Tutorial Sheet – III (Functions and Algorithms)

1. Let A and B be finite sets. Suppose A has m elements and B has n elements. State the relationship which must hold between m and n for each of the following to be true:
   (a) there exists a one-one (injection) map from A to B.
   (b) there exists an onto (surjection) map from A to B.
   (c) there exists a bijection (one-one onto) map from A to B.
2. Find the domain D and range of each of the following real-valued functions of a real variable:

   $(a)\ f(x) = \dfrac{1}{x-2}$ $\qquad$ $(c)\ f(x) = \sqrt{25 - x^2}$

   $(b)\ f(x) = x^2 - 3x - 4.$ $\qquad$ $(d)\ f(x) = x^2 \text{ where } 0 \le x \le 2.$

3. Define floor (or greatest integer), ceiling, absolute value, and remainder, exponential and logarithmic functions. Give examples for each.
4. Suppose that the set A contains 2 elements and the set B contains 5 elements.
   a) How many of the functions $f : A \to B$ are not onto?
   b) How many of the functions $f : A \to B$ are not one-to-one?

5. Let $f(x) = x + 2$, $g(x) = x - 2$, and $h(x) = 3x$ for $x \in R$, where R is set of Real numbers. Find $g \circ f, f \circ g, f \circ f, h \circ f, g \circ f \circ h$.
6. Let $f : R \to R$ and $g : R \to R$, where R is set of real numbers. Find $f \circ g, g \circ f$, where $f(x) = x^2 - 2$ and $g(x) = x + 4$. State whether these functions are injective, surjective or bijective.
7. Show that there exists a one-one and onto mapping from the set of naturals to the set of integers. Hence the cardinality of both the sets is same.
8. If $f : A \to B$ and $g : B \to C$ and both f and g are onto, show that $g \circ f$ is also onto. Is $g \circ f$ one-one if both g and f are one-one?
9. Let $f : R \to R$ be given by $f(x) = x^3 - 2$. Find f^{-1}.
10. Let f, g and h be functions from $\mathbb{N}$ to $\mathbb{N}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 100, \\ 2 & \text{if } x \le 100, \end{cases}$$

   $g(x) = x^2 + 1$ and $h(x) = 2x + 1$ for every $x \in \mathbb{N}$.
   a) Determine whether each function is one-to-one or onto.
   b) Find $h \circ (g \circ f)$ and $(h \circ g) \circ f$, and verify the Associative law for composition of functions.

11. Let a and b be positive integers, and suppose Q is defined recursively as follows:

$$Q(a,b) = \begin{cases} 0 & \text{if } a < b \\ Q(a-b, b) + 1 & \text{if } b \le a \end{cases}$$

   Find (i) Q (2, 5) (ii) Q (12, 5)
   (iii) What does this function Q do? Find Q (5861, 7).
12. Let n denote a positive integer. Suppose a function L is defined recursively as follows:

$$L(n) = \begin{cases} 0 & \text{if } n = 1 \\ L\left(\left\lfloor \dfrac{n}{2} \right\rfloor\right) & \text{if } n > 1 \end{cases}$$

   Find L (25).

13. Consider a recursive function G from set of positive integers to integers,

$$G(n) = \begin{cases} 1 & \text{if } n = 1 \\ 1 + G\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ G(3n - 1) & \text{if } n \text{ is odd and } n > 1. \end{cases}$$

Is G well defined? Justify.

14. Let x and y be two integers and suppose that g(x, y) is defined recursively by

$$g(x, y) = \begin{cases} 5 & \text{if } x < y \\ g(x - y, y + 2) + x & \text{if } x \geq y \end{cases}.$$

Find g (2, 7), g (5, 3) and g (15, 2).

15. Let a and b be integers, and suppose Q (a, b) is defined recursively by

$$Q(a, b) = \begin{cases} 5 & \text{if } a < b \\ Q(a - b, b + 2) + a & \text{if } a \geq b \end{cases}$$

Find Q (2, 7) and Q (5, 3).

16. A function $f : Z^+ \to Z$ is defined by

$$f(n) = \begin{cases} 1 & \text{if } n = 1 \\ f\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ 1 + f(5n - 9) & \text{if } n \text{ is odd and } n > 1. \end{cases}$$

Show that f is not well defined.

17. Suppose $P(n) = a_0 + a_1 n + a_2 n^2 + \dots\dots\dots + a_m n^m$ has degree m.Prove $P(n) = O(n^m)$.
18. Prove that $2x^4 + x^3 - x^2 + 3x + 4 = O(x^4)$.
19. Show that $9x^2$ is $O(x^3)$. Is it true that x^3 is $O(9x^2)$.
20. Show that x^3 is $O(x^4)$ but x^4 is not $O(x^3)$.
21. Using generating function solve the following recurrence relations:
    (i) $a_n = 8a_{n-1} + 10^{n-1}$ with initial condition $a_0 = 1$.
    (ii) $a_k = 7a_{k-1}$ with initial condition $a_0 = 5$.
    (iii) $a_k = 3a_{k-1} + 2;\ a_0 = 1$.
    (iv) $a_k = 7a_{k-1} - 12a_{k-2};\ a_0 = 1.;\ a_1 = 7$.
    (v) $a_k = ma_{k-1} - 1;\quad a_0 = m$.
22. Using Euclidean's Algorithm find the greatest common divisors of
    (i) 630 and 196 (ii) 1800 and 756.
    Also show that their gcd can be expressed as a linear integral combination of the numbers.
23. Show that 17,369 and 5472 are relatively prime.
24. Evaluate the following polynomial by using Horner's Method:
    (i) $f(x) = 4x^3 - 2x + 1$ at $x = -2$
    (ii) $f(x) = 17x^5 - 40x^3 + 16x - 7$ at $x = 3$
25. Using Russian Peasant Method multiply the following:
    (i) 461×973 (ii) 168×413 (iii) 141×141.