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Digital Signal Processing
EED

IEEE Signal Processing Magazines
↳ Implementation Difficulties
→ Describing concepts in a better way

Quizzes 10%
Lab 10% → completion of weekly lab
15% → lab exam
project - 10%

Midsem - 20%
Endsem - 35%

Midsem Topics covered in Class

→ DC Blocker

→ Introduction
→ System Modelling
→ Convolution
→ FIR
→ IIR
→ Block Convolution
→ Deconvolution
→ ~~[illegible]~~
→ Difference Equation
→ Cascaded and Parallel Forms
→ Similarity Measure → Auto correlation
Cross correlation
→ Z Transform
→ Eigen values and Eigen vectors
→ Frequency Response (DTFT)

Engineering ↔ Sets

Stegnography

Signals ↔ Sequence] → Discrete

Systems ↔ Functions

Voice (Speech) — Numbers — MP3 players — Numbers

Image, Video (Light Intensity variation) — "

Text (numbers) — "

Sensors o/p — "

Different types of signals converting into numbers will help in transmitting them together at the same time.

Numbers → Integers set → I

Real set R

Complex C

Real → 1 as 1.0 (write in decimals)

It is integers not decimal

Therefore signal is a sequence of numbers.

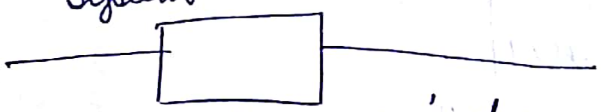
Order is important

default → Indexed number sequence → Integers (Discrete signal) $x[n]$
→ Real number (continuous signal) $x(t)$

We will use Integer indexed sequence.

System → manipulator

input $x[n]$



o/p $y[n]$

System is uniquely mapping using any manipulation

Mapping

Sample by Sample mapping

$x(0) \longrightarrow y(0)$

$x(1) \longrightarrow y(1)$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix} \longrightarrow \begin{bmatrix} y(0) \\ y(1) \\ y(2) \end{bmatrix}$$

Block by Block

$$x(n) \longrightarrow \boxed{} \longrightarrow y(n)$$

i/p $\qquad\qquad$ o/p

$y(n) = T(x(n))$

$\hookrightarrow$ Functional

One to one → reversible, you can get input from output

one to many → unidirectional

We will be looking at one to one, reversible mapping and linear systems

$T \rightarrow$ linear, $T \rightarrow$ non linear

Input sequence $(x(0), x(1), x(2))$

Output $(y(0), y(1), y(2), y(3))$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix}_{4\times1} \qquad \begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix}_{3\times1}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix}_{4\times1} = \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix}_{4\times3} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix}_{3\times1}$$

$\frac{10\times30}{10+30} = \frac{300}{40}$

57

$4\times1 \rightarrow$ output

$3\times1 \rightarrow$ input

Hence system $\rightarrow$ 4×3 matrix

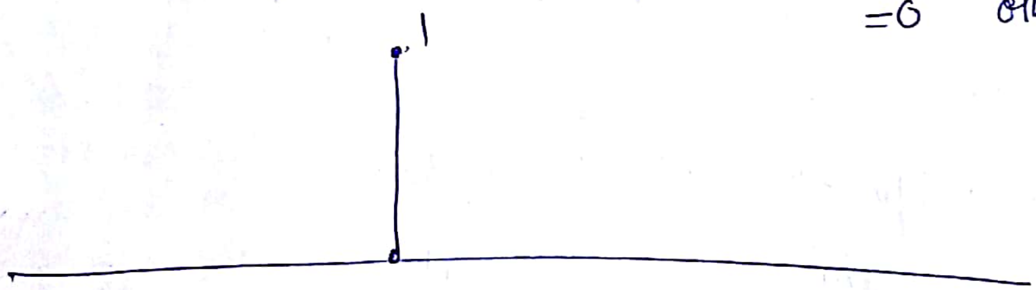
If we change the input to the system, we may or may not get the same system response.

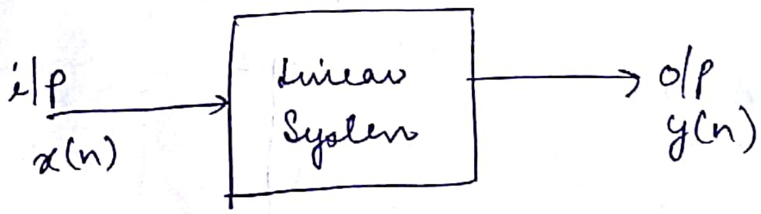
System Modelling $\rightarrow$ If we find a unique method to represent linear system which ~~gives a~~ is constant for all types of input then it is called system modelling.

Shift operation should exist.

Converting a system into numbers.

$$\delta(n) = 1 \quad n=0$$
$$= 0 \quad \text{otherwise}$$



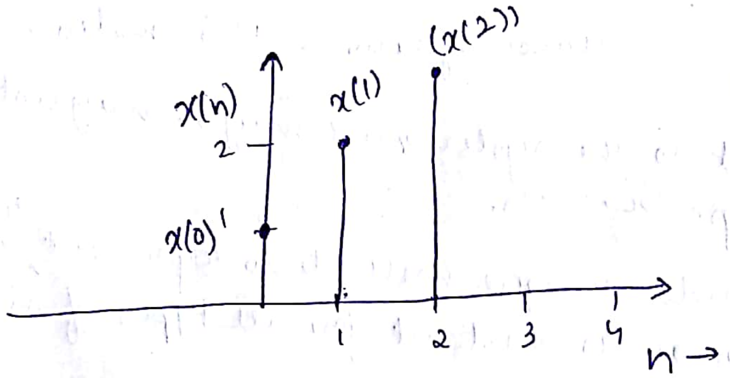


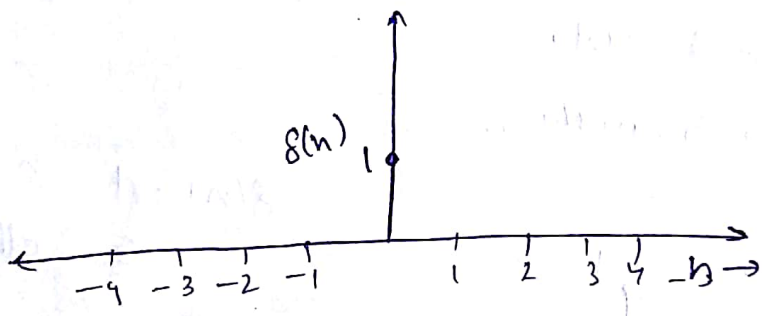
We want express input into some basic signals.

Ex:

$$\delta(n) = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$$

$$x(n) = \{\overset{(1)}{x(0)}, \overset{(2)}{x(1)}, \overset{(3)}{x(2)}\}$$

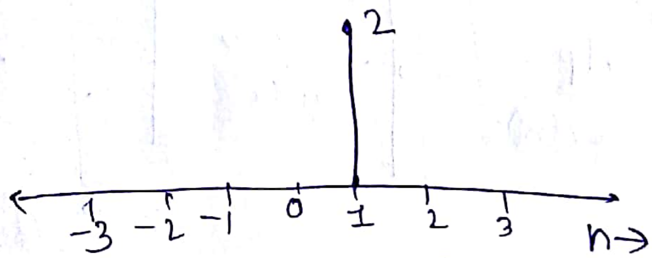




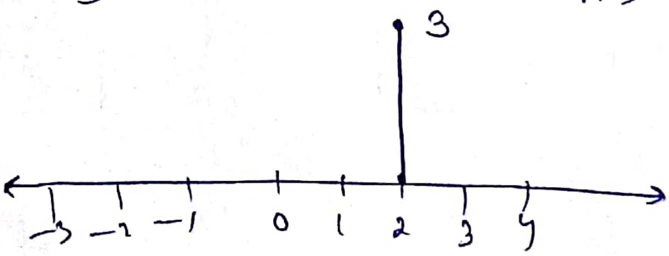
Split x(n) into three sub sequences.

1. $\delta(n)$

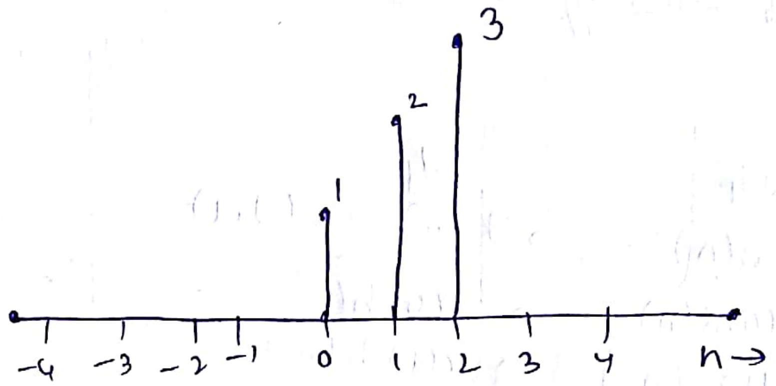
2. $\delta(n-1)$



3. $\delta(n-2)$



$1.\delta(n) + 2.\delta(n-1) + 3.\delta(n-2) = x(n)$



$x(n) = x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2)$

↳ weighted sum of delta sequence and its shifts

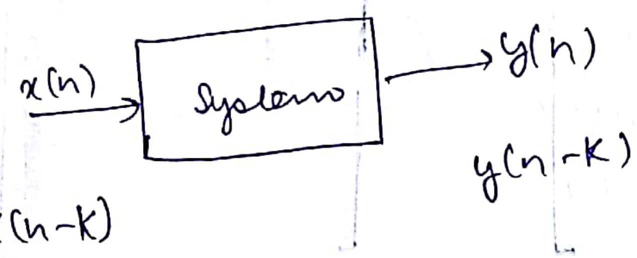
$$\boxed{x(n) = \sum_{k=0}^{k=\infty} x(K)\,\delta(n-K)}$$

$$\boxed{x(n) = \sum_{k=-\infty}^{\infty} x(K)\,\delta(n-K)}$$

Any input signal is expressible in terms of delta sequence and its shift in range [$-\infty,\infty$).

For a linear system, the system response will ~~case~~ be same for both individual parts of inputs and entire sequence of input i.e it satisfies superposition and homogenity.

Time invariant → Any system which does not vary with time, For ex if we delay the input, output is also delayed.



Combination of Linearity + Time invariance → LTI systems
Linear Time Invariant

I/P $\delta(n) \longrightarrow$ O/P $h(n)$ (Let)

$x(0)\delta(n) \longrightarrow x(0)h(n)$

$x(1)\delta(n-1) \longrightarrow x(1)h(n-1)$

$\vdots \qquad \vdots$

$$\boxed{y(n) = \sum_{K=-\infty}^{\infty} x(K)\,h(n-K)}$$

Advantage: So, We need to find h(n) once, then we can find all the outputs. i.e the system is unique.

To see this, we give specific input → δ(n). height is 1

$\delta(n) \to h(n)$ delta or impulse (discrete) response.
↳ response

dirac (continuous) response
height is undefined

impulse response → Linear Time Invariant represent model

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \begin{bmatrix} x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$x(n) \longrightarrow \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \longrightarrow y(n)$$

where $y(n) = \sum_{n=-\infty}^{-\infty} x(k)\,h(n-k)$

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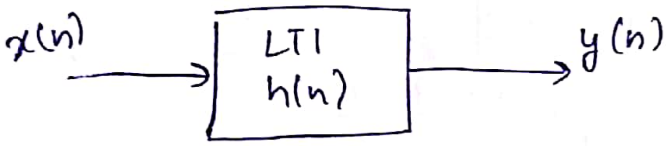
If $h(n) = (h(0))$ → infinite possibilities



$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$ → linear convolution sum

Causality of system

↳ depends on present and past values, not on future.

Real time → instantaneous
for a given input we get a output.

For a causal systems.

$$y(n) = \sum_{k=0}^{\infty} x(k)\, h(n-k)$$

Prove : $y(n) = \sum_{k=0}^{\infty} x(k)\, h(n-k)$

$$= \sum_{k=0}^{\infty} h(k)\, x(n-k)$$

$$y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)$$

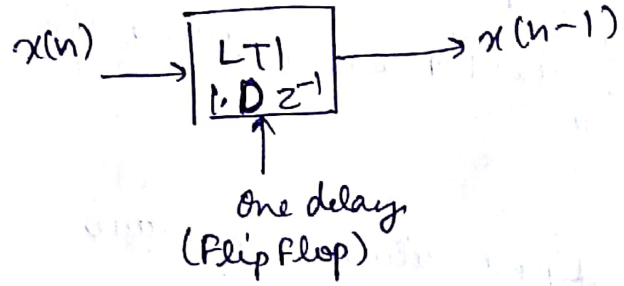
$$= \underbrace{h(0)x(n) + h(1)x(n-1) + h(2)x(n-2)}$$ ↳ weighted sum of input and its shifts

$h(n) = (h(0), h(1), h(2))$

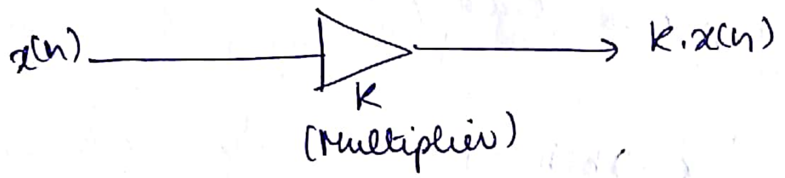
Impulse response is scaling the different shifts of input

Hardware requirement: ① Shift operator
② Scale operator
③ Adder/subtractor

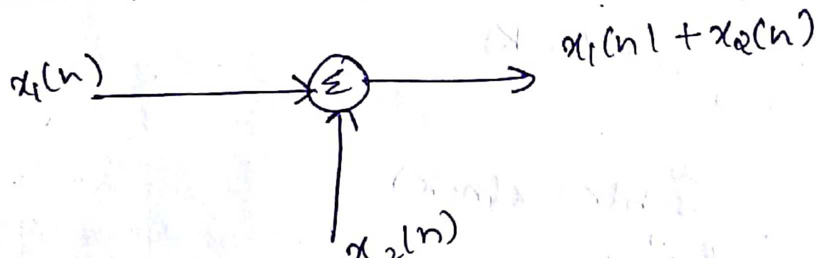
Shift operator



Scale operator:



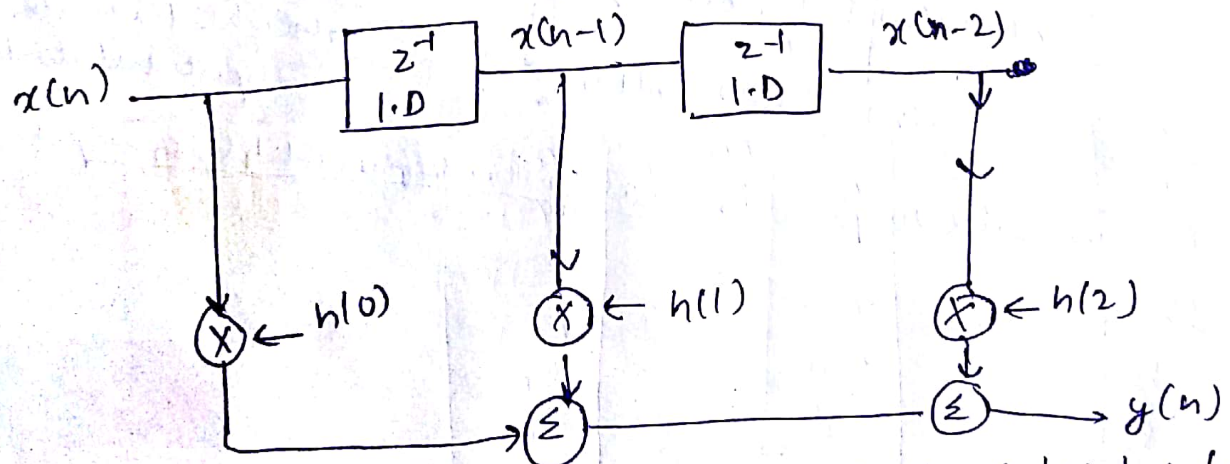
Adder / Subtractor:



(FIR)

Q. $y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2)$

$h(0)x(n) + h(1)x(n-1) + h(2)x(n-2)$



FIR Systems $\underbrace{(h'(0), h'(1), h'(2))}_{\text{characterisation}}$

If $h'(n) = (h'(0), h'(1), h'(2))$

we just need to change the three multipliers

Any LTI system can be of two types

① Finite length Impulse Response (FIR System)

② Infinite length Impulse Response (IIR System)

Transient output (the outputs we see before the final output)

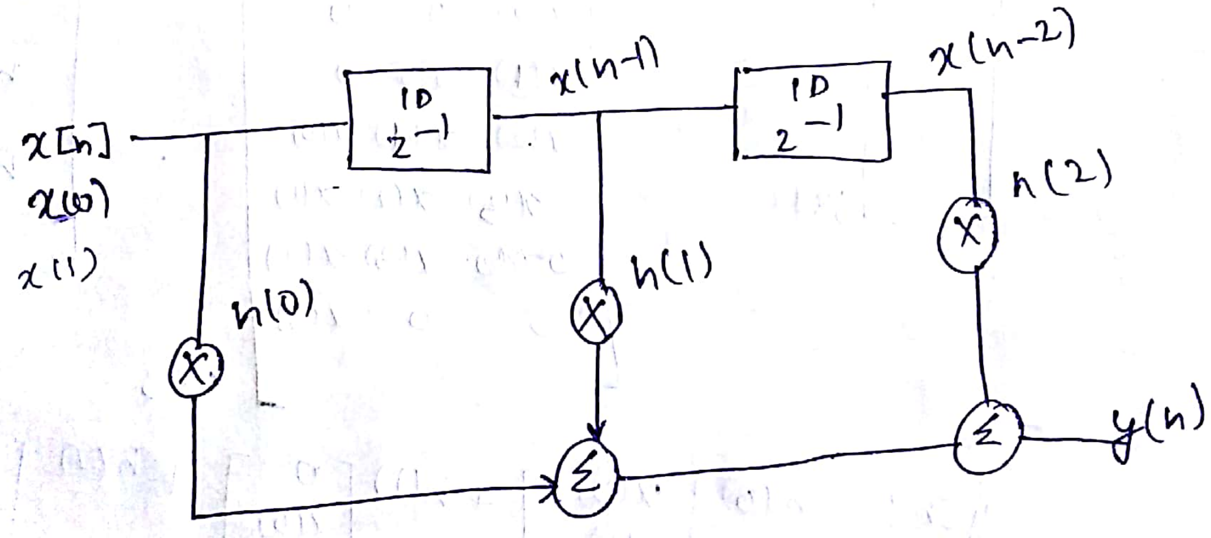
For Ex → ①

Till the three inputs are given, we won't be able to see the complete output.

Finite Impulse Response :

FIR $h(n) = [h(0), h(1), h(2)]$

IIR $x(n) = [x(0), x(1), x(2), x(3)]$



Input or Transient
$$y(0) = x(0)h(0)$$
$$y(1) = x(1)h(0) + x(0)h(1) + 0$$

No. of delays = on transient lengths

Steady State $\left\{ \begin{array}{l} y(2) = x(2)h(0) + x(1)h(1) + x(0)h(2) \\ y(3) = x(3)h(0) + x(2)h(1) + x(1)h(2) \end{array} \right.$

Output

Off Transient $y(4) = x(3)h(1) + x(2)h(2)$ Off transient length (2) = No. of delays

$y(5) = 0 + x(3)h(2)$

No. of delays = length of Impulse Response –

$y(6) = 0$

1. Output length of linear convolution sum is equal to input length + impulse response length – 1.
2. Identifying input on Transient output, steady state output, Input off transient output..

$$y[n] = \sum_{k=0}^{\infty} h(k)x(n-k)$$

$$= h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \alpha x(n-3)$$

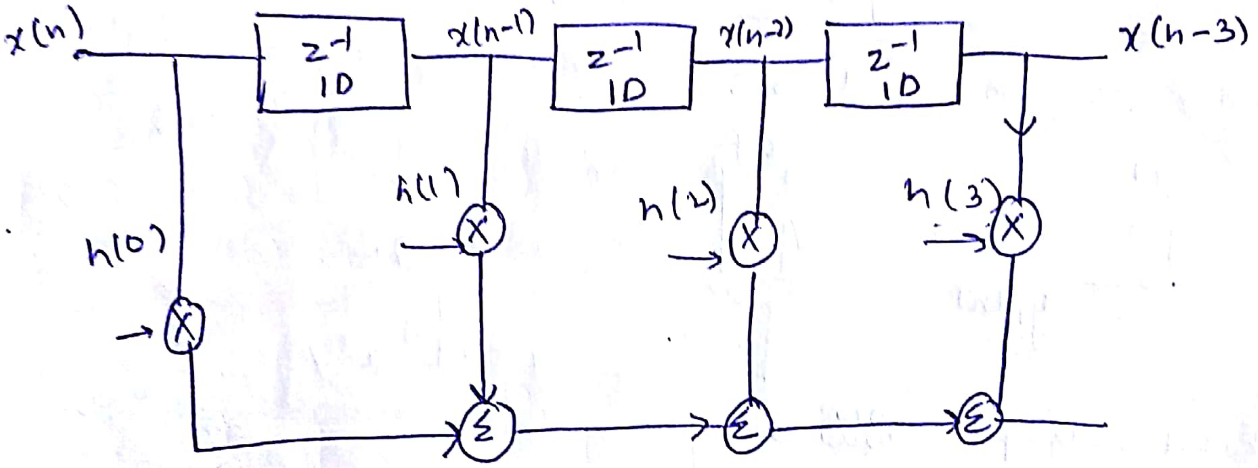
$$\underset{6\times1}{\underset{B}{\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}}} = \underset{6\times3}{\underset{A}{\begin{array}{ccc} x(n) & x(n-1) & x(n-2) \\ \hline \end{array}\begin{bmatrix} x(0) & 0 & 0 \\ x(1) & x(0) & 0 \\ x(2) & x(1) & x(0) \\ x(3) & x(2) & x(1) \\ 0 & x(3) & x(2) \\ 0 & 0 & x(3) \end{bmatrix}}} \underset{3\times1}{\underset{X}{\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}}}$$

$$y[n] = h(0)\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ 0 \\ 0 \end{bmatrix} + h(1)\begin{bmatrix} 0 \\ x(0) \\ x(1) \\ x(2) \\ x(3) \\ 0 \end{bmatrix} + h(2)\begin{bmatrix} 0 \\ 0 \\ x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \end{bmatrix} = \begin{bmatrix} h(0) & 0 & 0 & 0 \\ h(1) & h(0) & 0 & 0 \\ h(2) & h(1) & h(0) & 0 \\ 0 & h(2) & h(1) & h(0) \\ 0 & 0 & h(2) & h(1) \\ 0 & 0 & 0 & h(2) \end{bmatrix}_{6\times 4} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}_{4\times 1}$$

$h(n) = (h(0), h(1), h(2), h(3))$ FIR

$x(n) = (x(0), x(1))$



$\boxed{\text{No steady state}}$ as no output has effect of all components. Every output has only two inputs and corresponding response.

Minimum input length required is $\boxed{\text{on transient } +1}$ so that we can get steady state.

$h(n) = (h(0), h(1), h(2), h(3))$

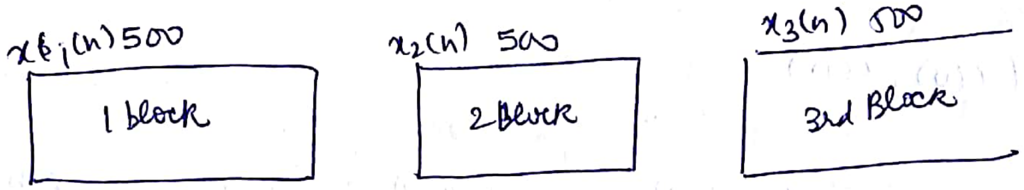
$x(n) = (x(0), x(1) \ldots \ldots \ldots)$

Real time processing → Output before completion of input

Off line processing / Stored → Output after completion of input

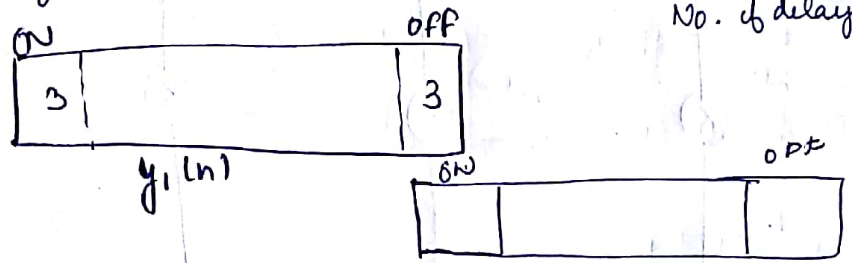
If we have infinite length input, divide it into blocks of length equal to tolerance limit

Let us say, we divide input into length of 500



Output length $500 + 4 - 1 = 503$ length of impulse response $= 4$

No. of delays $= 4 - 1 = 3$



First 3 samples → ON transient

last 3 samples – OFF transient

Since we forced input to stop after 500 samples there is an extra OFF transient for $(y_1(n))$ and ON transient for $(y_2(n))$.

OFF transient first term that goes away is $h(0)$

ON transient First term to come is $h(0)$

If we add these two we will not see any break in output and it will behave as a stored processing

This strategy is known as $\boxed{\text{overlap add method → Block Convolution}}$

Explore the other possibilities.

Block Convolutions.

$h(n) = (h(0), h(1), h(2)) = h(0)\delta(n) + h(1)\delta(n-1) + h(2)\delta(n-2)$

$x(n) = (x(0), x(1), x(2), x(3))$

$$y(n) = x(n) * h(n)$$

$$Y = \underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{x}$$

If $y(n)$ and $h(n)$ is given, finding $x(n)$ → <u>Source Identification Problem</u>
For ex. speech, image processing.

Given $y(n)$ $x(n)$, identify $h(n)$ → <u>System Identification Problem</u>

Given $x(n)$ $h(n)$, identify $y(n)$ → <u>Blind Convolution Problem</u>.

$$y(n) = \sum_{K=0}^{\infty} x(K) h(n-K) = \sum_{K=0}^{\infty} h(K) x(n-K)$$

$Y = HX$ → linear convolution

for above question

$$\underset{Y}{\begin{bmatrix} \quad \end{bmatrix}}_{6\times1} = \underset{H}{\begin{bmatrix} \quad \end{bmatrix}}_{6\times4} \underset{X}{\begin{bmatrix} \quad \end{bmatrix}}_{4\times1}$$

$$\begin{bmatrix} \quad \end{bmatrix}_{6\times1} = \underset{X}{\begin{bmatrix} \quad \end{bmatrix}}_{6\times3} \underset{H}{\begin{bmatrix} \quad \end{bmatrix}}_{3\times1}$$

Least Square Solution : (Linear Deconvolution) finding X from Y and H

$Y = HX$ multiplying by H^T 4×6 6×4

$H^TY = H^THX$

assuming H is a real number matrix

If H is a complex number matrix then multiply by $H^H = (H^*)^T$ ↳ (conjugate of H)^T

$$\underset{4\times6}{H^T}\,\underset{6\times1}{Y} = \underset{4\times6}{H^T}\,\underset{6\times4}{H}\,\underset{4\times1}{X}$$

$$(H^TY)_{4\times1} = (H^TH)_{4\times4}\,X_{4\times1}$$

$$(H^TH)^{-1}(H^TY) = (H^TH)^{-1}(H^TH)X$$

determinant of $|H^TH| \neq 0$

$$\boxed{\hat{X} = (H^TH)^{-1}H^TY}$$

Least Square solution

Now if x(n) y(n) are given,

$$\underset{6\times1}{Y} = \overset{X}{\begin{bmatrix} & \end{bmatrix}}_{6\times3}\overset{H}{\begin{bmatrix} \cdot \end{bmatrix}}_{3\times1}$$

$$(X^TY) = \overset{X^T}{\begin{bmatrix} & \end{bmatrix}}_{3\times6}\overset{X}{\begin{bmatrix} & \end{bmatrix}}_{6\times3}\overset{Y}{\begin{bmatrix} \end{bmatrix}}_{3\times1}$$

$$h = (X^TX)^{-1}(X^TY)$$

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Preambles → known information sent to the receiver.

Types of LTI System: finite Impulse Response (FIR)
Infinite Impulse Response (IIR)

Infinite Impulse <u>Respone</u> (IIR)

$h(n) = (h(0), \ldots\ldots)$ → gives infinite degree of freedom

→ infinite delays

→ No steady state

→ always in ON Transient State

if the coefficients of impulse response are not independent
For ex

$h(n) = (1, a, a^2, a^3 \ldots\ldots)$

→ GP should converge

→ only one delay required

independent terms of impulse response implies more degree of freedom

$h(n) = a^n \quad n \geq 0$ → causal system

$\ 0 \quad$ otherwise

Sequence can converge → to finite value
→ to infinite value (no use)

$|a|$ should be between 0 to 1

a can be 0
not 1

$0 \leq |a| < 1$

If $|a| = 0.9 \quad |a| = 0.1$

they both will converge to 0,

0.1 will Reach faster to 0 for less n

0.9 will reach 0 for more n

Impulse Response will change according to the value of a.

→ Such Representation gives variable finite impulse response.

→

$h_{effective}$

We can stop impulse response after it reaches 1% of its initial value,

$$a^{n_{eff}} = 0.01$$

→ it will help ~~xxx~~ us to find steady state by choosing appropriate value of n effective

2. $h(n) = a^n + b^n$ is the following sequence possible

both are convergent

convergence will depend on larger of a and b

→ a and b needs not be dependent.

→ 2 degree of freedom

→ we can take any sum of GP but it should be of finite length

→ Poles of a system

For ex: For $a^n + b^n$

Poles $= a, b$

dominated poles determines the length of system.

→ $h(n) = a^n \quad n \geq 0$

$= 0$

$$y(n) = a\,y(n-1) + x(n)$$

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$h(-1) = 0$

$h(n) = ah(n-1) + \delta(n)$

↳ Recursive Formula

$h(0) = 1$

$\delta(n) = 1 \quad n = 0$

$0 \quad$ otherwise

$h(1) = a + 0$

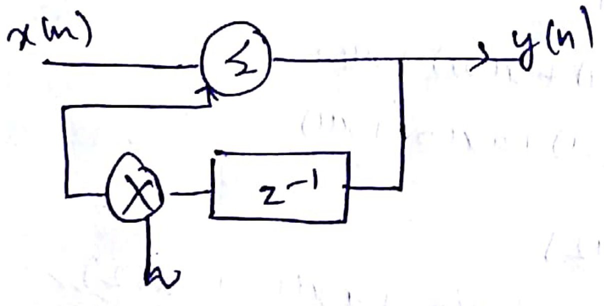
$h(2) = a(h(1)) + 0$

$h(2) = a^2$

If $x(n) = \delta(n)$

$y(n) = h(n)$

→ Choose appropriate value of a



effective length of h can be changed by changing a.

Hardware Implementation reduced to 1 delay

Steady state $= n^{eff} - 1$

IIR

→ dependent ~~sequence~~ series

→ convergent

→ recursive

Accumulation effect:

$|a| = 1$

↳ Creation of Accumulation (Overflow Condition)

due to

$y(n) = a\,y(n-1) + x(n)$ (Recursive Form)

recursions ⟶ Difference Equation ↳ (I st Order Constant Coefficient Linear Difference Equation)

→ as series is not convergent.

FIR → Non Recursive

IIR → Recursive Form

If $y(-1) \neq 0$

$$y(0) = a\,y(-1) + x(0)$$

$$y(1) = a\,y(0) + x(1)$$

$$= a(a y(-1) + x(0)) + x(1)$$

$$= a^2 y(-1) + a x(0) + x(1)$$

$$y(2) = a\,y(1) + x(2)$$

$$= a^3 y(-1) + a^2 x(0) + a x(1) + x(2)$$

$$y(n) = a^{n+1} y(-1) + a^n x(0) + a^{n-1} x(1) + \dots + x(n)$$

$$y(n) = \underbrace{a^{n+1} y(-1)}_{\uparrow} + \underbrace{\Sigma a^n x(0) + a^{n-1} x(1) + \dots + x(n)}_{\downarrow \text{ Inputs}}$$

Initial Condition of Difference Equation ↑ Zero Input Response
(Output even when input is zero)
(Natural Response of systems)

When initial conditions are zero, the response of input at that time is zero state response. (Forced system response)

If initial conditions are zero, we can characterise recursive response and nonrecursive form in a same way.

So in previous eqⁿ put $y(-1)=0$

$$y(n) = \sum a^k x(n-k)$$

$$= \sum_{0}^{\infty} h(k)\, x(n-k)$$

$$h(n) = a^k u(n)$$

Z - Transform

$$y(n) = ay(n-1) + x(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$z \rightarrow$ complex

Z has two variable
real imaginary

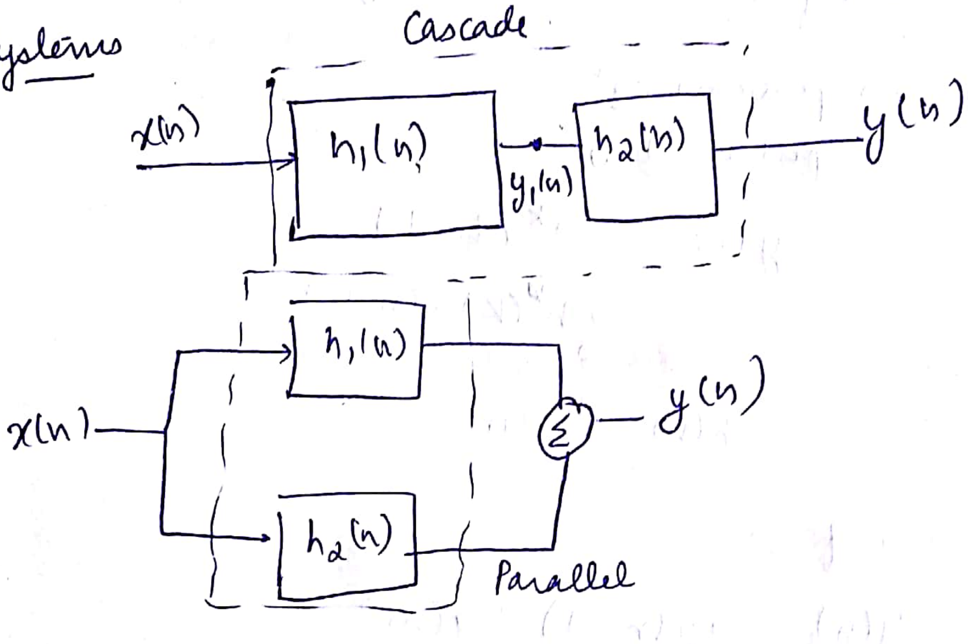
So $X(z) \rightarrow$ should be represented in 3-D graph
but to avoid that situation we draw two separate graphs
1. to represent magnitude
2. to represent phase.

$y = a^n + b^n$

$a\,y(n-1) + b\,y(n-2)$

Cascaded Systems

Parallel form



Effective Impulse response

$y_1(n) = x(n) * h_1(n)$

$y(n) = y_1(n) * h_2(n)$

$= (x(n) * h_1(n)) * h_2(n)$

$= x(n) * (h_1(n) * h_2(n))$

$= x(n) * h_3(n)$

$\boxed{h_3(n) = h_1(n) * h_2(n)}$

Effective Impulse Response

If systems are cascaded, their effective impulse responses will be convoluted.

$h_N(n) = h_1(n) * h_2(n) * \dots * h_N(n)$

Vice Versa, we can split inverse into n sums

For parallel form

$y_1(n) = x(n) \circledast h_1(n)$

$y_2(n) = x(n) * h_2(n)$

$$y(n) = y_1(n) + y_2(n)$$
$$= x(n) * h_1(n) + x(n) * h_2(n)$$
$$y(n) = x(n) * (h_1(n) + h_2(n))$$

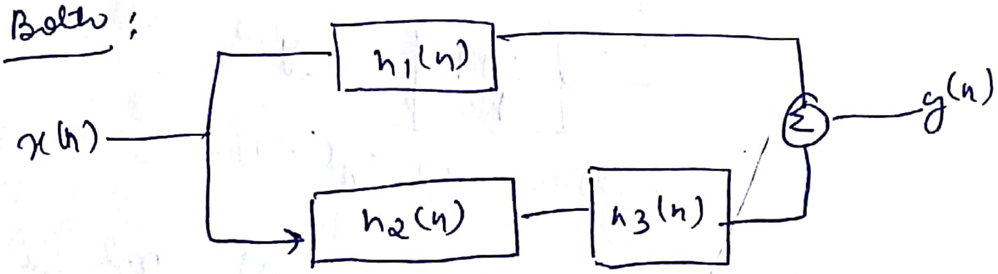
$$\boxed{h_3(n) = h_1(n) + h_2(n)}$$

Effective Impulse Response.

$$y(n) = x(n) * [h_1(n) + h_2(n) + \dots + h_N(n)]$$

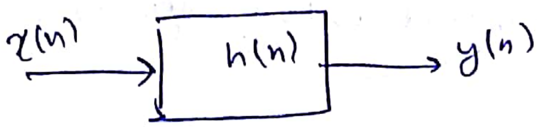
In parallel form, the impulse responses of different systems are added to form a single effective Impulse Response.

Combination of Both:



$$h_{eff} = h_1(n) + (h_2(n) * h_3(n))$$

Similarity Measure ↳ Correlation function



$$y(n) = h(n) * x(n)$$

Systems cannot tolerate ambiguity.

$$R_{xy}(l) = x(l) * y(-l)$$

↳ Linear convolution

Cross correlation

$$r_{xy}(l) = x(l) * y(-l)$$ ↳ Machine learning

Auto correlation ← $r_{xx}(l) = x(l) * x(-l)$

↳ extensively used in communication

Relation b/w $r_{yy}(l)$ $r_{xx}(l)$

Correlation → Auto Correlation function
Correlation → Cross correlation function

$$r_{xy}(l) = x(n) * y(-n)$$

$r_{xy}(l) = x(n) * y(-n)$ $n=l$

$r_{yx}(l) = y(n) * x(-n)$ $n=l$

$r_{xx}(l) = x(n) * x(-n)$ $n=l$

$r_{yy}(l) = y(n) * y(-n)$ $n=l$

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$r_{yx} = y(n) * x(-n) \quad n = l$

$r_{yx} = x(n) * h(n) * x(-n) \quad n = l$

$r_{yx}(l) = h(n) * x(n) * x(-n) \quad n = l$

$$\boxed{r_{yx}(l) = h(l) * r_{xx}(l)}$$

$r_{yy}(l) = y(n) * y(-n)$

$= x(n) * h(n) * (x(-n) * h(-n))$

$= \cancel{x * x}$

$= x(n) * x(-n) * h(n) * h(-n)$

$$\boxed{r_{yy}(l) = r_{xx}(l) * r_{hh}(l)} \rightarrow \text{Imp (used in Random process)}$$

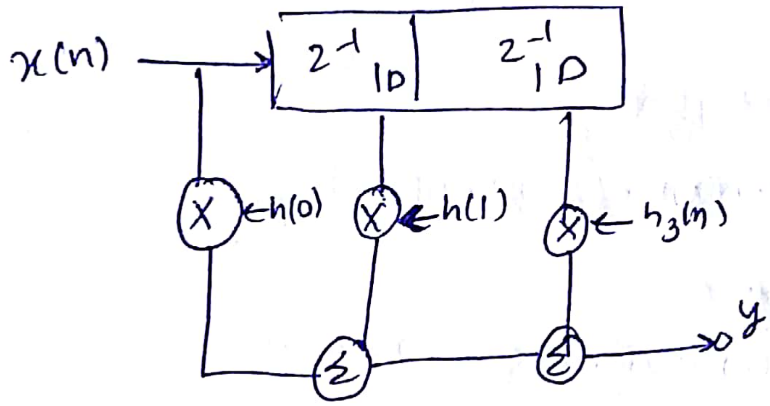
Normalised Auto correlation function $= \dfrac{r_{xx}(l)}{|r_{xx}(0)|}$

Normalised Cross correlation function

$$= \frac{r_{xy}(l)}{\sqrt{|r_{xy}(0) \cdot r_{yx}(0)|}}$$

This will ensure that function lies b/w -1 and 1.

$r_{xx}(0) \Rightarrow$ first element, source of energy of signal, Solutions of Linear Constant Coefficient Difference equation.



$$y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2)$$

↑

Second Order Non Recursive Linear constant coefficient Difference equation,

$$y = f(x(n), x(n-1), x(n-2))$$

$$y(n) = f(x(n), x(n-1) \ldots, y(n-1)\, y(n-2))$$

↳ Recursive

$$\boxed{\begin{array}{l} y(n) = a x(n-1) + x(n) \rightarrow \text{Non Recursive} \\ y(n) = a y(n-1) + x(n) \rightarrow \text{Recursive} \end{array}}$$

Z transform

$z = re^{j\theta}$ $\qquad r, \theta$ $\qquad s = \sigma + j\omega$

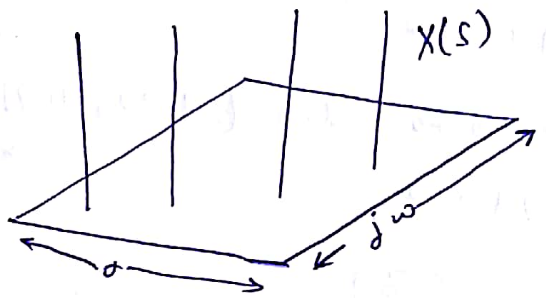
$-\infty \le \sigma \le \infty$

$-\infty \le \omega \le \infty$

$S = \sigma + j\omega$

$X(s) = X(\sigma + j\omega)$

→ Requires 3 dimension to represent



Cartesian Representation

We can have a mapping b/w the two as they are representing the same plane.

Domain of r and θ

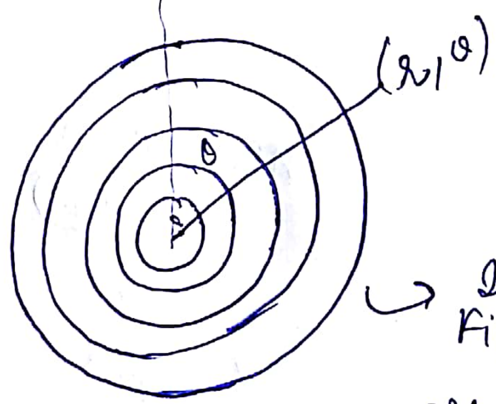
$0 \le r \le \infty$

$0 \le \theta \le 2\pi$

$-\pi \le \theta \le \pi$

$r = 0$

$\theta = 2\pi$



→ It cannot have fixed reference

→ If it is fixed it means its capabilities are limited

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$\rightarrow$ weighted sum of z^{-n}

$$x(n) \xleftrightarrow{Z} X(z)$$

$$x(n-1) \xleftrightarrow{Z} z^{-1} X(z)$$

$$x(n) * y(n) \longleftrightarrow X(z) Y(z)$$

Linear constant coefficient Difference eqⁿ $\rightarrow$ FIR (Recursive) / IIR (Non Recursive)

$$y(n) = ay(n-1) + x(n)$$

Z transform on both sides ($X(z)$, $Y(z)$ should exist)

$$Y(z) = az^{-1} Y(z) + X(z)$$

$$Y(z) - az^{-1} Y(z) = X(z)$$

$$Y(z) = \frac{X(z)}{(1-az^{-1})}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}} = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$$

Rational Polynomial

$$H(z) = \frac{B(z)}{A(z)}$$

$B(z) \rightarrow$ If we put roots of this polynomial over $H(z)$ will converge to zero

$A(z) \rightarrow$ If we put roots of this polynomial over $A(z)$ will go to infinity.

$$y(n) = a_1 y(n-1) + a_2 y(n-2) + x(n)$$

$$Y(z) = a_1 z^{-1} Y(z^{-1}) + a_2 z^{-2} Y(z^{-1}) + X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

Any recursive LCDE will result in ~~as~~ rational polynomial.

Non recursive

$$y(n) = a(x(n-1) + x(n)$$

$$\frac{Y(z)}{X(z)} = \frac{1 + a z^{-1}}{1}$$

For non recursive $A(z) = 1$
only one polynomial
so only one root

Roots of $B(z)$

$H(z) = 0 \Rightarrow$ Zeros of the Function $H(z)$

Roots of $A(z)$

$H(z) = \infty \longrightarrow$ Poles of the function $H(z)$

$$H(z) = \frac{1}{1 - a^{-1} z}$$

$H(z$

Z- ~~tr~~ transform

$$z = re^{j\theta}$$

$$h(n) = a^n u(n)$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$|az^{-1}| < 1$$

$$|z| > |a|$$

$$= \cancel{\frac{az^{-1}}{1-az^{-1}}}$$

$$= \frac{1}{1-az^{-1}}$$

Region of Convergence (ROC)

$$H(z) = \frac{B(z)}{A(z)}$$

System function

$$= \frac{(z-z_1)(z-z_2) \cdots (z-z_n)}{(z-z_1') \cdots (z-z_n')}$$

→ Splitting into partial fraction

→ finding roots.

$$H(z) = \frac{1}{(z-1)(z-2)}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)}$$

$$A(z-2) + B(z-1) = 1$$

$$z = 2 \qquad z = 1$$

$$B = 1 \qquad A = -1$$

$$= \frac{-1}{z-1} + \frac{1}{z-2}$$

$$= \frac{1}{1-z} + \frac{1}{2}\left(\frac{-1}{1-z/2}\right)$$

$a = 1$

$=$

$= \cancel{y(n) + -\frac{1}{2}\left(\frac{1}{2}\right)}$

eigen function of LTI system $\rightarrow x(n) = z^n$

$$x(n) \longrightarrow \underset{h(n)}{\boxed{LTI}} \longrightarrow y(n)$$

$$y(n) = x(n) * h(n)$$

$$= \sum_{K=-\infty}^{\infty} h(K) z^{n-K}$$

$$= \sum_{K=-\infty}^{\infty} h(K) z^{n} z^{-K}$$

$$= z^n \underbrace{\sum_{K=-\infty}^{\infty} h(K) z^{-K}}$$

let us say it converges to K

$=$

$$\underbrace{y(n) = x^n (K)}$$

Output is scalar multiple of input

If we change system response we will get different scaling factor,

can be compared with $\boxed{Ax = \lambda x}$

Transformation matrix

eigen value

eigenvector

$A_{3\times3}$ Matrix $\lambda_1\ \lambda_2\ \lambda_3$ are eigen values

$v_1\ v_2\ v_3$ are ~~corresponding~~ corresponding eigen vectors

$$Av_1 = \lambda_1 v_1$$

$$Av_2 = \lambda_2 v_2$$

$$Av_3 = \lambda_3 v_3$$

We can use eigen values and vectors to get A

$$A = \underbrace{\begin{bmatrix} v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}_{3\times3}}_{U} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_{3\times3}}_{\Lambda} \underbrace{\begin{bmatrix} - v_1 - \\ - v_2 - \\ - v_3 - \end{bmatrix}_{3\times3}}_{U^T}$$

$$A = U \Lambda U^T$$

We can get infinite A matrices because eigen vectors/vectors can have infinite values.

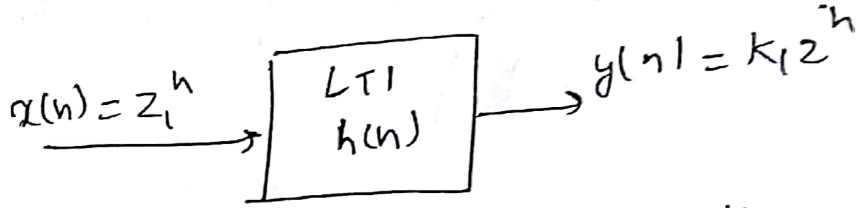
Three possibilities ⇒ eigen values 1. changing, eigen vector const; 2. constant, changing; 3. both changing

$$A = \underbrace{v_1 \lambda_1 v_1^T}_{3\times3} + \underbrace{v_2 \lambda_2 v_2^T}_{3\times3} + \underbrace{v_3 \lambda_3 v_3^T}_{3\times3}$$

Eigen Transformations

If we remove any one of eigen vector and eigenvalue, it will be impossible to get back A.

Completeness of Transformation → All eigen vectors and value should be there.



If $k_1 = \sum_{k=-\infty}^{\infty} h(k) z_1^{-k}$ is finite then

$y(n) = k_1 z^n$

We need all such k and z_1 whose sum is finite, to transform into A.

$$\left.\begin{matrix} - - - & k^2 & k_3 \\ & \vdots & \vdots \\ - - - & k & k_n \end{matrix}\right\}$$

If we do not have completeness, we cannot built that system.

$$\left.\begin{matrix} (z_1^n, z_2^n \cdots z_k^n) \\ (k_1, k_2 \cdots\cdots k_{z'}) \end{matrix}\right] \rightarrow \text{gives freedom of designing ed}$$

$z_1^n = r_1^n e^{j\theta_1}$

$|z_1| = 1$

$x(n) = 1 \cdot \underbrace{e^{j\theta_1 n}}_{z_1^n}$

$\sum_{k=-\infty}^{\infty} h(k) e^{-j\theta_1 k}$ → Finite → Infinite

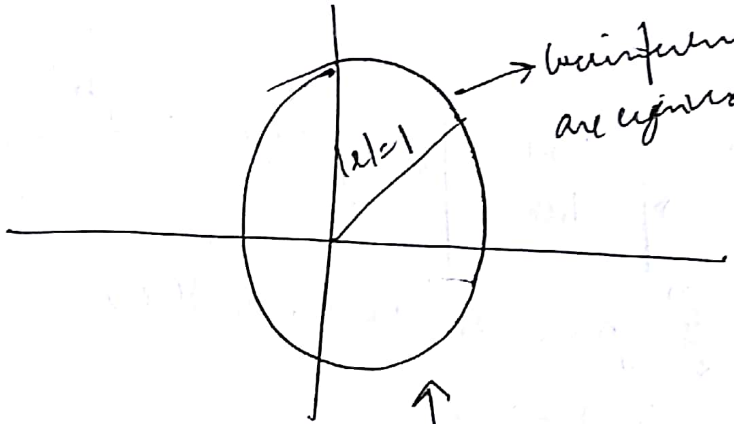
$y(n) = e^{j\theta_1 n} \underbrace{\sum_{k=-\infty}^{\infty} h(k) e^{-j\theta_1 k}}$

Laplace transform

Krelaintor

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

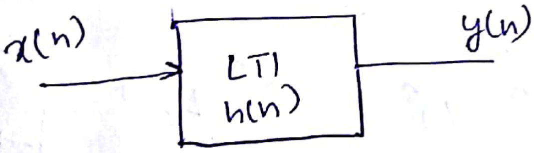
BIBO $\hookrightarrow$ Bounded Input Bounded Output



$\rightarrow$ circumference of circle are eigenvectors for x.

Frequency Response

$\rightarrow$ For any LTI System if $x(n) = z_1^n - K_1$ for all these we will get
$z_2^n - K_2$ scaled version of impulse response
⋮
$z_K^n - K_K \hookrightarrow$ degenerative type



$$z = re^{j\theta}$$

$$x = e^{j\theta_1 n}$$

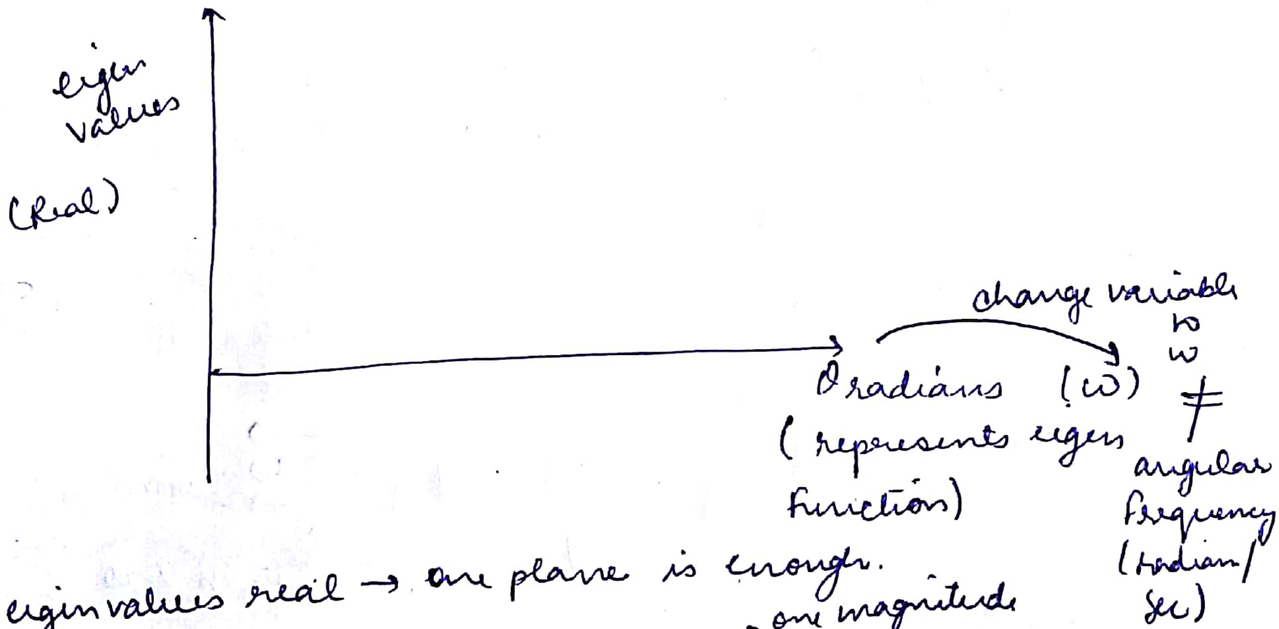
There are multiple sets of eigen functions possible of any LTI system. Ex $\rightarrow z^n$ and $e^{j\theta_1 n}$ both gives scaled version of LTI system.

If we get eigen values and eigen vectors we can build transformation matrix from here.

If we specific ROC of $z_1^n \ldots \ldots z_k^n \ldots$ sequence then we ~~cannot~~ can find the eigen vectors /functions.

$$x(n) = e^{j\theta_1 n}$$

They are not convergent
continous repetitive nature



If eigen values real → one plane is enough.

complex → two graphs → one magnitude
→ one phase

radian per second

$$-\infty \leq \Omega \leq \infty$$

Frequency Representation:

Discrete Time Fourier Transform

eigen values →

→ eigen Function

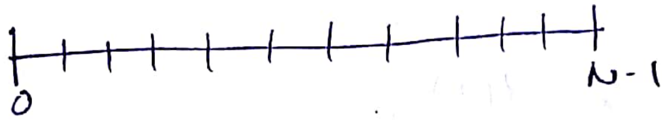
Limitation → Time Resolution as $-\infty \leq n \leq \infty$

they are non converging sequency with length 1.

Temporal Resolution does not exist.

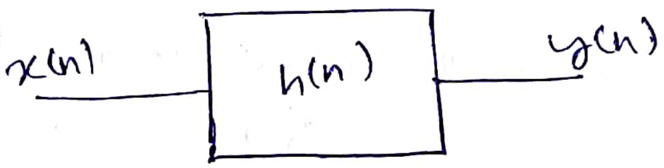
Finite length impulse

Sequence →



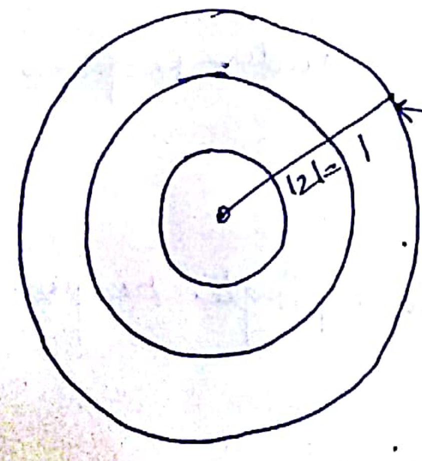
Simplest eigen functions for these
→ N dimensional representations
→ max only N functions

Frequency Response of an LTI System



$x(n) = e^{j\omega_i n}$

If we choose this input we will scaled functions of input as output.



This circle denotes all the values of ω (non decaying function)

Frequency Response is the set eigen vectors on $|z| = 1$

Example:

$$H(z) = 1 - z^{-1}$$

$$\frac{Y(z)}{X(z)} = 1 - z^{-1}$$

$$Y(z) = (1 - z^{-1}) X(z)$$
$$= X(z) - X(z) z^{-1}$$

* Inverse Z transform on both sides

$$\boxed{y(n) = x(n) - x(n-1)}$$

Non recursive as output is not a function of its delayed version.

ROC is entire Z plane ~~with exp~~ except $z=0$

We are interested on the values that lie of $|z|=1$ circle.
i.e Subset of ROC
for that we take $z = e^{j\omega}$.

$$H(z = e^{j\omega})$$

$$H(z = e^{j\omega}) = 1 - e^{-j\omega} = 2j e^{-j\omega/2} \frac{[e^{j\omega/2} - e^{-j\omega/2}]}{2j} = 2j e^{\omega/2 j} \sin \omega/2$$

$$|H(z = e^{j\omega})| = |2 \sin \omega/2|$$

↳ Magnitude Response

$$\angle H(z = e^{j\omega}) = \frac{\pi}{2} - \frac{\omega}{2}$$

↳ Phase | Angle response

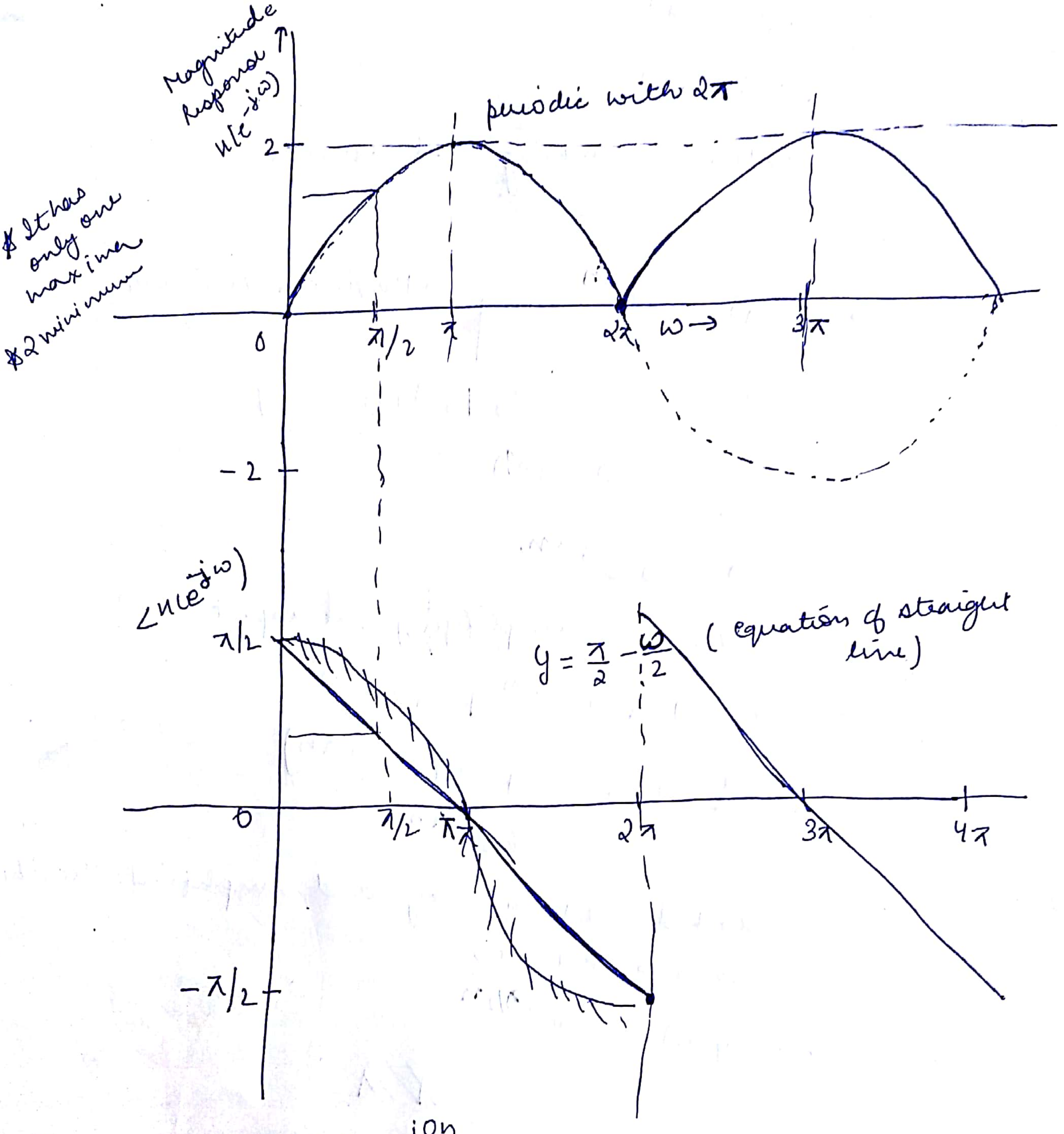
$$2j e^{\omega/2 j} \sin \omega/2$$

$$\cancel{2j} \cancel{\cos} = 1 - e^{-j\omega}$$

$$j = e^{j\pi/2}$$

$$2 e^{j\pi/2} e^{-j\omega/2} \sin \frac{\omega}{2}$$

Frequency Response = eigenvalue plot for $x(n) = e^{j\omega n}$



$x(n) = e^{j0n}$
↳ Sequence of 1's

$x(n) = \{\ldots 1, \ldots\}$

eigenvalue for $\omega = 0$ $= 0$

$y(n) = 0 \cdot e^{j\pi/2} = 0$

$\longrightarrow$ $H(n) = \delta(n) - \delta(n-1)$

$\longrightarrow$ $x(n) = e^{j0n}$

$y(n) = 0$

System is <u>blocking</u> that sequence

$\longrightarrow$ $x(n) = e^{j\pi n}$ → Maximum variation

$x(n) = \{ \ldots -1, 1, -1, 1, \ldots \}$

$y(n) = 2 e^{j0} \cdot e^{j\pi n}$

$y(n) = 2 e^{j\pi n}$

System is <u>amplifying</u> the input

$\longrightarrow$ $x_3(n) = 2e^{j0n} + 3e^{j\pi n}$

$y_3(n) = 0 + 3(2e^{j\pi n})$

$= 6 e^{j\pi n}$

Blocked one frequency and amplified the other

This is a <u>Filter</u>

$\longrightarrow$ $x_4(n) = e^{j\pi/2 n}$

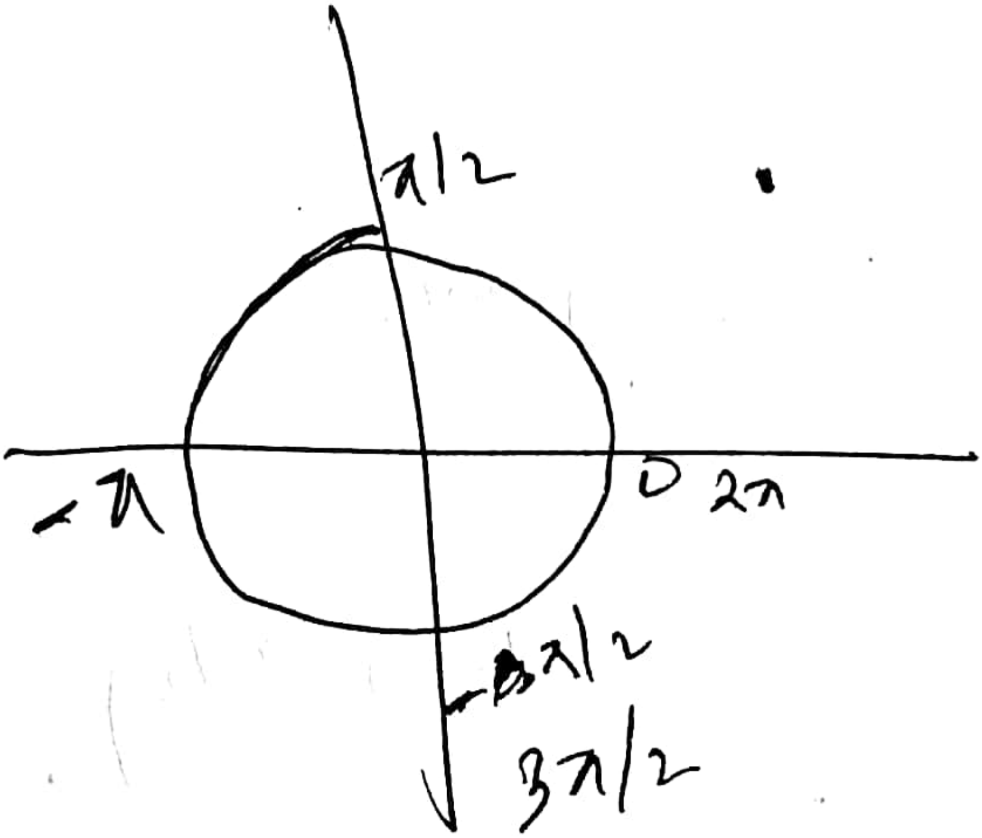
N/A

=

* No two eigen values are same.
* For every value of $0 \le \omega \le 2\pi$ the eigen value is different.
* eigen functions remain same only eigen values change.

DC BLOCKER

↑ Input is not varying


π/2
-π
0 2π
3π/2

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(Causal System)

$H(z) = 1 - z^{-m}$ m is an integer

$z = e^{j\omega}$

~~$H(e^{j\omega})$~~ =

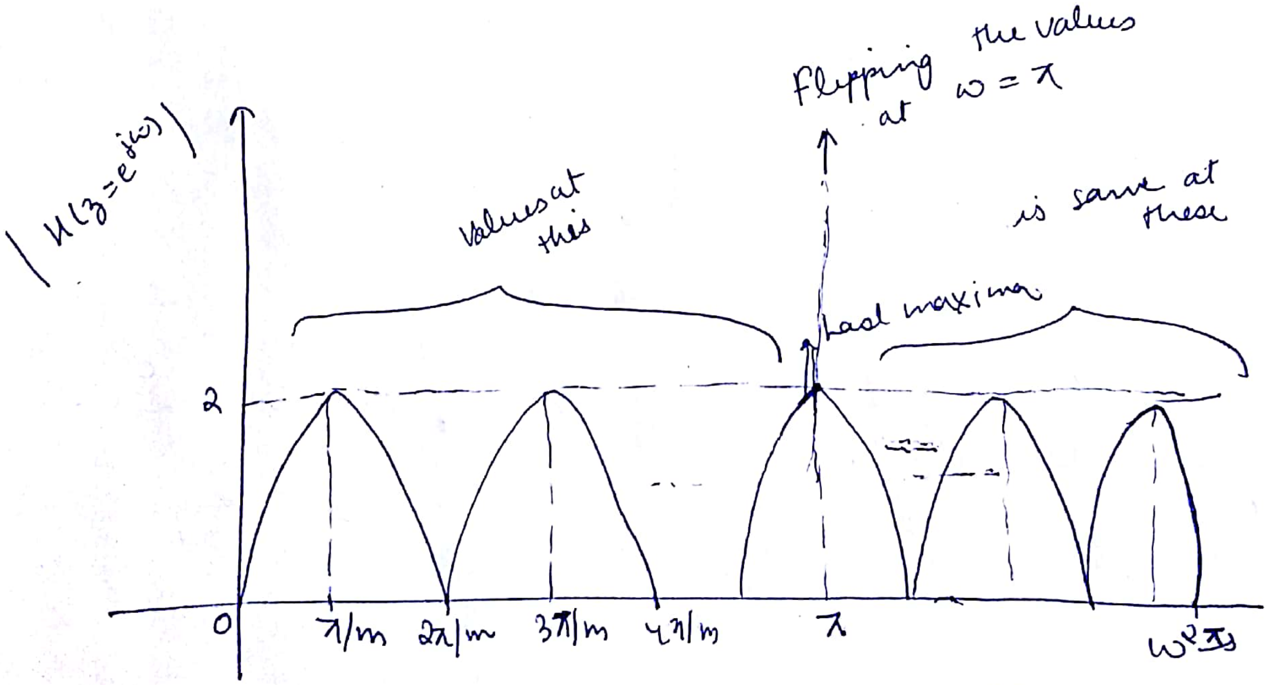
$$H(z = e^{j\omega}) = 1 - (e^{-j\omega})^m$$

$$= e^{-j\omega m/2} \frac{\left[e^{j\omega m/2} - e^{-j\omega m/2} \right]}{2j} \times 2j$$

$$= 2j e^{-j\omega m/2} \left(\sin \frac{\omega m}{2} \right)$$

$$= e^{j(\pi/2 - j\omega m/2)} \cdot 2 \sin \left(\frac{\omega m}{2} \right)$$

$$|H(z = e^{j\omega})| = |2 \sin(\omega m/2)|$$



for maximum equate $\frac{\omega m}{2}$ with $\pi/2$.

$$\frac{\omega m}{\cancel{2}} = \frac{\pi}{\cancel{2}}$$

$$\boxed{\omega = \frac{\pi}{m}}$$

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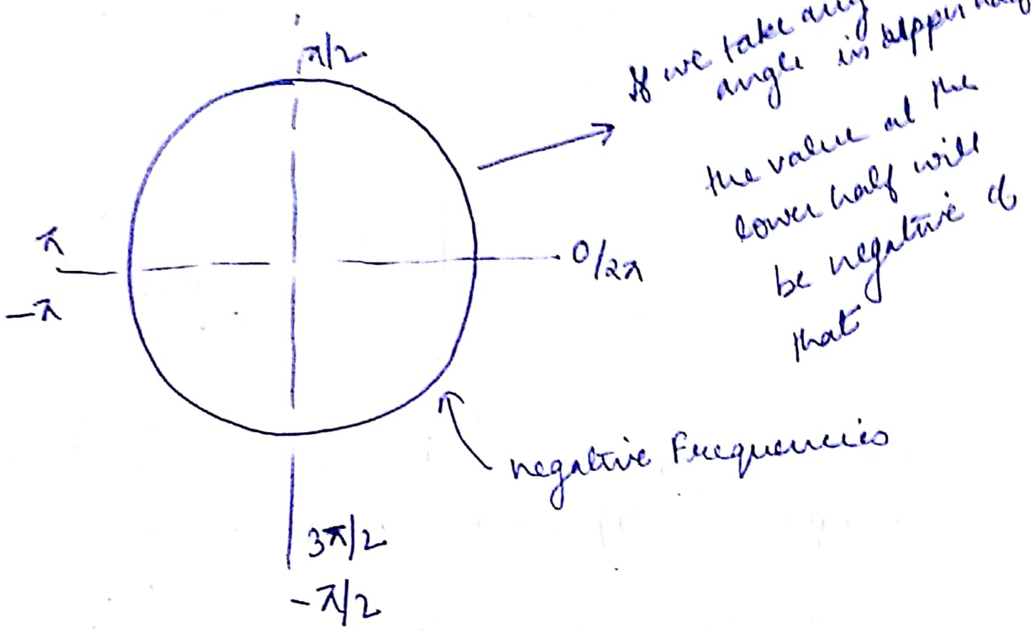
Minimum value

$$\frac{\omega m}{2} = 2\pi \quad \text{or} \quad \frac{\omega m}{2} = \pi$$

$$\omega = \frac{4\pi}{m} . \qquad \omega m = \frac{2\pi}{m}$$

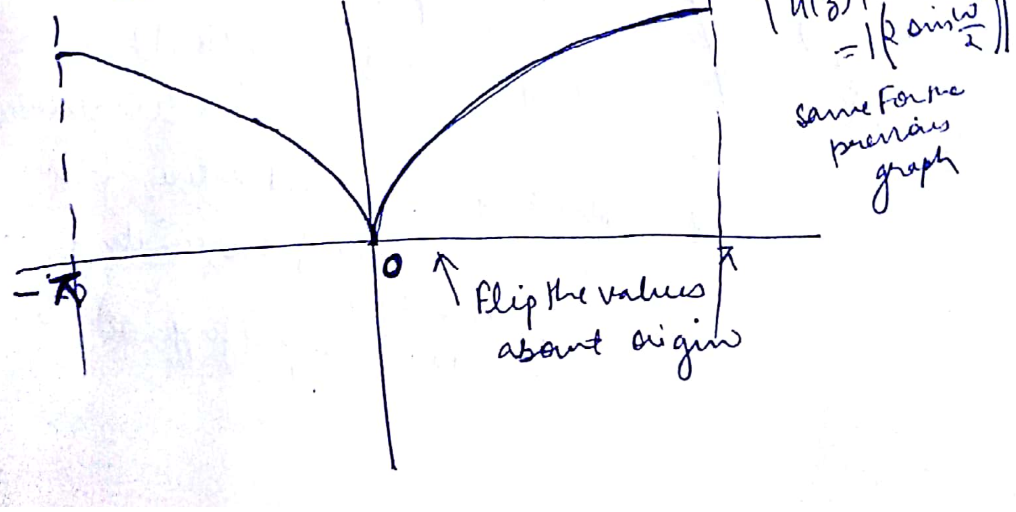


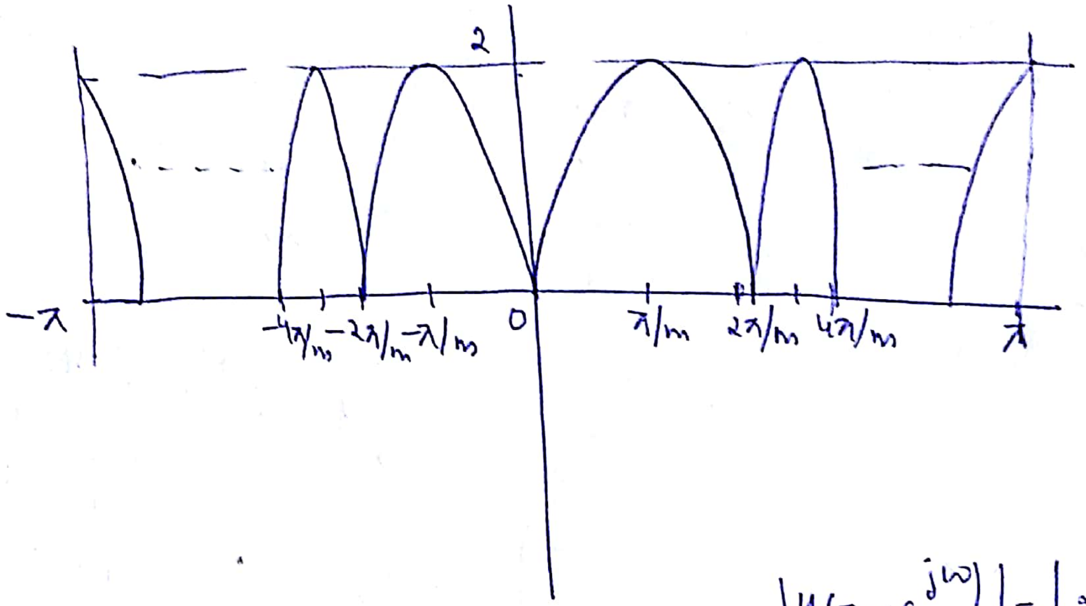
The maximum unique values available $= m/2$

Max. Frequency of Representation $= \pi$
$(2\pi = 0)$

Smallest Frequency $= 0$

Since we are writing magnitude value at $\frac{\pi}{2}$ will be equal $\frac{3\pi}{2}$





$\pi - \frac{\pi}{m}$

$\pi\left(\frac{m-1}{m}\right)$

$$\left|H(z=e^{j\omega})\right| = \left|2\sin\left(\omega m/2\right)\right|$$

In first example, we could only eliminate only one frequency (0). Now, in second we could eliminate

$0, 2\pi/m, 4\pi/m, 6\pi/m \ldots$

Same goes for max. values

$\pi/m, 3\pi/m, 5\pi/m \ldots$

Which means all even harmonics of π/m from 0 to π are eliminated

all odd harmonics of π/m are amplified from 0 to π.

$y(n) = x(n) - x(n-m) \rightarrow$ only ~~each~~ one substraction is required

last example we had

$y(n) = x(n) - x(n-1)$

↳ successive differences

Subtracting the current value from m^{th} sample

so this is capable of eliminating all even harmonics not only 0

same for amplifying odd harmonics

$H(z) = 1 - z^{-1} \longrightarrow$ coefficient is 1

↳ Infinite eigen value b/w 0 to π

but degree of freedom is 1

(only one 0)

$1 - z^{-m}$

↳ mth degree of polynomial

Find the magnitude response of the following system

$$H(z) = 1 + z^{-1} + z^{-2} + \dots + z^{-(m-1)}$$ where m is an integer

$$= \frac{1 - z^{-m}}{1 - z^{-1}}$$

Plot $|H(e^{j\omega})|$ Vs ω

$$H(z = e^{j\omega}) = e^{0j\omega} + e^{-j\omega} + e^{-2j\omega} + \dots + e^{-j\omega(m-1)}$$

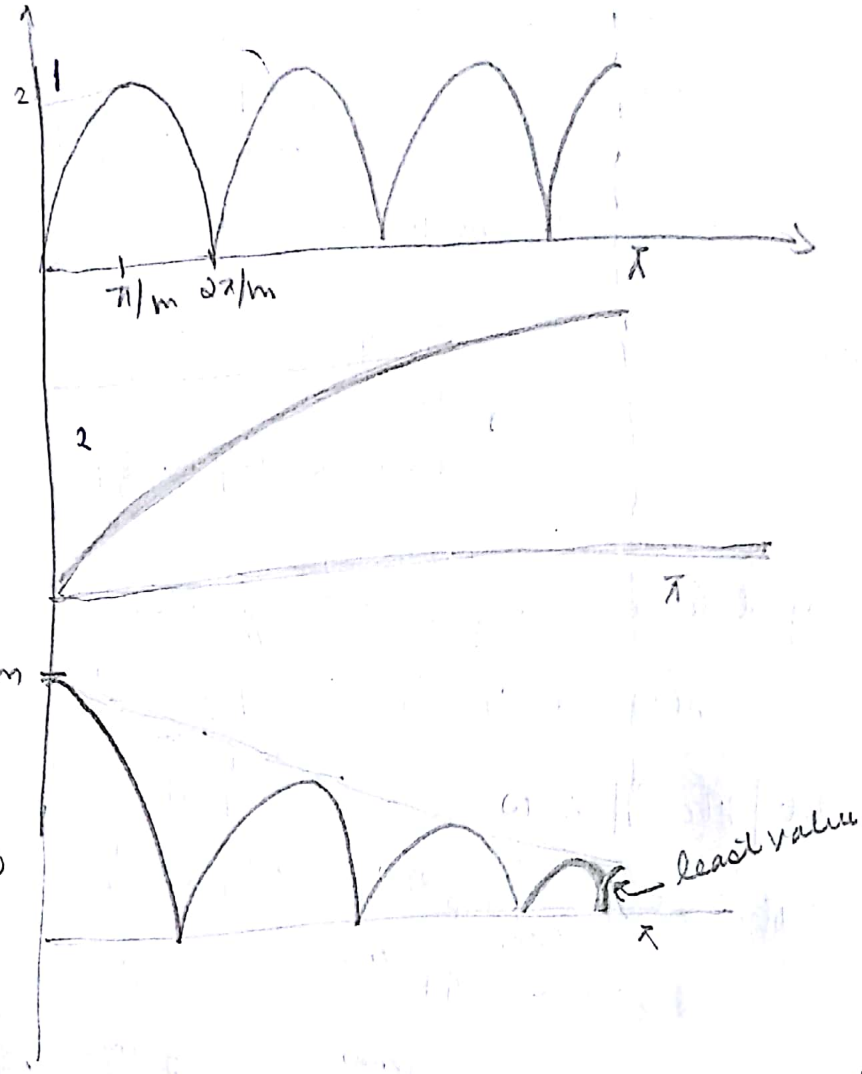
Infinite series G.P

$$H(z = e^{j\omega}) = \frac{1 - e^{-jm\omega}}{1 - e^{-j\omega}} = \frac{\cancel{2j} e^{-j\frac{\omega m}{2}} (\cancel{2} \sin(m\omega/2))}{\cancel{2j} e^{-j\omega/2} \cancel{2} \sin(\omega/2)}$$

$$|H(z = e^{j\omega})| = \left| \frac{\sin(m\omega/2)}{2\sin(\omega/2)} \right|$$

$$\angle H(e^{j\omega}) = $$

$$= -\frac{\omega}{2}(m-1)$$



Low pass Filter effect

For $\frac{①}{②}$ at 0 , the value of ① / ② is not defined

till ~~$\frac{\pi}{3}$ = $\frac{\omega}{2}$~~

$\sin\frac{\omega}{2} = \frac{1}{2}$

$\sin\frac{\omega}{2} = \sin\frac{\pi}{6}$

$\frac{\omega}{2} = \frac{\pi}{6}$

$\boxed{\omega = \frac{\pi}{3}}$

because $\omega = \pi/3$ increases
after $\omega = \pi/3$ decreases

0 to $\pi/3 \rightarrow$ amplifies
$\pi/3$ to $\pi \rightarrow$ decreases
} → creates a low pass filter

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Using L Hospital's Rule

$$|H(e^{j0})| = \frac{m\omega/2}{\omega/2} = m$$

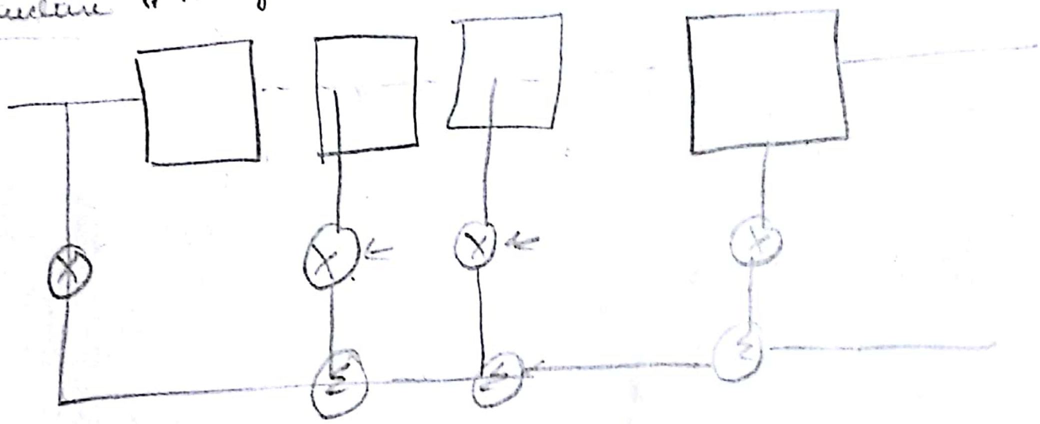
$$Y(z) = (1+z^{-1}+z^{-2}+ \cdots + z^{-(m-1)})X(z)$$

$$y(n) = x(n) + x(n-1) + \cdots + x(n-m+1)$$

We are picking up all the previous m samples to get the output

FIR

FIR structure (FIR) System



Till $(m-1)^{th}$ → On transient

m the sample – steady state

Moving sum filter → Frequency Domain

↓

Low Pass Filter

$$H(z) = 1 + z^{-1} + z^{-2} + \cdots + z^{-(m-1)}$$

$$= \frac{1-z^{-m}}{1-z^{-1}}$$

$$\frac{Y(z)}{X(z)} = \frac{1-z^{-m}}{1-z^{-1}} = Y(z)(1-z^{-1}) = X(z)(1-z^{-m})$$

$$Y(z)(1-z^{-1}) = (1-z^{-m})X(z)$$

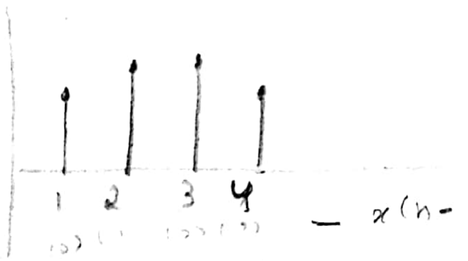
$$y(n) - y(n-1) = x(n) - x(n-m)$$

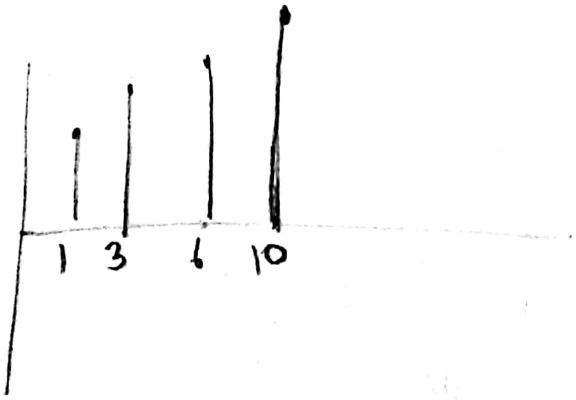
$$\boxed{y(n) = y(n-1) + x(n) - x(n-m)}$$ → Code for program

↓

IIR System

An FIR system is expressible in IIR system





We represented all the points from 0 to 2π in terms of polynomial or we took a polynomial and represented it within an finite range of 0 to π. (the graph repeats after this interval)

$$|H(e^{j\omega})| = |1 - e^{-j\omega}| \rightarrow \text{first degree}$$

$$H(e^{j\omega}) = 1 - e^{-j\omega m} \rightarrow m\text{th degree polynomial}$$

We can represent an uncountable set of number (polynomials) by countable set using Frequency representation.

Polynomial can be represented using → coefficients, → roots

$$H(e^{j\omega}) = 1 - e^{-j\omega m} \quad (m+1) \text{ coefficients}$$

different coefficients

coefficients = 1, (m-1) zeros, -1

$$H(e^{j\omega}) = 1 + e^{-j\omega} + e^{-2j\omega} + \dots + e^{-j(m-1)\omega}$$

$\hookrightarrow$ non zero coefficients

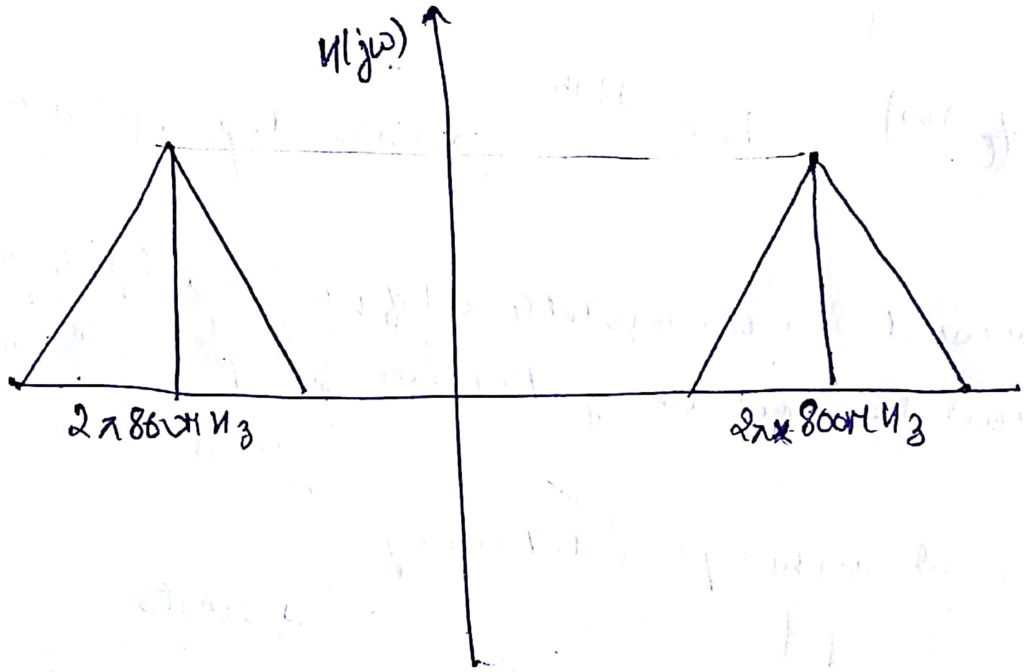
Eigenvalues → $H(e^{j\omega})$

$\hookrightarrow$ Alternative characterisation other than impulse response $h(n)$.

$H(z)$ → system function

Certain systems used in real life have representations in transform domain.

If we want to transmit at a carrier frequency of 800MHz then maximum should be at the Frequency not at zero,

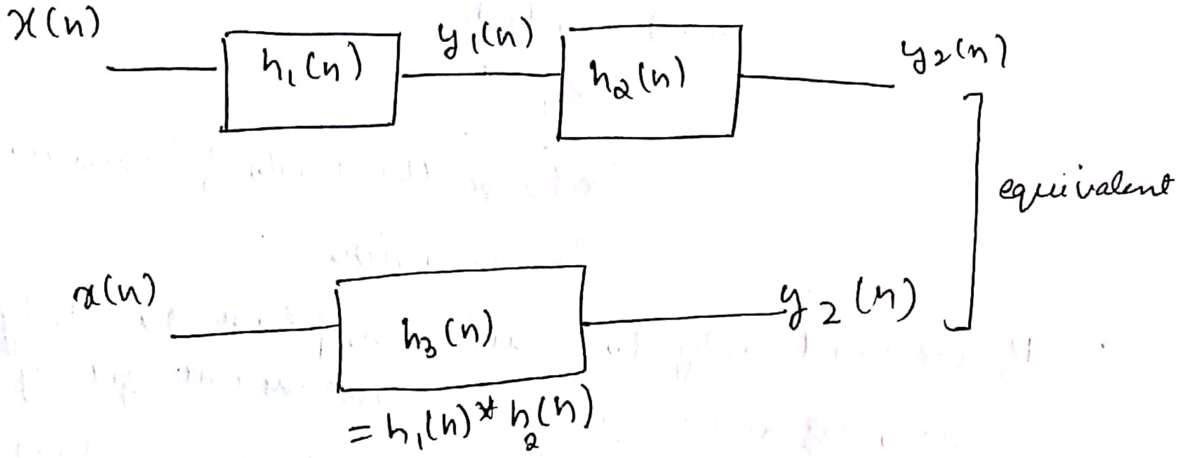


degree of polynomial = roots of the system,
Build a System for the above with 10 degree polynomial
↳ (10+1) coefficients
10 roots

Disadvantages of taking 100 degree (large degree polynomial) polynomial
→ more delay ~~because~~ ~~dec~~
transient time increases

Two ways to represent → polynomials

→ sample function



Is it possible ~~(foobar)~~ to get $y_2(n)$ in some of form $\tilde{x}(n)$

$$y_2(n) = \tilde{x}(n)$$

$h_2(n)$ should nullify the effect of $h_1(n)$

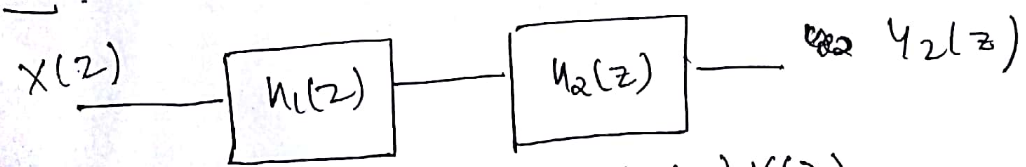
which means $h_3(n)$ should be some scaled function of del function.

This is called invertibility.

We can deconvolute the output to get the input back.

For this we should define a new impulse response that nullifies the effect of actual impulse response.

In Z Domain:



$$Y_2'(z) = H_1(z) H_2(z) X(z)$$

$$Y_2(e^{j\omega}) = H_1(e^{j\omega}) H_2(e^{j\omega}) X(e^{j\omega})$$

To get $Y_2(z) = X(z)$

$H_1(z) \cdot H_2(z) = 1$

$$\frac{B(z)}{A(z)} \cdot \underbrace{\frac{A(z)}{B(z)}}_{} = 1$$

Exchange the roots of numerator and denominator

If we need only to maintain phase or only magnitude then we use this → we cannot get by least square method

$$Y_2(e^{j\omega}) = H_1(e^{j\omega})\, H_2(e^{j\omega}) \cdot X(e^{j\omega})$$

For magnitude $\quad |H_1(e^{j\omega}) \cdot H_2(e^{j\omega})| = 1$

For phase $\quad \angle H_1 e^{j\omega} + \angle H_2 e^{j\omega} = 1$

↳ used in processing audio.