Dimensional Analysis

Q-1 The angular velocity Ω of a windmill is a function of windmill diameter *D*, wind velocity *V*, air density ρ, windmill height H as compared to atmospheric boundary layer height *L*, and the number of blades *N*: Ω = fcn(*D*, *V*, ρ, *H*/*L*, *N*). Viscosity effects are negligible. Rewrite this function in terms of dimensionless Pi groups.

Solution: We have $n = 6$ variables, $j = 3$ dimensions (M, L, T), thus expect $n - j = 3$ Pi groups. Since only ρ has *mass* dimensions, it drops out. After some thought, we realize that *H*/*L* and *N* are already dimensionless! The desired dimensionless function becomes:

$$\frac{\mathbf{\Omega D}}{\mathbf{V}} = fcn\left(\frac{\mathbf{H}}{\mathbf{L}}, \mathbf{N}\right) \quad \textit{Ans.}$$

Q-2 A dam spillway is to be tested by using Froude scaling with a one-thirtieth-scale model. The model flow has an average velocity of 0.6 m/s and a volume flow of 0.05 m^3/s. What will the velocity and flow of the prototype be? If the measured force on a certain part of the model is 1.5 N, what will the corresponding force on the prototype be?

Solution: Given $\alpha = L_m/L_p = 1/30$, Froude scaling requires that

$$V_p = \frac{V_m}{\sqrt{\alpha}} = \frac{0.6}{(1/30)^{1/2}} \approx \mathbf{3.3}\ \frac{\mathbf{m}}{\mathbf{s}}; \quad Q_p = \frac{Q_m}{\alpha^{5/2}} = \frac{0.05}{(1/30)^{5/2}} \approx \mathbf{246}\ \frac{\mathbf{m^3}}{\mathbf{s}} \quad \textit{Ans.}\ \text{(a)}$$

The force scales in similar manner, assuming that the density remains constant (water):

$$F_p = F_m\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2 = F_m(1)\left(\frac{1}{\sqrt{\alpha}}\right)^2\left(\frac{1}{\alpha}\right)^2 = (1.5)(30)^3 \approx \mathbf{40500\ N} \quad \textit{Ans.}\ \text{(b)}$$

Q-3 The power P generated by a certain windmill design depends upon its diameter D, the air density ρ, the wind velocity V, the rotation rate Ω, and the number of blades n. (a) Write this relationship in dimensionless form. A model windmill, of diameter 50 cm, develops 2.7 kW at sea level when $V = 40$ m/s and when rotating at 4800 rev/min. (b) What power will be developed by a geometrically and dynamically similar prototype, of diameter 5 m, in winds of 12 m/s at 2000 m standard altitude? (c) What is the appropriate rotation rate of the prototype?

Solution: (a) For the function $P = \text{fcn}(D, \rho, V, \Omega, n)$ the appropriate dimensions are $\{P\} = \{ML^2T^{-3}\}$, $\{D\} = \{L\}$, $\{\rho\} = \{ML^{-3}\}$, $\{V\} = \{L/T\}$, $\{\Omega\} = \{T^{-1}\}$, and $\{n\} = \{1\}$. Using (D, ρ, V) as repeating variables, we obtain the desired dimensionless function:

$$\frac{P}{\rho D^2 V^3} = fcn\left(\frac{\Omega D}{V}, n\right) \quad \textit{Ans. (a)}$$

(c) "Geometrically similar" means that n is the same for both windmills. For "dynamic similarity," the advance ratio $(\Omega D/V)$ must be the same:

$$\left(\frac{\Omega D}{V}\right)_{model} = \frac{(4800\ r/\text{min})(0.5\ m)}{(40\ m/s)} = 1.0 = \left(\frac{\Omega D}{V}\right)_{proto} = \frac{\Omega_{proto}(5\ m)}{12\ m/s},$$

$$\text{or:} \quad \Omega_{proto} = \mathbf{144\ \frac{rev}{min}} \quad \textit{Ans. (c)}$$

(b) At 2000 m altitude, $\rho = 1.0067$ kg/m^3. At sea level, $\rho = 1.2255$ kg/m^3. Since $\Omega D/V$ and n are the same, it follows that the power coefficients equal for model and prototype:

$$\frac{P}{\rho D^2 V^3} = \frac{2700W}{(1.2255)(0.5)^2(40)^3} = \frac{P_{proto}}{(1.0067)(5)^2(12)^3},$$

$$\textit{solve} \quad \mathbf{P_{proto}} = 5990\ W \approx \mathbf{6\ kW} \quad \textit{Ans. (b)}$$

Q-4 A simply supported beam of diameter D, length L, and modulus of elasticity E is subjected to a fluid crossflow of velocity V, density ρ, and viscosity μ. Its center deflection δ is assumed to be a function of all these variables. (a) Rewrite this proposed function in dimensionless form. (b) Suppose it is known that δ is independent of μ, inversely proportional to E, and dependent only upon ρV^2, not ρ and V separately. Simplify the dimensionless function accordingly.

Solution: Establish the variables and their dimensions:

$$\begin{array}{cccccccc} \delta & = \mathrm{fcn}(& \rho, & D, & L, & E, & V, & \mu\ ) \\ \{L\} & & \{M/L^3\} & \{L\} & \{L\} & \{M/LT^2\} & \{L/T\} & \{M/LT\} \end{array}$$

Then $n = 7$ and $j = 3$, hence we expect $n - j = 7 - 3 = \mathbf{4}$ Pi groups, capable of various arrangements and selected by myself, as follows (a):

$$\text{Well-posed final result:} \quad \boldsymbol{\frac{\delta}{L} = \mathrm{fcn}\left(\frac{L}{D}, \frac{\rho V D}{\mu}, \frac{E}{\rho V^2}\right)} \quad \textit{Ans. (a)}$$

(b) If μ is unimportant, then the Reynolds number ($\rho VD/\mu$) drops out, and we have already cleverly combined E with ρV^2, which we can now slip out upside down:

$$\text{If } \mu \text{ drops out and} \quad \delta \propto \frac{1}{E}, \quad \text{then} \quad \frac{\delta}{L} = \frac{\rho V^2}{E}\mathrm{fcn}\left(\frac{L}{D}\right),$$

$$\text{or:} \quad \boldsymbol{\frac{\delta E}{\rho V^2 L} = \mathrm{fcn}\left(\frac{L}{D}\right)} \quad \textit{Ans. (b)}$$