

# Aizenmann-Kesten-Newman proof for uniqueness of infinite cluster

Ishaan Bhadoo

## Abstract

We present Aizenman-Kesten-Newman's proof for uniqueness of infinite cluster following [GGR88]. Although the proof in their original paper is for  $\mathbb{Z}^d$ , it can be generalized to all transitive amenable graphs, we depict this generalization here.

## 1 Introduction

Usually, when encountering the theorem for the uniqueness of the infinite cluster, one sees it for  $\mathbb{Z}^d$  and using the Burton-Keane argument [BK89] (See [LP17] for a modern account). This article depicts an alternative proof of uniqueness, which was given by Aizenman, Kesten, and Newman in 1987 ([AKN87], before Burton-Keane) which provides an alternate perspective to uniqueness in percolation.

The proof language follows a modern style, and thus the notation differs. The paper follows [GGR88] simplification of the AKN proof for  $\mathbb{Z}^d$ . We complete the details in their paper and generalize it to the amenable setting.

We consider site percolation on an amenable, transitive graph  $G$ . Let  $\mathbb{P}_p$  be the Bernoulli percolation measure on  $\{0, 1\}^{V(G)}$ , with density  $p \in [0, 1]$ . Let  $\mathbb{E}_p$  be the expectation with respect to this measure. Further, let  $N_\infty$  denote the number of infinite clusters.

Recall that amenability captures the lattice-like behavior of a transitive graph. More formally we have:

**Definition 1.** We call a graph  $G$  amenable if there exists a sequence<sup>1</sup>  $\{B_n\}$  such that

$$\limsup_{n \rightarrow \infty} \frac{|\partial^{\text{int}} B_n|}{|B_n|} = 0$$

where  $\partial^{\text{int}} S = \{x \in S : \{x, y\} \in E, y \notin S\}$

We want to show the following theorem:

**Theorem 2.** Let  $p \in [0, 1]$ , consider site percolation on a transitive amenable graph such that  $\mathbb{P}_p(N_\infty \geq 1)$ . Then there exists almost surely a unique infinite open cluster.

Before diving into the proof, we set up some more notation. Let  $B_n$  denote the Folner sequence in our amenable graph. It is important to notice that  $\limsup_{n \rightarrow \infty} |B_n| = \infty$ , if not then there exists a  $M$  such that  $|B_n| \leq M \implies \frac{|\partial^{\text{int}} B_n|}{|B_n|} \geq \frac{1}{M} > 0$ , which contradicts amenability. Therefore we can treat  $|B_n|$  as an increasing sequence going to  $\infty$ .

We consider clusters contained in a finite set and later on take the limit. For  $x \in B_n$ , let  $C_n(x)$  be the cluster of  $x$  in  $B_n$ . Further, let  $\tilde{C}_n$  be the set of all clusters in  $B_n$  (i.e clusters inside  $B_n$ ).

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<sup>1</sup>This sequence is often referred to as the Folner sequence

Let  $\mathcal{C}_n \subseteq \tilde{\mathcal{C}}_n$  be the set of all clusters in  $B_n$  intersecting the internal boundary  $\partial_{\text{int}} B_n$  of  $B_n$ . Finally, let:

$$F_n(\omega) = \bigcup_{C \in \mathcal{C}_n} C \text{ and } G_n(\omega) = \bigcup_{C \in \mathcal{C}_n} \partial_{\text{ext}} C \cap B_n.$$

$$H_n(\omega) = \bigcup_{\substack{C_1, C_2 \in \mathcal{C}_n \\ C_1 \neq C_2}} \{ \{ \partial_{\text{ext}} C_1 \cap B_n \} \cap \{ \partial_{\text{ext}} C_2 \cap B_n \} \}$$

where for a set  $S$ ,  $\partial_{\text{ext}} S := \{x \in V(G) : x \sim S, x \notin S\}$ . Here  $F_n$  is the set of all open points in  $B_n$ .  $G_n$  is the set of all closed points adjacent to a cluster in  $B_n$ .  $H_n$  is the set of all points adjacent to two different clusters in  $B_n$ .

Define the event,  $L_x = \{x \text{ belongs to the external boundary of two or more infinite open clusters of } G\}$ . Then it is easy to see that:

**Lemma 3.** *For  $H_n(\omega)$ ,  $L_x$  as above, we have:*

$$\lim_{n \rightarrow \infty} \frac{1}{|B_n|} \mathbb{E}_p[|H_n|] = \mathbb{P}_p(L_x).$$

We also have the following lemma,

**Lemma 4.** ([AKN87]) *Suppose  $\mathbb{P}_p(L_x) = 0$ . Then the number of infinite clusters,  $N_\infty$ , is  $\leq 1$  almost surely.*

From Lemma 3 and Lemma 4, it suffices to show that

$$\lim_{n \rightarrow \infty} \frac{1}{|B_n|} \mathbb{E}_p[|H_n|] \leq 0. \quad (1)$$

Firstly, it is easy to see that:

$$|H_n(\omega)| \leq \left( \sum_{C \in \mathcal{C}_n} |\partial_{\text{ext}} C| \right) - |G_n(\omega)| \quad (2)$$

Also, if  $x \notin \partial_{\text{int}} B_n$  we have  $\mathbb{P}_p(x \in F_n \mid x \in F_n \cup G_n) = p$ , since  $\{x \in F_n \cup G_n\} = \text{event that } x \text{ is connected to } \partial_{\text{int}} B_n$ .

Thus,

$$\begin{aligned} \mathbb{E}_p \left( \sum_{C \in \mathcal{C}_n} |C| \right) &= \mathbb{E}_p(|F_n|) = \sum_{x \in B_n} \mathbb{P}_p(x \in F_n) \\ &= \sum_{x \in B_n} \mathbb{P}_p(x \in F_n \mid x \in G_n \cup F_n) \mathbb{P}_p(x \in G_n \cup F_n) \\ &= |\partial^{\text{int}} B_n| + \sum_{x \in B_n \setminus \partial^{\text{int}} B_n} p [\mathbb{P}_p(x \in G_n) + \mathbb{P}_p(x \in F_n)] \end{aligned}$$

Therefore we have,

$$\mathbb{E}_p(|F_n|) = |\partial^{\text{int}} B_n| + p \mathbb{E}_p(|G_n|) + p \mathbb{E}_p(|F_n|)$$

Which gives,

$$\mathbb{E}_p(|F_n|) = \frac{p}{1-p} \mathbb{E}_p(|G_n|) + \frac{1}{1-p} |\partial^{\text{int}} B_n| \quad (3)$$

Since  $\frac{|\partial_{\text{int}} B_n|}{|B_n|} \rightarrow 0$  we have, Thus by (2), (3),

$$\frac{\mathbb{E}_p(|H_n|)}{|B_n|} \leq \mathbb{E}_p \left( \sum_{C \in \mathcal{C}_n} \left\{ |\partial_{\text{ext}} C| - \frac{(1-p)}{p} |C| \right\} \right) + \frac{1}{p} \frac{|\partial^{\text{int}}(B_n)|}{|B_n|}.$$

Now let

$$h(C(x)) = |\partial_{\text{ext}} C(x)| - \frac{(1-p)|C(x)|}{p},$$

and

$$\overline{C}(x) = C(x) \cup \text{ext } C(x).$$

Then we have the following large deviation estimate,

**Lemma 5.**

$$\mathbb{P}_p(h(C(x)) \geq \varepsilon k, |\overline{C}(x)| = k) \leq e^{-k\varepsilon^2/4a},$$

where  $a = a(p)$  is strictly positive on  $(0, 1)$ .

*Proof.* Let  $a_{ml}(x)$  be the number of connected subgraphs of  $B_n$  containing  $x$  with  $m$  sites in their interior and  $l$  sites in their external boundary.

Then,  $\forall r > 0$ , by Chernoff's bound,

$$\begin{aligned} \mathbb{P}_p(h(C(x)) \geq \varepsilon k, |\overline{C}(x)| = k) &\leq e^{-\varepsilon k r} \mathbb{E}_p(e^{r h(C(x))}, |\overline{C}(x)| = k) \\ &= e^{-\varepsilon k r} \sum_{m+l=k} a_{ml}(x) (p e^{-r p^{-1}(1-p)})^m ((1-p)e^{-r})^l, \\ &\leq e^{-\varepsilon k r} (f(r, p))^k \mathbb{P}_{\tilde{p}}(|\overline{C}(x)| = k), \end{aligned}$$

where

$$f(r, p) = p e^{-r} p^{-1}(1-p) + (1-p)e^{-r},$$

and

$$\tilde{p} = \frac{p e^{-r p^{-1}(1-p)}}{f(r, p)} < 1.$$

It is easy to see that,

$$f(r, p) = 1 + O(r^2), \text{ as } r \rightarrow 0$$

.

$$\mathbb{P}_p(h(C(x)) \geq \varepsilon k \mid |\overline{C}(x)| = k) \leq e^{-\varepsilon k r + a k r^2},$$

for some  $a = a(p) > 0$  on  $(0, 1)$ . Now choose  $r$  such that Lemma 5 holds. □

*Proof of Theorem 2:* We now show (1), thereby implying Theorem 2.

Fix  $\varepsilon > 0$  and choose  $\delta$  such that  $0 < \delta < \frac{2}{\deg G} \leq 1$ .

Let

$$X_n(\omega) = \{C \in \mathcal{C}_n(\omega) : |\overline{C}| > |B_n|^\delta\},$$

$$Y_n = \{h(C) \leq \varepsilon |C| \mid \forall C \in X_n\}.$$

Then from Lemma 5,

$$1 - \mathbb{P}_p(B_n) \leq |B_n| \sum_{k=|B_n|^\delta}^{\infty} e^{-k\varepsilon^2/4a} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now, when  $B_n$  does **not** occur,  $\exists x \in B_n$  such that:

$$h(C(x)) > \varepsilon |C(x)|.$$

and

$$|\overline{C}(x)| > |B_n|^\delta.$$

Thus,

$$\frac{1}{|B_n|} \mathbb{E}_p \left( \sum_{C \in X_n} \left( |\partial_{\text{ext}} C| - \frac{1-p}{p} |C| \right) \right)$$

(using the tower law)

$$\leq \frac{1}{|B_n|} \mathbb{E}_p \left( \sum_{C \in X_n} \varepsilon |\overline{C}| \right) + \frac{(1 - \mathbb{P}_p(B_n)) \cdot 2|B_n|}{|B_n|} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \varepsilon \rightarrow 0.$$

Finally,

$$\frac{\sum_{C \in \mathcal{C}_n \setminus X_n} \left( |\partial_{\text{ext}} C| - \frac{1-p}{p} |C| \right)}{|B_n|} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since

$$|C| \leq |B_n|^\delta, \forall C \in \mathcal{C}_n \setminus X_n,$$

Combining, we get:

$$\lim_{n \rightarrow \infty} \frac{1}{|B_n|} \mathbb{E}_p(|H_n|) = 0.$$

Thus,  $\mathbb{P}_p(L_x) = 0$  which by our earlier comments implies that  $N_\infty \leq 1$ .

## References

- [AKN87] Michael Aizenman, Harry Kesten, and Charles M Newman. Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Communications in Mathematical Physics*, 111(4):505–531, 1987.
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- [LP17] Russell Lyons and Yuval Peres. *Probability on trees and networks*, volume 42. Cambridge University Press, 2017.