

Thm (Kesten, 1980) $p_c(2) = 1/2$.

Lemma (Square root trick)

Let A_1, A_2, \dots, A_n be increasing events with
 $\mathbb{P}_p(A_1) = \mathbb{P}_p(A_2) = \dots = \mathbb{P}_p(A_n)$ then

$$\mathbb{P}_p(A_1) \geq 1 - \left(1 - \mathbb{P}_p\left(\bigcup_{i=1}^n A_i^c\right)\right)^{\frac{1}{n}}$$

Pf: $1 - \mathbb{P}_p\left(\bigcup_{i=1}^n A_i^c\right) = \mathbb{P}_p\left(\bigcap_{i=1}^n A_i^c\right)$
 $\geq \prod_{i=1}^n \mathbb{P}_p(A_i^c)$
 $= (1 - \mathbb{P}_p(A_1))^n$

FKG holds for dec events also.

$$\Rightarrow \mathbb{P}_p(A_1) \geq 1 - \left(1 - \mathbb{P}_p\left(\bigcup_{i=1}^n A_i^c\right)\right)^{\frac{1}{n}}$$

□

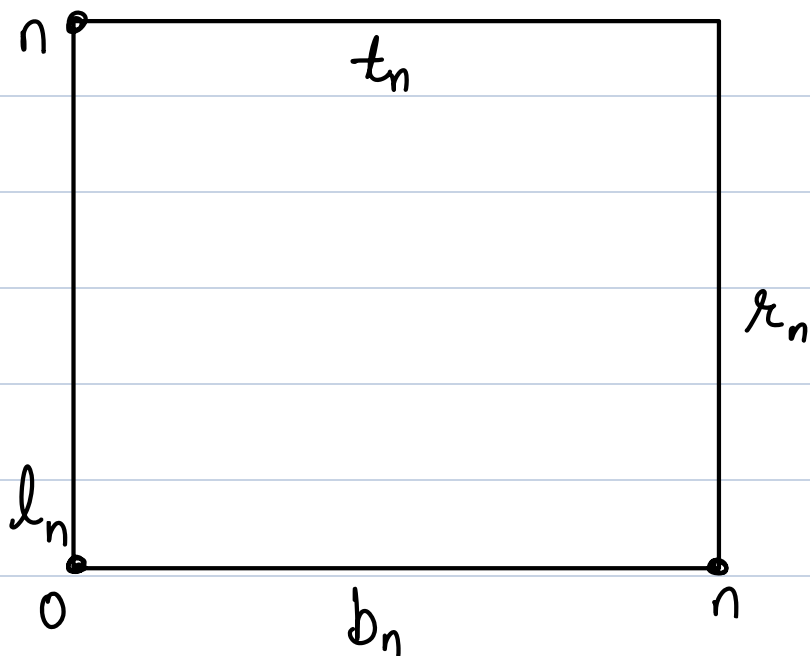
Proof of the theorem

Step ① For $d=2$, $\theta(1/2) = 0 \Rightarrow p_c \geq 1/2$

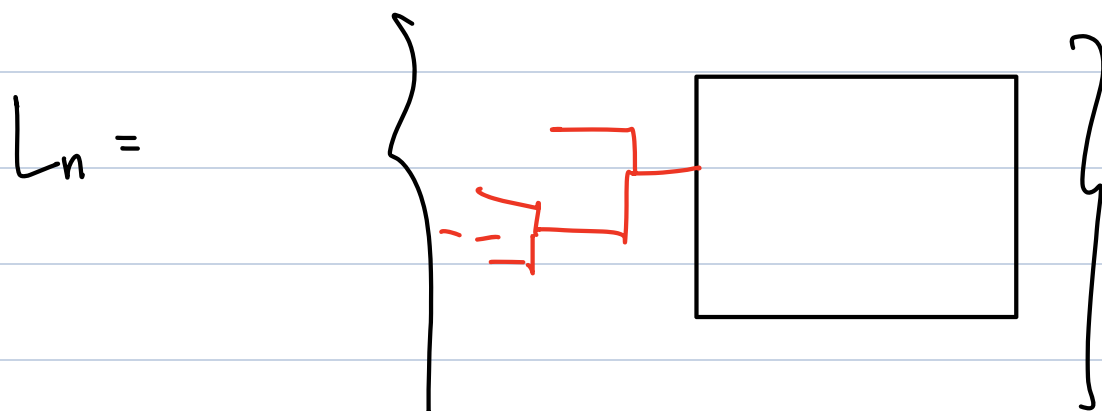
$[0, n]^2$

Let

$$L_n = \left\{ \exists \text{ an infinite open path from a vertex on } l_n \text{ which } \underline{\text{does not use}} \text{ any edge from } [0, n]^2 \right\}$$



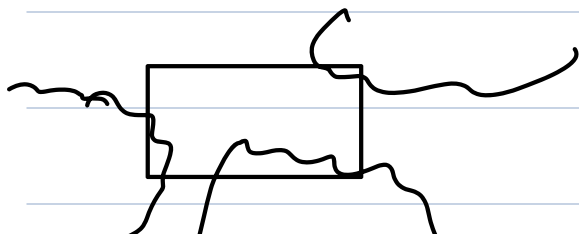
||^y define T_n, R_n, B_n



Suppose $\theta(\frac{1}{2}) > 0$, then

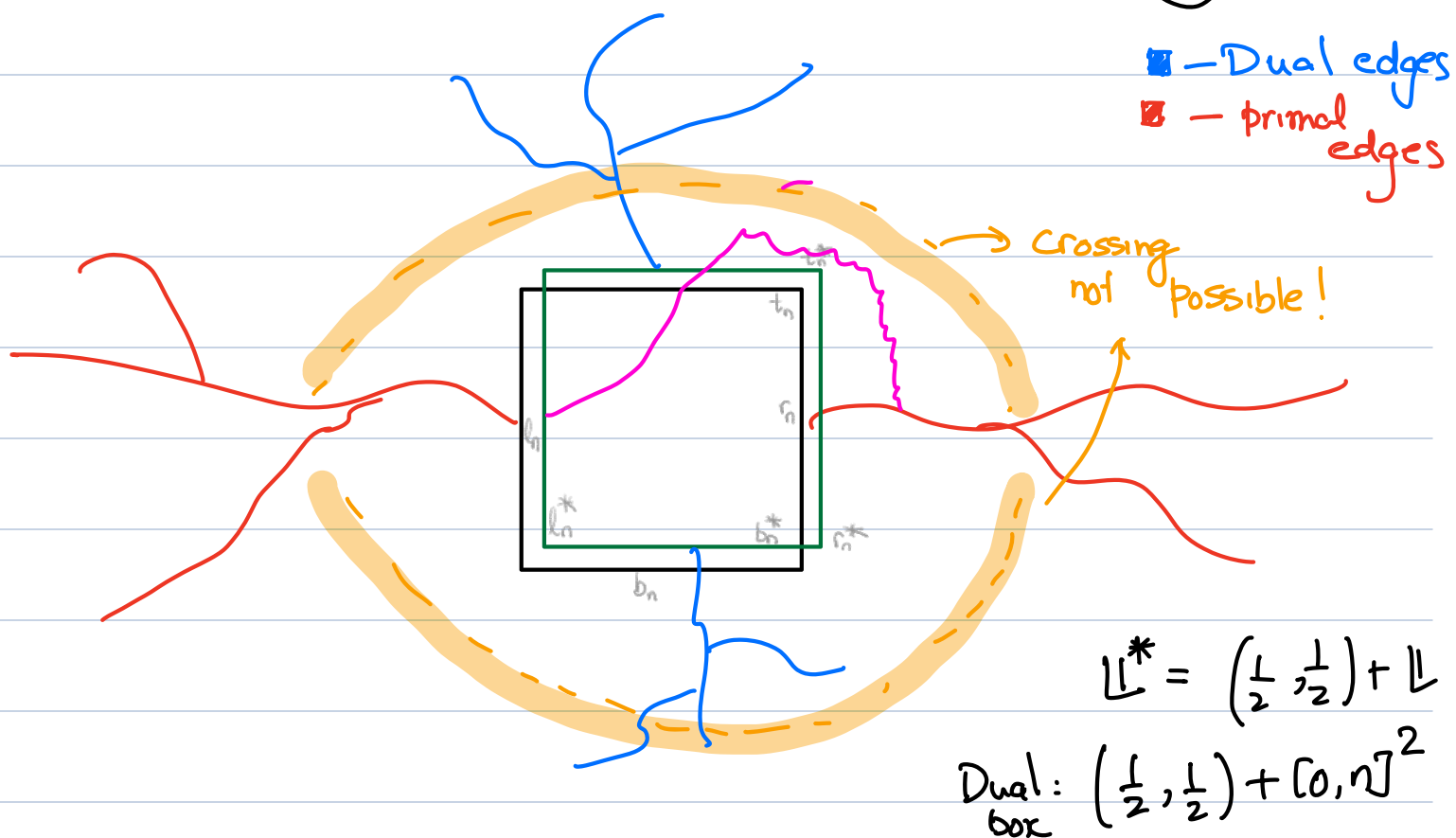
$$\mathbb{P}_{\frac{1}{2}} \left(\exists u \text{ in } \mathbb{Z}^d \text{ s.t. } C(u) \text{ is unbounded} \right) = 1$$

$$\Rightarrow \mathbb{P}_{\frac{1}{2}} (L_n \cup R_n \cup B_n \cup T_n) \xrightarrow{\text{as } n \rightarrow \infty} 1$$



\therefore by SRT, $P_{\frac{1}{2}}(L_n) \geq 1 - (1 - P(\cup_{i=1}^n))^{1/4}$
 $\longrightarrow 1$
 as $n \rightarrow \infty$

Now $P(L_n \cap R_n) \longrightarrow 1$ as $n \rightarrow \infty$
 $\longrightarrow (*)$



||^{ly} define $\{l_n^*, b_n^*, r_n^*, t_n^*\}$ and $\{T_n^*, B_n^*, L_n^*, R_n^*\}$

i.e. $L_n^* = \left\{ \begin{array}{l} \exists \text{ an unbounded closed path from} \\ l_n^* \text{ which does not use any edge} \\ \text{from } (\frac{1}{2}, \frac{1}{2}) + [0, n]^2 \end{array} \right\}$

$P = \frac{1}{2}$, the processes on \mathbb{L} and \mathbb{L}^* are exactly the same, thus by the same argument

$$\mathbb{P}_{\frac{1}{2}}(T_n^* \cap B_n^*) \longrightarrow 1 \text{ as } n \rightarrow \infty - \textcircled{+}$$

From $\textcircled{*}$ and $\textcircled{+}$,

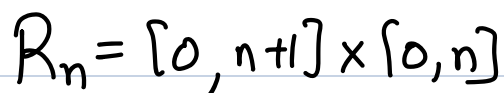
$$\mathbb{P}_{\frac{1}{2}}(L_n \cap R_n \cap T_n^* \cap B_n^*) \longrightarrow 1 \text{ as } n \rightarrow \infty$$

By uniqueness in \mathbb{Z}^2 the two red clusters must be connected by a path not going around, say the pink one.

But due to uniqueness in $(\mathbb{Z}^2)^*$ the two blue clusters must be connected, but due to the red cluster, the path can't go around, and due to the pink path they cannot be connected with help of the box also!

Thus either way we get 2 clusters contradicting

U

$$p_c(2) \leq \frac{1}{2}$$


$$S_n = \left(\frac{1}{2}, -\frac{1}{2} \right)$$

$$+ ([0, n] \times [0, n+1])$$

$$\subseteq (\mathbb{Z}^2)^*$$

$$A_n = \left\{ \begin{array}{l} \exists \text{ a left-right open crossing} \\ \text{of } R_n \text{ inside } R_n \end{array} \right\}$$
$$B_n = \left\{ \begin{array}{l} \exists \text{ a top-bottom closed dual} \\ \text{crossing of } S_n \text{ inside } S_n \end{array} \right\}$$

Whitney's Theorem (graph theory)

Either A_n occurs or B_n occurs not

both.

$$\mathbb{P}_p(A_n) + \mathbb{P}_p(B_n) = 1$$

! { hard part is that one of them }
must occur

$$\text{At } p = \frac{1}{2}, \quad \mathbb{P}_{\frac{1}{2}}(A_n) = \mathbb{P}(B_n) = \frac{1}{2}$$

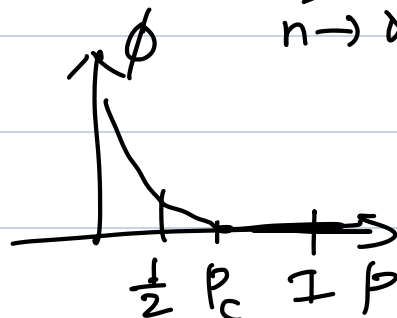
Suppose $p_c > \frac{1}{2}$,

$$\mathbb{P}_{\frac{1}{2}} \left\{ \exists \text{ an open path from } \omega \text{ to the line } x = n+1 \right\}$$

$$\leq \mathbb{P}_{\frac{1}{2}} \left\{ C_n \cap B_{n-1} \neq \emptyset \right\}$$

$$\leq c (n-1)^{d-1} e^{-(n-1)\phi(\frac{1}{2})} \rightarrow 0 \text{ as } n \rightarrow \infty$$

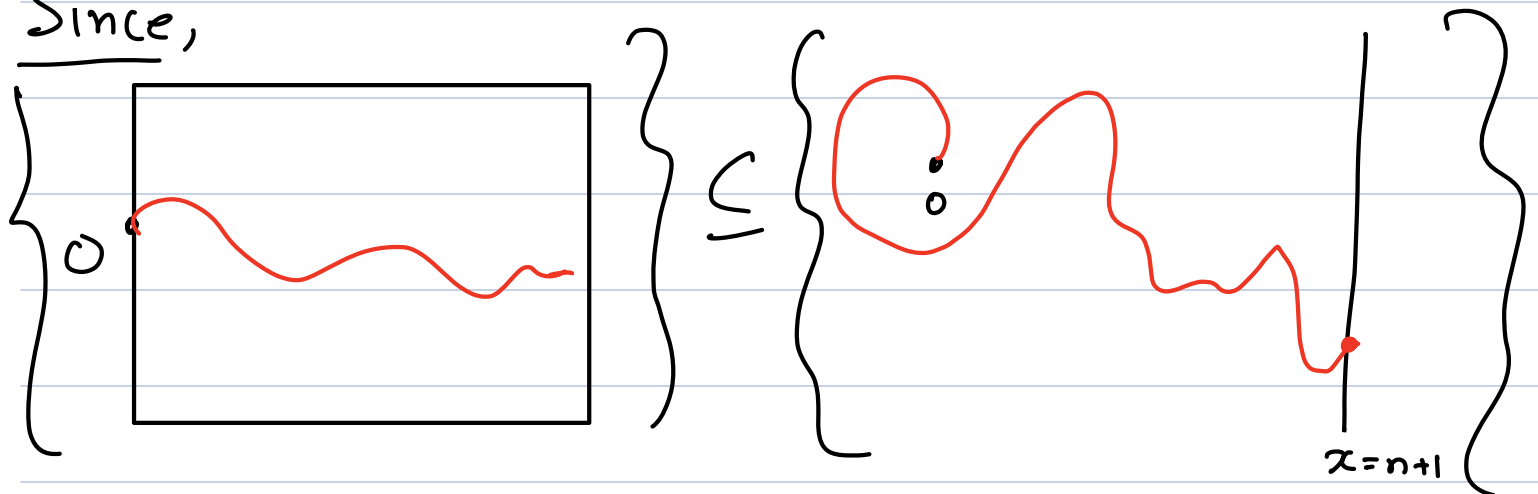
Since $\frac{1}{2} < p_c$, $\phi(\frac{1}{2}) > 0$.



Now by union bound

$$P_{\frac{1}{2}}(A_n) \leq n \sigma (n+1)^{d-1} e^{-n\phi(\frac{1}{2})}$$

Since,



and there are n points on the left side.

$$\text{Thus, } P_{\frac{1}{2}}(A_{\frac{1}{2}}) \longrightarrow 0 \text{ as } n \rightarrow \infty$$

$$\text{But } P_{\frac{1}{2}}(A_n) = \frac{1}{2} \Rightarrow \Leftarrow \text{ Thus } p_c(2) \leq \frac{1}{2}$$

Therefore we have

$$p_c(2) = \frac{1}{2}$$

→ Kesten's thm

Let $A_n = \{0 \longleftrightarrow \subseteq B_n\}$, $B_n = [-n, n]^2$

$$\Theta_n(p) = \mathbb{P}_p(A_n)$$

$\frac{d}{dp} \Theta_n(p)$ exists, since A_n depends on finitely many edges.

Duminil - Copin thm

$$\Theta_n'(p) \geq C \frac{n \Theta_n(p)}{\sum_{k=0}^{n-1} \Theta_k(p)}$$

Lemma: Let $\{f_n\}_{n \geq 0}$ be a seq. of inc and diff'ble functions satisfying

(i) $f_n : (a, b) \longrightarrow (0, M) \quad \forall n$

(ii) $\{f_n\}$ converges pointwise in (a, b)

(iii) $f_n' \geq \frac{n}{\sum_{k=0}^{n-1} f_k} f_n$

Then $\exists x_0 \in [a, b]$ s.t.

(a) $\forall x \in (a, x_0)$ and n large enough
s.t.
$$f_n(x) \leq M \exp\left(-\frac{\sqrt{n}(x_0 - x)}{2}\right)$$

(b) $\forall x \in (x_0, b)$

$f := \lim_{n \rightarrow \infty} f_n$ satisfies

$$f(x) \geq \frac{x - x_0}{2}.$$