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TOSHKENT AXBOROT TEXNOLOGIYALARI UNIVERSITETI

CHIZIQLI ALGEBRA

MTH 1234

MAVZU

MATRITSA RANGI. MATRITSA RANGINI HISOBLASH USULLARI

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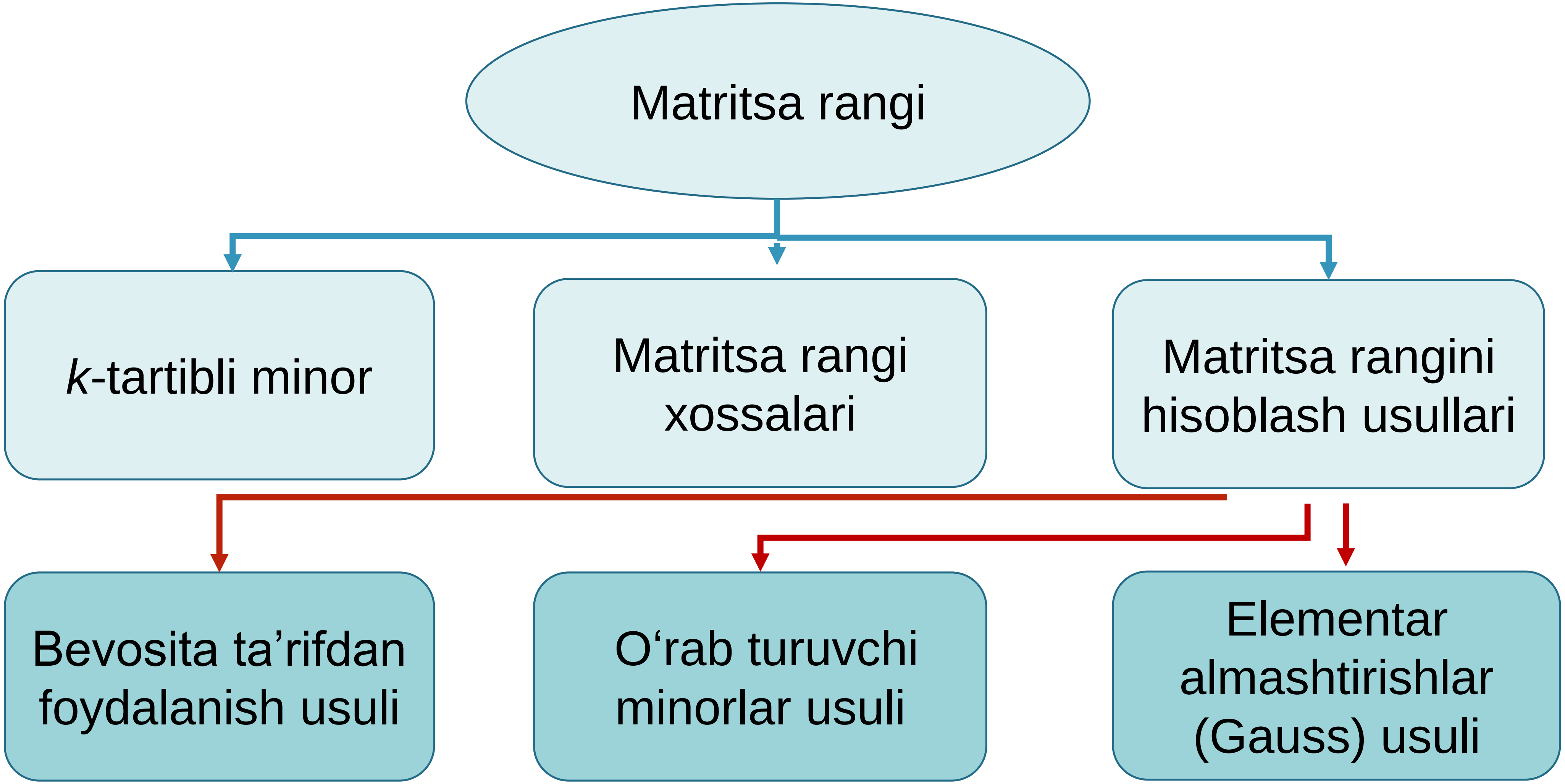
OLIY MATEMATIKA KAFEDRASI

Matritsa rangi va uni hisoblash usullari mavzusida quyidagilarni bilib olasiz:



1. *k*- tartibli minor

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1i} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2i} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mi} & \dots & a_{mn} \end{bmatrix}$$

Ixtiyoriy o‘lchamli matritsaning k ta satr va k ta ustunlari kesishmalarida joylashgan elementlaridan k -tartibli kvadrat matritsa tuzib olamiz.

Bu k -tartibli kvadrat matritsa determinantiga berilgan matritsaning k **-tartibli minori** deyiladi.

Eslab qoling!

Berilgan matritsa **kvadrat shaklda bo‘lsa**, uning eng katta tartibli minori o‘ziga teng.

Berilgan matritsa $n \times m$ o‘lchamli bo‘lsa, uning eng katta tartibli minori $k=\min(n,m)$ bo‘ladi.

k-tartibli minorlar sonini topish

$n \times m$ o‘lchamli matritsa berilgan bo‘lsin. Bu matritsadan ajratish mumkin bo‘lgan **k-tartibli minorlar soni:**

$$C_n^k \times C_m^k = \frac{n!}{k!(n-k)!} \cdot \frac{m!}{k!(m-k)!}$$

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$$

$$0! = 1$$

1-misol. $A = \begin{pmatrix} 4 & 5 & 7 & 7 \\ 2 & 1 & 4 & 3 \\ 3 & 7 & 0 & 8 \end{pmatrix}$ matritsaning nechta minori bor?

$$C_3^1 \times C_4^1 = \frac{3!}{1! \cdot 2!} \cdot \frac{4!}{1! \cdot 3!} = 3 \cdot 4 = 12$$

$$C_3^2 \times C_4^2 = \frac{3!}{2!(3-2)!} \cdot \frac{4!}{2!(4-2)!} = \frac{2! \cdot 3}{2! \cdot 1!} \cdot \frac{2! \cdot 3 \cdot 4}{2! \cdot 1 \cdot 2} = 3 \cdot 3 \cdot 2 = 18$$

$$C_3^3 \times C_4^3 = \frac{3!}{3!(3-3)!} \cdot \frac{4!}{3!(4-3)!} = \frac{1}{0!} \cdot \frac{3! \cdot 4}{3! \cdot 1!} = 4$$

1-tartibli minorlari 12 ta;

2-tartibli minorlari 18 ta;

3-tartibli minorlari 4 ta.

Jami 34 ta minor.

2. Matritsa rangi va uni hisoblash usullari

Matritsaning rangi deb, uning noldan farqli minorlarining eng yuqori tartibiga aytiladi va $\boldsymbol{rank\, A = r(A)}$ kabi belgilanadi.

$A_{n\times m}$ matritsa rangini topishning 3 xil usuli mavjud:

1. **Bevosita ta’rifdan foydalanish usuli**;
2. **O‘rab turuvchi minorlar usuli**;
3. **Elementar almashtirishlar usuli.**

3. Matritsa rangi xossalari

1⁰. Agar *A* matritsa $n \times m$ o‘lchovli bo‘lsa, u holda $rankA \leq \min(n, m)$ bo‘ladi.

2⁰. Agar *A* *n*- tartibli kvadrat matritsa va $|A| \neq 0$ bo‘lsa, u holda $rankA = n$.

3⁰. Agar *A* nol matritsa bo‘lsa, u holda $rankA = 0$;

$$\Theta = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \Rightarrow \quad rank\Theta = 0$$

4. Matritsa rangini bevosita ta’rifdan foydalanib topish

2-misol. $A=\begin{pmatrix} 1 & -2 \\ -2 & 4 \\ 3 & -7 \end{pmatrix}$ matritsa rangini aniqlang.

Yechilishi:

Bu matritsa 3×2 o‘lchamli bo‘lgani uchun $rank\ A \leq \min(3,\ 2) = 2.$

Matritsadan 2-tartibli minorlar ajratamiz va ularning qiymatini hisoblaymiz. Bu jarayonni noldan farqli 2-tartibli minor topilguncha davom ettiramiz: $M_1=\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix}=0,\quad M_2=\begin{vmatrix} 1 & -2 \\ 3 & -7 \end{vmatrix}=-1\neq 0.$ $M_3=\begin{vmatrix} -2 & 4 \\ 3 & -7 \end{vmatrix}$

$rankA=2$

5. Matritsa rangini o‘rab turuvchi minorlar usulida topish

k**-** tartibli minorni o‘z ichiga oluvchi barcha k+1 - tartibli minorlar **o‘rab turuvchi minorlar** deyiladi.

1-teorema. Agar $n \times m$ o‘lchovli matritsaning biror k- tartibli minorini o‘rab turuvchi k+1 - tartibli minorlar nolga teng bo‘lsa, u holda bu matritsadagi barcha k+1 - tartibli minorlar nolga teng bo‘ladi.

Natija: Agar $n \times m$ o‘lchovli matritsaning biror k- tartibli minorini o‘rab turuvchi k+1 - tartibli minorlar nolga teng bo‘lsa, u holda bu matritsaning rangi shu noldan farqli minor tartibiga teng bo‘ladi.

3-misol. $A=\begin{pmatrix} 2 & 1 & 0 & -1 & 3 \\ 4 & 2 & 1 & 0 & -1 \\ 2 & 1 & 1 & 1 & -4 \\ 0 & 0 & 2 & 4 & -14 \end{pmatrix}$ matritsa rangini o‘rab turuvchi minorlar usulida topamiz.

$\boldsymbol{a_{11}}\neq 0$, shu sababli uni 1- tartibli minor deb olamiz. Bu minorni o‘rab turuvchi, noldan farqli 2- tartibli minorni izlaymiz, bittasini topsak yetarli.

1). $\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0$; 2). $\begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = 2$

Endi bu minorni o‘rab turuvchi barcha 3-tartibli minorlarni qarab chiqamiz.

$$C_4^3 \cdot C_5^3 = \frac{4!}{3! \cdot 1!} \cdot \frac{5!}{3! \cdot 2!} = 4 \cdot 10 = 40$$

Jami 3-tartibli minorlar 40 ta ekan. Bizning minorni o‘rab turuvchi 3-tartibli minorlar esa 6 ta, shularni tekshiramiz.

$$A=\begin{pmatrix} 2 & 1 & 0 & -1 & 3 \\ 4 & 2 & 1 & 0 & -1 \\ 2 & 1 & 1 & 1 & -4 \\ 0 & 0 & 2 & 4 & -14 \end{pmatrix}$$

$$M_3=\begin{vmatrix} 2 & 1 & 0 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix}=4+2+0-0-4-2=0$$

$$M_3=\begin{vmatrix} 2 & 0 & -1 \\ 4 & 1 & 0 \\ 2 & 1 & 1 \end{vmatrix}=2-4+0+2-0-0=0$$

$$M_3=\begin{vmatrix} 2 & 0 & -1 \\ 4 & 1 & 0 \\ 0 & 2 & 4 \end{vmatrix}=8+0-8-0-0-0=0$$

$$M_3=\begin{vmatrix} 2 & 1 & 0 \\ 4 & 2 & 1 \\ 0 & 0 & 2 \end{vmatrix}=8+0+0-0-8-0=0$$

$$M_3=\begin{vmatrix} 2 & 0 & 3 \\ 4 & 1 & -1 \\ 2 & 1 & -4 \end{vmatrix}=-8+0+12-6-0+2=0$$

$$M_3=\begin{vmatrix} 2 & 0 & 3 \\ 4 & 1 & -1 \\ 0 & 2 & -14 \end{vmatrix}=-28+0+24-0-0+4=0$$

1-teoremaga ko‘ra, barcha 3 –tartibli minorlar nolga teng. Demak berilgan matritsaning rangi 2 ga teng.

Matritsa rangini ta’rifdan foydalanib topish usuli va o‘rab turuvchi minorlar usuli qanday farq qiladi?

Matritsa rangini ta’rifdan foydalanib topishda matritsaning eng katta tartibli minorlarini tekshirishdan boshlaymiz. Ya’ni tashqaridan boshlanadi.

Bu usulning kamchiligi: hisoblash ishlari juda ko‘p.

Matritsa rangini o‘rab turuvchi minorlar usulida topishda eng kichik tartibli, noldan farqli minordan boshlaymiz. Ya’ni ichkaridan boshlanadi.

Bu usulda hisoblash ishlari ancha kamaydi.

6. Matritsa rangini elementar almashtirishlar usulida topish

Matritsada **elementar almashtirishlar** deb, quyidagi almashtirishlarga aytiladi:

a) Ikkita parallel qatorni o‘rnini almashtirish;

b) Biror qatorning barcha elementlarini k≠0 songa ko‘paytirish;

c) Biror qatorning barcha elementlarini k≠0 songa ko‘paytirib, boshqa qatorning mos elementlariga qo‘shish;

Bu almashtirishlar natijasida hosil bo‘lgan matritsa berilgan matritsaga **ekvivalent matritsa** deyiladi

2-teorema. Matritsalar ustida elementar almashtirishlar natijasida uning rangi o‘zgarmaydi.

Matritsa rangini elementar almashtirishlar usulida topish algoritmi

Elementar almashtirishlar yordamida matritsa pog‘onasimon matritsa ko‘rinishiga keltiriladi;

Hosil bo‘lgan matritsaning noldan farqli satrlari soni berilgan matritsaning rangiga teng bo‘ladi.

3-teorema. Pog‘onasimon matritsaning rangi uning nolmas satrlari soniga teng.

4-misol. $A=\begin{pmatrix} 2 & 1 & 0 & -1 & 3 \\ 4 & 2 & 1 & 0 & -1 \\ 2 & 1 & 1 & 1 & -4 \\ 0 & 0 & 2 & 4 & -14 \end{pmatrix}$ matritsa rangini elementar almashtirishlar usulida topamiz.

$$\left.\begin{pmatrix} 2 & 1 & 0 & -1 & 3 \\ 4 & 2 & 1 & 0 & -1 \\ 2 & 1 & 1 & 1 & -4 \\ 0 & 0 & 2 & 4 & -14 \end{pmatrix}\right| \cdot(-2),(-1) \Rightarrow \begin{pmatrix} 2 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 2 & 4 & -14 \end{pmatrix} \div 2$$

$$\begin{pmatrix} 2 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 1 & 2 & -7 \end{pmatrix} \cdot(-1) \Rightarrow \begin{pmatrix} 2 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Pog‘onasimon matritsada 3- va 4-satrlarning barcha elementlari nolga teng.

Demak, $rankA=2$

Matritsa ustun(satr)larining chiziqli bog‘liqligi

Matritsa rangi uning satrlari yoki ustunlarining chiziqli bog‘liqligi (va erkliligi) bilan aniqlanadi. Matritsa ustunlarining chiziqli bog‘liqligini o‘rganamiz (Satrlar shunga o‘xshash aniqlanadi).

$A_{m\times n}$ matritsa berilgan bo‘lsin. Uning ustunlarini quyidagicha yozib olamiz:

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad e_1=\begin{pmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{pmatrix} \qquad e_2=\begin{pmatrix} a_{12} \\ a_{22} \\ \dots \\ a_{m2} \end{pmatrix} \qquad e_n=\begin{pmatrix} a_{1n} \\ a_{2n} \\ \dots \\ a_{mn} \end{pmatrix}$$

Agar matritsaning biror *k* ta ustuni uchun $e=\lambda_1 e_1+\lambda_2 e_2+\dots+\lambda_k e_k$ tenglik o‘rinli bo‘lsa, u holda e ustun $e_1, e_2, \dots, e_k$ ustunlarning **chiziqli kombinatsiyasi-dan tuzilgan** deyiladi, bunda $\lambda_1, \lambda_2, \dots, \lambda_k$ – ixtiyoriy haqiqiy sonlar.

Agar $\lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_k e_k = \theta$ tenglikni qanoatlantiruvchi $\lambda_1, \lambda_2, ..., \lambda_k$ sonlarning kamida bittasi noldan farqli bo‘lsa, u holda $e_1, e_2, ..., e_k$ ustunlar **chiziqli bog‘liq deyiladi,** bunda $\theta = (0 \quad 0 \ \dots 0)^{\mathrm{T}}$ nollar ustuni.

$$\lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_k e_k = \lambda_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{pmatrix} + \lambda_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \dots \\ a_{m2} \end{pmatrix} + ... + \lambda_k \begin{pmatrix} a_{1k} \\ a_{2k} \\ \dots \\ a_{mk} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}$$

Ustunlarning chiziqli bog‘liqligi matritsaning hech bo‘lmaganda bitta ustuni qolganlarining chiziqli kombinatsiyasidan iborat bo‘lishini bildiradi.

Matritsa ustun(satr)larining chiziqli bog‘liqligi

Agar $\lambda_1 = \lambda_2 = ... = \lambda_k = 0$ bo‘lganda $\lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_k e_k = \theta$ tenglik bajarilsa,

u holda $e_1, e_2, ..., e_k$ **ustunlar chiziqli erkli** deyiladi.

4-teorema. Agar matritsaning rangi $\boldsymbol{r}$ ga teng bo‘lsa, u holda unda $\boldsymbol{r}$ ta chiziqli erkli ustun (satr) topiladi, qolgan barcha ustun (satr)lar bu $\boldsymbol{r}$ ta ustun (satr)ning chiziqli kombinatsiyasi bo‘ladi.

Natija. Matritsaning rangi undagi chiziqli erkli satrlar(ustunlar) soniga teng.

Bazis minor

Tartibi matritsa rangiga teng bo‘lgan minor *bazis minor* deyiladi.

Kesishmasida bazis minor elementlari turgan satrlar va ustunlar *bazis satrlar va ustunlar* deyiladi.

5-teorema. Matritsaning istalgan satri (ustuni) uning bazis satr(ustun)larining chiziqli kombinatsiyasidan iborat bo‘ladi.
Bazis satr(ustun)lar chiziqli erkli bo‘ladi.

5-misol. $\begin{pmatrix} 1 & 2 & 3 & 0 \\ -1 & 0 & 4 & 1 \\ 0 & 2 & 6 & 2 \\ 2 & 4 & 5 & 1 \\ 1 & -1 & 5 & 0 \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{matrix}$ matritsada bazis satrlarni aniqlang.

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ -1 & 0 & 4 & 1 \\ 0 & 2 & 6 & 2 \\ 2 & 4 & 5 & 1 \\ 1 & -1 & 5 & 0 \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{matrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 7 & 1 \\ 0 & 2 & 6 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & -3 & 2 & 0 \end{pmatrix} \begin{matrix} e_1 \\ e_2 + e_1 \\ e_3 \\ e_4 - 2e_1 \\ e_5 - e_1 \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 7 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 25/2 & 3/2 \end{pmatrix} \begin{matrix} e_1 \\ e_2 + e_1 \\ e_3 - (e_2 + e_1) \\ e_4 - 2e_1 \\ e_5 - e_1 + 1{,}5(e_2 + e_1) \end{matrix} \sim$$

Oxirgi natija qayta ko‘chirib yozilgan:

$$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 7 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 25/2 & 3/2 \end{pmatrix} \begin{matrix} e_1 \\ e_2+e_1 \\ e_3-(e_2+e_1) \\ e_4-2e_1 \\ e_5-e_1+1{,}5(e_2+e_1) \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 7 & 1 \\ 0 & 0 & -1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 14 \end{pmatrix} \begin{matrix} e_1 \\ e_2+e_1 \\ e_3-e_2-e_1 \\ e_4-2e_1-(e_3-e_2-e_1) \\ e_5+0{,}5e_1+1{,}5e_2+12{,}5(e_3-e_2-e_1) \end{matrix}$$

Demak, e_1, e_2, e_3, e_5 $-$bazis satrlar va $rank(A) = 4.$

$$e_4-2e_1-(e_3-e_2-e_1)=0 \quad \Rightarrow e_4 = e_1 - e_2 + e_3$$

Determinantning nolga teng bo‘lishining yetarlilik sharti

6-teorema. Agar determinant nolga teng bo‘lsa, u holda uning ustunlari (va satrlari) chiziqli bog‘liqdir.

O‘z-o‘zini tekshirish uchun savollar:

1. Matritsa rangi deb nimaga aytiladi?
2. *k*-tartibli minor deb nimaga aytiladi?
3. $m \times n$ o‘lchamli matritsaning *k*- tartibli minorlari soni qaysi formuladan topiladi?
4. O‘rab turuvchi minor deb nimaga aytiladi?
5. Matritsa rangini hisoblashning qanday usullarini bilasiz?
6. Matritsa rangini topishning o‘rab turuvchi minorlar usuli algoritmini keltiring.
7. Matritsa ustida elementar almashtirishlar deb qanday almashtirishga aytiladi?
8. Matritsa ustida qanday amallarni bajarganda uning rangi o‘zgarmaydi?
9. Matritsa rangini topishning elementar almashtirishlar usuli algoritmini keltiring.

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E‘TIBORINGIZ UCHUN RAHMAT!

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