Biautomaticity of Coxeter groups

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Abstract

This article provides a comprehensive exposition of the results from Damian Osajda and Piotr Przytycki's work on the biautomaticity of Coxeter groups, as presented in their paper [OP22]. The content is based on a series of lectures delivered by Piotr Przytycki at the Geometry in Groups conference, held at the International Centre for Theoretical Sciences, Bangalore, which the author had the opportunity to attend.

Contents

1 Introduction

Coxeter groups were first introduced in [Cox34] as a generalization of reflection groups. They play a key role in algebra and geometry, arising as symmetry groups of regular polytopes, Weyl groups of simple Lie algebras, and triangle groups related to tessellations of both the Euclidean and hyperbolic planes.

Coxeter groups are also foundational in the theory of buildings—highly symmetric spaces connected to algebraic groups. Due to their symmetry these groups exhibit many important algebraic, geometric, and algorithmic properties. In this article, we focus on the property of biautomaticity. We begin by defining Coxeter groups.

Definition 1. Let S be a finite set of size k and $m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\}$ for $s \neq t$, and $m_{ss} = 1$. The *Coxeter group of rank k* with exponents m_{st} is the group presented by

$$\langle S \mid (st)^{m_{st}} = \mathrm{id} \rangle,$$

where $m_{st} = \infty$ means no relation.

Example: Let $S = \{s, t, p\}$ with $m_{st} = m_{tp} = 4$ and $m_{sp} = 2$. Then the group G is defined by

$$G = \langle s, t, p \mid (st)^4 = \mathrm{id}, (tp)^4 = \mathrm{id}, (sp)^2 = \mathrm{id} \rangle.$$

Relations, like $(st)^4 = \mathrm{id}$, signify that performing the operation st four times returns to the identity element, reflecting the symmetrical relationship between s and t.

2 Regular languages and biautomaticity

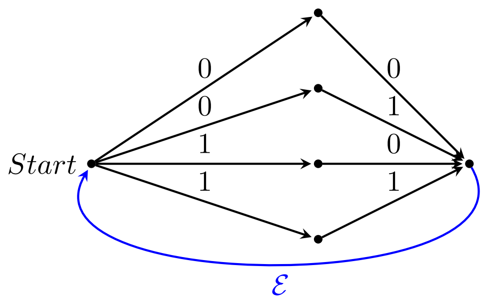
Let A be a finite set of letters. A *word* over A is a finite sequence of letters (possibly empty, denoted by $\mathcal{E}$). Words allow us to form a language.

Definition 2. A *language* over A is a subset of the set A^*, which is the set of all words over A.

Definition 3. A *finite state automaton* over A is a finite directed graph Γ with vertex set Γ^0, known as the set of states. The edges are labeled by $A \cup \{\mathcal{E}\}$.

Following [OP22], we use FSA to denote a finite state automaton. We distinguish a *start state* and *accept states* $Y \subset \Gamma^0$. The language of such an automaton is the set of words labeling a directed path from the start state to any state in Y.

For instance consider a FSA where the only accept state is *Start*, this gives a language over this FSA.
Example:



Y (accept states) = Start, A = {0,1}

Definition 4. A language $L \subseteq A^*$ is *regular* if it is the language of a finite-state automaton over A.

For instance in the above example $L = A^* = \{\text{all binary words}\}$. Let w be a word over A. For $t \geq 0$, we denote by $w(t)$ the prefix of w of length $\min\{t, \text{length}(w)\}$. Eg: $100(2) = 10$, $100(3) = 100$, $100(4) = 100$.

If $A = S \cup S^{-1}$ for a generating set S of a group G, and w is a word over A, then $\overline{w}$ denotes the group element represented by w.

Definition 5. Let X be the Cayley graph of a group G with finite generating set S. Let w, w' be words over $S \cup S^{-1}$. We say that w, w' k-fellow travel if for all $t \geq 0$, we have that $\overline{w(t)}$ and $\overline{w'(t)}$ are at distance $\leq k$ (In particular, $\overline{w}, \overline{w'}$ are at distance $\leq k$).

$$\begin{tikzcd}
 & \overline{w'(1)} \arrow[r] \arrow[d, <->, "\leq k"] & \overline{w'(2)} \arrow[d, <->, "\leq k"] \arrow[r, dashed, -] & {} \\
\mathrm{id} \arrow[ur, bend left] \arrow[r, bend right] & \overline{w(1)} \arrow[r] & \overline{w(2)} \arrow[r] & \overline{w(3)} \arrow[r, dashed, -] & {}
\end{tikzcd}$$

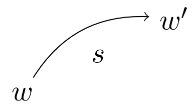
$w, \overline{w}$ k-fellow travel

We are now in a position to define biautomaticity.

Definition 6. A group G is *biautomatic* if there is a regular language L over $S \cup S^{-1}$ such that the map from L to G, sending w to $\overline{w}$, is surjective, and the following condition holds (we call them the fellow traveller properites, FTP(1) and FTP(2)):

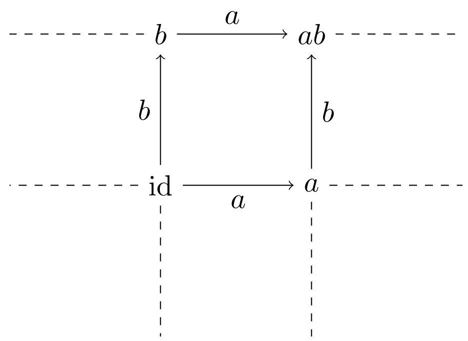
(1) For any $w, w' \in L$, with $\overline{w} = \overline{w'}$ or $\overline{w} = \overline{w'}s$ for some $s \in S$, we have that w and w' k-fellow travel.

(2) For any $w, w' \in L$ with $\overline{w} = s\overline{w'}$ for some $s \in S$, we have that w and w' k-fellow travel.

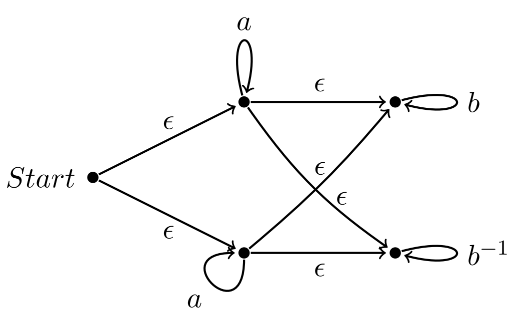


If (1) holds but not (2), then G is called *automatic*. If both hold, then G is called *biautomatic*.

Example:
As a starting example, lets try to prove that $\mathbb{Z}^2$ is biautomatic. Let $G = \mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$. $S = \{a, b\}$, and $A = \{a, b, a^{-1}, b^{-1}\}$. We will now show that $\mathbb{Z}^2$ is biautomatic.



Proof. To show biautomaticity we must exhibit a language (of a finite-state automaton) over $S \cup S^{-1}$ with properties as in definition 6. Let $L = \{a^k b^m : k, m \in \mathbb{Z}\}$. If we let $Y = \{\text{all states}\}$ then it is easy to see that L is regular with respect to the following automaton:



We claim that this regular language makes $\mathbb{Z}^2$ biautomatic. Firstly, the map $w \to \overline{w}$ is surjective due to $\mathbb{Z}^2$ being commutative (every element of $\mathbb{Z}^2$ be written as $a^k b^m$).

FTP: If $\overline{w} = \overline{w'}$ then we are done. Thus let $\overline{w} = \overline{w'}a$ in this case if we have $\overline{w} = a^k b^m$ then $\overline{w'} = a^{k-1} b^m$ from which it follows that w, w' 2-fellow travel. Similarly, if $\overline{w} = b\overline{w'}$ then we also have that w, w' 2-fellow travel.

This proves $\mathbb{Z}^2$ is biautomatic. Infact every finitely generated abelian group is biautomatic, this can be seen after proving that biautomaticity is preserved under direct products, see [ECH+92]. $\square$

One noteworthy property of biautomatic groups is that the word problem and conjugacy problem are solvable. This can be intuitively thought of as follows: proving biautomaticty roughly tells us how to move in the group, using the generators, this makes the above problems more tractable. See [ECH+92] for details. There are several examples of automatic, biautomatic groups. For instance the following groups are biautomatic:

- Euclidean groups
- Hyperbolic groups
- CAT(0) cubical groups
- Systolic groups

2.1 Properties of automatic groups—the quadratic isoperimetric inequality

We start with the following simple lemma which we will use in the future.

Lemma 1. *(bounded length difference) Let G be an automatic group with language L. Then there is a constant $N \in \mathbb{N}$ such that if $w \in L$ and $g \in G$ is at a distance 1 from $\overline{w}$ in the cayley graph, then g has a representative of length $\leq |w| + N$.*

Proof. Let Γ be the FSA, X be the cayley graph of G and k be the FTP constant. Further, let N be greater than the (number of states in Γ) $\times$ (number of vertices in B_k (the ball of radius k in X)). Let w and g be as in the statement of the theorem, and let w' be the shortest representative of g in L. If $|w'| > |w| + N$, then by applying FTP we get that all prefixes $w^{'}(t)$ for $t \in \{|w|+1, \cdots, |w|+N+1\}$ lie in $B_k(\overline{w})$, but by the choices of N we have that two prefixes must be the same, one can then remove this loop to get a smaller representative. Contradicting the minimality of w'. $\square$

An immediate corllary of the above is:

Corollary 2. *Any element of G represented by a word w over $S \cup S^{-1}$ can be represented by a better word in L of length $\leq N|w| + n_0$, where n_0 is the length of a representative of the identity in L.*

Biautomatic groups satisfy a quadratic isoperimetric inequality which helps us solve the word problem in polynomial (quadratic) time, we state it as a theorem.

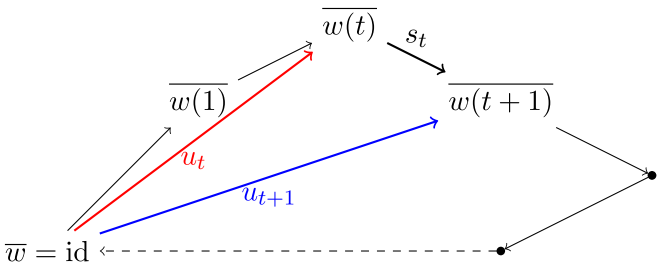
Theorem 3. *Every biautomatic group is finitely presented ($G = \langle S \mid R \rangle$, with R finite) and satisfies a quadratic isoperimetric inequality, i.e., there exists a constant C such that for any word w over $S \cup S^{-1}$ with $\overline{w} = e$, we have*

$$w = \prod_{i=1}^{n} g_i v_i g_i^{-1},$$

where $F(S)$ is the free group on S and $v_i \in R$, $g_i \in F(S)$, $n \leq C|w|^2$.

In particular, knowing some word e representing the identity element, we can solve the word problem in quadratic time, by finding a representative in L for the desired word, and feeding it and e to the equality recognizer.

We now turn to the proof of the theorem. Consider the following figure where u_t, u_{t+1} are paths.



Proof. By the corollary, the prefixes $\overline{w(t)}$ have representatives $u_t \in L$ of length $\leq N|w| + n_0$. Since $\overline{u_t}$ and $\overline{u_{t+1}}$ are at distance 1 by FTP (1), the paths labeled by u_t and u_{t+1} are at distance $\leq k$.
So, the loop labeled by $u_t s_t u_{t+1}^{-1}$ can be decomposed into $\leq N|w| + n_0$ loops of length $\leq 2k+2$. Thus, the loop labeled by w can be decomposed into $\leq N|w|^2 + n_0$ loops of length $\leq 2k+2$. So, we can take

$$R = \{\text{labels of all loops of length} \leq 2k+2.\}$$

This completes the proof. □

3 Walls and the voracious projection

We now define the *voracious* language [OP22] for a Coxeter group $G = \langle S \mid R \rangle$ with cayley graph X. This language will then be used to show that Coxeter groups are biautomatic.

Definition 7. Let $g \in G$ be a conjugate of an element of S. The wall W_g is the fixed point set of g in X. We call g the *reflection* in W_g.

Lemma 4 ([Ron09])**.** *Each wall separates X into two connected components called half-spaces. Further, a edge path is geodesic iff it intersects each wall at most once.*

Proof. See [Ron09] for the complete proof. □

Definition 8. An edge-path is a geodesic if it intersects each wall at most once.

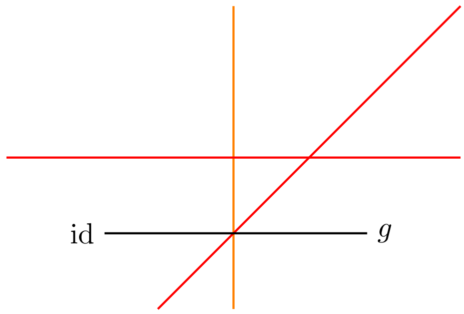
Many partial results before [OP22] were present. The automaticity of Coxeter was shown earlier by Brin-Haulett, Davis, Shapiro [DS91], further biautomaticity for 2-dimensional coxeter groups was shown before [MOP22]. In the light of this we state the main theorem:

Theorem 5 ([OP22])**.** *Coxeter groups are biautomatic.*

These theorems imply that Coxeter groups have significant structural properties that make them amenable to algorithmic solutions, which is crucial for the study of their geometry and combinatorial properties.

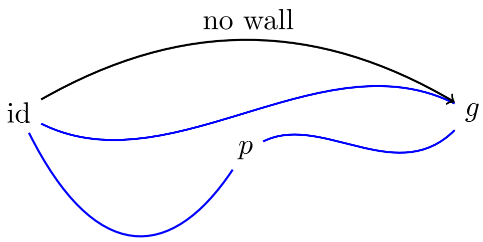
Definition 9. For $g \in G$, let $W(g)$ be the set of walls W separating g from the identity id, such that there is no wall separating g from W.

The figure illustrates the walls W separating the group element g from the identity. The diagram depicts various walls intersecting and the positioning of g in relation to these walls.



We consider the following partial order $(\leq)$ on G where $p \leq g$ if p lies on a geodesic from id to g in X. Equivalently there is no wall separating id, p and g

Equivalently, no wall separates point p from id to g in X:



Definition 10. For $g \in G$, let $\mathcal{P}(G)$ be the set of elements of G such that $p \leq g$ and such that there is no wall in $W(g)$ separating p from id.

Theorem 6. *For any Coxeter group G and $g \in G$, the set $\mathcal{P}(G)$ contains a largest element with respect to $\leq$, which we call $p(g)$ the Voracious Projection.*

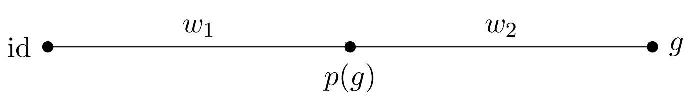
Theorem 7. *There is a constant $C = C(G)$ such that for any $g \in G$ we have:*

$$|g, p(g)|_X \leq C,$$

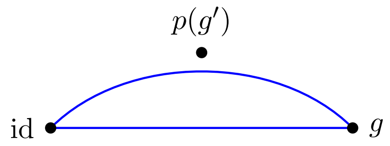
where $|\cdot|_X$ is the distance in the Cayley graph of G.

3.1 Defining the language

We define inductively with respect to $|g, id|_X$ all the words L_g of L representing g. For $g = id$, we let $L_{id} = \{\epsilon\}$, where ϵ denotes the empty word. For $g \neq$ id, let L_g be the set of concatenations $w_1 w_2$ where $w_1 \in L_{p(g)}$ and w_2 is the shortest word representing $p(g)^{-1}g$.



Lemma 8. *Let $g, g' \in G$ with $g' \leq g$, then $p(g') \leq p(g)$.*

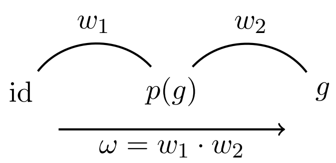


Proof. It suffices to prove that if $p(g') \in P(G)$ *then* $p(g') \leq g' \leq g$. If there is a wall W separating $p(g')$ from g, then $W \notin W(g')$ and thus there is a wall separating W from g' and hence from g, consequently $W \notin W(g)$. □

4 Fellow traveller property

We now show the Fellow Traveller Property:

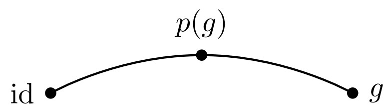
Suppose first that $\omega = \omega'$. Let w_1, w_2 such that $p(g) \leq g = \omega$ where $w_1 \in L_{p(g)}$ and w_2 satisfies $|w_2| = C$.



Now suppose that $\omega = \omega' s$, and denote $g = \omega$, $g' = \omega'$ and assume $g' < g$. By lemma, $p(g') \leq p(g)$ and consequently $p(p(g')) \leq p(g')$. We also have:
Corollary: Cayley graphs of these groups are bipartite.

Theorems

The following two theorems will now be use to regularity later. Theorem 1: $P(g)$ has a largest element $P(g)$.
Theorem 2: $|g, p(g)|_X \leq C$



The following important parallel wall theorem of Brink-Holt is also crucial later on:

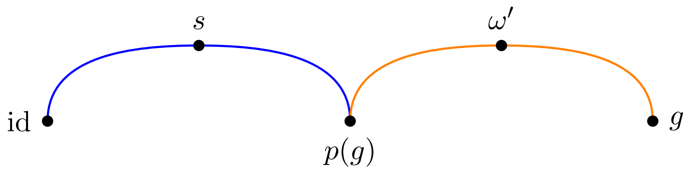
Theorem 9 (Parallel Wall Theorem of Brink-Holt)**.** *Let G be a Coxeter group. There is $D = D(G)$ such that for each wall W and $g \in G$ at a distance greater than D from W in X, there is a wall in X separating g from W.*

5 Regularity

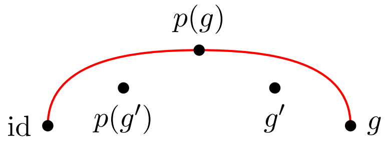
Definition 11. Let E be the set of walls not separated from id by any other wall.

Corollary: E is finite.

Lemma: If $T(PC)$ where $\omega = s\omega'$ and $w = w_1 w_2$ with $w_1 \in L_{p(g)}$ and w_2 ensuring $|w_2| \leq C$, then w_1, w_2 are such that w_1 is a shortest path from id to $p(g)$ and w_2 from $p(g)$ to g.



Thus, Theorem 1 and Theorem 2 combine to yield: If the projections are distinct ($p(g) \neq p(g')$), this implies $W_s \in W(g)$.

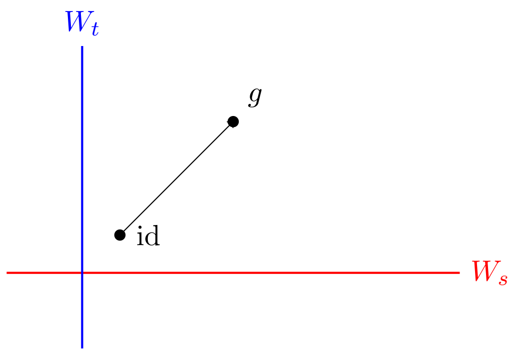


Theorem *(Bipodality by Brink-Holtz)* Let g be separated from the identity by walls W_s, W_t where $s \neq t$ in S, then the structure of the Cayley graph ensures bipodality.

Wall Separation and Projections

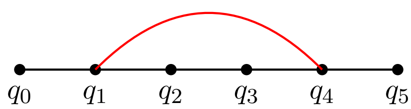
If g is separated from W_s (or W_t) by another wall, then g is separated from any wall in $\langle s, t \rangle W_s, U(s,t) W_t$ by another wall (except possibly for W_t).

Theorem 10 (P-Yau, Dyer-Fristoe-Housley-Neft)**.** *: Let X_t be a component of $X \pm t$. Then α has a smallest element.*



Proof of Theorem

It suffices to show that for each $q_0, q_n \in Q$, there is a q such that $q_i \geq q \leq q_n$.
Consider the geodesic $H = q_0, \ldots, q_n$. We will modify H so that there is no $q_{n-1} < q_i \geq q_{i+1}$.



We will consider the set $R = \langle \gamma_i, \gamma_i' \rangle q_i$, where γ_i are paths in H modified to respect the order constraints.

This concludes the presentation of Theorems for Lecture 3 on Wall Geometry and Group Theory.

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