Impartial Combinatorial Games and Sprague-Grundy Theorem

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Impartial Combinatorial Game (ICG)

- There are 2 players - both the players have the same set of legal moves at every state of the game.
  - Example: Chess doesn't fall under this category, since at one state of the game, both the players have the same set of moves but not all of them are legal.
- Both the players take turns alternately.
- There exists a set of **Final Positions** in the game, at which the game ends ie no legal moves beyond this position.

Winning and Losing States

- All terminal states are **Losing (L).**
- From every **Winning (W)** state, we can go to **at least one L state.**
- From every **L state, all moves** go to a **W state ie** we are **forced** to go to a Winning state irrespective of what we move.

How to check whether your strategy is correct?

Your strategy should fulfil all of the following 3 conditions:

- All final positions must be **L.**
- From any **W,** you should be able to go to a **L.**
- From any **L,** you can only go to a **W**.

Question

- Alice and Bob can remove {1, 3, 4} stones from a pile of size 10.
- Suggest a strategy and who wins the game. (Alice plays first)

Nim Game

- Alice and Bob are playing a game where:
    - N piles, ith pile has Ai number of stones.
    - At each move, can choose any pile, and remove any **non-zero** number of stones.

How we do we solve a Nim Game? - Sprague Grundy Theorem

- For a given Nim Game with N piles, each having Ai stones:
    - XOR of all Ai ie X = A1 ⊕ A2 ⊕ ..... ⊕ Ai:
        - If **X > 0, player 1 wins.**
        - If **X = 0, player 2 wins.**

How to convert an ICG to a Nim Game? - Part 1

- To convert an ICG to a Nim Game, we use Grundy Number
    - Grundy Numbers are defined as follows
        - Grundy Number of terminal states is 0
        - Grundy Number of any other state S = MEX { Grundy Number of all states reachable from S }
- MEX (**M**inimum **EX**cluded):
    - MEX of a set is the smallest value that is not included in that set
    - For e.g. :
        - MEX {0} = 1
        - MEX {0,1} = 2
        - MEX {0,1,2,4} = 3
        - MEX {1,2} = 0

How to convert an ICG to a Nim Game? - Part 2

- We find the Grundy number of each pile and convert it into an equivalent Nim Game.
- Then we apply Sprague-Grundy Theorem on the new equivalent nim game.

Question 1

Alice and Bob are playing a game with N piles, ith pile having Ai stones. In one move, they can choose any pile and divide it into 2 non-empty piles. For example: a pile of size 5 can be divided into (1,4), (2,3), (3,2), (4,1). The person who can not make any move loses (a pile of size 1 can not be divided further). Determine the winner, Alice plays first.

Input: N piles and their sizes.

1 <= N <= 1e5

1 <= Ai <= 1e18

Question 2

Alice and Bob are playing a game with N piles, ith pile having Ai stones. In one move, they can remove any non-zero number of stones from a pile. In one move, they can also add a non-zero number of stones to a pile. However, this operation can be done at most K times. Determine the winner, Alice plays first.

Input: N piles and their sizes.

$1 <= N <= 1e5$

$1 <= Ai <= 1e18$

$1 <= K <= 1e5$

Question 3

Alice and Bob are playing a game with N groups, each group having Mi piles, with each pile having stone Ki, Ki+1, ... Ki+Mi-1. One can pick any pile from any group and take any non-zero number of stones. Determine the winner, Alice plays first.

For example, N = 2, K1 = 2 M1 = 1 ie group 1 = [2], K2 = 3 M2 = 2 ie group 2 =[3, 4]

Input: N piles and their sizes.

1 <= N <= 1e5

1 <= K,M <= 1e18

Solution Question 3

To take XOR of a range ie [L, R] = (XOR [1, L-1]) XOR (XOR [1, R])

XOR for [1, N]:

Reference: [GFG XOR of [L, R]]

```
def XORN(n):
    mod = n % 4

    if (mod == 0):
        return n

    elif (mod == 1):
        return 1

    elif (mod == 2):
        return n + 1

    elif (mod == 3):
        return 0
```

Question 4

Alice and Bob are playing a game with N piles, ith pile having Ai stones. For a pile containing x stones (at the moment):

- If x is even:
  - Remove one stone from the pile OR
  - Replace this pile with K piles of size x/2
- If x is odd:
  - Remove one stone from the pile.

Determine the winner, Alice plays first.

Input: N piles and their sizes, K.

$1 <= N <= 1e5$; $1 <= Ai <= 1e18$; $1 <= K <= 1e5$

Question 5

Alice and Bob are given a rectangle of size W x H. In one move, one can divide the rectangle into 2 parts, and give the bigger part to their opponent.

For example:

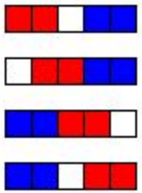
5 x 6 rectangle => 4 x 6 rectangle and 1 x 6 rectangle => 4 x 6 rectangle to the opponent.

Determine the winner, Alice plays first.

Input: 1 <= W, H <= 1e18

Question 6

Two players start with a strip of n white squares and they take alternate turns.
On each turn, a player picks two contiguous white squares and paints them black.
The first player who cannot make a move loses.



- n=1: No valid moves, so the first player loses automatically.
- n=2: Only one valid move, after which the second player loses.
- n=3: Two valid moves, but both leave a situation where the second player loses.
- n=4: Three valid moves for the first player, who is able to win the game by painting the two middle squares.
- n=5: Four valid moves for the first player (shown below in red), but no matter what the player does, the second player (blue) wins.

Constraints: n <= 1e3

Question 7

Alice and Bob are playing a game with N piles, ith pile having Ai stones.
In one move, they can pick any pile, say the ith pile having Ai stones and transform the number into $(Ai*j) / p$,
where p is any prime number which divides Ai and
j is an integer in range [1,p), that is $1 \le j < p$.

The player who can not make a move loses.