Bounds on the critical threshold for regular graphs

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Abstract

In this article, we study the critical percolation threshold for arbitrary d-regular graphs. It is well-known that $p_c \geq \frac{1}{d-1}$ for such graphs, we prove that among all quasi-transitive graphs, the equality $p_c(G) = \frac{1}{d-1}$ holds if and only if G is a tree. Furthermore, we provide counterexamples that illustrate the necessity of the quasi-transitive assumption.

1 Introduction

Consider independent (Bernoulli) bond percolation on a locally finite, connected, infinite simple graph G (all graphs are assumed to satisfy these conditions unless stated otherwise), i.e. we retain edges with probability p and throw them away with probability $1-p$. We use $\mathbb{P}_p$ to denote the corresponding percolation measure. An important function to consider in the context of percolation is $\psi(p) = \mathbb{P}_p(\exists$ an infinite connected component$)$, this leads to the definition of the critical parameter $p_c := \sup\{p : \psi(p) = 0\}$. It's very to hard to find the exact value of p_c for most graphs, however for $d-$regular trees one can show that $p_c = \frac{1}{d-1}$.

The proof follows two steps: By a first-moment argument one can show that, a graph G with maximal degree $d < \infty$ satisfies $p_c(G) \geq \frac{1}{d-1}$. For trees with degree d by a dual second moment upper bound we can get $p_c \leq \frac{1}{d-1}$, implying $p_c = \frac{1}{d-1}$ (see [Roc24, Claim 2.3.9] for details). This leads to the following question: consider percolation on a $d-$regular graph G, are trees the only graphs with $p_c = \frac{1}{d-1}$?

We start with some definitions. For a graph let $\mathrm{AUT}(G)$ be the set of all automorphisms (adjacency preserving bijections) of G. A graph is called *quasi-transitive* if the number of orbits for the action of $\mathrm{AUT}(G)$ on G is finite. A graph is called *transitive* if there is only one orbit. Under the assumption of quasi-transitivity we have the following theorem:

Theorem 1. *Let G be a quasi-transitive $d-$regular graph. Then $p_c(G) \geq \frac{1}{d-1}$ and equality holds if and only if G is a tree.*

It is important to note that being quasi-transitive is essential. In Section 2, using the general theory for percolation on trees we give a counterexample (for each d) when one drops this assumption. Next, we show the above theorem by constructing a covering of every quasi-transitive d regular graph using regular trees and then use the strict monotonicity result of [MS19]. The main tools we use are from [MS19] and [LP17]. For completeness, we cover the required background for percolation on trees and some techniques from the theory of coverings, in an effort to make this article self-contained.

1.1 Connection to the connective constant

A self-avoiding walk is a path that visits no vertex more than once. To define the connective constant, fix a starting vertex o, the set of all self-avoiding walks of length n starting at o is denoted as SAW_n. The connective constant $\mu(G)$ of a graph G is then defined as

$$\mu(G) := \lim_{n\to\infty} |\mathrm{SAW}_n|^{\frac{1}{n}}$$

By Fekete's lemma, it can be checked that this limit exists. The connective constant is closely related to the critical threshold by the following lemma.

Lemma 2. *For any connected infinite graph G, $p_c(G) \geq \frac{1}{\mu(G)}$.*

Proof. Let $\mathcal{C}(o)$ denote the connected component of o in Bernoulli percolation with parameter p. Define $S_n(o)$ as the set of self-avoiding walks of length n within $\mathcal{C}(o)$. If $\mathcal{C}(o)$ is infinite, then $S_n(o) \neq \emptyset$ for all n. From this, we deduce:

$$\mathbb{P}_p(o \leftrightarrow \infty) \leq \mathbb{P}_p[S_n(o) \neq \emptyset] \overset{\text{Markov's Ineq}}{\leq} \mathbb{E}_p[|S_n(o)|] = |\mathrm{SAW}_n| p^n.$$

By taking n-th roots we get, $1 \leq \mu(G)p$ whenever $\mathbb{P}_p(o \leftrightarrow \infty) > 0$. In particular, this holds for $p > p_c$. □

An analogous statement for the connective constant was previously shown by Grimmett and Li [GL15, Thm 4.2].

Theorem 3. *([GL15, Thm 4.2]) Let $G = (V, E)$ be a d-regular quasi-transitive graph and let $d \geq 3$. We have that $\mu(G) < d - 1$ if G has cycles.*

By using Lemma 2, this implies Theorem 1. The techniques used in the proof of Theorem 3 are entirely combinatorial and therefore differ from the covering method.

2 Percolation on trees

In this section for $d \geq 3$ we give an example of a d-regular graph with cycles such that $p_c = \frac{1}{d-1}$ (trivially such an example cannot exist for $d = 2$). To do this we use the theory of percolation on locally finite trees. We start by defining the branching number of a locally finite, infinite tree.

2.1 Branching number and the critical point for trees

Suppose $T = (V, E)$ is an infinite locally finite tree with root O. We imagine the tree T as growing downward from the root O. For $x, y \in V$, we write $x \leq y$ if x is on the shortest path from O to y; and T_x for the subtree of T containing all the vertices y with $y \geq x$. For a vertex $x \in V$ we denote by $d(x, O)$ the graph distance from O to x. We want to understand the critical point for a tree, motivated by the comparison from Galton-Watson branching processes this naturally leads us to the study of the average number of branches coming out of a vertex which is called the branching number of a tree. To rigorously define this we use conductances and flows on trees. For each edge e we define the conductance of an edge to be $c(e) := \lambda^{-|e|}$, where $|e|$ denotes the distance of the edge e from the root O. It is natural to define conductances decreasing exponentially with the distance since trees grow exponentially.

If λ is very small then due to large conductances there is a non-zero flow on the tree satisfying $0 \leq \theta(e) \leq \lambda^{-|e|}$. While increasing the value of λ we observe a critical value λ_c above which such a flow does not exists. This is precisely the branching number. Specifically,

$$br(T) := \sup\{\lambda : \exists \text{ a non-zero flow } \theta \text{ on } T \text{ such that } 0 \leq \theta(e) \leq \lambda^{-|e|} \ \forall \ e \in T\}$$

By using the max-flow min-cut theorem we get that,

$$br(T) = \sup\{\lambda : \inf_{\pi} \sum_{e \in \pi} \lambda^{-|e|} > 0\}$$

Where the infimum is over all cutsets π separating O from ∞. Using this as the definition it is easy to see that $p_c \geq \frac{1}{br(T)}$, indeed by using a first moment bound at $\lambda = \frac{1}{p}$ for $p < p_c$. By using a (weighted) second-moment method it can be shown that the reverse inequality also holds. In particular, we have the following result of Lyons.

Theorem 4. *(R. Lyons, [Lyo90]) Let T be a locally finite, infinite tree then, $p_c(T) = \frac{1}{br(T)}$ where $br(T)$ is the branching number of the tree.*

Proof: The proof essentially uses a lower bound on being connected to infinity in terms of conductances [Lyo92]. See [LP17] for the proof.

Thus, to find the critical threshold for a tree one needs to know how to compute its branching number. However, the definition of the branching number makes this in general hard, thankfully, for sub-periodic trees (defined below) we have a significantly easier method of calculating the branching number.

2.2 Superiodic trees

For a tree T we define its upper exponential growth rate as

$$\overline{grT} := \limsup_{n\to\infty} |T_n|^{\frac{1}{n}}$$

where T_n is the number of vertices at a distance n from O. Similarly one can define the lower exponential growth rate as

$$\underline{grT} := \liminf_{n\to\infty} |T_n|^{\frac{1}{n}}$$

We say that the exponential growth rate exists if $\overline{grT} = \underline{grT}$.

We now define subperiodic trees. Fix a $N \geq 0$. An infinite tree T is called N**- subperiodic** if $\forall x \in T$ there exits an adjacency preserving injection $f : T_x \to T_{f(x)}$ with $|f(x)| \leq N$ (where $|\cdot|$ is the distance from O). A tree is called **subperiodic** if there exits a N for which it is N-subperiodic. Since in general the growth rate is easier to calculate, the following theorem is the key to calculating p_c for subperiodic trees.

Theorem 5. *(Subperiodicity and Branching Number, [LP17]) For every subperiodic infinite tree T, the exponential growth rate exists and $brT = grT$.*

2.3 Non-quasi transitive counter-examples

We are now ready to give our counterexamples. Let T be a tree with root O such that every vertex in T has degree d, except two vertices X, Y that are adjacent to the root having degree $d-1$. Hence, $T = (V, E)$ is the graph formed by all black edges shown in Figure 1, now define $G := T \cup \{e\} = (V, E \cup \{e\})$ to be the graph obtained after adding the red edge e. Hence, G is a d-regular graph, we claim that $p_c(G) = \frac{1}{d-1}$.

For the tree T, $|T_1| = d$, $|T_2| = (d-2)(d-1) + 2(d-2) = (d-2)(d+1)$, after this point every point has $d-1$ branches coming out, so $|T_{2+n}| = (d-2)(d+1)(d-1)^n$. Therefore gr$T = d-1$.

T is clearly subperiodic since for all x such that $d_T(x, O) \geq 2$, T_x is exactly T_A so we can define the function $f(v) = \phi(v)$ where ϕ is the isomorphism between T_x and T_A. Thus T is 1-subperiodic. Now by theorem 3, br$T$ = gr$T = d-1$. Thus $p_c(T) = \frac{1}{d-1}$, since T is a subgraph of G, $p_c(G) \leq p_c(T) = \frac{1}{d-1}$. However, G is of degree d, thus by a standard first moment bound, $p_c(G) \geq \frac{1}{d-1}$. Therfore we have, $p_c(G) = \frac{1}{d-1}$. Thus G is a d-regular graph with cycles such that $p_c(G) = \frac{1}{d-1}$. The fact that G is not quasi-transitive follows from theorem 1.

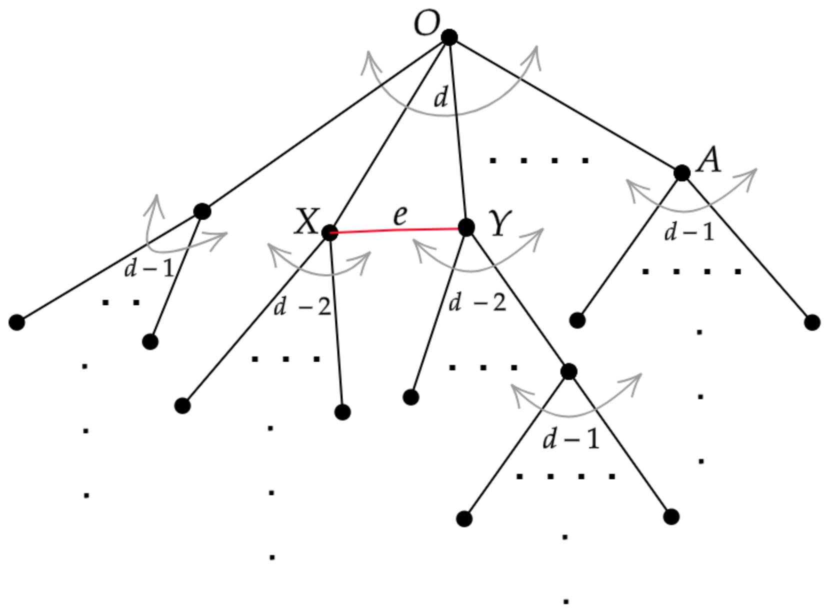

Figure 1: On removing the edge e we get a sub-periodic tree T with $\mathrm{gr}T = \mathrm{br}T = d-1$

3 Proof of the Theorem

We now show that if G is a quasi-transitive, d regular graph then $p_c(G) > \frac{1}{d-1}$. The key idea is to cover every quasi-transitive d regular graph (with cycles) by a d regular tree. We start by defining what a covering map means in the context of percolation, next we use the results of Martineau and Severo [MS19] about critical thresholds under coverings.

3.1 Critical points under coverings

The question of critical points under coverings was asked by Benjamini and Schramm in their celebrated paper " Percolation beyond $\mathbb{Z}^d$, Many Questions and a Few Answers" [BS96, Question 1]. They conjectured that if G, H are quasi-transitive graphs and G covers but is not isomorphic to H and $p_c(G) < 1$ then $p_c(G) < p_c(H)$. This conjecture was resolved by Martineau and Severo [MS19]. Following their paper we set up some definitions necessary to define a covering map.

Consider a map $\pi : V(G) \to V(H)$, we say that this map is a *strong covering map* if its $1-$Lipscitz (i.e. $d_H(\pi(x), \pi(y)) \leq d_G(x, y)$) and it has the *strong lifting property*: for every $x \in V(G)$, and for every neighbour u of $\pi(x)$ there is a unique neighbour of x that maps to u. Next we say that a map $\pi : V(G) \to V(H)$ has *uniformly non-trivial fibres* ([Sev20]) if there exists R such that for all $x \in V(G)$ there exists $y \in V(G)$ such that $\pi(x) = \pi(y)$ and $0 < d_G(x, y) \leq R$. We are now ready to state the main tool:

Theorem 6. *(F.Severo, S. Martineau, [MS19]) Let G and H be graphs of bounded such that there is a map $\pi : V(G) \to V(H)$ which is a strong covering map with uniformly non-trivial fibres. Then if $p_c(G) < 1$, we have $p_c(G) < p_c(H)$.*

The above result relies on the theory of enhancements. A technique first introduced by Aizenmann and Grimmett [AG91] as a recipe to prove strict inequalities between critical points of graphs, and is part of a more general idea of interpolation between percolation configurations [Sev20]. For background on the technique of enhancements see [Sev20], [BBR14]. We now show that this theorem holds for $G = d$ regular tree and H a d regular quasi-transitive graph with cycles. In particular we have the following:

Proposition 7. *Let T_d be the d regular tree and H be a quasi-transitive $d-$regular graph with cycles, then there exists a strong covering map π with uniformly non-trivial fibres from $V(T_d)$ to $V(H)$.*

Proof. We start by constructing a graph X from our graph H which covers H and is isomorphic to T_d. Fix a vertex $x_0 \in V(H)$. Define the vertices of X to be the non-backtracking paths $< x_0, x_1, \cdots x_n >$ starting at x_0 (a path $< x_0, x_1, \cdots x_n >$ is called non-backtracking if $x_{i+2} \neq x_i \ \forall i$). Two paths are connected in X if one is an extension of the other by an edge (this is precisely the universal cover). We claim that for a d regular H, X is isomorphic to T_d.

The fact that X is a tree is clear since all paths are non-backtracking and start at a fix vertex x_0. Now any point $< x_0, x_1, \cdots, x_n >$ has neighbours as $< x_0, x_1 \cdots, x_{n-1} >$ and $< x_0, x_1, \cdots, x_n, u >$ where u runs over all neighbours of x_n not equal to x_{n-1}, this shows d regularity. Therefore $X \cong T_d$.

For the covering map we let π be the map which projects every path to its last vertex, more formally define $\pi : V(T_d) \to V(H)$ such that $\pi(< x_0, x_1, \cdots x_n >) = x_n$ where we identify T_d with X. We now show that this is a strong covering map with uniformly non-trivial fibres.

Lipschitz property. Let $x = < x_0, \cdots, x_n >, y = < y_0, \cdots, y_m >$. We want to show that $d_H(\pi(x), \pi(y)) = d_H(x_n, y_m) \leq d_X(x, y)$. Let z be the common ancestor of x, y in X. Then $d_X(x, y) = d_X(x, z) + d_X(z, y)$. Since x is a descendant of z it is easy to see that $d_X(x, z) \geq d_H(\pi(x), \pi(z))$. Thus by the above equation $d_X(x, y) \geq d_H(\pi(x), \pi(y))$.

Uniformly non-trivial fibres. We show that there are uniformly non-trivial fibres. This is the only property that requires quasi-transitivity. Pick a $x =< x_0, x_1, \cdots, x_n >$. By quasi-transitivity, we can find a K (independent of x_n) such that there is a cycle (not necessarily simple) $C =< x_n, x_{n+1}, \cdots, x_{n+m} = x_n >$ of length $m \leq K$. If $x_{n-1} = x_{n+1}$, then $y =< x_0, \cdots, x_{n-1}, x_{n+2}, \cdots x_{n+m} = x >$ is a non-backtracking path satisfying $\pi(x) = \pi(y)$. Otherwise, consider the path $y =< x_0, x_1, \cdots, x_{n-1}, x_n, x_{n+1}, \cdots, x_n >$ since $x_{n-1} \neq x_{n+1}$, this is a non-backtracking path and gets mapped $\pi(x)$.

Strong lifting property. Pick a $x =< x_0, x_1, \cdots x_n > \in V(X)$, for any neighbour u of $\pi(x)$ we need to find a neighbour of x mapping to it. If $u = x_{n-1}$ then let that neighbour be $< x_0, x_1, \cdots x_{n-1} >$, otherwise let it be $< x_0, x_1, \cdots x_n, u >$, thus π is a strong covering map with uniformly non-trivial fibres. By our earlier comments, this also proves Theorem 1. $\square$

4 Concluding remarks

Even though we worked with quasi-transitive graphs, the same proof extends to graphs with *bounded local girth*. The concept of bounded local girth can be defined as follows: Consider a vertex x let $L_x = \inf\{l(C) : C \text{ cycle}[1], C \ni x\}$, where $l(C)$ is the length of the cycle C, we refer to L_x as the girth of x. We say that a graph G has bounded local girth if $\sup\limits_x L_x < \infty$

Following the same proof if one assumes bounded local girth then for any d regular graph G, $p_c > \frac{1}{d-1}$. Hence trees minimize p_c in the space of all graphs with bounded local girth or no cycles.

The above method uses coverings to give a characterisation of trees among quasi-transitive graphs, a similar question can be asked for the uniqueness threshold p_u, even though a theorem

[1]We are not excluding non-simple cycles, i.e. cycles which visit the same vertex multiple times.

similar to Theorem 6 has been shown for p_u (see [MS19]), we cannot apply the same technique since $p_u(T) = 1$ for a tree T.

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