Q2 (6)

A rectangular steel wire frame with length L and height H (as shown) has a linear mass density (i.e., mass per unit length) given by the following.

$$\rho(x,y) = \rho_0 \left(\frac{2H^2 - y^2}{H^2} \right)$$

Determine the total mass of the wire frame in terms of ρ_0, H, and L.

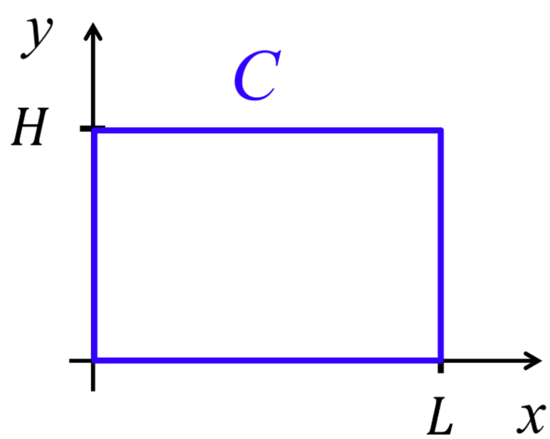

Figure 1: Graph of wire frame

The total mass of the wire frame is the sum of 4 line integrals which is shown below.

$$\begin{aligned}\oint_C \rho(x,y)\, d\ell &= \int\limits_{(0,0)\to(L,0)} \rho(x,y)\, d\ell + \int\limits_{(L,0)\to(L,H)} \rho(x,y)\, d\ell \\ &+ \int\limits_{(L,H)\to(0,H)} \rho(x,y)\, d\ell + \int\limits_{(0,H)\to(0,0)} \rho(x,y)\, d\ell\end{aligned}$$

Each curve can be parameterized with $t \in [0,1]$

$$\begin{aligned}(0,0) \to (L,0) &: \vec{r_1}(t) = (Lt, 0) & ||\vec{r_1}'(t)||dt = Ldt \\ (L,0) \to (L,H) &: \vec{r_2}(t) = (L, Ht) & ||\vec{r_2}'(t)||dt = Hdt \\ (L,H) \to (0,H) &: \vec{r_3}(t) = (L(1-t), H) & ||\vec{r_3}'(t)||dt = -Ldt \\ (0,H) \to (0,0) &: \vec{r_4}(t) = (0, H(1-t)) & ||\vec{r_4}'(t)||dt = -Hdt\end{aligned}$$

So, the total mass is computed with the following.

$$\begin{aligned}M = &\int_{t=0}^{1} \rho_0 \left(\frac{2H^2 - 0^2}{H^2} \right) Ldt &+ \int_{t=0}^{1} \rho_0 \left(\frac{2H^2 - H^2t^2}{H^2} \right) Hdt \\ &- \int_{t=0}^{1} \rho_0 \left(\frac{2H^2 - H^2}{H^2} \right) Ldt &- \int_{t=0}^{1} \rho_0 \left(\frac{2H^2 - H^2(1-t)^2}{H^2} \right) Hdt \\ &= \rho_0 L\end{aligned}$$