\cdot\vec{\boldsymbol{e}}_t\right)}{\left(\vec{\boldsymbol{e}}'_i\cdot\vec{\boldsymbol{e}}'_i\right)}\vec{\boldsymbol{e}}'_i,\quad t=2,3,\ldots,k$$

$$\vec{b}_1=\vec{a}_1=(2,1,3,-1)$$

$$\vec{b}_2=\vec{a}_2-\frac{\vec{b}_1\cdot\vec{a}_2}{\vec{b}_1\cdot\vec{b}_1}\vec{b}_1=\vec{a}_2-\frac{\vec{a}_1\cdot\vec{a}_2}{\vec{a}_1\cdot\vec{a}_1}\vec{a}_1=\vec{a}_2-\frac{2\cdot7+1\cdot4+3\cdot3+1\cdot3}{2^2+1^2+3^2+(-1)^2}\vec{a}_1=(7,4,3,-3)-\frac{30}{15}(2,1,3,-1)=(3,2,-3,-1)$$

$$\vec{b}_3=\vec{a}_4-\left(\frac{\vec{b}_1\cdot\vec{a}_4}{\vec{b}_1\cdot\vec{b}_1}\vec{b}_1+\frac{\vec{b}_2\cdot\vec{a}_4}{\vec{b}_2\cdot\vec{b}_2}\vec{b}_2\right)=\vec{a}_4-\left(2\vec{b}_1+\frac{0}{23}\vec{b}_2\right)=(1,5,1,10)$$

$$\vec{b}_1=(2,1,3,-1),\quad \vec{b}_2=(3,2,-3,-1),\quad \vec{b}_3=(1,5,1,10)$$

Hosil bo‘lgan vektorlar ortogonalmi? Tekshirib ko‘ramiz:

$$\vec{b}_1\cdot\vec{b}_2=0,$$
$$\vec{b}_1\cdot\vec{b}_3=0,$$
$$\vec{b}_3\cdot\vec{b}_2=0,$$

$$\vec{b}_1=(2,1,3,-1),\quad \vec{b}_2=(3,2,-3,-1),\quad \vec{b}_3=(1,5,1,10)$$

3) Bu ortogonal vektorlarni normallashtiramiz:

$$\vec{c}_1=\frac{\vec{b}_1}{\left|\vec{b}_1\right|}=\frac{(2,1,3,-1)}{\sqrt{4+1+9+1}}=\left(\frac{2}{\sqrt{15}},\frac{1}{\sqrt{15}},\frac{3}{\sqrt{15}},-\frac{1}{\sqrt{15}}\right),$$

$$\vec{c}_2=\frac{\vec{b}_2}{\left|\vec{b}_2\right|}=\frac{(3,2,-3,-1)}{\sqrt{9+4+9+1}}=\left(\frac{3}{\sqrt{23}},\frac{2}{\sqrt{23}},-\frac{3}{\sqrt{23}},-\frac{1}{\sqrt{23}}\right),$$

$$\vec{c}_3=\frac{\vec{b}_3}{\left|\vec{b}_3\right|}=\frac{(1,5,1,10)}{\sqrt{1+25+1+100}}=\left(\frac{1}{\sqrt{127}},\frac{5}{\sqrt{127}},\frac{1}{\sqrt{127}},\frac{10}{\sqrt{127}}\right),$$

$\vec{c}_1,\ \vec{c}_2,\ \vec{c}_3$ vektorlar ortonormallangan bazis vektorlar hisoblanadi.

Bir bazisdan boshqa bazisga o‘tish

7-misol. $\vec{x}(2;-1;\,5)$ vektor $\vec{e_1},\vec{e_2},\vec{e_3}$ bazisda berilgan. Uning quyidagi bazisdagi koordinatalarini toping:

$$\begin{cases} \vec{e}_1' = \vec{e}_1 \\ \vec{e}_2' = \vec{e}_1 + \vec{e}_2 \\ \vec{e}_3' = \vec{e}_1 + \vec{e}_3 \end{cases}$$

Yechilishi: $\vec{x} = 2\vec{e_1} - \vec{e_2} + 5\vec{e_3}$

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\boxed{\vec{x} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}}$$

$$\vec{e}_1' = \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\quad \vec{e}_2' = \vec{e}_1 + \vec{e}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\quad \vec{e}_3' = \vec{e}_1 + \vec{e}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

O‘tish matritsasini topishimiz kerak:

$$P \cdot \vec{x}' = \vec{x}$$

$$\vec{x}' = P^{-1} \cdot \vec{x}$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1}\vec{x} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix}$$

$$\boxed{\vec{x}' = \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix}}$$

O‘z-o‘zini tekshirish uchun savollar:

1. Chiziqli fazo deb nimaga aytiladi?
2. Chiziqli fazoning qism fazosi deb nimaga aytiladi? Misollar keltiring.
3. Chiziqli fazoda elementlarning chiziqli kombinatsiyasi nima?
4. *n*-o‘lchovli chiziqli fazo deb qanday chiziqli fazoga aytiladi?
5. Chiziqli fazo o‘lchovi deb nimaga aytiladi?
6. *n*-o‘lchovli chiziqli fazo bazisi deb nimaga aytiladi?
7. Har qanday vektorni fazoning bazisi orqali yoyish mumkinmi?
8. Evklid fazosi deb nimaga aytiladi?
9. Vektorning normasi nima?
10. Ortogonal vektorlar sistemasi haqida nima bilasiz?
11. Ortonormal bazis nima?

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