Assignment ②, Ishaan Bhadoo

② $\mathbb{T}$ — ~~rooted~~ binary tree

To find,

$p_c(\mathbb{T})$

Proof : Let $\partial\Lambda_n = \{x : \text{dist}(x, 0) = n\}$

Let's perform p-percolation on $\mathbb{T}$

Let Z_n be the number of vertices in $\partial\Lambda_n$ that are in $C(0)$.

That is, $Z_n = |C(0) \cap \mathbb{T}_n|$, Then

$$Z_n = \sum_{y \in C(0) \cap \partial\Lambda_{n-1}} \sum_{\substack{x \in \partial\Lambda_n \\ x \sim y}} \mathbb{1}(\{x, y\} \text{ is open})$$

Now $\sum_{\substack{x \in \partial\Lambda_n \\ x \sim y}} \mathbb{1}(\{x,y\} \text{ is open}) \sim \mathrm{Bin}(2,p)$

$\therefore Z_n = \sum_{j=1}^{Z_{n-1}} B_j$ where

$(B_j)_j$ are independent $\mathrm{Bin}(2,p)$ r.v.

Thus the seq $|C(0) \cap \partial\Lambda_n|$ is a GW process with offspring dist $\mathrm{Bin}(d-1,p)$

Now by the theorem for GW processes

$$p_c(T_1) = \frac{1}{\text{mean}} = \frac{1}{2} \qquad \blacksquare$$

(3) To show,

$\theta(p) > 0 \Rightarrow \mathbb{P}_p(\exists u \in V \text{ s.t. } \# C(u) = \infty)$

$= 1$

Proof :

Firstly $\mathbb{P}_p(\exists\, u \in V \text{ s.t. } \# C(u) = \infty)$

$$\geq \theta(p) > 0 \quad (1)$$

Now we show

$$\{\exists\, u \in V \text{ s.t. } \# C(u) = \infty\} \text{ is}$$

a tail event.

Let $\mathbb{E} = \{e_1, e_2 \ldots\ldots\}$

For a finite subset $E \subset \mathbb{E}$ let

$\tau_E = \sigma(\omega(e) : e \notin E)$, an event is a tail event if, it belongs to

$$\tau = \bigcap_{\substack{E \subset \mathbb{E} \\ \text{finite}}} \tau_E$$

So let $E \subset \mathbb{E}$ be a finite set of edges, then we'll show

$$\{w : \exists u \in V \text{ s.t. } \#C(u) = \infty\}$$

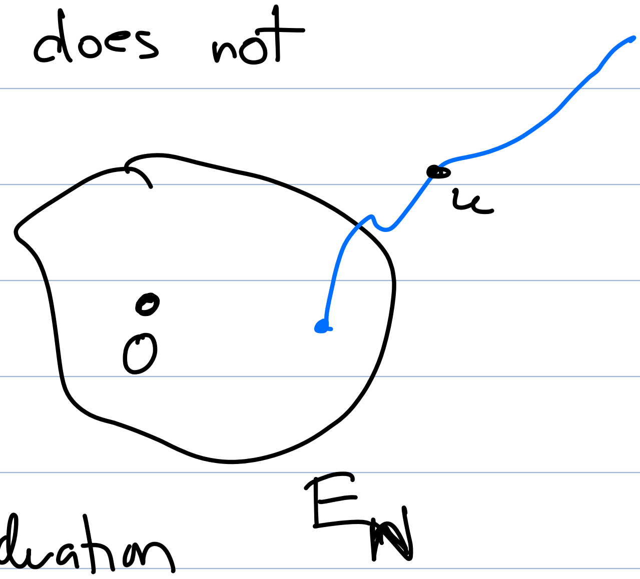
$$\in \mathcal{T}_E .$$

Let $E \subset E_N = \{e_1, e_2 \ldots e_N\}$ for some N.

Then $\mathcal{T}_{E_N} \subset \mathcal{T}_E$, consider the following figure.

Let $w \in \{w : \exists u \in V \text{ s.t. } \#C(u) = \infty\}$

then $\exists$ u s.t. $u \longleftrightarrow \infty$ does not touch E_N, and hence this $w \in \mathcal{T}_{E_N}$



Therefore the event in consideration is a tail event, and hence by ①, and Kolmogorov's 0-1 law,

$$\mathbb{P}_p(\exists u \in V \text{ s.t. } \#C(u) = \infty) = 1$$

④ <u>To show</u>

$\{u \longleftrightarrow v\}$ is inc

<u>Proof</u> Let $A = \{\omega : u \longleftrightarrow v\}$

A is inc if $\mathbb{1}_A$ is. Hence we need to show $\mathbb{1}_A(\omega) \geq \mathbb{1}_A(\omega')$ for $\omega \geq \omega'$.

If $\omega' \in \{u \longleftrightarrow v\}$ then trivially $\omega \in \{u \longleftrightarrow v\}$

Thus if $\mathbb{1}_A(\omega') = 1$ then $\mathbb{1}_A(\omega) = 1$
and if $\mathbb{1}_A(\omega') = 0$ then $\mathbb{1}_A(\omega) \geq \mathbb{1}_{A'}(\omega)$

Hence A is an inc event

$\boxtimes$