Q1.

Given: Approximation for x component of velocity in laminar boundary layer

$$u = U \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \qquad \text{where } \delta = c x^{1/2}$$

Show: $\dfrac{v}{U} = \dfrac{\delta}{\pi x}\left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + \frac{\pi}{2}\frac{y}{\delta}\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - 1\right]$ for incompressible flow.

Plot: v/U vs. y/δ to locate maximum value of v/U; evaluate at location where $x = 0.5\ m$ and $\delta = 5\ mm$.

<u>Solution</u>: Apply differential continuity for incompressible flow.

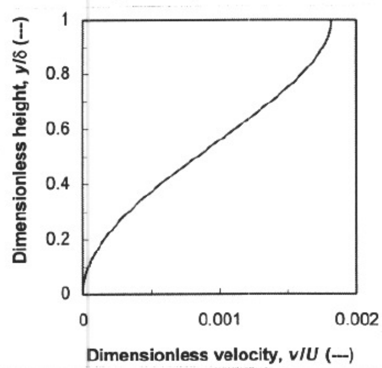
Basic equation: $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \underbrace{\dfrac{\partial w}{\partial z}}_{=0\ (2\text{-D flow})} = 0$

Thus $\dfrac{\partial v}{\partial y} = -\dfrac{\partial u}{\partial x} = -\dfrac{\partial u}{\partial \delta}\dfrac{d\delta}{dx} = -\left(\dfrac{\pi y}{2}\right)\left(-\dfrac{1}{\delta^2}\right)\cos\left(\dfrac{\pi y}{2\delta}\right)\dfrac{U}{2} c x^{-1/2} = \dfrac{U}{2x}\left(\dfrac{\pi y}{2\delta}\right)\cos\left(\dfrac{\pi y}{2\delta}\right)$

Integrating, $v = \displaystyle\int_0^y \frac{\partial v}{\partial y}dy + f(x) = \int_0^y \frac{U}{2x}\left(\frac{\pi}{2}\frac{y}{\delta}\right)\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)dy + f(x)$

$$v = \frac{2\delta}{\pi}\frac{U}{2x}\int_0^{\frac{\pi}{2}\frac{y}{\delta}} n\cos n\,dn + f(x) = \frac{\delta}{\pi}\frac{U}{x}\Big[\cos n + n\sin n\Big]_0^{\frac{\pi}{2}\frac{y}{\delta}} + \cancel{f(x)}^{\,0}$$

$$\frac{v}{U} = \frac{1}{\pi}\frac{\delta}{x}\left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + \left(\frac{\pi}{2}\frac{y}{\delta}\right)\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - 1\right]$$



This expression is a maximum at $y = \delta$, where

$$\frac{v}{U} = \frac{1}{\pi}\frac{\delta}{x}\left[\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) - 1\right] = \frac{\delta}{\pi x}\left(\frac{\pi}{2} - 1\right)$$

and

$$\left.\frac{v}{U}\right)_{max} = 0.182\,\frac{\delta}{x}$$

At the location given

$$\left.\frac{v}{U}\right)_{max} = 0.182 \times 0.005\ m \times \frac{1}{0.5\ m} = 0.00182 \quad \text{or} \quad 0.182 \text{ percent}$$

Q!2.

Given: Flow in xy plane, $v = -Bxy^3$ where $B = 0.2\ m^{-3}\cdot s^{-1}$ and coordinates are measured in meters; steady, $\rho = c$.

Find: (a) Simplest x component of velocity.
(b) Equation of streamlines.

Plot: streamlines through points (1,4) and (2,4).

Solution:

Basic equation: $\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \underbrace{\cancel{\frac{\partial}{\partial z}\rho w}}_{=0(1)} + \underbrace{\cancel{\frac{\partial \rho}{\partial t}}}_{=0(2)}$

Assumptions: (1) flow in the xy plane (given), $\frac{\partial}{\partial z} = 0$
(2) ρ = constant (given)

Then, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ or $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$

and $\frac{\partial u}{\partial x} = -\frac{\partial}{\partial y}(-Bxy^3) = 3Bxy^2$

Integrating,

$$u = \int \frac{\partial u}{\partial x}\,dx = \int 3Bxy^2 = \frac{3}{2}Bx^2y^2 + f(y).$$

The simplest expression is obtained with $f(y) = 0$

$$\therefore\ u = \frac{3}{2}Bx^2y^2 \qquad u$$

The equation of the streamlines is

$$\left.\frac{dy}{dx}\right)_{sl} = \frac{v}{u} = \frac{-Bxy^3}{\frac{3}{2}Bx^2y^2} = \frac{-2y}{3x}$$

Separating variables & integrating

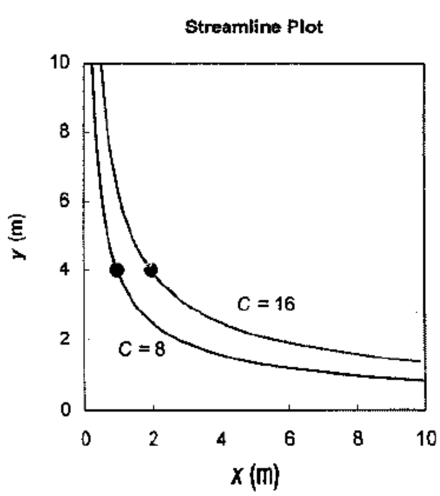
$$\frac{3}{2}\frac{dy}{y} + \frac{dx}{x} = 0$$

$$\frac{3}{2}\ln y + \ln x = \ln c$$

$$xy^{3/2} = c \quad \text{Streamline}$$

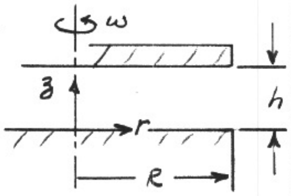
pt (1,4) $\quad xy^{3/2} = 8$

pt (2,4) $\quad xy^{3/2} = 16$



Q3.

Given: Flow between parallel disks as shown.



Velocity is purely tangential.
No-slip condition is satisfied, so velocity varies linearly with z.

Find: Expression for velocity field.

<u>Solution</u>: A general velocity field would be

$$\vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{k}$$

but velocity is purely tangential, so $V_r = V_z = 0$. Then we seek

$$V_\theta = V_\theta(r, \theta, z)$$

By symmetry, $\frac{\partial V_\theta}{\partial \theta} = 0$, so

$$V_\theta = V_\theta(r, z)$$

Since the variation with z is <u>linear</u>, $V_\theta = z f(r) + c$ at most, that is

$$\frac{\partial V_\theta}{\partial z} = f(r)$$

at most.

Along the surface $z = 0$, $V_\theta = 0$, so $c = 0$.

Along the surface $z = h$, $V_\theta = \omega r$, so

$$V_\theta(z = h) = \omega r = h f(r)$$

or

$$f(r) = \frac{\omega r}{h}$$

and

$$V_\theta = \omega r \frac{z}{h}$$

Thus

$$\vec{V} = \omega r \frac{z}{h} \hat{e}_\theta$$ $\vec{V}$

Q4.

Given: Velocity field $\vec{V} = (x^2 - y^2)\hat{i} - 2xy\hat{j}$

Find: Corresponding family of stream functions.

<u>Solution:</u> ψ may be defined only if flow is incompressible.

Basic equations: $$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \overbrace{\cancel{\frac{\partial \rho w}{\partial z}}}^{=0(1)} + \overbrace{\cancel{\frac{\partial \rho}{\partial t}}}^{=0(2)} = 0$$

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Assumptions: (1) $\vec{V} = \vec{V}(x,y)$, so $\frac{\partial}{\partial z} = 0$

(2) ρ = constant, so $\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial x} = 0$

Then
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2x - 2x = 0,$$ so flow is incompressible.

Thus
$$u = x^2 - y^2 = \frac{\partial \psi}{\partial y}; \quad \psi = \int u\,dy + f(x) = x^2 y - \frac{1}{3}y^3 + f(x)$$

$$v = -2xy = -\frac{\partial \psi}{\partial x}; \quad \psi = \int -v\,dx + g(y) = x^2 y + g(y)$$

Comparing these two expressions for ψ, we see that

$$f(x) = 0 \text{ and } g(y) = -\frac{1}{3}y^3$$

so $$\psi = x^2 y - \frac{1}{3}y^3$$ ψ

Q5.

Given: Velocity field represented by

$$\vec{V} = (Ax - B)\hat{i} + Cy\hat{j} + Dt\hat{k} \qquad (x, y \text{ in } m)$$

where $A = 2\ s^{-1}$, $B = 4\ m/s$, and $D = 5\ m/s^2$

Find: (a) Proper value of C for incompressible flow.
(b) Acceleration of particle at $(x, y) = (3, 2)$.
(c) Sketch streamlines in xy plane.

Solution: For incompressible flow, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$. Since $w = Dt$, $\partial w/\partial z = 0$, and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = C = -\frac{\partial u}{\partial x} = -A = -2\,s^{-1}$$ C

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

$$\vec{a}_p = (Ax - B)(A\hat{i}) + (Cy)(C\hat{j}) + (Dt)(0) + D\hat{k}$$

$$\vec{a}_p(3,2) = \left(\frac{2}{s} \times 3m - \frac{4m}{s}\right)\left(\frac{2}{s}\right)\hat{i} + \left(\frac{-2}{s} \times 2m\right)\left(-\frac{2}{s}\right)\hat{j} + \frac{5m}{s^2}\hat{k}$$

$$\vec{a}_p(3,2) = 4\hat{i} + 8\hat{j} + 5\hat{k} \ \ m/s^2$$ $\vec{a}_p(3,2$

In the xy plane, streamlines are $\frac{dy}{dx} = \frac{v}{u} = \frac{Cy}{Ax - B}$. Thus

$$\frac{dx}{Ax - B} = \frac{dy}{Cy} \quad \text{or} \quad \frac{dx}{Ax - B} = -\frac{dy}{Ay} \quad \text{or} \quad \frac{dx}{x - B/A} + \frac{dy}{y} = 0$$

Integrating

$$\ln(x - B/A) + \ln y = \ln C_0$$

$$\left(x - \frac{B}{A}\right)y = \text{const}$$