Q2. 2D fully developed steady flow, $V_z = 0$: $V_x, V_y\ (x)$

C.E. $\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \Rightarrow V_x = \text{const}.\quad x=0, V_x = 0 \Rightarrow V_x = 0$

N.S. x: $\frac{\partial p}{\partial x} = 0 \Rightarrow p = p(y)$. At $x = h$, $p = p_{atm}\ \forall y \Rightarrow p = p_{atm}\ \forall(x,y)$

$\frac{\partial p}{\partial y} = 0$ N.S. y dir. $\Rightarrow \mu \frac{\partial^2 V_y}{\partial x^2} = \rho g \Rightarrow V_y = \frac{\rho g}{\mu}\frac{x^2}{2} + C_1 x + C_2$

B.C. $x = 0, V_y = V_0 \Rightarrow C_2 = V_0$; $x = h, \frac{\partial V_y}{\partial x} = 0 \Rightarrow C_1 = -\rho g h/\mu$

$\Rightarrow V_y = V_0 - \frac{\rho g x}{\mu}\left(h - \frac{x}{2}\right)$; $\frac{Q}{w} = \int V_y dx = V_0 h - \rho g h^3/3\mu$

Q3. 2 flow regimes: $0 < y < \frac{h}{2}$: A ; $h/2 < y < h$: B $\Rightarrow$ steady state, 2D, fully developed $V_x^A, V_y^A = f(y)$, V_x^B, V_y^B

C.E. $\Rightarrow V_y^A = \text{const}.$, $V_y^B = \text{const}.$ B.C. at $y = 0 \Rightarrow V_y^A = 0$; $y = h$, $V_y^B = 0$

Given $\frac{\partial p}{\partial x} = 0$, $g f_x = 0$

N.S. in x direction $\Rightarrow \frac{d^2 V_x^A}{dy^2} = 0 \Rightarrow V_x^A = C_1 y + C_2$ B.C. $y = 0$ $V_x^A = 0$

$\frac{d^2 V_x^B}{dy^2} = 0 \Rightarrow V_x^B = C_3 y + C_4$ $y = h/2$ $V_x^A = V_x^B$, $\mu_A \frac{dV_x^A}{dy} = \mu_B \frac{dV_x^B}{dy}$

$y = h$ $V_x^B = 0$

$\Rightarrow C_2 = 0$
$V_0 = C_3 h + C_4$
$C_1 \frac{h}{2} = C_3 \frac{h}{2} + C_4$
$\mu_A C_1 = \mu_B C_3$

$\Rightarrow V_x^A = \frac{\mu_B}{\mu_A + \mu_B}\frac{2 V_0 y}{h}$

$V_x^B = \frac{\mu_A}{\mu_A + \mu_B} \frac{2 V_0 (y-h)}{h} + V_0$

(Notice: velocity continuous at interface but slopes are not)

Q4. Fully developed $\Rightarrow \frac{\partial \vec{V}}{\partial z} = 0$ Assume $V_\theta = 0$ (laminar flow). Axisymmetry $\Rightarrow \frac{\partial ()}{\partial \theta} = 0$

Steady $\Rightarrow \frac{\partial ()}{\partial t} = 0$; $\vec{V} = V_r \hat{e}_r + V_z \hat{e}_z$;

C.E. $\Rightarrow \frac{\partial}{\partial r}(r V_r) = 0 \Rightarrow r V_r = \text{Const}. \Rightarrow V_r = 0$ (B.C. at $r = R$)

N.S. r: $\frac{\partial p}{\partial r} = 0$; θ: $0 = 0$; z: $-\frac{\partial p}{\partial z} - \rho g + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V_z}{\partial r}\right) = 0$

$\frac{\partial p}{\partial r} = 0, \frac{\partial p}{\partial \theta} = 0 \Rightarrow p = f(z)$ at $r = R - h$, $p = p_{atm}\ \forall z \Rightarrow f(z) = p_{atm} \Rightarrow \frac{\partial p}{\partial z} = 0.$

$\Rightarrow \frac{\mu}{r}\frac{d}{dr}\left(r \frac{dV_z}{dr}\right) = \rho g \Rightarrow V_z = \frac{\rho g r^2}{4\mu} + C_1 \ln r + C_2$ at $r = R$, $V_z = 0$; $r = R - h$, $\tau_{rz} = \mu \frac{dV_z}{dr} = 0$

$\Rightarrow C_1 = -\frac{\rho g (R-h)^2}{2\mu}$; $C_2 = -\frac{\rho g R^2}{4\mu} + \frac{\rho g}{2\mu}(R-h)^2 \ln R \Rightarrow V_z = \frac{\rho g}{2\mu}\left\{(R-h)^2 \ln \frac{R}{r} - \frac{(R^2 - r^2)}{2}\right\}$

Q5. radial flow $\Rightarrow V_r \neq 0$, $V_\theta = V_z = 0$ $\frac{\partial ()}{\partial \theta} = 0$, $\frac{\partial ()}{\partial t} = 0$, $\frac{\partial \vec{V}}{\partial z} = 0$

C.E. $\Rightarrow \frac{\partial}{\partial r}(r V_r) = 0 \Rightarrow r V_r = \text{Const}.$ at $r = R_1$, $V_r = V_1 \Rightarrow V_r = \frac{R_1 V_1}{r}$

N.S. $\frac{\partial p}{\partial \theta} = 0$, $\frac{\partial p}{\partial z} = -\rho g$, $\frac{\partial p}{\partial r} = -\rho V_r \frac{dV_r}{dr}$; $P = p + \rho g z \Rightarrow \frac{\partial P}{\partial z} = \frac{\partial p}{\partial z} + \rho g = 0$

$P = P(r)$ $\frac{\partial p}{\partial r} = \frac{dP}{dr} \Rightarrow \frac{dP}{dr} = -\rho\left(\frac{R_1 V_1}{r}\right)\left(\frac{-R_1 V_1}{r^2}\right) = \frac{\rho V_1^2 R_1^2}{r^3} \Rightarrow P = -\frac{\rho V_1^2 R_1^2}{2 r^2} + C$

$P_1 = -\frac{\rho V_1^2}{2} + C \Rightarrow C = P_1 + \frac{\rho V_1^2}{2} \Rightarrow P = P_1 + \frac{\rho V_1^2}{2} - \frac{\rho V_1^2 R_1^2}{2 r^2} = P_1 + \frac{\rho V_1^2}{2}\left(1 - \frac{R_1^2}{r^2}\right)$

(Note by defining P we get rid of functionality of z in pressure)

Q6. Flow same as in Q2 of Tutorial Sheet with $\frac{\partial p}{\partial z} = 0 \Rightarrow V_z = C_1 \ln r + C_2$

B.C. at $r = R_1$, $V_z = V_0$, at $r = R_2$ $V_z = 0$ $\Rightarrow V_z = V_0 \frac{\ln r/R_2}{\ln \frac{R_1}{R_2}}$; Power = (shear force on wire) $\times V_0$ $= \tau_{rz}|_{R_1} 2\pi R_1 L \times V_0$

$\tau_{rz} = \mu \frac{dV_z}{dr}\Big|_{R_1} = \frac{\mu V_0}{\ln R_1/R_2}$

$\Rightarrow \text{Power} = \frac{\mu V_0^2}{\ln R_1/R_2} 2\pi R_1 L$