Aizenman-Kesten-Newman proof for uniqueness of infinite cluster

Ishaan Bhadoo

Abstract

We present Aizenman-Kesten-Newman's proof for uniqueness of infinite cluster following [GGR88]. Although the proof in their original paper is for $\mathbb{Z}^d$, it can be generalized to all transitive amenable graphs, we depict this generalization here.

1 Introduction

Usually, when encountering the theorem for the uniqueness of the infinite cluster, one sees it for $\mathbb{Z}^d$ and using the Burton-Keane argument [BK89] (See [LP17] for a modern account). This article depicts an alternative proof of uniqueness, which was given by Aizenman, Kesten, and Newman in 1987 [AKN87] (before Burton-Keane!) which provides an alternate perspective to uniqueness in percolation.

The proof language follows a modern style, and thus the notation differs. The paper follows [GGR88] simplification of the AKN proof for $\mathbb{Z}^d$. We complete the details in their paper and generalize it to the amenable setting.

We consider site percolation on an amenable, transitive graph G. Let $\mathbb{P}_p$ be the Bernoulli percolation measure on $\{0,1\}^{V(G)}$, with density $p \in [0,1]$. Let $\mathbb{E}_p$ be the expectation with respect to this measure. Further, let N_∞ denote the number of infinite clusters.

Recall that amenability captures the lattice-like behavior of a transitive graph. More formally we have:

Definition 1. *We call a graph G amenable if there exists a sequence*[1] $\{B_n\}$ *such that*

$$\limsup_{n\to\infty} \frac{|\partial^{int} B_n|}{|B_n|} = 0$$

where $\partial^{int} S = \{x \in S : \{x,y\} \in E, y \notin S\}$

We want to show the following theorem:

Theorem 2. *Let $p \in [0,1]$, consider site percolation on a transitive amenable graph such that $\mathbb{P}_p(N_\infty \geq 1)$. Then there exists almost surely a unique infinite open cluster.*

Before diving into the proof, we set up some more notation. Let B_n denote the Folner sequence in our amenable graph. It is important to notice that $\limsup_{n\to\infty} |B_n| = \infty$, if not then there exists a M such that $|B_n| \leq M \implies \frac{|\partial^{\text{int}} B_n|}{|B_n|} \geq \frac{1}{M} > 0$, which contradicts amenability. Therefore we can treat $|B_n|$ as an increasing sequence going to ∞.

We consider clusters contained in a finite set and later on take the limit. For $x \in B_n$, let $C_n(x)$ be the cluster of x in B_n. Further, let $\tilde{\mathcal{C}}_n$ be the set of all clusters in B_n (i.e clusters inside B_n).

Let $\mathcal{C}_n \subseteq \tilde{\mathcal{C}}_n$ be the set of all clusters in B_n intersecting the internal boundary $\partial_{\text{int}} B_n$ of B_n.

[1] This sequence is often refered to as the Folner sequence

Finally, let:

$$F_n(\omega) = \bigcup_{C \in \mathcal{C}_n} C \text{ and } G_n(\omega) = \bigcup_{C \in \mathcal{C}_n} \partial_{\text{ext}} C \cap B_n.$$

$$H_n(\omega) = \bigcup_{\substack{C_1, C_2 \in \mathcal{C}_n \\ C_1 \neq C_2}} \{\{\partial_{\text{ext}} C_1 \cap B_n\} \cap \{\partial_{\text{ext}} C_2 \cap B_n\}\}$$

where for a set S, $\partial_{\text{ext}} S := \{x \in V(G) : x \sim S, x \notin S\}$. Here F_n is the set of all open points in B_n. G_n is the set of all closed points adjacent to a cluster in B_n. H_n is the set of all points adjacent to two different clusters in B_n.

Define the event, $L_x = \{x$ belongs to the external boundary of two or more infinite open clusters of $G\}$.

Then it is easy to see that:

Lemma 3. *For $H_n(\omega)$, L_x as above, we have:*

$$\lim_{n \to \infty} \frac{1}{|B_n|} \mathbb{E}_p\big[|H_n|\big] = \mathbb{P}_p(L_x).$$

We also have the following lemma,

Lemma 4. *Suppose $\mathbb{P}_p(L_x) = 0$. Then the number of infinite clusters, N_∞, is ≤ 1 almost surely.*

Proof of Lemma 4. By Newman-Schulman, we know that if $N_\infty > 1$ then $N_\infty = \infty$. Thus for any x we can find a r such that $B_r(x)$ intersects at least 2 infinite clusters, and then it is easy to see by insertion-deletion tolerance that $\mathbb{P}_p(L_x) > 0$. $\square$

From Lemma 3 and Lemma 4, it suffices to show that

$$\lim_{n \to \infty} \frac{1}{|B_n|} \mathbb{E}_p\big[|H_n|\big] \leq 0. \tag{1}$$

Firstly, it is easy to see that:

$$|H_n(\omega)| \leq \left(\sum_{C \in \mathcal{C}_n} |\partial_{\text{ext}} C| \right) - |G_n(\omega)| \tag{2}$$

Also, if $x \notin \partial_{\text{int}} B_n$ we have $\mathbb{P}_p\big(x \in F_n \mid x \in F_n \cup G_n\big) = p$, since $\{x \in F_n \cup G_n\}$ = event that x is connected to $\partial_{\text{int}} B_n$.

Thus,

$$\mathbb{E}_p \left(\sum_{C \in \mathcal{C}_n} |C| \right) = \mathbb{E}_p(|F_n|) = \sum_{x \in B_n} \mathbb{P}_p(x \in F_n)$$

$$= \sum_{x \in B_n} \mathbb{P}_p\big(x \in F_n \mid x \in G_n \cup F_n\big) \mathbb{P}_p\big(x \in G_n \cup F_n\big)$$

$$= |\partial^{\text{int}} B_n| + \sum_{x \in B_n \setminus \partial^{\text{int}} B_n} p\big[\mathbb{P}_p(x \in G_n) + \mathbb{P}_p(x \in F_n)\big]$$

Therefore we have,

$$\mathbb{E}_p(|F_n|) = |\partial^{\text{int}} B_n| + p\mathbb{E}_p(|G_n|) + p\mathbb{E}_p(|F_n|)$$

Which gives,

$$\mathbb{E}_p(|F_n|) = \frac{p}{1-p}\mathbb{E}_p(|G_n|) + \frac{1}{1-p}|\partial^{\text{int}} B_n| \tag{3}$$

Thus by (2), (3),

$$\frac{\mathbb{E}_p(|H_n|)}{|B_n|} \leq \mathbb{E}_p\left(\sum_{C\in\mathcal{C}_n}\left\{|\partial_{\text{ext}}C| - \frac{(1-p)}{p}|C|\right\}\right) + \frac{1}{p}\frac{|\partial^{\text{int}}(B_n)|}{|B_n|}.$$

Now let

$$h(C(x)) = |\partial_{\text{ext}}C(x)| - \frac{(1-p)|C(x)|}{p},$$

and

$$\overline{C}(x) = C(x) \cup \text{ext } C(x).$$

Then we have the following large deviation estimate (taken from [GGR88]),

Lemma 5.

$$\mathbb{P}_p\big(h(C(x)) \geq \varepsilon k, |\overline{C}(x)| = k\big) \leq e^{-k\varepsilon^2/4a}$$

where $a = a(p)$ is strictly positive on $(0,1)$.

Proof. Let $\mathrm{a}_{ml}(x)$ be the number of connected subgraphs of B_n containing x with m sites in their interior and l sites in their external boundary.

Then, $\forall\, r > 0$, by Chernoff's bound,

$$\mathbb{P}_p\big(h(C(x)) \geq \varepsilon k, |\overline{C}(x)| = k\big) \leq e^{-\varepsilon kr}\mathbb{E}_p\big(e^{rh(C(x))}, |\overline{C}(x)| = k\big)$$

$$= e^{-\varepsilon kr}\sum_{m+l=k} a_{ml}(x)\big(pe^{-rp^{-1}(1-p)}\big)^m\big((1-p)e^{-r}\big)^l \leq e^{-\varepsilon kr}\big(f(r,p)\big)^k\mathbb{P}_{\tilde{p}}\big(|\overline{C}(x)| = k\big),$$

where

$$f(r,p) = pe^{-r}p^{-1}(1-p) + (1-p)e^{-r},$$

and

$$\tilde{p} = \frac{pe^{-rp^{-1}(1-p)}}{f(r,p)} < 1.$$

It is easy to see that,

$$f(r,p) = 1 + O(r^2), \text{ as } r \to 0$$

Thus,

$$\mathbb{P}_p\big(h(C(x)) \geq \varepsilon k \mid |\overline{C}(x)| = k\big) \leq e^{-\varepsilon kr + akr^2},$$

for some $a = a(p) > 0$ on $(0,1)$. Now choose r such that Lemma 5 holds. □

Proof of Theorem 2: We now show (1), thereby implying Theorem 2.

Fix $\varepsilon > 0$ and choose δ such that $0 < \delta < \frac{2}{\deg G} \leq 1$.

Let

$$X_n(\omega) = \big\{C \in C_n(\omega) : |\overline{C}| > |B_n|^\delta\big\},$$

and

$$Y_n = \big\{h(C) \leq \varepsilon|C| \;\; \forall\, C \in X_n\big\}.$$

Since, when Y_n does **not** occur, $\exists\, x \in B_n$ such that: $h(C(x)) > \varepsilon|C(x)|$ and $|\overline{C}(x)| > |B_n|^\delta$. By Lemma 5 we have,

$$1 - \mathbb{P}_p(Y_n) \leq |B_n| \sum_{k=|B_n|^\delta}^{\infty} e^{-k\varepsilon^2/4a} \to 0 \quad \text{as } n \to \infty.$$

Thus by the tower law,

$$\frac{1}{|B_n|}\mathbb{E}_p\left(\sum_{C \in X_n}\left(|\partial_{\text{ext}}C| - \frac{1-p}{p}|C|\right)\right) \leq \frac{1}{|B_n|}\mathbb{E}_p\left(\sum_{C \in X_n}\varepsilon|\overline{C}|\right) + \frac{(1-\mathbb{P}_p(B_n)) \cdot 2|B_n|}{|B_n|}$$

The RHS goes to 0 as $n \to \infty$, $\varepsilon \to 0$.

For the other clusters, since $|C| \leq |B_n|^\delta$, $\forall C \in \mathcal{C}_n \setminus X_n$, it is easy to see that,

$$\frac{\sum_{C \in \mathcal{C}_n \setminus X_n}\left(|\partial_{\text{ext}}C| - \frac{1-p}{p}|C|\right)}{|B_n|} \to 0 \quad \text{as } n \to \infty.$$

Combining, we get:

$$\lim_{n\to\infty} \frac{1}{|B_n|}\mathbb{E}_p(|H_n|) = 0.$$

Thus, $\mathbb{P}_p(L_x) = 0$ which by our earlier comments implies that $N_\infty \leq 1$. $\square$

References

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