Course - Convex Optimization

Slot - B21

## Tutorial - 1

Q.1) a) $f(x, y) = \frac{x^2}{y}$

$$\Rightarrow \frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x} \frac{x^2}{y} = \frac{2x}{y}$$

$$\Rightarrow \frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y} \frac{x^2}{y} = \frac{-x^2}{y}$$

$$\nabla f(x,y) = \left[ \frac{2x}{y}, \frac{-x^2}{y} \right]$$

b) $f(x,y) = \sin(x) + \cos(y)$

$$\Rightarrow \frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x} [\sin x + \cos y] = \cos x$$

$$\Rightarrow \frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y} [\sin x + \cos y] = -\sin y$$

Q.2) compute second order Taylor Approximation :-

a) $f(x,y) = \frac{x^2}{y}$ $\Rightarrow \frac{\partial}{\partial y} \frac{2x}{y} = \frac{-2x}{y^2}$

$$\nabla^2 f(x,y) = \begin{bmatrix} \frac{2}{y} & \frac{-2x}{y^2} \\ \frac{-2x}{y^2} & \frac{-4x}{y^3} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

b) $f(x,y) = \sin(x) + \cos(y)$

$$\frac{\partial}{\partial y} \cos x = 0 \quad , \quad \frac{\partial}{\partial x} - \sin y = 0$$

$$\nabla^2 f(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -\sin x & 0 \\ 0 & -\cos y \end{bmatrix}$$

Q.3) LU and QR decomposition of: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

LU:-

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 = R_2 - R_1 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Write coefficient 1 in matrix L at row 2, column 1 :

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 = R_3 + R_2 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Write the coefficient -1 in matrix L at row 3, column 2:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

The obtained matrix is U :-

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

QR :-

Orthonormalize set of vectors formed by columns of given matrix:

$$\left\{ \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{6}/6 \\ -\sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}, \begin{bmatrix} -\sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix} \right\}$$

The columns of matrix Q are the orthonormalized vectors:

$$Q = \begin{bmatrix} \sqrt{2}/2 & \sqrt{6}/6 & -\sqrt{3}/3 \\ \sqrt{2}/2 & -\sqrt{6}/6 & \sqrt{3}/3 \\ 0 & \sqrt{6}/3 & \sqrt{3}/3 \end{bmatrix} \Rightarrow Q^T = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ \sqrt{6}/6 & -\sqrt{6}/6 & \sqrt{6}/3 \\ -\sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \end{bmatrix}$$

finally :-

$$R = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ \sqrt{6}/6 & -\sqrt{6}/6 & \sqrt{6}/3 \\ -\sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{6}/2 & \sqrt{6}/6 \\ 0 & 0 & 2\sqrt{3}/3 \end{bmatrix}$$

Q.4) compute SVD of following matrix : $\begin{bmatrix} 3 & 3 & 2 \\ 2 & 3 & 2 \end{bmatrix}$

$$\text{Transpose} = \begin{bmatrix} 3 & 2 \\ 3 & 3 \\ 2 & 2 \end{bmatrix}$$

$$Y = \begin{bmatrix} 3 & 2 \\ 3 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 2 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 15 & 10 \\ 15 & 18 & 12 \\ 10 & 12 & 8 \end{bmatrix}$$

Eigen value of $Y$ : $\dfrac{\sqrt{1469}+39}{2} \Rightarrow \begin{bmatrix} \dfrac{-13+\sqrt{1469}}{20} \\ 3/2 \\ 1 \end{bmatrix}$

Eigen value $= 0$

Eigen vector : $\begin{bmatrix} 0 \\ -2/3 \\ 1 \end{bmatrix}$

Find square roots of non-zero eigen values ($\sigma_i$) :

$$\sigma_1 = \frac{\sqrt{2}\sqrt{39-\sqrt{1469}}}{2}, \quad \sigma_2 = \frac{\sqrt{2}\sqrt{\sqrt{1469}+39}}{2}$$

The $\Sigma$ matrix is zero matrix with $\sigma_i$ on its diagonal :

$$\Sigma = \begin{bmatrix} \dfrac{\sqrt{2}\sqrt{39-\sqrt{1469}}}{2} & 0 & 0 \\ 0 & \dfrac{\sqrt{2}\sqrt{\sqrt{1469}+39}}{2} & 0 \end{bmatrix}$$

The columns of matrix $V$ are normalized (unit) vectors :

$$V = \begin{bmatrix} \dfrac{-\sqrt{26}+\sqrt{226}}{2\sqrt{\sqrt{1469}+113}} & \dfrac{-\sqrt{26}+\sqrt{226}}{2\sqrt{113-\sqrt{1469}}} & 0 \\ \dfrac{30}{\sqrt{26\sqrt{1469}+2938}} & \dfrac{30}{\sqrt{2938-26\sqrt{1469}}} & -\dfrac{2\sqrt{13}}{13} \\ \dfrac{20}{\sqrt{26\sqrt{1469}+2938}} & \dfrac{20}{\sqrt{2938-26\sqrt{1469}}} & \dfrac{3\sqrt{13}}{13} \end{bmatrix}$$

$$u_i = \frac{1}{\sigma_i}\begin{bmatrix} 3 & 3 & 2 \\ 2 & 3 & 2 \end{bmatrix} \cdot v_i$$

$$= \frac{1}{\frac{\sqrt{2}\sqrt{39-\sqrt{1469}}}{2}} \cdot \begin{bmatrix} 3 & 3 & 2 \\ 2 & 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} \frac{-\sqrt{26}+\sqrt{226}}{2\sqrt{\sqrt{1469}+113}} \\ \frac{30}{\sqrt{26\sqrt{1469}+2938}} \\ \frac{20}{\sqrt{26\sqrt{1469}+2938}} \end{bmatrix}$$

$$u_1 = \begin{bmatrix} \frac{339-7\sqrt{1469}\sqrt{1469-37\sqrt{1469}}}{113(-39+\sqrt{1469})\sqrt{\sqrt{1469}+63}} \\ \frac{\left(2-\frac{8\sqrt{1469}}{113}\right)\sqrt{1469-37\sqrt{1469}}}{(-39+\sqrt{1469})\sqrt{\sqrt{1469}+63}} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2}\begin{bmatrix} 3 & 3 & 2 \\ 2 & 3 & 2 \end{bmatrix} \cdot v_2 = \frac{1}{\frac{\sqrt{2}\sqrt{\sqrt{1469}+39}}{2}} \cdot \begin{bmatrix} 3 & 3 & 2 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} \frac{-\sqrt{26}+\sqrt{226}}{2\sqrt{113-\sqrt{1469}}} \\ \frac{30}{\sqrt{2930-26\sqrt{1469}}} \\ \frac{20}{\sqrt{2938-26\sqrt{1469}}} \end{bmatrix}$$

$$v = \begin{bmatrix} \frac{(339-7\sqrt{1469})\sqrt{1469-37\sqrt{1469}}}{113(-39+\sqrt{1469})\sqrt{\sqrt{1469}+63}} & \frac{(7\sqrt{1469}+339)\sqrt{37\sqrt{1469}}}{113\sqrt{63-\sqrt{1469}}(\sqrt{1469}+39)} \\ \frac{\frac{2-8\sqrt{1469}}{113}\sqrt{1469-37\sqrt{1469}}}{(-39+\sqrt{1469})\sqrt{\sqrt{1469}+63}} & \frac{\left(\frac{2+8\sqrt{1469}}{113}\right)\sqrt{37\sqrt{1469}+1469}}{\sqrt{63-\sqrt{1469}}(\sqrt{1469}+39)} \end{bmatrix}$$

The matrices $u, \Sigma, v$ are such that: $\begin{bmatrix} 3 & 3 & 2 \\ 2 & 3 & 2 \end{bmatrix}$

$$u = \begin{bmatrix} -0.659 & 0.752 \\ 0.752 & 0.659 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 0.579 & 0 & 0 \\ 0 & 6.218 & 0 \end{bmatrix}$$

$$v = \begin{bmatrix} -0.818 & 0.575 & 0 \\ 0.478 & 0.680 & -0.554 \\ 0.318 & 0.454 & 0.832 \end{bmatrix}$$

Q.5) $f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x$

To find the critical point of f'x f(x,y), we generate and solve a system of equations in which each partial deriv-ative is set equal to 0.

The solutions to the systems are the critical points. Then, the second partial derivative test allows us to clarify each critical point as a maximum, minimum.

To implement the test, we compute the value $D(a,b)^2$
$f_{xx}(a,b) \quad f_{yy}(a,b) - (f_{xy}(a,b))^2$
for each critical point (a,b) using the expression for the second partial derivative.

Once we have their information, the following guidelines allow us to classify each critical point:

1. If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$ then $(a,b)$ is maximum.
2. If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$ then $(a,b)$ is minimum.
3. If $D(a,b) = 0$ then test is inconclusive.
4. If $D(a,b) < 0$ then $(a,b)$ is a saddle point.

$D.(-5,0) = 144 \times (-5)^2 - 216(-5)(0) - 144(0)^2$
$= 3600 > 0$

$f_{xx}(-5,0) = 12(-5) = -60 < 0$
$\therefore$

From classification guidelines, we obtain:

a) Local minima $= (5,0)$
b) Local maxima $= (-5,0)$