31/10/18

2 FFT Algorithms — DIT FFT / DIF FFT — are useful in reducing the complexity from $\{N^2\}$ to $\{\frac{N}{2} N \log_2 N\}$ For computing a N point DFT

exact terms

* where N is a power of 2 i.e $N = 2^x$ and x is any integer

(This assumption is also the limitation)

in O() representation

$$O(N^2) \longrightarrow O(N \log_2 N)$$

1 $\underbrace{2 \quad 4}_{1}$ 8 16 $\underbrace{32 \quad 64}_{32}$ 128 256 512

As the numbers are increasing the spacing b/w them is also increasing, which means now there are more numbers For which we cannot find DFT using these methods.

For ex: B/w 256 and 512, there are 256 numbers for which the assumption is not satisfied.

To use this algorithm at any cost and no restriction on input size.

Solution 1
→ Modulo

$x(0)$, $x(1)$ and $x(2)$. Apply Modulo operator, we will get a length ~~xxxx~~ = 2. $x(2)$ will get added to $x(0)$

Solution 2:

Add zeros to make a 3 point DFT into a 4 point DFT.

This is a better option.

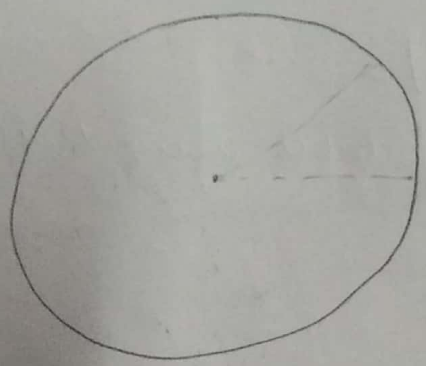
Scanned by CamScanner

Suppose we want to have a 17 point DFT then we will compute 32 point DFT. We will have to see which is a bigger module 17^2 or $16 \log_2 32$.

Error in the worst case scenario, using this algorithm is a useful option.

We need to go the upper band ceil 129—256. So this algorithm can be used as a default

Spectral Representation



DTFT
↓
Complete information

DFT
↳ Gives information about Specific N points

$-\pi \leq w \leq \pi$

When w is a continuous variable from $-\pi$ to π then it is a DTFT.

If we compute N point DFT then we divide the circle into N equal parts.

Scanned by CamScanner

Spectral component of the input signal on that particular angles.

$N \to \infty$ DFT is equal to DTFT

In a continuous signal we are looking for the number of cycles per second.

Sampling Frequency = f_s. This is for the signals of continuous type

f_s = Number of samples / sec

The connection point is Nyquist theorem.

This is a connection point b/w a continuous time signal to a discrete time signal.

Input cannot have more than $f_s/2$.

If the input is more than $f_s/2$. Then we get aliasing in the spectral domain.

Nyquist Interval :

$$-f_s/2 \qquad\qquad f_s/2$$

The highest frequency representable using Nyquist interval is $f_s/2$

$$-1/2 \qquad 0 \qquad 1/2 \quad \text{cycles / samples}$$

cycle = 2π radians

$$-2\pi/2 \qquad\qquad 2\pi \times 1/2 \quad \text{radians / sample}$$

$$-\pi \qquad\qquad \pi$$

This is the scale we are operating on

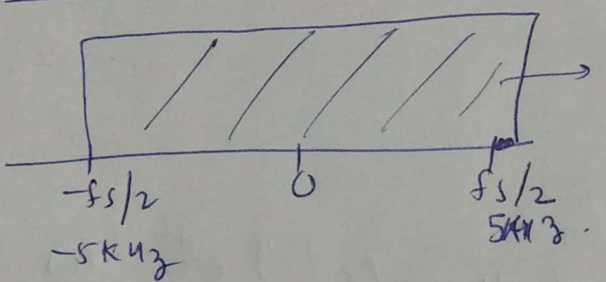
Using Nyquist Sampling theorem can be connected to digital.

In f_s is increased in analog, representable frequency will increase.

Digital representation is a fixed representation with respect to fixed f_s. If f_s is changed, there will be a different representation, They will map to different frequencies

f_s - sampling frequency

Nyquist <u>Interval</u>



All the frequencies should be passed.

All the frequencies after 5KHz should be blocked

$$f_s = 10KHz$$

~~If f_s = 20KHz~~

~~Reverse symmetric spectrum~~ because

w scale ranging from $-\frac{\pi f}{f_s}$ to $\frac{\pi f}{f_s}$

Low Pass Filter - It should pass all the frequencies.

Ideal low pass filter $\omega_c = \pi/5$ radian/samp

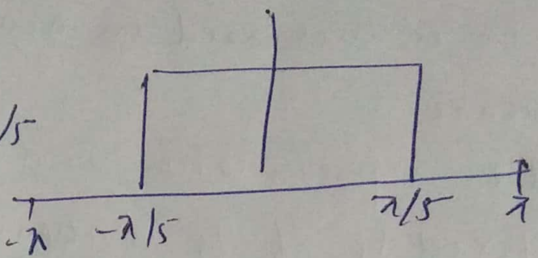
In terms of analog frequencies $f_c = 1KHz$ with $f_s = 10KHz$

$$\omega = \frac{2\pi f}{f_s}$$ → This is the conversion to get back to the digital scale.

$H(e^{j\omega}) = 1 \qquad -\frac{\pi}{5} \le \omega \le \frac{\pi}{5}$

$|H(e^{j\omega})| = 1 \qquad -\pi/5 \le \omega \le \pi/5$

$\quad\quad = 0 \qquad$ otherwise



$\angle H(e^{j\omega}) = 0 \qquad -\pi < \omega < \pi$

Zero phase response

An impulse response can be implemented using

→ Non Recursive FIR

→ Recursive IIR

Applying Inverse DTFT

$$h_{LPF}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LPF}(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$\frac{1}{2\pi}\int_{-\pi/5}^{\pi/5} 1 \cdot e^{j\omega n}\, d\omega = \left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/5}^{\pi/5}$$

Generalizing : For any general ω_c $\quad \left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c}$

Scanned by CamScanner

$$\frac{1}{2\pi}\left[\frac{e^{j\frac{\pi}{5}n}}{jn} - \frac{e^{-j\pi/5 n}}{jn}\right]$$

$$= \frac{1}{n}\sin\left(\frac{\pi}{5}n\right) = \frac{\sin\left(\pi/5\, n\right)}{n}$$

$$\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi}\left[\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right]$$

$$= \frac{\sin\omega_c n}{\pi n}$$

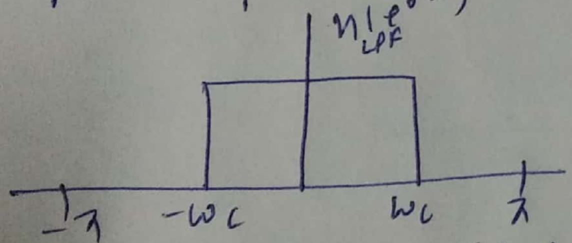
where ω_c is any frequency ranging from 0 to π

$h_{LPF}(n) = \frac{\sin\omega_c n}{\pi n}$ This function is not valid for $n=0$

Infinite Impulse Response $h_{LPF}(0) = \frac{\omega_c}{\pi}$ $n=0$

Perfect non causal system

phase response should be zero



We are calling this as ideal filter because there is a sharp gain.

Gain can be any value b/w 0 and 1. There is only one such point of uncertainity ω_c.

This filter has zero phase response.

If zero phase response then there is no need to wait for the output.

5/11/12

To build:

$\boxed{\text{N-length}}$ FIR System (LowPass Filter with cut off w_c)

It should form a causal system

Ideal System gives us IIR and noncausal system

Now, we have to leave some information to get FIR and causal system.

There can be infinite methods to convert IIR into FIR i.e ~~it~~ is the conversion is not unique.

We need to provide some extra information to ~~do~~ make it unique.

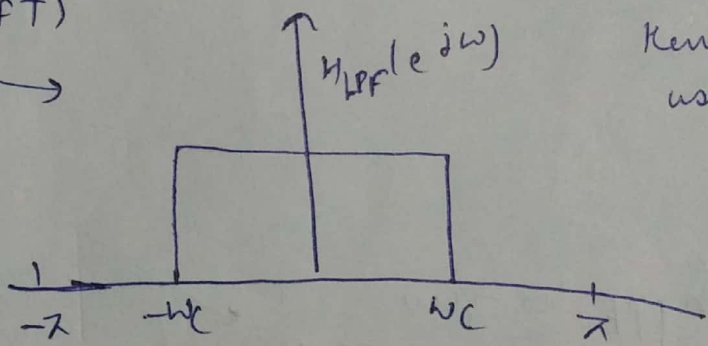
Evaluate each method and define a cost function.

~~of~~ cost function → minimum (unique)

→ maximum (unique)

How to determine the cost function?

Ideal LPF Filter (DTFT)



since it is a continous function Hence we are using DTFT not DFT

After converting it into FIR, find its DTFT coefficients and compute

$$\int_{-\pi}^{\pi} \left| H_{LPF}(e^{jw}) - \hat{H}_{LPF}(e^{jw}) \right|^2 . dw$$

→ error (Least Square error)

→ defined in transform domain

The one that gives the least solution is the best solution.

This cost function is the difference energy function

Scanned by CamScanner

By using Parseval's Relationship, converting the previous equation into time domain

$$\sum_{n=-\infty}^{\infty} |h_{LPF}(n) - \hat{h}_{LPF}(n)|^2$$

The least value using both will be equal

If we take $-2 \le n \le 2$ ↳ one function is defined in this range ($\hat{h}_{LPF}$)

$$SE = \underbrace{\sum_{n=-\infty}^{-3} |h_{LPF}(n) - \hat{h}_{LPF}(n)|^2}_{} + \underbrace{\sum_{n=-2}^{2} |[h_{LPF}(n) - \hat{h}_{LPF}(n)]|^2}_{} + \underbrace{\sum_{n=3}^{\infty} |(h_{LPF}(n) - \hat{h}_{LPF}(n))|^2}_{}$$

SE → square error

other is going from $-\infty$ to ∞

$\hat{h}_{LPF}$ is zero within these ranges. Hence the cost is fixed

This sum plays the major role. We want this sum to be zero.
↳ Best case

This has variable cost

In this $-2 \le n \le 2$, the value of $\hat{h}_{LPF}$ should be equivalent to ideal filter

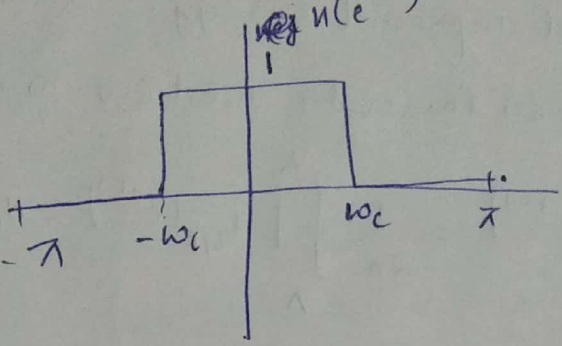
12
Scanned by CamScanner

Least Square Solution

$$\hat{h}_{LPF}(n) = h_{LPF}(n)$$

12/11/18

Ideal Frequency Response of LPF



$$H(e^{j\omega}) = 1 \qquad -\omega_c \le \omega \le \omega_c$$
$$= 0 \qquad \text{otherwise}$$

IDTFT $\quad \angle H(e^{j\omega}) = 0 \longrightarrow$ zero phase response is most appreciated systems and hence is ideal

$$h_{LPF}(n) = \frac{\sin \omega_c n}{\pi n} \qquad n \neq 0 \quad = h_{LPF}(-n)$$

↳ symmetric about y axis

(Infinite length)

$$= \frac{\omega_c}{\pi} \qquad n = 0$$

but we require ~~an~~ a N length FIR system

We can calculate the error in ① Time Domain
② Frequency Domain

These two representations have one thing equal which is the energy of the signal.

Now error signal $= \hat{h}_{LPF}(n) - h_{LPF}(n)$

does not represent energy.

We need to use Parseval's relation to equate the two

$$E_{LPF}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{H}_{LPF}(e^{j\omega}) - H_{LPF}(e^{j\omega})|^2 \, d\omega$$

$$e^2_{LPF}(n) = \sum_{-\infty}^{\infty} \left[\hat{h}_{LPF}(n) - h_{LPF}(n)\right]^2 \quad -①$$

Now we need to find the value of n for which e^2_{LPF} is minimum.

we can divide eq① into 3 parts

$$e^2_{LPF}(n) = \underbrace{\sum_{-\infty}^{(N/2-1)} (\;)}_{\mathrm{I}} + \underbrace{\sum_{-N/2}^{N/2} (\;)}_{\mathrm{II}} + \underbrace{\sum_{N/2+1}^{\infty} (\;)}_{\mathrm{III}}$$

but $\hat{h}_{LPF}(n)$ is defined only in range $\rightarrow -N/2 \le n \le N/2$

which makes Ist and IIIrd term to be of fixed cost and independent of $\hat{h}_{LPF}(n)$.

Now $e^2_{LPF}(n)$ will be minimum when IInd term is zero

which is possible when $\boxed{\hat{h}_{LPF}(n) = h_{LPF}(n)}$

This solution is still not unique as we can take any value of N

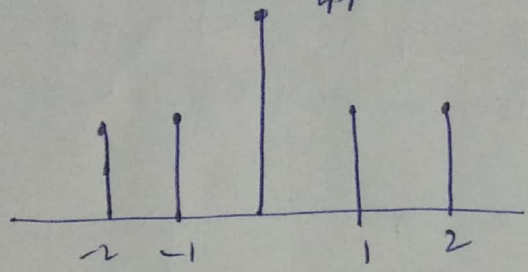
$\underbrace{-N/2 \le n \le N/2}$

by choosing this range we as it gives us symmetry about y axis.

Scanned by CamScanner

If we don't mention symmeticity about y axis then we will not achieve zero phase response.

Example
For $N = 5$



Now, the resultant system is non-causal (Finite non causality)
but we required a causal system.

Causal System $\rightarrow \hat{h}_{LPF}(n-2)$
phase will be changed $e^{-j2\omega}$

$\hat{h}_{LPF}(n) \xleftrightarrow{DTFT} \hat{H}_{LPF}(e^{j\omega})$

$\hat{h}_{LPF}(n-2) \xleftrightarrow{DTFT} e^{-j2\omega}\hat{H}_{LPF}(e^{j\omega})$ — Ist conclusion

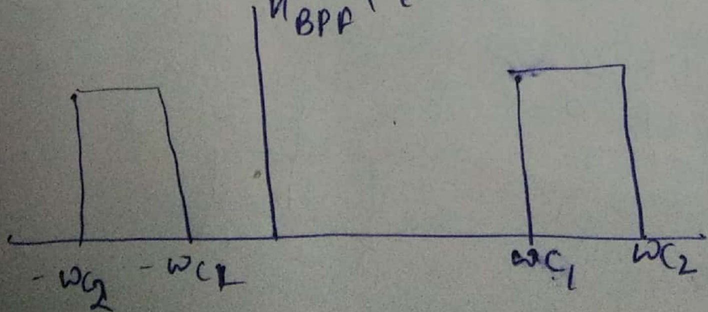
phase will be $\underbrace{0}_{\text{Initial phase}} + \underbrace{2\omega}_{\text{new}}$ $\rightarrow$ linear phase (this is implementable)

this will preserve the envelope of the signal.

$\boxed{\text{II}^{nd} \rightarrow \text{Our device was best in terms of least square method}}$

Result $\rightarrow$ Linear Phase N-length FIR lowpass filter

Derive for HPF, BPF, Band Stop Filter



Scanned by CamScanner

14/11/18

Format of Project Report

i) First page should include Title & Group member details

ii) Paper

iii) Individual Understanding (comprehension) of paper

iv) Implementation — Program (with sufficient comment)
— Results

v) Conclusion of Results

More weightage

---

① We for any impulse response, if it is symmetric then it has a linear phase in frequency domain

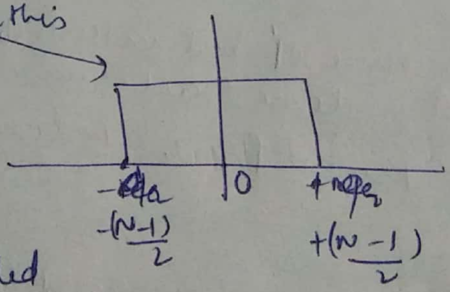
② The final frequency response of $\hat{h}_{LPF}(n)$

Two possible Symmetry → Even Symmetry
→ Odd symmetry

✲ N must be odd For this

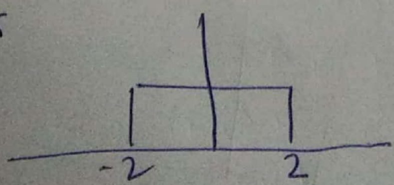
because when we plot from $-N/2$ to $N/2$

0 is also included which makes N odd.



Where N is the length of the system

For ex For $N=5$



Scanned by CamScanner

comes through the coefficients

Length of FIR system — even / Odd

Small number
comes through roots

Order of FIR System — even / Odd

FIR system can be represented in terms of polynomial of z.

No. of coefficients > No. of roots

For n^{th} order system

No. of coefficients $\rightarrow n+1$

roots $\rightarrow n$

Two possible combination

~~System~~ Symmetry & Order

Good way of representing

It has 1 dimension less

Symmetry & Length

N^{th} Order FIR SYSTEM (Low pass)

$N \rightarrow$ even
$\quad \rightarrow$ Odd

Symmetry $\rightarrow$ even   ex. (LPF, BPF, HPF)
$\quad \rightarrow$ Odd

4 possible combinations

Case 1: Even Order Even Symmetry
Case 2: Odd Order Even Symmetry
Case 3: Even order Odd Symmetry
Case 4: Odd order Odd symmetry

results in $\rightarrow$ ① Non causal zero phase
$\rightarrow$ ② Causal linear phase

Case 1: Even Order Even Symmetry // (Odd length Even Symmetry)
N represents Order here Not length

For Even Symmetry

$$h(n) = h(-n)$$

$N=4$ (length = 5)

$$h(n) = \{h(0), h(1), h(2), h(3), h(4)\}$$

$h(0)$: Index of Causal System starts from $n=0$

$h(2)$: Symmetry point

$H(z) \leq h(0)$

$$h(n) = h(0)\delta(n) + h(1)\delta(n-1) + h(2)\delta(n-2) + h(3)\delta(n-3) + h(4)\delta(n-4)$$

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}$$

$$= h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + h(3)e^{-j3\omega} + h(4)e^{-j4\omega}$$

Frequency Response $\leftarrow \left[H(z) \right]_{z=e^{j\omega}}$

For even symmetry about $h(2)$

$$h(1) = h(3)$$

$$h(0) = h(4)$$

Scanned by CamScanner

$$H(e^{j\omega}) = e^{-j2\omega}\left[h(0)e^{j2\omega} + h(1)e^{j\omega} + h(2) + h(3)e^{-j\omega} + h(4)e^{-j2\omega}\right]$$

$$= e^{-j2\omega}\left[h(0)e^{j2\omega} + h(1)e^{j\omega} + h(2) + h(1)e^{-j\omega} + h(0)e^{-j2\omega}\right]$$

$$= e^{-j2\omega}\left[2h(0)\left(\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right) + h(1)\left(e^{j\omega} + e^{-j\omega}\right) + h(2)\right]$$

$$H(e^{j\omega}) = e^{-j2\omega}\underbrace{\left[2h(0)\cos 2\omega + 2h(1)\cos\omega + h(2)\right]}_{\text{magnitude part}}$$

Phase part
which has linear phase
-2ω

$$\boxed{H(e^{j\omega}) = e^{-j\frac{N}{2}\omega}\left[h\left(\frac{N}{2}\right) + 2\sum_{n=1}^{N/2} h\left(\frac{N}{2} - n\right)\cos n\omega\right]}$$

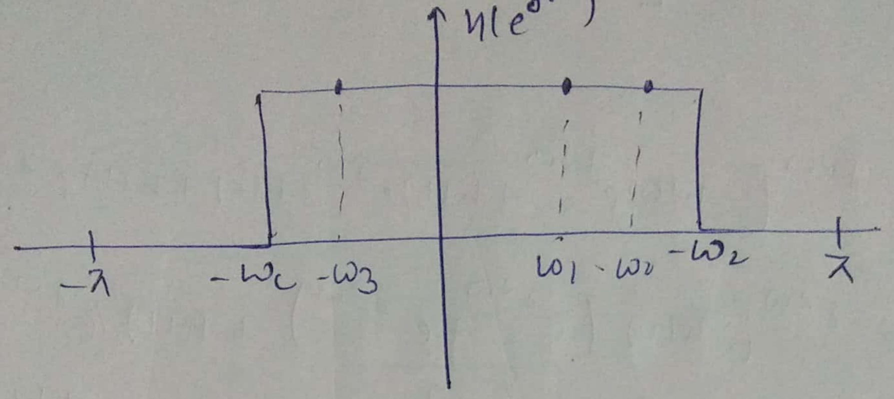
If we know the frequency response, by using the above equation we can find out the impulse response scale factors in time domain ($h(\frac{N}{2})$ etc)

$$H(e^{j\omega}) = e^{-j2\omega}\left[2h(0)\cos 2\omega + 2h(1)\cos\omega + h(2)\right]$$

we need to find $h(0)$, $h(1)$, $h(2)$

Scanned by CamScanner

For a LPF, the frequency response we have is



— ①

we have 3 unknowns ($h(0), h(1), h(2)$) , then minimum
three equations are required .

But, we have more than 3 equations here.
which means more equations , less number of unknowns.
Hence, there can be many solutions .

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(1)z^{-3} + h(0)z^{-4}$$

Order of polynomial = 4

Degree of freedom = No. of coefficients = 2 ($h(0), h(1)$)

We are designing a N order polynomial with
$\frac{N}{2}$ degree of freedom.

This is the limitation of this technique.
This is the cost we are paying for getting the FIR
LPF filter linear phase .

If we remove the linear phase condition , we will get
N order polynomial with N degree of freedom
but that will increase the no. of unknowns too.

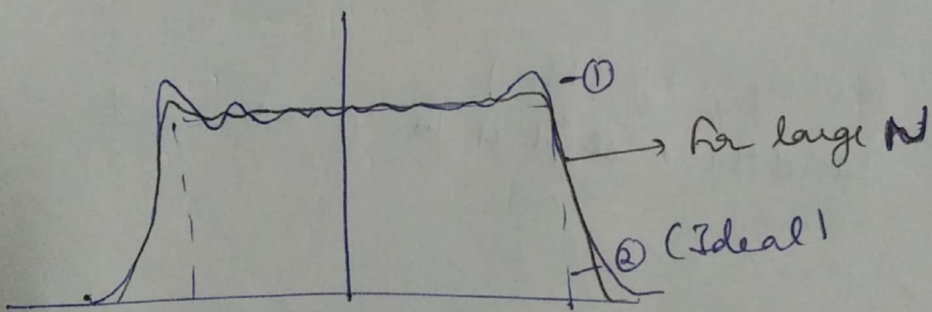
Scanned by CamScanner

Let us assume three frequencies w_1, w_2, w_3 on Graph ①
Using these we will get the ~~coep~~ coefficient which might not have minimum error.
In order to get minimum error, we should use least square method using all the three points available.

For least square

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega}) - \hat{H}(e^{j\omega})|^2 \, d\omega$$

For large N, the estimated ~~step~~ impulse response will have more coefficients matching with the ideal response, then it will be more close to ideal behaviour.



~~24/2/16~~

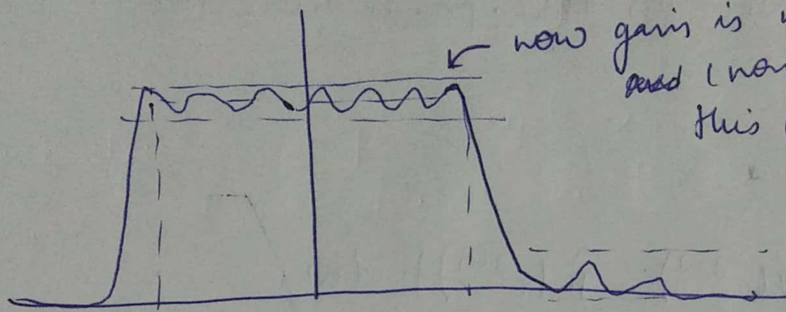
But even for $N \to \infty$, there will be some error, because of the discontinuity at two places.
Error can never be zero.

Scanned by CamScanner

Impulse Response with odd Symmetry

① Differentiator (FM Modulation / Demodulation)

② Hilbert Transform (SSB)



To get a Finite Response

Use a Window

$w(n) = 1 \quad 0 \le n \le N-1$

$w(n) = 1 \quad -N/2 \le n \le N/2$

multiply by ($h(n)$)

and then shift it to $0 \le n \le N-1$

In Frequency domain, the one with least number of oscillation gives the best solution

Scanned by CamScanner

19/11/18

For $N=2$

no. of coefficients $= 3$

$\{h(0), h(1), h(2)\}$

for even symmetry $h(0) = h(2)$

$$H(z) = h(0)z^{0} + h(1)z^{-1} + h(2)z^{-2}$$

$$H(e^{j\omega}) = h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega}$$

$$= h(0) + h(1)e^{-j\omega} + h(0)e^{-j2\omega}$$

$$= h(0)(1 + e^{-j2\omega}) + h(1)e^{-j\omega}$$

$$H(z^{-1}) = h(0) + h(1)z + h(2)z^{2}$$

multiply by z^{-2}

$$z^{-2}(H(z^{-1})) = h(0)z^{-2} + h(1)z^{-1} + h(2)$$

compare this with $\quad h(2) = h(0)$

$$H(z) = h(0) + h(1)z^{-1} + h(0)z^{-2}$$

These two equations are equal

$$\boxed{H(z) = z^{-N} H(z^{-1})}$$

This is called ~~[illegible]~~ mirror Image polynomial.

$H(z) = z^{-N}H(z^{-1})$ implies if we have a zero at some z then we will also have ~~[illegible]~~ zero at some z^{-1}

Scanned by CamScanner

Ex: $H(z_1) = 0$

z_1

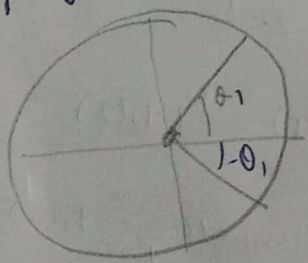
$z = z_1$ is the root of the polynomial

$\boxed{H\left(\frac{1}{z_1}\right) = 0}$

In order to make a linear phase systems, if we take a root z_1 as the other root. Then we should also take $\frac{1}{z_1}$

For a real polynomial → coefficients are real,
↳ roots are real

For a complex polynomial complex roots should be in conjugate pair. so that we get real coefficients



This will make the ~~coeffieno~~ polynomial real.

For a complex zero (root), we have total 4 zero

complex zero $(\alpha + \beta i)$ conjugate $(\alpha - \beta i)$

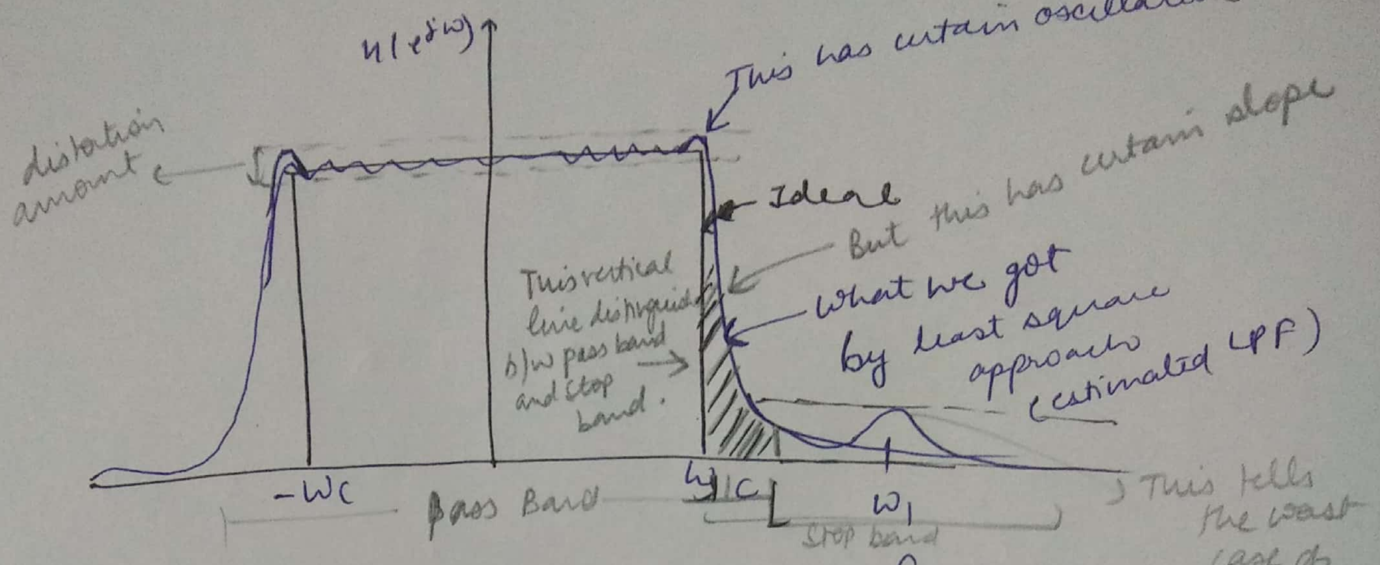
For symmetry and linear phase $\frac{1}{\alpha + \beta i}$

For symmetry and linear phase $\frac{1}{\alpha - \beta i}$

there, 4 ~~roots~~ zeros

Scanned by CamScanner

Ideal Lowpass filter



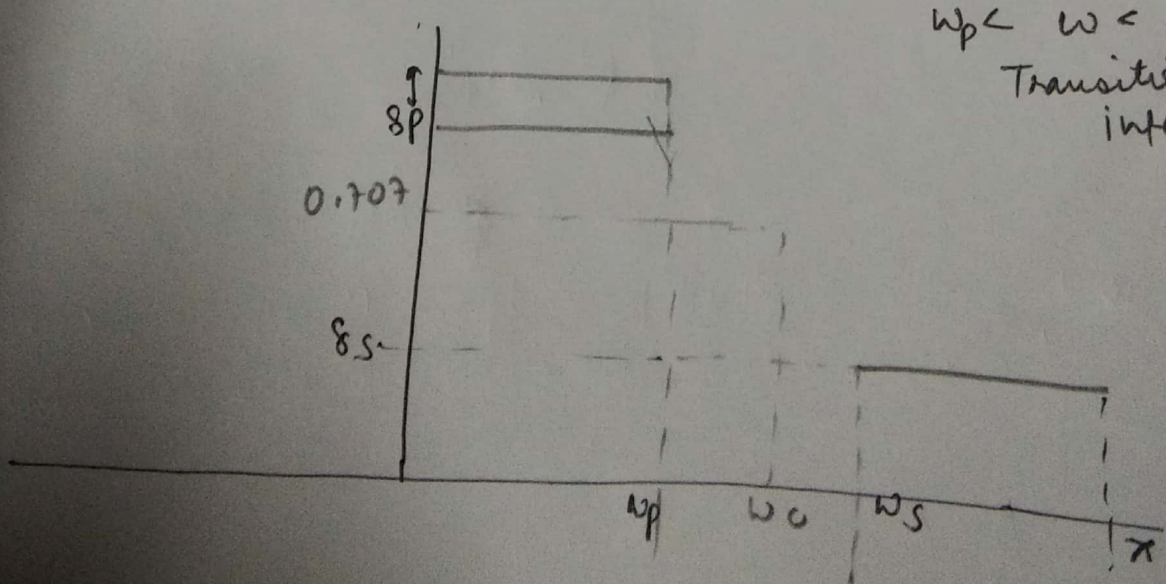
We will not be able to eliminate ω_1 frequency using our estimated LPF.

Highest gain ~~loop~~ is at the point of discontinuity

Highest gain in the ~~stop~~ band should be reduced to zero.

Highest gain in the pass band should be reduced to 1.

region marked like this → This region neither belongs to pass band nor stop band and is called transition region.



$\omega_p < \omega < \omega_s$
Transition interval