Topics: Probability Warm-up, Blocking and Statistical Multiplexing
First four problems are meant to brush up your basics of probability (EE3110 level). You must be able to solve these problems comfortably on your own.

Q. 1 Conditional probability. For two events A and B conditional probability of A given B is denoted by $\mathbb{P}(A|B)$ and is defined as

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)},$$

whenever $\mathbb{P}(B) > 0$, else $\mathbb{P}(A|B)$ is 0.
Independence. Two events A and B are independent of each other if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

This definition can be generalized to more than two events: $\{A_i : 1 \leq i \leq n\}$ are said to be mutually independent if for any subset K of $\{1, 2, \cdots, n\}$

$$\mathbb{P}(\cap_{i \in K} A_i) = \prod_{i \in K} \mathbb{P}(A_i).$$

Bayes' theorem. Let A_i, $1 \leq i \leq n$ be a collection of mutually exclusive (disjoint, i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$) and exhaustive ($\cup_{i=1}^n A_i = \Omega$) events, and B be another event then

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i \cap B)}{\sum_{i=1}^n \mathbb{P}(B|A_i)\mathbb{P}(A_i)}.$$

The beauty of this theorem is that it allows one to compute probabilties of complicated conditional events from that of "basic" events. In many real life scenarios you may only have the probabilities of some "basic" events estimated based on historical records. But, you need to know some other complicated scenarios.

1. Prove Bayes' theorem (on your own).
2. A couple has two children. What is the probability that one of the two is a girl given at least one is a boy? (use common sense to find the basic events and assign probabilities to them, and then use Bayes' theorem)

Q. 2 Independence of random variables. Two random variables X_1 and X_2 are said to be independent if events generated by them are independent, i.e., for any two subsets R_1 and R_2 (they should belong to the respective σ-algebras, but we chose not to be so rigorous about that in this course) of $\mathbb{R}$

$$\mathbb{P}\left(\{X \in R_1\} \cap \{X_2 \in R_2\}\right) = \mathbb{P}\{X \in R_1\}\mathbb{P}\{X_2 \in R_2\}.$$

Similarly following the notion of independence of n events one can define n random variables $X_1, X_2, \cdots, X_n$ to be mutually independent if for any $R_1, R_2, \cdots, R_n$ (again measurability required, but skipped for simplicity) $\{X_i \in R_i\}$ are independent.

1. Assuming $\{X_i : 1 \leq i \leq n\}$ to be either continuous or discrete random variables (try both cases) show that for any set of functions $f_1, f_2, \cdots, f_n$ (from $\mathbb{R}$ to $\mathbb{R}$) and any subset K of $[n] := \{1, 2, \cdots n\}$ we have
$$\mathbb{E}[\sum_{i=1}^n f_i(X_i)] = \sum_{i=1}^n \mathbb{E}[f_i(X_i)].$$
2. Assuming $\{X_i : 1 \leq i \leq n\}$ to be either continuous or discrete random variables (try both cases) show that for any set of functions $f_1, f_2, \cdots, f_n$ (from $\mathbb{R}$ to $\mathbb{R}$) and any subset K of $[n] := \{1, 2, \cdots n\}$ we have
$$\mathbb{E}[\prod_{i=1}^n f_i(X_i)] = \prod_{i=1}^n \mathbb{E}[f_i(X_i)],$$
if $\{X_i\}$ are mutually independent.

Q. 3 Understanding variance and risk. We learned about expectation of a random variable in class. But, as you can clearly see the random variable varies around its expectation (a.k.a mean). The variation of a random variable measures its expected square deviation around the mean:

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

If X is an outcome of a random phenomenon (e.g., return from an investment) then the variance is a useful measure of the associated "uncertainty".
For two random variables X and Y we define covariance (a.k.a. correlation coefficient) between X and Y as

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y]).$$

Let n be a positive integer. We are given $n + 1$ *independent* random variables $X, X_1, X_2 \cdots, X_n$ with the same mean and variance.

1. Which of the following has a lower variance :

   $$\mathrm{Var}(X_1 + X_2 \cdots + X_n) \quad \text{or} \quad \mathrm{Var}(nX)?$$

   Compute each of these to relate them.

2. The selling point of mutual funds is that they are less risky than buying a single stock. Explain this statement using the result to the previous question.

3. Suppose the $X_1, X_2, \ldots, X_n$ have correlation coefficient ρ. Now under what conditions does the sum of random variables have lower variance?

Q. 4 Positive correlation. We say that two events A and B are *positively correlated* if

$$P(A|B) > P(A).$$

Prove or disprove that if A and B are positively correlated then $P(B|A) > P(B)$.

Q. 5 Both the statistical condition and the multiplexing technique contribute to statistical multiplexing gain From class notes, recall the deterministic and the statistical conditions for supporting n calls at time t_0 in a fiber/cable of capacity C. Let $r_i(t_0)$ be i.i.d. for $i = 1, 2, \ldots n$ and have the following distribution: $\mathbb{P}(r_i(t_0) = 0) = p$, $\mathbb{P}(r_i(t_0) = r) = q$ and $\mathbb{P}(r_i(t_0) = R) = 1 - p - q$ for $0 < p < p + q < 1$ and $0 < r < R$.

- Prove that multiplexing cannot meet the deterministic condition.
- Prove that for any reasonable probability requirement δ, circuit switching cannot meet the statistical condition for number of users $\geq \frac{(1+\epsilon)C}{R}$ for any $\epsilon > 0$.
- Find the number of users that can be supported under statistical multiplexing (packet switching).

Q. 6 There are infinite number of users indexed by positive integers. User k requires an instantaneous rate X_k, which is a random variable upper bounded by 3. The average rate required by user k is $1 + f(k)$ for some positive function $f(\cdot)$. If users are allowed in the increasing order of their indices, how many users can be accommodated with probability $1 - \delta$ in a pipe of capacity C?
You may not get a nice closed form, but understand the process of getting the answer. Try different $f(\cdot)$ like $\frac{1}{k}$, $\frac{1}{k^2}$.