EE2703: Applied Programming Lab
Assignment No 6: The Laplace Transform

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EE20B046

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1 Introduction

The aim of this assignment is to analyse LTI systems using the `scipy.signal` library.

We majorly use the `scipy.signal.lsim()` and `scipy.signal.impulse()` functions to calculate the system response and the inverse Laplace transform respectively.

2 Questions

2.1 Question 1

We need to solve for the time response of a spring satisfying the equation:

$$\ddot{x} + 2.25x = f(t)$$

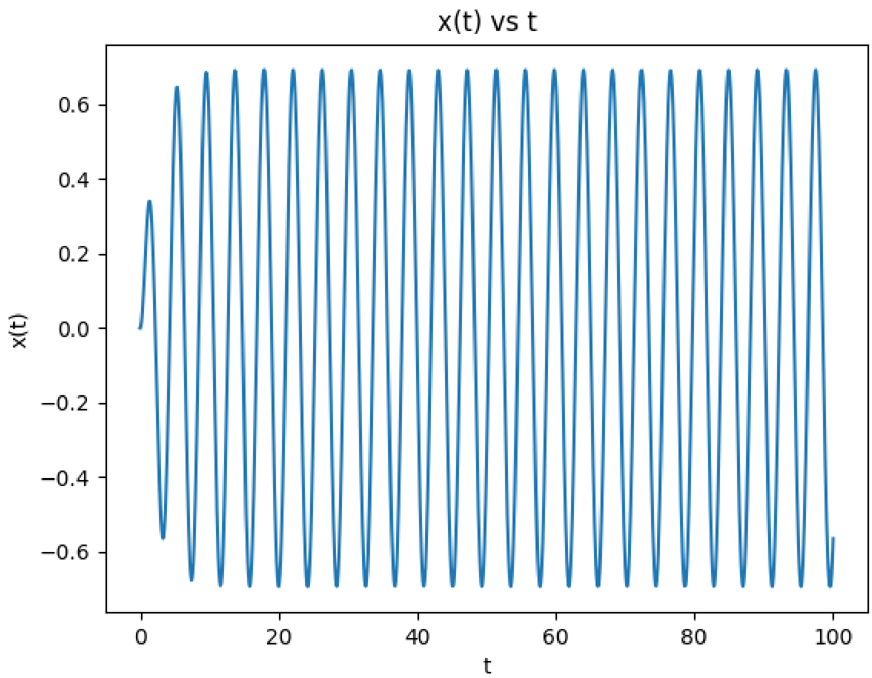
where

$$f(t) = \cos(1.5t)\exp(-0.5t)u(t)$$

We solve this by taking Laplace transform on both sides and then taking inverse Laplace of X(s) using `sp.impulse()` as explained in the code below.

```
#Question 1
#solving an equation using laplace transform

'''given system equation x''+2.25x = f(t)
where f(t) = cos(1.5t)e^(-0.5t)u(t)
and the Laplace transform of f(t) is
F(s) = (s+0.5)/((s+0.5)^2+2.25)

Thus, the given equation in Laplace domain is s^2 X(s) + 2.25 X
    (s) = F(s)
which implies X(s) = F(s)/(s^2+2.25)
which implies X(s) = (s+0.5) / ( [ (s+0.5)^2 + 2.25 ] [ s^2 +
    2.25 ] )
```

```
and x(t) = L^-1 (X(s))
'''
#X = (s+0.5) / ( [ s^2 + s + 2.5 ] [ s^2 + 2.25 ] )
X = sp.lti( np.poly1d([1,0.5]) , np.polymul( np.poly1d([1,1,2.5]) , np.poly1d([1,0,2.25]) ) )

#using sp.impulse to calculate the inverse laplace
t,x=sp.impulse(X, None, np.linspace(0,100,1001))

plt.plot(t,x)
plt.xlabel('t')
plt.ylabel('x(t)')
plt.title('x(t) vs t')
plt.show()
```

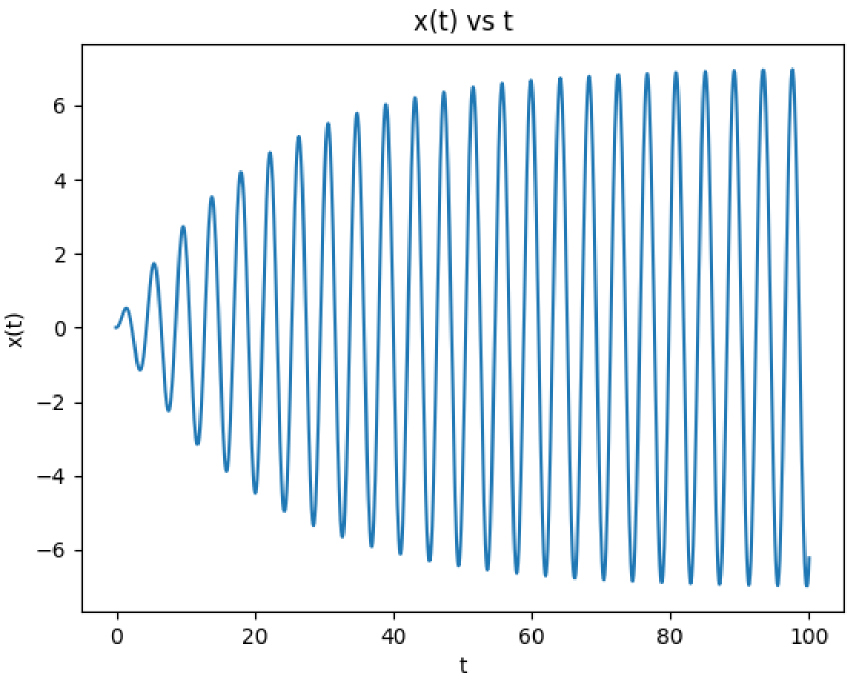


2.2 Question 2

We repeat the same problem with a smaller decay of 0.05 as explained below.

```
#Question 2
#Solving for a much smaller decay of 0.05

'''given system equation x''+2.25x = f(t)
where f(t) = cos(1.5t)e^(-0.05t)u(t)
and the Laplace transform of f(t) is
F(s) = (s+0.5)/((s+0.05)^2+2.25)

```

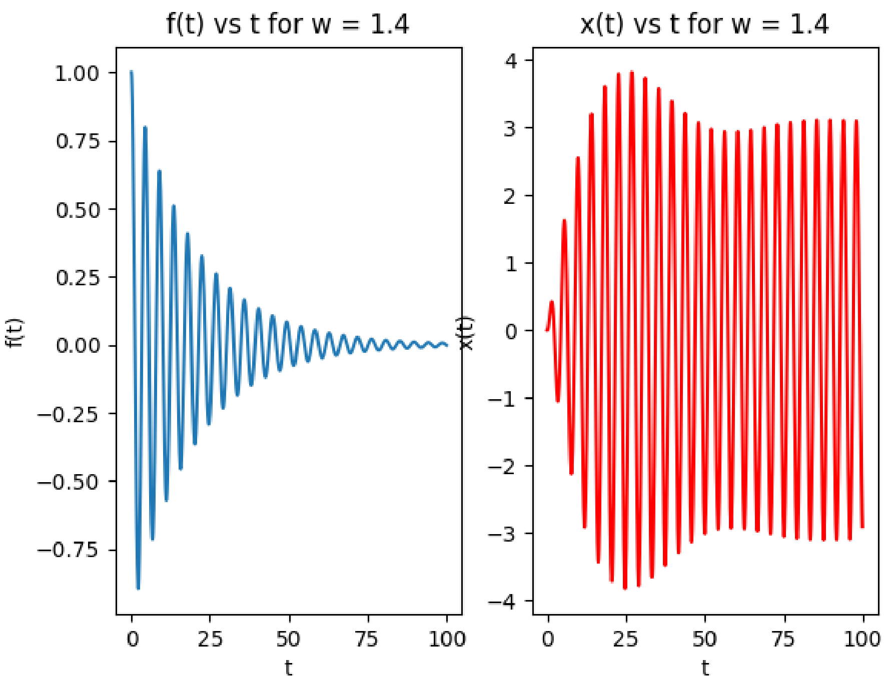
```
Thus, the given equation in Laplace domain is s^2 X(s) + 2.25 X
    (s) = F(s)
which implies X(s) = F(s)/(s^2+2.25)
which implies X(s) = (s+0.5) / ( [ (s+0.05)^2 + 2.25 ] [ s^2 +
    2.25 ] )
and x(t) = L^-1 (X(s))
'''
#X = (s+0.5) / ( [ s^2 + 0.1s + 2.2525 ] [ s^2 + 2.25 ] )
X = sp.lti( np.poly1d([1,0.5]) , np.polymul( np.poly1d
    ([1,0.1,2.2525]) , np.poly1d([1,0,2.25]) ) )

#using sp.impulse to calculate the inverse laplace
t,x=sp.impulse(X, None, np.linspace(0,100,1001))

plt.plot(t,x)
plt.xlabel('t')
plt.ylabel('x(t)')
plt.title('x(t) vs t')
plt.show()
```

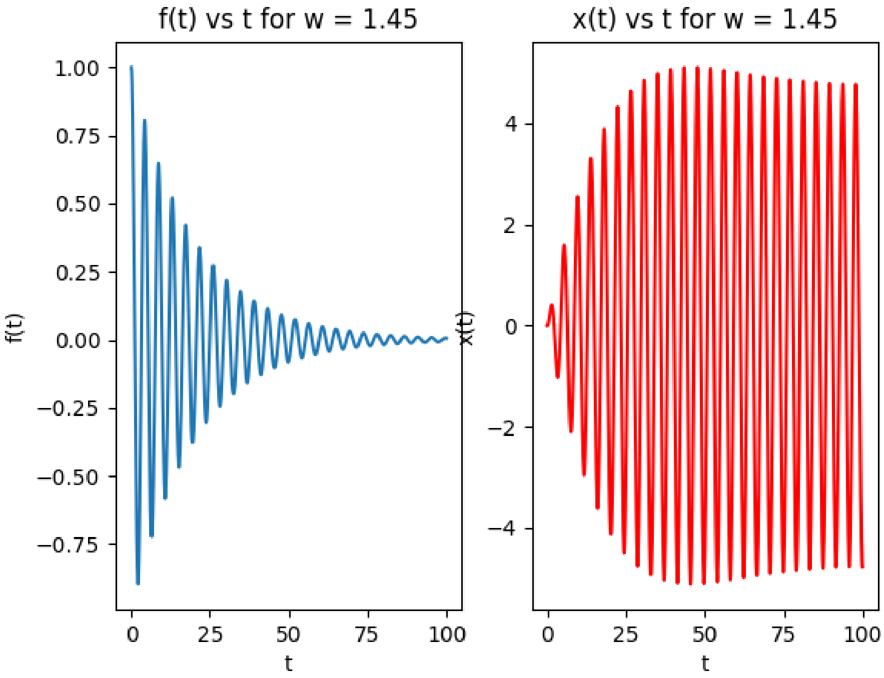


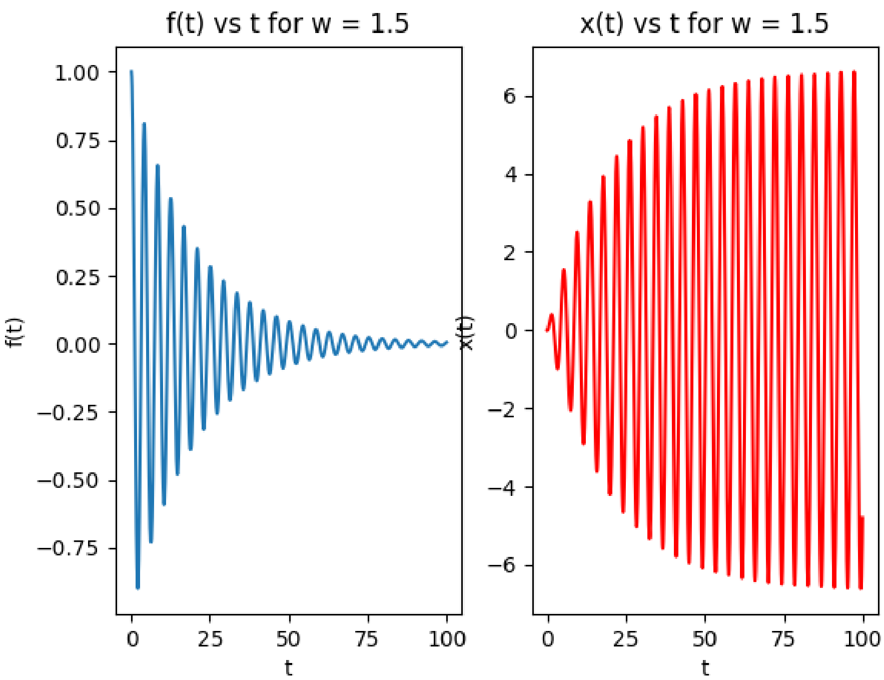
2.3 Question 3

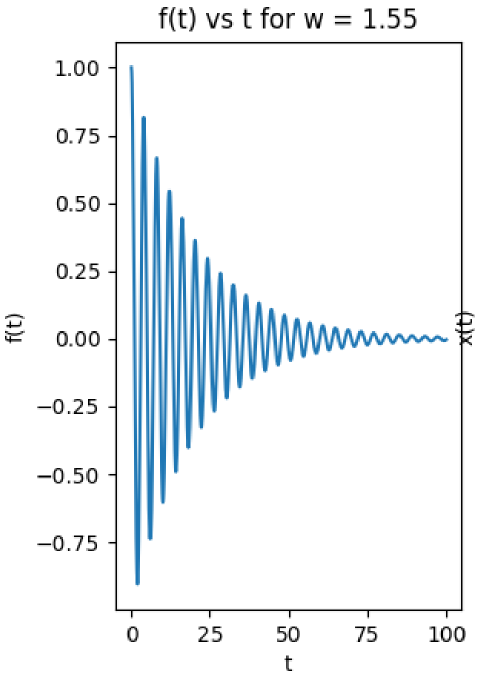
This time, we solve the problem for a range of frequencies i.e [1.4, 1.45, 1.5, 1.55, 1.6] by obtaining the transfer $H(s) = X(s)/F(s)$ and then using `sp.lsim` to calculate the output $x(t)$ for given input signals, as explained in the code below.

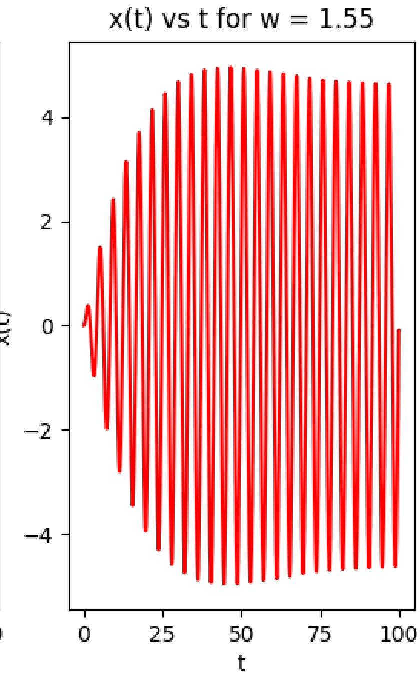
```python
#Question 3
#Finding the LTI response over a range of frequencies in 1.4 to 1.6 with step of 0.05
for w in np.arange(1.4, 1.6, 0.05):
    H = sp.lti([1], [1, 0, 2.25]) #H(s) = 1/(s^2+2.25)
    t = np.linspace(0, 100, 1000)
    f = np.cos(w * t) * np.exp(-0.05 * t) #f(t) = cos(wt)e^(-0.05t)u(t)

    #making subplots of the corresponging input and output

    plt.subplot(1, 2, 1)
    plt.plot(t, f)
    plt.xlabel('t')
    plt.ylabel('f(t)')
    plt.title('f(t) vs t for w = ' + str(w))

    t, x, svec = sp.lsim(H, f, t) # using lsim to calculate the output of the system
    plt.subplot(1, 2, 2)
    plt.plot(t, x, 'r')
    plt.xlabel('t')
    plt.ylabel('x(t)')
    plt.title('x(t) vs t for w = ' + str(w))
    plt.show()
```

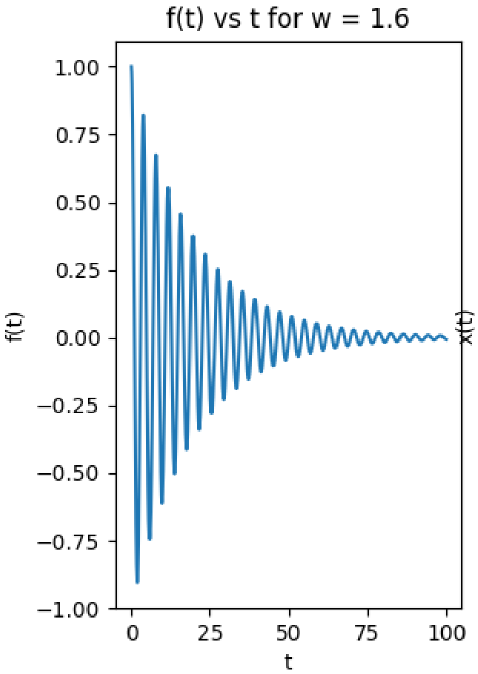


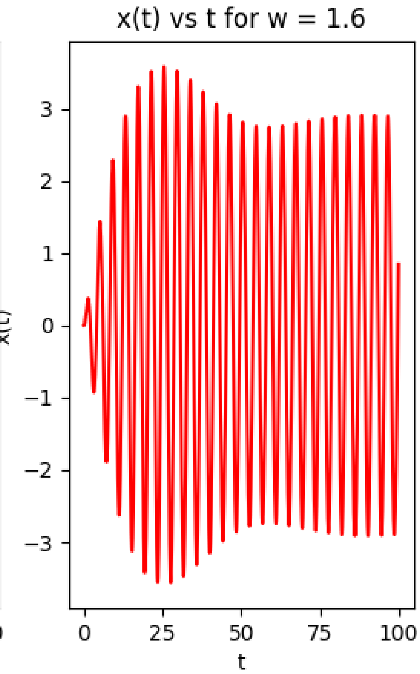

f(t) vs t for w = 1.45
x(t) vs t for w = 1.45
f(t)
x(t)
t
t


f(t) vs t for w = 1.5
x(t) vs t for w = 1.5
f(t)
x(t)
t
t


f(t) vs t for w = 1.55
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
f(t)
0
25
50
75
100
t


x(t) vs t for w = 1.55
4
2
0
−2
−4
x(t)
0
25
50
75
100
t


f(t) vs t for w = 1.6
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
f(t)
0
25
50
75
100
t


x(t) vs t for w = 1.6
3
2
1
0
−1
−2
−3
x(t)
0
25
50
75
100
t

From the given equation, we notice that the natural frequency of the system is 1.5. We notice that the plot for w = 1.5 has the highest amplitude, which is understandable from our knowledge of physics, since this is the case of resonance!

2.4 Question 4

We have a set of coupled spring equations:

$$\ddot{x} + (x - y) = 0$$

$$\ddot{y} + 2(y - x) = 0$$

On solving these in Laplace domain, with the given initial conditions, we get:

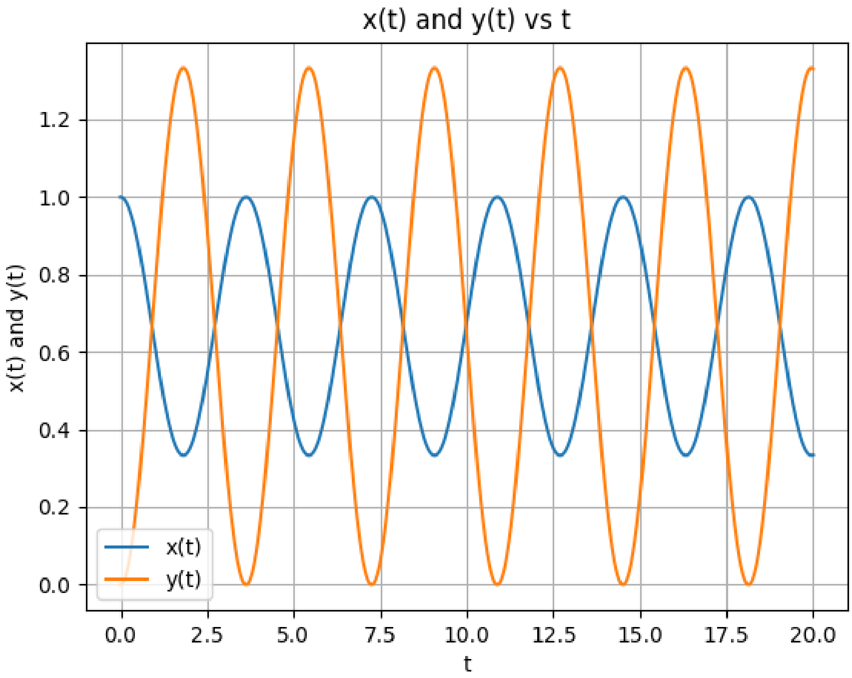
$$X(s) = \frac{s^2 + 2}{s^3 + 3s}$$

$$Y(s) = \frac{2}{s^3 + 3s}$$

Now using inverse Laplace, we obtain $x(t)$ and $y(t)$ and they are plotted in the time interval $0 < t < 20s$ as follows.

```
#Question 4
#Solving for a coupled spring problem
"""
Given Equations:
x'' + x - y =0
y'' + 2y - 2x = 0
subject to initial conditions
x(0) = 1
y(0) = 0
x'(0) = 0
y'(0) = 0
On substituting y from first equation into the second equation,
    we get
x"" + 3x'' = 0
Taking Laplace Transform keeping initial conditions in mind,
    this becomes,
X(s) = (s^2+2) / (s^3 + 3s)
Y(s) = 2/(s^3 + 3s)

"""

t = np.linspace(0, 20, 1001)
X = sp.lti([1, 0, 2], [1, 0, 3, 0])
Y = sp.lti([2], [1, 0, 3, 0])
#using sp.impulse to calculate the inverse laplace
t,x=sp.impulse(X, None, t)
t,y=sp.impulse(Y, None, t)

#plotting x and y
```

```
plt.plot(t,x)
plt.plot(t,y)
plt.xlabel('t')
plt.ylabel('x(t) and y(t)')
plt.title('x(t) and y(t) vs t')
plt.legend(['x(t)','y(t)'])
plt.grid()
plt.show()
```



2.5 Question 5

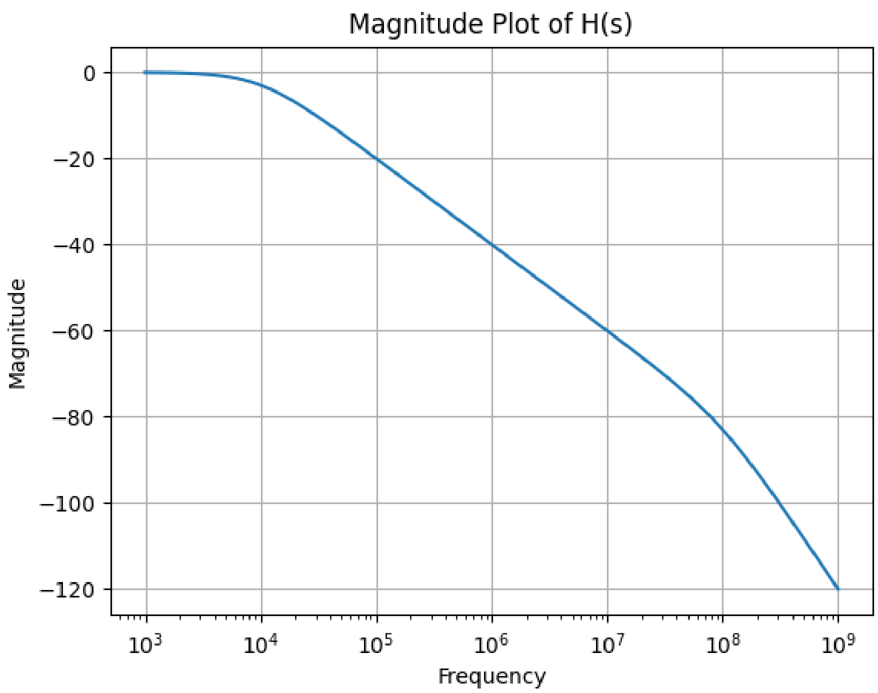
Here, we have an RLC filter with the filter transfer function value as

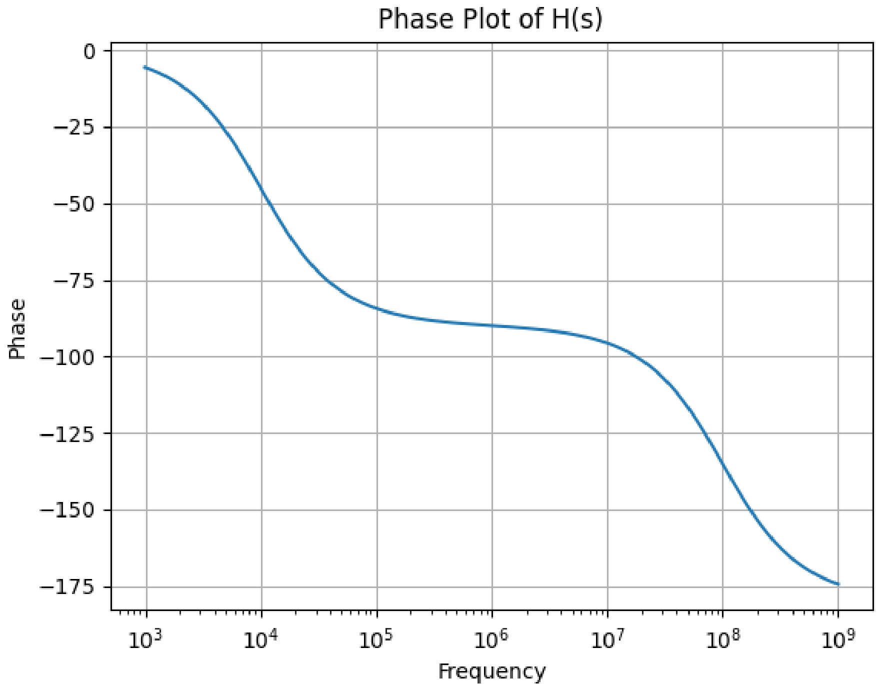
$$H(s) = \frac{10^{12}}{s^2 + 10^8 s + 10^{12}}$$

We now find and plot the magnitude and phase Bode plots of this filter using `sp.Bode()` as follows.

```
#Question 5
#To obtain the magnitude and phase of the steady state transfer
    function of the system
'''
Upon manual solving, we get,
H(s) = 10^12/( s^2 + 10^8 s + 10^12)
'''
```

```
H = sp.lti([10**12], [1, 10**8, 10**12])
#plotting Bode magnitude plot
w, mag, phase = sp.bode(H)
plt.semilogx(w, mag)
plt.xlabel('Frequency')
plt.ylabel('Magnitude')
plt.title('Magnitude Plot of H(s)')
plt.grid()
plt.show()

#plotting Bode phase plot
plt.semilogx(w, phase)
plt.xlabel('Frequency')
plt.ylabel('Phase')
plt.title('Phase Plot of H(s)')
plt.grid()
plt.show()
```



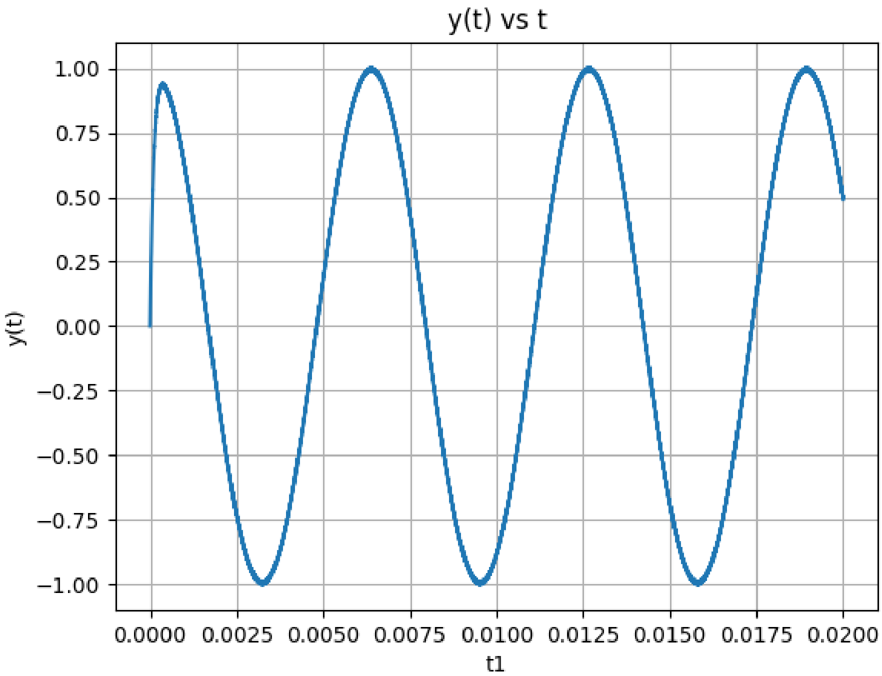


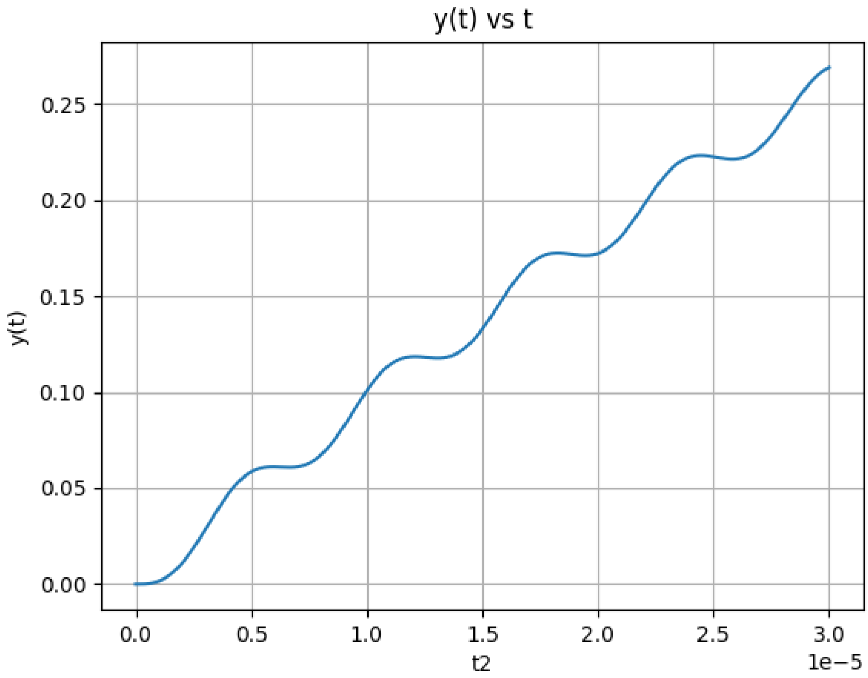
2.6 Question 6

As we have defined our filter in the previous problem, we use that to find the output of our system using `sp.lsim()`. To capture the fast variation in the beggining, we also solve the problem for the time interval of $0 < t < 30\mu s$.

```
#Question 6
# vi(t) = cos(10^3t)u(t) - cos(10^6t)u(t)
# To define the transfer function as a system and obtain the
    output
t1 = np.linspace(0, 0.02, 10001)
t2 = np.linspace(0, 30*10**-6, 10001) #initial time interval
    zoomed in
vi1 = np.cos(10**3 * t1) - np.cos(10**6 * t1)
vi2 = np.cos(10**3 * t2) - np.cos(10**6 * t2)
H = sp.lti([10**12], [1, 10**8, 10**12])
#Finding the output using sp.lsim
t1, y1, svec = sp.lsim(H, vi1, t1)
t2, y2, svec = sp.lsim(H, vi2, t2)


#plotting the output
plt.plot(t1, y1)
plt.xlabel('t1')
plt.ylabel('y(t)')
plt.title('y(t) vs t')
plt.grid()
```

```
plt.show()

plt.plot(t2, y2)
plt.xlabel('t2')
plt.ylabel('y(t)')
plt.title('y(t) vs t')
plt.grid()
plt.show()
```





By manual calculations, we see that the 3-dB bandwidth of the given RLC filter is 10^4 rad/s. We know that, the signal components with frequency more than the 3-dB bandwidth will be highly attenuated and thus the high frequency component can only be seen as a ripple in the small time interval plot, and is not visible in the large interval time plot.

3 Conclusion

We have learnt to analyse LTI systems using the Laplace tranforms and have learnt to do the same using the `scipy.signal` library. We have also learnt to calculate time domain response, Bode plots, Laplace tranforms etc. and also plotted graphs for the same.