Percolation on Hyperbolic Graphs

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Percolation on Hyperbolic Graphs*

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Abstract

In this report, we study percolation on transitive graphs. We begin by defining two phase transitions: one for the existence and the other for the uniqueness of infinite connected components in our graph after percolation. We present elementary results on these transitions on trees and establish their relation to the branching number. Next, we demonstrate Burton and Keane's famous argument, which shows the degeneracy of the non-uniqueness phase for amenable graphs. We then explore the concept of unimodularity of graphs, which leads to the elegant mass transport principle for these graphs — a crucial tool for our future study. Towards the end of our report, following Hutchcroft's 2019 paper, we present the proof of Benjamini and Schramm's famous conjecture, originally proposed in 1996, under the additional assumption of Gromov hyperbolicity, using newly developed techniques from functional analysis and hyperbolic geometry.

1 Introduction

In this section, we will define the process of percolation on graphs and the critical probabilities that we shall be dealing with in our report. We shall then state some results for percolation on trees concerning the connections of these probabilities to the branching number of a tree.

Definition 1. A **Percolation Measure Space** on a graph $G = (V, E)$ is a measure space $(\Omega, \mathfrak{F}, \mathbb{P})$ on the set of **configurations** $\Omega = \{0, 1\}^E$.
Usually, we consider the case where $\mathbb{P}$ is a product measure. That is for each edge e define a measure space $(\Omega_e, 2^{\{0,1\}}, \mathbb{P}_e)$ where $\Omega_e = \{0, 1\}$ such that $\mathbb{P}_e(1) = p_e$ and $\mathbb{P}_e(0) = 1 - p_e$, then $(\Omega, \bigotimes_{e\in E} 2^{\{0,1\}}, \bigotimes_{e\in E} \mathbb{P}_e)$ is a percolation measure space, such a measure space defines a process called **Bernoulli bond percolation** on the graph $G = (V, E)$. The edges marked with 0 are called **closed** whereas those marked with 1 are called **open.** One can think of $\omega = (V, \{e \in E : \omega(e) = 1\}) \in \Omega$ as a random subgraph of G, then we call the connected components of a configuration $\omega \in \Omega$ as **clusters** of ω. We denote K_x as the cluster of x in ω.

Now, we shall define two phase transitions for graphs: a phase transition for existence and then a phase transition for the uniqueness of infinite clusters.

To define such transitions, first define the function $\psi(p) := \mathbb{P}_p[\exists \text{ an infinite cluster}]$. Now the first critical parameter is $p_c := \sup\{p : \psi(p) = 0\}$ and the second critical parameter is $p_u := \inf\{p : \mathbb{P}_p[\exists \text{ an unique infinite cluster}] = 1\}$. By a standard coupling argument it is clear that $0 \le p_c \le p_u \le 1$.

The two standard questions in the field are :

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- **What are $p_c(G), p_u(G)$?**
- **When is $p_c(G) = p_u(G)$?**

This document mainly addresses the second question. Inequalities concerning p_c and p_u can be found in [LP17b, Chapter 7].

1.1 Organisation and Overview

This report has 3 parts; first, we shall be demonstrating many results concerning percolation on trees and answering the second question about trees. Next, we talk about the number of infinite clusters in the **amenable** setting and via the Burton-Keane argument settle the second question for these graphs as well. Finally, we move to the finale of the report, where we talk about percolation on hyperbolic spaces and discuss various results regarding critical probabilities on these graphs.

Notation and Conventions: Before starting we setup some conventions which we follow throughout the report.

- We only deal with connected, locally finite, transitive graphs.
- $\mathbb{P}_p$ will denote the standard Bernoulli measure with $p_e = p$ for all edges e.
- K_v denotes the cluster of the vertex v, we refer to $|K_v|$ as the size or the volume of the cluster of v.
- Expectations written with respect to the $\mathbb{P}_p$ measure will be denoted as $\mathbb{E}_p$
- We also refer to $\mathbb{E}_p(|K_v|)$ as the susceptibility (denoted, χ_p) of the graph under $\mathbb{P}_p$ percolation. This will not depend on the vertices due to transitivity.

2 Percolation on Trees

Both of these questions have been solved in the case of trees. Here, we state these results and move on. A very good source for percolation on trees is the famous book by Lyons and Peres [LP17b, Chapter 2].

We start by defining the branching number of a tree, one should think of this as a quantity telling us the average number of branches coming out of a vertex in a tree.

We define the branching number of the tree as follows. Start with a tree T and fix a vertex o which we refer to as the root of the tree. Now since trees grow exponentially fast, we define the conductances of an edge to decrease exponentially with the distance from o, say $c(e) := \lambda^{-|e|}$, where $|e|$ denotes the distance of the edge e from the root o.

If λ is very small then due to large conductances there is a non-zero flow on the tree satisfying $0 \leq \theta(e) \leq \lambda^{-|e|}$. While increasing the value of λ we observe a critical value λ_c above which "water" does not flow, this is precisely the branching number. Specifically,

$$br(T) := \sup\{\lambda : \exists \text{ a non-zero flow } \theta \text{ on } T \text{ such that } 0 \leq \theta(e) \leq \lambda^{-|e|} \ \forall\ e \in T\}$$

Now we state our first result,

Theorem 1. *Let T be a tree then, $p_c(T) = \frac{1}{br(T)}$ where $br(T)$ is the branching number of the tree.*

The proof of Thm.1 requires the basic knowledge of random walks on electrical networks, good sources for these concepts are [LP17b, Chapter 2] and [LP17a, Chapter 9].

The only way infinitely many infinite clusters become finite is if somehow they get connected as we increase p. However, due to the exponentially growing nature of trees, this is not possible. This vague heuristic tells us p_u should be 1 for a tree, this is in fact true :

Theorem 2. *Let T be a tree then $p_u(T) = 1$.*

In fact, this same heuristic will also be true in a much more general setting of non-amenability and will tell us the non-degeneracy of the non-uniqueness phase, that is the phase where there are infinitely many infinite clusters. This will be used in the proof of the famous Burton-Keane argument that we will see in the next section.

A special case of Thm.2 will be proved in Sec 3.2. For a general proof see [LP17b, Exc.7.37].

3 Number of Infinite Clusters

The number of infinite clusters is a random variable denoted N_∞ taking values in $\{0, 1, 2, ..\} \cup \{\infty\}$. In principle this can take any of these values with different probabilities, however in the transitive setting the values that it can take are pretty restricted. We will show that for transitive graphs N_∞ is constant a.s and $\in \{0, 1, \infty\}$. Then we demonstrate the famous argument by Burton and Keane which shows that under the additional assumption of **amenability** we actually have that $N_\infty \in \{0, 1\}$.

3.1 Properties of Bernoulli Measure

We start by stating some very general properties that a percolation measure might have on graphs. Next, we state the fact that the measure $\mathbb{P}_p$ indeed has these properties which makes proving the theorems given below easier in our case.

- Tolerance: Let A be an event and let $\Pi_e A = \{\Pi_e\omega : \omega \in A\}$ where $\Pi_e\omega = \omega \cup \{e\}$. Then one has **insertion tolerance** i.e. for all events such that $\mathbb{P}_p(A) > 0$ we have $\mathbb{P}_p(\Pi_e A) > 0$. In fact one has a stronger inequality $\mathbb{P}_p(\Pi_e A) \geq p\,\mathbb{P}_p(A)$. Similarly one has **deletion tolerance** i.e if $\mathbb{P}(A) > 0$ then $\mathbb{P}(\Pi_{\neg e} A) > 0$ where $\Pi_{\neg e} = \{\Pi_{\neg e}\omega : \omega \in A\}$ and $\Pi_{\neg e}\omega = \omega - \{e\}$.
- Invariance and Ergodicity : Suppose $\Gamma \subseteq \mathrm{AUT(G)}$ is a group of automorphisms of a graph G. A measure $\mathbb{P}$ is called $\Gamma-$invariant if $\mathbb{P}(\gamma A) = \mathbb{P}(A)$ for all $\gamma \in \Gamma$ and all events A. Where $\gamma A = \{\omega : \gamma^{-1}\omega \in A\}$. An event is called **invariant** if $\gamma A = A$, it is easy to see that the set of invariant events I_Γ is a σ field. Now a measure $\mathbb{P}$ is called Γ-ergodic if, for each $A \in I_\Gamma$ $\mathbb{P}(A) = 0$ or 1.

Theorem 3. *Say Γ acts transitively on a connected locally finite graph G. Then the measure $\mathbb{P}_p$ is both Γ ergodic and invariant. See [DC18, Lemma 2.8].*

3.2 Newman-Schulman Result

Now we prove the theorem by Newman and Schulman(1990) using both properties of insertion tolerance and ergodicity.

Theorem 4. *(Newman-Schulman) For a graph G the number of infinite clusters is constant $\mathbb{P}_p$ - a.s. and equal to 0,1, or ∞.*

Proof: The proof is in two parts; first we show the constancy of N_∞ then we show $N_\infty \in \{0, 1, \infty\}$.

- Constancy of N_∞ : For any $k \in \{0, 1, 2, .\} \cup \infty$ the event $\{N_\infty = k\}$ is an invariant event hence by ergodicity N_∞ is a constant.
- $N_\infty \in \{0, 1, \infty\}$: Assume that $N_\infty = k \in \{2, 3, 4..\}$ then consider the event $A_{x,y} = \{$ x and y lie in different infinite clusters$\}$. We have $\mathbb{P}_p(\bigcup_{x,y\in V} A_{x,y}) = 1 \implies \sum_{x,y\in V} \mathbb{P}_p(A_{x,y}) \geq 1 \implies \exists\ x, y$ such that $\mathbb{P}_p(A_{x,y}) > 0$ now add the path ζ between x and y by insertion tolerance $\mathbb{P}_p(\Pi_\zeta A_{x,y}) > 0$ but for $\omega \in \Pi_\zeta A_{x,y}$ we have $\omega \in \{N_\infty = k - 1\}$ hence $\mathbb{P}(N_\infty = k - 1) > 0$ contradicting constantcy of N_∞.

3.3 The Burton-Keane Argument

Now we demonstrate this elegant result given by Burton and Keane in 1990. First, we state an important combinatorial fact about trees.

Lemma 5. *Let T be a tree such that T has no vertices of degree 1 and suppose $B = \{x \in V(T) : deg(x) \geq 3\}$ then for any finite set K we have,*

$$|\partial_V(K)| \geq 2 + |K \cap B|$$

The above lemma tells that the number of ways to go "out" of K is at least the number of points of B in K.

Proof: Fix a finite set $K \subset V$ we shall prove this by induction on the number of vertices $K \cap T$. Let $L(K \cap T)$ be the set of leaves of $K \cap T$. Say $|L(K \cap T)| = 0$ then since $K \cap T$ is a forest we have that the connected components of $K \cap T$ are just points, in this case it is easy to see that $|\partial_V(K)| \geq 2 + |K \cap B|$. Now suppose that $|L(K \cap T)| = k$ then if x is a leaf and the connected component of x in $K \cap T$ does not contain any points from B we simply remove the connected component of x and then the result follows by the induction hypothesis. Otherwise, select a point ξ in B which is also in the connected component of x such that the path $x \longleftrightarrow \xi$ does not contain any points from B. Now remove this path, then again by the induction hypothesis we are done.

Theorem 6. *For any **amenable,** transitive, locally finite graph G one has that, $N_\infty \in \{0, 1\}$.*

Proof: We start by assuming that we have infinitely many infinite clusters. Now take a point o in the vertex set and call the set of trifurcation points of configuration ω as $\Lambda(\omega)$, one can show using some "surgical arguments" that each point o has a positive probability of being a trifurcation point, that is $c := \mathbb{P}(o \in \Lambda(\omega)) > 0$. Next, we take a finite set K, now the number of trifurcations points of K just by the linearity of expectation scales with the volume of K. However, taking a spanning tree η of ω and using the above lemma we see that the number of trifurcation points in a set K is at most the size of the boundary of K which will then contradict the amenability of the graph. For all the details see [LP17b, Thm.7.6].

4 The Mass Transport Principle

The mass transport principle (**MTP**) was introduced first by Häggström in 1997 [Hag97] and provides some elegant and magical proofs of many results in percolation.
We start by stating the result for Cayley graphs and then generalize it to all "Cayley-graph" like graphs.

4.1 Cayley Graphs

We shall assume that we have a countable group Γ then we have the MTP,

Theorem 7. *(Mass transport for countable groups.) Let Γ be a countable group and $f : \Gamma \times \Gamma \to [0, \infty]$ be a diagonally invariant function then*

$$\sum_{x \in \Gamma} f(o, x) = \sum_{x \in \Gamma} f(x, o). \tag{1}$$

We shall use it in the following form, Let $F(x, y; \omega)$ be a diagonally invariant function for invariant percolation. Then $f(x, y) = \mathbb{E}[F(x, y; \omega)]$ satisfies the above hypothesis.

4.2 Unimodular Graphs

In general, the mass transport principle does not hold for all graphs, see for example [LP17b, Ex.7.1]. However, there is a version of mass transport that holds for all transitive graphs given below.

Theorem 8. *(Mass transport for transitive graphs.) If Γ is a transitive group of automorphisms of a connected locally finite graph $G = (V, E)$, $f : V \times V \to [0, \infty]$ is invariant under the diagonal action of Γ and $o \in V$, then*

$$\sum_{x \in V} f(o, x) = \sum_{x \in V} f(x, o) \frac{|\mathrm{Stab_G}(x) o|}{|\mathrm{Stab_G}(o) x|}.$$

Now we call a group Γ **unimodular** if $|\mathrm{Stab_G(x)o}| = |\mathrm{Stab_G(o)x}|$ for all $x \in \Gamma o$. We say a graph G is unimodular if $\mathrm{AUT}(G)$ is unimodular. It is easy to see that the mass transport principle in the form we saw earlier holds for these graphs. Therefore all the applications of the MTP on Cayley graphs transfer to unimodular graphs as well.

Theorem 9. *Amenable graphs are unimodular.*

Proof : See [LP17b, Prop.8.14].

We give a small example of the power of the mass transport principle.

Theorem 10. *Let $n \geq 2$ and $\mathbb{T}_n$ be the n-regular tree, then we have $p_u(\mathbb{T}_n) = 1$.*

Proof: Firstly consider the group $\mathbb{Z}_2 = < a \mid a^2 = 1 >$, then, we have that the Cayley graph of the group, $A_n = \underbrace{\mathbb{Z}_2 * \mathbb{Z}_2 \cdots * \mathbb{Z}_2}_{n\text{-times}}$ is $\mathbb{T}_n$ for $n \geq 2$. Here $*$ denotes the free product of two groups.

Thus $\mathbb{T}_n$ is unimodular and we can apply the MTP. Let $K(\omega)$ be the number of infinite clusters in ω, now we define our mass transport as follows:

$$F(x, y; \omega) = \begin{cases} 0 & K(\omega) \neq 1 \\ 1 & K(\omega) = 1,\ x \notin K(\omega) \text{ and } y \in K(\omega) \text{ is closest to x in } K(\omega) \end{cases}$$

F is diagonally invariant therefore $f(x, y) = \mathbb{E}(F(x, y; \omega))$ satisfies, $\sum\limits_{x \in V(\mathbb{T}_n)} f(x, o) = \sum\limits_{x \in V(\mathbb{T}_n)} f(o, x)$.

Let $p < 1$ and consider $\mathbb{P}_p$ percolation on $\mathbb{T}_n$. Then we have that the percolation is $\mathrm{AUT}(\mathbb{T}_n)$ invariant. Assume the contrary i.e $\mathbb{P}(K(\omega) = 1) > 0$, Now consider a vertex at the boundary of the unique infinite cluster, it receives unit mass from all its descendants so the expected mass it receives is infinite. But the expected mass it sends out is at most 1, which contradicts the MTP! Hence for all $p < 1$ we have that $K(\omega) \in \{0, \infty\} \implies p_u = 1$.

For more on unimodularity, we refer the reader to [LP17b, Sec.8.2].

5 Percolation on Hyperbolic Spaces

We start by stating a famous conjecture by Benjamini and Schramm posed in 1996. This conjecture was first posed in their celebrated paper titled "Percolation beyond $\mathbb{Z}^d$, many questions, and a few answers." along with many other famous and interesting problems [BS96].

Conjecture 1. *(Benjamini-Schramm Conjecture)[BS96, Conjecture 6]: For a graph G with a positive cheeger constant, that is a non-amenable graph, one has $p_c(G) < p_u(G)$.*

This along with Thm.6 will give us a nice characterisation of non-amenable graphs:

Conjecture 2. *A graph G is non-amenable iff $p_c(G) < p_u(G)$.*

Now we base this section on Tom Hutchcroft's paper on the topic titled"Percolation on Hyperbolic Graphs." Following [Hut19] we present his proof for the Benjamin-Schramm conjecture under the additional assumption of hyperbolicity.

5.1 Introduction of a new parameter $p_{2\to 2}$ - A functional analytic approach

Setup: We have $G = (V, E)$ a connected, locally finite graph and let $\mathbb{R}^V$ be the space of functions on V. For each matrix $M \in [-\infty, \infty]^{V^2}$, define $D(M)$ to be $\{f \in \mathbb{R}^V : \sum_{v\in V} |f(v)||M(u, v)| < \infty \ \forall\ u \in V\}$. We shall be assuming some basic knowledge of functional analysis, good sources for relevant topics are [SS11], [EMT04], [Kre91].
We use the Riesz-Thorin interpolation theorem [SS11, pp. 52-57] with measure spaces $(V(G), 2^V, \#)$ and $(V(G), 2^V, \#)$ where $\#$ is the counting measure on the set of vertices.

Now think of M as a linear operator from $D(M) \to \mathbb{R}^V$ sending $f \to g$ where $g(u) = \sum_{v\in V} f(v)M(u, v)$. Now for p, $p' \in [1, \infty]$ the $p \to p'$ norm of M is,

$$||M||_{p\to p'} = \begin{cases} \infty & \text{if } \mathrm{L}^p(V) \not\subseteq D(M) \\ \sup\left\{\frac{||Mf||_{p'}}{||f||_p} : f \in L^p(V), f \neq 0\right\} & \text{if } \mathrm{L}^p(V) \subseteq D(M) \end{cases}$$

We will work with the **two point function**, $\tau(u, v) = \mathbb{P}_p[u \longleftrightarrow v]$, and define the linear operator $T_p(u, v) := \tau(u, v)$. It can be checked following the proof of duality of L_p spaces and the symmetricity of T that $||T_p||_{1\to 1} = ||T_p||_{\infty\to\infty} = \sup_{v\in V} \sum_{u\in V} \tau(u, v) = \sup_{v\in V} \chi_p(v)$.

We also define the critical parameter associated with the finiteness of $||T_p||_{q\to q}$ i.e. let $p_{q\to q} = \sup\{p : ||T_p||_{q\to q} < \infty\}$. By [DCT16],[AB87] it follows that $||T_p||_{1\to 1} < \infty$ iff $p < p_c(G)$. Again by duality and symmetricity, we have that $||T_p||_{q\to q} = ||T_p||_{\frac{q}{q-1}\to\frac{q}{q-1}}$. Hence it suffices to focus on $q \in [1, 2]$.

By the Riesz-Thorin theorem it follows that,

$$p_c(G) = p_{1\to 1} = p_{\infty\to\infty} \leq p_{q\to q} \leq p_{2\to 2}\ \forall\ q \in [1, 2].$$

In fact, we have,

$$p_{q\to q} \leq p_{2\to 2}\ \forall\ q \in [1, \infty].$$

Now the goal will be to show that $p_c(G) < p_{2\to 2} \leq p_u(G)$ for all transitive, non-amenable, hyperbolic graphs. This transforms our earlier problem into something more tangible, proving this will show that $p_c(G) < p_u(G)$ for all non-amenable, transitive, hyperbolic graphs (T-NAH).

We clearly have that $p_{2\to2} \leq p_u(G)$, since if not, let $p \in (p_u, p_{2\to2})$, for that p we have a unique infinite cluster and thus we have, $\tau_p(u,v) \geq \mathbb{P}(u \longleftrightarrow \infty)\mathbb{P}(v \longleftrightarrow \infty) = C^2$ where $C = \mathbb{P}(O \longleftrightarrow \infty)$ is the same for all points, and is positive. Thus the $L^2 \to L^2$ norm is unbounded giving us a contradiction(here we have established this for general transitive graphs).

Hence, now the main goal will be to show the following :

Theorem 11. *Let G be a connected, locally finite, transitive, nonamenable, Gromov hyperbolic graph. Then $p_c(G) < p_{2\to2}$.*

Before moving forward we state the so-called triangle condition. Define $\nabla_p = \sup_{v\in V} T_p^3(v,v)$. A graph is said to satisfy the triangle condition at p if $\nabla_p < \infty$. Here, we have by Cauchy-Schwarz that $\nabla_p < \infty$ for all $p < p_{2\to2}$. This will be very useful later on.

5.2 Two limit inequalities for susceptibily

We first start by giving a necessary and sufficient condition for $p_c < p_{2\to2}$ for general graphs and then prove the same in our setting.

If $p < p_{2\to2}$ then we know that $||T_p||_{2\to2} < \infty$. Our next proposition gives us an useful lower bound on the value $||T_p||_{2\to2}$ can take, further it tells us that $||T_p||_{2\to2} < \infty$ iff $p < p_{2\to2}$. Which tells us that $p_c < p_{2\to2}$ iff $||T_{p_c}||_{2\to2} < \infty$, this is often called the L^2 boundedness condition, a more detailed discussion about this can be found in [Hut20a].

Proposition 12. *[Hut19, Cor.2.6]: Let G be an infinite, connected, locally finite graph. Then,*

$$||T_p||_{2\to2}\frac{p_{2\to2}-p}{1-p} \geq \frac{1}{||A||_{2\to2}}$$

for every $p \in [0, p_{2\to2})$ and where A is the adjacency matrix of G. Moreover, one also has that $||T_{p_{2\to2}}||_{2\to2} = \infty$.

Proposition 13. *(Critical Behavior and Triangle condition.) If $p_c < p_{2\to2}$ then we have that $\nabla_{p_c} < \infty$ and $\exists\ C > 0$ such that,*

$$||T_p||_{1\to1} = \chi_p \leq C(p_c - p)^{-1}\ \forall\ p \in [0, p_c).$$

Proof: See [AN84] and [Hut20b, Sec.7].

To state an equivalent condition for $0 \leq p < p_c$, we define

$$C(T_p) = 1 - \sup\left\{\frac{\sum_{u,v\in K}\tau_p(u,v)}{\chi_p|K|} : K \subset V \text{ finite}\right\}$$

This can be thought of as the Cheeger constant for the random walk on G with the Markov matrix $\chi_p^{-1}T_p$, in the notation of [LP17b, Sec.6.1], we have $C(T_p) = \phi(G,c,d)$ where $c(x,y) := \tau_p(x,y), d(x) := \sum_{y\in V} c(x,y)$.

Proposition 14. *(Cheeger's Inequality.)[LP17b, Thm.6.7] Let G be a connected, locally finite, transitive graph. Then*

$$\chi_p(1 - C(T_p)) \leq ||T_p||_{2\to2} \leq \chi_p\sqrt{1 - C(T_p)^2}$$

For every $p \in (0, p_c)$.

Now we are ready to state and prove an equivalent condition for $p_c < p_{2\to 2}$.

Theorem 15. *Let G be a connected, locally finite, transitive graph. Then $p_c < p_{2\to 2}$ if and only if*

$$\liminf_{p\uparrow p_c} \frac{p_c - p}{1-p}\chi_p\sqrt{1-C(T_p)^2} < \frac{1}{||A||_{2\to 2}}.$$

Proof: If the given inequality holds and $p_c \not< p_{2\to 2}$ then $p_c = p_{2\to 2}$. Therefore from Prop.12 it follows that $||T_p||_{2\to 2}\frac{p_c-p}{1-p} \geq \frac{1}{||A||_{2\to 2}}$ but then by Cheeger's inequality (second inequality) one immediately gets a contradiction.

Now suppose $p_c < p_{2\to 2}$, then, $\nabla_{p_c} < \infty$. Hence by Prop.13 we know that $\chi_p(p_c - p)$ is "controlled", but by Cheeger's inequality (first inequality) and the fact that χ_p blows up as $p \uparrow p_c$ and $||T_{p_c}|| < \infty$ we get that $C(T_p) \to 1$ as $p \uparrow p_c$. Hence the left side of the inequality is 0 but the right side of the inequality is finite by the local finiteness of the graph G which proves the inequality.

Now this equivalence gives us something concrete to work with. The goal of the rest of the document will be to establish this equivalent condition for hyperbolic graphs, which will show $p_c < p_u$ and $\nabla_{p_c} < \infty$ for these graphs.
We also now assume the additional condition of unimodularity, since the non-unimodular case follows easily from [Hut20b] and the results given above. See [Hut19, Thm.2.9] for the proof.

5.3 Geometry of Hyperbolic Spaces

We start by defining what a hyperbolic graph is:

Definition 2. A graph G is said to be **Gromov hyperbolic** if it satisfies the **rips thin triangle condition**, that is, there exists a constant δ such that for any three vertices u, v, w of G and any three geodesics $[u, v], [v, w]$ and $[w, u]$ between them, every point in the geodesic $[u, v]$ is contained in the union of the δ-neighbourhoods of the geodesics $[v, w]$ and $[w, u]$.
Such a graph is called **δ-Hyperbolic.**

In this section, we state the relevant results from hyperbolic geometry which makes working with these graphs so special, as we will see it is the combination of these geometric properties combined with the probabilistic ones that make the proof possible in the hyperbolic setting.

The results in the section will be stated mainly from the following four references [Woe00, §22,§20B] [Hut19],[KB02],[Bow06]. The last reference in the above list is good for an introduction to hyperbolic graphs, whereas the first two are delightful references for understanding topological boundary theory.

5.3.1 The Hyperbolic d-space

We describe the hyperbolic d-space, as we will see the geometry of the hyperbolic d-space will be essential in deducing key properties about the geometry of hyperbolic graphs.

We shall be working with the **Poincaré half-space model** of the **hyperbolic d-space** $\mathbb{H}^d$ for $d \geq 2$. That is we identify $\mathbb{H}^d$ with $\mathbb{R}^{d-1} \times (0, \infty)$, with the metric explicitly given by,

$$d_{\mathbb{H}}((x_1, y_1), (x_2, y_2)) = 2\log \frac{\sqrt{||x_1 - x_2||_2^2 + (y_1 - y_2)^2} + \sqrt{||x_1 - x_2||_2^2 + (y_1 + y_2)^2}}{2\sqrt{y_1 y_2}}$$

where $x_i \in \mathbb{R}^{d-1}$ and $y_i \in (0, \infty)$

A **Half-space** in $\mathbb{H}^d$ is a set of the form $H(a,b) := \{x \in \mathbb{H}^d : d_{\mathbb{H}}(x,a) \leq d_{\mathbb{H}}(x,b)\}$ where $a \neq b$, the boundary of a half-space is known as a **hyperplane** which is isometric to $\mathbb{H}^{d-1}$. The hyperplanes are represented by Euclidean spheres that are orthogonal to $\mathbb{R}^{d-1}$.

5.3.2 The Bonk-Schramm Embedding

The following theorem is often very useful since it helps us comment about the geometry of arbitrary hyperbolic graphs using the geometry of the hyperbolic d-space. It was shown by Bonk and Schramm in 2000 [BS00].

Before stating the theorem we need the notion of a rough similarity.

Definition 3. (Rough Similarity) Let $f : (X, d_X) \to (Y, d_Y)$ be a map between metric spaces. It is called a rough similarity if $d_Y(f(X), y) \leq k$ for every $y \in Y$ and $\exists\, \lambda \in (0, \infty)$ and $k \in [0, \infty)$ such that

$$|\lambda d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| \leq k$$

for every $x_1, x_2 \in X$.

Theorem 16. *(Bonk-Schramm embedding theorem.) Let G be a bounded degree, connected, Hyperbolic graph. Then there exists $d \geq 1$ such that G is roughly similar to a closed convex subset $X \subseteq \mathbb{H}^d$, such subsets are, of course, always hyperbolic.*

We call a graph G **visible from infinity** if there is a universal constant C such that for every $x \in V(G)$ there is a doubly infinite geodesic ξ_x such that $d(\xi_x, x) \leq C$. It is easy to see that every non-amenable transitive graph is visible from infinity and that the space X in the Bonk-Schramm embedding is also visible from infinity. It is also intuitively clear that for any space which is visible from infinity $C(X) = X$, where $C(X)$ denotes the convex hull of X.

5.3.3 Discrete Half-Spaces and Degeneracy

A half space for a graph $G = (V, E)$ (denoted, $H_G(a, b)$) is called a discrete half-space. Such a half-space is called **proper** if there exist disjoint open sets U_1, U_2 such that $U_1 \cap cl(H_G(a, b)) = \emptyset$ where we have the topology given by [Woe00, §22] and closure is in $V \cup \delta G$ where δG is the Gromov boundary of the graph G.

Intuitively one can think of proper half-spaces as a set of points near the boundary of the convex hull of the graph G. Now we have the following basic fact.

Lemma 17. *([Hut19, Lemma 3.5]) Let G be a bounded degree, Gromov Hyperbolic graph that is visible from infinity then there exists a constant C such that if $d(a, b) \geq C$ then $H(a, b)$ is proper.*

Another important lemma which will be useful in the final proof later is the following :

Lemma 18. *([Hut19, Lemma 3.6]) Let G be a unimodular, non-amenable, hyperbolic graph then*

for every proper discrete half-space H of G there is an automorphism γ such that $H \cap \gamma H = \emptyset$.

A further assumption that we can make on our space X that comes from the Bonk-Schramm embedding is that we can take it to be **non-degenerate.** We define this now:

Definition 4. A space X is called **non-degenerate** if for every $r < \infty$ there is $R < \infty$ such that every $x \in X$ and every hyperplane ∂H in $\mathbb{H}^d$

$$B_X(x, R) \not\subseteq \bigcup_{y \in \partial H} B_{\mathbb{H}^d}(y, r)$$

In Bonk and Schramm's Embedding one can take X to be **non-degenerate** closed convex subset s.t. $X = C(X)$ [Hut19, Sec.3.9].

5.4 The Hyperbolic Magic Lemma

Now we shall turn our attention to what is perhaps one of the most important results of the paper **the Hyperbolic Magic lemma**. As the name suggests the "lemma" is truly magical in its use and will be highly important in the final proof.

Such a lemma was first proved by Benjamini and Schramm in the Euclidean setting, see [BS11, Lemma 2.3].

We now state this lemma and provide some intuition for what it tells us about the geometry of hyperbolic spaces. For a complete proof of the hyperbolic magic lemma, see [Hut19, Sec.4].

Theorem 19. *(Hyperbolic Magic lemma.) Let G be a hyperbolic graph with degrees bounded by a number M and suppose ϕ is the (λ, k) Bonk-Schramm embedding map to some $X \subseteq \mathbb{H}^d$ for $d \geq 1$. Then for every $\epsilon > 0$ one can find a $N(\epsilon) = N_{M,\lambda,k,d}(\epsilon)$ such that for every finite set $A \subseteq V$ there exist a subset A' with the following properties :*

$$|A'| \geq (1 - \epsilon)|A| \ (A' \text{ is not very small compared to } A) \tag{2}$$

For every v in A', there exists either a half-space or a pair of half-spaces such that,

$$d(\phi(v), \bigcup H_i) \geq \epsilon^{-1} \text{ and } |A - \phi^{-1} \bigcup H_i| \leq N(\epsilon). \tag{3}$$

Intuition: Intuitively the lemma tells us the following, take a finite set of vertices A and fix some Bonk-Schramm embedding map (here, we identify v with $\phi(v)$ in $\mathbb{H}^d$). Then from the viewpoint of most of the points of A (condition (2)) most points of A are either in one half-space or two half-spaces far away (part 1 of condition (3)). That is, the number of points that are not mapped to these half-spaces are small in number (part 2 of condition (3)).

5.5 Final Stretch

In this section, we show the final steps of the proof as shown in [Hut19].

The goal is to show $p_c < p_{2\to 2}$ for unimodular, transitive, non-amenable, hyperbolic graphs (we refer to these graphs as UT-NAH graphs). As discussed earlier this would be sufficient to show $p_c < p_u$ for all UT-NAH graphs. To show this we show the following two propositions:

Proposition 20. *Let G be a UT-NAH graph. Then,*

$$\limsup_{p \uparrow p_c} (p_c - p)\chi_p < \infty.$$

Proposition 21. *Let G be a UT-NAH graph. Then,*

$$\limsup_{p \uparrow p_c} (1 - C(T_p)) = 0.$$

In the next two sections, we state these two propositions and then end with the final proof.

5.5.1 Fraction of points near the boundary

To prove Prop.20 we use show the following lemma which tells us that in the UT-NAH setting, if one decomposes the susceptibility into two components, a constant proportion of the susceptibility comes from the event where the cluster lies in some proper half-space i.e cluster lies in half spaces that are near the boundary. This is a variant of a similar result first shown by Benjamini in [Ben16].

Lemma 22. *Let G be a UT-NAH graph. Then there exists positive constants c and r such that for each $0 \leq p < p_c$ there exists $u, v \in V$ with $d(u, v) \leq r$ such that $H_G(v, u)$ is proper and*

$$\mathbb{E}_p[|K_v| \mathbb{1}(K_v \subseteq H_G(u, v))] \geq c\chi_p.$$

The proof of the above lemma can be found in Hutchcroft's paper [Hut19, Lemma 5.3], and is quite easy to follow. We use this lemma to prove Prop.20.

Proof of Prop.20: The goal will be to show the following differential inequality :

$$\left(\frac{d}{dp}\right)_+ \chi_p \geq C^{-1}\chi_p^2 \text{ for all } p \in [\frac{p_c}{2}, p_c). \tag{4}$$

Where, $\left(\frac{d}{dp}\right)_+$ denotes the **lower right Dini derivative** given by

$$\left(\frac{d}{dp}\right)_+ f(p) := \liminf_{\epsilon \downarrow 0} \frac{f(p+\epsilon) - f(p)}{\epsilon}.$$

Proving (4) will then imply the result, since by integrating this from an arbitrary $p \in [\frac{p_c}{2}, p_c)$ to p_c we will get $\chi_p^{-1} \geq C^{-1}(p_c - p)$ which yields Prop.20.

We shall use Russo's formula to help us in showing Prop.20, we start by stating Russo's formula first.
Russo's Formula: If A is an increasing event, then we have

$$\left(\frac{d}{dp}\right)_+ \mathbb{P}_p(A) \geq \frac{1}{1-p} \sum_{e \in E} \mathbb{P}_p(e \text{ is closed and pivotal for } A).$$

First we start by writing $|K_v| = \sum_{u \in V} \mathbb{1}(u \longleftrightarrow v)$. Thus $\chi_p = \sum_{u \in V} \mathbb{P}_p(u \longleftrightarrow v)$, hence we have by Russo's formula that,

$$\left(\frac{d}{dp}\right)_+ \chi_p \geq \frac{1}{1-p} \sum_{e \in E} \mathbb{P}_p(e \text{ is closed and pivotal for } A = \{u \longleftrightarrow v\}).$$

Now any pivotal looks like the following, and thus we have,

$$\left(\frac{d}{dp}\right)_+ \chi_p \geq \frac{1}{1-p} \sum_{e \in E} \sum_{u \in V} \mathbb{P}_p(\{u \longleftrightarrow e^-\} \cup \{e^- \not\longleftrightarrow e^+\} \cup \{e^+ \longleftrightarrow v\}).$$

Now first summing over all edges with $e^- = w$ and then over w, we get,

$$\left(\frac{d}{dp}\right)_+ \chi_p \geq \frac{1}{1-p} \sum_{e^- = w} \sum_{w \in V} \left[\sum_{u \in V} \mathbb{P}_p(\{u \longleftrightarrow e^-\} \cup \{e^- \not\longleftrightarrow e^+\} \cup \{e^+ \longleftrightarrow v\}) \right].$$

Let $F(w,v,\omega) = \sum_{u \in V} \mathbb{1}_p(\{u \longleftrightarrow w\} \cup \{w \not\longleftrightarrow e^+\} \cup \{e^+ \longleftrightarrow v\})$, where e is an edge with $e^- = w$.
Now let $f(w,v) = \mathbb{E}_p(F(w,v,\omega))$ to get,

$$\left(\frac{d}{dp}\right)_+ \chi_p \geq \frac{1}{1-p} \sum_{e^-=w} \sum_{w \in V} f(w,v).$$

Using MTP we get,

$$\left(\frac{d}{dp}\right)_+ \chi_p \geq \frac{1}{1-p} \sum_{e^-=w} \sum_{w \in V} f(v,w). \tag{5}$$

Now let c, r be the constants of the last lemma then by lemma 18 we have that for any u, v such that $H = H_G(v,u)$ is proper and $d(v,u) \leq r$ we have an automorphism $\gamma = \gamma_{u,v}$ such that $\gamma \cap \gamma H = \emptyset$ and $d(v, \gamma v) \leq R$.

Let $0 < p < p_c$ and u, v, c, r as in lemma 22 and the corresponding γ as above. Then we have by independence,

$$\mathbb{E}_p(|K_v||K_{\gamma v}|\mathbb{1}(K_v \subseteq H, K_{\gamma v} \subseteq \gamma H) \geq c\chi_p^2.$$

Now it follows that there is a p-dependent constant c_p such that,

$$\mathbb{E}_p(|K_{e^-}||K_{e^+}|\mathbb{1}(e^- \not\longleftrightarrow e^+)) \geq c_p \mathbb{E}_p(|K_v||K_{\gamma v}|\mathbb{1}(K_v \subseteq H, K_{\gamma v} \subseteq \gamma H)). \tag{6}$$

Notice the left side of the last equation is exactly the right side of eq.(5). Hence we get,

$$\left(\frac{d}{dp}\right)_+ \chi_p \geq c_p c \chi_p^2.$$

Since $p_c \in (0,1)$, we see that the interval $[\frac{p_c}{2}, p_c)$ is away from both 0 and 1. Hence, c_p does not take values that are very small, thus we can find the suitable constant $C > 0$.

The proof of eq.(6) uses a standard finite energy argument, which relies on the important property of tolerance of the Bernoulli measure. Details can be found in [Hut19, pp. 28-30]

5.5.2 Influence of half-spaces far away

Now we come to the proof of Prop.21. Here we will use the following lemma which intuitively tells us that half-planes that are far away contribute negligibly to the susceptibility, which then, using the hyperbolic magic lemma which will help us prove Prop.21. The proof of the following lemma can be found in [Hut19, lemma 5.4].

Lemma 23. *Let $G = (V,E)$ be a UT-NAH graph and let ϕ be a Bonk-Schramm embedding map with parameters (d, λ, k) to a closed convex subset $X \subseteq \mathbb{H}^d$ which we take to be non-degenerate and visible from infinity. Then there exists a constant C such that for every vertex v, every half space $H \subseteq \mathbb{H}^d$ and $p \in [0, p_c)$ we have,*

$$\mathbb{E}_p|K_v \cap \phi^{-1}H| \leq \frac{C\chi_p}{d(\phi(v), H)}.$$

Proof of Prop.21: Let ϕ be our Bonk-Schramm embedding map with parameters (d, λ, k) to a closed, convex subset $X \subseteq \mathbb{H}^d$ which is a (λ, k) rough similarity. Now for each $\epsilon > 0$, let $N(\epsilon)$ be the constant given by the magic lemma and for each $0 < p < p_c$, let $\epsilon_p > 0 := \inf\{\epsilon : N(\epsilon) \leq \sqrt{\chi_p}\}$. So that we have $\epsilon_p \downarrow 0$ as $p \uparrow p_c$. Thus we have that for any finite set K we have a K' such that $|K'| \geq (1 - 2\epsilon_p)|K|$ and for each $u \in K'$ we have at most a pair of half-spaces $H_{i,u}$ in $\mathbb{H}^d$ satisfying the conditions of the magic lemma.

Consider,

$$\sum_{u,v\in K}\tau_p(u,v) \leq \sum_{u\in K-K',v\in V}\tau_p(u,v) + \sum_{u\in K',v\in K-\phi^{-1}(\bigcup H_{i,u})}\tau_p(u,v) + \sum_{u\in K',v\in\phi^{-1}(\bigcup H_{i,u})}\tau_p(u,v).$$

Now one can bound all of these three terms by our discussion above and lemma 23. Then we get,

$$\sum_{u,v\in K}\tau_p(u,v) \leq 2\epsilon_p|K|\chi_p + |K|\chi_p^{1/2} + 4C\epsilon_p|K|\chi_p.$$

Dividing through by χ_p and taking limits easily yields Prop.21.

5.5.3 Final Proof

Now we shall be proving Thm.15 using Prop.20 and Prop.21.

Proof of Thm 15: Since G is non-amenable it has exponential growth and in particular $p_c(G) < 1$. Therefore we have

$$\liminf_{p\uparrow p_c}\frac{p_c-p}{1-p}\chi_p\sqrt{1-C(T_p)^2} = 0.$$

Thus by local finiteness of the graph G, we have the result.

6 Final Remarks

The above proof of $p_c(G) < p_u(G)$ in the hyperbolic setting deeply used very special properties that these graphs had, and with a combination of various probabilistic and geometric tools we were able to give the proof. Although the proof outlined in this report only deals with transitive graphs, making some minor changes, we can easiliy extend this argument to the so-called **quasi-transitive** graphs, i.e., graphs for which the action of $\mathrm{AUT}(G)$ on G has only finitely many orbits. It should also be noted that several bounds given in the report can also be made stronger and by choosing various other constants in many of the results one can arrive at better results.

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