Conformal Invariance in 2D percolation

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What is Percolation?

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- Every edge of a graph G is retained with probability p and deleted with probability $1 - p$ independent of all the other edges. This random process is called **Bernoulli bond percolation.**
- This model was first introduced by Broadbent and Hammersley in 1957.

[629]

PERCOLATION PROCESSES

I. CRYSTALS AND MAZES

By S. R. BROADBENT and J. M. HAMMERSLEY

Received 15 August 1956

ABSTRACT. The paper studies, in a general way, how the random properties of a 'medium' influence the percolation of a 'fluid' through it. The treatment differs from conventional diffusion theory, in which it is the random properties of the fluid that matter. Fluid and medium bear general interpretations: for example, solute diffusing through solvent, electrons migrating over an atomic lattice, molecules penetrating a porous solid, disease infecting a community, etc.

1. *Introduction.* There are many physical phenomena in which a *fluid* spreads randomly through a *medium*. Here fluid and medium bear general interpretations: we may be concerned with a solute diffusing through a solvent, electrons migrating over an atomic lattice, molecules penetrating a porous solid, or disease infecting a community. Besides the random mechanism, external forces may govern the process, as with water percolating through limestone under gravity. According to the nature of the problem, it may be natural to ascribe the random mechanism either to the fluid or to the medium. Most mathematical analyses are confined to the former alternative, for which we retain the usual name of *diffusion process*: in contrast, there is (as far as we know) little published work on the latter alternative, which we shall call a *percolation process*. The present paper is a preliminary exploration of percolation processes; and, although our conclusions are somewhat scanty, we hope we may encourage others to investigate this terrain, which has both pure mathematical fascinations and many practical applications.

An Illustration

- Retained edges are called **open** and deleted edges **closed**, connected components are called **clusters.**

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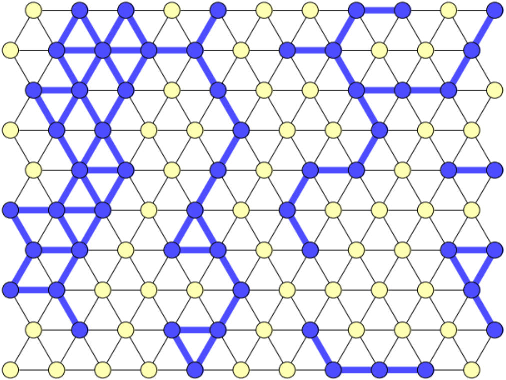
Probability on Trees and Networks - Rusell Lyons and Yuval Peres

Figure: Percolation on a 40×40 square grid graph at levels $p = 0.4, 0.5, 0.6$. Each cluster is given a different color.

Site Percolation

✧ Just like bond percolation, one can define a **site configuration** ω as a random subgraph of $\{0,1\}^V$. Where 0 is for closed vertices and 1 is for open vertices.

For example, consider site percolation on T:



Phase Transition and Critical probabilities

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Phase Transition and Critical probabilities

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- Then we can define the critical probability as $p_c = sup\{p : \theta(p) = 0\}$.

We use the notation p_c^s and p_c^b for site and bond percolation respectively.

- For a d-regular tree one has $p_c^b(\mathbb{T}_d) = \frac{1}{d-1}$ and for the integer square lattice $p_c^b(\mathbb{Z}^2) = \frac{1}{2}$ (**Kesten's Theorem**).
- From the above one gets that $0 < p_c^b(\mathbb{Z}^d) \leq \frac{1}{2}$. However the exact value of $p_c^b(\mathbb{Z}^d)$ for $d \geq 3$ is not known!

We also have a relation between the two critical probabilities:

$$\frac{1}{\Delta - 1} \leq p_c^b \leq p_c^s \leq 1 - (1 - p_c^b)^{\Delta - 1}$$

Conformal Invariance

- We want to explore the limiting structure of the percolation configuration in the scaling limit of increasingly fine lattice approximations to a fixed "nice" domain D (a nonempty proper open subset of C which is connected and simply connected).
- In a famous conjecture of Aizenmann and Langlands, Pouliot, and Saint-Aubin they claimed that if Λ is a planar lattice with suitable symmetry, and we consider critical percolation on Λ, then as the lattice spacing tends to zero certain limiting probabilities are invariant under conformal maps of the plane $\mathbb{C}$.
- We will be dealing with the most classical case where this conjecture has been resolved: independent site percolation on the triangular lattice T.

Conformal Invariance

- Fix "nice" domains $D, D' \subseteq \Lambda$. Now, we will call D and D' conformally equivalent if there exists a conformal bijection ϕ from D to D'.
- **Conformal Invariance** roughly states that, the limiting(random) behaviour of the model on D' is same (in law) as the image under ϕ of the limiting behaviour on D. This will become more concrete when we state Smirnov's theorem.

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- Now we state the conjecture formally.

4-marked domains and the conjecture

Let D be a domain and let a, b, c, d be four points on the boundary D in the counter-clockwise order. $(D, a, b, c, d) =: D_4$. D_4 is called a 4-marked domain.

For Λ a planar lattice and $\delta > 0$, by $\delta\Lambda$ we mean the lattice obtained by scailing Λ by δ about the origin. For instance, $\delta\mathbb{Z}^2 = \{(\delta a, \delta b) : a, b \in \mathbb{Z}\}$.

Define:
$\mathbb{P}_\delta(D_4, \Lambda, p) := \mathbb{P}_p(A_1$ and A_3 are connected via a open path in $\delta\Lambda)$.

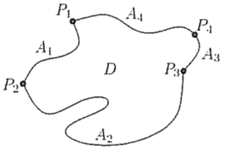


Figure 1 A 4-marked domain $D_4 = (D; P_1, P_2, P_3, P_4)$

Percolation - Bollobas, Riordan

The conjecture

Conjecture (Aizenmann, Langlands et.al)

Let Λ be a "nice" planar lattice. Then the limit $\lim_{\delta \to 0} \mathbb{P}_\delta(D_4, \Lambda, p_c) = \mathbb{P}(D_4)$ exists, lies in $(0, 1)$ and is independent of the lattice Λ. Furthermore, if D, D' are conformally equivalent such that the corresponding boundary points are mapped to each other in the natural order then, $\mathbb{P}(D_4) = \mathbb{P}(D'_4)$.

Let's narrow our focus to the case we are interested in: independent site percolation on the triangular lattice.

Firstly, it is clear that site percolation on $\mathbb{T}$ is equivalent to percolation on the faces of $\mathbb{H}$.

For percolation on the faces of the hexagonal lattice $\mathbb{H}$ (or, equivalently, site percolation on the triangular lattice $\mathbb{T}$), $p_c = \frac{1}{2}$.

Planar duality and the honeycomb lattice

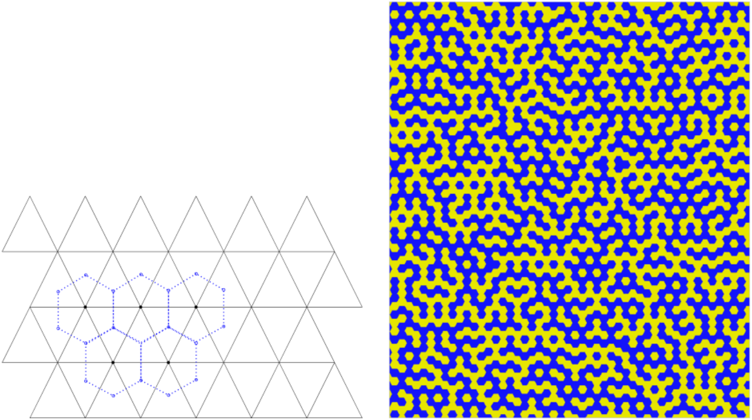


Figure 9.4: Left: The triangular lattice $\mathbb{T}$ and part of its dual lattice, the hexagonal (or honeycomb) lattice $\mathbb{H}$. Right: Percolation on the faces of $\mathbb{H}$ = site percolation on $\mathbb{T}$.

Percolation - Ariel Yadin

Cardy-Smirnov formula

Inspired by the conjecture, Cardy predicted the exact formula for $\mathbb{P}(D_4)$. For any marked domain D_4 we can define its cross-ratio η. Cardy's prediction was that

$$\mathbb{P}(D_4) = p(\eta) = \frac{3\Gamma(2/3)}{\Gamma(1/3)^2}\eta^{\frac{1}{3}}F_{1,2}(1/3, 2/3, 4/3; \eta).$$

This only depends upon the cross-ratio!
An easier way to put it (Carleson's form): For $D = T$ (an equilateral triangle) and a, b, c the vertices of T and $[c, a] = [0, 1]$, then for any $d \in (0, 1)$ $\mathbb{P}(T, a, b, c, d) = d$.

Cardy-Smirnov formula

Smirnov's Theorem

The conformal invariance conjecture holds for the lattice $\mathbb{T}$, $\mathbb{P}(D_4)$ exists and is given by Cardy's formula.

One would expect that due to conjectured **universality**, we can generalize this to all suitable planar lattices. But this is not the case!

Smirnov's proof depends very much on the underlying symmetries of the triangular lattice $\mathbb{T}$.

The basic idea of the proof is to construct "preharmonic" functions that encode the crossing probability and converge in the scaling limit to the conformal invariants of the domain.

Cardy-Smirnov formula

Some generalisations

The general principle of preharmonicity and preholomorphicity has been further developed by Chelkak, Hongler, Kemppainen, and Smirnov in establishing conformal invariance in the scaling limit of the critical Ising and FK models.

Discrete complex analysis is broadly the study of these functions and their properties and is very important tool in probability theory.

Discrete complex analysis appears also in the work of Duminil-Copin and Smirnov determining the connective constant of the hexagonal lattice, which makes substantial progress toward establishing a conformally invariant scaling limit for the self-avoiding walk (SAW).

Schramm-Loewner evolutions

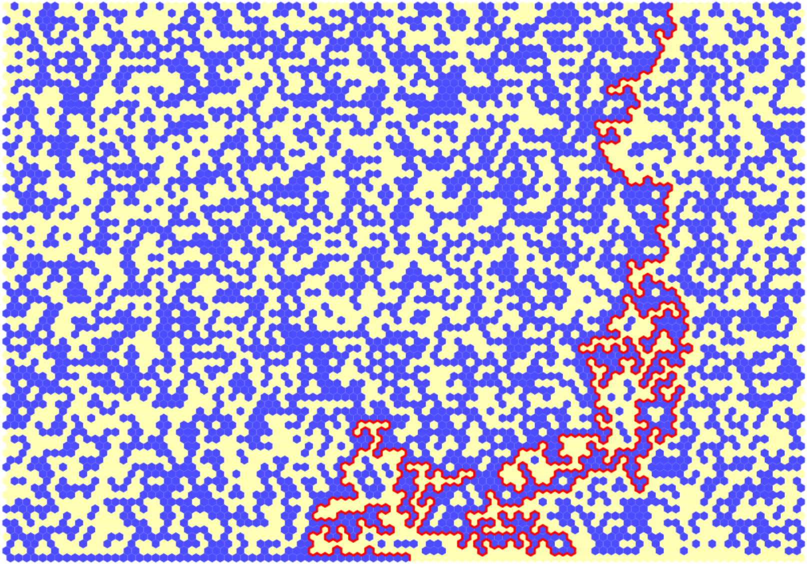
We discussed the notion of a limiting crossing probability, but, what are our "limiting percolation configurations"?

Earlier, no direct construction for the limiting object — that is, a construction not involving limits of discrete systems — was available.

Such a construction was discovered in 1999-2000 when, in the course of studying the scaling limit of the loop-erased random walk (LERW), Schramm gave an explicit mathematical description of a one-parameter family of conformally invariant random curves now called the Schramm-Loewner evolutions (SLE).

These curves are characterized by simple axioms which identify them as essentially the universal candidate for the scaling limits of macroscopic interfaces in planar models

Convergence to SLE_6



Thank You!