Last time : $\theta(p) = \mathbb{P}_p[0 \longleftrightarrow \infty]$

$$p_c = \inf\left\{ p \in [0,1] \;\middle|\; \theta(p) > 0 \right\}$$

For a transitive, locally finite connected G:

$$\varphi(p) = -\limsup_{n \to \infty} \frac{1}{n} \log\left(\mathbb{P}_p(n \le |E(K_v)| < \infty)\right)$$

Subcritical phase $0 \le p < p_c$

$\varphi(p) > 0$ $\rightarrow$ Aizenmann, Menshikov

Supercritical phase

- Difficult to understand

Some results known for asymptotics

- $$\mathbb{P}_p(0 \leftrightarrow \partial[n,r]^d, |K_0| < \infty) = \exp(-\Theta_p(n)) \text{ as } n \uparrow \infty$$

- $$\underbrace{\mathbb{P}_p(n \le |K_0| < \infty)}_{\text{Grimmet and Mastrand}} = \exp\left(-\Theta_p\left(n^{\frac{d-1}{d}}\right)\right) \text{ as } n \to \infty$$

Upper bound → Chayes, Chayes, Grimmet, Kesten and Schonmann
→ Kesten and Zhang

Main Theorem

$G=(V,E)$ be a connected, locally finite, non-amenable, transitive graph. Then

$$\theta(p) > 0 \quad \forall \quad p \in [0,1] \setminus \{p_c\}.$$

Amenability :

$$\Phi(G) = \inf \left\{ \frac{|\partial S|}{|S|} : S \subsetneq V, \text{ finite} \right\}$$

$\Phi(G)=0$

Cor of MT : TFAE

- G is Non-amenable
- $p_c(G)<1$ and $\theta(p)>0$ for some $p \in (p_c, 1)$.
- $p_c < 1$ and $\theta(p) > 0$ $\forall$ $p \in (p_c, 1)$.

Proof of MT :

Russo's Formula

let $X : \{0,1\}^E \longrightarrow \mathbb{R}$ be some R.V.

The derivative of X at e is defined as:

$$\partial_e X(\omega) = X(\omega^e) - X(\omega_e)$$

where $\omega^e(e') = \mathbb{1}_{\{e' \neq e\}} \omega(e') + \mathbb{1}_{\{e' = e\}}$

$$\omega_e(e') = \mathbb{1}_{(e' \neq e)} \omega(e')$$

$\mathbb{E}(\partial_e X(\omega))$ is called the influence of e on X.

Theorem (Russo's Formula) For any r.v. X that depends on finitely many edges, ($X \in \mathcal{F}_E$, $|E| < \infty$)

$$\frac{d}{dp} \mathbb{E}_p(X) = \sum_{e \in E} \mathbb{E}_p(\partial_e X).$$

— ①

Goal: $\mathbb{E}_p\left(e^{t|E(K_v)|} \mathbb{1}(|E(K_v)| < \infty)\right) < \infty$

If ① holds for some $t > 0$ then MT holds.

$$0 \leq \mathbb{E}_p\left(e^{tn} \mathbb{1}(n \leq |E(K_v)| < \infty)\right) \leq \left[\mathbb{E}_p\left(e^{t|E(K_v)|} \mathbb{1}(|E(K_v)| < \infty)\right)\right] < \infty$$

$$0 \leq e^{tn} \mathbb{P}_p(n \leq |E(K_v)| < \infty) \leq M$$

$$tn + \log\left(\mathbb{P}_p(n \leq |E(K_v)| < \infty)\right) \leq \log M$$

note that due to local finiteness $M > 0$.

$$\therefore\ t + \frac{1}{n} \log\left(\mathbb{P}_p(n \leq |E(K_v)| < \infty)\right) \leq \frac{\log M}{n}$$

$$0 < t \leq -\limsup_{n \to \infty} \frac{1}{n} \log\left(\mathbb{P}_p(n \leq |E(K_v)| < \infty)\right)$$

$$\therefore\ \varphi(p) > 0$$

Steps.

Use Russo's formula to express the p- derivative of the truncated exponential moment

$$\mathbb{E}_p\left(e^{t|E(K_v)|} \mathbb{1}(|E(K_v)| < \infty)\right)$$

(TEM)

Clusters growing while remaining finite.

as $\frac{d}{dp}(TEM(p)) =$ Positive term
+ Negative term

finite clusters becoming infinite.

Some Notation :

$G = (V, E)$ locally finite, connected. v a vertex of G.

For ω, $K_v(\omega) \to$ cluster of v, $E_v(\omega) = |E(K_v(\omega))|$

H_v = Set of all finite connected subgraphs of G, containing v.

For a function $F : H_V \to \mathbb{R}$, write:

$$\mathbb{E}_{p,n}(F(K_v)) := \mathbb{E}_p\left(F(K_v)\mathbb{1}(E_v \le n)\right)$$

and

$$\mathbb{E}_{p,\infty}(F(K_v)) = \mathbb{E}_p\left(F(K_v)\mathbb{1}(E_v < \infty)\right)$$

$\forall p \in [0,1], n \geq 1.$

It suffices to show,

$\forall \; p \in (p_c, 1) \quad \exists \; t > 0 \;$ s.t. $\mathbb{E}_{p,\infty}\left(E_v e^{tE_v}\right) < \infty$

- Given $F : H_V \to \mathbb{R}$, $n \geq 1$, $\mathbb{E}_{p,n}$ is a polynomial in p and is differentiable.

Express the derivative of $\mathbb{E}_{p,n}(F(K_v))$ in two ways

① Fluctuation

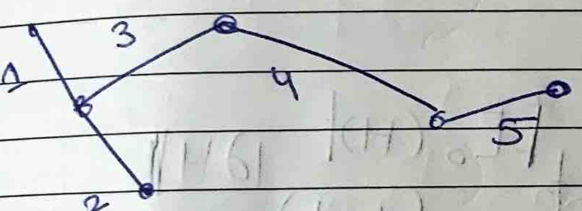
For a finite subgraph $H \subset G$ we define

$E(H)$ to be the set of edges of G with at least one endpoint in $V(H)$

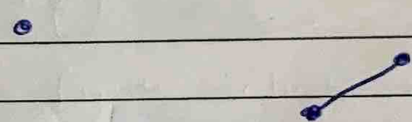
let $E_0(H) \longrightarrow$ set of edges of H

$\partial H := E(H) \setminus E_0(H)$

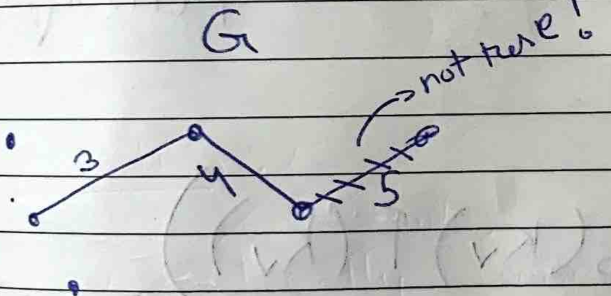
$\hookrightarrow$ Not the same as $\partial_E V(H)$!



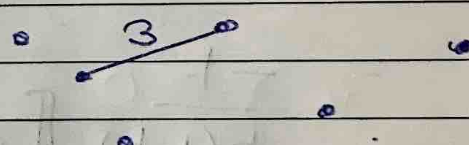
G



H



∂H



$\partial_E V(H)$

For each $p \in [0,1]$, we define the fluctuation

$$h_p(H) = p|\partial H| - (1-p)|E_0(H)|$$

So above, $h_p(H) = 3p - (1-p) = 4p - 1$

Now for $n \geq 1$, $p \in (0,1)$ and $F : H_V \to \mathbb{R}$

$$\frac{d}{dp}\left(\mathbb{E}_{p,n}\left(F(K_V)\right)\right)$$

$$= \sum_{H \in H_V} \frac{d}{dp}\left(p^{|E_0(H)|} (1-p)^{|\partial H|} F(H) \mathbb{1}\left(|E(H)| \leq n\right)\right)$$

$$= \frac{-1}{p(1-p)} \sum_{H \in H_V} h_p(H)\, p^{|E_0(H)|} (1-p)^{|\partial H|} F(H) \mathbb{1}\left(|E(H)| \leq n\right)$$

$$= \frac{-1}{p(1-p)} \mathbb{E}_{p,n}\left(h_p(K_V) F(K_V)\right) = -M_{p,n}\left(F(K_V)\right) \quad — (2)$$

(2) Russo's formula

Applying Russo' formula to $X(\omega) = F(K_V) \mathbb{1}(E_V \leq n)$

$$\frac{d}{dp} \mathbb{E}_{p,n}\left(F(K_V)\right) = \sum_{e \in E} \mathbb{E}_p(\partial_e X)$$

$$= \sum_{e \in E} \mathbb{E}_p \left(\left[F(K_v(\omega^e)) - F(K_v(\omega_e)) \right] \mathbb{1}\left(E_v(\omega^e) \leq n \right) \right)$$

$$- \sum_{e \in E} \mathbb{E}_p \left(F(K_v(\omega_e)) \mathbb{1}\left(E_v(\omega_e) \leq n < E_v(\omega^e) \right) \right)$$

Since

$$\mathbb{1}\left(E_v(\omega_e) \leq n \right) = \mathbb{1}\left(E_v(\omega^e) \leq n \right) + \mathbb{1}\left(E_v(\omega_e) \leq n < E_v(\omega^e) \right)$$

$$= U_{p,n}\left(F(K_v) \right) - D_{p,n}\left(F(K_v) \right)$$

where,

$$U_{p,n}\left(F(K_v) \right) = \sum_{e \in E} \mathbb{E}_p \left(F(K_v(\omega^e)) - F(K_v(\omega_e)) \right) \mathbb{1}\left(E_v(\omega^e) \leq n \right)$$

$$= \frac{1}{p} \sum_{e \in E} \mathbb{E}_{p,n} \left(F(K_v) - F(K_v(\omega_e)) \right) \mathbb{1}\left(\omega(e) = 1 \right)$$

$$D_{p,n} = \sum_{e\in E} \mathbb{E}_p \left(F(K_v(\omega^e)) \mathbb{1}(E_v(\omega_e) \leq n < E_v(\omega^e) \right)$$

$$= \frac{1}{1-p} \sum_{e\in E} \mathbb{E}_p \left(F(K_v) \mathbb{1}(\omega(e)=0, E_v \leq n < E_v(\omega^e) \right)$$

$$\therefore \quad -M_{p,n} (F(K_v)) = U_{p,n}(F(K_v)) - D_{p,n} (F(K_v))$$

For every F, $M_{p,n}$, $U_{p,n}$, $D_{p,n}$ are linearly dependent on F.

$D_{p,n}$ is ≥ 0 iff $F \geq 0$

$U_{p,n} \geq 0$ if F is inc

Controlling $M_{p,n}(e^{tE_v})$, $U_{p,n}(e^{tE_v})$, $D_{p,n}(e^{tE_v})$

The three propositions

only place where non-amenability (and transitivity) is used.

Proposition ① Let G be a connected, locally finite, non-amenable transitive graph.

Then for every $p_0 > p_c(G)$ $\exists\, C_{p_0} = C_{p_0}(G, p)$ s.t.

$$D_{p,n}(F(K_v)) \geq \frac{C_{p_0}}{1-p} E_{p,n}(E_v \cdot F(K_v))$$

for every non-negative $F : H_v \to [0, \infty)$, every $p_0 \leq p < 1$ and every $n \geq 1$.

(Note: $D_{p,n}(F(K_v)) \leq \frac{1}{1-p} E_{p,n}(F(K_v) \cdot E_v)$ always holds.)

Applying Prop ① to $F = e^{tE_v}$, we obtain $\forall\ p_c < p < 1$ $\exists\ C_p > 0$ s.t.

$$\frac{C_p}{1-p} E_{p,n}(E_v e^{tE_v}) \leq D_{p,n}(e^{tE_v})$$
$$= U_{p,n}(e^{tE_v}) + M_{p,n}(e^{tE_v}) \quad \forall\ n \geq 1,\ t \geq 0$$

Want: If $t>0$ is sufficiently small, then LHS is uniformly bounded in n.

$$M_{p,n}\left(e^{tE_v}\right) \leq \frac{C_p}{4(1-p)} \mathbb{E}_{p,n}\left(E_v e^{tE_v}\right)$$

$$+ \frac{1}{p(1-p)} \mathbb{E}_{p,n}\left(|h_p(K_v)| e^{tE_v} \mathbb{1}\left(|h_p(K_v)| \geq \frac{pC_p}{4} E_v\right)\right)$$

(obvious bond see formula)

$\longleftarrow$ 3

Via a martingale ~~andy~~ analysis for $t>0$ small enough:

Prop 2 Let $\alpha>0$. Then

$$\mathbb{E}_p^G\left(|h_p(K_v)| e^{tE_v} \mathbb{1}\left(\alpha E_v \leq |h_p(K_v)| < \infty\right)\right) < \infty$$

$\forall$ locally finite G, every $v \in V$ every $p \in [0,1]$ and every $0 \leq t < \alpha^2/2$

This helps bound $M_{p,n}$.

Bounding $U_{p,n}$ is tricky.

We need to set up some more notation:

Let H be a finite connected graph. For each edge e of H, let H_e denote the subgraph of H spanned by all edges of H other than e.

Given a vertex v, a set W of vertices in H, we write $\mathrm{Piv}(H, v, W)$ for the set of edges e s.t.

$$\exists\, w \in W \text{ s.t. } v \not\leftrightarrow w \text{ in } H_e$$

$$Br_k(H, v) = \max \{ |\mathrm{piv}(H, v, W)| : |W| \le k \}$$

Now

$$U_{p,n}(E_v^k) = \frac{1}{p} \sum_{e \in E} \left[E_v^k - E_v(\omega_e)^k \right] 1\!\!1(\omega(e) = 1)$$

$$= \frac{1}{p} \sum_{a_1, a_2 \cdots a_k \in E} \sum_{e \in E} \mathbb{P}_p\Big(\{a_1, a_2 \cdots a_k\} \subseteq E(K_v),\ (a_1, a_2 \cdots a_k) \not\subseteq E(K_v(\omega_e)),\ \omega(e) = 1 \text{ \& } E_v \le n \Big)$$

~~Since this holds~~ $A \subseteq E(K_v)$, $|A| \le k$

$$W = W(A) = \{\underbrace{w(e)}_{\text{endpt}} : e \in A\}$$

Then,

$$\sum_{e \in E} \mathbb{1}\left(A \not\subseteq E(K_v(w_e)) \text{ and } w(e) = 1\right)$$

$$\le |\mathrm{Piv}(K_v, v, W)| \le Br_k(K_v, v)$$

$$U_{p,n}(E_v^k) \le \frac{1}{p} \sum_{a_1, a_2 \ldots a_k \in E} \mathbb{E}_{p,n}\left(Br_k(K_v, v)\, \mathbb{1}\left(\{a_1, a_2 \ldots a_k\} \subseteq E(K_v)\right)\right)$$

$$= \frac{1}{p} \mathbb{E}_{p,n}\left(Br_k(K_v, v) E_v^k\right)$$

$\forall\, p \in (0,1)$ and $k \ge 1$

Summing over k

$$U_{p,n}(e^{tE_v}) = \sum_{k=0}^{\infty} \frac{t^k}{k!} U_{p,n}(E_v^k)$$

$$\le \frac{1}{p} \sum_{k=1}^{\infty} \frac{t^k}{k!} \mathbb{E}_{p,n}\left(Br_k(K_v, v) E_v^k\right) \quad \forall\, t > 0$$

Now trivially,

$$U_{p,n}\left(e^{tE_v}\right) \le \frac{C_p}{4(1-p)} \mathbb{E}_{p,n}\left(E_v e^{tE_v}\right)$$

$$+ \frac{1}{p} \sum_{k=1}^{\infty} \frac{t^k}{k!} \mathbb{E}_{p,n}\Big(Br_k(K_v, v)\, E_v^k\, \mathbb{1}\Big(Br_k(K_v, v) \ge \frac{pC_p}{4(1-p)} E_v\Big)\Big)$$

(4) ———

**

Prop (3) Let $\alpha > 0$, $p \in (0,1)$. Then $\exists$ $t_{\alpha,p} > 0$ s.t.

$$\sum_{k=1}^{\infty} \frac{t^k}{k!} \mathbb{E}_p^G\left(Br_k(K_v, v)\, E_v^k\, \mathbb{1}\left(\alpha E_v \le Br_k(K_v, v)\right)\right) < \infty$$

$< \infty$

$\forall$ locally finite graph $G = (V, E)$ every $v \in V$ and every $0 \le t < t_{\alpha,p}$

Proof of MT

Let G be a connected, locally finite graph and let $p > p_c(G)$

Let C_p be the constant from prop ①

by Eq^{ns} ②, ③, ④

$$\frac{C_p}{2(1-p)} \mathbb{E}_{p,n}\left(E_v e^{tE_v}\right) \le \frac{1}{p} \sum_{k=1}^{\infty} \frac{t^k}{k!} \mathbb{E}_{p,n}\left(A \right)$$

$$+ \frac{1}{p(1-p)} \mathbb{E}_{p,n}\left(|h_p(K_v)| e^{tE_v} \mathbb{1}\left(|h_p(K_v)| \ge \frac{pC_pE_v}{4}\right)\right)$$

$\forall$ $n \ge 1$ and $t \ge 0$.

$$\left(A = Br_k(K_v, v)\, E_v^k\, \mathbb{1}\left(Br_k(K_v, v) \ge \frac{pC_p E_v}{4(1-p)}\right) \right)$$

$$\frac{C_p}{2(1-p)} \mathbb{E}_{p,\infty}\left(E_v e^{tE_v}\right)$$

$$\le \frac{1}{p} \sum_{k=1}^{\infty} \frac{t^k}{k!} \mathbb{E}_{p,n}(A)$$

$$+ \frac{1}{p(1-p)} \mathbb{E}_{p,n}\left(|h_p(K_v)| e^{tE_v} \mathbb{1}\left(|h_p(K_v)| \ge \frac{pC_p}{4}\right.\right.$$

by prop 2, 3 $\exists\ t_0 = t_0(p, C_p) > 0$ s.t.

RHS is ~~uniformly bounded~~ finite

$\forall\ 0 \le t < t_0$ $\boxtimes$

Ev)