Tutorial Sheet – I (Set Theory & Mathematical Induction)

1. Represent the following set in tabular form:

   $(i)\ A = \{x : x^2 - 3x + 2 = 0\}$

   $(ii)\ B = \{x : x \text{ is an integer } \& 1 < x < 7\}$

   $(iii)\ C = \{x : x \text{ is a perfect square}, \ x < 30\}$

   $(iv)\ D = \{x : x \text{ is a positive integral divisor of } 60\}$

2. (a) Which of the following sets are equal

   $A = \{x : x \in N, x < 4\}$

   $B = \{x : x^2 - 2x + 1 = 0\}$

   $C = \{x : x^3 - 6x^2 + 11x - 6 = 0\}$

   (b) Let A = {1, 2… 8, 9}, B = {2, 4, 6, 8}, C = {1, 3, 5, 7, 9}, D = {3, 4, 5}, E = {3, 5}
   Which of the above sets can equal to X under each of the following conditions?

   (i) X and B are disjoint. (ii) $X \subseteq A$ but $X \not\subset C$.

   (iii) $X \subseteq D$ but $X \not\subset B$ (iv) $X \subseteq C$ but $X \not\subset A$.

3. Let $U = \mathbb{R}$, $A = \{x \in \mathbb{R} : x > 0\}$, $B = \{x \in \mathbb{R} : x > 1\}$ and $C = \{x \in \mathbb{R} : x < 2\}$. Find each of the following sets:

   a) $A \cup B$ b) $A \cup C$ c) $B \cup C$ d) $A \cap B$
   e) $A \cap C$ f) $B \cap C$ g) $\overline{A}$ h) $\overline{B}$
   i) $\overline{C}$ j) $A \setminus B$ k) $B \setminus C$

4. (a) If $A \cap B = A \cap C$ then is it necessary that $B = C$?

   (b) If $A \cup B = A \cup C$ then is it necessary that $B = C$?

5. If A= {1, 2, 3, 5}, B= {4, 5, 7} and C= {1, 6, 7} then prove that

   (a) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

   (b) $A - (A - B) \subseteq B$.

   (c) $A - (B \cap C) = (A - B) \cup (A - C)$.

   (d) $A \cap (B - C) = (A \cap B) - (A \cap C)$

   (e) $A - (B - C) = (A - B) \cup (A \cap C)$.

   (f) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

   (g) $(A - B) \times C = (A \times C) - (B \times C)$.

   (h) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

6. If $A_i = [0, i]$ where $i \in$ N(set of naturals), then find (a) $A_1 \cup A_2$ (b) $A_3 \cap A_4$ $(c) \cup_{i=5}^{i=10} A_i$

   and Verify that $(\bigcup_{i=1}^{3} A_i)^c = \bigcap_{i=1}^{3} A_i^c$.

7. Let $D_n = \left(0, \dfrac{1}{n}\right)$ where $n \in N$, the set of positive integers. Find (i) $D_3 \cup D_5$.

   (ii) $D_4 \cap D_{10}$ (iii) $\bigcup_{i \in A} D_i$ where A is a subset of N (iv) $\bigcap_{i \in N} D_i$

8. Prove the following (in general):

   $(i)\ A - (B \cap C) = (A - B) \cup (A - C)$

   $(ii)\ (A \cup B)^c = A^c \cap B^c$

   $(iii)\ (A \cup B) \cap (A \cup B^c) = A.$

   $(iv)\ (A \cup B) - (A \cap B) = (A - B) \cup (B - A).$ $[Note\ the\ def^n\ of\ symm.\ difference]$

   $(v)\ If\ A \subseteq B\ \&\ C \subseteq D\ \ then\ \ A \cup C \subseteq B \cup D\ \ \&\ \ A \cap C \subseteq B \cap D.$

9. If U={1,2,3,4,5,6,7}, A={1,2,3,5}, B={1,3,4,6}and C={1,2,4,7}find

   $(i)\ n(A \cup B \cup C)$ $\quad$ $(iv) A \Delta B$

   $(v)(B \Delta C) - A$

   $(ii)\ n(A \cap (B \cap C))$

   $(iii)\ n(A \cap B^c \cap C)$

10. A computer company must hire 20 programmers to handle system programming jobs and 30 programmers for applications programming. Of those hired,5 are expected to perform jobs of both types. How many programmers must be hired? Ans: 45

11. In a class of 25 students, 12 have taken Mathematics. 8 have taken Mathematics but not Biology. Find the number of students who have taken Mathematics and Biology and those who have taken Biology but not Mathematics. Ans: 4, 13

12. Out of 250 candidates who failed in an examination, it was revealed that 128 failed in Mathematics, 87 in physics, and 134 in aggregate.31 failed in Mathematics and in Physics, 54 failed in the aggregate and in Mathematics, 30 failed in the aggregate and in Physics. Find how many candidates failed:
   (a) in all the three subjects; Ans: 16
   (b) in Mathematics but not in Physics; Ans:97
   (c) in the aggregate but not in Mathematics; Ans: 80
   (d) in Physics but not in the aggregate or in Mathematics; Ans:42
   (e) in the aggregate or in Mathematics, but not in Physics; Ans:163

13. A survey of 500 television watchers produced the following information: 285 watch football games, 195 watch hockey games, 115 watch basketball games, 45 watch football and basketball games, 70 watch football and hockey games, 50 watch hockey and basketball games, and 50 do not watch any of the three kinds of games.
    a. How many people in the survey watch all three kinds of games?
    b. How many people watch exactly one of the sports?

14. If A and B are disjoint sets s.t. $|A \cup B| = |A|$, then what must be true about B?

15. Write the dual of each equation:

    $(i)\ A \cup B = (B^c \cap A^c)^c \quad (ii)\ A = (B^c \cap A) \cup (A \cap B)$

    $(iii)\ A \cup (A \cap B) = A \quad (iv)\ (A \cap B) \cup (A^c \cap B) \cup (A \cap B^c) \cup (A^c \cap B^c) = U$

16. Find the power set of S= {1, 2,{2},{1,2}, 3}.

17. If $|A| = n$ then prove that $|P(A)| = 2^n$.

18. The power set *P*(*A*) of a set *A* is the set of all subsets of *A*. Suppose that *A* = {1, 2, 3, 4, 5}.
    a) How many elements are there in *P*(*A*)?
    b) How many elements are there in $P(A \times P(A)) \cup A$?
    c) How many elements are there in $P(A \times P(A)) \cap A$?

19. Let X = {1, 2… 8, 9}. Determine whether or not each of the following is a partition of X:
    (a) [{1, 3, 6}, {2, 8}, {5,7,9}] (b) [{1,5,7},{2,4,8,9},{3,5,6}]
    (c) [{2,4,5,8},{1,9},{3,6,7}] (d) [{1,2,7},{3,5},{4,6,8,9},{3,5}]

20. Let S = {1,2,3,4,5,6}. Determine whether or not each of the following is a partition of S:
    (a) [{1,2,3},{1,4,5,6}] (b) [{1,2},{3,5,6}]
    (c) [{1,3,5},{2,4},{6}] (d) [{1,3,5,},{2,4,6,7}]

21. Let $[A_1, A_2, ..., A_m]$ *and* $[B_1, B_2, ..., B_n]$ be partitions of a set X. Show that the collection of sets

$$P = [A_i \cap B_j : i = 1,2,...,m,\ \ j = 1,2,...,n] - \phi$$

    is also a partition (called the cross partition) of X.(Observe that we have deleted the empty set)

22. Let X = { 1, 2, 3, …,8,9}. Find the cross partition P of the following partitions of X:

$$P_1 = [\{1,3,5,7,9\},\{2,4,6,8\}] \ \ and \ \ P_2 = [\{1,2,3,4\},\{5,7\},\{6,8,9\}]$$

Using the technique of mathematical induction:

23. Prove that $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta) \quad \forall n \in N$
24. Show that $9^n - 8n - 1$ is divisible by 64.
25. Prove that $1^3 + 2^3 + 3^3 + ----- + n^3 = \left(\frac{1}{2}n(n+1)\right)^2 \quad \forall n \in N$
26. Prove that $1^2 + 3^2 + -- + (2n-1)^2 = \frac{(4n^3 - n)}{3} \quad \forall n \in N$
27. $1^2 - 2^2 + 3^2 + ---- + (-1)^{n+1} n^2 = (-1)^{n+1} \frac{n(n+1)}{2} \quad \forall n \in N$
28. Prove that $n^3 + 5n$ is divisible by 6 $\quad \forall n \in N$
29. Prove that $2^n < n! \quad \forall n \geq 4, n \in N$
30. $n < 2^n \quad \forall n \in N$
31. If $r \in R, r \neq 1 \;\&\; n \in N$ then $1 + r + r^2 + --- + r^n = \frac{1 - r^{n+1}}{1 - r}$
32. Conjecture a formula for the sum of first n odd natural numbers 1+3+----+ (2n-1) and prove your formula by using mathematical induction.
33. Find the least n for which the statement is true and then prove that $10n < 3^n$.
34. Find the least n for which the statement is true and then prove that $(1 + n^2) < 2^n$.
35. Prove that $(1+1^2) + (2+2^2) + (3+3^2) + \ldots + (n+n^2) = \left(\frac{n(n+1)(n+2)}{3}\right) \quad \forall n \in N.$
36. Discuss the problem of Tower of Hanoi and prove it by the method of induction.
    (Hint: The minimum number of moves required to move n-discs from one peg to other (using intermediate peg) is 2^n-1).