Random Walk on Dynamical Percolation

Contents

1 Overview

Percolation has been a popular model of study ever since it was introduced in 1957 by Broadbent and Hammersley [BH57]. This essay focuses on a dynamical adaptation of percolation where the percolation environment evolves as a two-state Markov chain. Each edge refreshes at rate μ and becomes open with probability p and closed with probability $1-p$. We study a model of random walk introduced by Peres, Steif, and Stauffer [PSS15] on this evolving environment, where the walker chooses one of its neighbors at rate 1 and jumps only if that edge is open.

A key step towards understanding the dynamical walk is to relate it to the well-understood simple random walk (SRW). This is when the walker chooses one of its neighbors at rate 1 and moves there. In this spirit, Hermon and Sousi [HS20] established several comparison principles between important random walk quantities, such as the hitting times and mixing times of the dynamical walker and the SRW. They showed, for example,

$$t_{\mathsf{hit}}^{\mathsf{DP}} \leq \frac{C(\mu)}{p} t_{\mathsf{hit}}^{\mathsf{SRW}},$$

where μ is the refresh rate for the evolution of the configuration, and p is the probability that the edge is open after refreshment.

While [HS20] resolved many key aspects of hitting and mixing time behavior, they left open the question of whether a comparison principle could also be established for cover times [HS20, Question 1.12]. This essay addresses that open problem.

The goal of this essay is twofold. First, it systematically studies the key comparison results for hitting times developed in [HS20], beginning with a setup of the dynamical model and an introduction to the key techniques used in its analysis. Second, it presents new comparison theorems for cover times, extending the known framework to graph families where no such bounds were previously available. Specifically, we develop comparison theorems for the following classes:

Uniformly locally transient graphs. We show a cover time comparison for a certain class of finite graphs, which we call uniformly locally transient. We explain why this notion is a natural analog of transience in the finite setting and discuss the obstacles our method faces in large-diameter or recurrent cases.

High-dimensional graphs. Next, we study a special subclass of uniformly locally transient graphs called high-dimensional graphs, i.e., graph sequences G_n satisfying $\mathrm{diam}(G_n)^2 = o\left(\frac{n}{\log n}\right)$. These can be thought of as graph sequences with "dimension" at least 3. For example, one such sequence is $(\mathbb{Z}_n)^d_{n\geq 1}$, the d-dimensional tori with $d \geq 3$. For these families, we prove stronger, more quantitative comparison principles of the form

$$t_{\mathsf{cov}}^{\mathsf{DP}} \leq \frac{C}{\mu p} t_{\mathsf{cov}}^{\mathsf{SRW}}.$$

Rapidly mixing graphs. Next, we tackle graphs with rapid mixing, i.e., sequences (G_n) such that $t_{\mathsf{rel}}^{\mathsf{SRW}} = O(n^{1-\delta})$ for some $\delta > 0$. To obtain a cover time comparison for this class, we adapt Zuckerman's technique [Zuc90] to derive a general lower bound on the SRW cover time $t_{\mathsf{hit}}^{\mathsf{SRW}} \log n$, which, as we will see, allows us to extend the comparison.

Abelian groups. Finally, we show a comparison theorem for all Abelian groups using a result of Ding, Lee, and Peres [DLP11], following remark 3.7 of [HS20].

The remainder of the essay is structured as follows. Section 2 introduces the formal setup of percolation, dynamical percolation, and the random walk on dynamical percolation. Section 3 defines the key quantities of interest, including hitting times, mixing times, and cover times. Section 4 presents the proof of the main comparison result for hitting times. Section 5 presents the central contributions of this essay: new comparison principles for cover times between dynamical walk and SRW on various families of graphs, including high-dimensional graphs, Cayley graphs, and graphs with rapid mixing. Section 6 discusses comparison principles for random walk on dynamical percolation on the other side of the spectrum, namely, the graphs with moderate growth. In Section 7, we conclude with future directions, some questions, and the limitations of our techniques.

Acknowledgments

I thank Prof. Perla Sousi for suggesting this topic of study and for many helpful discussions. I also thank Maarten Markering for discussions about Matthews' bound in the dynamical percolation setting.

2 Introduction

2.1 Bernoulli Percolation

A classical model in probability theory is that of Bernoulli percolation. This model was introduced by Broadbent and Hammersley in [BH57], and is one of most studied models in probability. Defined simply, in this model, each edge is open (retained) with probability p and closed (not retained) with probability $1-p$ (we call the measure this gives as the percolation measure π_p). The connected components in this configuration are called clusters.

Performing Bernoulli percolation on an infinite graph G leads to several basic questions. The first question is the existence of a phase transition. For $p=0$ it is clear that the cluster of the origin is just the origin (say O), whereas for $p=1$ the cluster is the entire graph, and in particular, it's infinite. From a standard uniform coupling of the percolation configuration, it is easy to show that there is a critical point p_c such that for $p>p_c$ there is an infinite cluster and for $p<p_c$ there is no infinite cluster.

One can explicitly find the values of the critical point for certain graphs. For instance, a d-regular tree has $p_c(\mathbb{T}_d)=\frac{1}{d-1}$ (infact from [Bha24] in some sense, for a d-regular graph $p_c=\frac{1}{d-1}$ if and only if G is a tree). A classic theorem of Harris also tells us that for the integer square lattice $p_c(\mathbb{Z}^2)=\frac{1}{2}$. This however is essentially all the graphs for which we can determine p_c explicitly and in general it is a very hard problem to find p_c. Even though one gets that $0<p_c(\mathbb{Z}^d)\leq\frac{1}{2}$, the exact value of $p_c(\mathbb{Z}^d)$ for $d\geq 3$ is not known!

Another question is about the non-triviality of the phase transition (in the sense of emergence of an infinite cluster). More specifically, we can ask whether $p_c\in(0,1)$, it is easy to see that this is not true in general and that $p_c(\mathbb{Z})=1$. For most graphs, however, there is typically a phase transition. Usually, the bound $p_c>0$ is easier to establish.

Proposition 2.1. *Let G be an infinite graph of max degree Δ. Then, $p_c(G)\geq\frac{1}{\Delta-1}$, in particular $p_c(G)>0$.*

Proof. Let $\Gamma_n=\{\gamma:\gamma \text{ is a self avoiding walk of length } n\}$ and let $\partial\Lambda_n=\{x:d(x,O)=n\}$. Suppose that, $p_c(G)<\frac{1}{\Delta-1}$ and fix a $p_c<p<\frac{1}{\Delta-1}$. Consider, $X_n=\sum_{\gamma\in\Gamma_n}\mathbb{1}(\gamma \text{ is open})$. Then, the probability that we are connected to the n boundary is clearly contained in the event that $X_n>0$. Thus we have,

$$\pi_p(0\leftrightarrow\partial\Lambda_n)\leq\pi_p(X_n>0)\leq\mathbb{E}(X_n)=\sum_{\gamma\in\Gamma_n}\pi_p(\gamma \text{ is open})=p^n|\Gamma_n|$$

However, $|\Gamma_n|\leq\Delta(\Delta-1)^{n-1}$ so we get that, $\pi_p(X_n>0)\leq p^n\Delta(\Delta-1)^{n-1}\to 0$ as $n\to\infty$. This is a contradiction. Thus $p_c(G)\geq\frac{1}{\Delta-1}$. $\square$

The bound $p_c<1$ is, however, much harder. Intuitively, one expects that for graphs which grow fast enough $p_c<1$. Through a recent result of Duminil-Copin, Goswami, et.al [DCGR+20] (and even more recently by Easo, Severo, and Tassion [EST24]), we know that $p_c<1$ for all transitive graphs that grow at least quadratically. More specifically,

Definition 2.1. *A transitive graph $G=(V,E)$ with bounded geometry is said to have super-linear growth if $\limsup_{r\to\infty}\frac{|B_r(x)|}{r}=\infty$, where $B_r(x)$ is the ball of radius r.*

Theorem 2.1 (Duminil-Copin, Goswami, et.al 2020, Easo, Severo, Tassion 2024)**.** *Let G be a graph with super-linear growth then, $0<p_c(G)<1$.*

As a result of Aizenman, Kesten, and Newman [AKN87], it is known that for percolation $\mathbb{Z}^d$, there is a unique infinite cluster after p_c. Thus, the subcritical and supercritical phases are well understood; however, the problem of the existence of infinite cluster at criticality in a completely open. In particular, we have the following conjecture.

Conjecture 1. For all $d \geq 2$, there is no infinite cluster at criticality for percolation on $\mathbb{Z}^d$.

A breakthrough result of Hara and Slade [HS94] proves the above conjecture for $d \geq 19$, using the technique of lace expansion. Through Harris' theorem on the square lattice we also have the conjecture for $d = 2$. However, the conjecture for the intermediate dimension $d = 3$ is open and is one of the biggest open problems in the subject.

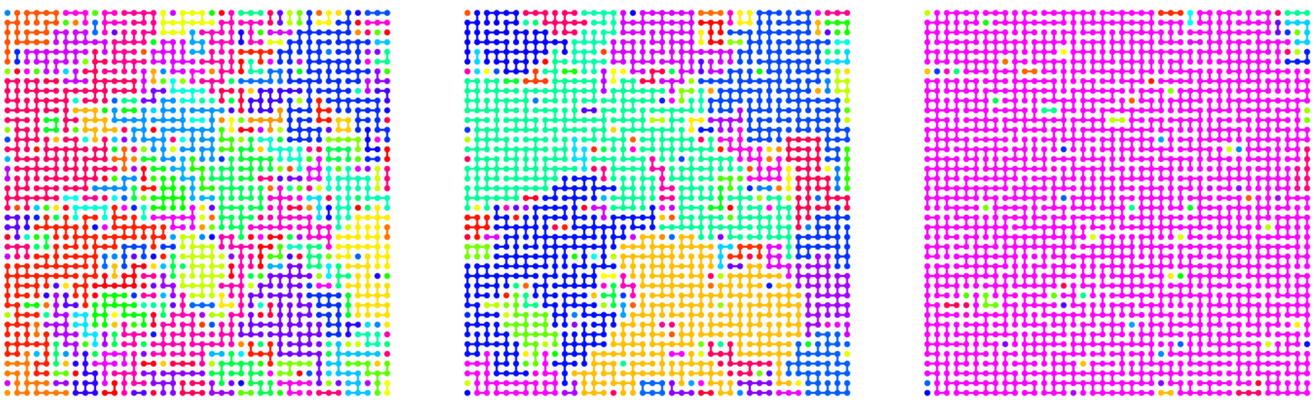

Figure 1: Percolation on a 40 × 40 square grid graph at $p = 0.4, 0.5, 0.6$. Each cluster is given a different color. © Probability on Trees and Networks - Russell Lyons and Yuval Peres. [LP17b]

2.2 Dynamical Percolation and Random Walk on Dynamical Percolation

In this essay, we consider a dynamical version of percolation on a graph $G = (V, E)$. This model was first introduced in [OYJ97] and is a model for an evolving percolation process. Given $p \in (0, 1)$ and a refresh rate $\mu > 0$, we construct a time-evolving random environment as follows. At time $t = 0$, the configuration is initialized according to some law on $\{0, 1\}^E$.

Each edge $e \in E(G)$ is equipped with an independent Poisson clock of rate μ. When the clock of edge e rings, the edge refreshes: it becomes open with probability p and closed with probability $1-p$, independently of its previous state and independently of the rest of the configuration. We denote the configuration at time t by η_t, where $\eta_t(e) = 1$ if edge e is open at time t and $\eta_t(e) = 0$ otherwise.

The process $(\eta_t)_{t\geq 0}$ forms a continuous-time Markov process on $\{0, 1\}^E$. It is easy to check that π_p (the percolation measure on edges) is a stationary distribution for this process. In particular, if the initial distribution is π_p, then $\eta_t \sim \pi_p$ for all $t \geq 0$.

In this essay, we later restrict to finite graphs, but for historical context, let us focus on an infinite graph. A central question in the study of dynamical percolation was (and still is) whether, starting from the distribution π_p, there exist atypical times (called exceptional times) at which the percolation configuration behaves significantly differently from what one typically expects at a fixed time. In most contexts, "significantly

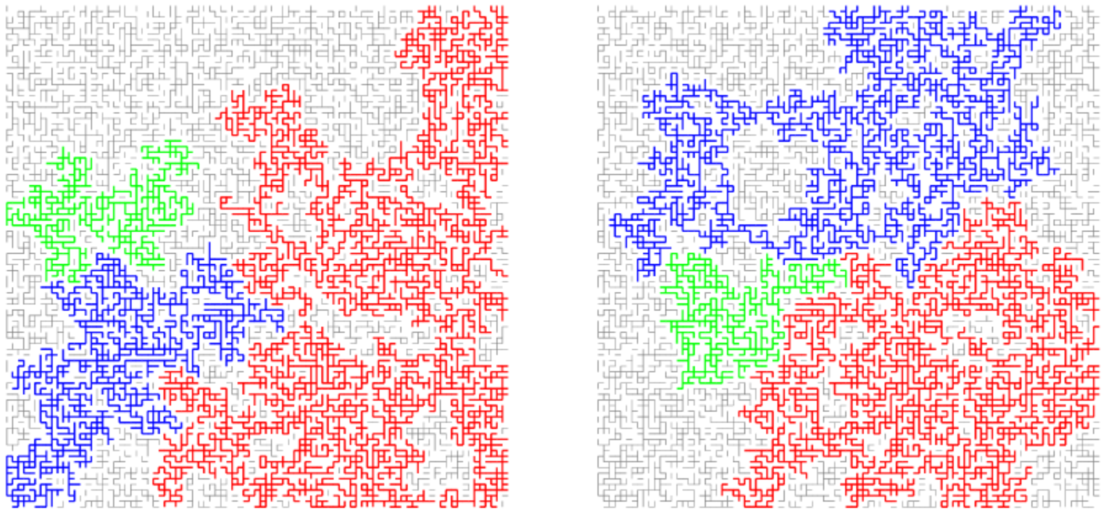

Figure 2: Dynamical Percolation on a square grid with p fixed at two different times. Observe how the clusters change. © Oded Schramm's contributions to noise sensitivity. [Gar11]

different" refers to the appearance or disappearance of an infinite connected cluster. We begin by stating a basic but important result, originally established in [OYJ97], which states that exceptional times do not exist when we are not at p_c. Let C_t denote the event that there exists an infinite cluster at time t, and let $\mathbb{P}^{(\mu,p)}$ represent the probability measure governing the full dynamical percolation process with parameter (μ, p) on a given graph.

Proposition 2.2. *For any infinite graph G, consider dynamical percolation started at π_p with $\mu = 1$ then we have:*

$$\mathbb{P}^{(1,p)}\left((C_t)^c \text{ occurs for every } t\right) = 1 \quad \text{if} \quad p < p_c(G),$$
$$\mathbb{P}^{(1,p)}\left(C_t \text{ occurs for every } t\right) = 1 \quad \text{if} \quad p > p_c(G).$$

Proof. We just give an idea of the proof. For details, we refer to reader to [OYJ97].

Fix $p < p_c(G)$. The goal is to show that with probability one, the event C_t (that there is an infinite open cluster at time t) does not occur for any $t \geq 0$.

To do this, consider a small time interval of the form $[t, t+\delta]$, and define a new percolation configuration where an edge is declared open if it was open at *any* point during this interval. Because the edge states evolve independently according to Poisson clocks, the probability that an edge is ever open during this interval is strictly greater than p but can be made arbitrarily close to p by choosing δ small enough. In particular, for small enough δ, the resulting configuration is still subcritical, and so with probability one, it contains no infinite cluster.

Now cover the time axis $[0, \infty)$ using a countable collection of overlapping intervals of the form $[k\delta/2, k\delta/2 + \delta]$ for integers $k \geq 0$. By the above argument, for each such interval, with probability one there is no time inside the interval when an infinite cluster exists. Taking a countable union over these intervals, we conclude that almost surely, there is no time $t \geq 0$ at which an infinite cluster appears.

The case $p > p_c(G)$ is similar: one considers an edge to be open if it is open throughout a small interval $[t, t+\delta]$. For small enough δ, the resulting configuration remains supercritical, and a similar covering argument shows that almost surely, an infinite cluster exists at all times. □

Thus in some sense the critical point is the most interesting case. Several results exists at $p = p_c$, in particular, we have the following results.

Theorem 2.2 ([OYJ97]). *For the graph $\mathbb{Z}^d$ with $d \geq 19$, and $\mu = 1$. Dynamical percolation started at π_p has no exceptional times at criticality.*

The proof of the above theorem uses several ideas that only work in high-dimensional percolation on $\mathbb{Z}^d$ for $d \geq 19$, inspired by Hara and Slade's work. Nonetheless, we have the following result for the triangular lattice and the $2d$ lattice.

Theorem 2.3 (O. Schramm, J. Steif, [SS11]). *Consider $\mu = 1$ and dynamical percolation on $\mathbb{Z}^2$ started from π_p. Then no exceptional times exist. Similarly, for dynamical site percolation (defined analogously but for vertices) on the triangular lattice started from π_p, no exceptional times exists.*

To keep this essay concise, we end our discussion on dynamical percolation. There is, of course, a lot more that be can be asked for this model and we refer the reader to Jeffrey Steif's excellent exposition [Ste09].

2.3 Random Walk on Dynamical Percolation

We now define the model that we discuss for the remainder of the essay: random walk on dynamical percolation, introduced in [PSS15]. Let $(X_t)_{t\geq 0}$ denote the position of the walker at time $t \geq 0$. The joint process $(X_t, \eta_t)_{t\geq 0}$ evolves as follows. The environment $(\eta_t)_{t\geq 0}$ evolves independently according to the dynamical percolation process described above.

Meanwhile, the walker (independently) evolves as a continuous-time random walk of rate 1. At each time when the walker's clock rings, the walker chooses one of its neighbors uniformly at random, jumps to that neighbour only if that edge is open at that moment (i.e., open in η_t). Otherwise, it remains at its current vertex.

This describes a Markov process (X_t, η_t) on $V \times \{0, 1\}^E$. We emphasize that while $(X_t, \eta_t)_{t\geq 0}$ is Markovian on its own, the process (X_t) alone is not Markovian due to the dependence on the evolving environment.

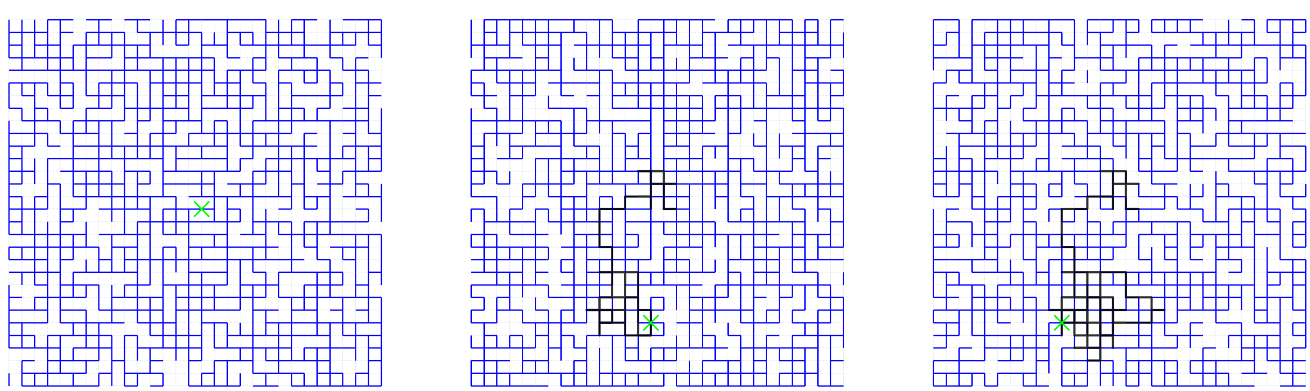

Figure 3: Snapshots of a random walker on dynamical percolation on a 30×30 grid at three different times. The blue edges are open, the black line shows the trajectory, and the green marker indicates the current walker position. The edge refresh rate is $\mu = 0.5$. Simulation coded in Python.

Different phases of μ. The interesting phase is when $\mu \leq 1$, in fact $\mu \ll 1$. This is because if $\mu > 1$ then at the time scale of the walker the environment does not evolve fast enough to be interesting (it essentially becomes close to the static case), and no interesting behaviour emerges. This case is also relevant to real-world scenarios, such as evolving networks where $\mu < 1$ is natural. Hence, we assume $\mu \leq 1$ for this entire essay, unless specified otherwise.

We have the following theorem from [PSS15], which says that the usual transience above $d \geq 3$ and recurrence below $d \leq 2$ hold for the dynamical walker.

Theorem 2.4. *Let $G = \mathbb{Z}^d$ with $d \in \{1, 2\}$ then for any $p \in [0, 1]$ and $\mu > 0$ and initial configuration η_0 we have for any $s_0 \geq 0$:*

$$\mathbb{P}(\cup_{s \geq s_0} \{X_s = 0\}) = 1.$$

For $G = \mathbb{Z}^d$ and $d \geq 3$ we have for any $p \in (0, 1]$ and $\mu > 0$ that $\lim_{t \to \infty} X_t = \infty$ a.s.

From now on we restrict ourselves to the case of a finite graph $G = (V, E)$. It is easy to check that the product measure $\pi_{\mathsf{full},p} = \pi \times \pi_p$ is a stationary distribution for the full process (X_t, η_t), where π denotes the measure on V given by

$$\pi(x) = \frac{\deg(x)}{2|E|} \text{ for each } x \in V.$$

Here, $\deg(x)$ denotes the degree of the vertex x in the graph G.

3 Hitting times, cover times, and related quantities

In this section, we introduce several important quantities associated with random walks on graphs, both in the static setting (simple random walk, or SRW) and the dynamic setting (random walk on dynamical percolation). Understanding these quantities will be crucial for the comparisons between the SRW and random walk on dynamical percolation that we develop later. We start by defining these concepts for a finite state Markov chain and then give specific notations for the case of SRW and random walk on dynamical percolation.

Definition 3.1 (Hitting times and Cover times)**.** *Consider an ergodic Markov chain $(X_t)_{t\geq 0}$ with state space Ω, transition matrix P and stationary distribution π. Then for each set $A \subset \Omega$ define, $T_A := \inf\{t : X_t \in A\}$ when A is a singleton $\{y\}$ we simply write T_y. The hitting time and cover time of the Markov chain are defined as:*

$$t_{\mathsf{hit}} := \max_{x,y} \mathbb{E}_x(T_y) \text{ and } t_{\mathsf{cov}} := \max_{x} \mathbb{E}_x(\tau_{\mathsf{cov}})$$

where $\tau_{\mathsf{cov}} = \inf\{t : \forall x \in \Omega \text{ there exists } \ s \leq t \text{ such that } X_s = x\}$ is the time taken to visit all points.

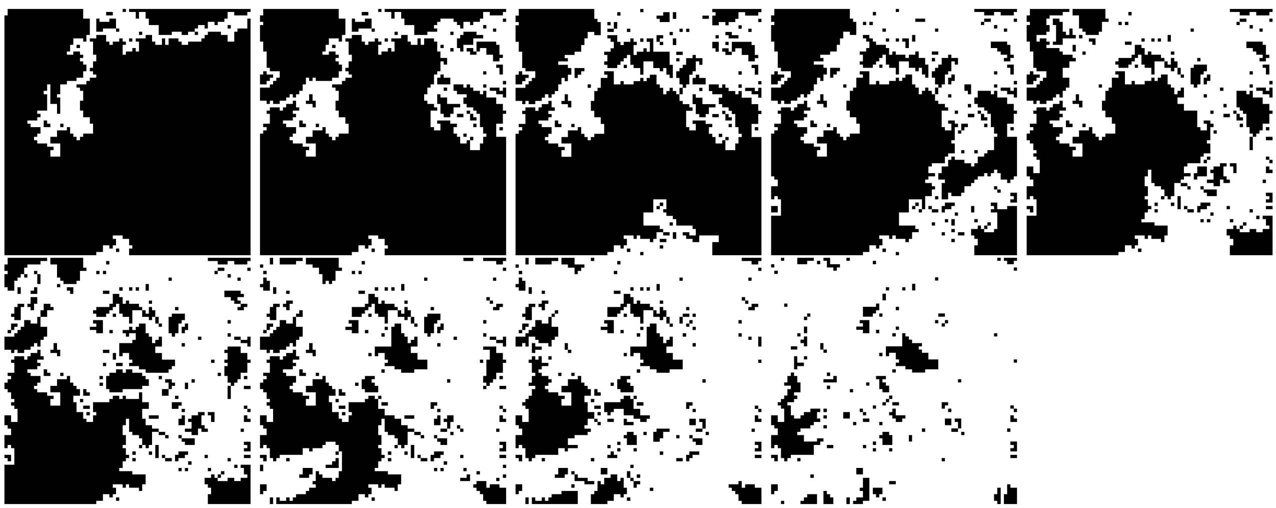

Figure 4: Figure to illustrate the cover time of a simple random walk on a 75×75 discrete torus. Unvisited vertices are colored black. Snapshots are taken at 10%τ_{cov}, 20%τ_{cov}, $\cdots$, 100%τ_{cov}, where τ_{cov} is the cover time. The trajectory required 136,387 steps to complete the cover. Simulation coded in Python.

Definition 3.2 (Mixing times)**.** *Consider an ergodic Markov chain $(X_t)_{t\geq 0}$ with state space Ω, transition matrix P, and stationary distribution π. Fix $\varepsilon > 0$. The total variation and L_∞ mixing times of the Markov chain are defined as*

$$t_{\mathsf{mix}}(\varepsilon) := \min\{t \geq 0 : \max_x \|P^t(x,\cdot) - \pi\|_{\mathsf{TV}} \leq \varepsilon\}$$

and

$$t^{(\infty)}_{\mathsf{mix}}(\varepsilon) := \min\{t \geq 0 : \max_x \|P^t(x,\cdot) - \pi\|_{\infty,\pi} \leq \varepsilon\},$$

where, recall that $\|\mu - \nu\|_{\mathsf{TV}} = \frac{1}{2}\sum_x |\mu(x) - \nu(x)|$ and $\|\mu - \nu\|_{\infty,c} = \sup_x \left|\frac{\mu(x)-\nu(x)}{c(x)}\right|$.

We refer to the markov chain $(X_t, \eta_t)_{t\geq 0}$ as the full process. Let $\mathbb{P}^{(\mu,p)}_{x,\eta}$ and $\mathbb{E}^{(\mu,p)}_{x,\eta}$ denote the law and the expectation of the full process started from (x, η). We want to compare a simple random walker on G to the random walker on dynamical percolation. We consider the continuous time version of the simple random walker with jump rate 1 to avoid ergodic issues. Let us start by defining the following hitting times.

$$t_{\text{hit}}^{\mathsf{SRW}} := \max_{x,y}\mathbb{E}_x(T_y) \text{ and } t_{\text{hit}}^{\mathsf{full},(\mu,p)} := \max_{x,y\in V,\eta\in\{0,1\}^E} \mathbb{E}_{x,\eta}^{\mathsf{full},(\mu,p)}\left(T_{y\times\{0,1\}^E}\right)$$

Similarly, we define the following cover times,

$$t_{\text{cov}}^{\mathsf{SRW}} := \max_x \mathbb{E}_x(\tau_{\mathsf{cov}}) \text{ and } t_{\text{cov}}^{\mathsf{full},(\mu,p)} := \max_{x\in V,\eta\in\{0,1\}^E} \mathbb{E}_{x,\eta}^{\mathsf{full},(\mu,p)}(\tau_{\mathsf{cov}}^{(\mu,p)})$$

where $\tau_{\mathsf{cov}}^{(\mu,p)}$ is the first time the random walk on dynamical percolation visits every vertex.

A note of caution: These are not the cover and hitting times in the usual sense of considering (X_t,η_t) a Markov chain. We define them in the above manner since we only care about the coordinate of the random walker.

We denote the corresponding mixing times for the simple random walk by $t_{\text{mix}}^{\mathsf{SRW}}(\varepsilon)$ and $t_{\text{mix}}^{\mathsf{SRW},(\infty)}(\varepsilon)$, and the mixing time for the random walk on dynamical percolation by $t_{\text{mix}}^{\mathsf{full},(\mu,p)}(\varepsilon)$ and $t_{\text{mix}}^{\mathsf{full},(\mu,p),(\infty)}(\varepsilon)$. Following standard convention we fix $\varepsilon = \frac{1}{4}$ and drop it from the notation, unless specified otherwise.

Peres, Stauffer, and Steif initiated the study of dynamical percolation in the subcritical regime for the special case of $(\mathbb{Z}_n^d)_{n\geq 1}$, the following comparison theorems were established in [PSS15] and [Mar23].

Theorem 3.1. *For all $d \geq 1$ and $p \in (0, p_c(\mathbb{Z}^d))$, there exists $C = C(d,p) < \infty$ and $c = c(d,p) > 0$ so that the following hold.*

(i) *For all n and for all $\mu \leq 1$,*

$$\frac{cn^2}{\mu} \leq t_{\text{hit}}^{\mathsf{full},(\mu,p)}(\mathbb{Z}_n) \leq \frac{Cn^2}{\mu} \text{ and } \frac{cn^2}{\mu} \leq t_{\text{cov}}^{\mathsf{full},(\mu,p)}(\mathbb{Z}_n) \leq \frac{Cn^2}{\mu}$$

(ii) *For all n and for all $\mu \leq 1$,*

$$\frac{cn^2\log n}{\mu} \leq t_{\text{hit}}^{\mathsf{full},(\mu,p)}(\mathbb{Z}_n^2) \leq \frac{Cn^2\log n}{\mu} \text{ and } \frac{cn^2(\log n)^2}{\mu} \leq t_{\text{cov}}^{\mathsf{full},(\mu,p)}(\mathbb{Z}_n^2) \leq \frac{Cn^2(\log n)^2}{\mu}$$

(iii) *For all $d \geq 3$, for all n and for all $\mu \leq 1$,*

$$\frac{cn^d}{\mu} \leq t_{\text{hit}}^{\mathsf{full},(\mu,p)}(\mathbb{Z}_n^d) \leq \frac{Cn^d}{\mu} \text{ and } \frac{cn^d\log n}{\mu} \leq t_{\text{cov}}^{\mathsf{full},(\mu,p)}(\mathbb{Z}_n^d) \leq \frac{Cn^d\log n}{\mu}$$

Given the hitting time comparisons, [Mar23] establishes the cover time comparison. The upper bound simply follows from the Matthew bound in the dynamic setting. The hard part of the theorem is to establish lower bounds. As we will see, from our methods, we can get upper bounds for $\mathbb{Z}_n^d$ for all values of p not necessarily subcritical. For the proof of the theorem, we refer the reader to [PSS15] and [Mar23].

We are now ready to state one of the main theorems from [HS20].

Theorem 3.2 (Theorem 1.1, [HS20])**.** *For every μ there exists a positive constant c_1' such that for all graphs G and all p we have that*

$$t_{\mathsf{hit}}^{\mathsf{full},(\mu,p)} \leq \frac{c_1'}{p}\, t_{\mathsf{hit}}^{\mathsf{SRW}} \tag{3.1}$$

Moreover, there exists a constant c_2' so that for all graphs G and all $(\mu,p) \in (0,1]^2$ we have that

$$t_{\mathsf{hit}}^{\mathsf{full},(\mu,p)} \leq c_2'\left(\mu^{-1} t_{\mathsf{hit}}^{\mathsf{full},(1,p)} + t_{\mathsf{mix}}^{\mathsf{full},(\mu,p)}\right) \tag{3.2}$$

Intuitively, one can think about the theorem in the following manner: suppose we want to hit y by starting from x, the simple random walker is slower than the dynamical walker due to potential bottlenecks in the graph, while the dynamical walker might also face these issues, due to the evolving nature of the process it can escape these challenges quicker. The appearance of $\frac{1}{p}$ can also be explained. It is the factor by which the walker is slowed down because at any point, it only sees a p-fraction of the edges that the simple random walk sees. In a similar fashion, one might also expect $\frac{1}{\mu}$ (expected time for an edge to refresh) as a factor slowing the dynamical walk; indeed, we later show that for certain graphs the constant in the theorem can be taken to be of the form $\frac{c}{\mu p}$, making this dependence explicit.

Remark. Hermon and Sousi discuss how their comparison theorem for hitting times implies that $t_{\text{hit}}^{\mathsf{full},(\mu,p)} \lesssim_{\mu,p} |V|^3$ (a natural analog of the same upper bound for the static hitting time). This leads to an analogous question for the $|V|^3$ upper bound for the cover time. This will be a consequence of the conjectured version of Theorem 3.2 for cover time. As we will see this will follow for certain families of graphs in Section 5.

Now, we recall a well-known method to relate cover times and hitting times for any graph is Matthews' inequality. We first state the static version and show its extension to the dynamic setting.

Proposition 3.1 (Matthews' Inequality, [Mat88]). *Simple random walk on any graph G satisfies*

$$\max_{A\subset V}\min_{x,y\in A}\mathbb{E}_x(T_y)\left(1+\ldots+\frac{1}{|A|-1}\right)\leq t_{\text{cov}}^{\mathsf{SRW}}\leq t_{\text{hit}}^{\mathsf{SRW}}\left(1+\ldots+\frac{1}{|V(G)|-1}\right) \tag{3.3}$$

For the proof of the above proposition, we refer the reader to [LP17a]. It is easy to see that Matthews' Inequality works for any irreducible Markov chain. In particular, we have the following.

Proposition 3.2 (Matthews' upper bound in the dynamic setting). *Dynamical random walk with parameters $p\in(0,1)$ and $\mu>0$ on any graph G satisfies*

$$t_{\text{cov}}^{\mathsf{full},(\mu,p)}\leq t_{\text{hit}}^{\mathsf{full},(\mu,p)}\left(1+\ldots+\frac{1}{|V(G)|}\right)\asymp t_{\text{hit}}^{\mathsf{full},(\mu,p)}\log(|V(G)|) \tag{3.4}$$

This proposition follows the same proof as the Matthew bound and was used in [Mar23] to establish matching upper bounds for cover times. We include it here for convenience.

Proof. Without loss of generality, let our vertex set be $\{1,\cdots,n\}$, further let n be the worst starting vertex for cover time, i.e. $t_{\mathsf{cov}}^{\mathsf{full},(\mu,p)}=\mathbb{E}_{n,\eta_0}^{(\mu,p)}(\tau_{\mathsf{cov}})$ for some η_0. Consider a uniform random permutation $\sigma\in S_{n-1}$ of the set $\{1,2,\ldots,n-1\}$, independent of the Markov chain. Now define T_k as the first time all of $\sigma(1),\ldots,\sigma(k)$ have been visited, and let $L_k:=X_{T_k}$ denote the last state among $\{\sigma(1),\ldots,\sigma(k)\}$ to be visited. Then clearly, $T_k=\sum_{i=1}^{k}(T_i-T_{i-1})$ with $T_0:=0$.

Fix the starting state to be n, we now show that:

$$t_{\mathsf{cov}}^{\mathsf{full},(\mu,p)}=\mathbb{E}_{n,\eta_0}^{(\mu,p)}[\tau_{\mathsf{cov}}]=\mathbb{E}_{n,\eta_0}^{(\mu,p)}[T_{n-1}]\leq t_{\mathsf{hit}}^{\mathsf{full},(\mu,p)}\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right),$$

We now show that for each $i\leq n-1$, $\mathbb{E}_n[T_i-T_{i-1}]\leq\frac{1}{i}t_{\text{hit}}^{\mathsf{full},(\mu,p)}$. To see this, observe that the event $L_i=\sigma(i)$ (i.e., the last state to be visited among $\{\sigma(1),\ldots,\sigma(i)\}$ is $\sigma(i)$) occurs with probability $1/i$ by symmetry: the labeling is uniformly random, so each state is equally likely to be last. Also,

$$\mathbb{E}_{n,\eta_0}^{(\mu,p)}[T_i-T_{i-1}\mid L_i=\sigma(i)]\leq t_{\text{hit}}^{\mathsf{full},(\mu,p)},$$

since in that case, the walk transitions from a state in $\{\sigma(1),\ldots,\sigma(i-1)\}$ to $\sigma(i)$, and hitting any state from any other takes at most t_{hit} in expectation.

Therefore,

$$\mathbb{E}_{n,\eta_0}^{(\mu,p)}[T_i - T_{i-1}] = \mathbb{E}_{n,\eta_0}^{(\mu,p)}[T_i - T_{i-1} \mid L_i = \sigma(i)] \cdot \mathbb{P}(L_i = \sigma(i)) \leq t_{\mathsf{hit}} \cdot \frac{1}{i}.$$

Thus,

$$\mathbb{E}_{n,\eta_0}^{(\mu,p)}[T_{n-1}] \leq t_{\text{hit}}^{\mathsf{full},(\mu,p)} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right),$$

as desired.

□

Notation: For functions f, g we will write $f(n) \lesssim g(n)$ if there is a constant $c > 0$ such that $f(n) \leq cg(n)$ for all n. We write $f(n) \asymp g(n)$ if both $f(n) \lesssim g(n)$ and $g(n) \lesssim f(n)$ holds. Finally, we write $\lesssim_c$ and $\asymp_c$ when the constant depends on c.

We now move our attention to spectral quantities associated with Markov chains. To start, we define the Dirichlet form associated to a generator $\mathcal{L}$.

$$\mathcal{E}_{\mathcal{L}}(f,f) = \frac{1}{2}\sum_{x,y} \pi(x)\mathcal{L}(x,y)(f(x)-f(y))^2 = \pi(-\mathcal{L}f \cdot f)$$

for all $f : \Omega \to \mathbb{R}$. Next, for $\varepsilon > 0$ define the spectral profile,

$$\Lambda(\varepsilon) = \min\{\mathcal{E}_{\mathcal{L}}(f,f) : f : \Omega \to \mathbb{R}, \mathrm{Var}_\pi(f) = 1, \pi(\mathrm{supp}(f)) \leq \varepsilon\}$$

where supp denotes the support of h and $\mathrm{Var}_\pi(h) = \mathbb{E}_\pi[(h - \mathbb{E}_\pi(h))^2]$ is the variance of h. This leads to the definition of the ε-spectral-profile for $\varepsilon > 0$.

$$t_{\mathrm{spectral-profile}}(\varepsilon) = \int_{4\min_x \pi(x)}^{\frac{4}{\varepsilon}} \frac{2}{\delta\Lambda(\delta)}\, d\delta$$

For transitve graphs, one can show $t_{\mathrm{spectral-profile}}^{\mathsf{SRW}} \lesssim \log n\ t_{\mathrm{rel}}^{\mathsf{SRW}}$ as follows. Since $\Lambda(\delta)$ minimizes over a restricted subset of functions (those supported on small sets), we have $\Lambda(\delta) \geq \gamma(P)$ (the Poincaré constant $= 1/t_{\mathrm{rel}}^{\mathsf{SRW}}$). Plugging this into the definition of $t_{\mathrm{spectral-profile}}^{\mathsf{SRW}}$, we get:

$$t_{\mathrm{spectral-profile}}^{\mathsf{SRW}}(\varepsilon) \leq 2t_{\mathrm{rel}}^{\mathsf{SRW}} \int_{4\pi_*}^{4/\varepsilon} \frac{1}{\delta}\, d\delta = 2t_{\mathrm{rel}} \log\left(\frac{4/\varepsilon}{4\pi_*}\right).$$

For transitive graphs, we have $\pi_* = \frac{1}{n}$, and therefore for $\varepsilon = \frac{1}{4}$,

$$t_{\text{spectral-profile}}\left(\frac{1}{4}\right) \leq 2t_{\mathrm{rel}} \log(16n) \lesssim t_{\mathsf{rel}} \log n \tag{3.5}$$

Let $t_{\mathrm{spectral-profile}}^{\mathsf{SRW}}$ be the spectral profile for the simple random walk. Then [HS20] show that it is possible to bound the L^∞ mixing time for the full process by the spectral profile as follows.

Theorem 3.3 (Theorem 1.2, [HS20].)**.** *There exists a positive constant c_3' such that for all graphs G and all $(\mu,p) \in (0,1]^2$ we have for all $\varepsilon \in (0,1)$*

$$t_{\mathsf{mix}}^{\mathsf{full},(\mu,p),(\infty)}(\varepsilon) \leq \frac{c_3'}{\mu p}\, t_{\text{spectral-profile}}^{\mathsf{SRW}}(\varepsilon) + \frac{c_3'}{\mu}|\log(1-p)| \tag{3.6}$$

In the context of the L^∞ mixing, the $\frac{|\log(1-p)|}{\mu}$ cannot be removed because the L^∞ mixing time is at least of this order. Nonetheless, through some effort, one can remove the term for the mixing time (i.e., the total variation mixing time) and get the following theorem.

Theorem 3.4. *There exists a positive constant c_3' (same as the above theorem) such that for all graphs G and all $(\mu, p) \in (0,1]^2$ we have for all $\varepsilon \in (0,1)$*

$$t_{\mathsf{mix}}^{\mathsf{full},(\mu,p)}(\varepsilon) \leq \frac{c_3'}{\mu p}\, t_{\mathsf{spectral\text{-}profile}}^{\mathsf{SRW}}(\varepsilon) \tag{3.7}$$

Another important quantity in our analysis is the relaxation time t_{rel}, which is defined as the inverse of the spectral gap (the smallest positive eigenvalue of $-\mathcal{L}$). We denote the relaxation time for the full process by $t_{\mathrm{rel}}^{\mathsf{full},(\mu,p)}$ and for the simple random walk as $t_{\mathrm{rel}}^{\mathsf{SRW}}$. The following result closely relates the relaxation time to mixing times.

Proposition 3.3. *Consider a reversible Markov chain on a finite state space Ω with stationary distribution π then for all $\varepsilon \in (0,1)$:*

$$t_{\mathsf{rel}} \log(\varepsilon) \leq t_{\mathsf{mix}}(\varepsilon/2) \leq t_{\mathsf{mix}}^{(\infty)}(\varepsilon) \leq t_{\mathsf{rel}}\ |\log((\min_x \pi(x))\varepsilon)|.$$

Remark. Before we move into the proofs, we would like to bring attention to the following point. Even though we assume the natural condition of $\mu \leq 1$, the same proofs work in the exact same way for $\mu > 1$ (both theorems from [HS20] and our theorems in Section 5), but here if there are terms of the form $\frac{1}{\mu}$ they will be replaced by $\frac{1+\mu}{\mu}$.

4 Comparison of Hitting Times

In this section, we prove the comparison theorem between hitting times. Before we dive into the proof, we give a high-level idea of the argument.

Firstly, let us assume that $\mu = 1$. We can do this because of the simple fact that the generator of the full process satisfies

$$\forall x \neq x' \in V, \eta, \eta' \in \{0,1\}^E, \mathcal{L}_{(1,p)}((x,\eta),(x',\eta')) \leq \frac{1}{\mu}\mathcal{L}_{(\mu,p)}((x,\eta),(x',\eta'))$$

As we will see, using this one can transfer estimate from $\mu = 1$ to $\mu < 1$ at a cost of the factor μ^{-1}.

We now show that we can also assume $\eta_0 \sim \pi_p$. One might try to argue this by waiting until all the edges have refreshed at least once, which takes time roughly $\log |E|$ (expected value of $|E|$ many exponentials). However, even though the edge refreshes are independent of the walker, the environment and the walker are coupled: the walker's motion depends on which edges are open, and this introduces a correlation between its current position and the current configuration of edges. For instance, the walker is more likely to be found in regions where open edges appeared more frequently in the recent past. In this sense, the environment "remembers" the walker, and we cannot yet treat it as independent.

Nonetheless, we show that after a small "cooling-off" period after $\log |E|$ the environment becomes stationary and independent of the walker. More specifically, we use the fact that $\mu = 1$ to show that at some stopping time slightly larger than the time after which each edge is likely to have refreshed, the environment is stationary and independent of the walker. This will then define a sequence of stopping times τ_i such that we would have: (A) $\eta_{\tau_i} \sim \pi_p$ and independent of X_{τ_i} for all i, (B)$\tau_0 = 0$, and $(\tau_{i+1} - \tau_i)_{i \geq 1}$ are i.i.d. random variables with well-behaved tails, i.e., such that $\mathbb{E}[e^{\delta \tau_1}] < \infty$, and (C) $Y_i := X_{\tau_i}$ is a Markov chain with the same stationary distribution as the simple random walk (this can be thought of as an embedding of the SRW in the path of the dynamical percolation walker). We call each τ_i a regeneration time and X_{τ_i} the auxiliary walk.

Since the auxiliary walk is embedded inside the trajectory of the dynamical walker, we use it as a bridge to relate the SRW and the dynamical walk. More specifically, since the difference of regeneration times have exponential tails (more precisely we will see that they have mean $e^{1/\mu}$) by Wald's identity, we would have that the hitting time from x, y in the dynamical walk satisfies: $\mathbb{E}^{\mathsf{full},(\mu,p)}_{x,\pi_p}(T_y) \leq e^{1/\mu}\mathbb{E}^{\mathsf{aux},(\mu,p)}_{x}(T_y)$. We use a variant of this for commute times. To finish, we need an upper bound on the auxiliary commute time by the simple random walk. We do this by noting that the transition matrix of the auxiliary upper bounds the simple random walk transition matrix by a constant factor, and conclude.

4.1 The auxiliary walk

We start towards formally defining the auxiliary chain. Let $G = (V, E)$ be a graph with vertex set V and edge set $E = \{e_1, e_2, \cdots e_n\}$. For every edge, we create an infinite number of (formal) copies $e_{i,1}, e_{i,2}, \cdots$. Now start with $(X_0, \eta_0) \sim \pi \times \pi_p$. For every time t we now define a time-evolving set which has all the memory of the walker; we call this set the infected set R_t. Initially, $R_0 = \emptyset$ and if $R_{t^-} = A$ and the exponential clock of X rings at time t we add to R_t the edge that X examines to cross at time t. If this edge already exists in A, then we add its lowest-numbered copy not in A.

Now we can order the edges of R_t as follows: assign label 1 to the edge with the lowest label in E, if some of its copies are also in R_t then simply use the ordering of the copies of that edge. Continue like this, assigning a label to the second-lowest edge and so on.

When $R_{t^-} = A$, then assign exponential clocks of rate μ to the edges of E that are not in A and generate an (single) exponential clock of rate $|A|\mu$ for the edges in A. When a clock of the edge in $E \setminus A$ rings, refresh that edge to open with probability p and closed with probability $1-p$. If the clock associated with A rings, then choose an index from $\{1, 2, \cdots |A|\}$ (given by the above ordering) pick one edge from A uniformly and remove that from R_{t^-} to obtain R_t. If that edge was a true edge (i.e., a member of E) refresh its state. This naturally defines a coupling of (X, η) with the same transition rates as the full process.

We are now ready to define the auxiliary walk.

Definition 4.1. *The auxiliary chain Y starting from x_0 is defined as follows: start $\eta_0 \sim \pi_p$ and $X_0 = x_0$. Set $R_0 = \emptyset$, and for every t consider the infected edges R_t. Define the regeneration times by $\tau_0 = 0$, and for $i \geq 0$ set*

$$\tau_{i+1} = \inf\{t \geq \tau_i + S_i : R_t = \emptyset\}$$

where $\tau_i + S_i$ is the first time after τ_i that R_t becomes empty. Finally, the auxiliary walk $(Y_i)_{i \in \mathbb{N}\cup\{0\}}$ is defined as the discrete chain $(X_{\tau_i})_{i\in\mathbb{N}\cup\{0\}}$

Remark. This definition, with our construction of the infected set, gives a natural coupling of the full process (δ_x, π_p), this observation will be very important later to compare the auxiliary chain and the full process.

Lemma 4.1. *We have the following.*

(1) *$(S_i)'s$ are i.i.d having exponential distribution with parameter 1 and independent of $(X_{\tau_i})_i$*

(2) *$(\tau_i - \tau_{i-1})_{i\geq 1}$ are i.i.d with mean $e^{\frac{1}{\mu}}$ and have exponential tails.*

(3) *The process (X_s, η_s, R_s) is positive recurrent.*

The above lemma, combined with a birth-death chain argument, can be used to show that the auxiliary walk looks like the simple random walk in the following sense (see Lemma 3.6, [HS20])

Theorem 4.1. *The invariant distribution of (Y_i) is π*

4.2 Proof of the hitting time comparison

We are now ready to prove the hitting time comparison. Recall the following notion of commute times and return times for an irreducible discrete-time Markov chain (Y_k).

$$T_a^+ = \inf\{t \geq 1 : Y_t = a\} \text{ and } T_{ab} = \inf\{t > T_b : Y_t = a\}$$

For reversible chains, we have the following well-known principle.

Theorem 4.2 (Dirichlet's Principle)**.** *Let Y be a reversible Markov chain on a finite state space Ω with transition matrix P and stationary distribution π. Then for all $a, b \in \Omega$ we have*

$$\frac{1}{\mathbb{E}_a(T_{ba})} = \inf_{f: f(a)=1, f(b)=0, 0\leq f\leq 1} \mathcal{E}_P(f,f)$$

Since the auxiliary chain is not proven to be reversible, in order to use the above result, we need to work with its symmetrization. Thankfully for its additive symmetrization, we have the following result of Doyle and Steiner [DS17].

Proposition 4.1. *Let Y be a Markov chain on a finite state space Ω with transition matrix P and stationary distribution π. Let P^* be its time reversal then its additive symmetrization is given by $S = \frac{P+P^*}{2}$, and for $a, b \in \Omega$ we have,*

$$\mathbb{E}_a^S(T_{ba}) \geq \mathbb{E}_a^P(T_{ba})$$

Proof. Let $v(x) = \mathbb{P}_x(T_a < T_b)$, then we know that v is harmonic on $\Omega \backslash \{a, b\}$ and $\mathbb{E}_a^P(T_{ba}) = \pi(a)\mathbb{P}_a^P(T_b < T_a^+)$. Since $v(b) = 0, v(a) = 1$ by Dirichlet's principle we get that

$$\frac{1}{\mathbb{E}_a^P(T_{ba})} = \mathcal{E}_P(v, v) = \mathcal{E}_S(v, v) \geq \frac{1}{\mathbb{E}_a^S(T_{ba})}$$

□

For showing the second part of the theorem, we would need the following lemma, which is similar to [Her18], page 12.

Lemma 4.2. *There exists a positive constant c so that for any reversible Markov chain on a finite state space Ω with stationary distribution π, for all $A \subset \Omega$ we have that*

$$\max_{x \in \Omega} \mathbb{E}_x(T_A) \leq \mathbb{E}_\pi(T_A) + ct_{\mathsf{mix}}$$

Proof. Aldous [Ald82] introduced the following notion of mixing. Define

$$t_{\mathsf{stop}} = \max_x \inf\{\mathbb{E}_x(T) : T \text{ randomised stopping time s.t. } \mathbb{P}_x(X_T = y) = \pi(y) \text{ for all } y\}.$$

From [Ald82] it can be shown that for every x there is a randomised stopping that achieves the infimum above. Let $x \in \Omega$ and let T_x be a stopping time such that $\mathbb{P}_x(X_T \in A) = \pi(A)$ for every A. Then, $\mathbb{E}_x(T_x) \leq t_{\mathsf{stop}}$ and

$$\mathbb{E}_x(T_A) \leq \mathbb{E}_x(T_x) + \mathbb{E}_x(\max(T_A - T_x, 0)) \leq t_{\mathsf{stop}} + \mathbb{E}_\pi(T_A)$$

Since, $\mathbb{E}_x(\max(T_A - T_x, 0)) \leq \mathbb{E}_x(\inf\{t \geq T_x : X_t \in A\} - T_x) = \mathbb{E}_\pi(T_A)$. Finally, [Ald82] also tells that $t_{\mathsf{stop}} \leq ct_{\mathsf{mix}}$. Therefore, $\max_{x \in \Omega} \mathbb{E}_x(T_A) \leq \mathbb{E}_\pi(T_A) + ct_{\mathsf{mix}}$. □

The plan is to compare the auxiliary walk with the simple random walk using the Dirichlet principle. For that, we need the following bounds on transition probabilities of the auxiliary walk.

Proposition 4.2. *Consider the auxiliary walk (Y_i) with parameters μ, p. Denote its transition matrix by $P_{\mathsf{aux},(\mu,p)}$. Then for all x, y such that for $\{x, y\} \in E$ we have*

$$P_{\mathsf{aux},(\mu,p)}(x, y) \geq P_{\mathsf{SRW}}(x, y)\frac{p\mu}{1+\mu}$$

We also have,

$$\frac{P_{\mathsf{aux},(\mu,p)}(x, y) + P^*_{\mathsf{aux},(\mu,p)}(x, y)}{2} \geq P_{\mathsf{SRW}}(x, y)\frac{p\mu}{1+\mu}$$

Proof. Let us say that at time 0 we start from the stationary distribution and and walk coordinate x. Consider the event that the first attempt of the walker is to jump to y and then the edge $\{x, y\}$ refreshes and then the walker jumps from y back to x. This event has probability atmost $P_{\mathsf{aux}}(x, y)$ and atleast $\mathbb{P}$(choosing to jump to y) $\times$ $\mathbb{P}$(the edge$\{x, y\}$ is open) times the probability that the edge refreshes and the infected set becomes empty before the walker's clock rings. This probability is $P_{\mathsf{SRW}}(x, y)p\frac{\mu}{1+\mu}$.

Since $\frac{\mu}{\mu+1}$ is the probability that a clock with rate μ rings before an independent clock of rate 1. □

Let us now show the proof of Theorem 3.2

Proof. We first start by showing that the regeneration time is approximately $\log |E|$ up to constants that depend on μ.

The worst-case regeneration time is achieved when we start with $X_0 = x$ and $\eta_0 = \eta$ and the infection process starts with all edges infected, i.e., $|R_0| = |E|$. Then the regeneration time τ will be the time taken for the number of infected edges to be 0. It is clear that $|R_t|$ evolves as a birth-death chain with parameters $q(i, i+1) = 1$ and $q(i, i-1) = \mu i$ for all x, η.

Thus we get that $\mathbb{E}(\tau) \asymp_\mu \log |E|$. Now η_τ is distributed according to π_p and is independent of X_τ. This is because in our edge set there are two kinds of edges, ones that have never been examined by the walker and ones that have been. The ones that have never been examined are independently distributed as Ber(p). The other ones are also Ber(p) by considering the last time it was examined before τ.

Therefore, we have,

$$t_{\text{hit}}^{\mathsf{full},(\mu,p)} \leq \mathbb{E}(\tau) + \max_{x,y} \mathbb{E}_{x,\pi_p}^{(\mu,p)}(T_{y\times\{0,1\}^E})$$

Consider the auxiliary chain with parameters μ, p and let S_{aux} denote its additive symmetrization. Then by Proposition 4.1 we have $\mathcal{E}_S(f,f) \geq \frac{p\mu}{1+\mu}\mathcal{E}_{P_{\mathsf{SRW}}}(f,f)$ for all $f \in \mathbb{R}^V$. This, combined with Theorem 4.2, gives us, for $x, y \in V$,

$$p\frac{\mu}{1+\mu}\mathbb{E}_x^{S,(\mu,p)}(T_{yx}) \leq \mathbb{E}_x^{\mathsf{SRW}}(T_{yx})$$

Thus by Proposition 4.1,

$$\mathbb{E}_x^{\mathsf{aux},(\mu,p)}(T_{yx}) \leq \mathbb{E}_x^{S,(\mu,p)}(T_{yx})$$

Now consider the commute time of the dynamical walker in the full process, i.e.

$$T_{yx}^{\mathsf{full}} = \inf\{t > T_{y\times\{0,1\}^E} : X_t = x\}$$

Now using the coupling of the full process started from (δ_x, π_p) with the auxiliary chain started from x, we obtain

$$T_{yx}^{\mathsf{full}} \leq \sum_{i=1}^{T_{yx}}(\tau_i - \tau_{i-1})$$

Thus by Wald's identity,

$$p\frac{\mu}{1+\mu}\mathbb{E}_{x,\pi_p}^{(\mu,p)}(T_{yx}^{\mathsf{full}}) \leq e^{1/\mu}\mathbb{E}_x(T_{yx})$$

So finally,

$$t_{\text{hit}}^{\mathsf{full},(\mu,p)} \leq_\mu \frac{1}{p}t_{\text{hit}}^{\mathsf{SRW}} + \log|E| \lesssim \frac{1}{p}t_{\text{hit}}^{\mathsf{SRW}}$$

Where the final inequality holds because $t_{\text{hit}}^{\mathsf{SRW}} \geq |V| - 1$ holds always (we see this later as well).

Proof of second inequality. We now show the second inequality. Let $x, y \in V$ and $\eta \in \{0,1\}^E$. Then by Lemma 4.2,

$$\mathbb{E}_{x,\eta}^{(\mu,p)}(T_{y\times\{0,1\}^E}) \leq ct_{\text{mix}}^{\mathsf{full},(\mu,p)} + \mathbb{E}_{\pi_{\mathsf{full}},p}^{(\mu,p)}(T_{y\times\{0,1\}^E})$$

Now, for any set A we know from the quasi-stationary distribution μ_A (see [AF02] for details) that

$$\mathbb{E}_{\pi_{\mathsf{full}},p}^{(\mu,p)}(T_A) \leq \mathbb{E}_{\mu_A}(T_A) = \frac{1}{\lambda_A} = \max_x \mathbb{E}_{x,p}^{(\mu,p)}(T_A)$$

Applying this to $A = y \times \{0,1\}^E$ and using the fact that $\lambda_{\mathbb{A}}(\mu) \geq \mu\lambda_A(1)$ by the extremal characterization of the eigenvalue, combined with our earlier comment about the generator. Thus we have

$$\mathbb{E}^{(\mu,p)}_{\pi_{\mathsf{full}},p}(T_{y\times\{0,1\}^E}) \leq \frac{1}{\lambda_{y\times\{0,1\}^E}} \leq \frac{1}{\mu} t_{\mathsf{hit}}^{\mathsf{full},(1,p)}$$

Therefore,

$$\mathbb{E}^{(\mu,p)}_{x,\eta}(T_{y\times\{0,1\}^E}) \leq ct_{\text{mix}}^{\mathsf{full},(\mu,p)} + \mathbb{E}^{(\mu,p)}_{\pi_{\mathsf{full}},p}(T_{y\times\{0,1\}^E}) \leq ct_{\text{mix}}^{\mathsf{full},(\mu,p)} + \frac{1}{\mu} t_{\mathsf{hit}}^{\mathsf{full},(1,p)}$$

□

4.3 Informal comments about proof of Theorem 3.3

In order to keep this essay concise, we do not go into the details of the proof establishing a comparison principle of the mixing time and the spectral profile of a simple random walk. Essentially, the authors in [HS20] want to use the auxiliary chain as a bridge to compare the simple random walk and the full process. Firstly, it is clear that even though the auxiliary chain might be non-reversible, due to the relation that we developed between the transition probabilities of the auxiliary walk and the simple random walk we get a relation between their Dirichlet forms, and so, we can compare their spectral profiles (and thus $t_{\text{spectral}-\text{profile}}$). This is step one of their proof, they compare the auxiliary walk with parameters $(1,p)$ and the simple random walk.

Now, it is important to realise that the same trick with the Dirichlet forms will not work directly when trying to compare the full process and the simple random walk directly. This is because that method fundamentally works only when you are on the same (or similar) state spaces. To get around this, the authors in [HS20] first compare the auxiliary walk with the simple random walk like above, and then compare the spectral profiles of the full process and the auxiliary walk directly using a probabilistic interpretation of the spectral profile in terms of hitting times. This interpretation relates the spectral profile at $\pi(A^c)$ to the exponential decay rate of the tail of the hitting time T_A, when the chain starts from the stationary distribution. More precisely, controlling $\mathbb{E}_{\pi_{\text{full}}}[\exp(cT_A)]$ allows us to control the spectral profile of the full process.

With the above idea in mind, for every $A \subseteq V \times \{0,1\}^E$ they consider the set $B \subseteq V$ given by $B = \left\{v \in V : \pi_p(\mathsf{Env}(v,A)) \geq \frac{1}{4}\right\}$ where $\mathsf{Env}(v,A) = \{\eta : (v,\eta) \in A\}$. Using the fact that $\pi(B^c) \lesssim \pi_{\mathsf{full},(\mu,p)}(A^c)$, the authors relate the hitting time of B to the hitting time of A and thus get a relation between the spectral profiles by the above interpretation. We omit rest of the technical details and refer the reader to [HS20].

Remark. Let $C_{x,y}$, $C_{x,y}^{\mathsf{full},(\mu,p)}$ be the commute times of the simple random walk and the random walk on dynamical percolation. Then, note that the first argument above shows in particular a comparison between commute times of the dynamical walk and the simple random walk

$$C_{x,y}^{\mathsf{full},(\mu,p)} \lesssim_\mu \frac{1}{p}\, C_{x,y}$$

In fact, we showed a comparison for the auxiliary walk and the simple random walk. Note that in the transitive graph setting, this leads to a pointwise comparison for any x, y. That is,

$$\mathbb{E}_x(T_y) \lesssim_\mu \frac{1}{p}\mathbb{E}_x^{\mathsf{aux},(\mu,p)}(T_y)$$

Note that from Theorem 3.3, we also have that

$$C_{x,y}^{\mathsf{full},(\mu,p)} \lesssim \frac{1}{\mu p}\left(C_{x,y} + t_{\text{spectral}-\text{profile}}^{\mathsf{SRW}}\right)$$

5 Comparison of Cover Times

In this section, we study the comparison principle between the cover times of the simple random walk (SRW) and the dynamical percolation walk. This addresses *Question 1.12* of [HS20].

In particular, we show that Question 1.12 holds (in the positive) for several natural families of graphs. That is, for each of the sequences (G_n) in the families below, there exists a constant $C(\mu) > 0$ such that for every graph in the sequence (G_n) and for all $p \in (0, 1]$, we have

$$t_{\mathsf{cov}}^{(\mu,p)}(G_n) \leq \frac{C(\mu)}{p} \cdot t_{\mathsf{cov}}^{\mathsf{SRW}}(G_n).$$

The constant $C(\mu)$ is uniform over all graphs in the sequences within the family. This confirms the comparison principle for the following classes:

- Uniformly locally transient graphs (ULT). These are graphs that are the finite analog of transient graphs.
- Cayley graphs of abelian groups

To illustrate the vast breadth of ULT graphs, we also focus on the following important subclasses.

- Rapidly mixing vertex-transitive graphs
- Vertex-ransitive graphs with diameter satisfying $D^2 \gtrsim n/\log n$ (For instance, $\mathbb{Z}_n^d$ for $d \geq 3$)

It is natural to ask whether there is a graph that is ULT but not in the above subclasses. We show that certain special graphs (namely, the product of a highly transient graph, Ramanujan graphs, and a highly recurrent graph, the cycle) are ULT graphs that do not belong in the above classes, despite also being vertex-transitive. We also give an example of a high-dimensional graph, which is not rapidly-mixing.

To end, we also give a more hands-on proof (avoiding the abelian group result) for: the tori $\mathbb{Z}_n$ and $\mathbb{Z}_n^2$.

5.1 Transient Graphs

The first class of graphs we work with are what we call uniformly locally transient (the nomenclature is inspired by [BHT22], and the reason for it is pretty obvious from our definition and the discussion below). Consider a sequence of graphs $(G_n)_{n\geq 0}$ of diverging sizes (i.e., $|V_n| \to \infty$). It is unclear what the correct analog of transience is for a sequence of finite graphs. Aldous [AF02, Chapter 4] discusses that a natural notion for transience in finite graphs is when $\mathbb{E}_\pi(T_\pi) = O(n)$, in this essay, we define the following stronger notion of transience.

Definition 5.1. *We call a sequence of graphs G_n of diverging sizes uniformly locally transient if*

$$\max_x \mathbb{E}_\pi(T_x) = O(n)$$

One can ask why we choose this to be the natural analog for transience in the finite setting. This can be explained as follows. The concept of transience in the infinite setting is typically defined by the condition

$$\int_0^\infty \mathbb{P}_x(X_t = x)dt < \infty,$$

which expresses the finiteness of expected returns to a point v over infinite time. Consider our definition for bounded degree graphs, (we have $\pi \asymp \frac{1}{n}$). It is not hard to see that for a vertex x,

$$n^{-1}\mathbb{E}_\pi(T_x) \asymp \int_0^\infty (\mathbb{P}_x(X_t = x) - \pi(x))$$

So the transience condition is equivalent to the boundedness of the above integral. Which can be thought of as a reasonable analog to the boundedness of the integral in the infinite setting if we think of "mixing" as "reaching infinity". Another advantage of our definition is that it also has the following equivalence.

Proposition 5.1. *A sequence of graphs G_n of diverging sizes uniformly locally transient if and only if $\mathcal{R}(G_n) = \max_{x,y} \mathcal{R}_{\mathsf{eff}}(x,y) \asymp \frac{1}{d^{(n)}_{\mathsf{ave}}}$ where $d^{(n)}_{\mathsf{ave}}$ is the average degree of G_n.*

Before going to the proof of this proposition, we denote $C_{x,y} = \mathbb{E}_x(T_y) + \mathbb{E}_y(T_x)$ as the commute time between x, y. We now list a series of lemmas that would be useful to prove the required result.

Lemma 5.1 (Commute time identity)**.** *$C_{x,y} = 2|E|\mathcal{R}_{x,y}$ where $\mathcal{R}_{x,y}$ is the effective resistance between x and y.*

We shall also need the following well-known resistance lower bound.

Lemma 5.2 (Resistance lower bound)**.** *Let x, y be vertices in a graph G, then*

(1) If x and y are not adjacent, then $\mathcal{R}_{x,y} \geq \frac{1}{\deg(x)} + \frac{1}{\deg(y)}$.

(2) If x, y are adjacent, $\mathcal{R}_{x,y} \geq \frac{1}{\deg(x)+1} + \frac{1}{\deg(y)+1}$.

Lemma 5.3 (See, [AF02])**.** *For any graph G,*

$$\max_x \mathbb{E}_\pi(T_x) \leq t^{\mathsf{SRW}}_{\mathsf{hit}}(G) \leq 4 \max_x \mathbb{E}_\pi(T_x)$$

Proof of Proposition 5.1. The commute time identity states that

$$\mathbb{E}_x(T_y) + \mathbb{E}_y(T_x) = 2|E|\mathcal{R}_{\mathsf{eff}}(x,y) = nd_{\mathsf{ave}}\mathcal{R}_{\mathsf{eff}}(x,y)$$

So $t^{\mathsf{SRW}}_{\mathsf{hit}} = \Theta(n)$ if and only if $\mathcal{R}(G) \asymp \frac{1}{d_n}$. Now by Lemma 5.3, $\max_x \mathbb{E}_\pi(T_x) \leq t^{\mathsf{SRW}}_{\mathsf{hit}} \leq 4\max_x \mathbb{E}_\pi(T_x)$, and by Lemmma 5.2, and by considering x, y such that one of them has degree $\geq d_{\mathsf{ave}}$ we have $t^{\mathsf{SRW}}_{\mathsf{hit}} \geq n$. Thus $t_{\mathsf{hit}} = \Theta(n)$ and the conclusion follows. □

In the vertex-transitive setting, it can be shown that our definition of transience matches with the following definitions. See the discussion in [Her18].

Theorem 5.1 (Equivalence of definitions)**.** *Consider a sequence of transitive graphs G_n. Then the following is equivalent.*

(1) *(G_n) is uniformly locally transient*

(2) $\min_{\{x,y \in V(G_n): x \neq y\}} P_x[T^+_x > T_y] \asymp 1$*, where $T^+_x := \inf\{t : X_t = x, X_s \neq x$ for some $s < t\}$.*

(3) *The local time upto relaxation is bounded:* $\max_{x,y \in V(G_n)} \int_0^{t_{\mathsf{rel}}(G_n)} P^{G_n}_t(x,x)\,dt = O(1)$

In [BHT22], Nathanaël Berestycki, Jonathan Hermon, and Lucas Teyssier discuss a notion of weakly uniformly transient graphs. These are graphs defined as having local time up to mixing uniformly bounded. More specifically,

Definition 5.2. *We say that a sequence of vertex-transitive finite graphs (G_n) of bounded degree, (with the SRW having transition matrices $P = P^{(n)}$ and stationary distributions $\pi = \pi_n$) is weakly uniformly transient (WUT), if*

$$\max_{x \in G_n} \mathbb{E}_x(L_x(t)) = O(1)$$

where $t = t_{\mathsf{mix}}(1/4)$, and $L_x(t) := \int_0^t \mathbf{1}_{\{X_s = x\}} \, \mathrm{d}s$ is the local time of the walk at x up to time t.

Remark. Let T_y be the hitting time of y. Since

$$\mathbb{E}_x[L_t(y)] = \mathbb{E}_x\left[\int_{T_y}^t \mathbf{1}_{\{X_s = y\}} \, ds\right] = \mathbb{E}_x\left[\mathbb{E}_y[L_{t-T_y}(y)]\right] \leq \mathbb{E}_x\left[\mathbb{E}_y[L_t(y)]\right] = \mathbb{E}_y[L_t(y)]$$

Therefore, above we can take the maximum over all pairs.

During our initial exploration of transience in the finite setting, we thought that they serve as a natural analog for finite transience. This is indeed the case: using methods from [TT20] it is easy to see that these graphs are equivalent to having uniformly bounded resistance and hence are uniformly locally transient. So they are indeed a subclass of the graphs that we are considering.

Our definition also fits well with early discussions about transience in the finite setting. From above, we can observe that the hitting time is always at least linear in terms of the number of vertices. Aldous and Fill [AF02] discuss that having a linear upper bound on the hitting time is a natural analogue of transience for finite graphs. Since we have by the commute identity that $\mathbb{E}_x(T_y) + \mathbb{E}_y(T_x) = n d_{\mathsf{ave}} \mathcal{R}_{x,y}$, the proposition above shows that the notion of locally uniform transience and the transience of Aldous and Fill are the same.

The following is the analog for the theorem on hitting time comparison from [HS20] and a step towards answering Question 1.12 of [HS20].

Theorem 5.2 (Cover time comparison with SRW)**.** *Let $\mu > 0$ and $(G_n)_{n\geq 0}$ be a uniformly locally transient sequence then there exists a constant $c_1(\mu)$ so that for any finite graph G which is part of $(G_n)_{n\geq 0}$ and all $p \in (0, 1]$ we have that*

$$t_{\text{cov}}^{\text{full},(\mu,p)} \leq c_1 p^{-1} t_{\text{cov}}^{\text{SRW}} \tag{5.1}$$

In fact, we will see that $t_{\text{cov}}^{\mathsf{SRW}} \asymp n \log n$ where $n = |V(G)|$ so, $t_{\text{cov}}^{\text{full},(\mu,p)} \leq c_1 p^{-1} n \log n$. We shall now prove the following result, which follows from Feige's lower bound for any n-vertex graph [Fei95]. To keep this essay concise, we do not go into the proof but refer the interested reader to his elegant proof [Fei95].

Theorem 5.3 (Feige, [Fei95])**.** *Let G be any graph with n vertices then $t_{\mathit{cov}} \geq n \log n$*

Proposition 5.2. *Let (G_n) be a uniformly locally transient sequence, then $t_{\mathsf{cov}} = \Theta(|V(G_n)| \log |V(G_n)|)$*

Proof. The upper bound follows from the fact that the hitting time is of order $|V(G_n)|$ combined with Matthews upper bound, and the lower bound follows from the above theorem. □

Let us first see how Theorem 5.2 readily follows.

Proof. Let (G_n) be a uniformly locally transient sequence. From propositions above, $t_{\mathsf{hit}} \asymp |V(G)|$. Now consider the following chain of inequalities

$$t_{\text{cov}}^{\mathsf{full},(\mu,p)} \lesssim t_{\text{hit}}^{\mathsf{full},(\mu,p)} \log(n) \leq \frac{c_1}{p} \log(n)\ t_{\mathsf{hit}}^{\mathsf{SRW}}$$

This is a consequence of Matthew's upper bound in dynamical percolation combined with the hitting time comparison of [HS20]. Now from Feige's result, since the hitting time is of the order n and the cover time is at least $n \log n$, we have that $t_{\text{cov}}^{\mathsf{SRW}} \geq t_{\text{hit}}^{\mathsf{SRW}} \log n$, therefore by Matthews bound, we get that

$$\log(n)\ t_{\mathsf{hit}}^{\mathsf{SRW}} \asymp t_{\text{cov}}^{\mathsf{SRW}}$$

Thus,

$$t_{\text{cov}}^{\mathsf{full},(\mu,p)} \lesssim t_{\text{hit}}^{\mathsf{full},(\mu,p)} \log(n) \leq \frac{c_1}{p} \log(n)\ t_{\mathsf{hit}}^{\mathsf{SRW}} \lesssim \frac{1}{p} t_{\text{cov}}^{\mathsf{SRW}}$$

□

5.2 Examples of Transient graphs

Let us now describe some examples of uniformly locally transient graphs. Throughout this subsection, let (G_n) be a transitive family. Let the degree of G_n be Δ_n. We start by recalling the following result from the remarkable paper of Tessera and Tointon [TT20].

Theorem 5.4. *Let G be a finite, connected, vertex-transitive graph of degree Δ. Then*

$$\frac{1}{\Delta} + \frac{\mathrm{diam}(G)^2}{\Delta |V(G)|} \lesssim \mathcal{R}_G = \max_{x,y} \mathcal{R}_{\mathsf{eff}}(x,y) \lesssim \frac{1}{\Delta} + \frac{\mathrm{diam}(G)^2 \log(|V(G)|/\Delta)}{|V(G)|}.$$

Definition 5.3. *We say that a sequence of transitive finite graphs G_n is high-dimensional, if the diameter of G_n satisfies,*

$$\mathsf{diam}(G_n)^2 = O\left(\frac{|V(G_n)|}{\log |V(G_n)|}\right).$$

It is clear from Theorem 5.4 that for these sequences $\mathcal{R}(G) \asymp \frac{1}{\Delta_n}$ and thus these sequences are uniformly locally transient and therefore the comparison theorem for cover times holds for these graphs.

An important example is the high-dimensional torus, i.e., $(\mathbb{Z}_n^d)_{n \geq 1}$ for all $d \geq 3$. If a graph sequence is not high-dimensional, the diameters roughly satisfy $\mathsf{diam}(G_n) = \Theta(\sqrt{|V(G_n)|})$, which we think of as 2-dimensional. Therefore, high-dimensional graphs can be thought of as graphs with "dimension" > 2.

It turns out that under the assumption of bounded degree for these high-dimensional graphs, we get much more refined comparison principles, where the dependence on μ becomes more explicit.

Theorem 5.5. *Let $\Delta > 0$. Then for a high-dimensional sequence (G_n), which is also transitive, with bounded degree Δ there exists a constant $c_2(\Delta)$ so that for all graphs in (G_n) and all $(\mu, p) \in (0,1]^2$ we have that*

$$t_{\text{hit}}^{\text{full},(\mu,p)} \leq c_2 (\mu p)^{-1} t_{\text{hit}}^{\text{SRW}} \tag{5.2}$$

We also have the following comparison of cover times, for some $c_3 = c_3(\Delta)$,

$$t_{\text{cov}}^{\text{full},(\mu,p)} \leq c_3 (\mu p)^{-1} t_{\text{cov}}^{\text{SRW}} \tag{5.3}$$

From our methods above, we know that the comparison principles for the cover time effortlessly go through for uniformly locally transient graphs given the comparison principle for hitting times. This is simply because $t_{\text{cov}}^{\mathsf{SRW}} \asymp \log |V(G_n)| t_{\text{hit}}^{\mathsf{SRW}}$. Thus, we focus on getting a comparison principle for the hitting times.

To the best of our knowledge, such a comparison is still open in full generality. However for we can show this for all high-dimensional graphs of bounded degree by the following theorem.

Theorem 5.6. *Let $\Delta > 0$. Then for a high-dimensional sequence (G_n), which is also transitive, with bounded degree Δ there exists a constant $c_6(\Delta)$ so that for all graphs in (G_n) we have that*

$$t_{\text{spectral-profile}}^{\mathsf{SRW}} \leq c_6 t_{\text{hit}}^{\mathsf{SRW}} \tag{5.4}$$

First, let us see how Theorem 5.5 follows from the above theorem.

Proof of Theorem 5.5 given Theorem 5.6. Suppose, $t_{\text{spectral-profile}}^{\mathsf{SRW}} \leq c_6 t_{\text{hit}}^{\mathsf{SRW}}$ then by Theorem 3.4 we have

$$t_{\text{mix}}^{\text{full},(\mu,p)} \lesssim (\mu p)^{-1} t_{\text{hit}}^{\mathsf{SRW}}$$

Now the result follows with the comparison theorem for mixing times (equation 3.2), i.e.

$$t_{\text{hit}}^{\text{full},(\mu,p)} \lesssim (\mu p)^{-1} t_{\text{hit}}^{\mathsf{SRW}}$$

□

Let us now prove Theorem 5.6.

Proof. Let (G_n) be a sequence of high-dimensional graphs of bounded degree Δ. Then, by an inequality of Chung, (see [Chu97]) we have $t_{\text{rel}}^{\mathsf{SRW}}(G_n) \leq \Delta \operatorname{diam}(G_n)^2$. Thus we get that $t_{\text{rel}}^{\mathsf{SRW}}(G_n) = O\left(\frac{|V(G_n)|}{\log |V(G_n)|}\right)$ and therefore $t_{\text{spectral}-\text{profile}}^{\mathsf{SRW}}(G_n) = O(|V(G_n)|) = O(t_{\text{hit}}^{\mathsf{SRW}})$. □

5.3 Rapidly mixing vertex transitive graphs

As we saw above, the main thing required for our method to work is that the Matthews inequality is tight. In the original paper [Mat88], Matthews believed that this should be the case for all rapidly mixing sequences. These are sequences (G_n) such that $t_{\text{rel}}^{\mathsf{SRW}} = O(n^{1-\delta})$ for some $\delta > 0$. Zuckermann [Zuc90] then showed that this is indeed the case for all vertex-transitive rapidly mixing graphs. Therefore for all those graphs we have,

$$t_{\text{hit}}^{\mathsf{SRW}} \log n \lesssim t_{\text{cov}}^{\mathsf{SRW}}$$

Therefore, by the same proof as above, we would have the following comparison theorem.

Theorem 5.7 (Cover time comparison with SRW)**.** *Let $\mu > 0$ and $(G_n)_{n\geq 0}$ be a rapidly mixing sequence then there exists a constant $c_7(\mu)$ so that for any finite graph G which is part of $(G_n)_{n\geq 0}$ and all $p \in (0,1]$ we have that*

$$t_{\text{cov}}^{\text{full},(\mu,p)} \leq c_7 p^{-1} t_{\text{cov}}^{\text{SRW}} \tag{5.5}$$

There are two ways to show that Matthew is tight for these graphs. First, a slightly less exciting way to use the following lemma.

Lemma 5.4 (Proposition 13.7., [LP17b]). *Consider random walk on a graph G with n vertices and diameter D. Let π denote the stationary distribution. Then, if $n \geq 64$*

$$t_{mix}(1/4) \geq \frac{D^2}{12 \log n + 4|\log \pi_{\min}|}$$

In particular, for transitive graphs, we get a bound of $D^2/\log n$ analogous to simple graphs.

Thus, all rapidly mixing graphs satisfy $D^2 \lesssim \log n \; n^{1-\delta}$, thus, they are high-dimensional and therefore uniformly locally transient, for which the Matthews bound is tight and comparison principles hold.

A more satisfying approach is to use the adaptation of Matthews lower bound and to prove it using a direct approach. We show this now.

Lemma 5.5 (Refinement of Matthews lower bound). *Let $G = (V, E)$ be a graph and let $S \subseteq V$. Let t be such that for each $x \in S$ at most m vertices $y \in S$ have a bad hitting time from x, i.e., they satisfy $\mathbb{E}_x(T_y) < t$. Then for each $z \in S$*

$$\mathbb{E}_z[\tau_{\mathsf{cov}}] \geq t \; (\log(|S|/m) - 2).$$

Therefore $t_{\mathsf{cov}} \geq t \; (\log(|S|/m) - 2)$.

Proof. We want to show that for any $z \in S$,

$$\mathbb{E}_z[\tau_{\mathsf{cov}}] \geq t(\log(|S|/m) - 2),$$

We fix the starting vertex $z \in S$. Let $\sigma = (\sigma(1), \cdots \sigma(|S| - 1))$ be a permutation of $S \setminus \{z\}$, chosen uniformly. Define T_k as the first time all of $\sigma_k := \{\sigma(1), \cdots \sigma(k)\}$ have been visited, and $R_k := T_k - T_{k-1}$. By symmetry, we have $\mathbb{P}[R_k \neq 0] = \frac{1}{k}$, since the probability that $\sigma(k)$ is the last of σ_k to be covered is $1/k$.

Condition on the walk up to time T_{k-1}, and σ_{k-1} and let $i := X_{T_{k-1}}$. We have

$$\mathbb{E}_z[R_k | R_k \neq 0, X_{T_{k-1}} = i] = \mathbb{E}_i[T_{\sigma(k)}]$$

and by assumption, the proportion of bad vertices is (this is very we deviate from Matthews proof),

$$\mathbb{P}[\mathbb{E}_i[T_{\sigma(k)}] < t] \leq \frac{m}{|S|}$$

Now, by writing, $\mathbb{E}_z(R_k) = \mathbb{E}_z(R_k | R_k \neq 0)$ we can lower bound, by Markov's inequality,

$$\mathbb{E}_z[R_k] \geq t(\mathbb{P}_z(R_k \neq 0) - \mathbb{P}(\mathbb{E}_i[T_{\sigma(k)}]) < t)) \geq \left(\frac{1}{k} - \frac{m}{|S|}\right)$$

Summing over $k \leq |m|/b$,

$$\sum_{k=1}^{|S|/m} \mathbb{E}_z[R_k] \geq t \sum_{k=1}^{|S|/m} \left(\frac{1}{k} - \frac{m}{|S|}\right) \geq t \left(\ln(|S|/m) - 2\right).$$

Thus, we obtain,

$$\mathbb{E}_z[\tau_{\mathsf{cov}}] \geq \sum_{k=1}^{|S|/m} \mathbb{E}[R_k] \geq t(\ln(|S|/m) - 2),$$

as desired. ∎

Let us now present the proof for the discrete simple random walk. This will not create any issues since in continuous time, the cover time satisfies

$$\tau_{\mathsf{cov}}^{\mathsf{cont}} = \sum_{i=0}^{\tau_{\mathsf{cov}}^{\mathsf{discrete}}} T_i$$

Where T_i is the clock of rate 1.

Then by Wald's identity, for any vertex v,

$$\mathbb{E}_v(\tau_{\mathsf{cov}}^{\mathsf{cont}}) = \mathbb{E}_v(\tau_{\mathsf{cov}}^{\mathsf{discrete}}) \implies t_{\mathsf{cov}}^{\mathsf{cont}} = t_{\mathsf{cov}}^{\mathsf{discrete}}$$

Therefore, we can safely assume that we are working with the discrete simple random walk. Let us now proceed with the proof.

Theorem 5.8. *Let (G_n) be a sequence of graphs such that $t_{\text{rel}}^{\mathsf{SRW}} = O(n^{1-\delta})$ then $t_{\text{cov}}^{\mathsf{SRW}} \asymp \log n \; t_{\text{hit}}^{\mathsf{SRW}}$*

We start by recalling the following spectral decomposition.

Theorem 5.9 (Spectral decomposition of reversible chains)**.** *Let P be reversible with respect to π. The inner product space $(\mathbb{R}^S, \langle \cdot, \cdot \rangle_\pi)$ has an orthonormal basis of real-valued eigenfunctions $(f_j)_{j \le |S|}$ corresponding to real eigenvalues $(\lambda_j)_{j \le |S|}$ and the eigenfunction f_1 corresponding to $\lambda_1 = 1$ can be taken to be the constant vector $(1, \ldots, 1)$. Moreover, the transition matrix P^t can be decomposed as*

$$\frac{P^t(x, y)}{\pi(y)} = 1 + \sum_{j=2}^{|S|} f_j(x) f_j(y) \lambda_j^t.$$

In particular, we have the above for the simple random walk on a graph. The above lemma helps us to conclude a bound on the L^1 distance of the transition probability and the stationary distribution.

Lemma 5.6. *For all $t \ge (k+1)t_{\mathsf{rel}} \log n$, we have*

$$\left| P^t(x, y) - \pi(y) \right| \le n^{-k}$$

Proof. By the above spectral decomposition we have that,

$$|P^t(x, y) - \pi(y)| \le \sum_j \pi(y) |f_j(x) f_j(y)| |\lambda_k|^t \le \sum_j \pi(y) (\lambda^*)^t \le n(\lambda^*)^t \le ne^{-t/t_{\mathsf{rel}}}$$

Where $\lambda^* = \max\{|\lambda_k| : k \le n\}$ and $t_{rel} = \frac{1}{1-\lambda^*}$ □

Lemma 5.7. *Suppose that $t_{\mathsf{rel}} \le n^{1-\delta}$ for $\delta > 0$. Then for any $\varepsilon > 0$ and any $x \in V$,*

$$\mathbb{E}_x(\tau_{\mathsf{cov}}) \ge (1 - o(1)) \delta \log n \cdot \min \left\{ \mathbb{E}_\pi(T_y) : y : \pi(y) < \frac{1+\varepsilon}{n} \right\}$$

Proof. We use the refinement of Matthews lower bound with $t = (1 - o(1)) \times \min \left\{ \mathbb{E}_\pi(T_y) : y : \pi(y) < \frac{1+\varepsilon}{n} \right\}$. Let $V' = \left\{ x : \pi(x) < \frac{1+\varepsilon}{n} \right\}$ be the set of vertices with small measure. Since $\sum_y \pi(y) = 1$, it is clear that $|V \setminus V'| \le \frac{n}{1+\varepsilon}$, so $|V'| \ge \left(1 - \frac{1}{1+\varepsilon}\right) n$. Now, for a fixed x, define $J_x = \left\{ y : \mathbb{P}_x(T_y \le 5t_{\mathsf{rel}} \log n) \ge \frac{1}{\log n} \right\}$. Since, the number of vertices visited till time t is obviously less than t, we have

$$\sum_y \mathbb{P}_x(T_y \le 5t_{\mathsf{rel}} \log n) \le 5t_{\mathsf{rel}} \log n$$

Thus $|J_x| \le 5t_{\mathsf{rel}}(\log n)^2$. Now for any $y \in V' \setminus J_x$, we have $\mathbb{E}_x(T_y) \ge \left(1 - \frac{1}{\log n}\right) \mathbb{E}_\nu(T_y)$ where ν is the distribution after the first $5t_{\mathsf{rel}} \log n$ steps starting from x. From the above lemma

$$|\nu(y) - \pi(y)| \le n^{-4} \implies \nu(y) \ge \pi(y) - \frac{1}{n^4} \ge \pi(y)\left(1 - \frac{1}{n^4 \pi(y)}\right)$$

Since, $\pi(y) \ge \frac{1}{n^3}$ we get, $\nu(y) \ge \pi(y)\left(1 - \frac{1}{n}\right)$. Finally, $\mathbb{E}_\nu(T_y) \ge \left(1 - \frac{1}{n}\right) \mathbb{E}_\pi(T_y)$, so $\mathbb{E}_x(T_y) \ge t$.

Therefore, V' is the set of vertices such that $|V'| \ge n(1 - \frac{1}{1+\varepsilon})$ more over at most $m = |J_x| \le 5t_{\mathsf{rel}}(\log n)^2 \le 5n^{1-\delta}(\log n)^2$ vertices y satisfy that $\mathbb{E}_x(T_y) \le t$. Thus, we have the result from the refinement of Mathews lower bound for V', m, t. □

In general this will show that

$$4 \log n \max\{\mathbb{E}_\pi(T_y) : y \in V\} \ge t_{\text{cov}}^{\mathsf{SRW}} \ge \delta \log n \min\left\{\mathbb{E}_\pi(T_y) : y : \pi(y) < \frac{1+\varepsilon}{n}\right\}$$

where the first inequality is the Matthews upper bound combined with $t_{\text{hit}}^{\mathsf{SRW}} \le 4 \max\{\mathbb{E}_\pi(T_y) : y \in V\}$. The first inequality is general, and the second holds in the rapidly mixing case. Combining our ideas above, we would get,

$$t_{\text{cov}}^{\mathsf{full},(\mu,p)} \le \log n \; t_{\text{hit}}^{\mathsf{full},(\mu,p)} \le \frac{C \log n}{\mu p} t_{\text{hit}}^{\mathsf{SRW}} \le \frac{4C \log n}{\mu p} \max\{\mathbb{E}_\pi(T_y) : y \in V\}$$

Therefore, to get a comparison principle, we would need an upper bound by the cover time of the maximum hitting time started from π, we have such a bound for the minimum over some vertices. If that's the same order as the maximum, then we are done. One class of graphs which smooths over these variation issues between expected hitting time is transitivity, in particular for transitive graphs we have that $\mathbb{E}_\pi(T_y)$ does not depend on y. Therefore, we get a cover time comparison for all transitive rapidly mixing graphs.

5.4 Cayley graph of Abelian groups

Let G be an abelian group. We also refer to its Cayley graph as G. Following Remark 3.7 of [HS20] and using the celebrated paper [DLP11], we show that the cover time comparison holds for abelian groups.

We first start by stating the result of Ding, Lee and Peres that gives us a comparison principle for cover times of two reversible walks on the same state space.

Theorem 5.10 (Comparison theorem for cover times, Theorem 1.6, [DLP11])**.** *Suppose G and G' are two graphs on the same set of nodes V, and C_G and $C_{G'}$ are the distances induced by respective commute times. If there exists a number $L \ge 1$ such that $C_G(u,v) \le L \cdot C_{G'}(u,v)$ for all $u, v \in V$, then*

$$t_{\mathsf{cov}}(G) \le O(L) \cdot t_{\mathsf{cov}}(G').$$

The adaptation of this theorem to our case of the auxiliary walk and the simple random walk is the following.

Theorem 5.11. *Suppose G be a graph with vertex set V such that the auxiliary chain is reversible. Let C_G and C_G^{aux} be the distances induced by the respective commute times of the SRW and the auxiliary walk. Then we know that $C_G(x,y) \lesssim_\mu \frac{1}{p} C_G^{\mathsf{aux}}(x,y)$ and thus from above we have comparison for auxiliary and simple random walk and thus*

$$t_{\text{cov}}^{\mathsf{full},(\mu,p)} \lesssim_\mu \frac{C}{p} t_{\text{cov}}^{\mathsf{SRW}}.$$

Where is reversibility of the auxiliary walk needed? We need the commute time to be a distance, which happens in the reversible case.

Lemma 5.8. *Let G be an Abelian group, then there is a graph automorphism ϕ such that for any $x, y, x \neq y$ $\phi(x) = y$ and $\phi(y) = x$. Thus, the auxiliary walk is reversible.*

Proof. Consider the map $\phi(g) = xyg^{-1}$. Let $G = \langle S|R\rangle$ be our Abelian group. It is clear that ϕ is a bijection, now if $gh^{-1} \in S$ we need to show that $\phi(g)\phi(h)^{-1} \in S$. Since $\phi(g)\phi(h)^{-1} = xyg^{-1}hx^{-1}y^{-1} = (gh^{-1})^{-1} \in S$, ϕ is a graph automorphism.

Since the dynamics and the transitions at the regeneration times are all automorphism-invariant we have $P^{\mathsf{aux}}(x, y) = P^{\mathsf{aux}}(\phi(x), \phi(y)) = P^{\mathsf{aux}}(y, x)$ and thus the auxiliary walk is reversible since the stationary distribution is same at each vertex. □

Therefore, we are done.

5.5 The one and two-dimensional tori

The cases for the one and two-dimensional tori follow from the result on the Cayley graph of Abelian groups. But this may seem like an injustice given the simplicity of these cases, therefore, we show a hands-on approach, which combined with our analysis of transient graphs, gives the comparison principle for $(\mathbb{Z}_n^d)_{n\geq 0}$ for any d.

The 1-dimensional torus. First, we show it for $(\mathbb{Z}_n)_{n\geq 0}$. We consider the auxiliary walk $(Y_i)_{i\geq 0} = (X_{\tau_i})_{i\geq 0}$ on the n-cycle $\mathbb{Z}_n$. Let x be a fixed vertex and y its antipodal point (i.e.,$y = x + \lfloor n/2\rfloor \pmod{n}$).

Define stopping times:

$$\zeta_0 = 0,\ \zeta_1 = \inf\{t > 0 : Y_t = y\},\ \zeta_2 = \inf\{t > \zeta_1 : Y_t = x\},\ \cdots$$

Each interval $[\zeta_{2i}, \zeta_{2i+2}]$ is a full commute: $x \to y \to x$. There are two arcs between x and y: the clockwise arc A and the anticlockwise arc B. To go from x to y, the walk must fully cover one arc. On the return $y \to x$, it again fully covers one arc. Thus, after one commute, if the walk switches arcs, it has covered the entire cycle. There are four direction combinations, in two of them (when you switch directions) we cover the cycle. So at each commute time between x, y by symmetry, there is a 50% chance of covering the cycle. Let N be the number of commutes until full coverage. Then:$N \preceq_{\mathsf{st}} \mathsf{Geom}(p)$, $\quad \mathbb{E}[N] \leq \frac{1}{p} = 2$. The expected time per commute is: $\mathbb{E}[\zeta_2 - \zeta_0] = \mathbb{E}_x[\tau_y] + \mathbb{E}_y[\tau_x]$, where τ_y is the hitting time of y from x. By Wald's identity:

$$\mathbb{E}_x^{\mathsf{aux}}[\tau_{\mathsf{cov}}] = \mathbb{E}[N] \cdot \left(\mathbb{E}_x^{\mathsf{aux}}[T_y] + \mathbb{E}_y^{\mathsf{aux}}(T_x)\right) \leq \frac{C'}{p} t_{\text{hit}}^{\mathsf{SRW}}$$

Since for the simple random walk on the cycle, the cover time and the hitting time are of the same order we get $t_{\text{cov}}^{\mathsf{full},(\mu,p)}(\mathbb{Z}_n) \leq \frac{C}{p} t_{\text{cov}}^{\mathsf{SRW}}(\mathbb{Z}_n)$.

The 2-dimensional torus The proof here follows from the fact that Matthews is tight! More specifically, $t_{\text{cov}}^{\mathsf{SRW}}(\mathbb{Z}_n^2) \asymp \log n\, t_{\text{hit}}^{\mathsf{SRW}}(\mathbb{Z}_n^2)$. In fact, the cover time is $n^2(\log n)^2$ and the hitting time is $n^2 \log n$. Therefore we have

$$t_{\text{cov}}^{\mathsf{full},(\mu,p)}(\mathbb{Z}_n^2) \leq \log n\ t_{\text{hit}}^{\mathsf{full},(\mu,p)}(\mathbb{Z}_n^2) \leq \frac{C \log n}{\mu p} t_{\text{hit}}^{\mathsf{SRW}}(\mathbb{Z}_n^2) \lesssim \frac{C(\mu)}{p} t_{\text{cov}}^{\mathsf{SRW}}(\mathbb{Z}_n^2)$$

We refer the reader to [LP17a] for a proof of the above classical fact. The idea to show that for the set $A = \{(x, y) :$ x,y are multiples of$\sqrt{n}\}$ and any two points $a, b \in A$ the hitting time is of the order t_{hit}.

Then, because this set is also big enough (order $\sim n^\alpha$), we get the above from Matthews inequality. This gives us a hint as to when Matthews can be shown to be tight, we need to find a large enough set with good hitting times for all pairs (this is exactly what we did in the rapidly mixing case).

5.6 Relation between different families

In the world of uniformly locally transient sequences, we studied 1.) High-dimensional graphs. 2.) Rapidly mixing graphs. In this section, we give two examples.

A high-dimensional graph sequence which is not rapidly mixing.

Consider $(G_n) = \mathbb{Z}_n^2 \times \mathbb{Z}_{\log n}$. Let $m = |V(G_n)| = n^2 \times \log n$. It is not hard to see that the mixing time for this graph will be of the order n^2. So clearly this sequence is not rapidly mixing, but it is high-dimensional, since the diameter is n and hence

$$\frac{D^2 \log m}{m} \asymp \frac{2\log n + \log\log n}{\log n} = O(1).$$

A uniformly locally transient sequence which is not a high-dimensional graph sequence.

This is a really interesting example of a sequence of vertex-transitive graphs that satisfy uniform local transience, but they are not high-dimensional! So they sit write at the border of transience. The idea is to consider a product of a very recurrent graph with a very transient graph. Let $m \geq 2$ be an even number and consider $G_n = C_m \times R_m$ where R_m is an expander Cayley graph of size $m \log\log m$. Thus $|V(G_n)| = m^2 \log\log m$. It is, in general, very hard to show that such expanders exist, but it can be shown that Ramanujan graphs of degree ≥ 3 do the job (see Morgenstern's generalization [Mor94] of the Lubotzky, Phillips, and Sarnak construction [LPS88]). The construction is beyond the scope of this essay, and so we assume that such a graph exists.

Part 1. G_n is uniformly locally transient.

We know that graphs that are weakly uniformly transient in the sense of [BHT22] are uniformly locally transient. So we show that G_n has bounded local time up to mixing.

The key idea is to decompose the vertex $x \in G_n$ into coordinates $(x^{(1)}, x^{(2)})$ where $x^{(1)} \in C_m$ and $x^{(2)} \in R_m$, so that the random walk on G_n becomes a product of independent random walks on C_m and R_m. The transition probability can thus be written as $p_t(x,y) = p_t^{(1)}(x^{(1)}, y^{(1)}) p_t^{(2)}(x^{(2)}, y^{(2)})$, where $p_t^{(1)}$ and $p_t^{(2)}$ are the respective transition probabilities on C_m and R_m.

For the cycle C_m, we have $p_t^{(1)}(x^{(1)}, x^{(1)}) \lesssim 1/\sqrt{t+1}$, while on the Ramanujan graph R_m, the spectral gap $\lambda > 0$ allows us to bound $\left| p_t^{(2)}(x^{(2)}, x^{(2)}) - \pi^{(2)}(x^{(2)}) \right| \leq e^{-\lambda t}$. Using these bounds and integrating $p_t(x,x)$ from s to t_{mix}, shows that $\mathbb{E}_x[L_x(t_{\mathsf{mix}}) - L_x(s)]$ tends to zero as $m \to \infty$, since $m^2 \lesssim t_{\mathsf{mix}}$ and $h = h_m \to \infty$. This establishes the desired transience condition uniformly in m. We omit the details and refer the reader to [BHT22].

Part 2. (G_n) is not high-dimensional

This is clear because the diameter of G_n is m, and so

$$\frac{D^2 \log n}{n} = \frac{(2\log m + \log\log m)m^2}{m^2 \log\log m} \to \infty$$

6 Random walk on dynamical percolation on moderate growth graphs

The graphs that we dealt with in the last two sections were uniformly locally transient graphs. We explained how graphs with small diameters were ULT, and thus, the only graphs that are left are the graphs with large diameters. Even though we are not able to give comparison principles for cover times in this setting. We present the discussion in [HS20] about these graphs. We shall be restricting our attention to vertex-transitive graphs.

Let $G = (V, E)$ be a graph with n vertices. Denote the volume of the ball of radius r by $V(r)$. Following [DSC94] we say that G has (c, a) moderate growth if $V(r) \geq cn(\frac{r}{\text{diam(G)}})^a$ for all r. Before starting, with our results, we state a result of Tessera and Tointon, which formalises the statement that these are exactly graphs with large diameters. More specifically, we have the following result of Tessera and Tointon from [TT21].

Theorem 6.1 (Corollary 2.8, [TT21]). *Let G be a finite connected vertex-transitive graph of degree d. If G has (c, a)-moderate growth, then*

$$\text{diam}(G) \geq \frac{1}{c^{1/a}} \left(\frac{|V(G)|)}{d+1} \right)^{1/a}. \tag{6.1}$$

Conversely, for every $\delta \geq 0$ there exists $n_0 = n_0(\delta)$ such that if $\text{diam}(G) \geq n_0$ *and*

$$\text{diam}(G) \geq \left(\frac{|V(G)|}{d+1} \right)^{\delta}, \tag{6.2}$$

then G has $(O_\delta(1), O_\delta(1))$-moderate growth.

The above result tells us that having a large diameter (more precisely at least polynomial in the number of vertices) is equivalent to being of moderate growth. We also recall the following result of Diaconis and Saloff-Coste.

Theorem 6.2 ([DSC94]). *Let G be a finite connected vertex-transitive graph of degree d. If G has (c, a)-moderate growth then,*

$$d^{-1} t_{\text{mix}}^{\text{SRW},(\infty)} \lesssim_{a,c} \text{diam(G)}^2 \lesssim_{a,c} t_{\text{rel}}^{\text{SRW}} \leq t_{\text{mix}}^{\text{SRW},(\infty)}$$

The aim of this section is to present a similar theorem of Hermon and Sousi which relates the relaxation time and mixing time of the random walk on dynamical percolation with the relaxation time of the simple random walk, under some mild conditions on the percolation cluster of G.

Let the percolation cluster of x be K_x and its edge boundary be ∂K_x. Define $M_p := \pi_p(|\partial K_x||K_x|^2) \leq \pi_p(|K_x|^3)$, and $N_p := \pi_p(|K_x|)$. These are precisely the expected sizes of $|\partial K_x||K_x|^2$and $|K_x|$ under the percolation measure π_p. Note that by vertex transitivity, they do not depend on x. Then we have the following theorem.

Theorem 6.3 ([HS20]). *For any $a, b, c \in \mathbb{R}_+$. Let $G = (V, E)$ be a connected vertex transitive graph of degree d, having (c, a)-moderate growth. Suppose that $|\log(1-p)| \leq$ $\text{diam}(G)^2$, $M_p \leq b$ and $N_p \leq$ $\text{diam}(G)/8$. Then,*

$$t_{\text{mix}}^{\text{full},(\mu,p),(\infty)} \asymp_{a,b,c,d} (\mu p)^{-1} t_{\text{rel}}^{\text{SRW}} \asymp_{a,b,c} t_{\text{rel}}^{\text{full},(\mu,p)}.$$

Moreover, even if p is not subcritical, we still have that

$$t_{\text{mix}}^{\text{full},(\mu,p),(\infty)} \lesssim_{a,c,d} (\mu p)^{-1} t_{\text{rel}}^{\text{SRW}}.$$

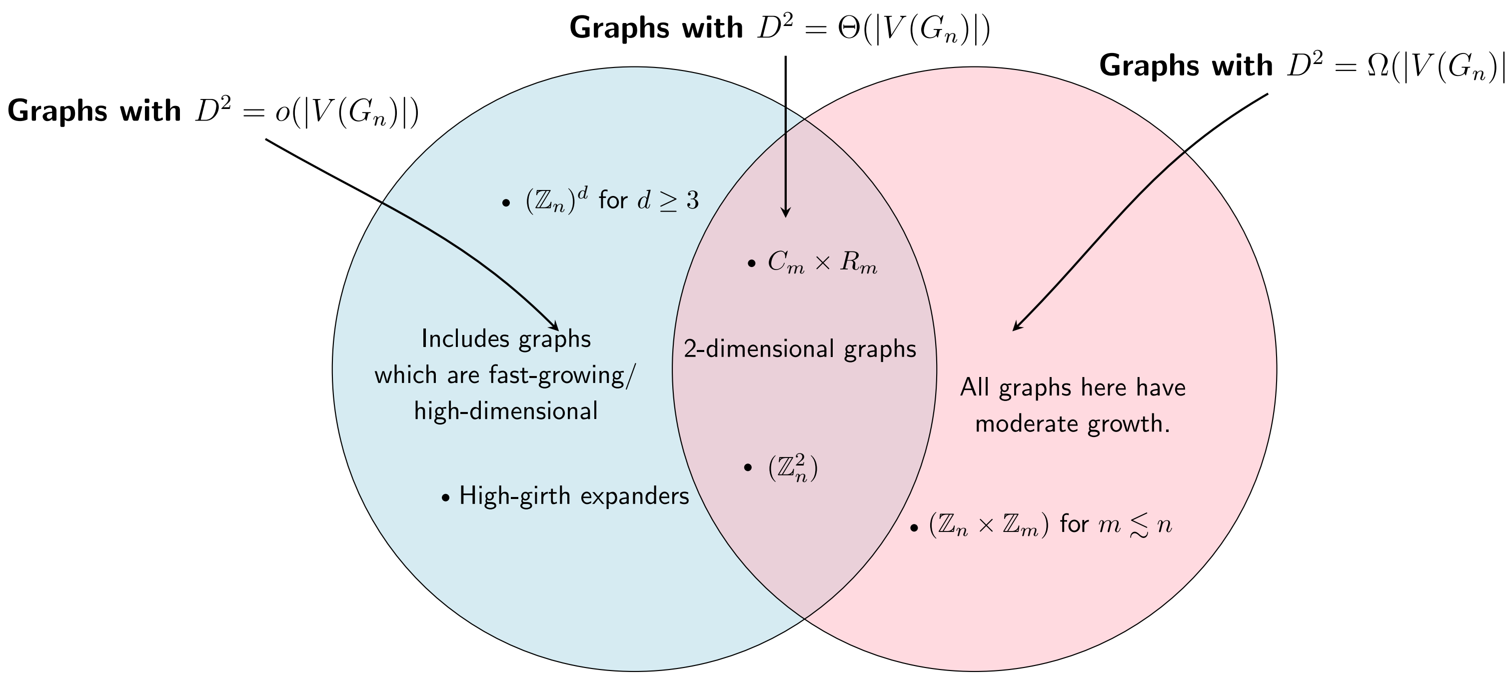

Figure 5: The universe of graph geometries. Blue represents all graphs with diameter $= O(|V(G_n)|)$. Pink represents all graphs with diameter $\Omega(|V(G_n)|)$. All high-dimensional graphs reside in the blue space, since $D^2 = O\left(\frac{|V(G_n)|}{\log|V(G_n)|}\right) = o(|V(G_n)|)$. By the equivalent diameter condition, all graphs in the pink space have moderate growth. We refer to graphs at the intersection as 2-dimensional due to their diameters being of the order $\sqrt{|V(G_n)|}$.

The condition on p should be thought of as being in the subcritical regime. We note that this condition is not that restrictive and usually follows under the assumption $\mathsf{diam}(G) \gg 1$.

Before going into the proof of the above theorem, we start with the following lemma, which is the main technical piece of the theorem. We know from Chung [Chu97] that the relaxation time is always upper bounded by the square of the diameter for all vertex transitive graphs. The lemma below tells us that the matching lower bound also holds for moderate growth graphs.

Lemma 6.1. *Let $c, a \in \mathbb{R}$. There exists a $c_1 = c_1(a, c) > 0$ such that for any G having (c, a)-moderate growth, and $N_p \leq \frac{\mathsf{diam}(G)}{4}$ we have,*

$$t_{\mathsf{rel}}^{\mathsf{full},(\mu,p)} \geq \frac{c_1(\mathsf{diam}(G) - 4N_p)^2}{\mu p M_p}$$

Proof of Lemma. Let $K_x(\eta)$ be the cluster of x in the configuration η, which we identify with the set of vertices in it. Fix some $o \in V$ and define $f(x, \eta) := \frac{1}{|K_x(\eta)|} \sum_{v \in K_x(\eta)} d_G(v, o)$, which is the average distance to o of a vertex in the cluster of x. For $\eta \in \{0,1\}^E$ let η^e be the percolation configuration defined as $\eta^e(e') := \eta(e')\mathbb{1}(e' \neq e) + \mathbb{1}(e = e')$. This is the configuration obtained from η by setting e to be open.

The idea is to use the extremal characterization of the relaxation time. We first start by computing $\mathcal{E}(f, f)$ for the f above. First note that,

$$\mathcal{E}(f, f) = \sum_{x \in V,\ \eta \in \{0,1\}^E,\ e \in \partial K_x(\eta)} \pi(x)\pi_p(\eta)\mu p(f(x, \eta) - f(x, \eta^e))^2$$

This is because the values of f do not change when just the walk coordinate changes. By simple computation, this is now less than

$$\mathcal{E}(f, f) \leq \mu p \sum_{x,\ \eta,\ e \in \partial K_x(\eta)} \pi(x)\pi_p(\eta)|K_x(\eta^e)|^2$$

Summing over x and using transitivity,

$$\mathcal{E}(f, f) \leq \mu p \sum_{\eta,\ e \in \partial K_x(\eta)} \pi_p(\eta)|K_x(\eta^e)|^2$$

Now for x, y such that $x, y \in E$ on the event that for$\{y \notin K_o,\ x \in K_o\}$ we have that, $|K_o^{\{x,y\}}| = |K_o| + |K_y|$. Therefore finally we get,

$$\mathcal{E}(f, f) = \mu p \sum_{x,y:\{x,y\} \in E} \pi_p[\mathbb{1}(y \notin K_o,\ x \in K_o)(|K_x| + |K_y|)^2]$$

It is easy to see that for all $A \subset V$ with $o, x \in A$ and $y \notin A$, if we are given that $K_o = A$ then due to independence, we have that $|K_y|$ is distributed as the size of a percolation cluster on the induced graph $V \setminus A$ which clearly is stochastically dominated by the law of $|K_y|$. Therefore, we have that $\mathbb{E}_{\pi_p}(|K_y|^a | K_o = A) \leq \mathbb{E}_{\pi_p}[|K_y|^a]$ and thus by law of total probability and transitivity,

$$\mathbb{E}_{\pi_p}(\mathbb{1}(y \notin A,\ x \in A)|K_y|^a | K_o) \leq \mathbb{E}_{\pi_p}(|K_y|^a) = \mathbb{E}_{\pi_p}(|K_o|^a)$$

Plugging this into what we showed above for $a = 2$ we get that $\mathcal{E}(f, f) \leq 4\mu p M_p$. Now to use the variational criterion we need to lower bound the variance of f under $\pi_{\mathsf{full},p}$. In particular, we now show that

$\mathsf{Var}_{\pi_{\mathsf{full},p}}(f) \geq c_{a,c}(\mathsf{diam}(G) - 4N_p)^2$. Firstly, we have that for all x

$$\left| d_G(x,o) - \sum_{\eta} \pi_p(\eta) f(x,\eta) \right| = \left| \sum_{\eta} \frac{\pi_p(\eta)}{|K_x(\eta)|} \sum_{v \in K_x(\eta)} (d_G(x,o) - d_G(v,o)) \right|$$

The maximal difference between $d_G(x,o)$ and the average over K_x is controlled by the diameter of K_x, which is at most proportional to its size $|K_x|$. Taking expectation, we get

$$|d_G(x,o) - \mathbb{E}_{\pi_p}[f(x,\eta)]| \leq \mathbb{E}_p(|K_x|) = N_p.$$

We lower bound the variance of f by considering two independent samples (X,η) and (Y,η') from stationarity. Define the event A that $d_G(Y,o) \leq \mathsf{diam}(G)/4$ and $d_G(X,o) \geq 3\ \mathsf{diam(G)}/4$. By moderate growth, we know that this event has a probability of at least some fixed $b > 0$. By vertex-transitivity and moderate growth, any ball of radius $\mathsf{diam}(G)/4$ contains a positive proportion of vertices, so both the set of vertices at distance at least $3\mathsf{diam}(G)/4$ and vertices within a $\mathsf{diam(G)}/4$ ball occupy a positive proportion of the graph.

We first start by writing, $2\mathsf{Var}_{\pi_{\mathsf{full},p}}(f) = \mathbb{E}\left((f(X,\eta) - f(Y,\eta'))^2\right)$. Now, the idea is that if one sample is close to o and the other is far, their corresponding f- values should be well separated. More specifically,

$$2\mathsf{Var}_{\pi_{\mathsf{full},p}}(f) \geq \mathbb{E}\big((f(X,\eta) - f(Y,\eta'))^2 \mathbf{1}(A)\big).$$

By Jensen's inequality, this is at least

$$(\mathbb{P}(A))^2 \left(\mathbb{E}[f(X,\eta)|A] - \mathbb{E}[f(Y,\eta')|A]\right)^2.$$

We know from earlier that

$$\mathbb{E}[f(X,\eta)|A] \geq 3\ \mathsf{diam}(G)/4 - N_p \ \text{ and } \quad \mathbb{E}[f(Y,\eta')|A] \leq \mathsf{diam}(G)/4 + N_p,$$

so their difference is at least $\mathsf{diam}(G)/2 - 2N_p$. Using $\mathbb{P}(A) \geq b$, we deduce

$$\mathsf{Var}_{\pi_{\mathsf{full},p}}(f) \geq \frac{b^2}{4}(\mathsf{diam}(G) - 4N_p)^2,$$

as required.

□

Now that we are done with our main technical piece, we are ready to show Theorem 6.3.

Proof. Let P be the transition matrix of the simple random walk on G. Then by a result of Diaconis and Saloff-Coste [DSC94] we have that for a vertex transitive (c,a) moderate growth graph,

$$c^2 \mathsf{diam}(G)^2 4^{-2a-1} \leq t_{\mathrm{rel}}^{\mathsf{SRW}} \lesssim t_{\mathsf{mix}}^{\mathsf{SRW},(\infty)} \lesssim_{c,a} \mathsf{diam}(G)^2$$

From the assumptions in the theorem and our previous lemma, we have that

$$t_{\mathrm{rel}}^{\mathsf{full},(\mu,p)} \gtrsim_{a,b,c} \frac{1}{\mu p} \mathsf{diam}(G)^2 \asymp_{a,c} \frac{1}{\mu p} t_{\mathrm{rel}}^{\mathsf{SRW}}$$

Now from the proof of proposition 8.1 in [HP20] we also have that for vertex-transitive graphs of degree d and (c,a) moderate growth that one has $t^{\mathsf{SRW}}_{\text{spectral}-\text{profile}} \lesssim_{a,c,d} \mathsf{diam}(G)^2$. Combining this with the comparison principle for mixing time with the spectral profile we get,

$$t_{\text{rel}}^{\mathsf{full},(\mu,p)} \lesssim t_{\text{mix}}^{\mathsf{full},(\mu,p),(\infty)} \lesssim_{a,c,d} \frac{1}{\mu p}\mathsf{diam}(G)^2 + \frac{1}{\mu}|\log(1-p)| \lesssim \frac{1}{\mu p}\mathsf{diam}(G)^2 \asymp \frac{1}{\mu p} t_{\text{rel}}^{\mathsf{SRW}}$$

$\square$

7 Conclusion, future directions and questions

In this essay, we addressed the problem of comparisons between random walk on dynamical percolation and simple random walk (SRW), as posed in [HS20]. We studied the known comparison results for hitting and mixing times, and extended these techniques to prove new cover time bounds for several graphs that are finite analogs of transient graphs.

The key idea was to use the hitting time comparison from [HS20] combined with a chain of inequalities that was true only when the cover time was of the order $t_{\text{hit}} \log n$. We showed that under the condition of minimal hitting time (or uniform local transience), this is indeed the case, and thus the comparison theorem for cover times holds. Next, we studied some important examples of uniformly transient graphs such as, $(\mathbb{Z}_n^d)_{n\geq 0}$ for $d \geq 3$, $(\mathbb{Z}_h \times \mathbb{Z}_n^2)_n \geq 0$ for $h \gtrsim \log n$, graph sequences (G_n) with diameters satisfying $D^2 = O(\log(|V(G_n)|/|V(G_n)|)$ or $t_{\text{rel}} = o(n^{1-\delta})$. For the last two classes, we got more refined comparison principles.

Several important questions remain open. Our proof only works in the case where the Matthews inequality is tight, in particular when $t_{\text{cov}} \asymp \log n \; t_{\text{hit}}$, it is natural to ask for what sequences Matthews is tight. In particular:

Question 1. Let (G_n) be a sequence of graphs with diverging sizes, is there an equivalent condition to $t_{\text{cov}} \asymp \log n \; t_{\text{hit}}$?

In the minimal hitting time case we know that $t_{\text{hit}} \asymp |V(G_n)|$ implies that $t_{\text{cov}} \asymp \log |V(G_n)||V(G_n)|$, we believe that the converse should also hold.

Question 2. Let (G_n) be a sequence of graphs with diverging sizes. If $t_{\text{cov}} \asymp |V(G_n)| \log |V(G_n)|$ then is $t_{\text{hit}} \asymp |V(G_n)|$?

Discussion about future directions: To start, note that the order of the Matthews bound is not important for the cover time comparison to work, specifically suppose we have sequences of graphs such that $t_{\text{cov}}(G_n) \asymp t_{\text{hit}}$ and $t_{\text{cov}}^{\text{full},(\mu,p)} \lesssim t_{\text{hit}}^{\text{full},(\mu,p)}$, (for e.g., for graphs like the cycle) then we have that

$$t_{\text{cov}}^{\text{full},(\mu,p)} \lesssim t_{\text{hit}}^{\text{full},(\mu,p)} \leq \frac{C(\mu)}{p} t_{\text{hit}}^{\text{SRW}} \lesssim \frac{C(\mu)}{p} t_{\text{cov}}^{\text{SRW}}$$

Now, informally, if the Matthews does not hold, can we show something of the above sort? We know that $t_{\text{hit}} \leq t_{\text{cov}} \ll \log n \; t_{\text{hit}}$, then can we show that $t_{\text{cov}} \asymp t_{\text{hit}}$ atleast in the transitive setting?

Another approach to showing the comparison principle is to use another auxiliary quantity, such as the blanket time, relaxation time etc, and to prove tight upper and lower bounds of the cover time with this quantity, i.e $t_{\text{cov}}^{\text{SRW}} \asymp M_n$ (say M_n) is the quantity. Kahn, Kim, Lovasz, Vu [KKLV00], showed that for $M_n = \max\{t_{\text{hit}}, \; \min_{x,y\in A} \mathbb{E}_x(T_y) \log |A|\}$ we have that

$$t_{\text{cov}}^{\text{SRW}} \asymp M_n (\log \log n)^2$$

It is quite possible that this gives us the comparison upto $(\log \log n)^2$ factor.

There are quantities like the blanket time, which are of the same order as the cover time (see [DLP11]). However, developing a comparison principle seems to be just as hard of a problem. An interesting exploration is the result of [DLP11] relating the cover time of a graph to the maximum of the square of the Gaussian free field, there could indeed be a more direct proof coming from this analysis.

For sequences (G_n), we are only left with the large diameter case, where things spectral quantities are simpler to analyse (in the sense that the mixing time and the relaxation time are both of the order diam^2), so one could try the above approaches just for the large diameter case where, perhaps, an auxiliary quantity is easier to find.

Finally, we note that further questions above the hierarchy of graph quantities can be asked. We refer the reader to [HS20] for interesting discussions. For instance, the authors in [HS20] believe that our bound for high-dimensional graphs of $t^{\mathsf{SRW}}_{\mathrm{spectral-profile}} \lesssim t^{\mathsf{SRW}}_{\mathrm{hit}}$ should hold for all transitive graphs. Overall, the theory of developing comparison principles between the dynamical walk and the simple random walk is a rich area of study with several interesting questions.

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