Percolation of arbitrary words on trees

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Abstract

In this article, we show that there are no exceptional words for percolation on connected, locally finite trees with $p_c(T) < \frac{1}{2}$.

1 Introduction

The proof follows the same two steps as the proof by Kesten-Benjamini for periodic trees. The proof is divided into two parts, firstly we show the finiteness of a certain sum and prove that it implies our result, next we show that the sum is indeed finite.

Before we dive into the proof, we study site percolation on trees. Define the quantity,

$$\widetilde{br}T = \sup\left\{\lambda : \exists \text{ non-zero flow } \theta \text{ on the vertex set such that } 0 \leq \theta(v) \leq \lambda^{-|v|}\right\}$$

Where flows on vertices is defined in a similar way as for edges, i.e. we have a root vertex o and every vertex other than o satisfies flow conservation. Here $|v|$ is the distance of v from o.

By max flow min cut theorem (which holds for countable locally finite graphs), we have that

$$\widetilde{br}T = \sup\left\{\lambda : \inf_{\Pi} \sum_{v\in\Pi} \lambda^{-|v|} > 0\right\}$$

where the infimum is over vertex cutsets Π, i.e. Π is a finite set of vertices such that o is not connected to infinity in $T \setminus \Pi$.

Lemma 1. *For site percolation on T,*

$$\mathbb{P}_p(o \leftrightarrow \infty) \leq \inf\left\{\sum_{v\in\Pi} p^{|v|} : \Pi \textit{ is a (vertex) cutset separating } o \textit{ and } \infty\right\}$$

Proof. Since, $\{o \leftrightarrow \infty\} \subseteq \bigcup_{v\in\Pi} \{o \leftrightarrow v\}$ for all (vertex) cutsets Π $\square$

We start by showing $p_c^{\text{site}} \geq \frac{1}{\widetilde{br}T}$. Consider $p > p_c$ and $\lambda = \frac{1}{p}$, then by Lemma 1, for all (vertex) cutsets Π

$$\sum_{v\in\Pi} \lambda^{-|v|} = \sum_{v\in\Pi} p^{|v|} > 0$$

So, $\widetilde{br}T \geq \frac{1}{p}$ for all $p > p_c$, which implies $p_c^{\text{site}} \geq \frac{1}{\widetilde{br}T}$.

For bond percolation on the tree T, it is a famous theorem of Lyons that $p_c^{\text{bond}} = \frac{1}{brT}$. Where brT is the branching number of the tree. By the same argument, a similar statement is true in our case. More specifically, we have:

Theorem 2. *Consider site percolation a rooted, locally finite, connected tree T. Then*

$$p_c^{site}(T) = \frac{1}{\widetilde{br}T}.$$

We want to show the following theorem,

Theorem 3. *Let T be a connected locally finite tree such that $p_c(T) < \frac{1}{2}$. Then $\mathbb{P}(S_\infty = \Xi) = 1$.*

First, we start by setting up some notation: Let T_n denote the set of vertices at distance n. For a word η, let $\eta^{(n)} = (\eta_1, \cdots, \eta_n)$.

For two vertices $v, w \in T$ and an infinite word $\eta \in \Xi$, we denote by $v \xleftrightarrow{\eta^{(n)}} w$ the event that there exists a self-avoiding path $v = v_0 \sim v_1 \sim \cdots \sim v_{n-1} = w$ from v to w along which $\eta^{(n)}$ is seen, i.e., such that $\omega_{v_i} = \eta_i$ for all $i \in \{0, \dots, n-1\}$.

Proof. Suppose $\widetilde{br}T = 2 + 3\varepsilon$ for some $\varepsilon > 0$. Then by definition there exists a non-zero flow θ such that $0 \le \theta(v) \le (2+2\varepsilon)^{-n}$. Fix this θ and let its strength be $||\theta|| = \delta > 0$.

Define random variables,

$$X_n(\eta^{(n)}) = \sum_{v \in T_n} (2+\varepsilon)^n \theta(v) \mathbb{1}(o \xleftrightarrow{\eta^{(n)}} v)$$

Also define the events,

$$E_n(\eta^{(n)}) = \left\{ X_n(\eta^{(n)}) \ge \delta \right\}$$

It is clear that if $E_n(\eta^{(n)})$ occurs the word $\eta^{(n)}$ must be seen from o. For the word $\mathbb{1} = (1, 1, \cdots)$, we set $E_n := E_n(\mathbb{1}^{(n)})$.

Lemma 4. *If $\sum_{n \ge 0} 2^n \ \mathbb{P}_{\frac{1}{2}}(E_n \setminus E_{n+1}) < \infty$ then all words are seen.*

Proof. Firstly note that due to symmetry $X_n(\eta^{(n)}) \overset{(d)}{=} X_n(\mathbb{1}^{(n)})$. Thus,

$$\mathbb{P}_{\frac{1}{2}}(E_n(\eta^{(n)}) \setminus E_{n+1}(\eta^{(n+1)})) = \mathbb{P}_{\frac{1}{2}}(E_n \setminus E_{n+1}) \ \forall \eta$$

Now suppose that the sum is finite. Consider the events,

$$A_n = \left\{ \text{there exists } \eta^{(n+1)} \text{ such that } E_n(\eta^{(n)}) \setminus E_{n+1}(\eta^{(n+1)}) \text{ holds} \right\}$$

We have, $\mathbb{P}(A_n) = \sum_{\eta^{(n+1)}} \mathbb{P}_{\frac{1}{2}}(E_n \setminus E_{n+1}) = 2^{n+1} \mathbb{P}_{\frac{1}{2}}(E_n \setminus E_{n+1}) \implies \sum_{n \ge 0} \mathbb{P}(A_n) < \infty$. Thus by Borel-Cantelli we get that $\mathbb{P}(A_n \text{ i.o.}) = 0$. So there exists $N = N(\omega)$ such that A_n does not happen a.s. for $n \ge N$ i.e., for all $n \ge N$ and for all $\eta^{(n+1)}$ we have that, $E_n(\eta^{(n)}) \setminus E_{n+1}(\eta^{(n+1)})$ does not occur.

Now we show that for all $n \ge 1$, there is a $\eta^{(n)}$ such that $E_n(\eta^{(n)})$ occurs. Consider

$$\sum_{\eta^{(n)}} X_n(\eta^{(n)}) = \sum_{\eta^{(n)}} \sum_{v \in T_n} (2+\varepsilon)^n \theta(v) \mathbb{1}(o \xleftrightarrow{\eta^{(n)}} v) = \sum_{v \in T_n} \sum_{\eta^{(n)}} (2+\varepsilon)^n \theta(v) \mathbb{1}(o \xleftrightarrow{\eta^{(n)}} v)$$

But $\sum_{\eta^{(n)}} \mathbb{1}(v \xleftrightarrow{\eta^{(n)}} w) = 1$. So,

$$\sum_{\eta^{(n)}} X_n(\eta^{(n)}) = \sum_{v \in T_n} (2+\varepsilon)^n \theta(v) = \delta(2+\varepsilon)^n$$

If $X_n(\eta^{(n)}) < \delta$ for all $\eta^{(n)}$, then the LHS $< \delta 2^n$, which is a contradiction. Thus there must exists a $\eta^{(n)}$ such that $E_n(\eta^{(n)})$ occurs.

Say $n > N$, then by the above observation $E_{n+m}(\eta^{(n)}, \xi^{(m)})$ must also occur for any ξ, thus all words are seen from the generation n.

□

Now we show that the sum under consideration is indeed finite, in the proof we use McDiarmid's inequality. We start by stating it:

Theorem 5. *(McDiarmid's Inequality) Let $X_1, \ldots, X_n$ be independent random variables where X_i is $\mathcal{X}_i$-valued for all i, and let $X = (X_1, \ldots, X_n)$. Assume $f : \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathbb{R}$ is a measurable function such that $\|D_i f\|_\infty < +\infty$ for all i.*
Then for all $\beta > 0$,

$$\mathbb{P}[f(X) - \mathbb{E}[f(X)] \leq -\beta] \leq \exp\left(-\frac{2\beta^2}{\sum_{i\leq n} \|D_i f\|_\infty^2}\right).$$

Here, for $x = (x_1, x_2, \cdots x_n)$ the definition of $D_i f$ is:

$$D_i f(x) := \sup_{y\in\mathcal{X}_i} f(x_1, x_2, \ldots, y, x_{i+1}, \ldots, x_n) - \sup_{y\in\mathcal{X}_i} f(x_1, x_2, \ldots, y', x_{i+1}, \ldots, x_n).$$

Lemma 6. *There exists $\varepsilon' > 0$ such that $\mathbb{P}(E_{n+1}^{\mathsf{c}}|E_n) \leq \exp(-(1+\varepsilon')^n)$*

It is clear that this lemma implies the summability condition in Lemma 4, so proving Lemma 6 suffices.

Proof of Lemma 6. Fix a configuration on $\bigcup\limits_{i\leq n} T_i$ such that E_n occurs. We show the above bound conditioned on this configuration. More formally, let $\tilde{\mathbb{P}}$ be the conditional measure given $E_n, v_1, \cdots v_k$ where $v_1, v_2, \cdots v_k$ are the vertices in $T_n \cap \mathcal{C}$ such that $\sum\limits_{1\leq i\leq k} (2+\varepsilon)^n \theta(v_i) \geq \delta$.

Write $X_{n+1} = \sum\limits_{i\leq k} Y_n(v_i)$ where $Y_n(v_i) = \sum\limits_{\substack{|x|=n+1\\ x\sim v_i}} (2+\varepsilon)^{n+1}\theta(x)\mathbb{1}(v_i \leftrightarrow x)$.

Firstly, note that $\tilde{\mathbb{E}}(X_{n+1}) = \frac{(2+\varepsilon)^{n+1}}{2} \sum\limits_i \theta(v_i)$. Now,

Let $f(y_1, y_2, \ldots, y_k) = \sum_{i=1}^k y_i$ then,

$$\|D_i f\|_\infty = \sup_{\omega,\omega'} \left|Y_n(v_i)(\omega) - Y_n(v_i)(\omega')\right| \leq 2\|Y_n(v_i)\|_\infty$$

Now,

$$Y_n(v_i) \leq \sum_{\substack{|x|=n+1\\ x\sim v_i}} (2+\varepsilon)^{n+1}\theta(x) = (2+\varepsilon)^{n+1}\theta(v_i).$$

Thus,

$$\|D_i f\|_\infty \leq 2(2+\varepsilon)^{n+1}\theta(v_i).$$

So,

$$\sum_{i\leq k} \|D_i f\|_\infty^2 \leq 4(2+\varepsilon)^{2(n+1)} \sum_{i\leq k} \theta(v_i)^2 \leq \frac{4(2+\varepsilon)^{2n+2}}{(2+2\varepsilon)^n} \sum_{i\leq k} \theta(v_i)$$

$$= \frac{4\,(2+\varepsilon)^{n+1}}{(2+2\varepsilon)^n} \cdot 2\,\tilde{\mathbb{E}}(X_{n+1})$$

$$= \frac{8\,(2+\varepsilon)^{n+1}}{(2+2\varepsilon)^n}\,\tilde{\mathbb{E}}(X_{n+1})$$

For $\beta = \frac{\varepsilon}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})$, By McDiarmid,

$$\tilde{\mathbb{P}}\left(X_{n+1} \leq \frac{2}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})\right) \leq \exp\left(-\frac{\left(\frac{\varepsilon}{2+\varepsilon}\right)^2 \tilde{\mathbb{E}}(X_{n+1})\,(2+2\varepsilon)^n}{8\,(2+\varepsilon)^{n+1}}\right).$$

Since $\tilde{\mathbb{E}}(X_{n+1}) \geq \frac{\delta\,(2+\varepsilon)}{2}$,

$$\tilde{\mathbb{P}}\left(X_{n+1} \leq \frac{2}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})\right) \leq \exp\left(-\frac{1}{16}\left(\frac{\varepsilon}{2+\varepsilon}\right)^2 \delta\left(1+\frac{\varepsilon}{2+\varepsilon}\right)^n\right)$$

$$= \exp\left(-C\left(1+\frac{\varepsilon}{2+\varepsilon}\right)^n\right).$$

Note that,

$$\left\{X_{n+1} \geq \frac{2}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})\right\} \implies E_{n+1}$$

So,

$$\tilde{\mathbb{P}}(E_{n+1} \text{ does not occur}) \leq \tilde{\mathbb{P}}\left(X_{n+1} \leq \frac{2}{2+\varepsilon}\tilde{\mathbb{E}}(X_{n+1})\right) \leq \exp\left(-C\left(1+\varepsilon'\right)^n\right)$$

Where $\varepsilon' = \frac{\varepsilon}{2+\varepsilon}$ and we are done. $\square$

References