APPENDIX - B

SOME USEFUL FORMULAE

B.1. Gradient of a Scalar, $\nabla\eta$

Cartesian

$$\nabla\eta = \frac{\partial\eta}{\partial x}\hat{i} + \frac{\partial\eta}{\partial y}\hat{j} + \frac{\partial\eta}{\partial z}\hat{k}$$

Cylindrical

$$\nabla\eta = \frac{\partial\eta}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\eta}{\partial\theta}\hat{\theta} + \frac{\partial\eta}{\partial z}\hat{k}$$

Spherical

$$\nabla\eta = \frac{\partial\eta}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\eta}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\eta}{\partial\phi}\hat{\phi}$$

B.2. Divergence of a Vector, $\nabla \cdot \mathbf{A}$

Cartesian

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Cylindrical

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial A_r}{\partial r} + \frac{A_r}{r} + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}$$

Spherical

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$= \frac{\partial A_r}{\partial r} + \frac{2A_r}{r} + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{A_\theta\cot\theta}{r} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

B.3. Curl of a Vector, $\nabla \times \mathbf{A}$

Cartesian

$$\nabla\times\mathbf{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Cylindrical

$$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{1}{r}\hat{r} & \hat{\theta} & \frac{1}{r}\hat{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$

Spherical

$$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{1}{r^2\sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

B.4. Laplace Operator, $\nabla^2\eta$

When η is a scalar, Φ

Cartesian $$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

Cylindrical $$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

Spherical $$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

When η is a vector **V**

The relevant form is the one contained within the curly brackets on the RHS of Navier-Stokes equation, Sec. B.9.

B.5. Material Rate of Change

$$\frac{D\eta}{Dt} = \frac{\partial\eta}{\partial t} + (\mathbf{V}\cdot\nabla)\eta$$

When η is a scalar Φ

Cartesian $$\frac{D\Phi}{Dt} = \frac{\partial\Phi}{\partial t} + V_x\frac{\partial\Phi}{\partial x} + V_y\frac{\partial\Phi}{\partial y} + V_z\frac{\partial\Phi}{\partial z}$$

Cylindrical $$\frac{D\Phi}{Dt} = \frac{\partial\Phi}{\partial t} + V_r\frac{\partial\Phi}{\partial r} + \frac{V_\theta}{r}\frac{\partial\Phi}{\partial\theta} + V_z\frac{\partial\Phi}{\partial z}$$

Spherical $$\frac{D\Phi}{Dt} = \frac{\partial\Phi}{\partial t} + V_r\frac{\partial\Phi}{\partial r} + \frac{V_\theta}{r}\frac{\partial\Phi}{\partial\theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

When η is a vector **V**

The relevant form is the one contained within the curly brackets on the LHS of the Navier-Stokes equation, Sec. B.9.

B.6. Rates of Deformations

Cartesian

$$\mathring{\epsilon}_x = \partial V_x/\partial x \qquad \mathring{\epsilon}_y = \partial V_y/\partial y \qquad \mathring{\epsilon}_z = \partial V_z/\partial z$$

$$\omega = \frac{1}{2}(\nabla\times\mathbf{V}) \qquad \text{(Sec. B.3)}$$

$$\mathring{\gamma}_{xy} = \mathring{\gamma}_{yx} = \frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}$$

$$\mathring{\gamma}_{yz} = \mathring{\gamma}_{zy} = \frac{\partial V_z}{\partial y} + \frac{\partial V_y}{\partial z}$$

$$\mathring{\gamma}_{zx} = \mathring{\gamma}_{xz} = \frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}$$

Cylindrical

$$\mathring{\epsilon}_r = \partial V_r/\partial r \qquad \mathring{\epsilon}_\theta = \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r} \qquad \mathring{\epsilon}_z = \partial V_z/\partial z$$

$$\omega = \frac{1}{2}(\nabla\times\mathbf{V}) \qquad \text{(Sec. B.3)}$$

$$\mathring{\gamma}_{r\theta} = \mathring{\gamma}_{\theta r} = \frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r} + \frac{1}{r}\frac{\partial V_r}{\partial_\theta}$$

$$\mathring{\gamma}_{\theta z} = \mathring{\gamma}_{z\theta} = \frac{1}{r}\frac{\partial V_z}{\partial\theta} + \frac{\partial V_\theta}{\partial z}$$

$$\mathring{\gamma}_{zr} = \mathring{\gamma}_{rz} = \frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r}$$

Spherical

$$\mathring{\epsilon}_r = \frac{\partial V_r}{\partial r} \qquad \mathring{\epsilon}_\phi = \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} + \frac{V_r}{r} + \frac{V_\theta\cot\theta}{r} \qquad \mathring{\epsilon}_\theta = \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r}$$

$$\omega = \frac{1}{2}(\nabla\times\mathbf{V}) \qquad \text{(Sec. B.3)}$$

$$\mathring{\gamma}_{r\theta} = \mathring{\gamma}_{\theta r} = \frac{r\partial(V_\theta/r)}{\partial r} + \frac{1}{r}\frac{\partial V_r}{\partial\theta}$$

$$\mathring{\gamma}_{\phi\theta} = \mathring{\gamma}_{\theta\phi} = \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{V_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial\phi}$$

$$\mathring{\gamma}_{\phi r} = \mathring{\gamma}_{r\phi} = \frac{1}{r\sin\theta}\frac{\partial V_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{V_\phi}{r}\right)$$

B.7. Newton-Stokes Law (see Sec. B.6. also)

Cartesian

$$\sigma_{xx} = -p + 2\mu\mathring{\epsilon}_x - \frac{2}{3}\mu\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

$$\sigma_{yy} = -p + 2\mu\mathring{\epsilon}_y - \frac{2}{3}\mu\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

$$\sigma_{zz} = -p + 2\mu\mathring{\epsilon}_z - \frac{2}{3}\mu\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

$$\tau_{xy} = \tau_{yx} = \mu\mathring{\gamma}_{xy}$$

$$\tau_{yz} = \tau_{zy} = \mu\mathring{\gamma}_{yz}$$

$$\tau_{zx} = \tau_{xz} = \mu\mathring{\gamma}_{zx}$$

Cylindrical

$$\sigma_{rr} = -p + 2\mu\mathring{\epsilon}_r - \frac{2}{3}\mu\left(\frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} + \frac{V_r}{r}\right)$$

$$\sigma_{\theta\theta} = -p + 2\mu\mathring{\epsilon}_\theta - \frac{2}{3}\mu\left(\frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} + \frac{V_r}{r}\right)$$

$$\sigma_{zz} = -p + 2\mu\mathring{\epsilon}_z - \frac{2}{3}\mu\left(\frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} + \frac{V_r}{r}\right)$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\mathring{\gamma}_{r\theta};\ \tau_{\theta z} = \tau_{z\theta} = \mu\mathring{\gamma}_{\theta z};\ \tau_{zr} = \tau_{rz} = \mu\mathring{\gamma}_{zr}$$

Spherical

$$\sigma_{rr} = -p + 2\mu\mathring{\epsilon}_r - \frac{2}{3}\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(V_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}\right]$$

$$\equiv -p + 2\mu\mathring{\epsilon}_r - \frac{2}{3}\mu\,(\nabla\cdot\mathbf{V})$$

$$\sigma_{\phi\phi} = -p + 2\mu\mathring{\epsilon}_\phi - \frac{2}{3}\mu\,(\nabla\cdot\mathbf{V})$$

$$\sigma_{\theta\theta} = -p + 2\mu\mathring{\epsilon}_\theta - \frac{2}{3}\mu\,(\nabla\cdot\mathbf{V})$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\mathring{\gamma}_{\theta r};\ \tau_{r\phi} = \tau_{\phi r} = \mu\mathring{\gamma}_{r\phi};\ \tau_{\theta\phi} = \tau_{\phi\theta} = \mu\mathring{\gamma}_{\phi\theta}$$

B.8. Continuity Equation

$$\boxed{\frac{\partial \rho}{\partial t} + \{\nabla.(\rho \mathbf{V})\} = 0}$$

Cartesian $$\frac{\partial \rho}{\partial t} + \left\{\frac{\partial}{\partial x}(\rho V_x) + \frac{\partial}{\partial y}(\rho V_y) + \frac{\partial}{\partial z}(\rho V_z)\right\} = 0$$

Cylindrical $$\frac{\partial \rho}{\partial t} + \left\{\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z)\right\} = 0$$

Spherical $$\frac{\partial \rho}{\partial t} + \left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho V_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\rho V_\phi\right)\right\} = 0$$

B.9. Navier-Stokes Equation

$$\boxed{\rho\left\{\frac{D\mathbf{V}}{Dt}\right\} \equiv \rho\left\{\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right\} = \rho\mathbf{f}\,\nabla p + \mu\left\{\nabla^2\mathbf{V}\right\}}$$

Cartesian

$$\rho\left\{\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z}\right\}$$

$$= \rho f_x - \frac{\partial p}{\partial x} + \mu\left\{\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right\}$$

$$\rho\left\{\frac{\partial V_y}{\partial t} + V_x\frac{\partial V_y}{\partial x} + V_y\frac{\partial V_y}{\partial y} + V_z\frac{\partial V_y}{\partial z}\right\}$$

$$= \rho f_y - \frac{\partial p}{\partial y} + \mu\left\{\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right\}$$

$$\rho\left\{\frac{\partial V_z}{\partial t} + V_x\frac{\partial V_z}{\partial x} + V_y\frac{\partial V_z}{\partial y} + V_z\frac{\partial V_z}{\partial z}\right\}$$

$$= \rho f_z - \frac{\partial p}{\partial z} + \mu\left\{\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right\}$$

Cylindrical

$$\rho\left\{\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right\}$$

$$= \rho f_r - \frac{\partial p}{\partial r} + \mu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right) + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial^2 V_r}{\partial z^2}\right\}$$

$$\rho\left\{\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}\right\}$$

$$= \rho f_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} + \frac{\partial^2 V_\theta}{\partial z^2}\right\}$$

$$\rho\left\{\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}\right\}$$

$$= \rho f_z - \frac{\partial p}{\partial z} + \mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2} + \frac{\partial^2 V_z}{\partial z^2}\right\}$$