Q3 (6)

A particle moves around the semicircular path C_1 from $(0, R)$ to $(0, -R)$ and then returns to $(0, R)$ along the straight path C_2. During this trip, the partice is subject to the force field

$$\vec{F}(\vec{r}) = k(xy, -x^2)$$

where k is a constant. Determine the total work done by the field on the particle in terms of k and R.

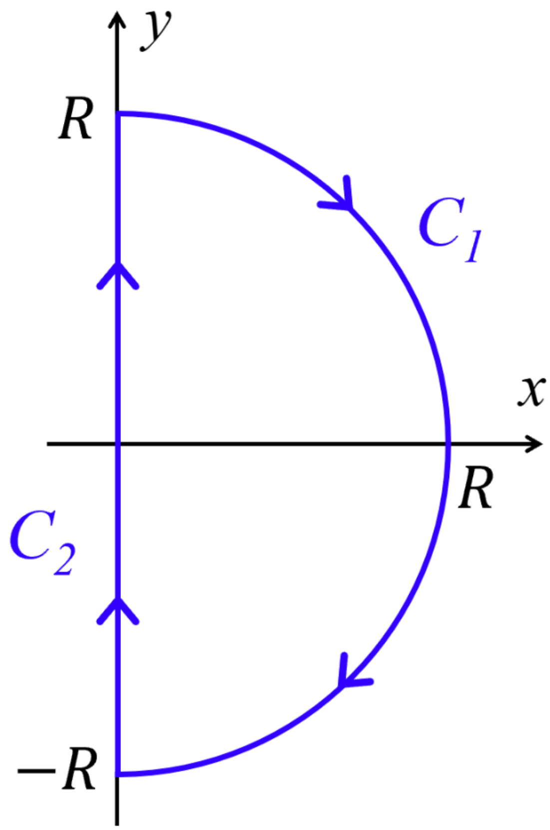

Figure 1: Graph of particle through force field

Work done by a force field is given by the following.

$$W = \int_a^b \vec{F}(\vec{r}(t)) \cdot d\vec{r}(t)$$

C_1 and C_2 can be parameterized with the following.

$$\begin{aligned} C_1 &: \vec{r}_1(t) = (R\sin(\pi t), R\cos(\pi t)), \quad && t \in [0, 1] \\ C_2 &: \vec{r}_2(t) = (0, R(2t-1)), && t \in [0, 1] \end{aligned}$$

So $\vec{F}(\vec{r_n}(t)) \cdot d\vec{r_n}(t)$ is equal to the following.

$$\begin{aligned} \vec{F}(\vec{r_1}(t)) \cdot d\vec{r_1}(t) &= k\pi R^3 \sin(\pi t) dt \\ \vec{F}(\vec{r_2}(t)) \cdot d\vec{r_2}(t) &= 0dt \end{aligned}$$

Which makes sense becuase $x = 0$ along C_2 and $\vec{F}$ has a factor of x in both components.

So, the total work done by the particle is given by the following.

$$\begin{aligned} W &= \int_{t=0}^{1} k\pi R^3 \sin(\pi t)\, dt \\ &= 2kR^3 \end{aligned}$$