EXERCISE 1-1

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Solution:-

(a)

$a_{ii} = a_{11} + a_{22} + a_{33} = 1+4+1 = 6$ (matrix)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}.$

$= 1+1+1+0+16+4+0+1+1 = 25$ (scalar)

$$a_{ij}a_{jk} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{pmatrix}$$

$\hookrightarrow$ matrix

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \text{(vector)}$$

$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3.$

$= (1+0+2+0+0+0+0+0+4) = 7$ (scalar)

$$b_ib_j = \begin{pmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{pmatrix}$$

$\hookrightarrow$ matrix.

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 1+0+4 = 5$ (scalar).

(b)

$a_{ii} = a_{11} + a_{22} + a_{33} = 1+2+2 = 5$ ~~(matrix)~~ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}.$

$= 1+4+0+0+4+1+0+16+4 = 30 \text{(scalar)}$

$$a_{ij}a_{jk} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{pmatrix} \text{(matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix} \text{(vector)}.$$

$$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3.$$

$$= 4+4+0+0+2+1+0+4+2 = 17 \text{(scalar)}.$$

$$b_ib_j = \begin{pmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{pmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$\hookrightarrow$ matrix.

$$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 4+1+1 = 6 \text{(scalar.}$$

(C)

$$a_{ii} = a_{11} + a_{22} + a_{33} = 1+0+4 = 5 \text{(scalar)}$$

$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}.$$

$$= 1+1+1+1+0+4+0+1+16 = 25 \text{(scalar)}$$

$$a_{ij}a_{jk} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & 4 & 18 \end{pmatrix}$$

$\hookrightarrow$ matrix.

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \to \text{(vector)}$$

$$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$$

$$= 1+1+0+1+0+0+0+0+0 = 3 \text{(scalar)}$$

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{matrix}$$

$$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 1+1+0 = 2 \text{(scalar)}.$$

EXERCISE 1-3:-

Solution:-

$$a_{ij}b_{ij} = -a_{ji}b_{ji} = -a_{ij}b_{ij} \Rightarrow 2a_{ij}b_{ij} = 0 \Rightarrow a_{ij}b_{ij} = 0.$$

$$(a) = a_{(ij)}a_{[ij]} = \frac{1}{4}\operatorname{tr}\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0.$$

$$(b) = a_{(ij)}a_{[ij]} = \frac{1}{4}\operatorname{tr}\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}^T\right) = 0$$

$$(c): a_{(ij)}a_{[ij]} = \frac{1}{4}\operatorname{tr}\left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$$

EXERCISE 1-5:-

Solution:-

$$\det(a_{ij}) = \varepsilon_{ijk}a_{1i}a_{2j}a_{3k} = \varepsilon_{123}a_{11}a_{22}a_{33} + \varepsilon_{231}a_{12}a_{23}a_{31} + \varepsilon_{312}a_{13}a_{21}a_{32} + \varepsilon_{321}a_{13}a_{22}a_{31} + \varepsilon_{132}a_{11}a_{23}a_{32} + \varepsilon_{213}a_{12}a_{21}a_{33}.$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{21})$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

EXERCISE 1-7:-

Solution:-

$$Q_{ij} = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) \\ \cos(x_2', x_1) & \cos(x_2', x_2) \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(90-\theta) \\ \cos(90-\theta) & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

$$b_i' = Q_{ij} b_j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1\cos\theta + b_2\sin\theta \\ -b_1\sin\theta + b_2\cos\theta \end{bmatrix}.$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\cos^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\sin^2\theta. & a_{12}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{21}\sin^2\theta. \\ a_{21}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{12}\sin^2\theta & a_{11}\sin^2\theta - (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\cos^2\theta \end{bmatrix}$$

EXERCISE 1-9:-

Solution:-

$$\alpha'\delta_{ij}\delta_{kl} + \beta'\delta_{ik}\delta_{jl} + \gamma'\delta_{il}\delta_{jk} = Q_{im}Q_{jn}Q_{kp}Q_{lq}(\alpha\delta_{mn}\delta_{pq} + \beta\delta_{mp}\delta_{nq} + \gamma\delta_{mq}\delta_{np}).$$

$$= \alpha Q_{im}Q_{jm}Q_{kp}Q_{lp} + \beta Q_{im}Q_{jn}Q_{km}Q_{ln} + \gamma Q_{im}Q_{in}Q_{..}Q_{..}$$

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$\alpha\delta_{ij}\delta_{kl}+\beta\delta_{ik}\delta_{jl}+\gamma\delta_{il}\delta_{jk}$

EXERCISE 1-11:-

Solution:-

If $a=\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$.

$I_a = a_{ii} = \lambda_1+\lambda_2+\lambda_3.$

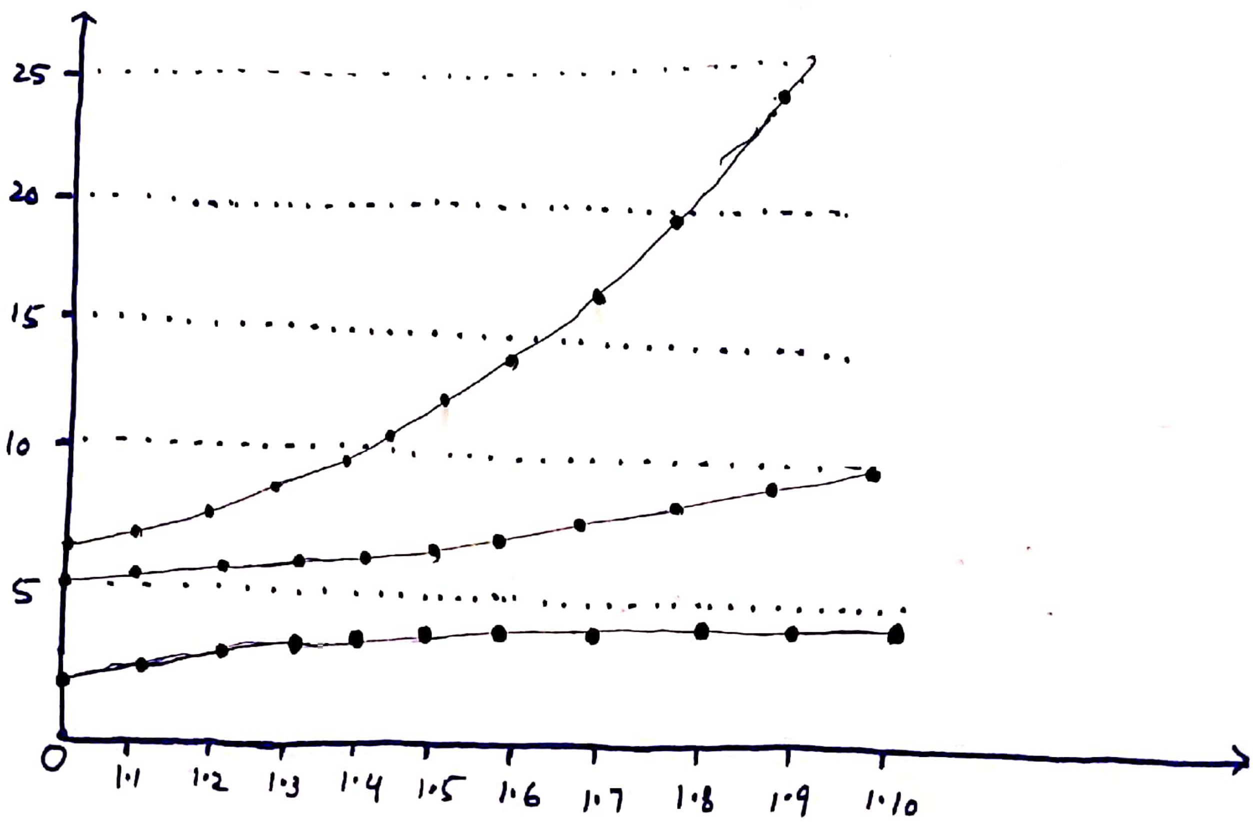
$II_a = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{vmatrix}$

$= \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3.$

$III_a = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2\lambda_3.$

EXERCISE 1-13:-

Solution:-



EXERCISE 1-15:-

Solution:-

$a_i = -\frac{1}{2}\varepsilon_{ijk} a_{jk}$.

$$\varepsilon_{imn} a_i = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{imn} a_{jk} = -\frac{1}{2}\begin{vmatrix} \delta_{ii} & \delta_{im} & \delta_{in} \\ \delta_{ji} & \delta_{jm} & \delta_{jn} \\ \delta_{ki} & \delta_{km} & \delta_{kn} \end{vmatrix} a_{jk}.$$

$$= -\frac{1}{2}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) a_{jk}.$$

$$= -\frac{1}{2}(a_{mn} - a_{nm}) = -\frac{1}{2}(a_{mn} + a_{mn}) = -a_{mn}.$$

$$\therefore a_{jk} = -\varepsilon_{ijk} a_i$$

EXERCISE 1-17

Cylindrical coordinates: $\xi^1 = r,\ \xi^2 = \theta,\ \xi^3 = z$.

$(ds)^2 = (dr)^2 + (rd\theta)^2 + (dz)^2 \Rightarrow h_1 = 1,\ h_2 = r,\ h_3 = 1$

$\hat{e}_r = \cos\theta e_1 + \sin\theta e_2,\ \hat{e}_\theta = -\sin\theta e_1 + \cos\theta e_2,\ \hat{e}_z = e_3$.

$$\frac{\partial \hat{e}_r}{\partial\theta} = \hat{e}_\theta,\ \frac{\partial \hat{e}_\theta}{\partial\theta} = -\hat{e}_r,\ \frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_\theta}{\partial r} = \frac{\partial \hat{e}_z}{\partial r} = \frac{\partial \hat{e}_z}{\partial\theta} = \frac{\partial \hat{e}_z}{\partial z} = 0.$$

$$\nabla = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}.$$

$$\nabla f = \hat{e}_r\frac{\partial f}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{e}_z\frac{\partial f}{\partial z}. \qquad \nabla\cdot u = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}.$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial f}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

$$\nabla\times u = \left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right)\hat{e}_z.$$

EXAMPLE PROOF:-

Suppose the basis $[e_1', e_2', e_3']$........

Solution:-

$$e_1' = \cos\theta e_1 + \sin\theta e_2.$$
$$e_2' = -\sin\theta e_1 + \cos\theta e_2.$$
$$e_3' = e_3.$$

$$[Q] = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$[A'] = [Q][A][Q^T]$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} A_{11}' & A_{12}' & A_{13} \\ A_{21}' & A_{22}' & A_{23} \\ A_{31}' & A_{32}' & A_{33} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_{11}\cos\theta + A_{12}\sin\theta. & -A_{11}\sin\theta + A_{12}\cos\theta & A_{13} \\ A_{21}\cos\theta + A_{22}\sin\theta. & -A_{21}\sin\theta + A_{22}\cos\theta \cdot & A_{23} \\ A_{31}\cos\theta + A_{32}\sin\theta & -A_{31}\sin\theta + A_{32}\cos\theta & A_{33} \end{pmatrix}$$

using half angles identities.

$$\left(\frac{A_{11}+A_{22}}{2}\right) + \left(\frac{A_{11}-A_{22}}{2}\right)\cos 2\theta + \left(\frac{A_{12}+A_{21}}{2}\right)\sin^2\theta.$$

$$\left(\frac{A_{21}-A_{12}}{2}\right) + \left(\frac{A_{21}+A_{12}}{2}\right)\cos 2\theta + \left(\frac{A_{22}-A_{11}}{2}\right)\sin 2\theta.$$

$$A_{31}\cos\theta + A_{32}\sin\theta.$$

$$\left(\frac{A_{12}-A_{21}}{2}\right) + \left(\frac{A_{12}+A_{21}}{2}\right)\cos 2\theta + \left(\frac{A_{22}-A_{11}}{2}\right)\sin 2\theta \qquad A_{13}\cos\theta + A_{23}\sin\theta$$

$$\left(\frac{A_{22}+A_{11}}{2}\right) + \left(\frac{A_{22}-A_{11}}{2}\right)\cos 2\theta - \left(\frac{A_{12}+A_{21}}{2}\right)\sin 2\theta \qquad A_{23}\cos\theta - A_{13}\sin\theta.$$

$$A_{32}\cos\theta - A_{31}\sin\theta.$$

Comparing b.s, we get:

$$A'_{11} = \frac{A_{11}+A_{22}}{2} + \frac{A_{11}-A_{22}}{2}\cos 2\theta + \frac{A_{12}+A_{21}}{2}\sin 2\theta.$$

$$A'_{12} = \frac{A_{12}-A_{21}}{2} + \frac{A_{12}+A_{21}}{2}\cos 2\theta + \frac{A_{22}-A_{11}}{2}\sin 2\theta.$$

$$A'_{13} = A_{13}\cos\theta + A_{23}\sin\theta.$$

$$A'_{21} = \frac{A_{21}-A_{12}}{2} + \frac{A_{21}+A_{12}}{2}\cos 2\theta + \frac{A_{12}-A_{11}}{2}\sin 2\theta.$$

$$A'_{22} = \frac{A_{22}+A_{11}}{2} + \frac{A_{22}-A_{11}}{2}\cos 2\theta - \frac{A_{12}+A_{21}}{2}\sin 2\theta.$$

$$A'_{23} = A_{23}\cos\theta - A_{13}\sin\theta.$$

$$A'_{31} = A_{31}\cos\theta + A_{32}\sin\theta.$$

$$A'_{32} = A_{32}\cos\theta - A_{31}\sin\theta.$$

$$A'_{33} = A_{33}.$$

In the special case when $[A]$ is symmetric in addition

$A_{13} = A_{23} = 0.$, so nine eq's simplify to.

$$A'_{11} = \frac{A_{11}+A_{22}}{2} + \frac{A_{11}-A_{22}}{2}\cos 2\theta + A_{12}\sin 2\theta.$$

$$A'_{22} = \frac{A_{11}+A_{22}}{2} - \frac{A_{11}-A_{22}}{2}\cos 2\theta - A_{12}\sin 2\theta$$

$$A'_{12} = \frac{-A_{11}-A_{21}}{2}\sin 2\theta.$$

together with $A'_{13} = A'_{23} = 0$ and $A'_{33} = A_{33}$. They are well known eq's underlying the Mohr's circle for transforming 2-tensor in 2D.