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Assignment
Parameter Estimation

Q1) $x_1, x_2 \ldots x_n$ → Random sample ; size = n
Normal population → parameter mean = θ_1, Var = θ_2.
Find max. likelihood estimation of these 2 parameters

Sol →
$$pdf = f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \times e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\mu = \theta_1 ; \sigma^2 = \theta_2$$

$$f(x_1) = \frac{1}{\sqrt{2\pi\sigma^2}} \times e^{-\frac{(x_1-\mu)^2}{2\sigma^2}}$$

$$f(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \times e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$f(x_i) = \frac{1}{\sqrt{2\pi\theta_2}} \times e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

Likelihood function
$$\hookrightarrow L(\theta_1,\theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} \times e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1,\theta_2) = \prod_{i=1}^{n}(\theta_2)^{-1/2} \prod_{i=1}^{n}(2\pi)^{-\frac{1}{2}} \prod_{i=1}^{n} e^{\left(-\frac{(x_i-\theta_1)^2}{2\theta_2}\right)}$$

$$L(\theta_1,\theta_2) = (\theta_2)^{-n/2}(2\pi)^{-n/2} e^{\left(-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2\right)} \quad \text{—(1)}$$

Taking log both sides

$$z = \ln(L(\theta_1,\theta_2)) = -\frac{n}{2}\ln(\theta_2) - \frac{n}{2}\ln 2\pi - \frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2 \quad \text{—(2)}$$

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diff. ② w.r.t θ_1 and equate to 0.

$$\frac{dz}{d\theta_1} = \frac{d}{d\theta_1}\left(\frac{-1}{\theta_2 \times 2}\sum_{i=1}^{n}(x_i-\theta_1)^2\right)$$

$$0 = \frac{-1}{2\theta_2}\sum_{i=1}^{n}\frac{d}{d\theta_1}(x_i-\theta_1)^2$$

$$0 = \sum_{i=1}^{n} 2(x_i-\theta_1)\times(-1)$$

$$0 = -2\sum_{i=1}^{n}(x_i-\theta_1)$$

$$0 = \sum_{i=1}^{n} x_i - n\theta_1$$

$$\theta_1 = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

$$\boxed{\theta_{1MLE} = \text{Sample mean} = \bar{x}}$$

for θ_2

$$\ln(L(\theta_1,\theta_2)) = -\frac{n}{2}\ln(2\pi\theta_2) - \frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2$$

diff. above eq. w.r.t. θ_2 and equate to 0.

$$0 = -\frac{n}{2}\frac{d}{d\theta_2}(\ln(2\pi\theta_2)) - \frac{1}{2}\sum_{i=1}^{n}(x_i-\theta_1)^2 \times \frac{d}{d\theta_2}\times\left(\frac{1}{\theta_2}\right)$$

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$$0 = -\frac{n}{2} \times \frac{1}{2\pi\theta_2} \times 2\pi - \frac{1}{2}\sum_{i=1}^{n}(x_i-\theta_1)^2 \times -\frac{1}{\theta_2^2}$$

$$\frac{n}{2\theta_2} = \frac{1}{2}\sum_{i=1}^{n}(x_i-\theta_1)^2 \times \frac{1}{\theta_2^2}$$

$$\theta_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\theta_1)^2$$

Also $\theta_1 = \bar{x}$

$$\boxed{\theta_{2_{MLE}} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = s^2 = \text{Sample variance}}$$

Q2) $x_1, x_2, \ldots x_n \to$ random sample; $B(m,\theta)$ $\theta \to$ unknown
$m \to$ known +ve integer.
Compute value of θ using MLE

pdf of Binomial $= f(x) = {}^mC_x\, \theta^x (1-\theta)^{m-x}$

joint density fn.

$$L(x_1, x_2 \ldots x_n) = \prod_{i=1}^{n} {}^mC_{x_i}\, \theta^{x_i} (1-\theta)^{m-x_i}$$

$$= \prod_{i=1}^{n} {}^mC_{x_i}\, \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{\sum_{i=1}^{n}(m-x_i)}$$

Taking log

$$\ln L = \ln\left(\prod_{i=1}^{n} {}^mC_{x_i}\right) + \ln\theta^{\sum_{i=1}^{n} x_i} + \ln(1-\theta)^{\sum_{i=1}^{n}(m-x_i)}$$

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$$\ln L = \ln\left(\prod_{i=1}^{n} {}^{m}C_{x_i}\right) + \sum_{i=1}^{n} (\ln\theta) x_i + \sum_{i=1}^{n} (m-x_i)\ln(1-\theta)$$

diff. w.r.t. θ and equate to 0

$$0 = \frac{d}{d\theta}\left(\sum_{i=1}^{n} x_i \ln\theta\right) + \sum_{i=1}^{n}(m-x_i)\frac{d}{d\theta}(\ln(1-\theta))$$

$$0 = \sum_{i=1}^{n} x_i \times \frac{1}{\theta} + \sum_{i=1}^{n}(m-x_i) \times \frac{-1}{1-\theta}$$

$$\frac{1}{\theta}\sum_{i=1}^{n} x_i = \left(nm - \sum_{i=1}^{n} x_i\right)\left(\frac{1}{1-\theta}\right)$$

$$\frac{1-\theta}{\theta} = \frac{nm - \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i}$$

$$\frac{1}{\theta} = \frac{nm}{\sum_{i=1}^{n} x_i}$$

$$\theta = \frac{\sum_{i=1}^{n} x_i}{nm}$$

$$\boxed{\theta = \frac{\bar{X}}{m}} \qquad \left[\bar{X} = \frac{\sum x_i}{n}\right]$$

$$\boxed{\theta_{MLE} \in (0,1) = \frac{\bar{X}}{m}}$$