Percolation of Words

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Abstract

This article documents progress on the problem of percolation of words.

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1 Introduction

Consider Bernoulli site percolation on a graph G with $p_c^{\text{site}}(G) < \frac{1}{2}$. Let $\Xi = \{0,1\}^{\mathbb{N}}$ be the set of all infinite binary sequences, referred to as the set of all words. We say that a word $\xi \in \Xi$ is *seen* from a vertex $v \in V$ if there exists a path (infinite self-avoiding path) $v = v_0 \sim v_1 \sim v_2 \sim \cdots$ such that: $\omega_i = \xi_i \quad \forall i \geq 0$.

Define, the set of words seen from a vertex, $S(v) = \{\xi \in \Xi : \xi \text{ is seen from } v\}$. We also define the set of all words seen as $S_\infty = \bigcup_{v\in V} S(v) = \{\xi \in \Xi : \xi \text{ is seen from some } v \in V\}$.

We want study S_∞. We operate on locally finite, connected graphs G with $p_c^{\text{site}}(G) < \frac{1}{2}$ at $p = \frac{1}{2}$. Let us start by stating the main problem we are interested in:

Conjecture 1 (Classification Conjecture II.(c), [HdNS14])**.** *Let G be an infinite graph, with uniformly bounded degree, and p_c^{site} denote its critical threshold for site percolation. If $p_c^{site} < p < 1 - p_c^{site}$, then all sequences can be embedded, i.e., $\mathbb{P}_p(S_\infty = \Xi) = 1$*

The goal of this project is to show the above in the case of transitive (or maybe general) non-amenable graphs. Before diving into non-amenability lets start by noting some past results.

1.1 Past Results

Theorem 1 (Kesten, Benjamini, [BK95])**.** *Consider site percolation on $G = \mathbb{Z}^d$ for $d \geq 10$, then $\mathbb{P}_{\frac{1}{2}}(S_\infty = \Xi) = 1$.*

It is known that $p_c^{\text{site}}(\mathbb{Z}^d) < \frac{1}{2}$ for $d \geq 3$, so its natural to ask if the above holds for $\mathbb{Z}^3$. This was shown in [NTT22].

Theorem 2 (Nolin, Tassion, Teixeira, [NTT22])**.** *Consider site percolation on $G = \mathbb{Z}^d$ for $d \geq 3$, then $\mathbb{P}_{\frac{1}{2}}(S_\infty = \Xi) = 1$.*

Even though Theorem 2 trivially implies Theorem 1, we chose to state them separately due to huge differences in the difficulty of proof. This highlights one of the reasons why Conjecture 1 may be easier to show than the general case.

Theorem 3 (Kesten, Sidoravicius,3 Zhang, [KSZ01])**.** *Let $\mathbb{Z}^2_{cp}$ be the closed-packed lattice obtained by adding diagonal edges to each face of $\mathbb{Z}^2$. Then $\mathbb{P}_{\frac{1}{2}}(S_\infty = \Xi) = 1$.*

2 Percolation of words beyond $\mathbb{Z}^d$

2.1 Percolation of words on trees

We start by stating Kesten and Benjamini's theorem for periodic trees.

Theorem 4 (Kesten, Benjamini, [BK95])**.** *Consider a tree T with period ν, such that $p_c(T) < \frac{1}{2}$. Then $\mathbb{P}_{\frac{1}{2}}(S_\infty = \Xi) = 1$.*

Recently, we managed to generalize the above theorem to cover all trees with $p_c(T) < \frac{1}{2}$. In particular we have,

Theorem 5. *Let (T, o) be a connected locally finite tree such that $p_c^{site}(T) < \frac{1}{2}$. Then $\mathbb{P}(S_\infty = \Xi) = 1$.*

The proof follows the same two steps as the proof by Kesten-Benjamini for periodic trees. The proof is divided into two parts, firstly we show the finiteness of a certain sum and prove that it implies our result, next we show that the sum is indeed finite. The key difference is that we use different events to do the above computation which capture more than just the growth rate of the tree.

Before we dive into the proof, we study site percolation on trees. Define the quantity,

$$\widetilde{br}T = \sup\left\{\lambda : \exists \text{ non-zero flow } \theta \text{ on the vertex set such that } 0 \le \theta(v) \le \lambda^{-|v|}\right\}$$

Where flows on vertices is defined in a similar way as for edges, i.e. we have a root vertex o and every vertex other than o satisfies flow conservation. Here $|v|$ is the distance of v from o.

By max flow min cut theorem (which holds for countable locally finite graphs), we have that

$$\widetilde{br}T = \sup\left\{\lambda : \inf_{\Pi}\sum_{v\in\Pi}\lambda^{-|v|} > 0\right\}$$

where the infimum is over vertex cutsets Π, i.e. Π is a finite set of vertices such that o is not connected to infinity in $T \setminus \Pi$.

Lemma 6. *For site percolation on T,*

$$\mathbb{P}_p(o \leftrightarrow \infty) \le \inf\left\{\sum_{v\in\Pi} p^{|v|} : \Pi \textit{ is a (vertex) cutset separating } o \textit{ and } \infty\right\}$$

Proof. Since, $\{o \leftrightarrow \infty\} \subseteq \bigcup_{v\in\Pi}\{o \leftrightarrow v\}$ for all (vertex) cutsets Π $\square$

We start by showing $p_c^{\text{site}} \ge \frac{1}{\widetilde{br}T}$. Consider $p > p_c$ and $\lambda = \frac{1}{p}$, then by Lemma 1, for all (vertex) cutsets Π

$$\sum_{v\in\Pi}\lambda^{-|v|} = \sum_{v\in\Pi} p^{|v|} > 0$$

So, $\widetilde{brT} \geq \frac{1}{p}$ for all $p > p_c$, which implies $p_c^{\text{site}} \geq \frac{1}{\widetilde{brT}}$.

For bond percolation on the tree T, it is a famous theorem of Lyons that $p_c^{\text{bond}} = \frac{1}{brT}$. Where brT is the branching number of the tree. By the same argument, a similar statement is true in our case. More specifically, we have:

Theorem 7. *Consider site percolation a rooted, locally finite, connected tree T. Then*

$$p_c^{site}(T) = \frac{1}{\widetilde{brT}}.$$

We want to show the following theorem,

Theorem 8. *Let T be a connected locally finite tree such that $p_c(T) < \frac{1}{2}$. Then $\mathbb{P}(S_\infty = \Xi) = 1$.*

First, we start by setting up some notation: Let T_n denote the set of vertices at distance n. For a word η, let $\eta^{(n)} = (\eta_1, \cdots, \eta_n)$.

For two vertices $v, w \in T$ and an infinite word $\eta \in \Xi$, we denote by $v \overset{\eta^{(n)}}{\longleftrightarrow} w$ the event that there exists a self-avoiding path $v = v_0 \sim v_1 \sim \cdots \sim v_{n-1} = w$ from v to w along which $\eta^{(n)}$ is seen, i.e., such that $\omega_{v_i} = \eta_i$ for all $i \in \{0, \ldots, n-1\}$. We simply write $\{v \leftrightarrow w\} := \{v \overset{\mathbb{1}^{(n)}}{\longleftrightarrow} w\}$ for the monochromatic word. For trees, since there is only one path between vertices, for clarity we write (only for trees) $v \overset{\eta}{\leftrightarrow} w$. We also write $x \geq S$ (for a vertex x and set of vertices S) if x is a descendant of S. We are now ready to show Theorem 8.

Proof. Suppose $\widetilde{brT} = 2 + 3\varepsilon$ for some $\varepsilon > 0$. Then by definition of the branching number there exists a non-zero flow θ such that $0 \leq \theta(v) \leq (2 + 2\varepsilon)^{-n}$. Fix this θ and let its strength be $||\theta|| = \delta > 0$.

Define random variables,

$$X_n(\eta) = \sum_{v \in T_n} (2 + \varepsilon)^n \theta(v) \mathbb{1}(o \overset{\eta}{\leftrightarrow} v)$$

Also define the events,

$$E_n(\eta) = \{X_n(\eta) \geq \delta\}$$

It is clear that if $E_n(\eta)$ occurs the word $\eta^{(n)}$ must be seen from o. For the word $\mathbb{1} = (1, 1, \cdots)$, we set $X_n := X_n(\mathbb{1})$ and $E_n := E_n(\mathbb{1})$.

Lemma 9. *If $\sum_{n \geq 0} 2^n \ \mathbb{P}_{\frac{1}{2}}(E_n \setminus E_{n+1}) < \infty$ then all words are seen.*

Proof. Firstly note that due to symmetry $X_n(\eta) \overset{(d)}{=} X_n(\mathbb{1})$. Thus,

$$\mathbb{P}_{\frac{1}{2}}(E_n(\eta) \setminus E_{n+1}(\eta)) = \mathbb{P}_{\frac{1}{2}}(E_n \setminus E_{n+1}) \ \forall \eta$$

Now suppose that the sum is finite. Consider the events,

$$A_n = \Big\{\text{there exists } \eta^{(n+1)} \text{ such that } E_n(\eta) \setminus E_{n+1}(\eta) \text{ holds}\Big\}$$

We have, $\mathbb{P}(A_n) = \sum_{\eta^{(n+1)}} \mathbb{P}_{\frac{1}{2}}(E_n(\eta) \setminus E_{n+1}(\eta)) = 2^{n+1}\mathbb{P}_{\frac{1}{2}}(E_n \setminus E_{n+1})$, the summability condition this implies that $\sum_{n \geq 0} \mathbb{P}(A_n) < \infty$.

Thus by Borel-Cantelli we get that $\mathbb{P}(A_n \text{ i.o.}) = 0$. So there exists $N = N(\omega)$ such that A_n does not happen a.s. for $n \geq N$ i.e., for all $n \geq N$ and for all $\eta^{(n+1)}$ we have that, $E_n(\eta) \setminus E_{n+1}(\eta)$ does not occur.

Now we show that for all $n \geq 1$, there is a $\eta^{(n)}$ such that $E_n(\eta)$ occurs. Consider

$$\sum_{\eta^{(n)}} X_n(\eta) = \sum_{\eta^{(n)}} \sum_{v \in T_n} (2+\varepsilon)^n \theta(v) \mathbb{1}(o \overset{\eta}{\leftrightarrow} v) = \sum_{v \in T_n} \sum_{\eta^{(n)}} (2+\varepsilon)^n \theta(v) \mathbb{1}(o \overset{\eta}{\leftrightarrow} v)$$

But $\sum\limits_{\eta^{(n)}} \mathbb{1}(v \overset{\eta}{\leftrightarrow} w) = 1$. So,

$$\sum_{\eta^{(n)}} X_n(\eta) = \sum_{v \in T_n} (2+\varepsilon)^n \theta(v) = \delta (2+\varepsilon)^n$$

If $X_n(\eta) < \delta$ for all $\eta^{(n)}$, then the LHS $< \delta 2^n$, which is a contradiction. Thus there must exists a $\eta^{(n)}$ such that $E_n(\eta)$ occurs. Let $n > N$, then by the above observation $E_{n+m}(\eta^{(n)}, \xi^{(m)})$ must also occur for any ξ, thus all words are seen from the generation n.

$\square$

Now we show that the sum under consideration is indeed finite, in the proof we use McDiarmid's inequality. We start by stating it:

Theorem 10 (McDiarmid's Inequality). *Let $X_1, \ldots, X_n$ be independent random variables where X_i is $\mathcal{X}_i$-valued for all i, and let $X = (X_1, \ldots, X_n)$. Assume $f : \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathbb{R}$ is a measurable function such that $\|D_i f\|_\infty < +\infty$ for all i. Then for all $\beta > 0$,*

$$\mathbb{P}[f(X) - \mathbb{E}[f(X)] \leq -\beta] \leq \exp\left(-\frac{2\beta^2}{\sum_{i \leq n} \|D_i f\|_\infty^2}\right).$$

Here, for $x = (x_1, x_2, \cdots x_n)$ the definition of $D_i f(x)$ is:

$$D_i f(x) := \sup_{y \in \mathcal{X}_i} f(x_1, x_2, \ldots, y, x_{i+1}, \ldots, x_n) - \sup_{y' \in \mathcal{X}_i} f(x_1, x_2, \ldots, y', x_{i+1}, \ldots, x_n).$$

Lemma 11. *There exists $\varepsilon' > 0$ and $C > 0$ such that $\mathbb{P}(E_{n+1}^{\mathsf{c}} | E_n) \leq \exp(-C(1+\varepsilon')^n)$*

It is clear that this lemma implies the summability condition in Lemma 4, so proving Lemma 6 suffices.

Proof of Lemma 6. Fix a configuration on $\bigcup\limits_{i \leq n} T_i$ such that E_n occurs. We show the above bound conditioned on this configuration. More formally, let $\tilde{\mathbb{P}}$ be the conditional measure given $E_n, U = \{u_1, \cdots u_k\}$ where U is the set of vertices in $T_n \cap \mathcal{C}$[1] such that $\sum\limits_{1 \leq i \leq k} (2+\varepsilon)^n \theta(u_i) \geq \delta$.

Clearly, $X_{n+1} = \sum\limits_{\substack{|x|=n+1 \\ x \geq U}} (2+\varepsilon)^{n+1} \theta(x) \mathbb{1}(U \leftrightarrow x)$ and thus $\tilde{\mathbb{E}}(X_{n+1}) = \frac{(2+\varepsilon)^{n+1}}{2} \sum\limits_i \theta(u_i)$.

Now, for each descendant of x of U in T_{n+1} consider the random variable $(2+\varepsilon)^{n+1}\theta(x)\mathbb{1}(U \leftrightarrow x) \leq (2+\varepsilon)^{n+1}\theta(x)$. Then by McDiarmid's inequality using f as the sum of these random variables, we get that

$$\mathbb{P}[X_{n+1} - \mathbb{E}[X_{n+1}] \leq -\beta] \leq \exp\left(-\frac{2\beta^2}{\sum_{i \leq n} \|D_i f\|_\infty^2}\right).$$

[1] cluster of o

Now,

$$\sum_i \|D_i f\|_\infty^2 \leq (2(2+\varepsilon)^{n+1})^2 \sum_{\substack{|x|=n+1 \\ x\geq U}} (2+\varepsilon)^{n+1}\theta(x)^2 \leq \frac{4(2+\varepsilon)^{2n+2}}{(2+2\varepsilon)^n} \sum_{i\leq k} \theta(u_i)$$

$$= \frac{4\,(2+\varepsilon)^{n+1}}{(2+2\varepsilon)^n} \cdot 2\,\tilde{\mathbb{E}}(X_{n+1})$$

$$= \frac{8\,(2+\varepsilon)^{n+1}}{(2+2\varepsilon)^n}\,\tilde{\mathbb{E}}(X_{n+1})$$

For $\beta = \frac{\varepsilon}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})$, by McDiarmid,

$$\tilde{\mathbb{P}}\left(X_{n+1} \leq \frac{2}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})\right) \leq \exp\left(-\frac{\left(\frac{\varepsilon}{2+\varepsilon}\right)^2 \tilde{\mathbb{E}}(X_{n+1})\,(2+2\varepsilon)^n}{4\,(2+\varepsilon)^{n+1}}\right).$$

Since $\tilde{\mathbb{E}}(X_{n+1}) \geq \frac{\delta\,(2+\varepsilon)}{2}$,

$$\tilde{\mathbb{P}}\left(X_{n+1} \leq \frac{2}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})\right) \leq \exp\left(-\frac{1}{8}\left(\frac{\varepsilon}{2+\varepsilon}\right)^2 \delta\left(1+\frac{\varepsilon}{2+\varepsilon}\right)^n\right)$$

$$= \exp\left(-C\left(1+\frac{\varepsilon}{2+\varepsilon}\right)^n\right).$$

Note that,

$$\left\{X_{n+1} \geq \frac{2}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})\right\} \implies E_{n+1}$$

So,

$$\tilde{\mathbb{P}}(E_{n+1} \text{ does not occur}) \leq \tilde{\mathbb{P}}\left(X_{n+1} \leq \frac{2}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})\right) \leq \exp\left(-C\left(1+\varepsilon'\right)^n\right)$$

Where $\varepsilon' = \frac{\varepsilon}{2+\varepsilon}$ and we are done. □

2.2 Perturbative result

Definition 12 (Non-amenability). *The Cheeger constant Φ_V is defined as:*

$$\Phi_V = \inf\left\{\frac{|\partial S|}{|S|} : S \subset V,\ S \text{ finite, connected}\right\}$$

A graph is called non-amenable if $\Phi_V > 0$.

For instance, $d-$regular trees are non-amenable for $d \geq 3$.

Using a theorem of Benjamini and Schramm one can show that all words are seen for non-amenable graphs with $\Phi_V \geq 2$. Consider the following theorem,

Theorem 13 (Benjamini-Schramm, [BS97]). *Let G be a locally finite graph with Cheeger constant $\Phi_V \geq 2$. Then G contains T_{n+1}, where T_{n+1} is a tree where the root has degree n and all other vertices have degree $n+2$.*

In particular since $p_c(T_{n+1}) = \frac{1}{n+1}$, this implies our result for a non-amenable graph with $\Phi_V \geq 2$.

Question: Let G be a non-amenable, transitive graph, and suppose $p_c(G) < \frac{1}{2}$. Can we find a tree T such that $p_c(T) < \frac{1}{2}$?

2.3 The truncation problem

A closely related problem to the problem of percolation of words is the so-called truncation problem. Which says the following:
Let $G = (V, E)$ be an infinite connected locally finite graph (not necessarily with $p_c^{\text{site}} < \frac{1}{2}$). Then for any $p \in (0,1)$ there exists a $N = N(p)$ such that $\mathbb{P}_p(S_\xi^N) = 1 \ \forall \xi$.

Where $S_\xi^N = \left\{\omega : \xi \text{ is seen in } \omega \text{ for the graph } G^N\right\}$. Where G^N is the $N-$fuzz of the graph G, i.e., $G^N = (V, E^N)$ where $E^N = \{\{x,y\} : 1 \le dist_G(x,y) \le N\}$. This was shown by B.N.B de Lima in [dL08] for $\mathbb{Z}^d$. We now show a simple proof for all non-amenable graphs.

First, define $S_\xi^N(v) = \left\{\omega : \xi \text{ is seen at v in } \omega \text{ for the graph } G^N\right\}$. Then clearly,

$$S_\xi^N(v) = \underset{v\in V}{\cup} S_\xi^N(v)$$

.

An important ingredient in the proof will be Weirman's coupling:

Theorem 14 (Weirman's coupling)**.** *Consider independent site percolation with parameter $p \in [0,1]$ on the graph G. Then, for any binary sequence $\xi \in \{0,1\}^{\mathbb{N}}$ and any vertex $v_0 \in V$, it holds that:*

$$P_p(S_\xi(v_0)) \ge \min\{P_p(S_0(v_0)), P_p(S_1(v_0))\}, \quad \forall \xi \in \{0,1\}^{\mathbb{N}}.$$

We want to show:

Theorem 15. *Consider site percolation on a non-amenable graph G. For every $p \in (0,1)$, there exists a positive integer $N = N(p)$ and a constant $c > 0$ such that*

$$P_p(S_\xi^N(v)) > c, \quad \forall \xi \in \{0,1\}^{\mathbb{N}}.$$

In particular,

$$P_p(S_\xi^N) = 1, \quad \forall \xi \in \{0,1\}^{\mathbb{N}}.$$

Consider the following two lemmas,

Lemma 16. *Let G be a non-amenable graph with Cheeger constant $\Phi_v(G)$ then,*

$$\Phi_v(G^N) \ge N\Phi_v(G).$$

Lemma 17. *For any graph G,*

$$p_c^{site}(G) \le \frac{1}{1+\Phi_v(G)}.$$

Lets start by showing that they imply Theorem 15.

Proof. Fix a $p \in (0,1)$. These two lemmas combined give us that:

$$\exists N = N(p) \text{ such that } p_c^{\text{site}}(G^N) < \min\{p_1, 1-p\}.$$

If that is the case, then

$$P_p(S_\xi^N) > 0 \quad \text{and} \quad P_p(S_\xi^N) > 0 \quad \forall \xi,$$

and thus by Weimann's coupling:

$$P_p(S_\xi^N) > 0 \quad \forall \xi.$$

Since S_ξ^N are all translation-invariant,

$$P_p(S_\xi^N) = 1 \quad \forall \xi.$$

This shows the truncation problem. $\square$

References

[BK95] Itai Benjamini and Harry Kesten. Percolation of arbitrary words in {0, 1} n. *The Annals of Probability*, pages 1024–1060, 1995.

[BS97] Itai Benjamini and Oded Schramm. Every graph with a positive cheeger constant contains a tree with a positive cheeger constant. *Geometric & Functional Analysis GAFA*, 7(3):403–419, 1997.

[dL08] Bernardo NB de Lima. A note about the truncation question in percolation of words. 2008.

[HdNS14] M.R. Hilário, B.N.B. de Lima, P. Nolin, and V. Sidoravicius. Embedding binary sequences into bernoulli site percolation on z3. *Stochastic Processes and their Applications*, 124(12):4171–4181, 2014.

[KSZ01] Harry Kesten, Vladas Sidoravicius, and Yu Zhang. Percolation of arbitrary words on the close-packed graph of zˆ2. 2001.

[NTT22] Pierre Nolin, Vincent Tassion, and Augusto Teixeira. No exceptional words for bernoulli percolation. *Journal of the European Mathematical Society*, 25(12):4841–4868, 2022.